Fučík spectra for vector equations

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Abstract. — *Let* $L : \mathrm{dom}L \subset L^2(\Omega;\mathbb{R}^N) \to L^2(\Omega;\mathbb{R}^N)$ *be a linear operator,* Ω *being open and bounded in* $\mathbb{R}^M$*. The aim of this paper is to study the Fučík spectrum for vector problems of the form* $Lu = \alpha Au^+ - \beta Au^-$*, where* A *is an* $N \times N$ *matrix,* α, β *are real numbers,* u^+ *a vector defined componentwise by* $(u^+)_i = \max\{u_i, 0\}$*,* u^- *being defined similarly. With* λ^* *an eigenvalue for the problem* $Lu = \lambda Au$ *, we describe (locally) curves in the Fučík spectrum passing through the point* (λ^*, λ^*)*, distinguishing different cases illustrated by examples, for which Fučík curves have been computed numerically.*

Key words:Nonlinear eigenvalue problems, asymmetric nonlinearities.

AMS Classification:34B15, 35P30.

1 Introduction

Since the pioneering work of Dancer [5] and Fučík [7, 8], problems with asymmetric nonlinearities, also called jumping nonlinearities, have been the subject of numerous studies. Most of these problems have the form

$$Lu = \alpha u^+ - \beta u^- + g(\cdot, u), \tag{1}$$

where L is a differential operator acting on a space of real-valued functions, $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$, and g is a supplementary nonlinear term, which typically is assumed to have sublinear growth in u, for $|u| \to \infty$.

The first question with respect to such problems is to determine the set of pairs $(\alpha, \beta) \in \mathbb{R}^2$ for which the homogeneous equation

$$Lu = \alpha u^+ - \beta u^- \tag{2}$$

has nontrivial solutions. These points form the so-called Fučík or Dancer-Fučík spectrum.

The degree of difficulty of the full problem (1) depends on whether (α, β) belongs or not to the Fučík spectrum. If not, the existence of solutions for (1) can be easily studied through the computation of a degree for the nonlinear operator associated to equation (2) (see [1] for some recent results concerning degree computations). The more difficult case, when (α, β) belongs to the Fučík spectrum, can be considered as a situation of resonance. Such problems have been studied, for instance, by Gallouët and Kavian [9], Schechter [16], Ben-Naoum, Fabry and Smets [2].

The aim of the present paper is to study Fučík spectrum for vector equations, the formulation being as follows. Let $L : \mathrm{dom}L \subset L^2(\Omega; \mathbb{R}^N) \to L^2(\Omega; \mathbb{R}^N)$ be a linear operator, Ω being open and bounded in $\mathbb{R}^M$ and A a real $N \times N$ matrix. We consider the equation

$$Lu = \alpha A u^+ - \beta A u^- \tag{3}$$

(we use the same notation for the matrix and the bounded operator on $L^2(\Omega; \mathbb{R}^N)$ associated to it). The notations u^+, u^- have to be understood componentwise, i.e. $(u^+)_i = \max\{u_i, 0\}$, $(u^-)_i = \max\{-u_i, 0\}$. The points $(\alpha, \beta) \in \mathbb{R}^2$ for which (3) has nontrivial solutions form the Fučík spectrum, which will be denoted by $\Sigma(L, A)$. With $\eta = (\alpha + \beta)/2$, $\epsilon = (\alpha - \beta)/2$, equation (3) can be rewritten as

$$Lu = \eta A u + \epsilon A |u|, \tag{4}$$

where the absolute value $|u|$ is again to be understood componentwise. Notice that, although we will consider below problem (3) or, equivalently (4), the discussion could easily be generalized by replacing (4) by

$$Lu = \eta A u + \epsilon B |u| \,,$$

B being another real $N \times N$ matrix.

Studying the Fučík spectrum of a system like (3) requires some preliminary steps and also calls for a distinction between different cases. Section 2 is devoted to a reduction of (3) to a finite-dimensional problem, which is valid for points (α, β) "not too far" from the diagonal in the (α, β)-plane. Using implicit function theorems, we present in Sections 3 and 4, local existence results for Fučík curves (i.e. curves which are contained in the Fučík spectrum) near the diagonal $\alpha = \beta$, distinguishing the case of curves which are transversal with respect to the diagonal, and the case of curves tangent to the diagonal; both cases are illustrated by examples. In Section 5, we consider hypotheses on L and A, under which a variational formulation can be given for the problem, the points of the Fučík spectrum being then associated to critical values of some functional, whereas in Section 6, we introduce a condition under which the Fučík spectrum, near an intersection with the diagonal $\alpha = \beta$, reduces locally to a single curve perpendicular to the diagonal at the point of intersection.

The information obtained concerning the slopes of the Fučík curves at their crossing point with the diagonal, is used as a starting point for numerical computations that have been performed to draw Fučík curves for the examples considered in the paper.

2 Reduction to an equivalent problem

The points of the form (α, α) clearly play a major role within the Fučík spectrum. They correspond to (generalized) eigenvalues λ for the equation

$$Lu = \lambda A u\,. \tag{5}$$

Let λ^* be a particular solution of the eigenvalue problem (5). The following hypotheses are assumed to hold throughout:

> (H_1) λ^* is an eigenvalue for the pair (L, A); $L - \lambda^* A$ is a Fredholm operator of index 0, which means that $\mathrm{Im}(L - \lambda^* A)$ is closed, $\ker(L - \lambda^* A)$ and $(\mathrm{Im}(L - \lambda^* A))^{\perp}$ having the same finite dimension.

Let the adjoint of L be denoted by L^* and the transpose of A by A^t; the orthogonal projections P, Q onto $\ker(L - \lambda^* A)$ and $\ker(L^* - \lambda^* A^t) = (\mathrm{Im}(L - \lambda^* A))^{\perp}$ respectively, will play an important role in the sequel. The scalar product in $\mathbb{R}^N$ will be denoted by $(\cdot, \cdot)$, whereas $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ will denote the scalar product and the norm in $L^2(\Omega; \mathbb{R}^N)$, respectively.

Using a Lyapunov-Schmidt decomposition, we will reduce equation (3) to a problem in a finite-dimensional space. Similar results can be found in [1] and [14] for scalar equations. Since the operator

$$\tilde{L}_{\lambda^* A} = (L - \lambda^* A)_{|(\ker(L-\lambda^* A))^\perp} : (\ker(L - \lambda^* A))^\perp \to \mathrm{Im}(L - \lambda^* A)$$

admits a bounded inverse, we can prove the following lemma, where $\|(\tilde{L}_{\lambda^* A})^{-1}\|$ and $\|A\|$ denote operator norms.

Lemma 1. *Assume that* (H_1) *holds and that* $\alpha, \beta \in \mathbb{R}$ *satisfy the condition*

$$(|\beta - \lambda^*| + |\alpha - \lambda^*|)\|(\tilde{L}_{\lambda^* A})^{-1}\|\|A\| < 1\,. \tag{6}$$

Then, for any $x \in \ker(L - \lambda^* A)$*, the problem*

$$Lu = \lambda^* A u + (I - Q)[\alpha A u^+ - \beta A u^- - \lambda^* A u], \tag{7}$$
$$Pu = x \tag{8}$$

has a unique solution $u_x = u_x(\alpha, \beta)$*. Moreover, the solution* $u_x(\alpha, \beta)$ *is locally Lipschitzian with respect to* x, α, β*.*

Proof. We see that the system (7), (8) is equivalent to

$$u = x + (\tilde{L}_{\lambda^* A})^{-1}(I - Q)[\alpha A u^+ - \beta A u^- - \lambda^* A u]$$

or

$$u = x + (\tilde{L}_{\lambda^* A})^{-1}(I - Q)[(\alpha - \lambda^*) A u^+ - (\beta - \lambda^*) A u^-]\,.$$

Condition (6) implies that, given $x \in \ker(L - \lambda^* A)$, the mapping appearing on the right-hand side of the above equation is a contraction mapping. The conclusion then follows from classical results concerning contraction mappings. □

Let $u_x = u_x(\alpha, \beta)$ be the solution of (7), (8). Define :

$$\begin{aligned} c_0(x, \alpha, \beta) &= -Q[\alpha A u_x^+ - \beta A u_x^- - \lambda^* A u_x] \\ &= -\frac{\alpha - \beta}{2} QA|u_x| - \left(\frac{\alpha + \beta}{2} - \lambda^*\right) QAu_x, \end{aligned}$$

so that u_x verifies

$$Lu_x = \alpha A u_x^+ - \beta A u_x^- + c_0(x, \alpha, \beta). \tag{9}$$

For given α, β, equation (3) admits a nontrivial solution if and only if there exists $x \in \ker(L - \lambda^* A), x \neq 0$, such that $c_0(x, \alpha, \beta) = 0$. The problem is therefore reduced to a problem in a finite-dimensional space.

Notice that

$$u_x = x + O(|\beta - \lambda^*| + |\alpha - \lambda^*|) \text{ for } (\alpha, \beta) \to (\lambda^*, \lambda^*),$$

uniformly for x in a bounded set, and that $c_0(rx, \alpha, \beta) = rc_0(x, \alpha, \beta)$ for all $r \geq 0$.

Determining the Fučík spectrum in the neighborhood of (λ^*, λ^*) thus consists in finding values of (α, β), close to (λ^*, λ^*), such that $c_0(x, \alpha, \beta) = 0$ for some $x \neq 0$. Using implicit function theorems, we treat that question in Sections 3 and 4, distinguishing two different cases.

3 Fučík curves transversal to the diagonal $\alpha = \beta$.

In order to determine curves in the neighborhood of (λ^*, λ^*), which belong to the Fučík spectrum $\Sigma(L, A)$, we consider the system

$$Lu = \alpha A u^+ - \beta A u^-, \|u\| = 1. \tag{10}$$

As we are interested in values of (α, β) in the Fučík spectrum, close to (λ^*, λ^*), we let $\varepsilon = (\alpha - \beta)/2, (\alpha + \beta)/2 = \lambda^* + \varepsilon\eta$; ε will be a small parameter. The system (10) can be written

$$Lu = \lambda^* A u + \varepsilon A|u| + \varepsilon\eta A u, \|u\| = 1\,;$$

we aim at determining, for ε "small", u, η as functions of ε satisfying the above equations. For $\varepsilon \neq 0$, it is equivalent to

$$\begin{aligned} u &= Pu + \varepsilon(\tilde{L}_{\lambda^* A})^{-1}(I - Q)[A|u| + \eta A u] + JQ[A|u| + \eta A u]\,, \\ & \|u\| = 1\,, \end{aligned} \tag{11}$$

where J is an isomorphism between $(\mathrm{Im}(L - \lambda^* A))^\perp$ and $\ker(L - \lambda^* A)$. Using the reduction of Section 2, it is also equivalent, for $\varepsilon \neq 0$, to the finite-dimensional problem

$$QA|u_x| + \eta QAu_x = 0, \|u_x\| = 1 \tag{12}$$

(in which u_x depends of course on ε, η). Working as in [1], we will apply an implicit function theorem to the problem (11), in order to solve it for u, η as functions of ε, for ε close to 0 (a similar approach has been used by Pope [14]; implicit function theorems also appear in [4] for studying the Fučík spectrum of fourth order differential operators). For $\varepsilon = 0$, the equation (11) or (12) reduces to

$$QA|x| + \eta QAx = 0,\ x \in \ker(L - \lambda^* A)\,, \|x\| = 1\,. \tag{13}$$

Notice that, if $(x_0, \eta_0) \in \ker(L - \lambda^* A) \times \mathbb{R}$ denotes a solution of (13), and if $QAx_0 \neq 0$, the value η_0 can be computed from

$$\eta_0 = -\frac{\langle A|x_0|, QAx_0\rangle}{\|QAx_0\|^2}\,. \tag{14}$$

In the application of an implicit function theorem, a difficulty lies in the fact that the set of points, at which the mapping $u \mapsto QA|u|$ is differentiable, need not be open in $L^2(\Omega; \mathbb{R}^N)$. That difficulty can be overcome by using a version of the implicit function theorem that only requires a (strong) Fréchet differentiability at one point (see [1]). The only term that needs to be considered in (11) is the term $JQA|u|$. The required differentiability property will be based on the fact, proved by adapting an argument of [17], that, with $\tilde{Q}$ a projection onto a finite dimensional space $X \subset L^2(\Omega; \mathbb{R})$, the mapping $L^2(\Omega; \mathbb{R}) \to X : U \mapsto \tilde{Q}|U|$, is strongly Fréchet-differentiable at the point U_0, provided that the real-valued function U_0 does not vanish on a set of non-zero measure. Since components of u that never contribute to QAu can be ignored, we introduce the following hypothesis:

> (H_2) for each $i = 1, \cdots, N$, the i^{th}-component of x_0 does not vanish on a set of positive measure, unless, for any $u \in L^2(\Omega; \mathbb{R}^N)$, the i^{th}-component of u brings no contribution to QAu.

With (H_2), the mapping $u \mapsto QA|u|$ is strongly differentiable at $x = x_0$, as a mapping from $L^2(\Omega; \mathbb{R}^N)$ to $(\mathrm{Im}(L - \lambda^* A))^\perp$. The application of the implicit function theorem yields the following result, in which (15) represents the invertibility of the Fréchet derivative and implies $QAx_0 \neq 0$; $\mathrm{sgn}(x_0)y$ is a vector whose i^{th} component is y_i, multiplied by the (not necessarily constant) sign of the i^{th} component of x_0. If a component $(x_0)_i$ of x_0 vanishes on a set of positive measure, $\mathrm{sgn}((x_0)_i)$ is undefined, but this does not matter since, by (H_2), $\mathrm{sgn}((x_0)_i)y_i$ then brings no contribution to $Q(A\,\mathrm{sgn}(x_0)y)$.

Theorem 1. *Let* $L : \mathrm{dom}L \subset L^2(\Omega;\mathbb{R}^N) \to L^2(\Omega;\mathbb{R}^N), A$ *and* λ^* *satisfy hypotheses* $(H_1), (H_2)$*. Let* (x_0, η_0) *be a solution of (13), such that*

$$\left.\begin{array}{l} y \in \ker(L - \lambda^* A), \langle y, x_0\rangle = 0, \\ Q(A\,\mathrm{sgn}(x_0)y) + \eta_0 QAy + \mu QAx_0 = 0 \end{array}\right\} \Longrightarrow y = 0, \mu = 0. \tag{15}$$

Then, there exist functions $\eta(\cdot), u(\cdot)$ *defined in a neighborhood* $\mathcal{E}$ *of* 0*, such that*

$$u(0) = x_0, \eta(0) = \eta_0 = -\frac{\langle A|x_0|, QAx_0\rangle}{\|QAx_0\|^2}, \tag{16}$$

$$Lu(\varepsilon) = (\lambda^* + \varepsilon\eta(\varepsilon))Au(\varepsilon) + \varepsilon A|u(\varepsilon)|, \|u(\varepsilon)\| = 1, \text{ for } \varepsilon \in \mathcal{E}. \tag{17}$$

Moreover, if

$$QAx \neq 0, \forall x \in \ker(L - \lambda^* A), x \neq 0, \tag{18}$$

and if (15) holds for all (η_0, x_0) *satisfying (13), the Fučík spectrum, within a neighborhood* $\mathcal{U}$ *of* (λ^*, λ^*)*, consists of a finite number of curves of the form* $(\lambda^* + \varepsilon(\eta(\varepsilon) + 1), \lambda^* + \varepsilon(\eta(\varepsilon) - 1))$*, where* $\eta(0)$ *is such that (13) has a solution* x_0 *for* $\eta = \eta(0)$*.*

Proof. As indicated above, the existence of functions $\eta(\cdot), u(\cdot)$ verifying (16), (17) follows from the application of an implicit function theorem to the system (12). For details, we refer to an analogous proof in [1].

For the last part of the theorem, let (α_n, β_n) belong to the Fučík spectrum, with $\alpha_n \to \lambda^*, \beta_n \to \lambda^*$ for $n \to \infty$. Let $\varepsilon_n = (\alpha_n - \beta_n)/2$. Because of (18), it is easily shown that λ^* is an isolated eigenvalue of (L, A); hence, for n large, we must have $\varepsilon_n \neq 0$. Let then η_n be defined by $(\alpha_n + \beta_n)/2 = \lambda^* + \varepsilon_n\eta_n$. The point (α_n, β_n) being in the Fučík spectrum, we must have

$$Q(A|u_{x_n}|) + \eta_n Q(Au_{x_n}) = 0,$$

for some x_n in $\ker(L - \lambda^* A), \|x_n\| = 1$. Passing to a subsequence, we can assume that $\{x_n\}$ converges to some $x^* \in \ker(L - \lambda^* A), \|x^*\| = 1$. It then follows that $\{\eta_n\}$ must converge to

$$\eta^* = -\frac{\langle A|x^*|, QAx^*\rangle}{\|QAx^*\|^2}$$

and that (η^*, x^*) must be a solution of (13). Because of (15), the solutions of (13) are isolated and therefore the number of solutions is finite. The conclusion then follows from the local uniqueness of solutions of (12), following from the implicit function theorem. ∎

When condition (15) is fulfilled, the above theorem asserts the existence of a curve in the Fučík spectrum, emanating from the point (λ^*, λ^*), with slope

$$\frac{\eta_0 - 1}{\eta_0 + 1} = -\frac{\|QAx_0\|^2 + \langle A|x_0|, QAx_0\rangle}{\|QAx_0\|^2 - \langle A|x_0|, QAx_0\rangle}\,. \tag{19}$$

If (15) holds for all (η_0, x_0) satisfying (13), the number of Fučík curves will be the number of solutions of (13). Moreover, it is clear that, if the conditions of Theorem 1 are satisfied, the same is true with η_0, x_0 replaced by $-\eta_0, -x_0$. Consequently, the Fučík curves can be grouped by pairs of curves of reciprocal slopes at (λ^*, λ^*), if their slopes are different from -1. Examples with multiple curves have been given in [1] for self-adjoint problems for scalar differential equations.

On the other hand, under (18), it follows from the last part of the proof ((15) is not required here) that, given $\delta > 0$, there is a neighborhood of (λ^*, λ^*), such that, within that set, no point of the Fučík spectrum, except (λ^*, λ^*), is to be found in the sector $|\alpha - \beta| \leq |\alpha + \beta - 2\lambda^*|/(M_0 + \delta)$, where

$$M_0 = \max_{x \in \ker(L - \lambda^* A),\, \|x\|=1} \frac{|\langle A|x|, QAx\rangle|}{\|QAx\|^2}$$

In the case where $\dim\ker(L-\lambda^*A) = \dim(\mathrm{Im}(L-\lambda^*A))^\perp = 1$, let $\ker(L-\lambda^*A) = \mathbb{R}x^*$, $(\mathrm{Im}(L-\lambda^*A))^\perp = \mathbb{R}y^*$. Condition (15) of Theorem 1 amounts to

$$QAx^* \neq 0 \text{ or } \langle Ax^*, y^*\rangle \neq 0,$$

and the equation (13) then gives two possibilities for η_0 :

$$\eta_0 = \pm\frac{\langle A|x^*|, y^*\rangle}{\langle Ax^*, y^*\rangle}\,.$$

We thus have, in the neighborhood of (λ^*, λ^*), two curves (not necessarily distinct) in the Fučík spectrum $\Sigma(L, A)$, whose slopes are given by

$$\frac{\pm\langle A|x^*|, y^*\rangle - \langle Ax^*, y^*\rangle}{\pm\langle A|x^*|, y^*\rangle + \langle Ax^*, y^*\rangle}\,. \tag{20}$$

Fučík spectra for scalar self-adjoint problems have been the most widely studied, starting from the classical periodic and Dirichlet boundary value problems for 2^{nd}-order ordinary differential equations [5, 7, 8]. Examples of 4^{th}-order problems are considered in [4, 12]. The following example, taken from Gaudenzi and Habets [10], illustrates the above results in the case of a non self-adjoint scalar problem.

EXAMPLE 1. Consider the scalar boundary value problem

$$u''' + \alpha u^+ - \beta u^- = 0, \tag{21}$$

$$u(0) = u'(0) = u(1) = 0\,. \tag{22}$$

Let the operator L be defined by

$$L : \mathrm{dom}L \subset L^2((0,1);\mathbb{R}) \to L^2((0,1);\mathbb{R}) : u \mapsto -u''',$$

where $\mathrm{dom}L$ is the set of functions in $H^3((0,1);\mathbb{R})$ verifying the boundary conditions; the matrix A is simply the number 1 here. The eigenvalues of L are simple, the i^{th} eigenvalue λ_i being equal to $-\tau_i^3$, where τ_i is the i^{th} zero of the function

$$z(s) = \frac{1}{3}\left[e^{-s} + 2e^{s/2}\sin\left(\frac{\sqrt{3}}{2}s - \frac{\pi}{6}\right)\right]\ ;$$

the corresponding eigenfunction is $u_i(t) = z(\lambda_i^{1/3}t)$. There are two Fučík curves emanating from each eigenvalue; they have been computed numerically in [10]. According to (20), the slopes of the Fučík curves starting from (λ_i, λ_i) are given by

$$\frac{\pm\langle |u_i|, w_i\rangle - \langle u_i, w_i\rangle}{\pm\langle |u_i|, w_i\rangle + \langle u_i, w_i\rangle}\,, \tag{23}$$

where w_i is an eigenfunction of the adjoint problem

$$-u''' - \lambda_i u = 0,$$

$$u(0) = u(1) = u'(1) = 0\,.$$

For the first eigenvalue, the problem (21), (22) has solutions of constant sign; hence, the Fučík spectrum contains the lines $\alpha = \lambda_1, \beta = \lambda_1$. For the second eigenvalue, the numerical values obtained from (23) for the slopes at (λ_2, λ_2) are -0.98411 and -1.01615 (the two curves are very close together, as shown in [10]); for the third eigenvalue, the values are -0.46108 and -2.16882.

4 Fučík curves tangent to the diagonal $\alpha = \beta$.

In the present section, we will establish the existence of Fučík curves tangent to the diagonal $\alpha = \beta$, under the hypothesis

$$(H_3) \qquad \dim[QA(\ker(L-\lambda^*A))] = \dim(\mathrm{Im}(L-\lambda^*A))^\perp - 1\,.$$

Taking into account the fact that $\dim\ker(L-\lambda^*A) = \dim(\mathrm{Im}(L-\lambda^*A))^\perp$, under (H_3), there exists $x^* \in \ker(L-\lambda^*A), \|x^*\| = 1, y^* \in (\mathrm{Im}(L-\lambda^*A))^\perp$, $\|y^*\| = 1$, such that

$$QAx^* = 0 \text{ and } \langle Ax, y^* \rangle = 0, \forall x \in \ker(L-\lambda^*A).$$

The definition of new variables will be slightly different from Section 3; we will let now $\varepsilon = (\alpha+\beta)/2 - \lambda^*, \varepsilon\eta = (\alpha-\beta)/2$, and again try to determine u, η as functions of ε, for ε "small". By analogy with (12), the equations to solve can be written, for $\varepsilon \neq 0$, as

$$\eta QA|u_x| + QAu_x = 0, \|u_x\| = 1. \tag{24}$$

For $\varepsilon = 0$, that system reduces to

$$\eta QA|x| + QAx = 0, x \in \ker(L-\lambda^*A), \|x\| = 1\,;$$

it admits the solutions $(x^*, 0)$ and $(-x^*, 0)$. Under the assumption

$$(H_4) \qquad \langle A|x^*|, y^* \rangle \neq 0,$$

we can apply an implicit function theorem and conclude that (24) can be solved in η, x for ε "small", leading to the following result.

Theorem 2. *Let $L : \mathrm{dom}L \subset L^2(\Omega;\mathbb{R}^N) \to L^2(\Omega;\mathbb{R}^N)$ and λ^* satisfy hypothesis $(H_1), (H_3)$ and (H_4). Then, there exist functions $\eta(\cdot), u(\cdot), v(\cdot)$, defined in a neighborhood $\mathcal{E}$ of 0, such that*

(i) $u(0) = x^*, v(0) = -x^*, \eta(0) = 0,$

(ii) $Lu(\varepsilon) = (\lambda^* + \varepsilon)Au(\varepsilon) + \varepsilon\eta(\varepsilon)A|u(\varepsilon)|,$
$Lv(\varepsilon) = (\lambda^* + \varepsilon)Av(\varepsilon) + \varepsilon\eta(\varepsilon)A|v(\varepsilon)|,$ *for* $\varepsilon \in \mathcal{E}$.

Under the hypotheses of Theorem 2, there are locally (at least) two Fučík curves emanating from (λ^*, λ^*) and tangent to the line $\alpha = \beta$, since $\eta(0) = 0$. This is illustrated by the example below, inspired by an example of scalar equation in [14]. Theorems 1 and 2 are complementary, in the sense that, in some cases, the Fučík curves whose existence is proved on the basis of one of the theorems can coexist with the Fučík curves obtained by the other.

However, it is clear that no Fučík curve transversal to the diagonal can exist when

$$QAx = 0, QA|x| \neq 0, \text{ for all } \in \ker(L - \lambda^* A), x \neq 0\,,$$

whereas, as already pointed above, there is no Fučík curve tangent to the diagonal when

$$QAx \neq 0, \forall x \in \ker(L - \lambda^* A), x \neq 0\,.$$

Example 1. Consider the system

$$u' = \alpha v^+ - \beta v^- \tag{25}$$

$$v' = -\alpha u^+ + \beta u^-\,, \tag{26}$$

with the boundary conditions

$$u(0) + u(\pi) = 0, v(0) = 0\,. \tag{27}$$

For the operator L, we take

$$L : \mathrm{dom}L \subset L^2((0,1);\mathbb{R}^2) \to L^2((0,1);\mathbb{R}^2) : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u' \\ v' \end{pmatrix},$$

$\mathrm{dom}L$ being the set of functions in $H^1((0,1);\mathbb{R}^2)$ verifying the boundary conditions. With

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

the eigenvalues for the problem $Lu = \lambda Au$ are $\lambda_n = (2n+1)$, with n an integer; they are simple, the corresponding eigenfunctions being given by

$$w_n(t) = \begin{pmatrix} \cos(2n+1)t \\ -\sin(2n+1)t \end{pmatrix}.$$

The same w_n are also eigenfunctions for the adjoint problem. It is easy to check that $\langle Aw_n, w_n\rangle = 0$, whereas

$$\langle A|w_n|, w_n\rangle = \int_0^\pi |\sin(2n+1)t| \cos(2n+1)t\, dt + \int_0^\pi |\cos(2n+1)t| \sin(2n+1)t\, dt \neq 0.$$

Hence, Theorem 2 applies and, consequently, two Fučík curves tangent to the diagonal $\alpha = \beta$ pass through the points (λ_n, λ_n). Two pairs of such curves have been computed and are represented in Figure 1 for $n = 0$ and $n = 1$.

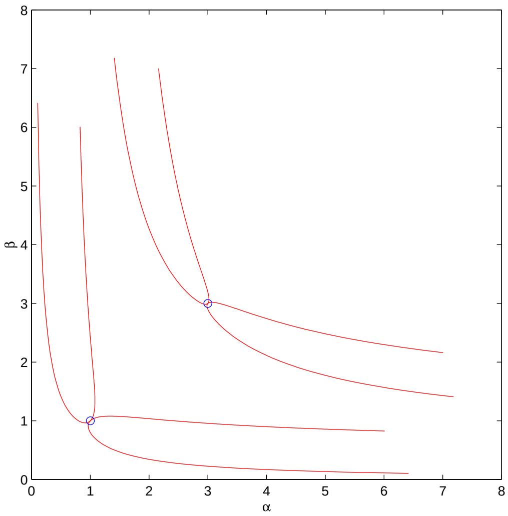

Figure 1: Fučík curves for the problem (25), (26).

5 Problems with a variational structure

We will now discuss situations where the problem has a variational structure and will study the Fučík spectrum in a square $I \times I$, where I is a closed interval such that the contraction condition (6) of Lemma 1 is satisfied for any α, β in I. We will assume that the operator L is self-adjoint, and the matrix A diagonal. These hypotheses may seem very restrictive, but the required structure may of course be obtained after a multiplication of equation (3) by an invertible matrix T chosen to make TL self-adjoint and TA diagonal. We define the functional

$$h_0 : \ker(L - \lambda^* A) \times I \times I \to \mathbb{R} : (x, \alpha, \beta) \mapsto \langle c_0(x, \alpha, \beta), x \rangle \ .$$

Notice that, $L - \lambda^* A$ being self-adjoint, the projectors $P = Q$ coincide, so that, using (9), we have

$$\begin{aligned} h_0(x, \alpha, \beta) &= \langle c_0(x, \alpha, \beta), x \rangle = \langle c_0(x, \alpha, \beta), u_x \rangle, \\ &= \langle L u_x, u_x \rangle - \alpha \langle A u_x^+, u_x \rangle + \beta \langle A u_x^-, u_x \rangle \ . \end{aligned}$$

or, since A is assumed diagonal,

$$h_0(x, \alpha, \beta) = \langle L u_x, u_x \rangle - \alpha \langle A u_x^+, u_x^+ \rangle - \beta \langle A u_x^-, u_x^- \rangle \ .$$

The next lemma presents a few properties of the functions c_0 and h_0.

Lemma 2. *Assume that the operator $L : \mathrm{dom} L \subset L^2(\Omega; \mathbb{R}^N) \to L^2(\Omega; \mathbb{R}^N)$ is self-adjoint, the matrix A diagonal and that condition (6) of Lemma 1 is satisfied for any $\alpha, \beta \in I$. Then, the function h_0 admits partial derivatives with respect to $\alpha, \beta \in I$, is differentiable with respect to $x \in \ker(L - \lambda^* A)$ and*

$$\frac{\partial}{\partial \alpha} h_0(x, \alpha, \beta) = -\langle A u_x^+, u_x^+ \rangle, \tag{28}$$

$$\frac{\partial}{\partial \beta} h_0(x, \alpha, \beta) = -\langle A u_x^-, u_x^- \rangle, \tag{29}$$

$$\nabla_x h_0(x, \alpha, \beta) = 2 c_0(x, \alpha, \beta). \tag{30}$$

Proof. To prove (28), we consider the solutions u_x, v_x corresponding to two different sets (α, β), (α', β) of coefficients; we thus have

$$\begin{aligned} L u_x &= \alpha A u_x^+ - \beta A u_x^- + c_0(x, \alpha, \beta), && (31) \\ L v_x &= \alpha' A v_x^+ - \beta A v_x^- + c_0(x, \alpha', \beta)\,. && (32) \end{aligned}$$

We will multiply the above equations respectively by v_x and u_x and subtract them. Since L is self-adjoint and A diagonal, we obtain

$$\begin{aligned} (\alpha - \alpha')\langle A u_x^+, v_x^+ \rangle - (\alpha - \beta)\langle A u_x^+, v_x^- \rangle + (\alpha' - \beta)\langle A u_x^-, v_x^+ \rangle \\ + \langle c_0(x, \alpha, \beta) - c_0(x, \alpha', \beta), x \rangle = 0. \quad (33) \end{aligned}$$

But, the matrix A being diagonal, the scalar product $\langle A u_x^+, v_x^- \rangle$ is the sum of multiples of terms of the form

$$\int_\Omega (u_x^+)_i (v_x^-)_i\,.$$

For such a term, we have

$$\begin{aligned} 0 \le \int_\Omega (u_x^+)_i (v_x^-)_i &= -\int_{(u_x)_i > 0, (v_x)_i < 0} (u_x)_i (v_x)_i \\ &\le \frac{1}{4} \int_{(u_x)_i > 0, (v_x)_i < 0} [(u_x)_i - (v_x)_i]^2 \\ &\le \frac{1}{4} \int_\Omega [(u_x)_i - (v_x)_i]^2\,. \end{aligned}$$

Since $u_x = u_x(\alpha, \beta)$ is Lipschitzian with respect to α, it then follows that there exists $C > 0$ such that

$$|\langle Au_x^+, v_x^- \rangle| \leq C|\alpha - \alpha'|^2.$$

A similar result holds for $\langle Au_x^-, v_x^+ \rangle$. Dividing (33) by $\alpha - \alpha'$ and letting α' tend to α, we obtain

$$\frac{\partial}{\partial \alpha} \langle c_0(x, \alpha, \beta), x \rangle = -\langle Au_x^+, u_x^+ \rangle.$$

The proof of (29) is similar.

For (30), let u_x and u_y be solutions given by Lemma 1 respectively for x and for y in X. We thus have

$$Lu_x = \alpha Au_x^+ - \beta Au_x^- + c_0(x, \alpha, \beta),$$

$$Lu_y = \alpha Au_y^+ - \beta Au_y^- + c_0(y, \alpha, \beta).$$

Multiplying the above equations respectively by u_y and by u_x, and working as above, it is easy to prove that

$$\begin{aligned} \langle c_0(x, \alpha, \beta) + c_0(y, \alpha, \beta), (x - y) \rangle &= \langle c_0(x, \alpha, \beta), x \rangle \\ -\langle c_0(y, \alpha, \beta), y \rangle &+ O(\|x - y\|^2). \end{aligned}$$

or, since $c_0(x, \alpha, \beta)$ is Lipschitzian with respect to x,

$$2\langle c_0(x, \alpha, \beta), (x - y) \rangle = \langle c_0(x, \alpha, \beta), x \rangle - \langle c_0(y, \alpha, \beta), y \rangle + O(\|x - y\|^2). \tag{34}$$

This shows that the function $h_0(\cdot, \alpha, \beta) : x \mapsto \langle c_0(x, \alpha, \beta), x \rangle$ is differentiable and that its gradient is given by (30). $\square$

In scalar problems, with L self-adjoint, A can be taken equal to 1. In that case, the conclusions (28), (29), (30) write

$$\frac{\partial}{\partial \alpha} h_0(x, \alpha, \beta) = -\|u_x^+\|^2, \ \frac{\partial}{\partial \beta} h_0(x, \alpha, \beta) = -\|u_x^-\|^2,$$

and

$$\nabla_x h_0(x, \alpha, \beta) = 2c_0(x, \alpha, \beta)$$

(see [1]).

Since (α, β) belongs to the Fučík spectrum if and only if $c_0(x, \alpha, \beta) = 0$ for some $x \neq 0$, the following theorem, which provides a variational characterization of that spectrum within $I \times I$, is an immediate consequence of the previous lemma; it generalizes well-known results for scalar equations (see [1], [13]).

Theorem 3. *Let the operator L and the matrix A satisfy the hypotheses of Lemma 2, let the closed interval $I \subset \mathbb{R}$ be such that condition (6) of Lemma 1 is satisfied for any α, β in I. Then the point $(\alpha, \beta) \in I \times I$ belongs to the Fučík spectrum for problem (3) if and only if 0 is a critical value of the functional*

$$h_0(\cdot, \alpha, \beta) : \ker(L - \lambda A) \to \mathbb{R} : x \mapsto \langle c_0(x, \alpha, \beta), x \rangle,$$

that critical value being reached at some point $x \neq 0$.

Theorem 3 can be used directly to characterize parts of the Fučík spectrum, which can be considered, under sign hypotheses on the matrix A, as the outermost parts of that spectrum within the square $I \times I$. Let us introduce the sets

$$\mathcal{F}^- = \{(\alpha, \beta) \in I \times I \mid \min_{x \in \ker(L - \lambda^* A),\, \|x\|=1} \langle c_0(x, \alpha, \beta), x \rangle = 0\}, \tag{35}$$

$$\mathcal{F}^+ = \{(\alpha, \beta) \in I \times I \mid \max_{x \in \ker(L - \lambda^* A),\, \|x\|=1} \langle c_0(x, \alpha, \beta), x \rangle = 0\}. \tag{36}$$

It results from Lemma 2 and Theorem 3 that $\mathcal{F}^-, \mathcal{F}^+$ are contained in the Fučík spectrum of L.

More precise conclusions can be obtained when the diagonal matrix A is positive (or negative) definite. In that case, since $h_0(x, \lambda^*, \lambda^*) = 0, \forall x \in \ker(L - \lambda^* A)$, it follows from (28), (29) that, for all $x \in \ker(L - \lambda^* A), x \neq 0$,

$$h_0(x, \alpha, \alpha) > 0, \text{ for } \alpha \in I, \alpha < \lambda^*, h_0(x, \alpha, \alpha) < 0, \text{ for } \alpha \in I, \alpha > \lambda^*.$$

Hence, the sets $\mathcal{F}^+, \mathcal{F}^-$ are non empty and separate the sets $\{(\alpha, \alpha) \in I \times I \mid \alpha < \lambda^*\}$ and $\{(\alpha, \alpha) \in I \times I \mid \alpha > \lambda^*\}$. Moreover, still under the assumption that A is positive definite, we have by (28), under the hypotheses of Theorem 3,

$$(\alpha, \beta) \in \mathcal{F}^- \Longrightarrow (\alpha', \beta') \notin \Sigma(L, A) \cap (I \times I) \text{ if } \alpha' < \alpha, \beta' < \beta\,;$$

A similar result holds for $\mathcal{F}^+$ or when A is negative definite, the roles of $\mathcal{F}^+, \mathcal{F}^-$ being then permuted. Under the above sign conditions for A, the sets $\mathcal{F}^-$ and $\mathcal{F}^+$ can thus be considered as the outermost parts of the Fučík spectrum within $I \times I$. Similar results have been described by Cac [3], Gonçalves and Magalhães [11], Magalhães [13], Schechter [15], for semilinear (scalar) elliptic boundary value problems. We collect the conclusions in the following proposition.

Corollary 1. *Let the operator L, the matrix A and the interval I satisfy the hypotheses of Theorem 3. Moreover, assume that the matrix A is positive (resp. negative) definite. Then, the Fučík spectrum $\Sigma(L, A)$ has an nonempty intersection with $I \times I$, containing the point (λ^*, λ^*) in its closure. That intersection contains the sets $\mathcal{F}^-, \mathcal{F}^+$, defined by (35), (36), and no point of $\Sigma(L, A) \cap (I \times I)$ is on the left (resp. on the right) of $\mathcal{F}^-$ or on the right (resp. on the left) of $\mathcal{F}^+$.*

Under the symmetry hypotheses made on L and A in this section and assuming A positive definite, it is possible to give a characterization of the Fučík spectrum near (λ^*, λ^*), which is slightly different from that of Theorem 1.

We start with the observation made in Section 2 that

$$u_x = x + O(|\beta - \lambda^*| + |\alpha - \lambda^*|) \text{ for } (\alpha, \beta) \to (\lambda^*, \lambda^*) \,. \tag{37}$$

More precisely, we have, for some $K > 0$,

$$\|u_x - x\| \le (|\beta - \lambda^*| + |\alpha - \lambda^*|)K\|x\|.$$

On the other hand, the matrix A being positive definite, we have $\langle Ax, x\rangle > 0, \forall x \in \ker(L - \lambda^* A), x \neq 0$ and, consequently, $PAx \neq 0, \forall x \in \ker(L - \lambda^* A), x \neq 0$. Adapting a remark following Theorem 1, we see that, for some $\delta > 0$, there is no point of the Fučík spectrum, near (λ^*, λ^*), in a sector $|\alpha - \beta| \le |\alpha + \beta - 2\lambda^*|/(M_0 + \delta)$, where

$$M_0 = \max_{x \in \ker(L - \lambda^* A),\, \|x\|=1} \frac{|\langle A|x|, x\rangle|}{\langle Ax, x\rangle.}$$

Hence, we can let $\varepsilon = (\alpha - \beta)/2, (\alpha + \beta)/2 = \lambda^* + \varepsilon\eta$, in the definition of c_0, which gives, taking into account the fact that $P = Q$,

$$c_0(x, \alpha, \beta) = -\varepsilon\, P(A|x|) + \varepsilon\eta PAx + O(\varepsilon^2) \text{ for } \varepsilon \to 0,$$

and

$$h_0(x,\alpha,\beta) = \langle c_0(x,\alpha,\beta), x\rangle \;=\; -\varepsilon\langle A|x|, x\rangle + \varepsilon\eta\langle Ax, x\rangle + O(\varepsilon^2) \quad \text{for } \varepsilon \to 0\,; \tag{38}$$

the last term in the above estimations is actually bounded by $\varepsilon^2 K'\|x\|$, for some $K' > 0$, if η belongs to a given compact set. By Theorem 3, (α,β) is a point of the Fučík spectrum if $h_0(\cdot,\alpha,\beta)$ has a local maximum or local minimum of value 0 at some point x^*, with $x \neq 0$. We will say that h_0 has a "true" (local) maximum on the set $S_A = \{x \in \ker(L-\lambda^* A) \mid \langle Ax, x\rangle = 1\}$, at the point x^*, if there exist a neighborhood $U \subset S_A$, of x^*, such that $\max\{h_0(x) \mid x \in U\} = h_0(x^*)$ and if $h_0(x) < h_0(x^*)$, for all $x \in \partial U$; a true minimum is defined similarly. The local extrema of $h_0(\cdot,\alpha,\beta)$ can be related to the local extrema of the function

$$G : \ker(L-\lambda^* A) \to \mathbb{R} : x \mapsto \langle A|x|, x\rangle$$

on the set S_A. More precisely, if G has a true maximum or minimum on the set S_A, at the point x^*, it is clear by a perturbation argument that, for η in a given bounded set, if $|\varepsilon|$ is sufficiently small, $h_0(\cdot,\alpha,\beta)$ will have a true maximum or minimum on S_A, of value close to

$$\varepsilon\, G(x^*) - \varepsilon\eta\,.$$

Let us assume, for instance, that $\varepsilon > 0$ and that G has a true maximum at the point x^*. By the definition of M_0, for $\eta = -M_0 - \delta$, the value of $\varepsilon\, G(x^*) - \varepsilon\eta$ is strictly negative; consequently, for ε sufficiently small, the function h_0 will have a true maximum of negative value near x^*. For $\eta = M_0 + \delta$, the sign of the maximum will be positive. Hence, keeping ε fixed, but small enough, we see that a value of η exists, close to $G(x^*)$, such that $h_0(\cdot,\alpha,\beta)$ will have a true maximum of value 0. The cases of a true minimum or of $\varepsilon < 0$ are treated similarly. In that way, we obtain, for $|\varepsilon|$ small enough, points (α,β) of the Fučík spectrum. As such points will depend continuously on $\varepsilon > 0$, a curve contained in the Fučík spectrum is derived locally, which is tangent to the line of equation

$$-\frac{\alpha-\beta}{2}\, G(x^*) + \left(\lambda^* - \frac{\alpha+\beta}{2}\right) = 0.$$

In other words, we have obtained the following result.

Theorem 4. *Let the operator L and the matrix A satisfy the hypotheses of Theorem 3, the matrix A being moreover positive definite. Assume that the function*

$$G : \ker(L - \lambda^* A) \to \mathbb{R} : x \mapsto \langle A|x|, x\rangle$$

has a true (local) maximum or minimum at x^*, subject to the constraint $\langle Ax, x\rangle = 1$. Then, in a neighborhood of (λ^*, λ^*), there is a curve in the Fučík spectrum of equation (3), emanating from the point (λ^*, λ^*), with slope

$$\frac{G(x^*) + 1}{G(x^*) - 1} \, . \tag{39}$$

Notice that, G being odd, to each nonzero maximum of G, corresponds a minimum (of opposite sign) and vice versa, so that the Fučík curves of Theorem 4 can be grouped by pairs if their slopes are different from -1.

EXAMPLE 1. We consider the following system of 2$^{\text{nd}}$-order ordinary differential equations:

$$u'' + k(u - v) + \alpha u^+ - \beta u^- = 0 \, , \tag{40}$$

$$v'' + k(v - u) + \alpha v^+ - \beta v^- = 0 \, , \tag{41}$$

with the Dirichlet boundary conditions

$$u(0) = u(\pi) = 0, v(0) = v(\pi) = 0 \, .$$

We take

$$L : \mathrm{dom}L \subset L^2((0, \pi); \mathbb{R}^2) \to L^2((0, \pi); \mathbb{R}^2) : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto - \begin{pmatrix} u'' + k(u - v) \\ v'' + k(v - u) \end{pmatrix} ,$$

dom$L$ being the set of functions in $H^2((0, \pi); \mathbb{R})$ verifying the boundary conditions; A will be the 2×2 identity matrix. It is easy to see that the numbers $\lambda = n^2$ ($n \in \mathbb{N}, n \neq 0$), and $\lambda = m^2 - 2k$ ($m \in \mathbb{N}, m \neq 0$) are the eigenvalues of the problem $Lu = \lambda Au$. If $n^2 \neq m^2 - 2k$, for all $m, n \in \mathbb{N}, m, n \neq 0$ all eigenvalues are simple whereas, if $n^2 = m^2 - 2k$ for some $m, n \in \mathbb{N}, m, n \neq 0$, this common value is an eigenvalue of multiplicity 2. In the latter case, the eigenspace is spanned by the eigenfunctions

$$w^{(1)}(t) = \begin{pmatrix} \sin nt \\ \sin nt \end{pmatrix} , \, w^{(2)}(t) = \begin{pmatrix} \sin mt \\ -\sin mt \end{pmatrix} \, .$$

We consider, for instance, the case $k = -5/2, \lambda^* = 9$ $(n = 3, m = 2)$. The conditions of Theorem 4 are satisfied, so that Fučík curves can be put in relation with the critical points of the function

$$G : \ker(L - \lambda^* A) \to \mathbb{R} : x \mapsto \langle |x|, x \rangle \,,$$

on the sphere $\|x\| = 1$ in $\ker(L - \lambda^* A)$. With $x_\theta = \cos\theta\, w^{(1)} + \sin\theta\, w^{(2)}$, we have

$$\begin{aligned} G(x_\theta) \;\; = \;\; & \int_0^\pi |\cos\theta \sin 3t + \sin\theta \sin 2t|(\cos\theta \sin 3t + \sin\theta \sin 2t) + \cdots \\ & |\cos\theta \sin 3t - \sin\theta \sin 2t|(\cos\theta \sin 3t - \sin\theta \sin 2t)\, dt \,. \end{aligned}$$

It is computed that, on the sphere $\|x\| = 1$, G has 6 extremal points, giving four extremal values. By Theorem 4, to these extremal values, cor-

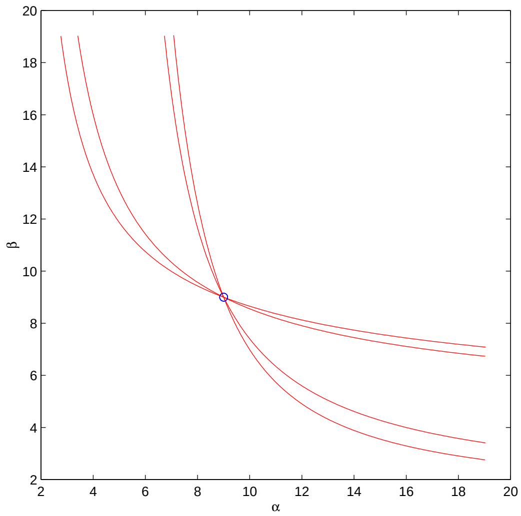

Figure 2: Fučík curves for the problem (40), (41).

respond four Fučík curves whose respective slopes, at the point $(9, 9)$, are $-2.6045, -2, -1/2,\ -0.3839$. The four Fučík curves are represented in Figure 2. The curves of slopes $-2, -1/2$ come from solutions with $u = v$ and are thus Fučík curves for the problem

$$\begin{aligned} & u'' + \alpha u^+ - \beta u^- = 0, \\ & u(0) = u(\pi) = 0\,. \end{aligned}$$

Their (well-known) equations are

$$\frac{2}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = 1\,, \frac{1}{\sqrt{\alpha}} + \frac{2}{\sqrt{\beta}} = 1\,.$$

6 Fučík spectrum reduced to a curve in a neighborhood of (λ^*, λ^*)

In this section, we discuss a situation where $\dim \ker(L-\lambda^* A) > 1$, where the nondegeneracy condition (15) of Theorem 1 fails to be satisfied, and which is nonetheless of practical interest. It concerns problems for which the following hypothesis is satisfied:

> (H_5) For α, β close to λ^*, if $c_0(x^*, \alpha, \beta) = 0$ for some $x^* \in \ker(L-\lambda^* A) \setminus \{0\}$, then, for each $x \in \ker(L - \lambda^* A), c_0(x, \alpha, \beta) = 0$.

Such a condition holds for many periodic boundary value problems for autonomous differential equations. More precisely, the following result is presented in [6] for scalar equations of order $2N$ (here, $A = I$).

Lemma 3. *Let $L : \operatorname{dom} L \subset L^2((0, 2\pi); \mathbb{R}) \to L^2((0, 2\pi); \mathbb{R})$ be a self-adjoint linear ordinary differential operator of order $2N$ with constant coefficients, where*

$$\begin{aligned} \operatorname{dom} L &= \{u \in H^{2N}((0, 2\pi); \mathbb{R}) \mid \\ &\quad u(0) = u(2\pi), \ldots , u^{(2N-1)}(0) = u^{(2N-1)}(2\pi)\}. \end{aligned}$$

If $\dim\ \ker(L - \lambda^* I) = 2$, *then* (H_4) *holds.*

When the operator L is self-adjoint, the matrix A diagonal and positive definite, and if (H_5) holds, it follows immediately from the discussion of the previous section that the sets $\mathcal{F}^-, \mathcal{F}^+$, defined by (35), (36), coincide and that no other point of the Fučík spectrum is contained in the set $I \times I$ of Corollary 1. Hence, we have the following local uniqueness result.

Corollary 2. *Let the hypotheses of Corollary 1 hold, as well as hypothesis (H_5). Then, there is a unique Fučík curve in the set $I \times I$; it crosses the diagonal $\alpha = \beta$ at the point (λ^*, λ^*) and its slope at that point is equal to -1.*

Example 1. The above corollary can be applied to the following system of 2^{nd}-order ordinary differential equations:

$$u'' + k(u-v) + \alpha u^+ - \beta u^- = 0\,, \tag{42}$$
$$v'' - k(u-v) + \alpha v^+ - \beta v^- = 0\,, \tag{43}$$

considered with the periodic boundary conditions

$$u(0) = u(2\pi), u'(0) = u'(2\pi), v(0) = v(2\pi), v'(0) = v'(2\pi)\,.$$

We take

$$L : \text{dom}L \subset L^2((0,\pi);\mathbb{R}^2) \to L^2((0,\pi);\mathbb{R}^2) : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto - \begin{pmatrix} u'' + k(u-v) \\ v'' - k(u-v) \end{pmatrix},$$

dom$L$ being the set of functions in $H^2((0,\pi);\mathbb{R}^2)$ verifying the boundary conditions; A will be the 2×2 identity matrix. The eigenvalues of L are of the form m^2, or n^2-2k, m,n being integers. Provided that $n^2 - 2k \neq m^2$ for all $m,n \in \mathbb{N}$, all eigenvalues, except 0 and $-2k$, are of multiplicity 2. The first system of eigenvalues then corresponds to solutions with $u = v$, the latter to solutions with $u = -v$. It is immediate that the nonlinear system (42), (43) has solutions with $u = v$, for which the Fučík curves are easily computed; they intersect the diagonal at a point (m^2, m^2). The Fučík curves passing through the points (n^2-2k, n^2-2k) are more interesting. Using arguments like the one used for the proof of Lemma 3, it can be shown that (H_5) holds for the function c_0 associated to such an eigenvalue. Hence, if a Fučík curve passes through the point (n^2-2k, n^2-2k), its slope at that point must be equal to -1. We have represented in Figure 3, for $k = -1.4$, the Fučík curves passing through the points $(3.8, 3.8)$ and $(4, 4)$ (obviously, in that figure, they are not restricted to the set $I \times I$ of Corollary 2).

The situation is not so clear when L is not self-adjoint. Assuming that (H_1) holds, we can make however, the following observation.

Proposition 1. *Let (H_1) hold. Assume that $QA(\ker(L-\lambda^* A)) = (\text{Im}(L-\lambda^* A))^\perp$, and that*

$$c_0(x,\alpha,\beta) = 0 \Longrightarrow c_0(-x,\alpha,\beta) = 0. \tag{44}$$

If the sequence $(\alpha_n, \beta_n) \in \Sigma(L,A)$ is such that $\alpha_n \to \lambda^*, \beta_n \to \lambda^*$, then $(\beta_n - \lambda^*)/(\alpha_n - \lambda^*) \to -1$.

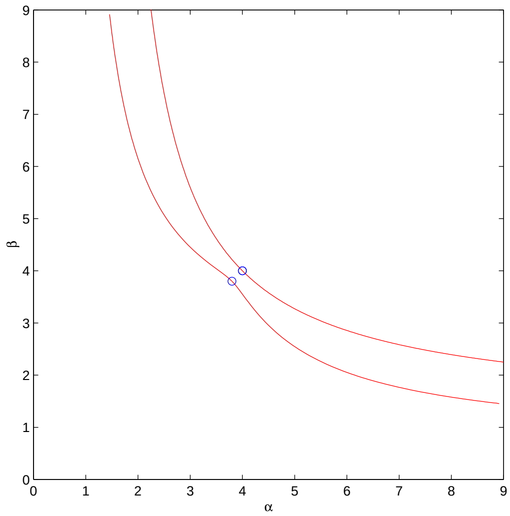

Figure 3: Fučík curves for the problem (42), (43).

In other words, under (44), if a Fučík curve passes through the point (λ^*, λ^*), its slope at that point must be equal to -1. Notice that the above proposition uses much weaker assumptions that Corollary 2.

Proof. Using arguments as in the last part of Theorem 1, we claim that there must exist $x^* \in \ker(L - \lambda^* A), x^* \neq 0$ and a number η^* such that

$$QA(|x^*|) + \eta^* QA(x^*) = 0.$$

By (44), we can replace x^* by $-x^*$ in the above equation, which implies $\eta^* QA(x^*) = 0$. Since QA is a bijection between $\ker(L - \lambda^* A)$ and $(\mathrm{Im}(L - \lambda^* A))^\perp$, $\eta^* = 0$ and the conclusion follows. □

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