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A SYSTEM OF DEGREE FOUR WITH AN INVARIANT TRIANGLE AND AT LEAST THREE SMALL AMPLITUDE LIMIT CYCLES

Z. H. LIU*, E. SÁEZ**, I. SZÁNTÓ**

*School of Mathematics and Computer Science, Guangxi University for Nationalities, Nanning, Guangxi 530006, P.R. China.
**Departamento de Matemática, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso - Chile.

Abstract. We show the existence of a polynomial system of degree four having three real invariant straight lines forming a triangle with at least three small amplitude limit cycles in the interior. Also, we obtain the necessary and sufficient conditions for the critical point at the interior of the bounded region to be a center.

1. Introduction

Let us consider a real autonomous system of ordinary differential equations on the plane with polynomial nonlinearities.

$$\dot{x} = P(x,y) = \sum_{i+j=0}^{n} a_{ij}x^iy^j, \quad \dot{y} = Q(x,y) = \sum_{i+j=0}^{n} b_{ij}x^iy^j \,, \text{ with } a_{ij}\,, b_{ij} \in \mathbb{R} \tag{1}$$

Suppose that the origin of (1) is a critical point of center-focus type. We are concerned with two closely related questions, both of which are significant elements in work on Hilbert's 16th Problem. The first is the number of limit cycles (that is, isolated periodic solutions) which bifurcate from a critical point and the second is the derivation of necessary and sufficient conditions for a critical point to be a center (that is, all orbits in the neighborhood of the critical point are closed).

In order to describe Hilbert's 16th Problem more precisely, let S_n be the collection of systems of form (1), with P and Q of degree at most n, and let $\pi(P,Q)$ be the number of limit cycles of (1). We let (P,Q) denote system (1) and define the so-called Hilbert numbers by

$$H_n = \ Sup\,\{\pi(P,Q); (P,Q) \in S_n\}\,.$$

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USM Grant No.12.09.05 and No.12.09.06 and NNSF of China Grant No.10971019.
E-mails: zhhliu@hotmail.com eduardo.saez@usm.cl ivan.szanto@usm.cl.

The problem consists of estimating H_n in terms of n and obtaining the possible relative configurations of the limit cycles.
This is the second part of the 16th Problem, which is contained in the famous list of problems proposed by Hilbert at the International Congress of Mathematicians held in Paris in 1900.

2. Limit cycles and center conditions

Let us assume that the origin is a critical point of (1) and transform the system to canonical form

$$\dot{x} = \lambda x + y + p(x, y), \quad \dot{y} = -x + \lambda y + q(x, y),$$

where p and q are polynomials without linear terms. For the origin to be a center we must have $\lambda = 0$. If $\lambda = 0$ and the origin is not a center, it is said to be a *fine focus.*

The necessary conditions for a center are obtained by computing the *focal values.* These are polynomials in the coefficients arising in P and Q and are defined as follows. There is a function V, analytic in a neighborhood of the origin, such that the rate of change along orbits, $\dot{V}$, is of the form $\eta_2 r^2 + \eta_4 r^4 + \cdots$, where $r^2 = x^2 + y^2$. The focal values are the η_{2k}, and the origin is a center if and only if they are all zero. However, since they are polynomials, the ideal they generate has a finite basis, so there is M such that $\eta_{2\ell} = 0$, for $\ell \leq M$, implies that $\eta_{2\ell} = 0$ for all ℓ. The value of M is not known *a priori*, so it is not clear in advance how many focal values should be calculated.

The software Mathematica [11] is used to calculate the first few focal values. These are then 'reduced' in the sense that each is computed modulo the ideal generated by the previous ones: that is, the relations $\eta_2 = \eta_4 = \cdots = \eta_{2k} = 0$ are used to eliminate some of the variables in η_{2k+2}. The reduced focal value η_{2k+2}, with strictly positive factors removed, is known as the *Lyapunov quantity* L_k. Common factors of the reduced focal values are removed and the computation proceeds until it can be shown that the remaining expressions cannot be zero simultaneously. The circumstances under which the calculated focal values are zero yield the necessary center conditions. The origin is a fine focus of *order* k if $L_i = 0$ for $i = 0, 1, ..., k - 1$ and $L_k \neq 0$. At most k limit cycles can bifurcate out of a fine focus of order k; these are called *small amplitude* limit cycles.

Various methods are used to prove the sufficiency of the possible center conditions. Of particular interest to us in this paper is the symmetry.

3. Polynomial systems having real invariant straight lines

As Hilbert's 16th Problem (second part) consists of estimating the number H_n of limit cycles of polynomial vector fields of degree n on the plane, and in order to make some progress on understanding this long-standing unsolved problem, researchers have considered several particular versions of it. For instance, some authors have considered the problem of estimating the number H_n of limit cycles of polynomial systems on the plane of degree n having l real invariant straight lines, where the real straight line $ax+by+c=0$ is invariant for the flow of (1) and is called a real invariant straight line if

$$aP(x,y)+bQ(x,y) = (ax+by+c)R(x,y)$$

for some real polynomial R.

In the literature several results are known for the existence of limit cycles in polynomial systems of degree n that have more than one real invariant straight line.

If we denote $H_n(l)$ as the number of limit cycles of (1) where l is the number of invariant straight lines and n the degree of the system, the following results are known

$H_2(2) = 0$ N. N .Bautin [1]

$H_2(1) \leq 1$ L. A. Cherkas and L. I. Zhilevich [2], [8]

$H_3(5) = 0$ Dai Guoren and Wo Songlin [3]

$H_n(\frac{(n-1)(n+2)}{2}) = 0$ Suo Guangjian and Sun Jifang [10]

$H_3(4) \leq 1$ R. Kooij [4]

$H_3(2) \geq 4$ N. G. Lloyd, J. M. Pearson, E. Sáez, and I. Szántó [6]

$H_3(2) \geq 6$ N.G.Lloyd, J. M. Pearson, E. Sáez, and I. Szántó [7]

$H_3(3) \geq 2$ Ye Yanqian and Ye Weiyin [13]

$H_3(3) \geq 4$ E. Sáez, I. Szántó and E. Gonzalez-Olivares [9]

In this paper, we discuss the particular problem of those systems that have three real invariant straight lines that form a triangle, called an invariant triangle. As the triangle is a graph, there is at least one singularity inside. We will assume that it is a fine focus type singularity.

In 1954, N. N Bautin in [1], shows that for a quadratic system ($n = 2$), the system

$$\dot{x} = x(a + bx + cy), \quad \dot{y} = y(d + ex + fy)$$

with two invariant straight lines has no limit cycles. Thus for a quadratic polynomial system with an invariant triangle has no limit cycles inside the triangle.

For $n = 3$, Ye Yanqian in [12] considers a class of systems

$$\begin{aligned} \dot{x} &= x[a_0(1 - x^2) + a_1 x(1 - x) + a_2 y + a_4 xy + a_5 y^2] \\ \dot{y} &= y[b_0(1 - y^2) + b_1 x + b_2 y(1 - y) + b_3 x^2 + b_4 xy] \end{aligned}$$

with an invariant triangle whose sides are real invariant straight lines $x = 0,\ y = 0$ and $x + y = 1$, and whose vertices are saddles.
Under certain conditions of the coefficients, the relative positions of other critical points of the cubic system and its invariant straight line with respect to the invariant triangle are determined. In all of the cases, no limit cycles are found.

Z. H. Liu et. al [5] consider a class of cubic systems given by

$$X_\mu : \begin{cases} \dot{x} &= (-1 + abx)(-cx - (a^2 + ab - abc - b^2c)x^2 + y + (2a^2 - \\ & \quad ab + a^2b - b^2 + ab^2 - abc)xy + (ab - 2a^2b)y^2) \\ \dot{y} &= (-1 + aby)(-x + (a^2 + ab^2 + b^3)xy - (a^2b + ab^2)y^2) \end{cases}$$

where $\mu = (a, b, c) \in \mathbb{R}^3$

These systems have three real invariant straight lines forming a triangle surrounding at least one limit cycle, where the small amplitude limit cycles are limit cycles which bifurcate out of a nonhyperbolic focus.
The existence of a cubic system with an invariant triangle containing more than one limit cycle remains an open problem.

4. Main Results ($n = 4$)

We show the existence of a polynomial system of degree four having three real invariant straight lines that form a triangle with at least three small amplitude limit cycles in the interior.

Let us consider a class of system of degree four

(2) $$X_\mu : \begin{cases} \dot{x} & = P(x,y) + bR(x,y) \\ \dot{y} & = Q(x,y) + bS(x,y) \end{cases}$$

where

$$\begin{array}{rcl} P(x,y) & = & (1+x)(a^3\lambda x - a^3 y - 4xy + a^2xy + a^3\lambda xy - 4y^2 - 4ay^2 + a^2y^2) \\ Q(x,y) & = & (1+y)(a^3x - a^2x^2 - a^2\lambda x^2 - a^3\lambda x^2 + 4xy + 4axy - a^2xy - \\ & & a^2\lambda xy + 4y^2) \end{array}$$

and

$$\begin{array}{rcl} R(x,y) & = & (1+x)y^2(1+y) \\ S(x,y) & = & (1+y)(ax^2 - x^3 - axy - y^2), \end{array}$$

with $\mu = (a, \lambda, b) \in \mathbb{R}^3$, $a > 0$.

Lemma 1 System (2) has three real invariant straight lines, namely,

$$x = -1 \ , \ y = -1 \quad \text{and} \quad x + y = a.$$

For $a > 0$ the straight lines form a triangle surrounding the origin.

Proof. It is clear that $x = -1$ and $y = -1$ are invariant straight lines, and for the line $x + y = a$, we have

$$\dot{x}+\dot{y} = (-a+x+y)(-a^2x - a^2Lx - ex^2 + a^2y - 4xy + exy - a^2Lxy - ex^2y + 4y^2 + exy^2).$$

Finally for $a > 0$, the lines form a triangle surrounding the origin and this proves the Lemma.

Theorem 1 If $\lambda = 0$, $a = 2$ and $b = \frac{24}{7}$, then system (2), at the singularity $(0,0)$, has a repelling fine focus of order 3 .

Proof. Rescaling the time $\{t \to \frac{1}{a^3}t\}$, the linear part of (2) at the singularity $(0,0)$ is

$$DX_\mu(0,0) = \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix}$$

If $\lambda = 0$, we consider $\tilde{\mu} = (a, 0, b)$, and we have that $divX_{\tilde{\mu}}(0,0) = 0$ and $detDX_{\tilde{\mu}}(0,0) = 1$, then the critical point $(0,0)$ is a fine focus.

Using the symbolic calculus of the Mathematica Software [11], we are able to compute the Lyapunov constants $L_k, k = 0,1,2,3$ and then to determine the topological type of the singular point at the origin.

If $\lambda = 0$ then

$$L_0 = 0 \ , \ L_1 = (a-2)(a+1)(b-4)(b-4-2a)$$

If $a = 2$ we have : $L_1 = 0$, $L_2 = b^2(b-4)(7b-24)$, and

$L_3 = b^2(b-4)(97b^3 - 481b^2 - 34b + 1680).$

If $b = \frac{24}{7}$, $L_2 = 0$ and $L_3 = \frac{243}{1882384}$ and this proves that system (2), at the singularity $(0,0)$, has a repelling fine focus of order 3 .

Theorem 2 System (2) with $a > 0$ has a center at the origin, if and only if, $\lambda = 0$ and $b = 4$.

Proof. If $\lambda = 0$ and $b = 4$, then system (2) is given by

$$X_{(a,0,4)} : \begin{cases} \dot{x} &= y(1+x)(-a^3 - 4x + a^2x - 4ay + a^2y + 4y^2) \\ \dot{y} &= x(1+y)(a^3 + 4ax - a^2x - 4x^2 + 4y - a^2y) \end{cases} \tag{3}$$

¿From the linear part at the origin, it is clear that the condition $\lambda = 0$ is necessary to have a center. On the other hand, for $b = 4, L_k = 0, \forall k$ and this shows that the conditions are necessary.

To show the sufficiency, we can take a rotation

$$x = \frac{u+v}{2}, \quad y = \frac{-u+v}{2}$$

and we obtain the new system

$$\begin{aligned} \dot{u} = P(u,v) &= (4u^2 - 4au^2 + a^3u^2 - u^4 - 2a^3v + 8u^2v + 2au^2v - \\ &\quad a^2u^2v - 4v^2 - 4av^2 + 2a^2v^2 - a^3v^2 - 2av^3 + a^2v^3 + v^4)/2 \\ \dot{v} = Q(u,v) &= 2u(-a+v)(-a^2 + u^2 - 4v - v^2) \end{aligned}$$

As the symmetries $P(-u,v) = P(u,v)$ and $Q(-u,v) = -Q(u,v)$ are satisfied, we prove the sufficiency that system (2) has a center at the origin.

Theorem 3 In the parameter space $\mathbb{R}^3$, there exists an open set $\mathcal{N}$, such that for all $(a, \lambda, b) \in \mathcal{N}$ with $b \neq 0$ and $a > 0$, then system (2) has three small-amplitude limit cycles coexisting with an invariant triangle.

Proof. It has been shown that systems X_μ have a invariant triangles surrounding the origin. If $a > 0$, then by Theorem 1, system (2) has at the singularity $(0,0)$ a repelling weak focus of order 3 if $\lambda = 0$, $a = 2$ and $b = \frac{24}{7}$. Perturbing the system by reversing the stabilities, a limit cycle is created (Hopf Bifurcation). By following the same process until all the Lyapunov constants are non-zero, three hyperbolic small amplitude limit cycles are created.

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