Electronic Journal of Qualitative Theory of Differential Equations
2012, No. **19**, 1-9; **http://www.math.u-szeged.hu/ejqtde/**

COMPACTNESS METHODS FOR HÖLDER ESTIMATES OF CERTAIN DEGENERATE ELLIPTIC EQUATIONS

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Abstract. In this paper we obtain the interior $C^{1,\alpha}$ regularity of the quasilinear elliptic equations of divergence form. Our basic tools are the elementary local L^∞ estimates and weak Harnack inequality for second-order linear elliptic equations, and the compactness method.

1. Introduction

In this paper we consider the following nonlinear elliptic problem

$$\operatorname{div}\left(g\left(|\nabla u|^2\right)\nabla u\right)=0 \quad \text{in } \Omega. \tag{1.1}$$

Here $g\in C^1([0,\infty))$ satisfies the following ellipticity condition

$$K^{-1}(Q+s)^{\frac{p}{2}-1}\le g(Q)+2g'(Q)Q\le K(Q+s)^{\frac{p}{2}-1}, \tag{1.2}$$

for $s\ge 0$ and $1<p<\infty$. In fact, condition (1.2) implies the following condition for a possibly larger constant K

$$K^{-1}(Q+s)^{\frac{p}{2}-1}\le g(Q)+2g'(Q)Q\le K(Q+s)^{\frac{p}{2}-1} \tag{1.3}$$

$$K^{-1}(Q+s)^{\frac{p}{2}-1}\le g(Q)\le K(Q+s)^{\frac{p}{2}-1} \tag{1.4}$$

$$|g'(Q)Q|\le K(Q+s)^{\frac{p}{2}-1}. \tag{1.5}$$

Especially when $g(x)=x^{\frac{p-2}{2}}$, (1.1) is reduced to

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)=0 \quad \text{in } \Omega, \tag{1.6}$$

which can be derived from the variational problem

$$\Phi(u)=\min_{v|_{\partial\Omega}=g}\Phi(v)=:\min_{v|_{\partial\Omega}=g}\int_\Omega|\nabla v|^p\,dx.$$

As usual, the solutions of (1.1) are taken in a weak sense. We now state the definition of weak solutions.

Definition 1.1. *A function $u\in W^{1,p}_{loc}(\Omega)$ is a local weak solution of (1.1) if for any $\varphi\in W^{1,p}_0(\Omega)$ we have*

$$\int_\Omega g\left(|\nabla u|^2\right)\nabla u\cdot\nabla\varphi\,dx=0.$$

2000 *Mathematics Subject Classification.* 35J60; 35J70.

Key words and phrases. Hölder estimates; Degenerate; Elliptic; Compactness Method.

*Corresponding author. This work is supported in part by the NSFC (11001165, 11101324), Research Fund for the Doctoral Program of Higher Education of China (20093108120003) and Shanghai Leading Academic Discipline Project (J50101).

Evans [6] have shown that ∇u is local Hölder continuous for weak solutions of (1.6) for $p \geq 2$ and then Lewis [9] extended the corresponding result to the case that $1 < p < \infty$. Moreover, Uhlenbeck [10] obtained the interior $C^{1,\alpha}$ regularity estimates for weak solutions of (1.1) with condition (1.2) and

$$|\rho'(Q_1)Q_1 - \rho'(Q_2)Q_2| \leq K\left(Q_1 + Q_2 + s\right)^{p/2-1-\beta}\left(Q_1 - Q_2\right)^{\beta}$$

for $s \geq 0$, $\beta > 0$ and $p \geq 2$, and DiBenedetto [3] considered the more general equations. Moreover, Wang [12] used compactness methods to give a quick proof of the interior $C^{1,\alpha}$ regularity for weak solutions of (1.6) for $1 < p < \infty$. Recently, Duzaar and Mingione [4,5] proved local Lipschitz regularity of the gradient for weak solutions of (1.1) for $1 < p < \infty$ and the more general equations. In this paper we will prove the interior $C^{1,\alpha}$ regularity for weak solutions of (1.1) with condition (1.2) by a compactness method, which is introduced by the authors (see [1, 11, 12, 13]). Our basic tools are the elementary local L^∞ estimates and weak Harnack inequality for second-order linear elliptic equations, and the compactness method.

The essence of $C^{1,\alpha}$ regularity of the solution is that the solution is almost a linear function. Actually, we can show that the difference between the solution and a linear function is like $|x|^{1+\alpha}$. Moreover, we can use the same method to prove $C^{k,\alpha}$ estimates for the solution if we replace the linear function by the k-th order polynomial function.

Definition 1.2. (1) *We call $u \in C_p^\alpha$ at the point $x = 0$ for $1 < p < \infty$ and $0 < \alpha < 1$ if*

$$[u]_{C_p^\alpha(0)} = \sup_{0<r\leq 1} \frac{1}{r^\alpha}\left(\fint_{B_r} |u - \overline{u}_{B_r}|^p \ dx\right)^{\frac{1}{p}} < \infty,$$

where $\overline{u}_{B_r} = \frac{1}{|B_r|}\int_{B_r} u \ dx$.

(2) *We call $u \in C_p^{1,\alpha}$ at the point $x = 0$ for $1 < p < \infty$ if there is a linear function $L(x) = Ax + B$ such that*

$$[u]_{C_p^{1,\alpha}(0)} = \sup_{0<r\leq 1} \frac{1}{r^{1+\alpha}}\left(\fint_{B_r} |u - L|^p \ dx\right)^{\frac{1}{p}} < \infty.$$

Now let us state the main result of this work.

Theorem 1.3. *If $u \in W_{loc}^{1,p}(B_1)$ is a weak solution of (1.1) with condition (1.2), then $u \in C_p^{1+\alpha}(0)$ for some $\alpha \in (0, 1)$.*

Remark 1.4. *If $u \in C_p^{1+\alpha}(0)$, then by Theorem 1.3, page 72 in [7], u is locally $C^{1,\alpha}$ in the classical sense.*

2. Compactness method

In this section we will finish the proof of Theorem 1.3 by the compactness method. We first consider the following approximation problem

$$\operatorname{div}\left(g\left(\epsilon + |\nabla u^\epsilon|^2\right)\nabla u^\epsilon\right) = 0, \qquad x \in \Omega,\ \epsilon \in (0, 1]. \tag{2.1}$$

We shall show uniform $C^{1,\alpha}$ estimates in Theorem 1.3 for u^ϵ for small $\epsilon > 0$. We will omit the index ϵ since the $C^{1,\alpha}$ estimates are uniform, and then $u^\epsilon \to u$ uniformly. Actually, from (2.1) we have

$$a_{ij}u_{ij} =: \left[g\left(\epsilon + |\nabla u|^2\right)\delta_{ij} + g'\left(\epsilon + |\nabla u|^2\right)2u_iu_j\right]u_{ij} = 0. \tag{2.2}$$

Now we denote $\widetilde{a_{ij}}$ by

$$\widetilde{a_{ij}} = \frac{g\left(\epsilon + |\nabla u|^2\right)\delta_{ij} + g'\left(\epsilon + |\nabla u|^2\right)2u_iu_j}{\left(s + \epsilon + |\nabla u|^2\right)^{\frac{p}{2}-1}}. \tag{2.3}$$

Then from (1.3)-(1.5) we have

$$K^{-1}|\xi|^2 \le \widetilde{a_{ij}}\xi_i\xi_j \le 3K|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^n,$$

and

$$\widetilde{a_{ij}}u_{ij} = 0.$$

Lemma 2.1. *If u is a local weak solution of (2.1) in B_1, then*

$$\|\nabla u\|_{L^\infty(B_{1/2})} \le C\left(\|\nabla u\|_{L^p(B_1)} + 1\right),$$

where C is independent of ϵ.

Proof. Let $v = \left(s + \epsilon + |\nabla u|^2\right)^{p/2}$. Then we find that

$$\left(\widetilde{a_{ij}}v_j\right)_i = \left(pa_{ij}u_{kj}u_k\right)_i. \tag{2.4}$$

Moreover, differentiating (2.1) with respect to x_k, we have

$$\left(a_{ij}u_{kj}\right)_i = 0.$$

Furthermore, (2.3) and (2.4) imply that

$$\left(\widetilde{a_{ij}}v_j\right)_i = pa_{ij}u_{kj}u_{ki} \ge 0. \tag{2.5}$$

Therefore, from the maximum principle (see Lemma 1.2, Chapter 4 in [2])) we obtain

$$\|\nabla u\|^p_{L^\infty(B_{1/2})} \le \|v\|_{L^\infty(B_{1/2})} \le C\left(\|\nabla u\|^p_{L^p\left(B_{3/4}\right)} + 1\right),$$

which finishes our proof. □

From the lemma above, we may as well assume that

$$|\nabla u| \le 1.$$

Lemma 2.2. *Let u be a local weak solution of (2.1) in B_1 and $|\nabla u| \le 1$. For any $\sigma > 0$, there exists an $\eta(\sigma) > 0$ such that if*

$$|\{x \in B_1 : |\nabla u| \le 1 - \eta\}| \le \eta |B_1|,$$

then there is a harmonic function v such that

$$\int_{B_{1/2}} |u - v|^p \; dx \le \sigma.$$

Proof. We prove it by contradiction. If the result is false, then there would exist $\sigma_0 > 0$, $\{\epsilon_k\}_{k=1}^{\infty}$ and $\{u_k\}_{k=1}^{\infty}$ satisfying

$$\begin{aligned}
&\int_{B_1} g\left(\epsilon_k + |\nabla u_k|^2\right) \nabla u_k \cdot \nabla \phi \, dx = 0 \quad \text{for any} \quad \phi \in C_0^{\infty}(B_1), \\
&|\nabla u_k| \leq 1, \\
&|D_k| \leq \frac{1}{2^k} |B_1|, \text{ where } D_k = \left\{ x \in B_1 : |\nabla u_k| \leq 1 - \frac{1}{2^k} \right\},
\end{aligned}$$

so that for any harmonic function v in $B_{1/2}$ we have

$$\int_{B_{1/2}} |u - v|^p \, dx \geq \sigma_0. \tag{2.6}$$

Hence, we may assume that

$$\begin{aligned}
&\epsilon_k \to \epsilon_0, \\
&u_k \to v \qquad \text{in } L^p(B_1), \\
&\nabla u_k \to \nabla v \quad \text{weakly in } L^p(B_1), \\
&|\nabla u_k| \to 1 \quad \text{in } B_1 \setminus D_k.
\end{aligned}$$

Since

$$\left\{ \int_{B_1 \setminus D_k} + \int_{D_k} \right\} g\left(\epsilon_k + |\nabla u_k|^2\right) \nabla u_k \cdot \nabla \phi \, dx = 0,$$

we deduce that

$$\int_{B_1} g(\epsilon_0 + 1) \nabla v \cdot \nabla \phi \, dx = 0$$

as $k \to \infty$. That is to say, v is a harmonic function, which is contradictory to (2.6). Thus, we complete the proof. □

Lemma 2.3. *Let u be a local weak solution of (2.1) in B_1 with $|\nabla u| \leq 1$. If*

$$|\{x \in B_1 : |\nabla u| \leq 1 - \eta\}| \geq \eta |B_1|,$$

then

$$|\nabla u| \leq 1 - \eta^2/C \quad \text{in } B_{1/2},$$

where C is independent of ϵ.

Proof. Let $w = (s + \epsilon + 1)^{p/2} - \left(s + \epsilon + |\nabla u|^2\right)^{p/2} \geq 0$. Then w is a local weak solution of

$$\left(\widetilde{a_{ij}} w_j\right)_i = -p a_{ij} u_{kj} u_{ki} \leq 0 \text{ in } B_1,$$

in view of (2.5). Therefore, from Theorem 8.18 in [8] we have

$$\inf_{B_{1/2}} w \geq \frac{1}{C} \int_{B_1} w \, dx,$$

which implies that

$$\inf_{B_{1/2}} \left((s + \epsilon + 1)^{p/2} - \left(s + \epsilon + |\nabla u|^2\right)^{p/2} \right)$$

$$\geq \frac{1}{C}\int_{B_1} (s+\epsilon+1)^{p/2} - \left(s+\epsilon+|\nabla u|^2\right)^{p/2} \, dx$$
$$\geq \frac{\eta}{C}\left((s+\epsilon+1)^{p/2} - \left(s+\epsilon+(1-\eta)^2\right)^{p/2}\right).$$

Thus we can easily obtain the desired result by using the elementary inequality $(1-x)^{\theta} \leq 1 - C\theta x$ for $0 < x < 1/2$ and $\theta > 0$. □

Corollary 2.4. *Let $\delta_0 = \eta^2/C$ as in the lemma above. Assume that u is a local weak solution of (2.1) in B_1 with $|\nabla u| \leq 1$. If*

$$\left|\left\{x \in B_{1/2^i} : |\nabla u| \leq (1-\eta)(1-\delta_0)^i\right\}\right| \geq \eta \left|B_{1/2^i}\right| \text{ for } i = 0, 1, ..., k,$$

then

$$|\nabla u| \leq (1-\delta_0)^i \quad \text{in } B_{1/2^i} \text{ for } i = 1, 2, ..., k+1,$$

where C is independent of ϵ.

Proof. We can prove by induction on i. From the lemma above, it is easy to check that our conclusion is valid for $i = 0$. Assume that the conclusion is valid for some i. We denote $w_1(x)$ by

$$w_1(x) = \frac{2^i}{(1-\delta_0)^i} u\left(\frac{x}{2^i}\right).$$

Then we can obtain the result from the lemma above. □

Lemma 2.5. *Let u be a local weak solution of (2.1) in B_1 with $|\nabla u| \leq 1$, $⨍_{B_1} |u|^p \, dx \leq 1$ and*

$$|\{x \in B_1 : |\nabla u| \leq 1-\eta\}| \leq \eta |B_1|.$$

(1) *For any $0 < \alpha < 1$ and $\theta > 0$, there exist $\eta > 0$ and $r_0 \in (0, 1/4)$ depending on θ, α, p, and a linear function $L_1(x) = A_1 x + B_1$ such that*

$$⨍_{B_{r_0}} |u - L_1|^p \, dx \leq \theta r_0^{p(1+\alpha)}.$$

(2) *For any $0 < \alpha < 1$, there exist $\eta > 0$ and $r_0 \in (0, 1/4)$ depending on α, p, and linear functions $L_k(x) = A_k x + B_k$ for $k = 0, 1, 2, 3, ...,$ with uniformly bounded coefficients such that*

$$⨍_{B_{r_0^k}} |u - L_k(x)|^p \, dx \leq r_0^{pk(1+\alpha)} \tag{2.7}$$

and

$$|A_{k+1} - A_k| \leq C r_0^{pk\alpha}, \tag{2.8}$$
$$|B_{k+1} - B_k| \leq C r_0^{pk(1+\alpha)}. \tag{2.9}$$

(3) *For any $0 < \alpha < 1$, there exist $\eta > 0$ depending on α, p, and a linear function $L(x) = Ax + B$ such that*

$$⨍_{B_r} |u - L|^p \, dx \leq C r^{p(1+\alpha)} \quad \text{for any } 0 < r \leq 1.$$

Proof. (1) For any $\sigma > 0$, from Lemma 2.2 there exists $\eta = \eta(\sigma) > 0$ such that

$$\int_{B_{1/2}} |u - v|^p \ dx \leq \sigma, \tag{2.10}$$

where v is a harmonic function in B_1. Since $u \in W^{1,p}_{loc}(B_1)$ is a weak solution of (2.1), then

$$\int_{B_{1/2}} |v|^p \ dx \leq C,$$

which implies that

$$\sup_{B_{1/4}} \left|D^2 v\right| \leq C.$$

Now, let $L_1(x) = A_1 x + B_1$ be the Taylor polynomial of v at 0. Then we have

$$\sup_{x \in B_{1/4}} |v - L_1| \leq C|x|^2.$$

Therefore, for any $0 < r < 1/4$ we have

$$\begin{aligned} ⨍_{B_r} |u - L_1|^p \ dx &\leq 2^{p-1} \left(⨍_{B_r} |u - v|^p \ dx + ⨍_{B_r} |v - L_1|^p \ dx \right) \\ &\leq 2^{p-1} \frac{\sigma}{|B_r|} + 2^{p-1} r^{2p}, \end{aligned}$$

which implies that

$$⨍_{B_{r_0}} |u - L_1|^p \ dx \leq 2^p r^{2p},$$

by taking σ small enough such that $\sigma \leq r^{2p} |B_r|$. Finally, choosing $r = r_0$ such that $2^p r_0^{p(1-\alpha)} = \theta$, we can finish the proof.

(2) We prove it by induction. From (1) we know the result is true for $k = 0, 1$, if we take $L_0 = 0$. Let us assume it is true for k. We denote $w(x)$ by

$$w(x) = \frac{(u - L_k)\left(r_0^k x\right)}{\theta r_0^{k(\alpha+1)}}.$$

Then w satisfies

$$\widetilde{a_{ij}}(w) w_{ij} = 0, \quad x \in B_1.$$

where

$$\begin{aligned} \widetilde{a_{ij}}(w) = & \frac{g\left(\epsilon + \left|\theta r_0^{k\alpha} \nabla w + A_k\right|^2\right) \delta_{ij}}{\left(s + \epsilon + \left|\theta r_0^{k\alpha} \nabla w + A_k\right|^2\right)^{\frac{p}{2}-1}} \\ & + \frac{g'\left(\epsilon + \left|\theta r_0^{k\alpha} \nabla w + L_k\right|^2\right) 2\left(\theta r_0^{k\alpha} w_i + (A_k)_i\right)\left(\theta r_0^{k\alpha} w_j + (A_k)_j\right)}{\left(s + \epsilon + \left|\theta r_0^{k\alpha} \nabla w + A_k\right|^2\right)^{\frac{p}{2}-1}}. \end{aligned}$$

Let v be the solution of

$$\widetilde{a_{ij}}(v) v_{ij} = 0,$$

with $v|_{B_{1/2}} = w$, where

$$\widetilde{a_{ij}}(v) = \frac{g\left(\epsilon + |A_k|^2\right)\delta_{ij}}{\left(s+\epsilon+|A_k|^2\right)^{\frac{p}{2}-1}} + \frac{g'\left(\epsilon + |A_k|^2\right)2(A_k)_i(A_k)_j}{\left(s+\epsilon+|A_k|^2\right)^{\frac{p}{2}-1}}.$$

Since $g \in C^1$, $\|\widetilde{a_{ij}}(w) - \widetilde{a_{ij}}(v)\|_{L^\infty(B_1)}$ is small enough if we choose θ small enough. For any $\tau > 0$, from Lemma 13 in [1] we can obtain

$$\|w - v\|_{L^\infty(B_{1/2})} \leq \tau,$$

by choosing θ small enough. Now, let $L^*(x) = A^*x + B^*$ be the Taylor polynomial of v at 0. Then we have

$$\sup_{x\in B_r} |v - L^*| \leq Cr^2 \quad \text{for any } r \in (0, 1/4).$$

Furthermore, choosing $\tau \leq r_0^{p(1+\alpha)}$, we find that

$$\fint_{B_{r_0}} |w - L^*|^p \ dx \leq \tau + Cr_0^{2p} \leq Cr_0^{p(1+\alpha)}.$$

Finally, from the definition of w we can obtain

$$\fint_{B_{r_0^{k+1}}} |w - L_{k+1}|^p \ dx \leq Cr_0^{p(k+1)(1+\alpha)},$$

by taking $L_{k+1} = L_k - \theta r_0^{k(\alpha+1)} L^*\left(\frac{x}{r_0^k}\right)$. Thus, (2.7)-(2.9) are true.

(3) From (2) it is easy to see that A_k, B_k converge to A_∞, B_∞ as $k \to \infty$ respectively. Now let $L(x) = A_\infty x + B_\infty$. Then we have

$$\fint_{B_{r_0^k}} |u - L(x)|^p \ dx \leq r_0^{pk(1+\alpha)} \quad \text{for } k = 0, 1, 2, \ldots.$$

Therefore, we have

$$\fint_{B_r} |u - L(x)|^p \ dx \leq r^{p(1+\alpha)} \quad \text{for any } 0 < r \leq 1,$$

which completes our proof. □

Now we are ready to prove the main result, Theorem 1.3.

Proof. We may as well assume that $u(0) = 0$ and $\fint_{B_1} |u|^p \ dx \leq 1$. We denote k by

$$\left|\left\{x \in B_{1/2^i} : |\nabla u| \leq (1-\eta)(1-\delta_0)^i\right\}\right| \geq \eta \left|B_{1/2^i}\right|, \quad i = 0, 1, 2, \ldots, k-1, \tag{2.11}$$

but,

$$\left|\left\{x \in B_{1/2^k} : |\nabla u| \leq (1-\eta)(1-\delta_0)^k\right\}\right| \leq \eta \left|B_{1/2^k}\right|. \tag{2.12}$$

We divide into two cases:

Case 1: $k = \infty$. That is to say, (2.11) is true for any i. Then, from Corollary 2.4 we find that

$$|\nabla u| \leq (1-\delta_0)^i \quad \text{in } B_{1/2^i},$$

which implies that

$$|u(x)| = |u(x) - u(0)| \leq |x|(1-\delta_0)^i \leq \frac{1}{1-\delta_0}|x|^{1+\alpha_0} \quad \text{for } |x| \leq 1,$$

where $\alpha_0 = -\log_2(1-\delta_0)$. Now fix an α and then determine δ_0 and α_0. Let $\alpha_1 = \min\{\alpha_0, \alpha\}$. Therefore, we have

$$|u(x)| \leq C|x|^{1+\alpha_0} \leq C|x|^{1+\alpha_1} \quad \text{for } |x| \leq 1.$$

Case 2: $k < \infty$. Similarly, Corollary 2.4 implies that

$$|\nabla u| \leq (1-\delta_0)^i \quad \text{in } B_{1/2^i} \quad \text{for } \ 0 \leq i \leq k, \tag{2.13}$$

which implies that

$$|u(x)| \leq C|x|^{1+\alpha_1} \quad \text{in } B_{1/2^i} \quad \text{for } \ 0 \leq i \leq k.$$

Now we denote w by

$$w(x) = \frac{2^k}{(1-\delta_0)^k} u\left(\frac{x}{2^k}\right).$$

Therefore, by Lemma 2.5 (3) and the definition of α_1, there is a linear function $L(x) = Ax + B$ such that

$$⨍_{B_r} |w - L|^p \ dx \leq Cr^{p(1+\alpha)} \leq Cr^{p(1+\alpha_1)}$$

for any $0 < r \leq 1$. Recalling the definition of w, we have

$$⨍_{B_r} \left| u\,(x) - (1-\delta_0)^k Ax - \frac{(1-\delta_0)^k B}{2^k} \right|^p \ dx \leq Cr^{p(1+\alpha_1)} \tag{2.14}$$

for any $0 < r \leq 1/2^k$. Moreover, for any $1/2^k < r \leq 1$ we have

$$\begin{aligned} & ⨍_{B_r} \left| u\,(x) - (1-\delta_0)^k Ax - \frac{(1-\delta_0)^k B}{2^k} \right|^p \ dx \\ & \leq C\left(\sup_{B_r} |u|^p + \left|(1-\delta_0)^k Ar\right|^p + \left| \frac{(1-\delta_0)^k B}{2^k} \right|^p \right) \\ & \leq Cr^{p(1+\alpha_1)}, \end{aligned}$$

since $(1-\delta_0)^k = 2^{-k\alpha_0} \leq r^{\alpha_0} \leq r^{\alpha_1}$. □

References

[1] L. A. Caffarelli, *Interior a priori estimates for solutions of fully nonlinear equations*, Ann. of Math. (2), 130(1)(1989), 189-213.

[2] Y. Chen & L. Wu, *Second order elliptic partial differential equations and elliptic systems*, American Mathematical Society, Providence, RI, 1998.

[3] E. DiBenedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal., 7(8)(1983), 827-850.

[4] F. Duzaar & G. Mingione, *Local Lipschitz regularity for degenerate elliptic systems*, Ann. Inst. H. Poincareé, 27(6)(2010), 1361-1396.

[5] F. Duzaar & G. Mingione, *Gradient estimates via linear and nonlinear potentials*, J. Funct. Anal., 259(11)(2010), 2961-2998.

[6] L. C. Evans, *A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic p.d.e.*, J. Differential Equations, 45(3)(1982), 356-373.
[7] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Princeton Univ. Press, 1983.
[8] D. Gilbarg & N. Trudinger, *Elliptic Partial Differential Equations of Second Order (3rd edition)*, Springer Verlag, Berlin, 1998.
[9] J. L. Lewis, *Regularity of the derivatives of solutions to certain degenerate elliptic equations*, Indiana Univ. Math. J., 32(6)(1983), 849-858.
[10] K. Uhlenbeck, *Regularity for a class of non-linear elliptic systems*, Acta Math., 138(1977), 219-240.
[11] M. Shaw & L. Wang, *Hölder and L^p estimates for $\Box_b$ on CR manifolds of arbitrary codimension*, Math. Ann., 331(2)(2005), 297-343.
[12] L. Wang, *Compactness methods for certain degenerate elliptic equations*, J. Differential Equations, 107(2)(1994), 341-350.
[13] L. Wang, *Hölder estimates for subelliptic operators*, J. Funct. Anal., 199(1)(2003), 228-242.

(Received August 7, 2011)

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