

**ON THE EXISTENCE OF MILD SOLUTIONS TO SOME
SEMILINEAR FRACTIONAL INTEGRO-DIFFERENTIAL
EQUATIONS**

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ABSTRACT. This paper deals with the existence of a mild solution for some fractional semilinear differential equations with non local conditions. Using a more appropriate definition of a mild solution than the one given in [12], we prove the existence and uniqueness of such solutions, assuming that the linear part is the infinitesimal generator of an analytic semigroup that is compact for $t > 0$ and the nonlinear part is a Lipschitz continuous function with respect to the norm of a certain interpolation space. An example is provided.

1. INTRODUCTION

Let \mathbb{X} be a Banach space and let $T > 0$. This paper is aimed at discussing about the existence and the uniqueness of a mild solution for the fractional semilinear integro-differential equation with nonlocal conditions in the form:

$$(1) \begin{cases} D^\beta x(t) = -Ax(t) + f(t, x(t)) + \int_0^t a(t-s)h(s, x(s)) ds, & t \in [0, T], \\ x(0) + g(x) = x_0, \end{cases}$$

where the fractional derivative D^β ($0 < \beta < 1$) is understood in the Caputo sense, the linear operator $-A$ is the infinitesimal generator of an analytic semigroup $(R(t))_{t \geq 0}$ that is uniformly bounded on \mathbb{X} and compact for $t > 0$, the function $a(\cdot)$ is real-valued such that

$$(2) \quad a_T = \int_0^T a(s) ds < \infty,$$

the functions f, g and h are continuous, and the non local condition

$$g(x) = \sum_{k=1}^p c_k x(t_k),$$

with $c_k, k = 1, 2, \dots, p$, are given constants and $0 < t_1 < t_2 < \dots < t_p \leq T$.

Let us recall that those nonlocal conditions were first utilized by K. Deng [4]. In his paper, K. Deng indicated that using the nonlocal condition $x(0) + g(x) = x_0$

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to describe for instance, the diffusion phenomenon of a small amount of gas in a transparent tube can give better result than using the usual local Cauchy Problem $x(0) = x_0$. Let us observe also that since Deng's paper, such problem has attracted several authors including A. Aizicovici, L. Byszewski, K. Ezzinbi, Z. Fan, J. Liu, J. Liang, Y. Lin, T.-J. Xiao, H. Lee, etc. (see for instance [1, 2, 3, 4, 9, 8, 7, 14, 11, 13] and the references therein).

This problem has been studied in Mophou and N'Guérékata [12]. In this paper, we revisit that work and use a more appropriate definition for mild solutions. Namely, we investigate the existence and the uniqueness of a mild solution for the fractional semilinear differential equation (1), assuming that f is defined on $[0, T] \times \mathbb{X}_\alpha \times \mathbb{X}_\alpha$ where $\mathbb{X}_\alpha = D(A^\alpha)$ ($0 < \alpha < 1$), the domain of the fractional powers of A .

The rest of this paper is organized as follows. In Section 2 we give some known preliminary results on the fractional powers of the generator of an analytic compact semigroup. In Section 3, we study the existence and the uniqueness of a mild solution for the fractional semilinear differential equation (1). We give an example to illustrate our abstract results.

2. PRELIMINARIES

Let $I = [0, T]$ for $T > 0$ and let \mathbb{X} be a Banach space with norm $\|\cdot\|$. Let $(\mathbb{B}(\mathbb{X}), \|\cdot\|_{\mathbb{B}(\mathbb{X})})$ be the Banach space of all linear bounded operators on \mathbb{X} and $A : D(A) \rightarrow \mathbb{X}$ be a linear operator such that $-A$ is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operators $(R(t))_{t \geq 0}$, which is compact for $t > 0$. In particular, this means that there exists $M > 1$ such that

$$(3) \quad \sup_{t > 0} \|R(t)\|_{\mathbb{B}(\mathbb{X})} \leq M.$$

Moreover, we assume without loss of generality that $0 \in \rho(A)$. This allows us to define the fractional power A^α for $0 < \alpha < 1$, as a closed linear operator on its domain $D(A^\alpha)$ with inverse $A^{-\alpha}$ (see [8]). We have the following basic properties for fractional powers A^α of A :

Theorem 2.1. ([15], pp. 69 -75). *Under previous assumptions, then:*

- (i) $\mathbb{X}_\alpha = D(A^\alpha)$ is a Banach space with the norm $\|x\|_\alpha := \|A^\alpha x\|$ for $x \in D(A^\alpha)$;
- (ii) $R(t) : \mathbb{X} \rightarrow \mathbb{X}_\alpha$ for each $t > 0$;
- (iii) $A^\alpha R(t)x = R(t)A^\alpha x$ for each $x \in D(A^\alpha)$ and $t \geq 0$;

(iv) For every $t > 0$, $A^\alpha R(t)$ is bounded on \mathbb{X} and there exist $M_\alpha > 0$ and $\delta > 0$ such that

$$(4) \quad \|A^\alpha R(t)\|_{\mathbb{B}(\mathbb{X})} \leq \frac{M_\alpha}{t^\alpha} e^{-\delta t};$$

(v) $A^{-\alpha}$ is a bounded linear operator in \mathbb{X} with $D(A^\alpha) = \text{Im}(A^{-\alpha})$; and

(vi) If $0 < \alpha \leq \nu$, then $D(A^\nu) \hookrightarrow D(A^\alpha)$.

Remark 2.2. Observe as in [9] that by Theorem 2.1 (ii) and (iii), the restriction $R_\alpha(t)$ of $R(t)$ to \mathbb{X}_α is exactly the part of $R(t)$ in \mathbb{X}_α .

Let $x \in \mathbb{X}_\alpha$. Since

$$\|R(t)x\|_\alpha = \|A^\alpha R(t)x\| = \|R(t)A^\alpha x\| \leq \|R(t)\|_{\mathbb{B}(\mathbb{X})} \|A^\alpha x\| = \|R(t)\|_{\mathbb{B}(\mathbb{X})} \|x\|_\alpha,$$

and as t decreases to 0

$$\|R(t)x - x\|_\alpha = \|A^\alpha R(t)x - A^\alpha x\| = \|R(t)A^\alpha x - A^\alpha x\| \rightarrow 0,$$

for all $x \in \mathbb{X}_\alpha$, it follows that $(R(t))_{t \geq 0}$ is a family of strongly continuous semigroup on \mathbb{X}_α and $\|R_\alpha(t)\|_{\mathbb{B}(\mathbb{X})} \leq \|R(t)\|_{\mathbb{B}(\mathbb{X})}$ for all $t \geq 0$.

Lemma 2.3. [9] *The restriction $R_\alpha(t)$ of $R(t)$ to \mathbb{X}_α is an immediately compact semigroup in \mathbb{X}_α , and hence it is immediately norm-continuous.*

Now, let Φ_β be the Mainardi function:

$$\Phi_\beta(z) = \sum_{n=0}^{+\infty} \frac{(-z)^n}{n! \Gamma(-\beta n + 1 - \beta)}.$$

Then

$$(5a) \quad \Phi_\beta(t) \geq 0 \text{ for all } t > 0;$$

$$(5b) \quad \int_0^\infty \Phi_\beta(t) dt = 1;$$

$$(5c) \quad \int_0^\infty t^\eta \Phi_\beta(t) dt = \frac{\Gamma(1 + \eta)}{\Gamma(1 + \beta\eta)}, \quad \forall \eta \in [0, 1].$$

For more details we refer to [10].

We set

$$(6) \quad \mathbb{S}_\beta(t) = \int_0^\infty \Phi_\beta(\theta) R(\theta t^\beta) d\theta,$$

$$(7) \quad \mathbb{P}_\beta(t) = \int_0^\infty \beta \theta \Phi_\beta(\theta) R(t^\beta \theta) d\theta$$

Then we have the following results

Lemma 2.4. [16] Let \mathbb{S}_β and \mathbb{P}_β be the operators defined respectively by (6) and (7). Then

- (i) $\|\mathbb{S}_\beta(t)x\| \leq M\|x\|$; $\|\mathbb{P}_\beta(t)x\| \leq M\frac{\beta}{\Gamma(\beta+1)}\|x\|$ for all $x \in \mathbb{X}$ and $t \geq 0$.
- (ii) The operators $(\mathbb{S}_\beta(t))_{t \geq 0}$ and $(\mathbb{P}_\beta(t))_{t \geq 0}$ are strongly continuous.
- (iii) The operators $(\mathbb{S}_\beta(t))_{t > 0}$ and $(\mathbb{P}_\beta(t))_{t > 0}$ are compact.

Lemma 2.5. Let \mathbb{S}_β and \mathbb{P}_β be the operators defined respectively by (6) and (7). Then

$$\|\mathbb{S}_\beta(t)x\|_\alpha \leq M\|x\|_\alpha, \forall x \in \mathbb{X}_\alpha, t \geq 0,$$

$$\|\mathbb{P}_\beta(t)x\|_\alpha \leq \begin{cases} \frac{\beta M_\alpha t^{-\beta\alpha}\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\|x\| & \text{if } x \in \mathbb{X}, t > 0, \\ \frac{\beta}{\Gamma(1+\beta)}\|x\|_\alpha & \text{if } x \in \mathbb{X}_\alpha, t > 0. \end{cases}$$

Proof. Using (3) and (5b) we have for any $x \in \mathbb{X}_\alpha$ and $t \geq 0$,

$$\begin{aligned} \|\mathbb{S}_\beta(t)x\|_\alpha &= \left\| \int_0^\infty \Phi_\beta(\theta)R(\theta t^\beta)x d\theta \right\|_\alpha \\ &\leq \int_0^\infty \Phi_\beta(\theta)\|A^\alpha R(\theta t^\beta)x\| d\theta \\ &\leq M \int_0^\infty \Phi_\beta(\theta)\|A^\alpha x\| d\theta \\ &= M\|x\|_\alpha, \forall x \in \mathbb{X}_\alpha. \end{aligned}$$

In view of (4) and (5c), we can write for any $t > 0$,

$$\begin{aligned} \|\mathbb{P}_\beta(t)x\|_\alpha &= \left\| \int_0^\infty \beta\theta\Phi_\beta(\theta)R(\theta t^\beta)x d\theta \right\|_\alpha \\ &\leq \int_0^\infty \beta\theta\Phi_\beta(\theta)\|A^\alpha R(\theta t^\beta)x\| d\theta \\ &\leq \int_0^\infty \beta\theta\Phi_\beta(\theta)\|A^\alpha R(\theta t^\beta)\|_{\mathbb{B}(\mathbb{X})}\|x\| d\theta \\ &\leq \beta M_\alpha t^{-\alpha\beta}\|x\| \int_0^\infty \theta^{1-\alpha}\Phi_\beta(\theta) d\theta \\ &\leq \frac{\beta M_\alpha t^{-\beta\alpha}\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\|x\|, \forall x \in \mathbb{X} \end{aligned}$$

and

$$\begin{aligned} \|\mathbb{P}_\beta(t)x\|_\alpha &= \left\| \int_0^\infty \beta\theta\Phi_\beta(\theta)R(\theta t^\beta)x d\theta \right\|_\alpha \\ &\leq \int_0^\infty \beta\theta\Phi_\beta(\theta)\|A^\alpha R(\theta t^\beta)x\| d\theta \\ &\leq M\|x\|_\alpha \int_0^\infty \beta\theta\Phi_\beta(\theta) d\theta \\ &= M\|x\|_\alpha \frac{\beta}{\Gamma(1+\beta)}, \forall x \in \mathbb{X}_\alpha. \end{aligned}$$

□

Definition 2.6. ([5, 6]) Let \mathbb{S}_β and \mathbb{P}_β be operators defined respectively by (6) and (7). Then a continuous function $x : I \rightarrow \mathbb{X}$ satisfying for any $t \in [0, T]$ the equation

$$(8) \quad \begin{aligned} x(t) &= \mathbb{S}_\beta(t)(x_0 - g(x)) \\ &+ \int_0^t (t-s)^{\beta-1} \mathbb{P}_\beta(t-s) \left[(f(s, x(s)) - \int_0^t a(t-s)h(s, x(s))) \right] ds, \end{aligned}$$

is called a mild solution of the equation (1).

In the sequel, we set

$$(9) \quad Kx(t) := \int_0^t a(t-s)h(s, x(s)) ds.$$

We set $\alpha \in (0, 1)$ and we will denote by \mathcal{C}_α , the Banach space $C([0, T], \mathbb{X}_\alpha)$ endowed with the supnorm given by

$$\|x\|_\infty := \sup_{t \in I} \|x\|_\alpha, \quad \text{for } x \in \mathcal{C}.$$

3. MAIN RESULTS

In addition to the previous assumptions, we assume that the following hold.

(H₁) The function $f : I \times \mathbb{X}_\alpha \rightarrow \mathbb{X}$ is continuous and satisfies the following condition: there exists a function $\mu_1(t) \in L^\infty(I, \mathbb{R}^+)$ such that

$$\|f(t, x) - f(t, y)\| \leq \mu_1(t) \|x - y\|_\alpha$$

for all $t \in I$, $x, y \in \mathbb{X}_\alpha$.

(H₂) The function $h : I \times \mathbb{X}_\alpha \rightarrow \mathbb{X}$ is continuous and satisfies the following condition: there exists a function $\mu_2(t) \in L^\infty(I, \mathbb{R}^+)$ such that

$$\|h(t, x) - h(t, y)\| \leq \mu_2(t) \|x - y\|_\alpha$$

for all $t \in I$, $x, y \in \mathbb{X}_\alpha$.

(H₃) The function $g : \mathcal{C}_\alpha \rightarrow \mathbb{X}_\alpha$ is continuous and there exists a constant b such that

$$\|g(x) - g(y)\|_\alpha \leq b \|x - y\|_\infty$$

for all $x, y \in \mathcal{C}_\alpha$.

Theorem 3.1. *Suppose assumptions (H₁)-(H₃) hold and that $\Omega_{\alpha, \beta, T} < 1$ where*

$$\Omega_{\alpha, \beta, T} = \left[Mb + \frac{\beta M_\alpha \Gamma(2 - \alpha) T^{\beta(1 - \alpha)}}{\Gamma(1 + \beta(1 - \alpha))(\beta(1 - \alpha))} \left(\|\mu_1\|_{L^\infty(I, \mathbb{R}_+)} + aT \|\mu_2\|_{L^\infty(I, \mathbb{R}_+)} \right) \right].$$

If $x_0 \in \mathbb{X}_\alpha$, then (1) has a unique mild solution $x \in \mathcal{C}_\alpha$.

Proof. Define the nonlinear integral operator $F : \mathcal{C}_\alpha \rightarrow \mathcal{C}_\alpha$ by

$$\begin{aligned} (Fx)(t) &= \mathbb{S}_\beta(t)(x_0 - g(x)), \\ &+ \int_0^t (t-s)^{\beta-1} \mathbb{P}_\beta(t-s) [f(s, x(s)) + Kx(s)] ds. \end{aligned}$$

where K is given by (9).

In view of Lemma 2.4- (ii), the integral operator F is well defined.

Now take $t \in I$ and $x, y \in \mathcal{C}_\alpha$. We have

$$\begin{aligned} \|(Fx)(t) - (Fy)(t)\|_\alpha &\leq \|\mathbb{S}_\beta(t)(g(x) - g(y))\|_\alpha \\ &+ \int_0^t (t-s)^{\beta-1} \|\mathbb{P}_\beta(t-s)(f(s, x(s)) - f(s, y(s)))\|_\alpha ds \\ &+ \int_0^t (t-s)^{\beta-1} \|\mathbb{P}_\beta(t-s)(Kx(s) - Ky(s))\|_\alpha ds \end{aligned}$$

which according to Lemma 2.5 and (H₃) gives

$$\begin{aligned} \|(Fx)(t) - (Fy)(t)\|_\alpha &\leq Mb\|x - y\|_\infty \\ &+ \frac{\beta M_\alpha \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} \int_0^t (t-s)^{\beta(1-\alpha)-1} \|(f(s, x(s)) - f(s, y(s)))\| ds \\ &+ \frac{\beta M_\alpha \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} \int_0^t (t-s)^{\beta(1-\alpha)-1} \|(Kx(s) - Ky(s))\| ds \end{aligned}$$

Since (H₂) and (2) hold, we can write

$$\begin{aligned} \|Kx(s) - Ky(s)\| &= \int_0^s a(s-\tau) \|h(\tau, x(\tau)) - h(\tau, y(\tau))\| d\tau \\ &\leq \int_0^s a(s-\tau) \mu_2(\tau) \|x(\tau) - y(\tau)\| d\tau \\ &\leq a_T \|\mu_2\|_{L^\infty(I, \mathbb{R}_+)} \|x - y\|_\infty. \end{aligned}$$

Thus, using (H₁) we obtain

$$\begin{aligned} \|(Fx)(t) - (Fy)(t)\|_\alpha &\leq Mb\|x - y\|_\infty \\ &+ \frac{\beta M_\alpha \Gamma(2 - \alpha) \|x - y\|_\infty}{\Gamma(1 + \beta(1 - \alpha))} \int_0^t (t-s)^{\beta(1-\alpha)-1} \mu_1(s) ds \\ &+ \frac{\beta M_\alpha \Gamma(2 - \alpha) T^{\beta(1-\alpha)}}{\Gamma(1 + \beta(1 - \alpha))(\beta(1 - \alpha))} a_T \|\mu_2\|_{L^\infty(I, \mathbb{R}_+)} \|x - y\|_\infty \\ &\leq Mb\|x - y\|_\infty \\ &+ \frac{\beta M_\alpha \Gamma(2 - \alpha) T^{\beta(1-\alpha)}}{\Gamma(1 + \beta(1 - \alpha))(\beta(1 - \alpha))} \|x - y\|_\infty \|\mu_1\|_{L^\infty(I, \mathbb{R}_+)} \\ &+ \frac{\beta M_\alpha \Gamma(2 - \alpha) T^{\beta(1-\alpha)} a_T}{\Gamma(1 + \beta(1 - \alpha))(\beta(1 - \alpha))} \|x - y\|_\infty \|\mu_2\|_{L^\infty(I, \mathbb{R}_+)} \\ &\leq \Omega_{\alpha, \beta, T} \|x - y\|_\infty \end{aligned}$$

So we get

$$\|(Fx)(t) - (Fy)(t)\|_\infty \leq \Omega_{\alpha,\beta,T}(t)\|x - y\|_\infty.$$

Since $\Omega_{\alpha,\beta,T} < 1$, the contraction mapping principle enables us to say that, F has a unique fixed point in \mathcal{C}_α ,

$$x(t) = \mathbb{S}_\beta(t)(x_0 - g(x)) + \int_0^t (t-s)^{\beta-1} \mathbb{P}_\beta(t-s) [f(s, x(s)) + Kx(s)] ds$$

which is the mild solution of (1). □

Now we assume that

(H₄) The function $f : I \times \mathbb{X}_\alpha \rightarrow \mathbb{X}$ is continuous and satisfies the following condition: there exists a positive function $\mu_1 \in L^\infty(I, \mathbb{R}^+)$ such that

$$\|f(t, x)\| \leq \mu_1(t),$$

(H₅) The function $h : I \times \mathbb{X}_\alpha \rightarrow \mathbb{X}$ is continuous and satisfies the following condition: there exists a positive function $\mu_2 \in L^\infty(I, \mathbb{R}^+)$ such that

$$\|h(t, x)\| \leq \mu_2(t),$$

(H₆) The function $g \in C(\mathcal{C}_\alpha, \mathbb{X}_\alpha)$ is completely continuous and there exist $\lambda, \gamma > 0$ such that

$$\|g(x)\|_\alpha \leq \lambda\|x\|_\infty + \gamma.$$

Theorem 3.2. *Suppose that assumptions (H₄)-(H₆) hold. If $x_0 \in \mathbb{X}_\alpha$ and*

$$(10) \quad M\lambda < \frac{1}{2}$$

then (1.1) has a mild solution on $[0, T]$.

Proof. Define the integral operator $F : \mathcal{C}_\alpha \rightarrow \mathcal{C}_\alpha$ by

$$\begin{aligned} (Fx)(t) &= \mathbb{S}_\beta(t)(x_0 - g(x)), \\ &+ \int_0^t (t-s)^{\beta-1} \mathbb{P}_\beta(t-s) [f(s, x(s)) + Kx(s)] ds, \end{aligned}$$

and choose r such that

$$\begin{aligned} r &\geq 2 \frac{T^{\beta(1-\alpha)} \beta M_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))(\beta(1-\alpha))} \left(\|\mu_1\|_{L^\infty(I, \mathbb{R}_+)} + a_T \|\mu_2\|_{L^\infty(I, \mathbb{R}_+)} \right) \\ &+ 2M(\|x_0\|_\alpha + \gamma). \end{aligned}$$

Let $B_r = \{x \in \mathcal{C}_\alpha : \|x\|_\infty \leq r\}$. We proceed in three main steps.

Step 1. We show that $F(B_r) \subset B_r$. For that, let $x \in B_r$. Then for $t \in I$, we have

$$\begin{aligned} \|(Fx)(t)\|_\alpha &\leq \|\mathbb{S}_\alpha(t)(x_0 - g(x))\|_\alpha \\ &+ \int_0^t (t-s)^{\beta-1} \|\mathbb{P}_\alpha(t-s)f(s, x(s))\|_\alpha ds \\ &+ \int_0^t (t-s)^{\beta-1} \|\mathbb{P}_\alpha(t-s)Kx(s)\|_\alpha ds \end{aligned}$$

which according to (H₄)-(H₆) and Lemma 2.5 gives

$$\begin{aligned} \|(Fx)(t)\|_\alpha &\leq M(\|x_0\|_\alpha + \lambda\|x\|_\infty + \gamma) \\ &+ \frac{\beta M_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} \int_0^t (t-s)^{\beta(1-\alpha)-1} \|f(s, x(s))\| ds \\ &+ \frac{\beta M_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} \int_0^t (t-s)^{\beta(1-\alpha)-1} \|Kx(s)\| ds \\ &\leq M(\|x_0\|_\alpha + \lambda\|x\|_\infty + \gamma) \\ &+ \frac{\beta M_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} \int_0^t (t-s)^{\beta(1-\alpha)-1} \mu_1(s) ds \\ &+ \frac{\beta M_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} \int_0^t (t-s)^{\beta(1-\alpha)-1} \int_0^s a(s-\tau) \mu_2(\tau) d\tau ds. \end{aligned}$$

Consequently, using the inequality $M\lambda < \frac{1}{2}$, which yields $M\lambda\|x\|_\infty < \frac{r}{2}$ and the choice of r above, we get

$$\begin{aligned} \|(Fx)(t)\|_\alpha &\leq M(\|x_0\|_\alpha + \lambda\|x\|_\infty + \gamma) \\ &+ \frac{\|\mu_1\|_{L^\infty(I, \mathbb{R}_+)} T^{\beta(1-\alpha)}}{(\beta(1-\alpha))} \frac{\beta M_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} \\ &+ \frac{\|\mu_2\|_{L^\infty(I, \mathbb{R}_+)} T^{\beta(1-\alpha)}}{(\beta(1-\alpha))} \frac{\beta M_\alpha \Gamma(2-\alpha) a_T}{\Gamma(1+\beta(1-\alpha))}. \end{aligned}$$

In view of (10) and the choice of r , we obtain

$$\|(Fx)\|_\infty \leq r.$$

Step 2. We prove that F is continuous. For that, let (x_n) be a sequence of B_r such that $x_n \rightarrow x$ in B_r . Then

$$\begin{aligned} f(s, x_n(s)) &\rightarrow f(s, x(s)), & n \rightarrow \infty, \\ h(t, x_n(s)) &\rightarrow h(t, x(s)), & n \rightarrow \infty \end{aligned}$$

as both f and h are jointly continuous on $I \times \mathbb{X}_\alpha$.

Now, for all $t \in I$, we have

$$\begin{aligned} \|Fx_n - Fx\|_\alpha &\leq \|\mathbb{S}_\beta(t)(g(x_n) - g(x))\|_\alpha \\ &+ \left\| \int_0^t (t-s)^{\beta-1} \mathbb{P}_\beta(t-s)(Kx_n(s) - Kx(s)) ds \right\|_\alpha \\ &+ \left\| \int_0^t (t-s)^{\beta-1} S(t-s)(f(s, x_n(s)) - f(s, x(s))) ds \right\|_\alpha, \end{aligned}$$

which in view of Lemma 2.5 gives

$$\begin{aligned} \|Fx_n - Fx\|_\alpha &\leq M\|g(x_n) - g(x)\|_\alpha \\ &+ \frac{\beta M_\alpha \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} \int_0^t (t - s)^{\beta(1 - \alpha) - 1} \|f(s, x_n(s)) - f(s, x(s))\| ds \\ &+ \frac{\beta M_\alpha \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} \int_0^t (t - s)^{\beta(1 - \alpha) - 1} \|Kx_n(s) - Kx(s)\| ds \end{aligned}$$

for all $t \in I$. Therefore, on the one hand using (2), (H₄) and (H₅), we get for each $t \in I$

$$\begin{aligned} \|f(s, x_n(s)) - f(s, x(s))\| &\leq 2\mu_1(s) \text{ for } s \in I, \\ \|Kx_n(s) - Kx(s)\| &\leq \int_0^s a(s - \tau) \|h(\tau, x_n(\tau)) - h(\tau, x(\tau))\| d\tau, \\ &\leq 2 \int_0^s a(s - \tau) \mu_2(\tau) d\tau \\ &\leq 2a_T \|\mu_2\|_{L^\infty(I, \mathbb{R}_+)} \text{ for } s \in I; \end{aligned}$$

and on the other hand using the fact that the functions $s \mapsto 2\mu_1(s)(t - s)^{\beta(1 - \alpha) - 1}$ and $s \mapsto (t - s)^{\beta(1 - \alpha) - 1}$ are integrable on I , by means of the Lebesgue Dominated Convergence Theorem yields

$$\begin{aligned} \int_0^t (t - s)^{\beta(1 - \alpha) - 1} \|f(s, x_n(s)) - f(s, x(s))\| ds &\rightarrow 0, \\ \int_0^t (t - s)^{\beta(1 - \alpha) - 1} \|Kx_n(s) - Kx(s)\| ds &\rightarrow 0. \end{aligned}$$

Hence, since $g(x_n) \rightarrow g(x)$ as $n \rightarrow \infty$ because g is completely continuous on \mathcal{C}_α , it can easily be shown that

$$\lim_{n \rightarrow \infty} \|(Fx_n) - (Fx)\|_\infty = 0,$$

as $n \rightarrow \infty$.

In other words, F is continuous.

Step 3. We show that F is compact. To this end, we use the Ascoli-Arzelà's theorem. For that, we first prove that $\{(Fx)(t) : x \in B_r\}$ is relatively compact in \mathbb{X}_α , for all $t \in I$. Obviously, $\{(Fx)(0) : x \in B_r\}$ is compact.

Let $t \in (0, T]$. For each $h \in (0, t)$, $\epsilon > 0$ and $x \in B_r$, we define the operator $F_{h,\epsilon}$ by

$$\begin{aligned}
 (F_{h,\epsilon}x)(t) &= \mathbb{S}_\beta(t)(x_0 - g(x)) \\
 &+ \int_0^{t-h} (t-s)^{\beta-1} \int_\epsilon^\infty \beta\theta\Phi_\beta(\theta)R((t-s)^\beta\theta)f(s, x(s))d\theta ds \\
 &+ \int_0^{t-h} (t-s)^{\beta-1} \int_\epsilon^\infty \beta\theta\Phi_\beta(\theta)R((t-s)^\beta\theta)Kx(s)d\theta ds \\
 &= \mathbb{S}_\beta(t)(x_0 - g(x)) \\
 &+ R(h^\beta\epsilon) \int_0^{t-h} (t-s)^{\beta-1} \int_\epsilon^\infty \beta\theta\Phi_\beta(\theta)R((t-s)^\beta\theta - h^\beta\epsilon)f(s, x(s))d\theta ds \\
 &+ R(h^\beta\epsilon) \int_0^{t-h} (t-s)^{\beta-1} \int_\epsilon^\infty \beta\theta\Phi_\beta(\theta)R((t-s)^\beta\theta - h^\beta\epsilon)Kx(s)d\theta ds.
 \end{aligned}$$

Then the sets $\{(F_{h,\epsilon}x)(t) : x \in B_r\}$ are relatively compact in \mathbb{X}_α since by Lemma 2.3, the operators $R_\alpha(t)$, $t > 0$ are compact on \mathbb{X}_α . Moreover, using (H_1) and (4) we have

$$\begin{aligned}
 \|(Fx)(t) - (F_{h,\epsilon}x)(t)\|_\alpha &\leq \\
 &\int_0^t (t-s)^{\beta-1} \int_0^\epsilon \beta\theta\Phi_\beta(\theta) \|R((t-s)^\beta\theta)f(s, x(s))\|_\alpha d\theta ds + \\
 &\int_{t-h}^t (t-s)^{\beta-1} \int_\epsilon^\infty \beta\theta\Phi_\beta(\theta) \|R((t-s)^\beta\theta)f(s, x(s))\|_\alpha d\theta ds + \\
 &\int_0^t (t-s)^{\beta-1} \int_0^\epsilon \beta\theta\Phi_\beta(\theta) \|R((t-s)^\beta\theta)Kx(s)\|_\alpha d\theta ds + \\
 &\int_{t-h}^t (t-s)^{\beta-1} \int_\epsilon^\infty \beta\theta\Phi_\beta(\theta) \|R((t-s)^\beta\theta)Kx(s)\|_\alpha d\theta ds.
 \end{aligned}$$

Then using (4) and (H_4) , we obtain

$$\begin{aligned}
 \|(Fx)(t) - (F_{h,\epsilon}x)(t)\|_\alpha &\leq \beta M_\alpha \int_0^t (t-s)^{\beta(1-\alpha)-1} \mu_1(s) \int_0^\epsilon \theta^{1-\alpha} \Phi_\beta(\theta) d\theta ds \\
 &+ \beta M_\alpha \int_{t-h}^t (t-s)^{\beta(1-\alpha)-1} \mu_1(s) \int_\epsilon^\infty \beta\theta^{1-\alpha} \Phi_\beta(\theta) d\theta ds \\
 &+ \beta M_\alpha \int_0^t (t-s)^{\beta(1-\alpha)-1} \int_0^\epsilon \beta\theta^{1-\alpha} \Phi_\beta(\theta) \|Kx(s)\| d\theta ds \\
 &+ \beta M_\alpha \int_{t-h}^t (t-s)^{\beta(1-\alpha)-1} \int_\epsilon^\infty \beta\theta^{1-\alpha} \Phi_\beta(\theta) \|Kx(s)\| d\theta ds.
 \end{aligned}$$

Since by (H_5) and (2),

$$\begin{aligned}
 \|Kx(s)\| &\leq \int_0^s a(s-\tau) \|h(\tau, x(\tau))\| d\tau \\
 &\leq \int_0^s a(s-\tau) \mu_2(\tau) d\tau \\
 &\leq a_T \|\mu_2\|_{L^\infty(I, \mathbb{R}_+)},
 \end{aligned}$$

using (5c), we deduce for all $\epsilon > 0$ that

$$\begin{aligned} \|(Fx)(t) - (F_{h,\epsilon}x)(t)\|_\alpha &\leq \frac{t^{\beta(1-\alpha)}\beta M_\alpha\|\mu_1\|_{L^\infty(I,\mathbb{R}_+)}}{\beta(1-\alpha)} \int_0^\epsilon \theta^{1-\alpha}\Phi_\beta(\theta)d\theta \\ &+ \frac{h^{\beta(1-\alpha)}\beta M_\alpha\Gamma(2-\alpha)\|\mu_1\|_{L^\infty(I,\mathbb{R}_+)}}{\beta(1-\alpha)\Gamma(1+\beta(1-\alpha))} \\ &+ \frac{t^{\beta(1-\alpha)}\beta M_\alpha\|\mu_2\|_{L^\infty(I,\mathbb{R}_+)a_T}}{\beta(1-\alpha)} \int_0^\epsilon \theta^{1-\alpha}\Phi_\beta(\theta)d\theta \\ &+ \frac{h^{\beta(1-\alpha)}\beta M_\alpha\Gamma(2-\alpha)a_T\|\mu_2\|_{L^\infty(I,\mathbb{R}_+)}}{\beta(1-\alpha)\Gamma(1+\beta(1-\alpha))}. \end{aligned}$$

In other words

$$\begin{aligned} \|(Fx)(t) - (F_{h,\epsilon}x)(t)\|_\alpha &\leq \frac{h^{\beta(1-\alpha)}\beta M_\alpha\Gamma(2-\alpha)\|\mu_1\|_{L^\infty(I,\mathbb{R}_+)}}{\beta(1-\alpha)\Gamma(1+\beta(1-\alpha))} \\ &+ \frac{h^{\beta(1-\alpha)}\beta M_\alpha\Gamma(2-\alpha)a_T\|\mu_2\|_{L^\infty(I,\mathbb{R}_+)}}{\beta(1-\alpha)\Gamma(1+\beta(1-\alpha))}. \end{aligned}$$

Therefore, the set $\{(Fx)(t) : x \in B_r\}$ is relatively compact in \mathbb{X}_α for all $t \in (0, T]$ and since it is compact at $t = 0$ we have the relatively compactness in \mathbb{X}_α for all $t \in I$. Now, let us prove that $F(B_r)$ is equicontinuous. By the compactness of the set $g(B_r)$, we can prove that the functions Fx , $x \in B_r$ are equicontinuous at $t = 0$. For $0 < t_2 < t_1 \leq T$, we have

$$\begin{aligned} \|(Fx)(t_1) - (Fx)(t_2)\|_\alpha &\leq \|(\mathbb{S}_\beta(t_1) - \mathbb{S}_\beta(t_2))(x_0 - g(x))\|_\alpha \\ &+ \left\| \int_0^{t_2} (t_1 - s)^{\beta-1} (\mathbb{P}_\beta(t_1 - s) - \mathbb{P}_\beta(t_2 - s)) (f(s, x(s)) + Kx(s)) ds \right\|_\alpha \\ &+ \left\| \int_0^{t_2} ((t_1 - s)^{\beta-1} - (t_2 - s)^{\beta-1}) \mathbb{P}_\beta(t_2 - s) (f(s, x(s)) + Kx(s)) ds \right\|_\alpha \\ &+ \left\| \int_{t_2}^{t_1} (t_1 - s)^{\beta-1} \mathbb{P}_\beta(t_1 - s) (f(s, x(s)) + Kx(s)) ds \right\|_\alpha \\ &\leq I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \|(\mathbb{S}_\beta(t_1) - \mathbb{S}_\beta(t_2))(x_0 - g(x))\|_\alpha \\ I_2 &= \left\| \int_0^{t_2} (t_1 - s)^{\beta-1} (\mathbb{P}_\beta(t_1 - s) - \mathbb{P}_\beta(t_2 - s)) (f(s, x(s)) + Kx(s)) ds \right\|_\alpha \\ I_3 &= \left\| \int_0^{t_2} ((t_1 - s)^{\beta-1} - (t_2 - s)^{\beta-1}) \mathbb{P}_\beta(t_2 - s) (f(s, x(s)) + Kx(s)) ds \right\|_\alpha \\ I_4 &= \left\| \int_{t_2}^{t_1} (t_1 - s)^{\beta-1} \mathbb{P}_\beta(t_1 - s) (f(s, x(s)) + Kx(s)) ds \right\|_\alpha \end{aligned}$$

Actually, I_1 , I_2 , I_3 and I_4 tend to 0 independently of $x \in B_r$ when $t_2 \rightarrow t_1$. Indeed, let $x \in B_r$ and $G = \sup_{x \in C_\alpha} \|g(x)\|_\alpha$. In view of Lemma 2.5, we have

$$\begin{aligned} I_1 &= \|(\mathbb{S}_\beta(t_1) - \mathbb{S}_\beta(t_2))(x_0 - g(x))\|_\alpha \\ &\leq \int_0^\infty \Phi_\beta(\theta) \left\| R(\theta t_1^\beta) - R(\theta t_2^\beta) \right\|_{\mathbb{B}(\mathbb{X})} \|x_0 - g(x)\|_\alpha d\theta \\ &\leq \int_0^\infty \Phi_\beta(\theta) \left\| R(\theta t_1^\beta) - R(\theta t_2^\beta) \right\|_{\mathbb{B}(\mathbb{X})} (\|x_0\|_\alpha + G) d\theta \end{aligned}$$

from which we deduce that $\lim_{t_2 \rightarrow t_1} I_1 = 0$ since by Lemma 2.3 the function $t \mapsto \|R_\alpha(t)\|_\alpha$ is continuous for $t \geq 0$

$$I_2 \leq \int_0^{t_2} \left\| (t_1 - s)^{\beta-1} (\mathbb{P}_\beta(t_1 - s) - \mathbb{P}_\beta(t_2 - s)) (f(s, x(s)) + Kx(s)) \right\|_\alpha ds.$$

Therefore using the continuity of $\mathbb{P}_\beta(t)$ (Lemma 2.4) and the fact that both f and K are bounded we conclude that $\lim_{t_2 \rightarrow t_1} I_2 = 0$

$$\begin{aligned} I_3 &\leq \int_0^{t_2} \left((t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1} \right) \left\| \mathbb{P}_\beta(t_2 - s) (f(s, x(s)) + Kx(s)) \right\|_\alpha ds \\ &\leq \frac{\beta M_\alpha \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} \int_0^{t_2} \left((t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1} \right) (t_2 - s)^{-\alpha\beta} \|f(s, x(s))\| ds \\ &\quad + \frac{\beta M_\alpha \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} \int_0^{t_2} \left((t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1} \right) (t_2 - s)^{-\alpha\beta} \|Kx(s)\| ds. \end{aligned}$$

Since $-(t_2 - s)^{-\alpha\beta} (t_1 - s)^{\beta-1} \leq -(t_1 - s)^{\beta(1-\alpha)-1}$ because $(t_1 - s)^{-\alpha\beta} \leq (t_2 - s)^{-\alpha\beta}$, we deduce that

$$\begin{aligned} I_3 &\leq \frac{\beta M_\alpha \Gamma(2 - \alpha) \|\mu_1\|_{L^\infty(I, \mathbb{R}_+)}}{\Gamma(1 + \beta(1 - \alpha))} \int_0^{t_2} \left((t_2 - s)^{\beta(1-\alpha)-1} - (t_1 - s)^{\beta(1-\alpha)-1} \right) ds \\ &\quad + \frac{a_T \beta M_\alpha \Gamma(2 - \alpha) \|\mu_1\|_{L^\infty(I, \mathbb{R}_+)}}{\Gamma(1 + \beta(1 - \alpha))} \int_0^{t_2} \left((t_2 - s)^{\beta(1-\alpha)-1} - (t_1 - s)^{\beta(1-\alpha)-1} \right) ds \\ &\leq \frac{\beta M_\alpha \Gamma(2 - \alpha) \|\mu_1\|_{L^\infty(I, \mathbb{R}_+)}}{\beta(1 - \alpha) \Gamma(1 + \beta(1 - \alpha))} (t_1 - t_2)^{\beta(1-\alpha)} \\ &\quad + \frac{a_T \beta M_\alpha \Gamma(2 - \alpha) \|\mu_1\|_{L^\infty(I, \mathbb{R}_+)}}{\beta(1 - \alpha) \Gamma(1 + \beta(1 - \alpha))} (t_1 - t_2)^{\beta(1-\alpha)}. \end{aligned}$$

Hence $\lim_{t_2 \rightarrow t_1} I_3 = 0$ since $\beta(1 - \alpha) > 0$.

$$\begin{aligned}
I_4 &\leq \int_{t_2}^{t_1} (t_1 - s)^{\beta-1} \|\mathbb{P}_\beta(t_1 - s)(f(s, x(s)) + Kx(s))\|_\alpha ds \\
&\leq \frac{\beta M_\alpha \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} \int_{t_2}^{t_1} (t_1 - s)^{\beta(1-\alpha)-1} \|f(s, x(s)) + Bx(s)\| ds \\
&\leq \frac{\beta M_\alpha \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} \int_{t_2}^{t_1} (t_1 - s)^{\beta(1-\alpha)-1} (\mu_1(s) + \int_0^s a(s - \tau) \|h(\tau, x(\tau))\| d\tau) ds \\
&\leq \frac{\beta M_\alpha \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} (\|\mu_1\|_{L^\infty(I, \mathbb{R}_+)} + a_T \|\mu_2\|_{L^\infty(I, \mathbb{R}_+)}) \int_{t_2}^{t_1} (t_1 - s)^{\beta(1-\alpha)-1} ds \\
&\leq \frac{(t_1 - t_2)^{\beta(1-\alpha)} \beta M_\alpha \Gamma(2 - \alpha)}{\beta(1 - \alpha) \Gamma(1 + \beta(1 - \alpha))} (\|\mu_1\|_{L^\infty(I, \mathbb{R}_+)} + a_T \|\mu_2\|_{L^\infty(I, \mathbb{R}_+)})
\end{aligned}$$

Since $\beta(1 - \alpha) > 0$, we deduce that $\lim_{t_2 \rightarrow t_1} I_4 = 0$.

In short, we have shown that $F(B_r)$ is relatively compact, for $t \in I$, $\{Fx : x \in B_r\}$ is a family of equicontinuous functions. Hence by the Arzela-Ascoli Theorem, F is compact. By Schauder fixed point theorem F has a fixed point $x \in B_r$, which obviously is a mild solution to (1). \square

4. EXAMPLE

Let $\mathbb{X} = L^2[0, \pi]$ equipped with its natural norm and inner product defined respectively for all $u, v \in L^2[0, \pi]$ by

$$\|u\|_{L^2[0, \pi]} = \left(\int_0^\pi |u(x)|^2 dx \right)^{1/2} \quad \text{and} \quad \langle u, v \rangle = \int_0^\pi u(x) \overline{v(x)} dx.$$

Consider the following integro-partial differential equation

$$(E) \quad \begin{cases} \frac{\partial^\beta u}{\partial t^\beta}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) + \frac{\cos(tx)}{1 + u^2(t, x)} + \int_0^t e^{-|t-s|} \cos(u(s, x)) ds, \\ u(t, 0) = u(t, \pi) = 0, \quad t \in [0, 1] \\ u(0, x) + \delta_0 \sum_{k=0}^N \int_0^\pi \cos(x - y) u(t_k, y) dy = u_0(x), \quad x \in [0, \pi] \end{cases}$$

where $t \in [0, 1]$, $x \in [0, \pi]$, $0 < t_1 < t_2 < \dots < t_N \leq 1$, and $\delta_0 > 0$.

First of all, note that f, h, a are given by

$$f(t, u(t, x)) = \frac{\cos(tx)}{1 + u^2(t, x)}, \quad a(t) = e^{-|t|}, \quad \text{and} \quad h(t, u(t, x)) = \cos(u(s, x)),$$

and hence in (H₄) and (H₅) we take $\mu_1(t) = \mu_2(t) = \pi$. Moreover, $a_1 = \int_0^1 e^{-|t|} dt = 1 - e^{-1}$.

Let A be the operator given by $Au = -u''$ with domain

$$D(A) := \{u \in L^2([0, \pi]) : u'' \in L^2([0, \pi]), u(0) = u(\pi) = 0\}.$$

It is well known that A has a discrete spectrum with eigenvalues of the form n^2 , $n \in \mathbb{N}$, and corresponding normalized eigenfunctions given by

$$z_n(\xi) := \sqrt{\frac{2}{\pi}} \sin(n\xi).$$

In addition to the above, the following properties hold:

- (a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis for $L^2[0, \pi]$;
- (b) The operator $-A$ is the infinitesimal generator of an analytic semigroup $R(t)$ which is compact for $t > 0$. The semigroup $R(t)$ is defined for $u \in L^2[0, \pi]$ by

$$R(t)u = \sum_{n=1}^{\infty} e^{-n^2 t} \langle u, z_n \rangle z_n.$$

- (c) The operator A can be rewritten as

$$Au = \sum_{n=1}^{\infty} n^2 \langle u, z_n \rangle z_n$$

for every $u \in D(A)$.

Moreover, it is possible to define fractional powers of A . In particular,

- (d) For $u \in L^2[0, \pi]$ and $\alpha \in (0, 1)$,

$$A^{-\alpha}u = \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} \langle u, z_n \rangle z_n;$$

- (e) The operator $A^\alpha : D(A^\alpha) \subseteq L^2[0, \pi] \mapsto L^2[0, \pi]$ given by

$$A^\alpha u = \sum_{n=1}^{\infty} n^{2\alpha} \langle u, z_n \rangle z_n, \quad \forall u \in D(A^\alpha),$$

where $D(A^\alpha) = \left\{ u \in L^2[0, \pi] : \sum_{n=1}^{\infty} n^{2\alpha} \langle u, z_n \rangle z_n \in L^2[0, \pi] \right\}$.

Clearly for all $t \geq 0$ and $0 \neq u \in L^2[0, \pi]$,

$$\begin{aligned} |R(t)u| &= \left| \sum_{n=1}^{\infty} e^{-n^2 t} \langle u, z_n \rangle z_n \right| \\ &\leq \sum_{n=1}^{\infty} e^{-t} |\langle u, z_n \rangle z_n| \\ &= e^{-t} \sum_{n=1}^{\infty} |\langle u, z_n \rangle z_n| \\ &\leq e^{-t} |u| \end{aligned}$$

and hence $\|R(t)\|_{B(L^2[0, \pi])} \leq 1$ for all $t \geq 0$. Here we take $M = 1$.

Set

$$g(u)(\xi) := \delta_0 \sum_{k=0}^N \int_0^\pi \cos(\xi - y) u(t_k, y) dy.$$

Suppose $\alpha \in (0, \frac{1}{2})$ and

$$(11) \quad \delta_0 < \frac{\sqrt{6}}{2\pi^2 N}.$$

Now

$$\begin{aligned} \|A^\alpha g(u)(\xi)\|_{L^2[0, \pi]}^2 &= \sum_{n \geq 1} n^{4\alpha} \|z_n\|_{L^2[0, \pi]}^2 |\langle g(u)(\xi), z_n \rangle|^2 \\ &\leq \sum_{n \geq 1} n^2 |\langle g(u)(\xi), z_n \rangle|^2 \\ &= \frac{2}{\pi} \sum_{n \geq 1} \left| \int_0^\pi g(u)(\xi) n \sin(n\xi) d\xi \right|^2 \\ &= \sum_{n \geq 1} \frac{1}{n^2} \left| \int_0^\pi \frac{\partial^2}{\partial \xi^2} g(u)(\xi) z_n(\xi) d\xi \right|^2 \\ &\leq \frac{\pi^2}{6} \left\| \frac{\partial^2}{\partial \xi^2} g(u)(\xi) \right\|_{L^2[0, \pi]}^2 \\ &\leq \frac{\pi^2}{6} \|g(u)(\xi)\|_{L^2[0, \pi]}^2 \\ &\leq \delta_0^2 \frac{\pi^2}{6} N^2 \pi^2 \|u\|_\infty^2 \end{aligned}$$

and hence $\|g(u)\|_\alpha \leq \lambda \|u\|_\infty + \mu$ where $\lambda = \frac{\delta_0 \pi^2 N}{\sqrt{6}}$ and $\mu = 0$. Therefore, the condition $M\lambda < \frac{1}{2}$ holds under assumption (11).

Using Theorem 3.2 and inequality Eq. (11) it follows that the system (E) at least one mild solution.

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