

# PERIODIC SOLUTIONS OF NEUTRAL DUFFING EQUATIONS

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**Abstract.** We consider the following neutral delay Duffing equation

$$ax''(t) + bx'(t) + cx(t) + g(x(t - \tau_1), x'(t - \tau_2), x''(t - \tau_3)) = p(t) = p(t + 2\pi),$$

where  $a$ ,  $b$  and  $c$  are constants,  $\tau_i$ ,  $i = 1, 2, 3$ , are nonnegative constants,  $g : R \times R \times R \rightarrow R$  is continuous, and  $p(t)$  is a continuous  $2\pi$ -periodic function. In this paper, combining the Brouwer degree theory with a continuation theorem based on Mawhin's coincidence degree, we obtain a sufficient condition for the existence of  $2\pi$ -periodic solution of above equation.

**Key words:** Periodic solution, Duffing equation, Brouwer degree, coincidence degree.

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## 1. Introduction

On the existence problem of periodic solutions for the Duffing equations

$$x''(t) + g(x) = p(t) = p(t + 2\pi), \tag{1.1}$$

so far there has been a wide literature since the interest in studying Eq.(1.1) comes from different sources. Under the conditions which exclude the resonance cases,

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many results have been obtained <sup>[1,2,3,4]</sup>. At resonance, many authors have paid much attention to the problem in recent years. [5] and [6] resolved the existence problem of  $2\pi$ -periodic solutions of Eq.(1.1) under some different conditions, respectively.

On the other hand, a few papers have appeared<sup>[7,8,9,10,11,12]</sup> which dealt with the existence problem of periodic solutions to the delay Duffing equations such as

$$x''(t) + g(x(t - \tau)) = p(t) = p(t + 2\pi). \quad (1.2)$$

Under some conditions which exclude the resonance cases, some results have been obtained<sup>[13,14,15]</sup>.

Next, [17] discussed the Duffing equations of the form

$$x''(t) + m^2x(t) + g(x(t - \tau)) = p(t) = p(t + 2\pi), \quad (1.3)$$

where  $m$  is a positive integer, and proved the existence of  $2\pi$ -periodic solutions of Eq.(1.3) under some conditions.

Jack Hale [21] and [22] put forward the Euler's equations which are of the form

$$x''(t) = f(t, x(t), x(t - r), x'(t), x'(t - r), x''(t - r)),$$

where  $r$  is a positive constant.

Motivated by above papers, in the present paper, we consider the neutral Duffing equations of the form

$$ax''(t) + bx'(t) + cx(t) + g(x(t - \tau_1), x'(t - \tau_2), x''(t - \tau_3)) = p(t) = p(t + 2\pi), \quad (1.4)$$

where  $a, b, c$  are constants,  $\tau_1, \tau_2, \tau_3$  are nonnegative constants,  $g : R \times R \times R \rightarrow R$  is continuous, and  $p(t)$  is a continuous  $2\pi$ -periodic function.

To the best of our knowledge, in this direction, few papers can be found in the literature. In this paper, combining the Brouwer degree theory with a continuation theorem based on Mawhin's coincidence degree<sup>[16]</sup>, we obtain a sufficient condition for the existence of  $2\pi$ -periodic solution of Eq.(1.4).

## 2. Existence of a Periodic Solution

In order to obtain the existence of a periodic solution of Eq. (1.4), we first make the following preparations.

Let  $X$  and  $Z$  be two Banach spaces. Consider an operator equation

$$Lx = \lambda Nx,$$

where  $L: \text{Dom } L \cap X \rightarrow Z$  is a linear operator and  $\lambda \in [0, 1]$  a parameter. Let  $P$  and  $Q$  denote two projectors such that

$$P : \text{Dom } L \cap X \rightarrow \text{Ker } L \quad \text{and} \quad Q : Z \rightarrow Z/\text{Im } L.$$

In the sequel, we will use the following result of Mawhin<sup>[16]</sup>.

**LEMMA 2.1.** *Let  $X$  and  $Z$  be two Banach spaces and  $L$  a Fredholm mapping of index 0. Assume that  $N : \bar{\Omega} \rightarrow Z$  is  $L$ -compact on  $\bar{\Omega}$  with  $\Omega$  open bounded in  $X$ . Furthermore suppose*

(a). *For each  $\lambda \in (0, 1)$ ,  $x \in \partial\Omega \cap \text{Dom } L$*

$$Lx \neq \lambda Nx.$$

(b). *For each  $x \in \partial\Omega \cap \text{Ker } L$ ,*

$$QNx \neq 0$$

and

$$\deg\{QN, \Omega \cap \text{Ker} L, 0\} \neq 0.$$

Then  $Lx = Nx$  has at least one solution in  $\bar{\Omega}$ .

Recall that a linear mapping  $L: \text{Dom } L \subset X \rightarrow Z$  with  $\text{Ker } L = L^{-1}(0)$  and  $\text{Im } L = L(\text{Dom} L)$ , will be called a Fredholm mapping if the following two conditions hold:

- (i).  $\text{Ker } L$  has a finite dimension;
- (ii).  $\text{Im } L$  is closed and has a finite codimension.

Recalled also that the codimension of  $\text{Im } L$  is the dimension of  $Z/\text{Im } L$ , i.e., the dimension of the cokernel  $\text{coker } L$  of  $L$ .

When  $L$  is a Fredholm mapping, its (Fredholm) index is the integer

$$\text{Ind } L = \dim \text{Ker } L - \text{codim } \text{Im } L.$$

We shall say that a mapping  $N$  is  $L$ -compact on  $\Omega$  if the mapping  $QN : \bar{\Omega} \rightarrow Z$  is continuous,  $QN(\bar{\Omega})$  is bounded, and  $K_P(I - Q)N : \bar{\Omega} \rightarrow X$  is compact, i.e., it is continuous and  $K_P(I - Q)N(\bar{\Omega})$  is relatively compact, where  $K_P: \text{Im } L \rightarrow \text{Dom } L \cap \text{Ker } P$  is a inverse of the restriction  $L_P$  of  $L$  to  $\text{Dom } L \cap \text{Ker } P$ , so that  $LK_P = I$  and  $K_PL = I - P$ .

**THEOREM 2.1.** *Assume that there exist a positive constant  $M$  and three non-negative constants  $\beta_1, \beta_2, \beta_3$  such that*

$$|g(x_1, x_2, x_3)| \leq M + \beta_1|x_1| + \beta_2|x_2| + \beta_3|x_3| \text{ for } \forall(x_1, x_2, x_3) \in R^3 \quad (2.1)$$

and

$$|abc| - |bc|\beta_3 - |ac|\beta_2 - (2|ab| + 2\pi|ac|)\beta_1 > \beta_3 \sqrt{(|ac| - \beta_3|c| - \beta_1|a|)|c|(|c| - \beta_1)}. \quad (2.2)$$

Then Eq.(1.4) has at least one  $2\pi$ -periodic solution.

*Proof.* In order to use Lemma 2.1 for Eq.(1.4), we take  $X = \{x(t) \in C^2(\mathbb{R}, \mathbb{R}) : x(t + 2\pi) = x(t)\}$  and  $Z = \{z(t) \in C(\mathbb{R}, \mathbb{R}) : z(t + 2\pi) = z(t)\}$ , and denote  $\|x\|_0 = \max_{t \in [0, 2\pi]} |x(t)|$  and  $\|x\|_2 = \max\{\|x\|_0, \|x'\|_0, \|x''\|_0\}$ . Then  $X$  and  $Z$  are Banach spaces when they are endowed with norms  $\|\cdot\|_2$  and  $\|\cdot\|_0$ , respectively.

Set

$$Lx = ax''(t), \quad Nx = -bx'(t) - cx(t) - g(x(t - \tau_1), x'(t - \tau_2), x''(t - \tau_3)) + p(t),$$

$$Px = \frac{1}{2\pi} \int_0^{2\pi} x(t) dt, \quad x \in X, \quad Qz = \frac{1}{2\pi} \int_0^{2\pi} z(t) dt, \quad z \in Z.$$

Since  $\text{Ker } L = \mathbb{R}$  and  $\text{Im } L = \{x \in Z : \int_0^{2\pi} x(t) dt = 0\}$ ,  $\text{Im } L$  is closed and  $\dim \text{Ker } L = \dim Z / \text{Im } L = 1$ . Therefore,  $L$  is a Fredholm mapping of index 0.

Corresponding to the operator equation

$$Lx = \lambda Nx, \quad \lambda \in (0, 1),$$

we have

$$ax''(t) + \lambda bx'(t) + \lambda cx(t) + \lambda g(x(t - \tau_1), x'(t - \tau_2), x''(t - \tau_3)) = \lambda p(t). \quad (2.3)$$

Let  $x(t) \in X$  is a solution of Eq.(2.3) for a certain  $\lambda \in (0, 1)$ . Integrating (2.3) from 0 to  $2\pi$ , we have

$$\int_0^{2\pi} cx(t) dt = \int_0^{2\pi} [p(t) - g(x(t - \tau_1), x'(t - \tau_2), x''(t - \tau_3))] dt,$$

from which, it implies that there exists a  $t^* \in (0, 2\pi)$  such that

$$2\pi cx(t^*) = \int_0^{2\pi} [p(t) - g(x(t - \tau_1), x'(t - \tau_2), x''(t - \tau_3))] dt.$$

Let  $m = \max_{t \in [0, 2\pi]} |p(t)|$ . Then

$$\begin{aligned} 2\pi |cx(t^*)| &\leq 2\pi(m + M) + \beta_1 \int_0^{2\pi} |x(t - \tau_1)| dt \\ &\quad + \beta_2 \int_0^{2\pi} |x'(t - \tau_2)| dt + \beta_3 \int_0^{2\pi} |x''(t - \tau_3)| dt \\ &= 2\pi(m + M) + \beta_1 \int_0^{2\pi} |x(t)| dt + \beta_2 \int_0^{2\pi} |x'(t)| dt + \beta_3 \int_0^{2\pi} |x''(t)| dt. \end{aligned}$$

Since for  $\forall t \in [0, 2\pi]$ ,

$$\begin{aligned} x(t) &= x(t^*) + \int_{t^*}^t x'(s) ds, \\ |x(t)| &\leq |x(t^*)| + \int_0^{2\pi} |x'(s)| ds \\ &\leq \frac{1}{\sqrt{2\pi}|c|} \left[ \sqrt{2\pi}(m + M) + \beta_1 \left( \int_0^{2\pi} |x(t)|^2 dt \right)^{\frac{1}{2}} \right. \\ &\quad \left. + (2\pi|c| + \beta_2) \left( \int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}} + \beta_3 \left( \int_0^{2\pi} |x''(t)|^2 dt \right)^{\frac{1}{2}} \right]. \end{aligned}$$

Thus

$$\begin{aligned} |c| \left( \int_0^{2\pi} |x(t)|^2 dt \right)^{\frac{1}{2}} &\leq \sqrt{2\pi}|c| \max_{t \in [0, 2\pi]} |x(t)| \\ &\leq \sqrt{2\pi}(m + M) + \beta_1 \left( \int_0^{2\pi} |x(t)|^2 dt \right)^{\frac{1}{2}} \\ &\quad + \beta_3 \left( \int_0^{2\pi} |x''(t)|^2 dt \right)^{\frac{1}{2}} \\ &\quad + (2\pi|c| + \beta_2) \left( \int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}} \end{aligned}$$

from which, it follows that

$$\begin{aligned} (|c| - \beta_1) \left( \int_0^{2\pi} |x(t)|^2 dt \right)^{\frac{1}{2}} &\leq \sqrt{2\pi}(m + M) + (2\pi|c| + \beta_2) \left( \int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}} \\ &\quad + \beta_3 \left( \int_0^{2\pi} |x''(t)|^2 dt \right)^{\frac{1}{2}}. \end{aligned} \tag{2.4}$$

Multiplying (2.3) by  $x''(t)$  and integrating from 0 to  $2\pi$ , we get

$$a \int_0^{2\pi} |x''(t)|^2 dt - \lambda c \int_0^{2\pi} |x'(t)|^2 dt + \lambda \int_0^{2\pi} x''(t)[g(x(t - \tau_1), x'(t - \tau_2), x''(t - \tau_3)) - p(t)] dt = 0,$$

from which, it implies that

$$\begin{aligned} & |a| \int_0^{2\pi} |x''(t)|^2 dt \\ & \leq |c| \int_0^{2\pi} |x'(t)|^2 dt + \int_0^{2\pi} |x''(t)| \left[ m + M \right. \\ & \quad \left. + \beta_1 |x(t - \tau_1)| + \beta_2 |x'(t - \tau_2)| + \beta_3 |x''(t - \tau_3)| \right] dt \\ & \leq |c| \int_0^{2\pi} |x'(t)|^2 dt + \left( \int_0^{2\pi} |x''(t)|^2 dt \right)^{\frac{1}{2}} \left[ \sqrt{2\pi}(m + M) \right. \\ & \quad \left. + \beta_1 \left( \int_0^{2\pi} |x(t)|^2 dt \right)^{\frac{1}{2}} + \beta_2 \left( \int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}} + \beta_3 \left( \int_0^{2\pi} |x''(t)|^2 dt \right)^{\frac{1}{2}} \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} (|a| - \beta_3) \int_0^{2\pi} |x''(t)|^2 dt & \leq |c| \int_0^{2\pi} |x'(t)|^2 dt + \left( \int_0^{2\pi} |x''(t)|^2 dt \right)^{\frac{1}{2}} \left[ \sqrt{2\pi}(m + M) \right. \\ & \quad \left. + \beta_1 \left( \int_0^{2\pi} |x(t)|^2 dt \right)^{\frac{1}{2}} + \beta_2 \left( \int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}} \right]. \quad (2.5) \end{aligned}$$

From (2.4) and (2.5), we have

$$\begin{aligned} & (|c| - \beta_1)(|a| - \beta_3) \int_0^{2\pi} |x''(t)|^2 dt \\ & \leq |c|(|c| - \beta_1) \int_0^{2\pi} |x'(t)|^2 dt \\ & \quad + \left( \int_0^{2\pi} |x''(t)|^2 dt \right)^{\frac{1}{2}} \left[ \sqrt{2\pi}|c|(m + M) \right. \\ & \quad \left. + \beta_1 \beta_3 \left( \int_0^{2\pi} |x''(t)|^2 dt \right)^{\frac{1}{2}} + (2\pi\beta_1 + \beta_2)|c| \left( \int_0^{2\pi} |x'(t)|^2 dt \right) \right], \end{aligned}$$

from which, it follows that

$$\begin{aligned}
 & (|ac| - \beta_3|c| - \beta_1|a|) \int_0^{2\pi} |x''(t)|^2 dt \\
 & \leq |c|(|c| - \beta_1) \int_0^{2\pi} |x'(t)|^2 dt \\
 & \quad + \left( \int_0^{2\pi} |x''(t)|^2 dt \right)^{\frac{1}{2}} \left[ \sqrt{2\pi}|c|(m + M) + |c|(2\pi\beta_1 + \beta_2) \left( \int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}} \right].
 \end{aligned}$$

Thus

$$\begin{aligned}
 & 2(|ac| - \beta_3|c| - \beta_1|a|) \left( \int_0^{2\pi} |x''(t)|^2 dt \right)^{\frac{1}{2}} \\
 & \leq \sqrt{2\pi}|c|(m + M) + |c|(2\pi\beta_1 + \beta_2) \left( \int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}} \\
 & \quad + \left\{ \left[ \sqrt{2\pi}|c|(m + M) + |c|(2\pi\beta_1 + \beta_2) \left( \int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}} \right]^2 \right. \\
 & \quad \left. + 4(|ac| - \beta_3|c| - \beta_1|a|)|c|(|c| - \beta_1) \int_0^{2\pi} |x'(t)|^2 dt \right\}^{\frac{1}{2}}. \tag{2.6}
 \end{aligned}$$

Using inequality  $(a + b)^{\frac{1}{2}} \leq a^{\frac{1}{2}} + b^{\frac{1}{2}}$ , for  $a \geq 0$  and  $b \geq 0$ , we have

$$\begin{aligned}
 & \left\{ \left[ \sqrt{2\pi}|c|(m + M) + |c|(2\pi\beta_1 + \beta_2) \left( \int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}} \right]^2 \right. \\
 & \quad \left. + 4(|ac| - \beta_3|c| - \beta_1|a|)|c|(|c| - \beta_1) \int_0^{2\pi} |x'(t)|^2 dt \right\}^{\frac{1}{2}} \\
 & \leq \sqrt{2\pi}|c|(m + M) + |c|(2\pi\beta_1 + \beta_2) \left( \int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}} \\
 & \quad + 2\sqrt{(|ac| - \beta_3|c| - \beta_1|a|)|c|(|c| - \beta_1)} \left( \int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}}. \tag{2.7}
 \end{aligned}$$



By (2.6) and (2.7), we have

$$\begin{aligned}
 & (|ac| - \beta_3|c| - \beta_1|a|) \left( \int_0^{2\pi} |x''(t)|^2 dt \right)^{\frac{1}{2}} \\
 & \leq \sqrt{2\pi}|c|(m+M) + \left[ |c|(2\pi\beta_1 + \beta_2) \right. \\
 & \quad \left. + \sqrt{(|ac| - \beta_3|c| - \beta_1|a|)|c|(|c| - \beta_1)} \right] \left( \int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}}.
 \end{aligned} \tag{2.8}$$

Multiplying (2.3) by  $x'(t)$  and integrating from 0 to  $2\pi$ , we obtain

$$b \int_0^{2\pi} |x'(t)|^2 dt + \int_0^{2\pi} x'(t)[g(x(t-\tau_1), x'(t-\tau_2), x''(t-\tau_3)) - p(t)] dt = 0,$$

from which, it implies that

$$\begin{aligned}
 |b| \int_0^{2\pi} |x'(t)|^2 dt & \leq \left( \int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}} \left[ \sqrt{2\pi}(m+M) + \beta_1 \left( \int_0^{2\pi} |x(t-\tau_1)|^2 dt \right)^{\frac{1}{2}} \right. \\
 & \quad \left. + \beta_2 \left( \int_0^{2\pi} |x'(t-\tau_2)|^2 dt \right)^{\frac{1}{2}} + \beta_3 \left( \int_0^{2\pi} |x''(t-\tau_3)|^2 dt \right)^{\frac{1}{2}} \right] \\
 & = \left( \int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}} \left[ \sqrt{2\pi}(m+M) + \beta_1 \left( \int_0^{2\pi} |x(t)|^2 dt \right)^{\frac{1}{2}} \right. \\
 & \quad \left. + \beta_2 \left( \int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}} + \beta_3 \left( \int_0^{2\pi} |x''(t)|^2 dt \right)^{\frac{1}{2}} \right].
 \end{aligned}$$

Thus

$$\begin{aligned}
 (|b| - \beta_2) \left( \int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}} & \leq \sqrt{2\pi}(m+M) + \beta_1 \left( \int_0^{2\pi} |x(t)|^2 dt \right)^{\frac{1}{2}} \\
 & \quad + \beta_3 \left( \int_0^{2\pi} |x''(t)|^2 dt \right)^{\frac{1}{2}},
 \end{aligned} \tag{2.9}$$

from which, together with (2.4), it implies that

$$\begin{aligned}
 & (|c| - \beta_1)(|b| - \beta_2) \left( \int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}} \\
 & \leq \sqrt{2\pi}|c|(m+M) + \beta_3|c| \left( \int_0^{2\pi} |x''(t)|^2 dt \right)^{\frac{1}{2}} \\
 & \quad + (2\pi\beta_1|c| + \beta_1\beta_2) \left( \int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}}.
 \end{aligned} \tag{2.10}$$

In view of (2.8) and (2.10), we can obtain

$$\begin{aligned}
 & (|c| - \beta_1)(|b| - \beta_2)(|ac| - \beta_3|c| - \beta_1|a|) \left( \int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}} \\
 & \leq \sqrt{2\pi}|c|(m + M)(|ac| - \beta_3|c| - \beta_1|a|) + \sqrt{2\pi}\beta_3c^2(m + M) \\
 & \quad + \left\{ (2\pi|c|\beta_1 + \beta_1\beta_2)(|ac| - \beta_3|c| - \beta_1|a|) + \beta_3|c| \left[ |c|(2\pi\beta_1 + \beta_2) \right. \right. \\
 & \quad \left. \left. + \sqrt{(|ac| - \beta_3|c| - \beta_1|a|)|c|(|c| - \beta_1)} \right] \right\} \left( \int_0^{2\pi} |x'(t)|^2 dt \right)^{\frac{1}{2}},
 \end{aligned}$$

from which, together with (2.2), it implies that there exists a positive constant  $R_1$  such that

$$\int_0^{2\pi} |x'(t)|^2 dt \leq R_1. \tag{2.11}$$

By (2.6) and (2.11), there exists a positive constant  $R_2$  such that

$$\int_0^{2\pi} |x''(t)|^2 dt \leq R_2. \tag{2.12}$$

From (2.4), (2.11) and (2.12), there exists a positive constant  $R_3$  such that

$$\int_0^{2\pi} |x(t)|^2 dt \leq R_3. \tag{2.13}$$

Therefore, there exist three positive constants  $R_1^*$ ,  $R_2^*$  and  $R_3^*$  such that  $\forall t \in [0, 2\pi]$ ,

$$|x(t)| \leq R_1^* \quad |x'(t)| \leq R_2^*, \quad |x''(t)| \leq R_3^*.$$

Let  $A = \max\{R_1^*, R_2^*, R_3^*, (m + M)/(|c| - \beta_1)\}$  and take  $\Omega = \{x(t) \in X : |x|_2 < A\}$ . We now will show that  $N$  is  $L$ -compact on  $\bar{\Omega}$ . For any  $x \in \bar{\Omega}$ ,

$$\begin{aligned}
 |QNx|_0 & \leq \frac{1}{2\pi} \int_0^{2\pi} [|b|R_2^* + |c|R_1^* + m + M + \beta_1R_1^* + \beta_2R_2^* + \beta_3R_3^*] dt \\
 & = M_1,
 \end{aligned}$$

where  $M_1 = |b|R_2^* + |c|R_1^* + m + M + \beta_1 R_1^* + \beta_2 R_2^* + \beta_3 R_3^*$ . Hence,  $QN(\bar{\Omega})$  is a bounded set in  $R$ . Obviously,  $QNx : \bar{\Omega} \rightarrow Z$  is continuous. For  $\forall z \in \text{Im } L \cap Z$ ,

$$(K_P z)(t) = \int_0^t ds \int_0^s z(u) du - \frac{1}{2\pi} \int_0^{2\pi} dt \int_0^t ds \int_0^s z(u) du$$

is continuous with respect to  $z$ , and

$$|K_P z|_0 \leq \frac{8}{3} \pi^2 \max_{t \in [0, 2\pi]} |z(t)|,$$

$$\begin{aligned} |K_P(I - Q)Nx|_0 &\leq \frac{8}{3} \pi^2 |Nx|_0 + \frac{8}{3} \pi^2 |QNx|_0 \\ &\leq \frac{16}{3} \pi^2 |Nx|_0 \\ &\leq \frac{16}{3} \pi^2 M_1. \end{aligned}$$

For  $\forall x \in \Omega$ , we have

$$\begin{aligned} \left| \frac{d}{dt} (K_P(I - Q)Nx) \right|_0 &\leq \int_0^t |[(I - Q)Nx](t)|_0 dt \\ &\leq 2\pi |[(I - Q)Nx](t)|_0 \\ &\leq 4\pi |Nx|_0 \leq 4\pi M_1. \end{aligned}$$

Thus, the set  $\{K_P(I - Q)Nx | x \in \bar{\Omega}\}$  is equicontinuous and uniformly bounded.

Consequently,  $N$  is  $L$ -compact. This satisfies condition (a) in Lemma 2.1.

When  $x \in \partial\Omega \cap \text{Ker } L = \partial\Omega \cap R$ ,  $x$  is a constant with  $|x| = A$ . Then

$$\begin{aligned} QNx &= \frac{1}{2\pi} \int_0^{2\pi} [-bx'(t) - cx(t) - g(x(t - \tau_1), x'(t - \tau_2), x''(t - \tau_3)) + p(t)] dt \\ &= -cx - g(x, 0, 0) + \frac{1}{2\pi} \int_0^{2\pi} p(t) dt. \end{aligned}$$

Thus

$$\begin{aligned} |QNx|_0 &\geq |c| \left( |x| - \frac{|g(x, 0, 0)| + m}{|c|} \right) \\ &\geq |c| \left( A - \frac{m + M + \beta_1 A}{|c|} \right) > 0. \end{aligned}$$

Therefore,  $QNx \neq 0, x \in \partial\Omega \cap R$ .

Set for  $0 \leq \mu \leq 1$

$$\phi(x, \mu) = \mu x(t) + (1 - \mu) \left[ x(t) + g(x(t - \tau_1), x'(t - \tau_2), x''(t - \tau_3)) - \frac{1}{2\pi} \int_0^{2\pi} p(t) dt \right].$$

When  $x \in \partial\Omega \cap \text{Ker } L$  and  $\mu \in [0, 1]$ ,  $x$  is a constant with  $|x| = A$ . Without loss of generality, we suppose  $x = A$ . Now we consider two possible cases: (1)  $x = A, c > 0$ ; (2)  $x = A, c < 0$ .

(1). When  $x = A$  and  $c > 0$ ,

$$\begin{aligned} \phi(x, \mu) &= cA + (1 - \mu) \left[ g(A, 0, 0) - \frac{1}{2\pi} \int_0^{2\pi} p(t) dt \right] \\ &\geq c \left[ A - \frac{1 - \mu}{c} \left( |g(A, 0, 0)| + \frac{1}{2\pi} \int_0^{2\pi} |p(t)| dt \right) \right] \\ &\geq c \left( A - \frac{m + M + \beta_1 A}{c} \right) > 0; \end{aligned}$$

(2). When  $x = A$  and  $c < 0$ ,

$$\phi(x, \mu) \leq c \left( A - \frac{m + M + \beta_1 A}{|c|} \right) < 0.$$

Thus when  $x = A, \phi(x, \mu) \neq 0$ . Therefore,

$$\begin{aligned} \deg(QN, \Omega \cap \text{Ker } L, 0) &= \deg \left\{ -cx(t) - g(x(t - \tau_1), x'(t - \tau_2), x''(t - \tau_3)) \right. \\ &\quad \left. + \frac{1}{2\pi} \int_0^{2\pi} p(t) dt, \Omega \cap \text{Ker } L, 0 \right\} \\ &= \deg(-cx, \Omega \cap \text{Ker } L, 0) \neq 0. \end{aligned}$$

By now we know that  $\Omega$  verifies all the requirements in Lemma 2.1. This completes the proof of Theorem 2.1.

**Example** The second order neutral delay differential equation

$$10x''(t) + 100x'(t) + 5x(t) + \frac{1 + \frac{1}{2}x(t-1) + \frac{1}{2}x'(t-2) + \frac{1}{100}x''(t-3)}{1 + x^2(t-1)} = \sin t, \quad (2.14)$$

satisfies all conditions in Theorem 2.1. Therefore, Eq.(2.14) has at least one  $2\pi$ -periodic solution.

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