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A JÓZSEF ATTILA TUDOMÁNYEGYETEM KÖZLEMÉNYEI

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JÓZSEF ATTILA TUDOMÁNYEGYETEM BOLYAI INTÉZETE

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$\varkappa$-products of slender modules

JOHN DAUNS and LÁSZLÓ FUCHS

Throughout, modules will be right unital over an arbitrary, but fixed ring R.

Let $\varkappa$ be an infinite cardinal, and H_j $(j\in J)$ a set of R-modules. By their *$\varkappa$-product* $\Pi^{<\varkappa}\{H_j|j\in J\}$ is meant the submodule of the direct product $H=\Pi\{H_j|j\in J\}$ consisting of all the elements $h=(h_j)$ whose support supp $h=\{j\in J|h_j\neq 0\}$ has cardinality $<\varkappa$. (We shall write supp $K=\bigcup\{\text{supp}\, h|h\in K\}$ for a subset K of H.) An R-module A is called *slender* if for R-modules $e_iR\cong R$ $(i<\omega)$ and for any R-homomorphism

$$\varphi\colon \Pi\{e_iR|i<\omega\}\to A$$

we have $\varphi e_i=0$ for almost all i. Slender modules behave in many respects like slender abelian groups; cf. DIMITRIĆ [2]. Slender modules need not be torsion-free, not even over commutative domains [3, p. 77].

In this note, our purpose is to investigate properties of $\varkappa$-products of slender modules over arbitrary rings. We shall concentrate on the problem of homomorphisms η of a product $\Pi\{G_i|i\in I\}$ of R-modules G_i with non-measurable index set I into the $\varkappa$-product $\Pi^{<\varkappa}\{H_j|j\in J\}$ of slender modules H_j. A generalization of a well-known theorem by Łoś [3, p. 52] guarantees that, for each $j\in J$, only finitely many ηG_i have nonzero projections in H_j, but virtually nothing is known about the global behavior of such a homomorphism η.

We study both the kernel and the image of η. Easy examples show that meaningful results on the image of η can only be obtained if the modules G_i are not too large: more precisely, if they can be generated by fewer elements than the cofinality cof $\varkappa$ of $\varkappa$. We shall prove the following theorem which also generalizes the well-known result that direct sums of slender modules are slender [3, p. 77].

Theorem. *Let I be a non-measurable index set and $\varkappa$ an infinite cardinal. Assume G_i $(i\in I)$ are R-modules each of which can be generated by strictly less than* $\lambda=\text{cof}\,\varkappa$

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elements, and H_j $(j\in J)$ *are slender R-modules. Given an R-homomorphism*

(1) $$\eta\colon G=\prod\{G_i\,|\,i\in I\}\to H=\prod^{<\varkappa}\{H_j\,|\,j\in J\},$$

define

(2) $$X=\{i\in I\,|\,\eta G_i\neq 0\}\quad\text{and}\quad Y=\bigcup\{\operatorname{supp}\eta G_i\,|\,i\in I\}.$$

Then we have:

(A) $|X|<\varkappa$;

(B) $\Pi\{G_i\,|\,i\in I\setminus X\}\leqq\ker\eta$;

(C) $|Y|<\varkappa$;

(D) $\operatorname{Im}\eta\leqq\Pi\{H_j\,|\,j\in Y\}\leqq H$.

For $\varkappa$ regular, conclusions (A) and (B) are actually proved without requiring the G_i to be less than $\operatorname{cof}\varkappa$ generated. The proof requires a more sophisticated argument if $\varkappa$ is a singular cardinal. We break down the proof into several lemmas and propositions dealing with portions of the Theorem.

For recent work on products of slender modules, and for applications of $\varkappa$-products in ring and module theory, see [3], [7], [8] and [1], as well as the literature quoted there.

1. Preliminaries

The symbols $\varkappa, \lambda$ will denote infinite cardinals (or ordinals); $\varkappa^+$ denotes the successor cardinal of $\varkappa$, and $|X|$ stands for the cardinality of a set X. For an R-module G, gen G means the minimum cardinality of generating sets of G. Here a cardinal $\varkappa$ is *measurable* if there exists a non-principal ultrafilter on $\varkappa$ which is closed under countable intersections. For the set theoretical concepts and results needed here, we refer to Jech [5; p. 27—28; p. 52].

By making use of [3, p. 52], the proof of the main theorem of Łoś on slenderness ([4; p. 161, Theorem 94.4]) can be modified so as to hold for slender R-modules, rather than for abelian groups.

1.1. J. Łoś Theorem. *Let I be a non-measurable index set and A a slender R-module. For any R-homomorphism $\varphi\colon\Pi\{G_i|i\in I\}\to A$ where the G_i are arbitrary R-modules we have:*

(i) $\varphi G_i=0$ *for almost all* $i\in I$; *and*

(ii) *if* $\varphi G_i=0$ *for all* $i\in I$, *then* $\varphi=0$.

It is straightforward to check that (1.1) continues to hold if the direct product ΠG_i is replaced by the $\varkappa$-product $\Pi^{<\varkappa}G_i$ with uncountable non-measurable $\varkappa$.

Using coordinate-wise arguments, we can at once derive the following corollary.

1.2. Corollary. *Let*

$$\eta\colon G = \Pi\{G_i \mid i\in I\} \to H = \Pi\{H_j \mid j\in J\} \tag{3}$$

be an R-homomorphism where the G_i are R-modules, all H_j are slender, and the index set I is non-measurable. If $\eta G_i = 0$ for each $i\in I$, then $\eta = 0$.

From now on we assume that the index sets are infinite.

In order to compare homomorphisms into products with those into $\varkappa$-products, we include the following result.

1.3. Proposition. *Assume the hypotheses of* (1.2). *Then the subset $X = \{i\in I \mid \eta G_i \neq 0\}$ of I satisfies:*

(i) $|X| \leqq |J|$, *and*

(ii) η *vanishes on* $\Pi\{G_i \mid i\in I\setminus X\}$.

Proof. Let $\varrho_j\colon H \to H_j$ be the jth coordinate projection. By the definition of slenderness, for each $j\in J$, the set

$$f(j) = \{i\in I \mid \varrho_j \eta G_i \neq 0\} \tag{4}$$

is finite. Evidently, $\bigcup\{f(j) \mid j\in J\} = X$ whence (i) is obvious. Since $\eta G_i = 0$ for all $i\in I\setminus X$, (ii) follows immediately from (1.2).

To facilitate proofs, we state here a lemma the proof of which is an easy exercise in set theory.

1.4. Lemma. *Let I be a set of infinite cardinality $\varkappa$ and $\bar{\varkappa}$ a cardinal $<\varkappa$. Suppose that $\{F_j \mid j\in J\}$ is a set of finite subsets of I such that, for each $i\in I$, the cardinality of $\{j\in J \mid i\in F_j\}$ does not exceed $\bar{\varkappa}$. Assume that $|J| = \varkappa$, which holds in particular if $\bigcup\{F_j \mid j\in J\} = I$. Then there is a subset $S\subset J$ such that*

(a) $|S| = \varkappa$;

(b) *the sets F_j $(j\in S)$ are pairwise disjoint.*

Note. If $\varkappa$ is regular, and in particular, weakly inaccessible, then it suffices to assume $|\{j\in J \mid i\in F_j\}| < \varkappa$ for each $i\in I$ to obtain (a)—(b).

2. Maps into $\varkappa$-products, regular $\varkappa$

In this section, we assume $\varkappa$ is a regular cardinal. Our first concern is the kernel of homomorphisms of products of modules into the $\varkappa$-products of slender modules. The following theorem gives fairly complete information about the kernel. (The restriction on gen G_i is not required for regular cardinals $\varkappa$.)

2.1. Proposition. *Let G_i $(i\in I)$ be R-modules, H_j $(j\in J)$ slender R-modules, and $|I|$ a non-measurable cardinal. Let* (1) *be an R-homomorphism where $\varkappa$ is a regular cardinal. Then* (*A*) *and* (*B*) *of Theorem hold.*

Proof. As (B) is a consequence of (1.2), only (A) requires a verification.

By way of contradiction, suppose that (A) is false. Without loss of generality, we may then assume that $X=I$ has cardinality $\varkappa$ (in particular, $\varkappa$ is non-measurable) and $G_i=g_iR$ are nonzero cyclic R-modules.

As in the proof of (1.3), we form the sets $f(j)$ [cf. (4)] which are finite for each $j\in J$. Setting $Y=\bigcup\{\operatorname{supp}\eta g_i|i\in I\}$, we have $I=\bigcup\{f(j)|j\in Y\}$ because of $X=I$. The finiteness of the $f(j)$ and $|I|=\varkappa$ imply $|Y|=\varkappa$. Since $\varrho_j\eta g_i=0$ for $j\in J\setminus Y$ and every $i\in I$, we may assume $Y=J$ and $f(j)\neq\emptyset$ for each $j\in J$.

The next step in our proof is to select a subset S of J such that the finite subsets $f(j)$ $(j\in S)$ are pairwise disjoint and $|S|=\varkappa$. This can be done with the aid of (1.4) (where $\bar{\varkappa}$ is the immediate predecessor of $\varkappa$ if such an ordinal exists; otherwise no such $\bar{\varkappa}$ is needed).

For each $j\in S$, set $C_j=\oplus\{G_i|i\in f(j)\}\neq 0$. Manifestly, $G^*=\Pi\{C_j|j\in S\}$ is a summand of G and $H^*=\Pi^{<\varkappa}\{H_j|j\in S\}$ is a summand of H. The restriction of η to G^* followed by the projection $H\to H^*$ yields a map $\eta^*\colon G^*\to H^*$ such that, for each $j\in S$, $0\neq\eta C_j\leqq H_j$. For every $j\in S$, pick a $c_j\in C_j$ satisfying $\eta^*c_j\neq 0$, and let

$$c=(\ldots,c_j,\ldots)\in G^* \quad (j\in S).$$

In view of $\varrho_j\eta^*C_k=0$ for all $j\neq k$ in S, the slenderness of H_j implies $\varrho_j\eta^*(c-c_j)=$ $=0$ for every $j\in S$ (recall the non-measurability of $\varkappa$). Consequently, $\varrho_j\eta^*c=$ $=\varrho_j\eta^*c_j\neq 0$ for all $j\in S$, contradicting the fact that the support of η^*c must have cardinality $<\varkappa$.

We turn our attention to the question as to when the image of η in (1) has to be contained in the $\varkappa$-product of a smaller subset of the H_j.

It is readily seen that some sort of restriction on the G_i is necessary in order to obtain such a conclusion. In fact, if one of the G_i's is the direct sum $\bigoplus_{j\in J} H_j$ and η acts on each H_j as the identity map, then the $\varkappa$-product of the H_j over the entire index set J is needed to accomodate $\operatorname{Im}\eta$. This example also shows that it won't be of any help to assume the slenderness of the G_i's. Cardinality restrictions on the G_i seem to be inevitable.

Accordingly, let us assume gen $G_i<\varkappa$ for each $i\in I$. If $\varkappa$ is a regular cardinal, then $|\operatorname{supp}\eta G_i|<\varkappa$. Keeping this in mind, we prove:

2.2. Proposition. *Let G_i $(i\in I)$ be R-modules with* gen $G_i<\varkappa$ *where I is non-measurable. If* (1) *is an R-homomorphism with H_j slender and $\varkappa$ regular, then both* (C) *and* (D) *of Theorem hold true.*

Proof. (2.1) shows that η acts non-trivially only on a subproduct $\Pi\{G_i|i\in X\}$ where $|X|<\varkappa$. By the regularity of $\varkappa$, likewise $Y=\bigcup\{\operatorname{supp}\eta G_i|i\in X\}$ has cardinality less than $\varkappa$. Assertion (D) is an obvious consequence of (C).

3. Maps into $\varkappa$-products, singular $\varkappa$

In this section, $\varkappa$ denotes a singular cardinal.

Let us start with a weak version of (2.1). Viewing (3) as a map into the $\varkappa^+$-product of the H_j, we derive:

3.1. Corollary. *Under the hypotheses of* (2.1), *but assuming $\varkappa$ is singular, we have:*

(i') $|X|\leqq\varkappa$, *and*

(ii') η *vanishes on* $\Pi\{G_i|i\in I\setminus X\}$.

We shall improve on (3.1) by limiting the sizes of the generating systems of the G_i.

The analogue of (2.2) fails if $\varkappa$ is singular, even if I is restricted to have cardinality $\lambda=\operatorname{cof}\varkappa$ — as is shown by the following example.

Let $\{J_\alpha|\alpha<\lambda\}$ be a set of pairwise disjoint subsets of J such that $|J_\alpha|<\varkappa$ for all $\alpha<\lambda$ and $\sup|J_\alpha|=\varkappa$. Let $G_\alpha=\oplus\{H_j|j\in J_\alpha\}$, $G=\Pi\{G_\alpha|\alpha<\lambda\}$ and $\eta\colon G\to H=\Pi^{<\varkappa}\{H_j|j\in J\}$ be induced by the identity maps on the H_j. This η exists (Im η is already in the λ^+-product of the H_j) and provides a counterexample.

Our best bet is cutting down the sizes of G_i to below λ. This enables us to obtain reasonably strong results. The point of departure is the following.

3.2. Lemma. *Let $\varkappa$ be a singular cardinal, I a non-measurable index set of cardinality* $\leqq\lambda=\operatorname{cof}\varkappa$, *and* H_j $(j\in J)$ *a family of slender modules. If* $\operatorname{gen} G_i<\lambda$ *for all the R-modules G_i, then for any R-homomorphism* (1) *conclusions* (C) *and* (D) *of Theorem hold true.*

Before entering into the proof of (3.2), we prove two auxiliary lemmas. In the next lemma, $\varkappa$ can be any infinite cardinal.

3.3. Lemma. *Suppose μ is a non-measurable ordinal $<\varkappa$ and*

$$\eta\colon G=\Pi\{G_\alpha|\alpha<\mu\}\to H=\Pi^{<\varkappa}\{H_j|j\in J\}$$

is an R-homomorphism where $G_\alpha=g_\alpha R$ are non-zero cyclic modules and H_j are slender. If $\varkappa_0$ is a cardinal number satisfying

$$\mu\leqq\varkappa_0<|\operatorname{supp}\eta g_0|,$$

then there exist a subset Y of J and an ordinal $\beta<\mu$ such that

(a) $Y \subset \operatorname{supp} \eta g_0$;
(b) $|Y| > \varkappa_0$;
(c) $Y \cap \operatorname{supp} \eta g_\alpha = \emptyset$ *for all* $\alpha \geqq \beta$.

Proof. Let ϱ_j denote the jth coordinate projection $H \to H_j$. Define a function $\psi: J \to \mu$ by letting $\psi(j)$ be the smallest ordinal $\gamma < \mu$ such that

$$\varrho_j \eta \Pi \{g_\alpha R | \gamma \leqq \alpha < \mu\} = 0.$$

Owing to the slenderness of H_j, such a $\psi(j)$ exists, so ψ is well-defined. For $\alpha < \mu$, we set

$$Y_\alpha = \{j \in \operatorname{supp} \eta g_0 \,|\, \psi(j) = \alpha\}. \tag{5}$$

Visibly, the Y_α are pairwise disjoint and their union for $\alpha < \mu$ is exactly $\operatorname{supp} \eta g_0$. Consequently,

$$\varkappa_0 < |\operatorname{supp} \eta g_0| = \sum_{\alpha < \mu} |Y_\alpha| = \max\{\mu, \sup_{\alpha < \mu} |Y_\alpha|\}.$$

Hence $\mu \leqq \varkappa_0$ implies $\varkappa_0 < \sup |Y_\alpha|$ which means that $|Y_\beta| > \varkappa_0$ for a suitable ordinal $\beta < \mu$. This β and $Y = Y_\beta$ are as desired.

The next lemma is more technical.

3.4. Lemma. *Assume $\varkappa$ is a singular cardinal, $\lambda = \operatorname{cof} \varkappa$ is non-measurable, G_α are non-zero cyclic and H_j are slender. If there are cardinals $\varkappa_\alpha (\alpha < \lambda)$ satisfying*
(a) $\varkappa_\alpha < |\operatorname{supp} \eta g_\alpha|$ *for* $\alpha < \lambda$;
(b) $\lambda \leqq \varkappa_0 < \varkappa_1 < \ldots < \varkappa_\alpha < \ldots \quad (\alpha < \lambda)$;
(c) $\sup \varkappa_\alpha = \varkappa$,
then there exist subsets J_α of J and ordinals $\mu(\alpha) < \lambda$ for all $\alpha < \lambda$ such that
(i) $\mu(0) < \mu(1) < \ldots < \mu(\alpha) < \ldots$;
(ii) $J_\alpha \subset \operatorname{supp} \eta g_{\mu(\alpha)}$;
(iii) $|J_\alpha| > \varkappa_\alpha$ *for* $\alpha < \lambda$;
(iv) *the sets* $J_0, J_1, \ldots, J_\alpha, \ldots$ *are pairwise disjoint.*

Proof. We take advantage of the function $\psi: J \to \lambda$ defined in the proof of (3.3), and in addition to (i)—(iv) we also require that J_α be of the form Y_β as in (5). More precisely, we impose an additional condition:

(v) $J_\alpha = \{j \in \operatorname{supp} \eta g_{\mu(\alpha)} | \psi(j) = \beta(\alpha)$ for some $\beta(\alpha) < \lambda\}$.

Right away we note that (v) implies

(vi) $\varrho_j \eta g_\delta = 0$ holds for every $j \in J_0 \cup \ldots \cup J_\alpha$ and for every $\delta > \operatorname{supp} \{\mu(\alpha), \beta(0), \ldots$ $\ldots, \beta(\alpha)\}$.

The J_α and $\mu(\alpha)$ will be constructed by transfinite induction. To start off, put $\mu(0) = 0$. Application of (3.3) yields a subset $Y \subset J$ and an ordinal $\beta(0) < \lambda$ satisfying (a)—(c) of (3.3) as well as (v). Define $J_0 = Y_{\beta(0)}$. Then for $\alpha = 0$, all of (i)—(v) hold.

Let $\gamma<\lambda$, and suppose that $\mu(\alpha)<\lambda$ and $J_\alpha\subset J$ have been selected for all $\alpha<\gamma$ satisfying conditions (i)—(v) for indices $<\gamma$. Define $\mu(\gamma)$ to be any ordinal $<\lambda$ exceeding $\mu(\alpha)$ and $\beta(\alpha)$ for all $\alpha<\gamma$; since $\lambda=\operatorname{cof}\varkappa$, such a $\mu(\gamma)$ does exist. Apply (3.3) to $g_{\mu(\gamma)}$ playing the role af g_0 and $\varkappa_\gamma$ the role of $\varkappa_0$, in order to obtain a set $Y=\{j\in\operatorname{supp}\eta g_{\mu(\gamma)}|\psi(j)=\delta\}$ for some $\delta<\lambda$ and an ordinal $\beta(\gamma)$ as stipulated by (3.3) (a)—(c). Setting $J_\gamma=Y$ and $\beta(\gamma)=\delta$ (see the proof of (3.3)), conditions (i)—(iii) and (v) will clearly hold for all indices $\leqq\gamma$. To convince ourselves that $J_\gamma\cap J_\alpha=\emptyset$ for every $\alpha<\gamma$, it suffices to note that (vi) implies $\varrho_j\eta e_{\mu(\gamma)}=0$ for every $j\in J_\alpha$. This completes the proof of (3.4).

Proof of (3.2). Since $\operatorname{gen} G_i<\lambda$ implies $|\operatorname{supp}\eta G_i|<\varkappa$, the assertion follows at once whenever $|I|<\lambda$. So let us assume $|I|=\lambda$ in which case we can think of I as consisting of the ordinals $<\lambda$.

From (1.2) we infer that $Y=\bigcup\{\operatorname{supp}\eta G_\alpha|\alpha<\lambda\}$ is the smallest subset of J with the property $\eta G\leqq\Pi^{<\varkappa}\{H_j|j\in Y\}$. Hence $|Y|\leqq\varkappa$ is immediate. By way of contradiction assume that $|Y|=\varkappa$.

Passing to a summand of G, we may assume that the cardinal numbers $|\operatorname{supp}\eta G_\alpha|$ are all different and $>\lambda$. Reindexing, we obtain an ascending chain

$$\lambda < |\operatorname{supp}\eta G_0| < |\operatorname{supp}\eta G_1| < \ldots < |\operatorname{supp}\eta G_\alpha| < \ldots \quad (\alpha < \lambda)$$

whose supremum is $\varkappa$. Since $\operatorname{gen} G_{\alpha+1}<\lambda$ and $|\operatorname{supp}\eta G_{\alpha+1}|>|\operatorname{sup}\eta G_\alpha|$, there must be a generator $g_\alpha\in G_{\alpha+1}$ whose image ηg_α has support of cardinality strictly $>|\operatorname{supp}\eta G_\alpha|$. Setting $\varkappa_\alpha=|\operatorname{supp}\eta G_\alpha|$, we obtain an ascending chain of cardinals,

$$\lambda = \varkappa_0 < \varkappa_1 < \ldots < \varkappa_\alpha < \ldots \quad (\alpha < \lambda)$$

with $\sup\varkappa_\alpha=\varkappa$, and with $\varkappa_\alpha<|\operatorname{supp}\eta g_\alpha|$ for all $\alpha<\lambda$.

Restricting η to the submodule $\bar{G}=\Pi\{g_\alpha R|a<\lambda\}$, (3.4) yields the existence of subsets $J_\alpha\subset J$ and ordinals $\mu(\alpha)<\lambda$ satisfying (i)—(iv) of (3.4). Define an element $\bar{g}=(\bar{g}_\alpha)_{\alpha<\lambda}\in\bar{G}$ as follows. Let $\bar{g}_\alpha=g_{\mu(\beta)}$ if $\alpha=\mu(\beta)$ for some $\beta<\lambda$, and let $\bar{g}_\alpha=0$ otherwise. From the definition of the $\mu(\beta)$ it is clear that $\varrho_j\eta(\bar{g}-g_{\mu(\alpha)})=0$ for all $j\in J_\alpha$. Hence $\varrho_j\eta\bar{g}=\varrho_j\eta g_{\mu(\beta)}\neq0$ for $j\in J_\alpha$, and we conclude that

$$|\operatorname{supp}\eta\bar{g}| \geqq \bigcup\{J_\alpha|\alpha < \lambda\} = \sup\varkappa_\alpha = \varkappa.$$

This contradiction completes the proof of (3.2).

We still need the following lemma.

3.5. Lemma. *Let $\varkappa$ be a non-measurable singular cardinal, and I an index set of cardinality $\varkappa$. If G_i are R-modules with* $\operatorname{gen} G_i<\lambda=\operatorname{cof}\varkappa$, *and if* (1) *is an R-homomorphism with H_j slender, then there exists a cardinal $\bar{\varkappa}<\varkappa$ such that, for each $i\in I$, the set* $\operatorname{supp}\eta G_i$ *has cardinality* $\leqq\bar{\varkappa}$.

Proof. First observe that gen $G_i<\lambda$ and $|\operatorname{supp}\eta g_i|<\varkappa$ for each $g_i\in G_i$ imply that $|\operatorname{supp}\eta G_i|<\varkappa$. Denying the existence of a $\bar{\varkappa}$ of the indicated type means that we can select a subset $\{G_\alpha|\alpha<\lambda\}$ of $\{G_i|i\in I\}$ such that the cardinalities $\varkappa_\alpha=$ $=|\operatorname{supp}\eta G_\alpha|$ form an increasing chain (with increasing α) whose supremum is $\varkappa$. The restriction of η to $G'=\Pi\{G_\alpha|\alpha<\lambda\}$ is a homomorphism satisfying the hypotheses of (3.2). Therefore, we conclude that $\eta G'$ has a support of cardinality $<\varkappa$, in contradiction to $\bigcup|\operatorname{supp}\eta G_\alpha|=\bigcup\varkappa_\alpha=\varkappa$.

Proof of Theorem. (2.1) and (2.2) take care of the case in which $\varkappa$ is a regular cardinal. So assume $\varkappa$ is singular.

(A) Without loss of generality, we may assume $|I|\geqq\varkappa$; otherwise there is nothing to prove. It suffices to verify (A) for $|I|=\varkappa$. By way of contradiction, assume $|X|=\varkappa$. We apply (1.4) to the set $\{f(j)|j\in J\}$ defined in (2) to obtain a subset $S\subset J$ of cardinality $\varkappa$ with $f(j)$ $(j\in S)$ pairwise disjoint; (3.5) assures the existence of a cardinal $\bar{\varkappa}$ needed in (1.4). Consider the following element of G: $\bar{g}=(g_i)$ where $g_i\in G_i$ with $\eta g_i\neq 0$ if $i\in\bigcup\{f(j)|j\in S\}$ and $g_i=0$ otherwise. An argument similar to the one used at the end of the proof of (2.1) leads us to the conclusion that $\eta\bar{g}$ must have a support of cardinality $\varkappa$ — a contradiction.

(B) follows from (A) in view of (1.2).

(C) Because of (3.5), we have $|\operatorname{supp}\eta G_i|\leqq\bar{\varkappa}<\varkappa$ for each i. This, together with (A), implies $|Y|<\varkappa$.

(D) is an immediate consequence of (C).

4. Embedding of μ-products in $\varkappa$-products

The case when the map η in (1) is a monomorphism deserves particular attention. In the following two corollaries, no restriction on gen G_i is needed.

4.1. Corollary. *Let $G_i\neq 0$ $(i\in I)$ and H_j $(j\in J)$ be R-modules, $|I|$ and $\varkappa$ non-measurable cardinals. If the H_j are slender and if, for some cardinal μ, there is a monomorphism*

$$\eta\colon G=\prod^{<\mu}\{G_i|i\in I\}\hookrightarrow H=\prod^{<\varkappa}\{H_j|j\in J\},$$

then either $|I|<\varkappa$ or $\mu\leqq\varkappa$.

Proof. If $|I|\geqq\varkappa$ and $\mu>\varkappa$, then G contains a submodule which is the product of $\varkappa$ cyclic submodules g_iR with $\eta g_i\neq 0$. This is impossible in view of (A), (B) in Theorem.

The next result is an immediate consequence of the preceding one. It generalizes a result on products and direct sums of slender groups, due to Łoś [6, p. 271].

4.2. Corollary. *Let both* G_i $(i \in I)$ *and* H_j $(j \in J)$ *be families of non-zero slender modules. If* $|I|$, $|J|$ *are non-measurable cardinals, and if* $\varkappa \leqq |J|$, $\mu \leqq |I|$, *then* $\Pi^{<\varkappa}\{G_i | i \in I\} \cong \Pi^{<\varkappa}\{H_j | j \in J\}$ *implies* $\varkappa = \mu$.

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DEPARTMENT OF MATHEMATICS
TULANE UNIVERSITY
NEW ORLEANS, LA 70118, U.S.A.

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Idempotent algebras with transitive automorphism groups

LÁSZLÓ SZABÓ

To Professor Béla Csákány on his 60th birthday

0. Introduction

As a rule a finite algebra with "large" automorphism group is functionally complete. The first general result was found by B. CSÁKÁNY [1], who proved that almost every nontrivial homogeneous algebra (i.e. an algebra whose automorphism group is the full symmetric group) is functionally complete; up to equivalence there are six exceptions. Csákány's theorem was first extended to algebras with triply transitive automorphism groups [9] and later to algebras with doubly transitive automorphism groups [4]; the exceptions are the affine spaces over finite fields. The most general result in this direction is proved in [5] where the structure of functionally incomplete algebras with primitive automorphism groups are completely discribed.

In this paper we investigate finite idempotent algebras with transitive automorphism groups. We show that if an at least three element finite idempotent algebra with transitive automorphism group is simple and has no compatible binary central relation then it is either functionally complete or affine (Theorem 3.1). Moreover, if an at least three element finite idempotent algebra with transitive automorphism group is simple and has a nontrivial semi-projection or a majority function among its term functions then it is functionally complete (Theorem 3.2).

1. Preliminaries

Let A be a fixed universe with $|A|>2$. For any positive integer n let $\mathbf{O}^{(n)}$ denote the set of all n-ary operations on A (i.e. maps $A^n \to A$) and let $\mathbf{O}=\bigcup_{n=1}^{\infty} \mathbf{O}^{(n)}$. An operation from $\mathbf{O}$ is *nontrivial* if it is not a projection. By a *clone* we mean a subset

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of $\mathbf{O}$ which is closed under superpositions and contains all projections. A ternary operation f on A is a *majority function* if for all $x, y \in A$ we have $f(x, x, y) = f(x, y, x) = f(y, x, x) = x$; f is a *Mal'tsev function* if $f(x, y, y) = f(y, y, x) = x$ for all $x, y \in A$. An n-ary operation t on A is said to be an *i-th semi-projection* $(n \geqq 3, 1 \leqq i \leqq n)$ if for all $x_1, \ldots, x_n \in A$ we have $t(x_1, \ldots, x_n) = x_i$ whenever at least two elements among $x_1, \ldots, x_n$ are equal.

A subset $F \subseteqq \mathbf{O}$ as well as the algebra (A, F) is *primal* or *complete* if the clone generated by F (i.e. the set of all term functions of (A, F)) is equal to $\mathbf{O}$; F as well as the algebra (A, F) is *functionally complete* if the clone generated by F together with all constant operations (i.e. the set of all algebraic functions of (A, F)) is equal to $\mathbf{O}$.

We are going to formulate Rosenberg's Completeness Theorem [6], [7] which is the main tool in proving our results. First, however, we need some further definitions.

Let $n, h \geqq 1$. An n-ary operation $f \in \mathbf{O}^{(n)}$ is said to *preserve the h-ary relation* $\varrho \subseteqq A^h$ if ϱ is a subalgebra of the h-th direct power of the algebra $(A; f)$; in other words, f preserves ϱ if for any $n \times h$ matrix with entries in A, whose rows belong to ϱ, the row of column values of f belong to ϱ. Then the set of operations preserving ϱ forms a clone, which is denoted by Pol ϱ. We say that a relation ϱ is a *compatible relation* of the algebra (A, F) if $F \subseteqq \mathrm{Pol}\, \varrho$. A binary relation is called *nontrivial* if it is distinct from the identity relation and the full relation.

An h-ary relation ϱ on A is called *central* if $\varrho \neq A^h$ and there exists a non-void proper subset C of A such that

(a) $(a_1, \ldots, a_h) \in \varrho$ whenever at least one $a_i \in C$ $(1 \leqq i \leqq h)$;

(b) ϱ is *totally symmetric*, i.e. $(a_1, \ldots, a_h) \in \varrho$ implies $(a_{1\pi}, \ldots, a_{h\pi}) \in \varrho$ for every permutation π of the indices $1, \ldots, h$;

(c) ϱ is *totally reflexive*, i.e. $(a_1, \ldots, a_h) \in \varrho$ if $a_i = a_j$ for some $i \neq j$ $(1 \leqq i, j \leqq h)$.

The set C is called the *center of* ϱ.

Let $h \geqq 3$. A family $T = \{\Theta_1, \ldots, \Theta_m\}$ $(m \geqq 1)$ of equivalence relations on A is called *h-regular* if each Θ_i $(1 \leqq i \leqq m)$ has exactly h blocks and $\Theta_T = \Theta_1 \cap \ldots \cap \Theta_m$ has exactly h^m blocks (i.e. the intersection $\bigcap_{i=1}^{m} B_i$ of arbitrary blocks B_i of Θ_i $(i = 1, \ldots, m)$ is nonempty). The relation determined by T is

$$\lambda_T = \{(a_1, \ldots, a_h) \in A^h \colon a_1, \ldots, a_h \text{ are not pairwise incongruent modulo } \Theta_i \text{ for all } i \ (1 \leqq i \leqq m)\}.$$

Note that h-regular relations are both totally reflexive and totally symmetric.

Now we are in the position to state Rosenberg's Theorem:

Theorem A (I. G. ROSENBERG [6], [7]). *A finite algebra* $\mathbf{A}=(A, F)$ *is primal if and only if* $F \subseteq \operatorname{Pol} \varrho$ *for no relation of any of the following six types:*

(1) *a bounded partial order;*

(2) *a binary relation* $\{(a, a\pi) | a \in A\}$ *where* π *is a permutation of* A *with* $|A|/p$ *cycles of the same length* p (p *is a prime number*);

(3) *a quaternary relation* $\{(a_1, a_2, a_3, a_4) \in A^4 | a_1+a_2=a_3+a_4\}$ *where* $(A; +)$ *is an elementary abelian p-group (p is a prime number);*

(4) *a nontrivial equivalence relation;*

(5) *a central relation;*

(6) *a relation determined by an h-regular family of equivalence relations.*

Let B be a finite set with $|B| \geqq 3$, and let $m>1, n \geqq 1$, $M=\{1, \ldots, m\}$, $N=\{1, \ldots, n\}$. An n-ary *wreath operation* on B^m is an operation w associated to permutations p_i of B $(i=1, \ldots, m)$, and maps r: $M \to N$, s: $M \to M$, as follows: For $x_i=(x_{i1}, \ldots, x_{im}) \in B^m$, $i=1, \ldots, n$ set

$$w(x_1, \ldots, x_n) = \big(p_1(x_{r(1)s(1)}), \ldots, p_m(x_{r(m)s(m)})\big).$$

Now an algebra is a *wreath algebra* if it is isomorphic to an algebra on B^m with wreath operations only.

In [8] I. G. ROSENBERG gave a functional completeness criterion for finite algebras whose operations are all surjective. Among others he proved the following:

Theorem B (I. G. ROSENBERG [8]). *Let* $\mathbf{A}$ *be a finite algebra whose operations are all surjective.*

(i) *If* $\mathbf{A}$ *has a compatible at least binary central relation then it also has a compatible binary central relation.*

(ii) *If* $\mathbf{A}$ *has an operation depending on at least two variables,* $\mathbf{A}$ *is simple and has a compatible relation determined by an h-regular family of equivalence relations then it is a wreath algebra.*

An algebra (A, F) is said to be *affine* with respect to an elementary abelian group $(A; +)$ if it has a compatible relation of type (3) in Theorem A determined by $(A: +)$. To any finite field K and natural number n we associate the following affine algebra:

$$\mathbf{A}_{K,n} = \big(K^n; x-y+z, \{rx+(1-r)y: r \in K_{n \times n}\}\big)$$

where $K_{n \times n}$ is the $n \times n$ matrix ring over K.

Theorem C (Á. SZENDREI [10]). *Let* $\mathbf{A}$ *be an at least three element simple finite idempotent algebra. If* $\mathbf{A}$ *is affine with respect to an elementary abelian group then it is equivalent to* $\mathbf{A}_{K,n}$ *for some finite field* K *and* $n \geqq 1$.

2. Lemmas

Lemma 2.1. *An idempotent wreath algebra cannot be simple.*

Proof. Let B be a finite set with $|B|\geqq 3$, $m>1$, and consider an n-ary wreath operation w on B^m associated to permutations p_i of B $(i=1, \ldots, m)$, and maps $r\colon M\to N$, $s\colon M\to M$ $(M=\{1, \ldots, m\}, N=\{1, \ldots, m\})$. It is easy to check that w is idempotent if and only if each p_i is the identity permutation on B, $i=1, \ldots, m$, and s is the identity permutation on M, i.e., for $x_i=(x_{i1}, \ldots, x_{im})\in B^m$, $i=1, \ldots, n$ we have

$$w(x_1, \ldots, x_n) = (x_{r(1)1}, \ldots, x_{r(m)m}).$$

Then w preserves the equivalence relations Θ_j $(j=1, \ldots, m)$ defined by

$$\Theta_j = \{((a_1, \ldots, a_m), (b_1, \ldots, b_m))\in(B^m)^2\colon a_j = b_j\},$$

Lemma 2.2. *If an at least three element finite algebra with transitive automorphism groups has a compatible bounded partial order then it has a nontrivial compatible binary reflexive and symmetric relation.*

Proof. *Let* $\mathbf{A}=(A, F)$ be an at least three element finite algebra with transitive automorphism group and let ϱ be a compatible bounded partial order of $\mathbf{A}$ with least element 0 and greatest element 1. Choose an automorphism π of $\mathbf{A}$ with $1\pi\neq 0, 1$. Then the relation $\sigma=(\varrho\cap\varrho\pi)\circ(\varrho\cap\varrho\pi)^{-1}$, where $\varrho\pi=\{(x\pi, y\pi)\colon (x, y)\in\varrho\}$, is a compatible binary reflexive and symmetric relation of A. Furthermore, σ is nontrivial, since $(0, 1\pi)\in\varrho\cap\varrho\pi\subseteqq\sigma$ and $(1,1\pi)\notin\sigma$.

Lemma 2.3. *If an at least three element finite algebra has a nontrivial compatible binary reflexive and symmetric relation then it has either a nontrivial congruence relation, or a compatible at least binary central relation, or a compatible relation determined by an h-regular family of equivalence relations.*

Proof. Let $\mathbf{A}=(A, F)$ be an at least three element finite algebra and let σ be a nontrivial binary reflexive and symmetric relation on A with $(a, b)\in\sigma$, $a\neq b$. Suppose that σ is a compatible relation of $\mathbf{A}$, i.e. $F\subseteqq \operatorname{Pol}\sigma$. Since σ is nontrivial we have $\operatorname{Pol}\sigma\neq\mathbf{O}$. Therefore, by Theorem A, there is a relation ϱ of one of the types (1), ..., (6) such that $\operatorname{Pol}\sigma\subseteqq\operatorname{Pol}\varrho$. Clearly, ϱ is a compatible relation of $\mathbf{A}$. We have to show that ϱ is of type (4) or (6) or an at least binary relation of type (5). Since $\operatorname{Pol}\varrho$ contains all constant operations, it cannot be of type (2) or a unary central relation. Suppose that ϱ is a bounded partial order with least element 0 and greatest element 1. Consider the unary operations f and g defined by $f(0)=a$, $f(x)=b$ if $x\neq 0$ and $g(1)=a$, $g(x)=b$ if $x\neq 1$. Then $f, g\in\operatorname{Pol}\sigma\subseteqq\operatorname{Pol}\varrho$. Therefore $(a, b)=(f(0), f(1))\in\varrho$ and $(b, a)=(g(0), g(1))\in\varrho$, a contradiction. Fi-

nally suppose that ϱ is of type (3), and let $c \in A$ with $c \neq a, b$. Consider the unary operation h defined by $h(a)=a$ and $h(x)=b$ if $x \neq a$. Then $h \in \operatorname{Pol} \varrho$, and $a+b-c \neq a$ as $c \neq a, b$. Therefore, $(a, b, c, a+b-c) \in \varrho$ implies $(a, b, b, b)=(h(a), h(b), h(c), h(a+b-c)) \in \varrho$, a contradiction.

3. Results and proofs

Theorem 3.1. *Let* **A** *be an at least three element finite idempotent algebra with transitive automorhism group. If* **A** *is simple and has no compatible binary central relation then it is either functionally complete or is equivalent to* $\mathbf{A}_{K,n}$ *for some finite field K and natural number n.*

Proof. Let **A** be a simple at least three element finite idempotent algebra with transitive automorphism group, and assume that A has no compatible binary central relation. If **A** is functionally incomplete then, by Theorem A, there is a relation ϱ of one of the types (1), ..., (6) such that Pol ϱ contains all algebraic functions of **A**. Since Pol ϱ contains all constant operations and **A** is simple, ϱ cannot be of type (2), (4) or a unary central relation. If ϱ is of type (6) then, by Theorem B, **A** is a wreath algebra and then, by Lemma 2.1, we have that **A** is not simple contrary to our assumption. If ϱ is an at least binary central relation then, again by Theorem B, **A** has a compatible binary central relation contrary to our assumption. Finally, if ϱ is a bounded partial order then taking into consideration Lemma 2.2 and 2.3, we obtain that **A** has a nontrivial congruence relation or an at least binary central relation or a compatible relation of type (6), which is a contradiction.

Hence ϱ is of type (3), i.e. A is affine with respect to an elementary abelian group and then, by Theorem C, we have that **A** is equivalent to $\mathbf{A}_{K,n}$ for some finite field K and $n \geqq 1$.

It is well-known (see e.g. [5] and [9]) that every nontrivial idempotent algebra has either a majority function or a Mal'tsev function or a nontrivial semi-projection or a nontrivial binary idempotent operation among its term functions.

Theorem 3.2. *If an at least there element finite idempotent algebra with transitive automorphism group is simple and has a majority function or a nontrivial semi-projection among its term functions then it is functionally complete.*

Proof. Let $\mathbf{A}=(A, F)$ be an at least three element simple finite idempotent algebra with transitive automorphism group that have a majority function or a nontrivial semi-projection among its term functions. It is well known (see e.g. [5] or [9]) that neither majority functions nor nontrivial semi-projections preserve a relation of type (3) and therefore **A** is not affine. Using Theorem 3.1, we have to

show only that **A** has no compatible binary central relations. Suppose that **A** has a compatible binary central relation ϱ with center C and let $c \in C$.

First consider the case when **A** has an n-ary nontrivial semi-projection t among its term functions ($n \geqq 3$). We can suppose that t is a first semi-projection. We call a subset $I \subseteq A$ an ideal iff $t(a_1, \ldots, a_n) \in I$ whenever $a_1 \in I$. Since an intersection of ideals is an ideal again, we may speak about an ideal generated by a subset of A. For any $a \in A$ denote by $I(a)$ the ideal generated by $\{a\}$. Clearly, if I is an ideal and $\pi \in \mathrm{Aut}\,\mathbf{A}$ then $I\pi$ is again an ideal, and $I(a)\pi = I(a\pi)$. Because of the transitivity of Aut **A** the cardinalities of the 1-generated ideals are equal, and greater then one since t is not the first projection. So the 1-generated ideals form an Aut **A**-invariant partition of A. Denote by θ the corresponding equivalence relation. Then θ is distinct from the identity relation and $\mathrm{Aut}\,\mathbf{A} \subseteq \mathrm{Pol}\,\theta$.

We show that $\theta \subseteq \varrho$, i.e. for any $a, b \in A$ we have $(a, b) \in \varrho$ if $I(a) = I(b)$. Let $a, b \in A$ with $I(a) = I(b)$. Consider the subset $I_a = \{x\colon (x, a) \in \varrho\}$. Then I_a is an ideal. Indeed, if $x_1 \in I_a$ and $x_2, \ldots, x_n \in A$ are arbitrary elements, then (x_1, a), $(x_2, c), \ldots, (x_n, c) \in \varrho$ implies that $(t(x_1, \ldots, x_n), a) = (t(x_1, x_2, \ldots, x_n), t(a, c, \ldots, c)) \in \varrho$, i.e. $t(x_1, \ldots, x_n) \in I_a$. Now, since I_a is an ideal with $a \in I_a$, we have $b \in I(b) = I(a) \subseteq I_a$ and $(b, a) \in \varrho$. Hence $\theta \subseteq \varrho$.

Consider the subalgebra σ of $\mathbf{A}^2$ generated by θ. Then $\theta \subseteq \sigma \subseteq \varrho$ and $F \cup \mathrm{Aut}\,\mathbf{A} \subseteq \mathrm{Pol}\,\sigma$, i.e., σ is a nontrivial compatible binary reflexive and symmetric relation of the algebra $\hat{\mathbf{A}} = (A; F \cup \mathrm{Aut}\,\mathbf{A})$. Taking into consideration Lemma 2.3, we have that $\hat{\mathbf{A}}$ has either a nontrivial congruence relation or an at least binary central relation or a relation of type (6). The first case cannot occur since $\hat{\mathbf{A}}$ is simple. In the third case, according to Theorem B, we obtain that $\hat{\mathbf{A}}$ and so **C** is a wreath algebra which, by Lemma 2.1, implies that **A** is not simple, a contradiction. In the second case let τ be an h-ary central relation of $\hat{\mathbf{A}}$, let u be an element in the center of τ, and let $a_1, \ldots, a_h \in A$ be arbitrary elements. Choose a $\pi \in \mathrm{Aut}\,\mathbf{A}$ such that $u\pi = a_1$. Then $(u, a_2\pi^{-1}, \ldots, a_h\pi^{-1}) \in \tau$ implies that $(a_1, \ldots, a_h) = (u\pi, (a_2\pi^{-1})\pi, \ldots, (a_h\pi^{-1})\pi) \in \tau$. Hence τ is the full relation A^h, which is a contradicton. This completes the proof in the case when **A** has a nontrivial semi-projection among its term functions.

Now consider the case when **A** has a majority term function d. From now on we call a subset $I \subseteq A$ an ideal iff $d(x, y, z) \in I$ whenever at least two of the arguments belong to I. A and the one-element subsets are obviously ideals. Since an intersection of ideals is an ideal again, we may speak about an ideal generated by a subset of A. For any $a \in A$ the set $I_a = \{x | (x, a) \in \varrho\}$ is an ideal. Indeed, if for example $x, y \in I_a$ and $z \in A$ is arbitrary element, then $(x, a), (y, a), (z, z) \in \varrho$ implies that $(d(x, y, z), a) = (d(x, y, z), d(a, a, z)) \in \varrho$, i.e. $d(x, y, z) \in I_a$. Clearly, if I is an ideal and $\pi \in \mathrm{Aut}\,\mathbf{A}$ then $I\pi$ is again an ideal.

Define a binary relation θ by setting $(a, b) \in \theta$ if and only if there is a minimal

ideal (i.e. an ideal properly containing one-element ideals only) containing a and b. Then θ is distinct from the identity relation and $\mathrm{Aut}\,\mathbf{A} \subseteq \mathrm{Pol}\,\theta$. We show that $\theta \subseteq \varrho$. Indeed, let $(a, b) \in \theta$. If $a = b$ then $(a, b) \in \varrho$, too. If $a \neq b$ then put $u = d(a, b, c)$ (c is a central element of ϱ) and let I be the minimal ideal with $a, b \in I$. Now $a = d(a, b, a)$, $b = d(a, b, b) \in I_u$. Since a and b are distinct, u is distinct from one of them, say $u \neq b$. By definition $u \in I$. We have $u, b \in I \cap I_b$, so by minimality of I, it follows that $I \subseteq I_b$, implying that $(a, b) \in \varrho$. Hence $\theta \subseteq \varrho$.

Consider the subalgebra σ of $\mathbf{A}^2$ generated by θ. Then $\theta \subseteq \sigma \subseteq \varrho$ and $F \cup \mathrm{Aut}\,\mathbf{A} \subseteq \mathrm{Pol}\,\sigma$, i.e., σ is a nontrivial compatible binary reflexive and symmetric relation of the algebra $\hat{\mathbf{A}} = (A; F \cup \mathrm{Aut}\,\mathbf{A})$. As we have seen above, this is impossible. This completes the proof in the case when A has a majority term function.

Theorem 3.3. *Every simple at least three element finite idempotent algebra with a Mal'tsev function among its term functions is either functionally complete or is equivalent to* $\mathbf{A}_{K,n}$ *for some finite* K *and natural number* n.

Proof. Let $\mathbf{A} = (A, F)$ be an at least three element simple finite idempotent algebra with a Mal'tsev function among its term functions. If $\mathbf{A}$ is functionally incomplete then, by the well-known Gumm—McKenzie Theorem (cf. e.g. in [2] and [3]) we have that $\mathbf{A}$ is affine. Finally apply Theorem C.

Problem. Is every at least three element finite simple idempotent algebra with transitive automorphism group either functionally complete or equivalent to $\mathbf{A}_{K,n}$ for some finite field K and natural number n?

As we have mentioned, every nontrivial idempotent algebra has either a majority function or a Mal'tsev function or a nontrivial semi-projection or a nontrivial binary idempotent operation among its term functions. Taking into consideration Theorem 3.2 and 3.3, the answer is positive if the algebra has either a majority function or a Mal'tsev function or a nontrivial semi-projection among its term functions. The remaining case is that, when the algebra has a nontrivial binary idempotent function and has neither a majority function nor a Mal'tsev function, nor a nontrivial semi-projection among its term functions.

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BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

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Central pattern functions

ENDRE VÁRMONOSTORY

To Professor Béla Csákány on his 60th birthday

A finite algebra $\mathfrak{A}$ with base set A is called *functionally complete* if every (finitary) operation on A is an algebraic function of $\mathfrak{A}$ (in GRÄTZER's sense [3]). WERNER [8] proved that every finite algebra $\langle A; t\rangle$ where t is the ternary discriminator function on A is functionally complete. FRIED and PIXLEY [2] showed that (in the case $|A|>2$) the algebra $\langle A; d\rangle$ with d the dual discriminator function on A is also functionally complete. The ternary discriminator and the dual discriminator are the most familiar examples of *pattern functions*. B. CSÁKÁNY [1] proved that for $|A|>2$ every finite algebra $\langle A; f\rangle$ where f is a non-trivial pattern function on A is functionally complete. B. Csákány suggested the following generalization of pattern function (see [6]). Consider an n-ary relation $\varrho\subseteq(A^n)$ on A. Two k-tuples $\langle x_1, \dots, x_k\rangle, \langle y_1, \dots, y_k\rangle\in A^k$ are of the same pattern with respect to ϱ if for $i_1, \dots, i_n\in\{1, \dots, k\}$, $\langle x_{i_1}, \dots, x_{i_n}\rangle\in\varrho$ and $\langle y_{i_1}, \dots, y_{i_n}\rangle\in\varrho$ mutually imply each other. An operation $f\colon A^k\to A$ is a *ϱ-pattern function* if $f(x_1, \dots, x_k)$ always equals some x_i, $i\in\{1, \dots, k\}$, where i depends on the ϱ-pattern of $\langle x_1, \dots, x_k\rangle$ only. The ϱ-pattern functions with ϱ the equality relation are the (usual) pattern functions.

The aim of this paper is to prove a functional completeness theorem on ϱ-pattern functions with ϱ central, which is analogous to the theorems mentioned above.

An n-ary relation ϱ on A is called *central* [5], if $\varrho\neq A^n$ and there exists a nonvoid proper subset C of A such that

(1) $\langle a_1, \dots, a_n\rangle\in\varrho$ whenever at least one $a_j\in C$ $(1\leqq j\leqq n)$;

(2) $\langle a_1, \dots, a_n\rangle\in\varrho$ implies $\langle a_{\sigma(1)}, \dots, a_{\sigma(n)}\rangle\in\varrho$ for every permutation σ of the indices $1, \dots, n$;

(3) $\langle a_1, \dots, a_n\rangle\in\varrho$ if $a_i=a_j$ for some $i\neq j$ $(1\leqq i, j\leqq n)$. Note that every unary relation C distinct from $\emptyset$ and A is central.

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Let ε be an equivalence and ϱ an arbitrary n-ary relation on A. If for $a_1, \ldots, a_n$ $b_1, \ldots, b_n \in A$, $(a_1, \ldots, a_n) \in \varrho$ and $(a_1, b_1) \in \varepsilon, \ldots, (a_n, b_n) \in \varepsilon$ together imply $(b_1, \ldots$ $\ldots, b_n) \in \varrho$, then ε is said to be *compatible* with ϱ. We say that ϱ is *simple*, if no non-trivial equivalence on A is compatible with ϱ. An operation f on A is said to *preserve* ϱ if ϱ is a subalgebra of the nth direct power of the algebra $\langle A; f\rangle$.

We will use the following version of ROSENBERG's completeness theorem (see [5]).

A finite algebra $\langle A; f\rangle$ with a single fundamental operation f is functionally complete iff

(a) f is a monotonic with respect to no bounded partial order on A,

(b) f preserves no non-trivial equivalence on A,

(c) f preserves no binary central relation on A,

(d) f is surjective and essentially at least binary,

(e) f preserves no quaternary relation.

$\theta = \{\langle a_0, a_1, a_2, a_3\rangle \in A^4 | a_0 + a_1 = a_2 + a_3\}$ where $\langle A; +\rangle$ is an elementary abelian p-group (p is prime number).

Let A be a finite set. For $k \geqq 2$ and for arbitrary $(k-1)$-ary, resp. l-ary $(1 \leqq$ $\leqq l \leqq k-1)$ relations τ and θ on A we define the k-ary τ-pattern functions f_k^τ, g_k^τ resp. the l-ary θ-pattern functions h_k^θ on A as follows

$$f_k^\tau(x_1, \ldots, x_k) = \begin{cases} x_k, & \text{if } (x_1, \ldots, x_{k-1}) \in \tau \\ x_1 & \text{otherwise,} \end{cases}$$

$$g_k^\tau(x_1, \ldots, x_k) = \begin{cases} x_1, & \text{if } (x_1, \ldots, x_{k-1}) \in \tau \\ x_k & \text{otherwise,} \end{cases}$$

$$h_k^\theta(x_1, \ldots, x_k) = \begin{cases} x_k, & \text{if } (x_{i_1}, \ldots, x_{i_l}) \in \theta \quad \text{for some} \quad 1 \leqq i_1 < \ldots < i_l \leqq k, \\ x_1 & \text{otherwise.} \end{cases}$$

If τ and θ are the equality relation on A, then f_3^τ is the ternary discriminator, g_3^τ is the dual discriminator and h_k^θ is a near projection.

Theorem. *Let τ and θ be arbitrary central relations on an at least three element finite set A. The algebras $\langle A; f\rangle$ with $f = f_k^\tau$ or g_k^τ are functionally complete if and only if τ is simple. The algebras $\langle A; h_k^\theta\rangle$ are not functionally complete.*

Remark 1. If $|A| = 2$, then τ and θ are unary. In this case f_k^τ and g_k^τ are monotone on $A(=\{0, 1\})$, and h_k^θ is a projection; therefore $\langle A; f\rangle$ with $f = f_k^\tau, g_k^\tau$, or h_k^θ is not functionally complete.

For the proof of Theorem 1 we need the following lemma.

Lemma. *Let τ be a relation and f an arbitrary τ-pattern function on A. If τ is not simple, then $\langle A; f\rangle$ is not functionally complete.*

Proof. If τ is not simple, then there exists an nontrivial equivalence ε on A which is compatible with τ. Clearly, ε is a congruence of $\langle A; f\rangle$. Hence $\langle A; f\rangle$ is not functionally complete.

Remark 2. If an at least binary arbitrary central relation τ on A has at least two central elements, then τ is not simple. In this case Lemma implies that, for an arbitrary τ-pattern function f, the algebra $\langle A; f\rangle$ is not functionally complete.

Proof of Theorem. First we prove that the algebras $\langle A; h_k^\theta\rangle$ are not functionally complete. If the centre of θ has at least two elements, this follows from Remark 2. If the centre of θ consists of a single element c, then the equivalence of A with blocks $\{c\}$ and $A\setminus\{c\}$ is an non-trivial congruence of $\langle A; h_k^\theta\rangle$. Therefore $\langle A; h_k^\theta\rangle$ is not functionally complete.

It remains to show that the algebras $\langle A; f\rangle$ with $f=f_k^\tau$ or g_k^τ and τ simple are functionally complete. Rosenberg's criterion will be used. Clearly, (d) is true for f_k^τ and g_k^τ. Furthermore, they depend on all of their variables and $f_k^\tau(x_1, \ldots, x_k)$, $g_k^\tau(x_1, \ldots, x_k)\in\{x_1, \ldots, x_k\}$ for $x_1, \ldots, x_k\in A$. Then, by Lemma 1 in [7], (e) also holds for them. Thus it is enough to prove that neither f_k^τ nor g_k^τ does preserve the relations ϱ in (a), (b), (c). Therefore we have to present a $k\times 2$ matrix with entries in A such that all rows belong to ϱ, but the row of column values does not belong to ϱ.

(a) Let $\leqq$ be a bounded partial order on A with least element 0 and greatest element 1 $(0,1\in A)$. In view of Remark 2, we can suppose that c is a unique central element of τ. We will use the following matrices to show that none of the functions f_k^τ, g_k^τ does preserve $\leqq$

$$\begin{array}{cccccccccccc}
h & h & t_1 & 1 & 0 & h & h & h & 0 & h & 1 & 1 \\
t_1 & 1 & 0 & h & t_1 & t_1 & t_1 & 1 & t_1 & 1 & 0 & t_1 \\
\cdot & \cdot & t_2 & t_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
t_{k-2} & 1 & t_{k-2} & t_{k-2} & t_{k-2} & t_{k-2} & t_{k-2} & 1 & t_{k-2} & 1 & 0 & t_{k-2} \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & h & 1 & 1 & h & h \\
\hline
h & 0 & t_1 & 0 & 1 & h & 1 & h & 1 & h & 1 & h
\end{array}.$$

Let h always denote an element of A distinct from 0 and 1. Consider the operation f_k^τ, and first suppose $c=1$. Since h is not a central, there exist $t_1, \ldots, t_{k-2}(\in A)$ for which $(h, t_1, \ldots, t_{k-2})\notin\tau$. Then the first matrix shows that f_k^τ does not preserve $\leqq$. Next suppose $c=h$. Since 0 is not central, there exist $t_1, \ldots, t_{k-2}(\in A)$ for which $(t_1, 0, t_2, \ldots, t_{k-2})\notin\tau$, and the second matrix applies. Finally, if $c=0$, then h is not a central, and there exist $t_1, \ldots, t_{k-2}(\in A)$ with $(h, t_1, \ldots, t_{k-2})\notin\tau$, and now the third matrix does the job. Now consider the operation g_k^τ, and first suppose $c=1$.

Since h is not central, there exist $t_1, \ldots, t_{k-2} (\in A)$ with $(h, t_1, \ldots, t_{k-2}) \notin \tau$. Then the fourth matrix shows that g_k^τ does not preserve $\leqq$. If $c=h$, then 0 is not central, and there exist $t_1, \ldots, t_{k-2} (\in A)$ with $(0, t_1, \ldots, t_{k-2}) \notin \tau$, and the fifth matrix is used. Finally, suppose $c=0$, then 1 is not central, and there exist $t_1, \ldots, t_{k-2} (\in A)$ with $(1, t_1, \ldots, t_{k-2}) \notin \tau$. In this case using the sixth matrix we also get that g_k^τ does not preserve $\leqq$.

(b) Let ε be an arbitrary non-trivial equivalence on A. We prove that the operations f_k^τ and g_k do not preserve ε. Since τ is simple, there exist elements $a_1, \ldots, a_{k-1}$, $b_1, \ldots, b_{k-1} (\in A)$ with $(a_1, \ldots, a_{k-1}) \in \tau, (a_1, b_1) \in \varepsilon, \ldots, (a_{k-1}, b_{k-1}) \in \varepsilon, (b_1, \ldots, b_{k-1}) \notin \tau$. Let $(t, b_1) \notin \varepsilon$, then $(a_1, t) \notin \varepsilon$ holds as well, and the matrix

$$\begin{array}{cc} a_1 & b_1 \\ \vdots & \vdots \\ a_{k-1} & b_{k-1} \\ t & t \\ \hline t & b_1 \\ a_1 & t \end{array}$$

shows that none of f_k^τ and g_k^τ do not preserve ε.

(c) Let ϱ be a binary central relation with centre C_ϱ. Let c be a unique central element of τ. To show that f_k^τ and g_k^τ do not preserve ϱ we use the following matrices

$$\begin{array}{cc} b & b \\ t_1 & c \\ \vdots & \vdots \\ t_{k-2} & c \\ a & a \\ \hline b & a \end{array} \quad \text{or} \quad \begin{array}{cc} a & b \end{array} \qquad \text{or} \qquad \begin{array}{cc} d & d \\ t_1 & l \\ \vdots & \vdots \\ t_{k-2} & l \\ c & c \\ \hline d & c \end{array} \qquad \text{or} \qquad \begin{array}{cc} d & d \\ t_1 & l \\ \vdots & \vdots \\ t_{k-2} & l \\ c & c \\ \hline d & c \end{array} \quad \begin{array}{cc} c & d. \end{array}$$

Now we have two cases.

(1) If $c \in C_\varrho$, then let $(a, b) \notin \varrho$. We can choose elements $t_1, \ldots, t_{k-2}$ with $(b, t_1, \ldots, t_{k-2}) \notin \tau$. Considering the first matrix we get that f_k^τ and g_k^τ do not preserve ϱ.

(2) If $c \notin C_\varrho$, then let d and l such that $(c, d) \notin \varrho$, and $l \in C_\varrho$. For $k=3$, if $(d, l) \in \tau$ then let t_1 such that $(d, t_1) \notin \tau$, and if $(d, l) \notin \tau$ then let $t_1 = d$. From the second matrix we get that f_k^τ and g_k^τ do not preserve ϱ. Finally, if $k \geqq 4$, there are elements $t_1, \ldots, t_{k-2}$ with $(d, t_1, \ldots, t_{k-2}) \notin \tau$ and the third matrix works.

Remark 3. Let A be a finite set, $|A| \geqq 3$. For an arbitrary relation ϱ on A

we define the following k-ary ϱ-pattern function on A

$$t_k^\varrho(x_1, x_2, \ldots, x_k) = \begin{cases} x_k, & \text{if } x_1 \varrho x_2 \varrho \ldots \varrho x_{k-1} \\ x_1 & \text{otherwise,} \end{cases}$$

$$s_k^\varrho(x_1, x_2, \ldots, x_k) = \begin{cases} x_1, & \text{if } x_1 \varrho x_2 \varrho \ldots \varrho x_{k-1} \\ x_k & \text{otherwise.} \end{cases}$$

We saw in [7] that $\langle A; f\rangle$ with $f=t_k^\varrho$ of $f=s_k^\varrho$ are functionally complete, if $k \geqq 3$, and ϱ is an arbitrary permutation on A or $\varrho=\delta \cup \delta^{-1}$ with an arbitrary permutation δ on A. If ϱ is an arbitrary central relation on A, then

$$t_k^\varrho(x_1, x_2, \ldots, x_2, x_3) = f_3^\varrho(x_1, x_2, x_3),$$

and

$$s_k^\varrho(x_1, x_2, \ldots, x_2, x_3) = g_3^\varrho(x_1, x_2, x_3).$$

Hence, using the Theorem, the following result follows.

$\langle A; f\rangle$ with $f=t_k^\varrho$ or $f=s_k^\varrho$ functionally complete if and only if ϱ is an arbitrary simple central relation.

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DEPARTMENT OF MATHEMATICS
TEACHER'S TRAINING COLLEGE
6725 SZEGED, HUNGARY

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Estimation of generalized moments of additive functions over the set of shifted primes

K.-H. INDLEKOFER and I. KÁTAI

1. Introduction. A function φ: $[0,\infty)\to[0,\infty)$ is called subadditive, if it is monotonically increasing, $\varphi(x)\to\infty$ as $x\to\infty$, and the condition

$$\varphi(x+y)\leqq c_1(\varphi(x)+\varphi(y)) \quad \text{for} \quad x,y\geqq 1 \tag{1.1}$$

holds with a suitable constant $c_1>0$.

It is clear that the functions $\log(1+x)$, x^Γ $(\Gamma>0)$ are subadditive. On the other hand (1.1) implies that $\varphi(x)=0(x^c)$ $(x\to 0)$ with some constant c.

We are interested in giving necessary and sufficient conditions for an additive function f for which

$$(P(x)=)\,P(x):=\sum_{p\leqq x}\varphi(|f(p+1)-\alpha(x)|)\ll \operatorname{li} x \tag{1.2}$$

holds true with a suitable function $\alpha(x)$. Here, and in what follows p runs over the set $\mathscr{P}$ of primes.

For the sake of simplicity we extend the domain of φ to the whole real line by $\varphi(-x):=\varphi(x)$. Then

$$\varphi(x+y)\leqq c_2+c_3(\varphi(x)+\varphi(y)) \tag{1.3}$$

obviously holds for $x,y\in\mathbf{R}$, where c_2, c_3 are suitable positive constants.

For an arbitrary additive function f let

$$A_f(x):=\sum_{\substack{p\leqq x\\ |f(p)|<1}}\frac{f(p)}{p} \tag{1.4}$$

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and (E_f), (E_f^*) denote the conditions:

$$(E_f) \qquad \sum_{\substack{p\in\mathcal{P}\\ |f(p)|<1}} \frac{f(p)}{p} \quad \text{is convergent (finite)}$$

$$(E_f^*) \qquad \sum_{\substack{|f(p)|\geqq 1\\ p\in\mathcal{P}}} 1/p<\infty.$$

An additive function f is said to be finitely distributed if (E_f), (E_f^*) hold true. Let $\pi(x, k, l)$ denote the number of the primes $q\leqq x$ for which $q\equiv l(\text{mod } k)$.

Theorem 1. *Let φ be subadditive. Assume that f is an additive function, for which there exists a real-valued function $\alpha(x)$ such that* (1.2) *holds. Then f can be written as $f=\lambda\log+h$, where h is finitely distributed, $\lambda\in\mathbf{R}$, furthermore*

$$(1.5)_1 \qquad \sum_{|h(p^m)|\geqq 1} \frac{\varphi(h(p^m))}{p^m}<\infty$$

and

$$(1.6) \qquad \sum_{\substack{|h(q^m)|\geqq 1\\ q^m\geqq x^{3/4}}} \varphi(h(q^m))\pi(x, q^m, -1) = 0(\text{li}\, x).$$

We have $\alpha(x)=\alpha^*(x)+0(1)$, where

$$(1.7) \qquad \alpha^*(x) := \lambda\log x+A_h(x).$$

In contrary, let h be a finitely distributed additive function for which $(1.5)_1$ *and* (1.6) *hold and let $f=\lambda\log+h$, $\lambda\in\mathbf{R}$. Then* (1.2) *holds.*

Remarks. A. Assume that

$$(1.5)_\Gamma \qquad \sum_{|h(p^m)|\geqq 1} \frac{\varphi^\Gamma(h(p^m))}{p^m}<\infty$$

holds with a suitable $\Gamma>1$. Then $(1.5)_1$ and (1.6) are satisfied.

B. It is not known whether condition (1.6) could be omitted or not. Let $P^*(n)$ denote the largest prime power divisor of n. Assume that

$$(1.8) \qquad \limsup_p \frac{(\log P^*(p+1))\log\log P^*(p+1)}{\log(p+1)}=\infty.$$

Then the condition (1.6) cannot be omitted in the Theorem, i.e. there exists such a finitely distributed h for which $(1.5)_1$ holds, but (1.6) does not hold.

Proof of Remark B. According to (1.8), there exists a sequence $p_1<p_2<\ldots$ of primes, $Q_1<Q_2<\ldots$ of prime powers, such that $p_i+1=a_iQ_i$, and

$$l_i := \frac{\log(p_i+1)}{a_i\cdot i^2}\to\infty.$$

Let now $h \geqq 0$ be defined on the set of prime powers q^m such that $h(q^m)=0$ if $q^m \in \{Q_i\}_{i \in \mathbf{N}}$, and $\varphi(h(Q_i))=Q_i/i^2$. Then $(1.5)_1$ holds, while

$$\sum_{p_i^{3/4}<q^m \leqq p_i} \varphi(h(q^m))\pi(p_i, q^m, -1) \geqq \varphi(h(Q_i)) = \frac{Q_i}{i^2} = \frac{p_i+1}{a_i \cdot i^2} = l_i \frac{p_i+1}{\log(p_i+1)}.$$

Thus (1.6) does not hold.

The theorem and Remark A will be proved in sections 3 and 4.

2. Lemmata. The main result of the proof of our theorem is a recent deep result of A. HILDEBRAND ([1], Theorem 4), which we state now as

Lemma 1. *There exist positive absolute constants δ and c such that if $x \geqq 2$, and f is a realvalued additive function satisfying*

$$\max_{a \in \mathbf{R}} \# \{p \leqq x : f(p+1) \in [a, a+1]\} \geqq (1-\delta)\pi(x), \tag{2.1}$$

then

$$\min_{|\lambda| \leqq c} \sum_{p \leqq x} \frac{1}{p} \min(1, |f(p)-\lambda \log p|^2) \leqq c. \tag{2.2}$$

Remark. Assume that (2.1) holds for an unbounded sequence x_ν of x. Then, for each x_ν there exists a $\lambda_\nu(=\lambda)$ for which (2.2) holds ture, $|\lambda_\nu| \leqq c$. Set λ be a limit point of the sequence $\{\lambda_\nu\}$. Then, from (2.2)

$$\sum_p \frac{1}{p} \min(1, |f(p)-\lambda \log p|^2) < \infty,$$

which implies that $h(n):=f(n)-\lambda \log n$ is finitely distributed. Another important tool is the following

Lemma 2. *Let α be a real number satisfying $0 \leqq \alpha < 2$. Then we have, for every $x \geqq 2$ and every additive f,*

$$\sum_{p \leqq x} |f(p+1)-E(x)|^\alpha \ll \frac{x}{\log x} B^\alpha(x),$$

where

$$E(x) = \sum_{p^m \leqq x} \frac{f(p^m)}{p^m}, \quad B(x) = \left(\sum_{p^m \leqq x} \frac{|f(p^m)|^2}{p^m}\right)^{1/2},$$

and the implied constant depends only on α.

Remark. This analogue of the Turán—Kubilius inequality was established by P. D. T. A. ELLIOTT for strongly additive functions (see [2], Lemma 4.18], the general case can be proved in the same way.

The following assertion due to ELLIOTT ([2], Lemma 4.19).

Lemma 3. *Let m be a non-negative integer, and δ a real number, $0<\delta\leqq 1/2$. Then there is a number c, depending upon n but not δ, so that the inequality*

$$(*)\qquad \sum_{x^{1-\delta}<Q\leqq x} p^{n-1}\pi(x, Q-1)^n \leqq \delta c\left(\frac{x}{\log x}\right)^n$$

holds for all sufficiently large values of x. Here Q runs over all prime powers.

Elliott proved this inequality letting Q to run over the set of primes only. $(*)$ can be proved in the same way.

Lemma 4. *The number of solutions of the equation $p+1=aq$ in prime variables $p, q<x$ is less than $\dfrac{cx}{l(a)\log^2 x}$ uniformly as $1\leqq a\leqq\sqrt{x}$.*

Lemma 5 (Titchmarsh inequality). *We have*

$$\pi(x, k, l) < \frac{cx}{\varphi(k)\log x/k} \quad \text{if} \quad 1\leqq k < x, \quad x\geqq 2.$$

For the proof of Lemma 4 and 5 see HALBERSTAM—RICHERT [3].

Lemma 6. *Let g be a strongly multiplicative function such that $0\leqq g(p)\leqq c$ holds for every prime p. Then*

$$\sum_{p\leqq x} g(p+1) \ll \pi(x)\exp\left(\sum_{p\leqq x}\frac{g(p)-1}{p}\right).$$

For the proof see [4], Lemma 1.

3. Proof of the Theorem. *Necessity.* Assume that (1.2) holds. Then the condition (2.1) of Lemma 1 is satisfied for every large x, consequently $f=\lambda\log+h$, where h is a finitely distributed function, $\lambda\in\mathbf{R}$. Let $\alpha_1(x)=\alpha(x)-\lambda\log x$. Since $h(n)-\alpha_1(x)=f(n)+\lambda\log\dfrac{x}{n}-\alpha(n)$, by (1.3), (1.2), and by

$$\sum_{p\leqq x}\varphi\left(\lambda\log\frac{x}{p+1}\right)\ll \operatorname{li} x,$$

we get

$$(3.1)\qquad \sum_{p\leqq x}\varphi\big(h(p+1)-\alpha_1(x)\big)\ll \operatorname{li} x.$$

Let h_1 be strongly additive defined for primes q such that

$$h_1(q)=\begin{cases} h(q) & \text{if } |h(q)|<1,\\ 0 & \text{otherwise,}\end{cases}$$

and let h_2 be defined so that $h_2(n):=h(n)-h_1(n)$.

Since $\varphi(x) \ll |x|^c < c_1 e^{|x|}$, and (E_{h^2}), (E_h^*) hold, we have

$$\sum_{p \leqq x} \varphi\big(h_1(p+1) - A(x)\big) \ll e^{-A(x)} \sum_{p \leqq x} e^{h_1(p+1)} + e^{A(x)} \sum_{p \leqq x} e^{-h_1(p+1)}.$$

By Lemma 6, the right hand side is bounded by

$$\ll (\operatorname{li} x) \exp\left(\sum_{q \leqq x} \frac{e^{h_1(q)} - 1 - h_1(q)}{q}\right) +$$

$$(\operatorname{li} x) \exp\left(\sum_{q \leqq x} \frac{e^{-h_1(q)} - 1 + h_1(q)}{q}\right) \ll \operatorname{li} x.$$

Thus

$$\sum_{p \leqq x} \varphi\big(h_2(p+1) - A(x)\big) \ll \operatorname{li} x, \tag{3.2}$$

whence

$$\sum_{p \leqq x} \varphi\big(h_2(p+1) - \alpha_2(x)\big) \ll \operatorname{li} x, \tag{3.3}$$

$$\alpha_2(x) = \alpha_1(x) - A(x)$$

immediately follows.

We shall prove that $\alpha_2(x)$ is bounded.

Let y be a large positive number. Let $h_2^{(1)}$, $h_2^{(2)}$ be additive functions, $h_2(n) = h_2^{(1)}(n) + h_2^{(2)}(n)$, and for prime powers q^m let

$$h_2^{(1)}(q^m) = \begin{cases} h(q^m) & \text{if } q^m \leqq y \\ 0 & \text{otherwise.} \end{cases}$$

Since $h_2^{(1)}(n)$ is bounded, therefore from (3.3) we have

$$\sum_{p \leqq x} \varphi\big(h_2^{(2)}(p+1) - \alpha_2(x)\big) \ll \operatorname{li} x. \tag{3.4}$$

Let Q_y denote the set of all prime powers $q^m \geqq y$ for which either $m \geqq 2$ or $|h(q)| \geqq 1$ holds. By using Lemma 4 and 5 and the Eratosthenian sieve one can get easily that there exists at least $cx/\log x$ prime p up to x, such that $q^m \| p+1$ implies that $q^m \notin Q_y$. For such a prime p we have $h_2^{(2)}(p+1) = 0$. Consequently, (3.4) can be held only if $\alpha_2(x) = 0(1)$.

Let $q^m \in Q_y$ and $S_q m$ be the set of those primes $p \leqq x$ for which $p+1 = q^m v$, where v is square free, $(v, q) = 1$, and v does not contain any prime factor $R > 2$ for which $|h(R)| > 1$. It is clear that $h_2^{(2)}(p+1) = h(q^m)$. By using the above argument and the prime number theorem for residue classes one get readily that

$$\#(S_q m) > \frac{c}{\varphi(q^m)} \frac{x}{\log x}$$

uniformly as $q^m \leqq \log\log x$, say. Hence, by (3.4), and by $\alpha_2(x) = 0(1)$, we get $(1.5)_1$ immediately, and even that $\alpha(x) = \lambda \log x + A(x) + 0(1)$.

Let $\mathcal{M}$ denote the set of those integers D, all exact prime-power factors of which belong to Q_1. Let us write $p+1$ in the form $p+1=Dv$, where v is square-free and does not contain any prime factors for which $h_2(q)\neq 0$, and $D\in\mathcal{M}$. This representation is unique. It is clear that $h_2(p+1)=h(D)$. Consequently

$$\text{(3.5)}\qquad \operatorname{li} x \gg \sum_{p\leqq x} \varphi\big(h_2(p+1)\big) = \sum_{D\in\mathcal{M}} \varphi\big(h_2(D)\big)\pi(x, D, -1),$$

and (1.6) holds true.

The necessity of the conditions is proved.

Sufficiency. Assume that $(1.5)_1$, (1.6), (1.7) are satisfied, where h is a finitely distributed function. We shall prove that (1.2) holds, if $f=\lambda\log+h$, $\lambda\in\mathbf{R}$. By using the subadditive property of φ, it is enough to prove it for $\lambda=0$, i.e. if $f=h$ is finitely distributed. We keep the notations $h_1, h_2, \mathcal{M}$.

It is enough to prove that

$$\text{(3.6)}\qquad \sum_{p\leqq x} \varphi\big(h_1(p+1)-A(x)\big) \ll \operatorname{li} x,$$

and that

$$\text{(3.7)}\qquad \sum_{p\leqq x} \varphi\big(h_2(p+1)\big) \ll \operatorname{li} x.$$

The first inequality was deduced from Lemma 6 earlier. It remains to prove only (3.7). We have

$$A := \sum_{p\leqq x} \varphi\big(h_2(p+1)\big) = \sum_{\substack{D\in\mathcal{M}\\ D<x}} \varphi\big(h_2(D)\big)\pi(x, D, -1) = \sum\nolimits_1 + \sum\nolimits_2,$$

where in $\sum_1$ we sum over $D\leqq x^{1-\delta}$, and in $\sum_2$ over the others. Here δ is a constant, $0<\delta<0,1$. By Lemma 5,

$$\sum\nolimits_1 \ll \operatorname{li} x \sum_{D\in\mathcal{M}} \frac{\varphi\big(h_2(D)\big)}{l(D)}.$$

Let us consider $\sum_2$. We split the sum $\sum_2=\sum_2'+\sum_2''$, where in $\sum_2'$ we sum over those $D>x^{1-\delta}$ which can be written as $D=D_1D_2$, where $(D_1, D_2)=1$ and $D_i<x^{1-\delta}$ $(i=1,2)$. Since $\varphi\big(h_2(D)\big)\ll\varphi\big(h_2(D_1)\big)+\varphi\big(h_2(D_2)\big)$, we can use Lemma 5 again,

$$\sum\nolimits_2' \ll \operatorname{li} x \sum_{D_1\in\mathcal{M}} \frac{\varphi\big(h_2(D_1)\big)}{l(D_1)} + \operatorname{li} x \sum_{D_2\in\mathcal{M}} \frac{\varphi\big(h_2(D_2)\big)}{l(D_2)}.$$

If D is considered in $\sum_2''$, then D has the form $D=D_1\cdot D_2$, where $D_1>x^{1-\delta}$ and D_1 is a prime or a prime power, $D_1=q^m$. Thus

$$\sum\nolimits_2'' \ll \sum_{D_2<\sqrt{x}} \varphi\big(h_2(D_2)\big)\pi(x, D_2, -1) + \sum_{\substack{q^m\in Q_1\\ q^m>x^{1-\delta}}} \varphi\big(h_2(q^m)\big)\pi(x, q^m, -1).$$

Collecting our inequalities, and taking into account (1.6), we have

$$A \ll \operatorname{li} x \sum_{D \in \mathcal{M}} \frac{\varphi(h_2(D))}{l(D)} + 0(\operatorname{li} x).$$

Finally we prove that the sum on the right hand side is convergent.

Indeed, iterating (1.1), we get that

$$\varphi(h_2(D)) \leqq \sum_{q^m \| D} c^{\omega(D/q^m)} \cdot \varphi(h_2(q^m)),$$

where $\omega(n)$ is the number of distinct prime divisors of n. Thus we have

$$\sum_{D \in \mathcal{M}} \frac{\varphi(h_2(D))}{l(D)} \leqq \sum_{D} \sum_{q^m \| D} \frac{\varphi(h_2(q^m))}{l(D)} \cdot \frac{c^{\omega(D/q^m)}}{l(D/q^m)} \leqq$$

$$\leqq \Big(\sum_{q^m \in Q_1} \frac{\varphi(h_2(q^m))}{l(q^m)} \Big) \Big(\sum_{H \in \mathcal{H}} \frac{c^{\omega(H)}}{l(H)} \Big).$$

$(1.5)_1$ implies the convergence of the first sum. The second sum is convergent as well.

The sufficiency part is proved.

4. Proof of Remark A. To estimate

$$S = \sum_{\substack{|h(q^m)| \geqq 1 \\ x^{3/4} \leqq q^m < x}} \varphi(h(q^m)) \pi(x, q, -1),$$

we apply Lemma 3, namely that

$$\sum_{x^{3/4} < q^m < x} \pi(x, q^m, -1)^{n+1} q^n < c_n (\operatorname{li} x)^{n+1}$$

holds for every integer $n \geqq 1$. Let n be so large that $\alpha_1 := 1 + 1/n \leqq \Gamma$, $\gamma = \dfrac{n}{n+1}$, β be defined from $\dfrac{1}{a_1} + \dfrac{1}{\beta} = 1$. Then, by Hölder's inequality,

$$S = \sum_{q^m > x^{3/4}} \frac{\varphi(h(q^m))}{q^{m\gamma}} \pi(x, q^m, -1) q^{\gamma m} \ll$$

$$\ll \Big(\sum_{|h(q^m)| \geqq 1} \frac{\varphi(h(q^m))^{\alpha_1}}{q^{\gamma \alpha_1}} \Big)^{1/\alpha_1} \Big(\sum (\pi(x, q^m, -1) q^{m\gamma})^{\beta} \Big)^{1/\beta}.$$

By Lemma 3, and by $(1.5)_\Gamma$ we get $S = 0(\operatorname{li} x)$. This finishes the proof.

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(K.-H. I.)
FACHBEREICH MATHEMATIK-INFORMATIK
UNIVERSITÄT-GH PADERBORN
WARBURGER STR. 100
D—4790 PADERBORN

(I. K.)
EÖTVÖS LORÁND UNIVERSITY
COMPUTER CENTER
BUDAPEST, H—1117
BOGDÁNFY ÚT 10/B.

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On a theorem of Kátai-Wirsing

BUI MINH PHONG

1. Introduction. An arithmetic function $f(n)$ is said to be additive if $(m, n)=1$ implies that

$$f(mn) = f(m)+f(n)$$

and it is completely additive if the above equality holds for all positive integers m and n. Let $\mathscr{A}$ and $\mathscr{A}^*$ denote the set of complex-valued additive and completely additive functions, respectively.

The problem concerning the characterization of $\log n$ as an additive arithmetic function was studied by several authors. The first such characterization is apparently that of P. ERDŐS [3]. He proved in 1946 that if a real valued additive function f satisfies the condition

$$f(n+1)-f(n) \to 0 \quad \text{as} \quad n \to \infty,$$

then $f(n)$ is a constant multiple of $\log n$. Later I. KÁTAI [4] and E. WIRSING [6] improving this result, proved that a function $f \in \mathscr{A}$ satisfying

$$\sum_{n \leq x} |f(n+1)-f(n)| = o(x) \quad \text{as} \quad x \to \infty$$

must be of the form $f=U \log$ for some complex constant U.

On the other hand, solving a conjecture of Kátai, P. D. T. A. ELLIOTT [1] showed that if a real function f is additive and satisfies the condition

$$f(An+B)-f(an+b) \to C \quad \text{as} \quad n \to \infty \tag{1}$$

for some integers $A>0, B, a>0, b$ with $Ab-aB \neq 0$ and for a real constant C, then $f(n)=U \log n$ holds for all positive integers n which are prime to $Aa(Ab-aB)$. In his proof Elliott relaxed the condition (1) to

$$\sum_{n \leq x} |f(An+B)-f(an+b)|^2 = o(x)$$

for the case $A \neq a$.

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Our purpose in this paper is to give a complete characterization of those functions $f, g \in \mathscr{A}$ for which the relation

$$\sum_{n \leq x} |g(an+b)-f(n)-d| = o(x) \tag{2}$$

holds for some fixed positive a, b and for a complex constant d.

We shall prove the following

Theorem 1. *Assume that $f, g \in \mathscr{A}$ satisfy* (2) *for some fixed positive integers a, b and for a complex constant d. Then there are a complex constant U and functions $F \in \mathscr{A}$, $G \in \mathscr{A}$ such that*

$$f(n) = U \log n + F(n)$$

$$g(n) = U \log n + G(n)$$

and

$$G(an+b)-F(n)-d+U \log a = 0$$

hold for all positive integers n.

Theorem 2. *Assume that $f \in \mathscr{A}$ satisfies the condition*

$$\sum_{n \leq x} |f(An+B)-f(Cn)-D| = o(x) \tag{3}$$

for some positive integers A, B, C and for a complex constant D. Then there are a complex constant U and a function $F \in \mathscr{A}$ such that

$$f(n) = U \log n + F(n)$$

and

$$F(n) = F[(n, BCC_A)]$$

hold for all positive integers n, where C_A denotes the product of all prime divisors of C which are prime to A.

We note that our theorems can be derived from a recent result due to P. D. T. A. Elliott [2], which was obtained with analytic methods. Here we shall prove our results by using elementary methods, which were used in [5].

2. Auxiliary results. In this section we assume that a function $f \in \mathscr{A}$ satisfies (3), i.e.

$$\sum_{n \leq x} |f(An+B)-f(Cn)-D| = o(x)$$

holds for some positive integers A, B, C and for a complex constant D.

Let C_A denote the product of all prime divisors of C which are prime to A. For an arbitrary positive integer n, let $E(n)=E_B(n)$ be the product of all prime power factors of B composed from the prime divisors of n, i.e. $E(n)|B$, $(E(n), B/E(n))=1$ and every prime divisor of $E(n)$ is a divisor of n.

Lemma 1. *For every fixed positive integer k and Q we have*

$$f(BCC_A Q^k) = kf(BCC_A Q) - (k-1)f(BCC_A), \tag{4}$$

furthermore

$$f(ACC_A^2 E(C)) = 2f(CC_A E(C)) - f(E(C)) + D. \tag{5}$$

Proof. For each positive integer Q we define the sequence

$$R = R(AC_A Q) = \{R_k\}_{k=1}^{\infty}$$

by the initial term $R_1 = 1$ and by the formula

$$R_k = R_k(AC_A Q) = 1 + AC_A Q + \ldots + (AC_A Q)^{k-1} \tag{6}$$

for all integers $k \geqq 2$. Moreover, let

$$T_k(n, Q) = (AC_A Q)^k E(CQ)n + BR_k(AC_A Q). \tag{7}$$

By using (6) and (7), we have

$$T_{k+1}(n, Q) = AC_A QT_k(n, Q) + B \tag{8}$$

and

$$(CC_A QE(CQ), T_k(n, Q)/E(CQ)) = 1 \tag{9}$$

for all integers $k \geqq 1$. Thus, using (3), (7), (8), (9) and the additivity of f, we have

$$\sum_{n \leqq x} |f(T_1(n, Q)) - f(CC_A QE(CQ)n) - D| = o(x)$$

and

$$\sum_{n \leqq x} |f(T_k(n, Q)) - f(T_{k-1}(n, Q)) - H(Q)| = o(x)$$

for all integers $k \geqq 2$, where

$$H(Q) := f(CC_A QE(CQ)) - f(E(CQ)) + D.$$

These imply that

$$\sum_{n \leqq x} |f(T_k(n, Q)) - f(CC_A QE(CQ)n) - (k-1)H(Q) - D| = o(x) \tag{10}$$

holds for every integer $k \geqq 1$.

We shall deduce from (10) that

$$f(A^{k-1}CC_A^k Q^k PE(CQ)) = (k-1)H(Q) + f(CC_A QPE(CQ)) \tag{11}$$

holds for every positive integer k, Q and P.

Let k, Q and P be positive integers. Considering

$$n := PR_k(AC_A Q)\{APCQR_k(AC_A Q)m + 1\} \tag{12}$$

and taking into account (10), it is easily seen that (11) holds if k, Q and P satisfy the

relation

(13) $$(P, R_k(AC_AQ)) = (PE(CQ)+B, R_k(AC_AQ)) = 1.$$

It is obvious that (13) is satisfied in the following cases:

$$P=1,\ Q=2B;\quad P=1,\ Q=2pB,$$

where p is a prime. Thus, we get from (11) that

$$f(p^k) = kf(p) \quad \text{if} \quad (p, 2ABC) = 1.$$

This with the additivity of f shows that

(14) $$f(nm) = f(n)+f(m) \quad \text{if} \quad (n, m, 2ABC) = 1.$$

Thus, by using (10), (12) and (14), we see that (11) also holds if we relax the condition (13) to

(15) $$(P, R_k(AC_AQ), 2B) = (PE(CQ)+B, R_k(AC_AQ), 2) = 1.$$

Assume that $(2, ABC)=1$ and k is an odd positive integer. In this case one can check that (15) holds for $P=Q=1$ and $P=1, Q=2$. Thus, we get from (11) that

(16) $$f(2^k) = kf(2) \quad \text{for all odd positive integers } k.$$

On the other hand, (15) also holds for $P=2^\nu$, $Q=2$ and $k=2$, where $\nu\geqq 0$ is an integer. From (11) we have

(17) $$f(ACC_A^2 2^{\nu+2}E(C)) = H(2)+f(CC_A 2^{\nu+1}E(C)).$$

Thus, we get from (17) that

$$f(2^k) = kf(2)+(k-1)\big(H(1)+f(CC_AE(C))-f(ACC_A^2E(C))\big)$$

holds for every positive integer k, which with (16) shows that

$$f(2^k) = kf(2) \quad (k = 1, 2, \ldots).$$

This with (14) implies that

(18) $$f(nm) = f(n)+f(m) \quad \text{if} \quad (n, m, ABC) = 1.$$

Similarly as above, by using (10), (12) and (18) we also have (11) if k, Q and P satisfy

(19) $$(P, R_k(AC_AQ), B) = 1.$$

Finally, let $P=P_1\cdot P_2$, where $(P_1, P_2)=(P_1, AC_AQ)=1$ and every prime divisor of P_2 is a divisor of AC_AQ. We have

$$(P_2, R_k(AC_AQ), B) = 1,$$

therefore by (11) and (19) it follows that

$$f(A^{k-1}CC_A^k Q^k P_2 E(CQ)) = (k-1)H(Q)+f(CC_A P_2 E(CQ)).$$

Since $(P_1, AC_A QP_2)=1$, by using the additivity of f, we get

$$f(A^{k-1}CC_A^k Q^k PE(CQ)) = f(A^{k-1}CC_A Q^k P_2 E(CQ))+f(P_1) =$$
$$= (k-1)H(Q)+f(CC_A QPE(CQ)),$$

which proves (11).

Applying (11) in the case $Q=1$, we obtain that

$$f(A^{k-1}CC_A^k PE(C)) = (k-1)H(1)+f(CC_A PE(C))$$

holds for every positive integer k and P, consequenly

$$f(A^{k-1}CC_A^k Q^k E(CQ)) = (k-1)H(1)+f(CC_A Q^k E(CQ)). \tag{20}$$

On the other hand, (11) with $P=1$ implies

$$f(A^{k-1}CC_A^k Q^k E(CQ)) = (k-1)H(Q)+f(CC_A QE(CQ)),$$

which with (20) gives

$$f(CC_A Q^k E(CQ)) = (k-1)(H(Q)-H(1))+f(CC_A QE(CQ)).$$

This, using the fact $(E(CQ), B/E(CQ))=1$ and the additivity of f, shows that

$$f(BCC_A Q^k) = kf(BCC_A Q)-(k-1)f(BCC_A).$$

So, we have proved Lemma 1, because (5) follows from (11) in the case $k=2$ and $P=Q=1$.

Lemma 2. *Let A, B be positive integers and D be a complex constant. If $f\in\mathscr{A}^*$ satisfies the condition*

$$\sum_{n\leqq x} |f(An+B)-f(n)-D| = o(x) \quad \text{as} \quad x\to\infty, \tag{21}$$

then there is a complex constant U such that

$$f(n) = U\log n \quad (n = 1, 2, 3, \ldots).$$

Proof. We first note that, by using (5) of Lemma 1 and the fact $f\in\mathscr{A}^*$, (21) implies

$$f(A) = D. \tag{22}$$

If $A=1$, then our assertion follows from the theorem of I. Kátai—E. Wirsing mentioned in Section 1. In the following we assume that $A\geqq 2$ and

$$\sum_{n\leqq x} |f(An+B)-f(An)| = o(x). \tag{23}$$

Let I_f denote those pairs (k, r) of positive integers for which

$$\sum_{n\leqq x} |f(kn+r)-f(kn)| = o(x).$$

Since $(A, B)\in I_f$ and $f\in\mathscr{A}^*$, we have $(A, 1)\in I_f$, furthermore if $(k_0, 1)\in I_f$, then $(k, 1)\in I_f$ for all integers $k\geqq k_0$, because

$$f((k+1)n+1)-f((k+1)n) = \{f(kn+1)-f(kn)\}-$$
$$-\{f[k((k+1)n+1)+1]-f[k((k+1)n+1)]\}.$$

Thus, we have $(k, 1)\in I_f$ for every integer $k\geqq A$.

We shall prove that if $(h+1, 1)\in I_f$ and integers k, r satisfy

$$0 < r < k/h \quad \text{and} \quad (k, r) = 1, \tag{24}$$

then $(k, r)\in I_f$. We prove this by using induction on r. For $r=1$ our assertion is true, because $1<k/h$ implies $k>h$. Assume that for every integer k, r satisfying (24) and $r<R$ we have $(k, r)\in I_f$. Let K be an integer such that

$$0 < R < K/h \quad \text{and} \quad (K, R) = 1. \tag{25}$$

Let k and r be positive integers which satisfy

$$Rk = Kr+1 \quad \text{and} \quad k < K,\ r < R. \tag{26}$$

It is easily seen by (25) and (26) that $(k, r)=1$, furthermore

$$Kr < Kr+1 = Rk < Kk/h,$$

which implies that $r<k/h$. Thus, k, r satisfy (24), and so $(k, r)\in I_f$.

On the other hand, we have

$$f(Kn+R)-f(Kn) = \{f[K(kn+r)+1]-f[K(kn+r)]\}+\{f(kn+r)-f(kn)\},$$

therefore, by using the fact $(K, 1)\in I_f$ and $(k, r)\in I_f$, we have $(K, R)\in I_f$. Thus we have proved (24).

We now deduce from (23) that $(2, 1)\in I_f$. To see this enough show that

$$(h+1, 1)\in I_f \quad \text{with} \quad h+1 > 2 \quad \text{implies} \quad (h, 1)\in I_f. \tag{27}$$

Assume that $(h+1, 1)\in I_f$ and $h+1>2$. Let

$$S(x) = \sum_{n\leqq x} |f(hn+1)-f(hn)|.$$

For each integer d with $0\leqq d\leqq h-1$ we can choose positive integers $K=K(d)$ and $R=R(d)$ such that

$$(hd+1)K = h^2R+1. \tag{28}$$

We have

(29)

$$S(x) = \sum_{n\leqq x} |f(hn+1)-f(hn)| = \sum_{d=0}^{h-1} \sum_{hm+d\leqq x} \left|f(h^2m+hd+1)-f\big(h(hm+d)\big)\right| =$$

$$= \sum_{d=0}^{h-1} \sum_{hm+d\leqq x} \Big|f\big(h^2(Km+R)+1\big)-f\big(h^2(Km+R)\big)+$$

$$+f\big(K(hm+d)+hR-Kd\big)-f\big(K(hm+d)\big)\Big|$$

and so $S(x)=o(x)$ if $hR-Kd=0$, because $(h+1, 1)\in I_f$ and $h+1>2$ implies $(h^2, 1)\in I_f$. If $hR-Kd\neq 0$, then we get from (28) that

$$0 < hR-Kd = (K-1)/h < K/h$$

and

$$(K, hR-Kd) = (K, hR) = 1.$$

Thus, $k:=K$ and $r:=hR-Kd$ satisfy the condition (24), and so $(K, hR-Kd)\in I_f$. By using this and the fact $(h^2, 1)\in I_f$, we also get from (29) that $S(x)=o(x)$. This shows that $(h, 1)\in I_f$, consequently $(2, 1)\in I_f$.

Assume now that

$$A(x) = \sum_{n\leqq x} |f(2n+1)-f(2n)| = o(x). \tag{30}$$

Let q be a fixed prime. As we have proved above, from (30) we have $(q, r)\in I_f$ if $0<r<q$ (see (24)). Let

$$T(x) := \sum_{n\leqq x} f(n).$$

Then, we have

$$\sum_{\substack{n\leqq x\\ n\equiv 0 \bmod q}} f(n) = \sum_{m\leqq x/q} \{f(q)+f(m)\} = \left[\frac{x}{q}\right] f(q)+T\left(\frac{x}{q}\right).$$

Let r be an integer for which $0<r<q$. Then $(q, r)\in I_f$, and so

$$\sum_{\substack{n\leqq x\\ n\equiv r \bmod q}} f(n) = \sum_{qm+r\leqq x} \{f(qm+r)-f(qm)\} + \sum_{qm+r\leqq x} f(qm) = o(x)+\left[\frac{x}{q}\right] f(q)+T\left(\frac{x}{q}\right).$$

These imply that

$$T(x) = q\left[\frac{x}{q}\right] f(q)+qT\left(\frac{x}{q}\right)+o(x) = xf(q)+qT\left(\frac{x}{q}\right)+o(x)$$

as $x\to\infty$, from which we get

$$\frac{f(q)}{\log q} = \lim_{x\to\infty} \frac{T(x)}{x\log x} =: U.$$

From this and using $f\in\mathscr{A}^*$ the proof of Lemma 2 is finished.

3. Proof of Theorem 2. Assume that $f \in \mathscr{A}$ satisfies the condition (3). Then from Lemma 1 we get that

$$f(BCC_A Q^k) = kf(BCC_A Q) - (k-1)f(BCC_A) \tag{31}$$

holds for every positive integer k and Q, where C_A denotes the product of all prime divisors of C which are prime to A.

For each prime p let $e = e(p)$ be a non-negative integer for which $p^e \| BCC_A$. Then for all integers $\beta \geqq e$ we deduce from (31) that

$$f(p^{\beta+1}) - f(p^\beta) = f(p^{e+1}) - f(p^e). \tag{32}$$

Now we write

$$f(n) = f_1(n) + F(n),$$

where f_1 is a completely additive function defined as follows:

$$f_1(p) := f(p^{e+1}) - f(p^e), \quad e = e(p). \tag{33}$$

Then, from (32) and (33) it follows that

$$F(p^{\beta+1}) = F(p^\beta),$$

which implies

$$F(p^k) = F[(p^k, BCC_A)] \quad (k = 0, 1, 2, \ldots).$$

Thus, we have

$$F(n) = F[(n, BCC_A)] \quad (n = 1, 2, 3, \ldots). \tag{34}$$

We shall prove that $f_1 = U \log$ for some constant U.

We note that by (3) we have

$$\sum_{n \leqq x} |f(ABC_A n + B) - f(BCC_A n) - D| = o(x). \tag{35}$$

By using $f = f_1 + F$ and (34) we get that

$$f(ABC_A n + B) - f(BCC_A n) - D = f_1(ABC_A n + B) - f_1(BCC_A n) + F(ABC_A n + B) -$$
$$- F(BCC_A n) - D = f_1(ABC_A n + B) - f_1(n) - \{f_1(BCC_A) - F(B) + F(BCC_A) + D\}$$

and so, by (35) and Lemma 2, there is a complex constant U such that $f_1 = U \log$. This completes the proof of Theorem 2.

4. Proof of Theorem 1. Assume that $f, g \in \mathscr{A}$ satisfy the condition (2), i.e.

$$\sum_{n \leqq x} |g(an + b) - f(n) - d| = o(x), \tag{36}$$

where a and b are positive integers and d is a complex constant.

For each positive integer N we have

$$(abN+1, a(abN+1)n+b) = 1$$

and

$$(abN+1)(a(abN+1)n+b) = a[(abN+1)^2 n+b^2 N]+b$$

for every positive integer n. Thus, by using the additivity of f, we get

$$f[(abN+1)^2 n+b^2 N]-f[(abN+1)n]-g(abN+1) =$$
$$= -\{g[(abN+1)(a(abN+1)n+b)]-f[(abN+1)^2 n+b^2 N]-d\}+$$
$$+\{g[a(abN+1)n+b]-f[(abN+1)n]-d\},$$

which with (36) implies that

$$\sum_{n\leq x} |f[(abN+1)^2 n+b^2 N]-f[(abN+1)n]-g(abN+1)| = o(x). \tag{37}$$

Applying Lemma 1 with $A=(abN+1)^2$, $B=b^2 N$ and $C=(abN+1)$ it follows from (37) that

$$f[b^2(abN+1)NQ^k] = kf[b^2(abN+1)NQ]-(k-1)f[b^2(abN+1)N] \tag{38}$$

holds for every positive integer k and Q. Since (38) holds for each fixed positive integer N, so (38) also holds for every positive integer N.

For each prime p, let N_p be the smallest positive integer for which $p \nmid abN_p+1$. It is obvious that $N_p \in \{1, 2\}$ for all primes p. We apply (38) with $Q=p$ and $N=N_p$ to get

$$f(b^2 N_p p^k) = kf(b^2 N_p p)-(k-1)f(b^2 N_p). \tag{39}$$

Similiarly, as in the proof of Theorem 2, we can deduce from (39) that there are functions $f_1 \in \mathscr{A}^*$ and $F \in \mathscr{A}$ such that

$$f = f_1 + F \tag{40}$$

and

$$F(p^k) = F[(p^k, b^2 N_p)] \quad (k = 0, 1, 2 \ldots), \tag{41}$$

where p is a prime number. Since $N_p \in \{1, 2\}$, one can check from (41) and the fact $(b, N_2)=1$ that

$$F(n) = F[(n, b^2)]+F[(n, N_2)] \quad (n = 1, 2, 3, \ldots). \tag{42}$$

By using (40) and (42), we have

$$f[(abN+1)^2 N_2 m+b^2 N]-f[(abN+1)N_2 m]-g(abN+1) = \tag{43}$$
$$= f_1[(abN+1)^2 N_2 m+b^2 N]-f_1(m)-D,$$

where

$$D := g(abN+1)+f_1[(abN+1)N_2]-F[(abN+1)^2N_2m+b^2N]+F[(abN+1)N_2m] =$$
$$= g(abN+1)+f_1[(abN+1)N_2]-\{F[(m,b^2)]+F[(N,N_2)]\}+\{F[(m,b^2)]+F(N_2)\} =$$
$$= g(abN+1)+f_1((abN+1)N_2)+F[(ab+1,N_2)].$$

Applying (37) with $n=N_2m$, by (43) we have

$$\sum_{n\leqq x} |f_1[(abN+1)^2N_2m+b^2N]-f_1(m)-D| = o(x),$$

which, by using Lemma 2, implies

(44) $$f_1 = U\log \quad \text{for some constant } U$$

and

$$g(abN+1)+F[(abN+1,N_2)] = f_1(abN+1) = U\log(abN+1).$$

The last equality holds for every positive integer N, consequently

$$g(m)+F[(m,N_2)] = U\log m$$

holds for all positive integers m which are prime to ab. Let

(45) $$G(m) := g(m)-U\log m \quad (m = 1, 2, 3, \ldots).$$

Then, we have

(46) $$G(m) = 0 \quad \text{if} \quad (m, 2ab) = 1.$$

Finally, we shall prove that

$$G(an+b)-F(n)-d+U\log a = 0 \quad (n = 1, 2, 3, \ldots),$$

which with (40), (44), (45) gives the proof of Theorem 1.

Since

$$G(an+b)-F(n)-d+U\log a =$$
$$= \{g(an+b)-f(n)-d\}-\{U\log(an+b)-U\log n-U\log a\}$$

we obtain from (36) that

(47) $$\sum_{n\leqq x} |G(an+b)-F(n)-d+U\log a| = o(x).$$

Let r be an arbitrary integer for which $0\leqq r<2b^2$. Then we get from (42) and (47) that

$$F(2b^2m+r) = F(r) \quad (m = 1, 2, \ldots)$$

and

(48) $$\sum_{m\leqq x} |G(2ab^2m+ar+b)-F(r)-d+U\log a| = o(x).$$

Let M be a positive integer. By (46), we have $G(2ab^2t+1)=0$ $(t=1, 2, \ldots)$, con-

sequently

$$G(2ab^2M+ar+b)-F(r)-d+U\log a = \tag{49}$$
$$= G(2ab^2M+ar+b)+G(2ab^2t+1)-F(r)-d+U\log a =$$
$$= G\big[2ab^2\big((2ab^2M+ar+b)t+M\big)+ar+b\big]-F(r)-d+U\log a$$

holds for every positive integer t. Thus, we get from (48) and (49) that

$$\sum_{t\leqq x} |G(2ab^2M+ar+b)-F(r)-d+U\log a| = o(x),$$

which implies

$$G(2ab^2M+ar+b)-F(r)-d+U\log a = 0 \tag{50}$$

for each positive integer M, i.e. (50) holds for every positive integer M. Since r is an arbitrary integer for which $0\leqq r<2b^2$, and (50) holds for every positive integer M, we have

$$G(an+b)-F(n)-d+U\log a = 0 \quad (n = 1, 2, \ldots).$$

This completes the proof of Theorem 1.

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EÖTVÖS LORÁND UNIVERSITY
COMPUTER CENTER
BOGDÁNFY U. 10/B
H—1117 BUDAPEST

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On additive functions with values in a compact Abelian group

BUI MINH PHONG

1. Introduction

Let G be an additively written, metrically compact Abelian topological group, $\mathbf{N}$ be the set of all positive integers. A function $f\colon \mathbf{N}\to G$ is called a completely additive, if

$$f(nm)=f(n)+f(m)$$

holds for all $n, m\in\mathbf{N}$. Let $\mathscr{A}_G^*$ denote the class of all completely additive functions $f\colon \mathbf{N}\to G$.

Let $A>0$ and $B\neq 0$ be fixed integers. We shall say that an infinite sequence $\{x_\nu\}_{\nu=1}^\infty$ in G is of property $D[A, B]$ if for any convergent subsequence $\{x_{\nu_n}\}_{n=1}^\infty$ the sequence $\{x_{A\nu_n+B}\}_{n=1}^\infty$ has a limit, too. We say that it is of property $E[A, B]$ if for any convergent subsequence $\{x_{A\nu_n+B}\}_{n=1}^\infty$ the sequence $\{x_{\nu_n}\}_{n=1}^\infty$ is convergent. We shall say that an infinite sequence $\{x_\nu\}_{\nu=1}^\infty$ in G is of property $\Delta[A, B]$ if the sequence $\{x_{A\nu+B}-x_\nu\}_{\nu=1}^\infty$ has a limit.

Let $\mathscr{A}_G^*(D[A, B])$, $\mathscr{A}_G^*(E[A, B])$ and $\mathscr{A}_G^*(\Delta[A, B])$ be the classes of those $f\in\mathscr{A}_G^*$ for which $\{x_\nu=f(\nu)\}_{\nu=1}^\infty$ is of property $D[A, B]$, $E[A, B]$ and $\Delta[A, B]$, respectively.

It is obvious that

$$\mathscr{A}_G^*(\Delta[A, B])\subseteqq\mathscr{A}_G^*(D[A, B])\quad\text{and}\quad\mathscr{A}_G^*(\Delta[A, B])\subseteqq\mathscr{A}_G^*(E[A, B]).$$

Z. Daróczy and I. Kátai proved in [1] that

$$\mathscr{A}_G^*(\Delta[1, 1])=\mathscr{A}_G^*(D[1, 1]),$$

and in [2] they deduced the following assertion: If $f\in\mathscr{A}_G^*(\Delta[1,1)$, then there exists a continuous homomorphism $\Psi\colon \mathbf{R}_x\to G$, $\mathbf{R}_x$ denotes the multiplicative group of the positive reals, such that $f(n)=\Psi(n)$ for all $n\in\mathbf{N}$.

For the case $A=2$ and $B=-1$ the complete characterization of $\mathscr{A}_G^*(D[2, -1])$ and $\mathscr{A}_G^*(\Delta[2, -1])$ has been given by Z. Daróczy and I. Kátai [3], [4].

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In a recent paper [5] we gave a complete characterization of $\mathscr{A}_G^*(E[A, B])$ and $\mathscr{A}_G^*(\Delta[A, B])$. Namely we showed that

$$\mathscr{A}_G^*(E[A, B]) = \mathscr{A}_G^*(\Delta[A, B])$$

and

$$\mathscr{A}_G^*(\Delta[A, B]) = \mathscr{A}_G^*(\Delta[1, 1]).$$

In the other words, if $f \in \mathscr{A}_G^*(E[A, B]) = \mathscr{A}_G^*(\Delta[A, B])$, then there is a continuous homomorphism Ψ: $\mathbf{R}_x \to G$ such that $f(n) = \Psi(n)$ for all $n \in \mathbf{N}$.

Our main purpose in this paper is to give a complete determination of $\mathscr{A}_G^*(D[A, B])$. We note that it is enough to characterize those classes $\mathscr{A}_G^*(D[A, B])$ for which $(A, B) = 1$, since

$$\mathscr{A}_G^*(D[Ad, Bd]) = \mathscr{A}_G^*(D[A, B])$$

holds for each $d \in \mathbf{N}$.

We shall prove the following

Theorem. *Let $A > 0$ and $B \neq 0$ be fixed integers for which $(A, B) = 1$ and let G be a metrically compact Abelian topological group. If $f \in \mathscr{A}_G^*(D[A, B])$, then there are $U \in \mathscr{A}_G^*$ and a continuous homomorphims Φ: $\mathbf{R}_x \to G$, $\mathbf{R}_x$ denotes the multiplicative group of positive reals, such that*

(I) $$f(n) = \Phi(n) + U(n) \quad \forall n \in \mathbf{N},$$

(II) $$U(n + A) = U(n) \quad \forall n \in \mathbf{N},\ (n, A) = 1,$$

(III) *If X_1, Γ denote the set of all limit points of $\{\Phi(n) | n \in \mathbf{N}\}$ and $\{U(n) | n \in \mathbf{N}\}$, respectively, then*

$$X_1 \cap \Gamma = \{0\}$$

and Γ is the smallest closed group generated by

$$\{U(m) | 1 \leqq m < A, (m, A) = 1\} \quad \textit{and} \quad \{U(p) | p \text{ is prime}, p | A\}.$$

Conversely, let Φ: $\mathbf{R}_x \to G$ be an arbitrary continuous homomorphism, X_1 be the smallest compact supgroup generated by $\{\Phi(n) | n \in \mathbf{N}\}$. Let $U \in \mathscr{A}_G^*$ be so chosen that $U(n + A) = U(n)$ for all $n \in \mathbf{N}$, $(n, A) = 1$ and the smallest closed group Γ generated by $U(\mathbf{N})$ has the property $X_1 \cap \Gamma = \{0\}$. Then the function

$$f(n) := \Phi(n) + U(n)$$

belongs to $\mathscr{A}_G^*(D[A, B])$.

2. Preliminary lemmas

In this section we shall prove some results which will be used in the proof of our theorem.

Lemma 1. *We have*

$$\mathscr{A}_G^*(D[A, B]) \subseteqq \mathscr{A}_G^*(D[A, 1])$$

for all fixed integers $A>0$ *and* $B\neq 0$.

Proof. Let $A>0$, $B\neq 0$ be fixed integers. Assume that

$$f\in\mathscr{A}_G^*(D[A, B]).$$

Let

$$n_1 < \ldots < n_\nu < \ldots \quad (n_\nu\in\mathbf{N})$$

be an infinite sequence for which the sequence $\{f(n_\nu)\}_{\nu=1}^\infty$ is convergent. Then, it is obvious that the sequence $\{f(|B|n_\nu)\}_{\nu=1}^\infty$ has also a limit, consequently we get from the definition of $\mathscr{A}_G^*(D[A, B])$ that

$$\lim_{\nu\to\infty} f\left[An_\nu+\frac{B}{|B|}\right] = \lim_{\nu\to\infty} f[A|B|n_\nu+B]-f(|B|)$$

exists as well. This implies in the case $B>0$ that $f\in\mathscr{A}_G^*(D[A, 1])$.

We now assume that $B<0$. In this case we have $f\in\mathscr{A}_G^*(D[A, -1])$. Since $\{f(n_\nu)\}_{\nu=1}^\infty$ is convergent, therefore the sequence $\{f(An_\nu^2)\}_{\nu=1}^\infty$ is convergent, too. Thus, by using the fact $f\in\mathscr{A}_G^*(D[A, -1])$, it follows that the following limit exists:

$$\lim_{\nu\to\infty} f(An_\nu+1) = \lim_{\nu\to\infty} f[(An_\nu)^2-1]-\lim_{\nu\to\infty} f[An_\nu-1].$$

This shows that $f\in\mathscr{A}_G^*(D[A, 1])$.

So we have proved Lemma 1.

In the following we assume that $A>0$, $B\neq 0$ are fixed integers and G is a metrically compact Abelian topological group. Let

$$f\in\mathscr{A}_G^*(D[A, B]).$$

We shall denote by X the set of limit points of $\{f(n)|n\in\mathbf{N}\}$, i.e. $g\in X$ if there exists a sequence

$$n_1 < \ldots < n_\nu < \ldots \quad (n_\nu\in\mathbf{N})$$

for which $f(n_\nu)\to g$. Let X_1 $(\subseteqq X)$ be the set of limit points of $\{f(An+1)|n\in\mathbf{N}\}$. Since $\mathbf{N}$ and the positive integers $m\equiv 1 \pmod{A}$ form semingroups, therefore $\{f(n)|n\in\mathbf{N}\}$ and $\{f(An+1)|n\in\mathbf{N}\}$ are semigroups as well. Thus, X and X_1 are closed semigroups in the compact group G, so by a known theorem (see [6], Theorem

(9.16)) they are compact subgroups in G. Since $0 \in X_1 \subseteqq X$ we have $f(n) \in X$ and $f(An+1) \in X_1$ for each $n \in \mathbf{N}$.

Let $g \in X$ and $f(n_\nu) \to g$ as $\nu \to \infty$. Then, by using Lemma 1, it follows that the sequence $\{f(An_\nu+1)\}_{\nu=1}^{\infty}$ is convergent. Let $f(An_\nu+1) \to g' (\in X_1)$. It is easily seen that g' is determined by g, and so the correspondence

$$H: g \to g' \quad (g \in X,\ g' \in X_1)$$

is a function.

Lemma 2. *The function* H: $X \to X_1$ *is continuous and*

$$H(X) = X_1.$$

Proof. We can prove Lemma 2 by the same method as was used in [1] (see Lemma 4 and Lemma 5), so we omit the proof.

Lemma 3. *We have*

$$H(g+h+f(A)) + H(g) = H(g+H(h+H(g))) \tag{2.1}$$

for all $g \in X$ *and* $h \in X$.

Proof. Let $g \in X$ and $h \in X$ be arbitrary elements. Let

$$n_1 < \ldots < n_\nu < \ldots \quad \text{and} \quad m_1 < \ldots < m_\nu < \ldots \quad (n_\nu, m_\nu \in \mathbf{N})$$

be such sequences for which $f(n_\nu) \to g$ and $f(m_\nu) \to h$. By using the following relation

$$(A^2 n_\nu m_\nu + 1)(An_\nu + 1) = An_\nu[Am_\nu(An_\nu + 1) + 1] + 1$$

and using the definition of H, we get immediately that (2.1) holds. So, we have proved Lemma 3.

Lemma 4. *Let*

$$E(f) := \{\varrho \in X \mid H(\varrho) = 0\}.$$

Then $E(f) \neq \emptyset$. *Furthermore, if* $\varrho \in E(f)$, *then*

$$H(k\varrho + (k-1)f(A)) = 0 \tag{2.2}$$

for every integer k. *In particular, we have*

$$H(-f(A)) = 0. \tag{2.3}$$

Proof. Since X_1 is a group, therefore $0 \in X_1$. Thus, it follows from $H(X) = X_1$ that there is at least one $\varrho \in X$ for which $H(\varrho) = 0$. Then $E(f) \neq \emptyset$. Furthermore, it is easily seen from (2.1) that

$$H(\varrho_1 + \varrho_2 + f(A)) = 0 \quad \text{if} \quad H(\varrho_1) = H(\varrho_2) = 0. \tag{2.4}$$

Assume that $\varrho\in E(f)$, i.e. $H(\varrho)=0$. By using (2.4) and induction on k we get immediately that (2.2) holds for every $k\in\mathbf{N}$. Let

$$V_\varrho = \{k(\varrho+f(A))\,|\,k\in\mathbf{N}\}.$$

Since (2.2) holds for every $k\in\mathbf{N}$, therefore we have

$$H(\delta-f(A)) = 0 \quad \text{for all} \quad \delta\in V_\varrho. \tag{2.5}$$

Let $\overline{V}_\varrho$ be the smallest closed set containing V_ϱ. It is clear that V_ϱ is a semigroup, therefore $\overline{V}_\varrho$ is a closed semigroup in G. Thus, by using a known theorem of [6], we get that $\overline{V}_\varrho$ is a compact group. Since H is continuous function and $\overline{V}_\varrho$ is the smallest closed set containing V_ϱ, it follows that (2.5) holds for all $\delta\in\overline{V}_\varrho$, consequently (2.2) holds for every integer k. So (2.2) is proved.

Finally, by applying (2.2) with $k=0$, we obtain (2.3).

The proof of Lemma 4 is finished.

Lemma 5. *We have*

$$H(g+\tau) = H(g)+\tau \tag{2.6}$$

for all $g\in X$ *and* $\tau\in X_1$.

Proof. We first prove that

$$H(\tau-f(A)) = \tau \quad \text{for all} \quad \tau\in X_1 \tag{2.7}$$

and

$$H(g-H(g)) = 0 \quad \text{for all} \quad g\in X. \tag{2.8}$$

Let $\tau\in X_1$. Then, it follows from $H(X)=X_1$ that there is one $h\in X$ such that $H(h)=\tau$. We apply (2.1) with $g=-f(A)$ and using (2.3), we have

$$H(H(h)-f(A)) = H(h),$$

which with $H(h)=\tau$ proves (2.7). It is clear that (2.8) is a consequence of (2.1) and (2.3) in the case $h+H(g)=-f(A)$.

We now prove Lemma 5.

Let $g\in X$ and $\tau\in X_1$ be arbitrary elements. By using (2.8), we have

$$H[(g+\tau)-H(g+\tau)] = 0$$

and

$$H[g-H(g)] = 0.$$

Applying Lemma 4 with $\varrho=g-H(g)$ and $k=-1$, we get that

$$H[-g+H(g)-2f(A)] = 0.$$

Let

$$\varrho_1 := g+\tau-H(g+\tau) \quad \text{and} \quad \varrho_2 := -g+H(g)-2f(A).$$

Then $H(\varrho_1)=H(\varrho_2)=0$, and so by (2.4) we have

$$H[(g+\tau-H(g+\tau))+(-g+H(g)-2f(A))+f(A)] = 0,$$

i.e.

$$H[(\tau-H(g+\tau)+H(g))-f(A)] = 0. \tag{2.9}$$

Since $\tau\in X_1$, $H(g+\tau)\in X_1$, $H(g)\in X_1$ and X_1 is a group, therefore

$$\tau-H(g+\tau)+H(g)\in X_1. \tag{2.10}$$

Finally, from (2.7), (2.9) and (2.10) we get that

$$\tau-H(g+\tau)+H(g) = 0,$$

which proves (2.6).

So we have proved Lemma 5.

Lemma 6. *We have*

$$H(g+h+f(A)) = H(g+h)+H(0) = H(g)+H(h) \tag{2.11}$$

for all $g\in X$ *and* $h\in X$.

Proof. Let $g\in X$ and $h\in X$. Since $H(h+H(g))\in X_1$ and $H(g)\in X_1$, by using Lemma 5, we have

$$H(g+H(h+H(g))) = H(g)+H(h+H(g)) = H(g)+H(h)+H(g).$$

This with (2.1) implies that

$$H(g+h+f(A)) = H(g)+H(h). \tag{2.12}$$

Thus, (2.12) holds for all $g\in X$ and $h\in X$.

On the other hand, we get from (2.12) that

$$H(g+h+f(A)) = H(g+h)+H(0).$$

This with (2.12) shows that (2.11) holds for all $g\in X$ and $h\in X$. The proof of Lemma 6 is finished.

3. Proof of the theorem

Assume that $A>0$ and $B\neq 0$ are fixed integers for which $(A,B)=1$ and G is a metrically compact Abelian topological group. Let

$$f\in\mathscr{A}_G^*(D[A,B]).$$

As in the Section 2, we denote by X and X_1 the set of limit points of $\{f(n)|\in\mathbf{N}\}$ and $\{f(An+1)|n\in\mathbf{N}\}$, respectively. Let $H\colon X\to X_1$ be a continuous function which is defined in Section 2, i.e., if $f(n_\nu)\to g$, then $f(An_\nu+1)\to H(g)$.

For an arbitrary $n \in \mathbf{N}$, let $S(n)$ be the product of all prime factors of n composed from the prime divisors of A, $R(n)$ be defined by $n = S(n) \cdot R(n)$, i.e. $(A, R(n)) = 1$ and every prime divisor of $S(n)$ is a divisor of A. Let $\bar{R}(n)$ be the smallest positive integer for which

$$\bar{R}(n) \equiv R(n) \pmod{A}.$$

It is obvious that $(\bar{R}(n), A) = 1$ and $1 \leqq \bar{R}(n) < A$.

Lemma 7. *Let*

$$U(n) := f[S(n) \cdot \bar{R}(n)] + H(0) - H(f[S(n) \cdot \bar{R}(n)]). \tag{3.1}$$

Then, we have

$$H(f(n)) - f(n) - H(0) + U(n) = 0 \tag{3.2}$$

for all $n \in \mathbf{N}$.

Proof. Let $\bar{H}$: $X \to X_1$ be the function which is defined by the relation $\bar{H}(g) = H(g) - H(0)$. Then, it is easily seen from Lemma 5 and Lemma 6 that

$$\bar{H}(g+h) = \bar{H}(g) + \bar{H}(h) \quad \forall g, h \in X, \tag{3.3}$$

$$\bar{H}(\tau) = \tau \quad \forall \tau \in X_1 \tag{3.4}$$

and

$$\bar{H}(X) = X_1. \tag{3.5}$$

For each $n \in \mathbf{N}$, let $c(n)$ be the smallest positive integer for which $R(n) \cdot c(n) \equiv 1 \pmod{A}$. Then, it is obvious that

$$f[R(n) \cdot c(n)] \in X_1 \quad \text{and} \quad f[\bar{R}(n) \cdot c(n)] \in X_1$$

hold for every $n \in \mathbf{N}$. By using (3.3) and (3.4), we deduce that

$$\bar{H}[f(n)] + \bar{H}[f(c(n))] = \bar{H}[f(n \cdot c(n))] = f[R(n) \cdot c(n)] + \bar{H}[f(S(n))]$$

and

$$\bar{H}[f(\bar{R}(n))] + \bar{H}[f(c(n))] = \bar{H}[f(\bar{R}(n) \cdot c(n))] = f(\bar{R}(n) \cdot c(n)).$$

These imply that

$$\bar{H}[f(n)] - \bar{H}[f(\bar{R}(n))] = f(R(n)) - f(\bar{R}(n)) + \bar{H}[f(S(n))],$$

consequently

$$\bar{H}[f(n)] - f(n) + \{f(S(n) \cdot \bar{R}(n)) - \bar{H}[f(S(n) \cdot \bar{R}(n))]\} = 0.$$

This with (3.1) proves (3.2)

Lemma 8. *We have*

(i) $U \in \mathscr{A}_G^*$,

(ii) $U(n+A) = U(n)$ *for all* $n \in \mathbf{N}$, $(n, A) = 1$,

(iii) *If* $\{a_1, \ldots, a_{\varphi(A)}\}$ *is a reduced residue system moduls* A, *then* $U(a_1), \ldots, U(a_{\varphi(A)})$ *form a group in* G.

(iv) *Let* Γ *denote the set of all limit points of* $\{U(n)|n\in\mathbf{N}\}$. *Then* Γ *is the smallest closed group generated by* $U(a_1), \ldots, U(a_{\varphi(A)}), U(p_1), \ldots, U(p_{\omega(A)})$, *where* $\{a_1, \ldots, a_{\varphi(A)}\}$ *is a reduced residue system moduls* A *and* $p_1, \ldots, p_{\omega(A)}$ *are all distinct prime factors of* A. *Furthermore, we have*

$$X_1 \cap \Gamma = \{0\}.$$

Proof. Parts (i) and (ii) follow at once from the definition of U and Lemma 7. The part (iii) is a consequence of (i) and (ii). To prove (iv) we first note that Γ is a closed semigroup in G, and so Γ is a group by Theorem (9.16) of [6]. Hence by (ii) it follows that Γ is the smallest closed group generated by $U(a_1), \ldots, U(a_{\varphi(A)}), U(p_1), \ldots$ $\ldots, U(p_{\omega(A)})$.

Since X_1, Γ are subgroups in G, therefore $0\in X_1\cap\Gamma$. Let us assume that $\delta\in$ $\in X_1\cap\Gamma$. Then there is a sequence $\{n_\nu\}_{\nu=1}^{\infty}$ for which $U(n_\nu)\to\delta$. Applying (3.2) with $n=n_\nu$, we have

$$H[f(n_\nu)]-f(n_\nu)-H(0)+U(n_\nu) = 0. \tag{3.6}$$

Since G is sequentially compact, therefore the sequence $\{f(n_\nu)\}_{\nu=1}^{\infty}$ contains at least one limit point. Let

$$f(n_{\nu_j}) \to g \quad (\in X).$$

Then, by (3.6) and using the fact H is continuous, we get

$$H(g)-g-H(0)+\delta = 0,$$

which with $H(g)-H(0)+\delta\in X_1$ implies that $g\in X_1$. So, by Lemma 5

$$\delta = g+H(0)-H(g) = 0.$$

Thus, we have proved that $X_1\cap\Gamma=\{0\}$. This completes the proof of (iv).

The proof of Lemma 8 is finished.

We now prove the theorem. We first show that

$$f(An+1)-H(f(n)) \to 0 \quad \text{as} \quad n\to\infty. \tag{3.7}$$

Assume the contrary. Let

$$f(An_\nu+1)-H(f(n_\nu)) \to \lambda \neq 0 \quad \text{as} \quad \nu\to\infty. \tag{3.8}$$

Since the sequence $\{f(n_\nu)\}_{\nu=1}^{\infty}$ contains at least one limit point, we can find a subsequence $\{n_{\nu j}\}_{j=1}^{\infty}$ of the sequence $\{n_\nu\}_{\nu=1}^{\infty}$ such that $f(n_{\nu_j})\to g$ $(\in X)$ as $j\to\infty$. Using the continuity of H, by (3.8) we have

$$H(g)-H(g) = \lambda,$$

which is contradiction. Thus, we have proved (3.7). From (3.2) and (3.7) we get

immediately

(3.9) $$f(An+1)-f(n)-H(0)+U(n)\to 0 \quad \text{as} \quad n\to\infty.$$

Let

$$F(n):=f(n)-U(n) \quad (n\in\mathbf{N}).$$

It is obvious by Lemma 8 that $F\in\mathscr{A}_G^*$ and

$$F(An+1)=f(An+1)-U(An+1)=f(An+1)$$

for all $n\in\mathbf{N}$. This with (3.9) implies

$$F(An+1)-F(n)-H(0)\to 0 \quad \text{as} \quad n\to\infty,$$

consequently $F\in\mathscr{A}_G^*(\Delta[A,1])$. It was proved in [5] that if $F\in\mathscr{A}_G^*(\Delta[A,1])$, then there is a continuous homomorphism Φ: $\mathbf{R}_x\to G$ such that $F(n)=\Phi(n)$ for all $n\in\mathbf{N}$, where $\mathbf{R}_x$ denotes the multiplicative group of positive reals. Thus, we have proved that

(3.10) $$f(n)=\Phi(n)+U(n),$$

where U satisfies the conditions (i)—(iv) of Lemma 8. By (3.2) and (3.10) we also have

$$\Phi(n)=H(f(n))-H(0) \quad \text{for all} \quad n\in\mathbf{N},$$

therefore it follows from (3.5) that the set of all limit points of $\{\Phi(n)|n\in\mathbf{N}\}$ is X_1.

So we have proved the first part of our theorem.

Finally, let Φ: $\mathbf{R}_x\to G$ be a continuous homomorphism and let $U\in\mathscr{A}_G^*$ be so chosen that

(3.11) $$U(n+A)=U(n) \quad \text{for all} \quad n\in\mathbf{N}, \quad (n,A)=1$$

and

$$X_1\cap\Gamma=\{0\}$$

where X_1,Γ denote the smallest closed groups in G which are generated by $\Phi(\mathbf{N})$ and $U(\mathbf{N})$, respectively.

Let

$$f(n):=\Phi(n)+U(n)\in\mathscr{A}_G^*.$$

Assume that for some subsequence $\{n_\nu\}_{\nu=1}^\infty$ of positive integers the sequence $\{f(n_\nu)_{\nu=1}^\infty$ converges. Then, by using $\Phi(n_\nu)\in X_1$, $U(n_\nu)\in\Gamma$ and $X_1\cap\Gamma=\{0\}$, we deduce that the sequences $\{\Phi(n_\nu)\}_{\nu=1}^\infty$ and $\{U(n_\nu)\}_{\nu=1}^\infty$ are convergent, therefore by (3.11) and $(A,B)=1$ we see that

$$\lim_{\nu\to\infty} f(An_\nu+B)=\lim_{\nu\to\infty}\{\Phi(An_\nu+B)+U(An_\nu+B)\}=$$
$$=\lim_{\nu\to\infty}\Phi(An_\nu+B)+U(B)=\Phi(A)+U(B)+\lim_{\nu\to\infty}\Phi(n_\nu)$$

exists as well. So we have proved that $f\in\mathscr{A}_G^*(D[A,B])$.

The proof of our theorem is finished.

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EÖTVÖS LORÁND UNIVERSITY
COMPUTER CENTER
BUDAPEST, H—1117
BOGDÁNFY ÚT 10/B.

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Moufang Lie loops and homogeneous spaces

PÉTER T. NAGY

0. Introduction

The classical model of the Moufang Lie loops is the multiplication on S^7 defined by Cayley numbers of norm one. This multiplication is in a close relation with the spherical geometry of S^7, which is a symmetric Riemannian space of constant curvature. This connection has been generalised by O. Loos in [9] to any Moufang loop by proving that the modified local multiplication $(x, y) \to x^{1/2} \cdot y \cdot x^{1/2}$ gives the (local) geodesic loop multiplication of a symmetric space (cf. [6]). The analogous correspondance gives a differential geometric machinery for the investigation of analytic Bol and Moufang loops (cf. [2], [4], [10]). For group multiplications, one has a 1-parameter family of modified local loop multiplications $x \underset{(\sigma)}{\circ} y = x^\sigma \cdot y \cdot x^{1-\sigma}$ $(\sigma \in \mathbf{R})$ investigated by M. A. Akivis [1], that are geodesic loops.

This paper is devoted to study geodesic loops of reductive homogeneous spaces associated with Moufang loops. Such a geodesic loop is gotten from the modified multiplication $x \underset{(\sigma)}{\circ} y = x^\sigma \cdot y \cdot x^{1-\sigma}$ for each $\sigma \in \mathbf{R}$ in the case of the groups. For Moufang loops one obtains geodesic loop of reductive homogeneous space only for $\sigma = \frac{1}{3}, \frac{1}{2}$, and $\frac{2}{3}$. For $\sigma = \frac{1}{2}$, the geodesic loop of the symmetric space was investigated by O. Loos in [9].

For $\sigma = \frac{2}{3}$, we give a description of the corresponding reductive space structure in this paper. An analogues description can be obtained for $\sigma = \frac{1}{3}$ using the right multiplication instead of the left one. Our method to describe this reductive space structure is to represent the original loop multiplication on the Moufang loop by a geodesic loop multiplication of an invariant connection on a reductive

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homogeneous space. Then we deform this latter geodesic loop multiplication as $x \underset{(\sigma)}{\circ} y = x^{\sigma} \cdot y \cdot x^{1-\sigma}$ $\left(\sigma = \frac{2}{3}\right)$ and prove that this gives the geodesic loop multiplication of the canonical connection of the reductive homogeneous space.

1. The local loops $x \underset{(\sigma)}{\circ} y = x^{\sigma} \cdot y \cdot x^{1-\sigma}$ $(\sigma \in \mathbf{R})$ on a group

Let $\mathcal{G}$ be a connected Lie group and let us consider the action of the group $\mathcal{K} = \mathcal{G} \times \mathcal{G}$ on $\mathcal{G}$ given by $((g_1, g_2), g) \in (\mathcal{G} \times \mathcal{G}) \times \mathcal{G} \to g_1 \cdot g \cdot g_2^{-1} \in \mathcal{G}$. The isotropy subgroup of this action is the diagonal $\mathcal{H} = \Delta(\mathcal{G} \times \mathcal{G}) = \{(g, g); g \in \mathcal{G}\}$. We denote by $\mathfrak{m}^{\sigma}$ the transversal subspace in the Lie algebra $\mathcal{T}_{(e,e)}(\mathcal{G} \times \mathcal{G}) = \mathfrak{g} + \mathfrak{g}$ of $\mathcal{K} = \mathcal{G} \times \mathcal{G}$ to the diagonal $\Delta(\mathfrak{g} + \mathfrak{g})$ defined by $\mathfrak{m}^{\sigma} = \{(\sigma X, (\sigma - 1) X), X \in \mathcal{G}\}$, where $\mathfrak{g} = \mathcal{T}_e \mathcal{G}$ and $\sigma \in \mathbf{R}$. It is clear that $\mathrm{Ad}_{\mathcal{H}} \mathfrak{m}^{\sigma} \subset \mathfrak{m}^{\sigma}$, hence the subspace $\mathfrak{m}^{\sigma}$ is a reductive complement of $\Delta(\mathfrak{g} + \mathfrak{g})$ in $\mathfrak{g} + \mathfrak{g}$. Let ∇^{σ} denote the canonical connection of the reductive homogeneous space $\mathcal{K}/\mathcal{H}$ given by $\mathfrak{m}^{\sigma}$.

Theorem 1.1. *The geodesic loop multiplication of the canonical connection ∇^{σ} of the reductive homogeneous space $\mathcal{K}/\mathcal{H}$ defined by*

$$\exp_x^{\sigma} \circ \tau_{e,x}^{\sigma} \circ (\exp_e^{\sigma})^{-1} y, \quad x, y \in \mathcal{G}$$

can be expressed in a normal neighbourhood of $e \in \mathcal{G}$ as

$$\exp_x^{\sigma} \circ \tau_{e,x}^{\sigma} \circ (\exp_e^{\sigma})^{-1} y = x \underset{(\sigma)}{\circ} y = x^{\sigma} \cdot y \cdot x^{1-\sigma},$$

where $\exp_x^{\sigma}$ denotes the exponential map at $x \in \mathcal{G}$ and $\tau_{e,x}^{\sigma}$ is the parallel translation $\mathcal{T}_e \mathcal{G} \to \mathcal{T}_x \mathcal{G}$ along the geodesic through e and x with respect to the connection ∇^{σ}.

Proof. If $(\sigma X, (1-\sigma) X) \in \mathfrak{m}^{\sigma}$ then the orbit of the 1—parameter subgroup $\exp t(\sigma X, (1-\sigma) X)$ through $e \in \mathcal{G}$ is

$$\exp t(\sigma X, (1-\sigma) X) e = (\exp t\sigma X, \exp t(\sigma - 1) X) e =$$
$$= \exp t\sigma X \cdot e \cdot \exp t(1-\sigma) X = \exp tX.$$

Using Proposition 2.4 in [7, Chap. X.] we obtain the parallel translation $\tau_{e,x}^{\sigma}$ in the form $\tau_{e,\exp tX}^{\sigma} = \mathcal{T}_e \lambda_{\exp t\sigma X} \circ \mathcal{T}_e \varrho_{\exp t(1-\sigma) X}$, where λ_x and ϱ_x denote the left and the right multiplication maps on $\mathcal{G}$, respectively. Since the mapping $\lambda_{\exp t\sigma X} \circ \varrho_{\exp t(1-\sigma) X}$ is an affine transformation of the connection ∇^{σ}, the geodesics of this connection have the form

$$\lambda_{\exp t\sigma X} \circ \varrho_{\exp t(1-\sigma) X} \exp sY = \exp t\sigma X \cdot \exp sY \cdot \exp t(1-\sigma) X =$$
$$= \exp X \cdot \exp s(\mathrm{Ad}_{\exp t(\sigma-1) X} Y).$$

It follows that the geodesics of the connections ∇^σ $(\sigma\in\mathbf{R})$ are independent of the parameter σ. Hence the geodesic loop multiplication of the connection ∇^σ satisfies the equations

$$\exp^\sigma_{\exp X}\circ\tau^\sigma_{e,\exp X}\circ(\exp^\sigma_{\exp Y})^{-1}\exp Y=\exp^\sigma_{\exp X}\circ\tau^\sigma_{e,\exp X}Y=$$

$$=\exp^1_{\exp X}\circ\mathcal{T}_e\lambda_{\exp\sigma X}\circ\mathcal{T}_e\varrho_{\exp(1-\sigma)X}Y=\exp^1_{\exp X}\circ\mathcal{T}_e\lambda_{\exp X}\circ\mathrm{Ad}_{\exp(\sigma-1)X}Y.$$

Since the group multiplication coincides with the geodesic loop multiplication of its left canonical connection ∇^1 we obtain

$$\exp^\sigma_{\exp X}\circ\tau^\sigma_{e,\exp X}\circ(\exp^\sigma_{\exp Y})^{-1}\exp Y=\exp X\cdot\exp\circ\mathrm{Ad}_{\exp(\sigma-1)X}Y=$$

$$=\exp\sigma X\cdot\exp Y\cdot\exp(1-\sigma)X,$$

that proves the theorem.

2. The left canonical connection of a loop

Let $\mathcal{L}$ be a smooth loop with identity element $e\in\mathcal{L}$. We define the translated loop multiplications centered at $a\in\mathcal{L}$ by the formula

$$x\underset{a}{\cdot}y:=x\cdot a\backslash y,\quad (a\in\mathcal{L})$$

where $x\cdot y=x\underset{e}{\cdot}y$ and $x\cdot x\backslash y=y$. This loop is isotopic to the original loop multiplication and has $a\in\mathcal{L}$ as identity element.

Let λ_x denote the left multiplication map of the loop $\mathcal{L}$ and $\mathcal{T}_e\lambda_x\colon\mathcal{T}_e\mathcal{L}\to\mathcal{T}_x\mathcal{L}$ be its tangent map.

Definition 2.1. The *left canonical connection* ∇ of the loop $\mathcal{L}$ is defined by the parallel vector fields

$$X(x):=\mathcal{T}_e\lambda_x X(e).$$

Since these vector fields are globally defined, the connection ∇ is obviously flat.

Proposition 2.2. *The left canonical connection of the translated loop multiplication $x\underset{a}{\cdot}y$ $(a\in\mathcal{L})$ on $\mathcal{L}$ coincides with the left canonical connection ∇ of the original loop.*

Proof. The assertion follows from the definition, because the left multiplication map of the translated loop multiplication $x\underset{a}{\cdot}y$ is $\lambda_x\lambda_a^{-1}$.

Proposition 2.3. *The covariant derivative* $(\nabla_Z T)(X, Y)$ *of the torsion tensor field* $T(X, Y)$ *of the connection* ∇ *is*

$$(\nabla_Z T)(X, Y) = \langle Z, Y, X\rangle - \langle Z, X, Y\rangle,$$

where $\langle X, Y, Z\rangle_a$ *is the associator of the translated loop multiplication* $x \underset{a}{\cdot} y$.

Proof. We denote by $\tau(x)$ the mapping $\mathscr{T}_e\lambda_x\colon \mathscr{T}_e\mathscr{L} \to \mathscr{T}_x\mathscr{L}$. Using the covariant constant vector fields $X(x)=\tau(x)X_e$ and $Y(x)=\tau(x)Y_e$, the torsion tensor field takes the form

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = -[X, Y] = -\dot{\tau}(x)Y_e\big(X(x)\big) + \dot{\tau}(x)X_e\big(Y(x)\big) =$$
$$= -\dot{\tau}(x)Y_e\big(\tau(x)X_e\big) + \dot{\tau}(x)X_e\big(\tau(x)Y_e\big),$$

where $\dot{\tau}(Y)$ denotes the derivative of the mapping τ by the variable x in the direction Y. It follows

$$(\nabla_Z T)(X, Y)_e = Z[\tau^{-1}(x)T(X, Y)]_e = -\dot{\tau}(e)T(X, Y)_e(Z_e) + Z[T(X, Y)]_e =$$
$$= -\dot{\tau}(e)\big[-\dot{\tau}(x)Y_e\big(\tau(x)X_e\big) + \dot{\tau}(x)X_e\big(\tau(x)Y_e\big)\big](Z) - \ddot{\tau}(e)Y_e(X_e, Z_e) -$$
$$-\dot{\tau}(e)Y_e\big(\dot{\tau}(e)X_e(Z_e)\big) + \ddot{\tau}(e)X_e(Y_e, Z_e) + \dot{\tau}(e)X_e\big(\dot{\tau}(e)Y_e(Z_e)\big).$$

We consider now a local coordinate system defined on a neighbourhood of the identity e on which the loop multiplication is of the form

$$x \cdot y = x + y + q(x, y) + r(x, x, y) + s(x, y, y) + \{\text{higher order terms}\},$$

where q is a bilinear, r and s are trilinear maps on the coordinate vector space. Then we can write

$$T(X, Y)_e = -q(X, Y) + q(Y, X),$$
$$(\nabla_Z T)(X, Y)_e = q\big(Z, q(X, Y)\big) - q\big(Z, q(Y, X)\big) - r(X, Z, Y) - r(Z, X, Y) -$$
$$-q\big(q(Z, X), Y\big) + r(Y, Z, X) + r(X, Z, Y) + q\big(q(Z, Y), X\big).$$

By the Theorem IX. 6.6. in [5], the commutator and the associator of the loop have the forms

$$[X, Y] = q(X, Y) - q(Y, X)$$

and

$$\langle X, Y, Z\rangle = q\big(q(X, Y), Z\big) - q\big(X, q(Y, Z)\big) + r(X, Y, Z) +$$
$$+ r(Y, X, Z) - s(X, Y, Z) - s(X, Z, Y),$$

respectively. Hence we obtain

$$(\nabla_Z T)(X, Y)_e = \langle Z, Y, X\rangle - \langle Z, X, Y\rangle,$$

which proves the assertion at $e \in \mathscr{L}$. For $a \neq e$ we consider in the same way the loop multiplication $x \underset{a}{\cdot} y$ instead of $x \cdot y$ to prove the assertion.

3. Alternative family of loops

Definition 3.1. The family of loop multiplications $x \underset{a}{\cdot} y$ defined on $\mathscr{L}$ is called *alternative* if the identities

$$x \underset{a}{\cdot} (x \underset{a}{\cdot} y) = (x \underset{a}{\cdot} x) \underset{a}{\cdot} y, \quad x \underset{a}{\cdot} (y \underset{a}{\cdot} y) = (x \underset{a}{\cdot} y) \underset{a}{\cdot} y, \quad x \underset{a}{\cdot} (y \underset{a}{\cdot} x) = (x \underset{a}{\cdot} y) \underset{a}{\cdot} x,$$

are satisfied for all $x, y, a \in \mathscr{L}$.

Proposition 3.2. *If the family of loop multiplications $x \underset{a}{\cdot} y$ defined on $\mathscr{L}$ is alternative then the torsion tensor field of its canonical connection ∇ satisfies*

$$(\nabla_Z T)(X, Y) = \frac{1}{3}\{T(T(X, Y), Z) + T(T(Y, Z), X) + T(T(Z, X), Y)\}.$$

Proof. By the assumption of the alternativity of the loop system the associator is

$$\langle X, Y, Z \rangle = \frac{1}{6}\{[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y]\}$$

(cf. Remark IX. 6.18. in [5]). Since $T(X, Y) = -[X, Y]$, the assertion follows from Proposition 1.3.

Proposition 3.3. *The connection $\mathring{\nabla}$ defined by $\mathring{\nabla}_X Y = \nabla_X Y + \frac{1}{3} T(X, Y)$ is complete. Its torsion and curvature tensor fields $\mathring{T}(X, Y)$, $\mathring{R}(X, Y)Z$ satisfy*

$$\mathring{\nabla}_Z \mathring{T} = 0, \quad \mathring{\nabla}_Z \mathring{R} = 0,$$

i.e. the manifold $\mathscr{L}$ with the connection $\mathring{\nabla}$ is a locally reductive homogeneous space.

Proof. Since the connections ∇ and $\mathring{\nabla}$ have the same geodesics $\mathring{\nabla}$ is complete. The relations $\mathring{\nabla}_Z \mathring{T} = 0, \mathring{\nabla}_Z \mathring{R} = 0$ follows by standard calculations from Proposition 3.2.

Theorem 3.4. *Let $\mathscr{L}$ be a connected and simply connected manifold equipped with an alternative family of loop multiplications $x \underset{a}{\cdot} y$. Then $\mathscr{L}$ can be represented as a global reductive homogeneous space $\mathscr{L} = \mathscr{G}/\mathscr{H}$, where the Lie algebras of the Lie groups $\mathscr{G}, \mathscr{H}$ satisfy $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ and $\mathrm{Ad}_{\mathscr{H}} \mathfrak{m} \subset \mathfrak{m}$. If $\mathring{\nabla}$ is the canonical connection of the homogeneous space $\mathscr{G}/\mathscr{H}$ and $\mathring{T}$ is its torsion tensor field, then the left canonical connection ∇ of the family of loop multiplications takes the form*

$$\nabla_X Y = \mathring{\nabla}_X Y - \mathring{T}(X, Y).$$

Proof. The assertion follows from the preceding propositions because a complete, connected and simply connected locally reductive homogeneous space is a global one (cf. Chap. X. Theorem 2.8. in [7]).

4. Geodesic loops of the left canonical connection

Proposition 4.1. *Let $\mathscr{L}$ be a Moufang loop and let ∇ denote its left canonical connection. If $x(t)$ is a geodesic through the point $a=x(0)$ with respect to ∇ then $x(t)$ is a one-parameter subgroup of the translated multiplication $x \underset{a}{\cdot} y$, and the parallel translation along $x(t)$ coincides with the map $\tau(x(t))\circ\tau(a)^{-1}$, where $\tau(x)=\mathscr{T}_e\lambda_x$. Consequently the loop multiplication can be written in the form*

$$x \underset{a}{\cdot} y = \exp_x\circ\tau(x)\circ\tau(a)^{-1}\circ\exp_a^{-1} y$$

in a neighbourhood of a.

Proof. Since the Moufang loops are alternative and their isotops are also Moufang loops, the family of loop multiplications $x \underset{a}{\cdot} y$ consists of diassociative loops (cf. Chap. VI. in [3]). The geodesic loop multiplication $x \underset{a}{\cdot} y=\exp_x\circ\tau(x)\circ$ $\circ\tau(a)^{-1}\circ\exp_a^{-1} y$ of the left canonical connection of the diassociative loops coincides with the original multiplication because the geodesics of the left canonical connection of the diassociative loops are the curves $x\cdot\exp tY\,(x\in\mathscr{L}, Y\in\mathscr{I}_e\mathscr{L})$, where $\exp tY$ is the 1-parameter subgroup of $\mathscr{L}$ tangent to $Y\in\mathscr{T}_e\mathscr{L}$. Hence the assertion follows from the results in the Section 1.

Proposition 4.2. *Let $x \underset{a}{\circ} y$ denote the local loop multiplication with identity element a defined by*

$$x \underset{a}{\circ} y = x^{2/3} \underset{a}{\cdot} (y \underset{a}{\cdot} x^{1/3})$$

on a Moufang loop $\mathscr{L}$. If $x(t)$ is a geodesic with respect to the left canonical connection ∇ through $x(0)=a$, then it is a one-parameter subgroup of the loop with multiplication $x \underset{a}{\circ} y$. The parallel translation along the one-parameter subgroup $x(t)$ with respect to the connection $\mathring{\nabla}_X Y=\nabla_X Y+\frac{1}{3}T(X, Y)$ coincides with the map $\mathscr{T}_a\mathring{\lambda}_a(x(t))$, where $\mathring{\lambda}_a$ denotes the left multiplication map $\mathring{\lambda}_a(x)y=x \underset{a}{\circ} y$. Consequently,

$$x \underset{a}{\circ} y = \exp_x\circ\mathscr{T}_a\mathring{\lambda}_a\circ\exp_a^{-1} y$$

in a neighbourhood of a.

Proof. We know, that the loop multiplications $x \underset{a}{\cdot} y\,(a\in\mathscr{L})$ have 1-parameter subgroups in each direction $X\in\mathscr{T}_a\mathscr{L}$, that are geodesics of the connection ∇. Since $a \underset{e}{\cdot} \exp_e tY$ is a 1-parameter subgroup of the multiplication $x \underset{a}{\cdot} y$, it can be written in the form

$$a \underset{e}{\cdot} \exp_e tY = \exp_a t\mathscr{T}_e\lambda_e(a)Y.$$

Similarly, $x^{-1} \underset{e}{\cdot} (\exp_e tY \underset{e}{\cdot} x)$ is a 1-parameter subgroup of the multiplication $x \underset{e}{\cdot} y$,

hence it can be written in the form $\exp_e t\mathscr{T}_e\lambda_e(x)^{-1}\circ\mathscr{T}_e\varrho_e(x)Y$. Thus we have

$$x\underset{e}{\circ}\exp_e tY = x\underset{e}{\cdot}\left(x^{-1/3}\underset{e}{\cdot}(\exp_e tY\underset{e}{\cdot}x^{1/3})\right) = x\underset{e}{\cdot}\left(\exp_e t\mathscr{T}_e\lambda_e(x^{1/3})^{-1}\circ\mathscr{T}_e\varrho_e(x^{1/3})Y\right) =$$

$$= \exp_x t\mathscr{T}_e\lambda_e(x^{2/3})\circ\mathscr{T}_e\varrho_e(x^{1/3})Y$$

and so

$$x\underset{e}{\circ}y = \exp_x\mathscr{T}_e\lambda_e(x^{2/3})\circ\mathscr{T}_e\varrho_e(x^{1/3})\exp^{-1}y$$

follows.

We prove now that the mapping $\mathscr{T}_e\lambda_e(x^{2/3})\circ\mathscr{T}_e\varrho_e(x^{1/3})$ is the parallel translation of the connection $\mathring{\nabla}$ along the geodesic segment $\exp_e tX\ (0\leqq t\leqq 1)$, where $\exp_e X=x$. First, we note that the 1-parameter groups of the multiplications $x\underset{a}{\cdot}y$ and $x\underset{a}{\circ}y$ coincide for all $a\in\mathscr{L}$. Let $Y(t)$ denote the vector field

$$Y(t) = \mathscr{T}_e\lambda_e\left(\exp_e\frac{2}{3}tX\right)\circ\mathscr{T}_e\varrho_e\left(\exp_e\frac{1}{3}tX\right)Y_0$$

along $\exp_e tX$, where $Y_0\in\mathscr{T}_e\mathscr{L}$. If $x_0=\exp_e t_0X$, $y_0=\exp_e\left(\frac{2}{3}t_0X\right)\underset{e}{\cdot}y\underset{e}{\cdot}\exp_e\left(\frac{1}{3}t_0X\right)$ and $X(t_0)=\mathscr{T}_e\lambda_e(x_0)X$ then we can write

$$= \exp_e\frac{2}{3}(t-t_0)X\underset{e}{\cdot}y_0\underset{e}{\cdot}\exp_e\frac{1}{3}(t-t_0)X = \exp_e\frac{2}{3}tX\underset{e}{\cdot}y\underset{e}{\cdot}\exp_e\frac{1}{3}tX.$$

Now

$$Y(t) = \mathscr{T}_{x_0}\lambda_{x_0}\left(\exp_{x_0}\frac{2}{3}(t-t_0)X(t_0)\right)\circ\mathscr{T}_{x_0}\varrho_{x_0}\left(\exp_{x_0}\frac{1}{3}(t-t_0)X(t_0)\right)Y(t_0)$$

follows, and then

$$(\nabla_X Y)(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\mathscr{T}_{x_0}\lambda_{x_0}^{-1}(\exp_{x_0}(t-t_0)X(t_0))Y(t_0)\right\}_{t_0} =$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left\{\mathscr{T}_{x_0}\lambda_{x_0}^{-1}\left(\exp_{x_0}\frac{1}{3}(t-t_0)X(t_0)\right)\circ\mathscr{T}_{x_0}\varrho_{x_0}\left(\exp_{x_0}\frac{1}{3}(t-t_0)X(t_0)\right)Y(t_0)\right\}_{t_0}.$$

We introduce a coordinate system around x_0 in which the multiplication $x\underset{x_0}{\cdot}y$ is of the form

$$x\underset{x_0}{\cdot}y = x+y+q(x,y)+r(x,x,y)+s(x,y,y)+\{\text{higher order terms}\}.$$

Then we obtain that

$$(\nabla_X Y)(t_0) = \frac{1}{3}q(X(t_0),Y(t_0))+\frac{1}{3}q(Y(t_0),X(t_0)) = \frac{1}{3}T(X(t_0),Y(t_0)),$$

or

$$(\mathring{\nabla}_X Y)(t_0) = (\nabla_X Y)(t_0)-\frac{1}{3}T(X(t_0),Y(t_0)) = 0.$$

Thus $\mathcal{T}_e\mathring{\lambda}_e=\mathcal{T}_e\lambda_e\left(\exp_e\frac{2}{3}tX\right)\circ\mathcal{T}_e\varrho_e\left(\exp_e\frac{1}{3}tX\right)$ gives the parallel translation along $\exp_e tX$ with respect to the connection $\mathring{\nabla}$, which was to be proved.

Theorem 4.3. *Let $\mathcal{L}$ be a connected Moufang loop with the multiplication $x\underset{e}{\cdot}y$ and $\mathcal{G}$ is the Lie transformation group generated by the maps $\mathring{\lambda}_e(x):=\lambda_e(x^{2/3})\varrho_e(x^{1/3})$ $(x\in\mathcal{L})$. Let $\mathcal{H}$ be the isotropy subgroup of $\mathcal{G}$ at $e\in\mathcal{L}$. The loop can be represented as a reductive homogeneous space $\mathcal{G}/\mathcal{H}$ with the reductive decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of $\mathcal{G}$ and $\mathcal{H}$, respectively. The complementary subspace $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$ consists of the tangent vectors of the one parameter subgroups $\{\mathring{\lambda}_e(x(t))\}$ at the identity of $\mathcal{G}$, where the curves $\{x(t)\}$ are the one parameter subgroups in the loop $\mathcal{L}$. Let $\mathring{T}$ be the torsion tensor field of the canonical connection $\mathring{\nabla}$ of $\mathcal{G}/\mathcal{H}$ and let ∇ be the invariant connection of the homogeneous space $\mathcal{G}/\mathcal{H}$ defined by*

$$\nabla_X Y=\mathring{\nabla}_X Y-\mathring{T}(X,Y).$$

Then the multiplication $x\underset{e}{\cdot}y$ locally coincides with the geodesic loop multiplication of ∇.

Proof. Let $\mathcal{L}$ be a connected Moufang loop. The translated multiplications $x\underset{a}{\cdot}y:=x\cdot a\backslash y$ locally coincide with the geodesic loop multiplications of the canonical connection ∇. Let $p\colon\bar{\mathcal{L}}\to\mathcal{L}$ be the universal covering of the loop $\mathcal{L}$. The kernel $p^{-1}(e)$ is a central abelian discrete subgroup of $\bar{\mathcal{L}}$, which is naturally isomorphic to the fundamental group of $\mathcal{L}$ (cf. Proposition IX. 1.24. in [5]). Since $p\colon\bar{\mathcal{L}}\to\mathcal{L}$ is a covering homomorphism it is covering homomorphism for the translated multiplications $x\underset{a}{\cdot}y:=x\cdot a\backslash y$ too. Let $\bar{\nabla}$ denote the covering connection of ∇ defined on the manifold $\bar{\mathcal{L}}$. It is clear from the construction of the covering loop multiplication on $\bar{\mathcal{L}}$ and of the covering connection $\bar{\nabla}$ that $\bar{\mathcal{L}}$ is a Moufang loop and the translated multiplications $x\underset{a}{\cdot}y$ on $\bar{\mathcal{L}}$ locally coincide with the geodesic loop multiplications of ∇. By Proposition 3.3. $\bar{\mathcal{L}}$ is a locally reductive homogeneous space with the connection $\bar{\nabla}$. Since it is simply connected, it can be represented as a global homogeneous space $\bar{\mathcal{L}}=\bar{\mathcal{G}}/\bar{\mathcal{H}}$, where $\bar{\mathcal{G}}$ is the transvection group (cf. Theorem I. 25. in [7]) of $\bar{\mathcal{L}}$ generated by the affine transformations having the local representation

$$\mathring{\bar{\lambda}}_x(y)=\exp_y\circ\mathcal{T}_x\mathring{\bar{\lambda}}_x(y)\circ\exp_x^{-1},$$

where

$$\mathring{\bar{\lambda}}_x(y):=\bar{\lambda}_x(y^{2/3})\bar{\varrho}_x(y^{1/3})$$

and $\bar{\lambda}_x(y)$ is the left multiplication map of the translated covering loop $x\underset{a}{\cdot}y$ on $\bar{\mathcal{L}}$. Since the mappings $\mathring{\bar{\lambda}}_e(z)$ are isomorphisms between the multiplications $\mathring{\bar{\lambda}}_e(x)y$ and

$\mathring{\lambda}_z(x)y$ we have $\mathring{\lambda}_e(z)\mathring{\lambda}_e(x)y=\mathring{\lambda}_z(\mathring{\lambda}_e(z)x)\mathring{\lambda}_e(z)y$. With the notation $u=\mathring{\lambda}_e(z)x$, we obtain

$$\mathring{\lambda}_z(u) = \mathring{\lambda}_e(z)\,\mathring{\lambda}_e(\mathring{\lambda}_e(z)^{-1}u)\,\mathring{\lambda}_e(z)^{-1}.$$

Thus the transvection group $\bar{\mathscr{G}}$ is generated by the maps $\mathring{\lambda}_e(z)$, $z\in\bar{\mathscr{L}}$. Consequently, the subgroup generated by the maps $\mathring{\lambda}_e(t)$, $t\in p^{-1}(e)$ in $\bar{\mathscr{G}}$ is central and the homomorphism $p\colon\bar{\mathscr{L}}\to\mathscr{L}$ can be extended to a homomorphism $\lambda(p)\colon\bar{\mathscr{G}}\to\mathscr{G}$ so that $\lambda(p)(\mathring{\lambda}_e(z))=\mathring{\lambda}_e(z)$ and the group $\mathscr{G}$ is generated by the maps $\mathring{\lambda}_e(z):=\lambda_e(x^{2/3})\varrho_e(x^{1/3})$ $(x\in\mathscr{L})$ and acts transitively on $\mathscr{L}$.

Since the complementary subspace $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$ correspond to the subspace spanned by the tangent vectors of the parallel translated frames in the linear frame bundle over $\mathscr{L}$, we obtain from Proposition 4.2. that $\mathfrak{m}$ consists of the tangent vectors of the one parameter subgroups $\{\mathring{\lambda}_e(x(t))\}$ at the identity of $\mathscr{G}$. Thus the assertion follows from Theorem 3.4.

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BOLYAI INSTITUTE
SZEGED UNIVERSITY
H—6720 SZEGED, HUNGARY

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Basic cohomology classes of compact Sasakian manifolds

HILLEL GAUCHMAN

1. Introduction and preliminaries. It was proved in [G1] that for any compact $(2m+1)$-dimensional Sasakian manifold M the following inequality is satisfied:

$$(1.1)\quad \int_M \left(|S|^2-\frac{1}{2}\varrho^2+2\varrho\right)dV+\frac{m-1}{2m\,\mathrm{Vol}(M)}\left(\int_M \varrho\, dV\right)^2 \geqq 2m(2m+1)\,\mathrm{Vol}(M),$$

where $|S|$, ϱ, $\mathrm{Vol}(M)$, and dV are the length of the Ricci tensor, the scalar curvature, the volume of M, and the Riemannian measure on M, respectively. Inequality (1.1) was applied in [G1] to a study of cohomologically Einstein—Sasakian manifolds. The purpose of this paper is to prove a set of inequalities for basic cohomology classes of compact Sasakian manifolds. The simplest of these inequalities is equivalent to inequality (1.1).

Let M be a $(2m+1)$-dimensional differentiable manifold (in what follows we assume the all manifolds, maps, differential forms, etc. are of class C^∞). Assume that M carries a global differential 1-form η such that $\eta\wedge(d\eta)^m\neq 0$ everywhere on M. Then we say that η defines a *contact structure* on M. A manifold M furnished with a contact structure η is called a *contact manifold.* It is known, [B], that a contact manifold (M,η) admits a unique global vector field X_0 satisfying $\eta(X_0)=1$ and $d\eta(X_0, X)=0$ for any tangent vector field X on M. X_0 is called the *characteristic vector field* of a contact manifold (M,η). Since vector field X_0 nowhere vanishes, M can be considered as a foliated manifold with 1-dimensional leaves. Let ω be a $\mathbf{F}$-valued differential k-form on a contact manifold (M,η), where $\mathbf{F}=\mathbf{R}$ or $\mathbf{C}$. We say that ω is *horizontal* if $i(X_0)\omega=0$, *invariant* if $L_{X_0}\omega=0$, and *basic* if it is horizontal and invariant. Here $i(X_0)$ and L_{X_0} are the inner product by X_0 and the Lie derivative, respectively. Denote by $A_B(M,\eta,\mathbf{F})$ (resp. $A_B^k(M,\eta,\mathbf{F})$) the set of all $\mathbf{F}$-valued basic forms (resp. basic k-forms), and by $C_B(M,\eta,\mathbf{F})$ (resp. $C_B^k(M,\eta,\mathbf{F})$) the set of all $\mathbf{F}$-valued closed basic forms (resp. closed basic k-forms) on M. It is easy to see that $dA_B^{k-1}(M,\eta,\mathbf{F})\subset C_B^k(M,\eta,\mathbf{F})$. Set $H_B^k(M,\eta,\mathbf{F})=C_B^k(M,\eta,\mathbf{F})/$

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$/dA_B^{k-1}(M, \eta, \mathbf{F})$. $H_B^k(M, \eta, \mathbf{F})$ is called the kth *basic cohomology group* of (M, η) over $\mathbf{F}$. In what follows we shall usually write $H_B^k(M)$ or $H_B^k(M, \mathbf{F})$ instead of $H_B^k(M, \eta, \mathbf{F})$, and similarly for $A_B^k(M, \eta, \mathbf{F})$ and $C_B^k(M, \eta, \mathbf{F})$. It is easy to see that if $\lambda \in C_B^k(M)$, $\mu \in C_B^l(M)$, then $\lambda \wedge \mu \in C_B^{k+l}(M)$, and if $\lambda \in C_B^k(M)$, $\mu \in dA_B^{l-1}(M)$, then $\lambda \wedge \mu \in dA_B^{k+l-1}(M)$. Therefore, for any $\alpha \in H_B^k(M)$, $\beta \in H_B^l(M)$, we have a well-defined product $\alpha \cdot \beta \in H_B^{k+l}$. Clearly,

$$H_B^0(M, \mathbf{F}) = \mathbf{F}, \quad H_B^k(M) = \{0\} \quad \text{for} \quad k \geqq 2m+1.$$

Generally, $\dim_{\mathbf{F}} H_B^k(M, \mathbf{F})$, $k=1, \ldots, 2m$, may be infinite. However, for "good" contact structures (such as K-structures or Sasakian structures) $\dim_{\mathbf{R}} H_B^k(M, \mathbf{R}) = \dim_{\mathbf{C}} H_B^k(M, \mathbf{C}) < \infty$.

A contact manifold (M, η) is called regular, [B], if X_0 is a regular vector field on M, that is every point $x \in M$ has a cubical coordinate neighborhood $\mathcal{U}$ such that the integral curves of X_0 passing through $\mathcal{U}$ pass through the neighborhood only once. It is known, [B], that any compact regular $(2m+1)$-dimensional contact manifold M is the bundle space of a principle circle bundle $\pi\colon M \to B$ over a $2m$-dimensional simplectic manifold B. It is easy to show that in the case of a compact regular contact manifold $H_B^k(M)$ is the pullback of $H^k(B)$, where $H^k(B)$ is the DeRham cohomology group of B.

Let (M, η) be a contact manifold. In what follows we will always use the following notation:

$$\Phi = d\eta. \tag{1.2}$$

Φ is a closed basic form. Therefore Φ represents a basic cohomology class. In what follows we will denote this cohomology class by Ω. $\Omega \in H_B^2(M)$ is called the *fundamental basic cohomology class.*

For a compact contact $(2m+1)$-dimensional manifold (M, η) we now define a linear function $I\colon A_B(M, \mathbf{F}) \to \mathbf{F}$ from the set of all basic $\mathbf{F}$-valued forms on M into $\mathbf{F}$ as follows: If $\omega \in A_B^{2k}(M, \mathbf{F})$, $k=0, 1, \ldots, m$, then

$$I(\omega) = \frac{1}{2^m m! \operatorname{Vol}(M)} \int_M \eta \wedge \Phi^{m-k} \wedge \omega. \tag{1.3}$$

If $\omega \in A_B^{2k+1}(M, \mathbf{F})$, $k=1, \ldots, m$, then $I(\omega)=0$. We shall denote by the same symbol I a function $I\colon H_B(M, \mathbf{F}) \to \mathbf{F}$ defined as follows: Let $\alpha \in H_B(M, \mathbf{F})$ and let ω be a closed basic form representing α. Then, by definition,

$$I(\alpha) = I(\omega). \tag{1.4}$$

We will show in Sec. 2 that $I(\alpha)$ is well-defined by formula (1.4), that is $I(\alpha)$ does not depend on a particular choice of a basic form ω representing α. It is clear from

the definition of I that

$$(1.5)\qquad \begin{aligned} I(\Phi^k\wedge\omega) &= I(\omega), \quad \text{if} \quad \omega\in A_B^{2l} \quad \text{and} \quad l+k\leqq m;\\ I(\Omega^k\cdot\alpha) &= I(\alpha), \quad \text{if} \quad \alpha\in H_B^{2l} \quad \text{and} \quad l+k\leqq m. \end{aligned}$$

By [S], page 3—4, $\int\eta\wedge\Phi^m=2^m m!\,\mathrm{Vol}\,(M)$. Therefore

$$(1.6)\qquad I(\Phi^k) = I(\Omega^k) = 1, \quad 0\leqq k\leqq m.$$

Let (M,η) be a contact manifold. An associated contact metric structure, [B], for a contact structure η is a collection (η, X_0, φ, g), where X_0 is the characteristic vector field, φ is a field of automorphisms of the tangent spaces of M, and g is a Riemannian metric on M such that

$$\begin{aligned} \varphi^2(X) &= -X+\eta(X)X_0,\\ \eta(X) &= g(X, X_0),\\ g(\varphi X, \varphi Y) &= g(X, Y)-\eta(X)\eta(Y),\\ \Phi(X, Y) &= g(X, \varphi Y), \end{aligned}$$

for any tangent vector fields X and Y on M. An associated contact metric structure for a contact structure η always exists, but is not unique, [B]. We say that a contact metric structure (η, X_0, φ, g) on M is *normal*, [B], if the almost complex structure T on $M\times\mathbf{R}$ defined by $T\left(X, f\dfrac{d}{dt}\right)=\left(\varphi X-fX_0, \eta(X)\dfrac{d}{dt}\right)$ is integrable. A differentiable manifold M furnished with a normal contact metric structure (η, X_0, φ, g) is called a *Sasakian manifold.*

Let $(M, \eta, X_0, \varphi, g)$ be a $(2m+1)$-dimensional Sasakian manifold. For $x\in M$, set

$$(1.7)\qquad D_x = \{X\in TM_x\colon \eta(X) = 0\}.$$

D_x is called the *horizontal subspace* at the point x. By (1.7), φ induces an almost complex structure (once more denoted dy φ) on D_x. Denote by $D_x^{\mathbf{C}}$ the complexification of D_x. Then $D_x^{\mathbf{C}}=D_x^{1,0}\oplus D_x^{0,1}$, where

$$(1.8)\qquad \begin{aligned} D_x^{1,0} &= \{X\in D_x^{\mathbf{C}}\colon \varphi X = \sqrt{-1}\,X\},\\ D_x^{0,1} &= \{X\in D_x^{\mathbf{C}}\colon \varphi X = -\sqrt{-1}\,X\}. \end{aligned}$$

It follows that the set $\mathrm{Hor}^p(M)$ of all $\mathbf{C}$-valued horizontal p-forms on M may be bigraded as follows;

$$\mathrm{Hor}^p(M) = \sum_{k+l=p}\mathrm{Hor}^{k,l}(M),$$

where $\mathrm{Hor}^{k,l}(M)$ is the set of all horizontal $(k+l)$-forms on M which can obtain

non-zero values only for sets of vectors $X_1, \ldots, X_{k+l} \in TM_x^{\mathbf{C}}$ among which k vectors belong to $D_x^{1,0}$ and l vectors belong to $D_x^{0,1}$.

Let $\alpha \in H_B^{k+l}(M, \mathbf{C})$. We say that α is *of the type* (k, l), if there is a basic form ω representing α, such that $\omega \in \mathrm{Hor}^{k,l}(M)$. We will see in Sec. 2 that for a $(2m+1)$-dimensional compact Sasakian manifold the notion for $\alpha \in H_B^p(M), (0 \leqq p \leqq m)$, to be of the type (k, l) is well defined. That means that if $\omega \in \mathrm{Hor}^{k,l}(M)$ and $\tau \in \mathrm{Hor}^{r,s}(M)$ represent the same basic cohomology class $\alpha \in H_B^p(M, \mathbf{C})$, then $k=r$ and $l=s$. For $0 \leqq k+l \leqq m$, set

$$H_B^{k,l}(M) = \{\alpha \in H_B^{k+l}(M, \mathbf{C}) \colon \alpha \text{ is of the type } (k, l)\}. \tag{1.9}$$

Then $H_B^{k,l}$ is a subgroup (as an additive group) of $H_B^{k+l}(M, \mathbf{C})$. We will show in Sec. 2 that for a compact Sasakian manifold there is a direct sum decomposition

$$H_B^p(M, \mathbf{C}) = \sum_{k+l=p} H_B^{k,l}(M), \quad 0 \leqq p \leqq m.$$

For $0 \leqq k+l \leqq m$, set

$$h_B^{k,l} = \dim_{\mathbf{C}} H_B^{k,l}(M).$$

$h_B^{k,l}$ will be called the *basic Hodge number of the type* (k, l). By (1.3), $\Phi \in \mathrm{Hor}^{1,1}(M)$. Hence $\Omega^k \in H_B^{k,k}(M)$. By (1.6), $\Omega^k \neq 0$. Therefore

$$h_B^{0,0} = 1, \quad h_B^{k,k} \geqq 1, \quad k = 1, \ldots, \left[\frac{m}{2}\right].$$

Moreover, we will show in Sec. 2 that

$$1 = h_B^{0,0} \leqq h_B^{1,1} \leqq \ldots \leqq h_B^{[m/2],[m/2]}.$$

In Sec. 3 we prove the main result of this paper:

Theorem 1.1. *Let $(M, \eta, X_0, \varphi, g)$ be a compact $(2m+1)$-dimensional Sasakian manifold and let k be an integer such that $1 \leqq k \leqq \frac{m}{2}$. Assume that $h_B^{k-1,k-1}=1$. Let $\alpha \in H_B^{k,k}(M)$. Then*

$$(-1)^k[I(\alpha \cdot \bar{\alpha}) - I(\alpha)I(\bar{\alpha})] \geqq 0, \tag{1.10}$$

and the equality holds if and only if $\alpha = t\Omega^k$, $t \in \mathbf{C}$. Here $\bar{\alpha}$ means the complex conjugate of α.

Taking $k=1$ in Theorem 1.1, we obtain

Corollary 1.2. *Let M be a compact Sasakian $(2m+1)$-dimensional $(m \geqq 2)$ manifold and let $\alpha \in H_B^{1,1}(M)$. Then*

$$I(\alpha \cdot \bar{\alpha}) - I(\alpha)I(\bar{\alpha}) \leqq 0 \tag{1.11}$$

and the equality holds if and only if $\alpha = t\Omega$, where $t \in \mathbf{C}$.

It follows easily from the results of Sec. 2 that if $b_2(M)=0$, where $b_2(M)$ is the second Betti number of M, then $h_B^{1,1}=1$. Hence, taking $k=2$ in Theorem 1.1, we obtain

Corollary 1.3. *Let M be a compact Sasakian $(2m+1)$-dimensional $(m\geqq 4)$ manifold and let $\alpha\in H_B^{2,2}(M)$. If $b_2(M)=0$, then*

$$I(\alpha\cdot\bar{\alpha})-I(\alpha)I(\bar{\alpha})\geqq 0, \tag{1.12}$$

and the equality holds if and only if $\alpha=t\Omega^2$, where $t\in\mathbf{C}$.

In Sec. 4 for any $(2m+1)$-dimensional Sasakian manifold and for any $k=1,\dots,m$ we introduce a canonical real closed basic form $C_k^{(B)}$ of bidegree (k,k). We will call this form the basic Chern form of a Sasakian manifold. Substituting $C_k^{(B)}$ instead of α in (1.10), we obtain an integral inequality similar to inequality (1.1). In the simplest case, when $k=1$, we obtain inequality (1.1).

If M is a regular Sasakian manifold, then M is a unit circle bundle over a Kaehler manifold B. It is easy to see that in this case the basic Chern form $C_k^{(B)}$ belongs to a basic cohomology class which is the pull-back of the Chern class $C_k(B)$. It was shown in [G2] that for $B=P^2(\mathbf{C})\times P^3(\mathbf{C})$,

$$I(C_2(B)\cdot C_2(B))-I(C_3(B))\cdot I(C_2(B))<0.$$

Hence, if a Sasakian manifold M is a unit circle bundle over $B=P^2(\mathbf{C})\times P^3(\mathbf{C})$, then

$$I(C_2^{(B)}(M)\cdot C_2^{(B)}(M))-I(C_2^{(B)}(M))\cdot I(C_2^{(B)}(M))<0.$$

Comparing this inequality with inequality (1.12), we see that the condition $b_2(M)=0$ in Corollary 1.3 cannot be omitted. More generally, this example shows that the condition $h_B^{k-1,k-1}=1$ in Theorem 1.1 is essential.

We conclude Sec. 4 by Remark showing how one can define basic Pontrjagin classes $P_k^{(B)}\in H_B^{4k}(M,\mathbf{R})$, $k=1,\dots,[m/2]$, on K-contact manifolds.

Finally we note that for Kaehler manifolds a theorem similar to Theorem 1.1 has been proved in [G2].

2. Decomposition theorems. For a compact metric manifold (M,η,X_0,φ,g) we will denote by $\langle\,,\rangle$ the local scalar product with respect to the Riemannian metric g, and by $(\lambda,\mu)=\int_M\langle\lambda,\mu\rangle\,dV$ the global scalar product, where λ and μ are differential forms of the same degree. As usual, $*$ will be the Hodge "star" operator and δ will be the adjoint of the operator of exterior differentiation, i.e. $(d\lambda,\mu)=(\lambda,\delta\mu)$, where λ and μ are forms of degrees p and $p+1$, respectively. We also will denote by $e(\eta)\lambda$ the exterior product by η, i.e. $e(\eta)\lambda=\eta\wedge\lambda$. Clearly, $(i(X_0)\lambda,\mu)=(\lambda,e(\eta)\mu)$ for any two differential forms λ and μ of degrees $p+1$ and p respectively.

Lemma 2.1. *Let* (M, η) *be a compact* $(2m+1)$*-dimensional contact manifold. Then the function* $I\colon H_B(M, \mathbf{F}) \to \mathbf{F}$ *given by formulas* (1.3) *and* (1.4) *is well-defined.*

Proof. Let λ and λ_1 be basic closed $2k$-forms representing the same basic cohomology class $\alpha \in H_B^{2k}(M)$. Then $\lambda - \lambda_1 = d\mu$ where μ is a basic $(2k-1)$-form.

We must prove that $\int_M \eta \wedge \Phi^{m-k} \wedge \lambda = \int_M \eta \wedge \Phi^{m-k} \lambda_1$. Therefore we must prove that $\int_M \eta \wedge d\omega = 0$, where $\omega = \Phi^{m-k} \wedge \mu$. Clearly, ω is a basic form. Let (η, X_0, φ, g) be a contract metric structure on M associated with contact structure η. By [S], page 3—4,

$$*1 = \frac{1}{2^m m!} \eta \wedge \Phi^m. \tag{2.1}$$

Hence,

$$\int_M \eta \wedge d\omega = (\eta \wedge d\omega, *1) = \frac{1}{2^m m!} (e(\eta)\, d\omega, e(\eta) \Phi^m) =$$

$$= \frac{1}{2^m m!} (d\omega, i(X_0) e(\eta) \Phi^m) = \frac{1}{2^m m!} (d\omega, \Phi^m) = \frac{1}{2^m m!} (\omega, \delta \Phi^m).$$

By [SH],

$$\delta \Phi^r = 4r(m-r+1) \eta \wedge \Phi^{r-1}. \tag{2.2}$$

Therefore,

$$\int_M \eta \wedge d\omega = \frac{4m}{2^m m!} (\omega, e(\eta) \Phi^{m-1}) = \frac{4m}{2^m m!} (i(X_0)\omega, \Phi^{m-1}) = 0,$$

since $i(X_0)\omega = 0$.

Corollary. *For any basic form* λ, $I(d\lambda) = 0$.

From now and to the end of this section let $(M, \eta, X_0, \varphi, g)$ be a compact $(2m+1)$-dimensional Sasakian manifold. Let us denote by d_B and $(\ ,\ )_B$ the restriction of the exterior differential and of the global scalar product on the space $A_B(M)$ of basic forms on M. Let $\delta_B\colon A_B(M) \to A_B(M)$ be the adjoint operator for d_B with respect to $(\ ,\ )_B$. Then $\Delta_B = \delta_B d_B + d_B \delta_B$ is called the *basic Laplacian*. The set $\mathfrak{H}_B^k$ of *basic harmonic k-forms* is the kernel of Δ_B on $A_B^k(M)$. Any Sasakian manifold M can be considered as a foliated manifold with 1-dimensional leaves. By the Main Theorem of [KT] (whose conditions are obviously satisfied for Sasakian manifolds), we have

$$A_B^k(M) \cong \Delta_B(A_B^k) \oplus \mathfrak{H}_B^k(M) \tag{2.3}$$

and $\dim_{\mathbf{C}} \mathfrak{H}_B^k < \infty$. It follows from (2.3) that

$$A_B^K(M, \mathbf{C}) = \operatorname{im} d_B \oplus \operatorname{im} \delta_B \oplus \mathfrak{H}_B^k(M). \tag{2.4}$$

As usual we obtain from (2.4) that $H_B^k(M, \mathbf{C}) \cong \mathfrak{H}_B^k(M)$.

Let $TM_x^{\mathbf{C}}$ be the complexified tangent space at the point $x \in M$. Then

$$TM_x^{\mathbf{C}} = D_x^{1,0} \oplus D_x^{0,1} \oplus \mathbf{C}X_0, \tag{2.5}$$

where $D_x^{1,0}$ and $D_x^{0,1}$ are defined by (1.8). It is known, [I], that the pair of complex distributions $(D_x^{1,0}, D_x^{0,1})$ defines a C—R structure on M. Hence each of the distributions $D_x^{1,0}$ and $D_x^{0,1}$ is integrable. Let $\{e_i, e_{\bar{i}}, X_0\}$, $i=1, \ldots, m$; $\bar{i}=m+1, \ldots, 2m$, be a local field of frames adapted to the decomposition (2.6). That means that at the point x each $e_i \in D_x^{1,0}$ and each $e_{\bar{i}} \in D_x^{0,1}$. Let $\{\theta^i, \theta^{\bar{i}}, \eta\}$ be the dual basis of $\mathbf{C}$-valued 1-forms on M. Then, by Frobenius' theorem

$$d\theta^i \equiv 0 \ (\mathrm{mod}\, \theta^j, \ j = 1, \ldots, m) \quad \text{and} \quad d\theta^{\bar{i}} \equiv 0 \ (\mathrm{mod}\, \theta^{\bar{j}}, \ \bar{j} = m+1, \ldots, 2m).$$

Therefore

$$d\theta^i = \sum a^i_{jk}\theta^j \wedge \theta^k + \sum a^i_{j\bar{k}}\theta^j \wedge \theta^{\bar{k}} + \sum b^i_j \eta \wedge \theta^j,$$

$$d\theta^{\bar{i}} = \sum a^{\bar{i}}_{\bar{j}k}\theta^{\bar{j}} \wedge \theta^k + \sum a^{\bar{i}}_{\bar{j}\bar{k}}\theta^{\bar{j}} \wedge \theta^{\bar{k}} + \sum b^{\bar{i}}_{\bar{j}} \eta \wedge \theta^{\bar{j}},$$

where $a^i_{jk}, a^i_{j\bar{k}}, a^{\bar{i}}_{\bar{j}k}, a^{\bar{i}}_{\bar{j}\bar{k}}, b^i_j, b^{\bar{i}}_{\bar{j}}$ are functions. It follows that for any horizontal form $\omega \in \mathrm{Hor}^{k,l}(M)$ of bidegree (k, l)

$$d\omega = \omega' + \omega'' + \eta \wedge \omega''', \tag{2.6}$$

where $\omega' \in \mathrm{Hor}^{k+1,l}(M)$, $\omega'' \in \mathrm{Hor}^{k,l+1}(M)$, $\omega''' \in \mathrm{Hor}^{k,l}(M)$. Assume now that ω is basic. Then $0 = i(X_0)d\omega = \omega'''$. Therefore $d\omega = \omega' + \omega''$. Set $d\omega' = \lambda' + \eta \wedge \mu'$, $d\omega'' = \lambda'' + \eta \wedge \mu''$, where λ', λ'', μ', and μ'' are horizontal. If follows that $0 = d\omega' + d\omega'' = (\lambda' + \lambda'') + \eta(\mu' + \mu'')$. Hence $\mu' + \mu'' = 0$. Since $\mu' \in \mathrm{Hor}^{k+1,l}(M)$ and $\mu'' \in \mathrm{Hor}^{k,l+1}(M)$, we obtain that $\mu' = \mu'' = 0$. Hence $d\omega'$ and $d\omega''$ are horizontal and therefore ω' and ω'' are basic. It follows that if $\omega \in A_B^{k,l}(M)$, where $A_B^{k,l}(M)$ is the set of basic forms on M of bidegree (k, l), then $d\omega = \omega' + \omega''$, where $\omega' \in A_B^{k+1,l}(M)$ and $\omega'' \in A_B^{k+1,l}(M)$. Set $d'_B\omega = \omega'$, $d''_B\omega = \omega''$. Then we obtain that $d_B = d'_B + d''_B$, where d'_B and d''_B are differential operators on $A_B(M, \mathbf{C})$ of bidegrees (1, 0) and (0, 1), respectively. Let δ'_B: $A_B(M, \mathbf{C}) \to A_B(M, \mathbf{C})$ and δ''_B: $A_B(M, \mathbf{C}) \to A_B(M, \mathbf{C})$ be the adjoint operators for d'_B and d''_B, respectively, with respect to the global scalar product $(\ ,\ )_B$. Then δ'_B and δ''_B are of bidegree $(-1, 0)$ and $(0, -1)$, respectively, and $\delta_B = \delta'_B + \delta''_B$. Set $\Delta'_B = \delta'_B d'_B + d'_B \delta'_B$, $\Delta''_B = \delta''_B d''_B + d''_B \delta''_B$.

Lemma 2.2. *Let ω be a basic p-form, $0 \leq p \leq m$. Then*

$$\Delta_B \omega = 2\Delta'_B \omega = 2\Delta''_B \omega.$$

Proof. This lemma is analogous to Theorem 3.7 of [W], Chapter V. A proof Lemma 2.2 can be obtained by repeating the arguments of the proof of the above mentioned theorem from [W], and we omit it.

Denote dy $\Pi_{k,l}$ the natural projection from $A_B(M, \mathbf{C})$ to $A_B^{k,l}(M)$. By Lemma 2.2,

$$\Delta_B \Pi_{k,l} = \Pi_{k,l} \Delta_B, \quad 0 \leqq k+l \leqq m. \tag{2.7}$$

For any differential form ω on M, set $L\omega = \Phi \wedge \omega$, where $\Phi = d\eta$. If ω is basic, then $L\omega$ is also basic. Therefore L induces the map L_B: $A_B(M) \to A_B(M)$. Denote by Λ the adjoint operator of L with respect to $(\ ,\ )$, and by Λ_B the adjoint operator of L_B with respect to $(\ ,\ )_B$. Clearly L_B and Λ_B are operators of bidegrees $(1, 1)$ and $(-1, -1)$, respectively.

Lemma 2.3.

(i) *If ω is basic, then $\Pi_{k,l}\omega$ is also basic.*

(ii) *If ω is a basic harmonic p-form and $0 \leqq p \leqq m$, then $\Pi_{k,l}\omega$ is also basic harmonic.*

(iii) *If ω is a harmonic p-form and $0 \leqq p \leqq m$, then $\Pi_{k,l}\omega$ is also harmonic.*

Remark. By [T1] and [Y], any harmonic p-form, $0 \leqq p \leqq m$, is basic harmonic. Therefore the operator $\Pi_{k,l}$ is well-defined on the set of harmonic p-forms, $0 \leqq p \leqq m$.

Proof. (i) Let $\omega \in A_B^p(M, \mathbf{C})$. Then $\omega = \omega_{0,p} + \omega_{1,p-1} + \ldots + \omega_{p,0}$, where $\omega_{k,l} = \Pi_{k,l}\omega$. By (2.6), $d\omega_{k,l} = \lambda + \eta \wedge \mu_{k,l}$, where λ is horizontal and $\mu_{k,l}$ is horizontal of bidegree (k, l). Since ω is basic,

$$0 = i(X_0)\, d\omega = i(X_0)(d\omega_{0,p} + \ldots + d\omega_{p,0}) = \mu_{0,p} + \mu_{1,p-1} + \ldots + \mu_{p,0},$$

Hence each $\mu_{k,l} = 0$. Therefore $i(X_0) d\omega_{k,l} = i(X_0)\lambda = 0$. Thus, $\omega_{k,l} = \Pi_{k,l}\omega$ is basic.

(ii) Let ω be a basic harmonic p-form, $0 \leqq p \leqq m$. By (2.7), $\Delta_B(\Pi_{k,l}\omega) = \Pi_{k,l}(\Delta_B \omega) = 0$. This proves (ii).

(iii) Let λ and μ be two basic forms on M. For Sasakian manifolds, formula (3.8) from [KT] gives

$$(\Delta\lambda, \mu) = \big((\Delta_B + L\Lambda)\lambda, \mu\big). \tag{2.8}$$

Let ω be a harmonic p-form, $0 \leqq p \leqq m$. Then ω and therefore $\Pi_{k,l}\omega$ are basic. Hence, by (2.7) and (2.8),

$$\begin{aligned}\big(\Delta(\Pi_{k,l}\omega), \Pi_{k,l}\omega\big) &= (\Delta_B \Pi_{k,l}\omega + L\Lambda\Pi_{k,l}\omega, \Pi_{k,l}\omega) = \\ &= (\Pi_{k,l}\Delta_B\omega + L\Pi_{k-1,l-1}\Lambda\omega, \Pi_{k,l}\omega).\end{aligned}$$

Since any harmonic p-form, $0 \leqq p \leqq m$, is basic harmonic, we have $\Delta_B \omega = 0$. By [T1], any harmonic p-form, $0 \leqq p \leqq m$, is effective, i.e. $\Lambda\omega = 0$. Therefore $\big(\Delta(\Pi_{k,l}\omega), \Pi_{k,l}\omega\big) = 0$. It follows that $\big(d(\Pi_{k,l}\omega), d(\Pi_{k,l}\omega)\big) + \big(\delta(\Pi_{k,l}\omega), \delta(\Pi_{k,l})\big) = 0$. Thus, $d(\Pi_{k,l}\omega) = \delta(\Pi_{k,l}\omega) = 0$. Therefore, $\Pi_{k,l}\omega$ is harmonic. This proves (iii).

Let ω be a closed basic form of bidegree (k, l), $0 \leqq k+l \leqq m$. Then, by (2.3), $\omega = \psi + \Delta_B \lambda$, where ψ is basic harmonic and λ is basic. By (2.7), $\omega = \Pi_{k,l}\omega =$

$=\Pi_{k,l}\psi+\Delta_B(\Pi_{k,l}\lambda)$. Since ψ is uniquely defined by ω and since, by Lemma 2.3, $\Pi_{k,l}\psi$ is basic harmonic, $\psi=\Pi_{k,l}\psi$. Therefore ψ is of bidegree (k, l). Thus, we obtain that if a basic cohomology class $\alpha\in H_B^{k+l}(M, \mathbf{C})$, $0\leqq k+l\leqq m$, contains a closed basic form of bidegree (k, l), its basic harmonic form is also of bidegree (k, l). Therefore, the cohomology group $H_B^{k,l}(M)$, defined by (1.9), is well-defined. By Lemma 2.3, we have a direct sum decomposition

$$H_B^p(M, \mathbf{C}) = H_B^{0,p}(M)\oplus H_B^{1,p-1}(M)\oplus \dots \oplus H_B^{p,0}(M), \quad 0 \leqq p \leqq m. \tag{2.9}$$

Similarly, let $H^p(M, \mathbf{C})$, $0\leqq p\leqq m$, be the pth DeRham cohomology group, and let $H^{k,l}(M)$ be the set of all elements of H^p which are represented by a harmonic p-form of bidegree (k, l). Then

$$H^p(M, \mathbf{C}) = H^{0,p}(M)\oplus H^{1,p-1}(M)\oplus \dots \oplus H^{p,0}(M), \quad 0 \leqq p \leqq m. \tag{2.10}$$

Let $0\leqq p\leqq m$, $0\leqq k\leqq l\leqq m$. Set

$$b_p = \dim_{\mathbf{C}} H^p(M, \mathbf{C}), \quad b_p^{(B)} = \dim_{\mathbf{C}} H_B^p(M, \mathbf{C}),$$
$$h^{k,l} = \dim_{\mathbf{C}} H^{k,l}(M, \mathbf{C}), \quad h_B^{k,l} = \dim_{\mathbf{C}} H_B^{k,l}(M, \mathbf{C}).$$

Here b_p are usual Betti numbers. We will call $b_p^{(B)}$, $h^{k,l}$ and $h_B^{k,l}$ the *basic Betti numbers*, the *Hodge numbers*, and the *basic Hodge numbers*, respectively. By (2.9) and (2.10),

$$\begin{aligned} b_p &= h^{0,p}+h^{1,p-1}+\dots+h^{p,0}, \quad 0 \leqq p \leqq m; \\ b_p^{(B)} &= h_B^{0,p}+h_B^{1,p-1}+\dots+h_B^{p,0}, \quad 0 \leqq p \leqq m. \end{aligned} \tag{2.11}$$

Denote by C a linear operator C: Hor $(M)\to$Hor (M) such that $C\omega=(\sqrt{-1})^{k-l}\omega$ if ω is of bidegree (k, l), where Hor (M) is the set of all horizontal forms on M. Let $*$ denote the Hodge "star" operator. Remind that ω is called effective, if $\Lambda\omega=0$.

Lemma 2.4. *Let ω be a horizontal and effective p-form, $0\leqq p\leqq m$, and let $0\leqq r\leqq m-p$. Then*

$$*(L^r\omega) = (-1)^{p(p-1)/2}\frac{r!}{2^{m-p-2r}(m-p-r)!}e(\eta)L^{m-p-r}C\omega.$$

Proof. This lemma is similar to Theorem 1.6 from [W], Chapter 5. The proof of Lemma 2.4 is just a repetition of the proof of the above mentioned theorem from [W], and we omit it.

We now prove a decomposition theorem for closed basic forms.

Theorem 2.5. *Let M be a compact $(2m+1)$-dimensional Sasakian manifold and let ω be a closed basic p-form on M. Then*

(i) *ω can be decomposed as*

$$\omega = \sum_{i=(p-m)^+}^{[p/2]} L^i\psi_i+d\lambda, \tag{2.12}$$

where $(p-m)^+=\max\{0,p-m\}$, ψ_i *is a harmonic* $(p-2i)$*-form, any* λ *is basic. In addition, harmonic forms* ψ_i, $i=(p-m)^+,\ldots,[p/2]$, *are uniquely defined by* ω.

(ii) *If* ω *is of bidegree* (k,l), *then for each* i, ψ_i *is of bidegree* $(k-i,l-i)$.

Proof. (i) We first consider the case $0\leqq p\leqq m$. By (2.4), $\omega=\psi+d\lambda$, where ψ is a basic harmonic p-form uniquely defined by ω, and λ is basic. Since ψ is basic harmonic, $d_B\psi=0$ and $\delta_B\psi=0$. For Sasakian manifolds formula 3.3 from [KT] takes the form $\delta\psi=\delta_B\psi+e(\eta)\Lambda\psi$. Therefore

$$d\psi = 0, \quad \delta\psi = e(\eta)\Lambda\psi. \tag{2.13}$$

Differential forms on Sasakian manifolds satisfying (2.13) were introduced in [0] and were called there *C-harmonic forms*. By the decomposition theorem for C-harmonic forms of degree p, $0\leqq p\leqq m$, [T2],

$$\psi = \sum_{i=0}^{[p/2]} L^i\psi_i,$$

where ψ_i are harmonic $(p-2i)$-forms uniquely defined by ψ. This proves (i) in the case $0\leqq p\leqq m$.

Let now $m+1\leqq p\leqq 2m$. Once more, $\omega=\psi+d\lambda$, where ψ is a basic harmonic form uniquely defined by ω, and λ is basic. Following [KT], for any basic q-form μ we set

$$\bar{*}\mu = (-1)^q i(X_0)*\mu.$$

Then $\bar{*}\bar{*}\mu=(-1)^q\mu$. By Lemma 2.4, for any horizontal and effective q-form μ, $0\leqq q\leqq m$, and for any r, $0\leqq r\leqq m-q$,

$$\bar{*}(L^r\mu) = (-1)^{q(q+1)/2}\frac{r!}{2^{m-q-2r}(m-q-r)!}L^{m-q-r}C\mu. \tag{2.14}$$

Set $\tilde{\psi}=\bar{*}\psi$. By [KT], $\bar{*}\Delta_B=\Delta_B\bar{*}$. Therefore $\tilde{\psi}$ is basic harmohic. Since $\tilde{\psi}$ is of degree $2m-p<m$, we have a decomposition

$$\tilde{\psi} = \sum_{j=0}^{[(2m-p)/2]} L^j\tilde{\psi}_j,$$

where $\tilde{\psi}_j$ are harmonic of degree $(2m-p-2j)$. By (2.24),

$$\psi = (-1)^{2m-p}\bar{*}\tilde{\psi} = \sum_{j=0}^{[(2m-p)/2]}\bar{*}(L^j\tilde{\psi}_j) =$$

$$= \sum_{j=0}^{[(2m-p)/2]}(-1)^{p(p+1)/2-m+j}\frac{j!}{2^{p-m}(p-m+j)!}L^{p-m+j}C\tilde{\psi}_j.$$

Set $i=p-m+j$,

$$\psi_i = (-1)^{p(p-1)/2+i}\frac{(i-p+m)!}{2^{p-m}i!}C\tilde{\psi}_{i-p+m}.$$

Then

$$\psi = \sum_{i=p-m}^{[p/2]} L^i\psi_i, \tag{2.15}$$

where we used the identity $p-m+\left[\frac{2m-p}{2}\right]=\left[\frac{p}{2}\right]$. The degree of $\tilde{\psi}_{i-p+m}$ is $(p-2i)$. Therefore deg $\tilde{\psi}_{i-p+m}\leqq m$ for $i=p-m, \ldots, [p/2]$. It follows by Lemma 2.3, that $\Pi_{k,l}\tilde{\psi}_{i-p+m}$ is harmonic. Since

$$C\tilde{\psi}_{i-p+m} = \sum_{k+l=p-2i}(\sqrt{-1})^{k-l}\Pi_{k,l}\tilde{\psi}_{i-p+m},$$

we obtain that ψ_i is harmonic of degree $(p-2i)$. By (2.15), $\bar{*}\psi=\sum_{i=p-n}^{[p/2]}\bar{*}(L^i\psi_i)$. Using Lemma 2.4, we easily obtain from the last equality that ψ_i, $i=p-m, \ldots, [p/2]$, are uniquely defined by ψ. This completes the proof of (i).

(ii) To prove (ii), assume that ω is of bidegree (k, l). Then, by (2.3), $\omega=\psi+$ $+\Delta_B\mu$, where ψ is basic harmonic and μ is basic. Then

$$\omega = \sum_{i=(p-m)^+}^{[p/2]} L^i\psi_i+\Delta_B\mu,$$

where ψ_i is harmonic of degree $(p-2i)$. It follows that

$$\omega = \Pi_{k,l}\omega = \sum_{i=(p-m)^+}^{[p/2]}\Pi_{k,l}L^i\psi_i+\Pi_{k,l}\Delta_B\mu = \sum_{i=(p-m)^+}^{[p/2]} L^i(\Pi_{k-i,l-i}\psi_i)+\Pi_{k,l}\Delta_B\mu.$$

If $p=k+l\leqq m$, then, by 2.7, $\Pi_{k,l}\Delta_B\mu=\Delta_B\Pi_{k,l}\mu$. Let $p=k+l\geqq m+1$. Since [KT] $\Delta_B\xi=(-1)^p\bar{*}\Delta_B\bar{*}\mu$. Therefore, since deg $(\bar{*}\mu)\leqq m$,

$$\Pi_{k,l}\Delta_B\mu = (-1)^p\Pi_{k,l}\bar{*}\Delta_B\bar{*}\mu = (-1)^p\bar{*}\Pi_{m-l,m-k}\Delta_B\bar{*}\mu =$$
$$= (-1)^p\bar{*}\Delta_B\Pi_{m-i,m-k}\bar{*}\mu = (-1)^p\bar{*}\Delta_B\bar{*}\Pi_{k,l}\mu = \Delta_B\Pi_{k,l}\mu.$$

Thus, for any p, $0\leqq p\leqq 2m$,

$$\omega = \sum_{i=(p-m)^+}^{[p/2]} L^i(\Pi_{k-i,l-i}\psi_i)+\Delta_B(\Pi_{k,l}\mu).$$

By Lemma 2.3, $\Pi_{k-i,l-i}\psi_i$ is harmonic and $\Pi_{k,l}\mu$ is basic. By uniqueness of decomposition (2.12), $\psi_i=\Pi_{k-i,l-i}\psi_i$. Hence ψ_i is of bidegree $(k-i, l-i)$. This proves (ii).

In course of the proof of Theorem 2.5 we saw that the notion of a basic harmonic form is the same as the notion of a C-harmonic form. Therefore we can use results of [0] and [T2] on C-harmonic forms. If ω is C-harmonic, then $L\omega$ is also

C-harmonic. Thus we obtain homomorphisms

$$L\colon H_B^{p-2}(M) \to H_B^p(M), \quad p \leqq m,$$

and

$$L\colon H_B^{k-1,l-1}(M) \to H_B^{k,l}(M), \quad k+l \leqq m.$$

These homomorphisms are one-to-one. In addition,

$$\begin{aligned} H_B^p(M) &= LH_B^{p-2}(M) \oplus H^p(M), \quad p \leqq m, \\ H_B^{k,l}(M) &= LH_B^{k-1,l-1}(M) \oplus H^{k,l}(M), \quad k+l \leqq m. \end{aligned} \tag{2.16}$$

It follows that

$$\begin{aligned} b_p &= b_p^{(B)} - b_{p-2}^{(B)}, \quad p \leqq m, \\ h^{k,l} &= h_B^{k,l} - h_B^{k-1,l-1}, \quad k+l \leqq m, \end{aligned} \tag{2.17}$$

where we set $b_{-2}^{(B)} = b_{-1}^{(B)} = h_B^{k,-1} = h_B^{-1,l} = 0$. It follows from (2.17) that

$$\begin{aligned} b_{p-2}^{(B)} &\leqq b_p^{(B)}, \quad p \leqq m, \\ h_B^{k-1,l-1} &\leqq h_B^{k,l}, \quad k+l \leqq m. \end{aligned} \tag{2.18}$$

In particular, we have

$$1 = h_B^{0,0} \leqq h_B^{1,1} \leqq \ldots \leqq h_B^{[m/2],[m/2]}. \tag{2.19}$$

Note that the mapping $\bar{*}$ induces the isomorphisms

$$\begin{aligned} H_B^p(M) &\cong H_B^{2m-p}(M), \quad 0 \leqq p \leqq 2m, \\ H_B^{k,l}(M) &\cong H_B^{m-l,m-k}(M), \quad 0 \leqq k, l \leqq m. \end{aligned} \tag{2.20}$$

In addition, the complex conjugation induces the isomorphism

$$H_B^{k,l}(M) \cong H_B^{l,k}(M), \quad 0 \leqq k, l \leqq m. \tag{2.21}$$

Therefore we have

$$\begin{aligned} b_p^{(B)} &= b_{2m-p}^{(B)}, \quad 0 \leqq p \leqq 2m, \\ h_B^{k,l} &= h_B^{l,k} = h_B^{m-k,m-l} = h_B^{m-l,m-k}, \quad 0 \leqq k, l \leqq m. \end{aligned} \tag{2.22}$$

3. Inequalities for basic cohomology classes. In this section we continue to assume that $(M, \eta, X_0, \varphi, g)$ is a $(2m+1)$-dimensional compact Sasakian manifold.

Lemma 3.1. *Let ω any τ be harmonic forms of degrees $2i$ and $2j$, respectlively. Assume that $0 \leqq i \leqq m/2$, $0 \leqq j \leqq m/2$, and $i \neq j$. Then $I(\omega \wedge \tau) = 0$, where I is defined by formula* (1.3). *In particular, if u is a harmonic form of degree $2i$, where $0 < i \leqq m/2$, then $I(u) = 0$.*

Proof. Let u be a harmonic p-form, $0 \leqq p \leqq m$. Then, by [T1], u is effective (i.e. $\Lambda u = 0$), and therefore, by [T2],

$$\Lambda^r L^{r+s} u = 2^{2r}(s+1) \ldots (s-r)(m-p-s-r+1) \ldots (m-p-s) L^s u.$$

Take $r=m+1$ and $s=m-p+1$ in this formula. Since $r+s>m$, we obtain that $L^{r+s}u=0$. Therefore,

$$L^{m-p+1}u = 0, \tag{3.1}$$

where u is a harmonic p-form, $0\leqq p\leqq m$. By (3.1) we obtain that if $i>j$, then

$$\Phi^{m-i-j}\wedge\omega\wedge\xi = (L^{m-2i+(i-j)}\omega)\wedge\tau = 0.$$

Similarly, if $i<j$, then

$$\Phi^{m-i-j}\wedge\omega\wedge\xi = \omega\wedge(L^{m-2j+(j-i)}\tau) = 0.$$

It follows that

$$I(\omega\wedge\tau) = \frac{1}{2^m m!\,\mathrm{Vol}\,(M)}\int_M \eta\wedge\Phi^{m-i-j}\wedge\omega\wedge\tau = 0.$$

Let ω and τ be closed basic $\mathbf{C}$-valued forms of bidegree (k, k), where $0\leqq k\leqq \leqq m/2$. By Theorem 2.5, we have

$$\omega = \sum_{i=0}^{k} L^i\omega_i + d\lambda, \quad \tau = \sum_{i=0}^{k} L^i\tau_i + d\mu,$$

where ω_i and τ_i are harmonic forms of bidegree $(k-i, k-i)$, and λ and μ are basic forms.

Lemma 3.2.

$$I(\omega\wedge\tau)-I(\omega)I(\tau) = \frac{1}{2^{2k}m!\,\mathrm{Vol}\,(M)}\sum_{i=0}^{k-1}(-1)^{k-i}2^{2i}(m-2k+2i)!\,(\omega_i, \bar{\tau}_i).$$

Proof.

$$I(\omega\wedge\tau) = I\Big(\Big(\sum_{i=0}^{k} L^i\omega_i + d\lambda\Big)\wedge\Big(\sum_{j=0}^{k} L^i\tau_j + d\mu\Big)\Big) = \sum_{i,j=0}^{k} I(L^{i+j}\omega_i\wedge\tau_j) + I(dv),$$

where

$$v = \lambda\wedge\Big(\sum_{j=0}^{k} L^i\tau_j\Big) + \Big(\sum_{i=0}^{k} L^i\omega_i\Big)\wedge\mu + \lambda\wedge d\mu$$

is a basic form of degree $4k-1$. By Corollary to Lemma 2.1, $I(dv)=0$. By (1.5) and by Lemma 3.1, $I(L^{i+j}\omega_i\wedge\omega_j)=I(\omega_i\wedge\omega_j)=0$, if $i\neq j$. Therefore

$$I(\omega\wedge\tau) = I(\omega_k\wedge\tau_k) + \sum_{i=0}^{k-1} I(\omega_i\wedge\tau_i). \tag{3.2}$$

Since $\deg\omega_k=\deg\tau_k=0$, we obtain that $I(\omega_k\wedge\tau_k)=\omega_k\tau_k I(1)=\omega_k\tau_k$. By Lemma 3.1, $I(\omega)=I\Big(\sum_{i=0}^{k} L^i\omega_i+d\lambda\Big)=\sum_{i=0}^{k} I(\omega_i)=I(\omega_k)=\omega_k$. Similarly, $I(\tau)=\tau_k$. Therefore,

by (3.2),

$$I(\omega\wedge\tau)-I(\omega)I(\tau)=\sum_{i=0}^{k-1} I(\omega_i\wedge\tau_i)=\frac{1}{2^m m!\,\mathrm{Vol}\,(M)}\sum_{i=0}^{k-1}\int_M \eta\wedge\Phi^{m-2k+2i}\wedge\omega_i\wedge\tau_i=$$

$$=\frac{1}{2^m m!\,\mathrm{Vol}\,(M)}\sum_{i=0}^{k-1}\big(\eta\wedge\omega_i,\ *(L^{m-2k+2i}\bar{\tau}_i)\big).$$

By Lemma 2.4,

$$*(L^{m-2k+2i}\bar{\tau}_i)=(-1)^{k-i}2^{m-2k+2i}(m-2k+2i)!\,\eta\wedge\bar{\tau}_i.$$

Hence

$$I(\omega\wedge\tau)-I(\omega)I(\tau)=\frac{1}{2^{2k}m!\,\mathrm{Vol}\,(M)}\sum_{i=0}^{k-1}(-1)^{k-i}2^{2i}(m-2k+2i)!\,(\omega_i,\bar{\tau}_i).$$

This proves the lemma.

Now we are able to prove Theorem 1.1.

Proof of Theorem 1.1. Let ω be a closed basic $\mathbf{C}$-valued form of bidegree (k,k), representing $\alpha\in H_B^{k,k}(M)$. By Theorem 2.5, $\omega=\sum_{i=0}^{k} L^i\omega_i+d\lambda$, where ω_i is a harmonic form of bidegree $(k-i,k-i)$, and λ is a basic form. Since $h_B^{k-1,k-1}=1$, we have by (2.19), that $h_B^{k-i,k-i}=1$ for $i=1,\dots,k$. Hence, by (2.17), $h^{k-i,k-i}=0$ for $i=1,\dots,k-1$. Therefore there is no harmonic forms of bidegree $(k-i,k-i)$ for $i=1,\dots,k-1$. By Lemma 3.2, we obtain that

$$(-1)^k[I(\omega\wedge\bar{\omega})-I(\omega)I(\bar{\omega})]=\frac{1}{m!\,\mathrm{Vol}\,(M)}(m-2k)!\,(\omega_0,\omega_0)\geqq 0.$$

The equality holds if and only if $\omega_0=0$. In this case $\omega=t\Phi^k+d\lambda$, where $t=\omega_k$. Therefore the equality holds if and only if $\alpha=t\Omega^k$. This proves the theorem.

4. Basic Chern forms. Let (M,η,X_0,φ,g) be a compact $(2m+1)$-dimensional Sasakian manifold and let ∇ be the Riemannian connection on (M,g). A linear connection on M given by the formula, [Ta]:

$$\bar{\nabla}_X Y=\nabla_X Y+\eta(X)\varphi Y+\eta(Y)\varphi X+\Phi(X,Y)X_0 \tag{4.1}$$

will be called *the canonical connection* on M. The following properties of the canonical connection are easily verified by direct computation:

$$\bar{\nabla}_X\eta=0,\quad \bar{\nabla}_X X_0=0,\quad \bar{\nabla}_X\varphi=0 \tag{4.2}$$

for any tangent vector X on M;

$$i(X_0)\tilde{\Theta}=0,\quad i(X_0)\tilde{T}=0, \tag{4.3}$$

where $\tilde{\Theta}$ and $\tilde{T}$ are the curvature form and the torsion form of $\bar{\nabla}$, respectively;

$$\begin{aligned}\tilde{R}(X,Y)Z = R(X,Y)Z+2\Phi(X,Y)\varphi Z+[\Phi(X,Z)-\eta(X)\eta(Z)]\varphi Y-\\ -[\Phi(Y,Z)-\eta(Y)\eta(Z)]\varphi X+[g(X,Z)\eta(Y)-g(Y,Z)\eta(X)]X_0,\end{aligned} \tag{4.4}$$

where R and $\tilde{R}$ are curvature tensors of ∇ and $\tilde{\nabla}$, respectively;

$$\tilde{R}(\varphi X, \varphi Y) = \tilde{R}(X,Y). \tag{4.5}$$

Consider M as a base of a vector bundle F with the fibre $D_x=\{X\in TM_x\colon \eta(X)=0\}$ at the point $x\in M$. The map $\varphi|_x\colon D_x\to D_x$ defines a complex structure on D_x. Hence F may be considered as a complex vector bundle over M. By (4.2), the canonical connection $\tilde{\nabla}$ induces a complex linear connection in the complex bundle F, which we will denote again by $\tilde{\nabla}$. Let $C_k^{(B)}$ be the kth Chern form of $\tilde{\nabla}$, [C]. $C_k^{(B)}$, $k=1, \ldots, m$, are defined by the formula

$$\det\left[tI+\frac{\sqrt{-1}}{2\pi}\tilde{\Theta}\right] = t^m+\sum_{k=1}^{m} C_k^{(B)} t^{m-k}. \tag{4.6}$$

$C_k^{(B)}$ is closed. By (4.3), $\tilde{\Theta}$ is horizontal. Therefore $C_k^{(B)}$ is horizontal. Hence $C_k^{(B)}$ is basic. Because of (4.5), $C_k^{(B)}$ is real and of bidegree (k, k). Thus, for any $k=1, \ldots, m$, $C_k^{(B)}$ is a canonically defined real closed basic $2k$-form of bidegree (k, k). We will call $C_k^{(B)}$ the kth *basic Chern form* of a Sasakian manifold. Substituting $C_k^{(B)}$ in (1.10) we obtain that in the case $h_B^{k-1,k-1}=1$ the following integral inequality is satisfied

$$(-1)^k[I(C_k^{(B)}\cdot C_k^{(B)})-I(C_k^{(B)})I(C_k^{(B)})] \geqq 0. \tag{4.7}$$

Using (4.4), we obtain by direct computation that in the case $k=1$ inequality (4.7) is the same as inequality (1.1).

Remark. Let (M, η) be a contact manifold. An associated contact metric structure (η, X_0, φ, g) is called an *associated K-metric structure*, [B], if X_0 is a Killing vector field with respect to g. If a contact manifold (M, η) admits an associated K-metric structure, (M, η) is called a K-contact manifold. We will show now how one can define basic Pontrjagin cohomology classes on a K-contact manifold.

Let (M, η) be a $(2m+1)$-dimensional contact manifold. A linear connection $\tilde{\nabla}$ on M will be called *basic* if

$$\tilde{\nabla}_X\eta = 0, \quad \tilde{\nabla}_X X_0 = 0, \quad i(X_0)\tilde{\Theta} = 0, \quad i(X_0)\tilde{T} = 0, \tag{4.8}$$

where $\tilde{\Theta}$ and $\tilde{T}$ are the curvature form and the torsion form of $\tilde{\nabla}$, respectively.

Assume that (M, η) admits a basic linear connection $\tilde{\nabla}$. Consider M as the base space of a real vector bundle with the $2m$-dimensional fibre $D_x=\{X\in TM_x\colon \eta(X)=0\}$ at the point $x\in M$. By (4.8), $\tilde{\nabla}$ can be considered as a connection in this vector

bundle. Put

$$\det\left[tI-\frac{1}{2\pi}\Theta\right] = t^m+\sum_{k=1}^{m} E_k(\tilde{\Theta})t^{m-k}.$$

Then $E_{2k}(\tilde{\Theta})$ is a closed and horizontal (since $\tilde{\Theta}$ is horizontal) $4k$-form, [C], p. 118. Hence $E_{2k}(\tilde{\Theta})$ is basic and therefore defines an element $p_k^{(B)}\in H_B^{4k}(M, \mathbf{R})$. We will show that $p_k^{(B)}$ does not depend on a choice of a basic linear connection. Indeed, let $\tilde{\nabla}'$ be another basic linear connection and let $\tilde{\Theta}'$ and T' be its curvature and torsion forms, respectively. Set $\alpha=\tilde{\nabla}'-\nabla$, $\tilde{\nabla}^t=\tilde{\nabla}+t\alpha$. Let $\tilde{\Theta}^t$ be the curvature form of $\tilde{\nabla}^t$. Then α is a linear form on M of the type ad GL $(2m, \mathbf{R})$, and by (4.8) and (4.9),

$$\alpha(X_0)X = \tilde{\nabla}'_{X_0}X-\tilde{\nabla}_{X_0}X = \tilde{\nabla}_X X_0+[X_0, X]+\tilde{T}'(X_0, X)-$$
$$-\tilde{\nabla}_X X_0-[X_0, X]-\tilde{T}(X_0, X) = 0.$$

Hence α is horizontal. By [C], p. 42, $\tilde{\Theta}^t=\tilde{\Theta}+tD\alpha-t^2\alpha\wedge\alpha$. Taking $t=1$, we obtain $D\alpha=\tilde{\Theta}'-\tilde{\Theta}+\alpha\wedge\alpha$. Therefore $\tilde{\Theta}=(1-t)\tilde{\Theta}+t\tilde{\Theta}'+t(1-t)\alpha\wedge\alpha$. It follows that $\tilde{\Theta}^t$ is horizontal for all t. By [C], p. 115, $E_{2k}(\tilde{\Theta}')-E_{2k}(\tilde{\Theta})=d\varrho$, where $\varrho=\int_0^1 \Psi(t)dt$ and where $\psi(t)$ is a polynomial function of α and $\tilde{\Theta}^t$. It follows that ϱ is horizontal. In addition,

$$L_{X_0}\varrho = [i(X_0)d+di(X_0)]\varrho = i(X_0)\,d\varrho = i(X_0)[E_{2k}(\tilde{\Theta}')-E_{2k}(\tilde{\Theta})] = 0.$$

Hence ϱ is basic. Thus, $E_{2k}(\tilde{\Theta}')$ and $E_{2k}(\tilde{\Theta})$ are homologous within basic forms. Therefore $E_{2k}(\tilde{\Theta}')$ and $E_{2k}(\tilde{\Theta})$ define the same element $p_k^{(B)}\in H_B^{4k}(M, \mathbf{R})$. If (M, η) is a contact manifold which admits a basic linear connection, then $p_k^{(B)}$, $k=1, \ldots$ $\ldots, [m/2]$, will be called *basic Pontrjagin classes* of (M, η).

Let now (M, η) be a K-contact manifold. Let (η, X_0, φ, g) be an associated K-metric structure and ∇ be the Riemannian connection on M with respect to g. Direct calculation shows that the connection

$$\tilde{\nabla}_X Y = \nabla_X Y+\eta(X)\varphi Y+\eta(Y)\varphi X+\Phi(X, Y)X_0$$

is a basic connection on (M, g). Hence the basic Pontrjagin classes $p_k^{(B)}\in H_B^{4k}(M, \mathbf{R})$, $k=1, \ldots, [m/2]$, are well-defined on each K-contact manifold (M, η).

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLIONIS
URBANA, IL 61801
USA

and

DEPARTMENT OF MATHEMATICS
EASTERN ILLIONIS UNIVERSITY
CHARLESTON, IL 61902
USA

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A note on the strong de la Vallée Poussin approximation

L. LEINDLER

1. Let $\{\varphi_n(x)\}$ be an orthonormal system on the finite interval (a, b). We shall consider series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \tag{1.1}$$

with real coefficients satisfying

$$\sum_{n=0}^{\infty} c_n^2 < \infty. \tag{1.2}$$

By the Riesz—Fischer theorem, series (1.1) converges in the metric L^2 to a square-integrable function $f(x)$. We denote the n-th partial sum of series (1.1) by $s_n(x)$.

It is well known that the notion of *strong summability* is due to HARDY and LITTLEWOOD [3], and the notion of *strong approximation* is due to ALEXITS [2].

Since the strong approximation investigations have started in the 1960s it has become more and more clear that most of the results concerning any property of ordinary approximation have an analogue in strong sense. In other words, we have the same rate of approximation for strong means as for ordinary ones if we consider any one of the most frequently used means. This is true in spite of the facts that, in general, neither strong summability nor strong approximation do not follow from the suitable general ordinary summability and approximation (see MÓRICZ [11] and LEINDLER [9]). Some sample theorems showing the great analogy between the ordinary and strong approximation results can be found e.g. in the works [1], [4], [5], [6], [8], [10].

Recently we have discovered that even in the case of the classical de la Vallée Poussin approximation there exists a result which has only a weaker analogue in strong sense. One of the aims of this note is to fill up this gap.

In order to formulate our statements precisely we recall some definitions and theorems.

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Now we define the ordinary, furthermore the strong and very strong de la Vallée Poussin means with exponent p $(p>0)$:

$$V_n(x) := \frac{1}{n} \sum_{\nu=n}^{2n-1} s_\nu(x), \quad (n \geqq 1),$$

$$V_n|p;\, x| := \left\{ \frac{1}{n} \sum_{\nu=n}^{2n-1} |s_\nu(x) - f(x)|^p \right\}^{1/p}$$

and

$$V_n|p,\, \nu;\, x| := \left\{ \frac{1}{n} \sum_{k=n}^{2n-1} |s_{\nu_k}(x) - f(x)|^p \right\}^{1/p},$$

where $\nu := \{\nu_k\}$ denotes an arbitrary increasing sequence of positive integers.

In [4] we proved

Theorem A. *Let $\{\lambda_n\}$ be a monotonic sequence of positive numbers such that*

$$\sum_{k=0}^{m} \lambda_{2^k}^2 \leqq K \lambda_{2^m}^2 . \text{ *)} \tag{1.3}$$

If

$$\sum_{n=0}^{\infty} c_n^2 \lambda_n^2 < \infty, \tag{1.4}$$

then we have that

$$V_n(x) - f(x) = o_x(\lambda_n^{-1})$$

holds almost everywhere (a.e.) in (a, b).

A similar, but weaker result in connection with the strong approximation $(p=1)$ was proved in [6] which reads as follows.

Theorem B. *Let $\{\lambda_n\}$ be a monotonic sequence of positive numbers with* (1.3). *If*

$$\sum_{n=0}^{\infty} c_n^2 \lambda_{2^n}^2 < \infty, \tag{1.5}$$

then

$$V_n|p, x| = o_x(\lambda_n^{-1}) \tag{1.6}$$

holds a.e. in (a, b) for any $0<p\leqq 2$.

It is easy to see that if $\lambda_n = n^\gamma$ with $\gamma>0$, then conditions (1.4) and (1.5) are equivalent; but if e.g. $\lambda_n = q^n$ with $q>1$, then (1.5) requires much more that (1.4) does in order to have the same order of approximation.

*) K will denote positive constant not necessarily the same one at each occurrence.

First we show that (1.4) also always implies (1.6), but this proof will be longer than that of the implication (1.5)⇒(1.6) in [6]. Thereafter, using a very general lemma proved only in 1982, we shall extend our result to the very strong approximation; i.e. we shall prove the following result.

Theorem. *Let* $\{\lambda_n\}$ *be a monotonic sequence of positive numbers with* (1.3). *If* (1.4) *holds, then*

$$V_n|p, \nu;\ x| = o_x(\lambda_n^{-1}) \tag{1.7}$$

a.e. in (a, b) *for any* $0<p\leqq 2$ *and for any increasing sequence* $\nu:=\{\nu_k\}$ *of positive integers.*

2. In order to prove our theorem we need two known lemmas.

Lemma 1 (Kronecker lemma, see e.g. [1] p. 68). *If* $s_n(a)$ *denotes the n-th partial sum of the numerical series* $\sum_{m=0}^{\infty} a_m$ *and* $\{\lambda_n\}$ *is an increasing sequence of positive numbers such that* $\lambda_n\to\infty$ *and the series* $\sum_{m=0}^{\infty} a_m\lambda_m^{-1}$ *converges, then* $s_n(a)=o(\lambda_n)$ *holds.*

Lemma 2 ([7]). *Let* $\delta>0$ *and* $\{\delta_n\}$ *an arbitrary sequence of non-negative numbers. Suppose that for any orthonormal system* $\{\varphi_n(x)\}$ *the condition*

$$\sum_{n=0}^{\infty} \delta_n\Big(\sum_{m=n}^{\infty} c_m^2\Big)^{\delta} < \infty \tag{2.1}$$

implies that the sequence $\{s_n(x)\}$ *of the partial sums of* (1.1) *possesses a property P, then any subsequence* $\{s_{\nu_n}(x)\}$ *also possesses property P.*

3. First we carry out the proof of (1.7) when $p=2$ and $\nu_k=k$. An elementary consideration shows that

$$V_n|2, x|^2 \leqq K\Big\{\frac{1}{n}\sum_{\nu=n}^{2n-1} |s_\nu(x)-V_\nu(x)|^2+\frac{1}{n}\sum_{\nu=n}^{2n-1} |V_\nu(x)-f(x)|^2\Big\}. \tag{3.1}$$

The second term on the right hand side of (3.1) is $o_x(\lambda_n^{-2})$ a.e. in (a, b) on account of Theorem A regarding the monotonicity of the sequence $\{\lambda_n\}$. Thus we have only to estimate the first term. For that purpose we first show that

$$\sum\nolimits_1 := \sum_{n=1}^{\infty} \lambda_n^2 n^{-1} \int_a^b (s_n(x)-V_n(x))^2\,dx < \infty. \tag{3.2}$$

Namely, an easy calculation gives that

$$\sum_1 = \sum_{n=1}^{\infty} \lambda_n^2 n^{-1} \frac{1}{n^2} \sum_{\nu=n+1}^{2n-1} (2n-\nu)^2 c_\nu^2 \leqq$$

$$\leqq \sum_{n=1}^{\infty} \lambda_n^2 \frac{1}{n} \sum_{\nu=n+1}^{2n-1} c_\nu^2 \leqq \sum_{\nu=1}^{\infty} c_\nu^2 \sum_{\nu/2<n<\nu} \lambda_n^2 n^{-1} \leqq \sum_{\nu=1}^{\infty} c_\nu^2 \lambda_\nu^2,$$

whence, by (1.4), (3.2) obviously follows. From (3.2), using B. Levi's theorem, we get that

$$\sum_{n=1}^{\infty} \lambda_n^2 n^{-1} (s_n(x) - V_n(x))^2 < \infty$$

a.e. in (a, b). Hence, by Lemma 1, the estimation

$$\sum_{n=1}^{2m} \lambda_n^2 (s_n(x) - V_n(x))^2 = o_x(2m) \tag{3.3}$$

holds a.e. in (a, b). But (3.3) clearly implies that

$$\frac{1}{m} \sum_{n=m}^{2m-1} |s_n(x) - V_n(x)|^2 = o_x(\lambda_m^{-2})$$

also holds a.e. in (a, b). Summing up our partial results we get that

$$V_n|2, x| = o_x(\lambda_n^{-1}). \tag{3.4}$$

On account of the Hölder's inequality, we get, for any $0<p\leqq 2$, that

$$V_n|p, x| \leqq V_n|2, x|$$

whence, by (3.4),

$$V_n|p, x| = o_x(\lambda_n^{-1}) \tag{3.5}$$

also holds a.e. in (a, b). This completes the proof when $\nu_k = k$.

Finally, the statement of Theorem in its generality, i.e. for arbitrary $\nu := \{\nu_k\}$, follows from (3.5) using Lemma 2 with $\delta = 1$ and $\delta_n := \lambda_n^2 - \lambda_{n-1}^2$ $(\lambda_{-1} = 0)$; furthermore the property P in this case will be just the estimation given by (3.5).

Theorem is hereby proved completely.

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BOLYAI INSTITUTE
JÓZSEF ATTILA UNIVERSITY
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

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On strong approximation by Cesàro means of negative order

L. LEINDLER

1. Let $\{\varphi_n(x)\}$ be an orthonormal system on the finite interval (a, b). We shall consider series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \tag{1.1}$$

with real coefficients satisfying

$$\sum_{n=0}^{\infty} c_n^2 < \infty. \tag{1.2}$$

By the Riesz—Fischer theorem, series (1.1) converges in the metric L^2 to a square-integrable function $f(x)$. We denote the partial sums and the (C, α)-means of series (1.1) by $s_n(x)$ and $\sigma_n^\alpha(x)$, respectively. Furthermore, T_n will denote a positive regular summation method determined by a triangular matrix (α_{nk}/A_n) $\left(\alpha_{nk} \geqq 0 \text{ and } A_n := \sum_{k=0}^{n} \alpha_{nk}\right.$, and if s_k tends to s, then

$$\left. T_n := \frac{1}{A_n} \sum_{k=0}^{n} \alpha_{nk} s_k \to s \right).$$

G. SUNOUCHI [7] proved the following result.

Theorem A. *If* $0<\gamma<1$ *and*

$$\sum_{n=1}^{\infty} c_n^2 n^{2\gamma} < \infty, \tag{1.3}$$

then

$$\left\{\frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |f(x) - s_\nu(x)|^p\right\}^{1/p} = o_x(n^{-\gamma}) \tag{1.4}$$

holds almost everywhere (a.e.) for any $\alpha>0$ *and* $0<p<\gamma^{-1}$, *where* $A_n^\alpha := \binom{n+\alpha}{n}$.

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In [3] we generalized this result in such a way that we replaced the partials in (1.4) by Cesàro means of negative order. Our theorem reads as follows, where and in the sequel K will denote positive constant, not necessarily the same one.

Theorem B. *Suppose that* $0<\gamma<1$, $0<p<\gamma^{-1}$ *and* (1.3) *holds, furthermore that there exists a number* $\varrho>1$ *such that*

$$\frac{\varrho p}{\varrho-1} \geqq 2, \tag{1.5}$$

and with this ϱ *for any* $0<\omega<1$ *and* $2^m<n\leqq 2^{m+1}$

$$\sum_{l=0}^{m}\Big\{\sum_{\nu=2^l-1}^{\min(2^{l+1},n)} \alpha_{n\nu}^{\varrho}(\nu+1)^{\varrho(1-\omega)-1}\Big\}^{1/\varrho} \leqq Kn^{-\omega}A_n. \tag{1.6}$$

Then, for arbitrary

$$d>1-\frac{\varrho-1}{\varrho p}, \tag{1.7}$$

$$\Big\{\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|f(x)-\sigma_\nu^{d-1}(x)|^p\Big\}^{1/p} = o_x(n^{-\gamma}) \tag{1.8}$$

holds a.e. in (a, b).

It is easy to verify that in the special case $\alpha_{n\nu}=A_{n-\nu}^{\alpha-1}$ $(\alpha>0)$ condition (1.6) is satisfied, thus Theorem B with $d=1$ reduces to Theorem A.

But if we set $\alpha_{n\nu}=(\nu+1)^{\beta-1}$ $(\beta>0)$, then condition (1.6) will be satisfied only if $\beta\geqq 1$. Consequently, for $0<\beta<1$, we cannot apply Theorem B to get an estimate for the following strong Riesz means

$$h_n(f, d, \beta, p;\ x):=\Big\{(n+1)^{-\beta}\sum_{\nu=0}^{n}(\nu+1)^{\beta-1}|f(x)-\sigma_\nu^{d-1}(x)|^p\Big\}^{1/p},$$

but the Riesz summability is a frequently used summation method in connection with strong approximation. Nevertheless, if we want to get the estimate

$$h_n(f, d, \beta, p;\ x) = o_x(n^{-\gamma}) \tag{1.9}$$

for some $0<\beta<1$, then, as a possible solution, we can try to weaken the requirement of (1.6).

One of our aims is to give such a generalization of Theorem B.

We mention that in the special case $d=1$, i.e. if we approximate the function $f(x)$ with the partial sums $s_\nu(x)$ $(=\sigma_\nu^0(x))$, then already an estimate of type (1.9) is known. Namely, in a joint paper with H. Schwinn [6], we proved among others:

Theorem C. *If* $\gamma>0$ *and* $0<p\gamma<\beta$ *then condition* (1.3) *implies*

$$(1.10)\quad H_n(f,\beta,p,\nu;\,x):=\Big\{(n+1)^{-\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|f(x)-s_{\nu_k}(x)|^p\Big\}^{1/p}=o_x(n^{-\gamma})$$

a.e. in (a,b) *for any increasing sequence* $\nu:=\{\nu_k\}$ *of natural numbers.*

In [5] we investigated the so-called *limit case* of the restriction of the parameters, i.e. if $\beta=p\gamma$. Among others we proved:

Theorem D. *If* p *and* β *are positive numbers then for any increasing sequence* $\nu:=\{\nu_k\}$

$$(1.11)\qquad \sum_{n=1}^{\infty}c_n^2n^{2\beta/p}<\infty\quad(\gamma=\beta/p)$$

implies

$$(1.12)\qquad H_n(f,\beta,p,\nu;\,x)=o_x\big(n^{-\beta/p}(\log n)^{1/p}\big)$$

a.e. in (a,b).

Theorem E. *If* α *and* p *are positive numbers then for any increasing sequence* $\nu:=\{\nu_k\}$

$$(1.13)\qquad \sum_{n=1}^{\infty}c_n^2n^{2/p}<\infty\quad(\gamma=1/p)$$

implies

$$(1.14)\quad C_n(f,\alpha,p,\nu;\,x):=\left\{\frac{1}{A_n^{\alpha}}\sum_{k=0}^{n}A_{n-k}^{\alpha-1}|f(x)-s_{\nu_k}(x)|^p\right\}^{1/p}=o_x\big(n^{-1/p}(\log n)^{1/p}\big)$$

a.e. in (a,b).

We want to point out that Theorems C, D and E contrary to Theorems A and B do not claim the extra restriction $\gamma<1$. This is a great advantage of these theorems, but they do not allow of approximating with Cesàro means of negative order.

The common kernel of the proof of Theorems A and B is a very interesting result of T. M. Flett [1] and a useful lemma of G. Sunouchi [7] (here Lemma 2 and Lemma 3, respectively) and they, unfortunately, require the assumption $0<\gamma<1$. The proofs of Theorems C, D and E run on a perfectly different line, and these proofs do not use the assumption $\gamma<1$.

In the present paper we prove such a general theorem which generalizes Theorem B and includes all of Theorems C, D and E if $\gamma<1$. Unfortunately, we cannot extend the validity of our result for $\gamma\geqq1$. This remains as an interesting open problem, in our view.

Using the notations introduced above we can formulate our results.

Theorem. *Suppose that* $p>0$ *and* $0<\gamma<1$. *Moreover let us suppose that there exists a number* $\varrho>1$ *with property* (1.5) *and that with this* ϱ *and with* $n(l):=$

$:=\min(2^l, n)$, $2^m<n\leqq 2^{m+1}$

$$\sum_{l=0}^{m}\Big\{\sum_{\nu=n(l)-1}^{n(l+1)}\alpha_{n\nu}^{\varrho}(\nu+1)^{\varrho(1-\gamma p)-1}\Big\}^{1/\varrho}\leqq Kg(n)A_n n^{-\gamma p}, \tag{1.15}$$

where $g(t)$ denotes a non-decreasing positive function defined for $0\leqq t<\infty$.

Then, for any d satisfying (1.7), (1.3) *implies*

$$\Big\{\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|f(x)-\sigma_\nu^{d-1}(x)|^p\Big\}^{1/p}=O_x\big(g(n)^{1/p}n^{-\gamma}\big) \tag{1.16}$$

a.e. in (a, b).

If, in addition, for every fixed l,

$$\Big\{\sum_{\nu=n(l)-1}^{n(l+1)}\alpha_{n\nu}^{\varrho}\Big\}^{1/\varrho}=o\big(g(n)A_n n^{-\gamma p}\big),\quad as\quad n\to\infty, \tag{1.17}$$

then the O_x in (1.16) *can be replaced by o_x.*

Hence, by a useful lemma (here Lemma 1) we easily get the following result.

Corollary 1. *Under the assumptions of Theorem we have the estimate*

$$\Big\{\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|f(x)-\sigma_\nu^{d-1}(\mu;\,x)|^p\Big\}^{1/p}=O_x\big(g(n)^{1/p}n^{-\gamma}\big) \tag{1.18}$$

a.e. in (a, b), where $\mu:=\{\mu_k\}$ is an increasing sequence of natural numbers and

$$\sigma_n^{\beta}(\mu;\,x):=\frac{1}{A_n^{\beta}}\sum_{k=0}^{n}A_{n-k}^{\beta-1}s_{\mu_k}(x).$$

If (1.17) *also holds, then the O_x in* (1.18) *can be replaced by o_x.*

From Corollary 1, by an easy consideration to be detailed later on, we get the following results:

Corollary 1.1. *If $0<\gamma<1$, $d>\max(1/2, (p-1)/p)$ and $0<p\gamma<\beta$, then* (1.3) *implies*

$$\Big\{(n+1)^{-\beta}\sum_{\nu=0}^{n}(\nu+1)^{\beta-1}|f(x)-\sigma_\nu^{d-1}(\mu;\,x)|^p\Big\}^{1/p}=o_x(n^{-\gamma})$$

a.e. in (a, b) for any increasing sequence $\mu:=\{\mu_k\}$.

Corollary 1.2. *If* $\alpha>0$, $0<\gamma<1$, $0<p<\gamma^{-1}$ *and* $d>\max(1/2,(p-1)/p,(p-\alpha)/p)$, *then* (1.3) *implies*

$$\left\{\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|f(x)-\sigma_\nu^{d-1}(\mu;\,x)|^p\right\}^{1/p} = o_x(x^{-\gamma}) \tag{1.19}$$

a.e. in (a,b) *for any increasing sequence* $\mu:=\{\mu_k\}$.

Corollary 1.3. *If* $0<\gamma<1$, $d>\max(1/2,(p-1)/p)$ *and* $\beta=p\gamma$, *then* (1.3) *implies*

$$\{(n+1)^{-\beta}\sum_{\nu=0}^{n}(\nu+1)^{\beta-1}|f(x)-\sigma_\nu^{d-1}(\mu;\,x)|^p\}^{1/p} = o_x((\log n)^{1/p}n^{-\gamma}) \tag{1.20}$$

a.e. in (a,b) *for any increasing sequence* $\mu:=\{\mu_k\}$.

Corollary 1.4. *If* $0<\gamma<1, p\gamma=1, \alpha>0$ *and* $d>\max(1/2,(p-1)/p,(p-\alpha)/p)$, *then* (1.3) *implies*

$$\left\{\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|f(x)-\sigma_\nu^{d-1}(\mu;\,x)|^p\right\}^{1/p} = o_x((\log n)^{1/p}n^{-\gamma}) \tag{1.21}$$

a.e. in (a,b) *for any increasing sequence* $\mu:=\{\mu_k\}$.

First we remark that Corollary 1.2 is a slight improvement of Theorem 1 proved in [2].

Furthermore we mention that since $\sigma_k^0(\mu;x)=s_{\mu_k}(x)$, thus the special case $d=1$ of Corollary 1.1 coincides with Theorem C if $0<\gamma<1$. But if $\gamma\geqq 1$ then Corollary 1.1 does not work, consequently, we cannot say that Corollary 1.1 is a generalization of Theorem C. So, we can say that Corollary 1.1 is a generalization of Theorem C if and only if the range of parameter γ is restricted to $0<\gamma<1$.

The same assertion can be made regarding Corollary 1.3 and Theorem D, moreover in connection with Corollary 1.4 and Theorem E.

Finally we deduce one more statement from Corollary 1.

Corollary 1.5. *If* $p>0$, $0<\gamma<1$ *and* $d>\max(1/2,(p-1)/p)$, *then* (1.3) *implies*

$$\left\{\frac{1}{n}\sum_{\nu=n+1}^{2n}|f(x)-\sigma_\nu^{d-1}(\mu;\,x)|^p\right\}^{1/p} = o_x(n^{-\gamma}) \tag{1.22}$$

a.e. in (a,b) *for any increasing sequence* $\mu:=\{\mu_k\}$.

We point out that Corollary 1.5 sharpens and generalizes Corollary 1 proved in [3]. It can be used for any positive p, not only if $p<\gamma^{-1}$ as in [3].

2. In order to prove our theorem and corollaries we need three known lemmas and one to be proved now.

Lemma 1 ([4]). *Let* $\delta>0$ *and* $\{\delta_n\}$ *an arbitrary sequence of non-negative numbers. Suppose that for any orthonormal system* $\{\varphi_n(x)\}$ *the condition*

$$\sum_{n=0}^{\infty}\delta_n\Big(\sum_{m=n}^{\infty}c_m^2\Big)^{\delta}<\infty \tag{2.1}$$

implies that the sequence $\{s_n(x)\}$ *of the partial sums of* (1.1) *possesses a property P, then any subsequence* $\{s_{\nu_n}(x)\}$ *also possesses property P.*

Lemma 2 ([1]). *Set*

$$\tau_n^{\alpha}:=\tau_n^{\alpha}(x):=n\{\sigma_n^{\alpha}(x)-\sigma_{n-1}^{\alpha}(x)\}\quad\big(=\alpha\{\sigma_n^{\alpha-1}(x)-\sigma_n^{\alpha}(x)\}\ \textit{if}\ \alpha>0\big).$$

Let $\bar p\geqq\bar q>1$, $\bar\varrho>0$, $\bar\alpha>\bar\varrho-1$ *and* $\bar\beta\geqq\bar\alpha+(\bar q)^{-1}-(\bar p)^{-1}$. *Then*

$$\Big\{\sum_{n=0}^{\infty}(n+1)^{\bar p\bar\varrho-1}|\tau_n^{\bar\beta}|^{\bar p}\Big\}^{1/\bar p}\leqq K\Big\{\sum_{n=0}^{\infty}(n+1)^{\bar q\bar\varrho-1}|\tau_n^{\bar\alpha}|^{\bar q}\Big\}^{1/\bar q}. \tag{2.2}$$

Lemma 3 ([7]). *If* $0<\gamma<1$ *and* (1.3) *holds, then*

$$\int_a^b\Big\{\sum_{n=0}^{\infty}(n+1)^{2\gamma-1}|\sigma_n^{\alpha-1}(x)-\sigma_n^{\alpha}(x)|^2\Big\}dx\leqq K\sum_{n=1}^{\infty}c_n^2n^{2\gamma}$$

for any $\alpha>1/2$.

Lemma 4. *Under the assumptions of Theorem we have the inequality*

$$\int_a^b\Big\{\sup_{0\leqq n<\infty}\Big(\frac{n^{p\gamma}}{g(n)A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_\nu^{d-1}(x)-\sigma_\nu^{d}(x)|^p\Big)^{1/p}\Big\}^2dx\leqq K\sum_{n=1}^{\infty}c_n^2n^{2\gamma}. \tag{2.3}$$

Proof. Set $\varrho':=\varrho/(\varrho-1)$, then by (1.5) and (1.7), we have

$$\varrho'p\geqq 2\quad\text{and}\quad d>1-(\varrho'p)^{-1}. \tag{2.4}$$

Applying Hölder's inequality, by (1.15) and $\varrho>1$, we obtain that

$$\sum_{\nu=0}^{n}\alpha_{n\nu}|\tau_\nu^d(x)|^p\leqq\Big\{\sum_{\nu=0}^{n}\alpha_{n\nu}^{\varrho}(\nu+1)^{(\varrho/\varrho')-\gamma p\varrho}\Big\}^{1/\varrho}\Big\{\sum_{\nu=0}^{n}(\nu+1)^{\gamma p\varrho'-1}|\tau_\nu^d(x)|^{p\varrho'}\Big\}^{1/\varrho'}\leqq \tag{2.5}$$

$$\leqq Kg(n)n^{-\gamma p}A_n\Big\{\sum_{\nu=0}^{n}(\nu+1)^{\gamma p\varrho'-1}|\tau_\nu^d(x)|^{p\varrho'}\Big\}^{1/\varrho'}.$$

By the second statement of (2.4) we can choose α^* such that

$$d-\frac{1}{2}+\frac{1}{\varrho'p}>\alpha^*>\frac{1}{2} \tag{2.6}$$

holds. By (2.6), $0<\gamma<1$ and $\varrho'\varrho\geqq 2$ the conditions of Lemma 2 are fulfilled with

$\bar{p}=\varrho' p$, $\bar{q}=2$, $\bar{\varrho}=\gamma$, $\bar{\alpha}=\alpha^*$ and $\bar{\beta}=d$. Using Lemma 2 we get

$$\left\{\sum_{\nu=0}^{\infty}(\nu+1)^{\gamma p\varrho'-1}|\tau_\nu^d(x)|^{p\varrho'}\right\}^{1/p\varrho'} \leqq K\left\{\sum_{\nu=0}^{\infty}(\nu+1)^{2\gamma-1}|\tau_\nu^{\alpha^*}(x)|^2\right\}^{1/2}. \tag{2.7}$$

Thus, by (2.5), (2.6), (2.7) and Lemma 3, we have

$$\int_a^b\left\{\sup_{0\leqq n<\infty}\left(\frac{n^{\gamma p}}{g(n)A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\tau_\nu^d(x)|^p\right)^{1/p}\right\}^2 dx \leqq$$

$$\leqq K\int_a^b\left\{\sum_{\nu=0}^{\infty}(\nu+1)^{2\gamma-1}|\tau_\nu^{\alpha^*}(x)|^2\right\}dx \leqq K\sum_{n=1}^{\infty}c_n^2 n^{2\gamma}<\infty,$$

which gives statement (2.3).

3. Proof of Theorem. It is clear that

$$\left\{\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|f(x)-\sigma_\nu^{d-1}(x)|^p\right\}^{1/p} \leqq \tag{3.1}$$

$$\leqq K\left(\left\{\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|f(x)-\sigma_\nu^d(x)|^p\right\}^{1/p}+\left\{\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_\nu^d(x)-\sigma_\nu^{d-1}(x)|^p\right\}^{1/p}\right).$$

First we show that the first term has the required order.

Since $d>1/2$, so, e.g. by Theorem A with $p=1$, we get that

$$f(x)-\sigma_n^d(x) = o_x(n^{-\gamma}) \tag{3.2}$$

a.e. in (a, b).

Let now $\varepsilon>0$ be given. If x is a point where (3.2) holds, then let $M(x)$ be a positive integer such that for $n>M(x)$ the inequality

$$|f(x)-\sigma_n^d(x)| < \varepsilon n^{-\gamma} \tag{3.3}$$

is valid. For such x we get that

$$\sum_{\nu=0}^{n}\alpha_{n\nu}|f(x)-\sigma_\nu^d(x)|^p \leqq \Big(\sum_{n(l)\leqq M(x)}+\sum_{\substack{n(l)>M(x)\\ l\leqq m}}\Big)\pi_l, \tag{3.4}$$

where

$$\pi_l:=\left\{\sum_{\nu=n(l)-1}^{n(l+1)}\alpha_{n\nu}^{\varrho}(\nu+1)^{\varrho(1-\gamma p)-1}\right\}^{1/\varrho}\left\{\sum_{\nu=n(l)-1}^{n(l+1)}\frac{1}{\nu+1}\left((\nu+1)^{\gamma p}|f(x)-\sigma_\nu^d(x)|^p\right)^{\varrho'}\right\}^{1/\varrho'}.$$

By (1.15) it is easy to see that the first sum remains $O_x(g(n)A_n n^{-\gamma p})$, but if (1.17) also holds, then its order $o_x(g(n)A_n n^{-\gamma p})$.

On the other hand, by (1.15) and (3.3), the second sum in (3.4) is always less than $O_x(1)\varepsilon^p g(n)A_n n^{-\gamma p}$, that is, its order is always $o_x(g(n)A_n n^{-\gamma p})$.

Consequently we have

$$(3.5)\qquad \left\{\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|f(x)-\sigma_\nu^d(x)|^p\right\}^{1/p} = \begin{cases} O_x\big(g(n)^{1/p}n^{-\gamma}\big), & \text{always,}\\ o_x\big(g(n)^{1/p}n^{-\gamma}\big), & \text{if (1.17) holds,}\end{cases}$$

a.e. in (a, b).

Next we show that the second term in (3.1) also has the suitable orders of (3.5), according as (1.17) is not or is satisfied.

Now let ε be any positive number. Let us choose N so large that

$$(3.6)\qquad \sum_{n=N+1}^{\infty} c_n^2 n^{2\gamma} < \varepsilon^3.$$

By means of N we split series (1.3) into

$$\sum_{n=1}^{N} c_n^2 n^{2\gamma} < \infty \quad\text{and}\quad \sum_{n=N+1}^{\infty} c_n^2 n^{2\gamma} < \varepsilon^3,$$

and consider the corresponding orthogonal series. More exactly, let

$$(3.7)\qquad \sum_{n=0}^{\infty} a_n\varphi_n(x) \quad\text{with}\quad a_n = \begin{cases} c_n & \text{for} \quad n \leqq N,\\ 0 & \text{for} \quad n > N;\end{cases}$$

and

$$(3.8)\qquad \sum_{n=0}^{\infty} b_n\varphi_n(x) \quad\text{with}\quad b_n = \begin{cases} 0 & \text{for} \quad n \leqq N,\\ c_n & \text{for} \quad n > N.\end{cases}$$

If $\sigma_n^\alpha(a; x)$ and $\sigma_n^\alpha(b; x)$ denote the (C, α)-means of series (3.7) and (3.8), respectively, then

$$(3.9)\qquad \sigma_n^\alpha(x) = \sigma_n^\alpha(a;\ x)+\sigma_n^\alpha(b;\ x).$$

Since the number of the coefficients $a_n \neq 0$ is finite,

$$\sigma_\nu^{d-1}(a;\ x)-\sigma_\nu^d(a;\ x) = \frac{1}{A_\nu^d}\sum_{k=0}^{N} kA_{\nu-k}^{d-1}c_k\varphi_k(x)$$

if $\nu > N$; and for any $k \leqq N$ $A_{\nu-k}^{d-1}/A_\nu^d = O(1/\nu)$, so

$$(3.10)\qquad \sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_\nu^d(a;\ x)-\sigma_\nu^{d-1}(a;\ x)|^p \leqq$$

$$\leqq \sum_{l=0}^{m}\Big\{\sum_{\nu=n(l)-1}^{n(l+1)}\alpha_{n\nu}^{\varrho}(\nu+1)^{\varrho(1-\gamma p)-1}\Big\}^{1/\varrho}\Big\{\sum_{\nu=n(l)-1}^{n(l+1)}(\nu+1)^{-1+\gamma p\varrho'}|\sigma_\nu^d-\sigma_\nu^{d-1}|^{p\varrho'}\Big\}^{1/\varrho'} \leqq$$

$$\leqq O_x(1)\sum_{l=0}^{m}\Big\{\sum_{\nu=n(l)-1}^{n(l+1)}\alpha_{n\nu}^{\varrho}(\nu+1)^{\varrho(1-\gamma p)-1}\Big\}^{1/\varrho}\Big\{\sum_{\nu=n(l)-1}^{n(l+1)}(\nu+1)^{p\varrho'(\gamma-1)-1}\Big\}^{1/\varrho'} :=$$

$$:= O_x(1)\sum_{l=0}^{m} A_l\cdot B_l.$$

By $\gamma<1$ and $B_l \leqq 2\cdot 2^{(l-1)p(\gamma-1)}(\leqq 2)$, for any $\varepsilon>0$, there exists a positive integer l_0 such that if $l>l_0$ then $B_l<\varepsilon$. Thus

$$\sum_{l=0}^{m} A_l B_l = \Big(\sum_{l=0}^{l_0} + \sum_{l=l_0+1}^{m}\Big) A_l B_l \leqq 2 \sum_{l=0}^{l_0} A_l + \varepsilon \sum_{l=l_0+1}^{m} A_l,$$

whence, by (1.15), we get that

$$\sum_{l=0}^{m} A_l B_l \leqq K g(n) A_n n^{-\gamma p}; \tag{3.11}$$

and if (1.17) also holds then we have the estimate

$$\sum_{l=0}^{m} A_l B_l = o\big(g(n) A_n n^{-\gamma p}\big). \tag{3.12}$$

Summing up estimates (3.10), (3.11) and (3.12) we obtain that

$$\left\{\frac{1}{A_n}\sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^d(a;\, x) - \sigma_\nu^{d-1}(a;\, x)|^p\right\}^{1/p} = \begin{cases} O_x\big(g(n)^{1/p} n^{-\gamma}\big), & \text{always,} \\ o_x\big(g(n)^{1/p} n^{-\gamma}\big), & \text{if (1.17) holds,} \end{cases} \tag{3.13}$$

a.e. in (a, b).

In order to estimate the suitable terms of series (3.8) we use Lemma 4 and (3.6). Then

$$\int_a^b \left\{\sup_{0\leqq n<\infty} \left(\frac{n^{p\gamma}}{g(n) A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^d(b;\, x) - \sigma_\nu^{d-1}(b;\, x)|^p\right)^{1/p}\right\}^2 dx \leqq K\varepsilon^3.$$

Hence

$$\operatorname{meas}\left\{x \,\middle|\, \limsup \left(\frac{n^{p\gamma}}{g(n) A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^d(b;\, x) - \sigma_\nu^{d-1}(b;\, x)|^p\right)^{1/p} > \varepsilon\right\} \leqq K\varepsilon.$$

This, (3.9) and (3.13) imply

$$\left\{\frac{1}{A_n}\sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^d(x) - \sigma_\nu^{d-1}(x)|^p\right\}^{1/p} = \begin{cases} O_x\big(g(n)^{1/p} n^{-\gamma}\big), & \text{always,} \\ o_x\big(g(n)^{1/p} n^{-\gamma}\big), & \text{if (1.17) holds,} \end{cases} \tag{3.14}$$

a.e. in (a, b).

Finally, (3.5) and (3.14) yield both statements of Theorem, so our proof is completed.

Proof of Corollary 1. The statements of Corollary 1 follow from the statements of Theorem and from (1.3) using Lemma 1 with $\delta=1$ and $\delta_n := n^{2\gamma} - (n-1)^{2\gamma}$. More precisely, now we have to use Lemma 1 twice. First the property P is that the means $\sigma_\nu^{d-1}(x)$ of the sequence $\{s_n(x)\}$ approximate $f(x)$, in strong sense regarding the matrix (α_{nk}/A_n) and the exponent p, at the order given in Theorem by (1.16) a.e. in (a, b). Secondly, if (1.17) is also satisfied, then the suitable property P is

that the order of the approximation by the means mentioned above is $o_x(g(n)^{1/p}n^{-\gamma})$ a.e. in (a, b).

Proof of Corollary 1.1. Set $\alpha_{n\nu}:=(\nu+1)^{\beta-1}$. Then, regarding the condition $\beta>\gamma p$, an elementary calculation shows that both (1.15) and (1.17) with $g(n)\equiv \equiv 1$ hold for any $\varrho>1$.

On the other hand, since $d>\max(1/2, (p-1)/p)$ and $(1-d)<\min(1/2, 1/p)$ are equivalent, we can give a number $\varrho'>1$ such that $(1-d)p<1/\varrho'<\min(1, p/2)$ and if $\varrho:=\varrho'/(\varrho'-1)$, then both (1.5) and (1.7) hold.

Consequently, with this ϱ, all of the assumptions of Corollary 1 can be satisfied, so, applying Corollary 1, we get (1.18) immediately.

Proof of Corollary 1.2. We set $\alpha_{n\nu}:=A_{n-\nu}^{\alpha-1}$ and follow a similar consideration as in the previous proof with the only change that now we choose ϱ' such that $(1-d)p<1/\varrho'<\min(1, \alpha, p/2)$. Using the suitable ϱ and the condition $p\gamma<1$, elementary calculations show that all of the assumptions of Corollary 1 are satisfied; and Corollary 1 yields (1.19).

The proofs of Corollaries 1.3 and 1.4 run parallel, therefore we detail only the proof of Corollary 1.4.

Proof of Corollary 1.4. Set $\alpha_{n\nu}:=A_{n-\nu}^{\alpha-1}$. Using the assumption $p\gamma=1$ and ϱ' chosen by $(1-d)p<1/\varrho'<\min(1, \alpha, p/2)$, we get that conditions (1.15) and (1.17) with $g(n):=\log n$ and $\varrho:=\varrho'/(\varrho'-1)$ hold, furthermore (1.5) and (1.7) are also fulfilled. Therefore, with these quantities, Corollary 1 can be applied, and we get (1.21).

Proof of Corollary 1.5. Now we set

$$\alpha_{2n,\nu}:=\begin{cases}0, & \text{if } \nu\leqq n,\\ 1, & \text{if } n<\nu\leqq 2n,\end{cases}$$

and

$$\alpha_{2n+1,\nu}:=\begin{cases}\alpha_{2n,\nu}, & \text{if } \nu\leqq 2n,\\ 0, & \text{if } \nu=2n+1.\end{cases}$$

An easy calculation shows that both (1.15) and (1.17) with $g(n)\equiv 1$ hold for any $\varrho>1$. Since the assumption on d yields to choose ϱ' such that $(1-d)p<1/\varrho'< <\min(1, p/2)$, therefore conditions (1.5) and (1.7) can be satisfied simultaneously. Consequently we can apply Corollary 1, whence (1.22) obviously follows.

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BOLYAI INSTITUTE
JÓZSEF ATTILA UNIVERSITY
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

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On Jessen's inequality

J. E. PEČARIĆ and I. RAŞA

1. Introduction

Let X be a compact Hausdorff space and let $C(X)$ be the space of all continuous real-valued functions on X, endowed with supremum norm and usual ordering. Let $M_+^1(X)$ be the set of all probability Radon measures on X. The following fact is well-know [7, Sect. 6]:

(1) If μ is a bounded linear functional on $C(X)$ such that $\|\mu\|=\mu(1)=1$ then μ is positive, i.e. $\mu\in M_+^1(X)$.

Let K be a compact convex subset of a locally convex Hausdorff real space B and let B' be the topological dual of B. Let $\mu\in M_+^1(K)$. Then (see [7, Sect. 1 and Sect. 4]):

(2) *There exists a unique* $b\in B$ *such that* $\mu(l)=l(b)$ *for all* $l\in B'$. (In fact, $b\in K$; it is called the barycenter of μ).

(3) $f(b)\leqq\mu(f)$ *for every convex function* $f\in C(K)$.

The inequality (3) is related to the Jessen's inequality (see [1], [4], [5], [6], [10]).

We shall use these results to prove a Jessen-type theorem similar to Theorem 5 of F. V. Husseinov [3] and we shall extend (1), (2), (3) by considering a class of sublinear functionals studied by V. Totik [12] instead of linear functionals $\mu\in M_+^1(K)$.

2. A Jessen-type theorem

Let E be a nonempty set and let L be a linear space of real-valued functions defined on E; suppose that the constant function 1 belongs to L. Let $M\colon L\to R$ be a linear isotonic (i.e., $M(f)\leqq M(g)$ whenever $f,g\in L, f\leqq g$) functional such that $M(1)=1$.

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Let B be a locally convex Hausdorff real space and let $\mathscr{L}$ be a set of B-valued functions defined on E such that $l\circ F\in L$ for all $F\in\mathscr{L}$ and all $l\in B'$. Let $\mathscr{M}\colon \mathscr{L}\to B$ be such that $l(\mathscr{M}F)=M(l\circ F)$ for all $l\in B'$ and $F\in\mathscr{L}$.

Let K be a compact convex subset of B, let $\varphi\in C(K)$ be a convex function and let $F\in\mathscr{L}$. Suppose that $F(E)\subset K$ and $\varphi\circ F\in L$. Denote $H=\{h\in C(K)\colon h\circ F\in L\}$.

The following theorem contains a Jessen-type inequality (see [1], [2], [6], [10] and, especially, [3]).

Theorem 1. a) *$\mathscr{M}F\in K$ and $\varphi(\mathscr{M}F)\leqq M(\varphi\circ F)$.*

b) *If φ is a strictly convex function then $\varphi(\mathscr{M}F)=M(\varphi\circ F)$ if and only if $h(\mathscr{M}F)=M(h\circ F)$ for all $h\in H$.*

Proof. Let us remark that $1\in H$, $\varphi\in H$ and $l\in H$ for all $l\in B'$. Consider $\lambda\colon H\to R$, $\lambda(h)=M(h\circ F)$ for all $h\in H$. Then λ is a positive linear functional with $\|\lambda\|=1$. Using the Hahn—Banach theorem we deduce that there exists a bounded linear functional μ on $C(K)$ such that μ coincides with λ on H and $\|\mu\|=\|\lambda\|$. It follows that $\|\mu\|=\mu(1)=1$. Using (1) we infer that $\mu\in M_+^1(K)$.

Now let $l\in B'$. Then $\mu(l)=\lambda(l)=M(l\circ F)=l(\mathscr{M}F)$ and (2) implies that $\mathscr{M}F$ is the barycenter of μ, hence $\mathscr{M}F\in K$. Using (3) we deduce $\varphi(\mathscr{M}F)\leqq\mu(\varphi)=\lambda(\varphi)=$ $=M(\varphi\circ F)$.

Suppose now that φ is strictly convex (such a function exists in $C(K)$ if and only if K is metrizable) and $\varphi(\mathscr{M}F)=M(\varphi\circ F)$. Then $\mu(\varphi)=\varphi(\mathscr{M}F)$. By virtue of [8, Lemma], μ is the Dirac measure corresponding to $\mathscr{M}F$. It follows that $M(h\circ F)=$ $=\lambda(h)=\mu(h)=h(\mathscr{M}F)$ for all $h\in H$.

3. Inequalities for sublinear functionals

Let X be a compact Hausdorff space. Denote by $\mathscr{T}(X)$ the set of all sublinear functionals $A\colon C(X)\to R$ such that $A(1)=1$, $A(-1)=-1$ and $\|A\|=1$ (i.e., $|A(f)|\leqq 1$ for all $f\in C(X)$ with $\|f\|\leqq 1$).

V. Totik has proved in [12] that if $A\in\mathscr{T}(X)$ then $A(f)\leqq\max_X f$ for all $f\in C(X)$; moreover, if X is metrizable then for every $A\in\mathscr{T}(X)$ there exists a sequence $(x_n)_{n\in\mathbf{N}}$ in X such that $A(f)=\limsup\,(f(x_1)+\ldots+f(x_n))/n$ for all $f\in C(X)$.

We shall extend (1), (2) and (3) replacing $M_+^1(X)$ by $\mathscr{T}(X)$.

Example 1. Let $\mu,\nu\in M_+^1(X)$. Define $A(f)=\int_X \max(\nu(f),f(x))\,d\mu(x)$ for all $f\in C(X)$. Then $A\in\mathscr{T}(X)$.

Example 2. Let (ν_i) be a net in $M_+^1(X)$. Define $A(f)=\limsup\,\nu_i(f)$ for all $f\in C(X)$. Then $A\in\mathscr{T}(X)$.

Although the following extension of (1) is a consequence of Proposition 2 below, we insert here a direct proof.

Proposition 1. *Every* $A\in\mathcal{T}(X)$ *is isotonic.*

Proof. Let $f,g\in C(X)$, $f\leqq g$, $A(f)>A(g)$. Then $0<A(f)-A(g)\leqq A(h)$ where $h=f-g\leqq 0$. Let $m=-\min_X h$. Clearly $m>0$. We have $0<A(h)\leqq A(-m)+$ $+A(h+m)$; this yields $A(h+m)>m$. On the other hand $0\leqq h+m\leqq m$ and $\|A\|=1$ imply $A(h+m)\leqq m$, a contradiction.

Propositon 2. $A\in\mathcal{T}(X)$ *if and only if there exists a nonempty set* $S\subset M_+^1(X)$ *such that* $A(f)=\sup\{\mu(f)\colon \mu\in S\}$ *for all* $f\in C(X)$.

Proof. Let $A\in\mathcal{T}(X)$ and let $S=\{\mu\colon C(X)\to R\colon \mu \text{ linear}, \mu\leqq A\}$. Using the Hahn—Banach theorem we deduce that $A(f)=\sup\{\mu(f)\colon \mu\in S\}$ for all $f\in C(X)$. But if $\mu\in S$, then $\|\mu\|=\mu(1)=1$; therefore $S\subset M_+^1(X)$.

The converse is easy to prove.

The following result extends (2) and (3).

Theorem 2. *Let K be a compact convex subset of a locally convex Hausdorff real space B and let $A\in\mathcal{T}(K)$. There exists a unique nonempty compact convex subset Q of K such that* $A(h)=\max_Q h$ for all $h\in B'$. Moreover:

a) *A is linear on B' if and only if Q reduces to one point.*

b) *Let $M\subset K$. Then $A(f)\geqq\sup_M f$ for every convex function $f\in C(K)$ if and only if $M\subset Q$.*

Proof. By Proposition 2 there exists $S\subset M_+^1(K)$ such that $A(f)=$ $=\sup\{\mu(f)\colon \mu\in S\}$ for all $f\in C(K)$. Let $C=\operatorname{conv}\{b(\mu)\colon \mu\in S\}$ where $b(\mu)$ is the barycenter of μ. Let Q be the closure of C. If $h\in B'$ we have $A(h)=$ $=\sup\{\mu(h)\colon \mu\in S\}=\sup\{h(b(\mu)\colon \mu\in S\}=\sup\{h(x)\colon x\in C\}=\max_Q h$. The uniqueness of Q is an easy consequence of separation theorems.

a) Clearly A is linear on B' if Q reduces to one point. Conversely, let A be linear on B'. For $h\in B'$ we have $\max_Q h=A(h)=-A(-h)=-\max_Q(-h)=\min_Q h$. It follows that every $h\in B'$ is constant on Q, hence Q reduces to one point.

b) Let $M\subset Q$ and let $f\in C(K)$ be convex. Then $A(f)=\sup\{\mu(f)\colon \mu\in S\}\geqq$ $\geqq\sup\{f(b(\mu))\colon \mu\in S\}=\sup\{f(x)\colon x\in C\}=\max_Q f\geqq\sup_M f$.

Conversely, suppose that $A(f)\geqq\sup_M f$ for every convex function $f\in C(K)$. If $t\in M$ and $t\notin Q$, there exists $h\in B'$ such that $h(t)>\max_Q h$. We have $\sup_M h\geqq$ $\geqq h(t)>\max_Q h=A(h)\geqq\sup_M h$, a contradiction. Hence $M\subset Q$.

Example 3. Let $a\in[0,1]$, $A\colon C[0,1]\to R$, $A(f)=\int_0^1 \max(f(a),f(x))dx$. Then $A\in\mathcal{T}([0,1])$ and $Q=[(1-(1-a)^2)/2,(1+a^2)/2]$.

Example 4. Let $K=[0,1]^2$ and $A\in\mathscr{T}(K)$,

$$A = \sup\left\{\frac{1}{2}(\varepsilon_{(0,0)}+\varepsilon_{(0,1)}),\ \frac{1}{2}(\varepsilon_{(1,0)}+\varepsilon_{(1,1)})\right\}$$

where ε_t is the Dirac measure corresponding to t.

Then

$$Q = \left\{\left(x, \frac{1}{2}\right): x\in[0,1]\right\}.$$

Example 5. Let $K=[0,1]^2$ and

$$A = \sup\left\{\frac{1}{2}(\varepsilon_{(0,0)}+\varepsilon_{(1,1)}),\ \frac{1}{2}(\varepsilon_{(0,1)}+\varepsilon_{(1,0)})\right\}.$$

Then Q reduces to the point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Remark 1. Let $\mu\in M_+^1(K)$ and let $b=b(\mu)$. For $g\in C(K)$ define

$$B_n g(b) = \int_{K^n} g\big((t_1+\ldots+t_n)/n\big)\, d(\mu\otimes\ldots\otimes\mu)(t_1, \ldots, t_n).$$

(B_n is a Bernstein—Schnabl type operator; see [11] and the references given there.)

For every convex function $f\in C(K)$ we have the following improvement of (3):

$$\mu(f) = B_1 f(b) \geqq B_2 f(b) \geqq \ldots \geqq f(b)$$

and $\lim B_n f(b)=f(b)$ (see [11]). In [9] it is shown that $\mu(f)=f(b)$ if and only if f is affine on the closure of conv (supp μ).

Remark 2. Let now K be a Choquet simplex. Let $\mu\in M_+^1(K)$ and $b=b(\mu)$. Let ε_b be the Dirac measure and let $\mu_b\in M_+^1(K)$ be the unique maximal measure which represents b. Then for every convex function $f\in C(K)$ we have (see [7, Sect. 9]):

$$\varepsilon_b(f) \leqq \mu(f) \leqq \mu_b(f). \tag{4}$$

For $K=[a,b]\subset R$ inequalities similar to the second inequality are studied and generalized in [1] and [6], Lemma 1.

Let $A\in\mathscr{T}(K)$, $S\subset M_+^1(K)$, $A=\sup\{\mu: \mu\in S\}$. Let $f\in C(K)$ be convex. Then $A(f)=\sup\{\mu(f): \mu\in S\}\leqq\sup\{\mu_b(f): b\in Q\}$. Hence

$$\sup\{\varepsilon_b(f): b\in Q\} \leqq A(f) \leqq \sup\{\mu_b(f): b\in Q\}. \tag{5}$$

This is an extension of (4).

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(J. E. P.)
FACULTY OF TECHNOLOGY
IVE LOLE RIBARA 126
41 000 ZAGREB
CROATIA

(I. R.)
CATEDRA DE MATEMATICA
INSTITUTUL POLITEHNIC
STR. E. ISAC 15
3400 CLUJ, ROMANIA

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Approximation theorems for modified Szász operators*)

ZHU-RUI GUO and DING-XUAN ZHOU

1. Introduction

The Bernstein operators on $C[0, 1]$ are given by

$$B_n(f, x) = \sum_{k=0}^{n} f(k/n)\binom{n}{k} x^k(1-x)^{n-k}. \tag{1.1}$$

In 1972, H. BERENS and G. G. LORENTZ [3] gave the pioneering theorem on Bernstein operators in the form

$$|B_n(f, x)-f(x)| \leqq M\big(x(1-x)/n\big)^{\alpha/2} \Leftrightarrow \omega_2(f, t) = O(t^\alpha), \tag{1.2}$$

where $0<\alpha<2$, and

$$\omega_2(f, t) = \sup_{0\leqq h\leqq t}\ \sup_{h\leqq x\leqq 1-h} |f(x-h)-2f(x)+f(x+h)|. \tag{1.3}$$

In 1978, M. BECKER [1], R. J. NESSEL [2] gave similar results for Szász and Baskakov operators, Meyer—König and Zeller operators.

Berens—Lorentz type theorems for all exponential-type operators were given by K. SATO [9] in 1982. All these exponential-type operators reproduce linear functions, however if this is not the case, then similar Berens—Lorentz type results for non-Feller modified exponential-type operators have not been obtained till now.

In this paper we shall give such a result for modified Szász operators defined in 1985 by S. M. MAZHAR and V. TOTIK [8]:

$$L_n(f, x) = \sum_{k=0}^{\infty}\Big(n\int_0^\infty f(t)p_{n,k}(t)\,dt\Big)p_{n,k}(x), \tag{1.4}$$

where

$$p_{n,k}(x) = e^{-nx}(nx)^k/k!. \tag{1.5}$$

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Concerning these operators, Mazhar and Totik stated: "However it is far less obvious how the analogues of Theorem 2—3 look like in the case of L_n." Our interest stems from this problem. In fact, we will show a result, which, together with some known theorems, yields for $0<\alpha<1$

$$|L_n(f,x)-f(x)| \leqq M\big(1/n+(x/n)^{1/2}\big)^\alpha \Leftrightarrow |S_n(f,x)-f(x)| \leqq M_1\big(1/n+(x/n)^{1/2}\big)^\alpha$$
$$|S_n(f,x)-f(x)| \leqq M_2(x/n)^{\alpha/2} \Leftrightarrow \omega_1(f,t) = O(t^\alpha),$$

where $S_n(f,x)$ are the Szász operators given by

$$S_n(f,x) = \sum_{k=0}^{\infty} f(k/n)\, p_{n,k}(x). \tag{1.6}$$

We shall also give an equivalence theorem involving the smoothness of functions and the derivatives of the modified Szász operators.

2. A Berens—Lorentz type theorem

First let us give some identities.

Lemma 1 [8]. *For $L_n(f(t),x)$ given by* (1.4), *we have*

$$L_n(t,x) = x+1/n;$$
$$L_n\big((t-x)^2,x\big) = 2x/n+2n^{-2}. \tag{2.1}$$

Mazhar and Totik [8] gave the following direct theorem for modified Szász operators:

$$|L_n(f,x)-f(x)| \leqq K\omega_1\big(((x+1/n)/n)^{1/2}\big), \tag{2.2}$$

here

$$\omega(f,t) = \sup_{0\leqq h\leqq t}\ \sup_{x\geqq h/2} |f(x+h/2)-f(x-h/2)|$$

is the usual modulus of smoothness of f.

We have the Berens—Lorentz type inverse result as follows:

Theorem 1. *Let $f\in C[0,\infty)$ be bounded. Then with $0<\alpha<1$,*

$$|L_n(f,x)-f(x)| \leqq M(x/n+n^{-2})^{\alpha/2} \quad (x\geqq 0,\ n\in\mathbf{N}) \tag{2.3}$$

holds if and only if

$$\omega_1(f,t) = O(t^\alpha) \quad (t>0). \tag{2.4}$$

Remark 1. The assumption that f is bounded is necessary, which can be seen from the following example: Let $f(x)=(x+1)\ln(x+1)-(x+1)$. Then $\omega_1(f,t)\neq$

$\neq O(t^\alpha)$ for $\alpha=1/2$. However (2.3) is satisfied: For $x \geqq 1/n$, we have

$$|L_n(f,x)-f(x)| = \Big|f'(x)\,L_n(t-x,x)+L_n\Big(\int_x^t (t-u)f''(u)\,du, x\Big)\Big| \leqq$$

$$\leqq \ln(x+1)/n+\|uf''(u)\|_\infty L_n\big((t-x)^2/x, x\big) \leqq$$

$$\leqq Mx^{1/4}/n+2/n+2/(n^2x) \leqq (4+M)(x/n+n^{-2})^{1/4}.$$

For $x<1/n$ we have

$$|L_n(f,x)-f(x)| = \Big|L_n\Big(\int_x^t \ln(u+1)\,du, x\Big)\Big| \leqq$$

$$\leqq L_n\Big(\int_x^t u\,du, x\Big) \leqq L_n(t^2+x^2, x) =$$

$$= 2x^2+4x/n+2n^{-2} \leqq 8(x/n+n^{-2})^{1/4}.$$

Thus we have proved that the boundedness cannot be dropped.

Proof of Theorem 1. By (2.2) we shall only prove the necessity. For $d>0$, let

$$f_d(x) = d^{-1}\int_0^d f(x+s)\,ds. \tag{2.5}$$

Then we have for $f\in C[0,\infty)\cap L_\infty[0,\infty)$

$$\|f_d-f\|_\infty \leqq \omega_1(f,d);$$

$$\|f_d'\|_\infty \leqq d^{-1}\omega_1(f,d). \tag{2.6}$$

Note that since

$$L_n'(f,x) = nx^{-1}\sum_{k=0}^\infty n\int_0^\infty f(t)\,p_{n,k}(t)\,dt\,(k/n-x)\,p_{n,k}(x), \tag{2.7}$$

$$= n\sum_{k=0}^\infty n\int_0^\infty f(t)\big(p_{n,k+1}(t)-p_{n,k}(t)\big)\,dt\,p_{n,k}(x), \tag{2.8}$$

we have

$$|L_n'(f_d-f,x)| \leqq nx^{-1}\|f-f_d\|_\infty S_n(|t-x|,x) \leqq (n/x)^{1/3}\omega_1(f,d);$$

$$|L_n'(f_d-f,x)| \leqq 2n\|f-f_d\|_\infty,$$

where we used that

$$S_n(|t-x|,x) \leqq \big(S_n((t-x)^2,x)\big)^{1/2} = (x/n)^{1/2} \text{ (see, e.g. [1, 12])}.$$

Hence

$$|L_n'(f_d-f,x)| \leqq 2\omega_1(f,d)\min\{(n/x)^{1/2}, n\}. \tag{2.9}$$

From (2.8) and [7] we can also derive

$$|L_n'(f_d, x)| = \left|n \sum_{k=0}^{\infty} \int_0^{\infty} p_{n,k+1}(t) f_d'(t)\, dt\, p_{n,k}(x)\right| \leqq d^{-1}\omega_1(f, d).$$

Now for any $t>0$ and $0<h\leqq t$, $x\in(0,\infty)$, we get from (2.3) for any $n\in\mathbf{N}$

$$(2.10)\qquad |f(x+h)-f(x)| \leqq |f(x+h)-L_n(f, x+h)|+|f(x)-L_n(f, x)|+$$
$$+\left|\int_0^h L_n'(f_d, x+u)\, du\right|+\left|\int_0^h L_n'(f-f_d, x+u)\, du\right| \leqq$$
$$\leqq 2M\big((x+h)/n+n^{-2}\big)^{\alpha/2}+d^{-1}\omega_1(f, d)\,h+2\omega_1(f, d)\int_0^h \min\big\{(n/(x+u))^{1/2}, n\big\}\, du \leqq$$
$$\leqq 2M\big(d(n, x, h)\big)^{\alpha}+8h\omega_1(f, d)\big(d^{-1}+1/d(n, x, h)\big),$$

where $d(n, x, h)=\big((x+h)/n+n^{-2}\big)^{1/2}$. Note that $d(n, x, h)\geqq d(n+1, x, h)\geqq$ $\geqq d(n, x, h)/2$ for any $n\in\mathbf{N}$, hence for any $1/2>\delta>0$ we can choose $n\in\mathbf{N}$ such that

$$2d(n, x, h) \geqq \delta > d(n, x, h).$$

With this choice we get from (2.10) the estimate

$$|f(x+h)-f(x)| \leqq 2M\delta^{\alpha}+36h\omega_1(f, \delta)/\delta,$$

hence

$$\omega_1(f, t) \leqq 2M\delta^{\alpha}+36t\omega_1(f, \delta)/\delta \leqq (2M+36)\big(\delta^{\alpha}+t\omega_1(f, \delta)/\delta\big), \quad (t, \delta > 0)$$

which implies $\omega_1(f, t)=O(t^{\alpha})$ (see [3, 6]). Our proof is complete.

Remark 2. The same statement is true for Szász operators, however we shall omit the proof since it is just the same.

Remark 3. In [14], we have proved for Bernstein—Durrmeyer operators

$$(2.11)\qquad D_n(f, x) = \sum_{k=0}^{n}\left((n+1)\int_0^1 f(t)\binom{n}{k} t^k(1-t)^{n-k}\, dt\right)\binom{n}{k} x^k(1-x)^{n-k},$$

that for $1<\alpha<2$ there exist no functions $\{\psi_{n,\alpha}(x)\}_{n\in\mathbf{N}}$ such that the following equivalence holds for $f\in C[0, 1]$

$$(2.12)\qquad \omega_2(f, t) = O(t^{\alpha}) \Leftrightarrow |D_n(f, x)-f(x)| \leqq M\psi_{n,\alpha}(x).$$

In view of this result we cannot expect a similar characterization theorem by the modified Szász operators for functions satisfying

$$\omega_2(f, t) = O(t^{\alpha}) \quad \text{with} \quad 1 < \alpha < 2.$$

3. Derivatives and smoothness

Some results on the relation between the order of derivatives and smoothness have been obtained in [5, 7], most of which characterize the Ditzian—Totik modulus of smoothness. Z. DITZIAN gave a result on the characterization of the usual modulus of smoothness by the derivatives of Bernstein polynomials [4]. Recently one of the authors gave similar results for higher order of smoothness [14].

Let

$$\omega_2(f, t) = \sup_{0<h\leqq t} \sup_{x\geqq 0} |f(x)-2f(x+h)+f(x+2h)|.$$

For the modified Szász operators, we can prove

Theorem 2. *For* $f\in C[0,\infty)\cap L_\infty[0,\infty)$, $0<\alpha<2$, *we have*

$$\omega_2(f, t) = O(t^\alpha) \Leftrightarrow |L_n''(f, x)| \leqq M(\min\{n^2, n/x\})^{(2-\alpha)/2}. \tag{3.1}$$

Theorem 3. *For* $f\in C[0,\infty)\cap L_\infty$, $0<\alpha<1$, *we have*

$$\omega_1(f, t) = O(t^\alpha) \Leftrightarrow |L_n'(f, x)| \leqq M(\min\{n^2, n/x\})^{(1-\alpha)/2}. \tag{3.2}$$

Proof of Theorem 2. Proof of the direction "$\Rightarrow$": Suppose $\omega_2(f, t)\leqq Mt^\alpha$. By simple calculation one can get

$$L_n''(g, x) = n^2 \sum_{k=0}^{\infty} \Big(n \int_0^\infty g(t)\big(p_{n,k}(t)-2p_{n,k+1}(t)+p_{n,k+2}(t)\big)\,dt\Big)\, p_{n,k}(x) \tag{3.3}$$

$$= n^2x^{-2} \sum_{k=0}^{\infty} \Big(n \int_0^\infty g(t)\, p_{n,k}(t)\,dt\Big)\big((k/n-x)^2-kn^{-2}\big)\, p_{n,k}(x), \tag{3.4}$$

hence

$$|L_n''(g, x)| \leqq 4n^2 \|g\|_\infty, \tag{3.5}$$

and

$$|L_n''(g, x)| \leqq 2n/x \|g\|_\infty. \tag{3.6}$$

Now for $f\in C[0,\infty)\cap L_\infty(0,\infty)$, let us define the Steklov function as

$$f_d(x) = 4d^{-2} \iint_0^{d/2} \big(2f(x+u+v)-f(x+2u+2v)\big)\,du\,dv. \tag{3.7}$$

Then

$$\|f-f_d\| \leqq \omega_2(f, d), \tag{3.8}$$

and

$$\|f_d''\| \leqq 9d^{-2}\omega_2(f, d). \tag{3.9}$$

For f_d one can verify (see [7])

$$|L_n''(f_d, x)| \leqq \Big|\sum_{k=0}^{\infty} n \int_0^\infty p_{n,k+2}(t) f_d''(t)\,dt\, p_{n,k}(x)\Big| \leqq 9d^{-2}\omega_2(f, d). \tag{3.10}$$

Thus, we have for $d=(\min\{n^2, n/x\})^{-1/2}$

$$\begin{aligned}|L_n''(f,x) &\leqq |L_n''(f_d,x)|+|L_n''(f-f_d,x)| \leqq \\ &\leqq 9d^{-2}\omega_2(f,d)+4\min\{n^2, n/x)\,\omega_2(f,d) \leqq \\ &\leqq 13M(\min\{n^2, n/x\})^{1-\alpha/2}.\end{aligned}$$

Proof of the direction "$\Leftarrow$": To prove this part, we need the combination of $\{L_n\}$ defined as

$$L_{n,1}(f,x) = a_0(n)\,L_{n_0}(f,x)+a_1(n)\,L_{n_1}(f,x), \tag{3.11}$$

where $|a_0(n)|+|a_1(n)|\leqq B$, $n=n_0<n_1\leqq An$, with A, B absolute constants, having the property

$$L_{n,1}(t^i,x) = x^i, \quad i=0,1 \text{ (see e.g. [7]).} \tag{3.12}$$

Then we have for $f\in C[0,\infty)\cap L_\infty[0,\infty)$ by the method from [1, 3]

$$|L_{n,1}(f,x)-f(x)| \leqq M\omega_2(f, 1/n+(x/n)^{1/2}). \tag{3.13}$$

Now we can give our proof, where the commutativity of $\{L_n\}$ is crucial (see [7]). For $n, m\in\mathbf{N}$, $x\in(0,\infty)$, $0<h\leqq t$, we have

(3.14)

$$\begin{aligned}|L_m(f,x)-2L_m(f,x+h)+L_m(f,x+2h)| &\leqq 4M\omega_2(L_m f, 1/n+((x+2h)/n)^{1/2})+ \\ &+\iint_0^h |L_n''(L_{n,1}(f), x+u+v)|\,du\,dv\end{aligned}$$

Now we shall estimate the second term. First we have

$$|L_m''(L_{n,1}(f), x+u+v)| \leqq \|L_{n,1}''(f)\|_\infty \leqq 2B(An)^{2-\alpha}.$$

On the other hand, note that by

$$|x^{1-\alpha/2}L_{n,1}''(f,x)| \leqq MABn^{1-\alpha/2}$$

we have

$$\begin{aligned}|x^{1-\alpha/2}L_m''(L_{n,1}(f),x)| &= \Big|x^{1-\alpha/2}\sum_{k=0}^\infty m\int_0^\infty p_{m,k+2}(t)\,L_{n,1}''(f,t)\,dt\,p_{m,k}(x)\Big| \leqq \\ &\leqq MABn^{1-\alpha/2}x^{1-\alpha/2}\sum_{k=0}^\infty m\Big(\int_0^\infty p_{m,k+2}(t)\,t^{-1}\,dt\Big)^{1-\alpha/2}\Big(\int_0^\infty p_{m,k+2}(t)\,dt\Big)^{\alpha/2}p_{m,k}(x) \leqq \\ &\leqq MABn^{1-\alpha/2}x^{1-\alpha/2}\Big(\sum_{k=0}^\infty mp_{m,k}(x)/(k+2)\Big)^{1-\alpha/2} \leqq MABn^{1-\alpha/2},\end{aligned}$$

hence

$$\begin{aligned}\iint_0^h |L_m''(L_{n,1}(f), x+u+v)|\,du\,dv &\leqq MABn^{1-\alpha/2}\iint_0^h (x+u+v)^{\alpha/2-1}\,du\,dv \leqq \\ &\leqq MABn^{1-\alpha/2}h^\alpha(M_1h^2/(x+2h))^{1-\alpha/2} \leqq M_2h^2(n/(x+2h))^{1-\alpha/2},\end{aligned}$$

here we have used the fact that

$$\iint_0^h 1/(x+u+v)\,du\,dv \leqq M_1 h^2/(x+2h), \quad 0 < h \leqq 1, \quad x \geqq 0$$

(see [1]). Thus, combining the above estimates with (3.14), we have

$$(3.15) \qquad |L_m(f,x)-2L_m(f,x+h)+L_m(f,x+2h)| \leqq$$
$$\leqq 4M\omega_2\big(L_m f,\, 1/n+((x+2h)/n)^{1/2}\big)+M_3h^2\big(1/n+((x+2h)/n)^{1/2}\big)^{\alpha-2},$$

where M_3 is a constant independent of n, x, h and m.

Let C be a constant which will be determined later. Since

$$1/n+\big((x+2h)/n\big)^{1/2} < 1/(n-1)+\big((x+2h)/(n-1)\big)^{1/2} \leqq 2\big(1/n+((x+2h)/n)^{1/2}\big),$$

we can choose $n\in\mathbf{N}$, such that

$$t/(2C) \leqq 1/n+\big((x+2h)/n\big)^{1/2} \leqq t/C.$$

Then we get from (3.15) by induction

$$\begin{aligned}\omega_2(L_m f,t) &\leqq 4M\omega_2(L_m f,t/C)+(2C)^{(2-\alpha)}M_3t^\alpha \leqq\\ &\leqq \dots\\ &\leqq (4M)^k\omega_2(L_m f,tC^{-k})+(2C)^{2-\alpha}M_3t^\alpha\sum_{l=0}^{k-1}(4MC^{-\alpha})^l \leqq\\ &\leqq t^2(4M)^kC^{-2k}\|(L_m f)''\|_\infty+(2C)^{2-\alpha}M_3t^\alpha C^\alpha/(C^\alpha-4M).\end{aligned}$$

Now if we take here $C=(1+4M)^{1/\alpha}$, and let $k\to\infty$, we obtain

$$\omega_2(L_m f,t) \leqq 4C^2M_3t^\alpha/(C^\alpha-4M),$$

which implies $\omega_2(f,t)=O(t^\alpha)$, since the constant $4C^2M_3/(C^\alpha-4M)$ is independent of $m\in\mathbf{N}$.

Our proof is complete. We shall omit the proof of Theorem 3, since it is almost the same as the proof of Theorem 2.

4. A direct theorem for uniform approximation

When treating uniform approximation we shall always assume the boundedness of the functions. Let C_B be the set of bounded and continuous functions on $[0,\infty)$.

Theorem 4. *For $f\in C_B$, $L_n(f,x)$ given by* (1.4), *we have*

$$(4.1) \qquad \|L_nf-f\|_\infty \leqq C\big(\omega_\varphi^2(f,n^{-1/2})_\infty+\omega_1(f,n^{-1})+n^{-1}\|f\|_\infty\big),$$

where C is a constant independent of n, and $\omega_\varphi^2(f,n^{-1/2})_\infty$ is the so-called Ditzian—

Totik modulus of smoothness defined as

$$\omega_\varphi^2(f,t)_\infty = \sup_{0<h\leqq t} \|\Delta_{h\varphi}^2 f\|_\infty, \tag{4.2}$$

$$\varphi(x) = x^{1/2},$$

$$\Delta_h^2 f(x) = f(x-h)-2f(x)+f(x+h), \quad x \geqq h;$$

$$\Delta_h^2 f(x) = 0, \quad \text{otherwise.}$$

Remark 4. In view of the characterization theorems of Mazhar and Totik [8] on the saturation and non-optimal approximation, we can see that our result is of some value.

Proof of Theorem 4. For Szász operators given by (1.6), we have for $f\in C_B$ (see [6])

$$\|S_n(f)-f\|_\infty \leqq M\big(\omega_\varphi^2(f, n^{-1/2})_\infty + n^{-1}\|f\|_\infty\big). \tag{4.3}$$

We now need the Szász—Kantorovich operators given by

$$S_n^*(f,x) = \sum_{k=0}^{\infty} n \int_{k/n}^{(k+1)/n} f(t)\,dt\, p_{n,k}(x). \tag{4.4}$$

For these operators, we can easily deduce from (4.3) the direct result as

$$\|S_n^*(f)-f\|_\infty \leqq M\big(\omega_\varphi^2(f, n^{-1/2})_\infty + \omega_1(f, 1/n) + n^{-1}\|f\|_\infty\big). \tag{4.5}$$

We shall use the identities

$$S_n^*(t,x) = x+1/(2n), \quad S_n^*\big((t-x)^2, x\big) = x/n+3^{-1}n^{-2}. \tag{4.6}$$

By (4.5), we only need to prove

$$\big\|2\big(f-S_n^*(f)\big)-(f-L_nf)\big\|_\infty \leqq C\big(\omega_\varphi^2(f, n^{-1/2})_\infty + n^{-1}\|f\|_\infty\big).$$

Using the Ditzian—Totik K-functional

$$K_{\varphi,2}(f,t)_\infty = \inf_{g\in D}\{\|f-g\|_\infty + t(\|g\|_\infty + \|\varphi^2 g''\|_\infty) + t^2\|g''\|_\infty\},$$

and its equivalence to ω_φ^2, it is sufficient for us to prove for $g\in D=\{g\in C_B\colon g'\in A.C._{\mathrm{loc}},\ g''\in L_\infty[0,\infty)\}$ the estimate

$$\big\|2\big(g-S_n^*(g)\big)-\big(g-L_n(g)\big)\big\|_\infty \leqq C\big((\|g\|_\infty + \|\varphi^2 g''\|_\infty)/n + n^{-2}\|g''\|_\infty\big). \tag{4.7}$$

From the above identities, we have for $g \in D$, $x \geqq n^{-1}$

$$\left|2\left(g(x)-S_n^*(g,x)\right)-\left(g(x)-L_n(g,x)\right)\right| =$$

$$= \left|L_n\Big(\int_x^t (t-u)g''(u)\,du, x\Big)-2S_n^*\Big(\int_x^t (t-u)g''(u)\,du, x\Big)\right| \leqq$$

$$\leqq \|\varphi^2 g''\|_\infty \left(L_n((t-x)^2/x, x)+2S_n^*((t-x)^2/x, x)\right) \leqq 12\,\|\varphi^2 g''\|_\infty/n.$$

For $0<x<1/n$, we have

$$\left|2\left(g(x)-S_n^*(g,x)\right)-\left(g(x)-L_n(g,x)\right)\right| \leqq$$

$$\leqq \|g''\|_\infty \left(L_n((t-x)^2, x)+2S_n^*((t-x)^2, x)\right) \leqq 12n^{-2}\,\|g''\|_\infty.$$

Thus we have obtained (4.7) and our proof of Theorem 4 is complete.

Remark 5. The second term on the rihgt of (4.1) is necessary, which can be seen from the following example: Let

$$f(x) = \begin{cases} x\ln x - x^2/2 + 1/2, & \text{for } x \in [0,1); \\ 0, & \text{otherwise.} \end{cases} \tag{4.8}$$

Then we have

$$f'(x) = \begin{cases} \ln x - x + 1, & \text{for } x \in (0,1); \\ 0, & \text{otherwise.} \end{cases} \tag{4.9}$$

$$f''(x) = \begin{cases} 1/x - 1, & \text{for } x \in (0,1); \\ 0, & \text{otherwise.} \end{cases} \tag{4.10}$$

Therefore we obtain a function $f \in C_B$ which satisfies

$$\omega_\varphi^2(f, n^{-1/2})_\infty = O(1/n) \quad n^{-1}\,\|f\|_\infty = O(1/n),$$

and

$$f' \notin L_\infty.$$

On the other hand, from the saturation class of the modified Szász operators $\{L_n\}$ [8] we have

$$\|L_n f - f\|_\infty \neq O(1/n).$$

Thus we have proved that the second term $\omega_1(f, 1/n)$ is necessary.

We can also give the weak-type inverse estimates for the moduli.

Lemma 2. *For $L_n(f, x)$ given by* (1.4), $n \in \mathbf{N}$, *we have*

$$\|(L_n f)''\|_\infty + \|L_n f\|_\infty \leqq M n^2 \|f\|_\infty, \quad f \in C_B; \tag{4.11}$$

$$\|\varphi^2 (L_n f)''\|_\infty + \|L_n f\|_\infty \leqq M n \|f\|_\infty, \quad f \in C_B; \tag{4.12}$$

$$\|(L_n f)''\|_\infty + \|L_n f\|_\infty \leqq M(\|f''\|_\infty + \|f\|_\infty), \quad f'' \in L_\infty; \tag{4.13}$$

$$\|\varphi^2 (L_n f)''\|_\infty + \|L_n f\|_\infty \leqq M(\|\varphi^2 f''\|_\infty + \|f\|_\infty), \quad \varphi^2 f'' \in L_\infty, \tag{4.14}$$

where M is a constant independent of n and f.

The proof can be easily obtained from the representations of the derivatives of $L_n f$ in [7].

Theorem 5. *For $f \in C_B$, $L_n(f, x)$ given by* (1.4), *we have*

$$\omega_\varphi^2(f, n^{-1/2})_\infty \leqq M_1 n^{-1} \sum_{k=1}^{\infty} \|L_k f - f\|_\infty + n^{-1} \|f\|_\infty; \tag{4.15}$$

$$\omega_1(f, n^{-1}) \leqq M_1 n^{-1} \Big(\sum_{k=1}^{n} \|L_k f - f\|_\infty + \|f\|_\infty\Big). \tag{4.16}$$

where M_1 is independent of f and $n \in \mathbf{N}$.

The proof is the same as given by Ditzian and Totik [6], (see also [10]), so we shall omit it.

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DEPARTMENT OF MATHEMATICS, ZHEJIANG UNIVERSITY;
ADVANCED INSTITUTE OF MATHEMATICS,
HANGZHOU
ZHEJIANG
310 027
PEOPLE'S REPUBLIC OF CHINA

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On the divergence phenomena of the differentiated trigonometric projection operators

S. P. ZHOU

1. Introduction

Let r be a nonnegative integer, $C_{2\pi}^r$ the class of 2π-periodic continuous functions which have r continuous derivatives, $C_{2\pi}=C_{2\pi}^0$, $\mathscr{T}_n$ the set of trigonometric polynomials of degree at most n. Let

$$P_n \in C_{2\pi} \to \mathscr{T}_n$$

be projection operators, that is, linear operators with the following properties:

(i) $P_n(f, x) \in \mathscr{T}_n$ if $f \in C_{2\pi}$,

(ii) $P_n(f, x) \equiv f(x)$ if $f \in \mathscr{T}_n$.

Furthermore,

$$\|P_n^{(r)}\| := \sup_{0 \not\equiv f \in C_{2\pi}} \frac{\|P_n^{(r)}(f, x)\|}{\|f\|}$$

is the norm of the r times differentiated operator, $\|\cdot\|$ denotes supremum norm over the real line, $E_n(f)$ is the best approximation of $f \in C_{2\pi}$ by trigonometric polynomials of degree n, and

$$\omega(f, \delta) = \max_{|h| \leq \delta} \|f(x+h) - f(x)\|.$$

Recently, P. O. RUNCK, J. SZABADOS and P. VÉRTESI [1] studied the convergence of differentiated trigonometric projection operators, they established

Theorem A. *If $f \in C_{2\pi}^r$ and $P_n \in C_{2\pi} \to \mathscr{T}_n$, then*

$$\|f^{(r)}(x) - P_n^{(r)}(f, x)\| = O\big(E_n(f^{(r)}) + E_n(f)\,\|P_n^{(r)}\|\big).$$

On the divergence phenomena [1] showed

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Theorem B. *Given* $r \geqq 0$, *a modulus of continuity* $\omega(t)$ *such that*

$$\lim_{t\to 0+} \frac{t}{\omega(t)} = 0, \tag{1}$$

and a sequence of projection operators $P_n \in C_{2\pi} \to \mathcal{T}_n$, *then there exists an* $f_r(x) \in C_r(\omega)$ *such that*

$$\limsup_{n\to\infty} \frac{\|f_r^{(r)}(x) - P_n^{(r)}(f_r, x)\|}{\omega(n^{-1}) \log n} > 0,$$

where

$$C_r(\omega) = \left\{ f(x) \colon f \in C_{2\pi}^r,\ \sup_{t>0} \frac{\omega(f^{(r)}, t)}{\omega(t)} < \infty \right\}.$$

Theorem C. *Given* $r \geqq 0$, *a sequence of projection operators* $P_n \in C_{2\pi} \to \mathcal{T}_n$ *and a sequence* $\varepsilon_1 \geqq \varepsilon_2 \geqq \ldots$, $\lim_{n\to\infty} \varepsilon_n = 0$, *then there exists an* $f_r(x) \in W^r \operatorname{Lip} 1$ *such that*

$$\limsup_{n\to\infty} \frac{\|f_r^{(r)}(x) - P_n^{(r)}(f_r, x)\|}{\varepsilon_n n^{-1} \log n} > 0,$$

where

$$W^r \operatorname{Lip} 1 = \{ f(x) \in C_{2\pi}^r \colon f^{(r)} \in \operatorname{Lip} 1 \}.$$

From the above discussions, a natural question thus arises: Can we remove condition (1) eventually while the same conclusion of Theorem B still holds? Using some new ideas, together with the basic methods given in [1] and careful calculation, the present paper will prove this fact.

2. Result and Proof

Lemma 1. *Let*

$$S_{N,n}(x) = \sum_{j=N}^{n} \frac{\sin jx}{j},$$

then for $|x| \leqq n^{-3/2}$,

$$\|S_{N,n}(x)\| = O(n^{-1/2}),$$

and for $|x| \geqq N^{-1/2}$,

$$\|S_{N,n}(x)\| = O(N^{-1/2}).$$

Proof. It is easy to see that

$$|S_{N,n}(x)| \leqq n|x| \tag{2}$$

since

$$|\sin jx| \leqq j|x|.$$

At the same time, by Abel transform,

$$\sum_{j=N}^{n} \frac{\sin jx}{j} = \sum_{j=N}^{n-1} \frac{1}{j(j+1)} \sum_{k=N}^{j} \sin kx + \frac{1}{n} \sum_{k=N}^{n} \sin kx,$$

while

$$\sum_{k=N}^{j} \sin kx = \frac{\cos\left(j+\frac{1}{2}\right)x + \cos\left(N-\frac{1}{2}\right)x}{2\sin\frac{x}{2}},$$

so [1])

$$|S_{N,n}(x)| \leqq C|x|^{-1}\left(\sum_{j=N}^{n-1} \frac{1}{j(j+1)} + \frac{1}{n}\right) = O(N^{-1}|x|^{-1}). \tag{3}$$

Now Lemma 1 follows from (2) and (3).

Lemma 2. *Given $r \geqq 0$ and $n \geqq 1$, there exists a function $g_{nr}(x) \in C_{2\pi}^{\infty}$ such that*

$$\|g_{nr}(x)\| \leqq Cn^r, \tag{4}$$

$$\|g_{nr}^{(j)}(x)\| \leqq Cn^j, \quad j = r, r+1, \tag{5}$$

$$|g_{nr}^{(r+1)}(x)| = \begin{cases} O(n^{r+2/3}), & |x| \leqq n^{-1} \\ O(n^{r+3/4}), & |x| \geqq n^{-1/4}, \end{cases} \tag{6}$$

and

$$\frac{1}{\pi} \int_{-\pi}^{\pi} g_{nr}^{(r)}(t) D_n(t)\, dt \geqq Cn^r \log n, \tag{7}$$

where

$$D_n(t) = \frac{\sin\frac{2n+1}{2}t}{2\sin\frac{t}{2}}$$

is the nth Dirichlet kernel.

Proof. Set

$$g_{nr}(x) = n^r\left(\sum_{j=[n^{1/2}]}^{[n^{2/3}]} \frac{\cos\left((n-j)x+\frac{r\pi}{2}\right)}{j(n-j)^r} - \sum_{j=[n^{1/2}]}^{[n^{2/3}]} \frac{\cos\left((n+j)x+\frac{r\pi}{2}\right)}{j(n+j)^r}\right),$$

then

$$g_{nr}^{(r)}(x) = n^r\left(\sum_{j=[n^{1/2}]}^{[n^{2/3}]} \frac{\cos(n-j)x}{j} - \sum_{j=[n^{1/2}]}^{[n^{2/3}]} \frac{\cos(n+j)x}{j}\right) = 2n^r \sin nx \sum_{j=[n^{1/2}]}^{[n^{2/3}]} \frac{\sin jx}{j}.$$

---

[1]) In the whole paper, C always indicates some positive constant depending upon r but independent of n which may have different values at different places.

Furthermore,

$$|g_{nr}^{(r+1)}(x)| = \left|2n^{r+1}\cos nx \sum_{j=[n^{1/2}]}^{[n^{2/3}]} \frac{\sin jx}{j} + 2n^r \sin nx \sum_{j=[n^{1/2}]}^{[n^{2/3}]} \cos jx\right| \leqq$$

$$\leqq 2n^r \left(n\left|\sum_{j=[n^{1/2}]}^{[n^{2/3}]} \frac{\sin jx}{j}\right| + |\sin nx| \left|\sum_{j=[n^{1/2}]}^{[n^{2/3}]} \cos jx\right|\right).$$

Altogether with Lemma 1 and

$$|\sin nx| \leqq n|x|,$$

$$\left|\sum_{j=[n^{1/2}]}^{[n^{2/3}]} \cos jx\right| = O(\min\{n^{2/3}, |x|^{-1}\}),$$

we then get for $|x| \leqq n^{-1}$,

$$g_{nr}^{(r+1)}(x) = O(n^{r+2/3}),$$

and for $|x| \geqq n^{-1/4}$,

$$g_{nr}^{(r+1)}(x) = O(n^{r+3/4}),$$

(6) is completed. By above discussions and the well-known estimate

$$\left|\sum_{j=1}^{n} \frac{\sin jx}{j}\right| = O(1)$$

for all n, (5) is trivial. The estimate (4) is also not difficult. If $r=0$, then (5) implies (4). Let $r \geqq 1$, we have

$$\|g_{nr}(x)\| \leqq 2n^r \sum_{j=[\sqrt{n}]}^{[n^{2/3}]} \frac{1}{j(n-j)^r} = o(n^r), \quad n \to \infty,$$

which implies (4). At last,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g_{nr}^{(r)}(t) D_n(t)\, dt = n^r \sum_{j=[n^{1/2}]}^{[n^{2/3}]} \frac{1}{j} \geqq Cn^r \log n,$$

that is (7).

Theorem. *Given $r \geqq 0$, a modulus of continuity $\omega(t)$ and a sequence of projection operators $P_n \in C_{2\pi} \to \mathcal{T}_n$, then there exists an $f_r(x) \in C_r(\omega)$ such that*

$$\limsup_{n\to\infty} \frac{\|f_r^{(r)}(x) - P_n^{(r)}(f_r, x)\|}{\omega(n^{-1})\log n} > 0.$$

Proof. Considering Theorem B, we only need to prove our theorem in Lip 1 case. If for any fixed N,

$$\limsup_{n\to\infty} \frac{\|g_{Nr}^{(r)}(x) - P_n^{(r)}(g_{Nr}, x)\|}{n^{-1}\log n} > 0,$$

then $g_{Nr}(x)$ is the required function. Now we suppose otherwise for any fixed N,

$$\lim_{n\to\infty} \frac{\|g_{Nr}^{(r)}(x)-P_n^{(r)}(g_{Nr},x)\|}{n^{-1}\log n} = 0. \tag{8}$$

Using the argument from [1], p. 291, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n^{(r)}\big(g_{nr}(\cdot+u),-u\big)\,du = \frac{1}{\pi}\int_{-\pi}^{\pi} g_{nr}^{(r)}(t)D_n(t)\,dt.$$

By (7) we get

$$\|P_n^{(r)}(g_{nr},x)\| \geqq Cn^r\log n. \tag{9}$$

We select a subsequence from natural numbers $n_1<n_2<\ldots$ by induction. Let $n_1=1$. After n_k, we choose n_{k+1} satisfying the following properties:

$$n_{k+1}^{-1} \leqq \frac{1}{2}n_k^{-4}, \tag{10}$$

$$n_{k+1}^{-1} \leqq \min\left\{\frac{1}{2}, \|P_{n_k}^{(r)}\|^{-1}\right\} n_k^{-1}, \tag{11}$$

$$\frac{\|g_{n_jr}^{(r)}(x)-P_{n_{k+1}}^{(r)}(g_{n_jr},x)\|}{n_{k+1}^{-1}\log n_{k+1}} \leqq k^{-1}\log^{-1}k, \quad j=1,2,\ldots,k. \tag{12}$$

Due to (8), (12) is possible. Define

$$f_r(x) = \sum_{j=1}^{\infty} g_{n_jr}(x)n_j^{-r-1}.$$

Clearly, $f_r\in C_{2\pi}^r$. For $\delta>0$, let $n_{k+1}^{-1}\leqq\delta<n_k^{-1}$. Then by mean value theorem there is a $\theta_k\in[0,1]$ such that

$$|f_r^{(r)}(x+\delta)-f_r^{(r)}(x)| \leqq \delta\sum_{j=1}^{k}|g_{n_jr}^{(r+1)}(x+\theta_k\delta)|\,n_j^{-r-1}+2\sum_{j=k+1}^{\infty}\|g_{n_jr}^{(r)}(x)\|\,n_j^{-r-1} :=$$

$$:= \Sigma_1+\Sigma_2.$$

Due to (5), (6) and (11),

$$\Sigma_2 \leqq C\sum_{j=k+1}^{\infty} n_j^{-1} = O(n_{k+1}^{-1}) = O(\delta).$$

Meanwhile since $(n_i^{-1}, n_i^{-1/4})\cap(n_j^{-1}, n_j^{-1/4})=\emptyset$, $1\leqq i<j$ (by (10)), if $x+\theta_k\delta\in(n_{i_0}^{-1}, n_{i_0}^{-1/4})$ for some i_0, $1\leqq i_0\leqq k$, then

$$x+\theta_k\delta\notin(n_j^{-1}, n_j^{-1/4})$$

for $1 \leqq j \leqq k, j \neq i_0$, that is, $x+\theta_k\delta$ satisfies (6). Therefore,

$$\Sigma_1 \leqq \delta\big(|g_{n_{i_0}r}^{(r+1)}(x+\theta_k\delta)|\, n_{i_0}^{-r-1} + \sum_{1\leqq j\leqq k, j\neq i_0} |g_{n_j r}^{(r+1)}(x+\theta_k\delta)|\, n_j^{-r-1}\big) \leqq$$

$$\leqq C\delta\big(1+\sum_{j=1}^{\infty} n_j^{-1/4}\big) = O(\delta),$$

thus we have proved $f_r \in W^r$ Lip 1. On the other hand,

$$\|f_r^{(r)}(x) - P_{n_k}^{(r)}(f_r, x)\| \geqq \|P_{n_k}^{(r)}(g_{n_k r}, x)\|\, n_k^{-r-1} -$$

$$-\sum_{j=1}^{k-1} \|g_{n_j r}^{(r)}(x) - P_{n_k}^{(r)}(g_{n_j r}, x)\|\, n_j^{-r-1} - \sum_{j=k}^{\infty} \|g_{n_j r}^{(r)}(x)\|\, n_j^{-r-1} -$$

$$-\sum_{j=k+1}^{\infty} \|P_{n_k}^{\gamma}(g_{n_j r}, x)\|\, n_j^{-r-1} := \Sigma_3 - \Sigma_4 - \Sigma_5 - \Sigma_6.$$

From (9),

$$\Sigma_3 \geqq C n_k^{-1} \log n_k, \tag{13}$$

while (5), (11) and (12) imply that

$$\Sigma_5 = O\big(\sum_{j=k}^{\infty} n_j^{-1}\big) = O(n_k^{-1}), \tag{14}$$

$$\Sigma_4 = O(k^{-1} n_k^{-1} \log n_k), \tag{15}$$

finally using (4) and (11) we get

$$\Sigma_6 = O\big(\|P_{n_k r}^{(r)}\| \sum_{j=k+1}^{\infty} n_j^{-1}\big) = O(\|P_{n_k r}^{(r)}\|\, n_{k+1}^{-1}) = O(n_k^{-1}). \tag{16}$$

Combining (13)—(16), we thus have

$$\|f_r^{(r)}(x) - P_{n_k}^{(r)}(f_r, x)\| \geqq C n_k^{-1} \log n_k$$

for sufficiently large k. Theorem is proved.

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DEPARTMENT OF MATHEMATICS,
STATISTICS AND COMPUTING SCIENCE
DALHOUSIE UNIVERSITY
HALIFAX, NOVA SCOTIA
CANADA B3H 3J5

Present address:
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA
CANADA T6G 2G1

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On the central limit theorem for series with respect to periodical multiplicative systems. II

S. V. LEVIZOV

Introduction. In [1] we studied the question of the central limit theorem (CLT) for lacunary subsystems of periodical multiplacitve orthonormal systems (PMONS), satisfying the so-called weak lacunarity condition.*) We also defined the concept of the subjection of subsystems to CTL and gave sufficient conditions for being a subjection, too. The sharpness of the given conditions was also shown. The assumption about the boundedness of the sequence $\{p_n\}$ generating the investigated PMONS had an essential importance. The purpose of the present work is to extend these results to the case when $\overline{\lim}_{n\to\infty} p_n=+\infty$. We shall also investigate the connection of the rate of the growth of $\{p_n\}$ and the "density" of the lacunary sequence $\{n_k\}$.

1. Sufficient conditions. Let the PMONS $X=\{\chi_n(x)\}$ be defined by means of the sequence $\{p_n\}$. Denote $\{\chi_{n_k}(x)\}$ a lacunary subsystem of X such that the sequence $\{n_k\}$ satisfies the conditions

$$\frac{n_{k+1}}{n_k} \geqq 1+\omega(k) \quad \text{for} \quad k \geqq k_o, \tag{1}$$

where $\omega(k)\downarrow 0$ and $k^\alpha\cdot\omega(k)\uparrow\infty$ for some $\alpha, 0<\alpha<1$.

For given sequence $\{n_k\}$ we define, as earlier, λ_k and $\lambda_k^l(q)$ as the quantity of the conjugate pairs and the (l,k)-adjoint numbers (with n_q for fixed q) in the $k-$th block of X, respectively. Also we put $\tilde{p}_k:=\max\{p_i\colon 1\leqq i\leqq k+1\}$; $k=1,2,\ldots$.

Without the restriction $p_n=O(1)$, first we shall give sufficient conditions for the validity of CLT in the case when all of the coefficients α_k are equal to 1.

*) Here and further, in order to avoid the repetitions, for concepts, notations and formulations we refer to [1].

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Theorem C. *Suppose that the sequences* $\{n_k\}$, $\{\omega(k)\}$ *and* $\{p_k\}$ *satisfy condition* (1) *and additionally:*

a)

(2) $$\ln \tilde{p}_k = o\left(\sqrt{f(k+1)}\cdot\omega(f(k+1))\right) \quad as \quad k\to\infty;$$

b) *there exists a real number* η, $0\leqq\eta\leqq 1$ *such that*

(3) $$\lim_{N\to\infty}\frac{1}{f(N+1)}\sum_{i=1}^{N}\lambda_i = \eta;$$

c) *there exists an absolute constant* $C>1$ *such that for any fixed* q *and for any* j, $0\leqq j\leqq k-1$

(4) $$\lambda_k^j(q)\cdot\omega(f(k)) = O(C^{j-k})\cdot\ln\tilde{p}_k \quad as \quad k\to\infty.$$

Then the subsystem $\{\chi_{n_k}(x)\}$ *is the subject to* CLT.

Proof. The line of the proof follows that of Theorem A in [1] (conditions (2)—(4) of Theorem C are the analogues of conditions (1.2)—(1.4) of Theorem A, respectively). All of the lemmas of § 2 ([1]) remain valid. Lemmas of § 3 need only some light modifications, caused by the estimation of the value $\delta_k=f(k+1)-f(k)$. In our case, using the arguments of [2], we have:

$$\delta_k = O\left\{\frac{\ln p_{k+1}}{\omega(f(k+1))}\right\}.$$

In addition, since $a_k=1$ for $k=1,2,\ldots$, then $b_k=\max\{|a_j|:f(k)<j\leqq\leqq f(k+1)\}=1$ and with condition (2) we conclude that

$$b_k = o\left\{\frac{B_k\cdot\omega(f(k+1))}{\ln\tilde{p}_k}\right\} \quad \text{as} \quad k\to\infty.$$

These facts simplify the proofs of Lemmas 3.2 and 3.3 (see [1]). The completion of Theorem C runs as in § 4 of [1].

Remark 1. It is easy to see that Theorem A is a corollary of Theorem C in the case $p_n=O(1)$ and $a_k=1$ for $k=1,2,\ldots$.

Remarks 2. The sharpness of condition b) of Theorem C follows from the proof of Theorem B ([1]). At the same time the sharpness of conditions a) and c) remains open.

If the coefficients a_k are not all equal we can formulate the following statement, which is a generalization of Theorems A and C.

Theorem D. *Let the sequences* $\{n_k\}$, $\{\omega(k)\}$ *any* $\{p_k\}$ *satisfy conditions* (1), (2) *and* (4). *Further let a sequence* $\{a_k\}$ *be such that*

$$A_k^2 = \sum_{i=1}^{k} a_i^2 \to \infty, \quad a_k = O\left(\frac{A_k}{\sqrt{k}}\right) \quad as \quad k \to \infty; \tag{5}$$

and there should exist the limit

$$\eta = \lim_{N\to\infty} \frac{1}{B_N^2} \sum_{j}^{f(N+1)} (a_j \cdot \hat{a}_j) < \infty. \tag{6}$$

Then the subsystem $\{a_k \chi_{n_k}(x)\}$ *is the subject to* CLT (*briefly*: $\{a_k \chi_{n_k}(x)\} \subset$ CLT).

The proof of Theorem D goes analogously to the foregoing reasons.

Remark 3. The additional condition $a_k = O\left(\frac{A_k}{\sqrt{k}}\right)$ in this case is essential. Namely, in the course of the proofs of Theorems A and C we used the estimation $|\Delta_k(x)| = o(B_k)$. In Theorems A and C this estimation arises from conditions (1.2) (see [1]) and (2), respectively. But in Theorem D it needs some supplementary calculations. Since the condition $a_k = O\left(\frac{A_k}{\sqrt{k}}\right)$ implies that $b_k = O\left(\frac{B_k}{\sqrt{f(k)}}\right)$, hence (2) gives that

$$|\Delta_k(x)| = \sum_{i=f(k)+1}^{f(k+1)} a_i \chi_{n_i}(x) \leqq b_k \cdot \delta_k = O\left(\frac{B_k}{\sqrt{f(k)}}\right) \cdot \frac{\ln p_{k+1}}{\omega(f(k+1))} =$$

$$= o\left(\frac{B_k \cdot \sqrt{f(k+1)}}{\sqrt{f(k)}}\right) = o(B_k) \quad \text{as} \quad k \to \infty,$$

because $f(k+1) \sim f(k)$ $\left(\text{by virtue of } \delta_k < \frac{\ln p_{k+1}}{\omega(f(k+1))} = o(\sqrt{f(k+1)})\right)$. The condition $a_k = O\left(\frac{A_k}{\sqrt{k}}\right)$ holds, e.g., for any non-increasing sequence $\{a_k\}$ $(a_k \geqq 0)$.

Using the methods of proof of the previous theorems it is possible to establish a further analogue of Theorem A.

Theorem E. *Let the sequences* $\{n_k\}$, $\{\omega(k)\}$, $\{a_k\}$ *and* $\{p_k\}$ *satisfy conditions* (1), (4), (6) *and*

$$A_k^2 = \sum_{i=1}^{k} a_i^2 \to \infty, \quad A_k = O(k \cdot a_k); \tag{7}$$

$$b_k = o\left(\frac{B_k \cdot \omega(f(k+1))}{\ln \tilde{p}_k}\right) \quad \text{as} \quad k \to \infty. \tag{8}$$

Then $\{a_k \chi_{n_k}(x)\} \subset CLT$.

Remark 4. The second condition of (7) assures the relation $f(k+1)\sim f(k)$ (in the proofs of Theorems A, C, D this follows form conditions (1.2) and (2), respectively). In the present case we have: $B_k=A_{f(k+1)}=O\big(f(k+1)\cdot a_{f(k+1)}\big)=$ $=O\big(b_k\cdot f(k+1)\big)$, whence (8) gives the required relation. The condition $A_k=$ $=O(k\cdot a_k)$ is fulfilled, e.g., for any non-decreasing sequence $\{a_k\}$.

Concluding the paragraph we note that in Theorems A, C, D, E conditions (1.4) and (4) can be replaced by the next one. Denote by p_k^j the quantity of pairs (n_q, n_r) such that $m_k\leqq n_q, n_r<m_{k+1}$ and $1\leqq q+r<m_{j+1}$ (j can be equal to 0, 1, ..., $k-1$). Then instead of conditions (1.4) and (4) it is possible to use the condition

$$\sum_{j=0}^{k-1} b_j^2\cdot \varrho_k^j\cdot \delta_j = o(B_k^2) \quad \text{as} \quad k\to\infty. \tag{9}$$

This condition is "cruder" than conditions (1.4) and (4), but, on the other hand, it simplifies the proof of Lemma 3.3 (see [1]) essentially. In this case we immediately obtain the estimation

$$\int_0^1 (\Delta_k^2\cdot \bar{\Delta}_j^2)\,dx \leqq b_k^2\cdot b_j^2\cdot \varrho_k^j\cdot \delta_j$$

(see the arguments in [1]). Therefore

$$L_N^{(2)} = \sum_{k=1}^{N}\sum_{j=0}^{k-1}\int_0^1 (\Delta_k^2\cdot \bar{\Delta}_j^2)\,dx \leqq \sum_{k=1}^{N} b_k^2\cdot \sum_{j=0}^{k-1} b_j^2\cdot \varrho_k^j\cdot \delta_j \leqq$$

$$\leqq \sum_{k=1}^{N}\Big(\sum_{i=f(k)+1}^{f(k+1)}\Big)\sum_{j=0}^{k-1} b_j^2\cdot \varrho_k^j\cdot \delta_j = {}^2_N B\, o(B_N^2) = o(B_N^4) \quad \text{as} \quad N\to\infty,$$

that was required. Then we can estimate $L_N^{(3)}$ similarly and the proof of Lemma 3.3 can be completed faster.

The sharpness of condition (9) can be demonstrated by the same counter-examples as the sharpness of (1.4) in Theorem A.

2. How fast can the numbers p_k grow? Comparing Theorem A with Theorems C, D, E we notice that the conditions of type (2) is a "key moment" in the question on CLT for the case $\overline{\lim\limits_{n\to\infty}}\, p_n=+\infty$. Now we consider the problem: What kind of growth of $\{p_k\}$ does the fulfilment of condition (2) assure even if $a_k=1$, $\omega(k)=$ $=\dfrac{1}{k^\alpha}$; $k=1, 2, \ldots$; $\alpha\geqq 0$? In this case we have:

$$\ln \tilde{p}_k = o\big(\sqrt{f(k+1)}\cdot (f(k+1))^{-\alpha}\big) = o\big((f(k+1))^{1/2-\alpha}\big). \tag{10}$$

By (10) it follows that when $p_k=O(1)$ then an admissible boundary of the lacunarity is $\alpha=\dfrac{1}{2}$ (this fact is well known for the Walsh system).

For convenience we assume that the sequence $\{n_k\}$ is "regularly" lacunary, i.e. there exist constants c and d such that $c \geqq d > 0$ and

$$1+\frac{c}{k^\alpha} \geqq \frac{n_{k+1}}{n_k} \geqq 1+\frac{d}{k^\alpha} \tag{11}$$

for some $\alpha > 0$ and $k=1, 2, \ldots$.

In this case we have:

$$m_k \leqq n_{f(k)+1} \leqq n_{f(k)} \cdot \left(1+\frac{c}{(f(k))^\alpha}\right) \leqq \ldots \leqq n_1 \cdot \prod_{i=0}^{f(k)} \left(1+\frac{c}{i^\alpha}\right).$$

Taking the logarithm, we obtain that

$$\ln m_k \leqq \ln n_1 + \sum_{i=1}^{f(k)} \ln\left(1+\frac{c}{i^\alpha}\right). \tag{12}$$

Since $f(k) \uparrow \infty$ and $\ln\left(1+\frac{c}{i^\alpha}\right) = O\left(\frac{1}{i^\alpha}\right)$, then (12) implies that

$$\ln m_k = O\left(\ln n_1 + \sum_{i=1}^{f(k)} \frac{1}{i^\alpha}\right) = O\left(f(k)^{1-\alpha}\right). \tag{13}$$

By (10) and (13) it follows that under assumptions (11) we used relation (13).

$$\ln \tilde{p}_k = \{(\ln m_k)^{(1/2-\alpha)/1-\alpha}\} \tag{14}$$

(here we use the relation $f(k+1) \sim f(k)$).

Since $\ln m_k = \sum_{i=1}^{k} \ln p_i$, thus (14) shows a correlation between the growth of $\{p_k\}$ and the index of the lacunarity. So, if $p_k \sim \exp(k^\beta)$, $\beta > 0$, then $\ln m_k \asymp k^{\beta+1}$ and from (14) we obtain the following sufficient condition

$$\beta < 1-2\alpha. \tag{15}$$

Inequality (15) shows that only an exponential growth of $\{p_k\}$ can force to move away from the boundary $\alpha = \frac{1}{2}$.

Investigating the critical case $\alpha = \frac{1}{2}$ we shall suppose that

$$\frac{n_{k+1}}{n_k} - 1 \asymp \frac{\varphi(k)}{\sqrt{k}},$$

where $\varphi(k) \uparrow \infty$ and $\varphi(k) = o(\sqrt{k})$.

Then for the fulfilment of (2) it is sufficient that

$$\ln \tilde{p}_k = o(\varphi(f(k))). \tag{16}$$

By simple calculations we obtain the relation

$$\ln m_k = o\left(\sqrt{f(k)}\cdot\varphi(f(k))\right),$$

which implies

$$\ln m_k = o(f(k)). \tag{17}$$

By (16) and (17) we conclude that if $p_k \sim k^\gamma$ $(\gamma>0)$, then the following functions $\varphi(k)$ are suitable: $\varphi(k)=(\ln k)^{1+\varepsilon}$ for any $\varepsilon>0$, $\varphi(k)=\ln k\cdot\ln\ln(k+1)$ and so on.

In particular, if $\{p_k\}$ consists of only the prime numbers (i.e. $p_1=2$, $p_2=3$, $p_3=5$, $p_4=7$, ... etc.) then $p_k\sim k\ln k$ and all foregoing statements are valid.

Finally, we remark that under condition (11), by the estimation $\max\limits_{i\leq k}\beta_i = O\left(\dfrac{\ln\tilde{p}_k}{\omega(f(k+1))}\right)$, condition (9) can be simplified. In the case $a_k=1$ $(k=1, 2, \ldots)$ it has the form

$$\sum_{j=0}^{k-1} \varrho_k^j = o(k),$$

where we used relaiton (13).

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POLYTECHNICAL INSTITUTE, CHAIR OF HIGHER MATHEMATICS

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On composition operators

ASHOK KUMAR

1. Preliminaries

Let $(X, \mathcal{S}, \lambda)$ be a σ-finite measure space and let Φ be a measurable non-singular $(\lambda\Phi^{-1}(E)=0$ whenever $\lambda(E)=0)$ transformation from X into itself. Then the *composition transformation* C_Φ on $L^2(X, \mathcal{S}, \lambda)$ is defined as

$$C_\Phi f = f\circ\Phi \quad \text{for every} \quad f\in L^2(X, \mathcal{S}, \lambda).$$

In case C_Φ is a bounded operator with range in $L^2(X, \mathcal{S}, \lambda)$ we call it a *composition operator*. In this paper we generalize the Theorem 1 [11] and prove that the result is also true in case $(X, \mathcal{S}, \lambda)$ is a σ-finite standard Borel space. In the subsequent sections we characterize composition operators with ascent 1 and descent 1 and give a criterion for partial isometry and co-isometry composition operator. In the last section hyponormal composition operators on $l^2(\mathbf{N}, \mathcal{S}, \lambda)$, the square summable weighted sequence space, have been characterized and a necessary condition for C_Φ to be hyponormal on $L^2(X, \mathcal{S}, \lambda)$ is given, where $(X, \mathcal{S}, \lambda)$ is a standard Borel space.

Let $B(L^2(\lambda))$, $R(C_\Phi)$, $R(C_\Phi)^\perp$ denote the Banach algebra of all bounded linear operators on $L^2(\lambda)$, the range of C_Φ and the orthogonal complement of the range of C_Φ respectively. We denote by

$$f_0 = \frac{d\lambda\Phi^{-1}}{d\lambda} \quad \text{and} \quad g_0 = \frac{d\lambda(\Phi\circ\Phi)^{-1}}{d\lambda}$$

the Radon—Nikodym derivative of the measure $\lambda\Phi^{-1}$ and $\lambda(\Phi\circ\Phi)^{-1}$ with respect to the measure λ, respectively. The symbols X_0 and X_0' will stand for the set $\{x: f_0(x)=0\}$ and $\{x: g_0(x)=0\}$ respectively. The multiplication operators induced by f_0 and g_0 are denoted by M_{f_0} and M_{g_0}.

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If E and F are two measurable sets, then "$E=F$" will indicate that their symmetric difference is of measure zero. We denote the characteristic function of a measurable set E by χ_E.

Definition. An operator A on a Hilbert space H is called a *Fredholm operator* if the range of A is closed and the dimensions of the kernel and co-kernel are finite.

Definition. A *standard Borel space* X is a Borel subset of a complete separable metric space T. The class $\mathcal{S}$ will consist of all the sets of form $X\cap B$, where B is a Borel subset of T.

2. Fredholm composition operators

Theorem 1. *Let $(X,\mathcal{S},\lambda)$ be a σ-finite non-atomic standard Borel space and $C_\Phi\in B(L^2(\lambda))$. Then C_Φ is a Fredholm operator if and only if it is invertible.*

Proof. If C_Φ is invertible, then C_Φ is clearly a Fredholm operator.

Since $\operatorname{Ker} C_\Phi=\operatorname{Ker} C_\Phi^* C_\Phi=\operatorname{Ker} M_{f_0}$ [17, p. 82], where M_{f_0} is the multiplication operator induced by f_0, and X is a non-atomic, the nullity of C_Φ is either zero or infinite. Suppose C_Φ is a Fredholm operator. Then, since C_Φ is one-to-one with closed range, to prove that C_Φ is invertible it is enough to show that Φ is one-to-one a.e. [18, Theorem 2; 13, Corollary 2.4]. Suppose Φ is not one-to-one. By Corollary 8.2 [22] there exist two Borel sets Y_1 and Z_1 such that Φ is one-to-one on Y_1 onto Z_1 and $\lambda(X\setminus Y_1)\neq 0$. Now $(X\setminus Y_1)$ is a Borel set. Let $\Phi_1=\Phi|(X\setminus Y_1)$. Again by Corollary 8.2 [22] we get two Borel sets $Y_2\subseteq(X\setminus Y_1)$ and Z_2 such that Φ is one-to-one on Y_2 onto Z_2. Since X is a non-atomic, we can write $Y_2=\bigcup_{n=1}^{\infty} E_n$ such that $0<\lambda(E_n)<\infty$, $E_n\cap E_m=\emptyset$ whenever $n\neq m$, and $\lambda(Y_1\cap\Phi^{-1}(\Phi[E_n]))\neq 0$. From the fact that $R(C_\Phi)=L^2(X,\Phi^{-1}(\mathcal{S}),\lambda)$ [13, Lemma 2.4], where $\Phi^{-1}(\mathcal{S})=\{\Phi^{-1}(E)\colon E\in\mathcal{S}\}$, it follows that, for every $n\in\mathbf{N}$, there exists $K_n\in\Phi^{-1}(\mathcal{S})$ such that $\langle\chi_{E_n},\chi_{K_n}\rangle\neq 0$. This shows that χ_{E_n} does not belong to $R(C_\Phi)^\perp$. Since $\lambda(Y_1\cap\Phi^{-1}(\Phi[E_n]))\neq 0$, we have $E_n\neq\Phi^{-1}(E)$ for any $E\in\mathcal{S}$, and hence χ_{E_n} can not belong to $R(C_\Phi)$ also. Let $L^2(\lambda)=R(C_\Phi)^\perp\oplus R(C_\Phi)$. Consider $\{\chi_{E_n}\}=\{f_n+g_n\}$, where $f_n\in R(C_\Phi)^\perp$ and $g_n\in R(C_\Phi)$. In view of the remark [22, page 3] $\Phi[E_n]=\{\Phi(x)\colon x\in E_n\}$ is a Borel set. Let $F_n=\Phi^{-1}(\Phi[E_n])$. Since $\Phi|Y_2$ is one-to-one, $\{F_n\}$ is a disjoint sequence of sets. We claim that $g_n=g_n\cdot\chi_{F_n}$. Suppose $g_n\neq g_n\cdot\chi_{F_n}$. Then $\lambda(G_n\setminus F_n)\neq 0$, wheer $G_n=\{x\colon g_n(x)\neq 0\}$. Since

$$f_n(x)=\begin{cases}1-g_n(x) & \text{for } x\in E_n,\\ -g_n(x) & \text{for } x\in X\setminus E_n.\end{cases}$$

We can find a set $G'\subseteq(G_n\setminus F_n)$ belonging to the σ-algebra $\Phi^{-1}(\mathcal{S})$ such that

$\langle f_n, \chi_{G'}\rangle \neq 0$ which is a contradiction. Thus $g_n = g_n \cdot \chi_{F_n}$. Since

$$\langle f_n, f_m\rangle = \langle \chi_{E_n} - g_n, \chi_{E_m} - g_m\rangle = 0 \quad \text{whenever} \quad n \neq m,$$

$\{f_n\}$ is an infinite orthogonal sequence in $R(C_\Phi)^\perp$ which contradicts the assumption that $\dim R(C_\Phi)^\perp$ is finite. Hence Φ is one-to-one (a.e.). Thus C_Φ has dense range and hence C_Φ is invertible.

3. Ascent and descent of a composition operator

Definition. Let A be an operator on a Hilbert space H. Then the *ascent* $\alpha(A)$ of A is the least non-negative integer such that $\operatorname{Ker} A^k = \operatorname{Ker} A^{k+1}$ for all $k \geqq \alpha(A)$ and the *descent* $\delta(A)$ of A is the least non-negative integer such that $\overline{R(A^k)} = \overline{R(A^{k+1})}$ for all $k \geqq \delta(A)$, where $\overline{R(A)}$ is the closure of the range of A.

We shall prove the following theorem which characterizes composition operators with ascent 1.

Theorem 2. *Let* $C_\Phi \in B(L^2(X, \mathcal{S}, \lambda))$. *Then* C_Φ *has ascent* 1 *if and only if* $\lambda(\Phi\circ\Phi)^{-1}(E) = 0$ *implies* $\lambda\Phi^{-1}(E) = 0$ *for* $E \in \mathcal{S}$.

Proof. Since C_Φ is a composition operator, there exists an $M < \infty$ such that

$$\lambda\Phi^{-1}(\Phi^{-1}(E)) \leqq M\lambda\Phi^{-1}(E) \leqq M^2\lambda(E)$$

for every $E \in \mathcal{S}$ [20, Theorem 1]. This shows that the measure $\lambda(\Phi\circ\Phi)^{-1}$ defined as $\lambda(\Phi\circ\Phi)^{-1}(E) = \lambda\Phi^{-1}(\Phi^{-1}(E))$ is absolutely continuous with respect to the measure λ, and consequently for every $E \in \mathcal{S}$ we have $\lambda(\Phi\circ\Phi)^{-1}(E) = \int_E g_0\, d\lambda$. Suppose $\lambda\Phi^{-1}(E) = 0$ whenever $\lambda(\Phi\circ\Phi)^{-1}(E) = 0$. Then, by absolute continuity of $\lambda(\Phi\circ\Phi)^{-1}$ and the equation $\lambda\Phi^{-1}(E) = \int_E f_0\, d\lambda$, it follows that $X_0 = X_0'$ and hence by [17, page 82] we conclude that

$$\operatorname{Ker} C_\Phi = \operatorname{Ker} M_{f_0} = L^2(X_0) = \operatorname{Ker} M_{g_0} = \operatorname{Ker} C_\Phi^2.$$

This shows that C_Φ is of ascent 1.

Conversely, suppose $\operatorname{Ker} C_\Phi = \operatorname{Ker} C_\Phi^2$. Since $\operatorname{Ker} C_\Phi = L^2(X_0)$ and $\operatorname{Ker} C_\Phi^2 = L^2(X_0')$, it follows that $X_0 = X_0'$. Since $\lambda\Phi^{-1}(E) = \int_E f_0\, d\lambda$ and $\lambda(\Phi\circ\Phi)^{-1}(E) = \int_E g_0\, d\lambda$, it follows that $\lambda(\Phi\circ\Phi)^{-1}(E) = 0$ implies $\lambda\Phi^{-1}(E) = 0$.

Theorem 3. *Let* X *be a* σ*-finite standard Borel space and let* $C_\Phi \in B(L^2(X, \mathcal{S}, \lambda))$. *Then the operator* C_Φ *has ascent* 1 *if and only if* $\Phi[X_1] \supseteqq X_1$, *where* $\Phi[X_1] = \{\Phi(x_1) : x_1 \in X_1\}$ *and* $X_1 = X \setminus X_0$.

Proof. Suppose $\Phi[X_1] \supseteqq X_1$. Then, since $\operatorname{Ker} C_\Phi = L^2(X_0)$ and $L^2(X) = L^2(X_0) \oplus L^2(X_1)$, every $f \in \operatorname{Ker} C_\Phi^2$ can be written as

$$f = f_1 + g_1,$$

where $f_1 \in \operatorname{Ker} C_\Phi$ and $g_1 \in L^2(X_1)$. Since

$$g_1 \circ \Phi \circ \Phi = C_\Phi^2 g_1 = C_\Phi^2 f = 0 \quad \text{and} \quad \Phi[X_1] \supseteqq X_1,$$

it follows that $g_1 = 0$ a.e. on X_1. Hence $f = f_1$. Thus $\operatorname{Ker} C_\Phi^2 \subseteqq \operatorname{Ker} C_\Phi$. The inclusion otherway is true in general for every operator. This shows that $\operatorname{Ker} C_\Phi = \operatorname{Ker} C_\Phi^2$.

Conversely, suppose $\Phi[X_1] \not\supseteqq X_1$. Now, if E is a measurable subset of $X_1 \setminus \Phi[X_1]$ of non-zero finite measure, then $C_\Phi^2 \chi_E = 0$. Since E is a subset of X_1, it follows that $C_\Phi \chi_E \neq 0$, which implies $\operatorname{Ker} C_\Phi \neq \operatorname{Ker} C_\Phi^2$. Hence the proof of the theorem is completed.

Corollary 4. *Let $C_\Phi \in B(l^2(\mathrm{N}, \mathcal{S}, \lambda))$. Then $\operatorname{Ker} C_\Phi = \operatorname{Ker} C_\Phi^2$ if and only if $(\Phi \circ \Phi)[\mathrm{N}] = \Phi[\mathrm{N}]$, where $\Phi[\mathrm{N}]$ is the range of Φ.*

Example 5. Let $X = \mathbf{R}$, the set of real numbers, and let $\Phi(x) = |x|$, $x \in \mathbf{R}$. Then C_Φ is a composition operator with ascent 1 on $L^2(\mathbf{R})$.

In the following theorem, we shall characterize composition operator with descent 1.

Theorem 6. *Let $C_\Phi \in B(L^2(\lambda))$. Then $\overline{R(C_\Phi)} = \overline{R(C_\Phi^2)}$ if and only if $\Phi^{-1}(\mathcal{S}) = (\Phi \circ \Phi)^{-1}(\mathcal{S})$, where $\mathcal{S}$ is the σ-algebra of measurable subsets of X.*

Proof. Suppose $\Phi^{-1}(\mathcal{S}) = (\Phi \circ \Phi)^{-1}(\mathcal{S})$. Then, since the ranges of C_Φ and C_Φ^2 are dense in $L^2(X, \Phi^{-1}(\mathcal{S}), \lambda)$ and $L^2(X, (\Phi \circ \Phi)^{-1}(\mathcal{S}), \lambda)$ respectively [13, Lemma 2.4], it follows that $\overline{R(C_\Phi)} = \overline{R(C_\Phi^2)}$.

Conversely, suppose $\overline{R(C_\Phi)} = \overline{R(C_\Phi^2)}$. Then

$$L^2(X, \Phi^{-1}(\mathcal{S}), \lambda) = L^2(X, (\Phi \circ \Phi)^{-1}(\mathcal{S}), \lambda).$$

We claim that $\Phi^{-1}(\mathcal{S}) = (\Phi \circ \Phi)^{-1}(\mathcal{S})$. Suppose $\Phi^{-1}(\mathcal{S}) \neq (\Phi \circ \Phi)^{-1}(\mathcal{S})$. Then, since $(\Phi \circ \Phi)^{-1}(\mathcal{S})$ is a subfamily of $\Phi^{-1}(\mathcal{S})$, there exists an element $E = \Phi^{-1}(F)$ which does not belong to $(\Phi \circ \Phi^{-1}(\mathcal{S})$. Since X is σ-finite, we can write

$$X = \bigcup_{i=1}^{\infty} X_i = \bigcup_{i=1}^{\infty} \Phi^{-1}(X_i) = \bigcup_{i=1}^{\infty} (\Phi \circ \Phi)^{-1}(X_i),$$

where $\{X_i\}$ is a disjoint sequence of sets of finite measure. Let $F_i = E \cap (\Phi \circ \Phi)^{-1}(X_i)$. Then F_i does not belong to $(\Phi \circ \Phi^{-1}(\mathcal{S})$ for some $i \in \mathrm{N}$ or otheswise E will be in

$(\Phi\circ\Phi)^{-1}(\mathscr{S})$. This shows that $\chi_{F_i}\notin\overline{R(C_\Phi^2)}$ for that $i\in\mathbf{N}$, which is a contradiction. Thus the proof of the theorem is complete.

In the following theorem we characterize composition operators with descent 1 in particular case.

Theorem 7. *Let X be a σ-finite standard Borel space and let $C_\Phi\in B(L^2(\lambda))$. Then $\overline{R(C_\Phi)}=\overline{R(C_\Phi^2)}$ if and only if $\Phi|X_1$ is injective a.e., where $X_1=\{x\colon f_0(x)\neq 0\}$.*

Proof. Since $\overline{R(A)}\supseteq\overline{R(A^2)}$ for any bounded operator A on a Hilbert space, it follows that $\overline{R(C_\Phi)}\supseteq\overline{R(C_\Phi^2)}$. We will show that if $\Phi|X_1$ is injective a.e., then the equality prevails. Suppose $\chi_E\in\overline{R(C_\Phi)}$. Then there exists a measurable subset $F\subseteq X_1$ such that $E=\Phi^{-1}(F)$. Since $\Phi|X_1$ is injective a.e. and X is a σ-finite standard Borel space, $\Phi[F]$ is a Borel set and $\Phi[F]=\bigcup_{n=1}^{\infty}E_n$ for some disjoint sequence $\{E_n\}$ of measurable subsets of finite measure. Consider the sum

$$\sum_{n=1}^{\infty}\chi_{E_n}\circ\Phi\circ\Phi=\sum_{n=1}^{\infty}C_\Phi^2\chi_{E_n}.$$

It is easy to see that the sum converges to χ_E a.e. By the Lebesgue dominated convergence theorem it converges to χ_E in L^2-norm. Hence χ_E belongs to the closure of $R(C_\Phi^2)$. From this it follows that all simple functions which belong to $\overline{R(C_\Phi)}$ also belongs to $\overline{R(C_\Phi^2)}$. This is enough to establish the equality $\overline{R(C_\Phi)}=\overline{R(C_\Phi^2)}$.

Conversely, suppose $\Phi|X_1$ is not injective a.e. Then, since X_1 is a Borel set, by Corollary 8.2 [22], there exists two Borel sets A and Z such that $\Phi_1=\Phi|X_1$ is one-to-one on A onto Z, $\lambda\Phi_1^{-1}(X_1\setminus Z)=0$ and $\lambda(X_1\setminus A)\neq 0$. Let $F\subseteq(X_1\setminus A)$ be a measurable set of finite measure such that $\lambda(A\cap\Phi^{-1}(\Phi[F]))\neq 0$. Then $\chi_{\Phi^{-1}(F)}=C_\Phi\chi_F\in R(C_\Phi)$. We claim that $\chi_{\Phi^{-1}(F)}$ does not belong to $\overline{R(C_\Phi^2)}$. If $\chi_{\Phi^{-1}(F)}\in\overline{R(C_\Phi^2)}=L^2(X,(\Phi\circ\Phi)^{-1}(\mathscr{S}),\lambda)$, then there exists $E\in\mathscr{S}$ such that $\Phi^{-1}(F)=(\Phi\circ\Phi)^{-1}(E)=\Phi^{-1}(G)=\Phi^{-1}(G\cap A)\cup\Phi^{-1}(G\setminus(G\cap A))$, where $G=\Phi^{-1}(E)$. Since $\lambda(A\cap\Phi^{-1}(\Phi[F]))\neq 0$, we can conclude that $\lambda(G\cap A)\neq 0$, and hence $\lambda(\Phi^{-1}(G\cap A))=\int_{G\cap A}f_0\neq 0$ which is a contradiction.

Corollary 8. *Let $\inf\{\lambda(n)\colon n\in\mathbf{N}\}\geq c>0$ and $\sup\{\lambda(n)\colon n\in\mathbf{N}\}<\infty$ and let $C_\Phi\in B(l^2(\mathbf{N},\mathscr{S},\lambda))$. Then $R(C_\Phi)=R(C_\Phi^2)$ if any only if $\Phi|\Phi[\mathbf{N}]$ is one-to-one.*

Example 9. Let $X=[-1,1]$ and λ be the Lebesgue measure on the Borel subsets of X. Let $\Phi(x)=|x|$. Then $C_\Phi\in B(L^2(\lambda))$ and $R(C_\Phi)=R(C_\Phi^2)$.

We shall give an example of a composition opeartor when $R(C_\Phi)\neq R(C_\Phi^2)$ but $\overline{R(C_\Phi)}=\overline{R(C_\Phi^2)}$.

Example 10. Let $X=\mathbf{R}$, and let $\mathscr{S}$ be the σ-algebra of Borel subsets of $\mathbf{R}$

wint λ as the Lebesgue measure. Define the measurable function Φ as follows

$$\Phi(x) = \begin{cases} 1/x & \text{if } x\in]0,1[, \\ -x-(n-2) & \text{if } x\in[1,\infty[\quad \text{and} \quad n\leqq x<n+1, \quad n=1,2,3,\ldots, \\ x-(n-1) & \text{if } x\in]-\infty,0] \quad \text{and} \quad n\leqq -x<n+1, \quad n=0,1,2,\ldots. \end{cases}$$

This Φ induces a composition operator C_Φ on $L^2(\mathbf{R})$. Since $\chi_{]-1,0]}\notin R(C_\Phi^2)$ and $\chi_{]-1,0]}\in R(C_\Phi)$, it follows that $R(C_\Phi)\neq R(C_\Phi^2)$. But, since $(\Phi\circ\Phi)^{-1}(\mathscr{S})=\Phi^{-1}(\mathscr{S})$, we have $\overline{R(C_\Phi)}=\overline{R(C_\Phi^2)}$.

4. Partial isometry and co-isometry

Definition. An operator A on a Hilbert space is said to be a *partial isometry* if it is an isometry on the orthogonal complement of its kernel.

Theorem 11. *Let C_Φ be a composition operator on $L^2(X,\mathscr{S},\lambda)$. Then C_Φ is a partial isometry if and only if f_0 is a characteristic function.*

Proof. Suppose C_Φ is a partial isometry. Then $C_\Phi=C_\Phi C_\Phi^* C_\Phi$, [8, Corollary 3, Problem 98] and it follows that $C_\Phi^* C_\Phi=C_\Phi^* C_\Phi C_\Phi^* C_\Phi$ which is equivalent to $M_{f_0}=M_{f_0}\cdot M_{f_0}=M_{f_0^2}$. From this we conclude that f_0 is a characteristic function.

Conversely, suppose f_0 is a characteristic function. Then, since $\operatorname{Ker} C_\Phi=$ $=L^2(X_0)$ and $(\operatorname{Ker} C_\Phi)^\perp=L^2(X_1,\mathscr{S}_1,\lambda)$, where $X_1=X\setminus X_0$ and $\mathscr{S}_1=\{E\cap X_1\colon E\in\mathscr{S}\}$, it follows that

$$C_\Phi^* C_\Phi f = M_{f_0} f = f \quad \text{for all} \quad f\in(\operatorname{Ker} C_\Phi)^\perp.$$

This shows that C_Φ is an isometry on the orthogonal complement of its kernel.

Corollary 12. *Let $C_\Phi\in B(l^2(N))$, where $l^2(N)=\{\{a_n\}\colon \Sigma|a_n|^2<\infty\}$. Then C_Φ is a partial isometry if and only if Φ is one-to-one.*

Proof. Since

$$f_0(n) = \frac{\lambda\Phi^{-1}(n)}{\lambda(n)} = \lambda\Phi^{-1}(n),$$

the Corollary follows.

Example 13. Let $X=[0,\infty[$ and λ be the Lebesgue measure on the Borel subsets of X. Let $\Phi_c(x)=x+c$, where $c\in X$. Then $C_{\Phi_c}\in B(L^2(\lambda))$; $f_0(x)=1$, for $c\leqq x<\infty$, and $f_0(x)=0$, for $0\leqq x<c$. Hence by the above theorem $\{C_{\Phi_c}\colon c\in X\}$ is a family of partial isometries on $L^2(X)$.

Definiton. An operator A on a Hilbert space is called a *co-isometry if* $AA^*=I$.

Theorem 14. *Let $C_\Phi \in B(L^2(\lambda))$. Then C_Φ is a co-isometry if and only if C_Φ is onto and $f_0 \circ \Phi = 1$ a.e.*

Proof. Since, for every $f \in R(C_\Phi)$,

$$f = C_\Phi C_\Phi^* f = C_\Phi C_\Phi^* C_\Phi g = C_\Phi M_{f_0} g = C_\Phi(f_0 \cdot g) = f_0 \circ \Phi \cdot f,$$

where $C_\Phi g = f$, C_Φ is is co-isometry if and only if C_Φ is onto and $f_0 \circ \Phi = 1$ a.e.

Corollary 15. *Let $C_\Phi \in B(l^2(N))$. Then the following statements are equivalent:*

(i) *C_Φ is partial isometry,*

(ii) *C_Φ is co-isometry,*

(iii) *C_Φ is onto,*

(iv) *Φ is one-to-one.*

5. Hyponormal composition operators

Definition. An operator A on a Hilbert space H is called *hyponormal* if $A^*A - AA^* \geqq 0$.

In [9] hyponormal composition operators have been studied but it remains an open problem to find measure theoretic condition which is both necessary and sufficient for the hyponormality of C_Φ.

Lemma 16. *Let C_Φ be a composition operator on $L^2(\lambda)$. Then C_Φ is hyponormal only if C_Φ is one-to-one.*

Proof. Suppose C_Φ is hyponormal. Then

$$\operatorname{Ker} C_\Phi C_\Phi^* \supseteqq \operatorname{Ker} C_\Phi^* C_\Phi = \operatorname{Ker} C_\Phi = L^2(X_0).$$

Since $\operatorname{Ker} C_\Phi C_\Phi^* = \operatorname{Ker} C_\Phi^* = R(C_\Phi)^\perp = L^2(X, \Phi^{-1}(\mathscr{S}), \lambda)^\perp$, then, for every measurable subset E of X_0 of non-zero finite measure, there exists an element F in $\mathscr{S}$ such that

$$(\chi_E, C_\Phi \chi_F) = \langle \chi_E, \chi_{\Phi^{-1}(F)} \rangle \neq 0,$$

which is contradiction. Thus it follows that the measure of X_0 is zero. This shows that C_Φ is one-to-one.

Corollary 17. *Let $C_\Phi \in l^2(\mathrm{N}, \mathscr{S}, \lambda)$. Then C_Φ is hyponormal only if Φ is onto.*

Lemma 18. *Let $C_\Phi \in B(l^2(\mathrm{N}, \mathscr{S}, \lambda))$. Then*

$$e_n = \frac{\lambda(n)}{\lambda\Phi^{-1}(\Phi(n))} \chi_{\Phi^{-1}(\Phi(n))} + e_n',$$

where e_n is the characteristic function of $\{n\}$, and

$$e_n' \in \left(l^2(\mathbf{N}, \Phi^{-1}(\mathscr{S}), \lambda)\right)^{\perp}, \quad \Phi^{-1}(\mathscr{S}) = \{\Phi^{-1}(E) \colon E \in \mathscr{S}\}.$$

Proof. Since

$$l^2(\mathbf{N}, \mathscr{S}, \lambda) = \overline{R(C_\Phi)} \oplus R(C_\Phi)^{\perp} = l^2(\mathbf{N}, \Phi^{-2}(\mathscr{S}), \lambda) \oplus \left(l^2(\mathbf{N}, \Phi^{-1}(\mathscr{S}), \lambda)\right)^{\perp},$$

e_n admits the form

$$e_n = c\chi_{\Phi^{-1}(\Phi(n))} + e_n', \quad e_n' \in R(C_\Phi)^{\perp},$$

and it follows that

$$e_n' = e_n - c\chi_{\Phi^{-1}(\Phi(n))}.$$

Since $c\chi_{\Phi^{-1}(\Phi(n))} \perp e_n'$,

$$\langle c\chi_{\Phi^{-1}(\Phi(n))}, e_n - c\chi_{\Phi^{-1}(\Phi(n))} \rangle = 0,$$

which implies that

$$c = \frac{\lambda(n)}{\lambda\Phi^{-1}(\Phi(n))}.$$

This completes the proof of the Lemma.

Using the notation

$$f_0(n) = \frac{\lambda\Phi^{-1}(n)}{\lambda(n)},$$

C_Φ^*, the adjoint of C_Φ, can be expressed as follows:

$$C_\Phi^* e_n = C_\Phi^* \left(\frac{\lambda(n)}{\lambda\Phi^{-1}(\Phi(n))} \chi_{\Phi^{-1}(\Phi(n))} + e_n' \right) = \frac{\lambda(n)}{\lambda\Phi^{-1}(\Phi(n))} C_\Phi^* C_\Phi \chi_{\Phi(n)} =$$

$$= \frac{\lambda(n)}{\lambda\Phi^{-1}(\Phi(n))} f_0 \cdot \chi_{\Phi(n)} = \frac{\lambda(n)}{\lambda\Phi^{-1}(\Phi(n))} \cdot f_0 \circ \Phi(n) \cdot \chi_{\Phi(n)} =$$

$$= \frac{\lambda(n)}{\lambda\Phi^{-1}(\Phi(n))} \cdot \frac{\lambda\Phi^{-1}(\Phi(n))}{\lambda(\Phi(n))} \cdot \chi_{\Phi(n)} = \frac{\lambda(n)}{\lambda\Phi(n)} \cdot \chi_{\Phi(n)}.$$

The proof of the following theorem is analogous to the proof of the Proposition 11.5 [2].

Theorem 19. *Let $C_\Phi \in B(l^2(\mathbf{N}, \mathscr{S}, \lambda))$. Then C_Φ is hyponormal if and only if Φ is onto and*

$$\sum_{m \in \Phi^{-1}(n)} \frac{(\lambda(m))^2}{\lambda\Phi^{-1}(m)} \leqq \lambda(n).$$

Proof. Suppose C_Φ is hyponormal. Then, by Corollary 17., Φ is onto. Let ζ_n be the subspace spanned by $\{e_m\}_{m \in \Phi^{-1}(n)}$, $f \in \zeta_n$ and $f = \sum c_m e_m$. Then

$$\int |f \circ \Phi|^2 \, d\lambda = \langle C_\Phi f, C_\Phi f \rangle \geqq \langle C_\Phi^* f, C_\Phi^* f \rangle = \langle C_\Phi^* \sum c_m e_m, C_\Phi^* \sum c_m e_m \rangle$$

and thus, by the above computation,

$$\int |f|^2 d\lambda\Phi^{-1} \geqq \left\langle \sum c_m \frac{\lambda(m)}{\lambda\Phi(m)} e_{\Phi(m)}, \sum c_m \frac{\lambda(m)}{\lambda\Phi(m)} e_{\Phi(m)} \right\rangle.$$

This implies

$$\|f\|^2_{\lambda\Phi^{-1}} \geqq \left| \sum c_m \frac{\lambda(m)}{\lambda\Phi(m)} \right|^2 \cdot \lambda(n) = \frac{1}{\lambda(n)} \left|\sum c_m \lambda(m)\right|^2 = \frac{1}{\lambda(n)} \left|\langle f, h|\Phi^{-1}(n)\rangle_{\lambda\Phi^{-1}}\right|^2,$$

where

$$h = \frac{d\lambda}{d\lambda\Phi^{-1}}.$$

Since the inner product with $h|\Phi^{-1}(n)$ in $L^2(\mathrm{N}, \mathscr{S}, \lambda\Phi^{-1})$ induces a linear functional,

$$\lambda(n) \geqq \|h|\Phi^{-1}(n)\|^2_{\lambda\Phi^{-1}} = \sum \left(\frac{\lambda(m)}{\lambda\Phi^{-1}(m)}\right)^2 \lambda\Phi^{-1}(m) = \sum_{m\in\Phi^{-1}(n)} \frac{(\lambda(m))^2}{\lambda\Phi^{-1}(m)}.$$

Conversely, suppose the hypothesis of the theorem holds. Then

$$\langle C_\Phi C_\Phi^* f, f\rangle = \langle C_\Phi^* f, C_\Phi^* f\rangle = \langle C_\Phi^* \sum c_n e_n, C_\Phi^* \sum c_n e_n\rangle =$$

$$= \left\langle \sum c_n \frac{\lambda(n)}{\lambda\Phi(n)} \chi_{\Phi(n)}, \sum c_n \frac{\lambda(n)}{\lambda\Phi(n)} \chi_{\Phi(n)} \right\rangle =$$

$$= \left\langle \sum_n \sum_{i\in\Phi^{-1}(n)} c_i \frac{\lambda(i)}{\lambda\Phi(i)} e_n, \sum_n \sum_{i\in\Phi^{-1}(n)} c_i \frac{\lambda(i)}{\lambda\Phi(i)} e_n \right\rangle =$$

$$= \sum_n \frac{1}{\lambda(n)} \left|\sum_{i\in\Phi^{-1}(n)} c_i \lambda(i)\right|^2 = \sum_n \left|\left\langle f|\Phi^{-1}(n), \frac{1}{\sqrt{\lambda(n)}} h\right\rangle_{\lambda\Phi^{-1}}\right|^2 \leqq$$

$$\leqq \sum \|f|\Phi^{-1}(n)\|^2_{\lambda\Phi^{-1}} = \|f\|^2_{\lambda\Phi^{-1}} = \langle C_\Phi f, C_\Phi f\rangle = \langle C_\Phi^* C_\Phi f, f\rangle,$$

which shows that C_Φ is hyponormal.

Let X be a σ-finite standard Borel space and X_1 be the maximal subset of X such that $\Phi^{-1}(\Phi(x)\cap(X_1\setminus\{x\}))\neq\emptyset$ for $x\in X_1$. Let $X_2=\Phi[X_1]=\{\Phi(x_1)\colon x_1\in X_1\}$. Then X_2 is a Borel set [22, page 3]. Let $f_0(x)=c_n$ for $x\in X_2^{(n)}$, where $\{X_2^{(n)}\}$ is a disjoint sequence of sets such that $\bigcup_n X_2^{(n)}=X_2$, and let $Y_2^{(n)}=\Phi^{-1}(X_2^{(n)})$.

In the following theorem we consider measurable transformation Φ on a σ-finite standard Borel space such that f_0 satisfies the above property and find necessary condition for C_Φ induced by such Φ to be hyponormal which would explain $f_0\geqq f_0\circ\Phi$ [9, Theorem 9, Example 16] is not a necessary condition for hyponormality of C_Φ.

Theorem 20. *Let C_Φ be a composition operator induced by above type measurable transformation. Then C_Φ is hyponormal only if*

$$\int_E f_0\, d\lambda \geqq \frac{\lambda(E)}{\lambda\Phi^{-1}(\Phi(E))} \int_E f_0\circ\Phi\, d\lambda, \quad E\subset Y_2^{(n)}$$

and

$$f_0(x) \geqq f_0\circ\Phi(x) \quad a.e.\ on \quad x\in X\setminus X_1,$$

where X_1 is the maximal subset of X, σ-finite standard Borel space, such that $\Phi^{-1}(\Phi(x))\cap(X_1\setminus\{x\})\neq\emptyset$ for $x\in X_1$.

Proof. It follows similarly as in Lemma 18 that

$$\chi_E = \frac{\lambda(E)}{\lambda\Phi^{-1}(\Phi(E))}\chi_{\Phi^{-1}(\Phi(E))}+g, \quad g\in R(C_\Phi)^\perp, \quad E\subset Y_2^{(n)}$$

and

$$C_\Phi^*\chi_E = \frac{\lambda(E)}{\lambda\Phi^{-1}(\Phi(E))} C_\Phi^* C_\Phi \chi_{\Phi(E)} = \frac{\lambda(E)}{\lambda\Phi^{-1}(\Phi(E))} f_0\cdot\chi_{\Phi(E)}.$$

Since C_Φ is hyponormal, then for $E\subset Y_2^{(n)}$

$$\int_E f_0\, d\lambda = \langle C_\Phi^* C_\Phi\chi_E, \chi_E\rangle \geqq \langle C_\Phi C_\Phi^*\chi_E, \chi_E\rangle = \frac{\lambda(E)}{\lambda\Phi^{-1}(\Phi(E))}\langle f_0\circ\Phi\chi_{\Phi^{-1}(\Phi(E))}, \chi_E\rangle =$$

$$= \frac{\lambda(E)}{\lambda\Phi^{-1}(\Phi(E))}\langle f_0\circ\Phi\cdot\chi_E, \chi_E\rangle = \frac{\lambda(E)}{\lambda\Phi^{-1}(\Phi(E))}\int_E f_0\circ\Phi\, d\lambda.$$

If $E=\{x: f_0(x)<f_0\circ\Phi(x),\ x\in X\setminus X_1\}$ has a positive measure, then for a finite set $F\subset E$.

$$\int_F f_0\, d\lambda = \langle C_\Phi^* C_\Phi\chi_F, \chi_F\rangle < \int_F f_0\circ\Phi\, d\lambda = \langle C_\Phi C_\Phi^*\chi_F, \chi_F\rangle,$$

which is a contradiction. This proves the theorem.

The above theorem explains why the function in Example 10 [9, p. 131] does not induce hyponormal composition operator.

Since in Example 10 [9, p. 131]

$$\int_{[1,3/2]} f_0\, d\lambda = 1/4\cdot 1/2 = 1/8 < \frac{\lambda[(1,3/2])}{\lambda\Phi^{-1}(\Phi[1,3/2]]}\int_{[1,3/2]} f_0\circ\Phi\, d\lambda =$$

$$= \frac{1/2}{1+1/2+1}\cdot 5/2\cdot 1/2 = 1/4,$$

C_Φ is not hyponormal.

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DEPARTMENT OF MATHEMATICS
PURDUE UNIVERSITY
WEST LAFAYETTE, INDIANA 47 907
U.S.A.

Current address
DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF JAMMU, NEW CAMPUS
JAMMU — 180 004, INDIA.

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On the coadjoint orbits of connected Lie groups

L. PUKANSZKY

Introduction. Let G be a connected Lie group with the Lie algebra $\mathfrak{g}$, O an orbit, of positive dimension, of the coadjoint representation and ω_0 the corresponding canonical 2-form (cf. [2], Proposition 5.2.2, p. 182). It is well-known, that pairs like (O, ω_O) play an important role in many questions of the unitary representation theory of G. The objective of the present paper is to analyse (O, ω_0) by aid of suitable ideals of $\mathfrak{g}$. In more details, given an element g of O, we define $B(x, y) \equiv ([x, y], g)$ $(x, y \in \mathfrak{g})$. Let $\mathfrak{m}$ be an ideal of $\mathfrak{g}$, different from $\mathfrak{g}$. We say, that it is admissible, if it contains its orthogonal complement, with respect to B, for one and hence for all g of O. Such ideals always exist if $\mathfrak{g}$ is nilpotent, and are of a common occurrence when $\mathfrak{g}$ is solvable (cf. Section 4 below). Let $\mathcal{O}$ be the projection of O on $\mathfrak{m}^*$, the dual of the underlying space of $\mathfrak{m}$. Then $\mathfrak{m}$ determines a subbundle $\mathfrak{M}$ of the tangent bundle $T(O)$ of O. Let O' be the subbundle, orthogonal to $\mathfrak{M}$, of the cotangent bundle $T^*(O)$ of O. O and O' carry canonically the structure of a principal bundle, with the structure group $\mathfrak{m}^\perp$, over $\mathcal{O}$; O is acted upon by $\mathfrak{m}^\perp$ through translations and both bundles are trivial. Let s be a global section of O; it determines an isomorphism φ of principal $\mathfrak{m}^\perp$-bundles over $\mathcal{O}$, from O onto O' (cf. Lemma 9 and Lemma 11 in Section 2). We set $\eta = s^*\omega_O \in Z^2(O)$ and write p for the canonical projection from O' onto O. Let ϑ be the canonical 1-form on $T^*(O)$. Our principal result (cf. Theorem 1 in Section 3) states, that

$$\omega_O = \varphi^*(p^*\eta - d\vartheta).$$

As an application, in Section 4 we give a new proof for the existence of global Darboux coordinates in the case, when G is is solvable and O is simply connected (cf. [5], Theorem 3, p. 208).

The organization of the paper is as follows. Section 1 discusses the bundle structure of O, and Section 2 the relation of O to O'. Section 3 contains the proof of the result quoted above, and Section 4 the discussion of the Darboux coodordinates.

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The reader is advised to consult the end of the paper, where some key notational conventions, employed throughout the paper, are explained.

1. As stated above, the objective of this section is the investigation of some bundle structure on O. The proof of the principal statement (cf. Proposition) could be abbreviated by the use of standard results (cf. in particular [1], 16.14.1, p. 87) but some elements of the proof below will be needed later.

Let G be a connected Lie group with the Lie algebra $\mathfrak{g}$. If $\mathfrak{a}$ is a subspace of g, $\mathfrak{a}_B^\perp \subset \mathfrak{g}$ will stand for its orthogonal complement with respect to B, belonging to some $g \in \mathfrak{g}^*$ specified by the context and $\mathfrak{a}^\perp$ for the orthogonal complement in $\mathfrak{g}^*$. We fix an orbit O, of positive dimension, of the coadjoint representation and an ideal $\mathfrak{m}$, admissible with respect to O, that is $\mathfrak{m}_B^\perp \subset \mathfrak{m}$ for one, and thus for all elements of O. Fixing an element g of O, we set $K = G_g$, and consider O as a C^∞-manifold by transfer from G/K. Let us note, that the identity map from O into $\mathfrak{g}^*$ is smooth. We write h for $g|\mathfrak{m}$, and set $T = G_h$; $\mathcal{O}$ has a differentiable structure as G/T. Let π be the restriction map $\mathfrak{g}^* \to \mathfrak{m}^*$. We recall (cf. [1], 16.14.9, p. 94) that with the above definitions $(O, \mathcal{O}, \pi)$ is a fiber bundle with a fiber diffeomorphic to T/K. In the following we show that this fibration is identical with the orbit space of $\mathfrak{m}^\perp$, acting on O by translation.

Lemma 1. *With the above notation we have:* $(G_h)_0\, g = g + \mathfrak{m}^\perp$.

Proof. (i) For $n = 2, 3, \ldots$, let $\{l_j : 1 \leqq j \leqq n\}$ be some subset of $\mathfrak{g}_h = \mathfrak{m}_B^\perp$. We claim, that $l_1 \ldots l_n g = 0$. In fact, let L be the left-hand side. Given an element $k \in \mathfrak{g}$, we put $l = (-1)^n [l_{n-1} \ldots [l, k] \ldots]$. Since $n \geqq 2$ and $\mathfrak{g}_h \subset \mathfrak{m}$, l belongs to $\mathfrak{m}$, and thus we conclude that $(k, L) = ([l_n, l], g) = 0$ by virtue of $l_n \in \mathfrak{g}_h$. Since $(G_h)_0$ is generated by elements of the form $\exp(l)$ $(l \in \mathfrak{g}_h)$ we conclude that $(G_h)_0 g = g + \mathfrak{g}_h g$. — (ii) This being so it is enough to prove that if $\mathfrak{m}$ is an ideal of $\mathfrak{g}$ containing $\mathfrak{g}_g$, we have $\mathfrak{m}^\perp = \mathfrak{g}_h g$. Note, that if $\mathfrak{a}$ is a subspace of $\mathfrak{m}$, then $\mathfrak{a}_B^\perp = (\mathfrak{a} g)^\perp$. We have therefore

$$\mathfrak{m} = \mathfrak{m} + \mathfrak{g}_g = (\mathfrak{g}_h)_B^\perp = (\mathfrak{g}_h g)^\perp,$$

whence $\mathfrak{g}_h g = \mathfrak{m}^\perp$. Summing up, we have proved that

$$(G_h)_0 g = g + \mathfrak{m}^\perp.$$

From here we can conclude

Lemma 2. *The triple* $(O, \mathcal{O}, \pi)$ *is a principal* $\mathfrak{m}^\perp$*-space.*

Lemma 3. *The map* $t \mapsto tg$ $(t \in T)$ *induces a diffeomorphism* $T/K \cong g + \mathfrak{m}^\perp$.

Proof. We recall (cf. [1], 16.10.7, p. 62) that if G acts smoothly on the C^∞-manifold X, and $x \in X$ is such that Gx is locally closed, then Gx carries a differentiable structure, well-determined by the condition that $s \mapsto sx$ be a diffeomorphism

from G/G_x onto Gx. We apply this by replacing X, G, x through $\mathfrak{g}^*$, G_h and g respectively. To conclude our proof it is enough to note that, by Lemma 1, we have: $Tg=g+m^\perp$, which is closed in $\mathfrak{g}^*$.

Lemma 4. *There is a global section* $s\colon \mathcal{O}\mapsto O$.

Proof. We recall (cf. [1], 16.12.2, p. 82) that if (X, B, π) is a fiber bundle, with a fiber diffeomorphic to $\mathbf{R}^N$, then there is a global section $s\colon B\mapsto X$. Thus it is enough to note that in our case, by Lemma 3, we have $T/K\cong g+\mathfrak{m}^\perp$.

For a fixed $s\in\Gamma(O)$, we define $f\colon O\mapsto\mathfrak{g}^*$ by $f(g)\equiv g-s(\pi(g))$. We can note at once that f is smooth, takes its values in $\mathfrak{m}^\perp$ and satisfies $f(g+v)=f(g)+v$ for any $g\in O$ and $v\in\mathfrak{m}^\perp$. We set $X=\mathcal{O}\times\mathfrak{m}^\perp$ and define $\Psi\colon O\mapsto X$ by $\Psi(g)=$ $=\pi\{(g),f(g)\}$ $(g\in O)$.

Lemma 5. *Ψ is a smooth bijection $O\to X$.*

Proof. Smoothness being evident, it is enough to show that is bijective. In fact, (i) Assume, that $\Psi(g)=\Psi(g')$. Then, in particular, $\pi(g)=\pi(g')$ and thus $g'=$ $=g+v$ with some $v\in\mathfrak{m}^\perp$. We have, however, also $f(g)=f(g')=f(g)+v$ and hence $v=0$ and $g=g'$. — (ii) We claim that Ψ is surjective. In fact, let $\{h, w\}\in X$ be given. Suppose that $g\in O$ satisfies $\pi(g)=h$. Defining $g'=g+w-f(g)$ we have clearly $\Psi(g')=\{h, w\}$. Summing up, we have shown, that Ψ is a smooth bijection $O\to X$. We recall that $K=G_g$, $h=g|m$ and $T=G_h$.

Lemma 6. *The restriction of the canonical map $G/K\to O$ to a fiber of $G/K\to$ $\to G/T$ is an isomorphism of this fiber to an $\mathfrak{m}^\perp$-orbit of O (the latter considered as a submanifold of g^*).*

Proof. Suppose that $g'\in O$ is given, and, say, $g'=ag$ $(a\in G)$. Then $a(T/K)$ is the fiber corresponding to g'. It is enough to show that the map $t\mapsto atg$ $(t\in T)$ induces an isomorphism $T/K\to g'+\mathfrak{m}^\perp$. But, by Lemma 3, the map of loc. cit. (h, say) from T/K onto $g+\mathfrak{m}^\perp$ induces an isomorphism and thus it suffices to observe that

$$\begin{tikzcd} a(T/K) \arrow[r] & g'+\mathfrak{m}^\perp \\ T/K \arrow[u, "a(\ )"] \arrow[r, "h"'] & g+\mathfrak{m}^\perp. \arrow[u, "a(\ )"'] \end{tikzcd}$$

Lemma 7. *With the above notation $\Psi\colon O\to X$ is an isomorphism of fiber bundles.*

Proof. By Lemma 5, Ψ is a smooth fiber preserving bijection $\Psi\colon O\to X$ and by Lemma 6, the restriction of Ψ to any fiber in O is an isomorphism with its image. Thus it is enough to recall (cf. [1], 16.21.2, p. 75) that (in particular) if (X, B, π) and (X', B, π') are fiber bundles and $f\colon X\to X'$ is a fiber-preserving smooth map, then it

is an isomorphism of fiber bundles, if its restriction to any fiber is an isomorphism with its image.

Proposition. *$(O, \mathcal{O}, \pi)$ is a principal bundle with the structure group $\mathfrak{m}^{\perp}$ acting on O by translations.*

Proof. By what we have seen above, it is enough to observe that Ψ is equivariant with respect to the action of $\mathfrak{m}^{\perp}$ on O and X respectively.

2. The objective of this section is to present some material needed in the next section for the proof of Theorem 1. We continue to assume that O is a fixed orbit, of positive dimension, of the coadjoint representation and $\mathfrak{m}$ is an admissible ideal (cf. Introduction). We start by introducing some notational conventions. 1) If Y is a left G-space, $m \in Y$, and $x \in \mathfrak{g} = \mathrm{Lie}\,(G)$, we set

$$\sigma_m(x) = (d/dt)\exp(tx)\,m|_{t=0}.$$

Given $g \in \mathfrak{g}^*$, we denote by τ_g the canonical translation $T_g(\mathfrak{g}^*) \to \mathfrak{g}^*$ (cf. [1], p. 22). Note that we have clearly: $\tau_g \sigma_g(x) = xg$. 3) With the above notation we can write for $x, y \in \mathfrak{g}$:

$$\omega_O(\sigma_g(x) \wedge \sigma_g(y)) = B(x, y).$$

We remark, that if $t = \sigma_g(x)$ and $v \in T_g(O)$, then $\omega_O(t \wedge v) = (x, \tau_g v)$. In fact, assuming $v = \sigma_g(y)$ we have $\omega_O(t \wedge v) = ([x, y], g) = (x, yg) = (x, \tau_g v)$. We denote by $\mathfrak{N}$ the distribution on O such that $\tau_g N_g = \mathfrak{m}g$. Let us observe, that if $v_g \in \mathfrak{N}$ and $t \in T_g(O)$ is such that $T_g(\pi)t = 0$, then $\omega_O(t \wedge v_g) = 0$. In fact, assuming $t = \sigma_g(x)$, we have $0 = T_g(\pi)t = \sigma_h(x)$, whence $x \in \mathfrak{g}_h = \mathfrak{m}_B^{\perp} \subseteq \mathfrak{m}$. If $v_g = \sigma_g(y)$ for a $y \in \mathfrak{m}$ we have: $\omega_O(t \wedge v_g) = ([x, y], g) = 0$ by $x \in \mathfrak{g}_h$. We conclude from all this that there is a map $P\colon \mathfrak{N}_g \to (T_h(\mathcal{O}))^*$ such that $P(v_g)(T_g(\pi)t) = \omega_O(t \wedge v_g)$ $(t \in T_g(O))$. Writing $p\colon T^*(\mathcal{O}) \to \mathcal{O}$ and $pr\colon T(O) \to O$ for the canonical projections, we note

$$\begin{array}{ccc} \mathfrak{N} & \xrightarrow{P} & T^*(\mathcal{O}) \\ {\scriptstyle pr}\downarrow & & \downarrow{\scriptstyle p} \\ O & \xrightarrow[\pi]{} & \mathcal{O}. \end{array}$$

Let σ be a section $\mathcal{O} \to O$ (cf. Lemma 4) and form as loc. cit. $f(g) \equiv g - \sigma(\pi(g))$ $(g \in O)$. If f is any smooth map $O \to \mathfrak{m}^{\perp}$ we can define $F(g) \in T_g(\mathfrak{g}^*)$ by $\tau_g F(g) = f(g)$, and note that F is a vector field on O taking its values in $\mathfrak{N}$. In fact, to see this, it is enough to have $\mathfrak{m}^{\perp} \subseteq \mathfrak{m}g$; but this is equivalent to $\mathfrak{m}_B^{\perp} = (\mathfrak{m}g)^{\perp} \subset \mathfrak{m}$ or $\mathfrak{m}_B^{\perp} \subset \mathfrak{m}$, which we assume. All this being so, for $g \in O$ we set: $\varphi(g) = P(F(g))$; we have clearly

$$\begin{array}{ccc} O & \xrightarrow{\varphi} & T^*(\mathcal{O}) \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle p} & \\ & \mathcal{O}. & \end{array}$$

Lemma 8. *With notation as above, we have for* $g\in O$ *and* $x\in\mathfrak{g}$:

$$\varphi(g)(\sigma_h(x)) = (x, f(g)).$$

Proof. Writing $t=\sigma_g(x)$, we obtain

$$\varphi(g)(\sigma_h(x)) = P(F(g))(T_g(\pi)t) = \omega_O(t\wedge F(g)) = (x, f(g))$$

and thus: $\varphi(g)(\sigma_h(x))=(x, f(g))$ $(g\in O, x\in\mathfrak{g})$.

We denote by $\mathfrak{M}$ the subbundle of $T(\mathcal{O})$ such that $\tau_h M_h=\mathfrak{m}h$. Recalling, that $X=\mathcal{O}\times\mathfrak{m}^\perp$, we note that there is a canonical identification between $\mathfrak{M}^\perp$ and X. In fact, given $\lambda\in\mathfrak{M}^\perp$ let us put $h=p(\lambda)$. We define $\lambda'\in\mathfrak{m}^\perp$ by $\lambda'(x)\equiv\lambda(\sigma_h(x))$ $(x\in\mathfrak{g})$. This being so, we set $\Phi(\lambda)=\{h, \lambda'\}$. We observe that Φ is a bijection $\mathfrak{M}^\perp\to X$. In fact, if $\Phi(\mu)=\Phi(\nu)=\{h, \lambda\}$, say, we have $\mu, \nu\in(T_h(\mathcal{O}))^*$ and $\mu(\sigma_h(x))\equiv\equiv\lambda(x)\equiv\nu(\sigma_h(x))$ $(x\in\mathfrak{g})$, and thus $\mu=\nu$ and Φ is injective. Let now $\{h, \lambda'\}\in X$ be given. If $\sigma_h(x)=0$, we have $x\in\mathfrak{g}_h\subseteq\mathfrak{m}$ and hence we can define $\lambda\in(T_h(\mathcal{O}))^*$ by $\lambda(\sigma_h(x))\equiv\lambda'(x)$ $(x\in\mathfrak{g})$. In this fashion $\Phi(\lambda)=\{h, \lambda'\}$, and Φ is surjective. Below we shall write O' for $\mathfrak{M}^\perp$. We can define on O' the structure of a principal $\mathfrak{m}^\perp$-bundle as follows. Given $v\in\mathfrak{m}^\perp$, let $A_h(v)\in(T_h(\mathcal{O}))^*$ such that $A_h(v)(\sigma_h(x))\equiv\equiv(x, v)$ $(x\in\mathfrak{g})$. Then if $\lambda\in O'$ and $p(\lambda)=h$, we can set $\lambda v=\lambda+A_h(v)$. We note, that $\Phi(\lambda v)=\Phi(\lambda)v$. — We remark that if $g\in O$, then we have: $\varphi(g)\in O'$. In fact, Lemma 8 implies, that $\varphi(g)(\sigma_h(x))\equiv(x, f(g))$ $(x\in\mathfrak{g})$; but by $f(g)\in\mathfrak{m}^\perp$, the right-hand-side vanishes for $x\in\mathfrak{m}$.

Lemma 9. $\varphi: O\to O'$ *is an isomorphism of principal* $\mathfrak{m}^\perp$*-bundles over* $\mathcal{O}$.

Proof. Let $\Psi: O\to X$ be as in Lemma 5, corresponding to the section $\mathcal{O}\to O$ employed in the definition of φ. To obtain the desired conclusion, it is enough to note that clearly $\Phi\circ\varphi=\Psi$.

Lemma 10. *Let* ϑ *be the canonical* 1*-form on* $T^*(\mathcal{O})$. *Then, with notation as above, we have:* $\varphi^*\vartheta=-\iota(F)\omega_O$.

Proof. Assume that $t\in T_g(O)$. We have

$$\begin{aligned}(\varphi^*\vartheta)(t) &= \vartheta(T_g(\varphi)t) = (T_{\varphi(g)}(p)T_g(\varphi)t, \varphi(g)) = \\ &= \varphi(g)(T_g(p\circ\varphi)t) = \varphi(g)(T_g(\pi)t) = P(F(g))(T_g(\pi)t) = \\ &= \omega_O(t\wedge F(g)) = -(\iota(F)\omega_O)(t)\end{aligned}$$

whence $\varphi^*\vartheta=-\iota(F)\omega_O$.

Lemma 11. *Let* $\varphi: O\to O'$ *be an isomorphism of principal* $\mathfrak{m}^\perp$*-bundles over* $\mathcal{O}$. *Then there is a section* $s\in\Gamma(O)$ *giving rise to* φ *as described before Lemma 8.*

Proof. (i) Given $t\in T_g(O)$, by virtue of the computation of the proof of Lemma 10 we have: $(\varphi^*\vartheta)(t)=\varphi(g)(T_g(\pi)t)$. — (ii) We define the vector field F on O by

$\varphi^*\vartheta = -\iota(F)\omega_O$ and set $f(g) \equiv \tau_g(F(g))$. We claim, that for all $x \in g$: $\varphi(g)(\sigma_h(x)) = (x, f(g))$. In fact, writing $t = \sigma_g(x)$ we have

$$\varphi(g)(\sigma_h(x)) = \varphi(g)(T_g(\pi)t) = (\varphi^*\vartheta)(t) = \omega_O(t \wedge F(g)) = (x, f(g))$$

whence $\varphi(g)(\sigma_h(x)) \equiv (x, f(g))$, as stated above. — (iii) a) We observe that f takes its values in $\mathfrak{m}^\perp$. In fact, we have $\varphi(g) \in O'$ and thus, by (ii) above: $(x, f(g)) = \varphi(g)(\sigma_h(x)) = 0$ for all $x \in \mathfrak{m}$. b) We note that for any $g \in O$ and $v \in \mathfrak{m}^\perp$: $f(g+v) = f(g) + v$. In fact, we have for all $x \in \mathfrak{g}$:

$$(x, f(g+v)) = \varphi(g+v)(\sigma_h(x)) =$$

$$= \varphi(g)(\sigma_h(x)) + A_h(v)(\sigma_h(x)) = (x, f(g)) + (x, v) = (x, f(g) + v)$$

and thus $f(g+v) = f(g) + v$. In this manner we can define $s \in \Gamma(O)$ by $s(\pi(g)) \equiv g - f(g)$ $(g \in O)$. — (iv) We observe that $F(g) \in N_g$. We have, in fact $\tau_g F(g) = f(g) \in \mathfrak{m}^\perp \subset \mathfrak{m}g$, since $\mathfrak{m}$ is admissible with respect to O. In this fashion we can form $\psi(g) \equiv P(F(g))$. — (v) We show finally, that $\varphi = \psi$. In fact, we have by Lemma 8 and (ii) above: $\psi(g)(\sigma_h(x)) = (x, f(g)) = \varphi(\sigma_h(x))$ $(x \in \mathfrak{g})$, providing the desired conclusion.

3. The principal objective of this section is Theorem 1. We start with the following definition. Let us write $\mathfrak{b}$ for the quotient algebra $\mathfrak{g}/\mathfrak{m}$ and α for the canonical morphism $\mathfrak{g} \to \mathfrak{b}$. Given $x \in \mathfrak{g}$, we write X for the vector field on O satisfying $X_g = \sigma_g(x)$. This being so, we define the $\mathfrak{b}$-valued 1-form δ by $\delta(t_g) = \alpha(x)$, if $t_g = \sigma_g(x)$.

Lemma 12. *With the above notation we have:* $d\delta = [\delta, \delta]$

Proof. Let t, t' be in $T_g(O)$, $t = \sigma_g(x)$, $t' = \sigma_g(y)$, say $(x, y \in \mathfrak{g})$. We have

$$d\delta(t \wedge t') = d\delta(X_g \wedge Y_g) = X_g\delta(Y) - Y_g\delta(X) - \delta([X, Y]_g).$$

But $\delta(Y_g) \equiv \delta(\sigma_g(y)) \equiv \alpha(y)$ and thus $X_g(\delta(Y)) \equiv 0$ and similarly, $Y_g\delta(X) \equiv 0$. Writing $z = [x, y]$, we have $Z = -[X, Y]$. From this we conclude that

$$d\delta(t \wedge t') = \delta(Z_g) = \alpha(z) = \alpha([x, y]) = [\alpha(x), \alpha(y)] = [\delta(t), \delta(t')],$$

and therefore: $d\delta(t \wedge t') = [\delta(t), \delta(t')]$ $(t, t' \in T_g(O))$.

We note that there is a canonical identification between the dual $\mathfrak{b}^*$ and $\mathfrak{m}^\perp$. Given a $\mathfrak{b}$-valued k-form γ on O, and a smooth map f: $O \to \mathfrak{m}^\perp$, we shall write γ_f for the numerical-valued k-form defined at $g \in O$ by $\gamma_f(\ ) = (\gamma(\ ), f(g))$. In particular, if $f(g) \equiv v \in \mathfrak{m}^\perp$ is fixed, we write γ_v for γ_f. — Below, given $v \in \mathfrak{m}^\perp$, we denote by L_v the map $L_v g = g + v$ $(g \in O)$.

Lemma 13. *With the above notation we have*

$$L_v^* \omega_O = \omega_O - (d\delta)_v.$$

Proof. Let t and t' be in $T_g(O)$ such that $t=\sigma_g(x)$, $t'=\sigma_g(y)$, say. There is an $\bar{x}\in\mathfrak{g}$ such that

$$xg = \tau_g t = \tau_{g+v}(T_g(L_v)t) = \bar{x}(g-v)$$

and analogously for y. From this we conclude, that

$$(L_v^*\omega_O)(t\wedge t') = \omega_O(T_g(L_v)t\wedge T_g(L_v)t') =$$
$$= ([\bar{x}, \bar{y}], g+v) = (\bar{x}, \bar{y}(g+v)) = (\bar{x}, yg) = ([\bar{x}, y], g)$$

and therefore: $(L_v^*\omega_O-\omega_O)(t\wedge t')=([\bar{x}-x, y], g)$. In this manner it will be enough to show that $([\bar{x}-x, y], g)=-(d\delta)_v(t\wedge t')$. To this end we note that a) $\bar{x}-x\in\mathfrak{g}_h$. In fact, we have by definition: $(\bar{x}-x)g=-\bar{x}v$, and thus for all $l\in\mathfrak{m}$: $([x-x, l], g)=$ $=(l, x\bar{v})=0$. Next we note, that $xv=\bar{x}v$. In fact, to see this, by a) it suffices to observe that $av=0$ for all $a\in\mathfrak{g}_n$. In this fashion we can conclude, that

$$([\bar{x}-x, y], g) = -(y, (\bar{x}-x)g) = (y, \bar{x}v) = (y, xv) =$$
$$= -([x, y], v) = -([\delta(t), \delta(t')], v) = -(d\delta)_v(t\wedge t')$$

where we have made use of Lemma 12. Summing up, we have thus obtained $L_v^*\omega_O=$ $=\omega_O-(d\delta)_v$, as claimed at the beginning.

Since, as we have seen in Section 1, O is a principal bundle with the structure group $\mathfrak{m}^\perp$, below, whenever convenient, we shall write gv in place of $g+v=L_v g$ ($v\in\mathfrak{m}^\perp$). Note that gv can stand also for

$$(d/d\tau)(g+\tau v)|_{v=0}\in T_g(O).$$

Let f: $O\to\mathfrak{m}^\perp$ be a smooth map satisfying $f(g+v)\equiv f(g)+v$ ($g\in O$, $v\in\mathfrak{m}^\perp$). We define the $\mathfrak{m}^\perp$-valued 1-form ζ by $\zeta(t)=\tau_{f(g)}(T_g(f)t)$. We have for $g\in O$, $t\in T_g(O)$ and $v\in\mathfrak{m}^\perp$: 1) $\zeta_{gv}(T_g(L_v)t)=\zeta_g(t)$, 2) $\zeta_g(gv)=v$. In this manner ζ defines a connection form on the principal $\mathfrak{m}^\perp$-bundle $(O, \mathcal{O}, \pi)$. We shall write $V_g(O)$ for the collection of all vertical vectors at g that is $V_g(O)=\{t;\ t\in T_g(O)$ such that $T_g(\pi)=0\}$. We recall that the dual $\mathfrak{b}^*$ of $\mathfrak{b}$ is canonically identifiable with $\mathfrak{m}^\perp$.

Lemma 14. *We have for* $t\in T_g(O)$, $w\in V_g(O)$:

$$\omega_O(t\wedge w) = (\delta(t), \zeta(v)).$$

Proof. To this end it is enough to note that, if $t=\sigma_g(x)$ and $w=\sigma_g(y)$ ($y\in\mathfrak{g}_h$), then we have

$$\omega_O(t\wedge w) = (x, yg) = (\delta(t), \zeta(w)).$$

Let us observe that, in particular, $V_g(O)$ is orthogonal to itself with respect to $(\omega_O)_g$.

Below we assume to be given a fixed choice of $s\in\Gamma(O)$; $\varphi: O\to O'$ will correspond to it as preceeding Lemma 8. — We recall that ϑ is the canonical 1-form on $T^*(O)$.

Lemma 15. *With the previous notation we have:* $\varphi^*\vartheta=\delta_f$.

Proof. Let $t\in T_g(O)$ be such that $t=\sigma_g(x)$ $(x\in\mathfrak{g})$. We have, as in the proof of Lemma 10, using Lemma 8,

$$(\varphi^*\vartheta)(t) = \varphi(g)(\sigma_h(x)) = (x, f(g)) = \delta_f(t)$$

and hence $(\varphi^*\vartheta)(t)=\delta_f(t)$.

Theorem 1. *With the previous noration let us put* $\eta=s^*\omega_O\in Z^2(\mathcal{O})$. *Then we have* $\omega_O=\varphi^*(p^*\eta-d\vartheta)$.

Proof. (i) Writing $L=\varphi^*p^*\eta\in Z^2(O)$, we have:

$$L = (p\circ\varphi)^*\eta = \tau^*s^*\omega_O = (s\circ\pi)^*\omega_O.$$

(ii) We have, by virtue of the flat connection, corresponding to ζ on the principal $\mathfrak{m}^\perp$-bundle $(O, \mathcal{O}, \pi)$, the following representation of $t_g\in T_g(O)$ as the sum of horizontal and vertical components

$$t_g = (T_g(s\circ\pi)(t_g))f(g)+g\zeta(t_g).$$

Denoting by P the horizontal projection, we thus obtain:

$$T_g(s\circ\pi)\,t_g = T_g(L_{-f(g)})(Pt_g).$$

Given $t, t'\in T_g(O)$, we have by Lemma 13:

$$L(t\wedge t') = \omega_O(T_g(L_{-f(g)})\,Pt\wedge T_g(L_{-f(g)})\,Pt') = (\omega_O)_P(t\wedge t')+(d\delta)_f(Pt\wedge Pt').$$

(iii) We claim that $d\delta(Pt\wedge Pt')=d\delta(t\wedge t')$. In fact, we have by Lemma 12: $d\delta(Pt\wedge Pt')=[\delta(Pt), \delta(Pt')]$ and thus it suffices to show, that $\delta(t)=\delta(Pt)$, or that $\delta(t)=0$ if $t\in V_g(O)$. To see this we can assume that $t=\sigma_g(x)$ $(x\in\mathfrak{g}_h)$. But then $\delta(t)=\alpha(x)=0$, by $\mathfrak{g}_h\subseteq\mathfrak{m}=\ker(\alpha)$. In this manner, by the end of (ii) above we obtain:

$$L(t\wedge t') = (\omega_O)_P(t\wedge t')+(d\delta)_f(t\wedge t').$$

(iv) For the definition to be used below, of the wedge product between two vector-valued 1-forms we refer to [1], 16.20.15.5, p. 141. — We maintain, that

$$(\omega_O)_P = \omega_O+\zeta\wedge\delta.$$

In fact, let us write P_v for the vertical projection. We have by Lemma 14:

$$(\omega_O)_P(t\wedge t') = \omega_O(Pt\wedge Pt') = \omega_O\big((t-P_v(t))\wedge(t'-P_v(t'))\big) =$$
$$= \omega_O(t\wedge t') - \omega_O\big(t\wedge P_v(t)\big) - \omega_O\big(P_v(t)\wedge t'\big) =$$
$$= \omega_O(t\wedge t') - \big(\delta(t), \zeta(t')\big) + \big(\delta(t'), \zeta(t)\big) = \omega_O(t\wedge t') + (\zeta\wedge\delta)(t\wedge t')$$

or $(\omega_O)_P(t\wedge t') = \omega_O(t\wedge t') + (\zeta\wedge\delta)(t\wedge t')$, proving our assertion. In this fashion we can conclude that $L = \omega_O + \zeta\wedge\delta + (d\delta)_f$. — (v) We assert next that $d(\delta_f) = \zeta\wedge\delta + (d\delta)_f$. In fact, this is implied by the following simple proposition. Let V be a real vector space of dimension m, M a C^∞-manifold, γ a V-valued 1-form and $f\colon M\to V^*$ a smooth map. Then we have $d(\gamma_f) = df\wedge\gamma + (d\gamma)_f$. In fact, let (v_j) be a basis in V and (v_j^*) the dual basis. Then we can write

$$\gamma = \sum_{j=1}^m \gamma_j v_j, \quad f = \sum_{j=1}^m f_j v_j^* \quad \text{where} \quad (\gamma_j)\subset \mathscr{E}_1(M).$$

We have

$$\gamma_f = \sum_{j=1}^m f_j\gamma_j \quad \text{and thus} \quad d(\gamma_f) = \sum_{j=1}^m (df_j\wedge v_j) + \sum_{j=1}^m f_j\cdot d\gamma_j.$$

Hence it is enough to note that for any pair h, k of tangent vectors we have: $(df\wedge\gamma)(h\wedge k) = \big(df(h), \gamma(k)\big) - \big(df(k), \gamma(h)\big)$, which concludes our proof. — (vi) Summing up, we have by (iv)—(v) above: $L = \omega_O + d(\delta_f)$. Lemma 15 asserts that $\delta_f = \varphi^*\vartheta$ and thus $d(\delta_f) = \varphi^*(d\vartheta)$. Since $L = \varphi^* p^*\eta$ we get finally

$$\omega_O = \varphi^*(p^*\eta - d\vartheta)$$

as claimed in Theorem 1.

4. The objective of this concluding section is an alternative approach to the following result, first proved in [5], Theorem 3, p. 208.

Theorem 2. *Let O be a simply connected coadjoint orbit of the connected and simply connected solvable Lie group G. Then there is a diffeomorphism $\beta\colon R^d\to O$ such that $\beta^*\omega_O$ is constant.*

Proof. We denote by $\mathfrak{g}$ the Lie algebra of G and proceed by induction according to $\dim(\mathfrak{g})$. We distinguish the following two major possibilities:

A. There is an ideal $\mathfrak{m}$ of codimension one such that $\mathfrak{g} = \mathfrak{g}_g + \mathfrak{m}$ for some g in O. Let π be the restriction map $g^*\to m^*$, and M the connected subgroup of G determined by $\mathfrak{m}$. Then $\pi(O)$ is a coadjoint orbit of M and $\pi|O$ is a diffeomorphism $O\to\pi(O)$. We have, in addition, that $\omega_O = (\pi|O)^*\omega_{\pi(O)}$. By virtue of the assumption of our induction there is a diffeomorphism $\gamma\colon R^\delta\to\pi(O)$ such that $\gamma^*(\omega_{\pi(O)})$ is constant. But then it is enough to take $\beta = (\pi^{-1}|O)\circ\gamma$.

B. Here we assume that the hypothesis of A cannot be realized. Let $\mathfrak{m}$ be a fixed ideal of codimension one. Then, for any $g\in\mathfrak{g}^*$, we have $\mathfrak{g}_g\subseteq\mathfrak{m}$. Putting $h = g|\mathfrak{m}$

we claim that $\mathfrak{g}_h \subseteq \mathfrak{m}$. In fact, if k is in $\mathfrak{g}_h - \mathfrak{m}$, k is orthogonal, with respect to B belonging to g, to $\mathfrak{m}$. But then k is orthogonal to $\mathfrak{g}$ and thus $k \in \mathfrak{g}_g$ and $\mathfrak{g} = \mathfrak{m} + \mathfrak{g}_g$, contrary to our assumption. We note, in particular, that in this case $\mathfrak{m}$ is admissible with respect to O.

(i) We fix an element y in O, and write $x = \pi(y)$ and $O_0 = Mx \subseteq \mathfrak{m}^*$. We claim that O_0 is simply connected. In fact, let us put $\mathcal{O} = \pi(O)$. We have, by Lemma 1, $O = \pi^{-1}(\mathcal{O})$, and thus $\mathcal{O}$ is simply connected and G_x is connected. But, by what we have seen above, $\mathfrak{g}_x \subseteq \mathfrak{m}$ and thus $G_x \subset M$, and $G_x = M_x$, and $O_0 = M/M_x$ is simply connected. — (ii) We omit the straightforward verification of the following result. Let G be an arbitrary connected and simply connected Lie group with the Lie algebra $\mathfrak{g}$. Let α be an automorphism of $\mathfrak{g}$; we set $\beta = (\alpha^{-1})^* \in \mathrm{End}\,(\mathfrak{g}^*)$. Then, if O is any coadjoint orbit, then so is $\beta(O)$ and $\beta^*(\omega_{\pi(O)}) = \omega_O$. — (iii) By virtue of the assumption of our inductive procedure, there is a diffeomorphism g_0 from R^δ onto O_0 such that $g_0^*(\omega_{O_0})$ is constant. We fix an element $k \in \mathfrak{g} - \mathfrak{m}$, write $\gamma(t) \equiv \exp(tk)$ and define a map $h\colon R^{\delta+1} \to \mathcal{O}$ by $h(t, T) \equiv \gamma(t) g_0(T)$ $(t \in \mathfrak{R}, T \in R^\delta)$. Then h is a diffeomorphism from $R^{\delta+1}$ onto $\mathcal{O}$. Let $\mathfrak{a}$ be the subspace spanned by k; we have $g = \mathfrak{m} + \mathfrak{a}$. Let j be the projection onto the second summand. We define $\iota\colon \mathfrak{m}^* \to \mathfrak{g}^*$ such for $h \in \mathfrak{m}^*$ we have $\iota(h)|\mathfrak{m} = h$ and $\iota(h)|\mathfrak{a} = 0$. We write ι also for $\iota|\mathcal{O} \in \Gamma(O)$ and set $\eta = \iota^* \omega_O$. In the following we shall proove that $h^*(\eta)$ is constant. a) Let $h \in O$ be fix and $g = \iota(h)$. Assuming that $t, t' \in T_h(O)$ are given and $t = \sigma_h(u)$, $t' = \sigma_h(v)$ $(u, v \in \mathfrak{g})$ we claim that $\eta(t \wedge t') = B(u, v) - B(ju, v) - B(u, jv)$. In fact, 1) we have for any real τ: $\exp(\tau u) g - i(\exp(\tau u) h) \in \mathfrak{m}^\perp$. Hence there is an $n \in \mathfrak{m}^\perp$ such that $ug = i(uh) + n$. 2) By virtue of (ii) in the proof of Lemma 1, we have $\mathfrak{g}_h g = \mathfrak{m}^\perp$, and thus there is $\bar{u} \in \mathfrak{g}_h$ with $n = \bar{u} g$. From this we can conclude that $\tau_g \iota_{*h}(t) = \iota(uh) = (u - \bar{u}) g$. Similarly, there is $\bar{v} \in \mathfrak{g}_h$ such that $\tau_g \iota_{*h}(t) = (v - \bar{v}) g$. 3) We conclude from this that

$$\eta(t \wedge t') = \omega_O(\iota_{*h}(t) \wedge \iota_{*h}(t')) = ([u - \bar{u}, v - \bar{v}], g) = ([u, v], g) - ([u, \bar{v}], g) - ([\bar{u}, v], g).$$

4) We note that $([u, \bar{v}], g) = (u, \bar{v} g) = (ju, \bar{v} g)$. But, by 2), $\bar{v} g = vg - i(uh)$ and the last term is orthogonal to $\mathfrak{a}$. Hence $([u, \bar{v}], g) = (ju, vg) = B(ju, v)$, and similarly $([\bar{u}, v], g) = B(u, jv)$. In this manner we obtain for $t = \sigma_h(u)$, $t' = \sigma_h(v)$: $\eta(t \wedge t') = B(u, v) - B(ju, v) - B(u, jv)$ as claimed above. — b) Let U be an M-orbit in $\mathcal{O}$. We claim that $(id_U)^* \eta = \omega_U$. In fact, suppose that $h \in U$ and $t, t' \in T_h(U)$. Then there are $u, v \in \mathfrak{m}$ such that $t = \sigma_h(u)$, $t' = \sigma_h(v)$. Since $ju = 0 = jv$, we have by a):

$$((id_U)^* \eta)(t \wedge t') = ([u, v], h) = \omega_U(t \wedge t')$$

and thus $(id_U)^* \eta = \omega_U$ as claimed above. For $T = (t_1, \ldots, t_\delta)$ we form vector fields on $\mathcal{O}$ by

$$D_0 = \partial/\partial t, \quad D_j = \partial/\partial t_j \quad (1 \leqq j \leqq \delta).$$

To prove that $h^*\eta$ is constant, it will be enough to show that $(h^*\eta)(D_i\wedge D_j)$ is constant for $0\leqq i,j\leqq\delta$. — c) We start by proving the last claim for i,j such that $1\leqq i,j\leqq\delta$. In fact, let h be an element of $\mathcal{O}$, $h=h(t,T)$, say. We write $h_0=h(0,T)\in O_0$, and thus $h=\gamma(t)h_0$. Putting $O_t=\gamma(t)O_0$ we recall (cf. (ii)), that this is an M-orbit in $\mathcal{O}$. We have also $D_j|_h=T_{h_0}(\gamma(t))(D_j|_{h_0})\in T_h(O_t)$. Using b) above we conclude from this that

$$\eta(D_i|_h\wedge D_j|_h) = \omega_{O_t}(D_i|_h\wedge D_j|_h) =$$
$$= \omega_{O_t}(T_{h_0}(\gamma(t))(D_i|_{h_0}))\wedge T_{h_0}(\gamma(t))(D_j|_{h_0}) = (\gamma(t)^*\omega_{O_t})(D_i|_{h_0}\wedge D_j|_{h_0}).$$

But the last expression, by virtue of (ii), is equal to

$$\omega_{O_0}(D_i|_{h_0}\wedge D_j|_{h_0})$$

which, by the choice of g_0: $R^\delta\to O_0$ is constant, as h_0 varies over O_0. — d) We claim now that $\eta(D_0|_h\wedge D_j|_h)\equiv 0$ $(1\leqq j\leqq\delta)$. To this end it is enough to show that $\eta(\sigma_h(k)\wedge\sigma_h(u))\equiv 0$ if $u\in\mathfrak{m}$. But, by $jk=k$ and $ju=0$ this is implied by a). In this manner we have completed proving that $h^*\eta$ is constant, as we claimed at the start of (iii). — (iv) Let $\zeta\in\mathcal{E}_1(\mathcal{O})$ be such that $h^*(\zeta)=dt$. We define f: $R^{\delta+2}\to O'$ by $f(u,t,T)\equiv(h(t,T),u\zeta)$. Then f is a diffeomorphism from $R^{\delta+2}$ onto O'. Also, $\vartheta'=u\zeta$ is the pullback of the canonical 1-form on $T^*(\mathcal{O})$, to O'. By virtue of what we have seen in (iii), $f^*(p^*\eta-d\vartheta')$ is constant. — (v) We recall that by Theorem 1, there is a diffeomorphism φ: $O\to O'$ such that $\omega_O=\varphi^*(p^*\eta-d\vartheta')$. Hence $\beta=\varphi^{-1}\circ f$ is a diffeomorphism $R^d\to O$ such that $\beta^*\omega_O$ is constant, completing the proof of Theorem 2.

Some notational conventions. 1) Given a Lie group G with the Lie algebra $\mathfrak{g}$, $\mathfrak{g}$ is considered as a G-module with respect to the adjoint representation. Similarly, $\mathfrak{g}$ is a $\mathfrak{g}$-module with respect to the adjoint representation of $\mathfrak{g}$. Also $\mathfrak{g}^*$, the dual of the underlying space of $\mathfrak{g}$, is a G or $\mathfrak{g}$-module with respect to the coadjoint representation and its differential respectively. — 2) If a Lie group G acts smoothly on a C^∞-manifold X, for $x\in X$, G_x stands for the stabilizer of x in G, and $\mathfrak{g}_x$ for the subalgebra corresponding to G_x. — 3) A distribution on X will be denoted by a capital German letter. If $\mathfrak{M}$ is such, $M_x\subset T_x(X)$ will denote its value at x X. — 4) Given a principal bundle $\mathcal{B}$ with the structure group G, given $x\in B$ and $g\in G$, we shall write sometimes xg even if the action of g derives from an abelian group structure.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF PENNSYLVANIA
PHILADELPHIA, PA., U. S. A.

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Compatibility and incompatibility of Calkin equivalence with the Nagy—Foias calculus

J. ESTERLE and F. ZAROUF

Introduction. We have shown in [2] that there exist two absolutely continuous (a.c.) contractions T_1 and T_2 which commute and are such that T_2-T_1 is compact, and such that there exists a function h in H^∞ with $h(T_2)-h(T_1)$ not compact.

In this article, we give sufficient conditions on T_1 and T_2 which guarantee that $h(T_2)-h(T_1)$ is compact for any h in H^∞. In the particular case where T_1 is a diagonal operator whose eigenvalues are simple we characterize the a.c. T_2 which commute with T_1 and which verify $h(T_2)-h(T_1)$ is compact for any h in H^∞.

Notations. Let H be a separable infinite-dimensional complex Hilbert space, $\mathscr{L}(H)$ the Banach algebra of bounded linear operators on H and $\mathscr{K}(H)$ the space of compact operators on H. For $T\in\mathscr{L}(H)$, we denote by $r(T)$ the spectral radius of T; if T is an absolutely continuous contraction, we denote by $h(T)$, $h\in H^\infty$, the image of h under the Sz.-Nagy—Foias functional calculus.

Proposition 1. *Let T_1 and T_2 be two a.c. contractions in $\mathscr{L}(H)$ such that $T_1T_2=T_2T_1$ and T_2-T_1 is of finite rank. Then for every $h\in H^\infty$, $h(T_2)-h(T_1)$ is of finite rank.*

Proof. Set $A=T_2-T_1$ and let k be the rank of A. Then, for $n\in\mathbf{N}$, we have $T_2^n=T_1^n+AV_n$, where V_n is an element of $\mathscr{L}(H)$. Now, let h be in H and (p_j) a polynomial sequence which converges to h in the weak*-topology. Then $p_j(T_2)=$ $=p_j(T_1)+AW_j$, $W_j\in\mathscr{L}(H)$. Since the rank of AW_j is less than k, by taking the limit in the weak*-topology, we obtain $h(T_2)=h(T_1)+W$, where W is an operator whose rank is less than k. (It is well-known and easy to see that the set of operators T whose rank is less than k is weak*-closed in $\mathscr{L}(H)$). This completes the proof of the proposition.

We have the following observation for T_1 and T_2 with compact difference whose spectral radii are less than 1.

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Observation 2. *Let T_1 and T_2 be two contractions satisfying $r(T_1)<1$, $r(T_2)<1$ and $T_2-T_1\in K(H)$. Then:*

$$h(T_2)-h(T_1)\in K(H), \quad h\in H^\infty.$$

Indeed, let $h(z)=\sum_{k=0}^{\infty} a_k z^k$ be a function in H^∞. Then:

$$h(T_2)-h(T_1) = \sum_{k=0}^{\infty} a_k(T_2^k-T_1^k)$$

and $T_2^k-T_1^k$ can be written in the form:

$$T_2^k-T_1^k = \sum_{j=0}^{k-1} T_2^j(T_2-T_1)T_1^{k-j-1}.$$

Hence $T_2^k-T_1^k$ is compact for every $k\geqq 1$ and so $h(T_2)-h(T_1)$ is a norm-limit of compact operators, hence, it is compact.

The following theorem gives another example of a.c. contractions T_1 and T_2 such that $h(T_2)-h(T_1)\in\mathscr{K}(H)$, $h\in H^\infty$.

Theorem 3. *Let T_1 and T_2 be two a.c. contractions such that $T_1=S\oplus 0$ and $T_2=S\oplus K$, $K\in\mathscr{K}(H)$. Then, S and K are a.c. contractions, $r(K)<1$ and $h(T_2)-$ $-h(T_1)\in\mathscr{K}(H)$ for every $h\in H^\infty$.*

Proof. It is clear that K is absolutely continuous. If $r(K)=1$, then K will have a eigenvalue of modulus 1 which contradicts the absolute continuity of T_2. Hence $r(K)<1$ and if $h(z)=\sum_{k=0}^{\infty} a_k z^k$ is in H^∞, we have:

$$h(T_2)-h(T_1) = \big(h(S)\oplus h(K)\big)-\big(h(S)\oplus h(0)\big) = \sum_{k=1}^{\infty} a_k K^k$$

which is compact.

We examine now the particular case where T_1 and T_2 are diagonal operators.

Let (e_n) be an orthonormal basis for H, let (α_n) and (β_n) be two sequences in the unit disc $\mathbf{D}$ and let T_α and T_β be the diagonal operators associated to (α_n) and (β_n) respectively. Then:

Theorem 4. *The following assertions are eguivalent:*

a) $\lim_{n\to\infty} \dfrac{\beta_n-\alpha_n}{1-|\beta_n|} = 0,$

b) $h(T_\beta)-h(T_\alpha)$ *is compact for every* $h\in H^\infty$.

The proof uses the following

Lemma 5. *Let (u_n) and (v_n) be two complex sequences.*

a) *If (u_n) and (v_n) are in* $\mathbf{D}$, *then:*

$$\lim_{n\to\infty} \frac{v_n-u_n}{1-\overline{v_n}\,u_n} = 0 \Leftrightarrow \lim_{n\to\infty} \frac{v_n-u_n}{1-|v_n|} = 0$$

b) *If* $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = 0$, *then there exists an increasing sequence* $(n_k)\subset\mathbf{N}$ *such that:*

$$\frac{|u_{n_i}|}{|v_{n_j}|} \leqq 2^{j-i-1} \quad \textit{if} \quad j<i \quad \textit{and} \quad \frac{|v_{n_j}|}{|u_{n_i}|} \leqq 2^{i-j-1} \quad \textit{if} \quad i<j.$$

Proof. Assertion a) results from:

$$\frac{|v_n-u_n|}{|1-\overline{v_n}\,u_n|} \leqq \frac{|v_n-u_n|}{1-|v_n|}$$

and

$$\frac{|v_n-u_n|}{1-|v_n|} = \frac{|v_n-u_n|(1+|v_n|)}{(1-\overline{v_n}\,u_n)\left(1+\overline{v_n}\,\dfrac{(u_n-v_n)}{1-\overline{v_n}\,u_n}\right)}.$$

Assertion b) can be obtained by using a simple induction.

Proof of Theorem 4. To prove a)$\Rightarrow$b), it is sufficient to show that if a) holds then:

$$\lim_{n\to\infty} |h(\beta_n)-h(\alpha_n)| = 0.$$

For $h\in H^\infty$ and $a\in\mathbf{D}$, we can write the function $g(z)=h(z)-h(a)$ under the form $g(z)=(z-a)g_a(z)$, $g_a\in H^\infty$ and $\|g_a\|\leqq 2\|h\|_\infty/(1-|a|)$. This implies that

$$|h(\beta_n)-h(\alpha_n)| \leqq 2\|h\|_\infty \frac{|\beta_n-\alpha_n|}{1-|\beta_n|}, \quad h\in H^\infty$$

and so a)$\Rightarrow$b).

Now, suppose that $h(T_\beta)-h(T_\alpha)$ is compact for every $h\in H^\infty$ and the sequence (v_n), $v_n=|(\beta_n-\alpha_n)/(1-\bar{\beta}_n\alpha_n)|$ does not converge to zero. Since the sequence (v_n) is bounded, it contains a subsequence (v_{n_k}) which converges to a positive limit l. As $T_\alpha-T_\beta$ is compact, we have $0\neq\beta_{n_k}-\alpha_{n_k}\to 0$ and so $|\alpha_{n_k}|\to 1$ and $|\beta_{n_k}|\to 1$. Therefore, for example, the sequence (β_{n_k}) contains a Blaschke subsequence $(\beta_{n_{k_l}})$ that is $\sum_{l=0}^{\infty}(1-|\beta_{n_{k_l}}|)<\infty$. From Lemma 5, by extracting another subsequence, we can suppose that the subsequence (β_{n_k}) is a Blaschke sequence and:

$$\frac{1-|\beta_{n_j}|}{1-|\alpha_{n_i}|} \leqq 2^{i-j-1} \quad \text{if} \quad i<j, \quad \text{and} \quad \frac{1-|\alpha_{n_j}|}{1-|\beta_{n_i}|} \leqq 2^{i-j-1} \quad \text{if} \quad i<j.$$

For $0 \neq a \in \mathbf{D}$, denote by e_a the function:

$$e_a(z) = \frac{|a|}{a} \frac{a-z}{1-\bar{a}z}, \quad z \in \mathbf{D}.$$

We have:

$$\big|1-|e_a(z)|\big| \leqq 2\frac{1-|a|}{1-|z|}$$

and as $|e_a(z)| = |e_z(a)|$ we have also:

$$\big|1-|e_a(z)|\big| \leqq 2\frac{1-|z|}{1-|a|}.$$

It results that:

$$\big|1-|e_{\beta_{n_i}}(\alpha_{n_j})|\big| = 2^{-|i-j|}, \ i \neq j \quad \text{so} \quad |e_{\beta_{n_j}}(\alpha_{n_i})| \geqq 1-2^{-|i-j|}, \ i \neq j$$

and for any fixed j

$$\prod_{i \neq j} |e_{\beta_{n_i}}(\alpha_{n_j})| \geqq \prod_{i<j} (1-2^{-|i-j|}) \prod_{i>j} (1-2^{-|i-j|}) \geqq \Big(\prod_{k=1}^{\infty} (1-2^{-k})\Big)^2 = c > 0.$$

Let

$$B(z) = \prod_{k=1}^{\infty} \frac{|\beta_{n_k}|}{\beta_{n_k}} \frac{\beta_{n_k}-z}{1-\overline{\beta_{n_k}}z}$$

be the Blaschke product associated to the sequence (β_{n_k}). Then:

$$|B(\alpha_{n_j})| = \left|\prod_{k \neq j} \frac{|\beta_{n_k}|}{\beta_{n_k}} \frac{\beta_{n_k}-\alpha_{n_j}}{1-\overline{\beta_{n_k}}\alpha_{n_j}}\right| |e_{\beta_{n_j}}(\alpha_{n_j})| \geqq c\,|e_{\beta_{n_j}}(\alpha_{n_j})| \to cl.$$

hence $B(\beta_{n_j}) - B(\alpha_{n_j}) = -B(\alpha_{n_j})$ does not converge to 0. This contradicts the compactness of $B(T_\beta) - B(T_\alpha)$, and the theorem is proved.

Remark 6. If $T = T_\alpha$, where $\alpha = (\alpha_n)$ is a sequence of distinct elements of $\mathbf{D}$, then every element S of the commutant of T can be written $S = T_\beta$, where $\beta = (\beta_n)$ is a sequence of complex numbers. If S is an a.c. contraction, then $\beta_n \in \mathbf{D}$, $n \in \mathbf{N}$. Therefore we see that $h(S) - h(T)$ is compact for every $h \in H^\infty$ if and and only if

$$\lim_{n\to\infty} \frac{\beta_n - \alpha_n}{1-|\alpha_n|} = 0.$$

If $\sup |\alpha_n| = 1$, T is a completely nonunitary contraction with $r(T) = 1$. Hence we see that there exist a.c. contractions $S \neq T$ such that $r(T) = 1$, $ST = TS$ and $h(S) - h(T) \in \mathscr{K}(H)$ for every $h \in H^\infty$ and a.c. contractions S' such that $r(S') = 1$, $S'T = TS'$, $S' - T \in \mathscr{K}(H)$ and $h(S') - h(T) \notin \mathscr{K}(H)$ for some $h \in H^\infty$.

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UNIVERSITÉ BORDEAUX I
U. F. R. DE MATHÉMATIQUES ET INFORMATIQUE
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On the joint Weyl spectrum. III

MUNEO CHŌ

Dedicated to Professor Tsuyoshi Ando on his 60th birthday

1. Introduction. In [3], we proved that the Weyl theorem holds for a commuting pair of normal operators on a Hilbert space. In this paper we show, by a simple proof, that the Weyl theorem holds for a commuting n-tuple of normal operators and, moreover, its Weyl spectrum coincides with the essential spectrum.

Let $\mathfrak{H}$ be a complex Hilbert space. Let $\mathscr{B}(\mathfrak{H})$ be the algebra of all bounded linear operators on $\mathfrak{H}$ and $\mathscr{K}(\mathfrak{H})$ be the ideal of all compact operators on $\mathfrak{H}$. Let $\mathscr{C}(\mathfrak{H})$ denote the Calkin algebra $\mathscr{B}(\mathfrak{H})/\mathscr{K}(\mathfrak{H})$, with corresponding Calkin map $\pi\colon \mathscr{B}(\mathfrak{H})\to\mathscr{C}(\mathfrak{H})$. Let $\mathbf{T}=(T_1, \ldots, T_n)$ be a commuting n-tuple of operators on $\mathfrak{H}$. Let $\sigma(\mathbf{T})$ be the (Taylor) joint spectrum of $\mathbf{T}$. We refer the reader to [9] for the definition of $\sigma(\mathbf{T})$.

The joint Weyl spectrum $\omega(\mathbf{T})$ of $\mathbf{T}=(T_1, \ldots, T_n)$ is defined as the set

$$\omega(\mathbf{T}) = \bigcap\{\sigma(\mathbf{T}+\mathbf{K})\colon \mathbf{T}+\mathbf{K} = (T_1+K_1, \ldots, T_n+K_n) \text{ is a commuting } n\text{-tuple for } K_1, \ldots, K_n\in\mathscr{K}(\mathfrak{H})\}.$$

The joint essential spectrum $\sigma_e(\mathbf{T})$ of $\mathbf{T}=(T_1, \ldots, T_n)$ is defined as the set

$$\sigma_e(\mathbf{T}) = \sigma(\pi(\mathbf{T})),$$

where $\pi(\mathbf{T})=(\pi(T_1), \ldots, \pi(T_n))$.

For a commuting n-tuple $\mathbf{T}=(T_1, \ldots, T_n)$, $\pi_{00}(\mathbf{T})$ is the set of all isolated points in $\sigma(\mathbf{T})$ which are joint eigenvalues of finite multiplicity.

2. Theorem. From Corollary 3.8 in [6] and Theorem 2.6 in [7], we have the following

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Theorem 1. *Let* $\mathbf{T}=(T_1, \ldots, T_n)$ *be a doubly commuting n-tuple of hyponormal operators on H. Then* $\mathbf{z}=(z_1, \ldots, z_n)\in\sigma_e(\mathbf{T})$ *if and only if there exists a sequence* $\{x_k\}$ *of unit vectors in* $\mathfrak{H}$ *with* $x_k\to 0$ *weakly such that* $(T_i-z_i)^*x_k\to 0$ *as* $k\to\infty$.

Immediately, we have the following result.

Theorem 2. *Let* $\mathbf{T}=(T_1, \ldots, T_n)$ *be a doubly commuting n-tuple of hyponormal operators on* $\mathfrak{H}$. *Then* $\sigma_e(\mathbf{T})\subset\omega(\mathbf{T})$.

Lemma 3. *Let* $\mathbf{T}=(T_1, \ldots, T_n)$ *be a doubly commuting n-tuple of hyponormal operators on* $\mathfrak{H}$. *If* $\boldsymbol{\alpha}=(\alpha_1, \ldots, \alpha_n)$ *is an isolated point of* $\sigma(\mathbf{T})$, *then* $\boldsymbol{\alpha}$ *is a joint eigenvalue of* $\mathbf{T}$.

Proof. Let Γ be a surface $|\mathbf{z}-\boldsymbol{\alpha}|=\varepsilon$ $(\varepsilon>0)$, whose interior has no point of $\sigma(\mathbf{T})$ except $\boldsymbol{\alpha}$. Define

$$P=\frac{1}{(2\pi i)^n}\int_\Gamma R_{\mathbf{z}-\mathbf{T}}\wedge dz_1\wedge\ldots\wedge dz_n.$$

Then P is a nonzero projection which commutes with every T_i $(i=1, \ldots, T_n)$ (see [10]). Let $\mathbf{T}_{|P}=(PT_1P, \ldots, PT_nP)$. Then $\mathbf{T}_{|P}$ is a doubly commuting n-tuple of hyponormal operators and $\sigma(\mathbf{T}_{|P})=\{\boldsymbol{\alpha}\}$. By Theorem 3.4 in [5], $\boldsymbol{\alpha}$ is a joint eigenvalue of $\mathbf{T}$.

Theorem 4. *Let* $\mathbf{T}=(T_1, \ldots, T_n)$ *be a doubly commuting n-tuple of hyponormal operators on* $\mathfrak{H}$. *Then* $\omega(\mathbf{T})\subset\sigma(\mathbf{T})-\pi_{00}(\mathbf{T})$.

Proof. For every $\mathbf{z}=(z_1, \ldots, z_n)\in\mathbf{C}^n$, $\mathbf{T}-\mathbf{z}=(T_1-z_1, \ldots, T_n-z_n)$ is a doubly commuting n-tuple of hyponormal operators. Hence we may only prove that if $0\in\pi_{00}(\mathbf{T})$, then $0\notin\omega(\mathbf{T})$. Let 0 be in $\pi_{00}(\mathbf{T})$. Then $\mathfrak{N}=\operatorname{Ker}(T_1^*T_1+\ldots+T_1^*T_n)$ is a finite dimensional subspace. Let P denote the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{N}$. Since then P is a compact operator and $PT_i=T_iP=0$ $(i=1, \ldots, n)$, $\mathbf{T}+\mathbf{P}=$ $=\left(T_1+\frac{1}{\sqrt{n}}\cdot P, \ldots, T_n+\frac{1}{\sqrt{n}}\cdot P\right)$ is a doubly commuting n-tuple of hyponormal operators. We let $\mathbf{R}=\left(\left(T_1+\frac{1}{\sqrt{n}}\cdot P\right)_{|\mathfrak{N}}, \ldots, \left(T_n+\frac{1}{\sqrt{n}}\cdot P\right)_{|\mathfrak{N}}\right)$ and $\mathbf{S}=\left(\left(T_1+\frac{1}{\sqrt{n}}\cdot P\right)_{|\mathfrak{N}^\perp}, \ldots\right.$ $\left.\ldots, \left(T_n+\frac{1}{\sqrt{n}}\cdot P\right)_{|\mathfrak{N}^\perp}\right)$. Since then $\mathfrak{N}$ is a reducing subspace for every T_i $(i=1, \ldots, n)$, it follows that $\mathbf{R}$ and $\mathbf{S}$ are doubly commuting n-tuples of hyponormal operators on $\mathfrak{N}$ and $\mathfrak{N}^\perp$ respectively and $\sigma(\mathbf{T}+\mathbf{P})=\sigma(\mathbf{R})\cup\sigma(\mathbf{S})$. It is clear that $0\notin\sigma(\mathbf{R})$. If $0\in\sigma(\mathbf{S})$, then 0 is an isolated point of $\sigma(\mathbf{S})$. Hence by Lemma 3, 0 is a joint eigenvalue of $\mathbf{S}$ and so of $\mathbf{T}$. So there exists a nonzero vector x in $\mathfrak{N}^\perp$ such that $T_ix=0$ $(i=1, \ldots, n)$. This is a contradiction. Therefore we have $0\notin\sigma(\mathbf{T}+\mathbf{P})$.

Theorem 5. *Let* $\mathbf{T}=(T_1, \ldots, T_n)$ *be a commuting n-tuple of normal operators on* $\mathfrak{H}$. *Then* $\sigma_e(\mathbf{T})=\omega(\mathbf{T})=\sigma(\mathbf{T})-\pi_{00}(\mathbf{T})$.

Proof. By Theorems 2 and 4, we may only prove that

$$\sigma(\mathbf{T})-\pi_{00}(\mathbf{T})\subset\sigma_e(\mathbf{T}).$$

In [8], FIALKOW proved that if γ is a nonisolated point of $\sigma(\mathbf{T})$, then $\gamma\in\sigma_e(\mathbf{T})$. It is also clear that if γ is a isolated point of $\sigma(\mathbf{T})$ with infinite multiplicity, then $\gamma\in\sigma_e(\mathbf{T})$.

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DEPARTMENT OF MATHEMATICS
JOETSU UNIVERSITY OF EDUCATION
JOETSU 943, JAPAN

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On the compositions of (α, β)-derivations of rings, and applications to von Neumann algebras

MATEJ BREŠAR

Introduction

There are two motivations for this research. The first one is an old and well-known result of E. POSNER [12]:

Theorem A. *Let R be a prime ring of characteristic not 2. If the composition of derivations d, g of R is a derivation, then either $d=0$ or $g=0$.*

A number of authors have proved extensions of this theorem; we refer the reader to some ring-theoretic results [3, 5, 9] and to some results from analysis [4, 10, 11].

The other motivation comes from the theory of von Neumann algebras. In a series of papers A. B. Thaheem and some other authors have studied the identity $\alpha+\alpha^{-1}=\beta+\beta^{-1}$ where α and β are automorphisms of a von Neumann algebra. This identity plays an important role in the Tomita—Takesaki theory (see, e.g., [6, 7, 8]). In [13 and 14] and in a joint paper with AWAMI [18], THAHEEM has given various proofs of the following theorem.

Theorem B. *Let R be a von Neumann algebra and α, β be *-automorphisms of R satisfying $\alpha+\alpha^{-1}=\beta+\beta^{-1}$. If α and β commute then there exists a central projection p in R such that $\alpha(p)=\beta(p)=p$, $\alpha=\beta$ on pR, and $\alpha=\beta^{-1}$ on $(1-p)R$.*

For other results concerning the identity $\alpha+\alpha^{-1}=\beta+\beta^{-1}$ we refer to some recent papers [1, 15, 16, 17] where further references can be found.

It is our aim in this paper to extend Theorem A to more general mappings on more general rings, so that the special case of this extension gives a generalization of Theorem B. In particular, our research can be viewed as a new, more elementary

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approach to the study of the identity $\alpha+\alpha^{-1}=\beta+\beta^{-1}$. In a subsequent paper we hope to consider this identity without assuming the commutativity of α and β.

Let R be a ring and α, β be automorphisms of R. An additive mapping d of R into itself is called an (α, β)-derivation if

$$d(xy) = \alpha(x)\,d(y)+d(x)\beta(y) \quad \text{for all} \quad x, y\in R.$$

An (α, β)-derivation d is said to be inner if there exists $a\in R$ such that $d(x)=\alpha(x)a-a\beta(x)$ for all $x\in R$. Of course, derivations are (1,1)-derivations where 1 is the identity on R. We will study the case where the composition of an (α, β)-derivation d and a (γ, δ)-derivation g is an $(\alpha\gamma, \beta\delta)$-derivation. We will first generalize Theorem A by proving that if R is prime of characteristic not 2 and g commutes with both γ and δ, then either $d=0$ or $g=0$ (Corollary 1). An abbreviated version of our main theorem reads as follows.

Theorem 1. *Let R be a 2-torsion free semiprime ring, d be an (α, β)-derivation of R, and g be a (γ, δ)-derivation of R. Suppose that d commutes with both α and β, and that g commutes with both γ and δ. If dg is an $(\alpha\gamma, \beta\delta)$-deriavtion then there exist ideals U and V of R such that $U\oplus V$ is an essential ideal of R, $d=0$ on V and $g=0$ on U. Moreover, if the annihilator of any ideal in R is a direct summand (in particular, if R is a von Neumann algebra), then $U\oplus V=R$.*

As an immediate consequence of Theorem 1 we obtain that the decomposition of Theorem B holds in arbitrary semiprime ring in which the annihilator of any ideal is a direct summand (Corollary 2). Moreover, the assumption that α and δ preserve adjoints is removed (in fact, we do not work in rings with involution).

Preliminaries

Throughout, R will represent an associative ring. Recall that R is prime if $aRb=0$ implies that $a=0$ or $b=0$. R is said to be semiprime if $aRa=0$ implies that $a=0$. Equivalently, R is semiprime if it has no nonzero nilpotent ideals. Every C^*-algebra is semiprime (for $0\neq aa^*a\in aRa$ if $a\neq 0$). A von Neumann algebra is prime if and only if it is a factor (i.e., its center consists of scalar multiples of the identity).

Let R be semiprime. Suppose that $aRb=0$ for some $a, b\in R$. Then we also have $(bRa)R(bRa)=0$, $abRab=0$, $baRba=0$, and therefore $bRa=0$, $ab=0$, $ba=0$ since R is semiprime. Note that the left and right and two-sided annihilators of an ideal U in R coincide. It will be denoted by Ann (U). Note also that $U\cap \mathrm{Ann}\,(U)=0$ and $U\oplus \mathrm{Ann}\,(U)$ is an essential ideal (i.e., $(U\oplus \mathrm{Ann}\,(U))\cap I\neq 0$ for every nonzero ideal I of R). We will be especially concerned with semiprime rings R in

which the annihilator of any ideal is a direct summand; that is, $\mathrm{Ann}(U)\oplus \oplus\mathrm{Ann}(\mathrm{Ann}(U))=R$ for any ideal U of R. Every von Neumann algebra has this property; namely, the annihilator of any ideal in a von Neumann algebra R is σ-weakly closed, therefore it is of the form pR for some central projection p in R. More generally, the same is true for Baer *-rings, and, therefore, for AW^*-algebras (see, e.g., [2]).

The results

Lemma 1. *Let R be a 2-torsion free semiprime ring, d be an (α, β)-derivation of R and g be a (γ, δ)-derviation of R. Suppose that the composition dg is an $(\alpha\gamma, \beta\delta)$-derivation, and suppose that g commutes with both γ and δ. Then $g(x)R(\alpha^{-1}d)(y)=0$ for all $x, y\in R$.*

Proof. We have $h=dg$ is a $(\alpha\gamma, \beta\delta)$-derivation. Consequently $(\beta^{-1}d)(g\delta^{-1})= =\beta^{-1}h\delta^{-1}$; that is, the composition of a $(\beta^{-1}\alpha, 1)$-derivation $\beta^{-1}d$ and a $(\gamma\delta^{-1}, 1)$-derivation $g\delta^{-1}$ is a $((\beta^{-1}\alpha)(\gamma\delta^{-1}), 1)$-derivation $\beta^{-1}h\delta^{-1}$. We will show that $g\delta^{-1}$ commutes with $\gamma\delta^{-1}$. Note that this implies that there is no loss of generality in assuming $\beta=1$ and $\delta=1$.

Thus, let us prove that $g\delta^{-1}$ and $\gamma\delta^{-1}$ commute. Since g commutes with γ and δ, it suffices to show that $g\gamma\delta^{-1}=g\delta^{-1}\gamma$. By the definition of (γ, δ)-derivations we have

$$(\gamma g)(xy) = \gamma^2(x)(\gamma g)(y)+(\gamma g)(x)(\gamma\delta)(y),$$

$$(g\gamma)(xy) = \gamma^2(x)(g\gamma)(y)+(g\gamma)(x)(\delta\gamma)(y).$$

Since we have assumed that $\gamma g=g\gamma$ the relations imply that $(g\gamma)(x)(\gamma\delta-\delta\gamma)(y)=0$ for all $x, y\in R$; but γ is onto, so we also have $g(x)(\gamma\delta-\delta\gamma)(y)=0$ for all $x, y\in R$. Substituting xz for x it follows easily that $g(x)R(\gamma\delta-\delta\gamma)(y)=0$ for all $x, y\in R$. In particular, $g((\gamma\delta-\delta\gamma)(x))R(\gamma\delta-\delta\gamma)(g(x))=0$ for every x in R. Since g commutes with $\gamma\delta-\delta\gamma$, and since R is semiprime, it follows that $g\gamma\delta=g\delta\gamma$. Multiplying this relation from the right and from the left by δ^{-1} we arrive at $g\delta^{-1}\gamma=g\gamma\delta^{-1}$.

Now, we may assume that $\beta=\delta=1$. A direct computation shows that

$$(dg)(xy) = (\alpha\gamma)(x)(dg)(y)+(d\gamma)(x)g(y)+(\alpha g)(x)d(y)+(dg)(x)y.$$

On the other hand, since dg is an $(\alpha\gamma, 1)$-derivation, we have

$$(dg)(xy) = (\alpha\gamma)(x)(dg)(y)+(dg)(x)y.$$

Comparing the two expressions so obtained for $(dg)(xy)$, we see that

$$(d\gamma)(x)g(y)+(\alpha g)(x)d(y) = 0 \quad \text{for all} \quad x, y\in R. \tag{1}$$

Replacing y by yz in (1) we obtain

$$(d\gamma)(x)\gamma(y)g(z)+(d\gamma)(x)g(y)z+(\alpha g)(x)\alpha(y)d(z)+(\alpha g)(x)d(y)z = 0.$$

By (1) this relation reduces to

$$(2) \qquad (d\gamma)(x)\gamma(y)g(z)+(\alpha g)(x)\alpha(y)d(z) = 0 \quad \text{for all} \quad x, y, z\in R.$$

Replacing y by $g(y)$ in (2) and using the assumption that g commutes with γ, we then get

$$(d\gamma)(x)g(\gamma(y))g(z)+(\alpha g)(x)(\alpha g)(y)d(z) = 0.$$

On the other hand, using (1) twice we obtain

$$\{(d\gamma)(x)g(\gamma(y))\}g(z) = -(\alpha g)(x)\{(d\gamma)(y)g(z)\} = (\alpha g)(x)(\alpha g)(y)d(z).$$

Comparing the last two relations we get $2(ag)(x)(\alpha g)(y)d(z)=0$ for all $x, y, z\in R$. Since R is 2-torsion free we then have

$$0 = \alpha^{-1}((\alpha g)(x)(\alpha g)(y)(d)(z)) = g(x)g(y)(\alpha^{-1}d)(z).$$

Thus $g(x)g(y)(\alpha^{-1}d)(z)=0$ for all $x, y, z\in R$. Replacing x by xu it follows at once that $g(x)Rg(y)(\alpha^{-1}d)(z)=0$; similarly we see that $g(x)Rg(y)R(\alpha^{-1}d)(z)=0$. The semiprimeness of R then yields $g(y)R(\alpha^{-1}d)(z)=0$ and so the lemma is proved.

As an immediate consequence of Lemma 1 we obtain the following generalization of Posner's theorem.

Corollary 1. *Let R be a prime ring of characteristic not 2, d be an (α, β)-derivation of R, and g be an (γ, δ)-derivation of R. Suppose that g commutes with both γ and δ. If the composition dg is an $(\alpha\gamma, \beta\delta)$-derivation then either $d=0$ or $g=0$.*

Example. The assumption that g commutes with both γ and δ is not superfluous. Moreover, the following simple example shows that it cannot be replaced by the assumption that d commutes with both α and β. Suppose that a prime ring with unit element 1 contains elements a and b such that $a^2=0$, $b^2=1$, $ab+ba=0$, and a, b do not lie in the center of R (for example, in the ring of 2×2 matrices the elements

$$a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

satisfy these conditions). Define the inner automorphism γ by $\gamma(x)=bxb$ and the $(\gamma, 1)$-derivation g by $g(x)=\gamma(x)ba-bax$; $g\neq 0$ since $g(x)a=-baxa\neq 0$ for some $x\in R$ by the primeness of R. If d is the inner derivation, $d(x)=ax-xa$, then $dg=0$.

We need two easy lemmas.

Lemma 2. *Let R be any ring and Θ be an automorphism of R. If Θ maps an ideal W onto itself then Θ maps* Ann (W) *onto itself.*

Proof. Given $w \in W$, $u \in \mathrm{Ann}(W)$ we have $0 = \Theta(uw) = \Theta(u)\Theta(w)$ and similarly, $\Theta(w)\Theta(u) = 0$. By assumption, $\Theta(w)$ is an arbitrary element in W, so it follows that $\Theta(u) \in \mathrm{Ann}(W)$. Thus Θ maps $\mathrm{Ann}(W)$ into itself. Analogously, Θ^{-1} maps $\mathrm{Ann}(W)$ into itself, which means that Θ is onto on $\mathrm{Ann}(W)$.

Lemma 3. *Let R be a semiprime ring, and let d be an (α, β)-derivation of R which commutes with both α and β. If d maps R into an ideal W of R, then d is zero on* $\mathrm{Ann}(W)$.

Proof. Given $w \in W$, $u \in \mathrm{Ann}(W)$ we have $u(\alpha^{-1}d)(w) = ud(\alpha^{-1}(w)) \in \mathrm{Ann}(W)W = 0$. Thus $\alpha(u)d(w) = \alpha(u(\alpha^{-1}d)(w)) = 0$. Hence $d(u)\beta(w) = \alpha(u)d(w) + d(u)\beta(w) = d(uw) = 0$. But then also $0 = \beta^{-1}(d(u)\beta(w)) = d(\beta^{-1}(u))w$. That is, $d(\beta^{-1}(u)) \in \mathrm{Ann}(W)$ for any $u \in \mathrm{Ann}(W)$. However, by assumption $d(\beta^{-1}(u))$ lies in W, so we are forced to conclude that $d(\beta^{-1}(u)) = 0$. Since d and β^{-1} commute, $d(u) = 0$ as well.

We now have enough information to prove the main theorem of this paper.

Theorem 1. *Let R be a 2-torsion free semiprime ring, d be an (α, β)-derivation of R, and g be an (γ, δ)-derivation of R. Suppose that d commutes with both α and β, and that g commutes with both γ and δ. If the composition dg is an $(\alpha\gamma, \beta\delta)$-derivation, then there exist ideals U and V of R such that:*

(i) $U \cap V = 0$ and $U \oplus V$ is an essential ideal of R. Moreover, if the annihilator of any ideal in R is a direct summand (in particular, if R is a von Neumann algebra), then $U \oplus V = R$,

(ii) If Θ is any automorphism of R which commutes with d then Θ maps U onto U and V onto V,

(iii) d maps R into U and d is zero on V,

(iv) g maps R into V and d is zero on U.

In particular, $dg = gd = 0$.

Proof. Let U_0 be the ideal of R generated by all $d(x)$, $x \in R$. Let $V = \mathrm{Ann}(U_0)$ and $U = \mathrm{Ann}(V)$. Thus (i) holds. If an automorphism Θ of R commutes with d, then $\Theta(xd(y)z) = \Theta(x)d(\Theta(y))\Theta(z) \in U_0$ for all $x, y, z \in R$. Similarly, $\Theta(xd(y)) \in U_0$, $\Theta(d(y)z) \in U_0$ and $\Theta(d(y)) \in U_0$. Thus U_0 is invariant under Θ. Likewise U_0 is invariant under Θ^{-1}. Hence Θ maps U_0 onto itself. From Lemma 2 it follows that Θ maps V onto V, and therefore also U onto U. Thus (ii) is proved. Since d maps R into $U_0 \subseteq U$, (iii) follows immediately from Lemma 3. It remains to prove (iv). In view of Lemma 3 it suffices to show that $g(x)$ lies in v for every $x \in R$. By Lemma 1, since d and α^{-1} commute, we have $g(x)Rd(y) = 0$ for all $x, y \in R$. Thus $g(x) \in \mathrm{Ann}(U_0) = V$. Combining (iii) and (iv) we see that $dg = gd = 0$. The proof of the theorem is complete.

Let R be any ring. Suppose that automorphisms α and β of R satisfy $\alpha+\alpha^{-1}=\beta+\beta^{-1}$ and $\alpha\beta=\beta\alpha$. Multiply the first relation by α, and observe that the relation which we obtain can be written in the form $(\alpha-\beta)(\alpha-\beta^{-1})=0$. That is, the composition of the (α, β)-derivation $\alpha-\beta$ and the (α, β^{-1})-derivation $\alpha-\beta^{-1}$ is equal to zero. Note that all the requirements of Theorem 1 are fulfilled. Thus we have

Corollary 2. *Let R be a 2-torsion free semiprime ring. Suppose that automorphisms α and β of R satisfy $\alpha+\alpha^{-1}=\beta+\beta^{-1}$. If α and β commute then there exist ideals U and V of R such that:*

(i) $U\cap V=0$ and $U\oplus V$ is an essential ideal. Moreover, if the annihilator of any ideal in R is a direct summand (in particular, if R is a von Neumann algebra) then $U\oplus V=R$,

(ii) α and β map U onto U and V onto V,

(iii) $\alpha=\beta$ on V,

(iv) $\alpha=\beta^{-1}$ on U.

We conclude this paper with the following direct consequence of Corollary 2.

Corollary 3. *Let R be a prime ring of characteristic not 2. Suppose that automorphisms α, β of R satisfy $\alpha+\alpha^{-1}=\beta+\beta^{-1}$. If α and β commute then either $\alpha=\beta$ or $\alpha=\beta^{-1}$.*

We leave as an open question whether or not the assumption that α and β commute can be removed in Corollary 3 (it certainly cannot be removed in the case R is semiprime, as Thaheem [17] has shown).

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UNIVERSITY OF MARIBOR
DEPARTMENT OF MATHEMATICS
PF, KOROŠKA 160
62 000 MARIBOR
SLOVENIA

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Bibliographie

K. Andersen, Brook Taylor's Work on Linear Perspective. A study of Taylor's Role in the History of Perspective Geometry. Including Facsimiles of Taylor's Two Books on Perspective. (With 114 Illustrations), (Sources in the History of Mathematics and Physical Sciences, 10), X+259 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong—Barcelona—Budapest, 1992.

In order to indicate the significance of Taylor's studies, the author cites Edward Noble's words: "...[Taylor's] fate has been to be more admired and celebrated than understood." K. Andersen explains this fact noticing two "paradoxes" (in the Concluding Remarks of the book). First, "... the part of Taylor's theory that was mostly applied — and presumably known as his principles — existed long before him. His real and impressive improvements of the theory of perspective — among which his wider use of vanishing lines, and his contributions to direct constructions, and the theory of inverse problems are especially significant — were, however, not much noticed." Secondly, "... precisely Taylor's studies, which are the most incomprehensible of the entire pre-nineteenth-century literature on perspective, evoked response in a circle of practitioners."

When Brook Taylor wrote his books, the theory of perspective and the everyday demands of the painting were in strong interrelation, and perspective was an independent theory in mathematics of its own. So that the linear perspective was the integrating part of the fine art — at least in Taylor's mind. His work reflects a typical viewpoint of some scientists'. Namely that the scientific understanding is a "sine qua non" for the appropriate practical problem (painting, design, etc.).

In the introductionary study K. Andersen presents Taylor's work as a comprehensive survey of the preceding and actual results of the early 18th century. Without this essay it would be almost impossible to establish the significance of the two works presented in facsimile form. The modern mathematician needs some help to understand the terminology and the treatment since the two books were written in the time of the evolution of the theory as a whole (and also in details). This task is entirely fulfilled (starting with the exposition of the basic concepts and methods, Taylor's "inheritance" and his contributions to development of the perspective geometry, proceeding to some historical overview).

After the preliminaries the author could restrict the deal of necessary remarks on the two facsimiles to 41 respectively 35 remarks (indicated by starts in the original texts).

A bibliography is also added, listing the most important works concerning this exciting material.

This volume is not only a useful book for any researcher in this field, but also an original contribution to the researches in the history of mathematics.

J. Kozma (Szeged)

R. R. Akhmerov—M. I. Kamenskii—A. S. Potapov—A. E. Rodkina—B. N. Sadovskii, Measures of Noncompactness and Condensing Operators, (Operator Theory: Advances and Applications, 55), VIII+249 pages, Birkhäuser Verlag, Basel—Boston—Berlin, 1992.

Kuratowski was the first who introduced (in 1930) a quantitative characteristic measuring the degree of noncompactness. In the mid Fifties in functional analysis various measures of noncompactness was applied to investigate condensing operators which map any set into a set which is in certain sense more compact than the original set. It turns out that condensing operators have similar properties as compact operators. The text is divided into four chapters. The first chapter introduces the notions of Kuratowski, Hausdorff and general measures of noncompactness, the notion of condensing operators and gives the basic properties. The second chapter is devoted to the characterization of linear condensing operators in spectral terms and studies the perturbation of the spectrum. The third chapter develops the theory of the index of fixed points for nonlinear condensing operators. The fourth chapter applies the theory to problems for differential equations in Banach space, stochastic differential equations, functional differential equations and integral equations.

The book can be offered to anyone who is interested in topological relation of functional analysis and has some background in functional analysis and general topology.

L. Gehér (Szeged)

R. Balian, From Microphysics to Macrophysics, Vol I., (Texts and Monographs in Physics), XXII+465 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong—Barcelona—Budapest, 1991.

This is the English translation of the original French book, which has grown out from lecture notes for students of the Ecole Polytechnique in Paris. It is an advanced textbook of statistical physics and thermodynamics. One of its great merits is that the subject is based on the laws of quantum physics, which is necessary if one wants to avoid the problems and contradictions, raised by classical statistical mechanics. This modern approach is made familiar by beginning with the simple example of an ensemble of two-level magnetic atoms. Then the introduction of the concepts of statistical mechanics with its foundations in quantum mechanics follows naturally. Classical systems are treated as a special case. The concept of entropy is also introduced from the quantum physical point of view, and its connection with information theory is presented too. The connection between thermodynamics and statistical physics is built up gradually, first by only referring to elementary facts from thermal physics, and later on in two separate chapters devoted to advanced thermodynamics. In the first one the traditional presentation in the form of the main laws can be found, in the other one the more modern approach is presented by postulating the existence and the properties of the entropy as a thermodynamic potential. Among the examples the very delicate and problematical questions of dielectric and magnetic substances are treated with due attention. The perfect gas, the real gas, and the gas-liquid phase transition are also discussed in this first volume, while the ideal quantum gas together with other non-traditional applications of statistical physics are left for the second one.

There are several interruptions of the main text. Vivid discussions on the historical evolution of the fundamental concepts of statistical physics, and also philosophical considerations concerning its paradoxes makes the reading a pleasant entertainment. The clarity of the presentation and the comprehensive content will certainly make this book together with the forthcoming second volume a standard reference of the field.

M. Benedict (Szeged)

P. Bamberg—S. Sternberg, A course in mathematics for students of physics: 1, Cambridge University Press, Cambridge—New York—Port Chester—Melbourne—Sydney, 1991.

We have already recommended this book when read its 2nd volume. However, the reviewer is in the position of presenting a recommendation anew after reading the first volume, too.

This volume is "neither more" nor less than a very good textbook for studying mathematics necessary to understand most important physical concepts, phenomena and lows. It provides, at the same time, the basis for any mathematical studies: affine and Euclidean planes, linear transformations, matrix representations, linear differential equations (in 2-dimensional planes), calculus in the plane, differential forms, line and double integrals, vector spaces, and determinants. The above list indicates a standard material, however the demands on the physical applications completely satisfied in each sections and on all levels.

The reader meet this requirement firstly in the examples (e.g. applications of differential equations to the well-known physical systems, normal modes — also in higher dimensions). On the other hand, some significant chapters of the classical and modern physics are explicitly discussed (special relativity, Poincaré group and the Galilean group, momentum, energy and mass, Gaussian optics).

This text examines the most important concepts (on undergraduate level), paying attention to both excellent exposition and demonstration by clear reasonings.

The redaction systematically goes back to the notions and facts previously introduced and proved, so that the volume is self-contained in this respect. Every section begins with some introduction, which give an outlook of the subsequent material, and closes with a brief summary which is used to take some emphases on the appropriate place.

Various topics are described in a uniform manner. This is a good help for the beginner to find the relations between new and previously discussed ideas.

This new classical book is recommended as an undergraduate text as well as a good reading for anybody interested in physics, but with some need of mathematical backgrounds.

J. Kozma (Szeged)

T. Banchoff—J. Wermer, Linear Algebra Through Geometry (Second Edition), with 92 Illustrations, (Undergraduate Texts in Mathematics), XII+305 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong—Barcelona—Budapest, 1992.

The second edition of this classical book has been enlarged.

The first four chapters remained unchanged except for some additional remarks in Chapter 4: simultaneously are discussed the 4-spaces and the possibility of their generalization for n-spaces, closed by the general definition of the determinant of an $n \times n$ matrix. These four chapters represent a good introduction into linear algebra through geometry. The leading idea of the authors is to consider the Euclidean plane (space) as a vector space, and find their properties independent from the concrete geometric meaning. The first two chapters provides a detailed analysis of the geometry of vectors in the line and the plane, starting with the notion of the vector and linear transformation, furthermore a development of the elementary properties of the commonly used binary operations, and proceeding in the third chapter to a deeper study of vectors in a 3-space (by means of linear transformations). There is also given the classification of conic sections and quadric surfaces.

The content of the next chapter of the first edition is now partly attached to Chapter 4 (homogeneous and inhomogeneous systems of linear equations). The new 5th chapter treats the notion of an abstract n-dimensional vector space. There is no more direct contact with the (visualizable)

space, however the uniform treatment helps the reader to remember the lower-dimensional analogues.

Chapter 6 is completely a new one, dealing with inner product vector spaces, the Gram-Schmidt orthonormalization process and orthogonal decomposition of a vector space.

In Chapter 7 we can find a brief summary on symmetric matrices in the necessary extent in order to prove the theorem on diagonalization.

Finally, Chapter 8 covers three applications: differential systems, least squares approximation and curvature of function graphs.

These latter new chapters contain welcome and useful material concerning the original topic.

In this new form the book can be recommended as an introductionary text-book. However, after studying algebra without parallel studies on geometry, every reader will find a strengthening of his or her former knowledge on both geometry and algebra.

J. Kozma (Szeged)

David M. Bressoud, Second Year Calculus (Undergraduate Texts in Mathematics), XI+386 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong—Barcelona, 1991.

This is an excellent textbook for multi-variable and vector calculus, emphasizing the historical physical problems from which the subject has grown, but couching much of it in the modern terminology of differential forms. The book guides the reader from the birth of the mechanized view of the world in Isaac Newton's *Mathematical Principles of Natural Philosophy* in which mathematics becomes the ultimate tool for modelling physical reality, to the dawn of a radically new and often counter-intuitive age in Albert Einstein's Special Theory of Relativity in which it is the mathematical model that suggests new aspects of that reality. The student learns to compute orbits and rocket-projections, model flows and force fields, and derives the laws of electricity and magnetizm. The languages of differential forms enables the reader to see how mathematical symmetry leads to the conclusion that matter and energy are interchangeable.

The chapter headings are: $F=ma$; Vector Algebra; Celestial Mechanics; Differential Forms; Line integrals; Linear Transformations; Differential Calculus; Integration by Pullback; Techniques of Differential Calculus; The Fundamental Theorem of Calculus; $E=mc^2$. Every chapter contains very good exercises helping the students to understand the text.

The style of the book is clear. It is highly recommended both to instructors and students.

J. Németh (Szeged)

Commutative Harmonic Analysis I, Edited by V. P. Khavin and N. K. Nikol'skij, (Encyclopaedia of Mathematical Sciences, 15), VI+268 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong—Barcelona—Budapest, 1991.

This volume is consisting of three parts: I. Methods and Structure of Commutative Harmonic Analysis (V. P. Khavin); II. Classical Themes of Fourier Analysis (S. V. Kislyakov); III. Methods of the Theory of Singular Integrals: Hilbert Transform and Calderon—Zygmund Theorem (E. M. Dyn'kin).

In the first part the following topics are detailed: A short course of Fourier analysis of periodic functions; Harmonic analysis in R^d; Harmonic analysis on groups; Historical survey on Fourier series; Spectral analysis and spectral synthesis.

The second part is dealing with the following materials: Fourier series (convergence and summability); The harmonic conjugation operator; Fourier coefficients; Absolutely convergent Fourier series; Fourier integrals.

The third part is devoted to the subjects as: Hilbert transform (in L^1, in L^2, in L^p and in Hölder classes); Calderon—Zygmund operators; L^2 estimates, L^p estimates; The maximal operator.

At the end of all parts rich references can be found. Furthermore it should be pointed out that numerous examples illustrate the connections to differential and integral equations, approximation theory, number theory, probability theory and physics.

This excellent well-written book should serve as a standard reference for researchers in the field but it can also be recommended to students who want to become researchers in mathematics.

J. Németh (Szeged)

Delay Differential Equations and Dynamical Systems, Edited by S. Busenberg and M. Martelli, (Lecture Notes in Mathematics, 1475), VIII+249 pages, Springer-Verlag, Berlin—Heidelberg—New York — London—Paris—Tokyo—Hong Kong—Barcelona—Budapest, 1991.

This is one of the Proceedings of a Conference in honor of Kenneth Cooke held in Claremont, California, Jan. 13—16, 1990 under the title International Conference on Differential Equations and Applications to Biology and Population Dynamics. A companion volume in the Biomathematics Lecture Notes series of Springer contains papers devoted to applications in biology and population dynamics. The contents of the present volume is summarized in the Preface by the editors as follows:

"The contributions in this volume are collected in two groups, the first consisting of survey articles and the second of research papers. The three survey articles are by Kenneth Cooke and Joseph Wiener who review the recently opened area of differential equations with piecewise continuous arguments; by Jack Hale who discusses a fascinating array of results in the stability of delay differential equations viewed as dynamical systems; and by Paul Waltman who presents an overview of useful new results on persistence in dynamical systems. The research contributions part of the volume consists of nineteen papers which present new results in delay differential equations and dynamical systems. The papers are united by the common thread of the underlying topic but, as is characteristic of this field, employ a wide array of deep mathematical theories and techniques. These include methods from linear and nonlinear functional analysis, a number of topological and topological degree techniques, as well as asymptotic and other classical analysis methods. Many of these mathematical techniques were originally created in order to address problems arising in the field of differential equations and are still being stimulated by challenges from this field."

Kenneth Cooke has been one of the most artful and original practitioners in the interdisciplinary research work involving delay differential equations, dynamical systems and their applications in biology and population dynamics. This volume is worthy of him, it will be very interesting and useful for scientists interested in the topic.

L. Hatvani (Szeged)

F. Digne—J. Michel, Finite Groups of Lie Type (London Mathematical Society, Student Texts, 21), 159 pages, Cambridge University Press, Cambridge—New York—Port Chester—Melbourne—Sydneay, 1991.

The aim of this volume is to present basic facts concerning a particular class of finite groups, called of Lie type. They are finite groups arising as groups of rational points of reductive groups over $\bar{F}_q$ defined over F_q.

The text follows a course given at the University of Paris VII in the academic year 1987—88, so that it contains a fairly complete picture of the topic on introductionary level in the style of a well-organized series of lectures. It is enough refer to the strict and consequent definition-proposition-corollary-remark structure which is completed by references at the end of each chapter (lecture). On the other hand the reader can find introductionary sentences at the beginning of the chapters, gaining perspective for the "audience". Furthermore, the proofs are well thought, and for the omitted ones (easy or standard) can be found a good reference.

The book is divided into 16 chapters. The introductionary chapter (ch. 0) is directed towards the basic knowledge on algebraic groups. Further chapters develop the theory step by step. The first three chapters provide a good introduction to this theory by explaining basic concepts as Bruhat decomposition, intersection of parabolic subgroups, rationality and Frobenius endomorphism. The subsequent chapters include a treatment of cohomological methods and Gelfand-Graev representations. Finally, the last chapter ensures numerous examples of finite groups of Lie type.

This volume is suitable by its design for introductionary courses or seminars on the subject.

J. Kozma (Szeged)

L. R. Foulds, Graph Theory Applications. (Univeritext), 385 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong, 1992.

The book is divided into two parts: the first one discusses the theory of graphs and the second one is dealing with the applications.

The first part begins with a historical background and the basic notations. The next chapters concerned with such a fundamental graph theoretical disciplines as the connectivity (Chapter 2), the trees as the most important class of the graphs (Chapter 3), Euler and Hamiltonian Graphs (Chapter 4) and the planarity. Chapter 6, on the matrices on graphs, is essential for a later discussion, on graph theoretic algorithms. Chapter 7 is an introduction to the directed graphs. These graphs are widely used in electrical ingeneering. The next chapter discusses the coverings and colouring which has applications in industrial ingeneering and other disciplines. Chapter 9 covers graph theoretic algorithms. The electrical ingeneering uses the results of the matroid theory which are introduced in Chapter 10.

Part II has mainly longer chapters explaining the applications of the abovementioned material in various branches of engineering, operation research and science. Due to limitation of space just a few applications have been presented in some depth: some exact and heuristic algorithms in operation research, the printed circuit design in electrical engineering, production planning and control, the facility layout (in which the author's research activity is well known) in industrial engineering. Some other algorithms are mentioned from the fields of physics, chemistry and biology. The last chapter covers such civil engineering applications as earthwork projects and traffic network design.

Since the book offered to different university courses each chapter has a separate subtitle with different exercises. This book, like the other works of the author, is written in clear style. The book is well organized and self-contained. Is is recommended as a textbook in teaching experience and for those students who are interested in the applications of graph theory in practice.

Gábor Galambos (Szeged)

Geometric aspects of Functional Analysis, Edited by J. Lindenstrauss and V. D. Milman (Lecture Notes in Mathematics, 1469), IX+207 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong—Barcelona—Budapest, 1991.

This is the fifth published volume of the proceedings of the Israel Seminar on Geometric Aspects of Functional Analysis. The program on the first page shows that in the period 1989—90, as in the previous years, the most outstanding representatives of the subject participated at the seminar. The papers collected in this volume are original research papers and some survey papers which also contain new results. From the contents:

L. Carleson, Stochastic models of some dynamical systems; V. Milman, Some applications of duality relations; Ya. G. Sinai, Mathematical problems in the theory of quantum chaos; P. M. Bleher, Quasi-classical expansions and the problem of quantum chaos; A. G. Reznikov, A strenghtened isoperimetric inequality for simplices; M. Talagrand, A new isoperimetric inequality and the concentration of measure phenomenon; P. F. X. Müller, Permutations of the Haar system; J. Bourgain, On the distribution of polynomials on high dimensional convex sets; J. Bourgain, J. Lindenstrauss, On covering a set in R^d by balls of the same diameter; M. Meyer, S. Reisner, Characterization of affinely-rotation-invariang log-concave measures by section-centroid location; J. Bourgain, Remarks on Mongomery's conjectures on Dirichlet sums; M. Schmuckenschläger, On the dependence on ε in a theorem of J. Bourgain, J. Lindenstrauss and V. D. Milman; G. Schechtman, M. Schmuckenschläger, Another remark on the volume of the intersection of two L_p^n balls; J. Bourgain, On the restriction and multiplier problem in R^3.

The papers prove taht the organizers of the seminar and the participants keep continue the developing of a new theory which is a combination of the very strong methods of probability theory, Banach space theory and convex geometry. The volume is recommended mainly to specialists who would like to follow the results of this subject.

J. Kincses (Szeged)

E. Hairer—G. Wanner, Solving Ordinary Differential Equations II, Stiff and Differential-Algebraic Problems (Springer Series in Computational Mathematics, 14), VIII+601 pages, Springer-Verlag, Berlin—New York—Budapest, 1991.

This book is the continuation of the excellent Part I. (published in 1987 as Vol. 8 of the same Series). The present volume has all the virtues of the first part plus even more up-to-date material, more references (from the last 3 centuries), more then 100 figures and more humour. Let me quote just one pun exercise from page 213:

"Interpret the meaning of the "N" in the definition for AN-stability. Check among

⋮

Nec plus ultra
Notre Dame
Nottinghamshire
No smoking

⋮

other

and send to the authors."

This second volume reconsiders and enlarges the material of Part I. Chapter IV investigates

Runge-Kutta methods for stiff problems. Chapter V is on multistep methods for stiff problems. The last Chapter VI introduces singular perturbations and differential-algebraic equations.

This book needs no special recommendations. Everybody opening it will read it, too. I think it will be 'the' book for my graduate courses in the next few years.

János Virágh (Szeged)

Y. Hino—S. Murakami—T. Naito, Functional Differential Equations with Infinite Delay, (Lecture Notes in Mathematics, 1473), X+317 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong—Barcelona—Budapest, 1991.

In several processes, e.g. in biology and population dynamics, it is typical that the velocity of the change of the state variable depends not only on the momentary values of the state variables but also on their earlier values, too. In other words, the future depends not only on the present but on the past, too. If one has to take into account only a finite segment of the past, then one has a functional equation with finite delay. In that case the single natural phase space is the space of continuous functions over a finite interval with the usual supremum norm. However, if one has to take into account the whole past, then the delay is infinite, and there is a big variety in the choice of the phase space among the linear spaces with seminorms. There are many facts which hold independently of each concrete phase space. It is a natural idea to summarize these results from the discussion of the equation on an abstract phase space defined by some axioms induced from many examples for the phase space. The authors develop the theory of the functional differential equations with infinite delay from such a point of view.

Chapter 1 contains the formulation of axioms of the phase space together with many examples. After a brief presentation of the basic theorems on the existence, uniqueness, continuous dependence of the solutions (Chapter 2) and an introduction to Stieltjes integrals (Chapter 3), the theory of linear equations is developed from Chapter 4 through Chapter 6. Chapter 7 is devoted to fading memory spaces. In Chapter 8 the stability problem in functional differential equations on a fading memory space is studied in connection with limiting equations. Chapter 9 discusses the existence of periodic and almost periodic solutions of functional differential equations.

This is a very important monograph; it should be on the shelf of every mathematician who makes research on functional differential equations.

L. Hatvani (Szeged)

I. S. Hughes, Elementary Particles, Third edition, XXII+431 pages, Cambridge University Press, Cambridge—New York—Port Chester—Melbourne—Sydney, 1991.

It would be difficult to find any other science that has developed in recent years as fast as particle physics. Therefore there is a great demand for textbooks presenting the subject in a comprehensive manner. This was the aim of the author and he fulfilled his task very well, by upgrading the earlier editions of the book.

The text is written for undergraduates in physics, but it can be interesting for a mathematical physicist as well, who deals with gauge field theory and related issues. The volume explains in simple terms the interesting interplay between actual physical experiments and theoretical concepts, that has led to the great revolution in particle physics in the last two decades. The reader gets some insight, how people do work in the huge laboratories of the few giant accelerator centers of the world, what

are the principal features of their complicated apparatus, and understands the common aims and efforts of theorists and experimentalists to understand and classify the different interactions of elementary particles.

The organization of the chapters follows the historical formation of particle physics, what is certainly the consequence of the fact, that this is already the third edition of the book. This can be advantageous from a pedagogical viewpoint, but leads also to a certain kind of unbalance in the exposition. For instance the detailed and common discussion of muons and pions — such very different particles — should have been possibly avoided. On the other hand the reader can find every important fact of the subject in this book, the theory of leptons, quarks, gluons, weak bosons, spontaneous symmetry breaking, supersymmetry and all that explained only with simple quantum mechanics. Especially remarkable is the last chapter, written for this third edition about the connection between particle physics and cosmology.

M. Benedict (Szeged)

Arthur Jones—Sidney A. Morris—Keneth R. Pearson, Abstract Algebra and Famous Impossibilities (Universitext) X+187 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong—Barcelona—Budapest, 1991.

In this book three of the oldest problems of mathematics are discussed. They are the cube duplication (the Delian problem), the angle trisection and the quadrature of the circle. These are construction problems performed by straightedge and compass, which are more than 2000 years old and have mytical origin known in several version. For example, one version of the problem of doubling the cube, found in a work of Eratosthenes (c. 284—192 B.C.) relates that the Delians, suffering from pestilence, consulted the oracle, who advised constructing an altar double size of the existing one. The Delian realized that doubling the side would not double the volume and therefore they turned to Plato, who told them that the god of the oracle had not so answered because he wanted or needed a doubled altar, but in order to censure the Greeks for their indifference to mathematics and their lack of respect for geometry. Plutach also gives this story.

Actually, these construction problems are extensions of problems already solved by the Greeks. Various explanations of the restriction to straightedge and compass have been given. The straight line and the circle were, in the Greek view, the basic figures, and the straightedge and compass are their physical analogues. Hence constructions with these tools were preferable. The reason is also given that Plato objected to other mechanical instruments because they involved too much of the world of sense rather than the world of ideas, which he regarded as primary.

The unsolvability of these problems was proved in the last century, based on the Galois theory and Lindemann's result on the transcendence of π.

It is very useful if these questions are in the curriculum of mathematics, but after a course of Galois theory (with a course on Group theory as prequisite) and after a course of Complex variables very few students can be involved in it.

The excellent book of Jones, Morris and Pearson solves this problem by giving a very simple and nearly self-contained treatment of the unsolvability of the three ancient constructing problems. Most of the material needs only some knowledge of linear algebra. This is the content of the first six chapters.

In the fairly independent Chapter 7 complete and elegant proofs of transcendence of e and π are given. In contrast to the most known proofs they need only elementary facts from the calculus.

A special feature of this volume that at the begining of the more complicated or long proofs there is an outline of the procedure.

Each chapter contains examples and exercises which makes the book more comfortable for teaching purposes. It is warmly recommended for second year university courses.

Lajos Klukovits (Szeged)

H. Jürgensen—F. Migliorini—J. Szép, Semigroups, 121 pages, Akadémiai Kiadó, Budapest, 1991.

The authors describe the book in the introduction as follows:

"This volume does not attempt to provide a "complete" presentation of semigroup theory. Instead, we focus on essentially one aspect: the classification of elements by properties of the induced translation and the related global structural properties of a semigroup. Several new results were found in particular, on increasing elements in semigroups and many new open problems were identified. In this sense, we hope that this book may serve not only as a summary but also a starting point for further research."

L. Megyesi (Szeged)

Frances Kirwan, Complex Algebraic Curves (London Mathematical Society Student Texts, 23), VIII+264 pages Cambridge University Press, Cambridge—New York—Port Chester—Melbourne—Sydney, 1992.

The purpose of this book is to give an introduction to the elementary methods of algebraic geometry and Riemann surface theory on the basis of the usual undergraduate courses of algebra, surface topology and complex analysis. Chapter 1 contains a collection of motivations from different areas of classical mathematics and historical background for the study of algebraic curves. Chapter 2 is devoted to the introduction of complex projective space and to the investigation of elementary properties of algebraic curves in this space. Chapter 3 studies the tangent and intersection properties of complex algebraic curves. Chapter 4 gives an investigation of the intuitive topological properties of algebraic curves and proves the degree-genus formula. In Chapters 5 and 6 the methods of holomorphic and meromorphic function theory are used for the study of the relations between complex algebraic curves and Riemann surfaces. There is given an introduction to the theory of abelian integrals and to the Riemann—Roch theory of nonsingular projective curves in the complex projective plane. Finally Chapter 7 is devoted to the study of the singularities of algebraic curves.

The book contains three appendices on the basic results of algebra, topology and complex analysis which are used in the treatment. Thus it is as self-contained as possible. There are given many exercises of different difficulties.

This well-organised book can be recommended to lecturers and students of universities and for mathematicians who are interested in the interrelation of algebra, geometry and analysis.

Péter T. Nagy (Szeged)

Helmut Koch, Introduction to Classical Mathematics I (Mathematics and Its Application, 70), XII+453 pages, Kluwer Academic Publishers, Dordrecht—Boston—London, 1991.

The main purpose of this volume is expressed by the author as follows: "This book is directed towards all those who have mastered two years of university mathematics. It aims to convey an overview of classical mathematics, particularly that of the 19th centrury and the first half of 20th century".

Here "classical" means that the methods and results discussed are the real classics of mathematics. Motivation, especially the original motivation, is a prime concern and so is clarity and, consequently, proofs and discussions are in terms of modern concepts and ideas. The order of the contents follows historical development beginning with Gauss', *Disquesitions Arithmeticae* and ending with the *Idee der Riemannschen Fläche* of Weyl.

The book is divided into 30 chapters, all of them end with exercises.

For further orientation here are some of the most characteristic chapter headings: Congruences; Quadratic forms; Theory of surfaces; Harmonic analysis; Prime numbers in arithmetic progressions; Theory of algebraic equations; The beginnings of complex function theory; Entire functions; Riemann surfaces; Elliptic functions; Riemann geometry; Field theory; Dedekind's theory of ideals; Theory of algebraic functions of one variable; Proof of the prime number theorem; Combinatorical topology.

The book is well written, the presentation of the material is clear. The necessary prerequisities are a basic knowledge of algebra and calculus.

This very valuable, excellent book is recommended to researchers, students and historian of mathematics interested in the classical development of mathematics.

J. Németh (Szeged)

Bernhard Korte—László Lovász—Rainer Schrader, Greedois. (Algorithms and Combinatorics, 4), 211 pages, Springer-Verlag, Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong, 1991.

This book is organized as follows. After an exhausting historical overview the authors review some basic concepts of matroid theory in Chapter II. Chapter III gives a comprehensive study of antimatroids and gives a large variety of examples. In the next three chapters the basic toolbox of matroid theory to greedoids have been extended. The chapters VII—X deal with special classes of greedoids. The first three of these answer the general question: which greedoids can be obtained from matroids and antimatroids by certain construction principles? In Chapter X the class of transposition greedoids is treated. Chapter XI was devoted to the optimization in greedoids and the last section deals with the connection between greedoids and topology.

The algorithmic principles play an ever increasing role in mathematics. The connection between the algorithms and the structure of the underlying mathematical object is obvious. The idea of greediness plays a fundamental role not only in discrete algorithms but in the design of continuous algorithms as well. This excellent book leads the reader to the current borderline of open research problems of greedoid theory. By unifying different approaches this self-contained book is an indispensable tool for all scientists interested in algorithmic aspects and computer science.

Gábor Galambos (Szeged)

P. Latiolais, Topology and Combinatorial Group Theory, (Lecture Notes in Mathematics, 1440), VI+207 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong—Barcelona, 1990.

In the last decades more and more nice and deep theorems and methods was born as a result of the interaction among topology, algebraic topology and combinatorial group theory. The aim of the editor of this proceeding was to present some of the most typical results of this subject. From the contents:

I. L. Anshel, On two relator groups; M. A. Bogley, When is the homotopy set $[X, Y]$ infinite?; R. N. Cruz, Periodic knots and desuspensions of free involutions on spheres; C. Droms, J. Lewin, H. Servatius, The Tits conjecture and the five — string braid group; C. Droms, B. Servatius, H. Servatius, The finite basis extension property and graph groups; B. Fine, Subgroup presentations without coset representatives; M. Frame, J. Hefferon, Fractal dimensions of limit sets of some Kleinian groups; R. Goldstein, Bounded cancellation of automorphisms of free products; C. Hog — Angeloni, A short topological proof of Cohn's theorem; C. Hog — Angeloni, On the homotopy type of 2 — complexes with a free product of cyclic groups as fundamental group; C. Hog — Angeloni, M. P. Latiolais, W. Metzler, Bias ideals and obstructions to simple — homotopy type; G. Huck, Embeddings of acyclic 2 — complexes in S^4 with contractible complement; W. Imrich, E. C. Tuener, Fixed subsets of homomorphisms of free groups; G. Lupton; A note on a conjecture of Stephen Halperins; M. Lustig, On the rank, the deficiency and the homological dimension of groups: the computation of a lower bound via Fox ideals; S. Rosenbrock, A reduced spherical diagram into a ribbon — disk complement and related examples; C. Schaufele, N. Zumoff, *-groups, graphs, and bases; T. W. Tucker, some topological graph theory for topologists: A sampler of covering space constructions;

The ideas and methods of these papers can be regarded as a kernel from which a new theory can be developed. This volume is recommended to everybody who is interested in this new subject of mathematics.

J. Kincses (Szeged)

Stanislaw Łojasiewicz, Introduction to Complex Analytic Geometry, XIV+523 pages, Birkhauser-Verlag, Basel—Boston—Berlin, 1991.

This monograph is a self-contained presentation of the basic results and methods of complex analytic geometry, i.e. the geometry of analytic spaces (sets) described by systems of analytic equations.

We can fully agree with the aim of the author: "It does not pretend to reflect the entire theory. Its aim is to familiarize the reader with the basic range of problems, using means as elementary as possible." So that it presents a number of the results and techniques in detail.

The first 138 pages develop most of the necessary background material on algebra, topology and complex analysis (on complex manifolds). The first chapter deals with rings of holomorphic functions, while the notion of analytic sets and germs can be found in the following chapter. The aim of the third chapter is to make clear the local structure of analytic sets. As a consequence of Rickert's descriptive lemma can be found the Hilbert Nullstellensatz. Chapter IV and V include some observations on local structure, singularity problems and holomorphic mappings (Rouche's theorem, Andreotti—Stoll theorem). Problem of normalization is considered in Chapter 6, based on Cartan-Oka theorem. The last chapter contains a comprehensive presentation of the ideas of Serre about the "necessary" algebraicity of analytical objects in projective spaces, including the

most important theorems on algebraicity and normality from the elementary discussion of the manifold structure on the projective and Grassmannian spaces up to the characterization of biholomorphic mappings of Grassmann manifolds.

This new edition is an important contribution to the (English language) literature. It is a slightly revised and extended version of the Polish edition (translated from Polish by Maciej Klimek). The important changes in chapter V and VI are, first of all, the Grauert—Remmert formula, Cartan's closedness theorem and Serre's normality criterion (among others). These changes call forth some minor corrections and additional remarks in the first chapters, as well.

The book is a clearly written excellent expository text on the theory. It is carefully organized so convenient for the reader for individual study or as a text-book of seminars.

J. Kozma (Szeged)

Mathematik, Realität and Asthetik — Eine Bilderfolge Zum VLSI Chip Desing —, Mathematics, Reality, and Aesthetics — A Picture Set on VLSI Chip Design —, Forschungsinstitut für Diskrete Mathematik Rheinische Friedrich — Wilhelms — Universität, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong—Barcelona, 1991.

About fifty years ago in the course of a discussion on the teaching of calculus in the secondary schools G. Alexits — a famous Hungarian mathematician — tried to sketch the intention of the teaching of mathematics in these schools. In his opinion one of the most important thing is the emphasis of the aesthetical features of the subject. G. H. Hardy wrote the following words in his booklet "A Mathematician's Apology": "The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colours or the words, must fit together in a harmonious way." The aesthetical features of the mathematics show a great variety. This (picture) book is produced by the Forschungsinstitut für Diskrete Mathematik Rheinische Friedrich — Wilhelms — Universität. The role of the discrete mathematics is constantly increasing. The partical and also the theoretical problems demand the changes. Perhaps the best is to cite some sentences of the (short) texts (the texts are in German and in English): "The Research Institute of Discrete Mathematics/Institute of Operations Research of the University of Bonn is engaged in the mathematical calculation and desingn of VLSI (very large scale integrated) logic chips within the framework of a scientific cooperation contract with IBM Germany.

The pictures shown here have been chosen to provide an insight into this design process. We have in particular tried to emphasize the contrast between the mathematical design (plotter plan) and physical reality (microphotograph of the chip).

For this purpose we have chosen a telecommunication chip with the code name ZORA.

It is especially satisfying for a mathematician interested in applications to see a direct relationship between the mathematical model and reality. We begin by showing several examples of the ZORA chip which can also be viewed in tenfold magnification. We next present a complete wiring and placement plan as calculated with methods of discrete mathematics. This is contrasted with a picture of the produced chip magnified 40fold. The pictures that follow show corresponding portions magnified 220fold to 4500fold.

As mathematics — and its applications — always has an aesthetic component, we have made the daring attempt to contrast some of our pictures which are a direct result of our desing algorithms with several chosen pictures of modern constructivist art. We hope that our artistically interested public as well as the artists themselves: De Stijl, Bauhaus, Mondrian, Albers, Bill, Lohse will forgive us."

This unusual book gives an interesting visual adventure to the reader and if he is a teacher then he will show these pictures to his students too, and perhaps everyone will see the connections of mathematics, arts and technology in another way.

L. Pintér (Szeged)

Matroid Applications. Edited by Neil White, 350 pages, Cambridge University Press, Cambridge—New York—Port Chester—Melbourne—Sydney, 1992.

This is the third volume of a series that began with Theory of Matroids and continued with Combinatorial Geometrics.

This volume begins with a chapter on the applications of matroid theory to the rigidity of frameworks (Walter Withley). In the next chapter M. Deza discusses the perfect matroid design problem which is one of the most beautiful application of the matroids. In Oxley's chapter different methods are considered for generalizing the matroid axioms to infinite ground sets. The next chapter (Simoes Pereira) is dealing with the matroidal families of graphs. Rival and Stanford consider two questions of algebraic aspects of partition lattices. T. Brylawski and J. Oxley discusses the matroid connection of the Tutte Polynomial and its applications. The last but one chapter (by A. Björner) desribes the homology and shellability properties of several simplicial complexes associated with a matroid. The book is concluded with an exposition by Björner and Ziegler on greedoids.

The book concentrates on the applications of matroid theory to a variety of topics from geometry, combinatorics and operation research. The contributors have written their articles to form a cohesive account so this volume is a valuable reference for research workers.

Gábor Galambos (Szeged)

Peter Meyer-Nieberg, Banach Lattices, XV+395 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Tokyo—Hong Kong—Barcelona—Budapest, 1991.

The book contains five chapters each of which is divided into sections. The first chapter introduces the notions of Riesz spaces and Banach lattices and develops the classical theory of these spaces. The second chapter is devoted to classical Banach lattices and contains technical results being essential for the remainder of the book. The third chapter studies operators which are defined on Riesz spaces or have values in a Riesz space from topological as well as lattice theoretical point of view. The fourth chapter is concerned with the spectral properties of positive operators on complex Banach lattices. In this chapter the so-called order spectrum of regular operators is also introduced. The last chapter investigates the structural properties of Banach lattices. At the end of each section a rich collection of exercises can be found. The familiarity of the reader with the Banach space theory is supposed.

L. Gehér (Szeged)

Microlocal Analysis and Nonlinear Waves, Edited by Michael Beals, Richard B. Melrose and Jefferey Rauch, (The IMA Vlumes in Mathematics and its Applications, 30), XI+199 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong—Barcelona, 1991.

The volume contains articles based on proceedings of a workshop which was a part of the 1988—89 IMA program on "Nonlinear Waves". Twenty years ago it was shown that some methods used for the investigation of the behaviour of linear hyperbolic waves can be applied to nonlinear

problems, too. The use of these relatively new techniques characterizes the articles of this volume. The titles and authors of the works: On the interaction of conormal waves for semilinear wave equations (A. S. Barreto); Regularity of nonlinear waves associated with a cusp (U. Beals); Evolution of a punctual singularity in an Eulerian flow (J. Y. Chemin); Water waves, Hamiltonian systems and Cauchy integrals (W. Craig); Infinite gain of regularity for dispersive evolution equations (W. Craig, T. Kappeler and W. Strauss); On the fully nonlinear Cauchy problem with small data. II (L. Härmander); Interacting weakly nonlinear hyperbolic and dispersive waves (J. K. Hunter); Nonlinear resonance can create dense oscillations (J-L. Joly and J. Rauch); Lower bounds of the life-span of small classical solutions for nonlinear wave equations (Li Ta-Tsien); Propagation of stronger singularities of sulutions to semilinear wave equations (Liu Lingi); Conormality, cusps and non-linear interaction (R. B. Melrose); Quasimodes for the Laplace operator and glancing hypersurfaces (G. S. Popov); A decay estimate for the three-dimensional inhomogeneous Klein—Gordon equation and global existence for nonlinear equations (T. C. Sideris); Interaction of singularities and propagation into shadow regions in semilinear boundary problems (U. Williams).

L. Pintér (Szeged)

Miscellanea Mathematica, Edited by P. Hilton, F. Hirzenbruch, R. Remmert, XIII+326 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong—Barcelona—Budapest, 1991.

Mathematics is surrounded by certain mysticism. It is mostly due to its exactness, abstractness and his own individual language which often disguise its origins in and connections with the physical world. Publishing mathematics, therefore, requires special efforts and talent. Dr. Heinz Götze, with his typical enthusiasm, took up this challenge and has dedicated his life to scientific publishing. He has made a unique and invaluable contribution to the spread of the mathematical culture.

A group of 22 eminent mathematicians, including the editors of this volume, has decided to publish a "Festschrift" for him (a series of papers dedicated to him).

The result of their efforts is this volume which contains 22 independent articles. They are not usual research papers rather contributions to the culture of mathematics. Most of them have strong historical and/or personal feature.

The reviewer is sure that this Festschrift, this series of essays will be enjoyed by mathematicians and a lot of non-mathematicians, teachers and students of mathematics, everybody who interested in the culture of mathematics.

Lajos Klukovits (Szeged)

Numerical Methods for Free Boundary Problems, Edited by P. Neittaanmäki (International Series of Numerical Mathematics, 99), XV+439 pages, Birkhäuser-Verlag, Basel—Boston—Berlin, 1991.

This volume contains 4 invited lectures and 35 contributed papers of a Conference held at the University of Jyväskylä, Finland on July 23—27, 1990.

The invited lectures were: H. W. Alt and I. Pawlow: A mathematical model and an existence theory for non-isothermal phase separation — V. Barbu: The approximate solvability of inverse one-phase Stefan problem — H. D. Mittelmann, C. C. Law, D. F. Jankowski and G. P. Neitzel:

Stability of thermocapillary convection in float-zone crystal growth — V. Rivkind: Numerical solution of coupled Navier-Stokes and Stefan equations.

The contributed papers can be grouped around the topics Stefan like problems, optimal control, optimal shape design, identification, dam and fluid flow problems.

Many participants of the Conference came from Eastern Countries and their papers — hitherto hardly accessible to Western scientists — could be of special interest to both mathematicians and applied scientists.

János Virágh (Szeged)

Jan Okniński, Semigroup Algebras (Monographs and Textbooks in Pure and Applied Mathematics, 138), IX+357 pages, Marcel Dekker, Inc., New York, 1991.

The present book is the first monograph on the theory of noncommutative semigroup rings. This branch of ring theory has grown rapidly during the last ten years, and has proved to be very useful not only for constructing examples in various domains of ring theory but also as a tool in theories like those of linear representations of semigroups, representations of finite dimensional algebras, growth and Gelfand—Kirillov dimension of algebras.

Here is the table of contents of the book:

Part I. Semigroups and their algebras: 1. Completely 0-simple and linear semigroups. 2. Semigroups with finiteness conditions. 3. Weakly periodic semigroups. 4. Semigroup algebras: general results and techniques. 5. Munn algebras. 6. Gradations. — Part II. Semigroup algebras of cancellative semigroups. 7. Groups of fractions. 8. Semigroups of polynomial growth. 9. Δ-methods. 10. Unique-/two-unique-product semigroups. 11. Subsemigroups of polycyclic-by-finite groups. — Part III. Finiteness conditions. 12. Noetherian semigroup algebras. 13. Spectral properties. 14. Descending chain conditions. 15. Regular algebras. 16. Self-injectivity. 17. Other finiteness conditions: a survey. — Part IV. Semigroup algebras satisfying polynomial identities. 18. Preliminaries on PI-algebras. 19. Semigroups satisfying permutational property. 20. PI-semigroup algebras. 21. The radical. 22. Prime PI-algebras. 23. Dimensions. 24. Monomial algebras. 25. Azumaya algebras. — Part V. Problems.

Most of the material comes from the literature of the past 10 years, and several new results are included. The author's main concern was ring theoretical properties for which a systematic treatment could be presented. The starting point is mostly results on group rings, in the case of PI-semigroup algebras also those on commutative semigroup rings. The approach is that of ring theory, no special class of semigroups (except cancellative ones) is considered for its own sake. In consequence of this approach, putting together the results from pure semigroup theory in the book, one gets a rather specific and unusual but interesting selection of material.

Each chapter ends with bibliographical notes and comments on related results appearing in the literature. The last part presents 37 open problems (many of them extracted from the main text) with information on partial results and sometimes comments on possible developments.

Summarizing: This book is a valuable contribution to the literature. It puts together an important collection of results, and will therefore certainly serve as a basic reference in the field. By developing various interesting topics up to the borders of our present-day knowledge, it will hopefully stimulate further research. The exposition is very clear, suitable also for graduate students who are familiar with the fundamental results in ring theory. For the reviewer it was a pleasure to read this book.

László Márki (Budapest)

Robert E. O'Malley, Jr., Singular Perturbation Methods for Ordinary Differential Equations, (Applied Mathematical Sciences, 89), VIII+225 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong—Barcelona—Budapest, 1991.

Perturbation theory provides a useful collection of methods for the study of equations close to equations of a specific (simpler) form. These equations are called *unperturbed,* and their solutions are assumed to be known. Perturbation theory studies the effect of small changes in the differential equations on the behaviour of solutions. The perturbed problem $P_\varepsilon(y_\varepsilon)=0$ (e.g. a boundary value problem, an integral or other operator equation) typically contains a small parameter $\varepsilon>0$ which represents the influence of many nearly negligible physical influences. The problem is a regular perturbation problem if its solution $y_\varepsilon(x)$ converges as $\varepsilon\to 0$ to the solution $y_0(x)$ of the unperturbed (limiting) problem $P_0(y_0)=0$. A singular perturbation is said to occur whenever the regular perturbation limit $y_\varepsilon(x)\to y_0(x)$ fails. This is the case e.g. if the small parameter ε is the coefficient of the highest derivative in the differential equation.

The book treats both the initial and boundary value problems, linear and non-linear ones. The methods are illustrated by interesting applications such as relaxation oscillations, a combustion model, semiconductor modeling, shocks and transition layers, nonlinear control problems. The numerous exercises closing the sections are extremely valuable.

This well-written and well-organized book can be highly recommended to both mathematicians and users of mathematics interested in ordinary differential equations.

L. Hatvani (Szeged)

Bruce P. Palka, An Introduction to Complex Function Theory (Undergraduate Texts in Mathematics), XVII+559 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong—Barcelona, 1991.

This book is the outgrowth of lectures held by the author at the University of Texas. The work is intended for a broad class of students, for students who are interested in practical questions and also for students who are primarily interested in theoretical problems. This great variety of the audience (and — I hope — the readers) requires a considerable effort from the author. To find the right level of mathematical rigor and to present the necessary details of considerations is such a task which does not go without rich pedagogical experiences. To offer the right emphasis on techniques and — on the other hand — on concepts and motivation is an important problem of the author. For the reviewer the most characteristic feature of this work is the excellent "mixing" of the conservative and modern discussions. Also the titles of the points of Chapter 5 (Cauchy's Theorem and its Consequences) give an insight into the attitude of this work: The Local Cauchy Theorem; Winding Numbers and the Local Cauchy Integral Formula. (Do you know where the expression "winding number" comes from? The author gives an answer to this question.); Consequences of the Local Cauchy Integral Formula; More about Logarithm and Power Functions; The Global Cauchy Theorems; Simply Connected Domains; Homotopy and Winding Numbers; Exercises (There are 83 exercises here, some of them with hints).

In the book you can find the customary themas of complex function theory, sequences and series of analytic functions, isolated singularities, conformal mapping and so on. My favourite one is the (short) chapter on harmonic functions. I am sure that the reader after investigating this chapter will be eager to read more about these problems.

The exercises worked out in the text and especially the proposed exercises at the end of the chapters represent an essential part of the work. Sometimes it is evident that the author spent as much effort in preparing these examples as he did on the corresponding main text itself.

At last I would like to mention the great number of the beautiful figures.

L. Pintér (Szeged)

Prospects in Complex Geometry. Proceedings of the 25th Taniguchi International Symposium held in Kata, and the Conference held Kyoto, July 31—August 9, 1989. Edited by J. Noguchy—T. Ohsawa (Lecture Notes in Mathematics, 1468), 421 pages, Springer-Verlag, Berlin—Heidelberg, 1991.

These contributions report on recent research on a wild spectrum of modern geometry. The central subject is complex structure with the point on geometric connections.

Each article is written by a prominent author specially for this volume.

Contents. Hyperkähler Structure on the Moduli Space of Flat Bundles (A. Fujiki), Hardy Spaces and BMO Riemann Surfaces (H. Shiga), Application of a certain Integral Formula to Complex Analysis (K. Takegoshi), On Inner Radii of Teichmüller Spaces (T. Nakanishi—T. Velling), On the Causal Structures of the Šilov Boundaries of Symmetric Bounded Domains (M. Taniguchi), A strong Harmonic Representation Theorem on Complex Spaces with Isolated Singularities (T. Ohsawa), Mordel-Weill Lattices of Type E_8 and Deformation of Singularities (T. Shioda), The Spectrum of a Riemann Surface with a Cusp (S. Wolpert), Moduli Spaces of Harmonic and Holonormic Mappings and Diophantine Geometry (T. Miyano), Global Nondeformability of the Complex Projective Space (Y.-T. Siu), Some Aspects of Hodge Theory on Non-Complete Algebraic Manifolds (I. Bauer—S. Kosarew), L^p-Cohomology and Satake Compactifications (S. Zucker), Harmonic Maps and Kähler Geometry (J. Jost—S. T. Yau), Complex-Analycity of Pluriharmonic Maps and their Constructions (Y. Ohnita—S. Udagawa), Higher Eichler Integrals and Vector Bundles over the Moduli of Spinned Riemann Surfaces (K. Saito).

J. Kozma (Szeged)

S. Prössdorf—B. Silbermann, Numerical Analysis for Integral and Related Operator Equations (Operator Theory: Advances and Applications, 52), 542 pages, Birkhäuser-Verlag, Basel—Boston—Berlin, 1991.

This monograph is devoted to the investigation of the 'boundary element methods' (sometimes referred to as 'boundary integral equation methods') for solving boundary value problems.

As the Authors state: "The book is addressed to a wide audience of readers. We hope that both the mathematician interested in theoretical aspects of numerical analysis and the engineer wishing to see practically realizable recipes for computations will find a few suggestions."

And now the bad news: "... The study of the equations we encounter... requires having recourse to a series of heavy guns from mathematical analysis." Chapters 1, 2 and 6 contain the theoretical background.

The primary aim of the book is to demonstrate the power of Banach algebra techniques in numerical analysis. In Chapter 7 they are introduced and applied to the finite section and collocation methods for singular integral operators. In Chapters 10—13 this approach is carried over to spline collocation and spline Galerkin methods. For further orientation here are a few characteristic

notions tackled throughout the book: the convergence manifold concept for Fredholm integral equations of the second type, Wiener—Hopf integral equations and convolution equations of the Mellin type.

The 'Notes and Comments' part at each chapter gives full references and historical remarks of the presented material. Equipped with Notation, Name and Subject indices this book is a valuable source of information for all specialists working in this field.

János Virágh (Szeged)

Jeffrey Rauch, Partial Differential Equations (Graduate Texts in Mathematics), X+263 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong—Barcelona—Budapest, 1991.

This work is based on a course given by the author at the University of Michigan. In our days perhaps one of the main problems of the lectures (writers) of partial differential equations is the following. Several important mathamatical notions appear in connections with problems concerning partial differential equations at first. See e.g. the Fourier series, various orthogonal systems and so on. But at the same time e.g. the application of the theory of Fourier series, Fourier transforms and other mathematical notions becomes an indispensable and powerful tool to our treatment. So e.g. the theory of the partial differential equations gives a natural introduction of the notion of the orthogonal series and at the same time it uses the results of the theory of these series. (Sometimes these results are relatively deep ones.) The author of this work assumes that the reader is trained in advanced calculus, real analysis, complex analysis and functional analysis. In an appendix there is a short introduction into the theory of distributions, but since from Chapter 2 the distribution theory is the basic language of the text. I hope that the reader has some knowledge from this theory, too. Although the aim of the author is to present such a text which requires no previous knowledge of differential equations, in my opinion only the reader who has some classical bases in differential equations will enjoy this work really. But for a qualified reader I can not recommend a better work in partial differential equations (taking into account the number of pages, too). I think this is a modern up to date discussion. The style is clear and inspiring. I like the remarks: "The reader is invited to give the generalizations by using the language of..."; "There are many ways of defining the notion of a function with derivative in $L^2(I)$. Most are equivalent and useful. One which is not good is that..." and the similar ones. Chapter headings and some titles of points are: Power series methods (The fully nonlinear Cauchy—Kovalevskaya theorem; F. John's global Holmgren theorem; Characteristics and singular solutions); Some harmonic analysis (Tempered distributions; L^2 derivatives and Sobolev spaces); Solution of initial value problems by Fourier synthesis (Schrödinger equation; Fourier synthesis for the heat equation; Fourier synthesis for the wave equation; Inhomogeneous equations, Duhamel's principle); Propagators and x-space methods (Applications of the heat propagator; The wave equation propagator for $d=1$, for $d=3$; The method of descent); The Dirichlet problem (Dirichlet's principle; The direct method of the calculus of variations; The Fredholm alternative; Maximum principles from potential theory).

L. Pintér (Szeged)

Reinhold Remmert, Funktionentheorie II (Grundwissen Mathematik, 6), XIX+299 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong—Barcelona—Budapest, 1991.

For a reader about to embark on research work in any field in which complex function theory plays a part this volume is a splendid and probably an essential introduction. The most characteristic

feature of this book is a strange phenomenon: the readers see the finished work (building) but at the same time they see the starting-points, the development of the work (the support of the building). This development may be followed closely step by step. E.g. in the first chapter the presentation of the well-known product form of $\sin \pi z$ is a mathematical gem. The presentation of the material reminds me of the best Pólya's works. I have seen only a few books where the historical remarks form such an essential and natural part of the work as in the present case. To mention an especially interesting citation see p. 60 where we find: "Demonstratio formulae $\int_0^1 w^{a-1}(1-w)^{b-1}dw...$" by C. G. J. Jocobi, in original form written in Latin. (Of course this is not the most significant citation but a strange one.) The book consists of three main parts: infinite products, theory of mappings and selecta, and these parts consist of chapters. Some themas from these chapters: products of holomorphic functions; the gamma function; entire functions; holomorphic functions with given zeros; functions with given rational singular part; theorems of Vitali and Montel; the Riemann's mapping theorem etc. (The chapter on Riemann's mapping theorem is my favourite one in this book. A citation from the book: Ahlfors: "Riemann's writings are full of almost cryptic messages of the future. For instance, Riemann's mapping theorem is ultimately formulated in terms which would defy any attempt of proof, even with modern methods." Here are some mathemeticians who worked on the mapping theorem: C. Neumann, H. A. Schwarz, H. Poincaré, D. Hilbert, P. Koebe, C. Carathéodory, L. Fejér and F. Riesz. What a list of names!) The third part (Selecta) consists of special important questions. In general one cannot find these problems gathering systematically in one volume. Some of these questions are: theorems of Bloch, Schottky, Picard (after the „small" Picard's theorem we have "two amusing applications"), Fatou, M. Riesz, Ostrowski and the theory of Runge. Perhaps a reference to the solution Bieberbach's conjecture by de Branges fails to me in this part. Naturally this subjective remark does not diminish the advantages of this excellent work.

L. Pintér (Szeged)

Representation Theory of Finite Groups and Finite-Dimensional Algebras, Edited by G. O. Michler and C. M. Ringle, Proceedings of the Conference at the University of Bielefeld from May 15—17, 1991, IX+520 pages, Birkhäuser-Verlag, Basel—Boston—Berlin, 1991.

Besides the seventeen research papers in this book the first 220 pages are devoted to seven survay articles, which are:

B. Fischer: Clifford matrices,

B. Huppert: Research in representation theory at Mainz (1984—1990),

K. Lux and H. Pahlings: Computational aspects of representation theory,

B. M. Matzat: Der Kenntnisstand in der konstruktiven Galoisschen Theorie,

G. O. Michler: Contributions to modular representation theory of finite groups,

C. M. Ringel: Recent advances in the representation theory of finite dimensional algebras, and

K. W. Roggenkamp: The isomorphism problem for integral group rings of finite groups.

These papers give a good account of what progress has been made in group representation theory recently and how are the recent developments related to classical results and problems.

I recommend this excellent book mainly for experts of group (representation) theory and related topics.

Gábor Czédli (Szeged)

Y. S. Samoilenko, Spectral Theory of Families of Self-Adjoint Operators (Mathematics and Its Applications, 57), XVI+293 pages, Kluwer Academic Publishers, Dordrecht—Boston—London, 1991.

This volume is a translation by E. V. Tisjachnij; the title of the original work is: »Элементы математической теории многочастотных колебаний. Инвариантные торы« and it was published by »Наука« in Moscow, 1987.

The book deals with finite and countable families of self-adjoint operators which are connected by various algebraic relations. Such families are closely connected with representation theory of Lie groups and Lie algebras and are applied in the mathematical models of quantum systems.

Part I is devoted to commutative families of self-adjoint operators and discusses their joint spectral properties, the connections of such families with the unitary representations of inductive limits of certain Lie groups and (as illustration) deals with differential operators on functions of countably many variables. In Part II countable dimensional Lie algebras are discussed which are inductive limits of finite dimensional ones. Dealing with their representations, families of self-adjoint operators are treated, which establish basises in these Lie algebras. In Part III some algebraic relations are exactly defined for unbounded self-adjoint operators, and collections satisfying such relations are considered. Among others spectral properties are studied and structure theorems are given. In Part IV constructive methods of description of non-commutative random sequences are presented. The Bibliography lists more than five hundred items connected with the contents of the book.

The reader needs (of course) some backgrounds. The prerequirements include the basic theory of $*$-unbounded self-adjoint operators, Lie groups and Lie algebras as well as some knowledge of algebras and their representations. The book can be recommended to mathematicians and physicists interested in spectral theory, Lie algebras, (non-commutatative) probability, statistical physics, physical systems with many degrees of freedom or quantum field theory.

E. Durszt (Szeged)

W. M. Schmidt, Diophantine Approximations and Diophantine Equations (Lecture Notes in Mathematics, 1467), VIII+217 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong—Barcelona—Budapest, 1991.

This is the printed version of the author's lecture at Columbia University in the fall of 1987 and at the University of Colorado 1988/1989.

The text is devided into five chapters. The main topics of the first chapter are: Siegel's Lemma and heights (or "field height"). The second chapter is devoted to Roth's theorem, and its some useful generalizations. The Thue equation is in the centre of the third chapter. Among others there are interesting new results given by Bombieri, Mueller and the author (on the number of solutions of such equation, furthermore on the number of solutions of Thue equation with few nonzero coefficients). The fourth chapter deals with the S-unit equations and hyperelliptic equations. One of the interesting equations is: $2^x+3^y=4^z$; Evertse's results for this equation are very useful. The final chapter is devoted to diaphantine equations in more than two variables.

The rich Bibliography includes more than hundred references.

The book is easy-to-read, it may be a useful piece of reading not only for experts but for students as well.

J. Németh (Szeged)

J. B. Seaborn, Hypergeometric Functions and Their Applications, (Texts in Applied Mathematics, 8), XI+250 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong—Barcelona, 1991.

The main purpose of this book is to develop the theory of special functions which often occurs in applied mathematics, engineering and in classical and quantum physics. These functions (gamma function, Bessel functions, Hermite, Lagrange and Laguerre polinomials etc.) are solutions of differential equations, but more equivalent ways of defining these functions can be found in the text. It is shown that these functions can be expressed in terms of special power series, called hypergeometric functions which is the most practical method to the study and numerical calculation of these functions. The text is divided into 12 chapters. It is assumed that the reader is familiar with classical analysis and has some knowledge of Schrödinger equation.

L. Gehér (Szeged)

V. A. Smirnov: Renormalization and Asymptotic Expansions (Progress in Physics, 14), X+380, pages, Birkhäuser-Verlag, Basel—Boston—Berlin, 1991.

The monograph treats the fundamental problem of quantum field theory, how to remove divergences in the perturbation expansion of Feynman amplitudes. This is a very important procedure in modern theoretical physics, and it has shown up considerable successes in calculating experimentally measurable quantities. Renormalization is not only a certain calculational method, however, but also a theoretical construction involving group theory, graph theory, and the author tries to introduce both the practical and the principial aspects of the subject. The book is divided into three parts. The first one outlines the general problem and characterizes the nature of divergences. Part two is devoted to the different regularization schemes the Bogoliubov — Parasiuk — dimensional, the analytic and the auxiliary mass renormalizations. The infrared counterpart of usual renormalization is examined in detail. Part three contains the methods of asymptotic expansions, when the relevant energies and momenta are large. It is a pity that the author does not place this very interesting theme into a somewhat wider scope, at least a more detailed introduction would have been very useful. The style and the presentation is rather technical. Therefore, the book can be recommended mainly to the experts in quantum field theory.

M. Benedict (Szeged)

André Weil, The Apprenticeship of a Mathematician, 198 pages, Birkhäuser-Verlag, Basel—Boston—Berlin, 1992.

This excellent book is the English edition of the author's autobiography. It shows the life of a great mathematician whose horisons have never been limited to mathematics. His career led him to a lot of countries: to Italy, Germany first of all; to India where he lived and thaught at a critical time in the history of that country; to Russia; to Princeton called at times a mathematician's paradise to Finland (to a prison, where he narrowly escaped execution); to France where he was convicted for dodging his military obligations (in the prison — like a lot of mathematicians in the history — he had time to write one of his best mathematical works); to England where lived through the Battle of London before returning to France and then to United States and finally to Brasil, scene of the last of his vicissitudes, before returning permanently to United States. Through these often pictures-

que episodes, the destiny of a mathematician is unfolded, of which perhaps the most important event was his participation in the foundation of the Bourbaki Group.

This very enjoyable reading is recommended to all mathematicians.

J. Németh (Szeged)

Anatoly A. Zhigljavsky, Theory of Global Random Search (Mathematics and Its Application, 65), Edited by J. Pintér, XVIII+341 pages, Kluwer Academic Publishers' Dordrecht—Boston—London, 1991.

The book is the English translation of an earlier work of the author written in Russian (Leningrad University Press, 1987). Beyond the general overview of global optimization methods, the majority of the volume deals with random search methods and their theoretical background.

In recent years, several review books and monographs have been published on global optimization. Dixon and Szegő edited two volumes of contributed papers of the Workshops Towards Global Optimisation 1 and 2 (North-Holland, 1975 and 1978). The first overview of the field was the book of Törn and Žilinskas (Global Optimization, Springer, 1989) followed by the volumes of Horst and Tuy (Global Optimization — Deterministic Approaches, Springer, 1990) and Floudas and Pardalos (Recent Advances in Global Optimization, Princeton, 1991). Specific parts of the field have been addressed by Pardalos and Rosen (Constrained Global Optimizations Algorithms and Applications, Springer, 1987), Mockus (Bayesian Approach to Global Optimization, Kluwer, 1989) and Floudas and Pardalos (A Collection of Test Problems for Constrained Global Optimization Algorithms, Springer, 1990).

The book of Zhigljavsky completes this series quite well: the random search and sampling methods have not been studied in such a detailed way. The reader will find an interesting comparison of present global optimization methods according to their conditions, type of information utilised, theoretical grounds and amount of numerical results available. The construction and convergence of global random search algorithms and the role of statistical inference in global optimization are investigated thoroughly together with some auxiliary results.

The strength of the bibliography including some 240 references is that special attention is devoted to the Russian language literature that remains usually hidden for the English-oriented part of the optimization community. The unusual typesetting (e.g. $\mathscr{R}^n$ and χ instead of the more common R^n and X causes an uneven line-spacing in the book that (together with many other errors) makes the reading somewhat tiring. The volume can be recommended for those working in the field of multiextremal nonlinear optimization and interested in stochastic methods.

T. Csendes (Szeged)

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Bibliographie

INFORMATIONS FOR AUTHORS

Acta Scientiarum Mathematicarum will publish carefully selected pure mathematical papers. Articles submitted (in duplicate) for publication should be typed double spaced on one side of each sheet (approximately 25 rows and 50 letters in each row). Authors should take the greatest possible care in preparing the manuscript. Hand-written symbols are satisfactory if clearly done; special instructions, when necessary, should be included on a separate sheet. Manuscripts longer than 30 pages will generally not be considered. Authors should consult recent issues of our Acta for general style conventions.

After the references give the author's affiliation.

Authors will receive only galley-proof. Joint authors should indicate which of them should receive the galley-proof.

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Readers, Authors, and Prospective Authors

Dear Colleagues:

With the consent of the Editorial Board I pass the editorial position of our Acta by January 1st, 1993, to Professor László Kérchy.

On this accasion I wish to express my heartfelt thanks to Everybody for helping me in this honouring and delightful, but sometimes also rather tiresome position.

I also sincerely wish our Acta and the new Editor much success.

Szeged, June 1992.

László Leindler

Please use the address:
Professor László Kérchy, editor,
Acta Sci. Math.,
Bolyai Institute
Aradi vértanúk tere 1
6720 Szeged, Hungary

INDEX — TARTALOM

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