OSCILLATION IN NEUTRAL PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS AND INEQUALITIES

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Abstract: We derive some sufficient conditions for certain classes of ordinary differential inequalities of neutral type with distributed delay not to have eventually positive or negative solutions. These, together with the technique of spatial average, the Green's Theorem and Jensen's inequality, yield some sufficient conditions for all solutions of a class of neutral partial functional differential equations to be oscillatory. An example is given to illustrate the result.

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1. INTRODUCTION

Consider the following scalar neutral partial functional differential equation

$$\begin{aligned}&\tfrac{\partial}{\partial t}[u(x,t)-\textstyle\sum_{i=1}^{m}\lambda_i(t)u(x,t-r_i)]+p(x,t)u(x,t)+\int_\alpha^\beta q(x,t,s)F[u(x,h(t,s))]d\sigma(s)\\&=a(t)\Delta u(x,t)+\textstyle\sum_{j=1}^{n}a_j(t)\Delta u(x,\tau_j(t)),\quad (x,t)\in\Omega\times[0,\infty)\end{aligned} \tag{1}$$

subject to either the Dirichlet boundary condition

$$u(x,t)=0,\quad (x,t)\in\partial\Omega\times[0,\infty) \tag{2}$$

or the Neumann boundary condition

$$\frac{\partial}{\partial\nu}u(x,t)+r(x,t)u(x,t)=0,\quad (x,t)\in\partial\Omega\times[0,\infty), \tag{3}$$

where Ω is a bounded domain in R^N with piecewise smooth boundary $\partial\Omega$, Δ is the Lapalcian operator in R^N, r_i are given positive constants, $\lambda_i\in C(R^+;R^+)$ for all $1\le i\le m$, $p\in C(\bar\Omega\times R^+;R^+), q\in C(\bar\Omega\times R^+\times J;R^+)$, $J=[\alpha,\beta]$ with two given reals $\alpha\le\beta$, $F\in C(R;R)$, $h\in C(R^+\times J;R)$ and $h(t,s)$ is nondecreasing with respect to t and s, respectively, with $h(t,s)<t$ for all $t\in R^+$ and $s\in J$ and $\lim_{t\to\infty}\min_{s\in J}h(t,s)=\infty$, $\sigma:J\to R$ is nondecreasing, $a,a_j\in C(R^+;R^+)$, $j=1\cdots,n$, $\tau_j\in C(R^+;R^+)$ is

nondecreasing, $\tau_j(t) < t$ and $\lim_{t\to\infty} \tau_j(t) = \infty$ for all $1 \le j \le n$, ν is the unit exterior normal vector to $\partial\Omega$ and $r \in C(\partial\Omega \times R^+; R^+)$.

We say a solution of (1)subject to either (2) or (3) is *oscillatory* in the domain $\Omega \times R^+$ if for every positive real τ there exists a point $(x_0, t_0) \in \Omega \times [\tau, \infty)$ such that $u(x_0, \tau_0) = 0$.

Our goal is to give sufficient conditions guaranteeing all solutions of the problem (1)-(2) or (1)-(3) are oscillatory. One of the typical results we will prove is as follows:

Theorem 1.1. Assume that

$$F : R \to R \text{ is odd, convex and } \inf_{y>0} F(y)/y > 0; \tag{4}$$

$$\sum_{i=1}^{m} \lambda_i(t) = 1, \text{ and there is } i_0 \text{ such that } \lim_{t\to\infty} \lambda_{i_0}(t) = \lambda_0 > 0. \tag{5}$$

$$\int^{\infty} \int_{\alpha}^{\beta} \min_{x\in\Omega} q(x, s, \chi) d\sigma(\chi) ds = \infty. \tag{6}$$

Then every solution of (1) subject to either (2) or (3) is oscillatory in $\Omega \times R^+$.

Our approach is to show that if u is a solution of (1) subject to, for example, the Neumann boundary condition, then

$$U(t) = \frac{1}{\int_\Omega \phi(x) ds} \int_\Omega u(x, t)\phi(x) de \tag{7}$$

with large t satisfies the scalar functional differential inequality of neutral type

$$\frac{d}{dt}[U(t) - \sum_{i=1}^{m} U(t - r_i)] + \int_{\alpha}^{\beta} \min_{x\in\bar{\Omega}} q(x, t, \chi) F[U(h(t, \chi))] d\sigma(\chi) \le 0, \tag{8}$$

where ϕ is a positive eigenfunction associated with the smallest eigenvalue of the Dirichlet problem

$$\Delta u + \mu u = 0 \text{ in } \Omega \text{ and } u|_{\partial\Omega} = 0. \tag{9}$$

This enables us to apply our general sufficient conditions for scalar differential inequalities of neutral type not to have eventually positive or negative solutions.

The rest of this paper is organized as follows: in Section 2, we collect several technical lemmas concerning the operator

$$z(t) = y(t) - \sum_{j=1}^{m} \lambda_i(t) y(t - r_i) \tag{10}$$

and the relation between oscillations and convergence for a class of ordinary differntial inequalities of neutral type. Section 3 contains our main results on sufficient conditions for the ordinary differential inequalities of neutral type under consideration not to have eventually positive solutions. Finally, in Section 4, we apply these general results to prove Theorem 1.1 and other general results for all solutions of the partial functional differential equation (1) subject to certain boundary conditions to be oscillatory.

2. TECHNICAL LEMMAS

We consider the following first order nonlinear differential inequality with distributed deviating arguments

$$\frac{d}{dt}[y(t) - \sum_{i=1}^{m} \lambda_i(t)y(t - r_i)] + \int_{\alpha}^{\beta} Q(t, s)F[y(h(t, s))]d\sigma(s) \leq 0, \tag{11}$$

where r_i are given positive constants, $\lambda_i \in C(R^+; R^+)$ for all $1 \leq i \leq m$, $Q \in C(R^+ \times J; R^+)$, $J = [\alpha, \beta]$ with two given reals $\alpha \leq \beta$, $F \in C(R; R)$, $h \in C(R^+ \times J; R)$ and $h(t, s)$ is nondecreasing with respect to t and s, respectively, with $h(t, s) < t$ for all $t \in R^+$ and $s \in J$ and $\lim_{t\to\infty} \min_{s\in J} h(t, s) = \infty$, and $\sigma : J \to R$ is nondecreasing.

We will derive some sufficient conditions for inequality (11) not to have any eventually positive solutions. Note that if we let $F^*(y) = -F(-y)$ for all $y \in R$, then any solution y of (11) yields a solution $-y$ for the differntial inequality

$$\frac{d}{dt}[y(t) - \sum_{i=1}^{m} \lambda_i(t)y(t - r_i)] + \int_{\alpha}^{\beta} Q(t, s)F^*[y(h(t, s))]d\sigma(s) \geq 0. \tag{12}$$

Therefore, one should easily obtain similar sufficient conditions for (12) not to have any eventually negative solutions. This also shows all of the sufficient conditions stated below for inequality (11) not to have eventually positive solutions can be modified to those sufficient conditions for all solutions of the following neutral functional differntial equation

$$\frac{d}{dt}[y(t) - \sum_{i=1}^{m} \lambda_i(t)y(t - r_i)] + \int_{\alpha}^{\beta} Q(t, s)F[y(h(t, s))]d\sigma(s) = 0 \tag{13}$$

to be oscillatory (i.e., to have arbitrarily large zeros).

LEMMA 2.1. Assume that

$$F(y) \geq 0 \quad \text{for any } y > 0 \tag{14}$$

and that there exist $i_0 \in \{1, \cdots, m\}$ such that

$$\lim_{t\to\infty} \sum_{i=1}^{m} \lambda_i(t) = \lambda \in (0, 1], \quad \lim_{t\to\infty} \lambda_{i_0}(t) = \lambda_0 > 0. \tag{15}$$

Consider a bounded and eventually positive solution y of inequality (11) and let

$$z(t) = y(t) - \sum_{i=1}^{m} \lambda_i(t)y(t - r_i), \quad t \in R^+. \tag{16}$$

Then we have

(i) $\lim_{t\to\infty} z(t) = 0$ in case where $\lambda = 1$;

(ii) $\lim_{t\to\infty} y(t)$ exists in case where $\lambda < 1$.

Proof. Clearly, z is bounded on R^+ and for large $t \geq 0$, we have

$$z'(t) \leq -\int_{\alpha}^{\beta} Q(t, s)F[y(h(t, s))]d\sigma(s) \leq 0.$$

Therefore, $\lim_{t\to\infty} z(t) = z(\infty)$ exists.

On the other hand, since y is bounded on R^+, we can find two sequences of nonnegative reals $\{t_n\}$ and $\{t_n^*\}$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} t_n^* = \infty$$

and

$$\lim_{n\to\infty} y(t_n) = \limsup_{t\to\infty} y(t) = A, \ \lim_{n\to\infty} y(t_n^*) = \liminf_{t\to\infty} y(t) = B.$$

For each $\epsilon > 0$ there is a positive integer N such that for all $n \geq N$, we have

$$\lambda_{i_0}(t_n) > 0, \quad \lambda_{i_0}(t_n^*) > 0$$

and

$$y(t_n - r_i) \leq A + \epsilon, \quad y(t_n^* - r_i) \geq B - \epsilon, i = 1, \cdots, m.$$

Therefore, for $n \geq N$, we get

$$\begin{aligned} y(t_n - r_{i_0}) &= \tfrac{1}{\lambda_{i_0}(t_n)}[y(t_n) - z(t_n) - \textstyle\sum_{i=1,i\neq i_0}^m \lambda_i(t_n) y(t_n - r_i)] \\ &\geq \tfrac{1}{\lambda_{i_0}(t_n)}[y(t_n) - z(t_n) - (A+\epsilon)\textstyle\sum_{i=1,i\neq i_0}^m \lambda_i(t_n)] \end{aligned}$$

and

$$\begin{aligned} y(t_n^* - r_{i_0}) &= \tfrac{1}{\lambda_{i_0}(t_n^*)}[y(t_n^*) - z(t_n^*) - \textstyle\sum_{i=1,i\neq i_0}^m \lambda_i(t_n^*) y(t_n^* - r_i)] \\ &\leq \tfrac{1}{\lambda_{i_0}(t_n^*)}[y(t_n^*) - z(t_n^*) - (B-\epsilon)\textstyle\sum_{i=1,i\neq i_0}^m \lambda_i(t_n^*)]. \end{aligned}$$

Taking $n \to \infty$, we obtain

$$A \geq \frac{1}{\lambda_0}[A - z(\infty) - (A+\epsilon)(\lambda - \lambda_0)]$$

and

$$B \leq \frac{1}{\lambda_0}[B - z(\infty) - (B-\epsilon)(\lambda - \lambda_0)].$$

Consequently,

$$A(1-\lambda) \leq z(\infty) \leq B(1-\lambda). \tag{17}$$

In case where $\lambda = 1$, we immediately obtain from (17) that $\lim_{t\to\infty} z(t) = 0$, establishing the conclusion (i).

In case where $\lambda < 1$, we obtaion from (17) that $A \leq B$. This yields $A = B$ and completes the proof for (ii).

LEMMA 2.2. Assume condition (15) holds with $\lambda < 1$, and assume

$$\inf_{y>0} F(y)/y = K > 0. \tag{18}$$

If

$$\int^\infty \int_\alpha^\beta Q(t,s) d\sigma(s) dt = \infty, \tag{19}$$

then every eventually positive solution of inequality (11) tends to zero as $t \to \infty$.

Proof: Consider an eventually positive solution y of (11) and set z as in (16). Since

$$z'(t) \leq -\int_\alpha^\beta Q(t,s) F[y(h(t,s))] d\sigma(s) \leq -K \int_\alpha^\beta Q(t,s) y(h(t,s)) d\sigma(s) \leq 0$$

for all sufficiently large t, z is eventually decreasing. Hence, there exists $t_1 \geq 0$ such that for $t \geq t_1$ and $s \in J$,

$$z(t) \leq z(h(t,s)) = y(h(t,s)) - \sum_{i=1}^{m} \lambda_i(h(t,s)) y(h(t,s) - r_i) \leq y(h(t,s)).$$

Thus, there exists $T \geq t_1$ so that for all $t \geq T$, one has

$$\begin{aligned} 0 &\geq z'(t) + \textstyle\int_\alpha^\beta Q(t,s) F[y(h(t,s))] d\sigma(s) \\ &\geq z'(t) + K \textstyle\int_\alpha^\beta Q(t,s) y(h(t,s)) d\sigma(s) \\ &\geq z'(t) + K z(t) \textstyle\int_\alpha^\beta Q(t,s) d\sigma(s) \end{aligned}$$

from which it follows that

$$z(t) \leq z(T) e^{-\int_T^t \int_\alpha^\beta KQ(t,s) d\sigma(s) dt}, \quad t \geq T.$$

Set

$$\mu = \max\{z(T), 0\}.$$

Then,

$$\begin{aligned} y(t) &= z(t) + \textstyle\sum_{i=1}^m \lambda_i(t) y(t - r_i) \\ &\leq \textstyle\sum_{i=1}^m \lambda_i(t) y(t - r_i) + \mu e^{-\int_T^t \int_\alpha^\beta KQ(t,s) d\sigma(s) dt}, \quad t \geq T. \end{aligned}$$

We are going to show that y is bounded on R^+. By way of contradiction, if y is unbounded, then there exists a sequence $\{t_n\} \subset [T, \infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and

$$\lim_{n\to\infty} y(t_n) = \infty, \quad y(t_n) = \max_{T \leq t_n \leq t_n} y(t). \tag{20}$$

Now we have

$$\begin{aligned} y(t_n) &\leq \textstyle\sum_{i=1}^m \lambda_i(t_n) y(t_n - r_i) + \mu e^{-\int_T^{t_n} \int_\alpha^\beta KQ(t,s) d\sigma(s) dt} \\ &\leq y(t_n) \textstyle\sum_{i=1}^m \lambda_i(t_n) + \mu e^{-\int_T^{t_n} \int_\alpha^\beta KQ(t,s) d\sigma(s) dt}. \end{aligned}$$

Since $\lambda < 1$, we can find some positive integer N such that

$$\sum_{i=1}^m \lambda_i(t_n) < 1 \quad \text{for } n \geq N.$$

Thus for $n \geq N$ we have

$$y(t_n) \leq \frac{\mu}{1 - \sum_{i=1}^m \lambda_i(t_n)} e^{-\int_T^{t_n} \int_\alpha^\beta KQ(t,s) d\sigma(s) dt}.$$

But as $n \to \infty$, the right hand side tends to $\frac{\mu}{1-\lambda} \cdot 0$ due to condition (19). This yields a contradiction to $\lim_{n\to\infty} y(t_n) = \infty$.

So, y is bounded and hence, Lemma 2.1 ensures that $\lim_{t\to\infty} y(t) = \eta$ exists, which implies that z is bounded as well. We claim that $\eta = 0$. Otherwise, $\eta > 0$ and $y(t) \geq \eta/2$ for large t. Therefore, there exists $T^* \geq T$ such that

$$\begin{aligned} 0 &\geq z'(t) + \textstyle\int_\alpha^\beta Q(t,s) F[y(h(t,s))] d\sigma(s) \\ &\geq z'(t) + \frac{\eta}{2} \textstyle\int_\alpha^\beta KQ(t,s) d\sigma(s) \end{aligned}$$

for all $t \geq T^*$, and hence

$$z(t) \leq z(T^*) - \frac{\eta}{2} \int_{T^*}^{t} \int_{\alpha}^{\beta} KQ(t,s)d\sigma(s)dt$$

which, due to the condition (19), leads to a contradiction to the boundedness of z. This completes the proof.

In what follows, we are going to state three results from [2,3] which will be used in oue work. Consider the retarded differential inequality

$$X'(t) + \int_{\alpha}^{\beta} P(t,s)X(h(t,s))d\sigma(s) \leq 0, \tag{21}$$

where $P \in C(R^+ \times J; R^+)$.

LEMMA 2.3. Assume that
(H0) there exists a function $g \in C(R^+ \times J, (0,\infty))$ such that $g(g(t,s),s) = h(t,s)$; $g(t,s)$ is nondecreasing with respect to $t \geq 0$ and $s \in J$ respectively; $t > g(t,s) > h(t,s)$ and $\lim_{t\to\infty} \min_{s\in J} g(t,s) = \infty$;
(H1) σ is not right continuous at $C_i (i = 0, 1, \cdots \nu)$ with $\alpha < C_1 < C_2 < \cdots < C_\nu < \beta$ and $\liminf_{t\to\infty} \int_{g(t,\chi^*)}^{t} \int_{C_j}^{\chi^*} P(\eta,s)d\sigma(s)d\eta > 0$ for all $\chi^* \in (C_j, C_{j+1}]$, $j = 0, \cdots \nu$, where

$$C_0 = \alpha,\ C_{\nu+1} = \beta,\ \sigma(\alpha - 0) = \sigma(\alpha),\ \sigma(\beta + 0) = \sigma(\beta);$$

(H2) $\liminf_{t\to\infty} \int_{h(t,\beta)}^{t} \int_{\alpha}^{\beta} P(\eta,s)d\sigma(s)dsd\eta > \frac{1}{e}$.
Then (21) does not have eventually positive solutions.

LEMMA 2.4. Assume that conditions (H0) and (H2) are satisfied and
(H3) $\liminf_{t\to\infty} \int_{g(t,\beta)}^{t} \int_{\alpha}^{\beta} P(\eta,s)d\sigma(s)d\eta > 0$.
Then (21) does not have eventually positive solutions.

LEMMA 2.5. Assume that (H1) and (H2) are satisfied. Then (21) has no eventually positive solutions if one of the following conditions is satisfied:
(H4) $\liminf_{t\to\infty} \int_{h(t,\alpha_i)}^{t} \int_{\alpha}^{\beta} P(\eta,s)d\sigma_i(s)d\eta > \frac{1}{e}$ for some $i \in \{1, \cdots \nu\}$, here we assume σ is continuous at $\alpha = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_\nu = \beta$ and set

$$\sigma_i(s) = \begin{cases} \sigma(s), & s \in [\alpha_{i-1}, \alpha_i], \\ \sigma(\alpha_{i-1}), & s < \alpha_{i-1}, \\ \sigma(\alpha_i), & s > \alpha_i. \end{cases}$$

(H5) $\int_{\alpha}^{\beta} \ln[\int_{\alpha}^{\beta} \liminf_{t\to\infty} \int_{h(t,\chi)}^{t} P(\theta,\eta)d\theta d\sigma(\chi)]d\sigma(\eta) + \int_{\alpha}^{\beta} d\sigma(\eta) > 0$;
(H6)

$$\int_{\alpha}^{\beta} \int_{\alpha}^{\beta} [\liminf_{t\to\infty} \int_{h(t,\eta)}^{t} P(\theta,\chi)d\theta]^{\frac{1}{2}} [\liminf_{t\to\infty} \int_{h(t,\chi)}^{t} |P(\theta,\eta)d\theta]^{\frac{1}{2}} d\sigma(\chi)d\sigma(\eta) > \frac{1}{e}.$$

3. MAIN RESULTS

THEOREM 3.1 Assume that (15), (18) and (19) are satisfied. In addition, assume that

$$\sum_{i=1}^{m} \lambda_i(t) = 1, \quad t \geq t_0 \geq 0. \tag{22}$$

Then (11) has no eventually positive solution.

Proof: Assume that (11) has an eventually positive solution y. Then there exists $t_1 \geq t_0 \geq 0$ so that $y(t) > 0$ for all $t \geq t_1 \geq t_0$. Set

$$z(t) = y(t) - \sum_{i=1}^{m} \lambda_i(t) y(t - r_i), \quad t \geq 0.$$

Since there exists $t_2 \geq t_1$ such that for $t \geq t_2$,

$$z'(t) \leq -\int_\alpha^\beta Q(t,s) F[y(h(t,s))] d\sigma(s) \leq -K \int_\alpha^\beta Q(t,s) y(h(t,s)) d\sigma(s) \leq 0,$$

z is nonincreasing on $[t_2, \infty)$.

We first consider the case where there exists $t_3 \geq t_2$ such that $z(t) < 0$ for all $t \geq t_3$. In this case, we have

$$z(t) \leq z(t_3) < 0, \quad t \geq t_3. \tag{23}$$

We want to show that y is bounded. If this is not true, then there exists a sequence $\{s_n\} \subset [t_4, \infty)$, $t_4 = t_3 + \max_{1 \leq i \leq m} r_i$, such that $s_n \to \infty$ and $y(s_n) \to \infty$ as $n \to \infty$, and $y(t_n) = \max_{t_3 \leq t \leq s_n} y(t)$ for all $n \geq 1$. Using (22) and (23), we get

$$\begin{aligned} y(s_n) &= z(s_n) + \textstyle\sum_{i=1}^m \lambda_i(s_n) y(s_n - r_i) \\ &\leq z(t_3) + y(s_n) \textstyle\sum_{i=1}^m \lambda_i(s_n) \\ &= z(t_3) + y(s_n), \end{aligned}$$

from which we get $z(t_3) \geq 0$, a contradiction to (23). Hence, y is bounded and thus, z is bounded. By Lemma 2.1, we have $\lim_{t\to\infty} z(t) = 0$, which contradicts the fact that $z'(t) \leq 0$ and $z(t) < 0$ for all $t \geq t_4$.

We next consider the remaining possible case that there exists $T \geq t_1$ such that $z(t) > 0$ for all $t \geq T$. Then we have

$$y(t) > \sum_{i=1}^{m} \lambda_i(t) y(t - r_i), \quad t \geq T. \tag{24}$$

If $\liminf_{t\to\infty} y(t) = 0$, then there exists a sequence $\{s_n\} \subset [T, \infty)$ such that $\lim_{n\to\infty} s_n = \infty$ and

$$\lim_{n\to\infty} y(s_n) = 0 \quad y(s_n) = \min_{t_1 \leq t \leq s_n} y(t), n \geq 1.$$

Using (22) and (24), we obtain the following contradiction

$$\begin{aligned} y(s_n) \quad &> \textstyle\sum_{i=1}^m \lambda_i(s_n) y(s_n - r_i) \\ &\geq y(s_n) \textstyle\sum_{i=1}^m \lambda_i(s_n) \\ &= y(s_n), \quad n \geq 1. \end{aligned}$$

Hence, $\liminf_{t\to\infty} y(t) > 0$. Therefore, there exist $T^* \geq T$ and a constant $M > 0$ such that

$$y(t) \geq M, \quad y(h(t,s)) > M \text{ for all } t \geq T^*, s \in J.$$

Thus,

$$z'(t) \leq -K \int_\alpha^\beta Q(t,s) y(h(t,s)) d\sigma(s) \leq -KM \int_\alpha^\beta Q(t,s) d\sigma(s), \quad t \geq T^*.$$

This yields

$$z(t) \le z(T^*) - KM \int_{T^*}^{t} \int_{\alpha}^{\beta} Q(\eta,\chi) d\sigma(\chi) d\eta \to -\infty$$

as $t \to \infty$, a contradiction to $z(t) > 0$ for $t \ge T^*$. This completes the proof.

THEOREM 3.2 Assume that (18) and (H0) are satisfied. In addition, we assume that

$$\sum_{i=1}^{m} \lambda_i(t) \le 1 \text{ for large } t; \tag{25}$$

and

$$\rho := \inf_{(t,s)\in R^+ \times J} Q(t,s) > 0 \tag{26}$$

and either

(H1*) σ is not right continuous at $C_i (i = 0, 1, \cdots \nu)$ with $\alpha < C_1 < C_2 < \cdots < C_\nu < \beta$ and $\liminf_{t\to\infty} \int_{g(t,\chi^*)}^{t} \int_{C_j}^{\chi^*} Q(\eta,\chi) d\sigma(\chi) d\eta > 0$ for $\chi^* \in (C_j, C_{j+1}]$, $j = 0, \cdots, \nu$, where

$$C_0 = \alpha,\ C_{\nu+1} = \beta,\ \sigma(\alpha - 0) = \sigma(\alpha),\ \sigma(\beta + 0) = \sigma(\beta); \textit{or}$$

(H2*) $\liminf_{t\to\infty} \int_{h(t,\beta)}^{t} \int_{\alpha}^{\beta} Q(\eta,\chi) d\sigma(\chi) d\eta > \frac{1}{Ke}$.

Then (11) has no eventually positive solutions.

Proof: Suppose that y is an eventually positive solution of (11). Then there exists $t_0 \ge 0$ such that for all $t \ge t_0, s \in J$ and $1 \le i \le m$,

$$y(t) > 0,\ y(t - r_i) > 0,\ y(h(t,s)) > 0,\ y(h(t,s) - r_i) > 0.$$

Let

$$z(t) = y(t) - \sum_{i=1}^{m} \lambda_i(t) y(t - r_i).$$

We have

$$z(h(t,s)) \le y(h(t,s)), \quad t \ge t_0, s \in J.$$

From (11) and (18), we have

$$z'(t) + \int_{\alpha}^{\beta} KQ(t,s) z(h(t,s)) d\sigma(s) \le 0, \quad t \ge t_0. \tag{27}$$

We shall show that $z(t) > 0$ for sufficiently large t. In fact, from (11) and (26) it follows that

$$z'(t) \le -\rho \int_{\alpha}^{\beta} K y(h(t,s)) d\sigma(s), \quad t \ge t_0.$$

Thus, $\lim_{t\to\infty} z(t) = z(\infty) \in [-\infty, \infty)$. If $z(\infty) = -\infty$, then from the definition of z and (25) it follows that y is unbounded. Hence there exists $t_1 \ge t_0$ with $z(t_1) < 0$ and $y(t_1) = \max_{t_0 \le t \le t_1} y(t)$. But we have

$$z(t_1) = y(t_1) - \sum_{i=1}^{m} \lambda_i(t_1) y(t_1 - r_i) \ge y(t_1)[1 - \sum_{i=1}^{m} \lambda_i(t_1)] \ge 0.$$

This yields a contradiction. Hence $z(\infty) > -\infty$. Using (26), we get

$$\begin{aligned}
&\rho K \int_{t_1}^t \int_\alpha^\beta z(h(\eta,\chi))d\sigma(\chi)d\eta \\
&\le \rho K \int_{t_1}^t \int_\alpha^\beta y(h(\eta,\chi))d\sigma(\chi)d\eta \\
&\le \int_{t_1}^t \int_\alpha^\beta KQ(\eta,\chi)y(h(\eta,\chi))d\sigma(\chi)d\eta \\
&\le -\int_{t_1}^t z'(s)ds = z(t_1) - z(t) \le z(t_1) - z(\infty) < \infty.
\end{aligned}$$

Since $z'(t) < 0$ for $t \ge t_0$ and $h(t,s) \le t$ for all $s \in J$, we have $t_2 \ge t_0$ such that

$$z(h(t,s)) > z(t) \text{ for all } t \ge t_2, s \in J.$$

Then it follows that

$$\rho K \int_\alpha^\beta d\sigma(\chi) \int_{t_2}^t z(s)ds < \infty,$$

i.e., $z \in L^1((t_2,\infty))$ and hence, $\lim_{t\to\infty} z(t) = 0$. As z is strictly decreasing, there must be $t_3 \ge t_1$ such that $z(t) > 0$ for all $t \ge t_3$. Therefore, z is an eventually positive solution of (27). However, Lemma 2.3 ensures that (27) has no eventually positive solutions. This contradiction shows that (11) does not have eventually positive solution. This completes the proof.

One of the key steps in the above proof is to show that if y is an eventually positive solution of (11), then z is an eventually positive solution (27). Using the same technique and Lemmas 2.4 and 2.5, we can also establieh the following two results:

THEOREM 3.3. Assume that conditions (18), (25), (26), (H0) and (H2*) are satisfied. Also assume that
(H3)* $\liminf_{t\to\infty} \int_{g(t,\beta)}^t \int_\alpha^\beta Q(\eta,\chi)d\sigma(\chi)d\eta > 0$.
Then (11) has no eventually positive solutions.

THEOREM 3.4. Assume that conditions (18), (25), (26), (H0) and (H1*) are satisfied. Then (11) has no eventually positive solutions if one of the following conditions holds:
(H4)* $\liminf_{t\to\infty} \int_{h(t,\alpha_i)}^t \int_\alpha^\beta Q(\eta,s)d\sigma_i(s)d\eta > \frac{1}{Ke}$ for some $i \in \{1,\cdots\nu\}$, here we assume σ is continuous at $\alpha = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_\nu = \beta$ and set

$$\sigma_i(s) = \begin{cases} \sigma(s), & s \in [\alpha_{i-1},\alpha_i], \\ \sigma(\alpha_{i-1}), & s < \alpha_{i-1}, \\ \sigma(\alpha_i), & s > \alpha_i. \end{cases};$$

(H5)* $\int_\alpha^\beta \ln[\int_\alpha^\beta \liminf_{t\to\infty} \int_{h(t,\chi)}^t KQ(\eta,\theta)d\eta d\sigma(\chi)]d\sigma(\theta) + \int_\alpha^\beta d\sigma(s) > 0$;
(H6)* $\int_\alpha^\beta \int_\alpha^\beta [\liminf_{t\to\infty} \int_{h(t,\chi)}^t KQ(\eta,\chi)d\eta]^{\frac{1}{2}} [\liminf_{t\to\infty} \int_{h(t,\chi)}^t KQ(\eta,\theta)d\eta]^{\frac{1}{2}} d\sigma(\eta)d\sigma(\theta) > \frac{1}{e}$.

We now consider the case where $\lambda < 1$.

THEOREM 3.5. Assume that conditions (18), (15), (26), (H0), (H1*) and (H2*) are satisfied and that $\lambda < 1$. Then (11) has no eventually positive solutions and (1.2) has no eventually negative solutions.

Proof: Let y be an eventually positive solution of (11). We can assume that there exists $t_0 \ge 0$ such that $y(t) > 0$ for all $t \ge t_0 \ge 0$. Note that condition (H2) implies (19), thus Lemma 2.2 implies that $\lim_{t\to\infty} y(t) = 0$. Therefore, for $z(t) = y(t) - \sum_{i=1}^m \lambda_i(t)t(t-r_i)$, we have $\lim_{t\to\infty} z(t) = 0$. From (11) and (26) it follows that there exists $t_1 \geq t_0$ such that

$$z'(t) \leq -\int_\alpha^\beta Q(t,s)F[y(h(t,s))]d\sigma(s) \leq -\rho K \int_\alpha^\beta y(h(t,s))d\sigma(s) < 0$$

for all $t \geq t_1 \geq t_0$. This shows that z is strictly increasing on $[t_1, \infty)$. Thus

$$z(t) > 0 \quad \text{for all } t \geq t_1. \tag{28}$$

Choose $t_2 \geq t_1$ so that $z(h(t,s)) \leq y(h(t,s))$ for all $t \geq t_2 \geq t_1$. Then we have

$$z'(t) + \int_\alpha^\beta KQ(t,s)z(h(t,s))d\sigma(s) \leq 0, \;\; t \geq t_2. \tag{29}$$

By Lemma 2.3, we see that (29) has no eventually positive solutions, which contradicts (28). This completes the proof.

Using the same argument and Lemmas 2.2 and 2.5, we can also get

THEOREM 3.6. Assume that all conditions (18), (15), (26), (H0), (H2*) and (H6*) are satisfied and that $\lambda < 1$, Then (11) has no eventually positive solution.

4. OSCILLATIONS FOR NEUTRAL PARTIAL FDES

We begin with the proof of Theorem 1.1:

Assume, by way of contradiction, that (1)-(2) has a solution u such that for some $\tau > 0$, $u(x,t)$ has no zeros in $\Omega \times [\tau, \infty)$. Since F is odd we may assume, without loss of generality, that $u(x,t) > 0$ for $(x,t) \in \Omega \times [\tau, \infty)$. Then there exists $t_1 > \tau$ such that

$$u(x, t-r_i) > 0, \;\; u(x, \tau_j(t)) > 0, \;\; x \in \Omega, t \geq t_1, 1 \leq i \leq m, 1 \leq j \leq n$$

and

$$u(x, h(x,t)) > 0, \;\; x \in \Omega, t \geq t_1, s \in J.$$

Multiplying both sides of (1) by the positive eigenfunction associated with the smallest eigenvalue $\mu_0 > 0$ of the Dirichlet problem (9) and integrating with respect to x over the domain Ω yield

$$\begin{array}{l} \frac{d}{dt}[\int_\Omega u(x,t)\phi(x)dx - \sum_{i=1}^m \lambda_i(t)\int_\Omega u(x,t-r_i)\phi(x)ds] + \int_\Omega p(x,t)u(x,t)\phi(x)dx \\ \;\; + \int_\Omega \int_\alpha^\beta q(x,t,\chi)F[u(x,h(t,\chi))]\phi(x)d\sigma(\chi)dx \\ = a(t)\int_\Omega \delta u\phi(x) + \sum_{i=1}^n a_j(t)\int_\Omega \Delta u(x,\tau_j(t))\phi(x)ds, \quad t \geq t_1. \end{array} \tag{30}$$

It is easy to see that

$$\begin{array}{l} \int_\Omega \int_\alpha^\beta q(x,t,\chi)F[u(x,h(t,\chi))]\phi(x)d\sigma(\chi)dx \\ = \int^\beta \int_\Omega q(x,t,\chi)F[u(x,h(t,\chi))]\phi(x)dxd\sigma(\chi). \end{array} \tag{31}$$

Using Green's Theorem, we obtain

$$\begin{array}{l} \int_\Omega \Delta u(x,t)\phi(x)dx \\ = \int_{\partial\Omega}[\phi(x)\frac{\partial}{\partial\nu}u(x,t) - u(x,t)\frac{\partial}{\partial\nu}\phi(x)]dS + \int_\Omega u(x,t)\Delta\phi(x)dx \\ = -\mu_0 \int_\Omega u(x,t)\phi(x)dx, \quad t \geq t_1, \end{array} \tag{32}$$

and

$$\begin{array}{l} \int_\Omega \Delta u(x,\tau_j(t))\phi(x)dx \\ = \int_{\partial\Omega}[\phi(x)\frac{\partial}{\partial\nu}u(x,\tau_j(t)) - u(x,\tau_j(t))\frac{\partial}{\partial\nu}\phi(x)]dS + \int_\Omega u(x,\tau_j(t))\Delta\phi(x)dx \\ = -\mu_0\int_\Omega u(x,\tau_j(t))\phi(x)dx, \quad t \geq t_1, 1 \leq j \leq n, \end{array} \tag{33}$$

where dS is the surface integral element on $\partial\Omega$. As F is convex, we can apply Jensen's inequality to get

$$\int_\Omega F[u(x,h(t,\chi))]\phi(x)dx \geq \int_\Omega \phi(x)dx \cdot F[\frac{1}{\int_\Omega \phi(x)dx}\int_\Omega u(x,h(t,\chi))\phi(x)dx]. \tag{34}$$

Combining (30)-(34) yields

$$\begin{array}{l} \frac{d}{dt}[\int_\Omega u(x,t)\phi(x)dx - \sum_{i=1}^m \lambda_i(t)\int_\Omega u(x,t-r_i)\phi(x)dx] + \int_\Omega p(x,t)u(x,t)\phi(x)dx \\ \ +[\int_\Omega \phi(x)dx]\int_\alpha^\beta \min_{x\in\bar\Omega} q(x,t,\chi)F(\frac{1}{\int_\Omega \phi(x)dx}\int_\Omega u(x,h(t,\chi))\phi(x)dx)d\sigma(\chi) \\ \leq -\mu_0 a(t)\int_\Omega u(x,t)\phi(x)dx - \mu_0\sum_{j=1}^n a_j(t)u(x,\tau_j(t))\phi(x)dx \leq 0, \quad t \geq t_1. \end{array} \tag{35}$$

So, the function

$$U(t) = \frac{1}{\int_\Omega \phi(x)dx}\int_\Omega u(x,t)\phi(x)dx, \ \ t \geq t_1$$

satisfies the following scalar ordinary neutral differntial inequality

$$\frac{d}{dt}[U(t) - \sum_{i=1}^m \lambda_i(t)U(t-r_i)] + \int_\alpha^\beta \min_{x\in\bar\Omega} q(x,t,\chi)F[U(h(t,\chi))]d\sigma(\chi) \leq 0, \quad t \geq t_1. \tag{36}$$

Theorem 3.1 shows that (36) can not have eventually positive solution, a contradiction to $U(t) > 0$ for all $t \geq t_1$. So, all solutions of (1)-(2) are oscillatory.

Now we assume, again by way of contradiction, that (1)-(3) has a solution which is positive in $\Omega \times [\tau,\infty)$ for some $\tau > 0$. Then there exists $t_1 > \tau$ such that

$$u(x,t-r_i) > 0, \ \ u(x,\tau_j(t)) > 0, \ \ x \in \Omega, t \geq t_1, 1 \leq i \leq m, 1 \leq j \leq n$$

and

$$u(x,h(x,t)) > 0, \ \ x \in \Omega, t \geq t_1, s \in J.$$

By Green's Theorem, we have

$$\int_\Omega \Delta u dx = \int_{\partial\Omega}\frac{\partial u}{\partial\nu}dS = -\int_{\partial\Omega} r(x,t)u(x,t)dS \leq 0, \tag{37}$$

and

$$\int_\Omega \Delta u(x,\tau_j(t))dx = \int_{\partial\Omega}\frac{\partial}{\partial\nu}u(x,\tau_j(t))dS = -\int_{\partial\Omega} r(x,\tau_j(t))u(x,\tau_j(t))dS \leq 0, 1 \leq j \leq n. \tag{38}$$

Using Jensen's inequality, we get

$$\int_\Omega F[u(x,h(t,\chi))]dx \geq |\Omega|F[\frac{1}{|\Omega|}\int_\Omega u(x,h(t,\chi))dx], \ \ t \geq t_1. \tag{39}$$

Integrating (1) with respect to x over Ω and applying (37)-(40) yield

$$\begin{aligned}&\tfrac{d}{dt}[\int_\Omega u(x,t)dx-\sum_{i=1}^m\lambda_i(t)\int_\Omega u(x,t-r_i)dx]+\int_\Omega p(x,t)u(x,t)dx\\ &+|\Omega|\int_\alpha^\beta \min_{x\in\bar\Omega} q(x,t,\chi)F[\tfrac{1}{|\omega|}\int_\Omega u(x,h(t,\chi))dx]d\sigma(\chi)\\ &\le -a(t)\int_{\partial\Omega} r(x,t)u(x,t)dS-\sum_{j=1}^n a_j(t)\int_{\partial\Omega} r(x,\tau_j(t))u(x,\tau_j(t))dS\le 0, t\ge t_1.\end{aligned}$$

Therefore, the function

$$V(t)=\frac{1}{|\Omega|}\int_\Omega u(x,t)dx,\quad t\ge t_1$$

satisfies the following scalar ordinary neutral differential inequality

$$\frac{d}{dt}[V(t)-\sum_{i=1}^m\lambda_i(t)V(t-r_i)]+\int_\alpha^\beta \min_{x\in\bar\Omega} F[V(x,h(t,\chi))]d\sigma(\chi)\le 0,\ t\ge t_1. \tag{40}$$

By Theorem 3.1, (40) can not have an eventually positive solution, a contradiction to the fact that $V(t)>0$ for all $t\ge t_1$. This completes the proof.

For the purpose, we give the following

Example. Consider

$$\begin{aligned}&\tfrac{\partial}{\partial t}[u(x,t)-e^{-\pi}u(x,t-\tfrac{\pi}{2})-(1-e^{-\pi})u(x,t-\pi)]+(\tfrac{t}{2}+e^{-\pi})u(x,t)\\ &+\int_{-\pi}^{-\pi/2}3tu(x,t+\chi)d\chi\\ &=\tfrac{5}{2}t\tfrac{\partial^2}{\partial x^2}u(x,t-\tfrac{\pi}{2})+(e^{-\pi}+4t)\tfrac{\partial^2}{\partial x^2}u(x,t-\tfrac{3\pi}{2}),\\ &\qquad (x,t)\in(0,\pi)\times(0,\infty)\end{aligned} \tag{41}$$

subject the Dirichlet boundary condition

$$u(0,t)=u(\pi,t)=0,\ \ t>0. \tag{42}$$

It is easy to verify all hopotheses of Theorem 1.1 for this case, and thus all solutions of (41)-(42) are oscillatory.

Using Theorems 3.2-3.5, we can get other sets of sufficient conditions for all solutions of (1)-(2) or (1)-(3) to be oscillatory. We will only state the results, since the proof is similar to that for Theorem 1.1.

THEOREM 4.1. Assume that (4), (25) and (H0) are satisfied. In addition, we assume that

$$\rho:=\inf_{(t,s)\in R^+\times J,x\in\bar\Omega} q(x,t,x)>0 \tag{43}$$

and either

(H1**) σ is not right continuous at $C_i(i=0,1,\cdots\nu)$ with $\alpha<C_1<C_2<\cdots<C_\nu<\beta$ and $\liminf_{t\to\infty}\int_{g(t,\chi^*)}^t\int_{C_j}^{\chi^*}\min x\in\bar\Omega q(x,\eta,\chi)d\sigma(\chi)d\eta>0$ for $\chi^*\in(C_j,C_{j+1}]$, $j=0,\cdots,\nu$, where

$$C_0=\alpha,\ C_{\nu+1}=\beta,\ \sigma(\alpha-0)=\sigma(\alpha),\ \sigma(\beta+0)=\sigma(\beta); or$$

(H2**) $\liminf_{t\to\infty}\int_{h(t,\beta)}^t\int_\alpha^\beta \min_{x\in\bar\Omega} q(x,\eta,\chi)d\sigma(\chi)d\eta>\frac{1}{Ke}$.

Then every solution of (1)-(2) or (1)-(3) is oscillatory.

THEOREM 4.2. Assume that conditions (4), (25), (43), (H0) and (H2**) are satisfied. Also assume that

(H3)** $\liminf_{t\to\infty}\int_{g(t,\beta)}^{t}\int_{\alpha}^{\beta}\min_{x\in\bar\Omega} q(x,\eta,\chi)Q(\eta,\chi)d\sigma(\chi)d\eta > 0$.
Then every solution of (1)-(2) or (1)-(3) is oscillatory.

THEOREM 4.3. Assume that conditions (4), (25), (43), (H0) and (H1**) are satisfied. Then every solution of 91)-(2) or (1)-(3) is oscillatory if one of the following conditions holds:
(H4)** $\liminf_{t\to\infty}\int_{h(t,\alpha_i)}^{t}\int_{\alpha}^{\beta}\min_{x\in\bar\Omega} q(x,\eta,s)d\sigma_i(s)d\eta > \frac{1}{Ke}$ for some $i\in\{1,\cdots\nu\}$, here we assume σ is continuous at $\alpha=\alpha_0<\alpha_1<\alpha_2<\cdots<\alpha_\nu=\beta$ and set

$$\sigma_i(s)=\begin{cases}\sigma(s), & s\in[\alpha_{i-1},\alpha_i],\\ \sigma(\alpha_{i-1}), & s<\alpha_{i-1},\\ \sigma(\alpha_i), & s>\alpha_i.\end{cases}$$

(H5)** $\int_{\alpha}^{\beta}\ln[\int_{\alpha}^{\beta}\liminf_{t\to\infty}\int_{h(t,\chi)}^{t}K\min_{x\in\bar\Omega}(x,\eta,\theta)d\eta d\sigma(\chi)]d\sigma(\theta)+\int_{\alpha}^{\beta}d\sigma(s)>0$;
(H6)**

$$\int_{\alpha}^{\beta}\int_{\alpha}^{\beta}[\liminf_{t\to\infty}\int_{h(t,\chi)}^{t}K\min_{x\in\bar\Omega}(x,\eta,\chi)d\eta]^{\frac{1}{2}}$$
$$[\liminf_{t\to\infty}\int_{h(t,\chi)}^{t}K\min_{x\in\bar\Omega}(x,\eta,\theta)d\eta]^{\frac{1}{2}}d\sigma(\eta)d\sigma(\theta)$$
$$>\frac{1}{e}.$$

THEOREM 4.4. Assume that conditions (4), (15), (43), (H0), (H1**) and (H2**) are satisfied and that $\lambda<1$. Then every solution of (1)-(2) or (1)-(3) is oscillatory.

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