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On the stability of a fractional-order differential equation with nonlocal initial condition

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Abstract

The topic of fractional calculus (integration and differentiation of fractional-order), which concerns singular integral and integro-differential operators, is enjoying interest among mathematicians, physicists and engineers (see [1]-[2] and [5]-[14] and the references therein). In this work, we investigate initial value problem of fractional-order differential equation with nonlocal condition. The stability (and some other properties concerning the existence and uniqueness) of the solution will be proved.

Key words: Fractional calculus; Banach contraction fixed point theorem; Nonlocal condition; Stability.

1 Introduction

Let $L_1[a,b]$ denote the space of all Lebesgue integrable functions on the interval $[a,b]$, $0 \leq a < b < \infty$, with the L_1-norm $||x||_{L_1} = \int_0^1 |x(t)|\ dt$.

Definition 1.1 *The fractional (arbitrary) order integral of the function $f \in L_1[a,b]$ of order $\beta \in R^+$ is defined by (see [11] - [14])*

$$I_a^\beta\ f(t)\ =\ \int_a^t \frac{(t\ -\ s)^{\beta\ -\ 1}}{\Gamma(\beta)}\ f(s)\ ds,$$

where $\Gamma(.)$ is the gamma function.

Definition 1.2 *The (Caputo) fractional-order derivative D^α of order $\alpha \in (0,1]$ of the function $g(t)$ is defined as (see [12] - [14])*

$$D_a^\alpha\ g(t)\ =\ I_a^{1\ -\ \alpha}\ \frac{d}{dt}\ g(t), \quad t\ \in\ [a,b].$$

Now the following theorem (some properties of the fractional-order integration and the fractional-order differentiation) can be easily proved.

Theorem 1.1 *Let $\beta,\ \gamma\ \in R^{+}$ and $\alpha\ \in (0,1]$. Then we have:*

(i) $I_a^{\beta}\ :\ L_1 \to L_1$, *and if* $f(t) \in L_1$, *then* $I_a^{\gamma}\ I_a^{\beta}\ f(t)\ =\ I_a^{\gamma+\beta}\ f(t)$.

(ii) $\lim_{\beta \to n}\ I_a^{\beta}\ f(t)\ =\ I_a^{n}\ f(t)\ ,\ n\ =\ 1,\ 2,\ 3,\ ...$ *uniformly.*

If $f(t)$ *is absolutely continuous on* $[a,b]$, *then*

(iii) $\lim_{\alpha \to 1}\ D_a^{\alpha}\ f(t)\ =\ D\ f(t)$

(iv) If $f(t) = k \neq 0$, k *is a constant, then* $D_a^{\alpha}\ k\ =\ 0$.

In ([3]) the nonlocal initial value problem for first-order differential inclusions:

$$\begin{cases} x'(t)\ \in\ F(t,x(t)), \quad t\ \in\ (0,1], \\ x(0)\ +\ \sum_{k=1}^{m}\ a_k\ x(t_k)\ =\ x_0, \end{cases}$$

was studied, where $F\ :\ J \times \Re \to 2^{\Re}$ is a set-valued map, $J = [0,1]$, $x_0 \in \Re$ is given, $0 < t_1 < t_2 < \cdots < t_m < 1$, and $a_k \neq 0$ for all $k = 1, 2, \cdots, m$.

Our objective in this paper is to investigate, by using the Banach contraction fixed point theorem, the existence of a unique solution of the following fractional-order differential equation:

$$D^{\alpha}\ x(t)\ =\ c(t)\ f(x(t))\ +\ b(t), \tag{1}$$

with the nonlocal condition:

$$x(0)\ +\ \sum_{k=1}^{m}\ a_k\ x(t_k)\ =\ x_0, \tag{2}$$

where $x_0 \in \Re$ and $0 < t_1 < t_2 < \cdots < t_m < 1$, and $a_k \neq 0$ for all $k = 1, 2, \cdots, m$. Then we will prove that this solution is uniformly stable.

2 Existence of solution

Here the space $C[0,1]$ denotes the space of all continuous functions on the interval $[0,1]$ with the supremum norm $||y|| = \sup_{t\in[0,1]} |y(t)|$.

To facilitate our discussion, let us first state the following assumptions:

(i) $|\frac{\partial f}{\partial x}| \leq k$,

(ii) $c(t)$ is a function which is absolutely continuous,

(iii) $b(t)$ is a function which is absolutely continuous.

Definition 2.1 *By a solution of the initial value Problem (1) - (2) we mean a function* $x \in C[0,1]$ *with* $\frac{dx}{dt} \in L_1[0,1]$.

Theorem 2.1 *If the above assumptions (i) - (iii) are satisfied such that*

$$1 + \sum_{k=1}^{m} a_k \neq 0 \quad \text{and} \quad A < \frac{\Gamma(1+\alpha)}{k\ ||c||},$$

$$\text{where } A = 1 + |a| \sum_{k=1}^{m} |a_k| \quad \text{and} \quad a = \left(1 + \sum_{k=1}^{m} a_k\right)^{-1},$$

then the initial value Problem (1) - (2) has a unique solution.

Proof: For simplicity let $c(t)f(x(t)) + b(t) = g(t, x(t))$.
If $x(t)$ satisfies (1) - (2), then by using the definitions and properties of the fractional-order integration and fractional-order differentiation equation (1) can be written as

$$I^{1-\alpha}\, x'(t) = g(t, x(t)).$$

Operating by I^α on both sides of the last equation, we obtain

$$x(t) - x(0) = I^\alpha\, g(t, x(t)),$$

by substituting for the value of $x(0)$ from (2), we get

$$x(t) = x_0 - \sum_{k=1}^{m} a_k\, x(t_k) + I^\alpha\, g(t, x(t)). \tag{3}$$

If we put $t = t_k$ in (3), we obtain

$$x(t_k) = x_0 - \sum_{k=1}^{m} a_k\, x(t_k) + I^\alpha\, g(t, x(t))|_{t=t_k}. \tag{4}$$

Then subtract (3) from (4) to get

$$x(t_k) = x(t) - I^\alpha\, g(t, x(t)) + I^\alpha\, g(t, x(t))|_{t=t_k}. \tag{5}$$

Substitute from (5) in (3), we get

$$\begin{aligned}
x(t) &= x_0 + I^\alpha\, g(t, x(t)) \\
&- \sum_{k=1}^{m} a_k\, \left(x(t) - I^\alpha\, g(t, x(t)) + I^\alpha\, g(t, x(t))|_{t=t_k} \right) \\
&= x_0 + I^\alpha\, g(t, x(t)) \\
&- \sum_{k=1}^{m} a_k\, x(t) + \sum_{k=1}^{m} a_k\, I^\alpha\, g(t, x(t)) - \sum_{k=1}^{m} a_k\, I^\alpha\, g(t, x(t))|_{t=t_k}, \\
\left(1 + \sum_{k=1}^{m} a_k\right) x(t) &= x_0 - \sum_{k=1}^{m} a_k\, I^\alpha\, g(t, x(t))|_{t=t_k} + \left(1 + \sum_{k=1}^{m} a_k\right) I^\alpha\, g(t, x(t)), \\
x(t) &= a \left(x_0 - \sum_{k=1}^{m} a_k\, I^\alpha\, g(t, x(t))|_{t=t_k} \right) + I^\alpha\, g(t, x(t)). \qquad (6)
\end{aligned}$$

Now define the operator $T : C \to C$ by

$$Tx(t) \ = \ a\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} \{c(s)f(x(s)) + b(s)\} \ ds\right) + I^{\alpha}\{c(s)f(x(s)) + b(s)\}. \tag{7}$$

Let $x, \ y \ \in \ C$, then

$$\begin{aligned}
Tx(t) - Ty(t) \ &= \ - \ a \ \sum_{k=1}^{m} a_k \ \int_0^{t_k} \ \frac{(t_k \ - \ s)^{\alpha-1}}{\Gamma(\alpha)} \ c(s) \ f(x(s)) \ ds \\
&+ \ a \ \sum_{k=1}^{m} a_k \ \int_0^{t_k} \ \frac{(t_k \ - \ s)^{\alpha-1}}{\Gamma(\alpha)} \ c(s) \ f(y(s)) \ ds \\
&+ \ \int_0^{t} \ \frac{(t \ - \ s)^{\alpha-1}}{\Gamma(\alpha)} \ c(s) \ \{f(x(s)) \ - \ f(y(s))\} \ ds \\
&= \ - \ a \ \sum_{k=1}^{m} a_k \ \int_0^{t_k} \ \frac{(t_k \ - \ s)^{\alpha-1}}{\Gamma(\alpha)} \ c(s) \ \{f(x(s)) \ - \ f(y(s))\} \ ds \\
&+ \ \int_0^{t} \ \frac{(t \ - \ s)^{\alpha-1}}{\Gamma(\alpha)} \ c(s) \ \{f(x(s)) \ - \ f(y(s))\} \ ds, \\
|Tx(t) - Ty(t)| \ &\leq \ k \ |a| \ \sum_{k=1}^{m} |a_k| \ \int_0^{t_k} \ \frac{(t_k \ - \ s)^{\alpha-1}}{\Gamma(\alpha)} \ |c(s)| \ |x(s) \ - \ y(s)| \ ds \\
&+ \ k \ \int_0^{t} \frac{(t \ - \ s)^{\alpha-1}}{\Gamma(\alpha)} \ |c(s)| \ |x(s) \ - \ y(s)| \ ds \\
&\leq \ k \ |a| \ \sum_{k=1}^{m} |a_k| \ \sup_t |c(t)| \ \sup_t |x(t) \ - \ y(t)| \ \int_0^{t_k} \frac{(t_k \ - \ s)^{\alpha-1}}{\Gamma(\alpha)} \ ds \\
&+ \ k \ \sup_t |c(t)| \ \sup_t |x(t) \ - \ y(t)| \ \int_0^{t} \frac{(t \ - \ s)^{\alpha-1}}{\Gamma(\alpha)} \ ds \\
&\leq \ k \ |a| \ \sum_{k=1}^{m} |a_k| \ \|c\| \ \|x \ - \ y\| \ \frac{t_k^{\alpha}}{\Gamma(1+\alpha)} \\
&+ \ k \ \|c\| \ \|x \ - \ y\| \ \frac{t^{\alpha}}{\Gamma(1+\alpha)} \\
&\leq \ \frac{k}{\Gamma(1+\alpha)} \ \left(1 \ + \ |a| \ \sum_{k=1}^{m} |a_k|\right) \ \|c\| \ \|x \ - \ y\| \\
&\leq \ \frac{k \ A \ \|c\|}{\Gamma(1+\alpha)} \ \|x \ - \ y\| \ = \ K||x \ - \ y||.
\end{aligned}$$

but since $K = \frac{kA\|c\|}{\Gamma(1+\alpha)} < 1$, then we get

$$||Tx \ - \ Ty|| \ < \ K \ ||x \ - \ y||,$$

which proves that the map $T : C \to C$ is contraction. Applying the Banach contraction fixed point theorem we deduce that (7) has a unique fixed point $x \in C[0, 1]$.

Now, differentiate (6) to obtain

$$
\begin{aligned}
x'(t) &= \frac{d}{dt}\, I^{\alpha}\ (\ c(t)\ f(x(t))\ +\ b(t)\ ) \\
&= (c(t)\ f(x(t))\ +\ b(t))|_{t=0}\ \frac{t^{\alpha-1}}{\Gamma(\alpha)}\ +\ I^{\alpha}\ \frac{d}{dt}\ (c(t)\ f(x(t))\ +\ b(t)) \\
&= K_1\ \frac{t^{\alpha-1}}{\Gamma(\alpha)} + I^{\alpha}\ (c'(t)\ f(x(t))\ +\ \frac{\partial f}{\partial x}\ x'(t)\ c(t)\ +\ b'(t)), \\
\int_0^1 |x'(t)|\ dt &\leq \frac{K_1}{\Gamma(1+\alpha)}\ t^{\alpha}|_0^1 \\
&+\ \int_0^1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left| c'(s)\ f(x(s))\ +\ \frac{\partial f}{\partial x}\ x'(s)\ c(s)\ +\ b'(s) \right| ds\ dt \\
&= \frac{K_1}{\Gamma(1+\alpha)} + \int_0^1 \left| c'(s)\ f(x(s)) + \frac{\partial f}{\partial x}\ x'(s)\ c(s) + b'(s) \right| \int_s^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ dt\ ds \\
&\leq \frac{K_1}{\Gamma(1+\alpha)}\ +\ \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left| c'(s)\ f(x(s))\ +\ \frac{\partial f}{\partial x}\ x'(s)\ c(s)\ +\ b'(s) \right| ds, \\
\|x'\|_{L_1} &\leq \frac{K_1}{\Gamma(1+\alpha)}\ +\ \frac{1}{\Gamma(1+\alpha)}\ (\|c'\|_{L_1}\ \|f\|\ +\ k\ \|x'\|_{L_1}\ \|c\|\ +\ \|b'\|_{L_1})\,, \\
\left(1 - \frac{k\|c\|}{\Gamma(1+\alpha)}\right) \|x'\|_{L_1} &\leq \frac{K_1}{\Gamma(1+\alpha)}\ +\ \frac{1}{\Gamma(1+\alpha)}\ (\|c'\|_{L_1}\ \|f\|\ +\ \|b'\|_{L_1})\,, \\
\|x'\|_{L_1} &\leq \left(1 - \frac{k\|c\|}{\Gamma(1+\alpha)}\right)^{-1} \left(\frac{K_1}{\Gamma(1+\alpha)}\ +\ \frac{1}{\Gamma(1+\alpha)}\ (\|c'\|_{L_1}\ \|f\|\ +\ \|b'\|_{L_1})\right).
\end{aligned}
$$

Therefore we obtain that $x' \in L_1[0,1]$.
To complete the equivalence of equation (6) with the initial value problem (1) - (2), let $x(t)$ be a solution of (6), differentiate both sides, and get

$$
\begin{aligned}
x'(t) &= \frac{d}{dt}\ I^{\alpha}\ g(t, x(t)) \\
&= g(t, x(t))|_{t=0}\ \frac{t^{\alpha-1}}{\Gamma(\alpha)}\ +\ I^{\alpha}\ \frac{d}{dt}\ g(t, x(t)).
\end{aligned}
$$

Then operate by $I^{1-\alpha}$ on both sides to obtain

$$
D^{\alpha}\ x(t)\ =\ g(t, x(t)).
$$

And if $t = 0$ we find that the nonlocal condition (2) is satisfied. Which proves the equivalence. ■

3 Stability

In this section we study the uniform stability (see [1], [4] and [6]) of the solution of the initial-value problem (1) - (2).

Theorem 3.1 *The solution of the initial-value problem (1) - (2) is uniformly stable*

Proof: Let $x(t)$ be a solution of

$$x(t) \;=\; a\left(x_0-\sum_{k=1}^{m} a_k\int_0^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}\{c(s)f(x(s))+b(s)\}\;ds\right)+I^{\alpha}\{c(s)f(x(s))+b(s)\} \tag{8}$$

and let $\widetilde{x}(t)$ be a solution of equation (8) such that $\widetilde{x}(0)=\widetilde{x}_0-\sum_{k=1}^{m} a_k\;\widetilde{x}(t_k)$. Then

$$\begin{aligned}
x(t)-\widetilde{x}(t) &= a\;(x_0-\widetilde{x}_0)-a\sum_{k=1}^{m}a_k\int_0^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}\;c(s)\;f(x(s))\;ds\\
&+ a\sum_{k=1}^{m}a_k\int_0^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}\;c(s)\;f(\widetilde{x}(s))\;ds\\
&+ \int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\;c(s)\;\{f(x(s))-f(\widetilde{x}(s))\}\;ds,\\
|x(t)-\widetilde{x}(t)| &\le |a|\;|x_0-\widetilde{x}_0|\\
&+ |a|\sum_{k=1}^{m}|a_k|\int_0^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}\;|c(s)|\;|f(x(s))-f(\widetilde{x}(s))|\;ds\\
&+ \int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\;|c(s)|\;|f(x(s))-f(\widetilde{x}(s))|\;ds\\
&\le |a|\;|x_0-\widetilde{x}_0|\\
&+ k\;|a|\sum_{k=1}^{m}|a_k|\;\sup_t|c(t)|\int_0^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}\;|(x(s)-\widetilde{x}(s)|\;ds\\
&+ k\;\sup_t|c(t)|\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\;|x(s)-\widetilde{x}(s)|\;ds\\
&\le |a|\;|x_0-\widetilde{x}_0|\\
&+ k\;|a|\;\|c\|\sum_{k=1}^{m}|a_k|\;\sup_t|x(t)-\widetilde{x}(t)|\int_0^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}\;ds\\
&+ k\;\|c\|\;\sup_t|x(t)-\widetilde{x}(t)|\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\;ds,\\
\|x-\widetilde{x}\| &\le |a|\;|x_0-\widetilde{x}_0|+k\;|a|\;\|c\|\sum_{k=1}^{m}|a_k|\;\|x-\widetilde{x}\|\;\frac{t_k^{\alpha}}{\Gamma(1+\alpha)}\\
&+ k\;\|c\|\;\|x-\widetilde{x}\|\;\frac{t^{\alpha}}{\Gamma(1+\alpha)}\\
&\le |a|\;|x_0-\widetilde{x}_0|+\frac{k\;\|c\|}{\Gamma(1+\alpha)}\left(1+|a|\sum_{k=1}^{m}|a_k|\right)\;\|x-\widetilde{x}\|
\end{aligned}$$

$$
\begin{aligned}
&= |a|\ |x_0 - \widetilde{x}_0| + \frac{k\ A\ \|c\|}{\Gamma(1+\alpha)}\ \|x - \widetilde{x}\|,\\
\left(1 - \frac{k\ A\ \|c\|}{\Gamma(1+\alpha)}\right) ||x - \widetilde{x}|| &\leq |a|\ |x_0 - \widetilde{x}_0|,\\
||x - \widetilde{x}|| &\leq \left(1 - \frac{k\ A\ \|c\|}{\Gamma(1+\alpha)}\right)^{-1} |a|\ |x_0 - \widetilde{x}_0|.
\end{aligned}
$$

Therefore, if $|\ x_0 - \widetilde{x}_0\ | < \delta(\varepsilon)$, then $||\ x - \widetilde{x}\ || < \varepsilon$, which complete the proof of the theorem. ■

References

[1] Abd El-Salam, Sh. A. and El-Sayed, A. M. A. On the stability of some fractional-order non-autonomous systems, *EJQTDE*, 6 (2007) 1-14.

[2] Ahmed, E., El-Sayed, A. M. A. and El-Saka, H. A. A. Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models, *Journal of Mathematical Analysis and Applications*, Vol. 325 (2007) 542-553.

[3] Boucherif, A. First-Order Differential Inclusions with Nonlocal Initial Conditions, *Applied Mathematics Letters*, 15 (2002) 409-414.

[4] Corduneanu, C. *Principles of Differential and Integral equations*, Allyn and Bacon, Inc., Boston, Massachusetts (1971).

[5] El-Sayed, A. M. A. and Gaafar, F. M. Fractional calculus and some intermediate physical processes, *Appl. Math. and Compute*, Vol. 144 (2003).

[6] El-Sayed, A. M. A. and Abd El-Salam, Sh. A. Weighted Cauchy-type problem of a functional differ-integral equation, *EJQTDE*, 30 (2007) 1-9.

[7] El-Sayed, A. M. A., El-Mesiry, A. E. M. and El-Saka H. A. A. On the fractional-order logistic equation, *J. Appl. Math. Letters*, Vol. 20 (2007) 817-823.

[8] Gorenflo, R. and Mainardy, F. Fractional calculus: Integral and Differential Equations of fractional-order, Vol. 378 *of CISM courses and lectures, Springer-Verlag, Berlin*, (1997), 223-276

[9] Hilfer, R. *Application of fractional calculus in physics* World Scientific, Singapore (2000).

[10] Kilbas, A. A., Srivastava and Trujillo, J. J. *Theory and applications of fractional differential equations*, ELSEVIER (2006)

[11] Miller, K. S. and Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley, New York (1993).

[12] Podlubny, I. and EL-Sayed, A. M. A. On two definitions of fractional calculus, *Preprint UEF 03-96 (ISBN 80-7099-252-2), Slovak Academy of Science-Institute of Experimental phys.* (1996).

[13] Podlubny, I. *Fractional Differential Equations*, Acad. press, San Diego-New York-London (1999).

[14] Samko, S., Kilbas, A. and Marichev, O. L. *Fractional Integrals and Derivatives*, Gordon and Breach Science Publisher, (1993).

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