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Oscillation and global asymptotic stability of a neuronic equation with two delays

Hassan A. El-Morshedy *

Department of Mathematics, Abha Teachers' College,

Abha P. O. Box 249, Saudi Arabia

(elmorshedy@yahoo.com)

B. M. El-Matary

Department of Mathematics, Damietta Faculty of Science, New Damietta 34517, Egypt

(bassantmarof@yahoo.com)

Abstract

In this paper we study the oscillatory and global asymptotic stability of a single neuron model with two delays and a general activation function. New sufficient conditions for the oscillation and nonoscillation of the model are given. We obtain both delay-dependent and delay-independent global asymptotic stability criteria. Some of our results are new even for models with one delay.

Keywords. neuronic equation; oscillation; stability; single neuron.
AMS Subject Classifications. 34K11; 34K20; 34K60; 92B20.

1 Introduction

Delay differential equations have been used to describe the dynamics of a single neuron to take into account the processing time. Pakdaman *et al* [15] considered a neuron that has a delayed self-connection with weight $a > 0$ and delay τ. Implementing a decay rate λ in the model, they found that the neuron activation at time t; say $x(t)$, follows the delay differential equation

$$\frac{dx(t)}{dt} = -\lambda x(t) + K + af(x(t-\tau)), \tag{1.1}$$

*Corresponding author. On leave from Department of Mathematics, Damietta Faculty of Science, New Damietta 34517, Egypt

where K is the constant input received by the neuron and the neuron transfer function f is defined by $f(x) = \frac{1}{1+e^{-x}}$.

In [4] the delay differential equation

$$\frac{dx(t)}{dt} = -x(t) + a\tanh(x(t) - bx(t-\tau) + C), \tag{1.2}$$

has been proposed to describe the behavior of the activation level $x(t)$ of a single neuron which is capable of self-activation modulated by a dynamic threshold C with a single delay τ. In the absence of the threshold effect, equation (1.2) has the form

$$\frac{dx(t)}{dt} = -x(t) + a\tanh(x(t) - bx(t-\tau)), \tag{1.3}$$

Letting $y(t) = x(t) - bx(t-\tau)$ in (1.3). Then y satisfies the equation

$$\frac{dy(t)}{dt} = -y(t) + a\tanh(y(t)) - ab\tanh(y(t-\tau)), \tag{1.4}$$

the stability and/or bifurcation analysis of (1.4) have been studied in [4, 16] and [13] but with more general activation function. It is also proved by [13, 14] that (1.4) is not capable of producing chaos violating the existence of chaos conjectured by [16]. Gopalsamy and Leung [4] proved that the unique equilibrium of (1.4) and hence of (1.3) is globally asymptotically stable if

$$a(1-b) < 1 \quad \text{and} \quad a(1+b) < 1$$

when $a > 0$ and $b \geq 0$, which agrees with the findings of [13, 16]. El-Morshedy and Gopalsamy [2] improves the above condition by allowing the equality signs to be non-strict. In fact, Theorem 3.1 in [2] is the best known absolute (delay-independent) global asymptotic stability criteria for (1.2).

Liao *et al* [10] considered a single neuron model with general activation function; namely,

$$\frac{dx(t)}{dt} = -x(t) + af(x(t) - bx(t-\tau) + C),$$

They discussed the local stability as well as the existence of Hopf bifurcation under the assumption that f has a continuous third derivative.

Based on [8]; Györi and Hartung [5] investigate the stability character of the single neuron model

$$\frac{dx(t)}{dt} = -\lambda x(t) + Af(x(t)) + Bf(x(t-\tau)) + C,$$

where $f(x) = 0.5(|x+1| - |x-1|)$.

As one may observe; all the above models contain only one delay. It has been demonstrated by [3, 7] that models of single neuron can contain many delays. In this work we investigate the oscillatory and asymptotic stability characters of a single neuron (neuronic) equation with two delays; namely,

$$\frac{dx(t)}{dt} = -\lambda x(t) + af(x(t) - bx(t-\tau) + cx(t-\sigma)), \quad \tau, \sigma > 0 \tag{1.5}$$

where $a, b, c \in R$, $\lambda > 0$ and with each solution of (1.5) an initial function $\phi \in C[-l, 0]$ is associated where $l = \max\{\tau, \sigma\}$. For generality reasons we will not assume that f is a tanh-like function only. Instead, we assume that f is continuous on R, $f(0) = 0$ and satisfies some or all of the following conditions

(H1) $0 < \frac{f(x)}{x} < 1$ for all $x \neq 0$.

(H2) f is differentiable near zero with $f'(0) = 1$.

(H3) $x \frac{d}{dx} \frac{f(x)}{x} < 0$ for all $x \neq 0$.

(H4) $|f(x)| < 1$ for all $x \in R$.

It can be seen that the substitution $y(t) = x(t) - bx(t-\tau) + cx(t-\sigma)$ transforms (1.5) into the equation,

$$\frac{dy(t)}{dt} = -\lambda y(t) + af(y(t)) - abf(y(t-\tau)) + acf(y(t-\sigma)), \quad t \geq l \tag{1.6}$$

In Section 2, we investigate the oscillatory character of (1.5). We say that a solution $x(t)$ of (1.5) is nonoscillatory if it is eventually positive or eventually negative, otherwise $x(t)$ is called oscillatory. Equation (1.5) is called oscillatory if all its solutions are oscillatory. If equation (1.5) has at least one nonoscillatory solution, then it is called nonoscillatory. The oscillation theory of the delay differential equations can be found in [3, 6]. In contrast with the stability of these equations, there are no absolute (delay-independent) oscillation criteria for first order delay differential equations. Although the oscillatory properties of models arising from many fields as mathematical biology is now completely characterized (see [3, 6] for more details), the oscillation of equations of the form (1.5) has not yet received the deserved attention. It seems that [2] is the only work on this type of equations.

The asymptotic behavior of the trivial solution of (1.5) will be considered in Section 3. Theorem 3.1 in [2] will be extended to (1.5) and interesting delay dependent global asymptotic stability criteria are obtained which are new even for the special case (1.2).

2 The Oscillatory Behavior

Suppose that $x(t)$ is a solution of (1.5). Define a function M as follows:

$$M(t) = \begin{cases} \frac{f(x(t)-bx(t-\tau)+cx(t-\sigma))}{x(t)-bx(t-\tau)+cx(t-\sigma)}, & \text{if } x(t) - bx(t-\tau) + cx(t-\sigma) \neq 0, \\ 1, & \text{if } x(t) - bx(t-\tau) + cx(t-\sigma) = 0, \end{cases}$$

for all $t \geq 0$. If (H1), (H2) hold, then M is continuous on $[0, \infty)$, $0 < M(t) \leq 1$ for all $t \geq 0$ and

$$f(x(t) - bx(t-\tau) + cx(t-\sigma)) = (x(t) - bx(t-\tau) + cx(t-\sigma))M(t).$$

Thus equation (1.5) can be rewritten in the form

$$\frac{dx(t)}{dt} = (aM(t) - \lambda)x(t) - abM(t)x(t-\tau) + acM(t)x(t-\sigma), \qquad t \geq 0. \tag{2.1}$$

It is not difficult to see that if $x(t)$ is a solution of (2.1), then $z(t) = x(t)e^{\int_0^t (\lambda - aM(s))ds}$ is a solution of the equation

$$\frac{dz(t)}{dt} + abM(t)e^{\int_{t-\tau}^t (\lambda - aM(s))ds} z(t-\tau) - acM(t)e^{\int_{t-\sigma}^t (\lambda - aM(s))ds} z(t-\sigma) = 0. \tag{2.2}$$

We will see that (2.1) and (2.2) play a key rule in the proofs of most of our results.

Theorem 2.1 *Assume that (H1), (H2) hold and*

$$bc \leq 0. \tag{2.3}$$

Equation (1.5) is nonoscillatory if one of the following conditions is satisfied:

(i) $ab \leq 0$.

(ii) $a(1 - b + c) > \lambda$ *and (H4) holds.*

Proof. Assume that $x(t)$ is a solution of (1.5) with $\phi(s) > 0$ for all $s \in [-\tau, 0]$. Let $t_0 > 0$ be such that $x(t) > 0$ for all $t \in [0, t_0)$ and $x(t_0) = 0$. From (2.2) we obtain

$$\frac{d}{dt}\left(x(t)e^{\int_0^t (\lambda - aM(s))ds}\right) = -abM(t)e^{\int_0^t (\lambda - aM(s))ds} x(t-\tau) + acM(t)e^{\int_0^t (\lambda - aM(s))ds} x(t-\sigma).$$

So if (i) holds, we get

$$\frac{d}{dt}\left(x(t)e^{\int_0^t (\lambda - aM(s))ds}\right) \geq 0 \text{ for all } t \in [0, t_0).$$

Integrating from 0 to t_0,

$$x(t_0)e^{\int_0^{t_0} (\lambda - aM(s))ds} \geq x(0) > 0,$$

which is impossible since $x(t_0) = 0$. Therefore $x(t) > 0$ for all $t \geq 0$; i.e., (1.5) is nonoscillatory.

Suppose that (ii) holds. Define a function F by

$$F(y) = -\lambda y + af((1 - b + c)y), \qquad y \in R.$$

Then

$$\frac{dF(y)}{dy}|_{y=0} = -\lambda + a(1-b+c)f'(0) = a(1-b+c) - \lambda > 0.$$

Consequently, $F(y)$ will be positive for all sufficiently small positive values of y. Since $\lim_{y\to\infty} F(y) = -\infty$, then there exists a positive value c such that $F(c) = 0$. Set $x(t) = c$, it follows that $x(t)$ satisfies (1.5); i.e., (1.5) is nonoscillatory. ■

In view of the idea used in the proof of Theorem 2.1(ii), a more general form from it can be obtained by replacing (ii) by the following phrase:

The equation $-\lambda y + af(1-b+c)y) = 0$ *has at least one nontrivial root.*

In the next oscillation results we will make use of the following theorem which is adapted from [6, Corollary 3.4.1] concerning the oscillation of the equation

$$\frac{dx(t)}{dt} + p(t)x(t-\tau) + q(t)x(t-\sigma) = 0, \qquad t \geq 0. \tag{2.4}$$

Theorem 2.2 *Assume that* $\tau,\ \sigma > 0$ *and* $p,\ q \in C\left([0,\ \infty),\ R^{+}\right)$ *such that* $\liminf_{t\to\infty}\ (p(t)+q(t)) > 0$. *Then each of the following two conditions is sufficient for the oscillation of equation (2.4):*

(a) $\liminf_{t\to\infty}\ (\tau p(t) + \sigma q(t)) > \frac{1}{e}$;

(b) $\liminf_{t\to\infty}\ (p(t)q(t))^{\frac{1}{2}}(\tau+\sigma) > \frac{1}{e}$.

We refer here to the fact that Theorem 2.2 is a consequence of [6, Theorem 3.4.2] but we use it here since (a) and (b) are practically easier to apply than the original condition provided in [6, Theorem 3.4.2].

Theorem 2.3 *Assume that (H1), (H2), (2.3) hold,*

$$ab > 0, \quad a \leq \lambda, \tag{2.5}$$

and either one of the conditions

$$\begin{gathered} ab\tau e^{(\lambda-a)\tau} - ac\sigma e^{(\lambda-a)\sigma} > \frac{1}{e}, \\ \sqrt{|bc|a^2}(\tau+\sigma)e^{\frac{1}{2}(\lambda-a)(\tau+\sigma)} > \frac{1}{e} \end{gathered} \tag{2.6}$$

is satisfied. Then (1.5) is oscillatory.

Proof. To the contrary let us assume that (1.5) is nonoscillatory. Without loss of generality one can assume that (2.1) has a solution $x(t)$ such that $x(t) > 0,\ t \geq t_0$ for some $t_0 \geq 0$. Recalling that $M(t) \leq 1$, we find from (2.1), (2.5) that

$$\frac{dx(t)}{dt} < 0 \ \text{ for all } t \geq t_0 + l.$$

Thus, there exists a real number $\mu \geq 0$ such that $\lim_{t\to\infty} x(t) = \mu$ and

$$x(t) > \mu \text{ for all } t \geq t_0 + l. \tag{2.7}$$

It follows from (2.1) and (2.7) that

$$\begin{aligned} \frac{dx(t)}{dt} &< (aM(t) - \lambda)\mu - abM(t)\mu + acM(t)\mu \\ &= ((aM(t) - \lambda) - abM(t) + acM(t))\mu \\ &= (a(1 - b + c)M(t) - \lambda)\mu, \quad t \geq t_0 + l. \end{aligned} \tag{2.8}$$

Suppose that $\mu \neq 0$. Then

$$\lim_{t\to\infty} M(t) = \begin{cases} \frac{f((1-b+c)\mu)}{(1-b+c)\mu}, & \text{if } b - c \neq 1, \\ 1, & \text{if } b - c = 1. \end{cases}$$

Taking into account that

$$a(1 - b + c) < \begin{cases} \lambda, & \text{if } a, b > 0, \\ 0, & \text{if } a, b < 0, \end{cases}$$

we obtain

$$\lim_{t\to\infty} (a(1 - b + c)M(t) - \lambda)\mu = -\beta,$$

where

$$0 < \beta = \begin{cases} \lambda\mu - af((1 - b + c)\mu), & \text{if } b - c \neq 1, \\ \lambda\mu, & \text{if } b - c = 1. \end{cases}$$

Thus (2.8) yields

$$\limsup_{t\to\infty} \frac{dx(t)}{dt} \leq -\beta.$$

The last inequality leads to the existence of a constant $\nu > 0$ such that $\frac{dx(t)}{dt} < -\nu$ for all sufficiently large t. Integrating the last inequality from a suitable large t (say $t_1 > t_0 + l$) to ∞, we get

$$\lim_{t\to\infty} x(t) = -\infty,$$

which is a contradiction. Therefore $\mu = 0$; that is,

$$\lim_{t\to\infty} x(t) = 0.$$

Consequently

$$\lim_{t\to\infty} M(t) = \lim_{t\to\infty} \frac{f(x(t) - bx(t - \tau) + cx(t - \sigma))}{x(t) - bx(t - \tau) + cx(t - \sigma)} = 1. \tag{2.9}$$

Set

$$z(t) = x(t)e^{\int_{t_0+\tau}^{t} (\lambda - aM(s))ds}, \quad t \geq t_0 + l.$$

Then it is easy to verify that $z(t)$ is a positive solution of equation (2.2) or equivalently equation (2.4), where

$$0 < p(t) = abM(t)e^{\int_{t-\tau}^{t}(\lambda - aM(s))ds},$$

and

$$0 < q(t) = -acM(t)e^{\int_{t-\sigma}^{t}(\lambda - aM(s))ds}.$$

Thus (2.2) is nonoscillatory. On the other hand (2.9) yields

$$\lim_{t\to\infty} p(t) = abe^{(\lambda-a)\tau} \quad \text{and} \quad \lim_{t\to\infty} q(t) = -ace^{(\lambda-a)\sigma}. \tag{2.10}$$

From (2.6) and (2.10) we see that all conditions of Theorem 2.2 are satisfied and hence equation (2.2) is oscillatory which is a contradiction. ■

The following oscillation results deal with the case $a > \lambda$ and hence complete, partially, Theorem 2.3.

Theorem 2.4 *Assume that (2.3), (H1) and (H2) are satisfied. If either one of the inequalities of (2.6) holds, $a > \lambda$ and*

$$e^{(a-\lambda)\tau} \le b, \tag{2.11}$$

then (1.5) is oscillatory.

Proof. Let equation (1.5) be nonoscillatory. One can assume that (1.5) has a solution $x(t) > 0,\ t \ge t_0 \ge 0$. It follows from (2.2) that

$$\frac{d}{dt}\left(x(t)e^{\int_{t_0+l}^{t}(\lambda - aM(s))ds}\right) < 0, \qquad t \ge t_0 + l.$$

Integrating the above inequality from $t-\tau$ to t we obtain,

$$x(t)e^{\int_{t_0+l}^{t}(\lambda - aM(s))ds} < x(t-\tau)e^{\int_{t_0+l}^{t-\tau}(\lambda - aM(s))ds}, \qquad t \ge t_0 + l.$$

Then

$$x(t) < x(t-\tau)e^{-\int_{t-\tau}^{t}(\lambda - aM(s))ds} < e^{(a-\lambda)\tau}x(t-\tau) \le bx(t-\tau), \qquad t \ge t_0 + l.$$

That is

$$x(t) - bx(t-\tau) \le 0, \qquad t \ge t_0 + l.$$

Also,

$$cx(t-\sigma) \le 0.$$

Hence

$$x(t) - bx(t-\tau) + cx(t-\sigma) \leq 0, \qquad t \geq t_0 + l.$$

It follows from (1.5) that

$$\frac{dx(t)}{dt} < -\lambda x(t), \qquad t \geq t_0 + l.$$

Therefore,

$$\frac{d}{dt}(x(t)e^{\lambda t}) < 0, \qquad t \geq t_0 + l,$$

which implies that $\lim_{t\to\infty} x(t) = 0$. Now the proof can be completed as in the proof of Theorem 2.3. ■

Notice that (2.3) and (2.11) imply that $c \leq 0$. This restriction is very important. In fact when $c > 0$, the conditions $a > \lambda$ and (2.3) imply that $ab \leq 0$ which leads to the nonoscillation of (1.5) according to Theorem 2.1.

Lemma 2.1 *Assume that $a \neq 0$ and (H4) holds. If x is any solution of (1.5), then there exists $t_0 \geq 0$ such that*

$$\frac{-|a|}{\lambda} < x(t) < \frac{|a|}{\lambda}, \qquad t \geq t_0. \tag{2.12}$$

Proof. From (1.5), we have

$$\frac{d}{dt}(x(t)e^{\lambda t}) = ae^{\lambda t} f(x(t) - bx(t-\tau) + cx(t-\sigma)).$$

Since $|f(x)| < 1$ for all $x \in R$,

$$-|a|e^{\lambda t} \leq \frac{d}{dt}(x(t)e^{\lambda t}) \leq |a|e^{\lambda t}, \qquad t \geq 0.$$

Integrating the above inequality from $s(\geq 0)$ to t, we obtain

$$-\frac{|a|}{\lambda}(e^{\lambda t} - e^{\lambda s}) \leq x(t)e^{\lambda t} - x(s)e^{\lambda s} \leq \frac{|a|}{\lambda}(e^{\lambda t} - e^{\lambda s}), \qquad \text{for all } t \geq s \geq 0.$$

The above inequality yields

$$e^{\lambda t}(x(t) - \frac{|a|}{\lambda}) \leq e^{\lambda s}(x(s) - \frac{|a|}{\lambda}), \qquad \text{for all } t \geq s \geq 0. \tag{2.13}$$

Therefore the function N, where $N(t) = e^{\lambda t}(x(t) - \frac{|a|}{\lambda})$, is nonincreasing on $[0, \infty)$. This implies that $N(t)$ must be eventually of one sign. We claim that $N(t)$ is eventually negative. Suppose not. Then there exists $T \geq 0$ such that $N(t) > 0$ for all $t \geq T$. But $N(t)$ is nonincreasing, then it has a nonnegative finite limit as $t \to \infty$. Also we have $e^{\lambda t} \to \infty$ as $t \to \infty$. It follows from (2.13) that

$$\lim_{t\to\infty} x(t) - \frac{|a|}{\lambda} = 0, \quad i.e., \quad \lim_{t\to\infty} x(t) = \frac{|a|}{\lambda}. \tag{2.14}$$

By making use of (2.14), equation (1.5) yields

$$\lim_{t\to\infty}\frac{dx(t)}{dt} = -|a| + af(\frac{|a|}{\lambda}((1-b+c))) < \ -|a|+|a| = 0. \tag{2.15}$$

As in the proof of Theorem 2.3 we conclude from (2.15) that $\lim\limits_{t\to\infty} x(t) = -\infty$ which contradicts (2.14). Thus $N(t)$ must be eventually negative as claimed; i.e., there exists $t_0 \geq 0$ such that $e^{\lambda t}(x(t) - \frac{|a|}{\lambda}) < 0, \quad t \geq t_0$. This inequality holds only if $x(t) < \frac{|a|}{\lambda}$ for all $t \geq t_0$. The left inequality of (2.12) can be proved similarly. ■

Theorem 2.5 *Assume that $b \geq 0$, $-c \geq 0$ and (H1)-(H4) hold. If either one of the conditions*

$$\begin{aligned}&(ab\tau e^{(\lambda-a)\tau} - ac\sigma e^{(\lambda-a)\sigma})r > \frac{1}{e},\\ &\sqrt{|bc|a^2}(\tau+\sigma)e^{\frac{1}{2}(\lambda-a)(\tau+\sigma)}r > \frac{1}{e}\end{aligned} \tag{2.16}$$

is satisfied where $r = \min\{\frac{f(\frac{a}{\lambda})}{\frac{a}{\lambda}}, \frac{f(\frac{-a(b-c)}{\lambda})}{\frac{-a(b-c)}{\lambda}}\}$, *then (1.5) is oscillatory.*

Proof. To the contrary let us assume that (1.5) has a solution $x(t)$ such that $x(t) > 0$, for all $t \geq t_0 \geq 0$. Then as in the previous proofs,

$$z(t) = x(t)e^{\int\limits_{t_0+l}^{t}(\lambda - aM(s))ds}$$

is a positive solution of equation (2.2). Set $u(t) = x(t) - bx(t-\tau) + cx(t-\sigma), \quad t \geq t_1 \geq t_0 + l$, where t_1 is so large that (2.12) is satisfied. Then

$$\frac{-a}{\lambda}(b-c) < u(t) < \frac{a}{\lambda}, \qquad t \geq t_1.$$

This inequality and (H3) yield

$$M(t) > \begin{cases} r & \text{if } u(t) \neq 0, \qquad t \geq t_1,\\ 1 & \text{if } u(t) = 0, \qquad t \geq t_1.\end{cases}$$

But (H1) implies

$$1 > r.$$

Then

$$M(t) > r, \qquad \text{for all } t \geq t_1. \tag{2.17}$$

Defining the functions p, q as in the proof of Theorem 2.3 and using similar arguments, then (2.16) and (2.17) imply that all conditions of Theorem 2.2 are satisfied and hence equation (2.2) is oscillatory which is a contradiction. ■

3 The Asymptotic Behavior

Theorem 3.1 *Assume that f is a nondecreasing function on R, (H1), (H4) holds and either*

$$bc \leq 0 \text{ and } \quad a(1+b-c) \leq \lambda, \qquad a(1-b+c) \leq \lambda, \tag{3.1}$$

or

$$bc \geq 0 \text{ and } \quad a(1+b+c) \leq \lambda, \qquad a(1-b-c) \leq \lambda \tag{3.2}$$

is satisfied. Then all solutions of (1.5) satisfy that

$$\lim_{t\to\infty} x(t) = 0. \tag{3.3}$$

Proof. We will prove the theorem when (3.1) holds. The proof when (3.2) holds is similar and will be omitted to avoid repetition. First we assume that x is a solution of (1.5). It follows from Lemma 2.1 that x is bounded. Therefore there exist $L, S \in R$ such that

$$L = \liminf_{t\to\infty} x(t) \leq S = \limsup_{t\to\infty} x(t).$$

Thus for any $\varepsilon > 0$ there exists $t_0 \geq 0$ such that

$$L - \varepsilon \leq x(t) \leq S + \varepsilon, \qquad t \geq t_0. \tag{3.4}$$

In view of the continuity of x one can choose two sequences $\{t_n\}$, $\{\tilde{t}_n\}$ such that t_n, $\tilde{t}_n \to \infty$ as $n \to \infty$,

$$\frac{d}{dt}x(t_n) = 0, \qquad \lim_{t\to\infty} x(t_n) = S, \tag{3.5}$$

and

$$\frac{d}{dt}x(\tilde{t}_n) = 0, \qquad \lim_{t\to\infty} x(\tilde{t}_n) = L. \tag{3.6}$$

Assume, for the sake of contradiction, that x does not satisfy (3.3); i.e., $L < S$. First consider the case when $a > 0$, $b > 0$ and $c < 0$. Using (3.4), (3.5) and (1.5) we obtain

$$\lambda x(t_n) \leq af(x(t_n) - b(L-\varepsilon) + c(L-\varepsilon)), \qquad t \geq t_0 + l. \tag{3.7}$$

Also (3.4), (3.6) and (1.5) imply

$$\lambda x(\tilde{t}_n) \geq af(x(\tilde{t}_n) - b(S+\varepsilon) + c(S+\varepsilon)), \qquad t \geq t_0 + l. \tag{3.8}$$

Let $n \to \infty$ in (3.7), (3.8). Then we obtain respectively that

$$\lambda S \leq af(S - b(L-\varepsilon) + c(L-\varepsilon)),$$

and

$$\lambda L \geq af(L - b(S+\varepsilon) + c(S+\varepsilon)).$$

Since ε is arbitrary small, we get

$$\lambda S \leq af(S-(b-c)L), \tag{3.9}$$

and

$$\lambda L \geq af(L-(b-c)S). \tag{3.10}$$

Assume that $S \leq 0$. Then $L < 0$ which, by (3.10), yields

$$f(L-(b-c)S) < 0.$$

From (H1) we conclude that

$$f(L-(b-c)S) > (L-(b-c)S).$$

Thus (3.10) yields

$$\lambda L > a(L-(b-c)S). \tag{3.11}$$

Hence,

$$L(\lambda - a) > -a(b-c)S. \tag{3.12}$$

But (3.1) leads to $\lambda - a \geq 0$, then $L > 0$ which is a contradiction. If $S > 0$, from (3.9) we get

$$0 < \lambda S \leq af(S-(b-c)L).$$

It follows that

$$f(S-(b-c)L) > 0.$$

From (H1), we conclude that

$$f(S-(b-c)L) < S-(b-c)L.$$

Thus

$$(\lambda - a)S < (-ab+ac)L. \tag{3.13}$$

Which implies that $L < 0$. Therefore, we obtain (3.11) and (3.12). Combining (3.12) and (3.13), we get

$$S((\lambda - a) + a(c-b)) < L((\lambda - a) + a(c-b)),$$

or equivalently

$$(S-L)(\lambda - a(1+b-c)) < 0,$$

which is impossible in view of (3.1) and the fact that $S > L$. This contradiction implies that $L = S$.

When $a<0$, $b<0$ and $c>0$, similar arguments as in the above case, imply easily (3.7), (3.8) as well as their consequences (3.9), (3.10). If $S\le 0$, then $L<0$ and hence (3.10) yields

$$f(L-(b-c)S)>0.$$

From (H1) and (3.10), we conclude that

$$\lambda L>a(L-(b-c)S). \tag{3.14}$$

Hence,

$$(\lambda-a)L>-a(b-c)S\ge 0.$$

Thus $L>0$, which is contradiction. If $S>0$, using similar arguments we obtain

$$0<(\lambda-a)S<(-ab+ac)L,$$

which implies that $L<0$. Therefore, (3.14) leads to

$$(\lambda-a)L>(-ab+ac)S.$$

From the previous two inequalities we get

$$(S-L)(\lambda-a(1+b-c))<0$$

which is impossible in view of (3.1) and the fact that $S>L$. This contradiction implies that $L=S$.

When $a<0$, $c<0$ and $b>0$, using similar arguments as before, we find

$$\lambda x(\tilde{t}_n)\ge af(x(\tilde{t}_n)-b(L-\varepsilon)+c(L-\varepsilon)). \tag{3.15}$$

Also

$$\lambda x(t_n)\le af(x(t_n)-b(S+\varepsilon)+c(S+\varepsilon)). \tag{3.16}$$

Let $n\to\infty$ in (3.15), (3.16). Then we obtain respectively that

$$\lambda L\ge af(L-b(L-\varepsilon)+c(L-\varepsilon)), \tag{3.17}$$

and

$$\lambda S\le af(S-b(S-\varepsilon)+c(S-\varepsilon)). \tag{3.18}$$

Since ε is arbitrary small, we get

$$\lambda L\ge af(L(1-b+c)), \tag{3.19}$$

and

$$\lambda S\le af(S(1-b+c)). \tag{3.20}$$

Suppose that $S > 0$ then (3.20) yields

$$f(S(1-b+c)) < 0.$$

So (H1) and (3.20) imply that

$$\lambda S \leq af(S(1-b+c)) < Sa(1-b+c) \leq \lambda S.$$

which is impossible. Therefore $S \leq 0$ and hence $L < 0$. From (3.19) we get

$$\lambda L \geq af(L(1-b+c)) > La(1-b+c) \geq \lambda L.$$

which is also impossible. Then $L = S$.

Consider now the last possible case; that is $a > 0,\ c > 0$ and $b < 0$,. The above reasoning implies similarly the inequalities (3.15)-(3.20). Suppose that $S > 0$. Then (3.20) yields

$$f(S(1-b+c)) > 0.$$

Thus

$$\lambda S \leq af(S(1-b+c)) < aS(1-b+c) \leq \lambda S.$$

It follows that

$$\lambda S < aS(1-b+c) \leq \lambda S,$$

which is impossible. Consequently we have $L < 0$. So, in view of (3.19), we obtain

$$\lambda L < aL(1-b+c) \leq \lambda L,$$

which is a contradiction. Hence $L = S$.

Since the trivial solution is the unique equilibrium, due to the second inequality of (3.1), we get $L = S = 0$. ∎

Lemma 3.1 *If all solutions of (2.2) are bounded and $\lambda > a > 0$ or $a \leq 0$, then the zero solution of (1.5) is globally exponentially stable.*

Proof. We know that

$$z(t) = x(t)e^{\int_0^t (\lambda - aM(s))ds}, \ t \geq 0. \tag{3.21}$$

is a solution of (2.2), for $t \geq l$, if $x(t)$ is a solution of equation (1.5). Since

$$e^{\int_0^t (\lambda - aM(s))ds} \geq \begin{cases} e^{\lambda t} & a \leq 0, \\ e^{(\lambda - a)t} & a > 0. \end{cases}$$

Then,

$$|x(t)| = |z(t)|e^{-\int_0^t (\lambda - aM(s))ds} \leq |z(t)| \begin{cases} e^{-\lambda t}, & a \leq 0 \\ e^{-(\lambda - a)t}, & \lambda > a > 0. \end{cases}$$

Since all solutions of (2.2) are bounded, then there exists a constant $A > 0$ such that $|z(t)| < A$ for all $t \geq 0$. It follows that

$$|x(t)| \leq A \begin{cases} e^{-\lambda t}, & a \leq 0 \\ e^{-(\lambda - a)t}, & \lambda > a > 0. \end{cases}$$

for all $t > 0$. Then $x(t)$ is exponentially stable. ■

Next we give some delay-dependent global asymptotic stability results. The first result is extracted from [11, Theorem 2.2] and [12, Theorem 2.2] for the equation

$$\frac{dy(t)}{dt} = -a(t)y(t) - \sum_{i=0}^{m} a_i(t) g_i(y(t - \tau_i)), \tag{3.22}$$

where as usual $y(t) = \phi(t)$ for all $t \in [-l, 0]$, $\phi \in C([-l, 0], R)$ and $l = \max\{\tau_i : i = 0, 1, \ldots, m\}$. Also, a, a_i are continuous bounded functions on $[0, \infty)$ such that $a(t) \geq 0$, $a_i(t) \geq 0$ for all $0 \leq i \leq m$, $t \geq 0$ and $\sum_{i=0}^{m} a_i(t) > 0$, $\int_0^\infty \sum_{i=0}^{m} a_i(t) dt = \infty$. The functions g_i are continuous on R with the following property;

$$g_i(0) = 0 \text{ and } 0 < \frac{g_i(x)}{x} \leq 1, \text{ for } x \neq 0, \text{ and all } 0 \leq i \leq m.$$

We use the following notations:

$$\mu = e^{(\inf_{t \geq l} \int_{t-l}^{t} a(s)ds)},$$

and

$$\bar{\lambda} = \begin{cases} \sup_{t \geq l} \int_{t-l}^{t} \sum_{i=0}^{m} a_i(t) \; dt, & a(t) \equiv 0 \\ \frac{\exp(\sup_{t \geq l} \int_{t-l}^{t} a(s)ds) - 1}{\inf_{t \geq 0} a(t)} (\sup_{t \geq 0} \sum_{i=0}^{m} a_i(t)), & \inf_{t \geq 0} a(t) > 0. \end{cases}$$

Now Muroyas' results [11, Theorem 2.2] and [12, Theorem 2.2] can be joint into the following result.

Theorem 3.2 *All Solutions of equation (3.22) are bounded from above and below if $\bar{\lambda} \leq \frac{1}{2} + \mu$, and the zero solution of (3.22) is globally asymptotically stable if $\bar{\lambda} < \frac{1}{2} + \mu$.*

The second result from literatures is the following one:

Theorem 3.3 *[9] Consider the delay differential equation (2.4) where τ and σ are non-negative constants and p, q are continuous functions satisfying the conditions*

$$p(t) + q(t - \tau + \sigma) \neq 0,$$

for τ sufficiently large and

$$2 \limsup_{t \to \infty} \left| \int_{t-\tau}^{t-\sigma} |q(s + \sigma)| ds \right| + \limsup_{t \to \infty} \int_{t-\tau}^{t} |p(s) + q(s - \tau + \sigma)| ds < 1.$$

Then every oscillatory solution of equation (2.4) tends to zero as $t \to \infty$.

To be able to apply Theorem 3.2, we need the following lemma.

Lemma 3.2 *Assume that $x(t)$ is a solution of equation (1.5) and $y(t) = x(t) - bx(t - \tau) + cx(t-\sigma)$. If $\lim_{t\to\infty} y(t) = 0$, then $\lim_{t\to\infty} x(t) = 0$. Furthermore, if $0 < \frac{f(x)}{x} \le 1$ and the trivial solution of (1.6) is stable, then the trivial solution of (1.5) is also stable.*

Proof. From (1.5), we obtain

$$\frac{d(e^{\lambda t}x(t))}{dt} = ae^{\lambda t}f(y(t)). \tag{3.23}$$

Integrating from $t-l$ to t and rearranging, we obtain

$$x(t) = e^{-\lambda l}x(t-l) + ae^{-\lambda t}\int_{t-l}^{t} e^{\lambda s}f(y(s))ds,$$

which yields

$$|x(t)| \le e^{-\lambda l}|x(t-l)| + \frac{|a|}{\lambda}(1-e^{-\lambda l})\max_{t-l\le s\le t}\{|f(y(s))|\}, \quad t \ge l. \tag{3.24}$$

Since $\lim_{t\to\infty} \max_{t-l\le s\le t}\{|f(y(s))|\} = 0$, then for any $\epsilon > 0$ there exists $t_\epsilon > l$ such that

$$|x(t)| \le (e^{-\lambda l} + \epsilon)|x(t-l)| + \epsilon, \quad \text{for } t \ge t_\epsilon.$$

Therefore $\lim_{t\to\infty} |x(t)| = 0$ according to [6, Lemma 1.5.3].

Now if $0 < \frac{f(x)}{x} \le 1$ and the trivial solution of (1.6) is stable, then for any $\epsilon_1 > 0$, there exists $\delta_1 > 0$ such that for the initial function ϕ_y associated with a solution y of (1.6) we have $|y(t)| < \epsilon_1$ for all $t \ge l$ when $||\phi_y|| < \delta_1$ where $||\phi_y|| = \max\{\phi_y(t) : 0 \le t \le l\}$. Let ϵ be an arbitrary positive number. Choose $\epsilon_1 = \frac{\lambda}{|a|}\epsilon$ and $\delta < \delta_1 < \epsilon_1$. Assume that ϕ_x is the initial function associated with a solution x of (1.5) such that $y(t) = x(t) - bx(t-\tau) + cx(t-\sigma)$. We claim that if $||\phi_x|| = \max\{\phi_x(t) : -l \le t \le 0\} < \delta < \epsilon$, then $|x(t)| < \epsilon$ for all $t > 0$. Suppose not, then there exists $t_0 > 0$ such that $|x(t)| < \epsilon$ for all $t < t_0$ and $|x(t_0)| = \epsilon$. Therefore, for $t_0 \ge l$, (3.24) yields

$$\epsilon \le \frac{|a|}{\lambda}\max_{t_0-l\le s\le t_0}\{|f(y(s))|\} \le \frac{|a|}{\lambda}\max_{t_0-l\le s\le t_0}\{|y(s)|\} < \epsilon,$$

which is impossible. When $t_0 < l$, we see from (3.23) that

$$x(t_0) = e^{-\lambda t_0}x(0) + ae^{-\lambda t_0}\int_0^{t_0} e^{\lambda s}f(y(s))ds,$$

Then

$$\epsilon \le \frac{|a|}{\lambda}\max_{0\le s\le t_0}\{|f(y(s))|\} \le \frac{|a|}{\lambda}\max_{0\le s\le t_0}\{|y(s)|\} < \frac{|a|}{\lambda}\delta_1 < \epsilon,$$

which is also impossible. Thus we get our claim which means that the trivial solution of (1.5) is stable. ■

Now assume that $m = 2$, $a(t) = \lambda$, $g_i = f$ for all $i = 0, 1, 2$, $a_0(t) = -a$, $a_1(t) = ab$ and $a_2(t) = -ac$. Then, applying Theorem 3.2 on (1.6) and using Lemma 3.2, we obtain the following result.

Theorem 3.4 *Assume that (H1), $a < 0$, $b \leq 0$ and $c \geq 0$ are satisfied. If*

$$-a(1-b+c)\frac{e^{l\lambda}-1}{\lambda} < \frac{1}{2} + e^{l\lambda},$$

then the zero solution of (1.5) is globally asymptotically stable.

The next result can also be obtained using Theorem 3.2 and Lemma 3.2. In this case we assume that $m = 3$, $a(t) = 0$, $g_0(x) = x$, $g_i = f$ for all $i = 1, 2, 3$, $a_0(t) = \lambda$, $a_1(t) = -a$, $a_2(t) = ab$, and $a_3(t) = -ac$.

Theorem 3.5 *Assume that (H1), $a < 0$, $b \leq 0$ and $c \geq 0$ are satisfied. If*

$$(\lambda - a(1-b+c))l < \frac{3}{2},$$

then the zero solution of (1.5) is globally asymptotically stable.

As in the proofs of Theorem 2.3 and Lemma 3.1; any solution $x(t)$ of (1.5) can be related to a solution $y(t)$ of equation (2.4) where

$$p(t) = abM(t)e^{\int_{t-\tau}^{t}(\lambda - aM(s))ds},$$

and

$$q(t) = -acM(t)e^{\int_{t-\sigma}^{t}(\lambda - aM(s))ds}.$$

If (H4) holds then the continuity of M and the boundedness of all solutions of (1.5) (according to Lemma 2.1) lead to the existence of a constant $B > 0$ such that $M(t) > B$ for all $t \geq 0$. If $ab > 0$, we have

$$p(t) > B\begin{cases} e^{(\lambda-a)\tau}, & \text{if } \lambda \geq a > 0, \\ e^{\lambda\tau}, & \text{if } a \leq 0. \end{cases}$$

Now when ab, $-ac > 0$, we have $p(t) + q(t) > 0$ for all $t \geq 0$, $\int_0^\infty (p(t)+q(t))dt = \infty$ and

$$\bar{\lambda} \leq \begin{cases} able^{(\lambda-a)\tau} - acle^{(\lambda-a)\sigma}, & \text{if } a < 0, \\ able^{\lambda\tau} - acle^{\lambda\sigma}, & \text{if } a \geq 0. \end{cases}$$

Applying Theorem 3.2 on equation (2.4) and using Lemma 3.1 we obtain the following result.

Theorem 3.6 *Assume that (2.3), (H1), (H2), (H4) hold and $ab > 0$. Then the zero solution of (1.5) is globally exponentially stable if either one of the following conditions is satisfied:*

$$able^{(\lambda-a)\tau} - acle^{(\lambda-a)\sigma} \leq \tfrac{3}{2}, \qquad \text{if } a \leq 0,$$
$$able^{\lambda\tau} - acle^{\lambda\sigma} \leq \tfrac{3}{2}, \qquad \text{if } \lambda > a > 0.$$

Now by making use of Theorem 3.3, we obtain the following result.

Theorem 3.7 *Assume that (H1), (H2), (2.3) and*

$$ab > 0 \tag{3.25}$$

are satisfied. If $x(t)$ is any solution of (1.5), then $\lim_{t\to\infty} x(t) = 0$ provided that either

$$ab\tau e^{(\lambda-a)\tau} - ace^{(\lambda-a)\sigma}(2|\tau-\sigma|+\tau) < 1, \quad \textit{when } a \le 0,$$

or

$$ab\tau e^{\lambda\tau} - ace^{\lambda\sigma}(2|\tau-\sigma|+\tau) < 1, \qquad \textit{when } \lambda \ge a > 0.$$

Proof. We know that (2.2) has the form (2.4) with

$$p(t) = abM(t)e^{\int\limits_{t-\tau}^{t}(\lambda-aM(s))ds}, \quad q(t) = -acM(t)e^{\int\limits_{t-\sigma}^{t}(\lambda aM(s))ds}.$$

Thus the functions p and q are continuous and

$$p(t)+q(t-\tau+\sigma) = abM(t)e^{\int\limits_{t-\tau}^{t}(\lambda-aM(s))ds} - acM(t-\tau+\sigma)e^{\int\limits_{t-\tau}^{t-\tau+\sigma}(\lambda aM(s))ds} \neq 0.$$

Taking into account that

$$\lambda - aM(s) < \begin{cases} \lambda - a, & \text{if } a \le 0, \\ \lambda, & \text{if } a > 0. \end{cases} \tag{3.26}$$

From (3.26) we have,

$$\begin{aligned} 2|q(s+\sigma)| \quad &\le 2\begin{cases} -ace^{\int\limits_{s}^{s+\sigma}(\lambda-a)ds}, & a \le 0, \\ -ace^{\int\limits_{s}^{s+\sigma}\lambda ds}, & a > 0. \end{cases} \\ &= 2\begin{cases} -ace^{(\lambda-a)\sigma}, & a \le 0, \\ -ace^{\lambda\sigma}, & a > 0. \end{cases} \end{aligned}$$

Also,

$$\begin{aligned} |p(s)+q(s-\tau+\sigma)| \quad &= |abM(s)e^{\int_{s-\tau}^{s}(\lambda-aM(s))ds} - acM(s-\tau+\sigma)e^{\int_{s-\tau}^{s-\tau+\sigma}(\lambda-aM(s))ds}| \\ &\le abe^{\int_{s-\tau}^{s}(\lambda-aM(s))ds} - ace^{\int_{s-\tau}^{s-\tau+\sigma}(\lambda-a\ M(s))ds}. \end{aligned}$$

Then

$$\begin{aligned} |p(s)+q(s-\tau+\sigma)| \quad &\le \begin{cases} abe^{\int_{s-\tau}^{s}(\lambda-a)ds} - ace^{\int_{s-\tau}^{s-\tau+\sigma}(\lambda-a)ds} & \text{if } a \le 0 \\ abe^{\int_{s-\tau}^{s}\lambda ds} - ace^{\int_{s-\tau}^{s-\tau+\sigma}\lambda ds} & \text{if } a > 0. \end{cases} \\ &= \begin{cases} abe^{(\lambda-a)\tau} - ace^{(\lambda-a)\sigma}, & \text{if } a \le 0 \\ abe^{\lambda\tau} - ace^{\lambda\sigma}, & \text{if } a > 0. \end{cases} \end{aligned}$$

Thus

$$\Lambda < \begin{cases} ab\tau e^{(\lambda-a)\tau} - ace^{(\lambda-a)\sigma}(2|\tau-\sigma|+\tau) < 1, & \text{if } a \le 0 \\ ab\tau e^{\lambda\tau} - ace^{\lambda\sigma}(2|\tau-\sigma|+\tau) < 1, & \text{if } a > 0. \end{cases}$$

where

$$\Lambda = 2\limsup_{t\to\infty} |\int_{t-\tau}^{t-\sigma} | \;\; q(s+\sigma)|ds| \;\; + \limsup_{t\to\infty} |\int_{t-\tau}^{t} | \; p(s) + q(s-\tau+\sigma)|ds|.$$

Then every oscillatory solution of equation (2.4) tends to zero as $t \to \infty$ according to Theorem 3.3. Therefore every oscillatory solution of equation (1.5) tends to zero as $t \to \infty$ according to Lemma 3.1. To complete the proof. We consider the case when equation (1.5) has a nonoscillatory solution, say $x(t)$. As usual, we assume that $x(t) > 0, \; t \ge t_0$ for some $t_0 \ge 0$. Therefore

$$y(t) = x(t)e^{\int_{t_0+l}^{t}(\lambda - aM(s))ds} > 0.$$

It follows from equation (2.4) that

$$\frac{dy(t)}{dt} = \frac{d}{dt}(x(t)e^{\int_{t_0+l}^{t}(\lambda - aM(s))ds}) < 0, \quad t \ge t_0 + l.$$

Then there is exists $L \ge 0$ such that $\lim_{t\to\infty} y(t) = L$. Thus $x(t)$ is bounded and hence, as before, a number $B > 0$ exists such that $M(t) > B$ for $t \ge t_0 + l$. Assume that $\sigma \ge \tau$. Then $t - \sigma \le t - \tau$ and

$$x(t-\sigma) > x(t-\tau)e^{\int_{t-\sigma}^{t-\tau}(\lambda - aM(s))ds}$$

If $L > 0$, there exists $t' \ge t_0$ such that $y(t) > L$ for all $t \ge t' + l$ and hence equation (2.4) yields

$$\begin{aligned} \frac{dy(t)}{dt} \quad &= -abM(t)x(t-\tau)e^{\int_{t_0+l}^{t}(\lambda - aM(s))ds} + acM(t)x(t-\sigma)e^{\int_{t_0+l}^{t}(\lambda - aM(s))ds}, \\ &\le B(-abe^{\int_{t_0+l}^{t}(\lambda - aM(s))ds} + ace^{\int_{t_0+l}^{t}(\lambda - aM(s))ds} e^{\int_{t-\sigma}^{t-\tau}(\lambda - aM(s))ds})x(t-\tau), \\ &= Be^{\int_{t-\tau}^{t}(\lambda - aM(s))ds}(-ab + ace^{\int_{t-\sigma}^{t-\tau}(\lambda - aM(s))ds})y(t-\tau), \\ &< BLe^{\int_{t-\tau}^{t}(\lambda - aM(s))ds}(-ab + ace^{\int_{t-\tau}^{t}(\lambda - aM(s))ds}). \end{aligned}$$

Thus

$$\frac{dy(t)}{dt} < -K, \quad \text{for all } t \ge t' + l$$

where $K = BL(ab - ac) > 0$. Integrating the last inequality from $t' + l$ to ∞ we get

$$\lim_{t\to\infty} y(t) = -\infty.$$

Which is a contradiction. Therefore $L = 0$.

When $\sigma < \tau$, similar arguments leads to the above conclusion ($L = 0$). Thus $\lim_{t\to\infty} y(t) = 0$ and $\lim_{t\to\infty} x(t) = 0$; i.e., every nonoscillatory solution of equation (1.5) tends to zero as $t \to \infty$. ■

Remark 3.1 *It should be noted that there are many interesting linear stability criteria that can be applied here (see, e.g., [1, 17] and the references cited therein) but of course it is not possible to apply all these results due to space limitation.*

We conclude our results with the following consequences of Theorems 3.4-3.6 (with $c = 0$, $l = \tau$) on the single delay model (1.3). As far as the authors know these results are new.

Corollary 3.1 *Assume that $a < 0$, $b \leq 0$. Then the zero solution of (1.3) is globally asymptotically stable if either one of the following conditions is satisfied:*

$$-a(1-b)(e^{\tau} - 1) < \frac{1}{2} + e^{\tau},$$

$$(1 - a(1-b))\tau < \frac{3}{2}.$$

Corollary 3.2 *Assume that $ab > 0$. Then the zero solution of (1.3) is globally exponentially stable if either one of the following conditions is satisfied:*

$$\begin{array}{ll} ab\tau e^{(1-a)\tau} \leq \frac{3}{2}, & \text{if } a \leq 0, \\ ab\tau e^{\tau} \leq \frac{3}{2}, & \text{if } 1 > a > 0. \end{array}$$

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