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Anti-periodic solutions for a class of fourth-order nonlinear differential equations with variable coefficients*

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Abstract

By applying the method of coincidence degree, some criteria are established for the existence of anti-periodic solutions for a class of fourth-order nonlinear differential equations with variable coefficients. Finally, an example is given to illustrate our result.

Key words: Anti-periodic solution; Fourth-order differential equation; Coincidence degree.

1 Introduction

In this paper, we should apply the method of coincidence degree to study the existence of anti-periodic solutions for a class of fourth-order nonlinear differential equations with variable coefficients in the form of

$$u''''(t) - a(t)u'''(t) - b(t)u''(t) - c(t)u'(t) - g(t, u(t)) = e(t), \tag{1.1}$$

where $a \in C^3(\mathbb{R}, \mathbb{R})$, $b \in C^2(\mathbb{R}, \mathbb{R})$ and $c \in C^1(\mathbb{R}, \mathbb{R})$ are $\frac{T}{2}$-periodic, $g \in C(\mathbb{R}^2, \mathbb{R})$ is T-periodic in its first argument, and $e \in C(\mathbb{R}, \mathbb{R})$ is T-periodic with $\int_0^T e(s)\,\mathrm{d}s = 0$.

During the past thirty years, there has been a great deal of work on the problem of the periodic solutions of fourth-order nonlinear differential equations, which have been used to describe nonlinear oscillations [1-5], and fluid mechanical and nonlinear elastic mechanical phenomena [6-12]. In [13], Bereanu discussed the existence of T-periodic solutions of the following fourth-order nonlinear differential equations:

$$u''''(t) - pu''(t) - g(t, u(t)) = e(t),$$

which can be regarded as a special case of Eq. (1.1) with $b(t) \equiv p$ and $a(t) = c(t) \equiv 0$.

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Arising from problems in applied sciences, it is well-known that the existence of anti-periodic solutions plays a key role in characterizing the behavior of nonlinear differential equations as a special periodic solution and have been extensively studied by many authors during the past twenty years, see [14-22] and references therein. For example, anti-periodic trigonometric polynomials are important in the study of interpolation problems [23, 24], and anti-periodic wavelets are discussed in [25]. However, to the best of our knowledge, there are few papers to investigate the existence of anti-periodic solutions to Eq. (1.1) by applying the method of coincidence degree.

The main purpose of this paper is to establish sufficient conditions for the existence of $\frac{T}{2}$-anti-periodic solutions to Eq. (1.1) by using the method of coincidence degree.

The organization of this paper is as follows. In Section 2, we make some preparations. In Section 3, by using the method of coincidence degree, we establish sufficient conditions for the existence of $\frac{T}{2}$-anti-periodic solutions to Eq. (1.1). An illustrative example is given in Section 4.

2 Preliminaries

For the readers' convenience, we first summarize a few concepts from [26].

Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces. Let $L : \operatorname{Dom} L \subset \mathbb{X} \to \mathbb{Y}$ be a linear mapping and $N : \mathbb{X} \to \mathbb{Y}$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\operatorname{Im} L$ is a closed subspace of $\mathbb{Y}$ and

$$\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty.$$

If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. It follows that

$$L|_{\operatorname{Dom} L \cap \operatorname{Ker} P} : (I - P)\mathbb{X} \to \operatorname{Im} L$$

is invertible and its inverse is denoted by K_P. If Ω is a bounded open subset of $\mathbb{X}$, the mapping N is called L-compact on $\mathbb{X}$, if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to \mathbb{X}$ is compact. Because $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

The following fixed point theorem of coincidence degree is crucial in the arguments of our main results.

Lemma 2.1. [26] *Let $\mathbb{X}$, $\mathbb{Y}$ be two Banach spaces, $\Omega \subset \mathbb{X}$ be open bounded and symmetric with $0 \in \Omega$. Suppose that $L : D(L) \subset \mathbb{X} \to \mathbb{Y}$ is a linear Fredholm operator of index zero with $D(L) \cap \bar{\Omega} \neq \emptyset$ and $N : \bar{\Omega} \to \mathbb{Y}$ is L-compact. Further, we also assume that*

(H) $Lx - Nx \neq \lambda(-Lx - N(-x))$ for all $x \in D(L) \cap \partial\Omega$, $\lambda \in (0, 1]$.

Then equation $Lx = Nx$ has at least one solution on $D(L) \cap \bar{\Omega}$.

Definition 2.1. *A continuous function $u : \mathbb{R} \to \mathbb{R}$ is said to be anti-periodic with anti-period $\frac{T}{2}$ on $\mathbb{R}$ if,*

$$u(t + T) = u(t), \quad u(t + \frac{T}{2}) = -u(t) \quad \text{for all } t \in \mathbb{R}.$$

Example 2.1. *The functions* $\sin x$ *and* $\cos x$ *are anti-periodic with anti-period* π (*as well as with anti-periods* 3π, 5π, *etc.*).

We will adopt the following notations:

$$C_T^k := \{u \in C(\mathbb{R}, \mathbb{R}) : u \text{ is } T\text{-periodic}\}, \;\; k \in \mathbb{N}, \quad |u|_\infty = \max_{t\in[0,T]} |u(t)|,$$

where u is a T-periodic function.

Lemma 2.2. [27] *For any* $u \in C_T^2$ *one has that*

$$\int_0^T |u'(s)|^2 \,\mathrm{d}s \leq \frac{T^2}{4\pi^2} \int_0^T |u''(s)|^2 \,\mathrm{d}s.$$

Lemma 2.3. [27] *For any* $u \in C_T^4$ *one has that*

$$|u^{(k)}|_\infty \leq T^{3-k} \left(\frac{1}{2}\right)^{4-k} \int_0^T |u''''(s)| \,\mathrm{d}s \, (k = 1, 2, 3).$$

3 Main result

Theorem 3.1. *Assume that the following conditions hold:*

(H_1) $a(t) \equiv 0$ *or* $|a(t)| \geq a^* > 0$ *for all* $t \in \mathbb{R}$, *where* a^* *is a constant.*

(H_2) $\max_{s\in[0,T]}[b(s) - \frac{3}{2}a'(s)] \leq 0$ *or* $\min_{s\in[0,T]}[b(s) - \frac{3}{2}a'(s)] \geq 0$.

If $\max_{s\in[0,T]}[b(s) - \frac{3}{2}a'(s)] \leq 0$, *then*

$$\frac{T\sqrt{T}}{4\pi} \int_0^T \left| \frac{a'''(s) - b''(s) + c'(s)}{2} \right| \mathrm{d}s < 1 + \min_{s\in[0,T]} \frac{T^2}{4\pi^2}[b(s) - \frac{3}{2}a'(s)];$$

If $\min_{s\in[0,T]}[b(s) - \frac{3}{2}a'(s)] \geq 0$, *then*

$$\frac{T\sqrt{T}}{4\pi} \int_0^T \left| \frac{a'''(s) - b''(s) + c'(s)}{2} \right| \mathrm{d}s < 1.$$

(H_3) *There exist* $N > 0$, $k \geq 0$ *and* $0 \leq \delta < 1$ *such that*

$$\max\{|g(t, u)|, |g(t, -u)|\} \leq N + k|u|^\delta \text{ for all } (t, u) \in \mathbb{R}^2.$$

(H_4) *For all* $(t, u) \in \mathbb{R}^2$,

$$g(t + \frac{T}{2}, -u) = -g(t, u), \quad e(t + \frac{T}{2}) = -e(t).$$

Then Eq. (1.1) *has at least one* $\frac{T}{2}$*-anti-periodic solution.*

Proof. Let

$$\mathbb{X} = \{u \in C_T^3 : u(t + \frac{T}{2}) = -u(t) \text{ for all } t \in \mathbb{R}\}$$

and

$$\mathbb{Y} = \{u \in C_T^0 : u(t + \frac{T}{2}) = -u(t) \text{ for all } t \in \mathbb{R}\}$$

be two Banach spaces with the norms

$$\|u\|_{\mathbb{X}} = \max\{|u|_\infty, |u'|_\infty, |u''|_\infty, |u'''|_\infty\} \quad \text{and} \quad \|u\|_{\mathbb{Y}} = |u|_\infty.$$

Define a linear operator $L : D(L) \subset \mathbb{X} \to \mathbb{Y}$ by setting

$$Lu = u'''' \text{ for all } u \in D(L),$$

where $D(L) = \{u \in \mathbb{X} : u'''' \in C(\mathbb{R}, \mathbb{R})\}$ and $N : \mathbb{X} \to \mathbb{Y}$ by setting

$$Nu = e(t) + a(t)u'''(t) + b(t)u''(t) + c(t))u'(t) + g(t, u(t)).$$

It is easy to see that

$$\operatorname{Ker} L = \{0\} \quad \text{and} \quad \operatorname{Im} L = \left\{u \in \mathbb{Y} : \int_0^T u(s)\,\mathrm{d}s = 0\right\} \equiv \mathbb{Y}.$$

Thus $\dim \operatorname{Ker} L = 0 = \operatorname{codim} \operatorname{Im} L$, and L is a linear Fredholm operator of index zero.

Define the continuous projector $P : \mathbb{X} \to \operatorname{Ker} L$ and the averaging projector $Q : \mathbb{Y} \to \mathbb{Y}$ by

$$Pu(t) = Qu(t) = \frac{1}{T}\int_0^T u(s)\,\mathrm{d}s \equiv 0.$$

Hence $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Denoting by $L_P^{-1} : \operatorname{Im} L \to D(L) \cap \operatorname{Ker} P$ the inverse of $L|_{D(L)\cap \operatorname{Ker} P}$, we have

$$\begin{aligned} L_P^{-1}u(t) = & \int_0^t \int_0^\gamma \int_0^\beta \int_0^\alpha u(s)\,\mathrm{d}s\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma - \frac{1}{2}\int_0^{\frac{T}{2}} \int_0^\gamma \int_0^\beta \int_0^\alpha u(s)\,\mathrm{d}s\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma \\ & + \frac{T-4t}{8}\int_0^{\frac{T}{2}} \int_0^\beta \int_0^\alpha u(s)\,\mathrm{d}s\,\mathrm{d}\alpha\,\mathrm{d}\beta + \frac{T^2-8t^2}{32}\int_0^{\frac{T}{2}} \int_0^\alpha u(s)\,\mathrm{d}s\,\mathrm{d}\alpha \\ & + \frac{T^3 - 6T^2t + 12Tt^2 - 16t^3}{192}\int_0^{\frac{T}{2}} u(s)\,\mathrm{d}s. \end{aligned}$$

Clearly, QN and $L_P^{-1}(I-Q)N$ are continuous. Using the Arzela-Ascoli theorem, it is not difficult to show that $QN(\bar{\Omega})$, $L_P^{-1}(I-Q)N(\bar{\Omega})$ are relatively compact for any open bounded set $\Omega \subset \mathbb{X}$. Therefore, N is L-compact on $\bar{\Omega}$ for any open bounded set $\Omega \subset \mathbb{X}$.

In order to apply Lemma 2.1, we need to find an appropriate open bounded subset Ω in $\mathbb{X}$. Corresponding to the operator equation $Lx - Nx = \lambda(-Lx - N(-x))$, $\lambda \in (0, 1]$, we have

$$u''''(t) = \frac{1}{1+\lambda}G(t, u) - \frac{\lambda}{1+\lambda}G(t, -u), \tag{3.1}$$

where

$$G(t,u)=e(t)+a(t)u'''(t)+b(t)u''(t)+c(t))u'(t)+g(t,u(t))$$

and

$$G(t,-u)=e(t)-a(t)u'''(t)-b(t)u''(t)-c(t))u'(t)+g(t,-u(t)).$$

Suppose that $u(t)\in\mathbb{X}$ is an arbitrary $\frac{T}{2}$-anti-periodic solution of system (3.1). Hence we have

$$\int_0^T u(s)\,\mathrm{d}s=\int_0^{\frac{T}{2}} u(s)\,\mathrm{d}s+\int_{\frac{T}{2}}^T u(s)\,\mathrm{d}s=\int_0^{\frac{T}{2}} u(s)\,\mathrm{d}s+\int_0^{\frac{T}{2}} u(s+\frac{T}{2})\,\mathrm{d}s=0$$

and

$$\int_0^T u'(s)\,\mathrm{d}s=\int_0^{\frac{T}{2}} u'(s)\,\mathrm{d}s+\int_{\frac{T}{2}}^T u'(s)\,\mathrm{d}s=\int_0^{\frac{T}{2}} u'(s)\,\mathrm{d}s+\int_0^{\frac{T}{2}} u'(s+\frac{T}{2})\,\mathrm{d}s=0.$$

Then there exists constant ξ, $\zeta\in[0,T]$ such that

$$u(\xi)=0\quad\text{and}\quad u'(\zeta)=0.$$

Therefore, we have

$$|u(t)|=\left|u(\xi)+\int_\xi^t u'(s)\,\mathrm{d}s\right|\le\int_\xi^t|u'(s)|\,\mathrm{d}s$$

and

$$|u(t)|=|u(t-T)|=\left|u(\xi)-\int_{t-T}^\xi u'(s)\,\mathrm{d}s\right|\le\int_{t-T}^\xi|u'(s)|\,\mathrm{d}s$$

for all $t\in[\xi,\xi+T]$. Combining the above two inequalities, we can get

$$\begin{aligned}|u|_\infty&=\max_{t\in[0,T]}|u(t)|\\&=\max_{t\in[\xi,\xi+T]}|u(t)|\\&\le\max_{t\in[\xi,\xi+T]}\left\{\frac{1}{2}\left(\int_\xi^t|u'(s)|\,\mathrm{d}s+\int_{t-T}^\xi|u'(s)|\,\mathrm{d}s\right)\right\}\\&\le\frac{1}{2}\int_0^T|u'(s)|\,\mathrm{d}s\\&\le\frac{1}{2}\sqrt{T}\left(\int_0^T|u'(s)|^2\,\mathrm{d}s\right)^{1/2}.\end{aligned}\tag{3.2}$$

By using a similar argument as that in the proof of (3.2), we can easily obtain

$$|u'|_\infty\le\frac{1}{2}\sqrt{T}\left(\int_0^T|u''(s)|^2\,\mathrm{d}s\right)^{1/2}.\tag{3.3}$$

In view of Lemma 2.2, we get from (3.2) that

$$|u|_\infty \le \frac{T\sqrt{T}}{4\pi}\left(\int_0^T |u''(s)|^2\,\mathrm{d}s\right)^{1/2}. \tag{3.4}$$

Thus

$$\left|\int_0^T e(s)u(s)\,\mathrm{d}s\right| \le \frac{T^2\sqrt{T}}{4\pi}|e|_\infty\left(\int_0^T |u''(s)|^2\,\mathrm{d}s\right)^{1/2}. \tag{3.5}$$

On the other hand, multiplying Eq. (3.1) by u and integrating it from 0 to T, it follows that

$$\begin{aligned}&\int_0^T |u''(s)|^2\,\mathrm{d}s + \int_0^T [b(s) - \frac{3}{2}a'(s)]|u'(s)|^2\,\mathrm{d}s\\ &= -\int_0^T \frac{a'''(s) - b''(s) + c'(s)}{2}|u(s)|^2\,\mathrm{d}s + \frac{1-\lambda}{1+\lambda}\int_0^T e(s)u(s)\,\mathrm{d}s\\ &+\frac{1}{1+\lambda}\int_0^T g(s,u(s))u(s)\,\mathrm{d}s - \frac{\lambda}{1+\lambda}\int_0^T g(s,-u(s))u(s)\,\mathrm{d}s.\end{aligned} \tag{3.6}$$

Assume that $\max_{s\in[0,T]}[b(s) - \frac{3}{2}a'(s)] \le 0$. Using (3.4), (3.5) and (H_3), we obtain

$$\begin{aligned}&\int_0^T \left(1 + \frac{T^2}{4\pi^2}[b(s) - \frac{3}{2}a'(s)]\right)|u''(s)|^2\,\mathrm{d}s\\ &\le \int_0^T \left|\frac{a'''(s) - b''(s) + c'(s)}{2}\right|\mathrm{d}s|u|_\infty^2 + \left|\int_0^T e(s)u(s)\,\mathrm{d}s\right|\\ &+\int_0^T \max\left\{|g(s,u(s))|, |g(s,-u(s))|\right\}|u(s)|\,\mathrm{d}s\\ &\le \frac{T\sqrt{T}}{4\pi}\int_0^T \left|\frac{a'''(s) - b''(s) + c'(s)}{2}\right|\mathrm{d}s\left(\int_0^T |u''(s)|^2\,\mathrm{d}s\right) + \frac{T^2\sqrt{T}}{4\pi}|e|_\infty\left(\int_0^T |u''(s)|^2\,\mathrm{d}s\right)^{1/2}\\ &+N\frac{T^2\sqrt{T}}{4\pi}\left(\int_0^T |u''(s)|^2\,\mathrm{d}s\right)^{1/2} + Tk\left\{\frac{T\sqrt{T}}{4\pi}\left(\int_0^T |u''(s)|^2\,\mathrm{d}s\right)^{1/2}\right\}^{\delta+1},\end{aligned}$$

in which together with (H_2) and $0 \le \delta < 1$ imply that there exists a positive constant M_1 satisfying

$$\int_0^T |u''(s)|^2\,\mathrm{d}s \le M_1 \quad\text{and}\quad \int_0^T |u''(s)|\,\mathrm{d}s \le \sqrt{TM_1}. \tag{3.7}$$

Therefore, from (3.2), (3.3) and (3.7), we can choose a constant M_2 such that

$$|u|_\infty \le M_2 \quad\text{and}\quad |u'|_\infty \le M_2. \tag{3.8}$$

If $a(t) \equiv 0$, by Eq. (3.1), (3.7) and (3.8) it follows that there exists a constant M_3 satisfying

$$\int_0^T |u''''(s)|\,\mathrm{d}s \le M_3.$$

If $|a(t)| \geq a^* > 0$ for all $t \in \mathbb{R}$, multiplying Eq. (3.1) by u''' and integrating it from 0 to T, it follows that

$$\begin{aligned}
a^* \int_0^T |u'''(s)|^2 \, \mathrm{d}s \leq & \sup_{s\in[0,T]} |b(s)| \int_0^T |u''(s)||u'''(s)| \, \mathrm{d}s + \sup_{s\in[0,T]} |c(s)| \int_0^T |u'(s)||u'''(s)| \, \mathrm{d}s \\
& + \sup_{s\in[0,T],|u|\leq M_2} \{|g(s,u)| + |g(s,-u)|\} \int_0^T |u'''(s)| \, \mathrm{d}s \\
& + |e|_\infty \int_0^T |u'''(s)| \, \mathrm{d}s \\
\leq & \sup_{s\in[0,T]} |b(s)| \left(\int_0^T |u''(s)|^2 \, \mathrm{d}s \right)^{1/2} \left(\int_0^T |u'''(s)|^2 \, \mathrm{d}s \right)^{1/2} \\
& + \sup_{s\in[0,T]} |c(s)| \left(\int_0^T |u'(s)|^2 \, \mathrm{d}s \right)^{1/2} \left(\int_0^T |u'''(s)|^2 \, \mathrm{d}s \right)^{1/2} \\
& + \sqrt{T} \sup_{s\in[0,T],|u|\leq M_2} \{|g(s,u)| + |g(s,-u)|\} \left(\int_0^T |u'''(s)|^2 \, \mathrm{d}s \right)^{1/2} \\
& + \sqrt{T} |e|_\infty \left(\int_0^T |u'''(s)|^2 \, \mathrm{d}s \right)^{1/2} \\
\leq & \Bigg(\sup_{s\in[0,T]} |b(s)| \sqrt{M_1} + \sup_{s\in[0,T]} |c(s)| \frac{T}{2\pi} \sqrt{M_1} \\
& + \sqrt{T} \sup_{s\in[0,T],|u|\leq M_2} \{|g(s,u)| + |g(s,-u)|\} + \sqrt{T} |e|_\infty \Bigg) \left(\int_0^T |u'''(s)|^2 \, \mathrm{d}s \right)^{1/2}.
\end{aligned}$$

Therefore, there exists a positive constant M_4 such that

$$\int_0^T |u'''(s)|^2 \, \mathrm{d}s \leq M_4 \quad \text{and} \quad \int_0^T |u'''(s)| \, \mathrm{d}s \leq \sqrt{TM_4}.$$

Then, we can easily find a positive constant M_5 satisfying

$$\int_0^T |u''''(s)| \, \mathrm{d}s \leq M_5.$$

Assume that $\min_{s\in[0,T]}[b(s) - \frac{3}{2}a'(s)] \geq 0$. In view of (3.6), we have

$$\begin{aligned}
\int_0^T |u''(s)|^2 \, \mathrm{d}s \leq & \frac{T\sqrt{T}}{4\pi} \int_0^T \left| \frac{a'''(s) - b''(s) + c'(s)}{2} \right| \mathrm{d}s \left(\int_0^T |u''(s)|^2 \, \mathrm{d}s \right) \\
& + \frac{T^2\sqrt{T}}{4\pi} |e|_\infty \left(\int_0^T |u''(s)|^2 \, \mathrm{d}s \right)^{1/2} + N \frac{T^2\sqrt{T}}{4\pi} \left(\int_0^T |u''(s)|^2 \, \mathrm{d}s \right)^{1/2} \\
& + Tk \left\{ \frac{T\sqrt{T}}{4\pi} \left(\int_0^T |u''(s)|^2 \, \mathrm{d}s \right)^{1/2} \right\}^{\delta+1}.
\end{aligned}$$

As in the preceding step, there must exist a positive constant M_6 such that

$$\int_0^T |u''''(s)|\,\mathrm{d}s \le M_6.$$

Set $M_7 = \max\{M_3, M_5, M_6\}$. Together with Lemma 2.3, there exists a positive constant M_8 satisfying

$$|u''|_\infty \le M_8 \quad \text{and} \quad |u'''|_\infty \le M_8.$$

Let

$$M = \max\{M_2, M_8\} + 1 \text{ (Clearly, } M \text{ is independent of } \lambda).$$

Take

$$\Omega = \{x \in \mathbb{X} : \|x\|_{\mathbb{X}} < M\}.$$

It is clear that Ω satisfies all the requirements in Lemma 2.1 and condition (H) is satisfied. In view of all the discussions above, we conclude from Lemma 2.1 that Eq. (1.1) has at least one $\frac{T}{2}$-anti-periodic solution. This completes the proof. ∎

Consider the following fourth-order nonlinear differential equations with delay:

$$u''''(t) - a(t)u'''(t) - b(t)u''(t) - c(t)u'(t) - g(t, u(t - \tau(t))) = e(t), \tag{3.9}$$

where $\tau \in C(\mathbb{R}, \mathbb{R})$, other coefficients are defined as that in Eq. (1.1).

Remark 3.1. *From the proof of Theorem* 3.1*, we can see that the delay term $\tau(t)$ in Eq.* (3.9) *has no effect on the result in Theorem* 3.1*. So the result in Theorem* 3.1 *also holds for Eq.* (3.9)*.*

4 An example

Example 4.1. *Let $0 \le r < (2\pi)^{-3/2}$. Then the following fourth-order differential equation*

$$u''''(t) - (r\sin^2 t + 1)u'''(t) + 50u''(t) - 100u'(t) + |\sin t| u^{\frac{3}{5}}(t-1) = \cos t$$

has at least one π-anti-periodic solution.

Proof. When $0 \le r < (2\pi)^{-3/2}$, it is easy to verify that (H_2) holds. Furthermore, it suffices to remark that the function $g(t, u) \equiv |\sin t| u^{\frac{3}{5}}$ satisfies

$$|g(t, u)| = |g(t, -u)| \le |u|^{\frac{3}{5}}$$

uniformly with respect to $t \in \mathbb{R}$. Hence (H_3) and (H_4) hold and the result follows from Theorem 3.1. This completes the proof. ∎

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