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On the Oscillatory Behavior for a Certain Class of Third Order Nonlinear Delay Difference Equations

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Abstract

By employing the generalized Riccati transformation technique, we will establish some new oscillation criteria for a certain class of third order nonlinear delay difference equations. Our results extend and improve some previously obtained ones. An example is worked out to demonstrate the validity of the proposed results.

Key Words and Phrases: Oscillation; Generalized Riccati transformation; Third order nonlinear delay difference equation.
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1 Introduction

The oscillation theory and asymptotic behavior of difference equations and their applications have been and still are receiving intensive attention over the last two decades. Indeed, the last few years have witnessed the appearance of several monographs and hundreds of research papers, see for example the references [1, 3, 6, 11]. Determination of oscillatory behavior for solutions of second order difference equations has occupied a great part of researchers' interest. Compared to this, however, the study of third order difference equations has received considerably less attention in the literature even though such equations often arise in the study of economics, mathematical biology and many other areas of mathematics whose discrete models are used, we refer to [2, 4, 5, 7, 8, 10, 12, 13, 14, 15, 16, 17, 18, 19]. Some of these results will be briefly stated below. Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we make a standing hypothesis that the equation under consideration does possess such solutions. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution x_n is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. We say that an equation is oscillatory if it has at least one oscillatory solution.

Here are some background details that may serve the readers and motivate the contents of this paper.

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For oscillation of linear difference equations: In [14], Smith considered the equation of the form

$$\Delta^3 x_n - p_n x_{n+2} = 0, \qquad n \geq n_0 \tag{1}$$

and studied the asymptotic and oscillatory behavior of the solutions subject to the condition $p_n > 0$ for $n \geq n_0$. Indeed, he proved that if

$$\sum_{n=n_0}^{\infty} p_n = \infty \tag{2}$$

then (1) is oscillatory. Further, the author considered the quasi-adjoint difference equation

$$\Delta^3 x_n + p_n x_{n+1} = 0, \qquad n \geq n_0 \tag{3}$$

and proved that (1) is oscillatory if and only if (3) is oscillatory. However, one can easily see that the results cannot be applied if $p_n = n^{-\alpha}$ for $\alpha > 1$.

In [12], the authors studied the difference equation of the form

$$\Delta^3 x_n + q_n x_n = 0, \qquad n \geq n_0 \tag{4}$$

and established some sufficient conditions for (4) to have monotonic and nonoscillatory solutions. They proved that if $q_n > 1$ for $n \geq n_0$ is a positive sequence then (4) is oscillatory.

In [13], it was proved that if

$$\sum_{l=n_0}^{\infty} \left[\sum_{t=n_0}^{l-1} \sum_{s=n_0}^{t-1} p_s \right] = \infty \tag{5}$$

and there exists a positive sequence ρ_n such that

$$\lim_{n\to\infty} \sup \sum_{s=n_0}^{n} \left[\rho_s p_s - \frac{(\Delta \rho_s)^2}{4\rho_s (s - n_0)} \right] = \infty \tag{6}$$

then the solution x_n of (3) either oscillates or satisfies $\lim_{n\to\infty} x_n = 0$. Results established in [13] provided substantial improvements for those obtained in [12] and [14].

In [16], the author considered the linear difference equation

$$\Delta^3 x_n - p_{n+1} \Delta x_{n+1} + q_{n+1} x_{n+1} = 0, \qquad n \geq n_0, \tag{7}$$

where p_n and q_n are nonnegative real sequences satisfying

$$\Delta p_n + q_{n+1} > 0 \tag{8}$$

and proved that if x_n is a nonoscillatory solution of (7) then there exists an integer N for which either $x_n \Delta x_n > 0$ or $x_n \Delta x_n < 0$ for all $n > N$.

In [15], the author investigated the linear difference equation

$$\Delta^3 x_n + p_{n+1} \Delta x_{n+2} + q_n x_{n+2} = 0, \qquad n \geq n_0, \tag{9}$$

where p_n and q_n are real sequences satisfying

$$p_n \geq 0, \;\; q_n < 0 \;\; \text{and} \;\; \sum_{n=n_0}^{\infty} (\Delta p_n - 2q_n) = \infty. \tag{10}$$

It was shown that if $p_{n+1} + q_n \leq 0$ for $n \geq n_0$ then $\text{sign} x_n = \text{sign} \Delta x_n = \text{sign} \Delta^2 x_n$ and (9) has both oscillatory and nonoscillatory solutions. Further, the author established a sufficient condition for the existence of oscillatory solutions. The main investigation is based on the value of the functional $F_1(x_n) = (\Delta x_n)^2 - 2x_{n+1}\Delta^2 x_n - p_n x_{n+2}^2$. In particular, it was proved that if there is a solution x_n of (9) such that $F(x_n) > 0$ then x_n is oscillatory. However, one can easily see that the condition depends on the solution itself whose determination might not be possible.

For oscillation of nonlinear difference equations: The authors in [18] considered the equation

$$\Delta(\Delta^2 x_n + p_n x_{n+1}) + p_n \Delta x_n + f(x_{n+1}) = 0, \qquad n \geq n_0, \tag{11}$$

where $f(x)/x \geq k > 0$ and p_n is a bounded real sequence such that

$$\sum_{s=n_0}^{\infty} p_n = \infty. \tag{12}$$

The authors studied the asymptotic behavior of the solutions and proved that if there exists a solution x_n of (11) satisfying $F_2(x_n) < 0$, where $F_2(x_n) = 2x_n(\Delta^2 x_n + p_n x_{n+1}) - (\Delta x_n)^2$, then x_n is oscillatory. On the other hand, the authors proved that if there exists a solution x_n of (11) satisfying $F_2(x_n) > 0$ then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \Delta x_n = \lim_{n\to\infty} \Delta^2 x_n = 0$. Nevertheless, due to condition (12) the results are no longer valid if $p_n = n^{-\alpha}$ for $\alpha > 1$.

In [16], the author investigated equation of form

$$\Delta(\Delta^2 x_n - p_{n+1}x_{n+1}) - q_{n+2}x_{n+2} = 0, \qquad n \geq n_0, \tag{13}$$

where p_n and q_n are nonnegative real sequences and satisfying (8). It was shown that there exists a solution x_n of (13) such that $x_n \Delta x_n \Delta^2 x_n \neq 0$, $x_n > 0$, $\Delta x_n > 0$ and $\Delta^2 x_n > 0$ for $n \geq n_0$ and if x_n is a nonoscillatory solution then there exists an integer N for which either $x_n \Delta x_n > 0$ or $x_n \Delta x_n < 0$ for all $n > N$. Furthermore, the author investigated the same result for equation (7) and proved that if v_n is a nonoscillatory solution of (13) then the two independent solutions of (7) satisfy the self-adjoint second order equation

$$\Delta\left(\frac{\Delta x_n}{v_n}\right) + \left(\frac{\Delta^2 v_{n-1} - p_n v_n}{v_n v_{n+1}}\right) x_{n+1} = 0. \tag{14}$$

In [8], the authors studied the oscillatory behavior of

$$\Delta(c_n \Delta(d_n \Delta(x_n))) + q_n f(x_{n-\sigma+1}) = 0, \qquad n \geq n_0, \tag{15}$$

where σ is a nonnegative integer and $f \in C(\mathbb{R}, \mathbb{R})$ such that $uf(u) > 0$ for $u \neq 0$ and satisfies

$$f(u) - f(v) = g(u,v)(u-v), \text{ for } u, v \neq 0 \text{ and } g(u,v) \geqslant \mu > 0 \tag{16}$$

and $q_n,\ c_n,\ d_n$ are positive sequences of real numbers such that

$$\sum_{n=n_0}^{\infty}\left(\frac{1}{c_n}\right) = \sum_{n=n_0}^{\infty}\left(\frac{1}{d_n}\right) = \infty \quad \text{and} \quad \Delta c_n \geqslant 0. \tag{17}$$

For the linear case, they used the Riccati transformation technique and established a sufficient condition for oscillation of equation (15). For the nonlinear case, however, some oscillation criteria were provided by reducing the oscillation of the equation to the existence of positive solution of a Riccati difference inequality. Nevertheless, one can easily see that condition (16) might not be satisfied when $f(u) = u^\gamma$ for $\gamma > 0$ and the results are valid only when $\Delta c_n \geqslant 0$. Therefore, one of our aims in this paper is to establish some sufficient conditions for oscillation bypassing condition (16) and removing the restriction in (17).

In [2], the authors considered the nonlinear delay difference equation

$$\Delta^3 x_n = p_n \Delta^2 x_{n+m} + q_n F(x_{n-g}, x_{n-h}) = 0, \qquad n \geq n_0, \tag{18}$$

where p_n and q_n are positive real sequences, p_n is nonincreasing, m, g, h are nonnegative integers and $F(x, y) = \operatorname{sign} x \geq |x|^{c_1} |y|^{c_2}$ where c_1 and c_2 are nonnegative constants such that $c_1 + c_2 > 0$. They established some sufficient conditions for the existence of oscillatory solutions. The main results are proved by reducing the order of the equation under consideration. Indeed, the oscillation of equation (18) reduces to the oscillation of a first order delay or advanced difference equations.

In [5], the authors considered the nonlinear difference equation

$$\Delta(c_n \Delta(d_n \Delta(x_n))) + q_n f(x_{n+\sigma}) = 0, \qquad n \geq n_0, \tag{19}$$

where c_n, d_n, q_n are sequences of nonneagtive real numbers and the function $f \in C(\mathbb{R}, \mathbb{R})$ such that $uf(u) > 0$ for $u \neq 0$. The main result in [5] was the classification of the nonoscillatory solutions with respect to the sign of their quasi differences.

In [7], the authors considered the nonlinear delay difference equation

$$\Delta(c_n \left(\Delta^2 x_n\right)^\gamma) + q_n f(x(\sigma_n)) = 0, \qquad n \geq n_0, \tag{20}$$

where c_n, σ_n, q_n are sequences of nonneagtive real numbers, $\sigma_n < n$, γ is quotient of odd positive integers, $f \in C(\mathbb{R}, \mathbb{R})$ such that $uf(u) > 0$ for $u \neq 0$, $f'(x) > 0$, $-f(-xy) \geq f(xy) \geq f(x)f(y)$ for $xy > 0$ and

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{c_n}\right)^\gamma < \infty.$$

The main approach of proving the results in [7] was also based on the reduction of the oscillation of (20) to the oscillation of first order delay difference equation. However, the results can only be applied in the case when $\sigma_n < n$. Further, the restriction $f'(x) > 0$ might not be satisfied. Indeed, if $f(x) = x\left(\frac{1}{9} + \frac{1}{1+x^2}\right)$ then $f'(x) = \frac{(x^2-2)(x^2-5)}{9(1+x^2)^2}$ changes sign four times.

Following this trend, we are concerned with the oscillation and the asymptotic behavior of solutions of the nonlinear delay difference equation of form

$$\Delta(c_n \Delta(d_n \Delta x_n)^\gamma) + q_n f(x_{n-\sigma}) = 0, \qquad n \geq n_0, \tag{21}$$

where $\gamma > 0$ is quotient of odd positive integers, $\sigma \in \mathbb{N}$ and

(h_1) c_n, d_n, q_n are positive sequences of real numbers;

(h_2) $f \in C(\mathbb{R}, \mathbb{R})$ such that $uf(u) > 0$ for $u \neq 0$ and $f(u)/u^\gamma \geqslant K > 0$.

Our attention is restricted to those solutions of (21) which exist on $[n_x, \infty)$ and satisfy $\sup\{|x(n)| : n > n_1\} > 0$ for any $n_1 \geq n_x$. It is to be noted that the results of the above mentioned papers provided several oscillation criteria under the conditions

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{c_n}\right)^\gamma = \infty \quad \text{and} \quad \sum_{n=n_0}^{\infty} \left(\frac{1}{d_n}\right) = \infty. \tag{22}$$

Therefore, it will be of great interest to establish oscillation criteria when

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{c_n}\right)^\gamma < \infty \quad \text{and} \quad \sum_{n=n_0}^{\infty} \left(\frac{1}{d_n}\right) < \infty. \tag{23}$$

The aim of the paper is to employ Riccati transformation technique to establish some new oscillation criteria for equation (21) under assumptions (22). We will prove our results bypassing condition (16) and removing the restriction $\Delta c_n \geqslant 0$. Unlike previously obtained results, new oscillation criteria are also obtained under assumptions (23). We will complement and improve the results in [8] and extend those in [13]. Some comparison between our theorems and those previously known ones are indicated throughout the paper.

The paper is organized as follows: In Section 2, we present some fundamental lemmas that will be useful in proving our main results. In Section 3, we will state and prove the main oscillation theorems. An example is given to demonstrate the validity of the results.

2 Some Fundamental Lemmas

In this section, we present some fundamental lemmas that will be used in the proofs of the main results. For equation (21), we define the quasi differences by

$$x_n^{[0]} = x_n, \ \ x_n^{[1]} = d_n \Delta x_n, \ \ x_n^{[2]} = c_n \Delta \left(x_n^{[1]} \right)^{\gamma} \ \text{ and } \ x_n^{[3]} = \Delta x_n^{[2]}. \tag{24}$$

It is to be noted that if x_n is a solution of (21) then $z = -x$ is also a solution of (21) since $uf(u) > 0$ for $u \neq 0$. Thus, concerning nonoscillatory solutions of (21), we will only restrict our attention to the positive ones.

We start with the following lemma which provides the signs of the quasi differences of the solution x_n of (21).

Lemma 1. *Let x_n be a nonoscillatory solution of (21). Assume that $(h_1) - (h_2)$ hold. Then there exists $N > n_0$ such that $x_n^{[i]} \neq 0$ for $i = 0, 1, 2$ and $n \geq N$.*

Proof. Without loss of generality, we assume that x_n is an eventually positive solution of (21) and there exists $n_1 \geq n_0$ such that x_n and $x_{n-\sigma} > 0$ for $n \geq n_1$. Since $q_n > 0$, then $x_n^{[3]} < 0$. Thus, there exists $n_2 \geq n_1$ such that $x_n^{[2]}$ is either positive or negative for $n \geq n_2$. It follows that $x_n^{[1]}$ is either increasing or decreasing for $n \geq n_2$ and so there exists $N \geq n_2$ such that $x_n^{[0]}$ is either positive or negative for $n \geq N$.

In view of Lemma 1, we deduce that all nonoscillatory solutions of (21) belong to the following classes:

$$\begin{aligned}
C_0 &= \{x_n : \exists \ N \text{ such that } x_n x_n^{[1]} < 0, \ x_n x_n^{[2]} > 0 \text{ for } n \geq N\}, \\
C_1 &= \{x_n : \exists \ N \text{ such that } x_n x_n^{[1]} > 0, \ x_n x_n^{[2]} < 0 \text{ for } n \geq N\}, \\
C_2 &= \{x_n : \exists \ N \text{ such that } x_n x_n^{[1]} > 0, \ x_n x_n^{[2]} > 0 \text{ for } n \geq N\}, \\
C_3 &= \{x_n : \exists \ N \text{ such that } x_n x_n^{[1]} < 0, \ x_n x_n^{[2]} < 0 \text{ for } n \geq N\}.
\end{aligned}$$

Lemma 2. *Let x_n be a nonoscillatory solution of (21). Assume that $(h_1) - (h_2)$ hold. If*

$$\sum_{n=n_0}^{\infty} \frac{1}{d_n} \sum_{s=n_0}^{n-1} \frac{1}{(c_s)^{\frac{1}{\gamma}}} = \infty. \tag{25}$$

Then C_3 is empty.

Proof. To prove that C_3 is empty, it is sufficient to show that if there is a positive solution x_n of (21), then the case $x_n x_n^{[1]} < 0$ and $x_n x_n^{[2]} < 0$ for $n \geq N$ is impossible. For the sake of contradiction, assume that there exists $n_1 > n_0$ such that $x_n^{[1]} < 0$ and $x_n^{[2]} < 0$ for $n \geq n_1$. Denote $a_0 = x_{n_1}^{[2]} < 0$. Then, since $x_n^{[2]}$ is decreasing we have $c_n \left(\Delta x_n^{[1]} \right)^{\gamma} < a_0$ for $n \geq n_1$. Thus by summing from n_1 to $n - 1$, we have

$$x_n^{[1]} < x_{n_1}^{[1]} + a_0^{\frac{1}{\gamma}} \sum_{s=n_1}^{n-1} \frac{1}{(c_s)^{\frac{1}{\gamma}}}.$$

Using that $x_{n_1}^{[1]} < 0$, we get

$$x_n^{[1]} < a_0^{\frac{1}{\gamma}} \sum_{s=n_1}^{n-1} \frac{1}{(c_s)^{\frac{1}{\gamma}}}.$$

Summing up from n_1 to $n-1$, we obtain

$$x_n < x_{n_1} + a_0^{\frac{1}{\gamma}} \sum_{s=n_1}^{n-1} \frac{1}{d_s} \sum_{u=n_1}^{s-1} \frac{1}{(c_u)^{\frac{1}{\gamma}}}.$$

Letting $n \to \infty$, then by (25) we deduce that $\lim_{n\to\infty} x_n = -\infty$ which contradicts that $x_n > 0$. The proof is complete.

Lemma 3. *Let x_n be a nonoscillatory solution of (21). Assume that $(h_1) - (h_2)$ hold. If (22) holds. Then $x_n \in C_0 \cup C_2$.*

Proof. Without loss of generality, we assume that x_n is an eventually positive solution of (21) and there exists $n_1 \geqslant n_0$ such that x_n and $x_{n-\sigma} > 0$ for $n \geqslant n_1$. In virtue of Lemma 1, we deduce that $x_n^{[0]}$, $x_n^{[1]}$ and $x_n^{[2]}$ are monotone and eventually of one sign. Therefore to complete the proof, we show that there are only the following two cases for $n \geqslant n_0$ sufficiently large:

(I) $x_n^{[0]} > 0,\ x_n^{[1]} > 0$ and $x_n^{[2]} > 0$;

(II) $x_n^{[0]} > 0,\ x_n^{[1]} < 0$ and $x_n^{[2]} > 0$.

In view of (h_2) and (21), we see that $x_n^{[3]} < 0$ for $n \geqslant n_1$. We claim that there is $n_2 \geqslant n_1$ such that for $n \geqslant n_2$, $x_n^{[2]} > 0$. Suppose to the contrary that $x_n^{[2]} \leq 0$ for $n \geqslant n_2$. Since $x_n^{[2]}$ is nonincreasing, there exists a negative constant L and $n_3 \geqslant n_2$ such that $x_n^{[2]} \leq L$ for $n \geqslant n_3$. Dividing by c_n and summing from n_3 to $n-1$, we obtain

$$x_n^{[1]} \leq x_{n_3}^{[1]} + L^{\frac{1}{\gamma}} \sum_{s=n_3}^{n-1} \frac{1}{(c_s)^{\frac{1}{\gamma}}}.$$

Letting $n \to \infty$, then by (22) we deduce that $x_n^{[1]} \to -\infty$. Thus, there is an integer $n_4 \geqslant n_3$ such that for $n \geqslant n_4$, $x_n^{[1]} \leq x_{n_4}^{[1]} < 0$. Dividing by d_n and summing from t_4 to t, we have

$$x_n - x_{n_4} \leq x_{n_4}^{[1]} \sum_{s=n_4}^{n-1} \frac{1}{d_s},$$

which implies that $x_n \to -\infty$ as $n \to \infty$. This contradicts the fact that $x_n > 0$. Then $x_n^{[2]} > 0$.

Lemma 4. *Let x_n be a nonsocillatory solution of (21) that belongs to C_0. Assume that $(h_1) - (h_2)$ and $n - \sigma \leq n$ hold. If*

$$\sum_{n=n_0}^{\infty} \frac{1}{d_n} \left[\sum_{t=n_0}^{n-1} \frac{1}{c_t} \sum_{s=n_0}^{t-1} q_s \right]^{\frac{1}{\gamma}} = \infty. \tag{26}$$

Then $\lim_{n\to\infty} x_n = 0$.

Proof. Without loss of generality, we assume that $x_{n-\sigma} > 0$ for $n \geq n_1$ where n_1 is chosen sufficiently large. In view of (h_2) and (21), we obtain

$$x_n^{[3]} + Kq_n x_{n-\sigma}^{\gamma} \leq 0, \qquad n \geq n_1. \tag{27}$$

Since x_n is positive and decreasing it follows that $\lim_{n\to\infty} x_n = b \geq 0$. Now we claim that $b = 0$. If $b \neq 0$ then $x_{n-\sigma}^{\gamma} \to b^{\gamma} > 0$ as $n \to \infty$. Hence there exists $n_2 \geq n_1$ such that $x_{n-\sigma}^{\gamma} \geq b^{\gamma}$. Therefore from (27), we have

$$x_n^{[3]} + Kq_n b^{\gamma} \leq 0, \qquad n \geq n_2.$$

Define the sequence $u_n = x_n^{[2]}$ for $n \geq n_2$. Then $\Delta x_n^{[2]} \leq -Aq_n$ where $A = Kb^{\gamma} > 0$. Summing the last inequality from n_2 to $n-1$, we get $x_n^{[2]} \leq x_{n_2}^{[2]} - A \sum\limits_{s=n_2}^{n-1} q_s$. In view of (26), it is possible to choose an integer n_3 sufficiently large such that $x_n^{[2]} \leq -\frac{A}{2} \sum\limits_{s=n_2}^{n-1} q_s$ for all $n \geq n_3$. Hence $\Delta \left(x_n^{[1]}\right)^{\gamma} \leq -\frac{A}{2}\frac{1}{c_n} \sum\limits_{s=n_2}^{n-1} q_s$. Summing the last inequality from n_3 to $n-1$, we obtain

$$\left(x_n^{[1]}\right)^{\gamma} \leq \left(x_{n_3}^{[1]}\right)^{\gamma} - \frac{A}{2} \sum_{t=n_3}^{n-1} \left(\frac{1}{c_t} \sum_{s=n_2}^{t-1} q_s \right).$$

Since $\Delta x_n < 0$ for $n \geq n_0$, the last inequality implies that

$$\Delta x_n \leq -\left(\frac{A}{2}\right)^{\frac{1}{\gamma}} \frac{1}{d_n} \left[\sum_{t=n_3}^{n-1} \frac{1}{c_t} \sum_{s=n_2}^{t-1} q_s \right]^{\frac{1}{\gamma}}.$$

Summing from n_4 to $n-1$, we have

$$x_n \leq x_{n_4} - \left(\frac{A}{2}\right)^{\frac{1}{\gamma}} \sum_{l=n_4}^{n-1} \frac{1}{d_l} \left[\sum_{t=n_3}^{l-1} \frac{1}{c_t} \sum_{s=n_2}^{t-1} q_s \right]^{\frac{1}{\gamma}}.$$

Condition (26) implies that $x_n \to -\infty$ as $n \to \infty$ which is a contradiction with the fact that $x_n > 0$. Then $b = 0$ and this completes the proof.

Lemma 5. *Let x_n be a nonoscillatory solution of (21) that belongs to C_2. Then there exists $n_1 \geq n_0$ such that*

$$\left(x_{n-\sigma}^{[1]}\right)^{\gamma} \geq \delta_{n-\sigma} x_n^{[2]}, \quad \textit{for } n \geq n_1,$$

where $\delta_n := \sum_{s=n_0}^{n-1} \frac{1}{c_s}$.

Proof. Since $x_n \in C_2$, then without loss of generality we can assume that there exists $N > n_0$ such that

$$x_n > 0,\ x_n^{[1]} > 0,\ x_n^{[2]} > 0 \ \text{ and } \ x_n^{[3]} \leq 0 \ \text{ for } \ n \geq N.$$

Hence

$$\left(x_n^{[1]}\right)^{\gamma} = \left(x_{n_1}^{[1]}\right)^{\gamma} + \sum_{s=n_1}^{n-1} \frac{c_s \Delta (x_s^{[1]})^{\gamma}}{c_s} \geq \delta_n x_n^{[2]}, \quad n \geq n_1. \tag{28}$$

Since $x_n^{[3]} \leq 0$, we have $x_{n-\sigma}^{[2]} \geq x_n^{[2]}$. This and (28) imply that

$$\left(x_{n-\sigma}^{[1]}\right)^{\gamma} \geq \delta_{n-\sigma} x_{n-\sigma}^{[2]} \geq \delta_{n-\sigma} x_n^{[2]}, \quad n \geq N_1 = N + \sigma.$$

Thus

$$\left(x_{n-\sigma}^{[1]}\right)^{\gamma} \geq \delta_{n-\sigma} x_n^{[2]}, \quad \text{for} \quad n \geq N_1.$$

3 Oscillation Criteria

In this section, we will establish some new sufficient conditions which guarantee that every solution x_n of (21) either oscillates or satisfies $\lim_{n\to\infty} x_n = 0$. In our analysis, we will present the proofs of our results under conditions (22) and (23) in two separate investigations.

3.1 Oscillation under condition (22)

Throughout this subsection we assume that there exists a double sequences $\{H_{m,n} : m \geq n \geq 0\}$ and $h_{m,n}$ such that:

(i) $H_{m,m} = 0$ for $m \geq 0$;

(ii) $H_{m,n} > 0$ for $m > n \geq 0$;

(iii) $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} \leq 0$ for $m \geq n \geq 0$;

(iv) $h_{m,n} = -\Delta_2 H_{m,n} (H_{m,n})^{-\frac{1}{\gamma+1}}$, $m > n \geq 0$.

For a given sequence ρ_n, we define

$$\begin{aligned}
\psi_n : &= K\rho_n q_n - \rho_n \Delta(c_n \alpha_n) + \frac{\rho_n \delta_{n-\sigma}^{\frac{1}{\gamma}} (c_{n+1})^{1+\frac{1}{\gamma}} (\alpha_{n+1})^{1+\frac{1}{\gamma}}}{d_{n-\sigma}} \\
\xi_n : &= \Delta\rho_n + \gamma\rho_n (1 + \frac{1}{\gamma})(c_{n+1}\alpha_{n+1}\delta_{n-\sigma})^{\frac{1}{\gamma}} d_{n-\sigma}^{-1}
\end{aligned}$$

and

$$\phi_{m,n} : = \frac{\rho_{n+1}^{1+\gamma}}{(1+\gamma)^{1+\gamma} \rho_n^{\gamma} \delta_{n-\sigma} d_{n-\sigma}^{-\gamma} H_{m,n}^{\gamma}} \left(\frac{\xi_n H_{m,n}}{\rho_{n+1}} - h_{m,n} H_{m,n}^{\frac{1}{\gamma+1}} \right)^{1+\gamma}.$$

Theorem 6. *Let x_n be a solution of (21) and ρ_n be a given positive sequence. Assume that $(h_1) - (h_2)$, (22) and (26) hold. If*

$$\lim_{m\to\infty} \sup \frac{1}{H_{m,n_0}} \sum_{n=n_0}^{m-1} [H_{m,n}\psi_n - \phi_{m,n}] = \infty. \tag{29}$$

Then x_n either oscillates or satisfies $\lim_{n\to\infty} x_n = 0$.

Proof. Suppose to the contrary that x_n is a nonoscillatory solution. Without loss of generality, we assume that $x_n > 0$ and $x_{n-\sigma} > 0$ for $n \geq n_1$ where n_1 is chosen so large. In view of Lemma 3, we deduce that condition (22) implies that $x_n \in C_0 \cup C_2$. If $x_n \in C_0$, then we are back to the proof of Lemma 4 to show that $\lim_{n\to\infty} x_n = 0$. We assume that the solution $x_n \in C_2$ and define the sequence ω_n by the generalized Riccati substitution

$$\omega_n := \rho_n \left[\frac{x_n^{[2]}}{x_{n-\sigma}^{\gamma}} + c_n \alpha_n \right], \quad n \geq n_1. \tag{30}$$

It follows that

$$\Delta\omega_n = \Delta(\rho_n c_n \alpha_n) + x_{n+1}^{[2]}\Delta\left[\frac{\rho_n}{x_{n-\sigma}^{\gamma}}\right] + \frac{\rho_n x_n^{[3]}}{x_{n-\sigma}^{\gamma}}.$$

In view of (27) and (30), the above equation can be written in the form

$$\Delta\omega_n \leq -K\rho_n q_n + \rho_n\Delta(c_n\alpha_n) + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - \frac{\rho_n x_{n+1}^{[2]}}{x_{n-\sigma}^{\gamma}x_{n-\sigma+1}^{\gamma}}\Delta(x_{n-\sigma}^{\gamma}). \tag{31}$$

First: we consider the case when $\gamma \geq 1$. By using the inequality ([9, see p. 39])

$$x^{\gamma} - y^{\gamma} \geq \gamma y^{\gamma-1}(x-y) \ \text{ for all } x \neq y > 0 \ \text{ and } \ \gamma \geq 1,$$

we may write

$$\Delta(x_{n-\sigma}^{\gamma}) = x_{n-\sigma+1}^{\gamma} - x_{n-\sigma}^{\gamma} \geq \gamma x_{n-\sigma}^{\gamma-1}\Delta x_{n-\sigma}, \quad \gamma \geq 1.$$

Substituting in (31), we find out

$$\Delta\omega_n \leq -K\rho_n q_n + \rho_n\Delta(c_n\alpha_n) + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - \frac{\gamma\rho_n x_{n+1}^{[2]}\Delta x_{n-\sigma}}{x_{n-\sigma}x_{n-\sigma+1}^{\gamma}}.$$

Since $x_n \in C_2$, it follows from Lemma 5 that there exists $n_2 \geq n_1$ such that

$$(\Delta x_{n-\sigma})^{\gamma} \geq \frac{\delta_{n-\sigma}}{d_{n-\sigma}^{\gamma}}x_n^{[2]} \quad \text{for} \quad n \geq n_2. \tag{32}$$

Using the fact that $x_{n-\sigma+1} \geq x_{n-\sigma}$, we obtain

$$\Delta\omega_n \leq -K\rho_n q_n + \rho_n\Delta(c_n\alpha_n) + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - \frac{\gamma\rho_n\delta_{n-\sigma}^{\frac{1}{\gamma}}x_{n+1}^{[2]}[x_n^{[2]}]^{\frac{1}{\gamma}}}{d_{n-\sigma}(x_{n-\sigma+1})^{\gamma+1}}. \tag{33}$$

Since $x_n^{[3]} < 0$, it follows that $x_{n+1}^{[2]} \leq x_n^{[2]}$ and thus $[x_{n+1}^{[2]}]^{\frac{1}{\gamma}} \leq [x_n^{[2]}]^{\frac{1}{\gamma}}$. This yields that

$$\Delta\omega_n \leq -K\rho_n q_n + \rho_n\Delta(c_n\alpha_n) + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - \frac{\gamma\rho_n\delta_{n-\sigma}^{\frac{1}{\gamma}}}{d_{n-\sigma}}\left(\frac{x_{n+1}^{[2]}}{x_{n-\sigma+1}^{\gamma}}\right)^{\frac{1+\gamma}{\gamma}}. \tag{34}$$

Second: we consider the case when $0 < \gamma < 1$. By using the inequality

$$x^{\gamma} - y^{\gamma} \geq \gamma x^{\gamma-1}(x-y) \ \text{ for all } x \neq y > 0,$$

we may write

$$\Delta(x_{n-\sigma}^{\gamma}) \geq \gamma x_{n-\sigma+1}^{\gamma-1}\Delta x_{n-\sigma}.$$

Substituting in (31), we have

$$\Delta\omega_n \leq -K\rho_n q_n + \rho_n\Delta(c_n\alpha_n) + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - \frac{\gamma\rho_n x_{n+1}^{[2]}\Delta x_{n-\sigma}}{x_{n-\sigma}^{\gamma}x_{n-\sigma+1}}.$$

By using the fact that x_n is increasing, we have

$$-\frac{\gamma\rho_n x_{n+1}^{[2]}\Delta x_{n-\sigma}}{x_{n-\sigma}^{\gamma}x_{n-\sigma+1}} \leq -\frac{\gamma\rho_n\delta_{n-\sigma}^{\frac{1}{\gamma}}}{d_{n-\sigma}}\left(\frac{x_{n+1}^{[2]}}{x_{n-\sigma+1}^{\gamma}}\right)^{\frac{1+\gamma}{\gamma}}. \tag{35}$$

Thus, we again obtain (34). However, from (30) we see that

$$\left(\frac{x_{n+1}^{[2]}}{x_{n-\sigma+1}^{\gamma}}\right)^{1+\frac{1}{\gamma}} = \left(\frac{\omega_{n+1}}{\rho_{n+1}} - c_{n+1}\alpha_{n+1}\right)^{1+\frac{1}{\gamma}}. \tag{36}$$

Then, by using the inequality [19, see p. 534]

$$(v-u)^{1+\frac{1}{\gamma}} \geq v^{1+\frac{1}{\gamma}} + \frac{1}{\gamma}u^{1+\frac{1}{\gamma}} - (1+\frac{1}{\gamma})u^{\frac{1}{\gamma}}v, \quad \gamma = \frac{odd}{odd} \geq 1,$$

we may write equation (36) as follows

$$\left(\frac{\omega_{n+1}}{\rho_{n+1}} - c_{n+1}\alpha_{n+1}\right)^{1+\frac{1}{\gamma}} \geq \left(\frac{\omega_{n+1}}{\rho_{n+1}}\right)^{1+\frac{1}{\gamma}} + \frac{(c_{n+1}\alpha_{n+1})^{1+\frac{1}{\gamma}}}{\gamma} - \frac{(1+\frac{1}{\gamma})(c_{n+1}\alpha_{n+1})^{\frac{1}{\gamma}}}{\rho_{n+1}}\omega_{n+1}.$$

Substituting back in (34), we have

$$\begin{aligned}\Delta\omega_n \quad \leq \quad & -K\rho_n q_n + \rho_n\Delta(c_n\alpha_n) - \frac{\rho_n\delta_{n-\sigma}^{\frac{1}{\gamma}}(c_{n+1})^{1+\frac{1}{\gamma}}(\alpha_{n+1})^{1+\frac{1}{\gamma}}}{d_{n-\sigma}} \\ & + \left(\frac{\Delta\rho_n}{\rho_{n+1}} + \frac{\gamma\rho_n(1+\frac{1}{\gamma})(c_{n+1}\delta_{n-\sigma}\alpha_{n+1})^{\frac{1}{\gamma}}}{d_{n-\sigma}\rho_{n+1}}\right)\omega_{n+1} \\ & - \left(\frac{\gamma\rho_n\delta_{n-\sigma}^{\frac{1}{\gamma}}}{d_{n-\sigma}(\rho_{n+1})^{1+\frac{1}{\gamma}}}\right)(\omega_{n+1})^{1+\frac{1}{\gamma}}. \end{aligned} \tag{37}$$

Thus,

$$\psi_n \leq -\Delta\omega_n + \frac{\xi_n}{\rho_{n+1}}\omega_{n+1} - \frac{\bar{\rho}_n}{(\rho_{n+1})^{1+\frac{1}{\gamma}}}(\omega_{n+1})^{1+\frac{1}{\gamma}}, \quad n \geq n_3,$$

where $\bar{\rho}_n = \gamma\rho_n\delta_{n-\sigma}^{\frac{1}{\gamma}}d_{n-\sigma}^{-1}$. Therefore, we have

$$\sum_{n=n_3}^{m-1} H_{m,n}\psi_n \leq -\sum_{n=n_3}^{m-1} H_{m,n}\Delta\omega_n + \sum_{n=n_3}^{m-1}\frac{\xi_n H_{m,n}}{\rho_{n+1}}\omega_{n+1} - \sum_{n=n_3}^{m-1}\frac{\bar{\rho}_n H_{m,n}}{(\rho_{n+1})^{1+\frac{1}{\gamma}}}(\omega_{n+1})^{1+\frac{1}{\gamma}},$$

which yields after summing by parts

$$\begin{aligned}\sum_{n=n_3}^{m-1} H_{m,n}\psi_n \quad \leq \quad & H_{m,n_3}\omega_{n_3} + \sum_{n=n_3}^{m-1}\omega_{n+1}\Delta_2 H_{m,n} + \sum_{n=n_3}^{m-1}\frac{\xi_n H_{m,n}}{\rho_{n+1}}\omega_{n+1} \\ & - \sum_{n=n_3}^{m-1}\frac{\bar{\rho}_n H_{m,n}}{(\rho_{n+1})^{1+\frac{1}{\gamma}}}(\omega_{n+1})^{1+\frac{1}{\gamma}}.\end{aligned}$$

Hence

$$\sum_{n=n_3}^{m-1} H_{m,n}\psi_n \leq H_{m,n_3}\omega_{n_3} + \sum_{n=n_3}^{m-1}\left(\frac{\xi_n H_{m,n}}{\rho_{n+1}} - h_{m,n}H_{m,n}^{\frac{1}{\gamma+1}}\right)\omega_{n+1} - \sum_{n=n_3}^{m-1}\frac{\bar{\rho}_n H_{m,n}}{(\rho_{n+1})^{1+\frac{1}{\gamma}}}(\omega_{n+1})^{1+\frac{1}{\gamma}}.$$

Using the fact that

$$Bu - Au^{1+\frac{1}{\beta}} \leq \frac{\beta^{\beta}}{(1+\beta)^{1+\beta}}\frac{B^{1+\beta}}{A^{\beta}}$$

for $A = \frac{\bar{\rho}_n H_{m,n}}{(\rho_{n+1})^{1+\frac{1}{\gamma}}}$ and $B = \left(\frac{\xi_n H_{m,n}}{\rho_{n+1}} - h_{m,n} H_{m,n}^{\frac{1}{\gamma+1}} \right)$, we obtain

$$\sum_{n=n_3}^{m-1} [H_{m,n}\psi_n - \phi_{m,n}] < H_{m,n_3}\omega_{n_3} \leq H_{m,n_0}\omega_{n_3},$$

which implies that

$$\sum_{n=n_0}^{m-1} [H_{m,n}\psi_n - \phi_{m,n}] < H_{m,n_0} \left(\omega_{n_3} + \sum_{n=n_0}^{n_3-1} \psi_n \right).$$

Hence

$$\lim_{m\to\infty} \sup \frac{1}{H_{m,n_0}} \sum_{n=n_0}^{m-1} [H_{m,n}\psi_n - \phi_{m,n}] < \infty,$$

which contradicts (29). The proof is complete.

The following result is an immediate consequence of Theorem 6.

Corollary 7. *Let x_n be a solution of (21) and assume that all the assumptions of Theorem 6 hold, except that the condition (29) is replaced by*

$$\lim_{m\to\infty} \sup \frac{1}{H_{m,n_0}} \sum_{n=n_0}^{m-1} H_{m,n}\psi_n = \infty \quad \text{and} \quad \lim_{m\to\infty} \sup \frac{1}{H_{m,n_0}} \sum_{n=n_0}^{m-1} \phi_{m,n} < \infty. \tag{38}$$

Then x_n either oscillates or satisfies $\lim_{n\to\infty} x_n = 0$.

In view of Theorem 6, if we choose $H_{m,n} = 1$ and

$$(\alpha_{n+1})^{\frac{1}{\gamma}} := -\frac{\gamma\Delta\rho_n}{(\gamma+1)\rho_n} d_{n-\sigma} c_{n+1}^{-\frac{1}{\gamma}} \delta_{n-\sigma}^{-\frac{1}{\gamma}} \tag{39}$$

we deduce that $\xi_n = 0$ and we have the following result.

Theorem 8. *Let x_n be a solution of (21) and ρ_n be a given positive sequence. Assume that $(h_1) - (h_2)$, (22) and (26) hold. If*

$$\lim_{n\to\infty} \sup \sum_{s=n_0}^{n} \psi_s = \infty. \tag{40}$$

Then x_n either oscillates or satisfies $\lim_{n\to\infty} x_n = 0$.

Theorem 8 improves Theorem 1 of Graef and Thandapani [8] in the sense that our results are proved for the nonlinear case and do not require condition (16) and that $\Delta c_n \geq 0$ for $n \geq n_0$. Moreover, we note that if $\gamma = 1$ and $\rho_n = 1$ then condition (40) reduces to condition 3 of Theorem 1 in [8]. This implies that Theorem 8 is an extension of Theorem 1 in [8].

Theorem 8 might provide different conditions for oscillation of all solutions of equation (21). This occurs upon choosing different values for ρ_n. For instance, let $\rho_n = n^\lambda$, $n \geq n_0$ where $\lambda > 1$ is a constant. Then, the next result follows.

Corollary 9. *Let x_n be solution of equation (21) and assume that all the assumptions of Theorem 6 hold, except that condition (40) is replaced by*

$$\lim_{n\to\infty} \sup \sum_{s=n_0}^{n} \left[K s^\lambda q_s - s^\lambda \Delta(c_s\alpha_s) + \frac{s^\lambda \delta_{s-\sigma}^{\frac{1}{\gamma}} (c_{s+1})^{1+\frac{1}{\gamma}} (\alpha_{s+1})^{1+\frac{1}{\gamma}}}{d_{s-\sigma}} \right] = \infty. \tag{41}$$

Then x_n *either oscillates or satisfies* $\lim_{n\to\infty} x_n = 0$.

By choosing the sequence $H_{m,n}$ in an appropriate form, one can derive several oscillation criteria for (21). Let us consider the double sequence $H_{m,n}$ defined by

$$H_{m,n} := (m-n)^{\lambda} \ \text{ or } \ H_{m,n} := \left(\log \frac{m+1}{n+1}\right)^{\lambda}, \lambda \geq 1, \ m \geq n \geq 0,$$

or

$$H_{m,n} := (m-n)^{(\lambda)} \quad \lambda \geq 1, \ m \geq n \geq 0,$$

where

$$(m-n)^{(\lambda)} = (m-n)(m-n+1)\dots(m-n+\lambda-1), \ (m-n)^{(0)} = 1$$

and

$$\Delta_2(m-n)^{(\lambda)} = (m-n-1)^{\lambda} - (m-n)^{\lambda} = -\lambda(m-n)^{(\lambda-1)}.$$

We observe that $H_{m,m} = 0$ for $m \geq 0$ and $H_{m,n} > 0$ and $\Delta_2 H_{m,n} \leq 0$ for $m > n \geq 0$. Then, the following results can be formulated.

Corollary 10. *Let* x_n *be a solution of (21) and assume that all the assumptions of Theorem 6 hold, except that the condition (29) is replaced by*

$$\lim_{m\to\infty} \sup \frac{1}{m^{\lambda}} \sum_{n=0}^{m-1} \left[(m-n)^{\lambda}\psi_n - \varphi_{m,n}\right] = \infty, \tag{42}$$

where

$$\varphi_{m,n} = \frac{\rho_{n+1}^{1+\gamma}\left(\frac{\xi_n (m-n)^{\lambda}}{\rho_{n+1}} - \lambda(m-n)^{\lambda-1}\right)^{1+\gamma}}{(1+\gamma)^{1+\gamma}\rho_n^{\gamma}\delta_{n-\sigma}d_{n-\sigma}^{-\gamma}(m-n)^{\lambda\gamma}}.$$

Then x_n *either oscillates or satisfies* $\lim_{n\to\infty} x_n = 0$.

Corollary 11. *Let* x_n *be a solution of (21) and assume that all the assumptions of Theorem 6 hold, except that the condition (29) is replaced by*

$$\lim_{m\to\infty} \sup \frac{1}{(\log(m+1))^{\lambda}} \sum_{n=0}^{m-1} \left[\left(\log \frac{m+1}{n+1}\right)^{\lambda} \psi_n - \vartheta_{m,n}\right]$$

where

$$\vartheta_{m,n} = \frac{\rho_{n+1}^{1+\gamma}\left(\frac{\xi_n (\log \frac{m+1}{n+1})^{\lambda}}{\rho_{n+1}} - [(\log \frac{m+1}{n+2})^{\lambda} - (\log \frac{m+1}{n+1})^{\lambda}]\right)}{(1+\gamma)^{1+\gamma}\rho_n^{\gamma}\delta_{n-\sigma}d_{n-\sigma}^{-\gamma}(\log \frac{m+1}{n+})^{\gamma\lambda}}.$$

Then x_n *either oscillates or satisfies* $\lim_{n\to\infty} x_n = 0$.

Corollary 12. *Let* x_n *be a solution of (21) and assume that all the assumptions of Theorem 6 hold, except that the condition (29) is replaced by*

$$\lim_{m\to\infty} \sup \frac{1}{m^{\lambda}} \sum_{n=0}^{m-1} (m-n)^{\lambda} \left[\psi_n - \frac{\rho_{n+1}^{1+\gamma}\left(\frac{\xi_n}{\rho_{n+1}} - \lambda(m-n)^{-1}\right)^{1+\gamma}}{(1+\gamma)^{1+\gamma}\rho_n^{\gamma}\delta_{n-\sigma}d_{n-\sigma}^{-\gamma}(m-n)^{\lambda\gamma}}\right] = \infty.$$

Then x_n *either oscillates or satisfies* $\lim_{n\to\infty} x_n = 0$.

Example 13. Consider the equation

$$\Delta(\frac{1}{n}\Delta(\sqrt[3]{n}\Delta x_n)) + n x_{n-1} = 0, \quad n \geq 1, \tag{43}$$

where $\gamma = 1$, $c_n = \frac{1}{n}$, $d_n = \sqrt[3]{n}$, $q_n = n$ and $n - \sigma = n - 1$. It follows that $\delta_n = \sum_{s=1}^{n-1} \frac{1}{c_s} = \frac{n(n-1)}{2}$. It is clear that the sequences c_n, d_n, q_n and the function f satisfy conditions $(h_1) - (h_2)$ and (22). It remains to check conditions (26) and (40). From the above assumptions, it follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}} \sum_{t=1}^{n-1} t \sum_{s=1}^{t-1} s = \infty.$$

This shows that condition (26) is satisfied. By choosing $\rho_n = n$, one can easily see that

$$\limsup_{n\to\infty} \sum_{l=1}^{n} \left[Kl^2 - l\Big(\frac{1}{(l-1)(l-2)^{\frac{2}{3}}(l-3)} - \frac{1}{l(l-1)^{\frac{2}{3}}(l-2)}\Big) + \frac{1}{2l(l-1)^{\frac{1}{3}}(l-2)} \right] = \infty.$$

Thus, condition (40) holds. Therefore, by Theorem 8 we conclude that every solution x_n of equation (43) either oscillates or satisfies $\lim_{n\to\infty} x_n = 0$.

Remark 14. It is obvious that results obtained in [8] can not be applied to equation (43).

3.2 Oscillation under condition (23)

Throughout this subsection, the sequences ρ_n, ψ_n and $(\alpha_{n+1})^{\frac{1}{\gamma}}$ are assumed in similar manner. In addition, we assume that (25) holds and thus in view of Lemma 2, we deduce that the class C_3 is empty. Therefore, if x_n is a solution of (21) then $x_n \in C_0 \cup C_1 \cup C_2$.

We define the sequence Q_n by

$$Q_n := Kq_n \left(\sum_{s=N}^{n-\sigma} \frac{1}{d_s} \right)^{\gamma},$$

where $n - \sigma > N$ for $N > n_0$.

Theorem 15. *Let x_n be a solution of (21) and ρ_n be a given positive sequence such that (40) holds. Assume that $(h_1) - (h_2)$, (23), (25)and (26) hold. If*

$$\lim_{n\to\infty} \sup \sum_{u=n_6}^{n-1} \frac{1}{d_u} \left[\sum_{s=n_5}^{u-1} \frac{1}{c_s} \sum_{t=n_4}^{s-1} Q_t \sum_{\tau=t-\sigma}^{\infty} \frac{1}{c_\tau} \right]^{\frac{1}{\gamma}} = \infty. \tag{44}$$

Then x_n either oscillates or satisfies $\lim_{n\to\infty} x_n = 0$.

Proof. Suppose to the contrary that x_n is a nonoscillatory solution of equation (21). Without loss of generality we may assume that $x_n > 0$ and $x_{n-\sigma} > 0$ for $n \geq n_1$ where n_1 is chosen so large. Condition (25) implies that the solution x_n belongs to the space $C_0 \cup C_1 \cup C_2$. If $x_n \in C_0$, then we are back to the proof of Lemma 4 to show that $\lim_{n\to\infty} x_n = 0$. If $x_n \in C_2$, then we are back to the proof of Theorem 6 to get a contradiction. To complete the proof, it is sufficient to show that under condition (44) there is no solution $x_n \in C_1$. Therefore, we suppose to the contrary that there exists $N > n_1$ such that $x_n^{[1]} > 0$ and $x_n^{[2]} < 0$ for $n \geq N$. In view of the quasi differences (24), we observe that

$$\Delta x_n = \frac{x_n^{[1]}}{d_n}.$$

Summing up from N to $n-1$, we have

$$x_n - x_N = \sum_{s=N}^{n-1} \frac{x_s^{[1]}}{d_s} \geq x_n^{[1]} \sum_{s=N}^{n-1} \frac{1}{d_s}. \tag{45}$$

Hence, there exists $n_3 > N$ such that

$$x_{n-\sigma} \geq x_{n-\sigma}^{[1]} \sum_{s=N}^{n-\sigma} \frac{1}{d_s}, \text{ for } n \geq n_3.$$

Using this in (21), we get

$$\Delta\left(c_n \Delta\left(x_n^{[1]}\right)^{\gamma}\right) + K q_n \left(\sum_{s=N}^{n-\sigma} \frac{1}{d_s}\right)^{\gamma} \left(x_{n-\sigma}^{[1]}\right)^{\gamma} \leq 0, \qquad n \geq n_3. \tag{46}$$

Setting $y_n = \left(x_n^{[1]}\right)^{\gamma} > 0$, we deduce that $\Delta y_n < 0$ and y_n satisfies the difference inequality

$$\Delta(c_n\left(\Delta y_n\right)) + Q_n y_{n-\sigma} \leq 0, \quad \text{for } n \geq n_3. \tag{47}$$

Since $n-\sigma \to \infty$ as $n \to \infty$, we can choose $n_4 > n_3$ such that $n-\sigma \geq n_4$ for $n \geq n_4$ and thus

$$\begin{aligned} y_\infty - y_{n-\sigma} &= \sum_{s=n-\sigma}^{\infty} \Delta y_s = \sum_{s=n-\sigma}^{\infty} c_s \Delta y_s \frac{1}{c_s} \\ &< c_{n-\sigma} \Delta y_{n-\sigma} \sum_{s=n-\sigma}^{\infty} \frac{1}{c_s} < c_{n_4} \Delta y_{n_4} \sum_{s=n-\sigma}^{\infty} \frac{1}{c_s}. \end{aligned}$$

Thus

$$-y_{n-\sigma} < c_{n_4} \Delta y_{n_4} \sum_{s=n-\sigma}^{\infty} \frac{1}{c_s}.$$

Substituting back in (47), we have

$$\Delta(c_n\left(\Delta y_n\right)) < L Q_n \left(\sum_{s=n-\sigma}^{\infty} \frac{1}{c_s}\right), \quad \text{for } n \geq n_4, \tag{48}$$

where $L = c_{n_4} \Delta y_{n_4} < 0$. Summing this inequality from n_4 to $n-1$, we see that

$$c_n\left(\Delta y_n\right) < c_n\left(\Delta y_n\right) - c_{n_4}\left(\Delta y_{n_4}\right) < L \sum_{s=n_4}^{n-1} Q_s \sum_{\tau=s-\sigma}^{\infty} \frac{1}{c_\tau}.$$

where $\Delta y_n < 0$. Summing again from n_5 to $n-1$, we have

$$y_n < L \sum_{s=n_5}^{n-1} \frac{1}{c_s} \sum_{t=n_4}^{s-1} Q_t \sum_{\tau=t-\sigma}^{\infty} \frac{1}{c_\tau}$$

or equivalently

$$\Delta x_n < (L)^{\frac{1}{\gamma}} \left(\frac{1}{d_n}\right) \left[\sum_{s=n_5}^{n-1} \frac{1}{c_s} \sum_{t=n_4}^{s-1} Q_t \sum_{\tau=t-\sigma}^{\infty} \frac{1}{c_\tau}\right]^{\frac{1}{\gamma}}.$$

Summing from n_6 to $n-1$, we have

$$x_n < x_{n_6} + (L)^{\frac{1}{\gamma}} \sum_{u=n_6}^{n-1} \frac{1}{d_u} \left[\sum_{s=n_5}^{u-1} \frac{1}{c_s} \sum_{t=n_4}^{s-1} Q_t \sum_{\tau=t-\sigma}^{\infty} \frac{1}{c_\tau} \right]^{\frac{1}{\gamma}}.$$

By condition (44), we have $\lim_{n\to\infty} x_n = -\infty$ which contradicts the fact that $x_n > 0$. The proof is complete.

Theorem 16. *Let x_n be a solution of (21). Let ρ_n be a positive sequence. Assume that $(h_1) - (h_2)$, (23), (25) and (26) hold. If (44) holds and there exist double sequences $H_{m,n}$ and $h_{m,n}$ satisfy (29), then x_n either oscillates or satisfies $\lim_{n\to\infty} x_n = 0$.*

Proof. Suppose to the contrary that x_n is a nonoscillatory solution of equation (21). Without loss of generality we may assume that $x_n > 0$ and $x_{n-\sigma} > 0$ for $n \geq n_1$ where n_1 is chosen so large. Condition (25) implies that the solution x_n belongs to the space $C_0 \cup C_1 \cup C_2$. If $x_n \in C_0$, then we are back to the proof of Lemma 4 to show that $\lim_{n\to\infty} x_n = 0$. If $x_n \in C_1$, then we are back to the proof of Theorem 15 to get a contradiction. To complete the proof, it is sufficient to show that under condition (44) there is no solution $x_n \in C_1$. Thus, we proceed as in the proof of Theorem 15 to get a contradiction. The proof is complete.

The following results are an immediate consequences of Theorem 16.

Corollary 17. *Let x_n be solution of equation (21) and assume that all the assumptions of Theorem 16 hold, except that condition (29) is replaced by (41). Then x_n either oscillates or satisfies $\lim_{n\to\infty} x_n = 0$.*

Corollary 18. *Let x_n be a solution of (21) and assume that all the assumptions of Theorem 16 hold, except that the condition (29) is replaced by (42). Then x_n either oscillates or satisfies $\lim_{n\to\infty} x_n = 0$.*

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