Electronic Journal of Qualitative Theory of Differential Equations
2011, No. **83**, 1–26; **http://www.math.u-szeged.hu/ejqtde/**

RAYLEIGH PRINCIPLE FOR TIME SCALE SYMPLECTIC SYSTEMS AND APPLICATIONS

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Abstract. In this paper we establish the Rayleigh principle, i.e., the variational characterization of the eigenvalues, for a general eigenvalue problem consisting of a time scale symplectic system and the Dirichlet boundary conditions. No normality or controllability assumption is imposed on the system. Applications of this result include the Sturmian comparison and separation theorems for time scale symplectic systems. This paper generalizes and unifies the corresponding results obtained recently for the discrete symplectic systems and continuous time linear Hamiltonian systems. The results are also new and particularly interesting for the case when the considered time scale is "special", that is, consisting of a union of finitely many disjoint compact real intervals and/or finitely many isolated points.

1. Introduction

In this paper we consider the eigenvalue problem

$$(\mathcal{S}_\lambda), \quad x(a) = 0 = x(b), \tag{E}$$

where $(\mathcal{S}_\lambda)$ is the time scale symplectic system

$$x^\Delta = \mathcal{A}(t)\, x + \mathcal{B}(t)\, u, \quad u^\Delta = \mathcal{C}(t)\, x + \mathcal{D}(t)\, u - \lambda\, W(t)\, x^\sigma, \quad t \in [a, \rho(b)]_{\mathbb{T}}, \tag{$\mathcal{S}_\lambda$}$$

and $\lambda \in \mathbb{R}$ is a spectral parameter. Here we consider a bounded time scale $\mathbb{T}$ and with $a := \min \mathbb{T}$ and $b := \max \mathbb{T}$ we represent $\mathbb{T}$ as the time scale interval $[a, b]_{\mathbb{T}}$. For the theory of dynamic equations on time scales and its basic notation we refer to [7, 8, 13]. The coefficients of system $(\mathcal{S}_\lambda)$ are piecewise rd-continuous ($\mathrm{C_{prd}}$) $n \times n$ matrix functions on $[a, \rho(b)]_{\mathbb{T}}$ satisfying

$$\mathcal{S}^T(t)\, \mathcal{J} + \mathcal{J}\mathcal{S}(t) + \mu(t)\, \mathcal{S}^T(t)\, \mathcal{J}\mathcal{S}(t) = 0, \quad W(t) \text{ symmetric}, \quad t \in [a, \rho(b)]_{\mathbb{T}}, \tag{1.1}$$

$$W(t) \geq 0 \quad \text{for all } t \in [a, \rho(b)]_{\mathbb{T}}, \tag{1.2}$$

$$\mathcal{S}(t) := \begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ \mathcal{C}(t) & \mathcal{D}(t) \end{pmatrix}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{1.3}$$

where 0 and I are the zero and identity matrices of appropriate dimensions. The word "symplectic" refers to the fact that under (1.1) the fundamental matrix of system $(\mathcal{S}_\lambda)$ is a symplectic $2n \times 2n$ matrix. In the present paper we require no controllability or normality of system $(\mathcal{S}_\lambda)$. This implies that solutions of $(\mathcal{S}_\lambda)$ may be singular on nontrivial subintervals of $[a, b]_{\mathbb{T}}$ or even on the whole interval $[a, b]_{\mathbb{T}}$, see Section 2 for more details.

2010 *Mathematics Subject Classification. Primary* 34N05. *Secondary* 34B05, 34C10, 39A12.

Key words and phrases. Rayleigh principle; Time scale symplectic system; Linear Hamiltonian system; Discrete symplectic system; Finite eigenvalue; Finite eigenfunction; Sturmian separation theorem; Sturmian comparison theorem; Quadratic functional; Positivity.

In the continuous time case, i.e., when $\mathbb{T} = [a, b]$ is a real connected interval, system $(\mathcal{S}_\lambda)$ is the linear Hamiltonian system

$$x' = A(t)\, x + B(t)\, u, \quad u' = C(t)\, x - A^T(t)\, u - \lambda\, W(t)\, x, \quad t \in [a, b], \qquad (\mathrm{H}_\lambda)$$

where the coefficients are piecewise continuous $(\mathrm{C_p})$ on $[a, b]$ with $B(\cdot)$ and $C(\cdot)$ symmetric. In the classical results (under normality), the eigenvalue problem for the system (H_λ) is considered in [17]. When the normality assumption is absent, the corresponding oscillation and eigenvalue theory was developed in [25] and more recently in [20, 23]. In particular, the latter two papers contain respectively the Rayleigh principle and the Sturmian theory for such possibly "abnormal" systems (H_λ).

The above results were motivated by the corresponding discrete time theory in [6]. Specifically, in the discrete time setting the system $(\mathcal{S}_\lambda)$ reduces to the discrete symplectic system

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k - \lambda\, W_k x_{k+1}, \quad k \in [0, N]_{\mathbb{Z}}, \qquad (1.4)$$

where $[0, N]_{\mathbb{Z}} := \{0, 1, \dots, N\}$, see also [1, 4, 11]. Since the interval $[0, N]_{\mathbb{Z}}$ contains only finitely many points, the normality assumption is naturally absent in the oscillation and eigenvalue theory for system (1.4).

The time scale eigenvalue problem (E) was introduced in [21]. In this reference, the oscillation theorem was proven, which relates the number of eigenvalues (called the finite eigenvalues, see Section 2) of (E) which are less or equal to a given number λ and the number of proper focal points of a special solution of the system $(\mathcal{S}_\lambda)$. In the present paper we first derive the corresponding Rayleigh principle for the eigenvalue problem (E), i.e., we prove the variational characterization of the finite eigenvalues (see Theorem 4.1). This result generalizes the continuous and discrete time statements in [20, Theorem 1.1] and [6, Theorem 4.6] to arbitrary time scales. In the second part of this paper we then apply the oscillation theorem from [21, Corollary 6.4] and the new Rayleigh principle to obtain the Sturmian comparison and separation theorems for time scale symplectic systems, thus generalizing the corresponding results in [6] and [23] to arbitrary time scales. The new results in this paper are important not only on their own. For example, the Rayleigh principle (Theorem 4.1) can be used as a tool for deriving further new results for problems with more general boundary conditions, see e.g. [16, pg. 453] for the description of such a method. We shall proceed in this way in our subsequent work.

Our results in this paper are new and interesting even in the case when the underlying time scale $[a, b]_{\mathbb{T}}$ is "special" in the sense that it is the union of finitely many disjoint real intervals and/or finitely many isolated points. In such a case, a certain assumption made for the general time scales reduces to a simple condition (the Legendre condition) on the coefficient $\mathcal{B}(\cdot)$ over the continuous parts of $[a, b]_{\mathbb{T}}$.

The paper is organized as follows. In Section 2 we recall the basic properties of the eigenvalue problem (E). In Section 3 we collect some technical calculations related to admissible pairs of functions. In Section 4 we state and prove the Rayleigh principle for problem (E), while in Section 5 we establish the Sturmian comparison and separation theorems for time scale symplectic systems. The final section contains the discussion related to the above mentioned "special" time scales.

2. Time scale symplectic systems

By using the expansion $x^\sigma = x + \mu x^\Delta$, the system $(\mathcal{S})$ can be written in the form

$$z^\Delta = [\mathcal{S}(t) - \lambda \mathcal{Q}(t)]\, z, \quad t \in [a, \rho(b)]_{\mathbb{T}}, \tag{2.1}$$

where we put

$$z := \begin{pmatrix} x \\ u \end{pmatrix}, \quad \mathcal{Q}(t) := \begin{pmatrix} 0 & 0 \\ W(I + \mu \mathcal{A}) & \mu\, W \mathcal{B} \end{pmatrix}(t).$$

By a direct calculation it follows that the matrix $\mathcal{S}(t) - \lambda \mathcal{Q}(t)$ also satisfies condition (1.1)(i) for all $t \in [a, \rho(b)]_{\mathbb{T}}$ and $\lambda \in \mathbb{R}$. As a consequence we have the coefficient matrix $\mathcal{S}(\cdot) - \lambda \mathcal{Q}(\cdot)$ regressive on $[a, \rho(b)]_{\mathbb{T}}$ and hence, the system $(\mathcal{S}_\lambda)$ possesses unique (vector or matrix) solutions on $[a, b]_{\mathbb{T}}$ once the initial conditions are prescribed at any point $t_0 \in [a, b]_{\mathbb{T}}$. The solutions of $(\mathcal{S}_\lambda)$ belong to the set $\mathrm{C}^1_{\mathrm{prd}}$ of piecewise rd-continuously delta-differentiable functions on $[a, b]_{\mathbb{T}}$, i.e., they are continuous on $[a, b]_{\mathbb{T}}$ and their Δ-derivative is in $\mathrm{C}_{\mathrm{prd}}$. We adopt a usual convention that the vector and matrix solutions of $(\mathcal{S}_\lambda)$ or equivalently of system (2.1) will be denoted by small and capital letters, respectively, typically by $z(\cdot, \lambda) = (x(\cdot, \lambda), u(\cdot, \lambda))$ and $Z(\cdot, \lambda) = (X(\cdot, \lambda), U(\cdot, \lambda))$.

Since the dependence on λ in system $(\mathcal{S}_\lambda)$ is linear, it follows by [14, Corollary 4.5] that the solutions of $(\mathcal{S}_\lambda)$ are entire functions in λ when their initial conditions at some fixed $t_0 \in [a, b]_{\mathbb{T}}$ are independent of λ. We shall utilize special matrix solutions of $(\mathcal{S}_\lambda)$ which are called the *conjoined bases* or prepared or isotropic solutions of $(\mathcal{S}_\lambda)$, see [9, 12, 22]. Such a matrix solution $Z(\cdot, \lambda) = (X(\cdot, \lambda), U(\cdot, \lambda))$ is defined by the symmetry of $(X^T U)(\cdot, \lambda)$ and by $\operatorname{rank}(X^T(\cdot, \lambda),\ U^T(\cdot, \lambda)) = n$. The *principal solution* $\hat{Z}(\cdot, \lambda) = (\hat{X}(\cdot, \lambda), \hat{U}(\cdot, \lambda))$ of $(\mathcal{S}_\lambda)$ given by the initial conditions

$$\hat{X}(a, \lambda) \equiv 0, \quad \hat{U}(a, \lambda) \equiv I \tag{2.2}$$

will play a prominent role in our investigations. Since the initial conditions in (2.2) do not depend on λ, the functions $\hat{X}(t, \cdot)$ and $\hat{U}(t, \cdot)$ are entire in the argument λ for every $t \in [a, b]_{\mathbb{T}}$. This and assumption (1.2) imply that the kernel of $\hat{X}(t, \cdot)$ is piecewise constant on $\mathbb{R}$ with the same values of the subspaces $\operatorname{Ker} \hat{X}(t, \lambda^+)$ and $\operatorname{Ker} \hat{X}(t, \lambda^-)$ for every $\lambda \in \mathbb{R}$, see [21, Proposition 4.5] and its proof. Based on the above, the following algebraic definition of (finite) eigenvalues of (E) was given in [21, Definition 2.4]. A number $\lambda_0 \in \mathbb{R}$ is a *finite eigenvalue* of the eigenvalue problem (E) if

$$\theta(\lambda_0) := r(b) - \operatorname{rank} \hat{X}(b, \lambda_0) \geq 1, \quad \text{where} \quad r(b) := \max_{\lambda \in \mathbb{R}} \operatorname{rank} \hat{X}(b, \lambda).$$

In this case we call $\theta(\lambda_0)$ the *algebraic multiplicity* of the finite eigenvalue λ_0. By [21, Theorem 5.2], for every finite eigenvalue λ_0 of (E) there is a corresponding *finite eigenfunction* $z(\cdot, \lambda_0) = (x(\cdot, \lambda_0), u(\cdot, \lambda_0))$ which solves (E) with $\lambda = \lambda_0$ and satisfies

$$W(\cdot)\, x^\sigma(\cdot, \lambda_0) \not\equiv 0 \quad \text{on } [a, \rho(b)]_{\mathbb{T}}. \tag{2.3}$$

Moreover, the *geometric multiplicity* of the finite eigenvalue λ_0, i.e., the dimension of the corresponding eigenspace

$$\big\{\, W(\cdot)\, x^\sigma(\cdot, \lambda_0) \text{ on } [a, \rho(b)]_{\mathbb{T}}, \text{ such that } (x, u) \text{ solves (E) with } \lambda = \lambda_0 \,\big\},$$

is equal to $\theta(\lambda_0)$. Under (1.2) the finite eigenvalues of (E) are real and the finite eigenfunctions corresponding to different finite eigenvalues are orthogonal with respect to the bilinear form

$$\langle z, \tilde{z} \rangle_W := \int_a^b [x^\sigma(t)]^T W(t)\, \tilde{x}^\sigma(t)\, \Delta t,$$

where $z = (x, u)$ and $\tilde{z} = (\tilde{x}, \tilde{u})$, see [21, Propositions 5.7 and 5.8]. By (2.3) it follows that the number $\|z\|_W := \sqrt{\langle z, z \rangle_W}$ is positive for every finite eigenfunction z of (E). Hence, the finite eigenfunctions of (E) can be orthonormalized by the standard Gram–Schmidt procedure.

Next we discuss the concept of proper focal points for the conjoined bases of system $(\mathcal{S}_\lambda)$ as it is given in [21, Definition 3.1 and Remark 3.3]. Let us define on $[a, \rho(b)]_{\mathbb{T}}$ the $n \times n$ matrices $M = M(t, \lambda)$, $T = T(t, \lambda)$, and $P = P(t, \lambda)$ by

$$M := [I - X^\sigma (X^\sigma)^\dagger]\, \mathcal{B}, \quad T := I - M^\dagger M, \quad P := T X (X^\sigma)^\dagger \mathcal{B} T, \tag{2.4}$$

where we suppress the arguments t and λ in the conjoined basis $Z(\cdot, \lambda) = (X(\cdot, \lambda), U(\cdot, \lambda))$ and the argument t in the coefficient $\mathcal{B}$, and where $X^\dagger$ denotes the Moore–Penrose generalized inverse of X, see [2, 3]. Let $t_0 \in (a, b]_{\mathbb{T}}$ and $\lambda \in \mathbb{R}$ be given. A conjoined basis $Z(\cdot, \lambda) = (X(\cdot, \lambda), U(\cdot, \lambda))$ of $(\mathcal{S}_\lambda)$ has a proper focal point of multiplicity $m(t_0) \geq 1$ at the point t_0 if t_0 is left-dense and

$$m(t_0) := \operatorname{def} X(t_0, \lambda) - \operatorname{def} X(t_0^-, \lambda) = \dim \big([\operatorname{Ker} X(t_0^-, \lambda)]^\perp \cap \operatorname{Ker} X(t_0, \lambda) \big), \tag{2.5}$$

while it has a proper focal point of multiplicity $m(t_0) \geq 1$ in the interval $(\rho(t_0), t_0]_{\mathbb{T}}$ if t_0 is left-scattered and

$$m(t_0) := \operatorname{rank} M(\rho(t_0), \lambda) + \operatorname{ind} P(\rho(t_0), \lambda). \tag{2.6}$$

Here $\operatorname{def} A$ and $\operatorname{ind} A$ denote the defect and index of a matrix A, i.e., the dimension of its kernel and the number of its negative eigenvalues, respectively. This means that the conjoined basis $Z(\cdot, \lambda)$ does not have any proper focal points in $(a, b]_{\mathbb{T}}$ if

$$\operatorname{Ker} X(t, \lambda) \subseteq \operatorname{Ker} X(\tau, \lambda) \quad \text{for all } t, \tau \in [a, b]_{\mathbb{T}},\ \tau \leq t, \tag{2.7}$$

$$P(t, \lambda) \geq 0 \quad \text{for all } t \in [a, \rho(b)]_{\mathbb{T}}, \tag{2.8}$$

see also [15, Definition 4.1]. In order to avoid infinitely many proper focal points in the interval $(a, b]_{\mathbb{T}}$, the following assumption was introduced in [21, pg. 95].

$$\left.\begin{array}{l} \text{For every } \lambda \in \mathbb{R}, \\ \quad \text{(i) } \operatorname{Ker} X(\cdot, \lambda) \text{ is piecewise constant on } [a, b]_{\mathbb{T}}, \\ \quad \text{(ii) the function } P(\cdot, \lambda) \text{ is nonnegative definite in some right neigh-} \\ \qquad \text{borhood of every right-dense point } t_0 \in [a, b)_{\mathbb{T}} \text{ and in some left} \\ \qquad \text{neighborhood of every left-dense point } t_0 \in (a, b]_{\mathbb{T}}. \end{array}\right\} \tag{2.9}$$

Assumption (2.9) implies that the number of proper focal points of $Z(\cdot, \lambda)$ in $(a, b]_{\mathbb{T}}$ is finite, because the numbers $m(t_0)$ defined in (2.5) and (2.6) can now be positive only at finitely many points. In addition, by [21, Remark 3.4(viii)] we have $m(t_0) \leq n$.

With the system $(\mathcal{S}_\lambda)$ we associate the quadratic functional

$$\mathcal{F}_\lambda(z) := \int_a^b \Omega(z, z)(t)\, \Delta t - \lambda\, \langle z, z \rangle_W,$$

where for $z = (x, u)$ and $\tilde{z} = (\tilde{x}, \tilde{u})$ we define (suppressing the argument t)

$$\Omega(z, \tilde{z}) := x^T \mathcal{C}^T (I + \mu\mathcal{A})\, \tilde{x} + \mu\, x^T \mathcal{C}^T \mathcal{B}\tilde{u} + \mu\, u^T \mathcal{B}^T \mathcal{C}\tilde{x} + u^T (I + \mu\mathcal{D}^T)\, \mathcal{B}\tilde{u}. \tag{2.10}$$

The pair $z = (x, u)$ is *admissible* if $x \in \mathrm{C}^1_{\mathrm{prd}}$ on $[a, b]_{\mathbb{T}}$, $\mathcal{B}u \in \mathrm{C}_{\mathrm{prd}}$ on $[a, \rho(b)]_{\mathbb{T}}$, and it satisfies the first equation in $(\mathcal{S}_\lambda)$ on $[a, \rho(b)]_{\mathbb{T}}$. The functional $\mathcal{F}_\lambda$ is *positive definite* (or shortly positive) if $\mathcal{F}_\lambda(z) > 0$ for every $z = (x, u) \in \mathbb{A}$ with $x(\cdot) \not\equiv 0$, where

$$\mathbb{A} := \{z = (x, u),\ z \text{ is admissible and } x(a) = 0 = x(b)\}.$$

Since the first equation in the system $(\mathcal{S}_\lambda)$ does not contain λ, the admissible set $\mathbb{A}$ is the same for all $\lambda \in \mathbb{R}$. Denote by

$n_1(\lambda) :=$ the number of proper focal points of $\hat{Z}(\cdot, \lambda)$ in $(a, b]_{\mathbb{T}}$,

$n_2(\lambda) :=$ the number of finite eigenvalues of (E) which are less or equal to λ,

where we recall $\hat{Z}(\cdot, \lambda) = (\hat{X}(\cdot, \lambda), \hat{U}(\cdot, \lambda))$ to be the principal solution of $(\mathcal{S}_\lambda)$. The quantities $n_1(\lambda)$ and $n_2(\lambda)$ include the multiplicities of proper focal points and finite eigenvalues. The following characterization of the positivity of $\mathcal{F}_\lambda$ was proven in [15, Theorem 4.1].

Proposition 2.1 (Positivity)**.** *Let $\lambda \in \mathbb{R}$ be fixed. The functional $\mathcal{F}_\lambda$ is positive definite if and only if the principal solution of $(\mathcal{S}_\lambda)$ has no proper focal points in $(a, b]_{\mathbb{T}}$, i.e., if and only if $n_1(\lambda) = 0$.*

The relationship between the numbers $n_1(\lambda)$ and $n_2(\lambda)$ is described in the following result from [21, Corollary 6.4] combined with Corollary 5.2 below.

Proposition 2.2 (Oscillation theorem)**.** *Assume that the principal solution $\hat{Z}(\cdot, \lambda) = (\hat{X}(\cdot, \lambda), \hat{U}(\cdot, \lambda))$ of $(\mathcal{S}_\lambda)$ satisfies condition* (2.9)*. Then*

$$n_1(\lambda) = n_2(\lambda) \quad \textit{for all } \lambda \in \mathbb{R} \tag{2.11}$$

if and only if there exists $\lambda_0 < 0$ such that the functional $\mathcal{F}_{\lambda_0}$ is positive definite.

3. Technical calculations

In this section we collect some technical results regarding admissible pairs, which are needed in the proofs of the Rayleigh principle in Section 4 and the Sturmian separation and comparison theorems in Section 5. Throughout this section we let $Z = (X, U)$ be a conjoined basis of $(\mathcal{S})$ with finitely many proper focal points in $(a, b]_{\mathbb{T}}$, and recall the definition of $\Omega(z, \hat{z})$ from (2.10).

Lemma 3.1. *Let $z = (x, u)$ be admissible and $\hat{z} = (\hat{x}, \hat{u})$ be such that $\hat{x} \in \mathrm{C}_{\mathrm{prd}}$ on $[a, b]_{\mathbb{T}}$ and $\hat{u} \in \mathrm{C}^1_{\mathrm{prd}}$ on $[a, b)_{\mathbb{T}}$. Then*

$$\int_a^b \Omega(z, \hat{z})(t)\, \Delta t = (x^T \hat{u})(t)\, \Big|_a^b - \int_a^b \{(x^\sigma)^T (\hat{u}^\Delta - \mathcal{C}\hat{x} - \mathcal{D}\hat{u})\}(t)\, \Delta t. \tag{3.1}$$

On the other hand, if $\hat{z} \in \mathrm{C}_{\mathrm{prd}}$ only, then for every $t_0 \in [a, \rho(b)]_{\mathbb{T}}$

$$\mu(t_0)\, \Omega(z, \hat{z})(t_0) = (x^T \hat{u})(t)\, \Big|_{t_0}^{\sigma(t_0)} - \big\{(x^\sigma)^T [\, \hat{u}^\sigma - \mu\mathcal{C}\hat{x} - (I + \mu\mathcal{D})\, \hat{u}\,]\big\}(t_0). \tag{3.2}$$

Proof. Identity (3.1) follows by the integration by parts formula using the equality $\mu \mathcal{C}^T \mathcal{B} = \mathcal{A}^T + \mathcal{D} + \mu \mathcal{A}^T \mathcal{D}$, obtained from (1.1). Formula (3.2) is proven in a similar way. ■

Lemma 3.2. *If there exist points $t_1, t_2 \in [a,b]_{\mathbb{T}}$, $t_1 < t_2$, such that $\operatorname{Ker} X(t_2) \not\subseteq \operatorname{Ker} X(t_1)$, then for each vector $d \in \operatorname{Ker} X(t_2) \setminus \operatorname{Ker} X(t_1)$ the pair $z = (x,u)$ defined by*

$$\big(x(t),\, u(t)\big) := \begin{cases} \big(X(t)\, d,\, U(t)\, d\big), & t \in [a, t_2)_{\mathbb{T}}, \\ (0,0), & t \in [t_2, b]_{\mathbb{T}}, \end{cases} \tag{3.3}$$

is admissible, $x(\cdot) \not\equiv 0$ on $[a,b]_{\mathbb{T}}$, and

$$\mathcal{F}_0(z) = -d^T X^T(a)\, U(a)\, d.$$

In particular, if $Z = \hat{Z}$ is the principal solution of $(\mathcal{S})$, then $z \in \mathbb{A}$ and $\mathcal{F}_0(z) = 0$.

Proof. See [15, Proposition 6.3] and its proof. ■

Lemma 3.3. *If there exists a left-scattered point $t_0 \in (a,b]_{\mathbb{T}}$ such that $P(\rho(t_0)) \not\geq 0$, then for each vector $c \in \mathbb{R}^n$ with $c^T P(\rho(t_0))\, c < 0$ the pair $z = (x,u)$ defined by*

$$\big(x(t),\, u(t)\big) := \begin{cases} \big(X(t)\, d,\, U(t)\, d\big), & t \in [a, \rho(t_0))_{\mathbb{T}}, \\ \big(X(t)\, d,\, U(t)\, d - T(t)\, c\big), & t = \rho(t_0), \\ (0,0), & t \in [t_0, b]_{\mathbb{T}}, \end{cases} \tag{3.4}$$

where $d := \{\mu (X^\sigma)^\dagger \mathcal{B} T c\}(\rho(t_0))$, is admissible, $x(\cdot) \not\equiv 0$ on $[a,b]_{\mathbb{T}}$, and

$$\mathcal{F}_0(z) = -d^T X^T(a)\, U(a)\, d + \mu(\rho(t_0))\; c^T P(\rho(t_0))\, c.$$

In particular, if $Z = \hat{Z}$ is the principal solution of $(\mathcal{S})$, then $z \in \mathbb{A}$ and $\mathcal{F}_0(z) < 0$.

Proof. See [15, Proposition 6.2] and Subcases IIa–IIb in its proof. Note that in the latter reference the definitions of the admissible pairs $z = (x,u)$ can be unified to have the form as in (3.4). ■

Remark 3.4. If $z_1 = (x_1, u_1)$ and $z_2 = (x_2, u_2)$ are two admissible pairs defined by formulas (3.3) and/or (3.4) through vectors d_1 and d_2, respectively, then the symmetry of $X^T(a)\, U(a)$ implies the identity

$$x_1^T(a)\, u_2(a) = d_1^T X^T(a)\, U(a)\, d_2 = d_1^T U^T(a)\, X(a)\, d_2 = u_1^T(a)\, x_2(a).$$

Next we calculate the value of the integral $\int_a^b \Omega(z_1, z_2)(t)\, \Delta t$ when the functions z_1 and z_2 in (3.3) and (3.4) correspond to proper focal points of the conjoined basis Z. Following the definition of proper focal points in (2.5)–(2.6), we distinguish the cases when Z has a proper focal point at some point t_0, meaning that either $\operatorname{def} X(t_0) - \operatorname{def} X(t_0^-) \geq 1$ if t_0 is left-dense or $\operatorname{rank} M(\rho(t_0)) \geq 1$ if t_0 is left-scattered, and when Z has a proper focal point in $(\rho(t_0), t_0)_{\mathbb{T}}$, meaning that $\operatorname{ind} P(\rho(t_0)) \geq 1$ if t_0 is left-scattered. Note that as in [18, Lemma 1(ii)] we have at all left-scattered points $t_0 \in (a,b]_{\mathbb{T}}$ the equivalence $M(\rho(t_0)) = 0$ if and only if $\operatorname{Ker} X(t_0) \subseteq \operatorname{Ker} X(\rho(t_0))$.

Lemma 3.5. *Suppose that Z has proper focal points at some (not necessarily distinct) points τ_1 and τ_2 in $(a,b]_{\mathbb{T}}$. Then there are vectors d_1, d_2 such that $d_j \in \operatorname{Ker} X(\tau_j)$ and either $d_j \notin \operatorname{Ker} X(\tau_j^-) \equiv \operatorname{Ker} X(\tau_j - \varepsilon)$ for some $\varepsilon > 0$ small enough if τ_j is left-dense,* *or $d_j \notin \operatorname{Ker} X(\rho(\tau_j))$ if τ_j is left-scattered, $j \in \{1,2\}$. In both cases the vectors d_1, d_2 satisfy the assumption of Lemma 3.2, so that for the admissible pairs $z_1 = (x_1, u_1)$ and $z_2 = (x_2, u_2)$ constructed through formula (3.3) with $t_2 := \tau_j$ and $t_1 := \tau_j - \varepsilon$ if τ_j is left-dense and $t_1 := \rho(\tau_j)$ if τ_j is left-scattered we have*

$$\int_a^b \Omega(z_1, z_2)(t)\,\Delta t = -u_1^T(a)\,x_2(a). \tag{3.5}$$

Proof. The result follows from identity (3.1) of Lemma 3.1 on the interval $[a, \min\{\tau_1, \tau_2\}]_{\mathbb{T}}$. The details of this calculation, as well as of similar calculations below, are here omitted. ■

Lemma 3.6. *Suppose that Z has proper focal points at some point τ_1 (which can be either left-dense or left-scattered) and in $(\rho(\tau_2), \tau_2)_{\mathbb{T}}$ where τ_2 is left-scattered. Then there is a vector $d_1 \in \mathbb{R}^n$ and an admissible $z_1 = (x_1, u_1)$ defined by (3.3) which satisfies Lemma 3.2 with $t_2 := \tau_1$ and $t_1 := \tau_1 - \varepsilon$ if τ_1 is left-dense and $t_1 := \rho(\tau_1)$ if τ_1 is left-scattered. Also, there are vectors $c_2, d_2 \in \mathbb{R}^n$ and an admissible $z_2 = (x_2, u_2)$ defined by (3.4) which satisfies Lemma 3.3 with $t_0 := \tau_2$. And in this case formula (3.5) holds.*

Proof. The result is proven by applying identity (3.1) of Lemma 3.1 on $[a, \tau_1]_{\mathbb{T}}$ if $\tau_1 < \tau_2$ or on $[a, \tau_2]_{\mathbb{T}}$ if $\tau_2 < \tau_1$, and when $\tau_1 = \tau_2$ by applying identity (3.1) on $[a, \rho(\tau_1)]_{\mathbb{T}}$ and identity (3.2) at $\rho(\tau_1)$. ■

Lemma 3.7. *Assume that Z has proper focal points in $(\rho(\tau_1), \tau_1)_{\mathbb{T}}$ and $(\rho(\tau_2), \tau_2)_{\mathbb{T}}$ where τ_1 and τ_2 are left-scattered. Then there are vectors $c_1, d_1, c_2, d_2 \in \mathbb{R}^n$ and admissible pairs $z_1 = (x_1, u_1)$ and $z_2 = (x_2, u_2)$ defined by (3.4) which satisfy Lemma 3.3 with $t_0 := \tau_1$ and $t_0 := \tau_2$, respectively. In addition, if $\tau_1 \neq \tau_2$, then formula (3.5) holds, while if $\tau_1 = \tau_2$, then we have*

$$\int_a^b \Omega(z_1, z_2)(t)\,\Delta t = -u_1^T(a)\,x_2(a) + \mu(\rho(\tau_1))\; c_1^T P(\rho(\tau_1))\,c_2. \tag{3.6}$$

Proof. The first part is proven by identity (3.1) of Lemma 3.1 on $[a, \tau_1]_{\mathbb{T}}$ if $\tau_1 < \tau_2$ or on $[a, \tau_2]_{\mathbb{T}}$ if $\tau_2 < \tau_1$. The second part, i.e, formula (3.6), follows by identity (3.1) on $[a, \tau_1]_{\mathbb{T}}$ and by identity (3.2) at $\rho(\tau_1)$. ■

The next result corresponds to the discrete time case in [10, Lemma 4].

Lemma 3.8. *Let $t_0 \in (a, b]_{\mathbb{T}}$ be left-scattered such that the conjoined basis $Z = (X, U)$ has a proper focal point of multiplicity $m = p + q$ in $(\rho(t_0), t_0]_{\mathbb{T}}$, where $p := \operatorname{rank} M(\rho(t_0))$ and $q := \operatorname{ind} P(\rho(t_0))$. Let $d_1, \dots, d_p \in \operatorname{Ker} X(t_0) \setminus \operatorname{Ker} X(\rho(t_0))$ be linearly independent vectors associated with the proper focal point at t_0 and let $c_1, \dots, c_q$ be the orthonormal eigenvectors corresponding to the negative eigenvalues of $P(\rho(t_0))$. Then with*

$$d_{p+j} := \mu(\rho(t_0))\,X^{\dagger}(t_0)\,\mathcal{B}(\rho(t_0))\,T(\rho(t_0))\,c_j \quad \text{for } j \in \{1, \dots, q\} \tag{3.7}$$

the vectors $d_1, \dots, d_p, d_{p+1}, \dots, d_{p+q}$ are (all together) linearly independent.

Proof. If $q = 0$, then the result is trivial by the assumed linear independence of $d_1, \dots, d_p$. Thus, we assume that $q \geq 1$. Denote by $\lambda_1, \dots, \lambda_q$ the negative eigenvalues of $P(\rho(t_0))$ associated with the orthonormal eigenvectors $c_1, \dots, c_q$. First we prove that the vectors

$d_{p+1},\dots,d_{p+q}$ are linearly independent. Let $g := \sum_{j=1}^{q} \alpha_j\, d_{p+j}$ and suppose that $g = 0$ for some $\alpha_1,\dots,\alpha_q \in \mathbb{R}$. If we define $c := \sum_{j=1}^{q} \alpha_j\, c_j$ and abbreviate by T_0, X_0, $\mathcal{B}_0$, P_0, and μ_0 the values of $T(t)$, $X(t)$, $\mathcal{B}(t)$, $P(t)$, and $\mu(t)$ at $t = \rho(t_0)$, respectively, then

$$\begin{aligned}
0 &= c^T T_0 X_0\, g = c^T T_0 X_0 \sum_{j=1}^{q} \alpha_j\, \mu_0 X^\dagger(t_0)\, \mathcal{B}_0 T_0\, c_j = \mu_0\, c^T T_0 X_0 X^\dagger(t_0)\, \mathcal{B}_0 T_0\, c = \mu_0\, c^T P_0\, c \\
&= \mu_0 \sum_{i=1}^{q} \sum_{j=1}^{q} \alpha_i\, \alpha_j\, c_i^T P_0\, c_j = \mu_0 \sum_{i=1}^{q} \sum_{j=1}^{q} \alpha_i\, \alpha_j\, \lambda_j\, c_i^T c_j = \mu_0 \sum_{j=1}^{q} \alpha_j^2\, \lambda_j \le 0.
\end{aligned}$$

This is possible only if $\alpha_1 = \dots = \alpha_q = 0$, which shows the linear independence of $d_{p+1},\dots,d_{p+q}$. Next, if $p = 0$, then the proof is finished, so we assume further on that $p \ge 1$. Let $e := f + g$, where for some $\alpha_1,\dots,\alpha_{p+q} \in \mathbb{R}$ we define $f := \sum_{j=1}^{p} \alpha_j\, d_j$ and $g := \sum_{j=1}^{q} \alpha_{p+j}\, d_{p+j}$ as above. Then $f \in \operatorname{Ker} X(t_0)$. If we assume that $e = 0$, then for $c := \sum_{j=1}^{q} \alpha_{p+j}\, c_j$ we have

$$\begin{aligned}
0 &= -c^T T_0 X_0 X^\dagger(t_0)\, X(t_0)\, f = c^T T_0 X_0 X^\dagger(t_0)\, X(t_0)\, g \\
&\overset{(3.7)}{=} c^T T_0 X_0 X^\dagger(t_0)\, X(t_0) \sum_{j=1}^{q} \alpha_{p+j}\, \mu_0\, X^\dagger(t_0)\, \mathcal{B}_0 T_0\, c_j = \mu_0\, c^T P_0\, c = \mu_0 \sum_{j=1}^{q} \alpha_{p+j}^2\, \lambda_j \le 0,
\end{aligned}$$

where we used the formula $X^\dagger = X^\dagger X X^\dagger$. This is however possible only if $\alpha_{p+1} = \dots = \alpha_{p+q} = 0$. Therefore, we have $g = 0$, and consequently also $f = -g = 0$. The linear independence of $d_1,\dots,d_p$ now implies that $\alpha_1 = \dots = \alpha_p = 0$ as well. Hence, the vectors $d_1,\dots,d_{p+q}$ are linearly independent. ∎

4. Rayleigh principle

In this section we prove the following variational characterization of the finite eigenvalues of the eigenvalue problem (E). This theorem is a time scale generalization of the continuous and discrete time results in [20, Theorem 1.1] and [6, Theorem 4.6]

Theorem 4.1 (Rayleigh principle)**.** *Assume that the principal solution $\hat{Z}(\cdot,\lambda)$ satisfies condition* (2.9)*, the functional $\mathcal{F}_{\lambda_0}$ is positive definite for some $\lambda_0 < 0$, and* (1.2) *holds. Let $\lambda_1 \le \dots \le \lambda_m \le \dots$ be the finite eigenvalues of the eigenvalue problem* (E) *with the corresponding orthonormal finite eigenfunctions $z_1,\dots,z_m,\dots$. Then for each $m \in \mathbb{N} \cup \{0\}$*

$$\lambda_{m+1} = \min \left\{ \frac{\mathcal{F}_0(z)}{\langle z, z\rangle_W},\ z \in \mathbb{A},\ (Wx^\sigma)(\cdot) \not\equiv 0,\ z \perp z_1,\dots,z_m \right\}. \tag{4.1}$$

The list of finite eigenvalues $\lambda_1 \le \dots \le \lambda_m \le \dots$ in Theorem 4.1 really makes sense, because by [21, Proposition 4.5 and Corollary 6.3] the finite eigenvalues of (E) are isolated and bounded below. In addition, when there are only finitely many (say $p < \infty$) finite eigenvalues of (E), then we put $\lambda_{p+1} := \infty$ in (4.1). Before proving Theorem 4.1 we shall develop some necessary auxiliary tools.

Lemma 4.2. *Let $z_1, \dots, z_m$ be orthonormal finite eigenfunctions of* (E) *corresponding to the (not necessarily distinct and not necessarily consecutive) finite eigenvalues $\lambda_1, \dots, \lambda_m$. For any $\beta_1, \dots, \beta_m \in \mathbb{R}$ we set $\hat{z} := \sum_{i=1}^m \beta_i z_i$. Then $\hat{z} = (\hat{x}, \hat{u}) \in \mathrm{C}^1_{\mathrm{prd}} \cap \mathbb{A}$ and*

$$\mathcal{F}_0(\hat{z}) = \sum_{i=1}^m \lambda_i \beta_i^2, \quad \|\hat{z}\|_W^2 = \sum_{i=1}^m \beta_i^2. \tag{4.2}$$

Proof. The identities in (4.2) follow by direct calculations by the aid of Lemma 3.1, compare also with the proof of [20, Lemma 2.13]. ■

Lemma 4.3 (Global Picone formula)**.** *Let $\lambda \in \mathbb{R}$ be fixed and suppose that $Z = (X, U)$ is a conjoined basis of $(\mathcal{S}_\lambda)$ satisfying conditions (i) and (ii) in* (2.9)*. Then for any admissible $z = (x, u)$ with $x(t) \in \operatorname{Im} X(t)$ on $[a, b]_{\mathbb{T}}$ we have*

$$\mathcal{F}_\lambda(z) \geq \int_a^b w^T(t)\, P(t)\, w(t)\, \Delta t + \{x^T U X^\dagger x\}(t)\, \Big|_a^b, \tag{4.3}$$

where $w := u - U X^\dagger x$. If, in addition, kernel condition (2.7) *holds, $\int_a^b w^T(t)\, P(t)\, w(t)\, \Delta t = 0$, and $x(b) = 0$, then $x(t) \equiv 0$ on $[a, b]_{\mathbb{T}}$.*

Proof. The result follows from [24, Theorem 3.19] and its proof. Note that the assumption $P(t) \geq 0$ used in the proof of [24, Theorem 3.19] is satisfied under conditions (i) and (ii) in (2.9). ■

Remark 4.4. In the global Picone formula (4.3) we have the equality sign if the kernel of $X(\cdot, \lambda)$ changes only at isolated points, which is for example the case of discrete time in [5, Proposition 2.1(iv)].

In the next result we extend Lemma 4.3 to include the finite eigenfunctions of (E). This statement is an extension of [6, Theorem 4.2] and [20, Theorem 3.1] to general time scales.

Theorem 4.5 (Extended global Picone formula)**.** *Assume* (1.2) *and fix $\lambda \in \mathbb{R}$. Let $Z = (X, U)$ be a a conjoined basis of $(\mathcal{S}_\lambda)$ satisfying conditions (i) and (ii) in* (2.9)*. Let $\lambda_1 \leq \cdots \leq \lambda_m$ be finite eigenvalues of* (E) *with the corresponding orthonormal finite eigenfunctions $z_1, \dots, z_m$. For any $\beta_1, \dots, \beta_m \in \mathbb{R}$ we set $\hat{z} := \sum_{i=1}^m \beta_i z_i$. Finally, let $z = (x, u) \in \mathbb{A}$ be such that $z \perp z_1, \dots z_m$ and such that $\tilde{z} = (\tilde{x}, \tilde{u}) := z + \hat{z}$ satisfies the image condition*

$$\tilde{x}(t) \in \operatorname{Im} X(t) \quad \text{for all } t \in [a, b]_{\mathbb{T}}.$$

Then with $\tilde{w} := \tilde{u} - U X^\dagger \tilde{x}$ on $[a, b]_{\mathbb{T}}$ we have the inequality

$$\mathcal{F}_\lambda(z) \geq \int_a^b \tilde{w}^T(t)\, P(t)\, \tilde{w}(t)\, \Delta t + \sum_{i=1}^m (\lambda - \lambda_i)\, \beta_i^2. \tag{4.4}$$

Proof. From $\tilde{z} = z + \hat{z}$ we have

$$\mathcal{F}_\lambda(\tilde{z}) = \mathcal{F}_\lambda(z) + \mathcal{F}_\lambda(\hat{z}) + 2 \int_a^b \Omega(z, \hat{z})(t)\, \Delta t. \tag{4.5}$$

We now evaluate the terms in (4.5) separately. By Lemma 4.3 and $\tilde{z} \in \mathbb{A}$ we have

$$\mathcal{F}_\lambda(\tilde{z}) \geq \int_a^b \tilde{w}^T(t)\, P(t)\, \tilde{w}(t)\, \Delta t, \tag{4.6}$$

while Lemma 4.2 yields

$$\mathcal{F}_\lambda(\hat{z}) = \mathcal{F}_0(\hat{z}) - \lambda\, \langle \hat{z}, \hat{z} \rangle_W = \sum_{i=1}^m (\lambda_i - \lambda)\, \beta_i^2. \tag{4.7}$$

For the last term in (4.5) we have by Lemma 3.1

$$\int_a^b \Omega(z, \hat{z})(t)\, \Delta t = \sum_{i=1}^m \beta_i \int_a^b \Omega(z, z_i)(t)\, \Delta t = \sum_{i=1}^m \beta_i\, \lambda_i\, \langle z, z_i \rangle_W = 0, \tag{4.8}$$

because $z \perp z_1, \dots z_m$. Upon inserting formulas (4.6)–(4.8) into equation (4.5) we obtain the result in (4.4). ■

We are now ready to establish the Rayleigh principle on time scales.

Proof of Theorem 4.1. Let $\hat{Z}(\cdot, \lambda) = (\hat{X}(\cdot, \lambda), \hat{U}(\cdot, \lambda))$ be the principal solution of $(\mathcal{S}_\lambda)$. Assumption (2.9) for the principal solution and $\mathcal{F}_{\lambda_0} > 0$ implies through Proposition 2.2 that equality (2.11) holds, i.e., $n_1(\lambda) = n_2(\lambda)$ for all $\lambda \in \mathbb{R}$. Moreover, by Proposition 2.1, we have $n_1(\lambda) \equiv 0$ for all $\lambda \leq \lambda_0$.

Let us fix $m \in \mathbb{N} \cup \{0\}$. Consider the first $m+1$ finite eigenvalues (including their multiplicities) $\lambda_1 \leq \dots \leq \lambda_{m+1}$ of (E) with the corresponding orthonormal finite eigenfunctions $z_1, \dots, z_{m+1}$. For convenience we put $\lambda_0 := -\infty$. Suppose that for a given $\lambda \in \mathbb{R}$ we have $\lambda \in (\lambda_m, \lambda_{m+1})$, that is, $n_2(\lambda) = m$ and λ is not a finite eigenvalue of (E). Hence, by the definition of finite eigenvalues,

$$\operatorname{rank} \hat{X}(b, \lambda_0) = r(b) = \max_{\kappa \in \mathbb{R}} \operatorname{rank} \hat{X}(b, \kappa).$$

This yields that $\operatorname{def} \hat{X}(b, \lambda) = \operatorname{def} \hat{X}(b, \lambda_0)$. And since $\mathcal{F}_{\lambda_0} > 0$ is assumed, it follows that b is not a proper focal point of $\hat{Z}(\cdot, \lambda)$, compare with the argument in the proof of [6, Theorem 4.6, pg. 3120]. Therefore, the principal solution $\hat{Z}(\cdot, \lambda)$ has exactly m proper focal points in the open interval $(a, b)_{\mathbb{T}}$ and $n_1(\lambda) = n_2(\lambda) = m$. Let us denote these proper focal points by $\tau_1 < \dots < \tau_l$, where $\tau_1 > a$ and $\tau_l < b$, and where the multiplicities of these proper focal points add up to m. By definition, if the point τ_j is left-dense, then its multiplicity as a proper focal point of $\hat{Z}(\cdot, \lambda)$ is equal to

$$m_j := \operatorname{def} \hat{X}(\tau_j, \lambda) - \operatorname{def} \hat{X}(\tau_j^-, \lambda) = \dim \left([\operatorname{Ker} \hat{X}(\tau_j^-, \lambda)]^\perp \cap \operatorname{Ker} \hat{X}(\tau_j, \lambda) \right),$$

while if the point τ_j is left-scattered, then its multiplicity is

$$m_j := \operatorname{rank} M(\rho(\tau_j), \lambda) + \operatorname{ind} P(\rho(\tau_j), \lambda).$$

Moreover, the numbers m_j satisfy $\sum_{j=1}^l m_j = m$.

Consider now a linear combination $\hat{z} = (\hat{x}, \hat{u})$ of the finite eigenfunctions $z_1, \dots, z_m$, that is, $\hat{z} = \sum_{i=1}^m \beta_i z_i$, where the coefficients $\beta_1, \dots, \beta_m \in \mathbb{R}$ are at this moment unspecified. Then $\hat{z}$ is admissible and $\hat{x}(a) = 0 = \hat{x}(b)$, i.e., $\hat{z} \in \mathbb{A}$.

For the function $\tilde{z} = (\tilde{x}, \tilde{u}) := \hat{z}$ we consider the homogeneous system of linear equations for the variables $\beta_1, \ldots, \beta_m$ determined by the conditions

$$\tilde{x}(\tau_j) \in \big([\operatorname{Ker} \hat{X}^T(\tau_j^-, \lambda)]^\perp \cap \operatorname{Ker} \hat{X}^T(\tau_j, \lambda) \big)^\perp \quad \text{if } \tau_j \text{ is left-dense}, \tag{4.9}$$

$$\left.\begin{aligned} & M^T(\rho(\tau_j), \lambda)\, \tilde{x}(\tau_j) = 0, \\ & \tilde{w}(\rho(\tau_j)) \perp \big\{\alpha \in \mathbb{R}^n,\ \ \alpha \text{ is an eigenvector} \\ & \qquad\qquad \text{corresponding to a negative} \\ & \qquad\qquad \text{finite eigenvalue of } P(\rho(\tau_j), \lambda)\big\} \end{aligned}\right\} \quad \text{if } \tau_j \text{ is left-scattered}, \tag{4.10}$$

where $\tilde{w}(t) := \tilde{u}(t) - \hat{U}(t, \lambda)\, \hat{X}^\dagger(t, \lambda)\, \tilde{x}(t)$. Since for each left-dense point τ_j we have

$$\begin{aligned} \operatorname{def} \hat{X}^T(\tau_j, \lambda) - \operatorname{def} \hat{X}^T(\tau_j^-, \lambda) &= \operatorname{rank} \hat{X}^T(\tau_j^-, \lambda) - \operatorname{rank} \hat{X}^T(\tau_j, \lambda) \\ &= \operatorname{rank} \hat{X}(\tau_j^-, \lambda) - \operatorname{rank} \hat{X}(\tau_j, \lambda) \\ &= \operatorname{def} \hat{X}(\tau_j, \lambda) - \operatorname{def} \hat{X}(\tau_j^-, \lambda) = m_j, \end{aligned}$$

the number of equations in (4.9) is exactly the sum of the multiplicities m_j corresponding to the left-dense proper focal points τ_j. Moreover, the numbers of linearly independent equations in (4.10)(i) and (4.10)(ii) are respectively $\operatorname{rank} M^T(\rho(\tau_j), \lambda) = \operatorname{rank} M(\rho(\tau_j), \lambda)$ and $\operatorname{ind} P(\rho(\tau_j), \lambda)$, so that the conditions in (4.10) represent in total exactly that many equations as is the sum of the multiplicities m_j corresponding to the left-scattered proper focal points τ_j. Altogether, there are exactly $\sum_{j=1}^{l} m_j = m$ linearly independent homogeneous equations in system (4.9)–(4.10) for the m variables $\beta_1, \ldots, \beta_m$.

We shall prove by the time scale induction principle, see [7, Theorem 1.7], that

$$\tilde{x}(t) \in \operatorname{Im} \hat{X}(t, \lambda) \quad \text{for all } t \in [a, b]_{\mathbb{T}}. \tag{4.11}$$

Therefore, for $t_0 \in [a, b]_{\mathbb{T}}$ we consider the statement

$$\mathbb{S}(t_0) :=\ \tilde{x}(t) \in \operatorname{Im} \hat{X}(t, \lambda) \ \text{ for all } t \in [a, t_0]_{\mathbb{T}}.$$

(I) Initial condition. Let $t_0 = a$. Then $\tilde{x}(a) = 0 \in \operatorname{Im} \hat{X}(a, \lambda)$, so that the statement $\mathbb{S}(a)$ holds true.

(II) Jump condition. Let $t_0 \in [a, \rho(b)]_{\mathbb{T}}$ be right-scattered and suppose that $\mathbb{S}(t_0)$ holds. Then $\tilde{x}(t_0) = \hat{X}(t_0, \lambda)\, c \in \operatorname{Im} \hat{X}(t_0, \lambda)$ for some $c \in \mathbb{R}^n$. If $\sigma(t_0)$ is not one of the proper focal points of $\hat{Z}(\cdot, \lambda)$, then we have $\operatorname{Ker} \hat{X}^\sigma(t_0, \lambda) \subseteq \operatorname{Ker} \hat{X}(t_0, \lambda)$, and then $\tilde{x}^\sigma(t_0) \in \operatorname{Im} \hat{X}^\sigma(t_0, \lambda)$ follows from [15, Proposition 5.2] on $[t_0, \sigma(t_0)]_{\mathbb{T}}$, i.e., from the relation between the kernel condition and the image condition. On the other hand, if $\sigma(t_0) = \tau_j$ for some $j \in \{1, \ldots, l\}$, i.e., if $(t_0, \sigma(t_0)]_{\mathbb{T}}$ contains a proper focal point of $\hat{Z}(\cdot, \lambda)$, then by (4.10)(i)

$$0 = M^T(t_0, \lambda)\, \tilde{x}^\sigma(t_0) = \mathcal{B}^T(t_0)\, [I - \hat{X}^\sigma(t_0, \lambda)\, \hat{X}^{\sigma\dagger}(t_0, \lambda)]\, \tilde{x}^\sigma(t_0). \tag{4.12}$$

The admissibility of $\tilde{z}$ yields as in [6, Lemma 4.3(ii)] that (suppressing the arguments t_0 and λ in the calculations below)

$$\tilde{x}^\sigma = (I + \mu \mathcal{A})\, \hat{X} c + \mu \mathcal{B} \tilde{u} = (\hat{X}^\sigma - \mu \mathcal{B} \hat{U})\, c + \mu \mathcal{B} \tilde{u} = \hat{X}^\sigma c + \mu \mathcal{B}(\tilde{u} - \hat{U} c). \tag{4.13}$$

Hence, by inserting (4.13) into (4.12) we obtain

$$\begin{aligned} 0 = M^T \tilde{x}^\sigma &\overset{(4.13)}{=} \mathcal{B}^T (I - \hat{X}^\sigma \hat{X}^{\sigma\dagger})\, [\hat{X}^\sigma c + \mu \mathcal{B}(\tilde{u} - \hat{U} c)] \\ &= \mu \mathcal{B}^T (I - \hat{X}^\sigma \hat{X}^{\sigma\dagger})\, \mathcal{B}(\tilde{u} - \hat{U} c). \end{aligned} \tag{4.14}$$

Since the matrix $I - \hat{X}^\sigma \hat{X}^{\sigma\dagger}$ is a projection, the multiplication of equation (4.14) from the left by the vector $\mu\,(\tilde{u} - \hat{U}c)^T$ yields

$$\|(I - \hat{X}^\sigma \hat{X}^{\sigma\dagger})\, \mathcal{B}(\tilde{u} - \hat{U}c)\|^2 = 0, \quad \text{i.e.,} \quad (I - \hat{X}^\sigma \hat{X}^{\sigma\dagger})\, \mathcal{B}(\tilde{u} - \hat{U}c) = 0. \tag{4.15}$$

Here we recall that a real $n \times n$ matrix A is a projection if it is symmetric and $A^2 = A$. Therefore,

$$\tilde{x}^\sigma \overset{(4.13)}{=} \hat{X}^\sigma c + \mu \mathcal{B}(\tilde{u} - \hat{U}c) \overset{(4.15)}{=} \hat{X}^\sigma c + \mu \hat{X}^\sigma \hat{X}^{\sigma\dagger} \mathcal{B}(\tilde{u} - \hat{U}c) \in \operatorname{Im} \hat{X}^\sigma.$$

This shows that the statement $\mathbb{S}(\sigma(t_0))$ holds true.

(III) Continuation condition. Let $t_0 \in [a, b)_{\mathbb{T}}$ be right-dense and suppose that $\mathbb{S}(t_0)$ holds. Then $\tilde{x}(t_0) \in \operatorname{Im} \hat{X}(t_0, \lambda)$ and, since the kernel of $\hat{X}(\cdot, \lambda)$ is piecewise constant on $[a, b]_{\mathbb{T}}$, $\operatorname{Ker} \hat{X}(t, \lambda) \equiv \operatorname{Ker} \hat{X}(t_0^+, \lambda) \subseteq \operatorname{Ker} \hat{X}(t_0, \lambda)$ for all $t \in (t_0, t_0 + \varepsilon]_{\mathbb{T}}$ for some $\varepsilon > 0$. Thus, by [15, Proposition 5.2] on $[t_0, t_0+\varepsilon]_{\mathbb{T}}$ we get $\tilde{x}(t) \in \operatorname{Im} \hat{X}(t, \lambda)$ for all $t \in [t_0, t_0+\varepsilon]_{\mathbb{T}}$. Consequently, the statement $\mathbb{S}(t)$ holds for all $t \in (t_0, t_0 + \varepsilon]_{\mathbb{T}}$, which we wanted to prove.

(IV) Closure condition. Let $t_0 \in (a, b]_{\mathbb{T}}$ be left-dense and suppose that $\mathbb{S}(t)$ holds for all $t \in [a, t_0)_{\mathbb{T}}$, i.e., $\tilde{x}(t) \in \operatorname{Im} \hat{X}(t, \lambda)$ for all $t \in [a, t_0)_{\mathbb{T}}$. If t_0 is not one of the proper focal points of $\hat{Z}(\cdot, \lambda)$, then $\operatorname{Ker} \hat{X}(t_0^-, \lambda) = \operatorname{Ker} \hat{X}(t_0, \lambda)$, and in this case the image condition $\tilde{x}(t_0) \in \operatorname{Im} \hat{X}(t_0, \lambda)$ follows from [15, Proposition 5.2] on the interval $[t_0 - \varepsilon, t_0]_{\mathbb{T}}$ for some $\varepsilon > 0$ small enough, since we know that $\tilde{x}(t_0 - \varepsilon) \in \operatorname{Im} \hat{X}(t_0 - \varepsilon, \lambda)$. On the other hand, if $t_0 = \tau_j$ is one of the proper focal points of $\hat{Z}(\cdot, \lambda)$, then $\operatorname{Ker} \hat{X}(t_0^-, \lambda) \subsetneqq \operatorname{Ker} \hat{X}(t_0, \lambda)$ and, by (4.9),

$$\tilde{x}(t_0) \in \big([\operatorname{Ker} \hat{X}^T(t_0^-, \lambda)]^\perp \cap \operatorname{Ker} \hat{X}^T(t_0, \lambda) \big)^\perp = \operatorname{Ker} \hat{X}^T(t_0^-, \lambda) + [\operatorname{Ker} \hat{X}^T(t_0, \lambda)]^\perp. \tag{4.16}$$

The continuity of $\hat{X}^T(\cdot, \lambda)$ yields that

$$\operatorname{Ker} \hat{X}^T(t_0^-, \lambda) \subseteq \operatorname{Ker} \hat{X}^T(t_0, \lambda), \quad \text{i.e.,} \quad [\operatorname{Ker} \hat{X}^T(t_0, \lambda)]^\perp \subseteq [\operatorname{Ker} \hat{X}^T(t_0^-, \lambda)]^\perp,$$

so that the sum of subspaces in (4.16) is a direct sum. And since $\tilde{x}(\cdot)$ is continuous, we have

$$\tilde{x}(t_0) = \tilde{x}(t_0^-) \in \operatorname{Im} \hat{X}(t_0^-, \lambda) = [\operatorname{Ker} \hat{X}^T(t_0^-, \lambda)]^\perp.$$

Hence, by (4.16), it follows that

$$\tilde{x}(t_0) \in [\operatorname{Ker} \hat{X}^T(t_0, \lambda)]^\perp = \operatorname{Im} \hat{X}(t_0, \lambda).$$

This means that the statement $\mathbb{S}(t_0)$ holds, which is what we wanted to prove.

By the induction principle we conclude that the image condition (4.11) is satisfied. We now apply the extended global Picone formula (Theorem 4.5) with $z := 0$ to obtain

$$0 = \mathcal{F}_\lambda(z) \geq \int_a^b \tilde{w}^T(t)\, P(t, \lambda)\, \tilde{w}(t)\, \Delta t + \sum_{i=1}^m (\lambda - \lambda_i)\, \beta_i^2. \tag{4.17}$$

The second term in (4.17) is nonnegative, because $\lambda > \lambda_m \geq \cdots \geq \lambda_1$ is now assumed. Concerning the first term, we note that by (2.9)(ii) the matrix $P(t,\lambda) \geq 0$ everywhere in $[a,\rho(b)]_{\mathbb{T}}$ except possibly at finitely many right-scattered points t_0. And in this case the principal solution $\hat{Z}(\cdot,\lambda)$ has a proper focal point in $(t_0,\sigma(t_0))_{\mathbb{T}}$, i.e., $t_0 = \rho(\tau_j)$ and $\sigma(t_0) = \tau_j$ for some $j \in \{1,\dots,l\}$. From the construction in (4.10)(ii) we can see that $\tilde{w}(t_0)$ is orthogonal to the eigenvectors corresponding to the negative eigenvalues of the matrix $P(t_0)$. This implies that $\tilde{w}^T(t_0)\,P(t_0)\,\tilde{w}(t_0) \geq 0$, and consequently $\int_{t_0}^{\sigma(t_0)} \tilde{w}^T(t)\,P(t,\lambda)\,\tilde{w}(t)\,\Delta t \geq 0$ for each such a point t_0. Therefore,

$$\int_a^b \tilde{w}^T(t)\,P(t,\lambda)\,\tilde{w}(t)\,\Delta t \geq 0. \tag{4.18}$$

Combining (4.18) and (4.17) we get the inequality $0 \geq \sum_{i=1}^m (\lambda - \lambda_i)\,\beta_i^2 \geq 0$, so that by using $\lambda > \lambda_i$ for every $i = 1,\dots,m$ we must necessarily have $\beta_1 = \cdots = \beta_m = 0$. Consequently, the linear system representing equations (4.9)–(4.10) possesses only the trivial solution $\beta_1 = \cdots = \beta_m = 0$. Therefore, the matrix of this linear system must be invertible.

Let now $z = (x,u) \in \mathbb{A}$ be such that $z \perp z_1,\dots,z_m$. Then for the function $\tilde{z} := z + \hat{z}$ the conditions in (4.9)–(4.10) represent a linear system for the coefficients $\beta_1,\dots,\beta_m$ (and in general this system may be inhomogeneous) with an invertible coefficient matrix, as we just proved. Therefore, there exist unique $\beta_1,\dots,\beta_m \in \mathbb{R}$ satisfying this system. By the same way as in the previous part of this proof (i.e., by the time scale induction principle) we conclude that the image condition (4.11) is now satisfied for this $\tilde{z} = (\tilde{x},\tilde{u})$. The extended global Picone formula (Theorem 4.5) then yields

$$\mathcal{F}_\lambda(z) \geq \int_a^b \tilde{w}^T(t)\,P(t,\lambda)\,\tilde{w}(t)\,\Delta t + \sum_{i=1}^m (\lambda - \lambda_i)\,\beta_i^2 \geq 0, \tag{4.19}$$

since $\lambda > \lambda_i$ for every $i = 1,\dots,m$, and since (4.18) holds as a consequence of assumption (2.9) for $\hat{Z}(\cdot,\lambda)$ and the construction of $\tilde{w}(\cdot)$ in (4.10)(ii). From (4.19) we get

$$\mathcal{F}_0(z) - \lambda\,\langle z,z\rangle_W = \mathcal{F}_\lambda(z) \geq 0, \quad \text{i.e.,} \quad \mathcal{F}_0(z) \geq \lambda\,\langle z,z\rangle_W. \tag{4.20}$$

Inequality (4.20) is therefore established for every $\lambda \in (\lambda_m,\lambda_{m+1})$. If we now take the limit as $\lambda \to \lambda_{m+1}^-$, we get from (4.20) the inequality

$$\mathcal{F}_0(z) \geq \lambda_{m+1}\,\langle z,z\rangle_W,$$

showing that the infimum of the Rayleigh quotient $\mathcal{F}_0(z)/\langle z,z\rangle_W$ in (4.1) does not exceed λ_{m+1}. Since $z_{m+1} = (x_{m+1},u_{m+1})$ is a finite eigenfunction of (E) corresponding to the finite eigenvalue λ_{m+1}, it follows that $z_{m+1} \in \mathbb{A}$ and $W(\cdot)\,x_{m+1}^\sigma(\cdot) \not\equiv 0$ on $[a,\rho(b)]_{\mathbb{T}}$, and $\mathcal{F}_{\lambda_{m+1}}(z_{m+1}) = 0$. Hence,

$$\mathcal{F}_0(z_{m+1}) = \lambda_{m+1}\,\langle z_{m+1},z_{m+1}\rangle_W.$$

Since by the construction of the finite eigenfunctions we have $z_{m+1} \perp z_1,\dots,z_m$, it follows that the minimum in (4.1) is indeed equal to λ_{m+1} and this minimum is attained at $z = z_{m+1}$.

Finally, if $\lambda_{m+1} = \cdots = \lambda_{m+p}$ is a multiple finite eigenvalue of (E) with multiplicity $p \geq 2$, then any function $z \in \mathbb{A}$ with $z \perp z_1, \dots, z_{m+q}$ (for any $q \in \{1, \dots, p\}$) satisfies automatically $z \perp z_1, \dots, z_m$. Therefore, by the previous argument we have for such z

$$\mathcal{F}_0(z) \geq \lambda_{m+1} \langle z, z \rangle_W = \cdots = \lambda_{m+q} \langle z, z \rangle_W, \quad q \in \{1, \dots, p\}.$$

This completes the proof of the Rayleigh principle on time scales (Theorem 4.1). ■

Similarly to [20, Corollary 4.1] we can characterize the existence of finitely or infinitely many finite eigenvalues in terms of the dimension of the space

$$\mathbb{W} := \{ (Wx^\sigma)(\cdot), \quad z = (x, u) \in \mathbb{A} \}.$$

The space $\mathbb{W}$ contains all the functions $(Wx_i^\sigma)(\cdot)$, where $z_i = (x_i, u_i)$ are the finite eigenfunctions of (E). Consequently, the number of finite eigenvalues cannot be larger than $\dim \mathbb{W}$. From Theorem 4.1 we can then conclude the following.

Corollary 4.6. *Assume that* (1.2) *holds, the principal solution* $\hat{Z}(\cdot, \lambda)$ *of* $(\mathcal{S}_\lambda)$ *satisfies condition* (2.9)*, and* $\mathcal{F}_\lambda$ *is positive definite for some* $\lambda < 0$.

(i) *The eigenvalue problem* (E) *has infinitely many finite eigenvalues* $-\infty < \lambda_1 \leq \lambda_2 \dots$ *with* $\lambda_m \to \infty$ *as* $m \to \infty$ *if and only if* $\dim \mathbb{W} = \infty$.

(ii) *The eigenvalue problem* (E) *has exactly* $p \in \mathbb{N} \cup \{0\}$ *finite eigenvalues if and only if* $\dim \mathbb{W} = p$.

In both cases (i) and (ii) in Corollary 4.6 the finite eigenvalues of (E) satisfy (4.1), where in (ii) we put $\lambda_{p+1} := \infty$. The final result of this section is a generalization of [20, Theorem 4.3] and [6, Theorem 4.7] to time scales.

Theorem 4.7 (Expansion theorem)**.** *Assume that* (1.2) *holds, the principal solution* $\hat{Z}(\cdot, \lambda)$ *of* $(\mathcal{S}_\lambda)$ *satisfies condition* (2.9)*, and that* $\mathcal{F}_\lambda$ *is positive definite for some* $\lambda < 0$. *Denote by* $\mathcal{I}$ *the index set which is equal to* $\mathbb{N}$ *if* $\dim \mathbb{W} = \infty$ *and which is equal to* $\{1, \dots, p\}$ *if* $\dim \mathbb{W} = p \geq 1$. *Let* $z = (x, u) \in \mathbb{A}$. *Then*

$$x = \sum_{i \in \mathcal{I}} c_i \, x_i, \quad \textit{i.e.,} \quad \lim_{m \to \infty} \left\| z - \sum_{i=1}^{m} c_i \, z_i \right\|_W = 0, \quad \textit{where } c_i := \langle z, z_i \rangle_W \textit{ for all } i \in \mathcal{I}. \tag{4.21}$$

Proof. The proof is the same as in the continuous and discrete time cases in [20, Theorem 4.3] and [6, Theorem 4.7] and it is therefore omitted. We need to mention that the argument in these proofs yields in the time scale setting that $x^\sigma(\cdot) = \sum_{i \in \mathcal{I}} c_i \, x_i^\sigma(\cdot)$ on $[a, \rho(b)]_{\mathbb{T}}$. But since the functions $x(\cdot)$ and $x_i(\cdot)$ are continuous on $[a, b]_{\mathbb{T}}$ and since $x(a) = 0 = x_i(a)$ for every $i \in \mathcal{I}$, it follows by [21, Lemma 5.10] that $x(t) = \sum_{i \in \mathcal{I}} c_i \, x_i(t)$ on $[a, b]_{\mathbb{T}}$, as it is claimed in (4.21). ■

5. Sturmian theorems

In this section we consider first the system $(\mathcal{S}_\lambda)$ and another time scale symplectic system of the same form

$$\underline{x}^\Delta = \underline{\mathcal{A}}(t) \, \underline{x} + \underline{\mathcal{B}}(t) \, \underline{u}, \quad \underline{u}^\Delta = \underline{\mathcal{C}}(t) \, \underline{x} + \underline{\mathcal{D}}(t) \, \underline{u} - \lambda \, \underline{W}(t) \, \underline{x}^\sigma, \quad t \in [a, \rho(b)]_{\mathbb{T}}, \tag{$\underline{\mathcal{S}}_\lambda$}$$

whose coefficients $\underline{\mathcal{A}}(\cdot)$, $\underline{\mathcal{B}}(\cdot)$, $\underline{\mathcal{C}}(\cdot)$, $\underline{\mathcal{D}}(\cdot)$, $\underline{W}(\cdot)$ satisfy the same assumptions (1.1) as the coefficients of system $(\mathcal{S}_\lambda)$. The quadratic functional corresponding to system $(\underline{\mathcal{S}}_\lambda)$ will be denoted by $\underline{\mathcal{F}}_\lambda$. Define on $[a,\rho(b)]_{\mathbb{T}}$ the symmetric matrices (suppressing the argument t in the notation)

$$\mathcal{G} := \begin{pmatrix} \mathcal{C}^T - \mu\,\mathcal{C}^T\mathcal{A} + \mathcal{A}^T\mathcal{E}\mathcal{A} & \mu\,\mathcal{C}^T - \mathcal{A}^T\mathcal{E} \\ \mu\,\mathcal{C} - \mathcal{E}\mathcal{A} & \mathcal{E} \end{pmatrix}, \quad \mathcal{E} := \mathcal{B}\mathcal{B}^\dagger(I+\mu\mathcal{D})\,\mathcal{B}^\dagger,$$

and similarly we define the matrices $\underline{\mathcal{G}}$ and $\underline{\mathcal{E}}$. Then a simple calculation shows that for an admissible $z=(x,u)$ we have

$$\Omega(z,z) = \begin{pmatrix} x \\ x^\Delta \end{pmatrix}^T \mathcal{G} \begin{pmatrix} x \\ x^\Delta \end{pmatrix}, \quad (I+\mu\mathcal{D}^T)\,\mathcal{B} = \mathcal{B}^T\mathcal{E}\mathcal{B}. \tag{5.1}$$

The following results gives a comparison of the definiteness of the functionals $\mathcal{F}_\lambda$ and $\underline{\mathcal{F}}_\lambda$.

Proposition 5.1 (Comparison theorem)**.** *Let $\lambda,\lambda_0\in\mathbb{R}$ with $\lambda\le\lambda_0$ and assume that*

$$\mathcal{G}(t)\ge\underline{\mathcal{G}}(t),\quad 0\le W(t)\le\underline{W}(t),\quad \operatorname{Im}\big(\mathcal{A}(t)-\underline{\mathcal{A}}(t)\quad \mathcal{B}(t)\big)\subseteq\operatorname{Im}\underline{\mathcal{B}}(t)\quad \text{on } [a,\rho(b)]_{\mathbb{T}}. \tag{5.2}$$

Then the positivity (nonnegativity) of the functional $\underline{\mathcal{F}}_{\lambda_0}$ implies the positivity (nonnegativity) of the functional $\mathcal{F}_\lambda$.

Proof. The proof is similar to the proof of [16, Theorem 3.2], so the details are here omitted. ∎

As a consequence we obtain a comparison of the definiteness of the functionals $\mathcal{F}_\lambda$ for different values of λ. It allows to replace the condition on the positivity of $\mathcal{F}_\lambda$ for all $\lambda\le\lambda_0$ used in the oscillation theorem in [21, Corollary 6.4] by the positivity of $\mathcal{F}_{\lambda_0}$ alone (compare the previous reference with Proposition 2.2).

Corollary 5.2. *Suppose that* (1.2) *holds and let $\lambda_0\in\mathbb{R}$ be fixed. The functional $\mathcal{F}_{\lambda_0}$ is positive definite (nonnegative) if and only if the functional $\mathcal{F}_\lambda$ is positive definite (nonnegative) for every $\lambda\le\lambda_0$.*

Proof. We take the coefficients in system $(\underline{\mathcal{S}}_\lambda)$ to be equal to the coefficients of $(\mathcal{S}_\lambda)$. Then the conditions in (5.2) are satisfied trivially and the result follows from Proposition 5.1. ∎

In the subsequent results we establish much more precise relationship between the numbers of proper focal points of conjoined bases of the two systems of the form $(\mathcal{S}_\lambda)$ and $(\underline{\mathcal{S}}_\lambda)$. Let us consider two generic time scale symplectic systems

$$x^\Delta = \mathcal{A}(t)\,x + \mathcal{B}(t)\,u, \quad u^\Delta = \mathcal{C}(t)\,x + \mathcal{D}(t)\,u, \quad t\in[a,\rho(b)]_{\mathbb{T}}, \tag{$\mathcal{S}$}$$

$$\underline{x}^\Delta = \underline{\mathcal{A}}(t)\,\underline{x} + \underline{\mathcal{B}}(t)\,\underline{u}, \quad \underline{u}^\Delta = \underline{\mathcal{C}}(t)\,\underline{x} + \underline{\mathcal{D}}(t)\,\underline{u}, \quad t\in[a,\rho(b)]_{\mathbb{T}}, \tag{$\underline{\mathcal{S}}$}$$

whose coefficients satisfy the assumptions in (1.1)(i). We shall now derive the Sturmian comparison and separation theorems for these two systems. Accordingly to the matrix $P(\cdot)$ in (2.4), we define the matrix $\underline{P}(\cdot)$ through a conjoined basis $\underline{Z}=(\underline{X},\underline{U})$ of system $(\underline{\mathcal{S}})$. And as in (2.9) we utilize similar hypotheses for the conjoined bases $Z=(X,U)$ of $(\mathcal{S})$ and $\underline{Z}=(\underline{X},\underline{U})$ of $(\underline{\mathcal{S}})$. The following result is a generalization of [10, Theorem 1] and [23, Theorem 1.1] to time scales.

Theorem 5.3 (Sturmian separation theorem). *Suppose that conditions (i) and (ii) in (2.9) holds for every conjoined basis of $(\mathcal{S})$. If there exists a conjoined basis of $(\mathcal{S})$ with no proper focal points in $(a,b]_{\mathbb{T}}$, then every other conjoined basis of $(\mathcal{S})$ has at most n proper focal points in $(a,b]_{\mathbb{T}}$.*

Proof. The first assumption yields that every conjoined basis of $(\mathcal{S})$ has finitely many proper focal points in $(a,b]_{\mathbb{T}}$. The existence of a conjoined basis of $(\mathcal{S})$ with no proper focal points in $(a,b]_{\mathbb{T}}$ implies by [16, Corollary 3.2] (or in fact it is equivalent to, by Proposition 2.1) the positivity of the functional $\mathcal{F}_0$. Let $Z=(X,U)$ be a conjoined basis of $(\mathcal{S})$ with totally $r>n$ proper focal points in $(a,b]_{\mathbb{T}}$ including the multiplicities, and let these proper focal points be identified as $a<\tau_1<\cdots<\tau_k\leq b$ with the multiplicities $m_1,\dots,m_k$ satisfying $\sum_{i=1}^k m_i=r$. The above notation of these proper focal points means that for every $i\in\{1,\dots,k\}$ the conjoined basis Z has a proper focal point of the multiplicity m_i at the point τ_i if τ_i is left-dense, where m_i is given by the number (2.5) with $t_0:=\tau_i$, and Z has a proper focal point of the multiplicity m_i in the interval $(\rho(\tau_i),\tau_i]_{\mathbb{T}}$ if τ_i is left-scattered, where m_i is given by the number (2.6) with $t_0:=\tau_i$.

For every index $i\in\{1,\dots,k\}$ such that τ_i is left-dense let $d_1^{[i]},\dots,d_{m_i}^{[i]}\in\mathbb{R}^n$ be a basis for the space $[\operatorname{Ker}X(\tau_i^-)]^\perp\cap\operatorname{Ker}X(\tau_i)$, whose dimension is exactly m_i, and then let $z_j^{[i]}=(x_j^{[i]},u_j^{[i]})$ for $j\in\{1,\dots,m_i\}$ be the corresponding admissible pairs defined by (3.3) from Lemma 3.2, for which

$$x_j^{[i]}(b)=0,\quad x_j^{[i]}(\cdot)\not\equiv 0\ \text{ on } [a,b]_{\mathbb{T}},\quad \mathcal{F}_0(z_j^{[i]})=-(d_j^{[i]})^T X^T(a)\,U(a)\,d_j^{[i]} \tag{5.3}$$

for all $j\in\{1,\dots,m_i\}$. Similarly, for every index $i\in\{1,\dots,k\}$ such that τ_i is left-scattered let $d_1^{[i]},\dots,d_{p_i}^{[i]}\in\operatorname{Ker}X(\tau_i)\setminus\operatorname{Ker}X(\rho(\tau_i))$, where $p_i:=\operatorname{rank}M(\rho(\tau_i))$, be linearly independent vectors, and let $c_1^{[i]},\dots,c_{q_i}^{[i]}\in\mathbb{R}^n$ be mutually orthogonal unit eigenvectors corresponding to the negative eigenvalues $\lambda_1^{[i]},\dots,\lambda_{q_i}^{[i]}$ of the symmetric matrix $P(\rho(\tau_i))$, i.e., $(c_j^{[i]})^T P(\rho(\tau_i))\,c_j^{[i]}=\lambda_j^{[i]}<0$ for $j\in\{1,\dots,q_i\}$, where $q_i:=\operatorname{ind}P(\rho(\tau_i))$, so that $p_i+q_i=m_i$. Let $z_j^{[i]}=(x_j^{[i]},u_j^{[i]})$ for $j\in\{1,\dots,p_i\}$ be the corresponding admissible pairs defined by (3.3) from Lemma 3.2, for which (5.3) holds for all $j\in\{1,\dots,p_i\}$. Furthermore, let $z_{j+p_i}^{[i]}=(x_{j+p_i}^{[i]},u_{j+p_i}^{[i]})$ and $d_{j+p_i}^{[i]}:=\{\mu(X^\sigma)^\dagger\mathcal{B}Tc_j^{[i]}\}(\rho(\tau_i))$ for $j\in\{1,\dots,q_i\}$ be the admissible pairs defined by (3.4) from Lemma 3.3, for which

$$\left.\begin{aligned} x_{j+p_i}^{[i]}(b)=0,\quad x_{j+p_i}^{[i]}(\cdot)\not\equiv 0\ \text{ on } [a,b]_{\mathbb{T}},\\ \mathcal{F}_0(z_{j+p_i}^{[i]})=-(d_{j+p_i}^{[i]})^T X^T(a)\,U(a)\,d_{j+p_i}^{[i]}+\mu(\rho(\tau_i))\,\lambda_j^{[i]} \end{aligned}\right\} \tag{5.4}$$

for all $j\in\{1,\dots,q_i\}$. Note that in both cases (τ_i left-dense or left-scattered) we have $z_j^{[i]}(\cdot)\equiv 0$ on $[\tau_i,b]_{\mathbb{T}}$ for all $j\in\{1,\dots,m_i\}$. We now order the admissible pairs $z_j^{[i]}$ as

$$z_1^{[1]},\dots,z_{m_1}^{[1]},z_1^{[2]},\dots,z_{m_2}^{[2]},\dots,z_1^{[k]},\dots,z_{m_k}^{[k]} \tag{5.5}$$

and denote these admissible pairs as $z^{(1)},\dots,z^{(r)}$, i.e., the functions in (5.5) are indexed as

$$z^{(1)},\dots,z^{(m_1)},z^{(m_1+1)},\dots,z^{(m_1+m_2)},\dots,z^{(r-m_k+1)},\dots,z^{(r)} \tag{5.6}$$

and they are determined by the corresponding vectors denoted by $d_1,\dots,d_r$.

Since each of the initial values $x^{(1)}(a), \dots, x^{(r)}(a)$ is an n-vector and since we assume that we have $r > n$ of these initial values, then they must be linearly dependent, i.e.,

$$\sum_{l=1}^{r} \alpha_l \, x^{(l)}(a) = 0 \quad \text{for some coefficients } \alpha_1, \dots, \alpha_r \in \mathbb{R} \text{ with some } \alpha_l \neq 0. \tag{5.7}$$

We now define the pair $z = (x, u)$ by

$$z(t) := \sum_{l=1}^{r} \alpha_l \, z^{(l)}(t) \quad \text{for all } t \in [a, b]_{\mathbb{T}}. \tag{5.8}$$

Then z is admissible,

$$x(a) = \sum_{l=1}^{r} \alpha_l \, x^{(l)}(a) \overset{(5.7)}{=} 0, \quad x(b) = \sum_{l=1}^{r} \alpha_l \, x^{(l)}(b) = \sum_{i=1}^{k} \sum_{j=1}^{m_i} \alpha_j^{[i]} \, x_j^{[i]}(b) \overset{(5.3),(5.4)}{=} 0, \tag{5.9}$$

where $\alpha_j^{[i]} := \alpha_{m_1 + \dots + m_i + j}$. Hence, $z \in \mathbb{A}$ and with notation (5.6) it follows that

$$\mathcal{F}_0(z) = \sum_{l=1}^{r} \sum_{m=1}^{r} \alpha_l \, \alpha_m \int_a^b \Omega(z^{(l)}, z^{(m)})(t) \, \Delta t.$$

The value of each of the above integrals is calculated by the aid of Lemmas 3.5–3.7 depending on the type of the proper focal point to which the admissible functions $z^{(l)}$ and $z^{(m)}$ belong. Denote by J the set of indices $i \in \{1, \dots, k\}$ such that the conjoined basis Z has a proper focal point in the interval $(\rho(\tau_i), \tau_i)_{\mathbb{T}}$, that is, $q_i = \operatorname{ind} P(\rho(\tau_i)) \geq 1$. Then by Lemmas 3.5–3.7 we get

$$\mathcal{F}_0(z) = Q - \sum_{l=1}^{r} \sum_{m=1}^{r} \alpha_l \, \alpha_m \, [x^{(m)}(a)]^T u^{(l)}(a) \overset{(5.8)}{=} Q - x^T(a) \, u(a) \overset{(5.9)}{=} Q, \tag{5.10}$$

$$\text{where} \quad Q := \sum_{i \in J} \sum_{j=1}^{q_i} \sum_{s=1}^{q_i} \alpha_j^{[i]} \, \alpha_s^{[i]} \, \mu(\rho(\tau_i)) \, (c_j^{[i]})^T P(\rho(\tau_i)) \, c_s^{[i]}.$$

Since for each $i \in J$ the vectors $c_1^{[i]}, \dots, c_{q_i}^{[i]}$ are mutually orthogonal unit eigenvectors corresponding to the negative eigenvalues $\lambda_1^{[i]}, \dots, \lambda_{q_i}^{[i]}$ of the symmetric matrix $P(\rho(\tau_i))$, it follows that for all $j, s \in \{1, \dots, q_i\}$ we have

$$(c_j^{[i]})^T P(\rho(\tau_i)) \, c_s^{[i]} = \lambda_j^{[i]} (c_j^{[i]})^T c_s^{[i]} = \begin{cases} \lambda_j^{[i]}, & \text{for } s = j, \\ 0, & \text{for } s \neq j, \end{cases}$$

Thus, by (5.10),

$$\mathcal{F}_0(z) = Q = \sum_{i \in J} \mu(\rho(\tau_i)) \sum_{j=1}^{q_i} \lambda_j^{[i]} \big(\alpha_j^{[i]}\big)^2 \leq 0, \tag{5.11}$$

and the inequality in (5.11) is strict if the set J is nonempty. Consequently, the positivity of the functional $\mathcal{F}_0$ implies that $x(\cdot) \equiv 0$ on $[a, b]_{\mathbb{T}}$. We will show that this necessarily leads to $\alpha_1, \dots, \alpha_r$ being zero.

Consider the last proper focal point τ_k. If τ_k is left-dense, then following the proof of [23, Theorem 1.1], the definition of the admissible functions $z^{(l)}(\cdot)$ in (3.3) yields that

$x^{(l)}(\cdot) \equiv 0$ on $[\tau_k - \varepsilon, \tau_k)_{\mathbb{T}}$ for some sufficiently small $\varepsilon > 0$ for every $l = 1, \dots, r - m_k$. Therefore, from equations (5.6) and (5.8) we obtain

$$\sum_{l=r-m_k+1}^{r} \alpha_l \, x^{(l)}(\cdot) = x(\cdot) \equiv 0, \quad \text{i.e.,} \quad \sum_{j=1}^{m_k} \alpha_j^{[k]} \, x_j^{[k]}(\cdot) \equiv 0 \ \text{ on } [\tau_k - \varepsilon, \tau_k)_{\mathbb{T}}. \tag{5.12}$$

The identity in (5.12) holds also at τ_k by the definition of $x_j^{[k]}(\tau_k)$ or just by the continuity of $x_j^{[k]}(\cdot)$. Hence, for every $t \in [\tau_k - \varepsilon, \tau_k)_{\mathbb{T}}$ we have

$$0 \overset{(5.12)}{=} \sum_{j=1}^{m_k} \alpha_j^{[k]} \, x_j^{[k]}(t) \overset{(3.3)}{=} \sum_{j=1}^{m_k} \alpha_j^{[k]} \, X(t) \, d_j^{[k]} = X(t) \, e, \quad \text{where } \ e := \sum_{j=1}^{m_k} \alpha_j^{[k]} \, d_j^{[k]}. \tag{5.13}$$

Since the vectors $d_1^{[k]}, \dots, d_{m_k}^{[k]}$ form a basis of the orthogonal complement of $\operatorname{Ker} X(\tau_k^-)$ in $\operatorname{Ker} X(\tau_k)$, it follows that $e \in [\operatorname{Ker} X(\tau_k^-)]^\perp$, since the vector e is by (5.13) a linear combination of $d_1^{[k]}, \dots, d_{m_k}^{[k]}$. On the other hand, formula (5.13) implies that $e \in \operatorname{Ker} X(\tau_k^-)$. Consequently, $e = 0$. The definition of e in (5.13) and the linear independence of $d_1^{[k]}, \dots, d_{m_k}^{[k]}$ now yields that $\alpha_1^{[k]} = \dots = \alpha_{m_k}^{[k]} = 0$, that is, $\alpha^{(r-m_k+1)} = \dots = \alpha^{(r)} = 0$.

If τ_k is left-scattered, then inspired by the proof of [10, Theorem 1] we have from (3.3) and (3.4) that $x^{(l)}(\rho(\tau_k)) = 0$ for all $l = 1, \dots, r - m_k$. Hence, by (5.6) and (5.8) we get

$$\sum_{l=r-m_k+1}^{r} \alpha_l \, x^{(l)}(\rho(\tau_k)) = x(\rho(\tau_k)) = 0, \quad \text{i.e.,} \quad \sum_{j=1}^{m_k} \alpha_j^{[k]} \, x_j^{[k]}(\rho(\tau_k)) = 0. \tag{5.14}$$

Consequently,

$$0 \overset{(5.14)}{=} \sum_{j=1}^{m_k} \alpha_j^{[k]} \, x_j^{[k]}(\rho(\tau_k)) \overset{(3.3),(3.4)}{=} \sum_{j=1}^{m_k} \alpha_j^{[k]} X(\rho(\tau_k)) \, d_j^{[k]} = X(\rho(\tau_k)) \, e, \tag{5.15}$$

where the vector e is defined by the second formula in (5.13). For brevity we set $p := p_k$ and $q := q_k$, and recall that the vectors $d_1^{[k]}, \dots, d_p^{[k]}$ "belong" to the focal point at τ_k, while the vectors $d_{p+1}^{[k]}, \dots, d_{p+q}^{[k]}$ "belong" to the focal interval $(\rho(\tau_k), \tau_k)_{\mathbb{T}}$, where from Lemmas 3.3 and 3.8 the vectors $d_{p+j}^{[k]}$ are given by formula (3.7) with $t_0 := \tau_k$ and $c_j := c_j^{[k]}$ for $j \in \{1, \dots, q\}$. From $\operatorname{Im} X^\dagger = \operatorname{Im} X^T$ we can see that $d_{p+1}^{[k]}, \dots, d_{p+q}^{[k]} \in \operatorname{Im} X^T(\rho(\tau_k)) = [\operatorname{Ker} X(\rho(\tau_k))]^\perp$. By splitting the vector e into the sum

$$e = f + g, \quad \text{where } \ f := \sum_{j=1}^{p} \alpha_j^{[k]} \, d_j^{[k]}, \quad g := \sum_{j=1}^{q} \alpha_{p+j}^{[k]} \, d_{p+j}^{[k]},$$

we get $f \in [\operatorname{Ker} X(\rho(\tau_k))]^\perp \cap \operatorname{Ker} X(\tau_k)$ and $g \in [\operatorname{Ker} X(\rho(\tau_k))]^\perp$. Hence, $e = f + g \in [\operatorname{Ker} X(\rho(\tau_k))]^\perp$. And since by (5.15) we have $f + g = e \in \operatorname{Ker} X(\rho(\tau_k))$, it follows that $e = 0$. But from Lemma 3.8 we know that the vectors $d_1^{[k]}, \dots, d_{p+q}^{[k]}$ are linearly independent, so that $e = 0$ implies $\alpha_1^{[k]} = \dots \alpha_{p+q}^{[k]} = 0$. Thus, as in the case of τ_k left-dense, we proved that $\alpha^{(r-m_k+1)} = \dots = \alpha^{(r)} = 0$.

Repeating the above argument with the proper focal points $\tau_{k-1}, \dots, \tau_1$ we obtain $\alpha_j^{[i]} = 0$ for every $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, m_i\}$, i.e., $\alpha_1 = \dots = \alpha_r = 0$. However, this contradicts condition (5.7), where at least one coefficient $\alpha_l \neq 0$. Therefore, the conjoined basis Z cannot have more than n proper focal points in $(a,b]_{\mathbb{T}}$ and the proof is complete. ■

In the proofs of the subsequent results we will utilize the eigenvalue problems (E) and

$$(\underline{\mathcal{S}}_\lambda), \quad \underline{x}(a) = 0 = \underline{x}(b), \tag{$\underline{\mathrm{E}}$}$$

in which the matrices $W(\cdot)$ and $\underline{W}(\cdot)$ are given by

$$W(t) \equiv I, \quad \underline{W}(t) \equiv I \quad \text{on } [a, \rho(b)]_{\mathbb{T}}. \tag{5.16}$$

The following result is a generalization of [23, Theorem 1.2] and [6, Theorem 1.2] to arbitrary time scales. Note that in view of Proposition 2.1 the choice of $m = 0$ in Theorem 5.4 yields the result of Theorem 5.3.

Theorem 5.4 (Sturmian comparison theorem). *Under* (5.16)*, suppose that the principal solution of* $(\underline{\mathcal{S}}_\lambda)$ *satisfies* (2.9)*, and conditions (i) and (ii) in* (2.9) *hold for every conjoined basis of* $(\mathcal{S})$*. Furthermore, let the functional* $\underline{\mathcal{F}}_\lambda$ *be positive definite for some* $\lambda < 0$ *and*

$$\mathcal{G}(t) \geq \underline{\mathcal{G}}(t), \quad \operatorname{Im}\big(\mathcal{A}(t) - \underline{\mathcal{A}}(t) \quad \mathcal{B}(t)\big) \subseteq \operatorname{Im} \underline{\mathcal{B}}(t) \quad \text{on } [a, \rho(b)]_{\mathbb{T}}. \tag{5.17}$$

If the principal solution of $(\underline{\mathcal{S}})$ *has* $m \in \mathbb{N} \cup \{0\}$ *proper focal points in* $(a,b]_{\mathbb{T}}$*, then every conjoined basis of* $(\mathcal{S})$ *has at most* $m+n$ *proper focal points in* $(a,b]_{\mathbb{T}}$*.*

Proof. The assumptions imply that there is finitely many proper focal points in $(a,b]_{\mathbb{T}}$ for every conjoined basis of $(\underline{\mathcal{S}}_\lambda)$ and $(\mathcal{S})$. Let $\hat{\underline{Z}} = (\hat{\underline{X}}, \hat{\underline{U}})$ be the principal solution of $(\underline{\mathcal{S}})$ and suppose that it has m proper focal points in $(a,b]_{\mathbb{T}}$. Let $Z = (X,U)$ be a conjoined basis of $(\mathcal{S})$ and let r be its number of proper focal points in $(a,b]_{\mathbb{T}}$. By Lemmas 3.2 and 3.3, for each proper focal point at τ_i and for each proper focal point in $(\rho(\tau_i), \tau_i)_{\mathbb{T}}$ of Z there is an $(\mathcal{A},\mathcal{B})$-admissible $z_i = (x_i, u_i)$ such that

$$x_i(b) = 0, \quad x_i(\cdot) \not\equiv 0, \quad \mathcal{F}_0(z_i) = -x_i^T(a)\, u_i(a), \tag{5.18}$$

$$x_i(b) = 0, \quad x_i(\cdot) \not\equiv 0, \quad \mathcal{F}_0(z_i) = -x_i^T(a)\, u_i(a) + \mu(\rho(\tau_i))\, \lambda_i, \tag{5.19}$$

respectively, where λ_i is a negative eigenvalue of the matrix $P(\rho(\tau_i))$. By Proposition 2.2, the finite eigenvalues of ($\underline{\mathrm{E}}$) are bounded below and

$$\underline{n}_1(\lambda) = \underline{n}_2(\lambda) \quad \text{for all } \lambda \in \mathbb{R}, \tag{5.20}$$

where

$\underline{n}_1(\lambda) :=$ the number of proper focal points of $\hat{\underline{Z}}(\cdot,\lambda)$ in $(a,b]_{\mathbb{T}}$,

$\underline{n}_2(\lambda) :=$ the number of finite eigenvalues of ($\underline{\mathrm{E}}$) which are less or equal to λ,

and where $\hat{\underline{Z}}(\cdot,\lambda) = (\hat{\underline{X}}(\cdot,\lambda), \hat{\underline{U}}(\cdot,\lambda))$ is the principal solution of $(\underline{\mathcal{S}}_\lambda)$, i.e., $\hat{\underline{X}}(a,\lambda) \equiv 0$ and $\hat{\underline{U}}(a,\lambda) \equiv I$ for all $\lambda \in \mathbb{R}$. Since we assume that the principal solution $\hat{\underline{Z}} = \hat{\underline{Z}}(\cdot,0)$ of $(\underline{\mathcal{S}})$ has m proper focal points in $(a,b]_{\mathbb{T}}$, i.e., $\underline{n}_1(0) = m$, formula (5.20) yields that the eigenvalue problem ($\underline{\mathrm{E}}$) has m finite eigenvalues $\lambda_1 \leq \cdots \leq \lambda_m \leq 0$. Let $\underline{z}_1 = (\underline{x}_1, \underline{u}_1), \ldots, \underline{z}_m = (\underline{x}_m, \underline{u}_m)$ be the corresponding orthonormal finite eigenfunctions. By the Rayleigh principle (Theorem 4.1), for every $(\underline{\mathcal{A}}, \underline{\mathcal{B}})$-admissible $\underline{z} = (\underline{x}, \underline{u})$ with $\underline{x}(a) = 0 = \underline{x}(b)$, $\underline{x}(\cdot) \not\equiv 0$, and $\underline{z} \perp \underline{z}_1, \dots, \underline{z}_m$ we have

$$\underline{\mathcal{F}}_0(\underline{z}) > 0 \cdot \langle \underline{z}, \underline{z} \rangle = 0, \qquad \langle \underline{z}, \tilde{\underline{z}} \rangle := \langle \underline{z}, \tilde{\underline{z}} \rangle_{\underline{W}} = \int_a^b [\underline{x}^\sigma(t)]^T \tilde{\underline{x}}^\sigma(t)\, \Delta t. \tag{5.21}$$

Consider the numbers

$$\beta_{ij} := \langle z_i, \underline{z}_j \rangle = \int_a^b [x_i^\sigma(t)]^T \underline{x}_j^\sigma(t)\, \Delta t \quad \text{for } i \in \{1, \dots, r\},\ j \in \{1, \dots, m\},$$

and the vectors

$$d_i := \begin{pmatrix} \beta_{i1} & \beta_{i2} & \dots & \beta_{im} & x_i^T(a) \end{pmatrix}^T \in \mathbb{R}^{m+n} \quad \text{for } i \in \{1, \dots, r\},$$

where the functions $z_i = (x_i, u_i)$ are from (5.18) and (5.19). If we assume that $r > m+n$, then these vectors $d_1, \dots, d_r$ must be linearly dependent, i.e., there are coefficients $c_1, \dots, c_r \in \mathbb{R}$ with some $c_i \neq 0$ such that

$$\sum_{i=1}^r c_i\, d_i = 0 \in \mathbb{R}^{m+n}. \tag{5.22}$$

Define now the pair $z = (x, u) := \sum_{i=1}^r c_i\, z_i$. Since each z_i is $(\mathcal{A}, \mathcal{B})$-admissible with $x_i(b) = 0$, it follows that z is also $(\mathcal{A}, \mathcal{B})$-admissible and $x(b) = 0$. Moreover, the definition of the vectors $d_1, \dots, d_r$ implies

$$x(a) = \sum_{i=1}^r c_i\, x_i(a) \overset{(5.22)}{=} 0.$$

Furthermore, for every $j \in \{1, \dots, m\}$ we have

$$0 \overset{(5.22)}{=} \sum_{i=1}^r c_i\, \beta_{ij} = \sum_{i=1}^r c_i\, \langle z_i, \underline{z}_j \rangle = \left\langle \sum_{i=1}^r c_i\, z_i,\ \underline{z}_j \right\rangle = \langle z, \underline{z}_j \rangle. \tag{5.23}$$

This yields that $z \perp \underline{z}_1, \dots, \underline{z}_m$. The fact that $x(\cdot) \not\equiv 0$ follows by the same argument as in the proof of Theorem 5.3, i.e., if $x(\cdot) \equiv 0$, then all the coefficients $c_1, \dots, c_r$ are zero, which is a contradiction. Moreover, from (5.18) and (5.19) we get similarly to the calculations in (5.10)–(5.11) that $\mathcal{F}_0(z) \leq 0$.

Define the function $\underline{x}(t) := x(t)$ on $[a, b]_{\mathbb{T}}$. Then (suppressing the argument t) we have $\underline{x}^\Delta - \underline{\mathcal{A}}\underline{x} = \mathcal{B}u + (\mathcal{A} - \underline{\mathcal{A}})\, x \in \operatorname{Im}\begin{pmatrix} \mathcal{A} - \underline{\mathcal{A}}, & \mathcal{B} \end{pmatrix}$ on $[a, \rho(b)]_{\mathbb{T}}$, so that by condition (5.17)(ii) for each $t \in [a, \rho(b)]_{\mathbb{T}}$ there exists a value $\underline{u}(t) \in \mathbb{R}^n$ such that $\underline{\mathcal{B}}\underline{u} = \underline{x}^\Delta - \underline{\mathcal{A}}\underline{x} \in \mathrm{C_{prd}}$ on $[a, \rho(b)]_{\mathbb{T}}$. This means that the pair $\underline{z} = (\underline{x}, \underline{u})$ is $(\underline{\mathcal{A}}, \underline{\mathcal{B}})$-admissible, $\underline{x}(a) = x(a) = 0$, $\underline{x}(b) = x(b) = 0$, and $\underline{x}(\cdot) = x(\cdot) \not\equiv 0$. Moreover, by (5.23) and the definition of $\langle \cdot, \cdot \rangle$ we have $\langle \underline{z}, \underline{z}_j \rangle = \langle z, \underline{z}_j \rangle = 0$ for each $j \in \{1, \dots, m\}$, that is, $\underline{z} \perp \underline{z}_1, \dots, \underline{z}_m$. Inequality (5.21) then implies that $\underline{\mathcal{F}}_0(\underline{z}) > 0$. On the other hand, by the definition of $\underline{x}$, $\mathcal{G}$, and $\underline{\mathcal{G}}$ we have (compare with [16, Theorem 3.2])

$$\underline{\mathcal{F}}_0(\underline{z}) = \int_a^b \begin{pmatrix} \underline{x}(t) \\ \underline{x}^\Delta(t) \end{pmatrix}^T \underline{\mathcal{G}}(t) \begin{pmatrix} \underline{x}(t) \\ \underline{x}^\Delta(t) \end{pmatrix} \Delta t \overset{(5.17)}{\leq} \int_a^b \begin{pmatrix} x(t) \\ x^\Delta(t) \end{pmatrix}^T \mathcal{G}(t) \begin{pmatrix} x(t) \\ x^\Delta(t) \end{pmatrix} \Delta t = \mathcal{F}_0(z) \leq 0.$$

This is a contradiction with the previously computed value $\underline{\mathcal{F}}_0(\underline{z}) > 0$. Hence, we must have $r \leq m + n$ and the proof is complete. ∎

Next we consider a generalization of [23, Theorem 1.3] and [6, Theorem 1.3] to time scales.

Theorem 5.5 (Sturmian comparison theorem)**.** *Under* (5.16)*, suppose that the principal solution of* ($\mathcal{S}_\lambda$) *and every conjoined basis of* ($\underline{\mathcal{S}}_\lambda$) *satisfy* (2.9)*. Furthermore, let the functional* $\mathcal{F}_\lambda$ *be positive definite for some* $\lambda < 0$ *and condition* (5.17) *satisfied. If the principal solution of* ($\mathcal{S}$) *has* $m \in \mathbb{N} \cup \{0\}$ *proper focal points in* $(a, b]_{\mathbb{T}}$*, then every conjoined basis of* ($\underline{\mathcal{S}}$) *has at least* m *proper focal points in* $(a, b]_{\mathbb{T}}$*.*

Proof. Let $\hat{Z} = (\hat{X}, \hat{U})$ be the principal solution of ($\mathcal{S}$) with m proper focal points in $(a, b]_{\mathbb{T}}$ and let $\underline{Z} = (\underline{X}, \underline{U})$ be a conjoined basis of ($\underline{\mathcal{S}}$). For any $\lambda \in \mathbb{R}$ let $\underline{Z}(\cdot, \lambda) = (\underline{X}(\cdot, \lambda), \underline{U}(\cdot, \lambda))$ be the conjoined basis of ($\underline{\mathcal{S}}_\lambda$) given by the initial conditions $\underline{X}(a, \lambda) \equiv \underline{X}(a)$ and $\underline{U}(a, \lambda) \equiv \underline{U}(a)$, and let $\hat{Z}(\cdot, \lambda) = (\hat{X}(\cdot, \lambda), \hat{U}(\cdot, \lambda))$ be the principal solution of ($\mathcal{S}_\lambda$). Then $\hat{Z}(\cdot, \lambda)$ and $\underline{Z}(\cdot, \lambda)$ have finitely many proper focal points in $(a, b]_{\mathbb{T}}$ for every $\lambda \in \mathbb{R}$, which we denote by $n_1(\lambda)$ and $p(\lambda)$, respectively. Then we need to prove $m = n_1(0) \le p(0)$. We shall prove a stronger result

$$n_1(\lambda) \le p(\lambda) \quad \text{for all } \lambda \in \mathbb{R}. \tag{5.24}$$

Under (5.16) we consider the eigenvalue problems (E) and ($\underline{\text{E}}$). By the oscillation theorem (Proposition 2.2), the finite eigenvalues of (E) are bounded below and equality (2.11) holds. Thus, there are $n_2(0) = n_1(0) = m$ nonpositive finite eigenvalues of (E). Let $\lambda_1 \le \cdots \le \lambda_r \le \lambda_{r+1} \le \dots$ be the finite eigenvalues of (E) with the corresponding orthonormal finite eigenfunctions $z_1, z_2, \dots$. Put $\lambda_0 := -\infty$ and, if the dimension of the admissible set $\mathbb{A}$ is finite, say $\dim \mathbb{A} = r < \infty$, then we put $\lambda_{r+j} := \infty$ for every $j \in \mathbb{N}$.

Fix any $\lambda \in \mathbb{R}$. Then $\lambda \in [\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N} \cup \{0\}$, and $k \le r$ if $\dim \mathbb{A} = r$. By Proposition 2.2, it follows that $n_1(\lambda) = n_2(\lambda) = k$. If $k = 0$, then the required inequality (5.24) holds trivially. Hence, we consider further on that $k \ge 1$. First suppose that λ is not a finite eigenvalue of (E), that is, $\lambda \in (\lambda_k, \lambda_{k+1})$. For constants $\beta_1, \dots, \beta_k \in \mathbb{R}$ we set $\tilde{z} = (\tilde{x}, \tilde{u}) := \sum_{i=1}^{k} \beta_i z_i$, so that $\tilde{z} \in \mathbb{A}$.

Let $\tilde{p}$ be the number of proper focal points of $\underline{Z}(\cdot, \lambda)$ in the interval $(a, b)_{\mathbb{T}}$. Then $\tilde{p} \le p(\lambda)$. Choose the coefficients $\beta_1, \dots, \beta_k$ so that the function $\tilde{z}$ satisfies $\tilde{p}$ linear homogeneous equations determined by the conditions

$$\tilde{x}(\tau_j) \in \big([\operatorname{Ker} \underline{X}^T(\tau_j^-, \lambda)]^\perp \cap \operatorname{Ker} \underline{X}^T(\tau_j, \lambda) \big)^\perp \quad \text{if } \tau_j \text{ is left-dense,} \tag{5.25}$$

$$\left.\begin{array}{l} \underline{M}^T(\rho(\tau_j), \lambda)\, \tilde{x}(\tau_j) = 0, \\ \tilde{w}(\rho(\tau_j)) \perp \{\alpha \in \mathbb{R}^n,\ \ \alpha \text{ is an eigenvector} \\ \qquad\qquad \text{corresponding to a negative} \\ \qquad\qquad \text{finite eigenvalue of } \underline{P}(\rho(\tau_j), \lambda)\} \end{array}\right\} \quad \text{if } \tau_j \text{ is left-scattered,} \tag{5.26}$$

where $\tilde{w}(t) := \tilde{u}(t) - \underline{U}(t, \lambda)\, \underline{X}^\dagger(t, \lambda)\, \tilde{x}(t)$ and where $\tau_1, \dots, \tau_l$ are the proper focal points of $\underline{Z}(\cdot, \lambda)$ in $(a, b)_{\mathbb{T}}$ whose multiplicities add up to $\tilde{p}$. The matrices $\underline{M}(\cdot, \lambda)$ and $\underline{P}(\cdot, \lambda)$ are defined by (2.4) through the conjoined basis $\underline{Z}(\cdot, \lambda)$.

Set $\underline{x}(t) := \tilde{x}(t)$ on $[a, b]_{\mathbb{T}}$. Since $\tilde{z}$ is $(\mathcal{A}, \mathcal{B})$-admissible, assumption (5.17)(ii) implies that $\underline{z} := (\underline{x}, \underline{u})$ is $(\underline{\mathcal{A}}, \underline{\mathcal{B}})$-admissible for some $\underline{u}(\cdot)$. Moreover, $\underline{x}(a) = \tilde{x}(a) = 0$, $\underline{x}(b) =$

$\tilde{x}(b)=0$, and for each $j\in\{1,\dots,k\}$

$$\langle \underline{z}, z_j\rangle = \langle \tilde{z}, z_j\rangle = \sum_{i=1}^{k} \beta_i \, \langle z_i, z_j\rangle = \beta_j.$$

By the time scale induction principle, following the proof of Theorem 4.1, one can verify that the image condition

$$\underline{x}(t) \in \operatorname{Im} \underline{X}(t,\lambda) \quad \text{for all } t\in[a,b]_{\mathbb{T}}$$

holds. Since by (5.1) the values of $\Omega(\tilde{z},\tilde{z})$ and $\langle \tilde{z},\tilde{z}\rangle$ do not depend on the component $\tilde{u}$, it follows that

$$\underline{\mathcal{F}}_\lambda(\underline{z}) = \underline{\mathcal{F}}_\lambda(\tilde{z}), \quad \text{where } \ \underline{\mathcal{F}}_\lambda(\underline{z}) := \underline{\mathcal{F}}_0(\underline{z}) - \lambda \, \langle \underline{z}, \underline{z}\rangle. \tag{5.27}$$

We now apply the extended global Picone formula (Theorem 4.5) with $z=0$, $m=k$, $\hat{z}=\underline{z}$ to obtain for $\tilde{w} := \tilde{u} - \underline{UX}^{\dagger}\tilde{x}$

$$\begin{aligned}\underline{\mathcal{F}}_\lambda(\underline{z}) \overset{(5.27)}{=} \underline{\mathcal{F}}_\lambda(\tilde{z}) \overset{(4.4)}{\geq} \int_a^b \tilde{w}^T(t)\, \underline{P}(t,\lambda)\, \tilde{w}(t)\,\Delta t + \sum_{i=1}^{k} (\lambda-\lambda_i)\,\beta_i^2 \\ \geq \sum_{i=1}^{k} (\lambda-\lambda_i)\,\beta_i^2 \geq 0,\end{aligned} \tag{5.28}$$

where we used that $\lambda>\lambda_i$ for every $i\in\{1,\dots,k\}$ and that $\tilde{w}$ is orthogonal to the eigenvectors corresponding to the negative eigenvalues of of $\underline{P}(\rho(\tau_i),\lambda)$, see (5.26). On the other hand, assumption (5.17) implies

$$\begin{aligned}\underline{\mathcal{F}}_0(\underline{z}) &= \int_a^b \begin{pmatrix} \underline{x}(t) \\ \underline{x}^{\Delta}(t)\end{pmatrix}^T \underline{\mathcal{G}}(t) \begin{pmatrix} \underline{x}(t) \\ \underline{x}^{\Delta}(t)\end{pmatrix} \Delta t \\ &\overset{(5.17)}{\leq} \int_a^b \begin{pmatrix} \tilde{x}(t) \\ \tilde{x}^{\Delta}(t)\end{pmatrix}^T \mathcal{G}(t) \begin{pmatrix} \tilde{x}(t) \\ \tilde{x}^{\Delta}(t)\end{pmatrix} \Delta t = \mathcal{F}_0(\tilde{z}).\end{aligned} \tag{5.29}$$

Therefore,

$$\underline{\mathcal{F}}_\lambda(\underline{z}) \overset{(5.27)}{=} \underline{\mathcal{F}}_0(\underline{z}) - \lambda\,\langle \underline{z},\underline{z}\rangle \overset{(5.29)}{\leq} \mathcal{F}_0(\tilde{z}) - \lambda\,\langle \tilde{z},\tilde{z}\rangle = \mathcal{F}_\lambda(\tilde{z}) \overset{(4.2)}{=} \sum_{i=1}^{k} (\lambda_i-\lambda)\,\beta_i^2, \tag{5.30}$$

where the value of $\mathcal{F}_\lambda(\tilde{z})$ is calculated separately by using Lemma 4.2. The combination of equations (5.28) and (5.30) then yields

$$0 \leq \sum_{i=1}^{k} (\lambda-\lambda_i)\,\beta_i^2 \overset{(5.28)}{\leq} \underline{\mathcal{F}}_\lambda(\underline{z}) \overset{(5.30)}{\leq} \sum_{i=1}^{k} (\lambda_i-\lambda)\,\beta_i^2 \leq 0,$$

where the last inequality follows from $\lambda>\lambda_i$ for every $i\in\{1,\dots,k\}$. This is however possible only if $\beta_1=\dots=\beta_k=0$. Therefore, system (5.25)–(5.26) of $\tilde{p}$ linear homogeneous equations has only the trivial solution. This implies that the number of equations (that is, $\tilde{p}$) must be bigger or equal to the number of variables $\beta_1,\dots,\beta_k$ (that is, k). Consequently,

$$n_1(\lambda) \overset{(2.11)}{=} n_2(\lambda) = k \leq \tilde{p} \leq p(\lambda). \tag{5.31}$$

This shows the result in (5.24) when λ is not a finite eigenvalue of (E). When $\lambda = \lambda_k$ is one of the finite eigenvalues of (E), then from the right-continuity of the functions $n_1(\cdot)$ and $p(\cdot)$ on $\mathbb{R}$, see [21, Theorem 6.1], it follows upon taking the limit in (5.31) as $\lambda \searrow \lambda_k^+$ that

$$n_1(\lambda_k) = n_1(\lambda_k^+) \overset{(5.31)}{\leq} p(\lambda_k^+) = p(\lambda_k).$$

Hence, inequality (5.24) holds also when $\lambda = \lambda_k$ is a finite eigenvalue of (E). ■

Remark 5.6. The proof of Theorem 5.5 shows that assumption (2.9) for every conjoined basis of $(\underline{\mathcal{S}}_\lambda)$ can be dropped. In that case we allow $p(\lambda) = \infty$, so that the inequality $n_1(\lambda) \leq p(\lambda)$, and in particular $n_1(0) \leq p(0)$, is satisfied automatically.

Let now $(\underline{\mathcal{S}}_\lambda)$=$(\mathcal{S}_\lambda)$, so that conditions (5.17) are trivially satisfied. This yields a generalization of [23, Theorem 1.4] and [6, Theorem 3.1] to time scales.

Theorem 5.7 (Sturmian separation theorem)**.** *Under* (5.16)*, suppose that for every conjoined basis of* $(\mathcal{S}_\lambda)$ *condition* (2.9) *holds and that* $\mathcal{F}_\lambda$ *is positive definite for some* $\lambda < 0$*. If the principal solution of* $(\mathcal{S})$ *has* $m \in \mathbb{N} \cup \{0\}$ *proper focal points in* $(a, b]_{\mathbb{T}}$*, then any other conjoined basis of* $(\mathcal{S})$ *has at least* m *and at most* $m+n$ *proper focal points in* $(a, b]_{\mathbb{T}}$*.*

Proof. By assumption (2.9), every conjoined basis of $(\mathcal{S})$ has finitely many proper focal points in $(a, b]_{\mathbb{T}}$. Let the principal solution of $(\mathcal{S})$ have m proper focal points in $(a, b]_{\mathbb{T}}$. Let $Z = (X, U)$ be any other conjoined basis of $(\mathcal{S})$ and denote by p its number of proper focal points in $(a, b]_{\mathbb{T}}$. Set $\underline{\mathcal{S}}(t) := \mathcal{S}(t)$ on $[a, \rho(b)]_{\mathbb{T}}$. Then by Theorem 5.4 we have $p \leq m + n$, while by Theorem 5.5 we get $p \geq m$. Thus, $m \leq p \leq m + n$ and the result is proven. ■

The final result of this section generalizes [23, Theorem 1.5] and [6, Theorem 1.1] to time scales.

Theorem 5.8 (Sturmian separation theorem)**.** *Under* (5.16)*, suppose that for every conjoined basis of* $(\mathcal{S}_\lambda)$ *condition* (2.9) *holds and that* $\mathcal{F}_\lambda$ *is positive definite for some* $\lambda < 0$*. Then the difference between the numbers of proper focal points in* $(a, b]_{\mathbb{T}}$ *of any two conjoined bases of* $(\mathcal{S})$ *is at most* n*.*

Proof. By assumption (2.9), every conjoined basis of $(\mathcal{S})$ has finitely many proper focal points in $(a, b]_{\mathbb{T}}$. Let $\hat{Z} = (\hat{X}, \hat{U})$ be the principal solution of $(\mathcal{S})$ and let $Z = (X, U)$ and $\tilde{Z} = (\tilde{X}, \tilde{U})$ be any two conjoined basis of $(\mathcal{S})$. Denote by $m, p, \tilde{p}$ their numbers of proper focal points in $(a, b]_{\mathbb{T}}$, respectively. Then by Theorem 5.7 we have $m \leq p \leq m + n$ and $m \leq \tilde{p} \leq m + n$. Upon subtracting m from both sides of these inequalities we obtain $0 \leq p - m \leq n$ and $0 \leq \tilde{p} - m \leq n$. Therefore, $p - \tilde{p} \leq n$ if $p \geq \tilde{p}$, or $\tilde{p} - p \leq n$ if $p \leq \tilde{p}$. Combining these two inequalities yields $|p - \tilde{p}| \leq n$, which is the statement of this theorem. ■

6. Special time scales

In this section we continue the study of the oscillation properties of symplectic systems $(\mathcal{S})$ and $(\underline{\mathcal{S}})$ on special time scales, which was initiated in [21, Section 9]. A time scale $\mathbb{T} = [a, b]_{\mathbb{T}}$ is called *special* if it consists of a finite union of disjoint closed and bounded real intervals and/or finitely many isolated points. That is, a special time scale $[a,b]_{\mathbb{T}}$ can be partitioned as

$$\left.\begin{aligned} a = t_0 < t_1 < \cdots < t_{N+1} = b, \quad \text{where} \\ [t_j, t_{j+1}]_{\mathbb{T}} = [t_j, t_{j+1}] \text{ or } [t_j, t_{j+1}]_{\mathbb{T}} = \{t_j, t_{j+1}\}, \end{aligned}\right\} \tag{6.1}$$

i.e., for every two consecutive partition points t_j and t_{j+1} the interval $[t_j, t_{j+1}]_{\mathbb{T}}$ is connected or $(t_j, t_{j+1})_{\mathbb{T}}$ is empty. Already such time scales unify the classical purely continuous and discrete time scales. We shall make the following standing hypothesis

$$\mathcal{B}(t) \geq 0 \quad \text{on continuous intervals } [t_j, t_{j+1}] \subseteq [a,b]_{\mathbb{T}}. \tag{6.2}$$

Then, by [19, Theorem 3], every conjoined basis $Z(\cdot,\lambda) = (X(\cdot,\lambda), U(\cdot,\lambda))$ of $(\mathcal{S}_\lambda)$ has the kernel of $X(\cdot,\lambda)$ piecewise constant on the continuous intervals $[t_j, t_{j+1}] \subseteq [a,b]_{\mathbb{T}}$, hence on $[a,b]_{\mathbb{T}}$. Moreover, by [15, Lemma 3.1] we have $P = T\,[(I+\mu\mathcal{D}^T)\,\mathcal{B} - \mu\mathcal{B}^T U^\sigma (X^\sigma)^\dagger \mathcal{B}]\,T$ on $[a,\rho(b)]_{\mathbb{T}} \times \mathbb{R}$, so that the matrix $P(t,\lambda) = T(t,\lambda)\,\mathcal{B}(t)\,T(t,\lambda)$ at every right-dense point. Therefore, we can see that in this case assumption (6.2) implies condition (2.9). In addition, on special time scales we have the result of [21, Theorem 9.5], saying that under (6.2) and $W(t) > 0$ on $[a,\rho(b)]_{\mathbb{T}}$, in particular for $W(\cdot) \equiv I$ used in the previous section, there exists $\lambda < 0$ such that the functional $\mathcal{F}_\lambda$ is positive definite. Therefore, the assumptions in the statements of Sections 4 and 5 significantly simplify for the special time scales.

The results below follow from the corresponding ones in Sections 4 and 5 and they are stated without the proofs.

Theorem 6.1 (Rayleigh principle)**.** *Assume* (6.1), (6.2), (1.2), *and the functional* $\mathcal{F}_\lambda$ *is positive definite for some* $\lambda < 0$. *Let* $\lambda_1 \leq \cdots \leq \lambda_m \leq \ldots$ *be the finite eigenvalues of the eigenvalue problem* (E) *with the corresponding orthonormal finite eigenfunctions* $z_1, \ldots, z_m, \ldots$. *Then for each* $m \in \mathbb{N} \cup \{0\}$ *equation* (4.1) *holds. Moreover, if* $W(t) > 0$ *for all* $t \in [a,\rho(b)]_{\mathbb{T}}$ *instead of* (1.2), *then the assumption on* $\mathcal{F}_\lambda$ *positive definite for some* $\lambda < 0$ *can be dropped.*

Theorem 6.2 (Sturmian comparison theorem)**.** *Assume* (6.1), (6.2), *and* (5.17). *If the principal solution of* $(\underline{\mathcal{S}})$ *has* $m \in \mathbb{N}\cup\{0\}$ *proper focal points in* $(a,b]_{\mathbb{T}}$, *then every conjoined basis of* $(\mathcal{S})$ *has at most* $m+n$ *proper focal points in* $(a,b]_{\mathbb{T}}$.

Theorem 6.3 (Sturmian comparison theorem)**.** *Assume* (6.1), (6.2), *and* (5.17). *If the principal solution of* $(\mathcal{S})$ *has* $m \in \mathbb{N}\cup\{0\}$ *proper focal points in* $(a,b]_{\mathbb{T}}$, *then every conjoined basis of* $(\underline{\mathcal{S}})$ *has at least* m *proper focal points in* $(a,b]_{\mathbb{T}}$.

Theorem 6.4 (Sturmian separation theorem)**.** *Assume* (6.1) *and* (6.2). *If the principal solution of* $(\mathcal{S})$ *has* $m \in \mathbb{N} \cup \{0\}$ *proper focal points in* $(a,b]_{\mathbb{T}}$, *then any other conjoined basis of* $(\mathcal{S})$ *has at least* m *and at most* $m+n$ *proper focal points in* $(a,b]_{\mathbb{T}}$.

Theorem 6.5 (Sturmian separation theorem)**.** *Assume* (6.1) *and* (6.2). *Then the difference between the numbers of proper focal points in* $(a,b]_{\mathbb{T}}$ *of any two conjoined bases of* $(\mathcal{S})$ *is at most* n.

Acknowledgement

The research of the first author was supported by the Czech Science Foundation under grant P201/10/1032 and by the research projects MSM 0021622409 and ME 891 (program Kontakt) of the Ministry of Education, Youth, and Sports of the Czech Republic. The research of the second author was supported by the National Science Foundation under grant DMS – 0707789.

The authors are grateful to an anonymous referee for detailed reading of the paper and valuable suggestions which helped improve its quality.

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(Received August 8, 2011)

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