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BOUNDS FOR THE SUMS OF ZEROS OF SOLUTIONS OF $u^{(m)} = P(z)u$ WHERE P IS A POLYNOMIAL

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Abstract. The main purpose of this paper is to consider the differential equation $u^{(m)} = P(z)u$ ($m \geq 2$) where P is a polynomial with complex, in general, coefficients. Let $z_k(u)$, $k = 1, 2, \ldots$ be the zeros of a nonzero solution u to that equation. We obtain bounds for the sums

$$\sum_{k=1}^{j} \frac{1}{|z_k(u)|} \quad (j \in \mathbb{N})$$

which extend some recent results proved by Gil'.

1. Introduction and main results

It is well known that Nevanlinna theory has appeared to be a powerful tool in the theory of ordinary differential equations in the complex plane $\mathbb{C}$. For the linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_1(z)f' + A_0(z)f = 0 \quad (k \geq 2) \tag{1}$$

whose coefficients $A_0(z), ..., A_{k-1}(z)$ are entire functions, and $A_0(z)$ is not equal to zero identically, it is well known that all solutions of (1) are entire functions, and that if some coefficients of (1) are transcendental then (1) has at least one solution with order $\rho(f) = \infty$. The active research of the asymptotic distribution of the zeros of linear differential equations in the complex plane was started by Bank and Laine [1]. They investigated the equation $f'' + A(z)f = 0$ with an entire function $A(z)$. We refer to the book [11] and some recent works [2, 3, 4, 5, 6, 12, 13, 14, 16] for the literature on asymptotic distribution and counting functions of zeros, and the growth of solutions of complex differential equations.

At the same time, bounds for the zeros of solutions are very important in various applications. Recently, Gil' [10] obtained some results on the bounds of the sums of the zeros of solutions for the second order differential equation $u'' = P(z)u$ with polynomial coefficients.

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In this paper, we consider the differential equation

$$u^{(m)} = P(z)u \quad (m \geq 2), \tag{2}$$

where

$$P(z) = \sum_{k=0}^{n} c_k z^k \quad (c_n \neq 0)$$

is a polynomial of degree n with in general complex coefficients. Denote by $z_k(u)$, $k = 1, 2, \ldots$ the zeros of the solution $u(z) = \sum_{k=0}^{\infty} u_k z^k$ of (2) with multiplicities taken into account. Without loss of generality we assume that the set of the zeros of u is infinite. If u has a finite number l of zeros, then we put

$$\frac{1}{z_k(u)} = 0 \quad (k = l+1, l+2, \ldots).$$

Rearrange the zeros of u in order of increasing modulus: $|z_k(u)| \leq |z_{k+1}(u)|$ $(k = 1, 2, \ldots)$. Put

$$\mu(P) := \exp\left[\sum_{j=0}^{n} \frac{|c_j|}{j+m}\right].$$

We obtain the following theorem which is an extension of Theorem 1.1 from [10].

Theorem 1.1. *If $u(0) \neq 0$ and $deg(P) = n \geq m-1 \geq 1$, then*

$$\sum_{k=1}^{j} \frac{1}{|z_k(u)|} \leq \left[\left(\sum_{k=0}^{m-1} |\frac{u_k}{u_0}| \sqrt[n+m]{k!}\right)^2 \cdot \frac{(m+1)^2}{4(m-1)^2}\mu^2(P) - 1\right]^{\frac{1}{2}} + \sum_{k=1}^{j} \frac{1}{\sqrt[n+m]{k+1}}$$

holds for any $j \in \mathbb{N}$.

Here and below in this section, we take $\gamma := \frac{1}{n+m}$. Denote by $\nu(u, r)$ $(r > 0)$ the counting function of the zeros of u in $|z| \leq r$. We can get the following corollary from Theorem 1.1 above and Corollary 2.2 from [10] (see also [7]). This is an extension of Corollary 4.1 in [10].

Corollary 1.1. *Under the hypothesis of Theorem 1.1, with the notation*

$$\widetilde{\eta}_j(u) := \frac{j(1-\gamma)}{\left[\left(\sum_{k=0}^{m-1} |\frac{u_k}{u_0}|(k!)^\gamma\right)^2 \cdot \frac{(m+1)^2}{4(m-1)^2}\mu^2(P) - 1\right]^{\frac{1}{2}}(1-\gamma) + (1+j)^{1-\gamma} - 1},$$

the inequality $|z_j(u)| \geq \widetilde{\eta}_j(u)$holds and thus $\nu(u, r) \leq j$ for any $r \leq \widetilde{\eta}_j(u)$ $(j = 1, 2, \ldots)$.

Furthermore, put

$$\widetilde{\vartheta}_1 = \left[\left(\sum_{k=0}^{m-1} |\frac{u_k}{u_0}|(k!)^{\gamma}\right)^2 \cdot \frac{(m+1)^2}{4(m-1)^2}\mu^2(P) - 1\right]^{\frac{1}{2}} + \frac{1}{2^{\gamma}}$$

and $\widetilde{\vartheta}_k = \frac{1}{(k+1)^{\gamma}}$ $(k = 2, 3, \ldots)$. Let $\zeta(.)$ be the Riemann Zeta function. Theorem 1.1 above and Corollary 2.3 from [10] (see also [7]) imply the following result which is an extension of Corollary 4.2 in [10].

Corollary 1.2. *Let $\phi(t)$ $(0 \le t < \infty)$ be a continuous convex scalar-valued function, such that $\phi(0) = 0$. Then under the hypothesis of Theorem 1.1,*

$$\sum_{k=1}^{j} \phi(|z_k(u)|^{-1}) \le \sum_{k=1}^{j} \phi(\widetilde{\vartheta}_k) \quad (j = 1, 2, \ldots).$$

In particular, for any $p \ge 1$ and $j = 2, 3, \ldots,$

$$\sum_{k=1}^{j} \frac{1}{|z_k(u)|^p} \le \sum_{k=1}^{j} \widetilde{\vartheta}^p$$

and therefore

$$\sum_{k=1}^{\infty} \frac{1}{|z_k(f)|^p} \le \left(\left[\left(\sum_{k=0}^{m-1} |\frac{u_k}{u_0}|(k!)^{\gamma}\right)^2 \cdot \frac{(m+1)^2}{4(m-1)^2}\mu^2(P) - 1\right]^{\frac{1}{2}} + \frac{1}{2^{p\gamma}}\right)^p + \zeta(p\gamma) - 2^{-\gamma} - 1$$

provided that $p\gamma > 1$.

Finally, in the light of Theorem 1.1 above and Corollary 2.4 from [10] (see also [7]) we obtain the following result which is an extension of Corollary 4.3 in [10].

Corollary 1.3. *Let $\Phi(t_1, t_2, \ldots, t_j)$ be a scalar-valued function with an integer j defined on the domain*

$$-\infty < t_j \le t_{j-1} \le \ldots \le t_2 \le t_1 < +\infty$$

and satisfying the condition

$$\frac{\partial \Phi}{\partial t_1} > \frac{\partial \Phi}{\partial t_2} > \ldots > \frac{\partial \Phi}{\partial t_j}$$

for $t_1 > t_2 > \ldots > t_j > -\infty$. Then under the hypothesis of Theorem 1.1,

$$\Phi(\frac{1}{|z_1(u)|}, \ldots, \frac{1}{|z_j(u)|}) \le \Phi(\widetilde{\vartheta}_1, \ldots, \widetilde{\vartheta}_j).$$

In particular, let $\{d_k\}_{k=1}^{\infty}$ *be a decreasing sequence of positive numbers with* $d_1 = 1$. *Then*

$$\sum_{k=1}^{j} \frac{d_k}{|z_k(u)|} \le \left[\left(\sum_{k=0}^{m-1} |\frac{u_k}{u_0}|(k!)^{\gamma} \right)^2 \cdot \frac{(m+1)^2}{4(m-1)^2}\mu^2(P) - 1 \right]^{\frac{1}{2}} + \sum_{k=1}^{j} \frac{d_k}{(k+1)^{\gamma}}$$

holds for $j = 2, 3, \ldots$.

2. Preliminaries and some lemmas

Consider the entire function

$$f(\lambda) = \sum_{k=0}^{\infty} c_k \lambda^k \quad (c_0 = 1) \tag{3}$$

with in general complex coefficients and finite order $\rho(f)$. Denote by $z_k(f)$, $k = 1, 2, \ldots$ the zeros of f with multiplicities taken into account. Similar discussion as in the first section, without loss of generality we assume that the set of the zeros of f is infinite. Enumerate the zeros of f in order of increasing modulus: $|z_k(f)| \le |z_{k+1}(f)|$ $(k = 1, 2, \ldots)$. The entire function f can be rewritten in the form

$$f(\lambda) = \sum_{k=0}^{\infty} \frac{a_k \lambda^k}{(k!)^{\widetilde{\gamma}}} \quad (\widetilde{\gamma} \in (0,1), \lambda \in \mathbb{C}, a_0 = 1). \tag{4}$$

Assume that

$$\Theta(f) := \left[\sum_{k=1}^{\infty} |a_k|^2 \right]^{\frac{1}{2}} < \infty. \tag{5}$$

The following result is proved by Gil' in [7] (see also in Section 5.1 from [9]).

Lemma 2.1. [7] *Let* f *be defined by* (3) *and condition* (5) *hold. Then*

$$\sum_{k=1}^{j} \frac{1}{|z_k(f)|} \le \Theta(f) + \sum_{k=1}^{j} \frac{1}{(k+1)^{\widetilde{\gamma}}} \quad (j = 1, 2, \ldots).$$

We below extend a result of Gil' (Lemma 3.1 in [10]).

Lemma 2.2. *Let* $n \ge m - 1 \ge 1$. *A nonzero solution* u *of equation* (2) *can be represented as*

$$u(z) = \sum_{k=0}^{\infty} \frac{\nu_k z^k}{\sqrt[n+m]{k!}},$$

where the numbers ν_k, $k = 0, 1, \ldots$ *satisfy the condition*

$$\sum_{k=0}^{\infty} |\nu_k|^2 \le \left(\sum_{k=0}^{m-1} |u_k| \sqrt[n+m]{k!} \right)^2 \cdot \frac{(m+1)^2}{4(m-1)^2}\mu^2(P).$$

Proof. It follows from the Wiman-Valiron theory (see page 281 from [15]) that any nonzero solution u of (2) is of order $\rho(u) = \frac{n+m}{m} < \infty$. Put $u(z) = \sum_{k=0}^{\infty} u_k z^k$, and then (2) yields

$$\sum_{k=m}^{\infty} k(k-1)\cdots(k-m+1)u_k z^{k-m} = \sum_{k=0}^{\infty}(\sum_{j=0}^{k} c_{k-j}u_j)z^k.$$

Here and below we put $c_j = 0$ for $j > n$. It follows from the above equality that

$$(k+m)(k+m-1)\cdots(k+1)u_{k+m} = \sum_{j=0}^{k} c_{k-j}u_j.$$

Take $\gamma := \frac{1}{n+m}$ and $\nu_k := (k!)^\gamma u_k$. Then we have

$$(k+m)(k+m-1)\cdots(k+1)\frac{\nu_{k+m}}{[(k+m)!]^\gamma} = \sum_{j=0}^{k} c_{k-j}\frac{\nu_j}{(j!)^\gamma}. \tag{6}$$

We now take into account two cases as follows.

In the case $k > n$, it follows from (6) that

$$\begin{aligned}
&(k+m)|\nu_{k+m}| \\
\leq\ & \frac{[(k+m)!]^\gamma}{(k+m-1)(k+m-2)\cdots(k+1)} \cdot (\sum_{j=0}^{k-n-1}\frac{|c_{k-j}||\nu_j|}{((j)!)^\gamma} + \sum_{j=k-n}^{k}\frac{|c_{k-j}||\nu_j|}{((j)!)^\gamma}) \\
=\ & \frac{[(k+m)!]^\gamma}{(k+m-1)(k+m-2)\cdots(k+1)} \cdot \sum_{j=k-n}^{k}\frac{|c_{k-j}||\nu_j|}{((j)!)^\gamma} \quad (c_j = 0 \text{ for } j > n) \\
\leq\ & \frac{1}{(m-1)(k+1)} \cdot \left(\frac{(k+m)!}{(k-n)!}\right)^\gamma \cdot \sum_{j=k-n}^{k}|c_{k-j}||\nu_j|.
\end{aligned}$$

Using the inequality between the arithmetic and geometric means and similar discussion as in [10],

$$\begin{aligned}
\left[\frac{(k+m)!}{(k-n)!}\right]^\gamma &= [(t+1)\cdots(t+n+m)]^\gamma \quad (t := k-n) \\
&\leq \left(\frac{(n+m)(t+\frac{n+m+1}{2})}{n+m}\right)^{\gamma(n+m)} \\
&= k + \frac{m-n+1}{2} \\
&\leq k+1 \qquad (n \geq m-1).
\end{aligned}$$

So if $k > n$, then

$$(k+m)|\nu_{k+m}| \le \frac{1}{m-1} \sum_{j=k-n}^{k} |c_{k-j}||\nu_j|. \tag{7}$$

For the other case where $k \le n$, it follows from (6) that

$$(k+m)(k+m-1)\cdots(k+1)|\nu_{k+m}| \le [(k+m)!]^\gamma \cdot \sum_{j=0}^{k} |c_{k-j}||\nu_j|. \tag{8}$$

Again by the inequality between the arithmetic and geometric means,

$$\begin{aligned} [(k+m)!]^\gamma &\le \left[\frac{(k+m)(0+\frac{k+m+1}{2})}{k+m} \right]^{\gamma(k+m)} \\ &= \frac{k+m+1}{2} \qquad (k \le n). \end{aligned}$$

Thus (8) gives that

$$(k+m)|\nu_{k+m}| \le \frac{1}{m-1} \cdot \frac{k+m+1}{2(k+1)} \sum_{j=0}^{k} |c_{k-j}||\nu_j|.$$

If $m-1 \le k \le n$, then we also have inequality

$$(k+m)|\nu_{k+m}| \le \frac{1}{m-1} \sum_{j=0}^{k} |c_{k-j}||\nu_j|. \tag{9}$$

If $0 \le k < m-1 \le n$, then $\frac{k+m+1}{2(k+1)} \le \frac{m+1}{2}$, and thus

$$(k+m)|\nu_{k+m}| \le \frac{m+1}{2(m-1)} \sum_{j=0}^{k} |c_{k-j}||\nu_j|. \tag{10}$$

Similar discussion as in [10], by (7), (9)-(10) and the comparison theorem (see section 1.6 in [8]), we have $|\nu_j| \le w_j$, where w_j is a solution of the equation

$$(k+m)w_{k+m} = \frac{m+1}{2(m-1)} \sum_{j=0}^{k} |c_{k-j}|w_j \tag{11}$$

with $w_0 = |\nu_0|$, $w_1 = |\nu_1|$, …, $w_{m-1} = |\nu_{m-1}|$. Put

$$F(z) := \sum_{j=0}^{\infty} w_j z^j. \tag{12}$$

Then

$$F^{'}(z)=\sum_{j=1}^{m-1}jw_jz^{j-1}+z^{m-1}\sum_{k=0}^{\infty}(k+m)w_{k+m}z^k.$$

Note that $c_j=0$ for $j>n$, and in consideration of (11) and (12),

$$\sum_{k=0}^{\infty}(k+m)w_{k+m}z^k=\frac{m+1}{2(m-1)}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}|c_{k-j}|w_j\right)z^k=\frac{m+1}{2(m-1)}\widehat{P}(z)F(z),$$

where $\widehat{P}(z)=\sum_{j=0}^{n}|c_j|z^j$. Hence,

$$F^{'}(z)=\sum_{j=1}^{m-1}jw_jz^{j-1}+z^{m-1}\frac{m+1}{2(m-1)}\widehat{P}(z)F(z)\quad (F(0)=w_0). \tag{13}$$

Let $z=re^{i\theta}$ for a fixed $\theta\in[0,2\pi)$ and $f(r)=F(re^{i\theta})$, thus (13) yields

$$e^{-i\theta}\frac{df(r)}{dr}=\sum_{j=1}^{m-1}jw_jr^{j-1}e^{i\theta(j-1)}+r^{m-1}e^{i\theta(m-1)}\frac{m+1}{2(m-1)}\widehat{P}(re^{i\theta})f(r)\quad (f(0)=w_0),$$

and therefore,

$$|f(r)|\le\sum_{j=0}^{m-1}w_jr^j+\frac{m+1}{2(m-1)}\int_0^r s^{m-1}|\widehat{P}(se^{i\theta})f(s)|ds.$$

By the Gronwall lemma,

$$|f(r)|\le\sum_{j=0}^{m-1}w_jr^j\cdot\frac{m+1}{2(m-1)}\exp\left[\int_0^r s^{m-1}|\widehat{P}(se^{i\theta})|ds\right].$$

But

$$\begin{aligned}\int_0^1 s^{m-1}|\widehat{P}(se^{i\theta})|ds &\le \int_0^1\sum_{j=0}^{n}|c_j|s^{j+m-1}ds\\ &= \sum_{j=0}^{n}|c_j|\int_0^1 s^{j+m-1}ds\\ &= \sum_{j=0}^{n}\frac{1}{j+m}|c_j|,\end{aligned}$$

and thus we get that

$$\max_{|z|=1}|F(z)|\le\max_{r=1}|f(r)|\le\frac{m+1}{2(m-1)}\sum_{j=0}^{m-1}w_j\cdot\mu(P),$$

where

$$\mu(P) := \exp\left[\sum_{j=0}^{n} \frac{|c_j|}{j+m}\right].$$

By making use of the Parseval equality,

$$\sum_{k=0}^{\infty} w_k^2 = \frac{1}{2\pi}\int_0^{2\pi} |F(e^{i\theta})|^2 d\theta \le \max_{|z|=1} |F(z)|^2 \le \left(\sum_{j=0}^{m-1} w_j\right)^2 \cdot \frac{(m+1)^2}{4(m-1)^2}\mu^2(P).$$

Recall that $|u_j| = \frac{|\nu_j|}{(j!)^\gamma} = \frac{w_j}{(j!)^\gamma}$ $(j = 0, 1, \ldots, m-1)$, where $\gamma = \frac{1}{n+m}$. In view of $|\nu_k| \le w_k$, we get the required inequality

$$\sum_{k=0}^{\infty} |\nu_k|^2 \le \left(\sum_{k=0}^{m-1} |u_k|(k!)^\gamma\right)^2 \cdot \frac{(m+1)^2}{4(m-1)^2}\mu^2(P).$$

□

3. Proof of Theorem 1.1

Let $u = \sum_{k=0}^{\infty} u_k z^k$ be a solution of (2) such that $u_0 = u(0) \neq 0$. Under the assumption of $n \ge m-1 \ge 1$, by Lemma 2.2 we get that

$$u(z) = \sum_{k=0}^{\infty} \frac{\nu_k z^k}{\sqrt[n+m]{k!}},$$

where the numbers ν_k, $k = 0, 1, \ldots$ satisfy the condition

$$\sum_{k=0}^{\infty} |\nu_k|^2 \le \left(\sum_{k=0}^{m-1} |u_k| \sqrt[n+m]{k!}\right)^2 \cdot \frac{(m+1)^2}{4(m-1)^2}\mu^2(P).$$

Put $f(z) := \frac{u(z)}{u_0}$. Then $f(0) = 1$, $f(z) = \sum_{k=0}^{\infty} \frac{\frac{\nu_k}{u_0} z^k}{\sqrt[n+m]{k!}}$, and

$$\left[\sum_{k=1}^{\infty} |\frac{\nu_k}{u_0}|^2\right]^{\frac{1}{2}} \le \left[\left(\sum_{k=0}^{m-1} |\frac{u_k}{u_0}| \sqrt[n+m]{k!}\right)^2 \cdot \frac{(m+1)^2}{4(m-1)^2}\mu^2(P) - 1\right]^{\frac{1}{2}} < \infty.$$

Hence, by Lemma 2.1,

$$\sum_{k=1}^{j} \frac{1}{|z_k(u)|} = \sum_{k=1}^{j} \frac{1}{|z_k(f)|}$$

$$\le \left[\left(\sum_{k=0}^{m-1} |\frac{u_k}{u_0}| \sqrt[n+m]{k!}\right)^2 \cdot \frac{(m+1)^2}{4(m-1)^2}\mu^2(P) - 1\right]^{\frac{1}{2}} + \sum_{k=1}^{j} \frac{1}{\sqrt[n+m]{k+1}}$$

holds for $j = 1, 2, \ldots$.

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