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Fixed point theorem utilizing operators and functionals

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Abstract

This paper presents a fixed point theorem utilizing operators and functionals in the spirit of the original Leggett-Williams fixed point theorem which is void of any invariance-like conditions. The underlying sets in the Leggett-Williams fixed point theorem that were defined using the total order of the real numbers are replaced by sets that are defined using an ordering generated by a border-symmetric set, that is, the sets that were defined using functionals in the original Leggett-Williams fixed point theorem are replaced by sets that are defined using operators.

Key words: Multiple fixed-point theorems, Leggett-Williams, expansion, compression.

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1 Introduction

Mavridis [8] attempted to generalize the Leggett-Williams [7] fixed point theorem by replacing arguments that involved concave and convex functionals with arguments

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that involved concave and convex operators. Some of the arguments went through seamlessly due to the antisymmetric property of the partial order generated by the cone, while others were dealt away with invariance-like conditions.

Anderson, Avery, Henderson and Liu [1] removed the invariance-like conditions when working in a cone P of a real Banach space E which is a subset of $F(K)$, the set of real valued functions defined on a set K. The key to this result was developing the notion of an operator being comparable to a function on a compact set. That is, if J_R is a compact subset of K and $x_R \in E$, we say that R is comparable to x_R on P relative to J_R if, given any $y \in P$, either $R(y) <_{J_R} x_R$ or $x_R \leq_{J_R} R(y)$. This did solve the problem of creating a fixed point theorem in the spirit of the original Leggett-Williams fixed point theorem that avoided invariance-like conditions with the underlying sets being defined using operators; however, the comparability criterion is very restrictive (very few operators satisfy it) and the theorem is valid only in a subset of real-valued functions.

By introducing an ordering through a border-symmetric set we are able to remove the comparability criterion–which also removed the restriction of working in subsets of real valued functions–while maintaining the spirit of the original Leggett-Williams fixed point theorem in regards to avoiding any invariance-like conditions. We are also able to replace the underlying sets of the Leggett-Williams fixed point theorem defined using functionals (applying a total ordering) with sets that are defined using operators (applying an ordering with boundary properties) which was the goal in the Mavridis manuscript.

Note that when $J_R = \{r\}$ and $x_R(t) = s$ then the comparability criterion of [1] says that for each $x \in P$ either $x(r) < s$ or $s \leq x(r)$. The same result can be obtained by defining the linear functional (hence the functional is both concave and convex) α by $\alpha(x) = x(r)$. Also, instead of defining the ordering based on evaluation, that is $y <_{J_R} z$ which means that $y(r) < z(r)$ (since r is the only element in J_R) as was done in the example found in [1], the ordering is defined in terms of sets, that is $y \ll_T z$ where $T := \{y \in E \mid y(r) \geq 0\}$ means that $z - y \in T^\circ$ hence $y(r) < z(r)$. Thus the envisioned applications of the Operator Type Expansion-Compression Fixed Point Theorem can be proven using the new result which is less restrictive and easier to interpret. We conclude with an illustration of the techniques introduced in this manuscript by revisiting the example found in [1] and providing a justification based on our main result (the statement of the theorem is essentially the same, however the justification is much different - based on the new result using border-symmetric sets and functionals instead of the restrictive techniques [1] based on the comparability criterion).

2 Preliminaries

In this section we will state the definitions that are used in the remainder of the paper.

Definition 1 *Let E be a real Banach space. A nonempty, closed, convex set $P \subset E$ is called a* cone *if, for all $x \in P$ and $\lambda \geq 0$, $\lambda x \in P$, and if $x, -x \in P$ then $x = 0$.*

Every subset C of a Banach space E induces an ordering in E given by $x \leq_C y$ if and only if $y - x \in C$, and we say that $x <_C y$ whenever $x \leq_C y$ and $x \neq y$. Furthermore, if the interior of C, which we denote as C°, is nonempty then we say that $x \ll_C y$ if and only if $y - x \in C^\circ$. Note that if C and D are subsets of a Banach space E with $C \subseteq D$ then

$$x \leq_C y \text{ implies } x \leq_D y$$

since $y - x \in C \subseteq D$. Also note that if P is a cone in the Banach space E then the ordering induced by P is a partial ordering on E. Since the closure and boundary of sets in our main results will be relative to the cone P, the definition of a border-symmetric set which follows is stated in terms of the interior which will refer to the interior relative to the entire Banach space E in our main results.

Definition 2 *A closed, convex subset M of a Banach space E with nonempty interior is said to be a* border-symmetric *subset of E if for all $x \in M$ and $\lambda \geq 0$, $\lambda x \in M$, and if the order induced by M satisfies the property that $x \leq_M y$ and $y \leq_M x$ implies that $x - y \notin M^\circ$ and $y - x \notin M^\circ$.*

Note that every nontrivial (not just the identity) cone P of a Banach space E is a border-symmetric subset of E if it has a nonempty interior since if $x \leq_P y$ and $y \leq_P x$ then $y - x, -(y - x) \in P$ thus $y - x = 0$ and $0 \notin P^\circ$. The border-symmetric property is a less restrictive replacement of the antisymmetric property of a partial order. Our main results rely on interior arguments of our border-symmetric subsets as well as the lack of symmetry in the interior of a border-symmetric subset.

Definition 3 *An operator is called* completely continuous *if it is continuous and maps bounded sets into precompact sets.*

Definition 4 *Let P be a cone in a real Banach space E. Then we say that $A : P \to P$ is a* continuous concave operator *on P if $A : P \to P$ is continuous and*

$$tA(x) + (1 - t)A(y) \leq_P A(tx + (1 - t)y)$$

for all $x, y \in P$ and $t \in [0, 1]$. Similarly we say that $B : P \to P$ is a continuous convex operator *on P if $B : P \to P$ is continuous and*

$$B(tx + (1 - t)y) \leq_P tB(x) + (1 - t)B(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Definition 5 *A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if*

$$\alpha : P \to [0, \infty)$$

is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0,1]$. Similarly we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E if

$$\beta : P \to [0, \infty)$$

is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $t \in [0,1]$.

Let R and S be operators on a cone P of a real Banach space E, Q and M be subsets of E that contain P, with $x_R, x_S \in E$, then we define the sets,

$$P_Q(R, x_R) = \{y \in P : R(y) \ll_Q x_R\}$$

and

$$P(R, S, x_R, x_S, Q, M) = P_Q(R, x_R) - \overline{P_M(S, x_S)}.$$

Definition 6 *Let D be a subset of a real Banach space E. If $r : E \to D$ is continuous with $r(x) = x$ for all $x \in D$, then D is a* retract *of E, and the map r is a* retraction. *The* convex hull *of a subset D of a real Banach space X is given by*

$$conv(D) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in D,\ \lambda_i \in [0,1],\ \sum_{i=1}^{n} \lambda_i = 1,\ \text{and } n \in \mathbb{N} \right\}.$$

The following theorem is due to Dugundji and its proof can be found in [4, *p* 44].

Theorem 7 *For Banach spaces X and Y, let $D \subset X$ be closed and let $F : D \to Y$ be continuous. Then F has a continuous extension $\tilde{F} : X \to Y$ such that $\tilde{F}(X) \subset \overline{conv(F(D))}$.*

Corollary 8 *Every closed convex set of a Banach space is a retract of the Banach space.*

The following theorem, which establishes the existence and uniqueness of the fixed point index, is from [5, *pp* 82-86]; an elementary proof can be found in [4, *pp* 58 & 238]. The proof of our main result in the next section will invoke the properties of the fixed point index.

Theorem 9 *Let X be a retract of a real Banach space E. Then, for every bounded relatively open subset U of X and every completely continuous operator $A : \overline{U} \to X$ which has no fixed points on ∂U (relative to X), there exists an integer $i(A, U, X)$ satisfying the following conditions:*

$(G1)$ *Normality: $i(A, U, X) = 1$ if $Ax \equiv y_0 \in U$ for any $x \in \overline{U}$;*

$(G2)$ *Additivity: $i(A, U, X) = i(A, U_1, X) + i(A, U_2, X)$ whenever U_1 and U_2 are disjoint open subsets of U such that A has no fixed points on $\overline{U} - (U_1 \cup U_2)$;*

$(G3)$ *Homotopy Invariance: $i(H(t, \cdot), U, X)$ is independent of $t \in [0, 1]$ whenever $H : [0, 1] \times \overline{U} \to X$ is completely continuous and $H(t, x) \neq x$ for any $(t, x) \in [0, 1] \times \partial U$;*

$(G4)$ *Solution: If $i(A, U, X) \neq 0$, then A has at least one fixed point in U.*

Moreover, $i(A, U, X)$ is uniquely defined.

3 Main Results

In the following lemmas we prove the criteria for an operator to be LW-inward and LW-outward (see the definitions that follow) which will be the basis of our compression-expansion fixed point theorem involving operators and functionals. All references to the boundary and closure of sets is relative to the cone P for the application of the fixed point index stated in Theorem 9 and references to the interior of sets are relative to the entire Banach space E.

Lemma 10 *Suppose P is a cone in real Banach space E, Q is a border-symmetric subset of E with $P \subset Q$, α is a non-negative continuous concave functional on P, B is a continuous convex operator on P, a is a nonnegative real number, and $y_B \in E$. Furthermore, suppose that $T : P \to P$ is completely continuous and that the following conditions hold:*

$(B1)$ $\{y \in P\ : a < \alpha(y)$ *and* $B(y) \ll_Q y_B\} \neq \emptyset$;

$(B2)$ *if $y \in \partial P_Q(B, y_B)$ and $a \leq \alpha(y)$, then $B(Ty) \ll_Q y_B$;*

(B3) *if* $y \in \partial P_Q(B, y_B)$ *and* $\alpha(Ty) < a$, *then* $B(Ty) \ll_Q y_B$.

If $\overline{P_Q(B, y_B)}$ *is bounded, then* $i(T, P_Q(B, y_B), P) = 1$.

Proof. By Corollary 8, P is a retract of the Banach space E since it is closed and convex.

Claim 1: $Ty \neq y$ for all $y \in \partial P_Q(B, y_B)$.

Suppose to the contrary, that is, there is a $z_0 \in \partial P_Q(B, y_B)$ with $Tz_0 = z_0$. Since $z_0 \in \partial P_Q(B, y_B)$, we have that $B(z_0) \not\ll_Q y_B$ (that is, $y_B - B(z_0) \notin Q^\circ$). Either $\alpha(Tz_0) < a$ or $a \leq \alpha(Tz_0)$. If $\alpha(Tz_0) < a$, then $B(Tz_0) \ll_Q y_B$ by condition $(B3)$, and if $a \leq \alpha(Tz_0) = \alpha(z_0)$, then $B(Tz_0) \ll_Q y_B$ by condition $(B2)$. Hence, in either case we have that $B(z_0) = B(Tz_0) \ll_Q y_B$ which is a contradiction since $z_0 \in \partial P_Q(B, y_B)$. Thus $Tz_0 \neq z_0$ and we have verified that T does not have any fixed points on $\partial P_Q(B, y_B)$.

Let $z_1 \in \{y \in P \ : a < \alpha(y) \ \text{ and } \ B(y) \ll_Q y_B\} \neq \emptyset$ (see condition $(B1)$), and let $H_1 : [0,1] \times \overline{P_Q(B, y_B)} \to P$ be defined by $H_1(t, y) = (1-t)Ty + tz_1$. Clearly, H_1 is continuous and $H_1([0,1] \times \overline{P_Q(B, y_B)})$ is relatively compact.

Claim 2: $H_1(t, y) \neq y$ for all $(t, y) \in [0,1] \times \partial P_Q(B, y_B)$.

Suppose not; that is, suppose there exists $(t_1, y_1) \in [0,1] \times \partial P_Q(B, y_B)$ such that $H(t_1, y_1) = y_1$. Since $y_1 \in \partial P_Q(B, y_B)$ we have that $B(y_1) \not\ll_Q y_B$, which together with $B(z_1) \ll_Q y_B$ implies $t_1 \neq 1$. From Claim 1 we have $t_1 \neq 0$. Either $\alpha(Ty_1) < a$ or $a \leq \alpha(Ty_1)$.

Case 1 : $\alpha(Ty_1) < a$.

By condition $(B3)$, we have

$$B(Ty_1) \ll_Q y_B$$

which implies that

$$(1-t_1)B(Ty_1) \ll_Q (1-t_1)y_B$$

since $t_1 \neq 1$ ($z_1 \notin \partial P_Q(B, y_B)$), thus we have

$$(1-t_1)B(Ty_1) + t_1 B(z_1) \ll_Q (1-t_1)y_B + t_1 B(z_1) \ll_Q (1-t_1)y_B + t_1 y_B = y_B,$$

since $t_1 \neq 0$.

Since B is a convex operator on P,

$$B(y_1) = B((1-t_1)Ty_1 + t_1 z_1) \leq_P (1-t_1)B(Ty_1) + t_1 B(z_1)$$

and since $P \subset Q$ we have

$$B(y_1) = B((1-t_1)Ty_1 + t_1 z_1) \leq_Q (1-t_1)B(Ty_1) + t_1 B(z_1).$$

Therefore,
$$B(y_1) \le_Q (1-t_1)B(Ty_1) + t_1 B(z_1) \ll_Q y_B,$$
which contradicts $B(y_1) \not\ll_Q y_B$.

Case 2 : $a \le \alpha(Ty_1)$.
We have

$$a = (1-t_1)a + t_1 a \le (1-t_1)\alpha(Ty_1) + t_1\alpha(z_1) \le \alpha((1-t_1)Ty_1 + t_1 z_1) = \alpha(y_1)$$

and thus by condition $(B2)$, we have $B(y_1) \ll_Q y_B$. This is the same contradiction we reached in the previous case.

Therefore, we have shown that $H_1(t,y) \ne y$ for all $(t,y) \in [0,1] \times \partial P_Q(B, y_B)$, and thus by the homotopy invariance property $(G3)$ of the fixed point index, $i(T, P_Q(B, y_B), P) = i(z_1, P_Q(B, y_B), P)$. And by the normality property $(G1)$ of the fixed point index, $i(T, P_Q(B, y_B), P) = i(z_1, P_Q(B, y_B), P) = 1$.

□

Lemma 11 *Suppose P is a cone in a real Banach space E, M is a border-symmetric subset of E with $P \subset M$, β is a non-negative continuous convex functional on P, A is a continuous concave operator on P, b is a nonnegative real number, and $y_A \in E$. Furthermore, suppose that $T : P \to P$ is completely continuous and that the following conditions hold:*

(A1) $\{y \in P \ : y_A \ll_M A(y) \ \textit{and} \ \beta(y) < b\} \ne \emptyset$;

(A2) if $y \in \partial P_M(A, y_A)$ *and* $\beta(y) \le b$, *then* $y_A \ll_M A(Ty)$;

(A3) if $y \in \partial P_M(A, y_A)$ *and* $b < \beta(Ty)$, *then* $y_A \ll_M A(Ty)$.

If $\overline{P_M(A, y_A)}$ *is bounded, then* $i(T, P_M(A, y_A), P) = 0$.

Proof. By Corollary 8, P is a retract of the Banach space E since it is closed and convex.

Claim 1: $Ty \ne y$ for all $y \in \partial P_M(A, y_A)$.

Suppose to the contrary, that is, there is a $w_0 \in \partial P_M(A, y_A)$ with $Tw_0 = w_0$. Since $w_0 \in \partial P_M(A, y_A)$, we have that $A(w_0) \not\ll_M y_A$ (that is, $y_A - A(w_0) \notin M^\circ$). Either $\beta(Tw_0) \le b$ or $\beta(Tw_0) > b$. If $\beta(Tw_0) > b$, then $y_A \ll_M A(Tw_0) = A(w_0)$ by condition $(A3)$, and if $\beta(w_0) = \beta(Tw_0) \le b$, then $y_A \ll_M A(Tw_0) = A(w_0)$ by condition $(A2)$. Hence, in either case we have that $y_A \ll_M A(Tw_0) = A(w_0)$ thus $A(w_0) - y_A \in M^\circ$ which is a contradiction since $A(w_0) - y_A \in M$ and $y_A - A(w_0) = -(A(w_0) - y_A) \in M$

implies that $A(w_0) - y_A \notin M^\circ$ since M is a border-symmetric subset of E. Thus $Tw_0 \neq w_0$ and we have verified that T does not have any fixed points on $\partial P_M(A, y_A)$.

Let $w_1 \in \{y \in P : y_A \ll_M A(y) \text{ and } \beta(y) < b\}$ (see condition $(A1)$), and let $H_0 : [0,1] \times \overline{P_M(A, y_A)} \to P$ be defined by $H_0(t, y) = (1-t)Ty + tw_1$. Clearly, H_0 is continuous and $H_0([0,1] \times \overline{P_M(A, y_A)}$ is relatively compact.

Claim 2: $H_0(t, y) \neq y$ for all $(t, y) \in [0,1] \times \partial P_M(A, y_A)$.

Suppose not; that is, suppose there exists $(t_0, y_0) \in [0,1] \times \partial P_M(A, y_A)$ such that $H_0(t_0, y_0) = y_0$. Since $y_0 \in \partial P_M(A, y_A)$ we have that $A(y_0) \leq_M y_A$. From Claim 1, we have $t_0 \neq 0$. Since $A(w_1) - y_A \in M^\circ$, $A(y_0) \leq_M y_A$ and M is a border-symmetric subset, we have $t_0 \neq 1$. Either $b < \beta(Ty_0)$ or $\beta(Ty_0) \leq b$.

Case 1 : $b < \beta(Ty_0)$.
By condition $(A3)$, we have $y_A \ll_M A(Ty_0)$ which implies that

$$(1-t_0)y_A \ll_M (1-t_0)A(Ty_0)$$

since $t_0 \neq 1$, thus we have

$$y_A = (1-t_0)y_A + t_0 y_A \ll_M (1-t_0)A(Ty_0) + t_0 A(w_1),$$

since $t_0 \neq 0$.

Since A is a concave operator on P,

$$(1-t_0)A(Ty_0) + t_0 A(w_1) \leq_P A((1-t_0)Ty_0 + t_0 w_1) = A(y_0)$$

and since $P \subset M$ we have

$$(1-t_0)A(Ty_0) + t_0 A(w_1) \leq_M A((1-t_0)Ty_0 + t_0 w_1).$$

Therefore,

$$y_A \ll_M (1-t_0)A(Ty_0) + t_0 A(w_1) \leq_M A(y_0)$$

hence $A(y_0) - y_A \in M^\circ$ and we have that $y_A - A(y_0) = -(A(y_0) - y_A) \in M$ which is a contradiction since M is a border-symmetric subset of E thus $A(y_0) - y_A \notin M^\circ$.

Case 2 : $\beta(Ty_0) \leq b$.
We have

$$\beta((1-t_0)Ty_0 + t_0 w_1) \leq (1-t_0)\beta(Ty_0) + t_0\beta(w_1) \leq (1-t_0)b + t_0 b = b$$

and thus by condition $(A2)$, we have $y_A \ll_M A(Ty_0)$. This is the same contradiction we reached in the previous case.

Therefore, we have shown that $H_0(t,y) \neq y$ for all $(t,y) \in [0,1] \times \partial P_M(A,y_A)$, and thus by the homotopy invariance property $(G3)$ of the fixed point index, $i(T, P_M(A,y_A), P) = i(w_1, P_M(A,y_A), P)$. And by the normality property $(G1)$ of the fixed point index, $i(T, P_M(A,y_A), P) = i(w_1, P_M(A,y_A), P) = 0$ since $w_1 \notin P_M(A,y_A)$.

$\square$

Definition 12 *Suppose P is a cone in a real Banach space E, Q is a subset of E with $P \subset Q$, α is a non-negative continuous concave functional on P, B is a continuous convex operator on P, a is a nonnegative real number, $y_B \in E$ and $T : P \to P$ is a completely continuous operator then we say that T is* LW-inward *with respect to $P_Q(B,\alpha,y_B,a)$ if the conditions $(B1), (B2)$, and $(B3)$ of Lemma 10, and the boundedness of $\overline{P_Q(B,y_B)}$ are satisfied.*

Definition 13 *Suppose P is a cone in a real Banach space E, M is a border-symmetric subset of E with $P \subset M$, β is a non-negative continuous convex functional on P, A is a continuous concave operator on P, b is a nonnegative real number, $y_A \in E$ and $T : P \to P$ is a completely continuous operator then we say that T is* LW-outward *with respect to $P_M(\beta, A, b, y_A)$ if the conditions $(A1), (A2)$, and $(A3)$ of Lemma 11, and the boundedness of $\overline{P_M(A,y_A)}$ are satisfied.*

Theorem 14 *Suppose P is a cone in a real Banach space E, Q is a subset of E with $P \subset Q$, M is a border-symmetric subset of E with $P \subset M$, α is a non-negative continuous concave functional on P, β is a non-negative continuous convex functional on P, B is a continuous convex operator on P, A is a continuous concave operator on P, a and b are nonnegative real numbers, and y_A and y_B are elements of E. Furthermore, suppose that $T : P \to P$ is completely continuous and*

$(D1)$ *T is* LW-inward *with respect to $P_Q(B,\alpha,y_B,a)$;*

$(D2)$ *T is* LW-outward *with respect to $P_M(\beta, A, b, y_A)$.*

If

$(H1)$ *$\overline{P_M(A,y_A)} \subsetneq P_Q(B,y_B)$, then T has a fixed point $y \in P(B,A,y_B,y_A,Q,M)$,*

whereas, if

$(H2)$ *$\overline{P_Q(B,y_B)} \subsetneq P_M(A,y_A)$, then T has a fixed point $y \in P(A,B,y_A,y_B,M,Q)$.*

Proof. We will prove the expansive result $(H2)$, as the proof of the compressive result $(H1)$ is nearly identical. To prove the existence of a fixed point for our operator T in

$P(A, B, y_A, y_B, M, Q)$, it is enough for us to show that $i(T, P(A, B, y_A, y_B, M, Q), P) \neq 0$.

Since T is LW-inward with respect to $P_Q(B, \alpha, y_B, a)$, we have by Lemma 10 that $i(T, P_Q(B, y_B), P) = 1$, and since T is LW-outward with respect to $P_M(\beta, A, b, y_A)$, we have by Lemma 11 that $i(T, P_M(A, y_A), P) = 0$.

In Lemma 10 we verified that T has no fixed points on $\partial P_Q(B, y_B)$ and in Lemma 11 we verified that T has no fixed points on $\partial P_M(A, y_A)$ thus T has no fixed points on $\overline{P_M(A, y_A)} - (P_Q(B, y_B) \cup P(A, B, y_A, y_B, M, Q))$. Also, the sets $P_Q(B, y_B)$ and $P(A, B, y_A, y_B, M, Q)$ are nonempty, disjoint, open subsets of $P_M(A, y_A)$, since $\overline{P_Q(B, y_B)} \subsetneq P_M(A, y_A)$ implies that $P(A, B, y_A, y_B, M, Q) = P_M(A, y_A) - \overline{P_Q(B, y_B)} \neq \emptyset$. Therefore, by the additivity property $(G2)$ of the fixed point index

$$i(T, P_M(A, y_A), P) = i(T, P_Q(B, y_B), P) + i(T, P(A, B, y_A, y_B, M, Q), P).$$

Consequently, we have $i(T, P(A, B, y_A, y_B, M, Q), P) = -1$, and thus by the solution property $(G4)$ of the fixed point index, the operator T has a fixed point $y \in P(A, B, y_A, y_B, M, Q)$.

□

4 Application

As an application of our main results, we consider the following second order nonlinear right focal boundary value problem,

$$x'' + g(t)f(x, x') = 0, \quad t \in [0, 1], \tag{1}$$

$$x(0) = x'(1) = 0, \tag{2}$$

where $g : [0, 1] \to [0, \infty)$ and $f : \mathbb{R}^2 \to [0, \infty)$ are continuous.

Let the Banach space $E = C^1[0, 1]$ with the norm of $\|x\| = \max_{t\in[0,1]} |x(t)| + \max_{t\in[0,1]} |x'(t)|$, and define the cone $P \subset E$ by

$$P := \{x \in E \mid x(t) \geq 0,\ x'(t) \geq 0,\ \text{for } t \in [0, 1],\ x \text{ is concave, and } x(0) = 0\}.$$

Then for any $x \in P$, we have $\|x\| = x(1) + x'(0)$. And from the concavity of any $x \in P$, we have that $x(t) \geq tx(1)$ and $x(t) \leq x'(0)t$ for $t \in [0, 1]$.

It is well known that the Green's function for $-x'' = 0$ and satisfying (2) is given by

$$G(t, s) = \min\{t, s\}, \quad (t, s) \in [0, 1] \times [0, 1].$$

We note that, for any $s \in [0,1]$, $G(t,s) \geq tG(1,s)$ and $G(t,s)$ is nondecreasing in t.

By using properties of the Green's function, solutions of (1), (2) are the fixed points of the completely continuous operator $T : P \to P$ defined by

$$Tx(t) = \int_0^1 G(t,s)g(s)f(x(s), x'(s))ds.$$

Since $(Tx)''(t) = -g(t)f(x,x') \leq 0$ on $[0,1]$ and $(Tx)(0) = (Tx)'(1) = 0$, we have $T : P \to P$.

For $y \in P$, we define the following operators:

$$(Ay)(t) = y'(0)t \text{ and } (By)(t) = \left(\frac{y'(0) + y(1)}{2}\right) t.$$

The above operators are continuous linear operators mapping P to P, and are convex or concave continuous operators as well.

In the following theorem, we demonstrate how to apply the compressive condition of Theorem 14 to prove the existence of at least one positive solution to (1), (2).

Theorem 15 *Suppose there is some $\tau \in (0,1)$ and $0 < a < b$ such that g and f satisfy*

(a) $f(u_1,u_2) > \frac{a}{\int_0^\tau g(s)ds}$, *for* $(u_1,u_2) \in [0, a\tau] \times [0,a]$,

(b) $f(u_1,u_2) < \frac{2b}{\int_0^1 (1+s)g(s)ds}$, *for* $(u_1,u_2) \in [0,b] \times [0,2b)$.

Then the right focal problem (1), (2) *has at least one positive solution $y \in P$ with $y'(0) > a$ and $y'(0) + y(1) < 2b$.*

Proof. Let $y_B(t) = bt$, $\alpha : P \to [0,\infty)$ be defined by $\alpha(y) = y'(0)$, and $Q := \{y \in E \mid y(1) \geq 0\}$ thus Q is a border-symmetric subset of E and $P \subset Q$.

Claim 1: T is LW-inward with respect to $P_Q(B, \alpha, y_B, b)$.

Subclaim 1.1: $\{y \in P\ : b < \alpha(y) \text{ and } B(y) \ll_Q y_B\} \neq \emptyset$.

Let

$$y_0(t) := \frac{7bt(2-t)}{12} \in P$$

then

$$\alpha(y_0) = y_0'(0) = \frac{7b}{6} > b$$

and

$$(By_0)(1) = \frac{y_0'(0) + y_0(1)}{2} = \frac{\frac{7b}{6} + \frac{7b}{12}}{2} = \frac{21b}{24} < b = y_B(1)$$

hence $By \ll_Q y_B$ and we have shown that $y_0 \in \{y \in P\ : b < \alpha(y)\ \text{ and }\ B(y) \ll_Q y_B\}$ thus it is nonempty.

Subclaim 1.2: If $y \in \partial P_Q(B, y_B)$ and $b \leq \alpha(y)$, then $B(Ty) \ll_Q y_B$.

Let $y \in \partial P_Q(B, y_B)$ with $b \leq \alpha(y)$, thus

$$(By)(1) = \frac{y'(0) + y(1)}{2} \leq b$$

and

$$b \leq \alpha(y) = y'(0)$$

thus $0 < y(1) \leq b$ and $b \leq y'(0) < 2b$ which implies that $0 \leq y(t) \leq b$ and $0 \leq y'(t) < 2b$ for $t \in [0, 1]$. Thus by property (b),

$$f(y(t), y'(t)) < \frac{2b}{\int_0^1 (1 + s)g(s)ds} \quad \text{for } \ t \in [0, 1]$$

therefore

$$\begin{aligned}
(BTy)(1) &= \frac{(Ty)'(0) + (Ty)(1)}{2} \\
&= \frac{1}{2}\int_0^1 g(s)f(y(s), y'(s))\ ds + \frac{1}{2}\int_0^1 sg(s)f(y(s), y'(s))\ ds \\
&= \frac{1}{2}\int_0^1 (1 + s)g(s)f(y(s), y'(s))\ ds \\
&< \left(\frac{b}{\int_0^1 (1 + s)g(s)ds}\right)\int_0^1 (1 + s)g(s)\ ds \\
&= b = y_B(1).
\end{aligned}$$

Hence, $(BTy)(1) < y_B(1)$ which verifies that $B(Ty) \ll_Q y_B$.

Subclaim 1.3: If $y \in \partial P_Q(B, y_B)$ and $\alpha(Ty) < b$, then $B(Ty) \ll_Q y_B$.

Let $y \in \partial P_Q(B, y_B)$ with $\alpha(Ty) < b$, thus

$$(Ty)'(0) < b$$

and by the concavity of Ty we have that

$$(Ty)(1) \leq (Ty)'(0) < b$$

hence

$$\begin{aligned}(BTy)(1) &= \frac{(Ty)'(0)+(Ty)(1)}{2}\\ &< \frac{b+b}{2} = b = y_B(1).\end{aligned}$$

Hence, $(BTy)(1) < y_B(1)$ which verifies that $B(Ty) \ll_Q y_B$.

It is easy to see that $\overline{P_Q(B, y_B)}$ is bounded, thus T is LW-inward with respect to $P_Q(B, \alpha, y_B, b)$. Let $y_A(t) = at$, $\beta : P \to [0, \infty)$ be defined by $\beta(y) = \frac{y(\tau)}{\tau}$, and $M = \{y \in E \mid y(t) \geq 0 \text{ for } t \in [\tau, 1]\}$ thus M is a border-symmetric subset of E and $P \subset M$.

Claim 2: T is LW-outward with respect to $P_M(\beta, A, a, y_A)$.

Subclaim 2.1: $\{y \in P \ : y_A \ll_M A(y) \text{ and } \beta(y) < a\} \neq \emptyset$.

Let

$$y_1(t) := \frac{a(4-\tau)t(2-t)}{4(2-\tau)} \in P$$

then

$$\beta(y_1) = \frac{y_1(\tau)}{\tau} = \frac{a(4-\tau)}{4} < a$$

and

$$y_1'(0) = 2\left(\frac{a(4-\tau)}{4(2-\tau)}\right) = \frac{a(4-\tau)}{4-2\tau} > a$$

hence for all $t \in (0, 1]$ $y_A(t) = at < (y_1'(0))t = (Ay_1)(t)$, thus we have that $y_A \ll_M Ay_1$ and we have shown that $y_1 \in \{y \in P \ : y_A \ll_M A(y) \text{ and } \beta(y) < a\}$ thus it is nonempty.

Subclaim 2.2: If $y \in \partial P_M(A, y_A)$ and $\beta(y) \leq a$, then $y_A \ll_M A(Ty)$.

Let $y \in \partial P_M(A, y_A)$ with $\beta(y) \leq a$, thus

$$A(y) \leq_M y_A \text{ which implies } 0 \leq y'(s) \leq y'(0) \leq a$$

for all $s \in [0, 1]$ and

$$\frac{y(\tau)}{\tau} = \beta(y) \leq a \text{ which implies } 0 \leq y(s) \leq y(\tau) \leq a\tau \text{ for } s \in [0, \tau].$$

Hence, for $t \in [\tau, 1]$

$$(ATy)(t) = ((Ty)'(0))t = t\int_0^1 g(s)f(y(s), y'(s))\, ds$$

$$\begin{aligned}
&= t\int_0^{\tau} g(s)f(y(s),y'(s))\,ds + t\int_{\tau}^{1} g(s)f(y(s),y'(s))\,ds\\
&\geq t\int_0^{\tau} g(s)f(y(s),y'(s))\,ds\\
&> t\left(\frac{a}{\int_0^{\tau} g(s)\,ds}\right)\int_0^{\tau} g(s)\,ds\\
&= at
\end{aligned}$$

thus for all $t \in [\tau, 1]$

$$y_A(t) = at < (Ty)'(0)t = A(Ty)(t)$$

therefore $y_A \ll_M A(Ty)$.

Subclaim 2.3: If $y \in \partial P_M(A, y_A)$ and $a < \beta(Ty)$, then $y_A \ll_M A(Ty)$.

Let $y \in \partial P_M(A, y_A)$ with $a < \beta(Ty)$, then since

$$\begin{aligned}
(Ty)'(0) &= \int_0^1 g(s)f(y(s),y'(s))\,ds\\
&= \int_0^{\tau} g(s)f(y(s),y'(s))\,ds + \int_{\tau}^{1} g(s)f(y(s),y'(s))\,ds\\
&\geq \int_0^{\tau} \left(\frac{s}{\tau}\right) g(s)f(y(s),y'(s))\,ds + \int_{\tau}^{1} g(s)f(y(s),y'(s))\,ds\\
&= \left(\frac{1}{\tau}\right)\left(\int_0^{\tau} sg(s)f(y(s),y'(s))\,ds + \int_{\tau}^{1} \tau g(s)f(y(s),y'(s))\,ds\right)\\
&= \frac{(Ty)(\tau)}{\tau}
\end{aligned}$$

we have that

$$(Ty)'(0) \geq \frac{(Ty)(\tau)}{\tau} = \beta(Ty) > a$$

thus for all $t \in (0, 1]$

$$y_A(t) = at < (Ty)'(0)t = A(Ty)(t)$$

therefore, for all $t \in [\tau, 1]$ we have $y_A(t) < A(Ty)(t)$ thus $y_A \ll_M A(Ty)$.

It is easy to see that $P_M(A, y_A)$ is bounded, thus T is LW-outward with respect to LW-outward with respect to $P_M(\beta, A, b, y_A)$.

Claim 3: $\overline{P_M(A, y_A)} \subsetneq P_Q(B, y_B)$.

Let $y \in \overline{P_M(A, y_A)}$, thus for $t \in [\tau, 1]$

$$(Ay)(t) = y'(0)t \leq at = y_A(t)$$

hence $y'(0) \leq a$. Since $y \in P$, y is concave and thus

$$y(1) = \frac{y(1) - y(0)}{1 - 0} \leq y'(0)$$

which implies that

$$\frac{y'(0) + y(1)}{2} \leq \frac{y'(0) + y'(0)}{2} \leq a < b$$

therefore for $t \in [\tau, 1]$,

$$(By)(t) \leq at < bt = y_B(t)$$

and we have shown that $y \in P_Q(B, y_B)$. Also, $y_{\frac{A+B}{2}} = \frac{y_A + y_B}{2} \in P_Q(B, y_B) - \overline{P_M(A, y_A)}$, hence we have verified that $\overline{P_M(A, y_A)} \subsetneq P_Q(B, y_B)$.

Therefore, by Theorem 14, T has a fixed point y^* in $P(B, A, x_B, x_A, Q, M)$. Hence, $y'(0) > a$ and $y'(0) + y(1) < 2b$. □

Example. The right focal boundary value problem,

$$x'' + t\left(1.2 + \frac{x}{x' + 1}\right) = 0, \quad t \in [0, 1], \tag{3}$$

$$x(0) = x'(1) = 0, \tag{4}$$

satisfies Theorem 15 with $a = \frac{1}{8}$, $b = 1$ and $\tau = \frac{1}{2}$ and thus has a solution x^* such that

$$x'(0) > \frac{1}{8} \quad \text{and} \quad x'(0) + x(1) < 2.$$

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