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q-Karamata functions and second order q-difference equations

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Abstract

In this paper we introduce and study q-rapidly varying functions on the lattice $q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$, $q > 1$, which naturally extend the recently established concept of q-regularly varying functions. These types of functions together form the class of the so-called q-Karamata functions. The theory of q-Karamata functions is then applied to half-linear q-difference equations to get information about asymptotic behavior of nonoscillatory solutions. The obtained results can be seen as q-versions of the existing ones in the linear and half-linear differential equation case. However two important aspects need to be emphasized. First, a new method of the proof is presented. This method is designed just for the q-calculus case and turns out to be an elegant and powerful tool also for the examination of the asymptotic behavior to many other q-difference equations, which then may serve to predict how their (trickily detectable) continuous counterparts look like. Second, our results show that $q^{\mathbb{N}_0}$ is a very natural setting for the theory of q-rapidly and q-regularly varying functions and its applications, and reveal some interesting phenomena, which are not known from the continuous theory.

Keywords: regularly varying functions, rapidly varying functions, q-difference equations, asymptotic behavior

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1 Introduction

We are interested in a description of certain asymptotic behavior of real functions defined on the lattice $q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$, $q > 1$. In particular, we introduce the concept of q-rapidly varying functions, which is naturally connected to q-regularly varying functions; the latter ones were studied in [27]. The theory of q-rapidly and q-regularly varying functions (which form the so-called q-Karamata functions) is then applied to examine asymptotic behavior of half-linear q-difference equations. A comparison with the continuous and other counterparts is made.

Recall that a measurable function of real variable $f : [a, \infty) \to (0, \infty)$ is said to be *rapidly varying of index* ∞, resp. *of index* $-\infty$ if it satisfies

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \begin{cases} \infty \ \text{ resp. } 0 & \text{for } \lambda > 1 \\ 0 \ \text{ resp. } \infty & \text{for } 0 < \lambda < 1. \end{cases} \tag{1}$$

Relations in (1) do not necessarily need to include both cases $\lambda > 1$ and $0 < \lambda < 1$. Indeed, it can be easily shown that

$$\lim_{x\to\infty} f(\lambda x)/f(x) = \infty \text{ resp. } 0 \ \ (\lambda > 1) \text{ iff } \lim_{x\to\infty} f(\lambda x)/f(x) = 0 \text{ resp. } \infty \ \ (0 < \lambda < 1).$$

For more information about rapid variation on $\mathbb{R}$, see, for example, [3, 17, 18] and references therein. In [22] the concept of rapidly varying sequences was defined in the following way. Let $[u]$ denote the integer part of u. A positive sequence $\{f_k\}$, $k \in \{a, a+1, \dots\} \subset \mathbb{Z}$, is said to be *rapidly varying of index* ∞, resp. *of index* $-\infty$ if it satisfies

$$\lim_{k\to\infty} \frac{f_{[\lambda k]}}{f_k} = \begin{cases} \infty \ \text{ resp. } 0 & \text{for } \lambda > 1 \\ 0 \ \text{ resp. } \infty & \text{for } 0 < \lambda < 1. \end{cases} \tag{2}$$

Note that the concept of rapidly varying sequence of index ∞ was introduced in [9] as $\lim_{k\to\infty} f_{[\lambda k]}/f_k = 0$ for $\lambda \in (0, 1)$. In [9] it was also shown that

$$\lim_{k\to\infty} f_{[\lambda k]}/f_k = \infty \text{ for } \lambda > 1 \text{ iff } \lim_{k\to\infty} f_{[\lambda k]}/f_{[k]} = 0 \text{ for } 0 < \lambda < 1$$

and the embedding theorem was proved, which somehow enables us to use the continuous theory in the discrete setting. These types of definitions of rapidly varying functions (1) and rapidly varying sequences (2), which include a parameter λ, correspond to the classic Karamata type definition of regularly varying functions, see [3, 14, 17, 18, 20, 30] and references therein. For further reading on rapid and regular variation in the discrete case we refer to [7, 9, 10, 12, 22, 23, 24, 32] and references therein. In [21] it was shown that for any rapidly varying function $f(t)$ of index $-\infty$ such that $f'(t)$ exists and increases one has

$$\lim_{t\to\infty} \frac{tf'(t)}{f(t)} = -\infty. \tag{3}$$

Conversely, if a continuously differentiable function f satisfies (3), then it is rapidly varying of index $-\infty$. A discrete version of these relations was shown in [22]. These results show that it is possible to use (3) as an alternative, and more practical, definition of rapid variation, at least in suitable settings, for instance in studying asymptotic properties of differential or difference equations. Having on mind relation (3), in [31] it was introduced a concept of rapid variation in the following way: Let $f : \mathbb{T} \to (0, \infty)$ be an rd-continuously differentiable function, where $\mathbb{T}$ is a time scale, i.e., a nonempty closed subset of reals. Then f is said to be *rapidly varying of index* ∞ (resp. $-\infty$), if

$$\lim_{t\to\infty} \frac{tf^{\Delta}(t)}{f(t)} = \infty \qquad\qquad \left(\text{resp.} \quad \lim_{t\to\infty} \frac{tf^{\Delta}(t)}{f(t)} = -\infty\right). \tag{4}$$

Moreover, under the condition that f^{Δ} increases, it was shown that this definition is equivalent to the following Karamata type definition: Let $\tau : \mathbb{R} \to \mathbb{T}$ be defined as $\tau(t) = \max\{s \in \mathbb{T} : s \le t\}$. A measurable function $f : \mathbb{T} \to (0, \infty)$ satisfying

$$\lim_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} = \begin{cases} \infty \ \text{ resp. } 0 & \text{for } \lambda > 1 \\ 0 \ \text{ resp. } \infty & \text{for } 0 < \lambda < 1, \end{cases} \tag{5}$$

is said to be *rapidly varying of index* ∞, resp. *of index* $-\infty$ *in the sense of Karamata.* Further it was shown that the assumption $\mu(t) = o(t)$ on the graininess in the theory of rapid variation on time scales cannot be omitted provided one wishes to obtain a continuous-like theory. Note that in the case when $\mathbb{T} = q^{\mathbb{N}_0}$, i.e., with $\mu(t) = (q-1)t$, this condition is not satisfied and some nontypical behaviors occur. Hence, a Karamata type theory on $q^{\mathbb{N}_0}$ cannot be reasonably included in a general theory on time scales. Moreover, as observed below, this theory on $q^{\mathbb{N}_0}$ shows substantial simplifications (in comparison with its continuous counterpart), and so we have another reason why it should be be examined separately.

The Karamata theory in general has many applications, see, e.g., [3]. We are particularly interested in its application to dynamic (differential, difference, ...) equations. For applications of the theory of rapid and regular variation in the theory of differential and difference equations see e.g. [11, 16, 20, 21, 22, 23, 24].

The aim of this paper is twofold. First we introduce the concept of q-rapid variation, which extends the existing related theories, and derive some important properties of q-rapidly varying functions. As we will see, thanks to its structure, $q^{\mathbb{N}_0}$ is a very natural setting for the theory of rapid variation and leads to surprising observations and simplifications when compared with its continuous counterpart. The second part of the paper is devoted to the study of asymptotic properties of the half-linear q-difference equation

$$D_q(\Phi(D_q y(t))) = p(t)\Phi(y(qt)), \tag{6}$$

on $q^{\mathbb{N}_0}$, where $\Phi(u) = |u|^{\alpha-1} \operatorname{sgn} u$ with $\alpha > 1$ and $p(t) > 0$. We are interested in obtaining necessary and sufficient conditions guaranteeing that positive solutions of (6) are q-regularly or q-rapidly varying. The special linear case of (6), i.e., when $\alpha = 2$, was already studied in the framework of q-regular variation in [27]. Here we generalize those results to equation (6) and supplement them by information about q-rapidly varying behavior of its solutions, which are new also in the linear case. Moreover, we present a new method which is suitable for the investigation of q-regularly and q-rapidly varying solutions of q-difference equations. Such a method is very effective and enables us to prove the results which in the corresponding continuous or classical discrete cases require more complicated approaches. Our approach seems to be very promising also for the examination of more general or other q-difference equations. Such results may then serve to predict how their continuous counterpart, which can be difficult to be examined, could look like.

The paper is organized as follows. In the next section we recall the theory of q-regularly varying functions, which was established in [27]. Then, we naturally extend this theory to q-rapidly varying functions and show some of their important properties. In the fourth section we establish the lemmas which are needed to prove the main results concerning asymptotic theory of q-difference equations. In Sections 5 and 6 we establish necessary and sufficient conditions for all positive solutions of equation (6) to be q-regularly or q-rapidly varying, respectively. In Section 7 we present a discussion on integral resp. nonintegral form of our conditions. Further, a comparison of obtained results with the existing ones in the differential or difference or q-difference equations case is made. The paper is concluded by showing relations with the "classical" classification of nonoscillatory solutions of (6). Some directions for a future research are also indicated at appropriate places.

2 Preliminaries

We start with some preliminaries on q-calculus. For material on this topic see [2, 8, 13]. See also [5] for the calculus on time scales which somehow contains q-calculus. A reader may note that some of the below defined "q-concepts" may slightly differ from how they are introduced in the classical q-literature. It is because we work on the lattice $q^{\mathbb{N}_0}$, and so we may follow a "time scale dialect" of q-calculus. Of course, all our notions are equivalent with the classically introduced q-concepts. The q-derivative of a function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is defined by $D_q f(t) = [f(qt) - f(t)]/[(q-1)t]$. Here are some useful rules: $D_q(fg)(t) = g(qt)D_q f(t) + f(t)D_q g(t) = f(qt)D_q g(t) + g(t)D_q f(t)$, $D_q(f/g)(t) = [g(t)D_q f(t) - f(t)D_q g(t)]/[g(t)g(qt)]$, $f(qt) = f(t) + (q-1)tD_q f(t)$. The q-integral of a function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is defined by

$$\int_a^b f(t)\, d_q t = \begin{cases} (q-1)\sum_{t\in[a,b)\cap q^{\mathbb{N}_0}} tf(t) & \text{if } a < b \\ 0 & \text{if } a = b \\ (1-q)\sum_{t\in[b,a)\cap q^{\mathbb{N}_0}} tf(t) & \text{if } a > b, \end{cases}$$

$a, b \in q^{\mathbb{N}_0}$. The improper q-integral is defined by $\int_a^\infty f(t)\, d_q t = \lim_{b\to\infty} \int_a^b f(t)\, d_q t$. Since the fraction $(q^a - 1)/(q-1)$ appears quite frequently, we use the notation

$$[a]_q = \frac{q^a - 1}{q - 1} \quad \text{for } a \in \mathbb{R}. \tag{7}$$

Note that $\lim_{q\to 1^+} [a]_q = a$. In view of (7), it is natural to introduce the notation

$$[\infty]_q = \infty \quad \text{and} \quad [-\infty]_q = \frac{1}{1-q}.$$

Let $\mathcal{R} = \{p : q^{\mathbb{N}_0} \to \mathbb{R} : 1 + (q-1)tp(t) \neq 0 \text{ for } t \in q^{\mathbb{N}_0}\}$. For $p \in \mathcal{R}$ we denote $e_p(t,s) = \prod_{u \in [s,t) \cap q^{\mathbb{N}_0}} [(q-1)up(u) + 1]$ for $s < t$, $e_p(t,s) = 1/e_p(s,t)$ for $s > t$, and $e_p(t,t) = 1$, where $s, t \in q^{\mathbb{N}_0}$. Here are some useful properties of $e_p(t,s)$: For $p \in \mathcal{R}$, $e(\cdot, a)$ is a solution of the IVP $D_q y = p(t)y$, $y(a) = 1$, $t \in q^{\mathbb{N}_0}$. If $s \in q^{\mathbb{N}_0}$ and $p \in \mathcal{R}^+$, where $\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + (q-1)tp(t) > 0 \text{ for all } t \in q^{\mathbb{N}_0}\}$, then $e_p(t,s) > 0$ for all $t \in q^{\mathbb{N}_0}$. If $p, r \in \mathcal{R}$, then $e_p(t,s)e_p(s,u) = e_p(t,u)$ and $e_p(t,s)e_r(t,s) = e_{p+r+t(q-1)pr}(t,s)$. Intervals having the subscript q denote the intervals in $q^{\mathbb{N}_0}$, e.g., $[a,\infty)_q = \{a, aq, aq^2, \dots\}$ with $a \in q^{\mathbb{N}_0}$.

Equation (6) can be viewed as a q-version of the half-linear differential equation studied e.g. in [11] or as a special case of the half-linear dynamic equation studied e.g. in [25] or as a generalization of the linear q-difference equation studied e.g. in [2, 4, 6, 19].

In [27] we introduced the concept of q-regular variation in the following way.

Definition 1. A function $f : q^{\mathbb{N}_0} \to (0,\infty)$ is said to be *q-regularly varying of index ϑ*, $\vartheta \in \mathbb{R}$, if there exists a function $\omega : q^{\mathbb{N}_0} \to (0,\infty)$ satisfying

$$f(t) \sim C\omega(t), \quad \text{and} \quad \lim_{t\to\infty} \frac{tD_q\omega(t)}{\omega(t)} = [\vartheta]_q, \tag{8}$$

C being a positive constant. If $\vartheta = 0$, then f is said to be *q-slowly varying.*

The totality of q-regularly varying functions of index ϑ is denoted by $\mathcal{RV}_q(\vartheta)$. The totality of q-slowly varying functions is denoted by $\mathcal{SV}_q$. Some important properties of q-regularly functions are listed in the following proposition.

Proposition 1 ([27])**.**

(i) The following statements are equivalent:

- $f \in \mathcal{RV}_q(\vartheta)$.
- *f is positive and satisfies*

$$\lim_{t\to\infty} \frac{tD_q f(t)}{f(t)} = [\vartheta]_q. \tag{9}$$

- *f is positive and satisfies*

$$\lim_{t\to\infty} \frac{f(qt)}{f(t)} = q^{\vartheta}. \tag{10}$$

- $f(t) = t^{\vartheta}\varphi(t)e_{\psi}(t,1)$, *where $\varphi : q^{\mathbb{N}_0} \to (0,\infty)$ tends to a positive constant and $\psi : q^{\mathbb{N}_0} \to \mathbb{R}$ satisfies $\lim_{t\to\infty} t\psi(t) = 0$ and $\psi \in \mathcal{R}^+$. Without loss of generality, the function φ can be replaced by a positive constant.*
- *f is positive and satisfies*

$$\lim_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} = (\tau(\lambda))^{\vartheta} \quad \textit{for } \lambda \geq 1 \tag{11}$$

where $\tau : [1,\infty) \to q^{\mathbb{N}_0}$ is defined as $\tau(x) = \max\{s \in q^{\mathbb{N}_0} : s \leq x\}$.

- $f(t) = t^\vartheta L(t)$, *where* $L \in \mathcal{SV}_q$.

(ii) *Let* $f \in \mathcal{RV}_q(\vartheta)$. *Then* $\lim_{t\to\infty} \log f(t)/\log t = \vartheta$. *This implies* $\lim_{t\to\infty} f(t) = 0$ *if* $\vartheta < 0$ *and* $\lim_{t\to\infty} f(t) = \infty$ *if* $\vartheta > 0$

(iii) *Let* $f \in \mathcal{RV}_q(\vartheta)$. *Then* $\lim_{t\to\infty} f(t)/t^{\vartheta-\varepsilon} = \infty$ *and* $\lim_{t\to\infty} f(t)/t^{\vartheta+\varepsilon} = 0$ *for every* $\varepsilon > 0$.

(iv) *Let* $f \in \mathcal{RV}_q(\vartheta)$. *Then* $f^\gamma \in \mathcal{RV}_q(\gamma\vartheta)$.

(v) *Let* $f \in \mathcal{RV}_q(\vartheta_1)$ *and* $g \in \mathcal{RV}_q(\vartheta_2)$. *Then* $fg \in \mathcal{RV}_q(\vartheta_1 + \vartheta_2)$ *and* $1/f \in \mathcal{RV}_q(-\vartheta_1)$.

(vi) *Let* $f \in \mathcal{RV}_q(\vartheta)$. *Then* f *is decreasing provided* $\vartheta < 0$, *and it is increasing provided* $\vartheta > 0$. *A concave* f *is increasing. If* $f \in \mathcal{SV}_q$ *is convex, then it is decreasing.*

We have defined q-regular variation at infinity. If we consider a function $f : q^{\mathbb{Z}} \to (0,\infty)$, $q^{\mathbb{Z}} := \{q^k : k \in \mathbb{Z}\}$, then $f(t)$ is said to be q-regularly varying at zero if $f(1/t)$ is q-regularly varying at infinity. Note that from the continuous or the discrete theory it is known also the concept of normalized regular variation. Because of (9), there is no need to introduce a normality in q-calculus case, since every q-regularly varying function is automatically normalized.

3 q-rapid variation

Looking at the values on the right hand sides of (9) and (10) it is natural to be interested in situations where these values attain their extremal values, i.e., $[-\infty]_q$ and $[\infty]_q$ in (9) and 0 and ∞ in (10). This leads to the concept of q-rapid variation.

Definition 2. A function $f : q^{\mathbb{N}_0} \to (0,\infty)$ is said to be *q-rapidly varying of index* ∞, resp. *of index* $-\infty$ if

$$\lim_{t\to\infty} \frac{tD_q f(t)}{f(t)} = [\infty]_q, \quad \text{resp.} \quad \lim_{t\to\infty} \frac{tD_q f(t)}{f(t)} = [-\infty]_q.$$

The totality of q-rapidly varying functions of index $\pm\infty$ is denoted by $\mathcal{RPV}_q(\pm\infty)$. In fact, we have defined q-rapid variation at infinity. If we consider a function $f : q^{\mathbb{Z}} \to (0,\infty)$, $q^{\mathbb{Z}} := \{q^k : k \in \mathbb{Z}\}$, then $f(t)$ is said to be q-rapidly varying at zero if $f(1/t)$ is q-rapidly varying at infinity. Therefore it is sufficient to develop just the theory of q-rapid variation at infinity. It is easy to see that the function $f(t) = b^t$ with $b > 1$ is a typical representative of the class $\mathcal{RPV}_q(\infty)$, while the function $f(t) = b^t$ with $b \in (0,1)$ is a typical representative of the class $\mathcal{RPV}_q(-\infty)$. Of course, these classes are much wider as can be seen also from the simple representations given in the following proposition, where we present important properties of q-rapidly varying functions.

Proposition 2. *(i) (Simple characterization) For a function $f \in q^{\mathbb{N}_0} \to (0,\infty)$, $f \in \mathcal{RPV}_q(\infty)$, resp. $f \in \mathcal{RPV}_q(-\infty)$, if and only if f satisfies*

$$\lim_{t\to\infty} \frac{f(qt)}{f(t)} = \infty, \quad \textit{resp.} \quad \lim_{t\to\infty} \frac{f(qt)}{f(t)} = 0.$$

(ii) (Karamata type definition) Define $\tau : [1,\infty) \to q^{\mathbb{N}_0}$ by $\tau(x) = \max\{s \in q^{\mathbb{N}_0} : s \le x\}$. For a function $f \in q^{\mathbb{N}_0} \to (0,\infty)$, $f \in \mathcal{RPV}_q(\infty)$, resp. $f \in \mathcal{RPV}_q(-\infty)$, if and only if f satisfies

$$\lim_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} = \infty, \quad \textit{resp.} \quad \lim_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} = 0, \quad \textit{for every } \lambda \in [q,\infty), \tag{12}$$

which holds if and only if f satisfies

$$\lim_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} = 0, \quad \textit{resp.} \quad \lim_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} = \infty, \quad \textit{for every } \lambda \in (0,1).$$

(iii) It holds $f \in \mathcal{RPV}_q(\infty)$ if and only if $1/f \in \mathcal{RPV}_q(-\infty)$.

(iv) If $f \in \mathcal{RPV}_q(\infty)$, then for each $\vartheta \in [0,\infty)$ the function $f(t)/t^\vartheta$ is eventually increasing and $\lim_{t\to\infty} f(t)/t^\vartheta = \infty$.

(v) If $f \in \mathcal{RPV}_q(-\infty)$, then for each $\vartheta \in [0,\infty)$ the function $f(t)t^\vartheta$ is eventually decreasing and $\lim_{t\to\infty} f(t)t^\vartheta = 0$.

Proof. (i) It follows from the identity

$$\frac{tD_q f(t)}{f(t)} = \frac{1}{q-1}\left(\frac{f(qt)}{f(t)} - 1\right).$$

(ii) We prove that $f \in \mathcal{RPV}_q(\infty)$ if and only if the first condition in (12) holds. Other cases follow similarly; we use also (iii). Assume that $f \in \mathcal{RPV}_q(\infty)$. As shown in (iv), f is eventually increasing. Thus, $f(\tau(\lambda t))/f(t) \ge f(qt)/f(t)$ for large t and $\lambda \in [q,\infty)$. Hence, thanks to (i), the result follows. The opposite implication is trivial.

(iii) In view of (i), the proof is trivial.

(iv) We have

$$\begin{aligned} D_q\left(\frac{f(t)}{t^\vartheta}\right) &= \frac{D_q f(t) t^\vartheta - f(t)\frac{(qt)^\vartheta - t^\vartheta}{(q-1)t}}{t^\vartheta (qt)^\vartheta} \\ &= \frac{D_q f(t) - \frac{f(t)}{t}[\vartheta]_q}{(qt)^\vartheta}. \end{aligned} \tag{13}$$

Since $f \in \mathcal{RPV}_q(\infty)$, for each $M > 0$ there exists $t_0 \in q^{\mathbb{N}_0}$ such that $tD_q f(t)/f(t) > M$ for $t \ge t_0$. Hence, $D_q f(t) > f(t)[\vartheta]_q/t$ for large t, and so $D_q(f(t)/t^\vartheta)$ is eventually positive in view of (13). Consequently, $f(t)/t^\vartheta$ is eventually increasing, and so its limit, as $t \to \infty$, must exist (finite positive or infinite). By a contradiction assume that $\lim_{t\to\infty} f(t)/t^\vartheta = K \in (0,\infty)$. Then $f(t) \sim Kt^\vartheta$, which implies $\lim_{t\to\infty} f(qt)/f(t) = q^\vartheta \ne \infty$, i.e., $f \notin \mathcal{RPV}_q(\infty)$, a contradiction.

(v) It follows from (iii) and (iv). ∎

Remark 1. (i) In contrast to the classical continuous or discrete theories, see e.g. [3, 9, 22], we can observe that the Karamata type definition can be (equivalently) expressed in a very simple form. Indeed, compare (ii) and (i) of Proposition 2 to see that, without loss of generality, we can restrict our consideration just to one value of the parameter λ, namely $\lambda = q$. The reason for this simply looking condition may be that rapid variation can be based on a product in the argument characterization which is quite natural for the q-calculus case. Observe that we do not consider the values of λ in the sets $(0, 1)$ and $(1, \infty)$ as in classical theories but in the sets $(0, 1)$ and $[q, \infty)$. It is because $\tau(\lambda t) = t$ for $\lambda \in (1, q)$.

(ii) Another simplification, in comparison with the classical theories, is that for showing the equivalence between the Karamata type definition (or the simple characterization (i) of Proposition 2) and Definition 2 we do not need additional assumptions like convexity, see (3).

(iii) In view of (iv) and (v) of Proposition 2, $\mathcal{RPV}_q(\infty)$ functions are always eventually increasing to ∞, while $\mathcal{RPV}_q(-\infty)$ functions are always eventually decreasing to zero.

(iv) It is not difficult to see that the concept of normalized q-rapid variation somehow misses point. Indeed, let "normalized" q-rapidly varying functions are defined as in Definition 2. Let us define the concept of q-rapid variation in a seemingly more general way: A function $g : q^{\mathbb{N}_0} \to (0, \infty)$ is q-rapidly varying of index ∞, resp. of index $-\infty$ if there are positive functions φ and ω satisfying $g = \varphi\omega$, $\lim_{t\to\infty} \varphi(t) = C \in (0, \infty)$, $\lim_{t\to\infty} tD_q\omega(t)/\omega(t) = [\infty]_q$ resp. $= [-\infty]_q$. By Proposition 2 we then get $\lim_{t\to\infty} g(qt)/g(t) = (\varphi(qt)/\varphi(t))(\omega(qt)/\omega(t)) = \infty$ resp. $= 0$, and so g is also "normalized" q-rapidly varying of index ∞, resp. $-\infty$. Note that these observations remain valid even under another generalization, where the condition $\lim_{t\to\infty} \varphi(t) = C \in (0, \infty)$ is relaxed to the inequalities $0 < C_1 \le \varphi(qt)/\varphi(t) \le C_2 < \infty$ for large t.

4 Auxiliary statements

The next lemmas will play important roles in showing q-regularly and q-rapidly varying behavior of solutions to (6).

Lemma 1. *Define the function* $f : (0, \infty) \to \mathbb{R}$ *by*

$$f(x) = \Phi\left(\frac{x}{q} - \frac{1}{q}\right) - \Phi\left(1 - \frac{1}{x}\right).$$

Then $x \mapsto f(x)$ *is strictly increasing for* $x > q^{1-1/\alpha}$ *and strictly decreasing for* $0 < x < q^{1-1/\alpha}$.

Proof. The statement follows from the equality

$$f'(x) = \frac{\alpha - 1}{(x-1)^2}\left(q\left|\frac{x-1}{q}\right|^\alpha - \left|\frac{x-1}{x}\right|^\alpha\right).$$

□

The next lemma shows that (6) can be viewed in terms of fractions which appear in characterization of q-regular or q-rapid variation.

Lemma 2. *For $y \neq 0$ define the operator $\mathcal{L}$ by*

$$\mathcal{L}[y](t) = \Phi\left(\frac{y(q^2t)}{qy(qt)} - \frac{1}{q}\right) - \Phi\left(1 - \frac{y(t)}{y(qt)}\right).$$

Then

$$D_q(\Phi(D_q y(t))) = \frac{\Phi(y(qt))}{(q-1)^\alpha t^\alpha}\mathcal{L}[y](t)$$

and equation (6) *can be written as $\mathcal{L}[y](t) = (q-1)^\alpha t^\alpha p(t)$ for $y \neq 0$.*

Proof. The statement is an easy consequence of the formula for q-derivative. Indeed,

$$\begin{aligned}
D_q(\Phi(D_q y(t)))) &= D_q\left(\Phi\left(\frac{y(qt)-y(t)}{(q-1)t}\right)\right)\\
&= \frac{1}{(q-1)t}\left(\Phi\left(\frac{y(q^2t)-y(qt)}{(q-1)qt}\right) - \Phi\left(\frac{y(qt)-y(t)}{(q-1)t}\right)\right)\\
&= \frac{1}{(q-1)^\alpha t^\alpha}\left(\Phi\left(\frac{y(qt)}{q}\left(\frac{y(q^2t)}{y(qt)} - 1\right)\right) - \Phi\left(y(qt)\left(1 - \frac{y(t)}{y(qt)}\right)\right)\right)\\
&= \frac{\Phi(y(qt))}{(q-1)^\alpha t^\alpha}\mathcal{L}[y](t).
\end{aligned}$$

□

The following lemma will play an important role when dealing with the indices of regular variation of solutions to (6).

Lemma 3. *Define the function $h_q : (\Phi(1/(1-q)), \infty) \to \mathbb{R}$ by*

$$h_q(x) = \frac{x}{1-q^{1-\alpha}}\left(1 - \left(1 + (q-1)\Phi^{-1}(x)\right)^{1-\alpha}\right).$$

Then the graph of $x \mapsto h_q(x)$ is a parabola like curve with the minimum at the origin. If $C > 0$, then the equation $h_q(x) - x - C = 0$ has two real roots $x_1 < 0$ and $x_2 > 1$ on $(\Phi(1/(1-q)), \infty)$. If $C = 0$, then the algebraic equation has the roots $x_1 = 0$ and $x_2 = 1$. Taking the limit in h_q as $q \to 1^+$ it holds $h_1(x) = |x|^\beta$, where $\beta = \alpha/(\alpha-1)$ is the conjugate number of α.

Proof. The shape of the curve follows from the facts that

$$h_q'(x)\operatorname{sgn}(x) = \frac{1}{1-q^{1-\alpha}}(1 - (1 + (q-1)\Phi^{-1}(x))^{-\alpha})\operatorname{sgn}(x) > 0$$

and

$$h_q''(x) = \frac{\beta(q-1)}{1-q^{1-\alpha}}|x|^{\beta-2}(1 + (q-1)\Phi^{-1}(x))^{-\alpha-1} > 0$$

for admissible x, $x \neq 0$, and $h_q(0) = h'_q(0) = h''_q(0) = 0$. The statement concerning the roots x_1, x_2 then easily follows from observing the intersections of the graphs of the line $x \mapsto x$ and the function $x \mapsto h_q(x) - C$, view of $h_q(1) = 1$. The equality $h_1(x) = |x|^\beta$ follows either by direct using the L'Hospital rule to h_q with the respect to q or from the identity (14), in view of the fact that $[a]_q$ tends to $[a]_1 = a$ as $q \to 1^+$. $\square$

We will need to rewrite the expression in the algebraic equation from the previous lemma in other terms; such a relation is described in the next statement.

Lemma 4. *For $\vartheta \in \mathbb{R}$ it holds*

$$\Phi([\vartheta]_q)[1-\vartheta]_{q^{\alpha-1}} = \Phi([\vartheta]_q) - h_q(\Phi([\vartheta]_q)), \tag{14}$$

where $1 + (q-1)\Phi^{-1}(\Phi([\vartheta]_q)) > 0$.

Proof. We have $1 + (q-1)\Phi^{-1}(\Phi([\vartheta]_q)) = 1 + (q-1)[\vartheta]_q = q^\vartheta > 0$. Further,

$$\begin{aligned}
\Phi([\vartheta]_q) - h_q(\Phi([\vartheta]_q)) &= \Phi([\vartheta]_q)\left(1 - \frac{1}{1-q^{1-\alpha}}\left(1 - q^{\vartheta(1-\alpha)}\right)\right) \\
&= \Phi([\vartheta]_q)\frac{q^{\vartheta(1-\alpha)} - q^{1-\alpha}}{1-q^{1-\alpha}} \cdot \frac{q^{\alpha-1}}{q^{\alpha-1}} \\
&= \Phi([\vartheta]_q)\frac{q^{(\alpha-1)(1-\vartheta)} - 1}{q^{\alpha-1} - 1} \\
&= \Phi([\vartheta]_q)[1-\vartheta]_{q^{\alpha-1}}.
\end{aligned}$$

$\square$

Next is described an important relation between the expression from the previous lemma and the function f from Lemma 1.

Lemma 5. *For $\vartheta \in \mathbb{R}$ it holds*

$$(q-1)^{-\alpha} f(q^\vartheta) = \Phi([\vartheta]_q)[1-\vartheta]_{q^{\alpha-1}}[1-\alpha]_q, \tag{15}$$

where f is defined in Lemma 1.

Proof. In view of $(q^{1-\alpha} - 1)/(q^{\alpha-1} - 1) = -1/q^{\alpha-1}$, we have

$$\begin{aligned}
(q-1)^\alpha \Phi([\vartheta]_q)[1-\vartheta]_{q^{\alpha-1}}[1-\alpha]_q &= \Phi\left(\frac{q^\vartheta - 1}{q-1}\right)\frac{(q^{(\alpha-1)(1-\vartheta)} - 1)(q-1)^\alpha(q^{1-\alpha} - 1)}{(q^{\alpha-1} - 1)(q-1)} \\
&= \Phi(q^\vartheta - 1)\frac{1 - q^{(\alpha-1)(1-\vartheta)}}{q^{\alpha-1}} \\
&= \frac{\Phi(q^\vartheta - 1)}{q^{\alpha-1}} - \frac{\Phi(q^\vartheta - 1)}{q^{(\alpha-1)-(\alpha-1)(1-\vartheta)}} \\
&= \Phi\left(\frac{q^\vartheta - 1}{q}\right) - \Phi\left(\frac{q^\vartheta - 1}{q^\vartheta}\right) \\
&= f(q^\vartheta).
\end{aligned}$$

$\square$

5 Rapidly varying solutions of second order half-linear q-difference equations

In this section we establish conditions guaranteeing that positive solutions of equation (6) are q-rapidly varying. Recall that all positive solutions of (6) are eventually monotone and convex. Furthermore, both, increasing and decreasing positive solutions always exist. More detailed information about classification of solutions to (6) will be presented in Section 8.

Theorem 1. *Equation* (6) *has solutions*

$$u \in \mathcal{RPV}_q(-\infty) \quad \textit{and} \quad v \in \mathcal{RPV}_q(\infty) \tag{16}$$

if and only if

$$\lim_{t\to\infty} t^\alpha p(t) = \infty. \tag{17}$$

Further, if condition (17) *holds, then all positive decreasing solutions of* (6) *belong to* $\mathcal{RPV}_q(-\infty)$ *and all positive increasing solutions of* (6) *belong to* $\mathcal{RPV}_q(\infty)$*. Each of the two conditions in* (16) *implies* (17)

Proof. Necessity. Let u be a solution of (6) such that $u \in \mathcal{RPV}_q(-\infty)$. Then u is eventually decreasing (towards zero) and $\lim_{t\to\infty} u(qt)/u(t) = 0$ by Proposition 2. Hence, $\lim_{t\to\infty} u(q^2t)/u(qt) = 0$, $\lim_{t\to\infty} u(t)/u(qt) = \infty$, and so $\lim_{t\to\infty} t^\alpha p(t) = \lim_{t\to\infty}(q-1)^{-\alpha}\mathcal{L}[u](t) = \infty$, in view of Lemma 2. Similarly, for a solution v of (6) with $v \in \mathcal{RPV}_q(\infty)$, we have v is eventually increasing (towards ∞) and $\lim_{t\to\infty} v(qt)/v(t) = \infty$, and so $\lim_{t\to\infty} t^\alpha p(t) = \lim_{t\to\infty}(q-1)^{-\alpha}\mathcal{L}[v](t) = \infty$.

Sufficiency. Let (17) hold and let u be a positive decreasing solution of (6) on $[a,\infty)_q$, a sufficiently large. Since u is decreasing, we have $u(qt) \le u(t)$, and in view of Lemma 2,

$$\infty = \lim_{t\to\infty} t^\alpha p(t) = \lim_{t\to\infty}(q-1)^{-\alpha}\mathcal{L}[u](t) \le \lim_{t\to\infty}(q-1)^{-\alpha}\Phi(u(t)/u(qt)-1).$$

Consequently, $\lim_{t\to\infty} u(t)/u(qt) = \infty$, and so $u \in \mathcal{RPV}_q(-\infty)$ by Proposition 2. Now assume that v is a positive increasing solution of (6) on $[a,\infty)_q$. Then $v(qt) \ge v(t)$, and, similarly as above,

$$\infty = \lim_{t\to\infty} t^\alpha p(t) \le (q-1)^{-\alpha}\Phi(v(q^2t)/(qv(qt)) - 1/q),$$

which implies $\lim_{t\to\infty} v(q^2t)/v(qt) = \infty$, and, consequently, $v \in \mathcal{RPV}_q(\infty)$. □

Remark 2. The results presented in Theorem 1, except of the necessity part for an increasing solution, are q-versions of the results presented for the corresponding differential, resp. difference, resp. dynamic equations (with $\mu(t) = o(t)$). See [24] for the discrete case and [31] for the time scale case. Concerning rapidly varying solutions of differential equations, we refer to Marić's book [20] or to [21] for the corresponding results in the linear case; however, according to the best of our knowledge, the corresponding case of rapid variation in the theory of half-linear differential equations has not been specially processed in the literature, with the note that the time scale results from [31] contain the continuous results. Also note that a necessity part for increasing solutions has not been proved in the differential (resp. difference or dynamic) equations setting yet. Finally we remark that our approach in q-calculus case is fairly different (and much simpler) in comparison with the corresponding cases in other settings.

6 Regularly varying solutions of second order half-linear q-difference equations

The results of this section can be understood as a generalization of the "linear" results or as an q-extension of the "continuous" results. At the same time we present here a new method of the proof, different from that in previous works. This method is designed just for the q-calculus case and turns out to be more effective than the method from [27], which can be seen as a q-version of the differential equations case ([20, 21]). Note that one idea of this method was used also in the previous section.

Theorem 2. *(i) Equation* (6) *has solutions*

$$u \in \mathcal{SV}_q \quad \text{and} \quad v \in \mathcal{RV}_q(1) \tag{18}$$

if and only if

$$\lim_{t\to\infty} t^\alpha p(t) = 0. \tag{19}$$

Further, if condition (19) *holds, then all positive decreasing solutions of* (6) *belong to* $\mathcal{SV}_q$ *and all positive increasing solutions of* (6) *belong to* $\mathcal{RV}_q(1)$. *Each of the two conditions in* (18) *implies* (19).

(ii) Equation (6) *has solutions*

$$u \in \mathcal{RV}_q(\vartheta_1) \quad \text{and} \quad v \in \mathcal{RV}_q(\vartheta_2) \tag{20}$$

if and only if

$$\lim_{t\to\infty} t^\alpha p(t) = B > 0, \tag{21}$$

where $\vartheta_i = \log_q[(q-1)\Phi^{-1}(\lambda_i)+1]$, $i = 1, 2$, $\lambda_1 < \lambda_2$ *being the roots of the equation* $h_q(\lambda) - \lambda + B/[1-\alpha]_q = 0$; *these roots satisfy* $\lambda_1 \in (\Phi(1/(1-q)), 0)$, $\lambda_2 > 1$, *and* ϑ_1, ϑ_2 *satisfy* $\vartheta_1 \in (-\infty, 0)$ *and* $\vartheta_2 > 1$. *Further, if condition* (21) *holds, then all positive decreasing solutions of* (6) *belong to* $\mathcal{RV}_q(\vartheta_1)$ *and all positive increasing solutions of* (6) *belong to* $\mathcal{RV}_q(\vartheta_2)$. *Each of the two conditions in* (20) *implies* (21).

Proof. First note that the intervals of allowed values for λ_1 and λ_2 follows from Lemma 3. The intervals for ϑ_1, ϑ_2 are then consequences of the relations $q^{\vartheta_i} = (q-1)\Phi^{-1}(\lambda_i) + 1$, $i = 1, 2$.

Parts (i) and (ii) will be proved simultaneously, assuming $B \geq 0$ in (21) and, consequently, having $\lambda_1 \in (\Phi(1/(1-q)), 0]$ and $\lambda_2 \geq 1$.

Necessity. Assume $u \in \mathcal{RV}_q(\vartheta_1)$. Using Lemmas 2, 4, and 5, we get

$$\begin{aligned}
\lim_{t\to\infty} t^\alpha p(t) &= (q-1)^{-\alpha} \lim_{t\to\infty} \mathcal{L}[u](t) \\
&= (q-1)^{-\alpha} \left(\Phi\left(\frac{q^\vartheta}{q} - \frac{1}{q}\right) - \Phi\left(1 - \frac{1}{q^\vartheta}\right)\right) \\
&= (q-1)^{-\alpha} f(q^{\vartheta_1}) \\
&= \Phi([\vartheta_1]_q)[1-\vartheta_1]_{q^{\alpha-1}}[1-\alpha]_q \\
&= [1-\alpha]_q \left(\Phi([\vartheta_1]_q) - h_q(\Phi([\vartheta_1]_q))\right) \\
&= [1-\alpha]_q \left(\lambda_1 - h_q(\lambda_1)\right) \\
&= [1-\alpha]_q \frac{B}{[1-\alpha]_q} \\
&= B.
\end{aligned}$$

The same arguments work for $v \in \mathcal{RV}_q(\vartheta_2)$.

In view of Proposition 1, solutions in $\mathcal{RV}_q(\vartheta_1)$ necessarily decrease (this includes also $\mathcal{SV}_q$ solutions because of their convexity) and solutions in $\mathcal{RV}_q(\vartheta_2)$ necessarily increase.

Sufficiency. Assume $\lim_{t\to\infty} t^\alpha p(t) = B \geq 0$ and let u be a positive decreasing solution of (6) on $[a,\infty)_q$. Let us write B as $B = [1-\alpha]_q(\Phi([\vartheta_1]_q) - h_q(\Phi([\vartheta_1]_q)))$. In view of Lemmas 2, 4, 5, we have

$$\lim_{t\to\infty} \mathcal{L}[u](t) = (q-1)^\alpha \lim_{t\to\infty} t^\alpha p(t) = (q-1)^\alpha B = f(q^{\vartheta_1}). \tag{22}$$

We will show that $\lim_{t\to\infty} u(qt)/u(t) = q^{\vartheta_1}$. Denote $M_* = \liminf_{t\to\infty} u(qt)/u(t)$ and $M^* = \limsup_{t\to\infty} u(qt)/u(t)$. First note that the case $M_* = 0$ cannot happen. Indeed, if $M_* = 0$, then $\limsup_{t\to\infty} \mathcal{L}[u](t) = \infty$, which is in contradiction with (22). We also have u decreasing, and hence $M_*, M^* \in (0,1]$. Taking lim sup and lim inf as $t \to \infty$ in

$$\Phi\left(\frac{u(q^2t)}{qu(qt)} - \frac{1}{q}\right) = \Phi\left(1 - \frac{u(t)}{u(qt)}\right) + (q-1)^\alpha t^\alpha p(t),$$

which arises by simple moving the terms in $\mathcal{L}[u](t) = (q-1)^\alpha t^\alpha p(t)$, we get $\Phi(M^*/q - 1/q) = \Phi(1 - 1/M^*) + f(q^{\vartheta_1})$ and $\Phi(M_*/q - 1/q) = \Phi(1 - 1/M_*) + f(q^{\vartheta_1})$. Hence, $f(M^*) = f(q^{\vartheta_1}) = f(M_*)$. Since $M_*, M^*, q^{\vartheta_1} \in (0,1]$ and f is strictly decreasing on $(0, q^{1-1/\alpha})$ (see Lemma 1), we get $M_* = M^* = q^{\vartheta_1}$, and so $u \in \mathcal{RV}_q(\vartheta_1)$. Similarly we proceed with a positive increasing solution v of (6). However, one additional observation is needed. First note that for $N_* = \liminf_{t\to\infty} v(qt)/v(t)$ and $N^* = \limsup_{t\to\infty} v(qt)/v(t)$ we have $N_*, N^* \in [1,\infty)$. In general, it may happen that $f(N_*) = f(N^*)$ with $N_* < N^*$. But this is possible only when $N_*, N^* \in [1,q]$, and then $f(N_*) = f(N^*) \leq f(1) = f(q)$. However, in our case we have $f(N_*) = f(N^*) = f(q^{\vartheta_2}) > f(q)$, and so $N_* = N^* = q^{\vartheta_2}$, which implies $v \in \mathcal{RV}_q(\vartheta_2)$. □

Remark 3. (i) Theorem 2 can be viewed as a q-version of the continuous results from [16] and the discrete results from [22]. See also [29, 28] for the time scale case. Theorem 2 can be understood also as a half-linear extension of the results from

[27]. More detailed information concerning a comparison with all these results and a discussion on the form of the conditions can be found in the next section.

(ii) It should be pointed out that the method of the proof is different from the one used in previous works. This method is suitable for investigating q-difference equations and we believe that it will enable us to prove other results which are q-versions of existing or nonexisting continuous results; in the latter case, such results may serve to predict the form of the continuous counterpart (we simply take, formally, the limit as $q \to 1^+$).

(iii) Concerning directions for a future research, we mention here the study of asymptotic behavior of some other q-difference equations in the framework of q-Karamata theory, e.g., of equation (6) with no sign condition on p, or of the nonlinear equation $D_q(\Phi(D_q y(t))) + f(t, y(qt)) = 0$, or of q-recurrence relations like $y(q^n t) + a_1(t)y(q^{n-1}t) + \cdots + a_{n-1}(t)y(qt) + a_n(t)y(t) = 0$, etc. Information about these equations can be useful either from a numerical point of view (in the theory of differential equations), or in many situations concerning various aspects of the extensive theory of q-calculus. In addition to q-regularly or q-rapidly varying functions, e.g. the concept of q-regular boundedness could be possibly introduced and applied.

7 Integral versus nonintegral conditions and comparison with other results

In this section we discuss the form of conditions guaranteeing q-regular resp. q-rapid variation of solutions. We compare our results with the existing linear ones and with the results from the differential equations case; we reveal substantial differences between the classical calculus case and the q-calculus case.

First recall the main statement of [27, Theorem 2]: The linear q-difference equation $D_q^2 y(t) = p(t)y(qt)$, $p(t) > 0$, has a fundamental set of solutions $u \in \mathcal{RV}_q(\vartheta_1)$ and $v \in \mathcal{RV}_q(\vartheta_2)$ iff $\lim_{t\to\infty} t\int_t^\infty p(s)\, d_q s = A \in [0, \infty)$, where $\vartheta_i = \log_q[(q-1)\lambda_i]$, $i = 1, 2$, $\lambda_1 < \lambda_2$ being the roots of the equation $\lambda^2 - [A(q-1)+1]\lambda - A = 0$. Comparing this result with that from Theorem 2, the following questions may come in our minds: 1. Are our new results on regularly varying solutions really extensions of the previous ones? 2. Why do we have expressed regularly varying behavior in terms of nonintegral condition involving the coefficient p while the previous results involve integral conditions? 3. How are these conditions related? The next lemma shows that in the case of existence of a proper limit, integral and nonintegral conditions are actually equivalent. Hence our new results really extend the existing ones. Moreover, from that lemma it follows that while the integral condition is not suitable for a unified characterization of rapid or regularly varying behavior of solutions to (6), this can be done via the nonintegral condition, see also Corollary 1. Later we show that these relations are specific just for q-calculus and differ from what is known in the continuous case. We stress that there is no sign condition on p in the next lemma.

Lemma 6. *Let $p : q^{\mathbb{N}_0} \to \mathbb{R}$ and $\alpha > 1$. It holds*

$$\lim_{t\to\infty} t^{\alpha-1} \int_t^\infty p(s)\, d_q s = A \in \mathbb{R} \quad \textit{iff} \quad \lim_{t\to\infty} t^\alpha p(t) = -[1-\alpha]_q A \in \mathbb{R}.$$

Moreover,

$$\textit{if} \quad \lim_{t\to\infty} t^\alpha p(t) = \pm\infty, \quad \textit{then} \quad \lim_{t\to\infty} t \int_t^\infty p(s)\, d_q s = \pm\infty,$$

but the opposite implication does not hold in general.

Proof. If. Assume $\lim_{t\to\infty} t^\alpha p(t) = -[1-\alpha]_q A$, where $A \in \mathbb{R} \cup \{\pm\infty\}$. Using the q-L'Hospital rule, we get

$$\begin{aligned} \lim_{t\to\infty} t^{\alpha-1} \int_t^\infty p(s)\, d_q s &= \lim_{t\to\infty} \frac{-p(t)}{((qt)^{1-\alpha} - t^{1-\alpha})/((q-1)t)} \\ &= \lim_{t\to\infty} \frac{t^\alpha p(t)}{-[1-\alpha]_q} \\ &= A. \end{aligned}$$

Only if. Assume $\lim_{t\to\infty} t^{\alpha-1} \int_t^\infty p(s)\, d_q s = A \in \mathbb{R}$. We have

$$\begin{aligned} t^{\alpha-1} \int_t^\infty p(s)\, d_q s &= t^{\alpha-1} \left(\int_t^{qt} p(s)\, d_q s + \int_{qt}^\infty p(s)\, d_q s \right) \\ &= (q-1) t^\alpha p(t) + \frac{1}{q^{\alpha-1}} (qt)^{\alpha-1} \int_{qt}^\infty p(s)\, d_q s. \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{t\to\infty} t^\alpha p(t) &= \lim_{t\to\infty} \frac{1}{q-1} \left(t^{\alpha-1} \int_t^\infty p(s)\, d_q s - \frac{1}{q^{\alpha-1}} (qt)^{\alpha-1} \int_{qt}^\infty p(s)\, d_q s \right) \\ &= \frac{1}{q-1} \left(A - \frac{A}{q^{\alpha-1}} \right) \\ &= -A[1-\alpha]_q. \end{aligned}$$

It remains to find a function p such that $\lim_{t\to\infty} t \int_t^\infty p(s)\, d_q s = \infty$, but $\lim_{t\to\infty} t^\alpha p(t)$ does not exist. For simplicity we present an example corresponding with the case $\alpha = 2$. Define the function

$$p(t) = \begin{cases} t^{-2}/q & \text{for } t = q^{2n}, \\ t^{-2}/q + t^{-3/2} & \text{for } t = q^{2n+1}, \end{cases}$$

where $n \in \mathbb{N} \cup \{0\}$. Then

$$t^2 p(t) = \begin{cases} 1/q & \text{for } t = q^{2n}, \\ \sqrt{t} + 1/q & \text{for } t = q^{2n+1}. \end{cases}$$

Thus we see that $\liminf_{t\to\infty} t^2 p(t) = 1/q < \infty = \limsup_{t\to\infty} t^2 p(t)$. Further, with $t = q^n$, we have $\int_t^\infty p(s)\,d_q s = (q-1)\sum_{j=n}^\infty q^j p(q^j)$. Hence, summing the appropriate geometric series, we obtain

$$t\int_t^\infty p(s)\,d_q s = \begin{cases} q^{n/2}\sqrt{q}+1 = \sqrt{qt}+1 & \text{for } t = q^{2n}, \\ q^{n/2}q+1 = q\sqrt{t}+1 & \text{for } t = q^{2n+1}. \end{cases}$$

Consequently, $\lim_{t\to\infty} t\int_t^\infty p(s)\,d_q s \geq \lim_{t\to\infty}\sqrt{qt} = \infty$. □

To see an interesting specific character of the results in the q-calculus case, let us recall some of their existing continuous counterparts, see [16, 20, 28, 31]. Positive solutions of the equation

$$(\Phi(y'(t)))' - p(t)\Phi(y(t)) = 0, \tag{23}$$

$p(t) > 0$, are regularly varying iff the limit $\lim_{t\to\infty} t^{\alpha-1}\int_t^\infty p(s)\,ds = C$ exists finite, and are rapidly varying iff $\lim_{t\to\infty} t^{\alpha-1}\int_t^{\lambda t} p(s)\,ds = \infty$ for all $\lambda > 1$. The indices of regular variation are given here by $\Phi^{-1}(\gamma_i)$, $i = 1, 2$, where $\gamma_1 < \gamma_2$ are the roots of $|\gamma|^\beta - \gamma - C = 0$. With the use of Lemma 3, observe how this result matches the one from Theorem 2 as $q \to 1^+$. Recall that $\lim_{t\to\infty} t^{\alpha-1}\int_t^\infty p(s)\,ds = C_1$ exists finite iff $\lim_{t\to\infty} t^{\alpha-1}\int_t^{\lambda t} p(s)\,ds = C_2(\lambda)$ exists finite for all $\lambda > 1$ with $C_2(\lambda) = C_1(\lambda^{\alpha-1} - 1)/\lambda^{\alpha-1}$; therefore all positive solutions of (23) are rapidly or regularly varying iff for every $\lambda > 1$ $\lim_{t\to\infty} t^{\alpha-1}\int_t^{\lambda t} p(s)\,ds$ exists finite or infinite. The expression

$$t^{\alpha-1}\int_t^{\tau(\lambda t)} p(s)\,d_q s, \tag{24}$$

considered on $q^{\mathbb{N}_0}$, can be understood as a q-version of $t^{\alpha-1}\int_t^{\lambda t} p(s)\,ds$. Further note that the expression $t^\alpha p(t)$ in q-calculus can be viewed in two ways: First, simply as a nonintegral expression. Second, up to certain constant multiple, as $t^{\alpha-1}\int_t^{qt} p(s)\,d_q s$, which is equal to (24) where $\lambda = q$. While the existence of a (finite or infinite) limit $\lim_{t\to\infty} t^\alpha p(t)$ clearly cannot serve to guarantee regularly or rapidly varying behavior of solutions to (23) (in the sense of sufficiency and necessity), this is possible in the q-calculus case. It is because in q-calculus the relations between the limits $\lim_{t\to\infty} t^{\alpha-1}\int_t^\infty p(s)\,d_q s$, $\lim_{t\to\infty} t^{\alpha-1}\int_t^{qt} p(s)\,d_q s$, and $\lim_{t\to\infty} t^\alpha p(t)$, are "closer" than in the classical calculus. Also note that while in the continuous case we need the limit $\lim_{t\to\infty} t^{\alpha-1}\int_t^{\lambda t} p(s)\,ds$ to exist for all parameters $\lambda > 1$, in the q-calculus case we require its existence just for one parameter, namely $\lambda = q$; compare also with the Karamata type definitions in the cases of both calculi. Finally we point out that the situation in the classical discrete case, see [22], or, of course, in a general time scale case (with the graininess μ such that $\mu(t) = o(t)$ as $t \to \infty$), see [31], is similar to that in the continuous case, and so the q-calculus case is indeed exceptional.

8 Classification of nonoscillatory solutions in the framework of q-Karamata theory

In this section we provide information about asymptotic behavior of all positive solutions of (6) as $t \to \infty$. All nontrivial solutions of (6) are nonoscillatory (i.e., of one sign for large t) and monotone for large t. Note that the solution space of (6) has just one half of the properties which characterize linearity, namely homogeneity (but not additivity). Because of the homogeneity, without loss of generality, we may restrict our consideration only to positive solutions of (6); we denote this set as $\mathbb{M}$. Thanks to the monotonicity, the set $\mathbb{M}$ can be further split in the two classes $\mathbb{M}^+$ and $\mathbb{M}^-$, where

$$\begin{aligned}
\mathbb{M}^+ &= \{x \in \mathbb{M} : \exists t_x \in q^{\mathbb{N}_0} \text{ such that } x(t) > 0, D_q x(t) > 0 \text{ for } t \ge t_x\},\\
\mathbb{M}^- &= \{x \in \mathbb{M} : x(t) > 0, D_q x(t) < 0\}.
\end{aligned}$$

It is not difficult to see that these classes are always nonempty. To see it, the reader can follow the continuous ideas described e.g. in [11, Chapter 4] or understand this equation as a special case of a more general quasi-linear dynamic equation; its asymptotic behavior is discussed e.g. in [1]. Basic properties of half-linear dynamic equations which include (6) as a special case, were studied e.g. in [25].

A positive function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is said to be a *q-Karamata function*, if f is q-slowly or q-regularly or q-rapidly varying; we write $f \in \mathcal{KF}_q$. We introduce the following notation:

$$\begin{aligned}
\mathbb{M}^-_{SV} &= \mathbb{M}^- \cap \mathcal{SV}_q,\\
\mathbb{M}^-_{RV}(\vartheta_1) &= \mathbb{M}^- \cap \mathcal{RV}_q(\vartheta_1), \vartheta_1 < 0,\\
\mathbb{M}^+_{RV}(\vartheta_2) &= \mathbb{M}^+ \cap \mathcal{RV}_q(\vartheta_2), \vartheta_2 \ge 1,\\
\mathbb{M}^-_{RPV}(-\infty) &= \mathbb{M}^- \cap \mathcal{RPV}_q(-\infty),\\
\mathbb{M}^+_{RPV}(\infty) &= \mathbb{M}^+ \cap \mathcal{RPV}_q(\infty),\\
\mathbb{M}^-_0 &= \{y \in \mathbb{M}^- : \lim_{t\to\infty} y(t) = 0\},\\
\mathbb{M}^+_\infty &= \{y \in \mathbb{M}^+ : \lim_{t\to\infty} y(t) = \infty\}
\end{aligned}$$

We distinguish three cases for the behavior of the coefficient $p(t)$ in equation (6):

$$\lim_{t\to\infty} t^\alpha p(t) = 0, \tag{25}$$

$$\lim_{t\to\infty} t^\alpha p(t) = B > 0, \tag{26}$$

$$\lim_{t\to\infty} t^\alpha p(t) = \infty. \tag{27}$$

With the use of the results of this paper we can claim:

$$\begin{aligned}
\mathbb{M}^- = \mathbb{M}^-_{SV} &\iff (25) \iff \mathbb{M}^+ = \mathbb{M}^+_{RV}(1) = \mathbb{M}^+_\infty,\\
\mathbb{M}^- = \mathbb{M}^-_{RV}(\vartheta_1) = \mathbb{M}^-_0 &\iff (26) \iff \mathbb{M}^+ = \mathbb{M}^+_{RV}(\vartheta_2) = \mathbb{M}^+_\infty,\\
\mathbb{M}^- = \mathbb{M}^-_{RPV}(-\infty) = \mathbb{M}^-_0 &\iff (27) \iff \mathbb{M}^+ = \mathbb{M}^+_{RPV}(\infty) = \mathbb{M}^+_\infty.
\end{aligned}$$

In view of previous results, we get the following statement.

Corollary 1. *The following statements are equivalent:*

- *There exists* $u \in \mathbb{M}$ *such that* $u \in \mathcal{KF}_q$.
- *For every* $u \in \mathbb{M}$, *it holds* $u \in \mathcal{KF}_q$.
- *There exists the (finite or infinite) limit*

$$\lim_{t\to\infty} t^{\alpha} p(t). \tag{28}$$

Because of this corollary and thanks to the example from the proof of Lemma 6, equation (6) may possess a positive solution, which is not in $\mathcal{KF}_q$. In fact, such a case happens if and only if the limit (28) does not exist, and then necessarily no positive solution of (6) is an element of $\mathcal{KF}_q$.

We finish the paper with mentioning some other directions for a future research. The above relations between the $\mathbb{M}$-classification and Karamata like behavior of solutions to (6) could be refined, provided more detailed information on the existence in all subclasses (in the sense of effective conditions) would be at disposal. Or, possibly, all observations can be extended to equations with no sign condition on p or to some other q-difference equations.

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