Electronic Journal of Qualitative Theory of Differential Equations
2009, No. **18**, 1-9; **http://www.math.u-szeged.hu/ejqtde/**

A generalized Fucik type eigenvalue problem for p-Laplacian

Yuanji Cheng
School of Technology
Malmö University, SE-205 06 Malmö, Sweden
Email: yuanji.cheng@mah.se

Abstract

In this paper we study the generalized Fucik type eigenvalue for the boundary value problem of one dimensional $p-$Laplace type differential equations

$$\begin{cases} -(\varphi(u'))' = \psi(u), \quad -T < x < T; \\ u(-T) = 0, \quad u(T) = 0 \end{cases} \tag{$*$}$$

where $\varphi(s) = \alpha s_+^{p-1} - \beta s_-^{p-1}, \psi(s) = \lambda s_+^{p-1} - \mu s_-^{p-1}, p > 1$. We obtain a explicit characterization of Fucik spectrum $(\alpha, \beta, \lambda, \mu)$, i.e., for which the (*) has a nontrivial solution.

(1991) AMS Subject Classification: 35J65, 34B15, 49K20.

1 Introduction

In the study of nonhomogeneous semilinear boundary problem

$$\begin{cases} -\Delta u = f(u) + g(x), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

it has been discovered in [3, 7] that the solvability of another boundary value problem

$$\begin{cases} -\Delta u = \lambda u_+ - \mu u_-, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

where $u_+ = \max\{u, 0\}, \quad u_- = \max\{-u, 0\}$ plays an important role. Since then there are many works devoted to this subject [5, 13, 14, references therein], and the study has also been extended to the p-Laplacian

$$\begin{cases} -\Delta_p u = \lambda u_+^{p-1} - \mu u_-^{p-1}, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

where $\Delta_p =: \operatorname{div}\{|\nabla u|^{p-2}\nabla u\}, p > 1$ [2, 4, 6, 10, 12] and even associated trigonometrical $p-$sine and cosine functions have been studied [9]. In this paper, we are interested in generalization of such Fucik spectrum and will consider one dimensional boundary value problem

$$\begin{cases} -(\alpha(u')_+^{p-1} - \beta(u')_-^{p-1}))' = \lambda u_+^{p-1} - \mu u_-^{p-1}, & -T < x < T \\ u(-T) = 0, \quad u(T) = 0 \end{cases} \tag{1.1}$$

where $\alpha, \beta, \lambda, \mu > 0$ are parameters, and call $(\alpha, \beta, \lambda, \mu)$ the generalized Fucik spectrum, if (1.1) has a non-trivial solution. The problem is motivated by the study of two-point boundary value problem

$$\begin{cases} -(\varphi(u')))' = \psi(x, u), & -T < x < T \\ u(-T) = 0, \quad u(T) = 0 \end{cases} \tag{1.2}$$

and to our knowledge it is always assumed in the literature that φ is an odd function. Thus a natural question arises: what would happen, if the function φ is merely a homeomorphism, not necessarily odd function on $\mathbf{R}$? Here we shall first investigate the autonomous eigenvalue type problem and in the forthcoming treat non-resonance problem.

By a solution of (1.2) we mean that $u(x)$ is of C^1 such that $\varphi(u'(x))$ is differentiable and the equation (1.2) is satisfied pointwise almost everywhere. The main results of this paper are complete characterization of Fucik type eigenvalues, their associated eigenfunctions and observations of changes of frequency, amplitude of solutions, when they pass the mini- and maximum points respective change their signs (see the figures below and (3.8) in details). Let $\pi_p = \frac{2\pi}{p\sin(\pi/p)}$, then we have

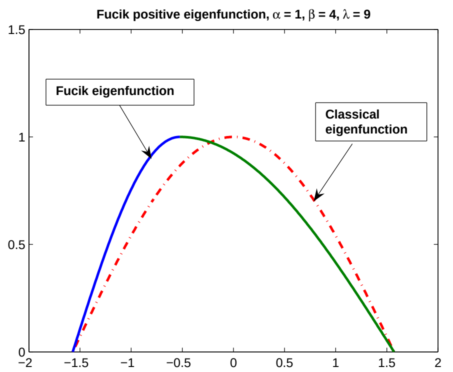
Theorem 1 *$(\alpha, \beta, \lambda, \mu)$ belongs to the generalized Fucik spectrum of (1.1), if and only if for some integer $k \geq 0$*

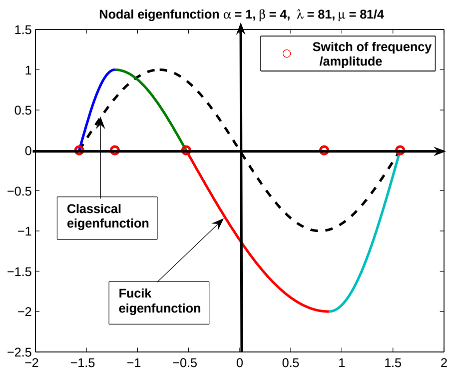
1) $(\sqrt[p]{\alpha} + \sqrt[p]{\beta})((k+1)\sqrt[p]{\lambda^{-1}} + k\sqrt[p]{\mu^{-1}})\pi_p = 2T$ and corresponding eigenfunction u is initially positive and has precisely $2k$ nodes.

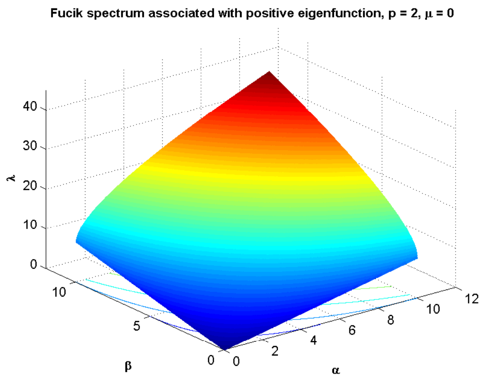
2) $(\sqrt[p]{\alpha} + \sqrt[p]{\beta})(k\sqrt[p]{\lambda^{-1}} + (k+1)\sqrt[p]{\mu^{-1}})\pi_p = 2T$ and the corresponding eigenfunction u is initially negative and has also $2k$ nodes

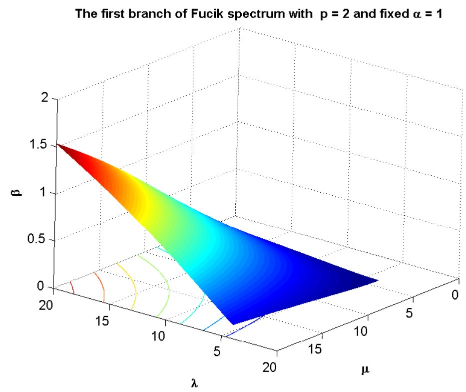
3) $(k+1)(\sqrt[p]{\alpha} + \sqrt[p]{\beta})(\sqrt[p]{\lambda^{-1}} + \sqrt[p]{\mu^{-1}})\pi_p = 2T$ and the corresponding eigenfunction u_1, u_2 has exact $2k+1$ nodes and u_1 is initially positive and u_2 is negative.

Moreover, the eigenfunctions are piecewise $p-$sine functions (see part 2 for definitions).









2 Review on 1D $p-$Laplacian

We shall review some basic results about eigenvalues and associated eigenfunctions for one dimensional $p-$Laplacian. Eigenfunctions are also called $p-$sine and -cosine functions, $\sin_p(x)$, $\cos_p(x)$, which have been discussed in details in [9, 11], but for our purpose we adopt the version in [1]. Let $\pi_p = \frac{2\pi}{p\sin\frac{\pi}{p}}$, the $p-$sine, $p-$cosine functions $\sin_p(x), \cos_p(x)$ are defined via

$$x = \int_0^{\sin_p(x)} \frac{dt}{\sqrt[p]{1-t^p}}, \quad 0 \le x \le \pi_p/2 \tag{2.1}$$

and extended to $[\pi_p/2, \pi_p]$ by $\sin_p(\pi_p/2+x) = \sin_p(\pi_p/2-x)$ and to $[-\pi_p, 0]$ by $\sin_p(x) = -\sin_p(-x)$ then finally extended to a $2\pi_p$ periodic function on the whole real line; Then $p-$cosine function is defined as $\cos_p x = \frac{d}{dx}(\sin_p x)$ and they have the properties:

$$\sin_p 0 = 0, \sin_p \pi_p/2 = 1; \quad \cos_p 0 = 1, \cos_p \pi_p/2 = 0.$$

They share several remarkable relations as ordinary trigonometric functions, for instance

$$|\sin_p x|^p + |\cos_p x|^p = 1.$$

But

$$\frac{d}{dx}(\cos_p x) = -|\tan_p x|^{p-2}\sin_p x \neq -\sin_p x, \quad \text{where } \tan_p x = \sin_p x/\cos_p x.$$

The eigenvalues of one dimensional p-Laplace operator

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' = \lambda |u(x)|^{p-2}u(x), & 0 \le x \le \pi_p \\ u(0) = 0, \quad u(\pi_p) = 0 \end{cases} \tag{2.2}$$

are $1^p, 2^p, 3^p, \cdots$ and the corresponding eigenfunctions are precisely

$$\sin_p(x), \quad \sin_p(2x), \quad \sin_p(3x), \cdots$$

and therefore we have another relation between $p-$cosine and sine functions

$$-(|\cos_p(kx)|^{p-2}\cos_p(kx))' = k|\sin_p(kx)|^{p-2}\sin_p(kx), \quad k = 1, 2, \cdots$$

Here we note that function $\sin_p(x)$ is a solution to the following problem

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' = |u(x)|^{p-2}u(x), & 0 < x < \pi_p/2 \\ u(0) = 0, \quad u'(\pi_p/2) = 0 \end{cases} \tag{2.3}$$

and any solution is of form $C\sin_p(x)$ for some constant C.

If the $p-$Laplacian is associated to another interval $[a, b]$, different from $[0, \pi_p]$, then by change of variables we see that eigenvalues and the associated eigenfunctions are

$$(\frac{k\pi_p}{b-a})^p, \quad \sin_p(k\frac{x-a}{b-a}\pi_p), \quad k = 1, 2, 3, \cdots. \tag{2.4}$$

3 Proof of Theorem 1

To understand the generalized Fucik spectrum of (1.1), we need to examine the following Dirichlet-Neumann boundary value problem

$$\begin{cases} -(\alpha(u')_+^{p-1} - \beta(u')_-^{p-1}))' = \lambda u_+^{p-1} - \mu u_-^{p-1}, & a < x < b \\ u(a) = 0, \quad u'(b) = 0 \end{cases} \tag{3.1}$$

We shall focus only on the constant sign solutions of (3.1) and note that there exist essentially only 'two' solutions, one positive and another negative, due to the positive homogeneity of (3.1).

If $u(x)$ is a positive solution of (3.1), then u must be increasing on $[a, b)$, because the equation (3.1) says that the function $g(x) =: \varphi(u'(x))$ is decreasing on $(a, b]$ and satisfies $g(b) = 0$, thus $g(x)$ is positive for all $x \in [a, b)$ and thus $u'(x)$ has to be positive, due to strict monotonicity of function $\varphi(s)$ and $\varphi(0) = 0$. It follows that u satisfies

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' = \frac{\lambda}{\alpha}|u(x)|^{p-2}u(x), & a < x < b \\ u(a) = 0, \quad u'(b) = 0 \end{cases} \tag{3.2}$$

It follows from (2.3) that $u(x) = C\sin_p(\frac{x-a}{2}\sqrt[p]{\frac{\lambda}{\alpha}})$ and α, λ satisfy $\pi_p\sqrt[p]{\alpha/\lambda} = b - a$. Likely if u is negative solution to (3.1), then

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' = \frac{\mu}{\beta}|u(x)|^{p-2}u(x), & a < x < b \\ u(a) = 0, \quad u'(b) = 0 \end{cases} \tag{3.3}$$

$u(x) = -C\sin_p(\frac{x-a}{2}\sqrt[p]{\frac{\mu}{\beta}})$ and β, μ satisfy $\pi_p\sqrt[p]{\beta/\mu} = b - a$.

In analogy we see that for the following boundary value problem

$$\begin{cases} -(\alpha(u')_+^{p-1} - \beta(u')_-^{p-1}))' = \lambda u_+^{p-1} - \mu u_-^{p-1}, & a < x < b \\ u'(a) = 0, \quad u(b) = 0 \end{cases} \tag{3.4}$$

the positive and negative solutions are $u(x) = D\sin_p(\frac{b-x}{b-a}\frac{\pi_p}{2})$ respectively $u(x) = -D\sin_p(\frac{b-x}{b-a}\frac{\pi_p}{2})$ and $\alpha, \beta, \lambda, \mu$ satisfy $\pi_p\sqrt[p]{\beta/\lambda} = b - a$ or $\pi_p\sqrt[p]{\alpha/\mu} = b - a$.

It follows from the above analysis that if u is a positive solution to (1.1) and $u(x_0) = \max u(x) := C$, then x_0 is determined by

$$\sqrt[p]{\alpha/\lambda}\pi_p = x_0 + T$$

and $\alpha, \beta, \lambda, \mu$ satisfy

$$L^+ := (\sqrt[p]{\alpha/\lambda} + \sqrt[p]{\beta/\lambda})\pi_p = 2T.$$

Furthermore the solution $u(x)$ is given by

$$u(x) = \begin{cases} C\sin_p(\frac{x+T}{2}\sqrt[p]{\frac{\lambda}{\alpha}}), & -T \le x \le -T + \pi_p\sqrt[p]{\frac{\alpha}{\lambda}} \\ C\sin_p(\frac{T-x}{2}\sqrt[p]{\frac{\lambda}{\beta}}), & T - \pi_p\sqrt[p]{\frac{\beta}{\lambda}} \le x \le T \end{cases} \tag{3.5}$$

and for the negative solution u of (1.1), then it holds

$$L^- := (\sqrt[p]{\alpha/\mu} + \sqrt[p]{\beta/\mu})\pi_p = 2T$$

$$u(x) = \begin{cases} -D\sin_p(\frac{x+T}{2}\sqrt[p]{\frac{\mu}{\beta}}), & -T \le x \le -T + \pi_p\sqrt[p]{\frac{\beta}{\mu}} \\ -D\sin_p(\frac{T-x}{2}\sqrt[p]{\frac{\mu}{\alpha}}), & T - \pi_p\sqrt[p]{\frac{\alpha}{\mu}} \le x \le T \end{cases} \tag{3.5'}$$

It is clear from (3.5) that $u(x)$ changes its frequency, when it passes its maximum point and is not symmetric anymore, which is in contrast to the symmetry principle of Gidas, Ni and Nirenberg [8].

For an initially positive nodal solution u to (1.1) with only one node at T_1, we get that $T_1, \alpha, \beta, \lambda, \nu$ satisfy

$$(\sqrt[p]{\alpha/\lambda} + \sqrt[p]{\beta/\lambda})\pi_p = T_1 + T.$$

$$(\sqrt[p]{\alpha} + \sqrt[p]{\beta})(\sqrt[p]{\lambda^{-1}} + \sqrt[p]{\mu^{-1}})\pi_p = 2T \tag{3.6}$$

If $C = \max_{x\in[-T,T_1]} u(x), -D = \max_{x\in[T,T]} |u(x)|$, then in view of identity

$$\frac{\alpha}{p'}(u'(x))^p_+ + \frac{\beta}{p'}(u'(x))^p_+ + \frac{\lambda}{p}(u(x))^p_+ + \frac{\mu}{p}(u'(x))^p_- = \text{ constant}, \forall x \in [-T, T] \tag{3.7}$$

we deduce

$$\lambda C^p = \mu D^p, \quad C = \sqrt[p]{\mu}t, \quad D = \sqrt[p]{\lambda}t, \quad \text{for some } t > 0.$$

Using the positive homogeneity of (1.1) we derive that the solution $u(x)$ is given in by

$$u(x) = \begin{cases} t\sqrt[p]{\mu}\sin_p(\frac{x+T}{2}\sqrt[p]{\frac{\lambda}{\alpha}}), & -T \le x \le -T + \pi_p\sqrt[p]{\frac{\alpha}{\lambda}} \\ t\sqrt[p]{\mu}\sin_p(\frac{T_1-x}{2}\sqrt[p]{\frac{\lambda}{\beta}}), & T_1 - \pi_p\sqrt[p]{\frac{\beta}{\lambda}} \le x \le T_1 \\ -t\sqrt[p]{\lambda}\sin_p(\frac{x-T_1}{2}\sqrt[p]{\frac{\mu}{\beta}}), & T_1 \le x \le T_1 + \pi_p\sqrt[p]{\frac{\beta}{\mu}} \\ -t\sqrt[p]{\lambda}\sin_p(\frac{T-x}{2}\sqrt[p]{\frac{\mu}{\alpha}}), & T - \pi_p\sqrt[p]{\frac{\alpha}{\mu}} \le x \le T \end{cases} \tag{3.8}$$

It follows from (3.8) that the differences between α and β are reflected by change of amplitudes between positive and negative waves at the switch between plus and minus.

For the switch from negative wave to positive wave, it holds

$$u(x) = \begin{cases} -\sqrt[p]{\lambda}\sin_p(\frac{x+T}{2}\sqrt[p]{\frac{\mu}{\beta}}), & -T \le x \le -T + \pi_p\sqrt[p]{\frac{\beta}{\mu}} \\ -\sqrt[p]{\lambda}\sin_p(\frac{T_1-x}{2}\sqrt[p]{\frac{\mu}{\alpha}}), & T_1 - \pi_p\sqrt[p]{\frac{\alpha}{\mu}} \le x \le T_1 \\ \sqrt[p]{\mu}\sin_p(\frac{x-T_1}{2}\sqrt[p]{\frac{\lambda}{\alpha}}), & T_1 \le x \le T_1 + \pi_p\sqrt[p]{\frac{\alpha}{\lambda}} \\ \sqrt[p]{\mu}\sin_p(\frac{T_1-x}{2}\sqrt[p]{\frac{\lambda}{\beta}}), & T - \pi_p\sqrt[p]{\frac{\beta}{\lambda}} \le x \le T \end{cases} \tag{3.8'}$$

where $T_1 = -T + (\sqrt[p]{\alpha/\mu} + \sqrt[p]{\beta/\mu})\pi_p$. In view of (3.8) and (3.8') we see that $u'(-T) = u'(T)$ and therefore can extend $u(x)$ to a $2T$-periodic function on the whole real line $\mathbf{R}$.

For any given integer $k \ge 1$. If u is a solution of (1.1) with $(2k+1)$ nodes, then it must have equal number of positive and negative (k+1) waves.

Let $T_1 < T_2 < \cdots < T_{2k+1}$ be the nodal points of u, it follows from (3.7) that all positive waves of u have same height and so are the same for negative waves. Moreover similarly as deriving (3.6) we get

$$(\sqrt[p]{\alpha} + \sqrt[p]{\beta})(\sqrt[p]{\lambda^{-1}} + \sqrt[p]{\mu^{-1}})\pi_p = T_{i+2} - T_i, \quad i = 1, \cdots, 2k-1.$$

Thereby $\alpha, \beta, \lambda, \mu$ satisfy

$$(k+1)(\sqrt[p]{\alpha} + \sqrt[p]{\beta})(\sqrt[p]{\lambda^{-1}} + \sqrt[p]{\mu^{-1}})\pi_p = 2T \tag{3.6'}$$

and the nodes are $T_{1+2i} = -T + (1+i)L^+ + iL^-, T_{2i} = -T + i(L^+ + L^-), i = 0, 1, 2, \cdots, k$. Furthermore let $T_0 = -T, T_{2k+2} = T$ then the initially positive solution $u(x)$ on $[T_{2i}, T_{2i+2}], i = 0, 1, 2, \cdots k$, is given by

$$u(x) = \begin{cases} t\sqrt[p]{\mu}\sin_p(\frac{x-T_i}{2}\sqrt[p]{\frac{\lambda}{\alpha}}), & T_i \le x \le T_i + \pi_p\sqrt[p]{\frac{\alpha}{\lambda}} \\ t\sqrt[p]{\mu}\sin_p(\frac{T_{i+1}-x}{2}\sqrt[p]{\frac{\lambda}{\beta}}), & T_{i+1} - \pi_p\sqrt[p]{\frac{\beta}{\lambda}} \le x \le T_{i+1} \\ -t\sqrt[p]{\lambda}\sin_p(\frac{x-T_1}{2}\sqrt[p]{\frac{\mu}{\beta}}), & T_{i+1} \le x \le T_{i+1} + \pi_p\sqrt[p]{\frac{\beta}{\mu}} \\ -t\sqrt[p]{\lambda}\sin_p(\frac{T_{i+2}-x}{2}\sqrt[p]{\frac{\mu}{\alpha}}), & T_{i+2} - \pi_p\sqrt[p]{\frac{\alpha}{\mu}} \le x \le T_{i+2} \end{cases} \tag{3.9}$$

and the initially negative solution u by

$$u(x) = \begin{cases} -t\sqrt[p]{\lambda}\sin_p(\frac{x-T_i}{2}\sqrt[p]{\frac{\mu}{\beta}}), & T_i \le x \le T_i + \pi_p\sqrt[p]{\frac{\beta}{\mu}} \\ -t\sqrt[p]{\lambda}\sin_p(\frac{T_{i+1}-x}{2}\sqrt[p]{\frac{\mu}{\alpha}}), & T_{i+1} - \pi_p\sqrt[p]{\frac{\alpha}{\mu}} \le x \le T_{i+1} \\ t\sqrt[p]{\mu}\sin_p(\frac{x-T_{i+1}}{2}\sqrt[p]{\frac{\lambda}{\alpha}}), & T_{i+1} \le x \le T_{i+1} + \pi_p\sqrt[p]{\frac{\alpha}{\lambda}} \\ t\sqrt[p]{\mu}\sin_p(\frac{T_{i+2}-x}{2}\sqrt[p]{\frac{\lambda}{\beta}}), & T_{i+2} - \pi_p\sqrt[p]{\frac{\beta}{\lambda}} \le x \le T_{i+2} \end{cases} \tag{3.9'}$$

where $T_{1+2i} = -T + (1+i)L^- + iL^+, T_{2i} = -T + i(L^+ + L^-), i = 0, 1, 2, \cdots, k$.

If u has $2k$ nodes, then there are two possibilities 1) $(k+1)$ positive waves and k negative waves, 2) k positive waves and $(k+1)$ negative waves. In 1) the solution should be initially positive and be initially negative in 2). It follows then

1)

$$(\sqrt[p]{\alpha}+\sqrt[p]{\beta})((k+1)\sqrt[p]{\lambda^{-1}}+k\sqrt[p]{\mu^{-1}})\pi_p=2T \tag{3.10}$$

and the solution u on $[-T,T-L^-]$ is $(2T-L_-)/k$-periodic and is given by (3.9) on $[T_i,T_{i+2}], i=0,1,\cdots,2k-2$, and on $[T_{2k},T], u(x)$ is given by

$$u(x)=\begin{cases} t\sqrt[p]{\mu}\sin_p(\frac{x-T_{2k}}{2}\sqrt[p]{\frac{\lambda}{\alpha}}), & T_{2k}\le x\le T_{2k}+\pi_p\sqrt[p]{\frac{\alpha}{\lambda}} \\ t\sqrt[p]{\mu}\sin_p(\frac{T-x}{2}\sqrt[p]{\frac{\lambda}{\beta}}), & T-\pi_p\sqrt[p]{\frac{\beta}{\lambda}}\le x\le T \end{cases} \tag{3.11}$$

2) For an initially negative solution with $2k$ nodes, then

$$(\sqrt[p]{\alpha}+\sqrt[p]{\beta})(k\sqrt[p]{\lambda^{-1}}+(k+1)\sqrt[p]{\mu^{-1}})\pi_p=2T \tag{3.12}$$

and on $[T_i,T_{i+2}]$ the solution u is given by (3.9') for $i=0,1,\cdots,2k-2$, and on $[T_{2k},T], u(x)$ is given by

$$u(x)=\begin{cases} -t\sqrt[p]{\lambda}\sin_p(\frac{x-T_{2k}}{2}\sqrt[p]{\frac{\mu}{\beta}}), & T_{2k}\le x\le T_{2k}+\pi_p\sqrt[p]{\frac{\beta}{\mu}} \\ -t\sqrt[p]{\lambda}\sin_p(\frac{T-x}{2}\sqrt[p]{\frac{\mu}{\alpha}}), & T-\pi_p\sqrt[p]{\frac{\alpha}{\mu}}\le x\le T. \end{cases} \tag{3.11'}$$

So the proof is complete.

4 A final remark

In the study of nontrivial solutions to one dimensional nonlinear differential equation

$$-(\varphi(x,u'))'=f(x,u) \tag{4.1}$$

one usually adopt the notation of solution by (4.1) being satisfied pointwise, which in turn ensures per definition only C^1 smoothness of solution. Of course, one expects higher order smoothness of the solutions. Here we shall examine this question for a very special case, namely $\varphi(x,s)=\alpha(x)s_+^{p-1}-\beta(x)s_-^{q-1}, p,q>1$ e.g.,

$$-(\alpha(x)(u')_+^{p-1}-\beta(x)(u')_-^{q-1})'=f(x,u) \tag{4.2}$$

If we assume that the equation (4.2) is satisfied pointwise and moreover $\alpha,\beta>0$ are also C^1, then any solution u is obviously C^2 for any point x where $u'(x)\neq 0$. So in order to get differentiability of u we need a closer examination at those points where $u'(x)=0$.

Let x_0 be a critical point of $u(x), u_0=u(x_0)$, if $p=q=2$, then we easily deduce from the equation (4.2) that for small $\delta>0$

$$u'(x_0-\delta)=f(x_0,u_0)/\alpha(x_0)(\delta+o(\delta)); \quad -u'(x_0+\delta)=f(x_0,u_0)/\beta(x_0)(\delta+o(\delta))$$

thereafter $u''(x)$ has a jump at x_0 since $\alpha \neq \beta$ and thus $u \in C^{1,1}$, but not C^2.

In general, for any critical point $x = x_0$ of $u(x)$, we have the following asymptotic as $\delta \to 0$

$$\begin{cases} u'(x_0 - \delta) &= C_1 \delta^{1/(p-1)}(1 + o(1)) \\ u'(x_0 + \delta) &= -C_2 \delta^{1/(q-1)}(1 + o(1)) \end{cases}$$

where $C_1 = \sqrt[p-1]{f(x_0, u_0)/\alpha(x_0)}$, $C_2 = \sqrt[q-1]{f(x_0, u_0)/\beta(x_0)}$. In view of the above estimates, we deduce that

1. If $1 < p, q < 2$ then $u \in C^2$
2. If $\max\{p, q\} = 2$ then $u \in C^{1,1}$
3. If $2 < p, q$ then $u \in C^{1,\varepsilon}$, $\varepsilon = \min\{\frac{1}{p-1}, \frac{1}{q-1}\}$.

Acknowledgement

This work is partially supported by the project Nr 20-06/120 for promotion of research at Malmö University and also the G S Magnusons fond, the Swedish Royal Academy of Science.

The author thanks the referee for careful reading and suggestions which lead to improvement of the presentation.

References

[1] Bennewitz C. and Saito Y. An embedding norm and the Lindqvist trigonometric functions. *Electronic J. Diff. Equa.* (2002) No. 86, 1-6.

[2] Cuesta M., de Figueriredo D., Gossez, J.-P., The beginning of the Fucik spektrum for the $p-$Laplacian. *J. Diff. Equa.* , 159:2 (1999) 212-238.

[3] Dancer N., On the Dirichlet problem for weakly nonlinear elliptic differential equations. *Proc. Royal Soci. Edinburgh*, 76 (1977) 283-300.

[4] Dancer, E. N., Remarks on jumping nonlinearities. *Topics in nonlinear analysis,* Birkhäuser, Basel, 1999, pp. 101-116

[5] de Figueiredo D., Gossez J.-P., On the first curve of the Fucik spectrum of an elliptic operator. *Diff. and Integral Equa.* 7:5 (1994), 1285–1302.

[6] Drábek P., Solvability and bifurcations of nonlinear equations. *Pitman Research Notes in Math.* vol 264 Longman, New York, 1992.

[7] Fucik S., Boundary value problems with jumping nonlinearities, *Casopis Pest. Mat.* 101 (1976) 69–87.

[8] Gidas B., Ni N. and Nirenberg L., Symmetry and related properties via the maximum principle. *Comm. Math. Phys.* 68 (1979) 209-243.

[9] Lindqvist, P. Some remarkable sine and cosine functions, *Ricerche di Matematica*, XLIV (1995) 269-290.

[10] Manasévich, R. and Mawhin J., The spectrum of p-Laplacian systems under Dirichlet, Neumann and periodic boundary conditions. Morse theory, minimax theory and their applications to nonlinear differential equations, 201–216, *New Stud. Adv. Math.*, 1, Int. Press, Somerville, MA, 2003.

[11] Otani M., A remark on certain nonlinear elliptic equations, *Proc. of the Faculty of Science, Tokai Univ.*, 19(1984) 23-28.

[12] Perera K., On the Fucik Spectrum of the p-Laplacian *NoDEA* 11:2 (2004) 259-270.

[13] Sadyrbaev F. and Gritsans A. Nolinear spectra for parameter dependent ordinary differential equations, *Nonlinear analysis: modelling and control,* 12:2(2007) 253-267.

[14] Schechter M. The Fucik spectrum. *Indiana Univ. Math. J.,* 43(1994) 1139-1157.

(Received January 15, 2008)