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Existence of solutions for $p(x)$–Laplacian equations[1]

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Abstract

We discuss the problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \lambda(a(x)\,|u|^{q(x)-2}u + b(x)\,|u|^{h(x)-2}u), & \text{for } x\in\Omega, \\ u=0, & \text{for } x\in\partial\Omega. \end{cases}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$ $(N\geq 2)$ and p is Lipschitz continuous, q and h are continuous functions on $\overline{\Omega}$ such that $1<q(x)<p(x)<h(x)<p^*(x)$ and $p(x)<N$. We show the existence of at least one nontrivial weak solution. Our approach relies on the variable exponent theory of Lebesgue and Sobolev spaces combined with adequate variational methods and the Mountain Pass Theorem.

Keywords and Phrases: variable exponent Lebesgue and Sobolev spaces; $p(x)$-Laplacian; variational methods; Mountain Pass Theorem.

AMS Subject Classifications (2000): 35J60, 35B30, 35B40

1. Introduction

The study of partial differential equations and variational problems involving $p(x)$-growth conditions has captured special attention in the last decades. This is a consequence of the fact that such equations can be used to model phenomena which arise in mathematical physics, for example:

- Electrorheological fluids: see Acebri and Mingione [1], Zhikov [25] and Růžička [20], Fan and Zhang [12], Mihăilescu and Rădulescu [16], Chabrowski and Fu [7], Hästö [14], Diening [8].
- Nonlinear porous medium: see Antontsev and Rodrigues [2], Buhrii and Mashiyev [4], and Songzhe, Wenjie, Chunling and Hongjun [21].
- Image Processing: Chen, Levine and Rao [6].

A typical model of an elliptic equation with $p(x)$-growth conditions is

$$-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = f(x,u). \tag{1.1}$$

The operator $-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$ is called the $p(x)$-Laplace operator and it is a natural generalization of the $p-$Laplace operator, in which $p(x)\equiv p>1$ is a constant. The $p(x)$-Laplacian

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processes have more complicated nonlinearity, for example, it is nonhomogeneous, so in the discussions some special techniques will be needed.

Problems like (1.1) with Dirichlet boundary condition have been largely considered in the literature in the recent years. We give in what follows a concise but complete image of the actual stage of research on this topic. We will use the notations such as p_1 and p_2 where

$$p_1 := ess \inf_{x\in\Omega} p(x) \leq p(x) \leq p_2 := ess \sup_{x\in\Omega} p(x) < \infty.$$

In the case $f(x,u) = \lambda |u|^{p(x)-2} u$ in [13] the authors established the existence of infinitely many eigenvalues for problem (1.1) by using an argument based on the Ljusternik-Schnirelmann critical point theory. Denoting by Λ the set of all nonnegative eigenvalues, they showed that Λ is discrete, $\sup \Lambda = \infty$ and pointed out $\inf \Lambda = 0$ for general $p(x)$, and only under some special conditions $\inf \Lambda > 0$. In the case $f(x,u) = \lambda |u|^{q(x)-2} u$, there are different papers, for example, in [12] the same authors proved that any $\lambda > 0$ is an eigenvalue of problem (1.1) when $p_2 < q_1$ and also when $q_2 < p_1$. In [18] the authors proved the existence of a continuous family of eigenvalues which lies in a neighborhood of the origin when $q_1 < p_1$ and $q(x)$ has subcritical growth in problem (1.1).

In the case $f(x,u) = A|u|^{a-2} u + B|u|^{b-2} u$ with $1 < a < p_1 < p_2 < b < \min\{N, \frac{Np_1}{N-p_1}\}$ and $A, B > 0$, in [17] Mihăilescu show that there exists $\lambda > 0$ such that, for any $A, B \in (0, \lambda)$, problem (1.1) has at least two distinct nontrivial weak solutions.

The aim of this paper is to discuss the existence of a weak solution of the $p(x)$-Laplacian equation

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right) = \lambda(a(x)|u|^{q(x)-2} u + b(x)|u|^{h(x)-2} u), & \text{for } x \in \Omega, \\ u = 0, & \text{for } x \in \partial\Omega. \end{cases} \qquad (P_\lambda)$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary, λ is a positive real number, p is Lipschitz continuous on $\overline{\Omega}$, and $q, h \in C_+(\overline{\Omega})$, $a(x), b(x) > 0$ for $x \in \overline{\Omega}$ such that $a \in L^{\beta(x)}(\Omega)$, $\beta(x) = \frac{p(x)}{p(x)-q(x)}$ and $b \in L^{\gamma(x)}(\Omega)$, $\gamma(x) = \frac{p^*(x)}{p^*(x)-h(x)}$. Here $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ or $p^*(x) = \infty$ if $p(x) \geq N$.

In the present paper, assuming the condition

$$1 < q_1 \leq q_2 < p_1 \leq p_2 < h_1 \leq h_2 < p_1^* \text{ and } p_2 < N \qquad (1.2)$$

and using the the variable exponent theory of Lebesgue and Sobolev spaces combined with adequate variational methods and the Mountain Pass Theorem, we show the existence of at least one nontrivial weak solution of problem (P_λ).

2.Preliminaries

We recall in what follows some definitions and basic properties of variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. In that context we refer to [9, 10, 15] for the fundamental properties of these spaces.

Set

$$L_{+}^{\infty}(\Omega)=\left\{p;\ p\in L^{\infty}(\Omega),\underset{x\in\Omega}{ess\inf}\ p(x)>1\right\}.$$

For $p\in L_{+}^{\infty}(\Omega)$, we define the *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u:\Omega\to\mathbb{R}$ for which the modular

$$\rho_{p(x)}(u)=\int_{\Omega}|u(x)|^{p(x)}\,dx$$

is finite. We define the *Luxembourg norm* on this space by the formula

$$\|u\|_{p(x)}=\inf\left\{\delta>0:\rho_{p(x)}\left(\frac{u}{\delta}\right)\leq 1\right\}.$$

Equipped with this norm, $L^{p(\cdot)}(\Omega)$ is a separable and reflexive Banach space. Define the *variable exponent Sobolev space* $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega)=\{u\in L^{p(x)}(\Omega);\ |\nabla u|\in L^{p(x)}(\Omega)\},$$

and the norm

$$\|u\|_{1,p(x)}=\|u\|_{p(x)}+\|\nabla u\|_{p(x)},\forall u\in W^{1,p(x)}(\Omega)$$

makes $W^{1,p(x)}(\Omega)$ a separable and reflexive Banach space. The space $W_0^{1,p(x)}(\Omega)$ is denoted by the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(x)}(\Omega)$. $W_0^{1,p(x)}(\Omega)$ is a separable and reflexive Banach space.

Proposition 2.1 [10, 15] *The conjugate space of* $L^{p(x)}(\Omega)$ *is* $L^{q(x)}(\Omega)$, *where* $\frac{1}{p(x)}+\frac{1}{q(x)}=1$. *For any* $u\in L^{p(x)}(\Omega)$ *and* $v\in L^{q(x)}(\Omega)$, *we have*

$$\left|\int_{\Omega}uvdx\right|\leq\left(\frac{1}{p_1}+\frac{1}{q_1}\right)\|u\|_{p(x)}\|v\|_{q(x)}\leq 2\|u\|_{p(x)}\|v\|_{q(x)}.$$

The next proposition illuminates the close relation between the $\|\cdot\|_{p(x)}$ and the convex modular $\rho_{p(x)}$:

Proposition 2.2 [9, 10, 15] *If* $u\in L^{p(x)}(\Omega)$ *and* $p_2<\infty$ *then we have*

$$\begin{aligned}
i)\ \|u\|_{p(x)} &< 1\,(=1;>1)\Leftrightarrow\rho_{p(x)}(u)<1\,(=1;>1),\\
ii)\ \|u\|_{p(x)} &> 1\implies\|u\|_{p(x)}^{p_1}\leq\rho_{p(x)}(u)\leq\|u\|_{p(x)}^{p_2},\\
iii)\ \|u\|_{p(x)} &< 1\implies\|u\|_{p(x)}^{p_2}\leq\rho_{p(x)}(u)\leq\|u\|_{p(x)}^{p_1},\\
iv)\ \|u\|_{p(x)} &= a>0\iff\rho_{p(x)}\left(\frac{u}{a}\right)=1
\end{aligned}$$

Proposition 2.3 [9, 10, 15] *If* $u,u_n\in L^{p(x)}(\Omega)$, $n=1,2,...$, *then the following statements are equivalent*

(1) $\lim_{n\to\infty}\|u_n-u\|_{p(x)}=0$;

(2) $\lim_{n\to\infty}\rho_{p(x)}(u_n-u)=0$;

(3) $u_n\to u$ *in measure in* Ω *and* $\lim_{n\to\infty}\rho_{p(x)}(u_n)=\rho_{p(x)}(u)$.

Lemma 2.4 [5] *Assume that* $r \in L^{\infty}_{+}(\Omega)$ *and* $p \in C_+(\overline{\Omega}) := \left\{ m \in C(\overline{\Omega}) : m_1 > 1 \right\}$. *If* $|u|^{r(x)} \in L^{p(x)}(\Omega)$*, then we have*

$$\min\left\{ \|u\|^{r_1}_{r(x)p(x)}, \|u\|^{r_2}_{r(x)p(x)} \right\} \le \left\| |u|^{r(x)} \right\|_{p(x)} \le \max\left\{ \|u\|^{r_1}_{r(x)p(x)}, \|u\|^{r_2}_{r(x)p(x)} \right\}.$$

Remark 2.5 If $r(x) \equiv r$, $r \in \mathbb{R}$ then

$$\| |u|^r \|_{p(x)} = \|u\|^r_{rp(x)}.$$

Given two Banach spaces X and Y, the symbol $X \hookrightarrow Y$ means that X is *continuously* imbedded in Y and the symbol $X \hookrightarrow\hookrightarrow Y$ means that there is a *compact* embedding of X in Y.

Proposition 2.6 [9, 10, 11, 15] *Assume that* Ω *is bounded and smooth.*

(i) Let $q, h \in C_+(\overline{\Omega})$*. If* $q(x) \le h(x)$ *for all* $x \in \overline{\Omega}$*, then* $L^{h(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

(ii) Let p *is Lipschitz continuous and* $p_2 < N$*, then for* $h \in L^{\infty}_{+}(\Omega)$ *with* $p(x) \le h(x) \le p^*(x)$ *there is a continuous imbedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{h(x)}(\Omega)$, *and also there is a constant* $C_1 > 0$ *such that* $\|u\|_{h(x)} \le C_1 \|u\|_{1,p(x)}$.

(iii) Let $p, q \in C_+(\overline{\Omega})$*. If* $p(x) \le q(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$*, then* $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$.

(iv) (Poincaré inequality) If $p \in C_+(\overline{\Omega})$*, then there is a constant* $C_2 > 0$ *such that*

$$\|u\|_{p(x)} \le C_2 \| |\nabla u| \|_{p(x)}, \ \forall\, u \in W^{1,p(x)}_0(\Omega).$$

Consequently, $\|u\| := \| |\nabla u| \|_{p(x)}$ *and* $\|u\|_{1,p(x)}$ *are equivalent norms on* $W^{1,p(x)}_0(\Omega)$*. In what follows,* $W^{1,p(x)}_0(\Omega)$*, with* $p \in C_+(\overline{\Omega})$*, will be considered as endowed with the norm* $\|u\|_{1,p(x)}$. We will use $\|u\| = \|\nabla u\|_{p(x)}$ for $u \in W^{1,p(x)}_0(\Omega)$ in the following discussions.

Finally, we introduce Mountain-Pass Theorem which is the main tool of the present paper.

Palais-Smale condition [24] Let E be a Banach space and $I \in C^1(E, \mathbb{R})$. If $\{u_n\} \subset E$ is a sequence which satisfies conditions

$$|I_\lambda(u_n)| < M,$$

$$I'_\lambda(u_n) \to 0 \text{ as } n \to \infty \text{ in } E^*$$

where M is a positive constant and E^* is the dual space of E, then $\{u_n\}$ possesses a convergent subsequence.

Mountain-Pass Theorem [24] *Let* E *be a Banach space, and let* $I \in C^1(E, \mathbb{R})$ *satisfy the Palais-Smale condition. Assume that* $I(0) = 0$, *and there exists a positive real number* ρ *and* $u, \upsilon \in E$ *such that*

(i) $\|\upsilon\| > \rho$, $I(\upsilon) \le I(0)$.

(ii) $\alpha = \inf\{I(u) : u \in E, \ \|u\| = \rho\} > 0$.

Put $G = \{g \in C([0,1], E) : g(0) = 0, \ g(1) = \upsilon\} \ne \varnothing$*. Set* $\beta = \inf_{g \in G} \sup_{t \in [0,1]} I(g(t))$.

Then, $\beta \geq \alpha$ and β is a critical value of I.

3. Main Results

The energy functional corresponding to problem (P_λ) is defined as $J_\lambda : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$,

$$J_\lambda(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx - \lambda \int_\Omega \frac{a(x)}{q(x)} |u|^{q(x)} dx - \lambda \int_\Omega \frac{b(x)}{h(x)} |u|^{h(x)} dx. \tag{3.1}$$

We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution for problem (P_λ) provided

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx = \lambda \int_\Omega a(x) |u|^{q(x)-2} uv dx + \lambda \int_\Omega b(x) |u|^{h(x)-2} uv dx$$

for all $v \in W_0^{1,p(x)}(\Omega)$.

Standard arguments imply that $J_\lambda \in C^1\left(W_0^{1,p(x)}(\Omega), \mathbb{R}\right)$ with

$$\langle J'_\lambda(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx - \lambda \int_\Omega a(x) |u|^{q(x)-2} uv dx - \lambda \int_\Omega b(x) |u|^{h(x)-2} uv dx \tag{3.2}$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$. Thus the weak solution of (P_λ) are exactly the critical points of J_λ.

The main result of the present paper is the following theorem.

Theorem 3.1 *Assume p is Lipschitz continuous, $q, h \in C_+(\overline{\Omega})$ and condition (1.2) is fulfilled. If*

$$\lambda \in \left(0, \min\left\{\frac{q_1 \rho^{p_2-q_1}}{4C_1^{q_1} p_2 \|a\|_{\beta(x)}}, \frac{h_1 \rho^{p_2-h_1}}{4C_1^{h_1} p_2 \|b\|_{\gamma(x)}}\right\}\right),$$

then the problem (P_λ) has at least one nontrivial solution, where $\rho \in (0,1)$.

To obtain the proof of Theorem 3.1, we use Mountain-Pass theorem. Therefore, we must show J_λ satisfies Palais-Smale condition in the first place.

Lemma 3.2 *Let λ satisfies the condition of Theorem 3.1. If $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ is a sequence which satisfies conditions*

$$|J_\lambda(u_n)| < M, \tag{3.3}$$

$$J'_\lambda(u_n) \to 0 \text{ as } n \to \infty \text{ in } \left(W_0^{1,p(x)}(\Omega)\right)^* \tag{3.4}$$

where M is a positive constant, then $\{u_n\}$ possesses a convergent subsequence.

Proof: First, we show that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Assume the contrary. Then, passing to a subsequence if necessary, we may assume that $\|u_n\| \to \infty$ as $n \to \infty$. Thus, we may consider that $\|u_n\| > 1$ for any integer n. By (3.4) we deduce that there exists $N_1 > 0$ such that for any $n > N_1$, we have

$$\|J'_\lambda(u_n)\| \leq 1.$$

On the other hand, for any $n > N_1$ fixed, the application

$$W_0^{1,p(x)}(\Omega) \ni \upsilon \to \langle J'_\lambda(u_n), \upsilon\rangle$$

is linear and continuous. The above information implies

$$|\langle J'_\lambda(u_n), \upsilon\rangle| \leq \|J'_\lambda(u_n)\|_{W_0^{-1,p'(x)}(\Omega)} \|\upsilon\| \leq \|\upsilon\|, \forall \upsilon \in W_0^{1,p(x)}(\Omega), \; n > N_1.$$

Setting $\upsilon = u_n$ we have

$$-\|u_n\| \leq \int_\Omega |\nabla u_n|^{p(x)} dx - \lambda \int_\Omega a(x)|u_n|^{q(x)} dx - \lambda \int_\Omega b(x)|u_n|^{h(x)} dx \leq \|u_n\|$$

for any $n > N_1$.

Using the assumption $\|u_n\| > 1$, relations (3.3), (3.4), Proposition 2.1, Lemma 2.4 and Proposition 2.6 (ii) we have

$$\begin{aligned} M &> J_\lambda(u_n) - \frac{1}{h_1}\langle J'_\lambda(u_n), u_n\rangle \\ &= \int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)} dx - \lambda \int_\Omega \frac{a(x)}{q(x)}|u_n|^{q(x)} dx - \lambda \int_\Omega \frac{b(x)}{h(x)}|u_n|^{h(x)} dx \\ &\quad -\frac{1}{h_1}\int_\Omega |\nabla u_n|^{p(x)} dx + \frac{\lambda}{h_1}\int_\Omega a(x)|u_n|^{q(x)} dx + \frac{\lambda}{h_1}\int_\Omega b(x)|u_n|^{h(x)} dx \\ &\geq \frac{1}{p_2}\int_\Omega |\nabla u_n|^{p(x)} dx - \frac{\lambda}{q_1}\int_\Omega a(x)|u_n|^{q(x)} dx - \frac{\lambda}{h_1}\int_\Omega b(x)|u_n|^{h(x)} dx \\ &\quad -\frac{1}{h_1}\int_\Omega |\nabla u_n|^{p(x)} dx + \frac{\lambda}{h_1}\int_\Omega a(x)|u_n|^{q(x)} dx + \frac{\lambda}{h_1}\int_\Omega b(x)|u_n|^{h(x)} dx \\ &\geq \left(\frac{1}{p_2} - \frac{1}{h_1}\right)\|u_n\|^{p_1} - \lambda\left(\frac{1}{h_1} - \frac{1}{q_1}\right) C_3 \|a\|_{\beta(x)} \|u_n\|^{q_2}, \end{aligned} \tag{3.5}$$

where $C_3 > 0$ is a constant independent of u_n and x, for n large enough. Dividing (3.5) by $\|u_n\|^{p_1}$ and passing to the limit as $n \to \infty$ we obtain $\frac{1}{p_2} - \frac{1}{h_1} < 0$.

Since $q_1 \leq q_2 < p_1 \leq p_2 < h_1$, this is a contradiction. It follows $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$.)

Next, we show the strong convergence of $\{u_n\}$ in $W_0^{1,p(x)}(\Omega)$. Since $\{u_n\}$ is bounded, up to a subsequence (which we still denote by $\{u_n\}$), we may assume that there exists $u \in W_0^{1,p(x)}(\Omega)$ such that

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,p(x)}(\Omega) \quad \text{as } n \to \infty.$$

By Proposition 2.6 (iii) we obtain

$$u_n \to u \text{ strongly in } L^{p(x)}(\Omega) \quad \text{as } n \to \infty. \tag{3.6}$$

Furthermore, from [3, 23] we have

$$u_n \to u \text{ strongly in } L^{p^*(x)}(K) \quad \text{as } n \to \infty, \tag{3.7}$$

where K is compact subset of Ω.

The above information and relation (3.4) imply

$$\langle J'_\lambda(u_n) - J'_\lambda(u), u_n - u\rangle \to 0 \text{ as } n \to \infty.$$

On the other hand, we have

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)(\nabla u_n - \nabla u)\,dx \\
=\;& \langle J'_\lambda(u_n) - J'_\lambda(u), u_n - u\rangle - \lambda\int_\Omega a(x)\left(|u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u\right)(u_n - u)\,dx \\
&-\lambda\int_\Omega b(x)\left(|u_n|^{h(x)-2}u_n - |u|^{h(x)-2}u\right)(u_n - u)\,dx.
\end{aligned}$$

Propositions 2.1, 2.3 and Lemma 2.4 we have

$$\begin{aligned}
&\lambda\left|\int_\Omega a(x)\left(|u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u_1\right)(u_n - u)\,dx\right| \\
\leq\;& \lambda\left|\int_\Omega a(x)|u_n|^{q(x)-2}u_n(u_n - u)\,dx\right| + \lambda\left|\int_\Omega a(x)|u_1|^{q(x)-2}u(u_n - u)\,dx\right| \\
\leq\;& C_4\|a\|_{\beta(x)}\left\||u_n|^{q(x)-1}\right\|_{\frac{p(x)}{q(x)-1}}\|u_n - u\|_{p(x)} + C_5\|a\|_{\beta(x)}\left\||u|^{q(x)-1}\right\|_{\frac{p(x)}{q(x)-1}}\|u_n - u\|_{p(x)} \\
\leq\;& C_4\|a\|_{\beta(x)}\|u_n\|_{p(x)}^{q_2-1}\|u_n - u\|_{p(x)} + C_5\|a\|_{\beta(x)}\|u\|_{p(x)}^{q_2-1}\|u_n - u\|_{p(x)},
\end{aligned}$$

where C_4, $C_5 > 0$ and $\frac{1}{\beta(x)} + \frac{q(x)-1}{p(x)} + \frac{1}{p(x)} = 1$.

Similarly, Propositions 2.1, 2.3 and Lemma 2.4 we have

$$\begin{aligned}
&\lambda\left|\int_\Omega b(x)\left(|u_n|^{h(x)-2}u_n - |u|^{h(x)-2}u\right)(u_n - u)\,dx\right| \\
\leq\;& C_6\|b\|_{\gamma(x)}\left\||u_n|^{h(x)-1}\right\|_{\frac{p^*(x)}{h(x)-1}}\|u_n - u\|_{p^*(x)} + C_7\|b\|_{\gamma(x)}\left\||u|^{h(x)-1}\right\|_{\frac{p^*(x)}{h(x)-1}}\|u_n - u\|_{p^*(x)} \\
\leq\;& C_6\|b\|_{\gamma(x)}||u_n||_{p^*(x)}^{h_2-1}\|u_n - u\|_{p^*(x)} + C_7\|b\|_{\gamma(x)}\|u\|_{p^*(x)}^{h_2-1}\|u_n - u\|_{p^*(x)},
\end{aligned}$$

where C_6, $C_7 > 0$ and $\frac{1}{\gamma(x)} + \frac{h(x)-1}{p^*(x)} + \frac{1}{p^*(x)} = 1$. By (3.6) and (3.7) we have

$$\|u_n - u\|_{p(x)} \to 0 \text{ as } n \to \infty$$

and

$$\|u_n - u\|_{p^*(x)} \to 0 \text{ as } n \to \infty.$$

Therefore, from above inequalities we deduce that

$$\lim_{n\to\infty} \int_\Omega b(x) \left(|u_n|^{h(x)-2} u_n - |u|^{h(x)-2} u \right) (u_n - u) = 0, \tag{3.8}$$

and

$$\lim_{n\to\infty} \int_\Omega a(x) \left(|u_n|^{q(x)-2} u_n - |u|^{q(x)-2} u \right) (u_n - u)\, dx = 0, \tag{3.9}$$

respectively. By (3.8) and (3.9) we obtain

$$\lim_{n\to\infty} \int_\Omega \left(|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u \right) (\nabla u_n - \nabla u) = 0. \tag{3.10}$$

This result and the following inequality [23, Lemma 2.2]

$$\left(|\xi|^{r-2} \xi - |\eta|^{r-2} \eta \right) (\xi - \eta) \geq 2^{-r} |\xi - \eta|, \; \forall r \geq 2; \; \xi, \eta \in \mathbb{R}^N. \tag{3.11}$$

yield

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n - \nabla u|^{p(x)}\, dx = 0. \tag{3.12}$$

This fact and Proposition 2.3 imply $\|\nabla u_n - \nabla u\|_{p(x)} \to 0$ as $n \to \infty$. Relation (3.12) and fact that $u_n \rightharpoonup u$ (weakly) in $W_0^{1,p(x)}(\Omega)$ enable us to apply [12] in order to obtain that $u_n \rightharpoonup u$ (strongly) in $W_0^{1,p(x)}(\Omega)$. Thus, Lemma 3.2 is proved.

Now, we show that the Mountain-Pass theorem can be applied in this case.

Lemma 3.3 *Assume $p, q, h \in C_+(\overline{\Omega})$ and condition (1.2) is fulfilled. The following assertions hold.*

(i) There exist $\lambda > 0$, $\alpha > 0$ and $\rho \in (0,1)$ such that

$$J_\lambda(u) \geq \alpha, \; \forall u \in W_0^{1,p(x)}(\Omega) \text{ with } \|u\| = \rho. \tag{3.13}$$

(ii) There exists $\omega \in W_0^{1,p(x)}(\Omega)$ such that

$$\lim_{t\to\infty} J_\lambda(t\omega) = -\infty. \tag{3.14}$$

(iii) There exists $\varphi \in W_0^{1,p(x)}(\Omega)$ such that $\varphi \geq 0, \varphi \neq 0$ and

$$J_\lambda(t\varphi) < 0, \tag{3.15}$$

for $t > 0$ small enough.

Proof. (i) Using Propositions 2.1, 2.2, 2.6 (ii) and Lemma 2.4 we deduce that for any $u \in W_0^{1,p(x)}(\Omega)$ with $\rho \in (0,1)$ we have

$$\begin{aligned}
J_\lambda(u) &\geq \frac{1}{p_2}\int_\Omega |\nabla u|^{p(x)}\,dx - \frac{\lambda}{q_1}\int_\Omega a(x)\,|u|^{q(x)}\,dx - \frac{\lambda}{h_1}\int_\Omega b(x)\,|u|^{h(x)}\,dx \\
&\geq \frac{1}{p_2}\|u\|^{p_2} - \frac{\lambda}{q_1}\|a\|_{\beta(x)}\left\||u|^{q(x)}\right\|_{\frac{p(x)}{q(x)}} - \frac{\lambda}{h_1}\|b\|_{\gamma(x)}\left\||u|^{h(x)}\right\|_{\frac{p^*(x)}{h(x)}} \\
&\geq \frac{1}{p_2}\|u\|^{p_2} - \frac{\lambda}{q_1}\|a\|_{\beta(x)}\ \|u\|_{p(x)}^{q_1} - \frac{\lambda}{h_1}\|b\|_{\gamma(x)}\|u\|_{p^*(x)}^{h_1} \\
&\geq \frac{1}{p_2}\|u\|^{p_2} - C_1^{q_1}\frac{\lambda}{q_1}\|a\|_{\beta(x)}\ \|u\|^{q_1} - C_1^{h_1}\frac{\lambda}{h_1}\|b\|_{\gamma(x)}\|u\|^{h_1}.
\end{aligned}$$

Taking

$$\lambda = \min\left\{\frac{q_1\rho^{p_2-q_1}}{4C_1^{q_1}p_2\|a\|_{\beta(x)}}, \frac{h_1\rho^{p_2-h_1}}{4C_1^{h_1}p_2\|b\|_{\gamma(x)}}\right\}$$

we obtain

$$J_\lambda(u) \geq \frac{\rho^{p_2}}{2p_2},\ \forall u \in W_0^{1,p(x)}(\Omega)\ \text{ with }\ \|u\| = \rho.$$

Thus Lemma 3.3 (i) is proved.

(ii) Let $\omega \in C_0^\infty(\Omega),\ \omega \geq 0,\ \omega \neq 0$ and $t > 1$.We have

$$\begin{aligned}
J_\lambda(t\omega) &= \int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla\omega|^{p(x)}\,dx - \lambda\int_\Omega \frac{t^{q(x)}}{q(x)}a(x)\,|\omega|^{q(x)}\,dx - \lambda\int_\Omega \frac{t^{h(x)}}{h(x)}b(x)\,|\omega|^{h(x)}\,dx \\
&\leq \frac{t^{p_2}}{p_1}\int_\Omega |\nabla\omega|^{p(x)}\,dx - \lambda\frac{t^{q_2}}{q_1}\int_\Omega a(x)\,|\omega|^{q(x)}\,dx - \lambda\frac{t^{h_2}}{h_1}\int_\Omega b(x)\,|\omega|^{h(x)}\,dx.
\end{aligned}$$

Since $q_2,\ p_2 < h_2$ we have $J_\lambda(t\omega) \to -\infty$. Thus Lemma 3.3 (ii) is proved.

(iii) Let $\varphi \in C_0^\infty(\Omega), \varphi \geq 0, \varphi \neq 0$ and $t \in (0,1)$. We have

$$J_\lambda(t\varphi) \leq \frac{t^{p_1}}{p_1}\int_\Omega |\nabla\varphi|^{p(x)}\,dx - \lambda\frac{t^{q_1}}{q_1}\int_\Omega a(x)\,|\varphi|^{q(x)}\,dx - \lambda\frac{t^{h_1}}{h_1}\int_\Omega b(x)\,|\varphi|^{h(x)}\,dx.$$

Since

$$\begin{aligned}
&\frac{\lambda t^{q_1}}{q_1}\int_\Omega a(x)\,|\varphi|^{q(x)}\,dx + \frac{\lambda t^{h_1}}{h_1}\int_\Omega b(x)\,|\varphi|^{h(x)}\,dx \\
< \ &\frac{\lambda t^{q_1}}{q_1}\left(\int_\Omega a(x)\,|\varphi|^{q(x)}\,dx + \int_\Omega b(x)\,|\varphi|^{h(x)}\,dx\right),
\end{aligned}$$

we have

$$J_\lambda(t\varphi) \le \frac{t^{p_1}}{p_1}\int_\Omega |\nabla\varphi|^{p(x)}\,dx - \frac{\lambda t^{q_1}}{q_1}\left(\int_\Omega a(x)\,|\varphi|^{q(x)}\,dx + \int_\Omega b(x)\,|\varphi|^{h(x)}\,dx\right) < 0,$$

for $t < \delta^{\frac{1}{p_1-q_1}}$ with

$$0 < \delta < \min\left\{1, \frac{\lambda p_1\left(\int_\Omega a(x)\,|\varphi|^{q(x)}\,dx + \int_\Omega b(x)\,|\varphi|^{h(x)}\,dx\right)}{q_1\int_\Omega |\nabla\varphi|^{p(x)}\,dx}\right\}.$$

Lemma 3.3 (iii) is proved.

Proof of Theorem 3.1 We set

$$G = \left\{g \in C([0,1], W_0^{1,p(x)}(\Omega)) : g(0) = 0,\ g(1) = \upsilon\right\},$$

where $\upsilon \in W_0^{1,p(x)}(\Omega)$ is determined by Lemma 3.3 (ii) and (iii), and

$$\beta := \inf_{g\in G}\sup_{t\in[0,1]} J_\lambda(g(t)).$$

According to Lemma 3.3 (ii) and (iii), we know that $\|\upsilon\| > \rho$, so every path $g \in G$ intersects the sphere $\|\upsilon\| = \rho$. Then Lemma 3.3 (i) implies

$$\alpha \le \inf_{\|u\|=\rho} J_\lambda(u) \le \beta,$$

with the constant $\alpha > 0$ in Lemma 3.3 (i), thus $\beta > 0$. By the Mountain-Pass theorem J_λ admits a critical value $\beta \ge \alpha$.

Since $J_\lambda \in C^1\left(W_0^{1,p(x)}(\Omega), \mathbb{R}\right)$, from Lemma 3.2 we conclude

$$J'_\lambda(u_n) \to J'_\lambda(u) \tag{3.16}$$

as $n \to \infty$.

Relations (3.3), $(3,4)$ and (3.16) show that $J'_\lambda(u) = 0$ and thus u is a weak solutions for problem (P_λ). Moreover, by relations (3.3), $(3,4)$ it follows that $J_\lambda(u) > 0$ and thus, u is a nontrivial weak solutions for problem (P_λ). The proof is completed.

Remark 3.4 If (u, λ) is a solution of (P_λ) and $u \ne 0$, as usual, we call λ and u an eigenvalue eigenfunction corresponding to λ of (P_λ), respectively. If (u, λ) is a solution of (P_λ) and $u \ne 0$, then

$$\lambda = \lambda(u) = \frac{\int_\Omega |\nabla u|^{p(x)}\,dx}{\int_\Omega (a(x)\,|u|^{q(x)} + b(x)\,|u|^{h(x)})dx}$$

and hence $\lambda > 0$.

Theorem 3.1 ensures that problem (P_λ) has a continuous family of positive eigenvalues that lie in a neighborhood of the origin. Furthermore, we obtain

$$\inf_{u\in W_0^{1,p(x)}\setminus\{0\}} \frac{\int\limits_\Omega |\nabla u|^{p(x)}\,dx}{\int\limits_\Omega (a(x)\,|u|^{q(x)} + b(x)\,|u|^{h(x)})dx} = 0.$$

Remark 3.5 Furthermore, we can consider the equation

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\,\nabla u\right) = \lambda(a\,(x)\,|u|^{q(x)-2}\,u - b(x)\,|u|^{h(x)-2}\,u), & \text{for } x\in\Omega, \\ u = 0, & \text{for } x\in\partial\Omega. \end{cases} \qquad (P_\lambda^{/})$$

where p, q, h, a, b and λ are the same in problem (P_λ).

The energy functional corresponding to problem $(P_\lambda^{/})$ is defined as $I_\lambda : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$,

$$I_\lambda(u) = \int\limits_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx - \lambda\int\limits_\Omega \frac{a(x)}{q(x)}\,|u|^{q(x)}\,dx + \lambda\int\limits_\Omega \frac{b(x)}{h(x)}\,|u|^{h(x)}\,dx.$$

We infer that for any $x \in \Omega$ and $u \in W_0^{1,p(x)}(\Omega)$

$$\begin{aligned} & \frac{\lambda a(x)}{q(x)}\,|u(x)|^{q(x)} - \frac{\lambda b(x)}{h(x)}\,|u(x)|^{h(x)} \\ \le\;& \frac{\lambda a_2}{q_1}\,|u(x)|^{q(x)} - \frac{\lambda b_1}{h_2}\,|u(x)|^{h(x)} \\ \le\;& \frac{\lambda a_2}{q_1}\left(\frac{a_2 h_2}{b_1 q_1}\right)^{\frac{q(x)}{h(x)-q(x)}} \\ \le\;& \frac{\lambda a_2}{q_1}\left[\left(\frac{a_2 h_2}{b_1 q_1}\right)^{\frac{q_2}{h_1-q_2}} + \left(\frac{a_2 h_2}{b_1 q_1}\right)^{\frac{q_1}{h_2-q_1}}\right] := A \end{aligned} \qquad (3.17)$$

where A is a positive constant independent of u and x, by using the following elementary inequality [19, Lemma 4]

$$at^k - bt^s \le a\left(\frac{a}{b}\right)^{\frac{k}{s-k}}, \quad \text{for all } t \ge 0$$

where $a, b > 0$ and $0 < k < s$.

Integrating (3.17) over Ω, we get

$$\lambda\int\limits_\Omega \frac{a(x)}{q(x)}\,|u(x)|^{q(x)}\,dx - \lambda\int\limits_\Omega \frac{b(x)}{h(x)}\,|u(x)|^{h(x)}\,dx \le D$$

where D is a positive constant independent of u. Thus,

$$
\begin{aligned}
J_\lambda(u) &\geq \frac{1}{p_2}\int_\Omega |\nabla u|^{p(x)}\,dx - \lambda\int_\Omega \frac{a(x)}{q(x)}\,|u|^{q(x)}\,dx + \lambda\int_\Omega \frac{b(x)}{h(x)}\,|u|^{h(x)}\,dx \\
&\geq \frac{1}{p_2}\,\|u\|^{p_1} - D,
\end{aligned}
$$

for all $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\| > 1$. We infer that $J_\lambda(u) \to \infty$ as $\|u\| \to \infty$. Therefore the energy functional I_λ is coercive on $W_0^{1,p(x)}(\Omega)$. Moreover, a similar argument as the one used in the proof of [16, Lemma 3.4] shows that I_λ is also weakly lower semi-continuous in $W_0^{1,p(x)}(\Omega)$. These facts enable us to apply [22, Theorem 1.2] in order to find that there exits $u_\lambda \in W_0^{1,p(x)}(\Omega)$ a global minimizer of I_λ and thus, a weak solution of problem $(P_\lambda^{/})$.

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References

[1] E. Acerbi and G. Mingione, Regularity results for stationary electrorheological fluids, Arch. Ration. Mech. Anal. **164** (2002), 213-259. MR1930392 (2003g:35020).

[2] S. N. Antontsev and J. F. Rodrigues, On stationary thermo-rheological viscous flows, Ann. Univ. Ferrara, Sez., Sci. Mat. **52** (2006), pp. 19–36.

[3] J. F. Bonder and A. Silva, The concentration-compactness principle for variable exponent spaces and application, (2009) arXiv:0906.1922v2.

[4] O. M. Buhrii and R. A. Mashiyev, Uniqueness of solutions of the parabolic variation inequality with variable exponent of nonlinearity, Nonlinear Anal., **70** (2009), 2325–2331.

[5] B. Cekic and R. A. Mashiyev, Existence and localization results for $p(x)-$Laplacian via topological methods, Fixed Point Theory and Applications, Volume 2010 (2010), Article ID 120646, 7 pages, doi:10.1155/2010/120646.

[6] Y. Chen, S. Levine and M. Rao, Variable exponent, linear growth functionals in image restoration, SIAM J. Appl. Math. **66** (4) (2006), pp. 1383–1406.

[7] J. Chabrowski and Y. Fu, Existence of solutions for $p(x)$-Laplacian problems on a bounded domain, J. Math. Anal. Appl. **306** (2005), 604-618. MR2136336 (2006e:35087).

[8] L. Diening, *Theoretical and Numerical Results for Electrorheological Fluids*, Ph.D. thesis, University of Freiburg, Germany, 2002.

[9] D. Edmunds and J. Răkosnik, Sobolev embeddings with variable exponent, Studia Math. **143** (2000), 267-293.

[10] X. L. Fan and D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl. **263** (2001), 424–446.

[11] X. Fan, J. Shen and D. Zhao, Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$, J. Math. Anal. Appl. **262** (2001), 749-760.

[12] X. L. Fan and Q. H. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problems, Nonlinear Anal. **52** (2003),1843-1852. MR1954585 (2004f:35060).

[13] X. L. Fan, Q.H. Zhang, D. Zhao, Eigenvalues of $p(x)$-Laplacian Dirichlet problem, J. Math. Anal. Appl. **302** (2005) 306–317.

[14] P. Hästö, The $p(x)$-Laplacian and applications, J. Anal. **15** (2007), 53-62 (Special proceedings volume).

[15] O. Kováčik and J. Rákosnik, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J. **41** (**116**) (1991), 592–618.

[16] M. Mihăilescu and V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **462** (2006), 2625-2641.

[17] M. Mihăilescu, Existence and multiplicity of solutions for an elliptic equation with $p(x)-$growth condition, Glasgow Math. J. **48** (2006) 411-418.

[18] M. Mihăilescu and V. Rădulescu, On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, Proc. Amer. Math. Soc. **135** (2007) 2929–2937.

[19] M. Mihăilescu, On a class of nonlinear problems involving a $p(x)$-Laplace type operator, Czechoslovak Math. J. **58** (2008), 155-172.

[20] M. Růžička, Electrorheological Fluids: Modeling and Mathematical Theory, Lecture Notes in Mathematics, 1748, Springer-Verlag, Berlin, 2000. MR1810360 (2002a:76004).

[21] L. Songzhe, G. Wenjie, C. Chunling and Y. Hongjun, Study of the solutions to a model porous medium equation with variable exponent of nonlinearity. J. Math. Anal. Appl.**342** (2008), 27-38.

[22] M. Struwe, Variational Methods: *Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer, Heidelberg, 1996.

[23] F. Yongqiang, The principle of concentration compactness in $L^{p(x)}$ spaces and its application, Nonlinear Analysis (2009), doi:10.1016/j.na.2009.01.023.

[24] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.

[25] V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), 675-710. MR0864171 (88a:49026).

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