

A NOTE ON THE THEOREM ON DIFFERENTIAL INEQUALITIES

H. ŠTĚPÁNKOVÁ

Abstract. It is proved that if a linear operator $\ell : C([a, b]; \mathbb{R}) \rightarrow L([a, b]; \mathbb{R})$ is nonpositive and for the initial value problem

$$u''(t) = \ell(u)(t) + q(t), \quad u(a) = c_1, \quad u'(a) = c_2$$

the theorem on *differential inequalities* is valid, then ℓ is an a -Volterra operator.

2000 Mathematics Subject Classification. 34K06, 34K12.

Key words and phrases: Second order linear functional differential equation, initial value problem, Volterra operator.

The following notation is used throughout the paper.

\mathbb{N} is the set of natural numbers.

\mathbb{R} is the set of real numbers, $\mathbb{R}_+ = [0, +\infty[$.

$C([a, b]; \mathbb{R})$ is the Banach space of continuous functions $u : [a, b] \rightarrow \mathbb{R}$ with the norm $\|u\|_C = \max\{|u(t)| : t \in [a, b]\}$.

$C([a, b]; \mathbb{R}_+) = \{u \in C([a, b]; \mathbb{R}) : u(t) \geq 0 \text{ for } t \in [a, b]\}$.

$\tilde{C}([a, b]; \mathbb{R})$ is the set of absolutely continuous functions $u : [a, b] \rightarrow \mathbb{R}$.

$\tilde{C}'([a, b]; \mathbb{R})$ is the set of functions $u \in \tilde{C}([a, b]; \mathbb{R})$ such that $u' \in \tilde{C}([a, b]; \mathbb{R})$.

$\tilde{C}'_{loc}([a, b]; \mathbb{R})$ is the set of functions $u \in \tilde{C}([a, b]; \mathbb{R})$ such that $u' \in \tilde{C}([a, \beta]; \mathbb{R})$ for every $\beta \in]a, b[$.

$\tilde{C}'_{loc}(\]a, b[; \mathbb{R})$ is the set of functions $u \in \tilde{C}(\]a, b[; \mathbb{R})$ such that $u' \in \tilde{C}([\alpha, \beta]; \mathbb{R})$ for every $[\alpha, \beta] \subset \]a, b[$.

$L([a, b]; \mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : [a, b] \rightarrow \mathbb{R}$ with the norm $\|p\|_L = \int_a^b |p(s)| ds$.

$L([a, b]; \mathbb{R}_+) = \{p \in L([a, b]; \mathbb{R}) : p(t) \geq 0 \text{ for almost all } t \in [a, b]\}$.

\mathcal{L}_{ab} is the set of linear bounded operators $\ell : C([a, b]; \mathbb{R}) \rightarrow L([a, b]; \mathbb{R})$.

P_{ab} is the set of operators $\ell \in \mathcal{L}_{ab}$ transforming the set $C([a, b]; \mathbb{R}_+)$ into the set $L([a, b]; \mathbb{R}_+)$.

We will say that $\ell \in \mathcal{L}_{ab}$ is an a -Volterra operator if for arbitrary $b_0 \in]a, b]$ and $v \in C([a, b]; \mathbb{R})$ satisfying the condition

$$v(t) = 0 \quad \text{for } t \in [a, b_0]$$

we have

$$\ell(v)(t) = 0 \quad \text{for almost all } t \in [a, b_0].$$

We will say that an operator $\Omega : L([a, b]; \mathbb{R}) \rightarrow C([a, b]; \mathbb{R})$ is an a -Volterra operator, if for arbitrary $b_0 \in]a, b]$ and $q \in L([a, b]; \mathbb{R})$ satisfying the condition

$$q(t) = 0 \quad \text{for almost all } t \in [a, b_0]$$

we have

$$\Omega(q)(t) = 0 \quad \text{for } t \in [a, b_0].$$

In what follows, the equalities and inequalities with integrable functions are understood to hold almost everywhere.

Consider the problem on the existence and uniqueness of a solution of the equation

$$u''(t) = \ell(u)(t) + q(t) \tag{1}$$

satisfying the initial conditions

$$u(a) = c_0, \quad u'(a) = c_1, \tag{2}$$

where $\ell \in \mathcal{L}_{ab}$, $q \in L([a, b]; \mathbb{R})$ and $c_0, c_1 \in \mathbb{R}$. By a solution of the equation (1) we understand a function $u \in \widetilde{C}'([a, b]; \mathbb{R})$ satisfying this equation (almost everywhere) in $[a, b]$.

Along with the problem (1), (2) consider the corresponding homogeneous problem

$$u''(t) = \ell(u)(t), \tag{1_0}$$

$$u(a) = 0, \quad u'(a) = 0. \tag{2_0}$$

The following result is well-known from the general theory of boundary value problems for functional differential equations (see, e.g., [1, 2, 4, 5, 8]).

Theorem 1. *The problem (1), (2) is uniquely solvable iff the corresponding homogeneous problem (1₀), (2₀) has only the trivial solution.*

Definition 1. We will say that an operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $\tilde{H}_{ab}(a)$ if for every function $u \in \tilde{C}'([a, b]; \mathbb{R})$ satisfying

$$u''(t) \geq \ell(u)(t) \quad \text{for } t \in [a, b], \quad (3)$$

and (2₀), the inequality

$$u(t) \geq 0 \quad \text{for } t \in [a, b] \quad (4)$$

holds.

Remark 1. It follows from Definition 1 that if $\ell \in \tilde{H}_{ab}(a)$, then the homogeneous problem (1₀), (2₀) has only the trivial solution. Therefore, according to Theorem 1 the problem (1), (2) is uniquely solvable. Moreover, the inclusion $\ell \in \tilde{H}_{ab}(a)$ guarantees that if $q \in L([a, b]; \mathbb{R}_+)$, then the unique solution of the problem (1), (2₀) is nonnegative.

Note also that $\ell \in \tilde{H}_{ab}(a)$ iff a certain theorem on differential inequalities hold. More precisely, whenever $u, v \in \tilde{C}'([a, b]; \mathbb{R})$ satisfy the inequalities

$$\begin{aligned} u''(t) &\leq \ell(u)(t) + q(t), & v''(t) &\geq \ell(v)(t) + q(t) & \text{for } t \in [a, b], \\ u(a) &= v(a), & u'(a) &= v'(a), \end{aligned}$$

then

$$u(t) \leq v(t) \quad \text{for } t \in [a, b].$$

In the paper [7], sufficient conditions are established guaranteeing the inclusion $\ell \in \tilde{H}_{ab}(a)$. In particular, in [7, Theorem 1.3] the following proposition is proved.

Proposition 1. *Let $-\ell \in P_{ab}$ be an a -Volterra operator and let there exist a function $\gamma \in \tilde{C}'_{loc}(]a, b[; \mathbb{R})$ satisfying*

$$\gamma''(t) \leq \ell(\gamma)(t) \quad \text{for } t \in [a, b], \quad (5)$$

$$\gamma(t) > 0 \quad \text{for } t \in]a, b[, \quad (6)$$

$$\gamma(a) + \lim_{t \rightarrow a^+} \gamma'(t) \neq 0. \quad (7)$$

Then $\ell \in \tilde{H}_{ab}(a)$.

Below we will prove (see Theorem 3) that in Proposition 1 the condition on ℓ to be a -Volterra operator is necessary. Analogous result for first order functional differential equations is proved in [3].

Before we formulate the main results, let us introduce the following definition.

Definition 2. Let the problem (1₀), (2₀) have only the trivial solution. Denote by Ω the operator, which assigns to every function $q \in L([a, b]; \mathbb{R})$ the solution of the problem (1), (2₀).

Remark 2. From Theorem 1 it follows that the operator Ω is well defined. It is also clear that Ω is a linear operator which maps the set $L([a, b]; \mathbb{R})$ into the set $C([a, b]; \mathbb{R})$.

Remark 3. It follows from [5, Theorem 1.4.1] that the operator Ω is continuous (bounded) (see also [1, 4, 6]).

Remark 4. It immediately follows from Definitions 1 and 2 that if $\ell \in \tilde{H}_{ab}(a)$, then the operator Ω is nonnegative, i.e., it transforms the set $L([a, b]; \mathbb{R}_+)$ into the set $C([a, b]; \mathbb{R}_+)$.

Theorem 2. Let $-\ell \in P_{ab}$ and $\ell \in \tilde{H}_{ab}(a)$. Then Ω is an a -Volterra operator.

Proof. Let $t_0 \in]a, b[$ and let the function $q \in L([a, b]; \mathbb{R})$ be such that

$$q(t) = 0 \quad \text{for } t \in [a, t_0]. \quad (8)$$

We will show that

$$\Omega(q)(t) = 0 \quad \text{for } t \in [a, t_0]. \quad (9)$$

Denote by u the solution of the problem (1), (2₀) and by v the solution of the problem

$$v''(t) = \ell(v)(t) + |q(t)|, \quad (10)$$

$$v(a) = 0, \quad v'(a) = 0. \quad (11)$$

According to Remark 1 (see also Remark 4) and the assumption $-\ell \in \tilde{H}_{ab}(a)$, we have

$$v(t) \geq 0 \quad \text{for } t \in [a, b], \quad (12)$$

$$u(t) \leq v(t) \quad \text{for } t \in [a, b]. \quad (13)$$

Since $-\ell \in P_{ab}$, it follows from (10) and (12) that

$$v''(t) \leq |q(t)| \quad \text{for } t \in [a, t_0]$$

Hence, on account of (8), (11) and (12), we obtain

$$v(t) = 0 \quad \text{for } t \in [a, t_0]. \quad (14)$$

On the other hand, by virtue of (1), (10), (13), and the assumption $-\ell \in P_{ab}$, we get

$$(u(t) - v(t))'' = \ell(u - v)(t) + q(t) - |q(t)| \geq q(t) - |q(t)| \quad \text{for } t \in [a, b].$$

Hence in view of (8) and (14) we get

$$u''(t) \geq v''(t) = 0 \quad \text{for } t \in [a, t_0].$$

The latter inequality, together with (13), (14) and (2₀), implies

$$u(t) = 0 \quad \text{for } t \in [a, t_0].$$

Consequently (since $u(t) = \Omega(q)(t)$ for $t \in [a, b]$), the equality (9) is fulfilled. \square

Theorem 3. *Let $-\ell \in P_{ab}$ and $\ell \in \tilde{H}_{ab}(a)$. Then ℓ is an a -Volterra operator.*

Proof. Assume the contrary, let ℓ be not an a -Volterra operator. Then there exist $v_0 \in C([a, b]; \mathbb{R})$ and $b_0 \in]a, b[$ such that

$$v_0(t) = 0 \quad \text{for } t \in [a, b_0]$$

and

$$\text{mes}\{t \in [a, b_0] : \ell(v_0)(t) \neq 0\} > 0.$$

Without loss of generality we can assume that

$$\text{mes}\{t \in [a, b_0] : \ell(v_0)(t) < 0\} > 0. \quad (15)$$

First we will show that

$$\Omega(\ell(|v_0|))(t) = 0 \quad \text{for } t \in [a, b_0], \quad (16)$$

where Ω is the operator introduced in Definition 2.

Choose a sequence of functions $v_k \in \tilde{C}'([a, b]; \mathbb{R})$, $k \in \mathbb{N}$, such that

$$\lim_{k \rightarrow +\infty} \|v_k - |v_0|\|_C = 0 \quad (17)$$

and

$$v_k(t) = 0 \quad \text{for } t \in [a, b_0], \quad k \in \mathbb{N}. \quad (18)$$

According to Remark 3 and (17), we get

$$\lim_{k \rightarrow +\infty} \|\Omega(\ell(v_k)) - \Omega(\ell(|v_0|))\|_C = 0. \quad (19)$$

It is clear that

$$v_k''(t) = \ell(v_k)(t) + q_k(t) \quad \text{for } t \in [a, b], \quad k \in \mathbb{N}, \quad (20)$$

where

$$q_k(t) \stackrel{\text{def}}{=} v_k''(t) - \ell(v_k)(t) \quad \text{for } t \in [a, b], \quad k \in \mathbb{N}. \quad (21)$$

Consequently,

$$v_k(t) = \Omega(q_k)(t) \quad \text{for } t \in [a, b], \quad k \in \mathbb{N}. \quad (22)$$

It follows from (20)–(22) that

$$v_k(t) = \Omega(v_k''(t) - \ell(v_k)(t)) \quad \text{for } t \in [a, b], \quad k \in \mathbb{N}. \quad (23)$$

Hence, taking into account the fact that Ω is an a -Volterra operator (see Theorem 2) and the condition (18), we obtain

$$\Omega(\ell(v_k))(t) = -v_k(t) = 0 \quad \text{for } t \in [a, b_0], \quad k \in \mathbb{N}.$$

Thus, in view of (19), we get the equality (16).

Let u be a solution of the problem (1), (2₀), where

$$q(t) = \begin{cases} -\ell(|v_0|)(t) & \text{for } t \in [a, b_0[\\ 0 & \text{for } t \in [b_0, b] \end{cases}. \quad (24)$$

It is evident that

$$u(t) = \Omega(q)(t) \quad \text{for } t \in [a, b] \quad (25)$$

and

$$q(t) \geq 0 \quad \text{for } t \in [a, b]. \quad (26)$$

Moreover, on account of the assumption $-\ell \in P_{ab}$, the inequality

$$\ell(|v_0|)(t) \leq \ell(v_0)(t) \quad \text{for } t \in [a, b]$$

holds. Consequently, due to (15) and (24)

$$\text{mes}\{t \in [a, b_0] : q(t) > 0\} > 0. \quad (27)$$

According to Theorem 2, Ω is an a -Volterra operator. Hence by virtue of (16) and (24) we get from (25) that

$$u(t) = 0 \quad \text{for } t \in [a, b_0]. \quad (28)$$

On the other hand, the inequality (26) and the assumption $\ell \in \tilde{H}_{ab}(a)$ imply

$$u(t) \geq 0 \quad \text{for } t \in [a, b]. \quad (29)$$

In view of (29) and the assumption $-\ell \in P_{ab}$, it follows from (1) that

$$u''(t) \leq q(t) \quad \text{for } t \in [a, b].$$

Hence, on account of (24), we obtain

$$u''(t) \leq 0 \quad \text{for } t \in [b_0, b].$$

The latter inequality, together with (28) and (29), yields

$$u(t) = 0 \quad \text{for } t \in [a, b].$$

Therefore, it follows from (1) that $q \equiv 0$, which contradicts (27).

□

References

- [1] N. V. Azbelev, V. P. Maksimov, L. F. Rakhmatullina, *Introduction to the theory of functional differential equations* (in Russian), Nauka, Moscow, 1991.
- [2] E. Bravyi, *A note on the Fredholm property of boundary value problems for linear functional differential equations*, Mem. Differential Equations Math. Phys. **20** (2000), 133–135.
- [3] E. Bravyi, A. Lomtatidze, B. Půža, *A note on the theorem on differential inequalities*, Georgian Math. J. **7** (2000), No. 4, 627–631.
- [4] R. Hakl, A. Lomtatidze, I. P. Stavroulakis, *On a boundary value problem for scalar linear functional differential equations*, Abstract and Appl. Anal. (2004), No. 1, 45–67.
- [5] I. Kiguradze, B. Půža, *Boundary value problems for systems of linear functional differential equations*, Folia, Facult. Scient. Natur. Univ. Masarykianae Brunensis, Mathematica **12**, 2003.
- [6] I. Kiguradze, B. Půža, *On boundary value problems for systems of linear functional differential equations*, Czechoslovak Math. J. **47** (1997), No. 2, 341–373.
- [7] A. Lomtatidze, H. Štěpánková, *On sign constant and monotone solutions of second order linear functional differential equations*, Mem. Differential Equations Math. Phys. **33** (2004), 10–45.
- [8] Š. Schwabik, M. Tvrdý, O. Vejvoda, *Differential and integral equations: boundary value problems and adjoints*, Academia, Praha, 1979.

(Received February 7, 2005)

Author's address:

Hana Štěpánková
Faculty of Education, University of South Bohemia,
Jeronýmova 10, 371 15 České Budějovice,
Czech Republic
E-mail: stepanh@pf.jcu.cz