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Anti-Periodic Solutions for a Class of Third-Order Nonlinear Differential Equations with a Deviating Argument*

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Abstract In this paper, we study a class of third-order nonlinear differential equations with a deviating argument and establish some sufficient conditions for the existence and exponential stability of anti-periodic solutions of the equation. These conditions are new and complement to previously known results.

Keywords: Third-order nonlinear differential equation; Exponential stability; Anti-periodic solutions; Deviating argument.

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1. INTRODUCTION

Consider the following third-order nonlinear differential equations with a deviating argument

$$x'''(t) + a(t)x''(t) + b(t)x'(t) + g_1(t, x(t)) + g_2(t, x(t - \tau(t))) = p(t), \tag{1.1}$$

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where a, b and p are continuous functions on $R = (-\infty, +\infty)$, g_1 and g_2 are continuous functions on R^2, $\tau \geq 0$ is a continuous function on R, and there exists a constant $\bar{\tau}$ such that $\bar{\tau} = \sup_{t \in R} \tau(t)$.

In applied science some practical problems are associated with equation (1.1), such as nonlinear oscillations [1,2,3], electronic theory [4], biological model and other models [5, 6]. Just as above, in the past few decades, the study for third order differential equation has been paid attention to by many scholars. Many results relative to the stability, instability of solutions, boundedness of solutions, convergence of solutions and existence of periodic solutions for equation (1.1) and its analogue equations have been obtained (see [7,8] and references therein). However, as pointed out in [8], the results about the existence of anti-periodic solutions for nonlinear differential equations whose orders are more than two are relatively scarce. Moreover, it is well known that the existence of anti-periodic solutions play a key role in characterizing the behavior of nonlinear differential equations (See [9−12]). Thus, it is worthwhile to continue to investigate the existence and stability of anti-periodic solutions of Eq. (1.1).

A primary purpose of this paper is to study the problem of anti-periodic solutions of (1.1). We will establish some sufficient conditions for the existence and exponential stability of the anti-periodic solutions of (1.1). Our results are new and complement to previously known results. In particular, an example is also provided to illustrate the effectiveness of the new results.

Let d_1 and d_2 be constants. Define

$$y(t) = \frac{dx(t)}{dt} + d_1 x(t), z(t) = \frac{dy(t)}{dt} + d_2 y(t), \tag{1.2}$$

then we can transform (1.1) into the following equivalent system

$$\begin{cases} \frac{dx(t)}{dt} = -d_1 x(t) + y(t), \\ \frac{dy(t)}{dt} = -d_2 y(t) + z(t), \\ \frac{dz(t)}{dt} = -(a(t) - d_1 - d_2) z(t) + (-(a(t) - d_1) d_1^2 + b(t) d_1) x(t) - g_1(t, x(t)) \\ \qquad -g_2(t, x(t - \tau(t))) + ((a(t) - d_1)(d_1 + d_2) - b(t) - d_2^2) y(t) + p(t). \end{cases} \tag{1.3}$$

Throughout this article, it will be assumed that there exists a constant $T > 0$ such that

$$p(t + T) = -p(t), \ g_1(t + T, u) = -g_1(t, -u), \ g_2(t + T, u) = -g_2(t, -u),$$

$$a(t+T)=a(t),\ b(t+T)=b(t),\ \tau(t+T)=\tau(t), \forall\ t,u\in R. \qquad (1.4)$$

We suppose that there exists a constant L^+ such that

$$L^+ > \sup_{t\in R}|p(t)|. \qquad (1.5)$$

It is known in [14–16] that for g_1, g_2, a, b, τ and p continuous, given a continuous initial function $\varphi\in C([-\bar\tau,0],R)$ and a vector $(y_0,z_0)\in R^2$, then there exists a solution of (1.3) on an interval $[0,T)$ satisfying the initial condition and satisfying (1.3) on $[0,T)$. If the solution remains bounded, then $T=+\infty$. We denote such a solution by $(x(t),y(t),z(t))=(x(t,\varphi,y_0,z_0),y(t,\varphi,y_0,z_0),z(t,\varphi,y_0,z_0))$. Let $y(s)=y(0)$ and $z(s)=z(0)$ for all $s\in[-\bar\tau,0]$. It follows that $(x(t),y(t),z(t))$ can be defined on $[-\bar\tau,+\infty)$.

Definition 1. Let $u(t):R\longrightarrow R$ be continuous in t. $u(t)$ is said to be T-anti-periodic on R, if

$$u(t+T)=-u(t) \ \text{ for all } \ t\in R.$$

Definition 2. Let $Z^*(t)=(x^*(t),\ y^*(t),\ z^*(t))$ be a T-anti-periodic solution of system (1.3) with initial value $(\varphi^*(t),\ y_0^*,\ z_0^*)\in C([-\bar\tau,0],R)\times R\times R$. If there exist constants $\lambda>0$ and $M>1$ such that for every solution $Z(t)=(x(t),y(t),z(t))$ of system (1.3) with any initial value $\varphi=(\varphi(t),\ y_0,\ z_0)\in C([-\bar\tau,0],R)\times R\times R$,

$$\max\{|x(t)-x^*(t)|,|y(t)-y^*(t)|,|z(t)-z^*(t)|\}\le M\max\{\|\varphi(t)-\varphi^*(t)\|,\ |y_0-y_0^*|,\ |z_0-z_0^*|\}e^{-\lambda t},$$

for all $t>0$ and $\|\varphi(t)-\varphi^*(t)\|=\sup_{t\in[-\bar\tau,\ 0]}|\varphi(t)-\varphi^*(t)|$. Then $Z^*(t)$ is said to be globally exponentially stable.

We also assume that the following condition holds.

(C_1) There exist constants $L_1\ge 0, L_2\ge 0, d_1>1,\ d_2>1$ and $d_3>0$ such that

(i) $|((-(a(t)-d_1)d_1^2+b(t)d_1)u-g_1(t,u))-((-(a(t)-d_1)d_1^2+b(t)d_1)v-g_1(t,v))|$
$\le L_1|u-v|,$ for all $t,u,v\in R$,

(ii) $|g_2(t,u))-g_2(t,v))|\le L_2|u-v|,$ for all $t,u,v\in R$,

(iii) $d_3=\inf_{t\in R}(a(t)-d_1-d_2)-\sup_{t\in R}|(a(t)-d_1)(d_1+d_2)-b(t)-d_2^2|>L_1+L_2.$

The paper is organized as follows. In Section 2, we establish some preliminary results, which are important in the proofs of our main results. Based on the preparations in Section 2, we state and prove our main results in Section 3. Moreover, an illustrative example is given in Section 4.

2. Preliminary Results

The following lemmas will be useful to prove our main results in Section 3.

Lemma 2.1. Let (C_1) hold. Suppose that $(\widetilde{x}(t), \widetilde{y}(t), \widetilde{z}(t))$ is a solution of system (1.3) with initial conditions

$$\widetilde{x}(s) = \widetilde{\varphi}(s),\ \widetilde{y}(0) = y_0,\ \widetilde{z}(0) = z_0,\ \max\{|\widetilde{\varphi}(s)|, |y_0|, |z_0|\} < \frac{L^+}{\eta},\ s \in [-\bar{\tau},\ 0], \tag{2.1}$$

where $\eta = \min\{d_1 - 1,\ d_2 - 1, d_3 - (L_1 + L_2)\}$. Then

$$\max\{|\widetilde{x}(t)|, |\widetilde{y}(t)|, |\widetilde{z}(t)|\} < \frac{L^+}{\eta} \text{ for all } t \geq 0. \tag{2.2}$$

Proof. Assume, by way of contradiction, that (2.2) does not hold. Then, one of the following cases must occur.

Case 1: There exists $t_1 > 0$ such that

$$\max\{|\widetilde{x}(t_1)|, |\widetilde{y}(t_1)|, |\widetilde{z}(t_1)|\} = |\widetilde{x}(t_1)| = \frac{L^+}{\eta} \text{ and } \max\{|\widetilde{x}(t)|, |\widetilde{y}(t)|, |\widetilde{z}(t)|\} < \frac{L^+}{\eta}, \tag{2.3}$$

where $t \in [-\bar{\tau}, t_1)$.

Case 2: There exists $t_2 > 0$ such that

$$\max\{|\widetilde{x}(t_2)|, |\widetilde{y}(t_2)|, |\widetilde{z}(t_2)|\} = |\widetilde{y}(t_2)| = \frac{L^+}{\eta} \text{ and } \max\{|\widetilde{x}(t)|, |\widetilde{y}(t)|, |\widetilde{z}(t)|\} < \frac{L^+}{\eta}, \tag{2.4}$$

where $t \in [-\bar{\tau}, t_2)$.

Case 3: There exists $t_3 > 0$ such that

$$\max\{|\widetilde{x}(t_3)|, |\widetilde{y}(t_3)|, |\widetilde{z}(t_3)|\} = |\widetilde{z}(t_3)| = \frac{L^+}{\eta} \text{ and } \max\{|\widetilde{x}(t)|, |\widetilde{y}(t)|, |\widetilde{z}(t)|\} < \frac{L^+}{\eta}, \tag{2.5}$$

where $t \in [-\bar{\tau}, t_3)$.

If **Case 1** holds, calculating the upper left derivative of $|\widetilde{x}(t)|$, together with (C_1), (1.3) and (2.3) imply that

$$0 \leq D^+(|\widetilde{x}(t_1)|) \leq -d_1|\widetilde{x}(t_1)| + |\widetilde{y}(t_1)| \leq -(d_1 - 1)\frac{L^+}{\eta} < 0,$$

which is a contradiction and implies that (2.2) holds.

If **Case 2** holds, calculating the upper left derivative of $|\widetilde{y}(t)|$, together with (C_1), (1.3) and (2.4) imply that

$$0 \leq D^+(|\widetilde{y}(t_2)|) \leq -d_2|\widetilde{y}(t_2)| + |\widetilde{z}(t_2)| \leq -(d_2 - 1)\frac{L^+}{\eta} < 0,$$

which is a contradiction and implies that (2.2) holds.

If **Case 3** holds, calculating the upper left derivative of $|\widetilde{z}(t)|$, together with (C_1), (1.3) and (2.5) imply that

$$\begin{aligned}
0 &\leq D^+(|\widetilde{y}(t_3)|) \\
&\leq -(a(t_3)-d_1-d_2)|\widetilde{z}(t_3)| + |(-(a(t)-d_1)d_1^2 + b(t_3)d_1)\widetilde{x}(t_3) - g_1(t_3,\widetilde{x}(t_3)) \\
&\quad -g_2(t_3,\widetilde{x}(t_3-\tau(t_3))) + ((a(t_3)-d_1)(d_1+d_2) - b(t) - d_2^2)\widetilde{y}(t_3) + p(t_3)| \\
&\leq -\inf_{t\in R}(a(t)-d_1-d_2)|\widetilde{z}(t_3)| + L_1|\widetilde{x}(t_3)| + L_2|\widetilde{x}(t_3-\tau(t_3))| \\
&\quad +\sup_{\in R}|(a(t)-d_1)(d_1+d_2) - b(t) - d_2^2||\widetilde{y}(t_3)| + |p(t_3)| \\
&\leq -(d_3-(L_1+L_2))\frac{L^+}{\eta} + |p(t_3)| \\
&< 0,
\end{aligned}$$

which is a contradiction and implies that (2.2) holds. The proof of Lemma 2.1 is now complete.

Remark 2.1. In view of the boundedness of this solution, from the theory of functional differential equations in [15], it follows that $(\widetilde{x}(t),\widetilde{y}(t),\widetilde{z}(t))$ can be defined on $[0,\infty)$.

Lemma 2.2. Let (C_1) hold. Moreover, assume that $Z^*(t)=(x^*(t),\ y^*(t),\ z^*(t))$ is a solution of system (1.3) with initial value $(\varphi^*(t),\ y_0^*,\ z_0^*)\in C([-\bar{\tau},0],R)\times R\times R$. Then, there exist constants $\lambda>0$ and $M>1$ such that for every solution $Z(t)=(x(t),y(t),z(t))$ of system (1.3) with any initial value $\varphi=(\varphi(t),\ y_0,\ z_0)\in C([-\bar{\tau},0],R)\times R\times R$,

$$\max\{|x(t)-x^*(t)|,|y(t)-y^*(t)|,|z(t)-z^*(t)|\}\leq M\max\{\|\varphi(t)-\varphi^*(t)\|,\ |y_0-y_0^*|,\ |z_0-z_0^*|\}e^{-\lambda t},$$

for all $t>0$.

Proof. Since $\min\{d_1-1,\ d_2-1,d_3-(L_1+L_2)\}>0$, it follows that there exist constants $\lambda>0$ and $\gamma>0$ such that

$$\gamma=\min\{((d_1-1)-\lambda,\ (d_2-1)-\lambda,d_3-L_1-L_2e^{\lambda\bar{\tau}}-\lambda\}>0. \tag{2.6}$$

Let $Z^*(t)=(x^*(t),\ y^*(t),\ z^*(t))$ be the solution of system (1.3) with initial value $(\varphi^*(t),\ y_0^*,\ z_0^*)\in C([-\bar{\tau},0],R)\times R\times R$, and $Z(t)=(x(t),y(t),z(t))$ be an arbitrary solution of system (1.3) with any initial value $\varphi=(\varphi(t),\ y_0,\ z_0)\in C([-\bar{\tau},0],R)\times R\times R$. Set $\bar{u}(t)=x(t)-x^*(t),\bar{v}(t)=$

$y(t)-y^*(t), \bar{w}(t)=z(t)-z^*(t)$. Then

$$\begin{cases} \dfrac{d\bar{u}(t)}{dt}=-d_1\bar{u}(t)+\bar{v}(t), \\ \dfrac{d\bar{v}(t)}{dt}=-d_2\bar{v}(t)+\bar{w}(t), \\ \dfrac{d\bar{w}(t)}{dt}=(-(a(t)-d_1-d_2)z(t)+(-(a(t)-d_1)d_1^2+b(t)d_1)x(t)-g_1(t,x(t)) \\ \qquad -g_2(t,x(t-\tau(t)))+((a(t)-d_1)(d_1+d_2)-b(t)-d_2^2)y(t)) \\ \qquad -(-(a(t)-d_1-d_2)z^*(t)+(-(a(t)-d_1)d_1^2+b(t)d_1)x^*(t)-g_1(t,x^*(t)) \\ \qquad -g_2(t,x^*(t-\tau(t)))+((a(t)-d_1)(d_1+d_2)-b(t)-d_2^2)y^*(t)). \end{cases} \tag{2.7}$$

We consider the Lyapunov functional

$$V_1(t)=|\bar{u}(t)|e^{\lambda t}, V_2(t)=|\bar{v}(t)|e^{\lambda t}, V_3(t)=|\bar{w}(t)|e^{\lambda t}. \tag{2.8}$$

Calculating the upper left derivative of $V_i(t)$ $(i=1,2,3)$ along the solution $(\bar{u}(t),\bar{v}(t),\bar{w}(t))$ of system (2.7) with the initial value $(\varphi(t)-\varphi^*(t),\ y_0-y_0^*,\ z_0-z_0^*)$, we have

$$\begin{aligned} D^+(V_1(t)) &= \lambda e^{\lambda t}|\bar{u}(t)|+e^{\lambda t}\mathrm{sign}(\bar{u}(t))\{-d_1\bar{u}(t)+\bar{v}(t)\} \\ &\le e^{\lambda t}\{(\lambda-d_1)|\bar{u}(t)|+|\bar{v}(t)|\}, \end{aligned} \tag{2.9}$$

$$\begin{aligned} D^+(V_2(t)) &= \lambda e^{\lambda t}|\bar{v}(t)|+e^{\lambda t}\mathrm{sign}(\bar{v}(t))\{-d_2\bar{v}(t)+\bar{w}(t)\} \\ &\le e^{\lambda t}\{(\lambda-d_2)|\bar{v}(t)|+|\bar{w}(t)|\}, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} & D^+(V_3(t)) \\ = {} & \lambda e^{\lambda t}|\bar{w}(t)|+e^{\lambda t}\mathrm{sign}(\bar{w}(t))\{(-(a(t)-d_1-d_2)z(t)+(-(a(t)-d_1)d_1^2+b(t)d_1)x(t) \\ & -g_1(t,x(t))-g_2(t,x(t-\tau(t)))+((a(t)-d_1)(d_1+d_2)-b(t)-d_2^2)y(t)) \\ & -(-(a(t)-d_1-d_2)z^*(t)+(-(a(t)-d_1)d_1^2+b(t)d_1)x^*(t)-g_1(t,x^*(t)) \\ & -g_2(t,x^*(t-\tau(t)))+((a(t)-d_1)(d_1+d_2)-b(t)-d_2^2)y^*(t))\} \\ \le {} & e^{\lambda t}\{(\lambda-(\inf_{t\in R}(a(t)-d_1-d_2)))|\bar{w}(t)|+L_1|\bar{u}(t)|+L_2|\bar{u}(t-\tau(t))| \\ & +\sup_{t\in R}|(a(t)-d_1)(d_1+d_2)-b(t)-d_2^2||\bar{v}(t)|\}. \end{aligned} \tag{2.11}$$

Let $M>1$ denote an arbitrary real number and set

$$\Theta=\max\{\|\varphi-\varphi^*\|, |y_0-y_0^*|, |z_0-z_0^*|\}>0.$$

It follows from (2.8) that

$$V_1(t) = |\bar{u}(t)|e^{\lambda t} < M\Theta,\ V_2(t) = |\bar{v}(t)|e^{\lambda t} < M\Theta,\ \text{and}\ \ V_3(t) = |\bar{w}(t)|e^{\lambda t} < M\Theta,$$

for all $t \in [-\bar{\tau},\ 0]$.

We claim that

$$V_1(t) = |\bar{u}(t)|e^{\lambda t} < M\Theta,\ V_2(t) = |\bar{v}(t)|e^{\lambda t} < M\Theta,\ \text{and}\ \ V_3(t) = |\bar{w}(t)|e^{\lambda t} < M\Theta, \tag{2.12}$$

for all $t > 0$. Contrarily, one of the following cases must occur.

Case I: There exists $T_1 > 0$ such that

$$V_1(T_1) = M\Theta, \quad \text{and} \quad V_i(t) < M\Theta, \quad \text{for all} \quad t \in [-\bar{\tau}, T_1), i = 1, 2, 3. \tag{2.13}$$

Case II: There exists $T_2 > 0$ such that

$$V_2(T_2) = M\Theta, \quad \text{and} \quad V_i(t) < M\Theta, \quad \text{for all} \quad t \in [-\bar{\tau}, T_2), i = 1, 2, 3. \tag{2.14}$$

Case III: There exists $T_3 > 0$ such that

$$V_3(T_3) = M\Theta, \quad \text{and} \quad V_i(t) < M\Theta, \quad \text{for all} \quad t \in [-\bar{\tau}, T_3), i = 1, 2, 3. \tag{2.15}$$

If **Case I** holds, together with (C_1) and (2.9), (2.13) implies that

$$0 \leq D^+(V_1(T_1)) \leq (\lambda - d_1)|\bar{u}(T_1)|e^{\lambda T_1} + |\bar{v}(T_1)|e^{\lambda T_1} \leq [\lambda - (d_1 - 1)]M\Theta. \tag{2.16}$$

Thus,

$$0 \leq \lambda - (d_1 - 1),$$

which contradicts (2.6). Hence, (2.12) holds.

If **Case II** holds, together with (C_1) and (2.10), (2.14) implies that

$$0 \leq D^+(V_2(T_2)) \leq (\lambda - d_2)|\bar{v}(T_2)|e^{\lambda T_2} + |\bar{w}(T_2)|e^{\lambda T_2} \leq [\lambda - (d_2 - 1)]M\Theta. \tag{2.17}$$

Thus,

$$0 \leq \lambda - (d_2 - 1),$$

which contradicts (2.6). Hence, (2.12) holds.

If **Case III** holds, together with (C_1) and (2.11), (2.15) implies that

$$\begin{aligned} 0 &\leq D^+(V_3(T_3)) \\ &\leq (\lambda - \inf_{t\in R}(a(t)-d_1-d_2))|\bar{w}(T_3)|e^{\lambda T_3} + L_1|\bar{u}(T_3)|e^{\lambda T_3} + L_2|\bar{u}(T_3-\tau(T_3)| \\ &\quad \cdot e^{\lambda(T_3-\tau(T_3))}e^{\lambda\tau(T_3)} + \sup_{t\in R}|(a(t)-d_1)(d_1+d_2)-b(t)-d_2^2||\bar{v}(T_3)|e^{\lambda T_3} \\ &\leq [\lambda - (d_3 - L_1 - L_2e^{\lambda\bar{\tau}})]M\Theta. \end{aligned} \tag{2.18}$$

Hence,

$$0 \leq \lambda - (d_3 - L_1 - L_2e^{\lambda\bar{\tau}}),$$

which contradicts (2.6). Hence, (2.12) holds. It follows that

$$\max\{|x(t)-x^*(t)|, |y(t)-y^*(t)|, |z(t)-z^*(t)|\}$$

$$\leq M\max\{\|\varphi(t)-\varphi^*(t)\|,\ |y_0-y_0^*|,\ |z_0-z_0^*|\}e^{-\lambda t}, \forall t>0. \tag{2.19}$$

This completes the proof of Lemma 2.2.

Remark 2.2. If $Z^*(t) = (x^*(t),\ y^*(t),\ z^*(t))$ be the T-anti-periodic solution of system (1.3), it follows from Lemma 2.2 and Definition 2 that $Z^*(t)$ is globally exponentially stable.

3. Main Results

In this section, we establish some results for the existence, uniqueness and exponential stability of the T-anti-periodic solution of (1.3).

Theorem 3.1. Suppose that (C_1) is satisfied. Then system (1.3) has exactly one T-anti-periodic solution $Z^*(t) = (x^*(t),\ y^*(t),\ z^*(t))$. Moreover, $Z^*(t)$ is globally exponentially stable.

Proof. Let $v(t) = (v_1(t), v_2(t), v_3(t)) = (x(t), y(t), z(t))$ be a solution of system (1.3) with initial conditions (2.1). By Lemma 2.1, the solution $(x(t), y(t), z(t))$ is bounded and (2.2) holds. From (1.4), for any natural number k, we obtain

$$\begin{aligned} ((-1)^{k+1}x(t+(k+1)T))' &= (-1)^{k+1}x'(t+(k+1)T) \\ &= (-1)^{k+1}[-d_1x(t+(k+1)T) + y(t+(k+1)T)] \\ &= -d_1((-1)^{k+1}x(t+(k+1)T)) + (-1)^{k+1}y(t+(k+1)T), \end{aligned} \tag{3.1}$$

$$
\begin{aligned}
((-1)^{k+1}y(t+(k+1)T))' &= (-1)^{k+1}y'(t+(k+1)T) \\
&= (-1)^{k+1}[-d_2y(t+(k+1)T)+z(t+(k+1)T)] \\
&= -d_2((-1)^{k+1}y(t+(k+1)T))+(-1)^{k+1}z(t+(k+1)T), \quad (3.2)
\end{aligned}
$$

and

$$
\begin{aligned}
& ((-1)^{k+1}z(t+(k+1)T))' \\
= \; & (-1)^{k+1}z'(t+(k+1)T) \\
= \; & (-1)^{k+1}[-(a(t+(k+1)T)-d_1-d_2)z(t+(k+1)T)+(-(a(t+(k+1)T)-d_1)d_1^2 \\
& +b(t+(k+1)T)d_1)x(t+(k+1)T)-g_1(t+(k+1)T,x(t+(k+1)T)) \\
& -g_2(t+(k+1)T,x(t+(k+1)T-\tau(t+(k+1)T)))+((a(t+(k+1)T)-d_1)(d_1+d_2) \\
& -b(t+(k+1)T)-d_2^2)y(t+(k+1)T)+p(t+(k+1)T)] \\
= \; & -(a(t)-d_1-d_2)((-1)^{k+1}z(t+(k+1)T))+(-(a(t)-d_1)d_1^2 \\
& +b(t)d_1)((-1)^{k+1}x(t+(k+1)T))-g_1(t,(-1)^{k+1}x(t+(k+1)T)) \\
& -g_2(t,(-1)^{k+1}x(t+(k+1)T-\tau(t)))+((a(t)-d_1)(d_1+d_2) \\
& -b(t)-d_2^2)((-1)^{k+1}y(t+(k+1)T))+p(t). \qquad (3.3)
\end{aligned}
$$

Thus, for any natural number k, $(-1)^{k+1}v(t+(k+1)T)$ are the solutions of system (1.3) on R. Then, by Lemma 2.2, there exists a constant $M>0$ such that

$$
\begin{aligned}
& |(-1)^{k+1}v_i(t+(k+1)T)-(-1)^kv_i(t+kT)| \\
\leq \; & Me^{-\lambda(t+kT)} \sup_{-\bar{\tau}\leq s\leq 0} \max_{1\leq i\leq 3} |v_i(s+T)+v_i(s)| \\
\leq \; & 2e^{-\lambda(t+kT)}M\frac{L^+}{\eta} \text{ for all } \; t+kT>0, \;\; i=1,\ 2,\ 3. \qquad (3.4)
\end{aligned}
$$

Hence, for any natural number m, we obtain

$$
(-1)^{m+1}v_i(t+(m+1)T)=v_i(t)+\sum_{k=0}^{m}[(-1)^{k+1}v_i(t+(k+1)T)-(-1)^kv_i(t+kT)], \qquad (3.5)
$$

where $i=1,\ 2,\ 3$.

In view of (3.4), we can choose a sufficiently large constant $N>0$ and a positive constant α such that

$$
|(-1)^{k+1}v_i(t+(k+1)T)-(-1)^kv_i(t+kT)| \leq \alpha(e^{-\lambda T})^k \text{ for all } \; k>N, \;\; i=1,\ 2,3, \qquad (3.6)
$$

on any compact set of R. It follows from (3.5) and (3.6) that $\{(-1)^m v(t+mT)\}$ uniformly converges to a continuous function $Z^*(t) = (x^*(t), y^*(t), z^*(t))^T$ on any compact set of R.

Now we will show that $Z^*(t)$ is T-anti-periodic solution of system (1.3). First, $Z^*(t)$ is T-anti-periodic, since

$$Z^*(t+T) = \lim_{m\to\infty} (-1)^m v(t+T+mT) = -\lim_{(m+1)\to\infty} (-1)^{m+1} v(t+(m+1)T) = -Z^*(t).$$

Next, we prove that $Z^*(t)$ is a solution of (1.1). In fact, together with the continuity of the right side of (1.3), (3.1), (3.2) and (3.3) imply that$\{((-1)^{m+1} v(t+(m+1)T))'\}$ uniformly converges to a continuous function on any compact set of R. Thus, letting $m \longrightarrow \infty$, we obtain

$$\begin{cases} \dfrac{dx^*(t)}{dt} = -d_1 x^*(t) + y^*(t), \\ \dfrac{dy^*(t)}{dt} = -d_2 y^*(t) + z^*(t), \\ \dfrac{dz^*(t)}{dt} = -(a(t)-d_1-d_2)z^*(t) + (-(a(t)-d_1)d_1^2 + b(t)d_1)x^*(t) - g_1(t, x^*(t)) \\ \qquad -g_2(t, x^*(t-\tau(t))) + ((a(t)-d_1)(d_1+d_2) - b(t) - d_2^2)y^*(t) + p(t). \end{cases} \tag{3.7}$$

Therefore, $Z^*(t)$ is a solution of (1.3). Finally, by Lemma 2.2, we can prove that $Z^*(t)$ is globally exponentially stable. This completes the proof.

4. An Example

Example 4.1. The following third-order nonlinear differential equation

$$x'''(t) + (9 - \frac{1}{|\sin t|+1})x''(t) + (\frac{-4}{|\sin t|+1} + 23)x'(t) + (18 - \frac{4}{|\sin t|+1})x(t)$$
$$+\sin x(t) + (\cos t)\cos x(t - e^{2|\sin t|}) = \sin t, \tag{4.1}$$

has exactly one π-anti-periodic solution, which is globally exponentially stable.

Proof. Set

$$y(t) = \frac{dx(t)}{dt} + 2x(t), z(t) = \frac{dy(t)}{dt} + 2y(t) \tag{4.2}$$

then we can transform (3.1) into the following equivalent system

$$\begin{cases} \dfrac{dx(t)}{dt} = -2x(t) + y(t), \\ \dfrac{dy(t)}{dt} = -2y(t) + z(t), \\ \dfrac{dz(t)}{dt} = -(5 - \dfrac{1}{|\sin t|+1})z(t) - \sin x(t) - (\cos t)\cos x(t - e^{2|\sin t|}) + y(t) + \sin t. \end{cases} \tag{4.3}$$

Then

(i) $|((-(a(t)-d_1)d_1^2+b(t)d_1)u-g_1(t,u))-((-(a(t)-d_1)d_1^2+b(t)d_1)v-g_1(t,v))|$
$= |(-\sin u)-(-\sin v)| \leq |u-v|,$ for all $t,u,v \in R$,

(ii) $|g_2(t,u))-g_2(t,v))| = |(\cos t)\cos u-(\cos t)\cos v| \leq |u-v|,$ for all $t,u,v \in R$,

(iii) $d_3 = \inf_{t\in R}(a(t)-d_1-d_2)-\sup_{t\in R}|(a(t)-d_1)(d_1+d_2)-b(t)-d_2^2|$
$= \inf_{t\in R}(5-\frac{1}{|\sin t|+1})-1| > 1+1.$

This implies that all assumptions needed in Theorem 3.1 are satisfied. Hence, system (4.3) has exactly one π-anti-periodic solution. Moreover, the π-anti-periodic solution is globally exponentially stable. It follows that equation (4.1) has exactly one π-antiperiodic solution, and all solutions of Eq. (4.1) exponentially converge to this π-anti-periodic solution.

Remark 4.1. Since Eq. (4.1) is a form of third-order nonlinear differential equation with varying time delays. One can observe that all the results in [8-13] and the references cited therein can not be applicable to prove that Eq. (4.1) has a unique anti-periodic periodic solution which is globally exponentially stable. Moreover, we propose a totally new approach to proving the existence of anti-periodic solutions of third-order nonlinear differential equation, which is different from [8] and the references therein. This implies that the results of this paper are essentially new.

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