Self–similar solutions to a convection–diffusion processes

M. Guedda
LAMFA, CNRS UPRES-A 6119, Université de Picardie Jules Verne,
Faculté de Mathématiques et d'Informatique, 33, rue Saint-Leu 80039 Amiens, France

Zaïna, in memorium

Abstract

Geometric properties of self–similar solutions to the equation $u_t = u_{xx} + \gamma(u^q)_x,\ x > 0,\ t > 0$ are studied, q is positive and $\gamma \in \mathbb{R} \setminus \{0\}$. Two critical values of q (namely 1 and 2) appear the corresponding shapes are of very different nature.

AMS(MOS) Subject Classification: 35K55, 35K65.

1. Introduction

IN THIS paper we shall derive properties of solutions to the equation

$$\begin{cases} u_t = u_{xx} + \gamma(u^q)_x, & \text{for } (x,t) \in (0,+\infty) \times (0,+\infty), \\ u_x(0,t) = 0, & \text{for } t > 0, \end{cases} \tag{1.1}$$

having the form

$$u(x,t) = t^\alpha g(xt^{-1/2}) =: t^\alpha g(\xi), \tag{1.2}$$

where $q > 0,\ \alpha = -\frac{1}{2(q-1)},\ \gamma \in \mathbb{R} \setminus \{0\}$, and $u > 0$ in the half space for appropriate nonnegative initial data.

If we substitute (1.2) into (1.1) we obtain for $q \neq 1$, the ODE

$$g'' + \gamma(g^q)' = \alpha g - \frac{1}{2}\xi g', \quad \xi > 0, \tag{1.3}$$

subject to the condition

$$g'(0) = 0. \tag{1.4}$$

Setting $\gamma = \pm 1$ we are led to the problem

$$\begin{cases} g'' + \varepsilon(g^q)' + \frac{1}{2}\xi g' - \alpha g = 0, & \xi > 0, \\ g'(0) = 0, \quad g(0) = \lambda, \end{cases} \tag{1.5}$$

where $\varepsilon = \pm 1, \lambda > 0$ and $q \neq 1$ is a positive number. This problem has a unique local solution for every $\lambda > 0$. We shall be interested in possible extension of solutions and their properties. A more general equation with $\gamma \neq 0$ can be transformed to (1.5) by introducing a new function $|\gamma|^{\frac{1}{q-1}} g$ which solves (1.5) with $\lambda|\gamma|^{\frac{1}{q-1}}$ instead of λ. When $\varepsilon = 1$ and $q > 1$ problem (1.5) was investigated in detail by Peletier and Serafini [12]. It is shown that there exists λ_c such that problem (1.5) has a unique global solution $g > 0$ such that $\xi^{-2\alpha} g(\xi)$ goes to 0 if and only if $1 < q < 2$ and $g(0) = \lambda_c$ and its asymptotic behavior at infinity is given by

$$g(\xi) = L\xi^{-2\alpha-1} e^{-\xi^2/4} \Big\{ 1 + 2(2\alpha + 1)(\alpha - 1)\xi^{-2} + o(\xi^{-2}) \Big\}$$

as $\xi \to +\infty$, for some positive constant L. The paper by Biler and Karch [3] is devoted to study the large-time behavior of solutions to (1.1) with $(u|u|^{q-1})_x, q > 1$ instead of $\gamma(u^q)_x$, where initial data satisfying $\lim_{x\to\infty} x^\beta u(x, 0) = A$ for some $A \in \mathbb{R}$ and $0 < \beta < 1$. In this paper we shall show that if $\varepsilon = -1$ and $1 < q < 2$ the solution of (1.5) changes the sign for every $\lambda > 0$. Also we are interested on values on q and $\lambda > 0$ which guarantee that problem (1.5) has a global positive solution with given behavior at infinity. The basic method used here is due to [5]. We analyze problem (1.5) in the phase plane. Somes results can be found in [3]
Equation (1.3) does not belong to the class of well–studied second order nonlinear ODE's. If we write it in the standard (from point of view of nonlinear oscillation theory) form

$$g'' + \left(q\varepsilon g^{q-1} + \frac{1}{2}\xi \right) g' - \alpha g = 0, \tag{1.6}$$

we can see that the "friction coefficient" which depends nonlinearly on g and on position, can change sign if $\varepsilon = -1$. The sign of α depends on q: if $q < 1$ then $\alpha > 0$ and $\alpha < 0$ for $1 < q$.

Remark 1.1. The function $w(x, t) = x^a h(tx^{-b}) =: x^a h(\eta)$ satisfies (1.1) if and only if $b = 2$ and $a = (q - 2)/(q - 1)$. The corresponding ODE is

$$h'' = (a - 3/2)\frac{h'}{\eta} + \frac{\varepsilon}{2}(h^q)' + \frac{h'}{\eta^2} + \frac{h}{\eta}(1 + qa\gamma\varepsilon h^{q-1}), \quad \eta > 0.$$

We shall not deal here with it.

The plan of the paper is the following:
Section 2 : Preliminary results.
Section 3 : Large ξ behaviour of all global solutions for $q > 2$ and $\varepsilon = -1$.
Section 4 : The case $0 < q < 1$.

2. Preliminaries

Rather than studying (1.5), we will deal here with the slightly more general ODE

$$g'' + q\varepsilon |g|^{q-1} g' = \alpha g - \frac{1}{2}\xi g', \quad \xi > 0, \tag{2.1}$$

$$g(0) = \lambda, \quad g'(0) = 0, \tag{2.2}$$

in which the nonnegative number q is not equal to 1, $\alpha = -\frac{1}{2(q-1)}, \lambda > 0$ and $\varepsilon \in \{-1, 1\}$.
Problem (1.5) is a special case of (2.1)–(2.2) in which $g > 0$. As it was mentioned before, for any $\lambda > 0$, problem (2.1)–(2.2) has a unique maximal solution $g(., \lambda) \in C^2([0, \xi_{max}))$. Furthermore $g(\xi, \lambda) > 0$ for small $\xi > 0$. An important objective is to find values of λ and q which insure that $g(., \lambda)$ is global, nonnegative and to give the asymptotic behavior as ξ tends to infinity. In this section we shall derive some properties of g which will be useful in the proof of the main results.

LEMMA 2.1. *Assume that $\alpha < 0$. Let g be a solution to (2.1)–(2.2) such that $g > 0$ on $[0, \xi_0)$. Then $g'(\xi) < 0$, for all $0 < \xi < \xi_0$.*

PROOF. As $g''(0) = \alpha\lambda < 0$ and $g'(0) = 0$, the function g is decreasing for small ξ. Suppose that there exists $\xi_1 \in (0, \xi_0)$ such that $g'(\xi) < 0$ on $(0, \xi_1)$ and $g'(\xi_1) = 0$. Using (2.1) one sees $g''(\xi_1) < 0$. Therefore we get a contradiction. □

LEMMA 2.2. *Assume that $\varepsilon = -1$ and $\alpha \le -\frac{1}{2}$. Then $g(., \lambda)$ changes the sign for every $\lambda > 0$.*

PROOF. Suppose in the contrary that $g(., \lambda) > 0$. Then $g'(., \lambda) < 0$ (and then $g(., \lambda)$ is global). On the other hand Equation (2.1) can be written as

$$g'' + \frac{1}{2}(\xi g)' = (\alpha + \frac{1}{2})g + (g^q)'.$$

Thus we have

$$g'(\xi) + \frac{1}{2}\xi g(\xi) = (\alpha + \frac{1}{2}) \int_0^\xi g(\eta) d\eta + g^q(\xi) - \lambda^q.$$

This implies that $g(\xi) \le e^{-\frac{\xi^2}{4}}$, for all $\xi \ge 0$. Then passing to the limit, $\xi \to +\infty$, we infer

$$0 = (\alpha + \frac{1}{2}) \int_0^\infty g(\eta) d\eta - \lambda^q.$$

This is impossible.

□

REMARK 2.1. The situation is different if $\varepsilon = 1$. Peletier and Serafini [12] showed that if $\alpha < -\frac{1}{2}$ the solution changes the sign for λ sufficiently small. And if $0 > \alpha \ge -\frac{1}{2}$, the solution g is nonnegative on $[0, +\infty[$.

Finally a standard analysis gives the following

LEMMA 2.3. *Assume that $\alpha > 0$. Let g be a solution to (2.1)–(2.2) defined on $[0, \xi_0[$, where $\varepsilon = \pm 1$. Then $g'(\xi) > 0$ for all $0 < \xi < \xi_0$.*

In fact we shall show in Section 4 that the solution g cannot blow-up for finite ξ. In the next sections we shall give the asymptotic behavior of all possible positive solutions to (2.1)–(2.2) in the following cases : $\varepsilon = -1$ and $q > 2$ and $\varepsilon = \pm 1$ and $0 < q < 1$.

3. Global behavior for $q > 2$ and $\varepsilon = -1$

The first simple consequence of the fact that $q > 2$ is that $0 > \alpha > -\frac{1}{2}$, and then if $g(\xi) > 0$ on $(0, +\infty)$ we have $g'(\xi) < 0$ for all $\xi > 0$. It is also clear that

$$g(\xi) \leq \lambda, \quad \forall\, \xi \geq 0. \tag{3.1}$$

Actually $g(\xi) \leq \lambda$, for small ξ, in order to be bigger that λ, the solution g has to return at some $\xi_1 > 0$, and at this point $g'(\xi_1) = 0$ and $g''(\xi_1) \geq 0$ in contradiction with (2.1) for $g(\xi_1) \geq 0$. If $g(\xi_1) < 0$ then g cannot cross the line $\xi = 0$ again : suppose "yes" at point ξ_2 : $g(\xi_2) = 0$. Here we have that $g < 0$ on (ξ_1, ξ_2) and by a uniqueness argument we may conclude that $g'(\xi_1) < 0$ and $g'(\xi_2) > 0$. Now observe that (2.1) can be written as

$$g'' + (|g|^q)' + \frac{1}{2}(\xi g)' = (\alpha + \frac{1}{2})g, \quad \xi \in (\xi_1, \xi_2).$$

Integrate the last equation over (ξ_1, ξ_2) :

$$g'(\xi_2) = (\alpha + \frac{1}{2}) \int_{\xi_1}^{\xi_2} g d\xi + g'(\xi_1) < 0,$$

while the left hand side is positive. We get a contradiction. So $g(\xi)$ is bounded from above by λ. And we can conclude that

LEMMA 3.1. *For any $\lambda > 0$, and $\varepsilon = \pm 1$, the solution $g(., \lambda)$ to (2.1)–(2.2) can have at most one zero on* $(0, \infty)$.

Peletier and Serafini proved in fact that for $\varepsilon = 1$ any solution is nonnegative.

The following lemma shows that all global positive solutions decay to 0.

LEMMA 3.2. *Let g be the solution to* (2.1)–(2.2) *where $q > 2$. Assume that $g(\xi) > 0$ for any $\xi > 0$. Then*

$$\lim_{\xi \to +\infty} g(\xi) = 0, \quad \lim_{\xi \to +\infty} g'(\xi) = 0.$$

PROOF. Since $g' < 0$ and g is bounded below by 0 g has a finite limit at ∞; say g_0. First there exists (ξ_n) such that $g'(\xi_n)$ goes to 0 as ξ_n tends to ∞ with n.
Now as the energy $E = (g')^2 - \alpha g^2$ is monotone decreasing for a large ξ we deduce that g' tends to 0 as $\xi \to \infty$. Now suppose that $g_0 > 0$. Equation (2.1) gives

$$g'' + \frac{1}{2}\xi g' < \alpha g_0.$$

Multiply this by $e^{\frac{\xi^2}{4}}$ and integrate :

$$g'(\xi) < \alpha g_0 e^{-\frac{\xi^2}{4}} \int_0^{\xi} e^{\frac{\tau^2}{4}} d\tau. \tag{3.2}$$

Since

$$\lim_{\xi\to+\infty}\frac{\displaystyle\int_0^\xi e^{\frac{\tau 2}{4}}d\tau}{\frac{1}{\xi}e^{\frac{\xi^2}{4}}}=2,$$

thanks to l'Hôpital's rule, we infer

$$g'(\xi)<\alpha g_0\frac{1}{\xi},\ \text{for } \xi \text{ large},$$

which implies that g goes to $-\infty$ as $\xi\to+\infty$, this is impossible. □

In [3] it is shown that

$$g(\xi)\le\lambda\xi^{2\alpha}\left\{1-2\alpha\int_0^{+\infty}\tau^{-2\alpha-1}e^{-\frac{\tau^2}{4}}d\tau\right\}, \tag{3.3}$$

for all $\xi>0$. Therefore $g(\xi)$ goes to 0 as $\xi\to+\infty$ since $\alpha<0$.

Lemma 3.3. *Assume $\varepsilon=-1$. Then there exists $\lambda_1>0$ such that the solution $g(.,\lambda)$ to (2.1)–(2.2), where $q>2$ and $\lambda>\lambda_1$, has exactly one positive zero.*

Proof. Assume that for all $\lambda>0$ the solution, $g(.,\lambda)$, to (2.1)–(2.2) is positive on $(0,+\infty)$ and then $g'(.,\lambda)<0$.
Set $g=g(.,\lambda)$.
Integrating of (2.1) over $(0,\xi)$ yields

$$g'(\xi)+\frac{1}{2}\xi g(\xi)=(\alpha+\frac{1}{2})\int_0^\xi g(\tau)d\tau+g^q(\xi)-\lambda^q.$$

Then

$$g'(\xi)\le(\alpha+\frac{1}{2})\lambda\xi-\lambda^q+g^q(\xi).$$

From the last inequality and (3.3), we deduce that

$$g(\xi)\le\lambda+\frac{1}{2}(\alpha+\frac{1}{2})\lambda\xi^2-\lambda^q\xi+C\lambda^q,$$

for some positive constant C, which is independent of λ.
Setting $\xi=2C$ we infer

$$g(2C)\le\lambda+2(\alpha+\frac{1}{2})C^2\lambda-C\lambda^q.$$

This shows that $g(2C)<0$ if λ is large, a contradiction. □

Let us now investigate in more detail how (g,g') behaves in the phase plane as ξ increases. We proceed as in [5]. Set $h=g'$, equation (2.1) is reduced to the following first order system

$$\left\{\begin{array}{lcl} g' & = & h,\\ \\ h' & = & \alpha g+q|g|^{q-1}h-\frac{1}{2}\xi h,\end{array}\right. \tag{3.4}$$

with the initial condition

$$g(0) = \lambda, \quad h(0) = 0. \tag{3.5}$$

This system has only one critical point $(0,0)$ and since $q > 1$, problem (3.4)–(3.5) has a unique local solution (g, h) for every $\lambda > 0$.
For each $\gamma > 0$ we define

$$P_\gamma = \Big\{(g,h); g > 0, h < 0, h \geq -\gamma g\Big\},$$

and we introduce

$$\xi(\lambda,\gamma) = 2(-\frac{\alpha}{\gamma} + \gamma) + 2q\lambda^{q-1}.$$

Arguing as in [5, 9] we obtain

Lemma 3.4. *For any fixed $\gamma > 0$, the set P_γ is positively invariant for $\xi_0 > \xi(\lambda,\gamma)$; that is if $\xi_0 > \xi(\lambda,\gamma)$ and $(g(\xi_0), h(\xi_0)) \in P_\gamma$, then $(g(\xi), h(\xi)) \in P_\gamma$ for all $\xi \geq \xi_0$.*

According to Lemmas 3.2 and 3.4 we have

Lemma 3.5. *Let g be the solution to (2.1). Assume that $g(\xi) > 0$ for all $\xi > 0$. Then*

$$\text{either} \quad \lim_{\xi\to+\infty} \frac{g'(\xi)}{g(\xi)} = 0 \quad \text{or} \quad \lim_{\xi\to+\infty} \frac{g'(\xi)}{g(\xi)} = -\infty. \tag{3.6}$$

The proof is similar to the proof of the corresponding results in [5, 9, 12].

Setting

$$L^\star(\lambda) = \lambda\left\{1 - 2\alpha\int_0^{+\infty} \tau^{-2\alpha-1} e^{-\frac{\tau^2}{4}} d\tau\right\}.$$

Proposition 3.1. *Let g be the solution to (2.1)–2.2) such that $g > 0$. Then the limit*

$$L(\lambda) = \lim_{\xi\to\infty} \xi^{-2\alpha} g(\xi),$$

exists in $[0, L^\star(\lambda)]$ and we have

$$\lim_{\xi\to\infty} \frac{g'(\xi)}{g(\xi)} = -\infty \Rightarrow L(\lambda) = 0,$$

$$\lim_{\xi\to\infty} \frac{g'(\xi)}{g(\xi)} = 0 \Rightarrow L(\lambda) > 0.$$

Proof. If $\lim_{\xi\to\infty} \frac{g'(\xi)}{g(\xi)} = -\infty$, then $g(\xi) = O(e^{-k\xi})$ as $\xi \to \infty$, so that $\xi^{-2\alpha}g(\xi)$ goes to 0 as $\xi \to \infty$. Now assume that

$$\lim_{\xi\to\infty} \frac{g'(\xi)}{g(\xi)} = 0.$$

Set

$$u(\xi) = \frac{g'(\xi)}{g(\xi)}.$$

Thus

$$u' + \frac{1}{2}\xi u = -\frac{1}{2}\frac{2-q}{q-1} + \varphi(\xi), \quad u(0) = 0, \tag{3.7}$$

where $\varphi(\xi) = qg^{q-1}u - u^2$.
Note that φ goes to 0 as $\xi \to +\infty$ and u can be defined by

$$u(\xi) = e^{-\frac{\xi^2}{4}} \int_0^{\xi} \{\alpha + \varphi(\tau)\} e^{\frac{\tau^2}{4}} d\tau, \tag{3.8}$$

hence

$$\xi u(\xi) = \frac{\displaystyle\int_0^{\xi} \{\alpha + \varphi(\tau)\} e^{\frac{\tau^2}{4}} \} d\tau}{\frac{1}{\xi} e^{\frac{\xi^2}{4}}}, \quad \forall\, \xi > 0. \tag{3.9}$$

Applying the l'Hôpital's rule to the right–hand side of (3.9), we infer

$$\lim_{\xi \to \infty} \xi u(\xi) = 2\alpha. \tag{3.10}$$

This shows in particular that for any $\tau > 0$ there exists $K_\tau > 0$ such that

$$g(\xi) \le K_\tau \xi^{(2\alpha+\tau)}, \text{ for all } \xi \ge 0. \tag{3.11}$$

Now given $1 \le k < 2 - \tau(q-1)$. Since

$$\xi^k \varphi(\xi) = qg^{q-1}\xi^k u - \xi^k u^2,$$

we get

$$\xi^k |\varphi(\xi)| \le qK_\tau^{q-1} \xi^{k-2+\tau(q-1)} + \xi^{k-2}(\xi u)^2.$$

According to the choice of k and to (3.10) we deduce $\lim_{\xi \to +\infty} \xi^k \varphi(\xi) = 0$.

On the other hand u satisfies:

$$\xi^k \{\xi u(\xi) - 2\alpha\} = \frac{\displaystyle\int_0^{\xi} (\alpha + \varphi(\tau))\, e^{\frac{\tau^2}{4}} - 2\alpha e^{\frac{\xi^2}{4}} \xi^{-1}}{e^{\frac{\tau^2}{4}} \xi^{-1-k}}.$$

Then, by l'Hôpital's rule, we get that

$$\lim_{\xi \to +\infty} \xi^k \{\xi u(\xi) - 2\alpha\} = 2 \lim_{\xi \to +\infty} \xi^k \varphi(\xi) = 0.$$

It follows from this that

$$\frac{g'}{g} = 2\alpha\frac{1}{\xi} + \frac{\epsilon(\xi)}{\xi^{k+1}},$$

for all $\xi > \xi_0$.
Hence

$$g(\xi) = L(\lambda)\xi^{2\alpha}\left\{1 + o(\frac{1}{\xi})\right\}, L(\lambda) > 0.$$

□

Now we are in position to give the asymptotic behavior of $g(., \lambda)$.

THEOREM 3.1. *Let g be the solution to (2.1)–(2.2) such that $g(\xi) > 0$ for all $\xi > 0$.*

1. *If $L(\lambda) = 0$, there exists $A > 0$ such that*

$$g(\xi) = Ae^{-\frac{\xi^2}{4}}\xi^{\frac{2-q}{q-1}}\left\{1 - \frac{b}{\xi^2} + o(\frac{1}{\xi^2})\right\},$$

2. *if $L(\lambda) > 0$, then*

$$g(\xi) = L(\lambda)\xi^{-\frac{1}{q-1}}\left\{1 - \frac{c}{\xi^2} + o(\frac{1}{\xi^2})\right\},$$

as $\xi \to \infty$, where $b = \dfrac{(2q-3)(q-2)}{(q-1)^2}$ and $c = 2q\alpha(L(\lambda))^{q-1} + 2\alpha(1-2\alpha)$.

PROOF. For the proof of item 2 it is sufficient to settle $\lim_{\xi\to+\infty} \xi^2(\xi u(\xi) - 2\alpha)$. Same as above we have

$$\lim_{\xi\to+\infty} \xi^2\left[\xi u(\xi) - 2\alpha\right] = 2\lim_{\xi\to+\infty} \xi^2\varphi(\xi) + 4\alpha.$$

This yields that

$$\lim_{\xi\to+\infty} \xi^2\left[\xi u(\xi) - 2\alpha\right] = 4q\alpha(L(\lambda))^{q-1} - 2(2\alpha)^2 + 4\alpha.$$

Consequently

$$\frac{g'}{g} = -2\frac{c}{\xi^3} - \frac{1}{q-1}\frac{1}{\xi} + \frac{\epsilon(\xi)}{\xi^3}, \tag{3.12}$$

where $c = 2q\alpha(L(\lambda))^{q-1} + 2\alpha(1-2\alpha)$.
A simple integration of (3.12) yields the desired asymptotic.

Now we shall prove item 1. Here we assume that $L(\lambda) = 0$. By Equation (2.1) one sees

$$\frac{g''}{\xi g'(\xi) + g} = \frac{-\dfrac{1}{2} + \alpha\dfrac{g}{\xi g'} + \dfrac{qg^{q-1}}{\xi}}{1 + \dfrac{g}{\xi g'}}.$$

Now using the l'Hôpital's rule and the fact that $g/g' \to 0$ at infinity we obtain

$$\lim_{\xi\to\infty} \frac{g'}{\xi g} = -\frac{1}{2}. \tag{3.13}$$

Next define

$$G(\xi) = \xi g' + \frac{1}{2}\xi^2 g, \quad F(\xi) = \xi^2 G - a\xi^2 g,$$

where

$$a = -(2\alpha+1) = \frac{q-2}{q-1}.$$

A simple computation shows that

$$\frac{G'}{g'} = 1 + \frac{\xi g}{g'} + q\xi g^{q-1} + \alpha\frac{\xi g}{g'}, \tag{3.14}$$

and

$$\frac{F'(\xi)}{g'(\xi)} = 2(\alpha+1)\xi\frac{g}{g'}\frac{G}{g} + q\xi^3 g^{q-1} + 2\frac{\xi g}{g'}\left[\frac{G}{g} - a\right]. \tag{3.15}$$

Applying again the l'Hôpital's rule to (3.14)–(3.15) we deduce successively

$$\lim_{\xi\to+\infty} \frac{G(\xi)}{g(\xi)} = a,$$

and

$$\lim_{\xi\to+\infty} \frac{F(\xi)}{g(\xi)} = 2b,$$

where $b = \frac{(2q-3)(q-2)}{(q-1)^2}$. Same as above results we get item 1 by an easy integration. □

Remark 3.1. The results of Theorem 3.1 have been recently obtained independently by P. Biler and G. Karch in [3].

Remark 3.2. It follows from Theorem 3.1 that if $g \in L^1((0,+\infty))$ and is positive then g satisfies item 1;

$$g(\xi) = Ae^{-\frac{\eta^2}{4}}\xi^{\frac{2-q}{q-1}}\left\{1 - \frac{b}{\xi^2} + o(\frac{1}{\xi^2})\right\}.$$

Integrating (2.1) over $(0,\xi)$ yields

$$g'(\xi) + \frac{1}{2}\xi g(\xi) - g^q(\xi) + \lambda^q = (\alpha + \frac{1}{2})\int_0^\xi g(\eta)d\eta.$$

Passing to the limit, as $\xi \to \infty$, we deduce

$$\lambda^q\frac{2(q-1)}{q-2} = \int_0^\infty g(\xi)d\xi.$$

This shows in particular the following uniqueness result.

Proposition 3.2. *Let $q > 2$. Let f and h be two solutions to*

$$\begin{cases} g'' - (g^q)' = \alpha g - \frac{1}{2}\xi g', \text{ on } (0,+\infty), \\ g'(0) = 0, \quad g(\xi) > 0, \text{ for all } \xi \geq 0, \end{cases}$$

such that

$$\int_0^\infty f(\xi)d\xi = \int_0^\infty h(\xi)d\xi.$$

Then $f \equiv h$.

Now we shall show that problem (2.1)–(2.2) has a positive solution satisfying item 2 provided that the initial data $g(0)$ is sufficiently small. To this end we set

$$f = g/\lambda.$$

Therefore f satisfies

$$\begin{cases} f'' + \frac{1}{2}\xi f' - q\lambda^{q-1}|f|^{q-1}f' - \alpha f = 0, \\ f'(0) = 0, \quad f(0) = 1. \end{cases} \tag{3.16}$$

If we now let $\lambda \to 0$, we formally obtain

$$\begin{cases} f'' + \frac{1}{2}\xi f' - \alpha f = 0, \\ f'(0) = 0, \quad f(0) = 1. \end{cases} \tag{3.17}$$

Since the energy function $H = (f')^2 - \alpha f^2$ is nonincreasing and uniformly bounded by $-\alpha > 0$, f is global and goes to 0. Moreover $f > 0, f' < 0$ and satisfies item 2 of Theorem 3.1 (otherwise we get $0 = \|f\|_1$, a contradiction -see Remark 3.1-). Since (3.16) is a regular perturbation of (3.17) it follows that the solution to (3.16) is global, positive and satisfies item 1 for λ sufficiently small. Results of the present section gives us quite a good picture of the main properties of solutions to (2.1)–(2.2). We have one of the following properties:

a) $g(\xi) > 0$ on some $(0, \xi_0)$ and $g(\xi_0) = 0$,

b) $g(\xi) > 0$ for all $\xi \geq 0$ and $g(\xi) = L(\lambda)\xi^{-\frac{1}{q-1}}\left\{1 - \frac{c}{\xi^2} + o(\frac{1}{\xi^2})\right\}$,

c) $g(\xi) > 0$ for all $\xi \geq 0$ and $g(\xi) = Ae^{-\frac{\xi^2}{4}}\xi^{\frac{2-q}{q-1}}\left\{1 - \frac{b}{\xi^2} + o(\frac{1}{\xi^2})\right\}$.

Returning to the original variables u and x we can see that the asymptotics behavior given in a) and b) yield the following two possibilities

a1) either

$$\int_0^\infty u(x,t)dx = +\infty, \qquad \text{for any } t > 0,$$

b1) or

$$\int_0^\infty u(x,t)dx = Mt^{\frac{1}{2}\frac{q-2}{q-1}}, \qquad \text{for any } t > 0.$$

4. Case $0 < q < 1$ and $\varepsilon = \pm 1$

In this section we consider

$$g'' + \varepsilon q|g|^q g' = \alpha g - \frac{1}{2}\xi g', \quad \xi > 0, \tag{4.1}$$

$$g(0) = \lambda > 0, \quad g'(0) = 0, \tag{4.2}$$

in which $\alpha = -\frac{1}{2}\frac{1}{q-1}, 0 < q < 1$ and $\varepsilon = \pm 1$. We study the asymptotic behavior of global solutions to (4.1)–(4.2). Note that $\alpha > 0$ and the standard theory of initial value problems implies the existence and uniqueness of such solutions in a neighbourhood of the origin. At $\xi = 0$ $g''(0) = \alpha\lambda > 0$. So in a small neighbourhood of 0 g is increasing. In order to show that problem (4.1)–(4.2) has a unique global solution, it is sufficient to show the following

LEMMA 4.1. *The solution $g(\xi)$ to (4.1)–(4.2) cannot blow-up for finite ξ; moreover $g'(\xi) > 0$ for all $\xi > 0$.*

PROOF. Let $\xi_0 > 0$ be the first positive zero for g'. At this point $g > 0$ so is g'' which is impossible in a small left neighbourhood of ξ_0.
Now assume that g blows-up at $\bar{\xi}$. Set

$$E = (g')^2 - \alpha g^2. \tag{4.3}$$

Using (4.1)–(4.2) one sees that $E'(\xi) = -2(g')^2(\xi)\left[\frac{1}{2}\xi + \varepsilon q g^{q-1}\right]$. Since $g^{q-1}(\xi)$ goes to 0 as $\xi \to \bar{\xi}$ we deduce that the limit $\lim_{\xi\to\bar{\xi}} E(\xi) = L$ exits in $[-\infty, A], A < +\infty$. This implies that

$$\frac{g'}{g} \le \sqrt{\alpha} + \gamma, \gamma > 0$$

for all $\xi \in (\xi_\gamma, \bar{\xi})$. And the last inequality yields that

$$g(\xi) \le g(\xi_\gamma)e^{(\sqrt{\alpha}+\gamma)(\xi-\xi_\gamma)}.$$

Therefore we get a contradiction. This means that g is bounded and then is global. □

LEMMA 4.2. $\lim_{\xi\to+\infty} g(\xi) = +\infty$.

PROOF. Suppose to the contrary that g is bounded. In that case, because of the monotonicity of g, we have $g(\xi) \to g_0, 0 < g_0 < +\infty$ and $g'(\xi_m) \to 0$ for some sequence ξ_m converging to $+\infty$ with m. Using E we can see that $\lim_{\xi\to+\infty} g'(\xi) = 0$. Therefore

$$\lim_{\xi\to+\infty} g'' + \frac{1}{2}\xi g' = \alpha g_0,$$

thanks to equation (4.1).
Arguing as in the proof of Lemma 3.2 we get

$$g' > \frac{C}{\xi}, \quad \text{for large } \xi,$$

and then g goes to infinity which leads to a contradiction. $\square$

Now we shall study the large ξ behaviour of g. First we prove that $u = g'/g$ decays to 0 as $\xi \to \infty$. Recall that u is bounded and satisfies

$$u' + \frac{1}{2}\xi u = \alpha + \varphi(\xi), \tag{4.4}$$

where

$$\varphi(\xi) = \varepsilon q u g^{q-1} - u^2. \tag{4.5}$$

A standard analysis of (4.4) implies that $u(\xi)$ converges to 0 as $\xi \to \infty$, and then $\varphi(\xi) \to 0$.

THEOREM 4.1. *Assume that* $0 < q < 1$. *Let* g *be the solution to* (4.1)–(4.2). *Then there exists* $L(\lambda) > 0$ *such that*

$$g(\xi) = L(\lambda)\xi^{2\alpha}\left\{1 - \frac{c}{\xi^2} + o(\frac{1}{\xi^2})\right\}, \quad \text{as } \xi \to +\infty, \tag{4.6}$$

where $c = 2\alpha(1-2\alpha) + 2\varepsilon q\alpha(L(\lambda))^{q-1}$.

The proof is similar as in Section 3. We show that $u = \dfrac{g'}{g}$ satisfies

$$\xi u = 2\alpha + 2\frac{c}{\xi^2} + o(\frac{1}{\xi^2}), \tag{4.7}$$

which leads to (4.6). $\square$

The following result gives a more precise estimate of g as ξ goes to infinity.

PROPOSITION 4.1. *Let* g *be the solution to* (4.1)–(4.2). *Assume that* $0 < q < 1$, *then*

$$g(\xi) = L(\lambda)\xi^{2\alpha}\left\{1 - \frac{c}{\xi^2} - \frac{d}{\xi^4} + o(\frac{1}{\xi^4})\right\}, \quad \text{as } \xi \to +\infty, \tag{4.8}$$

where

$$c = 2\alpha(1-2\alpha) + 2\varepsilon q\alpha(L(\lambda))^{q-1} \quad \text{and} \quad d = 3 - 4\alpha - 2\varepsilon q L^{q-1}(\lambda)c.$$

PROOF. It is sufficient to calculate

$$\lim_{\xi\to+\infty} \xi^2\Big[\xi^2(\xi u(\xi) - 2\alpha) - 2c\Big].$$

In fact by (4.4) we deduce that

$$\xi^2\Big[\xi^2(\xi u(\xi) - 2\alpha) - 2c\Big] = \frac{\displaystyle\int_0^\xi (\alpha + \varphi(s)e^{\frac{s^2}{4}}ds - 2\alpha\xi^{-1}e^{\frac{\xi^2}{4}} - 2c\xi^{-3}e^{\frac{\xi^2}{4}}}{e^{\frac{\xi^2}{4}}\xi^{-5}}.$$

Thus

$$\lim_{\xi\to+\infty} \xi^2\Big[\xi^2(\xi u(\xi) - 2\alpha) - 2c\Big] = 12c + 2\lim_{\xi\to+\infty} \xi^2\Big[\xi^2\varphi(\xi) + 2\alpha - c\Big],$$

thanks to l'Hôpital's rule. Define

$$A(\xi) = \xi^2 \varphi(\xi) + 2\alpha - c.$$

Thus

$$\begin{aligned} A(\xi) &= (2\alpha)^2 - (\xi u)^2 - \varepsilon q \left\{ \xi^2 g^{q-1} u - 2\alpha (L(\lambda))^{q-1} \right\}, \\ A(\xi) &= (2\alpha - \xi u)(2\alpha + \xi u) - \varepsilon q \left\{ \xi^2 g^{q-1} u - 2\alpha (L(\lambda))^{q-1} \right\}. \end{aligned}$$

By (4.7) and (4.8), we conclude that

$$\xi^2 A(\xi) = -8\alpha c - 4\varepsilon q (L(\lambda))^{q-1} c + o(1),$$

as $\xi \to 0$. Therefore

$$\lim_{\xi \to +\infty} \xi^2 \Big[\xi^2 (\xi u(\xi) - 2\alpha) - 2c \Big] = \Big(12 - 16\alpha - 8\varepsilon q (L(\lambda))^{q-1} \Big) c =: 4d.$$

The proof is completed as in the proof of the Theorem 3.1. □

In what follows we give some properties of $L(\lambda)$ in the case where $\varepsilon = 1$. We shall establish in particular that $L(\lambda)$ is strictly increasing with respect to λ, $L(\lambda)$ goes to 0 with λ and

$$L(\lambda) = l.\lambda + o(1), \; l > 0, \qquad \text{as } \lambda \to +\infty.$$

This is a consequence of the following

Theorem 4.2. *The function $\lambda \to L(\lambda)$ is continuous. Moreover for any $\lambda_2 > \lambda_1$ we have*

$$\frac{L(\lambda_2)}{\lambda_2} \geq \frac{L(\lambda_1)}{\lambda_1}$$

and there exists $L^\star > 0$ such that $L(\lambda) < \lambda L^\star$, for any $\lambda > 0$.

Proof. First we claim that if g_1 and g_2 are two solutions to problem (4.1)–(4.2) with initial values $\lambda_1 < \lambda_2$, then

$$\frac{g_2(\xi)}{g_1(\xi)} \geq \frac{\lambda_2}{\lambda_1}.$$

This leads in particular to

$$\frac{L(\lambda_2)}{L(\lambda_1)} \geq \frac{\lambda_2}{\lambda_1}.$$

Proof of the claim.

We show that the quotient $v = \dfrac{g_2}{g_1}$ is an increasing function. To this end we study the sign of the Wronskian

$$W = g_1 g_2' - g_1' g_2.$$

Using (4.1)–(4.2) one sees that W satisfies

$$\left(e^{h(\xi)}W\right)' = -qg_2'g_1e^{h(\xi)}\left[g_2^{q-1} - g_1^{q-1}\right], \quad W(0) = 0, \tag{4.9}$$

where

$$h(\xi) := \frac{\xi^2}{4} + q\int_0^\xi g_1^{q-1}(\tau)d\tau.$$

By assumption $\lambda_2 > \lambda_1$ the number

$$\xi_0 := \sup\left\{\xi, g_2(\tau) > g_1(\tau) \text{ on } [0, \xi]\right\}$$

is nonnegative. Suppose that $\xi_0 < +\infty$. It is clear that $g_1(\xi_0) = g_2(\xi_0)$ and $g_1'(\xi_0) > g_2'(\xi_0)$, so $W(\xi_0) < 0$. But since $q < 1$ we have

$$\left(e^{h(\xi)}W\right)' > 0,$$

on $(0, \xi_0)$. This implies that

$$e^{h(\xi)}W(\xi) > W(0) = 0,$$

for any $0 < \xi < \xi_0$. By continuity of W we deduce that $W(\xi_0) \geq 0$. We get a contradiction. □

This means that $\xi_0 = +\infty$ and $W(\xi) > 0$ for any $\xi > 0$. Therefore v is increasing. Now to prove that $L(\lambda)/\lambda$ is bounded, we consider problem (3.16) :

$$\begin{cases} f'' + \frac{1}{2}\xi f' + q\lambda^{q-1}|f|^{q-1}f' - \alpha f = 0, \\ f'(0) = 0, \quad f(0) = 1. \end{cases} \tag{4.10}$$

If we now let $\lambda \to \infty$, we get

$$\begin{cases} f'' + \frac{1}{2}\xi f' - \alpha f = 0, \\ f'(0) = 0, \quad f(0) = 1. \end{cases} \tag{4.11}$$

Let f_0 be the solution to (4.11). Arguing as above we deduce that

$$f_0(\xi) = L^\star \xi^{2\alpha}\left\{1 - \frac{c^\star}{\xi^2} + o(\frac{1}{\xi^2})\right\}.$$

Thus we conclude that $L(\lambda) < \lambda L^\star$ for any $\lambda > 0$.

Now we are in position to prove the continuity of the function $\lambda \to L(\lambda)$. We follow an idea due to [12]. Fix $\lambda_0 > 0, \xi_0 > 0$ and let $\delta > 0$ be a constant to be specified later.
Set $\lambda_1 = \lambda_0 - \delta,\ \lambda_2 = \lambda_0 + \delta$. For any $\lambda_1 \leq \lambda \leq \lambda_2$ we have

$$\frac{g'(\xi, \lambda)}{g(\xi, \lambda}) = \frac{2\alpha}{\xi} + r(\xi, \lambda), \quad \xi \geq \xi_0,$$

where

$$r(\xi,\lambda)=2\frac{c}{\xi^3}+o(\frac{1}{\xi^3}), c=2\alpha(1-2\alpha)+2q\alpha(L(\lambda))^{q-1}$$

thanks to (4.7). As $L(\lambda)$ is bounded on $[\lambda_1,\lambda_2]$ there exists $\bar{c}$, which depends only on λ_1,λ_2 and ξ_0 such that

$$|r(\xi,\lambda)|\leq \bar{c}\frac{1}{\xi^3}, \quad \forall\, \xi\geq\xi_0.$$

This yields that

$$\xi^{-2\alpha}g(\xi,\lambda)=\xi_0^{-2\alpha}g(\xi_0,\lambda)\exp\left(\int_{\xi_0}^{+\infty}r(\tau,\lambda)d\tau\right),$$

and for any $\beta>0$

$$\left|\exp\left(\int_{\xi_0}^{+\infty}r(\tau,\lambda)d\tau\right)-1\right|<\beta,$$

if $\xi_0>\xi_1(\beta)$. This implies that for $\xi_0>\xi_1(\beta)$ and $\lambda_1\leq\lambda\leq\lambda_2$

$$\left|L(\lambda)-\xi_0^{-2\alpha}g(\xi_0,\lambda)\right|<\beta\xi_0^{-2\alpha}g(\xi_0,\lambda),$$

therefore

$$\xi_0^{-2\alpha}g(\xi_0,\lambda)<\frac{L(\lambda)}{1-\beta}\leq\frac{L(\lambda_2)}{1-\beta}.$$

Consequently if $\beta=\dfrac{\varepsilon}{8L(\lambda_2)}<\dfrac{1}{2}$, for ε small, we get

$$\left|L(\lambda)-\xi_0^{-2\alpha}g(\xi_0,\lambda)\right|<\frac{\varepsilon}{4},$$

for any $\lambda_1\leq\lambda\leq\lambda_2$.
Hence

$$\left|L(\lambda)-L(\lambda_0)\right|\leq\left|L(\lambda)-\xi_0^{-2\alpha}g(\xi_0,\lambda)\right|+\left|\xi_0^{-2\alpha}g(\xi_0,\lambda)-\xi_0^{-2\alpha}g(\xi_0,\lambda_0)\right|+\left|L(\lambda_0)-\xi_0^{-2\alpha}g(\xi_0,\lambda_0)\right|.$$

Now if we choose for fixed $\xi_0>\xi_1$ a $\delta>0$ such that

$$\left|g(\xi_0,\lambda)-g(\xi_0,\lambda_0)\right|<\frac{\varepsilon}{2}\xi_0^{-2\alpha},$$

for any $|\lambda-\lambda_0|<\delta$ we infer

$$\left|L(\lambda)-L(\lambda_0)\right|<\varepsilon,$$

if $|\lambda-\lambda_0|<\delta$.

This completes the proof of Theorem 4.2. □

Corollary 4.1. *For any $L > 0$ the problem*

$$\begin{cases} g'' + \frac{1}{2}\xi g' + q|g|^{q-1}g' - \alpha g = 0, & \text{on } (0,+\infty), \\ g'(0) = 0, g > 0, \quad \xi^{-2\alpha}g(\xi) \to L, \end{cases}$$

has a unique solution.

Corollary 4.2. *Let $\alpha > -\frac{1}{2}$. For any $A > 0$ the function $f = \dfrac{A}{L^\star} f_0$ is the unique solution to*

$$\begin{cases} f'' + \frac{1}{2}\xi f' - \alpha f = 0, \\ f'(0) = 0, \quad \lim_{\xi\to+\infty} \xi^{2\alpha} f(\xi) = A, \end{cases} \tag{4.12}$$

where f_0 is the solution to (4.11).

Acknowledgments

The author thanks P. Biler, G. Karch and J. Goncerzewicz for interesting duscussions during his stay in Wroclaw. This work is supported by "Polonium-97074" and DAI-UPJV Amiens, France.

References

[1] A. Abramowitz & A. Stegun, Eds., Handbook of mathematical functions, National Bureau of Standars, 1964.

[2] G.I. Barenblatt,V.M. Entov & V.M. Ryzhik, Theory of fluids flows through natural rocks, Kluwer Academic Publishers, 1990.

[3] P. Biler & G. Karch, *A nonlinear Neumann problem on the half-line,* (to appear).

[4] G.I. Barenblatt & G.I. Sivashinski, *Self-similar solutions of the second kind in nonlinear filtration,* (translated from Russian PMM, pages 861–870), Applied Math. Mech. **33**, (1969), pp. 836–845.

[5] H. Brezis, L. A. Peletier & D. Terman, *A very singular solution of the heat equation with absorption*, Arch. Rat. Mech. Analysis, **95**, (1986), pp. 185–209.

[6] S. Claudi, *Asymptotic behavior for a diffusion-convection equation with rapidly decreasing initial data*, Adv. Diff. Eqns, **3**, (1998), pp. 361–386.

[7] M. Escobedo, J.L. Vazquez & E. Zuazua, *Asymptotic behaviour and source-type solutions for a diffusion-convection equation*, Arch. Rational Mech. Anal. **124**, (1993), pp.43–65.

[8] A. Gmira & L. Veron, *Large time behavior of a semilinear parabolic equation in $\mathbb{R}^n$,* J. Diff. Eqns, **53**, (1984), pp. 258–276.

[9] M. Guedda & R. Kersner, *Self–similar solutions to the generalized deterministic KPZ equation,* to appear

[10] A. Gmira, M. Guedda & L. Veron, *Source–Type solution for the one dimensional diffusion–convection equation*, to appear

[11] S. Kamin, L. A. Peletier & J. L. Vazquez, *On the Barenblat Equation of Elasto- Plastic Filtration*, Indiana Univ. Math. Journal, Vol **40**, No 4, (1991), pp. 1333–1362.

[12] L. A. Peletier & H. C. Serafini, *A very singular solution and self–similar solutions of the heat equation with convection,* Nonlinear Anal. Meth. Appl. Vol. **24**, 1, (1995), pp. 29–49.

[13] L. A. Peletier, D. Terman, & F. B. Weissler, *On the equation* $\Delta u + \frac{1}{2}x.\nabla u + f(u) = 0$, Arch. Rat. Mech. Analysis, **94**, (1986), pp. 83–99.

[14] A. A. Samarski, V. A. Galaktionov, S. Kurdyumov & A.P. Mikhailov, Blow–up in Quasilinear Parabolic Equations, De Gruyter Expositions in Mathematics, **19**, 1995.