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# On Csákány's problem concerning affine spaces

J. DUDEK

*Dedicated to Professor Béla Csákány on his 60th birthday*

In [1] B. CSÁKÁNY proved that for any prime $p$ an algebra $\mathfrak{A}=(A,f)$ where $f$ is at most 4-ary, is equivalent to an affine space over $\mathrm{GF}(p)$ if and only if

$$(*) \qquad p_n(\mathfrak{A}) = \frac{1}{p}((p-1)^n-(-1)^n)$$

for all $n\leqq 4$, and

$(**)$ *There exists no subalgebra $B$ of $A$ with* $1<\operatorname{card} B<p$

(in this case, the formula $(*)$ is valid for all $n\geqq 0$). In this connection, he posed the problem whether the condition $(**)$ can be dropped for some or all $p$. Earlier G. GRÄTZER and R. PADMANABHAN [11] showed that if $\mathfrak{A}$ is a groupoid and $p=3$, then actually the single condition $(*)$ is sufficient. Our result is a further step in this problem

Theorem. *If $G$ is a groupoid, then $G$ is equivalent to an affine space over* $\mathrm{GF}(5)$ *if and only if* $p_n(G)=\frac{1}{5}(4^n-(-1)^n)$ *for all* $n\geqq 0$.

(Of course, as in CSÁKÁNY's result [1], by an affine space we mean a nontrivial, i.e. containing more than one element, affine space.) In the sequel equivalent algebras are treated as identical and "an algebra" means always "a nontrivial algebra". Our terminology and notation are standard (see in [9] and [10]).

To prove our theorem we need among others the following results:

Fact 1 (Theorem 4.1 of [5]). If $(G,\cdot)$ is a nonmedial commutative idempotent groupoid, then

$$p_n(G,\cdot)\geqq\frac{7}{8}n! \quad \text{for all} \quad n\geqq 5.$$

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Recall that the groupoid $(G, \cdot)$ is medial if $(G, \cdot)$ satisfies $(xy)(uv)=(xu)(yv)$ for all $x, y, u, v \in G$.

Fact 2 (cf. [6]). Let $(G, \cdot)$ be a medial idempotent groupoid with card $G>1$. Then $p_2(G, \cdot)=3$ if and only if $(G, \cdot)$ is either a (nontrivial) affine space over GF(5) or a nontrivial Płonka sum of some affine spaces over GF(3) which are not all singletons (for the definition of a Płonka sum see [12]).

Fact 3. If $(A, +, \cdot)$ is a proper commutative idempotent algebra of type (2, 2) satisfying (cf. [7], also [8])

$$(x+z)z = (x+z)y \quad \text{(or the dual)}$$

then $(A, +, \cdot)$ is polynomially infinite, i.e., $p_n(A, +, \cdot)$ is infinite for all $n \geqq 2$.
A proper algebra here means that $x+y$ and $xy$ act on $A$ differently.

Fact 4 (cf. Theorem II of [5]). Let $(G, \cdot)$ be a commutative idempotent groupoid. Then $(G, \cdot)$ is a nontrivial Płonka sum of affine spaces over GF(3) being not all one-element if and only if $p_n(G, \cdot)=3^{n-1}$ for all $n$.

**1. General remarks.** First observe that if an algebra $\mathfrak{A}$ satisfies $(*)$ for $p=5$, then $\mathfrak{A}$ represents the sequence $\langle 0, 1, 3, 13\rangle$, i.e., $\mathfrak{A}$ is an idempotent algebra satisfying

$$p_2(\mathfrak{A}) = 3 \quad \text{and} \quad p_3(\mathfrak{A}) = 13.$$

Lemma 1.1. *If $(A, F)$ represents the sequence*

$$\langle 0, 1, 3, 13\rangle$$

*then $(A, F)$ contains as a reduct a proper idempotent algebra $(A, +, *)$ of type (2, 2) such that $+$ is commutative and $*$ is noncommutative. Moreover the polynomials $x+y$, $x*y$ and $y*x$ are the only essentially binary polynomials over $(A, F)$.*

Proof. Since $p_2(A, F)$ is odd we infer that the algebra $(A, F)$ contains at least one commutative and essentially binary operation, say, $+$. If all binary polynomials over $(A, F)$ are commutative, then we infer that $(A, F)$ contains as a reduct a proper commutative idempotent algebra $(A, +, \cdot, \circ)$ of type (2, 2, 2). Examining the symmetry groups of the following essentially ternary polynomials: $(x+y)+z$, $(xy)z$, $(x\circ y)\circ z$, $(x+y)z$, $xy+z$, $(x+y)\circ z$, $x\circ y+z$, $xy\circ z$ and $(x\circ y)z$ and using the Fact 3 we deduce that $p_3(A, +, \cdot, \circ)\geqq 21$ which is impossible. (Recall that an algebra $(A, \{f_t\}_{t\in T})$ of type $\tau=(n_t)_{t\in T}$ is called proper if the mapping $t\to n_t$ is one-to-one and every operation $f_t$ is essentially $n_t$-ary provided $n_t\geqq 1$, cf. [5].)

Lemma 1.2. *If an algebra $\mathfrak{A}=(A, F)$ satisfies $(*)$ for some $p\geqq 3$ and all $n\geqq 0$, then $\mathfrak{A}$ contains at least one commutative idempotent binary polynomial, say, $+$ and each every such a polynomial is medial.*

Proof. The first statement is clear since $p_2(\mathfrak{A})=p-2$ and hence $p_2(\mathfrak{A})$ is an odd number. Assume now that $(A, +)$ is nonmedial. Thus $(A, +)$ is a nonmedial commutative idempotent groupoid (being a reduct of $\mathfrak{A}$). Applying Fact 1 we get

$$\frac{(p-1)^n-(-1)^n}{p} = p_n(\mathfrak{A}) \geqq p_n(A, +) \geqq \frac{7}{8}n!$$

for all $n\geqq 5$. This yields

$$\frac{n!}{(p-1)^n-(-1)^n} \leqq \frac{8}{7p}$$

for all $n\geqq 5$ which is impossible. This completes the proof of the lemma.

Proposition 1.3. *Let $(G, \cdot)$ be a commutative groupoid. Then $(G, \cdot)$ is a nontrivial affine space over* GF(5) *if and only if $(G, \cdot)$ satisfies $(*)$ for $p=5$ and all $n\geqq 0$.*

Proof. It is clear that $(G, \cdot)$ is a nontrivial affine space over GF(5), then $p_n(G, \cdot)=\frac{4^n-(-1)^n}{5}$ for all $n$ (see e.g., [1]).

If $(G, \cdot)$ is a commutative groupoid such that $p_n(G, \cdot)=\frac{4^n-(-1)^n}{5}$ for all $n$, then using Lemma 1.2 we infer that $(G, \cdot)$ is a medial commutative idempotent groupoid. Since $p_2(G, \cdot)=3$ and $(G, \cdot)$ satisfies $(*)$ for all $n$ we infer, applying Fact 2 and Fact 4 that $(G, \cdot)$ is an affine space over GF(5).

Lemma 1.4. *If an idempotent algebra $\mathfrak{A}=(A, F)$ with $p_2(\mathfrak{A})>1$ contains as a reduct a Steiner quasigroup $(A, +)$, then $p_2(\mathfrak{A})\geqq 5$.*

Proof. Since $p_2(\mathfrak{A})>1$ we infer that $\mathfrak{A}$ contains as a reduct a proper binary idempotent algebra $(A, +, \cdot)$ of type $(2, 2)$ such that $(A, +)$ is a Steiner quasigroup. If $xy$ is commutative then the following polynomials

$$x+y, \quad xy, \quad x\circ y = (x+y)+(xy), \quad x*y = xy+y \quad \textit{and} \quad y*x$$

are essentially binary and pairwise distinct.

Assume now that $\cdot$ is noncommutative. Take into account the polynomial $x*y=xy+y$. It is easy to prove that $x*y$ is essentially binary and different from the polynomials $x+y$, $xy$ and $yx$. If $x*y\neq y*x$, then we clearly get $p_2(\mathfrak{A})\geqq 5$.

Suppose that $x*y=y*x$. First observe that $x*y \neq x+y$. Further the polynomials

$$x+y, \quad x*y, \quad x \square y=(x+y)+x*y, \quad xy \quad \text{and} \quad yx$$

yield five essentially binary and pairwise distinct polynomials and hence $p_2(\mathfrak{A}) \geqq 5$.

Lemma 1.5. *If $(G, \cdot)$ is a noncommutative groupoid satisfying $(*)$ for $p=5$ and all $n \geqq 0$, and $(G, \cdot)$ is not (polynomially) equivalent to a commutative groupoid, then the unique commutative polynomial $+$ over $(G, \cdot)$ is a semilattice polynomial.*

Proof. Let us add that the uniqueness of the polynomial $+$ follows from Lemma 1.1. Consider the reduct $(G, +)$. Put $x \circ y=x+2y$ (in general, $xy^k$ stands for $(\ldots(xy)\cdot \ldots \cdot)y$ where $y$ occurs $k$-times and $x+ky$ in the commutative case respectively). According to Theorem 1 of [1] we see that $x \circ y \neq y$. If $x \circ y=x$, then $(G, +)$ is a Steiner quasigroup and then applying Lemma 1.4 we get $p_2(G, \cdot) \geqq 5$, a contradiction. If $x \circ y$ is commutative, then $x \circ y=x+y$ and hence applying Lemma 1.2 we deduce that $(G, +)$ is medial. According to Theorem 8 of [4] the groupoid $(G, +)$ is a semilattice. If $x \circ y$ is noncommutative (of course, essentially binary), then either $x \circ y=xy$ or $x \circ y=yx$. Both cases prove that the groupoids $(G, \cdot)$ and $(G, +)$ are polynomially equivalent which contradicts the assumption. This completes the proof of the lemma.

**2. Noncommutative idempotent groupoids.** In this section we prove the theorem for the noncommutative case. We start with

Lemma 2.1. *If $(G, \cdot)$ is a noncommutative idempotent groupoid having a commutative binary polynomial, then the following polynomials*

$$f(x, y, z)=(xy)z \quad \textit{and} \quad g(x, y, z)=x(yz)$$

*are different and essentially ternary.*

Proof. Since $(G, \cdot)$ contains a commutative binary polynomial we infer that $(G, \cdot)$ is not a diagonal semigroup. Applying Lemma 3 of [2] we deduce that at least one of the polynomials $f$ and $g$ is essentially ternary. Further without loss of generality we may assume that $f$ is not essentially ternary and $g$ is essentially ternary. Since $(G, \cdot)$ contains a commutative polynomial we infer that $xy$ is essentially binary, i.e., $(G, \cdot)$ is proper. Thus we infer that $(G, \cdot)$ satisfies either

$$(xy)z=xz \quad \text{or} \quad (xy)z=yz$$

If $(G, \cdot)$ satisfies $(xy)z=xz$, then $(G, \cdot)$ also satisfies the identities $xy=(xy)y= =x(xy)$ and $x=(xy)x$ and every binary polynomial $p(x, y)$ over $(G, \cdot)$ is of the form:

$$x, y, xy, yx, y(xy), x(yx), x(y(xy)), y(x(yx)) \quad \text{and so on.}$$

If $p(x, y)=p(y, x)$ holds in $(G, \cdot)$, then using the identity $(xy)z=xz$ we get $xz=yz$ which proves that $(G, \cdot)$ is improper — a contradiction.

If the groupoid $(G, \cdot)$ satisfies $(xy)z=yz$, then the proof runs similarly and will be omitted. To complete the proof one can easily show that there are no noncommutative idempotent semigroups with a commutative binary polynomial:

Lemma 2.2. *If $(G, \cdot)$ is a noncommutative idempotent groupoid having a semilattice polynomial, say, $+$ and the symmetry groups of the polynomials $f$ and $g$ are trivial, then $p_3(G, \cdot)\geqq 19$.*

Proof. According to the preceding lemma we infer that $f$ and $g$ are essentially ternary and different. Consider now the following polynomials

$$(xy)z, \quad x(yz), \quad (x+y)z, z(x+y) \quad \text{and} \quad x+y+z.$$

It is routine to prove that all these polynomials are essentially ternary and consequently permuting variables in them we get 19 different essentially ternary polynomials, as required.

Lemma 2.3. *If $(G, \cdot)$ is a noncommutative idempotent groupoid satisfying $(*)$ for $p=5$ and all $n$ such that $(G, \cdot)$ is not polynomially equivalent to a commutative groupoid, then either the symmetry group of $f$ is nontrivial or the symmetry group of $g$ is nontrivial.*

Proof. An immediate consequence of Lemmas 1.5 and 2.2.

Lemma 2.4. *If $(G, \cdot)$ is a proper noncommutative idempotent groupoid such that the symmetry group of the polynomial*

$$f(x, y, z) = (xy)z$$

*is nontrivial, then $(G, \cdot)$ satisfies either*

$$(xy)z = (zy)x \quad \text{or} \quad (xy)z = (yx)z \quad \text{or} \quad (xy)z = (xz)y.$$

*(The same is true for $g(x, y, z)=x(yz)$.)*

Proof. Trivial since the identity $(xy)z=(yz)x$ proves that $(G, \cdot)$ is a semilattice.

Proposition 2.5. *Let $(G, \cdot)$ be a noncommutative idempotent groupoid satisfying*

$$(xy)z = (zy)x \quad (\text{or the dual}).$$

*Then $p_2(G, \cdot)=3$ if and only if $(G, \cdot)$ is a nontrivial affine space over* GF(5).

Proof. It is clear that a nontrivial affine space over GF(5), i.e., a groupoid $(G, \cdot)$ where $xy=2x+4y$ and $(G, +)$ is an abelian group of exponent 5, satisfies $p_2(G, \cdot)=3$ and $(G, \cdot)$ satisfies $(xy)z=(zy)x$.

Assume now that $p_2(G, \cdot)=3$. It is easy to see that the identity $(xy)z=(zy)x$ implies the medial law for the groupoid $(G, \cdot)$. Using Fact 2 we infer that $(G, \cdot)$ is either a nontrivial affine space over GF(5) or a nontrivial Płonka sum of some affine spaces over GF (3) being not all one-element. The second algebra is a commutative idempotent groupoid which cannot be polynomially equivalent to a noncommutative groupoid. This follows from the fact that the only noncommutative binary polynomial in this groupoid is a $P$-function, but for $P$-functions we have $p_2 \leqq 2$ (for details see [12]). Thus we have proved that $(G, \cdot)$ is an affine space over GF(5) which completes the proof.

Proposition 2.6. *If* $(G, \cdot)$ *is a noncommutative idempotent groupoid satisfying*

$$(xy)z = (yx)z \quad \text{(or the dual)}$$

*then* $p_2(G, \cdot) \neq 3$.

Proof. The assertion is obvious for improper groupoids.

First, we prove that if $(G, \cdot)$ is a proper such groupoid, then the polynomial $x \circ y=(xy)y$ is essentially binary and noncommutative.

If $(xy)y=x$, then $(G, \cdot)$ is right cancellative and the identity $(xy)z=(yx)z$ gives the commutativity of $\cdot$, a contradiction.

If $(xy)y=y$ holds, then we obtain

$$xy = \big(x(xy)\big)(xy) = \big((yx)x\big)(xy) = x(xy).$$

Hence we get

$$y = (xy)y = \big(x(xy)\big)y = \big((yx)x\big)y = xy.$$

Thus $xy=y$ which is impossible.

Assume now that $(xy)y=(yx)x$ and denote $(xy)y$ by $x+y$. Compute the polynomial $xy+yx$. We have

$$xy+yx = \big((xy)(yx)\big)(yx) = \big((yx)(yx)\big)(yx) = yx.$$

Thus $(G, \cdot)$ is a commutative groupoid, a contradiction.

If $x \circ y$ is essentially binary, noncommutative and $x \circ y \notin \{xy, yx\}$, then $p_2(G, \cdot) \geqq 4$ and therefore $p_2(G, \cdot) \neq 3$. Further assume that $x \circ y=xy$. Then we have $xy=(xy)y=(yx)y$. Putting $yx$ for $y$ in $xy=(xy)y$ we get

$$x(yx) = \big(x(yx)\big)(yx) = \big((yx)x\big)(yx) = yx.$$

Analogously we get $xy=x(xy)$. This proves that

$$xy = (xy)y = (yx)y = x(xy) = y(xy)$$

and consequently $p_2(G, \cdot)=2$, as required.

Similarly one proves that if $(G, \cdot)$ satisfies $(xy)y=yx$, then $(G, \cdot)$ also satisfies

$$xy = (yx)x = x(xy) = (xy)x = x(yx)$$

and therefore $p_2(G, \cdot)=2$. The proof is completed.

Now we deal with the last identity appearing in Lemma 2.4, namely the identity $(xy)z=(xz)y$ (the dual identity i.e., $x(yz)=y(xz)$ will be omitted in our considerations).

Lemma 2.7. *If $(G, \cdot)$ is a proper noncommutative groupoid satisfying*

$$(xy)z = (xz)y,$$

*then the polynomial $x\circ y=x(xy)$ is noncommutative and different from $y$ and $yx$.*

Proof. If $x(xy)=y(yx)$ holds in $(G, \cdot)$, then

$$x(xy) = (xx)(xy) = \big(x(xy)\big)x = \big(y(yx)\big)x = (yx)(yx) = yx.$$

Thus we get $x(xy)=yx$ which proves that $(G, \cdot)$ is commutative, a contradiction.

If $x(xy)=y$, then $y=yy=\big(x(xy)\big)y=xyxy=xy$, again a contradiction. If $x(xy)=yx$, then

$$xy = (xy)(xy) = \big(x(xy)\big)y = (yx)y = yx.$$

Thus $xy=yx$ which is impossible.

Lemma 2.8. *There is no (noncommutative) idempotent groupoid $(G, \cdot)$ satisfying $p_2(G, \cdot)=3$ and the identities*

$$(xy)z = (xz)y \quad \text{and} \quad x(xy) = x.$$

Proof. First we prove that the groupoid $(G, \cdot)$ satisfies either

$$(xy)y = x \quad \text{or} \quad (xy)y = xy.$$

Indeed, if $(xy)y=y$, then

$$yx = \big((xy)y\big)x = \big((xy)x\big)y = \big((xx)y\big)y = y$$

which is impossible. If $(xy)y=yx$, then

$$xy = (yx)x = \big((xy)y\big)x = (xy)y = yx$$

which gives $xy=yx$, a contradiction.

If $(xy)y=(yx)x$, then putting $yx$ for $x$ we get

$$y = y(yx) = \big(y(yx)\big)(yx) = \big((yx)y\big)y = (yx)y = yx,$$

which is impossible. Hence we proved that $(G, \cdot)$ satisfies either

$$(xy)y = x \quad \text{or} \quad (xy)y = xy.$$

Assume that $(G, \cdot)$ satisfies $(xy)y=x$. Consider the polynomial $x*y=x(yx)$. If $x*y=y*x$, then

$$x = \big(x(yx)\big)(yx) = \big(y(xy)\big)(yx) = \big(y(yx)\big)(xy) = y(xy).$$

Hence $y(xy)=x$, a contradiction.

Further it is easy to see that

$$x(yx) \neq y \quad \text{and} \quad x(yx) \neq yx.$$

According to the assumption $p_2(G, \cdot)=3$ we infer that $(G, \cdot)$ satisfies either

$$x(yx) = x \quad \text{or} \quad x(yx) = xy.$$

If so, then in both cases we get $p_2(G, \cdot)=2$ which contradicts the assumption.

To complete the proof we must consider one more case, namely, the groupoid $(G, \cdot)$ satisfies

$$(xy)z = (xz)y, \quad x(xy) = x \quad \text{and} \quad (xy)y = xy.$$

As above considering the polynomial $x*y=x(yx)$ one proves that $x*y$ is noncommutative and therefore the polynomial $x*y$ is one of the following polynomials: $x, y, xy, yx$. In any case one can easily check that the considered groupoid satisfies $p_2(G, \cdot)=2$ which is impossible. The proof of the lemma is completed.

Lemma 2.9. *Let $(G, \cdot)$ be a proper noncommutative idempotent groupoid satisfying $(xy)z=(xz)y$. Then $p_2(G, \cdot)=3$ if and only if $(G, \cdot)$ satisfies the identities*

$$xy = x(xy) \quad \textit{and} \quad x(yx) = y(xy).$$

*Moreover if an idempotent groupoid satisfies*

$$(xy)z = (xz)y, \quad xy = x(xy) \quad \textit{and} \quad x(yx) = y(xy),$$

*then the polynomial $x+y=x(yx)$ is a near-semilattice polynomial* (i.e., $x+x=x$, $x+y=y+x$ *and* $x+y=(x+y)+y$; cf. [5]).

Proof. Let $p_2(G, \cdot)=3$. Consider the polynomial $x\circ y=x(xy)$. Applying Lemma 2.7 we infer that $(G, \cdot)$ satisfies either

$$x(xy) = x \quad \text{or} \quad x(xy) = xy.$$

According to Lemma 2.8, the first case cannot occur. Thus $(G, \cdot)$ satisfies $x(xy)=$

$=xy$. Consider the polynomial $x+y=x(yx)$. If $x+y\in\{x, y, xy, yx\}$, then one gets $p_2(G, \cdot)=2$, a contradiction. If $x+y$ is essentially binary noncommutative and different from $xy$, $yx$, then clearly $p_2(G, \cdot)\geqq 4$ which contradicts the assumption. Thus we have proved that $x+y=y+x$. Further we have

$$(x+y)+y = x(yx)+y = \big(x(yx)\big)\big(y(x(yx))\big) =$$
$$= \big(x(yx)\big)\big(y(y(xy))\big) = \big(x(yx)\big)\big(y(xy)\big) = x+y.$$

Hence $x+y=(x+y)+y$ which proves that $(G, +)$ is a near-semilattice.

Assume now that $(G, \cdot)$ is noncommutative idempotent, satisfying $xy=x(xy)$, $x(yx)=y(xy)$, and $(xy)z=(xz)y$. Since $xy=(xy)y=(xy)x=x(xy)$ and $(xy)(yx)=$ $=x(yx)$ we infer that (in a proper noncommutative groupoid) we have $p_2(G, \cdot)=3$. This completes the proof of the lemma.

Lemma 2.10. *If $(G, \cdot)$ is an idempotent groupoid satisfying $(xy)z=(xz)y$ and $p_2(G, \cdot)=3$, then the symmetry group of the polynomial $g(x, y, z)=x(yz)$ is trivial.*

Proof. It is clear that $g$ does not admit any cycle of its variables ($(G, \cdot)$ is not a semilattice). If $(G, \cdot)$ satisfies $x(yz)=x(zy)$, then using Proposition 2.6 we infer that $p_2(G, \cdot)\neq 3$.

If $x(yz)=z(yx)$ holds in $(G, \cdot)$, then we obtain

$$xy = (xy)(xy) = \big(x(xy)\big)y = \big(y(xx)\big)y = (yx)y = yx.$$

Thus $xy=yx$ which proves that $(G, \cdot)$ is a semilattice, a contradiction. Assume now that $(G, \cdot)$ satisfies $x(yz)=y(xz)$. Applying Lemma 2.9 we get $x(yx)=y(xy)$ and hence using the identity $x(yz)=y(xz)$ we get $xy=yx$, a contradiction.

Proposition 2.11. *If an idempotent groupoid $(G, \cdot)$ satisfies $(xy)z=(xz)y$ (or the dual identity) and $p_2(G, \cdot)=3$, then $p_3(G, \cdot)\geqq 16$.*

Proof. According to Lemma 2.1 the polynomials $f(x, y, z)=(xy)z$ and $g(x, y, z)=x(yz)$ are essentially ternary and different. Applying Lemma 2.9 we see that $x+y=x(yx)$ is a near-semilattice polynomial. It is clear that $(G, \cdot)$ is a proper noncommutative idempotent groupoid and further the polynomials

$$q_1 = (x+y)z \quad \text{and} \quad q_2 = z(x+y)$$

are essentially ternary and their symmetry groups are of order 2. Consider now the following essentially ternary polynomials over $(G, \cdot)$:

$$f = (xy)z, \quad g = x(yz), \quad q_1 = (x+y)z, \quad q_2 = z(x+y) \quad \text{and} \quad s = (x+y)+z.$$

By the assumption and Lemma 2.10 we see that card $G(f)=2$ and card $G(g)=1$.

We also have $\operatorname{card} G(q_1)=\operatorname{card} G(q_2)=2$. Further observe that

$$(xy)z \neq x(y+z) \quad \text{and} \quad (xy)z \neq (y+z)x.$$

Indeed, if $(xy)z=x(y+z)$, then

$$xy = (xy)x = x(x+y) = x\big(x(yx)\big) = x(yx) = x+y$$

which proves that $(G, \cdot)$ is commutative, a contradiction (we use also the identity $x(xy)=xy$, see Lemma 2.9). The proof of the inequality $(xy)z\neq(y+z)x$ runs similarly. Further for the groupoid $(G, \cdot)$ we have

$$p_3(G, \cdot) \geqq \frac{3!}{\operatorname{card} G(f)}+\frac{3!}{\operatorname{card} G(g)}+\frac{3!}{\operatorname{card} G(q_1)}+\frac{3!}{\operatorname{card} G(q_2)}+$$

$$+\frac{3!}{\operatorname{card} G(s)} \geqq 3+6+3+3+1 = 16$$

which finishes the proof of the lemma.

**3. The proof of the Theorem.** In this section we prove the theorem. First if $(G, \cdot)$ is an nontrivial affine space over GF(5), then clearly using the formula from [1] we see

$$p_n(G, \cdot) = \frac{4^n-(-1)^n}{5}$$

for all $n$ (see also in [9]).

Let now $(G, \cdot)$ satisfy $(*)$ for all $n$ and $p=5$.

If $(G, \cdot)$ is commutative, then the proof follows from Proposition 1.3.

If $(G, \cdot)$ is noncommutative but the groupoid $(G, \cdot)$ is polynomially equivalent to a commutative groupoid, then the proof again follows from Proposition 1.3.

Assume that $(G, \cdot)$ is a (proper) noncommutative idempotent groupoid being not polynomially equivalent to a commutative groupoid. Then applying Lemma 1.5 we infer that $(G, \cdot)$ contains a semilattice polynomial, say, $+$.

Consider now the following polynomials

$$s = (x+y)+z, \quad f = (xy)z, \quad g = x(yz)$$

$$q_1 = (x+y)z \quad \text{and} \quad q_2 = z(x+y).$$

All these polynomials are essentially ternary (see Lemma 2.1). According to Lemma 2.2 we infer that at least one of the symmetry groups of the polynomials $f$ and $g$ is nontrivial, say, the symmetry group $G(f)$. Then applying Lemma 2.4 we deduce that $(G, \cdot)$ satisfies either

$$(xy)z = (zy)x \quad \text{or} \quad (xy)z = (yx)z \quad \text{or} \quad (xy)z = (xz)y.$$

If $(G, \cdot)$ satisfies $(xy)z=(zy)x$, then using Proposition 2.5 we infer that $(G, \cdot)$

is a nontrivial affine space over GF (5) but such algebras are polynomially equivalent to a commutative groupoid which contradicts the assumption.

Since $P_2(G, \cdot)=3$, applying Proposition 2.6 the identity $(xy)z=(yx)z$ does not hold in the groupoid $(G, \cdot)$.

Analogously, using Proposition 2.11 we conclude that the identity $(xy)z=(xz)y$ also does not hold in $(G, \cdot)$ which completes the proof of the Theorem.

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# Some nontrivial implications in congruence varieties

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*Dedicated to Professor Béla Csákány on his 60th birtday*

A congruence variety is a lattice variety generated by the class of congruence lattices of all members of some variety of algebras. The most known examples are $\mathscr{V}(R)$, the lattice varieties generated by congruence (or submodule) lattices of $R$-modules for rings $R$ with 1. Given a lattice identity $\alpha$ and a set $\Gamma$ of lattice identities, we write $\Gamma \models_c \alpha$ if every congruence variety satisfying $\Gamma$ also satisfies $\alpha$ (cf. JÓNSSON [8]). The implication $\Gamma \models_c \alpha$ is called nontrivial if $\Gamma \not\models \alpha$ (in the class of all lattices). For $\Gamma=\{\gamma\}$ we will write $\gamma$ rather than $\{\gamma\}$.

There are many results stating that $\gamma \models_c \alpha$ without $\gamma \models \alpha$ for certain pairs $(\gamma, \alpha)$ of lattice identities. These results are surveyed in JÓNSSON [8]; for a further development cf. FREESE, HERRMANN and HUHN [3]. However, all the known results are located at distributivity or modularity in the sense that either $\gamma \models_c \alpha \models_c$ distributivity $\models_c \gamma$ or $\gamma \models_c \alpha \models_c$ modularity $\models_c \gamma$. Now [1] offers an easy way to achieve $\gamma \models_c \alpha$ results of a different nature.

For an integer $n > 2$ and a modular lattice $L$, a system

$$\bar{f} = (a_i, c_{ij}\colon\ 1 \leqq i \leqq n,\ 1 \leqq j \leqq n,\ i \neq j)$$

of elements of $L$ is called a (von Neumann) $n$-frame in $L$ if $a_j \sum_{i \neq j} a_i = 0_{\bar{f}}$, $c_{jk} = c_{kj}$, $a_j c_{jk} = 0_{\bar{f}}$, $a_j + c_{jk} = a_j + a_k$ and $c_{jk} = (a_j + a_k)(c_{jl} + c_{lk})$ for all distinct $j, k, l \in \{1, 2, \ldots, n\}$ where $0_{\bar{f}}$ resp. $1_{\bar{f}}$ are the meet resp. join of all elements of $\bar{f}$ (cf. VON NEUMANN [9]). We write $x+y$ and $xy$ for the join and meet of $x$ and $y$.

Given $m \geqq 0$ and $n \geqq 1$, a lattice identity $\Delta(m, n)$ is defined in [7, page 289] such that, for any ring $R$ with 1, $\Delta(m, n)$ holds in $\mathscr{V}(R)$ iff the divisibility condition $(\exists r)(m \cdot r = n \cdot 1)$, abbreviated by $D(m, n)$, holds in $R$ (cf. [7, Prop. 6]). What else

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we need to know about $\Delta(m, n)$ is that $\Delta(m, n)$ is of the form

$$(x_1+x_2)(x_3+x_4) \leqq q_{m,n}(x_1, x_2, x_3, x_4).$$

Frames are projective in the variety of modular lattices. This was proved in two steps; first for (Huhn) diamonds in HUHN [6] (for a more explicit statement cf. FREESE [2]) and then frames and diamonds turned out to be equivalent in HERRMANN and HUHN [5, page 104]. Therefore there are lattice terms $b_i(\vec{x})$ and $d_{ij}(\vec{x})$ in variables $\vec{x}=(x_i, x_{ij}\colon 1\leqq i,j\leqq k,\ i\neq j)$ such that these terms produce a $k$-frame $(b_i(\vec{y}), d_{ij}(\vec{y})\colon 1\leqq i,\ j\leqq k,\ i\neq j)$ from any system $\vec{y}$ of elements of a modular lattice $L$ and, in addition, if $\vec{f}=(a_i, c_{ij}\colon 1\leqq i,j\leqq k,\ i\neq j)$ is a $k$-frame in $L$ then $b_i(\vec{f})=a_i$ and $d_{ij}(\vec{f})=c_{ij}$ for every $i\neq j$.

For $k\geqq 4$ the conjugation of the modular law and the identity

$$\big(d_{13}(\vec{x})+d_{23}(\vec{x})\big)\big(d_{14}(\vec{x})+d_{24}(\vec{x})\big) \leqq q_{m,n}\big(d_{13}(\vec{x}), d_{23}(\vec{x}), d_{14}(\vec{x}), d_{24}(\vec{x})\big),$$

where $\vec{x}=(x_i, x_{ij}\colon 1\leqq i,\ j\leqq k,\ i\neq j)$, will be denoted by $\Delta(m, n, k)$. Clearly, $\Delta(m, n, k)$ is equivalent to a single lattice identity modulo lattice theory.

Theorem. *Consider arbitrary integers* $m', m_i\geqq 0, n', n_i\geqq 1$, *and* $k', k_i\geqq 4$ $(i\in I)$ *where* $I$ *is an index set. Then* $\{\Delta(m_i, n_i, k_i)\colon i\in I\}\models_c \Delta(m', n', k')$ *if and only if* $\{D(m_i, n_i)\colon i\in I\}$ *implies* $D(m', n')$ *in the class of rings with* 1.

In particular, if $m\nmid n$ and $k\geqq 5$ then $\Delta(m, n, k)\models_c \Delta(m, n, k-1)$. This is a nontrivial implication, for we have the following

Proposition. *If* $m\nmid n$, $m\geqq 0$, $n\geqq 1$ *and* $k\geqq 5$ *then* $\Delta(m, n, k)\not\models \Delta(m, n, k-1)$.

To point out that the $\Delta(m, n, k)$ in the proposition are essentially distinct we present the following.

Remark. The set $\{\Delta(p, 1, \mathrm{k})\colon p \text{ prime}\}$, where $k\geqq 4$, is independent in congruence varieties in the sense that for every prime $q$

$$\{\Delta(p, 1, k)\colon p \text{ prime},\ p\neq q\}\not\models_c \Delta(q, 1, k).$$

Proof of the theorem. Since frames and diamonds are equivalent (cf. HERRMANN and HUHN [5, page 104]), the identities $\Delta(m, n, k)$ are diamond identities in the sense of [1]. What we need from [1] is only its Theorem 2, which we reformulate less technically as follows: For any diamond identity $\alpha$, $\Gamma\models_c\alpha$ iff for any ring $R$ with 1 $\Gamma$ implies $\alpha$ in $\mathscr{V}(R)$. Therefore it suffices to show that $\Delta(m, n, k)$ and $\Delta(m, n)$ are equivalent in any $\mathscr{V}(R)$. Clearly, $\Delta(m, n)$ implies $\Delta(m, n, k)$ and $\Delta(m, n)$ are equivalent in any $\mathscr{V}(R)$. Clearly, $\Delta(m, n)$ implies $\Delta(m, n, k)$ in $\mathscr{V}(R)$. Conversely, assume that $\Delta(m, n, k)$ holds in $\mathscr{V}(R)$. Let $M=M(u_1, u_2, \ldots, u_k)$

denote the $R$-module freely generated by $\{u_1, u_2, \ldots, u_k\}$. Then $\Delta(m, n, k)$ holds Sub$(M)$, the submodule lattice of $M$. It is easy to see (or cf. NEUMANN [9]) that the cyclic submodules $(Ru_i, R(u_i-u_j)\colon 1 \leqq i, j \leqq k,\ i \neq j)$ constitute a $k$-frame in Sub$(M)$. (In fact, this is the most typical example of a $k$-frame.) Therefore

$$(1)\qquad \big(R(u_1-u_3)+R(u_2-u_3)\big)\big(R(u_1-u_4)+R(u_2-u_4)\big) \leqq$$
$$\leqq q_{m,n}\big(R(u_1-u_3), R(u_2-u_3), R(u_1-u_4), R(u_2-u_4)\big)$$

holds in Sub$(M)$ and even in Sub$(M(u_1, u_2, u_3, u_4))$. Now the theory of Mal'tsev conditions (cf. WILLE [11] or PIXLEY [10]) together with the canonical isomorphism between Sub$(M(u_1, u_2, u_3, u_4))$ and the congruence lattice of $M(u_1, u_2, u_3, u_4)$ yield easily that $\Delta(m, n)$ holds in $\mathcal{V}(R)$. (Note that the first nine rows in the proof of [7, Prop. 6] supply a detailed proof of the fact that (1) implies the satisfaction of $\Delta(m, n)$ in $\mathcal{V}(R)$.)

Proof of the proposition. Let $\underset{\sim}{Z}$ denote the ring of integers. Since $m \nmid n$ and $\Delta(m, n, k-1)$ implies $\Delta(m, n)$ in $\mathcal{V}(\underset{\sim}{Z})$ by the proof above, $\Delta(m, n, k-1)$ fails in $\mathcal{V}(\underset{\sim}{Z})$. It is shown in HERRMANN and HUHN [4, Satz 7] that $\mathcal{V}(\underset{\sim}{Z})$ is generated by its finite members. Therefore there is a finite modular lattice $L$ with minimal number of elements such that $\Delta(m, n, k-1)$ fails in $L$. We intend to show that $\Delta(m, n, k)$ holds in $L$. Assume the contrary. Then there is a $k$-frame $\vec{f}=(a_i, c_{ij}\colon 1 \leqq i, j \leqq k,\ i \neq j)$ such that $\Delta(m, n)$ fails when $c_{13}, c_{23}, c_{14}, c_{24}$ are substituted for its variables. It is known that either all elements of a frame are equal or $a_1, a_2, \ldots, a_k$ are distinct atoms of a Boolean sublattice of length $k$ (cf., e.g., HERRMANN and HUHN [5, (iii) on page 101 and page 104]). Now only the latter is possible since the one element lattice satisfies any identity. Hence the subframe $\vec{g}=(a_i, c_{ij}\colon 1 \leqq i, j \leqq k-1,\ i \neq j)$ lies in the interval $L'=[0_{\vec{g}}, 1_{\vec{g}}]$. From $1_{\vec{g}}=a_1+\ldots+a_{k-1}<a_1+\ldots+a_k=1_{\vec{f}}$ we obtain $|L'|<|L|$. The frame $\vec{g}$ witnesses that $\Delta(m, n, k-1)$ fails in $L'$, which contradicts the choice of $L$.

The remark is concluded from the theorem quite easily; we need only to consider the ring of those rational numbers whose denominator is not divisible by $q$.

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# Algebras and varieties satisfying the congruence extension property

IVAN CHAJDA

*Dedicated to Professor Béla Csákány on his 60th birthday*

Algebras and varieties satisfying CEP were firstly investigated in [1]. There was proven the equivalence of CEP and PCEP in varieties and it was shown that such varieties cannot be characterized by Mal'cev type conditions. The aim of this paper is to give a term characterization of permutable varieties satisfying CEP and to give one modification of CEP.

Recall that an algebra $A$ satisfies the *Congruence Extension Property* (briefly CEP) if for every subalgebra $B$ of $A$ and each $\Theta\in\mathrm{Con}\,B$ there exists $\Phi\in\mathrm{Con}\,A$ such that $\Phi|_B=\Theta$. A class $\mathscr{C}$ of algebras satisfies CEP if each $A\in\mathscr{C}$ has this property.

Some notations: denote by $\Theta_A(x,y)$ the principal congruence on $A$ generated by $\langle x,y\rangle\in A^2$. If $z_1,\dots,z_n\in A$ and $s$ is an $n$-ary term over $A$, denote by $s(\vec{z})$ the expression $s(z_1,\dots,z_n)$. If $B$ is a subalgebra of $A$ and $\Theta\in\mathrm{Con}\,A$, denote by $\Theta|_B$ the restriction of $\Theta$ onto $B$, i.e. $\Theta|_B=\Theta\cap(B\times B)$. For $b\in A$ and $\Theta\in\mathrm{Con}\,A$, $[b]_\Theta$ is a congruence class of $\Theta$ containing $b$.

Theorem 1. *For a variety $\mathscr{V}$, the following conditions are equivalent:*

(1) *$\mathscr{V}$ is congruence-permutable and satisfies* CEP;

(2) *For every $(1+n)$-ary term $p$ there exists a 5-ary term $q$ such that*

$$p(x,\vec{z})=q(y,x,y,p(x,\vec{z}),p(y,\vec{z})),\quad p(y,\vec{z})=q(x,x,y,p(x,\vec{z}),p(y,\vec{z})).$$

Proof. (1)⇒(2): Let $\mathscr{V}$ be a congruence-permutable variety satisfying CEP and $A=F_{\mathscr{V}}(x,y,z_1,\dots,z_n)$ be a free algebra of $\mathscr{V}$ with $2+n$ free generators $x,y,z_1,\dots,z_n$. Let $p$ be a $(1+n)$-ary term of $\mathscr{V}$. Let $B$ be an algebra of $\mathscr{V}$ generated by four generators: $x$, $y$, $p(x,\vec{z})$, $p(y,\vec{z})$. Then clearly $B$ is a subalgebra of $A$ and

$$\langle p(y,\vec{z}),p(x,\vec{z})\rangle\in\Theta_A(x,y)\in\mathrm{Con}\,A.$$

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Since $\mathcal{V}$ satisfies CEP, it implies immediately

$$\langle p(y, \vec{z}), p(x, \vec{z})\rangle \in \Theta_B(x, y) \in \mathrm{Con}\, B.$$

Thanks to the congruence-permutability of $\mathcal{V}$, it implies the existence of a unary algebraic function $\tau$ over $B$ such that

$$p(y, \vec{z}) = \tau(x), \quad p(x, \vec{z}) = \tau(y).$$

Since $B$ has four generators, there exists a 5-ary term $q$ such that

$$\tau(w) = q\big(w, x, y, p(x, \vec{z}), p(y, \vec{z})\big),$$

whence (2) is evident.

(2)$\Rightarrow$(1): Suppose $\mathcal{V}$ satisfies (2). At first, we can choose the term $p$ to be the first projection, i.e.

$$p(x, z_1, \ldots, z_n) = x.$$

By (2), there exists a 5-ary term $q$ such that

$$x = q(y, x, y, x, y) \quad \text{and} \quad y = q(x, x, y, x, y).$$

Put $t(x, z, y) = q(z, x, y, x, y)$. Then

$$t(x, x, y) = q(x, x, y, x, y) = y$$

$$t(x, y, y) = q(y, x, y, x, y) = x,$$

thus $t(x, y, z)$ is a Mal'cev term proving congruence-permutability of $\mathcal{V}$.

Now, suppose $A \in \mathcal{V}$, $B$ is a subalgebra of $A$, $a$, $b$, $c$, $d$ are elements of $B$ and $\langle c, d\rangle \in \Theta_A(a, b)|_B$. Since $\mathcal{V}$ is congruence-permutable, there exists an $(1+n)$-ary term $p$ and elements $e_1, \ldots, e_n \in A$ such that

$$c = p(a, e_1, \ldots, e_n), \quad d = p(b, e_1, \ldots, e_n).$$

By (2), there exists a 5-ary term $q$ with

$$c = q(b, a, b, c, d), \quad d = q(a, a, b, c, d),$$

thus $\langle c, d\rangle \in \Theta_B(a, b)$ proving PCEP. By the Theorem in [1], PCEP and CEP are equivalent conditions in varieties, hence CEP is proved.

Example 1. Let $\mathcal{V}$ be a variety of abelian groups. Then every $(1+n)$-ary term $p(x, z_1, \ldots, z_n)$ can be written in the form

$$p(x, \vec{z}) = x^a \cdot z,$$

where $z = r(z_1, \ldots, z_n)$ for some $n$-ary term $r$ and $a \in Z$. Put

$$q(x_1, x_2, x_3, x_4, x_5) = x_2^a \cdot x_1^{-a} \cdot x_5.$$

Then (2) of Theorem 1 is satisfied:

$$q(y, x, y, p(x, \vec{z}), p(y, \vec{z})) = x^a \cdot y^{-a} \cdot (y^a \cdot z) = x^a \cdot z = p(x, \vec{z}),$$
$$q(x, x, y, p(x, \vec{z}), p(y, \vec{z})) = x^a \cdot x^{-a} \cdot (y^a \cdot z) = y^a \cdot z = p(y, \vec{z}).$$

Now, we will investigate some modification of CEP.

Definition. An algebra $A$ satisfies the *Strong Congruence Extension Property* (briefly SCEP) if for every subalgebra $B$ of $A$ and each $\Theta \in \mathrm{Con}\, B$ there $\Phi \in \mathrm{Con}\, A$ with $[b]_\Theta = [b]_\Phi$ for each $b \in B$. A class $\mathscr{C}$ of algebras satisfies SCEP if each $A \in \mathscr{C}$ has this property.

It is evident that SCEP implies CEP.

Example 2. Every Hamiltonian group satisfies SCEP.

This example can be generalized for algebras:

Lemma. *Every algebra satisfying* SCEP *is Hamiltonian.*

Proof. Let $B$ be a subalgebra of $A$. Put $\Theta = B \times B \in \mathrm{Con}\, B$. By SCEP, there exists $\Phi \in \mathrm{Con}\, A$ with $[b]_\Phi = B$ for each $b \in B$, thus $A$ is Hamiltonian.

Theorem 2. *Let $\mathscr{V}$ be a variety. $\mathscr{V}$ satisfies* SCEP *if and only if $\mathscr{V}$ is Hamiltonian.*

Proof. Let $\mathscr{V}$ be Hamiltonian and $B$ be a subalgebra of $A \in \mathscr{V}$, $b \in B$ and $\Theta \in \mathrm{Con}\, B$. By [2], $\mathscr{V}$ satisfies CEP, thus there exists $\Phi \in \mathrm{Con}\, A$ extending $\Theta$. Since $\mathscr{V}$ is Hamiltonian, $B$ is a block of some $\Psi \in \mathrm{Con}\, A$. For $\Theta^* = \Phi \cap \Psi$ we have $[b]_{\Theta^*} = [b]_\Phi \cap B = [b]_\Theta$ proving SCEP. The converse implication follows by the Lemma.

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# Two-dimensional real algebras with zero divisors

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BENJAMIN PEIRCE in his seminal work on linear associative algebras [6] classified all two-dimensional real associative (pure) algebras. He reported on his results from this work in talks delivered to the National Academy of Sciences during the period from 1867 to 1870, but they were not published until 1881, posthumously. (See [3], [4].) In 1958 LUCHIAN [4] began a classification of all two-dimensional real algebras with zero divisors, and in 1970 WALLACE [8] classified all two-dimensional power associative real algebras. Finally, in 1983 ALTHOEN and KUGLER [1] gave canonical forms for all two-dimensional real division algebras.

This paper complements all these previous works by presenting a list of canonical forms for multiplication tables of all two-dimensional real algebras which are not division algebras (i.e., which have zero divisors). The presentation is algorithmic: given any such algebra, one can easily derive one of our forms. Except in the case of the final table, as is clarified below, the tables presented are uniquely determined; the proof of this fact is not difficult and is omitted for purposes of brevity.

Consider the real algebra $\mathscr{A}$ with basis $\{\eta_1, \eta_2\}$ with respect to which multiplication is given by the following table:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & a_1\eta_1+a_2\eta_2 & b_1\eta_1+b_2\eta_2 \\ \eta_2 & c_1\eta_1+c_2\eta_2 & d_1\eta_1+d_2\eta_2 \end{array} .$$

We use the convention that Roman letters represent real numbers and Greek letters represent elements of $\mathscr{A}$. We can write the table in the abbreviated form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & \alpha & \beta \\ \eta_2 & \gamma & \delta \end{array}$$

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by setting $\alpha=a_1\eta_1+a_2\eta_2$, etc. Let $L_\chi$ and $R_\chi$ denote, respectively, left and right translations by the element $\chi\in\mathscr{A}$:

$$L_\chi(\alpha) = \chi\alpha \quad \text{and} \quad R_\chi(\alpha) = \alpha\chi, \quad \text{for all} \quad \alpha\in\mathscr{A}.$$

Then for $\chi=x_1\eta_1+x_2\eta_2$, with $x_1, x_2\in\mathbf{R}$, we find that

$$\det(L_\chi) = Q_L(x_1, x_2) = M_L x_1^2+N_L x_1 x_2+P_L x_2^2 \quad \text{and}$$

$$\det(R_\chi) = Q_R(x_1, x_2) = M_R x_1^2+N_R x_1 x_2+P_R x_2^2,$$

where

$$M_L = |\alpha, \beta|, \quad N_L = |\alpha, \delta|+|\gamma, \beta|, \quad P_L = |\gamma, \delta|,$$

$$M_R = |\alpha, \gamma|, \quad N_R = |\alpha, \delta|+|\beta, \gamma|, \quad \text{and} \quad P_R = |\beta, \delta|,$$

given, for example, that the notation $|\alpha, \beta|$ denotes the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

Let $\Delta=N_L^2-4M_LP_L=N_R^2-4M_RP_R$. By definition, $\mathscr{A}$ is a division algebra if and only if $L_\chi$ is nonsingular for all nonzero elements $\chi\in\mathscr{A}$ (or equivalently, $R_\chi$ is nonsingular for all nonzero elements $\chi\in\mathscr{A}$); this is the case if and only if the quadratic form $Q_L(x_1, x_2)$ is positive definite (or equivalently, $Q_R(x_1, x_2)$ is positive definite). Thus $\mathscr{A}$ is a division algebra if and only if $\Delta<0$, and $\mathscr{A}$ is an algebra with zero divisors if and only if $\Delta\geqq 0$.

As we proceed, we will subdivide the two-dimensional algebras with zero divisors into subclasses. One of the sets of criteria we use is the classification scheme of LUCHIAN [4]:

Definition. A two-dimensional algebra $\mathscr{A}$ belongs to the class:

| | |
|---|---|
| $L_1$, if $\Delta > 0$, | $R_1$, if $\Delta > 0$, |
| $L_2$, if $\Delta = 0$ but $Q_L(x_1, x_2) \not\equiv 0$, | $R_2$, if $\Delta = 0$ but $Q_R(x_1, x_2) \not\equiv 0$, |
| $L_3$, if $Q_L(x_1, x_2) \equiv 0$, | $R_3$, if $Q_R(x_1, x_2) \equiv 0$. |

Remarks. (1) The quadratic forms $Q_L(x_1, x_2)$ and $Q_R(x_1, x_2)$ and the quantity $\Delta$ are dependent upon the basis $\{\eta_1, \eta_2\}$ of $\mathscr{A}$, and at first glance it appears that the subclasses $L_i$ and $R_j$ are as well. However, this is not the case. In fact, the algebra $\mathscr{A}$ is in the class:

a) $L_1(R_1)$ if and only if there are two independent left (right) zero divisors in $\mathscr{A}$, but not every nonzero element of $\mathscr{A}$ is a left (right) zero divisor;

b) $L_2(R_2)$ if and only if there exists a left (right) zero divisor $\chi\in\mathscr{A}$ and every other left (right) zero divisor in $\mathscr{A}$ is a multiple of $\chi$;

c) $L_3(R_3)$ if and only if every nonzero element of $\mathscr{A}$ is a left (right) zero divisor.

(2) From the theory of quadratic forms (see [5], pp. 85—86), we know that if $\{\nu_1, \nu_2\}$ is another basis for $\mathscr{A}$, generating corresponding quadratic forms $Q_L^0(x_1, x_2)$ and $Q_R^0(x_1, x_2)$ and quantity $\Delta^0$, and if $T\colon \mathscr{A}\to\mathscr{A}$ is a linear transformation such

that $T(\eta_i)=v_i$, then $\Delta^0=|T|^2\Delta$. This provides a direct proof that the class $L_1=R_1$ is independent of basis.

(3) The triplets $L_1$, $L_2$, $L_3$ and $R_1$, $R_2$, $R_3$ each form partitions of the class of algebras with zero divisors. Following the notation of Luchian, we let $L_iL_j$ denote the intersection $L_i\cap L_j$. It is clear that $L_1=R_1$, and hence $L_1R_2=L_1R_3=L_2R_1=$ $=L_3R_1=\emptyset$; all other intersections, however, are nonempty.

As seen above, if $\Delta\geqq 0$, there exist nontrivial elements $\chi, \psi\in\mathscr{A}$ such that $L_\chi(\psi)=\chi\psi=0$. In fact, in some cases there exists a nontrivial element $\chi\in\mathscr{A}$ such that $L_\chi(\chi)=\chi^2=0$. The second set of criteria we use for classification of the two-dimensional algebras with zero divisors is based upon the number of such elements (if any exist).

Definition. A two-dimensional algebra $\mathscr{A}$ belongs to the class:

$S$, if there exists a nontrivial element $\chi\in\mathscr{A}$ whose square is 0,

$N$, if no such element exists.

The following proposition gives a criterion which determines whether an algebra lies in the class $N$ or the class $S$.

Proposition. *A two-dimensional real algebra $\mathscr{A}$ lies in the class N if and only if given any multiplication table for $\mathscr{A}$:*

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & \alpha & \beta \\ \eta_2 & \gamma & \delta \end{array}$$

*we have*

$$\begin{vmatrix} a_1 & b_1+c_1 & d_1 & 0 \\ 0 & a_1 & b_1+c_1 & d_1 \\ a_2 & b_2+c_2 & d_2 & 0 \\ 0 & a_2 & b_2+c_2 & d_2 \end{vmatrix} \neq 0.$$

Proof. There exists a nontrivial element $\chi=x_1\eta_1+x_2\eta_2$ in $\mathscr{A}$ with $\chi^2=0$ if and only if there exists a nontrivial solution $(x_1, x_2)$ to the system:

$$a_1x_1^2+(b_1+c_1)x_1x_2+d_1x_2^2 = 0,$$

$$a_2x_1^2+(b_2+c_2)x_1x_2+d_2x_2^2 = 0.$$

This will be the case if and only if the two polynomials

$$a_1z^2+(b_1+c_1)z+d_1 \quad \text{and} \quad a_2z^2+(b_2+c_2)z+d_2$$

have a common root, and this is true if and only if their resultant, given by the determinant above, is 0. (See [7] for details.)

Suppose that $\mathscr{A}$ is an algebra in which there exists a nontrivial element $\chi\in\mathscr{A}$ such that $L_\chi(\chi)=\chi^2=0$. In this case, we can set $\eta_1=\chi$ and let $\eta_2$ be any element independent of $\chi$ to obtain a multiplication table in the form of Table $S$ below.

Otherwise, $\chi\psi=0$ implies that $\chi$ and $\psi$ are independent elements of $\mathscr{A}$, and setting $\eta_1=\chi$ and $\eta_2=\psi$, we arrive at a multiplication table in the form of Table N below. Note that since $\mathscr{A}$ has no nontrivial elements which square to 0, necessarily $x_1^2\alpha+x_1x_2\gamma+x_2^2\gamma^2=0$ if and only if $x_1=x_2=0$. (This implies, in particular, that $\alpha, \delta\neq 0$.)

| | $\eta_1$ | $\eta_2$ |
|---|---|---|
| $\eta_1$ | $0$ | $\beta$ |
| $\eta_2$ | $\gamma$ | $\delta$ |

*Table S*

| | $\eta_1$ | $\eta_2$ |
|---|---|---|
| $\eta_1$ | $\alpha$ | $0$ |
| $\eta_2$ | $\gamma$ | $\delta$ |

*Table N*

$(x_1^2\alpha+x_1x_2\gamma+x_2^2\delta = 0 \Leftrightarrow x_1 = x_2 = 0).$

The classes $N$ and $S$ clearly also form a partition of the class of two-dimensional algebras with zero divisors. The multiplication tables given above are particularly useful in that they simplify the definitions of the $L_j$ and $R_j$ classes given above, as is shown by the following proposition. The proof follows immediately from the definitions of the quadratic forms $Q_L$ and $Q_R$ and the quantity $\Delta$.

Proposition. *If an algebra $\mathscr{A}$ has a basis with respect to which its multiplication table has the form of Table S, then $\mathscr{A}$ belongs to:*

$L_1$, *if* $|\beta,\gamma|\neq 0$, $\qquad R_1$, *if* $|\beta,\gamma|\neq 0$,

$L_2$, *if* $|\beta,\gamma|=0$ *but* $|\gamma,\delta|\neq 0$, $\qquad R_2$, *if* $|\beta,\gamma|=0$ *but* $|\beta,\delta|\neq 0$,

$L_3$, *if* $|\beta,\gamma|=|\gamma,\delta|=0$, $\qquad R_3$, *if* $|\beta,\gamma|=|\beta,\delta|=0$.

*If an algebra $\mathscr{A}$ has a basis with respect to which its multiplication table has the form of Table N, then $\mathscr{A}$ belongs to:*

$L_1$, *if* $|\alpha,\delta|\neq 0$, $\qquad R_1$, *if* $|\alpha,\delta|\neq 0$,

$L_2$, *if* $|\alpha,\delta|=0$ *but* $|\gamma,\delta|\neq 0$, $\qquad R_2$, *if* $|\alpha,\delta|=0$ *but* $|\alpha,\gamma|\neq 0$,

$L_3$, *if* $|\alpha,\delta|=|\gamma,\delta|=0$, $\qquad R_3$, *if* $|\alpha,\gamma|=|\alpha,\delta|=0$.

We now find a set of canonical forms for multiplication tables of two-dimensional algebras with zero divisors. As we proceed, we further partition the classes $L_iR_j$ defined by Luchian via their intersections with the classes $N$ and $S$. Following the previously established notation, we let concatenation denote intersection.

*The Case $L_3R_3$*: It is clear from the original definitions of the classes $L$ and $R$ that $\mathscr{A}$ belongs to the class $L_3R_3$ if and only if in any multiplication table for $\mathscr{A}$, the elements $\alpha$, $\beta$, $\gamma$ and $\delta$ are pairwise dependent. This implies that any multiplication for $\mathscr{A}$ has the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & \alpha v & bv \\ \eta_2 & cv & dv \end{array} .$$

*The Case $L_3R_3S$:* In this case we can begin with a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & bv \\ \eta_2 & cv & dv \end{array} .$$

The zero algebra is clearly one element of this class:

$$(L_3R_3S/1) \quad \begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & 0 \\ \eta_2 & 0 & 0 \end{array} .$$

This algebra is not catalogued by PEIRCE since it is "mixed". (See [6], p. 100.) It is Wallace's power associative algebra $A_1$.

Assume in what follows that the multiplication on $\mathscr{A}$ is nontrivial, so that $v\neq 0$ and at least one of $b$, $c$ and $d$ is nonzero. The element $v$ is determined up to scalar multiples by the fact that it spans the ranges of the left and right multiplication maps $L_\chi$ and $R_\chi$ of any element $\chi\in\mathscr{A}$. We proceed by considering two cases: $v^2=0$ and $v^2\neq 0$.

(1) $v^2=0$: Take $\eta_1=v$ and $\eta_2$ any element independent of $v$ to get a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & b\eta_1 \\ \eta_2 & c\eta_1 & d\eta_1 \end{array} .$$

(1a) $b\neq 0$, $b+c\neq 0$: Set $\zeta_1=\eta_1$, $\zeta_2=(-d/b[b+c])\eta_1+(1/b)\eta_2$. Then with $c'=c/b$, we get the table:

$$\begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & c'\zeta_1 & 0 \end{array} \quad (c'\neq -1).$$

(1b) $b\neq 0$, $b+c=d=0$: Set $\zeta_1=\eta_1$, $\zeta_2=(1/b)\eta_2$ to get:

$$\begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & -\zeta_1 & 0 \end{array} .$$

In the two preceding cases we have shown that we can get a table in the form:

$$(L_3R_3S/2)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & c'\zeta_1 & 0 \end{array}\ .$$

When $c'=-1$, this is Wallace's algebra $A(-1)$.

(1c) $b\neq 0$, $b+c=0$, $d\neq 0$: Set $\zeta_1=(d/b^2)\eta_1$, $\zeta_2=(1/b)\eta_2$ to get:

$$(L_3R_3S/3)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & -\zeta_1 & \zeta_1 \end{array}\ .$$

(1d) $b=c=0$ $(d\neq 0)$: Set $\zeta_1=d\eta_1$, $\zeta_2=\eta_2$ to get:

$$(L_3R_3S/4)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & 0 \\ \zeta_2 & 0 & \zeta_1 \end{array}\ .$$

This is Peirce's associative algebra $(c_2)$ and Wallace's algebra $B_1$.

(1e) $b=0$, $c\neq 0$: Set $\zeta_1=\eta_1$, $\zeta_2=(-d/c^2)\eta_1+(1/c)\eta_2$ to get:

$$(L_3R_3S/5)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & 0 \\ \zeta_2 & \zeta_1 & 0 \end{array}\ .$$

(2) $v^2\neq 0$: Take $\eta_2=v$ and $\eta_1$ any element such that $\eta_1^2=0$ (so that $\eta_1$ is necessarily independent of $v$). We get a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & b\eta_2 \\ \eta_2 & c\eta_2 & d\eta_2 \end{array}\qquad (d\neq 0).$$

(2a) $b\neq 0$, $|c/b|\leqq 1$: Set $\zeta_1=(1/b)\eta_1$, $\zeta_2=(1/d)\eta_2$. Then with $c'=c/b$, we get:

$$(L_3R_3S/6)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_2 \\ \zeta_2 & c'\zeta_2 & \zeta_2 \end{array}\qquad (|c'|\leqq 1).$$

(2b) $b\neq 0$, $|c/b|\geqq 1$: Set $\zeta_1=(-1/c)\eta_1+([b+c]/cd)\eta_2$ and $\zeta_2=(1/d)\eta_2$. With $c'=b/c$, we again get a table in the form of $(L_3R_3S/6)$.

(2c) $b=0$, $c\neq 0$: Set $\zeta_1=(-1/c)\eta_1+(1/d)\eta_2$, $\zeta_2=(1/d)\eta_2$, and $c'=0$ to get yet another table in the form of $(L_3R_3S/6)$.

(2d) $b=c=0$: Set $\zeta_1=\eta_1$, $\zeta_2=(1/d)\eta_2$ to get:

$$(L_3 R_3 S/7) \qquad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & 0 \\ \zeta_2 & 0 & \zeta_2 \end{array} \; .$$

Although this algebra is associative, it is not catalogued by Peirce because it is "mixed". ([6], p. 100.) It is Wallace's algebra $B_2$.

*The Case* $L_3 R_3 N$: We begin with a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & av & 0 \\ \eta_2 & cv & dv \end{array}$$

where $v\neq 0$ and $(0, 0)$ is the unique solution of $ax_1^2+cx_1x_2+dx_2^2=0$. Once again, the element $v$ is determined up to scalar multiples by the fact that it spans the ranges of the left and right multiplication maps $L_\chi$ and $R_\chi$ of any element $\chi\in\mathscr{A}$. If $\{\eta_1, v\}$ is dependent, we replace $\eta_1$ by $v$ to arrive at the table:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & a\eta_1 & 0 \\ \eta_2 & c\eta_1 & d\eta_1 \end{array} \; .$$

Otherwise, $v=n_1\eta_1+n_2\eta_2$, where $n_2\neq 0$. In this case, if $\chi=(-dn_2)\eta_1+(an_1+cn_2)\eta_2$, then $v\chi=0$ and $|v, \chi|=\begin{vmatrix} n_1 & n_2 \\ -dn_2 & an_1+cn_2 \end{vmatrix}=an_1^2+cn_1n_2+dn_2^2\neq 0$, so that $\{v, \chi\}$ is independent. Thus by replacing $\eta_1$ by $v$ and $\eta_2$ by $\chi$, we can again assume that we have a table in the preceding form.

To proceed we need a definition:

Definition.

$$\operatorname{sgn} x = \begin{cases} 1 & \text{for} \quad x \geqq 0 \\ -1 & \text{for} \quad x < 0. \end{cases}$$

(Note that we do not define $\operatorname{sgn} 0=0$, as is customary.)

The fact that $(0, 0)$ is the unique solution of $ax_1^2+cx_1x_2+dx_2^2=0$ implies that $c^2-4ad<0$ and hence $ad>0$. Thus setting $\zeta_1=(1/a)\eta_1$, $\zeta_2=([\operatorname{sgn} c]/\sqrt{ad})\eta_2$, and $c'=|c|/\sqrt{ad}$, we arrive at the table:

$$(L_3 R_3 N) \qquad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & \zeta_1 & 0 \\ \zeta_2 & c'\zeta_1 & \zeta_1 \end{array} \qquad (0 \leqq c' < 2).$$

Thus there are eight distinct table forms in the $L_3 R_3$ case.

*The Case* $L_3R_2S$: In this case we begin with a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & \beta \\ \eta_2 & \gamma & \delta \end{array}$$

with the $L_3$ and $R_2$ conditions

$$|\beta, \gamma| = |\gamma, \delta| = 0, \quad |\beta, \delta| \neq 0.$$

These imply that $\gamma=0$ and the table in fact has the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & \beta \\ \eta_2 & 0 & \delta \end{array} \quad (|\beta, \delta| \neq 0).$$

There are two cases to consider: that $\{\eta_1, \beta\}$ is dependent, and that $\{\eta_1, \beta\}$ is independent.

(1) $\{\eta_1, \beta\}$ is dependent: Our table becomes:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & b\eta_1 \\ \eta_2 & 0 & d_1\eta_1+d_2\eta_2 \end{array} \quad (bd_2 \neq 0).$$

(1a) $b-d_2\neq 0$: Take $\zeta_1=\eta_1$ and $\zeta_2=[d_1/b(d_2-b)]\eta_1+(1/b)\eta_2$. Then with $d'= =d_2/b$, we arrive at the table:

$$\begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & 0 & d'\zeta_2 \end{array} \quad (d' \neq 0, 1).$$

(1b) $b-d_2=0=d_1$: Set $\zeta_1=\eta_1$ and $\zeta_2=(1/b)\eta_2$ to arrive at:

$$\begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & 0 & \zeta_2 \end{array} .$$

Both of the preceding tables are of the form:

$$(L_3R_2S/1) \quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & 0 & d'\zeta_2 \end{array} \quad (d' \neq 0).$$

If $d'=1$, this algebra is associative, but it was missed by Peirce. It is Wallace's algebra $A_3$.

(1c) $b-d_2=0,\ d_1\neq 0$: Set $\zeta_1=(d_1/b^2)\eta_1$ and $\zeta_2=(1/b)\eta_2$ to arrive at:

$$(L_3R_2S/2)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & 0 & \zeta_1+\zeta_2 \end{array}\quad .$$

(2) $\{\eta_1, \beta\}$ is independent: Replace $\eta_2$ by $\beta$ to get a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & b\eta_2 \\ \eta_2 & 0 & d_1\eta_1+d_2\eta_2 \end{array}\quad (bd_1\neq 0).$$

Taking $\zeta_1=(1/b)\eta_1$, $\zeta_2=[\operatorname{sgn}(d_2)/\sqrt{|bd_1|}]\eta_2$, and $d'=|d_2|/\sqrt{|bd_1|}$, we arrive at the table:

$$(L_3R_2S/3)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_2 \\ \zeta_2 & 0 & \pm\zeta_1+d'\zeta_2 \end{array}\quad (d'\geqq 0).$$

*The Case $L_3R_2N$:* In this case we begin with a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & \alpha & 0 \\ \eta_2 & \gamma & \delta \end{array}\quad (\alpha, \delta\neq 0)$$

where the $L_3$ and $R_2$ conditions on such a table imply that

$$|\alpha, \delta| = |\gamma, \delta| = 0 \quad\text{and}\quad |\alpha, \gamma|\neq 0.$$

The only way these can hold is if $\delta=0$, but this is not allowed in Case $N$. Thus $L_3R_2N=\emptyset$.

*The Case $L_2R_3$*: Recall that every algebra $\mathscr{A}$ has a corresponding algebra $\mathscr{A}^{\text{opp}}$ whose underlying set and addition is the same as for $\mathscr{A}$ and whose multiplication $\circ$ is defined by $\alpha\circ\beta=\beta\alpha$. It follows that $\{\eta_1, \eta_2\}$ is a basis for $\mathscr{A}$ giving the multiplication table:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & \alpha & \beta \\ \eta_2 & \gamma & \delta \end{array}$$

if and only if $\{\eta_1, \eta_2\}$ is also a basis for $\mathscr{A}^{\text{opp}}$ giving the multiplication table:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & \alpha & \gamma \\ \eta_2 & \beta & \delta \end{array}\ .$$

Moreover, $\mathscr{A}$ lies in the class $L_i(R_i)$ if and only if $\mathscr{A}^{\text{opp}}$ lies in the class $R_i(L_i)$, for

$i=1, 2, 3$. Thus if $\mathscr{A}$ lies in the class $L_2R_3$, $\mathscr{A}^{\text{opp}}$ must lie in the class $L_3R_2$. In particular, it follows that $L_2R_3N=\emptyset$.

*The Case $L_2R_3S$:* If $\mathscr{A}$ lies in the class $L_2R_3S$, then $\mathscr{A}^{\text{opp}}$ lies in the class $L_3R_2S$ and hence must have a basis giving a multiplication table in one of the canonical forms just presented. Thus with respect to the same basis, $\mathscr{A}$ has a table in one of the following forms:

$$(L_2R_3S/1)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & 0 \\ \zeta_2 & \zeta_1 & d'\zeta_2 \end{array} \quad (d' \neq 0).$$

If $d'=1$, this is algebra is associative. It is Peirce's algebra $(b_2)$ and Wallace's algebra $A_2=A(0)$.

$$(L_2R_3S/2)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & 0 \\ \zeta_2 & \zeta_1 & \zeta_1+\zeta_2 \end{array}$$

$$(L_2R_3S/3)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & 0 \\ \zeta_2 & \zeta_2 & \pm\zeta_1+d'\zeta_2 \end{array} \quad (d' \geqq 0).$$

We have shown that there are three distinct table forms in each of the $L_3R_2$ and $L_2R_3$ cases.

*The Case $L_2R_2S$:* In this case we begin with a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & \beta \\ \eta_2 & \gamma & \delta \end{array}$$

where the $L_2$ and $R_2$ conditions imply that

$$|\beta, \gamma| = 0 \quad \text{and} \quad |\beta, \delta|, |\gamma, \delta| \neq 0.$$

This implies that the table must actually have the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & \beta \\ \eta_2 & c\beta & \delta \end{array} \quad (c \neq 0,\ |\beta, \delta| \neq 0).$$

Note that in this case $(x_1\eta_1+x_2\eta_2)^2=x_2[(1+c)x_1\beta+x_2\delta]$. Since $\{\beta, \delta\}$ is independent, this implies that $(x_1\eta_1+x_2\eta_2)^2=0$ if and only if $x_2=0$. This means that $\eta_1$ is determined up to scalars, as is $\beta$, since $\beta$ spans the range of $L_{\eta_1}$. Again there are two cases to consider: when $\{\eta_1, \beta\}$ is dependent, and when $\{\eta_1, \beta\}$ is independent.

(1) $\{\eta_1, \beta\}$ is dependent: Our table becomes:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & b\eta_1 \\ \eta_2 & c\eta_1 & d_1\eta_1+d_2\eta_2 \end{array} \quad (bcd_2 \neq 1).$$

(1a) $b+c-d_2 \neq 0$: Take $\zeta_1=\eta_1$ and $\zeta_2=[d_1/b(d_2-b-c)]\eta_1+(1/b)\eta_2$. With $c'=c/b$ and $d'=d_2/b$, we arrive at the table:

$$\begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & c'\zeta_1 & d'\zeta_2 \end{array} \quad (c'd' \neq 0,\ d'-c' \neq 0).$$

(1b) $b+c-d_2=0=d_1$: Set $\zeta_1=\eta_1$ and $\zeta_2=(1/b)\eta_2$. With $c'=c/b$ and $d'= =d_2/b$, we arrive at:

$$\begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & c'\zeta_1 & d'\zeta_2 \end{array} \quad (c'd' \neq 0,\ d' = c'+1).$$

Both of the preceding tables are of the form:

$$(L_2R_2S/1) \qquad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & c'\zeta_1 & d'\zeta_2 \end{array} \quad (c'd' \neq 0).$$

When $c'=d'=1$, we obtain Peirce's associative algebra $(a_2)$ and Wallace's algebra $B_5$. When $c'=1/\sigma$ and $d'=(1+\sigma)/\sigma$ $(\sigma \neq 0, -1)$, we obtain Wallace's algebra $A(\sigma)$.

(1c) $b+c-d_2=0$, $d_1 \neq 0$: Set $\zeta_1=(d_1/b^2)\eta_1$ and $\zeta_2=(1/b)\eta_2$. Then with $c'= =c/b$, we get:

$$(L_2R_2S/2) \qquad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & c'\zeta_1 & \zeta_1+(1+c')\zeta_2 \end{array} \quad (c'(1+c') \neq 0).$$

2) $\{\eta_1, \beta\}$ is independent: Here we can replace $\eta_2$ by $\beta$ to get a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & b\eta_2 \\ \eta_2 & c\eta_2 & d_1\eta_1+d_2\eta_2 \end{array} \quad (bcd_1 \neq 0).$$

In this case, take $\zeta_1=(1/b)\eta_1$ and $\zeta_2=[\operatorname{sgn}(d_2)/\sqrt{|bd_1|}]\eta_2$. With $c'=c/b$ and $d'=|d_2|/\sqrt{|bd_1|}$ we arrive at the table:

$$(L_2R_2S/3)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_2 \\ \zeta_2 & c'\zeta_2 & \pm\zeta_1+d'\zeta_2 \end{array} \quad (c'\neq 0,\ d'\geqq 0).$$

*The Case $L_2R_2N$:* Here we begin with a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & \alpha & 0 \\ \eta_2 & \gamma & \delta \end{array} \quad (\alpha, \delta \neq 0)$$

where the $L_2$ and $R_2$ definitions imply that

$$|\alpha, \delta| = 0 \quad \text{and} \quad |\alpha, \gamma|, |\gamma, \delta| \neq 0.$$

Since $\alpha\neq 0$ and $\{\alpha, \delta\}$ is dependent, we may rewrite the table as:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & \alpha & 0 \\ \eta_2 & \gamma & d\alpha \end{array} \quad (d\neq 0,\ |\alpha, \gamma|\neq 0).$$

For $\chi=x_1\eta_1+x_2\eta_2$ and $\xi=y_1\eta_1+y_2\eta_2$, $\chi\xi=(x_1y_1+dx_2y_2)\alpha+x_2y_1\gamma$. Thus since $\{\alpha, \gamma\}$ is independent, $\chi\xi=0$ if and only if $x_1y_1+dx_2y_2=x_2y_1=0$. When $\chi, \xi\neq 0$, this is the case if and only if $x_2=y_1=0$. Hence the only left zero divisors in $\mathscr{A}$ are multiples of $\eta_1$ and the only right zero divisors in $\mathscr{A}$ are multiples of $\eta_2$. (It follows that $\eta_1$ and $\eta_2$ are determined up to scalar multiples.)

We now rewrite the table as:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & a_1\eta_1+a_2\eta_2 & 0 \\ \eta_2 & c_1\eta_1+c_2\eta_2 & d(a_1\eta_1+a_2\eta_2) \end{array} \quad \left(d\neq 0,\ \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}\neq 0\right).$$

(1) $a_1, a_2\neq 0$: Take $\zeta_1=(1/a_1)\eta_1$, $\zeta_2=(a_2/a_1^2)\eta_2$. With $c_1'=c_1a_2/a_1^2$, $c_2'=c_2/a_1$, and $d'=da_2^2/a_1^2$, we get the table:

$$(L_2R_2N/1)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & \zeta_1+\zeta_2 & 0 \\ \zeta_2 & c_1\zeta_1+c_2'\zeta_2 & d'(\zeta_1+\zeta_2) \end{array} \quad (d'\neq 0,\ c_1\neq c_2').$$

(2) $a_2=0\,(a_1\neq 0)$: Let $\varepsilon=\operatorname{sgn}[c_1/a_1]/(a_1\sqrt{|d|})$ and take $\zeta_1=(1/a_1)\eta_1$ and $\zeta_2=\varepsilon\eta_2$. With $c_1'=|c_1/a_1|/\sqrt{|d|}$ and $c_2'=c_2/a_1$, we get:

$$(L_2R_2N/2)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & \zeta_1 & 0 \\ \zeta_2 & c_1'\zeta_1+c_2'\zeta_2 & \pm\zeta_1 \end{array}\quad (c_1'\geqq 0,\ c_2'\neq 0).$$

(3) $a_1=0\ (a_2\neq 0)$: Let $\varepsilon=\operatorname{sgn}[c_2/a_2]/(a_2\sqrt{|d|})$, $\zeta_1=\varepsilon\eta_1$ and $\zeta_2=(1/a_2|d|)\eta_2$, $c_1'=c_1/a_2|d|$ and $c_2'=|c_2/a_2|/\sqrt{|d|}$ to get the table:

$$(L_2R_2N/3)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & \zeta_2 & 0 \\ \zeta_2 & c_1'\zeta_1+c_2'\zeta_2 & \pm\zeta_2 \end{array}\quad (c_1'\neq 0,\ c_2'>0).$$

Thus there are six table forms in the $L_2R_2$ case.

*The Case $L_1R_1S$:* In this case we begin with a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & \beta \\ \eta_2 & \gamma & \delta \end{array}$$

where the $L_1$ and $R_1$ definitions imply that $|\beta,\gamma|\neq 0$. Here $(x\eta_1+y\eta_2)^2=$ $=y[x(\beta+\gamma)+y\delta]$. Thus if $\{\beta+\gamma,\delta\}$ is independent, only scalar multiples of $\eta_1$ square to 0, while if $\{\beta+\gamma,\delta\}$ is dependent, there are two independent elements with this property. As above, we procced by considering cases: that $\{\beta+\gamma,\delta\}$ is independent, and that $\{\beta+\gamma,\delta\}$ is dependent.

(1) Assume first that $\{\beta+\gamma,\delta\}$ is independent.

(1a) $\{\eta_1,\beta\}$ is independent: Replacing $\eta_2$ by $\beta$, we arrive at a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & b\eta_2 \\ \eta_2 & c_1\eta_1+c_2\eta_2 & d_1\eta_1+d_2\eta_2 \end{array}\quad \left(bc_1\neq 0,\ \begin{vmatrix} c_1 & b+c_2 \\ d_1 & d_2 \end{vmatrix}\neq 0\right).$$

Taking $\zeta_1=(1/b)\eta_1$, $\zeta_2=(1/c_1)\eta_2$, $c'=c_2b$, $d_1'=bd_1/c_1^2$ and $d_2'=d_2/c_1$, we arrive at the table:

$$(L_1R_1S/1)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_2 \\ \zeta_2 & \zeta_1+c'\zeta_2 & d_1'\zeta_1+d_2'\zeta_2 \end{array}\quad (d_2'-(1+c')d_1'\neq 0).$$

When $c'=-1$, $d_1'=0$, and $d_2'=1$, this is Wallace's algebra $A_4$.

1b) $\{\eta_1, \beta\}$ is dependent: Since $|\beta, \gamma| \neq 0$, we must have $\beta = b\eta_2$, $b \neq 0$, and hence $\{\eta_1, \gamma\}$ is independent. In this case replacing $\eta_2$ by $\gamma$ yields a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & b\eta_1 \\ \eta_2 & c\eta_2 & d_1\eta_1 + d_2\eta_2 \end{array} \qquad \left( \begin{vmatrix} b & c \\ d_1 & d_2 \end{vmatrix} \neq 0 \right).$$

Here, taking $\zeta_1 = (1/c)\eta_1$, $\zeta_2 = (1/b)\eta_2$, $d_1' = cd_1/b^2$ and $d_2' = d_2/b$, we arrive at the table:

$$(L_1 R_1 S/2) \quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & \zeta_1 & d_1'\zeta_1 + d_2'\zeta_2 \end{array} \quad (d_1' \neq d_2').$$

(2) $\{\beta+\gamma, \delta\}$ is dependent: Since $|\beta, \gamma| \neq 0$ implies that $\beta + \gamma \neq 0$, $\delta = k(\beta+\gamma)$ for some $k \in \mathbf{R}$. Then $(-k\eta_1 + \eta_2)^2 = -k(\beta+\gamma) + \delta = 0$, and replacing $\eta_2$ by $-k\eta_1 + \eta_2$ yields a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & b_1\eta_1 + b_2\eta_2 \\ \eta_2 & c_1\eta_1 + c_2\eta_2 & 0 \end{array} \qquad \left( \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \neq 0 \right).$$

(2a) $b_2, c_1 \neq 0$, $b_1/c_1 < b_2/c_2$: Taking $\zeta_1 = (1/b_2)\eta_1$, $\zeta_2 = (1/c_1)\eta_2$, $b' = b_1/c_1$, and $c' = c_2/b_2$, we arrive at the table:

$$(L_1 R_1 S/3) \quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & b'\zeta_1 + \zeta_2 \\ \zeta_2 & \zeta_1 + c'\zeta_2 & 0 \end{array} \quad (b'c' < 1).$$

(2b) $b_2, c_1 \neq 0$, $b_1/c_1 > b_2/c_2$: Taking $\zeta_1 = (1/c_1)\eta_2$, $\zeta_2 = (1/b_2)\eta_1$, $b' = c_2/b_2$, and $c' = b_1/c_1$, we again arrive at a table of the form $(L_1 R_1 S/3)$.

(2c) $b_2 = 0$ $(b_1, c_2 \neq 0)$, $c_1 \neq 0$: Taking $\zeta_1 = (1/c_2)\eta_1$, $\zeta_2 = (1/b_1)\eta_2$, and $c' = c_1/b_1$ yields the table:

$$(L_1 R_1 S/4) \quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & c'\zeta_1 + \zeta_2 & 0 \end{array} \quad (c' \neq 0).$$

(2d) $c_1 = 0$ $(b_1, c_2 \neq 0)$, $b_2 \neq 0$: Taking $\zeta_1 = (1/b_1)\eta_2$, $\zeta_2 = (1/c_2)\eta_1$, and $c' = b_2/c_2$ also yields a table of the form $(L_1 R_1 S/4)$.

(2e) $b_2 = c_1 = 0$ $(b_1, c_2 \neq 0)$: We take $\zeta_1 = (1/c_2)\eta_1$ and $\zeta_2 = (1/b_1)\eta_2$ to get the table:

$$(L_1 R_1 S/5) \quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & \zeta_2 & 0 \end{array}.$$

*The Case $L_1R_1N$:* In this case we begin with a table in the form:

| | $\eta_1$ | $\eta_2$ |
|---|---|---|
| $\eta_1$ | $\alpha$ | $0$ |
| $\eta_2$ | $\gamma$ | $\delta$ |

where the $L_1$ and $R_1$ definitions imply that $|\alpha, \delta| \neq 0$ and hence $\{\alpha, \delta\}$ is independent. It follows that $\gamma = k\alpha + \ell\delta$, where $k = |\gamma, \delta|/|\alpha, \delta|$ and $\ell = |\alpha, \gamma|/|\alpha, \delta|$, and one sees that the $N$ condition holds if and only if $k\ell \neq 1$, i.e., if and only if

$$|\gamma, \delta|\,|\alpha, \gamma| \neq |\alpha, \delta|^2.$$

Excepting scalar multiples, there are exactly two pairs of left and right zero divisors in the algebra: $\eta_1$, $\eta_2$ and $\bar{\eta}_1$, $\bar{\eta}_2$, where $\bar{\eta}_1 = -k\eta_1 + \eta_2$ and $\bar{\eta}_2 = \eta_1 - \ell\eta_2$; both are pairs of independent elements since $k\ell \neq 1$. With respect to $\{\eta_1, \eta_2\}$ the algebra has the multiplication table:

| | $\eta_1$ | $\eta_2$ | |
|---|---|---|---|
| $\eta_1$ | $a_1\eta_1 + a_2\eta_2$ | $0$ | |
| $\eta_2$ | $c_1\eta_1 + c_2\eta_2$ | $d_1\eta_1 + d_2\eta_2$ | $(a_1d_2 - a_2d_1 \neq 0)$ |

and with respect to $\{\bar{\eta}_1, \bar{\eta}_2\}$, it has the table:

| | $\bar{\eta}_1$ | $\bar{\eta}_2$ | |
|---|---|---|---|
| $\bar{\eta}_1$ | $\bar{a}_1\bar{\eta}_1 + \bar{a}_2\bar{\eta}_2$ | $0$ | |
| $\bar{\eta}_2$ | $\bar{c}_1\bar{\eta}_1 + \bar{c}_2\bar{\eta}_2$ | $\bar{d}_1\bar{\eta}_1 + \bar{d}_2\bar{\eta}_2$ | $(\bar{a}_1\bar{d}_2 - \bar{a}_2\bar{d}_1 \neq 0)$ |

where $\bar{a}_1 = \ell d_1 + d_2$, $\bar{a}_2 = d_1 + kd_2$, $\bar{c}_1 = -\ell c_1 - c_2$, $\bar{c}_2 = -c_1 - kc_2$, $\bar{d}_1 = \ell a_1 + a_2$, $\bar{d}_2 = a_1 + ka_2$, and hence $\begin{vmatrix} \bar{a}_1 & \bar{a}_2 \\ \bar{d}_1 & \bar{d}_2 \end{vmatrix} = (1 - k\ell)\begin{vmatrix} a_1 & a_2 \\ d_1 & d_2 \end{vmatrix} \neq 0.$

In this case we proceed by considering cases determined by whether the numbers $c_1$, $c_2$, $\bar{c}_1$, and $\bar{c}_2$ are zero or nonzero. While at first glance it would appear that there are sixteen such cases, the formulas $\bar{c}_1 = -\ell c_1 - c_2$ and $\bar{c}_2 = -c_1 - kd_2$ imply that only eight are actually possible. The remaining cases can be split into four groups as follows:

(1) $c_1 = c_2 = \bar{c}_1 = \bar{c}_2 = 0$.
(2) Exactly two of $c_1$, $c_2$, $\bar{c}_1$ and $\bar{c}_2$ are zero.
(These are necessarily either $c_1$ and $\bar{c}_2$, or $c_2$ and $\bar{c}_1$.)
(3) Exactly one of $c_1$, $c_2$, $\bar{c}_1$ and $\bar{c}_2$ is zero.
(4) All of $c_1$, $c_2$, $\bar{c}_1$ and $\bar{c}_2$ are nonzero.

(1) Assume first that $c_1 = c_2 = 0 = \bar{c}_1 = \bar{c}_2 = 0$.

(1a) $a_1=0$ $(a_2, d_1\neq 0)$: Taking $\zeta_1=a_2^{-2/3}d_1^{-1/3}\eta_1$, $\zeta_2=a_2^{-1/3}d_1^{-2/3}\eta_2$ and $d=a_2^{-1/3}d_1^{-2/3}d_2$, we arrive at a table in the form:

$$(L_1R_1N/1)\qquad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & \zeta_2 & 0 \\ \zeta_2 & 0 & \zeta_1+d\zeta_2. \end{array}$$

(1b) $a_1\neq 0$, $d_2=0$ $(a_2, d_1\neq 0)$: Take $\zeta_1=a_2^{-1/3}d_1^{-2/3}\eta_2$, $\zeta_2=a_2^{-2/3}d_1^{-1/3}\eta_1$ and $d=a_1a_2^{-2/3}d_1^{-1/3}$ to again obtain a table in the form $(L_1R_1N/1)$.

(1c) $a_1$, $d_2\neq 0$: Taking $\zeta_1=(1/a_1)\eta_1$, $\zeta_2=(1/d_2)\eta_2$, $a=a_2d_2/a_1^2$ and $d=a_1d_1/d_2^2$, we arrive at the table:

$$(L_1R_1N/2)\qquad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & \zeta_1+a\zeta_2 & 0 \\ \zeta_2 & 0 & d\zeta_1+\zeta_2 \end{array}\qquad (ad\neq 1).$$

(2) Next assume that exactly two of $c_1$, $c_2$, $\bar{c}_1$ and $\bar{c}_2$ are zero.

(2a) $c_1\neq 0$, $c_2=0$, $\bar{c}_1=0$ $(a_2=0,\ a_1\neq 0)$: Take $\zeta_1=(1/a_1)\eta_1$, $\zeta_2=(1/c_1)\eta_2$, $d_1'=a_1d_1/c_1^2$ and $d_2'=d_2/c_1$ to arrive at the table:

$$(L_1R_1N/3)\qquad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & \zeta_1 & 0 \\ \zeta_2 & \zeta_1 & d_1'\zeta_1+d_2'\zeta_2 \end{array}\qquad (d_2'\neq 0).$$

(2b) $c_1=0$, $c_2\neq 0$, $\bar{c}_2=0$ $(d_1=0,\ d_2\neq 0)$: Take $\zeta=(1/d_2)\eta_2$, $\zeta_2=(-1/c_2)\eta_1+(1/d_2)\eta_2$, $d_1'=(a_1c_2-a_1d_2)/c_2^2$ and $d_2'=-a_1/c_2$ to again get a table in the form $(L_1R_1N/3)$.

(3) Next assume that exactly one of $c_1$, $c_2$, $\bar{c}_1$ and $\bar{c}_2$ is zero.

(3a) $c_1=0$, $c_2\neq 0$, $\bar{c}_1\neq 0$, $\bar{c}_2\neq 0$ $(d_1\neq 0)$: Take

$$\begin{aligned}
\zeta_1 &= (1/c_2)\eta_1+[(a_1d_2-a_2d_1)/c_2^2d_1]\eta_2,\\
\zeta_2 &= (-1/c_2)\eta_1+[a_1/(a_1d_2-a_2d_1)]\eta_2,\\
a_1' &= (a_1c_2d_1+a_1d_2^2-a_2d_1d_2)/c_2^2d_1,\\
a_2' &= (a_1d_2-a_2d_1)(-a_1d_1d_2+a_2d_1^2+c_2d_1d_2)/c_2^3d_1^2,\\
d_1' &= (a_1^2c_2d_1+a_1a_2d_1d_2-a_2^2d_1^2)/(a_1d_2-a_2d_1)^2 \quad\text{and}\\
d_2' &= (-a_1^2d_2+a_1a_2d_1+a_2c_2d_1)/c_2(a_1d_2-a_2d_1)
\end{aligned}$$

to arrive at the table:

$$(^*)\qquad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & a_1'\zeta_1+a_2'\zeta_2 & 0 \\ \zeta_2 & \zeta_1+\zeta_2 & d_1'\zeta_1+d_2'\zeta_2 \end{array}\qquad [a_1'(d_2'+1)-a_2'(d_1'+1)=0,\ a_1'-a_2'\neq d_2'-d_1']$$

which is of the form:

$$(L_1R_1N/4)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & a_1\zeta_1+a_2\zeta_2 & 0 \\ \zeta_2 & \zeta_1+\zeta_2 & d_1\zeta_1+d_2\zeta_2 \end{array} \quad (|\alpha,\delta|\neq 0,\ (k+1)(\ell+1)=0,\ k\ell\neq 1)$$

where, as defined above,

$$k=(d_2-d_1)/(a_1d_2-a_2d_1) \quad\text{and}\quad \ell=(a_1-a_2)/(a_1d_2-a_2d_1).$$

(3b) $c_1\neq 0$, $c_2=0$, $\bar{c}_1\neq 0$, $\bar{c}_2\neq 0$ $(a_2\neq 0)$: Taking

$$\zeta_1=[d_2/(a_1d_2-a_2d_1)]\eta_1-(1/c_1)\eta_2,$$
$$\zeta_2=[(a_1d_2-a_2d_1)/a_2c_1^2]\eta_1+(1/c_1)\eta_2,$$
$$a_1'=(-a_1d_2^2+a_2c_1d_1+a_2d_1d_2)/c_1(a_1d_2-a_2d_1),$$
$$a_2'=(a_1a_2d_1d_2-a_2^2d_1^2+a_2c_1d_2^2)/(a_1d_2-a_2d_1)^2,$$
$$d_1'=(a_1d_2-a_2d_1)(a_1c_1-a_1d_2+a_2d_1)/a_2c_1^3, \quad\text{and}$$
$$d_2'=(a_1^2d_2-a_1a_2d_1+a_2c_1d_2)/a_2c_1^2,$$

we arrive at the table:

$$(^{**})\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & a_1'\zeta_1+a_2'\zeta_2 & 0 \\ \zeta_2 & \zeta_1+\zeta_2 & d_1'\zeta_1+d_2'\zeta_2 \end{array} \quad [(a_1'+1)d_2'-(a_2'+1)d_1'=0,\ a_1'-a_2'\neq d_2'-d_1']$$

which is again in the form $(L_1R_1N/4)$.

(3c) $c_1\neq 0$, $c_2\neq 0$, $\bar{c}_1=0$, $\bar{c}_2\neq 0$ $(a_1c_1c_2-a_2c_1^2+a_1c_2d_2-a_2c_2d_1=0)$:

(3ci) $a_1c_1+a_1d_2-a_2d_1\neq 0$ $(c_2=a_2c_1^2/(a_1c_1+a_1d_2-a_2d_1),\ a_2\neq 0)$:

Take

$$\zeta_1=[(a_1c_1+a_1d_2-a_2d_1)/a_2c_1^2]\eta_1, \quad \zeta_2=(1/c_1)\eta_2,$$
$$a_1'=a_1(a_1c_1+a_1d_2-a_2d_1)/a_1c_1^2, \quad a_2'=(a_1c_1+a_1d_2-a_2d_1)^2/a_1c_1^3,$$
$$d_1'=d_1a_2/(a_1c_1+a_1d_2-a_2d_1), \quad\text{and}\quad d_2'=d_2/c_1$$

to arrive at a table in the form (*) and hence of the form $(L_1R_1N/4)$.

(3cii) $a_1c_1+a_1d_2-a_2d_1=0$ $(a_2=0,\ a_1\neq 0,\ d_2\neq 0,\ c_1=-d_2)$: Take $\zeta_1=(1/c_2)\eta_1$, $\zeta_2=(-1/d_2)\eta_2$, $a'=a_1/c_2$, $d_1'=c_2d_1/d_2^2$ to arrive at a table in the form:

$$(^{***})\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & a'\zeta_1 & 0 \\ \zeta_2 & \zeta_1+\zeta_2 & d'\zeta_1-\zeta_2 \end{array} \quad (a'd'\neq 0,\ a'+d'\neq -1)$$

which is also of the form $(L_1R_1N/4)$.

(3d) $c_1\neq 0,\ c_2\neq 0,\ \bar{c}_1\neq 0,\ \bar{c}_2=0\ (a_1c_1d_2-a_2c_1d_1+c_1c_2d_2-c_2^2d_1=0)$:
(3di) $a_1d_2-a_2d_1+c_2d_2\neq 0\ (c_1=c_2^2d_1/(a_1d_2-a_2d_1+c_2d_2),\ d_1\neq 0)$: Take

$$\zeta_1=(1/c_2)\eta_1,\quad \zeta_2=[(a_1d_2-a_2d_1+c_2d_2)/c_2^2d_1]\eta_2,$$
$$a_1'=a_1/c_2,\quad a_2'=a_2d_1/(a_1d_2-a_2d_1+c_2d_2),$$
$$d_1'=(a_1d_2-a_2d_1+c_2d_2)^2/c_2^3d_1,\quad\text{and}\quad d_2'=d_2(a_1d_2-a_2d_1+c_2d_2)/c_2^2d_1$$

to arrive at a table in the form (**) and hence of the form $(L_1R_1N/4)$.

(3dii) $a_1d_2-a_2d_1+c_2d_2=0\ (d_1=0,\ a_1\neq 0,\ d_2\neq 0,\ c_2=-a_1)$: Take $\zeta_1=(-1/a_1)\eta_1$, $\zeta_2=(1/c_1)\eta_2$, $d'=a_2c_1/a_1^2$, and $d'=d_2/c_1$ to arrive at the table:

$$(****)\qquad \begin{array}{c|cc} & \zeta_1 & \zeta_2\\ \hline \zeta_1 & -\zeta_1+a'\zeta_2 & 0\\ \zeta_2 & \zeta_1+\zeta_2 & d'\zeta_2\end{array}\qquad (a'\neq 0,\ d'\neq 0,\ d\neq -1-a')$$

which is also of the form $(L_1R_1N/4)$.

(4) Finally, assume that $c_1, c_2, \bar{c}_1, \bar{c}_2\neq 0$. Then $a_1c_1c_2-a_2c_1^2+a_1c_2d_2-$ $-a_2c_2d_1\neq 0$, $a_1c_1d_2-a_2c_1d_1+c_1c_2d_2-c_2^2d_1\neq 0$. Taking $\zeta_1=(1/c_2)\eta_1$, $\zeta_2=(1/c_1)\eta_2$, $a_1'=a_1/c_2$, $a_2'=a_2c_1/c_1^2$, $d_1'=c_2d_1/c_1^2$, and $d_2'=d_2/c_1$, we arrive at the table:

$$(L_1R_1N/5)\qquad \begin{array}{c|cc} & \zeta_1 & \zeta_2\\ \hline \zeta_1 & a_1'\zeta_1+a_2'\zeta_2 & 0\\ \zeta_2 & \zeta_1+\zeta_2 & d_1'\zeta_1+d_2'\zeta_2\end{array}\qquad \begin{array}{l}[a_1'(d_1'+1)-a_2'(d_1'+1)\neq 0,\\ (a_1'+1)d_2'-(a_2'+1)d_1'\neq 0,\\ a_1'd_2'-a_2'd_1'\neq 0].\end{array}$$

On the other hand, because $k\ell\neq 1$, we may also use the basis change:

$$\zeta_1=[(c_1d_2-c_2d_1)/(a_1c_1d_2-a_2c_1d_1+c_1c_2d_2-c_2^2d_1)]\eta_1-$$
$$-[(a_1d_2-a_2d_1)/(a_1c_1d_2-a_2c_1d_1+c_1c_2d_2-c_2^2d_1)]\eta_2\quad\text{and}$$
$$\zeta_2=-[(a_1d_2-a_2d_1)/(a_1c_1c_2-a_2c_1^2+a_1c_2d_2-a_2c_2d_1)]\eta_1-$$
$$+(a_1c_2-a_2c_1)/(a_1c_1c_2-a_2c_1^2+a_1c_2d_2-a_2c_2d_1)]\eta_2$$

to get another table of the form $(L_1R_1N/5)$, where in this case,

$$a_1'=-(a_1c_2d_1-a_2c_1d_1+a_1d_2^2-a_2d_1d_2)/(a_1c_1d_2-a_2c_1d_1+c_1c_2d_2-c_2^2d_1),$$
$$a_2'=-(a_1c_1c_2-a_2c_1^2+a_1c_2d_2-a_2c_2d_1)(a_1d_1d_2-a_2d_1^2+c_1d_2^2-c_2d_1d_2)\cdot$$
$$\cdot(a_1c_1d_2-a_2c_1d_1+c_1c_2d_2-c_2^2d_1)^{-2},$$
$$d_1'=-(a_1^2c_2-a_1a_2c_1+a_1a_2d_2-a_2^2d_1)(a_1c_1d_2-a_2c_1d_1+c_1c_2d_3-c_2^2d_1)\cdot$$
$$\cdot(a_1c_1c_2-a_2c_1^2+a_1c_2d_2-a_2c_2d_1)^{-2},$$
$$\text{and}\quad d_2'=-(a_1^2d_2-a_1a_2d_1+a_2c_1d_2-a_2c_2d_1)/(a_1c_1c_2-a_2c_1^2+a_1c_2d_2-a_2c_2d_1).$$

In this case, two tables:

| | $\zeta_1$ | $\zeta_2$ |
|---|---|---|
| $\zeta_1$ | $a_1\zeta_1+a_2\zeta_2$ | $0$ |
| $\zeta_2$ | $\zeta_1+\zeta_2$ | $d_1\zeta_1+d_2\zeta_2$ |

| | $\zeta_1'$ | $\zeta_2'$ |
|---|---|---|
| $\zeta_1'$ | $a_1'\zeta_1+a_2'\zeta_2$ | $0$ |
| $\zeta_2'$ | $\zeta_1'+\zeta_2'$ | $d_1'\zeta_1'+d_2'\zeta_2'$ |

satisfying the side conditions defining the $(L_1R_1N/5)$ case are tables for isomorphic algebras if and only if

$$(^{\#})\quad a_1' = a_1,\quad a_2' = a_2,\quad d_1' = d_1,\quad d_2' = d_2,\quad \text{or}$$

$$(^{\#\#})\quad a_1' = -(a_1d_1-a_2d_1+a_1d_2^2-a_2d_1d_2)/(a_1d_2-a_2d_1+d_2-d_1),$$

$$a_2' = -(a_1-a_2+a_1d_2-a_2d_1)(a_1d_1d_2-a_2d_1^2+d_2^2-d_1d_2)/(a_1d_2-a_2d_1+d_2-d_1)^2,$$

$$d_1' = -(a_1^2-a_1a_2+a_1a_2d_2-a_2^2d_1)(a_1d_2-a_2d_1+d_2-d_1)/(a_1-a_2+a_1d_2-a_2d_1)^2,$$

$$\text{and}\quad d_2' = -(a_1^2d_2-a_1a_2d_1+a_2d_2-a_2d_1)/(a_1-a_2+a_1d_2-a_2d_1).$$

The case $(L_1R_1N/5)$ need not necessarily result in two distinct tables since the systems $(^{\#})$ and $(^{\#\#})$ may yield the same solutions, as is true, for example, in the case that $a_1=2$, $a_2=-10$, $d_1=3$, and $d_2=-3$. Clearly $(^{\#})$ and $(^{\#\#})$ have the same solutions if and only if

$$a_1 = -(a_1d_1-a_2d_1+a_1d_2^2-a_2d_1d_2)/(a_1d_2-a_2d_1+d_2-d_1),$$

$$a_2 = -(a_1-a_2+a_1d_2-a_2d_1)(a_1d_1d_2-a_2d_1^2+d_2^2-d_1d_2)/(a_1d_2-a_2d_1+d_2-d_1)^2,$$

$$d_1 = -(a_1^2-a_1a_2+a_1a_2d_2-a_2^2d_1)(a_1d_2-a_2d_1+d_2-d_1)/(a_1-a_2+a_1d_2-a_2d_1)^2,$$

$$\text{and}\quad d_2 = -(a_1^2d_2-a_1a_2d_1+a_2d_2-a_2d_1)/(a_1-a_2+a_1d_2-a_2d_1).$$

Each of the first and fourth of these equations is equivalent to

$$a_1+d_2+1 = 0,$$

and assuming this equation holds, the second and third equation each can be shown to be equivalent to

$$(a_2+d_1+1)\,|\alpha,\delta|+2(a_1-a_2)(d_1-d_2) = 0.$$

If these two equations are not both true, an algebra in the case $(L_1R_1N/5)$ will have two distinct tables fitting the canonical form; if they do hold, there is only one such table.

We have shown that there are ten distinct canonical forms for tables in the $L_1\,R_1$ case.

Summary. This paper proves that there are 34 distinct table forms two-dimensionel real algebras. Four of these describe division algebras and are classified in [1]. The other 30 describe algebras with zero divisors and appear above.

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# On the additive groups of $m$-rings

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Notation.

$Z(n)$ a cyclic group of order $n$.
$R$ a ring.
$R^+$ the additive group of $R$.
$R_p$ the $p$-primary component of $R^+$, $p$ a prime.
$P_m$ $\{p \text{ a prime} | m \equiv 1 \pmod{p-1}, m > 1\}$, $m$ a positive integer.

Definition. Let $m>1$ be a positive integer. A ring $R$ is said to be an $m$-ring if $a^m = a$ for all $a \in R$.

PIERCE [2, Corollary 12.5, and following comments] showed that an $m$-ring $R$ with unity satisfies $R = \bigoplus_{p \in P_m} R_p$, with $R_p$ of characteristic $p$ for each $p \in P_m$. The existence of a unity in $R$ is essential to Pierce's proof, as is the sheaf representation theory for commutative regular rings. In this note $m$-rings are not assumed to possess a unity. A complete description of the additive groups of $m$-rings will be obtained by elementary means. This classification contains the Pierce result.

Using Fermat's little theorem, and the existence of a primitive root of unity modulo $p$ for every prime $p$, (see [1]), one can prove:

Lemma 1. *Let $m>1$ be a positive integer. A prime $p$ satisfies $p | q^m - q$ for all positive integers $q$ and $m$ if and only if $p \in P_m$.*

Lemma 2. *Let $R$ be a ring which does not possess non-zero nilpotent elements. Then $R_p = \bigoplus_{\alpha_p} Z(p)$ with $\alpha_p$ a cardinal, for every prime $p$.*

Proof. Let $a \in R_p$ with $|a| = p^k$. Then $(pa)^k = (p^k a) a^{k-1} = 0$, and so $k=1$.

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Theorem 3. *Let* $m>1$ *be a positive integer, and let* $G$ *be an additive abelian group. There exists an m-ring* $R$ *with* $R^+=G$ *if and only if*

$$G=\bigoplus_{p\in P_m}\bigoplus_{\alpha_p} Z(p)$$

*with* $\alpha_p$ *an arbitrary cardinal for each* $p\in P_m$.

Proof. Let $R$ be an $m$-ring, $a\in R$, and $q>1$ be an arbitrary integer. Then $q^m a=q^m a^m=(qa)^m=qa$, i.e., $(q^m-q)a=0$. Therefore $R^+$ is a torsion group, and by Lemma 1 it follows that $R=\bigoplus_{p\in P_m} R_p$. Clearly $R$ does not possess non-zero nilpotent elements, and so Lemma 2 yields the result.

Conversely, let $G=\bigoplus_{p\in P_m}\bigoplus_{\alpha_p} Z(p)$ with $\alpha_p$ an arbitrary cardinal for each $p\in P_m$. Let $F_p$ be a field of order $p$. Every non-zero element $a\in F_p$ satisfies $a^{p-1}=1$. If $p\in P_m$, then $a^{m-1}=1$, and so $a^m=a$. Clearly $R=\bigoplus_{p\in P_m}\bigoplus_{\alpha_p} F_p$ is an $m$-ring with $R^+\cong G$.

The $m$-ring $R$ with additive group $G=\bigoplus_{p\in P_m}\bigoplus_{\alpha_p} Z(p)$ is not unital if $\alpha_p$ is an infinite cardinal for some prime $p$. To construct a unital $m$-ring with additive group $G$, it clearly suffices to consider $G=\bigoplus_{\alpha} Z(p)$, with $p$ a prime.

R. S. Pierce communicated to us the following example:

View $F_p$ as a topological space with the discrete topology, and let $X_p$ be the one point compactification of a discretely topologized set of cardinality $\alpha$. Then $C(X_p, F_p)$, the ring of $F_p$-valued continuous functions, is a unital $m$-ring with additive group isomorphic to $G$.

Another example of a unital $m$-ring with additive group $\bigoplus_{\alpha} Z(p)$ is the following:

Let $I$ be an index set, $|I|=\alpha$, and let $S=\prod_{|I|} F_p$, with elements of $S$ regarded as generalized sequences $(a_i)_{i\in I}$. Let $R$ be the subring of $S$ consisting of $a\in S$ for which there exists a finite subset $J\subseteq I$ such that $a_i=a_j$ for all $i,j\in I\setminus J$. Clearly $R$ is a unital $m$-ring, with $R^+=\bigoplus_{\alpha} Z(p)$.

An argument similar to that used in proving Theorem 3 yields:

Theorem 4. *Let* $R$ *be a ring such that for every* $a\in R$ *there exists a positive integer* $m(a)>1$, *depending on* $a$, *with* $a^{m(a)}=a$. *Then* $R^+=\bigoplus_{p\in P}\bigoplus_{\alpha_p} Z(p)$ *with* $P$ *a set of primes. Conversely, every such group is the additive group of a ring with the above property.*

For a different elementary approach to $m$-rings see [3].

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# On rings satisfying $(x, y, z)=(y, z, x)$

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Let $R$ be an arbitrary nonassociative ring having a Peirce decomposition, into a direct sum of $Z$ modules, relative to nonzero idempotent $e$ in the nucleus of $R$. If $R$ satisfies the identity $(x, y, z)=(y, z, x)$ then (i) under certain conditions on the $Z$ modules, $R$ is associative and (ii) if $R$ is prime then $e$ is the identity element of $R$.

As usual, the associator $(x, y, z)$ denotes $(xy)z-x(yz)$ and the commutator $(x, y)=xy-yx$. STERLING [3] has shown that semiprime rings satisfying

$$(x, y, z) = (y, z, x) \tag{1}$$

are alternative. The nucleus $N(R)$ of an arbitrary nonassociative ring $R$ consists of all elements $n$ in $R$ such that

$$(n, r, s) = 0 = (r, s, n) = (s, n, r) \quad \text{for all} \quad r, s \quad \text{in} \quad R.$$

It is well known, see SCHAFER [2] p. 18, that $N(R)$ is an associative subring of $R$. We need an identity valid in all rings, the so-called Teichmüller identity

$$(wx, y, z)-(w, xy, z)+(w, x, yz) = w(x, y, z)+(w, x, y)z. \tag{2}$$

We now take turns letting one of the four elements in (2) be in the nucleus. Thus

$$(nx, y, z) = n(x, y, z) \tag{3}$$

$$(wn, y, z) = (w, ny, z) \tag{4}$$

$$(w, xn, z) = (w, x, nz) \tag{5}$$

$$(w, x, yn) = (w, x, y)n \tag{6}$$

for any element $n$ in $N(R)$ and the rest of the elements arbitrary in $R$.

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Lemma. *Let $R$ be a ring satisfying* (1). *Then*

$$(R, N(R)) \subseteqq N(R).$$

Proof. Let $x$, $y$, $z$ be arbitrary elements of $R$ and $n$ be an element of $N(R)$. Using (1), (3), (5) and (6) we get

$$n(x, y, z) = (nx, y, z) = (y, z, nx) = (y, zn, x) = (x, y, zn) = (x, y, z)n.$$

Thus

$$n(x, y, z) = (x, y, z)n. \tag{7}$$

Again using (1), (3), (6) and (7) we get

$$(nx, y, z) = n(x, y, z) = (x, y, z)n = (y, z, x)n = (y, z, xn) = (xn, y, z).$$

Thus $((x, n), y, z)=0$. This implies that $(R, N(R))\subseteqq N(R)$.

A ring $R$ is said to have a Peirce decomposition relative to the idempotent $e\in R$ if $R$ can be decomposed into a direct sum of $Z$ modules $R_{ij}$ $(i, j=0, 1)$ where

$$R_{ij} = \{x\in R\colon\ xe = jx \text{ and } ex = ix\}.$$

It is known, see Jacobson [1], that if $R$ is an associative ring and if $e$ is an idempotent in $R$ then $R$ has a Peirce decomposition relative to $e$. Also, if $R$ has an identity element 1 and if we write $e_1=e$ and $e_0=1-e$ then $R_{ij}=e_iRe_j$.

Let $e\in N(R)$. Embed $R$ into the ring $R'=Z+R$ which contains an identity element 1. Clearly, $e$ and $1-e$ are in $N(R')$. It follows that $R=\oplus R_{ij}$ and $R_{ij}=e_iRe_j$ for $i, j=0, 1$. Thus

$$R_{ij}R_{kl} = (e_iRe_j)(e_kRe_l) = e_iR(e_ie_k)Re_l \subseteqq \delta_{jk}e_iRe_l = \delta_{jk}R_{il}$$

for $i, j, k, l=0, 1$ ($\delta$ denotes the Kronecker delta).

Theorem. *Let $R$ be a ring satisfying* (1) *with an idempotent $e\neq 0$ in $N(R)$.*

(i) *If $R$ satisfies the condition*

$$R_{ij}R_{ji} = R_{ii} \quad \textit{when} \quad i\neq j$$

*then $R$ is associative.*

(ii) *If $R$ is prime then $e$ is the identity element of $R$.*

Proof. (i) $R_{10}=(e, R_{10})=-(R_{10}, e)$ and $R_{01}=(R_{01}, e)$. Since $e\in N(R)$, by the above Lemma, $R_{10}$ and $R_{01}\subseteqq N(R)$. $N(R$ is an associative subring of $R$. So $R_{10}R_{01}$ and $R_{01}R_{10}\subseteqq N(R)$. By the given condition $R_{11}$ and $R_{00}\subseteqq N(R)$. It follows that

$$R = R_{11}+R_{10}+R_{01}+R_{00} \subseteqq N(R).$$

Hence $R$ is associative.

(ii) $R_{10}R_{01}+R_{01}R_{10}\subseteq N(R)$. This, together with the property $R_{ij}R_{kl}\subseteq\delta_{jk}R_{il}$, allows us to conclude that $B=R_{10}R_{01}+R_{10}+R_{01}+R_{01}R_{10}$ is an ideal of $R$ contained in $N(R)$. All rings $R$ have an ideal $A$, called the associater ideal. It is defined as the smallest ideal which contains all associators. It actually consists of all finite sums of associators and right multiples of associators. The associator ideal is never zero, except when $R$ is associative. We shall show that $BA=(0)$. Let $b\in B$. Then using Teichmüller identity (2) we get

$$(bx, y, z)-(b, xy, z)+(b, x, yz) = b(x, y, z)+(b, x, y)z.$$

Since $B$ is an ideal contained in $N(R)$ and $b\in B$ we have

$$(bx, y, z) = (b, xy, z) = (b, x, yz) = (b, x, y) = 0.$$

Thus, from the above equation, we get

$$b(x, y, z) = 0.$$

Also, since $b\in N(R)$,

$$b\big((x, y, z)w\big) = \big(b(x, y, z)\big)w = 0.$$

Thus we have proved that $bA=(0)$ for all $b$ in $B$. Hence $BA=(0)$. But $R$ is prime and nonassociative. This implies that $B=(0)$. So we have

$$R = R_{11}\oplus R_{00}.$$

Thus, $R_{11}$ and $R_{00}$ are ideals of $R$ such that

$$R_{11}R_{00} = (0).$$

From the primeness of $R$ again $R_{11}=(0)$ or $R_{00}=(0)$. But $0\neq e\in R_{11}$ so that $R_{11}\neq(0)$. We must have $R_{00}=(0)$. This implies that $e$ is the identity element of $R$.

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# On commutativity of left $s$-unital rings

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**1. Introduction.** In this paper we study the commutativity of a left $s$-unital ring $R$ satisfying the polynomial identity

$$(1) \qquad x^t[x^n, y] = \pm y^r[x, y^m]y^s \quad \text{for all} \quad x, y \in R,$$

where $m$, $n$, $r$, $s$ and $t$ are fixed non-negative integers. To establish commutativity, we need some extra conditions. The results of this paper generalize some of the well-known commutativity theorems.

**2. Preliminary results.** Throughout the present paper, $R$ will represent an associative ring (not necessarily with unity 1), $Z(R)$ the center of $R$, $C(R)$ the commutator ideal of $R$, $N(R)$ the set of all nilpotent elements in $R$, $N'(R)$ the set of all zero-divisors in $R$, and $R^+$ the additive group of $R$. As usual, for each $x, y \in R$, we write $[x, y] = xy - yx$. By $GF(q)$ we mean the Galois field (finite field) with $q$ elements, and $(GF(q))_2$ the ring of all $2\times 2$ matrices over $GF(q)$. Set

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

in $(GF(p))_2$, for a prime $p$.

Definition 1. A ring $R$ is called *left* (resp. *right*) *s-unital* if $x \in Rx$ (resp. $x \in xR$) for each $x \in R$. Further, $R$ is called *s-unital* if it is both left as well as right $s$-unital, that is, $x \in xR \cap Rx$ for each $x \in R$.

Definition 2. If $R$ is an $s$-unital (resp. a left or right $s$-unital) ring, then for any finite subset $F$ of $R$, there exists an element $e \in R$ such that $ex = xe = x$ (resp. $ex = x$ or $xe = x$) for all $x \in F$. Such an element $e$ is called the *pseudo* (resp. *pseudo left* or *pseudo right*) *identity* of $F$ in $R$.

Definition 3. For any positive integer $n$, the ring $R$ is said to have *property* $Q(n)$ if for all $x, y \in R$, $n[x, y] = 0$ implies $[x, y] = 0$.

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The property $Q(n)$ is an $H$-property in the sense of [9]. It is obvious that every $n$-torsion free ring $R$ has the property $Q(n)$, and every ring has the property $Q(1)$. Also, it is clear that if a ring $R$ has the property $Q(n)$, then $R$ has the property $Q(m)$ for every divisor $m$ of $n$.

In the proof of our results, we shall require the following well-known results.

Lemma 1 ([3, Lemma 2]). *Let $R$ be a ring with unity* 1, *and let $x$ and $y$ be elements in $R$. If $kx^m[x,y]=0$ and $k(x+1)^m[x,y]=0$ for some integers $m\geqq 1$ and $k\geqq 1$, then necessarily $k[x,y]=0$.*

Lemma 2 ([14, Lemma 3]). *Let $x$ and $y$ be elements in a ring $R$. If $[x,[x,y]]=0$, then $[x^k,y]=kx^{k-1}[x,y]$ for all integers $k\geqq 1$.*

Lemma 3 ([18, Lemma 3]). *Let $R$ be a ring with unity* 1, *and let $x$ and $y$ be elements in $R$. If $(1-y^k)x=0$, then $(1-y^{km})x=0$ for some integers $k>0$ and $m>0$.*

Lemma 4. *Let $x$ and $y$ be elements in a ring $R$. Suppose that there exists relatively prime positive integers $m$ and $n$ such that $m[x,y]=0$ and $n[x,y]=0$. Then $[x,y]=0$.*

Lemma 5 ([4, Theorem 4 $(C)$]). *Let $R$ be a ring with unity* 1. *Suppose that for each $x\in R$ there exists a pair $n$ and $m$ of relatively prime positive integers for which $x^n\in Z(R)$ and $x^m\in Z(R)$. Then $R$ is commutative.*

Following results play an important role in proving the main results of this paper. The first is due to KEZLAN [10, Theorem] and BELL [3, Theorem 1] (also see [9, Proposition 2]), the second and third are due to Herstein.

Theorem KB. *Let $f$ be a polynomial in $n$ non-commuting indeterminates $x_1,\dots,x_n$ with relatively prime integral coefficients. Then the following are equivalent:*

(1) *For any ring satisfying the polynomial identity $f=0$, $C(R)$ is a nil ideal.*

(2) *For every prime $p$, $(GF(p))_2$ fails to satisfy $f=0$.*

(3) *Every semi-prime ring satisfying $f=0$ is commutative.*

Theorem H ([7, Theorem 18]). *Let $R$ be a ring and let $n>1$ be an integer. Suppose that $x^n-x\in Z(R)$ for all $x\in R$. Then $R$ is commutative.*

Theorem H′ ([8, Theorem]). *If for every $x$ and $y$ in a ring $R$ we can find a polynomial $p_{x,y}(t)$ with integral coefficients which depends on $x$ and $y$ such that $[x^2p_{x,y}(x)-x,y]=0$, then $R$ is commutative.*

**3. Main Results.** Now, we present our results.

Theorem 1. *Let $n>1$, $m$, $r$, $s$ and $t$ be fixed non-negative integers, and let $R$ be a left s-unital ring satisfying the polynomial identity* (1). *Further, if $R$ possesses $Q(n)$, then $R$ is commutative.*

Following lemma shows that the ring considered in Theorem 1 is in fact an $s$-unital ring. According to Proposition 1 of [9] this lemma enables us to reduce the proof of Theorem 1 to a ring with unity 1.

Lemma 6. *Let $n>0$, $m$, $r$, $s$ and $t$ be fixed non-negative integers such that $(r, n, s, m, t)\neq(0, 1, 0, 1, 0)$, and let $R$ be a left s-unital ring satisfying the polynomial identity* (1). *Then $R$ is s-unital.*

Proof. Let $x$ and $y$ be arbitrary elements in $R$. Suppose that $R$ is a left $s$-unital ring. Then there exists an element $e\in R$ such that $ex=x$ and $ey=y$. Replace $x$ by $e$ in (1). Then $e^{t+n}y-e^tye^n=\pm(y^rey^{m+s}-y^{r+m}ey^s)$. Thus $y=ye^n\in yR$ for all $y\in R$. Therefore, $R$ is $s$-unital.

Lemma 7. *Let $n>0$, $m$, $r$, $s$ and $t$ be fixed non-negative integers, and let $R$ be a ring satisfying the polynomial identity* (1). *Then $C(R)$ is nil.*

Proof. Let $x=e_{11}$ and $y=e_{12}$. Then $x$ and $y$ fail to satisfy the polynomial identity (1) whenever $n>0$ except for $r=s=0$, $m=1$. In this later case one can choose $x=e_{12}$ and $y=e_{21}$. By Theorem KB,

(2) $$C(R)\subseteqq N(R).$$

Combining Lemma 7 with Theorem KB gives the following commutativity theorem for semi-prime rings.

Theorem 2. *Let $n>0$, $m$, $r$, $s$ and $t$ be fixed non-negative integers. If $R$ is a semi-prime ring satisfies the polynomial identity* (1), *then $R$ is commutative.*

Lemma 8. *Let $n>1$, $m$, $r$, $s$ and $t$ be fixed non-negative integers, and let $R$ be a ring with unity* 1. *Suppose that $R$ satisfies the polynomial identity* (1). *Further, if $R$ has $Q(n)$, then $N(R)\subseteqq Z(R)$.*

Proof. If $a\in N(R)$, then there exists a positive integer $p$ such that

(3) $$a^k\in Z(R) \quad \text{for all } k\geqq p, \quad \text{and } p \text{ minimal.}$$

Let $p=1$. Then $a\in Z(R)$. Suppose that $p>1$ and $b=a^{p-1}$. Replace $x$ by $b$ in (1) to get $b^t[b^n, y]=\pm y^r[b, y^m]y^s$. In view of (3) and the fact that $(p-1)n\geqq p$ for $n>1$,

(4) $$y^r[b, y^m]y^s=0 \quad \text{for all } y\in R.$$

Now, replace $x$ by $1+b$ in (1) to get $(1+b)^t[(1+b)^n, y]=\pm y^r[1+b, y]y^s$. As $1+b$

is invertible, using (4), the last identity gives

$$[(1+b)^n, y] = 0 \quad \text{for all} \quad y \in R. \tag{5}$$

Combining (3) and (5) yield $0=[(1+b)^n, y]=[1+nb, y]=n[b, y]$. Now, $Q(n)$ implies that $[b, y]=0$ for all $y \in R$, that is $a^{p-1} \in Z(R)$. This contradicts the minimality of $p$. So, $p=1$ and $a \in Z(R)$. Therefore,

$$N(R) \subseteqq Z(R). \tag{2'}$$

Remark 1. Combining (2) and (2'), one gets

$$C(R) \subseteqq N(R) \subseteqq Z(R), \tag{6}$$

for any ring $R$ with unity 1 which satisfies the polynomial identity (1) for all fixed non-negative integers $n>1$, $m$, $r$, $s$ and $t$ and whenever $R$ has $Q(n)$. Hence, in view of (6), $[x, [x, y]]=0$ for all $x, y \in R$ and thus the conclusion of Lemma 2 holds. In the proof of Theorem 1, we shall therefore routinely use Lemma 2 without explicit mention.

Proof of Theorem 1. According to Lemma 6, $R$ is $s$-unital. Therefore, in view of Proposition 1 of [9], it suffices to prove the theorem for $R$ with unity 1.

If $m=0$, then (1) gives $x^t[x^n, y]=0$. Thus, $nx^{t+n-1}[x, y]=0$. Replace $x$ by $x+1$ and apply Lemma 1 to obtain $n[x, y]=0$ which by $Q(n)$, we get $[x, y]=0$ for all $x, y \in R$. Therefore, $R$ is commutative.

Suppose that $m \geqq 1$. Let $q=(p^{t+n}-p)$ (for a prime $p$). Then from (1) we have $qx^t[x^n, y]=(p^{t+n}-p)x^t[x^n, y]=p^{t+n}x^t[x^n, y]-px^t[x^n, y]=(px)^t[(px)^n, y] \mp \mp py^r[x, y^m]y^s=(px)^t[(px)^n, y] \mp y^r[(px), y^m]y^s=0$. Therefore, $qnx^{t+n-1}[x, y]=0$. If we set $k=qn$, then $k[x, y]=0$ and thus $[x^k, y]=kx^{k-1}[x, y]=0$. So

$$x^k \in Z(R) \quad \text{for all} \quad x \in R. \tag{7}$$

We consider two cases:

*Case* (a): If $m>1$, then $x^t[x^n, y]=\pm m[x, y]y^{r+s+m-1}$ and $x^t[x^n, y^m]=\pm \pm m[x, y^m]y^{m(r+s+m-1)}$. So $mx^t[x^n, y]y^{m-1}=-m[x, y^m]y^{m(r+t+m-1)}$. By using (1), we obtain $my^r[x, y^m]y^{s+m-1}=m[x, y]y^{m(r+s+m-1)}$. Using Lemma 3, we get

$$m[x, y^m]y^{r+s+m-1}(1-y^{k(m-1)(r+s+m-1)}) = 0 \quad \text{for all} \quad x, y \in R. \tag{8}$$

Now, by (6), the polynomial identity (1) becomes

$$nx^{t+n-1}[x, y] = \pm my^{r+s+m-1}[x, y] = \pm m[x, y]y^{r+s+m-1} \quad \text{for all} \quad x, y \in R. \tag{9}$$

It is well-known that $R$ is isomorphic to a subdirect sum of subdirectly irreducible rings $R_i$ ($i \in I$, the index set). Each $R_i$ satisfies (1), (6), (7), (8) and (9). We consider the ring $R_i$ ($i \in I$). Let $S$ be the intersection of all non-zero ideals of $R_i$. Then $S \neq (0)$, and $Sd=0$ for any central zero-divisor $d$.

Let $a\in N'(R_i)$. By (8), $m[x, a^m]a^{r+s+m-1}(1-a^{k(m-1)(r+s+m-1)})=0$. If $m[x, a^m]a^{r+s+m-1}\neq 0$, then $a^{k(m-1)(r+s+m-1)}$ and $1-a^{k(m-1)(r+s+m-1)}$ are central zero-divisors. So $(0)=S(1-a^{k(m-1)(r+s+m-1)})=S\neq 0$, which is a contradiction. Thus

$$(10)\qquad m[x, a^m]a^{r+s+m-1}=0 \quad \text{for all} \quad x\in R_i.$$

From (9) and (10), $nx^{t+n-1}[x, a^m]=\pm m[x, a^m]a^{m(r+s+m-1)}=0$ and $n[x, a^m]=0$. Therefore, $nm[x, a]a^{m-1}=0$. Now,

$$n^2x^{t+n-1}[x, a]=n(nx^{t+n-1}[x, a])=\pm nm[x, a]a^{r+s+m-1}=0$$

and $n^2[x, a]=0$. But $[x^{n^2}, a]=n^2x^{n^2-1}[x, a]$. Therefore,

$$(11)\qquad [x^{n^2}, a]=0 \quad \text{for all} \quad x\in R_i.$$

If $c\in Z(R_i)$, then by (1), $(c^{t+n}-c)x^t[x^n, y]=(cx)^t[(cx)^n, y]-cx^t[x^n, y]=(cx)^t[(cx)^n, y]\mp y^r[(cx), y^m]y^s=0$ and thus $n(c^{t+n}-c)x^{t+n-1}[x, y]=0$. By Lemma 1 $n(c^{t+n}-c)[x, y]=0$. So

$$(12)\qquad (c^{t+n}-c)[x^n, y]=0 \quad \text{for all} \quad x, y\in R_i.$$

Using (7), we get

$$(13)\qquad (y^{k(t+n)}-y^k)[x^n, y]=0 \quad \text{for all} \quad x, y\in R_i.$$

Suppose that $y\in R_i$. If $[x^n, y]=0$, then $[x^{n^2}, y^j-y]=0$ for all positive integers $j$ and $x\in R_i$. If $[x^{n^2}, y]\neq 0$, then $[x^n, y]\neq 0$, for $[x^n, y]=0$ implies that $[x^{n^2}, y]=0$, which is a contradiction. If $[x^n, y]\neq 0$, then (13) implies that $y^{k(t+n)}-y^k$ is a zero-divisor. Therefore, $y^{k(t+n-1)+1}-y$ is also a zero-divisor. By (11), we have

$$(14)\qquad [x^{n^2}, y^{k(t+n-1)+1}-y]=0 \quad \text{for all} \quad x, y\in R_i.$$

As each $R$ satisfies (14), the original ring $R$ also satisfies (14). But $R$ has $Q(n)$. Combining (14) with Lemma 2, we obtain $[x, y^{k(t+n-1)+1}-y]=0$. Therefore, $R$ is commutative by Theorem H.

*Case* (b): Let $m=1$. Then $x^t[x^n, y]=\pm y^r[x, y]y^s$ and $nx^{t+n-1}[x, y]=\pm[x, y]y^{r+s}$. Replace $x$ by $x^n$ to get

$$nx^{n(t+n-1)}[x^n, y]=\pm[x^n, y]y^{r+s}=\pm nx^{n-1}[x, y]y^{r+s}=\pm nx^{t+n-1}[x^n, y].$$

Thus $n(1-x^{(n-1)(t+n-1)})x^{t+n-1}[x^n, y]=0$, which in view of Lemma 3, we get

$$(15)\qquad n(1-x^{k(n-1)(t+n-1)})x^{t+n-1}[x^n, y]=0 \quad \text{for all} \quad x, y\in R.$$

As in case (a) if $a\in N'(R\ )$, then by (15), $n(1-a^{k(n-1)(t+n-1)})a^{t+n-1}[a^n, y]=0$. Also, we can prove that

$$(16)\qquad na^{t+n-1}[a^n, y]=0 \quad \text{for all} \quad y\in R_i.$$

Now, we have $\pm[a^n, y]y^{r+s}=na^{n(t+n-1)}[a^n, y]=0$, and thus $[a^n, y]=0$. Therefore, $[a, y]y^{r+s}=a^t[a^n, y]=0$. So

$$(17) \qquad [a, y] = 0 \quad \text{for all} \quad y\in R_i.$$

If $c\in Z(R_i)$, then as in case (a), we get $(c^{t+n}-c)[x, y]=0$. In particular, by (7), $(x^{k(t+n)}-x^k)[x, y]=0$ for all $x, y\in R_i$. If $[x, y]=0$ for all $x, y\in R_i$, then $R$ satisfies $[x, y]=0$ for all $x, y\in R$. Therefore, $R$ is commutative. Now, if for each $x, y\in R_i$, $[x, y]\neq 0$, then $x^{k(t+n-1)+1}-x\in N'(R_i)$, and hence $x^{k(t+n-1)+1}-x\in N'(R)$. But the identity (17) is satisfied by $R$. So $[x^{k(t+n-1)+1}-x, y]=0$ for each $x, y\in R$. Therefore, $R$ is commutative by Theorem H.

In Theorem 1, $Q(n)$ is essential. To see this, we consider the following example:

Example 1. Let

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad C_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

be elements of the ring of all $3\times 3$ matrices over $\mathbf{Z}_2$, the ring of integers $\bmod 2$. If $R$ is the ring generated by the matrices $A_1$, $B_1$ and $C_1$, then using Dorroh construction with $\mathbf{Z}_2$ (see [4, Remark]), we obtain a ring $R$ with unity 1. Then $R$ is non-commutative and satisfies $[x^2, y]=[x, y^2]$ for all $x, y\in R$.

The presence of the identity in Theorem 1 is not superfluous. To see this we consider the following example.

Example 2. Let

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

be elements of the ring of all $3\times 3$ matrices over $\mathbf{Z}_2$. If $R$ is the ring generated by the matrices $A_2$, $B_2$ and $C_2$, then for each integer $n\geqq 1$, the ring $R$ satisfies the identity $[x^n, y]=[x, y^n]$ for all $x, y\in R$, but $R$ is not commutative.

Corollary 1 ([4, Theorem 5]). *Let $R$ be a ring with unity 1, and $n>1$ be a fixed integer. If $R^+$ is n-torsion free and $R$ satisfies the identity $[x^n, y]=[x, y^n]$ for all $x, y\in R$, then $R$ is commutative.*

Corollary 2 ([15, Theorem 2]). *Let $n\geqq m\geqq 1$ be fixed integers such that $mn>1$, and let $R$ be an s-unital ring. Suppose that every commutator in $R$ is $m!$-torsion free.*

*Further, if R satisfies the polynomial identity*

$$(18) \qquad [x^n, y] = [x, y^m] \quad \text{for all} \quad x, y \in R,$$

*then R is commutative.*

Corollary 3 ([16, Theorem 1]). *Let $n>1$ and m be positive integers, and let s and t be any non-negative integers. Let R be an associative ring with unity* 1. *Suppose*

$$(19) \qquad x^t[x^n, y] = [x, y^m]y^s \quad \textit{for all} \quad x, y \in R.$$

*Further, if R is n-torsion free, then R is commutative.*

In the following result we show that the conclusion of Theorem 1 is still valid if $Q(n)$ is replaced by requiring $m$ and $n$ to be relatively prime positive integers.

Theorem 3. *Let $m>1$, and $n>1$ be relatively prime integers, and let r, s, and t be non-negative integers. If R is a left s-unital ring satisfies the polynomial identity* (1), *then R is commutative.*

Proof. According to Lemma 6, $R$ is $s$-unital. Therefore, in view of Proposition 1 of [9], it is sufficient to prove the theorem for $R$ with unity 1.

Without loss of generality, we assume that $R$ is subdirectly irreducible. Let $a \in N(R)$. Consider $p$ and $b$ as in Lemma 7. Following the proof of Lemma 7, we obtain $n[b, y]=0$ and $m[b, y]=0$. By Lemma 4, $[b, y]=0$. So $a^{p-1} \in Z(R)$, which contradicts the minimality of $p$. Therefore $p=1$ and $a \in Z(R)$. Thus $N(R) \subseteqq Z(R)$. By Lemma 6,

$$(20) \qquad C(R) \subseteqq N(R) \subseteqq Z(R).$$

The proof of (7) also works in the present situation. So there exists an integer $k$ (as in the proof of Theorem 1) such that

$$(21) \qquad x^k \in Z(R) \quad \text{for all} \quad x \in R.$$

Let $u \in N'(R)$. Using argument similar to one as in the proof of Theorem 1 (see (11)), we get $[x^{n^2}, u]=0$ and $[x^{m^2}, u]=0$. By Lemma 4,

$$(22) \qquad [x, u] = 0 \quad \text{for all} \quad x \in R.$$

If $c \in Z(R)$, then, as observed in the proof followed by (11), we can prove that $n(c^{t+n}-c)[x, y]=0$ and $m(c^{t+n}-c)[x, y]=0$. By Lemma 4,

$$(23) \qquad (c^{t+n}-c)[x, y] = 0 \quad \text{for all} \quad x, y \in R.$$

By (21), $(y^{k(t+n)}-y^k)[x, y]=0$. Arguing as in the proof of Theorem 1, we finally get $y^{k(t+n-1)+1}-y \in N'(R)$. Hence (22) yields $y^{k(t+n-1)+1}-y \in Z(R)$ for all $y \in R$. By Theorem H, $R$ is commutative.

Corollary 4 ([16, Theorem 2]). *Let $m$ and $n$ be relatively prime positive integers, and let $s$ and $t$ be any non-negative integers. Suppose that $R$ is an associative ring with unity 1 satisfies the polynomial identity* (19). *Then $R$ is commutative.*

Next result deals with the commutativity of $R$ satisfying (1) for the case $n=1$.

Theorem 4. *Let $R$ be a left $s$-unital ring, and let $m$, $r$, $s$ and $t$ be fixed non-negative integers such that $(t, m, r, s)\neq(0, 1, 0, 0)$. If $R$ satisfies the polynomial identity*

(24) $$x^t[x, y] = \pm y^r[x, y^m]y^s \quad \textit{for all} \quad x, y\in R,$$

*then $R$ is commutative.*

Proof. According to Lemma 6, $R$ is an $s$-unital ring. In view of proposition 1 of [9], we prove the result for $R$ with unity 1.

*Case* (I): If $m=0$, then the identity (24) becomes $x^t[x, y]=0$. By Lemma 1, $[x, y]=0$ for each $x, y\in R$. Therefore, $R$ is commutative.

*Case* (II): Let $m>1$, $x=e_{11}$, and $y=e_{12}$. Then $x$ and $y$ fail to satisfy the identity (24). By Theorem KB, $C(R)\subseteq N(R)$. If $a\in N(R)$, then there exists a positive integer $p$ such that

(25) $$a^k\in Z(R) \quad \text{for all } k \geqq p, \text{ and } p \text{ minimal.}$$

If $p=1$, then $a\in Z(R)$. Now, let $p>1$, and let $b=a^{p-1}$. Replace $y$ by $b$ in (24) to get $x^t[x, b]=\pm b^r[x, b^m]b^s$. In view of (25), $x^t[x, b]=0$. By Lemma 1, $[x, b]=0$ for all $x\in R$. Therefore, $a^{p-1}\in Z(R)$ which is a contradiction. Thus $p=1$, and hence $N(R)\subseteq Z(R)$. So $C(R)\subseteq N(R)\subseteq Z(R)$. The method of proof of Theorem 1 enables us to establish the commutativity of $R$.

*Case* (III): Let $m=1$. Then (24) becomes

(26) $$x^t[x, y] = \pm y^r[x, y]y^s \quad \text{for all} \quad x, y\in R.$$

We consider the following cases.

(i): Let $r=0$. Then (26) becomes

(27) $$x^t[x, y] = \pm [x, y]y^s \quad \text{for all} \quad x, y\in R.$$

If $s=0$, then $t>0$. Thus, $x^t[x, y]=\pm[x, y]$ for all $x, y\in R$. Therefore, $R$ is commutative by [11, Theorem]. Similarly, if $t=0$ in (27), then $R$ is commutative by [11, Theorem]. Let $t>0$ and $s>0$. Then $x=e_{11}$, and $y=e_{12}$ fail to satisfy the identity (27). By Theorem KB, $C(R)\subseteq N(R)$. Now, for any positive integer $q$, we can easily see that

(28) $$x^{qt}[x, y] = \pm [x, y]y^{qt} \quad \text{for all} \quad x, y\in R.$$

If $a \in N(R)$, then for sufficiently large $q$, we get $x^{qt}[x, a]=0$ for all $x, y \in R$. By Lemma 1, $a \in Z(R)$. Therefore $C(R) \subseteqq N(R) \subseteqq Z(R)$.

Let $l=(p^{s+1}-p)>0$ for $s>0$ ($p$ is a prime). Then we can prove that

(29) $$x^l \in Z(R) \quad \text{for all} \quad x \in R.$$

By (28) and (29), $[x^{lt+1}, y]=\pm[x, y^{ls+1}]$ for all $x, y \in R$. In view of Proposition 3 (ii) of [9], there exists positive integer $j$ such that $[x, y^{(ls+1)^j}]=0$ for each $x, y \in R$. But $(ls+1)^j=lk+1$. Then (28) yields $[x, y]y^{lk}=0$, and so by Lemma 1, we obtain $[x, y]=0$ for all $x, y \in R$. Therefore, $R$ is commutative.

(ii): If $s=0$, then (26) becomes

(30) $$x^t[x, y] = \pm y^r[x, y] \quad \text{for all} \quad x, y \in R,$$

and so either $t>0$ or $r>0$. Without loss of generality, we can suppose that $r>0$. Clearly, $x=e_{11}$ and $y=e_{12}$ fail to satisfy (30). By Theorem KB, $C(R) \subseteqq N(R)$. Following the same argument as in (i) we can prove the commutativity of $R$.

(iii): If $t=0$, then (26) gives

(31) $$[x, y] = \pm y^r[x, y]y^s \quad \text{for all} \quad x, y \in R.$$

Then either $r>0$ or $s>0$. Clearly $x=e_{11}$ and $y=e_{12}$ fail to satisfy (31). Therefore, $C(R) \subseteqq N(R)$. Let $p$ and $b$ as defined in case (II). Then (31) holds and $[x, b]= =\pm b^r[x, b]b^s=0$ for all $x \in R$, which is a contradiction. Therefore $a \in Z(R)$ and $N(R) \subseteqq Z(R)$. Thus

(32) $$C(R) \subseteqq N(R) \subseteqq Z(R).$$

By (32) and Lemma 2, $[x, y]=\pm y^{r+s}[x, y]$ for all $x, y \in R$. Therefore, $R$ is commutative by [11, Theorem].

(iv): Let $r>0, s>0$ and $t>0$. Then $x=e_{11}$ and $y=e_{12}$ fail to satisfy (26). Therefore, $C(R) \subseteqq N(R)$. If $p$ and $b$ are as defined in case (II), then $x^t[x, b]=\pm \pm b^r[x, b]b^s=0$. So by Lemma 1, $[x, b]=0$, which contradicts the minimality of $p$. Therefore, $N(R) \subseteqq Z(R)$, and thus

(33) $$C(R) \subseteqq N(R) \subseteqq Z(R).$$

By (33), the identity (26) becomes

(34) $$x^t[x, y] = \pm[x, y]y^{r+s} \quad \text{for all} \quad x, y \in R.$$

Following the proof of (i), we can establish the commutativity of $R$.

Corollary 5 ([12, Theorem]). *Let $t$ and $m$ be two fixed non-negative integers. Suppose that $R$ satisfies the polynomial identity*

(35) $$x^t[x, y] = [x, y^m] \quad \textit{for all} \quad x, y \in R.$$

(i) *If $R$ is left s-unital, then $R$ is commutative except when* $(m, t)=(1, 0)$.

(ii) *If $R$ is right s-unital, then $R$ is commutative except when* $(m, t)=(1, 0)$; *and* $m=0$ *and* $t>0$.

Remark 2. In Corollary 5, for $m>1$, $R$ is commutative by Theorem 1. However, for $m=0$ (resp. $m=1$ and $t>0$), it is easy to prove the commutativity of $R$.

Corollary 6. *Let $n>0$ and $m$ (resp. $m>0$, and $n$) be fixed non-negative integers. Suppose that a left (resp. right) s-unital ring $R$ satisfies the polynomial identity*

(36) $$[xy, x^n \pm y^m] = 0 \quad \text{for all} \quad x, y \in R.$$

*If $R$ has $Q(n)$, then $R$ is commutative.*

Proof. Actually, $R$ satisfies the identity $x[x^n, y]=\pm[x, y^m]y$ for all $x, y \in R$. Therefore, $R$ is commutative.

Corollary 7. *Let $m>1$ and $n>1$ be relatively prime integers, and let $R$ be a left s-unital ring satisfying the polynomial identity* (36). *Then $R$ is commutative.*

In [6, Theorem B], Harmanci proved that "If $n>1$ is a fixed integer and $R$ is a ring with unity 1 which satisfies the identities $[x^n, y]=[x, y^n]$ and $[x^{n+1}, y]=[x, y^{n+1}]$ for each $x, y \in R$, then $R$ must be commutative." In [5, Theorem 6] Bell generalized this result. The following theorem further extends the result of Bell.

Theorem 5. *Let $m>1$ and $n>1$ be fixed relatively prime integers, and let $r$, $s$ and $t$ be fixed non-negative integers. If $R$ is a left s-unital ring satisfies both the identities*

(37) $$x^t[x^n, y] = \pm y^r[x, y^n]y^s \quad \text{and} \quad x^t[x^m, y] = \pm y^r[x, y^m]y^s \quad \text{for all} \quad x, y \in R,$$

*then $R$ is commutative.*

Proof. According to Proposition 1 of [9], we prove the theorem for $R$ with 1. Let $b$ as in the proof of Lemma 8. Following the proof of Theorem 1 and Theorem 2 of [16], we can prove that $n[b, y]=0$ and $m[b, y]=0$. By Lemma 4, $[b, y]=0$ for all $y \in R$. The argument in the proof of Lemma 8, gives $N(R) \subseteq Z(R)$. Also, $x=e_{22}$ and $y=e_{21}+e_{22}$ fail to satisfy the polynomial identities in (37). Hence, by Theorem KB, $C(R) \subseteq N(R)$, and thus $C(R) \subseteq N(R) \subseteq Z(R)$. The argument of subdirectly irreducible rings can then be carried out for $n$ and $m$, yielding integers $j>1$ and $k>1$ such that $[x^j-x, y^{n^2}]=0$ and $[x^k-x, y^{m^2}]=0$ for all $x, y \in R$. Let $f(x)=(x^j-x)^k-(x^j-x)$. Then $0=[f(x), y^{n^2}]=n^2[f(x), y]y^{n^2-1}$, and $0=[f(x), y^{m^2}]=m^2[f(x), y]y^{m^2-1}$. By Lemma 4 and Lemma 5, $[f(x), y]y^r=0$ for all $x, y \in R$, and $r=\max\{m^2-1, n^2-1\}$. Therefore, $f(x) \in Z(R)$. Since $f(x)=x^2g(x)-x$ with $g(x)$ having integral coefficients, Theorem H$'$ shows that $R$ is commutative.

Corollary 8 ([4, Theorem 6]). *Let $m>1$ and $n>1$ be relatively prime positive integers. If $R$ is any ring with unity 1 satisfies both the identities $[x^m, y]=[x, y^m]$ and $[x^n, y]=[x, y^n]$ for all $x, y \in R$, then $R$ is commutative.*

Remark 3. In case $m=0$ and $n \geqq 1$, Theorem 1 need not be true for right $s$-unital ring. Also, when $m=0$ and $t=1$, Corollary 4 is not valid for $s$-unital ring. In fact we have the following example.

Example 3. Let $K$ be a field. Then, the non-commutative ring $R=\begin{pmatrix} K & 0 \\ K & 0 \end{pmatrix}$, has a right identity element and satisfies the polynomial identity $x[x, y]=0$ for all $x, y \in R$. Hence, in the case $m=0$ and $n>0$, Theorem 1 need not be true for right $s$-unital rings. As a matter of fact, Example 3 disproves Theorems 1, 3, 4, and 5 for right $s$-unital case whenever both $r$ and $t$ are positive.

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# Additive functions satisfying congruences

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*Dedicated to János Galambos on his fiftieth birthday*

**1.** Let $A$ ($A^*$) denote the class of additive (completely additive) functions having real values, $A_{\mathbf{G}}$ ($A_{\mathbf{G}}^*$) be the set of additive (completely additive) functions defined on the set of nonzero Gaussian integers and taking on complex values.

It seems to us very probable that a condition

$$\sum_{j=0}^{k} F_j(n+j) \equiv 0 \pmod 1 \quad (\forall n\in\mathbf{N}) \tag{1.1}$$

for $F_j\in A^*$ ($j=0, \dots, k$) implies that the $F_j$ may take on only integer values, and similarly if $G_0, G_1, \dots, G_k \in A_{\mathbf{G}}^*$ and if

$$\sum_{j=0}^{k} G_j(\alpha+j)\in G \tag{1.2}$$

holds for all $a\in\mathbf{G}$ with the exception of $\alpha=0, -1, \dots, -k$, then $G_j(\alpha)\in\mathbf{G}$ for every $\alpha\in G\setminus\{0\}$ and $j=0, \dots, k$. In [1] the rational case was considered for $k=3$, while in [2] the Gaussian case for $k=3$, and the results support the above conjectures.

We should like to mention that our conjecture is not true in general for the wider class of additive functions. We say that an additive function $F$ is of a finite support mod 1, if $F(p^\alpha)\equiv 0 \pmod 1$ holds for all but finitely many primes $p$ and every $\alpha\geqq 1$. Similarly, we say that a function $G\in A_{\mathbf{G}}$ is of a finite support mod $G$ if $F(\Pi^a)\in\mathbf{G}$ holds for all prime powers $\Pi^a$ with the exception of at most finitely many primes $\Pi\in G$. We guess that the conditions (1.1), (1.2) for additive functions imply that the $F_j$ are of finite support mod 1, and $G_j$ are of finite support mod $\mathbf{G}$. It is quite easy to determine the additive functions $F$ or $G$ having finite support under

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the conditions (1.1), (1.2), respectively. A recent result due to Robert Styer supports this conjecture (case $k=2$, (1.1) is assumed, $F_0, F_1, F_2 \in A$).

If $k=1$ then much more is known. Several years ago it was proved by E. Wirsing that $F \in A$, $\|F(n+1)-F(n)\| \to 0$ $(n \to \infty)$ implies that $F(n)=\tau \log n + H(n)$, where $\tau$ is a suitable real number and $H(n)$ is an integer valued additive function. Here $\|x\|$ denotes the distance of $x$ to the nearest integer. This was a conjecture of the first named author. By his method one can get that $\|F_0(n)+F_1(n+1)\| \to 0$ $(n \to \infty)$, $F_0, F_1 \in A$ implies that $F_0(n)=\tau \log n + H_0(n)$, $F_1(n)=-\tau \log n + H_1(n)$ and $H_0(n) \equiv 0 \pmod 1$ identically.

It is quite plausible to believe that $F_0, F_1, \dots, F_k \in A$,

$$\Big\|\sum_{j=0}^{k} F_j(n+j)\Big\| \to 0 \quad \text{as} \quad n \to \infty$$

implies that

$$F_j(n) = \tau_j \log n + H_j(n) \quad (j = 0, \dots, k),$$

$$\sum_{j=0}^{k} \tau_j = 0, \quad \text{and} \quad \sum_{j=0}^{k} H_j(n+j) \equiv 0 \pmod 1.$$

Our purpose in this paper is to determine all those functions $G_0, \dots, G_5 \in A_G^*$ for which (1.2) $(k=5)$ holds true. This is formulated in Theorem 3 which is an easy consequence of Theorem 1 and 2.

Theorem 1. *Let* $H_j \in A^*$ $(j=0, 1, 2)$,

(1.3) $V(n) := H_0(n)+H_1(n+1)+H_2(n+2)-H_2(n+4)-H_1(n+5)-H_0(n+6).$

*Assume that* $V(n) \equiv 0 \pmod 1$ *for every* $n \in \mathbb{N}$. *Then* $H_j(n) \equiv 0 \pmod 1$ *holds for every* $j=0, 1, 2$ *and* $n \in \mathbb{N}$.

Theorem 2. *Let* $H_j \in A_G^*$ $(j=0, 1, 2)$, *and assume that*

(1.4) $V(\alpha) := H_0(\alpha)+H_1(\alpha+1)+H_2(\alpha+2)-H_2(\alpha+4)-H_1(\alpha+5)-H_0(\alpha+6)$

*is a Gaussian integer for all* $\alpha \in G \setminus \{0, -1, -2, -4, -5, -6\}$. *Then* $H_j(\alpha) \in G$ *for all* $\alpha \in G \setminus \{0\}$ *and* $j=0, 1, 2$.

**2. Proof of Theorem 1.**

Lemma 1. *If* $V(n) \equiv 0 \pmod 1$ *holds for every* $n \in \mathbb{N}$, *then* $H_j(n) \equiv 0 \pmod 1$ *holds for* $n \leq 17$ *and* $j=0, 1, 2$.

Proof. The following ten expressions are integers and they are linear combinations of $H_j(2)$, $H_j(3)$ and $H_j(5)$ for $j=0, 1, 2$.

1, $V(4)$

2, $V(10)-V(3)+V(6)+V(2)$

3, $V(20)+3V(1)+5V(2)-2V(3)+2V(6)-V(7)-2V(8)-2V(9)-$
$-2V(12)-2V(50)$

4, $V(28)+V(1)-V(2)-2V(3)-V(5)-V(6)-V(11)+V(24)$

5, $V(34)+3V(1)+2V(2)-2V(3)+V(5)+V(6)-V(7)-2V(8)-V(9)+V(11)-$
$-2V(12)-V(13)-V(15)-V(19)-V(21)-2V(50)$

6, $V(86)+8V(1)+4V(2)-3V(3)+3V(6)-V(7)-4V(8)-3V(9)-4V(12)-$
$-V(13)-V(14)-V(15)-V(17)-V(18)-V(19)-V(21)+V(24)-V(43)-$
$-V(45)-4V(50)$

7, $V(90)-7V(1)-3V(2)-V(3)-V(5)-3V(6)+4V(8)+V(9)-V(11)+2V(12)+$
$+V(18)-V(21)+V(45)+2V(50)$

8, $V(110)+2V(1)+5V(2)+V(3)+3V(6)-V(7)-V(8)-2V(9)-V(11)-V(12)-$
$-V(13)-V(14)-2V(15)-V(17)-V(18)-V(23)+V(32)-V(50)$

9, $V(184)+4V(1)+2V(2)+3V(3)-V(5)-2V(6)+V(7)+V(9)-V(11)+3V(12)+$
$+V(13)+V(14)+V(18)+3V(19)+3V(21)-V(23)-V(29)+V(32)+$
$+V(45)+2V(50)$

10, $V(203)+6V(1)-V(2)-2V(3)-V(5)+2V(6)+2V(7)-3V(8)+3V(9)+V(11)+$
$+V(12)+2V(13)+2V(14)+2V(15)+2V(17)+V(18)+3V(19)+4V(21)+$
$+2V(23)+V(25)+V(27)+V(29)+V(31)+V(37)+V(43)+V(45)+V(50)$

These conditions can be written in matrix form as $RH^T \equiv 0 \pmod 1$, where $H^T$ is the transpose of the vector

$$H = [H_0(2), H_0(3), H_0(5), H_1(2), H_1(3), H_1(5), H_2(2), H_2(3), H_2(5)]$$

and **R** is the matrix with integer entries given by

$$R = \begin{bmatrix} 1 & 0 & -1 & 0 & -2 & 1 & -2 & 1 & 0 \\ -6 & 1 & 1 & 1 & 0 & -1 & 4 & 0 & -2 \\ -13 & 2 & -1 & 1 & -2 & -2 & 13 & 3 & -9 \\ 6 & 2 & -2 & 4 & -5 & 3 & -8 & 6 & 0 \\ -9 & 5 & -2 & -2 & -2 & 1 & 11 & 4 & -8 \\ -11 & 4 & -5 & 3 & -13 & 4 & 19 & 15 & -20 \\ 9 & 5 & 4 & 3 & 12 & -3 & -15 & -11 & 17 \\ -8 & -3 & 0 & -5 & 0 & -1 & 12 & 2 & -8 \\ -4 & -9 & -4 & -4 & -11 & 2 & 1 & 6 & -8 \\ -11 & 0 & -6 & 1 & -16 & 4 & 11 & 9 & -14 \end{bmatrix}.$$

Using Gaussian elimination over the integers, it follows that the third row is linearly dependent upon the others (but needed to perform the Gaussian elimination) and that $H_j(2)$, $H_j(3)$ and $H_j(5)$ are integer valued for $j=0, 1, 2$.

To show that the same is true for the other primes less than or equal to 17 we consider the following expressions, which are linear combinations of $H_j(2)$, $H_j(3)$ and $H_j(5)$ alone and are therefore integer valued. Since $V(n)\equiv 0 \pmod 1$ it follows that $H_j(p)\equiv 0 \pmod 1$ for $j=0, 1, 2$, $p=7, 11, 13, 17$.

i, $V(1)+H_0(7)$

ii, $V(2)+H_1(7)$

iii, $V(3)+H_2(7)$

iv, $V(5)-H_2(7)+H_0(11)$

v, $V(6)-H_1(7)+H_1(11)$

vi, $V(8)+H_0(7)+H_1(13)$

vii, $V(12)-H_2(7)-H_1(13)+H_1(17)$

viii, $V(50)+H_0(7)+H_1(11)-H_1(17)-H_2(13)$

ix, $V(11)-H_0(11)-H_2(13)+H_0(17)$

x, $V(18)+V(22)+V(14)-H_0(11)+H_2(13)+H_2(11)$

xi, $V(20)-H_1(7)-H_2(11)+H_0(13)$

xii, $V(26)+V(30)-H_0(13)+H_1(7)-H_2(7)+H_2(17)$.

This proves Lemma 1.

The proof of Theorem 1 is completed by verifying the inductive step which is done in the following

Lemma 2. *If* $V(n)\equiv 0\ (1)$ *for all* $n\in N$ *and* $H_j(n)\equiv 0\ (1)$ *for all* $n\leqq 17$ *and* $j=0, 1, 2$, *then* $H_j(n)\equiv 0\ (1)$ *for all* $n\in N$ *and* $j=0, 1, 2$.

Proof. We prove the lemma indirectly. Assume that there is some $n$ which is smallest possible for which $H_j(n)\not\equiv 0\ (1)$ for $j=0$ or $j=1$ or $j=2$. Then $n=p$, $p$ is a prime. Since $V(p-6)\equiv 0\ (1)$, and $V(p-5)\equiv 0\ (1)$, it follows that $H_2(p)\not\equiv 0\ (1)$. From $V(p-4)\equiv 0\ (1)$ it follows that $p$ and $p+2$ must be prime, and therefore $p\equiv 2 \pmod 3$. From $V(p)\equiv 0\ (1)$ it follows that $p+6$ is a prime as well. Since $(p+10)$ is divisible by 3, $V(p+4)\equiv 0$ implies that $(p+8)$ must be a prime and therefore $p\equiv 1 \pmod 5$. We now consider $V(4p+6)$, which is equal to

$$H_0(4p+6)+H_1(4p+7)+H_2(4p+9)-H_2(4p+10)-H_1(4p+11)-H_0(4p+12).$$

But $(4p+6)\equiv 0 \pmod 5$ and $(4p+11)=0 \pmod 5$, while $(4p+10)=2(2p+5)$ and

$(2p+5)\equiv 0 \pmod 3$. Also $(4p+12)=4(p+3)$ with $(p+3)$ composite and $(4p+8)=$ $=4(p+2)$. Therefore $V(4p+6)\equiv 0$ (1) means that $H_1(4p+7)+H_2(p+2)\equiv 0$ (1). If $H_1(4p+7)\equiv 0$ (1) the lemma is proved, since $H_2(p)-H_2(p+2)\equiv 0$ (1), which follows from $V(p-2)\equiv 0$ (1). Since $p\equiv 2 \pmod 3$, it follows that $(4p+7)=3n$. If $n$ is composite, $H_1(4p+7)\equiv 0$.

If $n$ is prime, then $(n+9)<2p$ since $p>17$. Therefore $(n+k)$ is composite and less than $2p$ for $k=-1, 1, 3, 5, 7, 9$, from which it follows that $H_j(n+k)\equiv 0$ (1) for these values of $k$ and $j=0, 1, 2$. Since $V(n-1)\equiv 0$ (1) and $V(n+3)\equiv 0$ (1) one concludes that $H_1(n)\not\equiv 0$ (1) means that $n$, $n+4$ and $n+8$ must all be prime which is impossible. Hence $H_1(n)\equiv 0$ (1) and therefore $H_2(p)\equiv 0$ (1), which concludes the proof of Lemma 2 and therefore the proof of Theorem 1.

**3. Proof of Theorem 2.** To prove Theorem 2, clearly we may assume that $H_j$ are real valued functions. Let us observe furthermore that $H(\varepsilon\alpha)=H(\alpha)$ for each $H\in A_{\mathbf{G}}^*$ and $\varepsilon=-1, i, -i$. We introduce the notations

$$V^{+1}(\alpha):=H_0(\alpha)+H_1(\alpha+1)+H_2(\alpha+2)-H_2(\alpha+4)-H_1(\alpha+5)-H_0(\alpha+6),$$

$$V^{+1}(\alpha):=H_0(\alpha)+H_1(\alpha+i)+H_2(\alpha+2i)-H_2(\alpha+4i)-H_1(\alpha+5i)-H_0(\alpha+6i)$$

The norm of $\alpha$ is defined by $N(\alpha)=\alpha\bar{\alpha}$. The proof of Theorem 2 is also done by induction, this time using the norm of $\alpha$. Because of Theorem 1, we need to prove Theorem 2 only for elements of **G** which are not real nor purely imaginary. The following lemma lists some properties of such elements.

Two Gaussian integers $\beta=u+iv$ and $\gamma=c+di$ are congruent mod 5 in the arithmetic of **G** if $u\equiv c \pmod 5$ and $v\equiv d \pmod 5$ hold simultaneously. This is denoted by $(u, v)\equiv(c, d) \pmod 5$.

Lemma 3. *Let $\alpha$ be a Gaussian integer such that:*

(i) *$\alpha$ is a prime number,*

(ii) *$\alpha=a+bi$ with $a>b>0$;*

(iii) $N(\alpha)>13$;

(iv) $(a, b)\not\equiv(4, 1) \pmod 5$.

*Then*

(A) *$N(\alpha-n)<N(\alpha)$ for $n=1, 2, 3, 4, 5, 6$*

*and*

(B) *both of $\alpha+1$ and $\alpha+i$ are composite numbers, and their norms are strictly less than $2N(\alpha)$.*

*In addition at least one of the assertions* C, D, E *holds true:*

(C) *$\alpha+2$ is composite and $N(\alpha+2)<2N(\alpha)$;*

(D) *$\alpha+2i$ is composite and $N(\alpha+2i)<2N(\alpha)$, furthermore $N(\alpha-4i)\leqq N(\alpha)$ and $N(\alpha-ni)<N(\alpha)$ is true for $n=2$ and $n=3$;*

(E) $N(\alpha-n+2i)<N(\alpha)$ *for* $n=2, 3, 4, 5$ *and* 6, *while* $\alpha-1+2i$ *is composite and* $N(\alpha-1+2i)<2N(\alpha)$. *Moreover* $N(\alpha-4i)\leqq N(\alpha)$ *while* $N(\alpha-ki)<N(\alpha)$ *for* $k=2$ *and* 3.

Proof. Since $N(\alpha)>13$ and (ii) holds, therefore $a\geqq 4$. Hence (A) follows easily. Also (B) is obviously true. Since $\alpha$ is a prime, $\alpha$ is not an associate of $1+i$, therefore $1+i$ is a divisor of $\alpha+1$ and of $\alpha+i$, and so they are composite numbers. $N(\alpha+1)=a^2+b^2+2a+1$, $N(\alpha+i)=a^2+b^2+2b+1$, therefore the second assertion in (B) holds as well.

We shall classify $\alpha$ according to its residue (mod 5). Let

$$M(C) = \{(0, 1), (0, 4), (1, 1), (1, 4), (2, 2). (2, 3), (3, 0)\};$$

$$M(D) = \{(0, 3), (1, 0), (3, 2), (4, 0)\};$$

$$M(E) = \{(0, 2), (2, 0), (3, 3), (4, 4).$$

Since $\alpha$ is a prime, therefore $(a, b)$ (mod 5) belongs to exactly one of the sets $M(C)$, $M(D)$, $M(E)$, $(4, 1)$ (mod 5) is excluded by the condition (iv). We shall prove that the assertions (C), (D) and (E) are true if $(a, b)$ (mod 5) belongs to $M(C)$, $M(D)$, $M(E)$, respectively.

*Case* $M(C)$. If $(a, b)\in M(C)$ (mod 5), then $5|N(\alpha+2)$, which can be seen easily. This implies that $2+i|\alpha+2$, and so $\alpha+2$ is composite. Since $a>b>0$, therefore $a\geqq 4$. But $a=4$ cannot occur, therefore $a>4$ and $N(\alpha+2)=a^2+b^2+4(a+1)<$ $<2N(\alpha)$ obviously holds.

*Case* $M(D)$. If $(a, b)\in M(D)$ (mod 5), then $5|N(\alpha+2i)$ which implies that $\alpha+2$ is composite. Since $b\neq 1$, therefore $N(\alpha-ni)=N(\alpha)+n(n-2b)$, and so $n-2b$ is negative for $n=2$ and 3, nonpositive if $n=4$. This completes the proof of Case $M(D)$.

*Case* $M(E)$. If $(a, b)\in M(E)$ (mod 5), then $a\geqq 5$, and $a\not\equiv b$ (mod 2), since $\alpha$ is a prime. The case $(a-b)=1$ cannot occur, furthermore $b\neq 1$, whence we have that $b\geqq 2$ and $a-b>2$. By using these inequalities, we can prove (E) easily.

Since the functions $H_j$ under the condition (1.4) satisfy the conditions of Theorem 1, therefore we have that $H_j(\alpha\bar{\alpha})=H_j(\alpha)+H_j(\bar{\alpha})\equiv 0$ (mod 1). This implies that it is enough to prove Theorem 2 either for $\alpha$ or for $\bar{\alpha}$.

Lemma 4. *If* $V(\alpha)\equiv 0$ (mod 1) *for all* $\alpha\in \mathbf{G}\setminus\{0, -1, -2, -4, -5, -6\}$ *then* $H_j(\alpha)\equiv H_j(\bar{\alpha})$ (mod 1) *for all* $\alpha\in\mathbf{G}\setminus\{0\}$ *and* $j=0, 1, 2$.

Proof. Let $h_j(\alpha):=H_j(\alpha)-H_j(\bar{\alpha})$. Then $h_j(\bar{\alpha})=-h_j(\alpha)$. To prove the lemma, we prove that $h_j(\alpha)\equiv 0$ (mod 1) for $j=0, 1, 2$. We observe that, for $j=0, 1, 2$, $h_j(1\pm\ )=0$ and $h_j(n)=0$ for all rational integers.

The complete additivity of the function $H_j$ and the fact that $H_j(\varepsilon\alpha)=H_j(\alpha)$ for $\varepsilon=-1$, , $-$ allows us to obtain the following 9 congruences modulo 1, which prove the assertion for those Gaussian primes with norm less than 20. The 9 congruences are:

i, $h_2(2+i) \equiv V^{+i}(2-2i)+V^{+i}(2-4i)+V^{+i}(3-3i)-V^{+1}(-3+i) \equiv 0 \pmod 1$

ii, $h_0(2-3i) \equiv V^{+i}(2-3i) \equiv 0 \pmod 1$

iii, $h_1(3-2i) \equiv V^{+i}(3-3i) \equiv 0 \pmod 1$

vi, $h_1(4-i) \equiv V^{+1}(-1+i)-V^{+1}(-1-i) \equiv 0 \pmod 1$

v, $h_1(2+i) \equiv V^{+i}(4-4i)+V^{+i}(4-2i)-V^{+i}(1-3i) \equiv 0 \pmod 1$

vi, $h_0(2-i) \equiv V^{+i}(4-4i)+V^{+i}(4-2i) \equiv 0 \pmod 1$

vii, $h_2(4-i) \equiv V^{+i}(4-3i) \equiv 0 \pmod 1$

viii, $h_2(3-2i) \equiv V^{+i}(3-4i)+V^{+i}(3-2i) \equiv 0 \pmod 1$

ix, $h_0(4-i) \equiv V^{+i}(3-5i)+V^{+i}(3-i) \equiv 0 \pmod 1$.

We finish the proof by using induction. Let us assume that our Lemma 4 is not true. Let $\alpha$ be such an integer for which $h_j(\alpha)\not\equiv 0 \pmod 1$ for at least one of the $j\in\{0, 1, 2\}$. Let us choose that $\alpha$ for which $N(\alpha)$ is the smallest one. Then $N(\alpha)\geqq 20$, and $\alpha$ is a Gaussian prime. We may assume furthermore that condition (ii) of Lemma 3 true also.

It is clear that

$$0 \equiv V^{+1}(\alpha-6)-V^{+1}(\bar{\alpha}-6) \equiv h_0(\alpha-6)+h_1(\alpha-5)+$$
$$+h_2(\alpha-4)-h_2(\alpha-2)-h_1(\alpha-1)-h_0(\alpha) \pmod 1,$$
$$0 \equiv V^{+1}(\alpha-5)-V^{+1}(\bar{\alpha}-5) \equiv h_0(\alpha-5)+h_1(\alpha-4)+$$
$$+h_2(\alpha-3)-h_2(\alpha-1)-h_1(\alpha)+h_0(\alpha+1) \pmod 1.$$

Since $\alpha+1$ is a composite number, and $N(\alpha-k)<N(\alpha)$ for $1\leqq k\leqq 6$, we conclude that $h_0(\alpha)\equiv 0 \pmod 1$, and $h_1(\alpha)\equiv 0 \pmod 1$.

To prove that $h_2(\alpha)\equiv 0 \pmod 1$, we assume first that (iv) in Lemma 3 holds, i.e. that $(a, b)\not\equiv(4, 1) \pmod 5$. We observe that

$$0 \equiv V^{+1}(\alpha-4)-V^{+1}(\bar{\alpha}-4) \equiv -h_2(\alpha) \pmod 1$$

in Case $M(C)$,

(3.2) $$0 \equiv V^{+i}(\alpha-4i)+V^{+i}(\bar{\alpha}-2i) \equiv$$
$$\equiv h_0(\alpha-4i)-h_0(\alpha+2i)-h_2(\alpha)+h_2(\alpha-2i)+h_1(\alpha-3i)-h_1(\alpha+i) \pmod 1.$$

which implies that $h_2(\alpha)\equiv 0 \pmod 1$ in Case D. In Case E we start from the relation

$$-h_0(\alpha+2i) \equiv V^{+1}(\alpha-6+2i)-V^{+1}(\overline{\alpha-6+2i}) \equiv 0 \pmod 1$$

whence, by (3.2) we deduce that $h_2(\alpha)\equiv 0 \pmod 1$.

Finally, we consider the case $(a, b)\equiv(4, 1) \pmod 5$. Since $N(\alpha)\geqq 20$, $\alpha\neq 4+i$. Since 5 is a divisor of $N(\alpha+2i)$, in the case $b\neq 1$, $\alpha+2i$ is a composite number and $N(\alpha+2i)<2N(\alpha)$, $N(\alpha-ki)<N(\alpha)$ $(k=1, 2, 3, 4)$ are satisfied. Hence, by (3.2) we obtain that $h_2(\alpha)\equiv 0 \pmod 1$. If $b=1$, then $a\geqq 14$. In this case $N(\alpha-k+4i)<N(\alpha)$ holds for every integer $k$ in $1\leqq k\leqq 6$, and from $0\equiv V^{+1}(\alpha-6+4i)-V^{+1}(\overline{a-6+4i})\equiv 0 \pmod 1$ we deduce that $h_0(\alpha+4i)\equiv 0 \pmod 1$. Since $\alpha+3i$ and $\alpha+2i$ are composite numbers, and $N(\alpha+3i)<2N(\alpha)$, $N(\alpha+2i)<2N(\alpha)$, substituting first $\alpha$ by $\bar{\alpha}$, in (3.2), we get that $h_2(\alpha)\equiv 0 \pmod 1$.

By this the proof of Lemma 4 is completed.

Lemma 5. *If* $V(\alpha)\equiv 0 \pmod 1$ *holds for every* $\alpha\in G\setminus\{0, -1, -2, -4, -5, -6\}$ *then* $H_j(\alpha)\equiv 0 \pmod 1$ *for every* $\alpha\in G\setminus\{0\}$, *with* $N(\alpha)\leqq 13$, $j=0, 1, 2$. *Furthermore,* $H_2(4+i)\equiv 0 \pmod 1$.

Proof. The Gaussian primes $\pi$ with $N(\pi)<17$ are $(1\pm i)$, $(2\pm i)$ and $(3\pm 2\ )$. By Lemma 4 it suffices to consider either $\pi$ or $\bar{\pi}$. Also by Lemma 4,

$$H_j(\alpha)-H_j(\bar{\alpha}) \equiv H_j(\alpha)+H_j(\bar{\alpha}) \equiv 2H_j(\alpha) \equiv 0 \pmod 1.$$

This allows us to replace $H_j(\alpha)$ by $\pm H_j(\bar{\alpha})$, it means also that $2H_j(\alpha)\equiv\equiv 0 \pmod 1$ holds for every $\alpha$.

The additivity of the functions $H_j$, together with the factorization of Gaussian integers, allows us to obtain the following ten congruences, in the given order, which, as can be seen easily, prove the lemma:

i, $H_1(2+i) \equiv V^{+i}(3-i)+V^{+1}(-2+i) \equiv 0 \pmod 1$.

ii, $H_2(2+i) \equiv V^{+1}(4+6i)+V^{+i}(5+i)+V^{+1}(2i)+V^{+1}(i)+$
$+V^{+1}(-2+i) \equiv 0 \pmod 1$

iii, $H_2(3-2i) \equiv V^{+1}(6+2i)+V^{+1}(i)+V^{+1}(-2+i)+V^{+1}(2+2i) \equiv 0 \pmod 1$

iv, $H_0(3+2i) \equiv V^{+i}(9+i)+V^{+1}(3+2i)+V^{+i}(5-2i)+$
$+V^{+1}(-2+i)+V^{+1}(-1+2i)+V^{+1}(-1+i) \equiv 0 \pmod 1$

v, $H_1(2+3i) \equiv V^{+1}(4+2i)+V^{+1}(2i)+V^{+1}(1+3i)+$
$+V^{+1}(-1+2i)+V^{+1}(-1+i) \equiv 0 \pmod 1$

vi, $H_0(2-i) \equiv V^{+i}(6)+V^{+1}(1+i)+V^{+i}(5+i)+V^{+1}(2i)+V^{+1}(i) \equiv 0 \pmod 1$

vii, $H_0(1+i) \equiv V^{+i}(4-2i)+V^{+1}(-1+i) \equiv 0 \pmod 1$

viii, $H_1(1+i) \equiv V^{+1}(3+i)+V^{+1}(-1+i)+V^{+1}(-1+2i)+V^{+i}(4-i)+$
$$+V^{+1}(1+3i)+V^{+1}(-2+i) \equiv 0 \pmod 1$$
ix, $H_2(1+i) = V^{+i}(5-i)+V^{+1}(1+3i)+V^{+i}(4)+V^{+1}(-1+i)+$
$$+V^{+1}(-1+2i) \equiv 0 \pmod 1$$
x, $H_2(4+i) \equiv V^{+i}(4)+V^{+1}(-1+i)+V^{+i}(5-i) \equiv 0 \pmod 1$.

The final step of the proof of Theorem 2 is contained in the next

Lemma 6. *If* $V(\alpha)\equiv 0 \pmod 1$ *holds for all* $\alpha \in \mathbf{G}\setminus\{0, -1, -2, -4, -5, -6\}$, *and* $H_j(\alpha)\equiv 0 \pmod 1$ *for all nonzero* $\alpha$ *with* $N(\alpha)<17$ *and* $j=0, 1, 2,$ *then* $H_j(\alpha)\equiv 0 \pmod 1$ $(j=0, 1, 2)$ *holds for all nonzero Gaussian integer.*

Proof. Assume that the assertion is not true. Let $\alpha$ be such a Gaussian integer with smallest norm for which $H_j(\alpha)\not\equiv 0 \pmod 1$ for at least one $j$. By Lemma 4, we may assume that $\alpha=a+bi$, $a>b>0$. It is clear that $\alpha$ is a Gaussian prime.

Since $N(\alpha)>13$, taking into account the relations, $V^{+1}(\alpha-6)\equiv 0$, $V^{+1}(\alpha-5)\equiv \equiv 0 \pmod 1$, by Lemma 3 we deduce that $H_2(\alpha)\not\equiv 0 \pmod 1$.

Let us consider first the case $(a, b)\equiv(4, 1) \pmod 5$ which was excluded in Lemma 3. If $(a, b)\equiv(4, 1) \pmod 5$, then $\alpha+2i$ is composite and $N(\alpha+2i)< <2N(\alpha)$. If $b\neq 1$, then $b\geqq 6$ and $V^{+i}(\alpha-4i)\equiv 0 \pmod 1$ implies that $H_2(\alpha)\equiv \equiv 0 \pmod 1$. The case $\alpha=4+i$ was treated in Lemma 5, so we may assume that $\alpha\neq 4+i$. Thus we may assume that $b=1$ and $a\geqq 14$. Then $N(\alpha-k+4i)<N(\alpha)$ for $k=1, 2, 3, 4, 5, 6$, and $V^{+1}(\alpha-6+4i)\equiv 0 \pmod 1$ implies that $H_0(\alpha+4i)\equiv 0 \pmod 1$. Since $\alpha+2i$ and $\alpha+3i$ are composite numbers with norm less than $2N(\alpha)$, $V^{+i}(\alpha-2i)\equiv 0 \pmod 1$ implies that $H_2(\alpha)\equiv 0 \pmod 1$.

In all remaining cases Lemma 3 enables us to apply the induction hypothesis.

In Case C we consider $V^{+1}(\alpha-4)\equiv 0 \pmod 1$, while in Case D we take $V^{+i}(\alpha-4i)\equiv 0 \pmod 1$, and hence deduce immediately that $H_2(\alpha)\equiv 0 \pmod 1$. If Case E is satisfied, then we start from $V^{+1}(\alpha-6+2i)\equiv 0 \pmod 1$, which implies that $H_0(\alpha+2i)\equiv 0 \pmod 1$, and consider $V^{+i}(\alpha-4i)\equiv 0$, whence we have that $H_2(\alpha)\equiv 0 \pmod 1$.

By this the proof of Lemma 6 and therefore of Theorem 2 is completed.

**4.** The next theorem is an easy consequence of our Theorem 2.

Theorem 3. *Let* $F_0, \ldots, F_5 \in A_{\mathbf{G}}^*$ *which satisfy the relations*

$$F_0(\alpha)+F_1(\alpha+1)+F_2(\alpha+2)+F_3(\alpha+3)+F_4(\alpha+4)+F_5(\alpha+5) \equiv 0 \pmod{\mathbf{G}} \tag{4.1}$$

*for all* $\alpha \in G\setminus\{0, -1, -2, -3, -4, -5\}$. *Then* $F_j(\alpha)\equiv 0 \pmod{\mathbf{G}}$ *holds for all* $\alpha \in \mathbf{G}\setminus\{0\}$ *and* $j=0, \ldots, 5$.

Proof. It is enough to prove our theorem for functions $F_j$ which take on real values.

Let us write (4.1) in the form

$$U(\alpha):=F_0(\alpha)+F_1(\alpha+1)+F_2(\alpha+2)+F_3(\alpha+3)+F_4(\alpha+4)+F_5(\alpha+5)+F_6(\alpha+6),$$

where $F_6\in A_G^*$ is identically zero. Then

$$0\equiv U(-6-\alpha)\equiv F_6(\alpha)+F_5(\alpha+1)+F_4(\alpha+2)+F_3(\alpha+3)+$$
$$+F_2(\alpha+4)+F_1(\alpha+5)+F_0(\alpha+6) \pmod 1.$$

Let

$$H_0(\alpha):=F_0(\alpha)-F_6(\alpha),\quad H_1(\alpha):=F_1(\alpha)-F_5(\alpha),\quad H_2(\alpha)=F_2(\alpha)-F_4(\alpha).$$

Since $U(\alpha)-U(-\alpha-6)\equiv 0 \pmod 1$, therefore

(4.2)

$$H_0(\alpha)+H_1(\alpha+1)+H_2(\alpha+2)-H_2(\alpha+4)-H_1(\alpha+5)-H_0(\alpha+6)\equiv 0 \pmod 1$$

is satisfied for all Gaussian integers $\alpha$ for which the sequence $\alpha$, $\alpha+1$, $\alpha+2$, $\alpha+4$, $\alpha+5$, $\alpha+6$ does not contain the zero. Thus the conditions of Theorem 2 are satisfied, consequently $F_0(\alpha)-F_6(\alpha)\equiv 0$, $F_1(\alpha)\equiv F_5(\alpha)$, $F_2(\alpha)\equiv F_4(\alpha) \pmod 1$ holds for all nonzero Gaussian integers $\alpha$. Especially $F_0(\alpha)\equiv 0 \pmod 1$. If we write now

$$V(\alpha):\equiv F_{-1}(\alpha)+F_0(\alpha)+\dots+F_5(\alpha+5)$$

with $F_{-1}\in A_G^*$, $F_{-1}(\alpha)\equiv 0 \pmod 1$ identically, then we get similarly, that $F_{-1}(\alpha)\equiv\equiv F_5(\alpha)$, $F_0(\alpha)\equiv F_4(\alpha)$, $F_1(\alpha)\equiv F_3(\alpha) \pmod 1$ which implies that $F_5(\alpha)\equiv F_4(\alpha)\equiv F_1(\alpha)\equiv\equiv F_2(\alpha)\equiv 0 \pmod 1$, and the recursion (4.1) finally implies that $F_3(\alpha)\equiv 0 \pmod 1$ true as well.

By this our theorem is proved.

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# On equivalence of two variational problems in $k$-Lagrange spaces

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**1. Introduction.** In [3] we have considered generalization of the equivalence of two variational problems for single integrals treated by A. MOÓR ([7]) in Lagrange spaces $\mathbf{L}^{*n}=(M, \mathscr{L}^*)$ and $\mathbf{L}^n=(M, \mathscr{L})$ ([6]). This problem has the following form

(1.1)
$$\mathbf{E}_i(\mathscr{L}^*(x, y)) = \lambda(x, y)\,\mathbf{E}_i(\mathscr{L}(x, y)), \quad \left(\mathbf{E}_i := \frac{d}{dt}\dot{\partial}_i - \partial_i,\ \dot{\partial}_i := \partial/\partial y^i,\ \partial_i := \partial/\partial x^i\right),$$
$$\lambda(x, y) \neq 0,$$

where $y$ stands for $\dot{x}$, $\mathscr{L}$ and $\mathscr{L}^*$ are the two Lagrangians, and $\lambda$ depends not only on $x$ but on $y$ too. We have given the transformation of the Lagrangians as a necessary and sufficient condition for this equivalence. Moreover, we have shown geometrical consequences of the equivalence relation (1.1).

In 1975 A. MOÓR ([8]) gave a definition of equivalence of two variational problems for multiple integrals with the following relation

$$\mathbf{E}_i(\mathscr{L}^*(x^s, y^s_\alpha)) = \lambda^j_i(x^s)\,\mathbf{E}_j(\mathscr{L}(x^s, y^s_\alpha)), \quad \operatorname{rank}\|\lambda^j_i(x)\| = n \tag{1.2}$$

$$\left(y^s_\alpha := \partial x^s/\partial t^\alpha,\ \mathbf{E}_i := \frac{\partial}{\partial t^\alpha}\dot{\partial}^\alpha_i - \partial_i \text{ (summation over } \alpha);\ i, j, s = \overline{1, n};\ \alpha = \overline{1, k},\ k < n\right).$$

He investigated the properties of this relation but not in geometrical manner.

In [4] and [5] we have constructed a *geometrical model* for multiple integrals in the calculus of variations. Now we study a generalization of the Moór equivalence in geometrical manner using the theory of $k$-Lagrange geometry.

**2. The Moór equivalence of multiple integral variational problems in $k$-Lagrange spaces.** Consider the total space $E=\bigoplus\limits_{1}^{k} TM = \underset{\smile}{\overset{1}{TM}} \oplus \underset{\smile}{\overset{2}{TM}} \oplus \ldots \oplus \underset{\smile}{\overset{k}{TM}}$ of the vector

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bundle $\eta = (\overset{k}{\underset{1}{\oplus}} TM, \pi, M)$ with canonical coordinates $(x^i, y^i_\alpha)$ where $i$ runs from 1 to $n$ and $\alpha$ runs from 1 to $k$. By the theory of $k$-Lagrange spaces $\mathbf{L}^n_k$ ([4], [5]) we have a *regular* Lagrangian $\mathscr{L}\colon \overset{k}{\underset{1}{\oplus}} TM \to \mathbf{R}$ with the metric tensor field

$$(2.1)\qquad g^{\alpha\beta}_{ij}(x, y) = \partial^\alpha_i \partial^\beta_j \mathscr{L}(x, y); \ \operatorname{rank} \|g^{\alpha\beta}_{ij}\| = nk \quad (\partial^\alpha_i := \partial/\partial y^i_\alpha).$$

Now let $\mathscr{L}$ be defined on class $C^2$ of the *admissible submanifold* $C_k, \bar{C}_k, \dots$ on $M$, where

$$(2.2)\qquad C_k\colon x^i = x^i(t^\alpha), \quad \bar{C}_k\colon \bar{x}^j = \bar{x}^j(t^\alpha), \dots$$

and they coincide with each other on the boundary $\partial G_t$ of the parameter domain $G_t$ ([9], [10]).

Then we can construct the $k$-fold integral

(2.3)

$$I(C_k) = \int_{G_t} \mathscr{L}(x^i(t^\beta), y^i_\alpha(t^\beta))\, d(t); \ d(t) := dt^1 \dots dt^k; \ y^i_\alpha(t^\beta) := \partial x^i/\partial t^\alpha, \quad (\beta = \overline{1, k}).$$

This integral depends on the submanifold $C_k$ by means of which it is defined. It is known from the classical calculus of variations of multiple integrals ([9]) that if a submanifold $C_k$ is to afford an extreme value to $I$ relative to other admissible submanifold it is necessary that the first variation $\delta I$ of (2.3) should vanish. This implies that $C_k$ must satisfy the system of $n$ second order partial differential equations:

$$(2.4)\qquad \mathbf{E}_i(\mathscr{L}) := \frac{\partial}{\partial t^\alpha}(\partial^\alpha_i \mathscr{L}) - \partial_i \mathscr{L} = 0 \quad (\partial_i := \partial/\partial x^i),$$

where $\mathbf{E}_i$ are the components of the *Euler—Lagrange covariant vector* ([10]).

Let us consider a pair $\mathbf{L}^n_k = (M, \mathscr{L})$ and $\mathbf{L}^{*n}_k = (M, \mathscr{L}^*)$ of $k$-Lagrange spaces with the same base manifold $M$.

Definition 2.1. Two variational problems in $\mathbf{L}^n_k = (M, \mathscr{L})$ and $\mathbf{L}^{*n}_k = (M, \mathscr{L}^*)$ are called *equivalent in the sense of Moór* if

$$(2.5)\qquad \mathbf{E}_i(\mathscr{L}^*(x^j, y^j_\alpha)) = \lambda^s_i(x^j, y^j_\alpha)\, \mathbf{E}_s(\mathscr{L}(x^j, y^j_\alpha)); \ \det \|\lambda^s_i(x, y)\| \neq 0$$

hold identically.

Remark. In (2.5) the tensor field $\lambda$ depends on $y$ too.

**3. Some geometrical characters of the equivalence.** Relation (2.5) has the following explicit form:

$$(3.1)\quad (\partial^\alpha_i \partial^\beta_s \mathscr{L}^* - \lambda^j_i \partial^\alpha_j \partial^\beta_s \mathscr{L}) y^s_{\alpha\beta} + (\partial^\alpha_i \partial_s \mathscr{L}^* - \lambda^j_i \partial^\alpha_j \partial_s \mathscr{L}) y^s_\alpha - (\partial_i \mathscr{L}^* - \lambda^j_i \partial_j \mathscr{L}) = 0,$$

$$y^s_{\alpha\beta} := \frac{\partial^2 x^s}{\partial t^\alpha \partial t^\beta}.$$

Using condition (2.1) for $\mathscr{L}$ and $\mathscr{L}^*$ we get from (3.1)

$$(3.2)\qquad (\overset{*}{g}{}^{\alpha\beta}_{is}-\lambda^j_i g^{\alpha\beta}_{js})y^s_{\alpha\beta}+(\partial^\alpha_i\partial_s\mathscr{L}^*-\lambda^j_i\partial^\alpha_j\partial_s\mathscr{L})y^s_\alpha-(\partial_i\mathscr{L}^*-\lambda^j_i\partial_j\mathscr{L})=0.$$

Since (3.2) is an identity in $(x, y)$ it is necessary that the coefficients of $y^s_{\alpha\beta}$ should vanish. Hence we obtain from (3.2):

Theorem 3.1. *A necessary geometrical condition for equivalence of two variational problems of multiple integrals is that the k-Lagrange spaces* $(M, \mathscr{L}^*)$ *and* $(M, \mathscr{L})$ *be in „k-conformal" correspondence:*

$$(3.3)\qquad \overset{*}{g}{}^{\alpha\beta}_{is}(x,y)=\lambda^j_i(x,y)g^{\alpha\beta}_{js}(x,y).$$

From this Theorem it directly follows that the *metrical d-connections* (cf. [4])

$$LD^*=(\overset{*}{L}{}^i_{jm},\overset{*}{L}{}^{\alpha j}_{i\beta m},\overset{*}{C}{}^{i\alpha}_{jm},\overset{*}{C}{}^{\gamma i\beta}_{m\alpha j})\quad\text{and}\quad LD=(L^i_{jm},L^{\alpha j}_{i\beta m},C^{i\alpha}_{jm},C^{\gamma i\beta}_{m\alpha j}),$$

respectively, are related by the geometrical condition (3.3).

Proposition 3.1. *The d-tensor fields* $\overset{*}{\tilde{C}}{}^{\alpha\beta\gamma}_{ijm}$ *and* $\tilde{C}^{\alpha\beta\gamma}_{ijm}$ *are in the following relation*

$$(3.4)\qquad 2\overset{*}{\tilde{C}}{}^{\alpha\beta\gamma}_{ijm}=(\partial^\gamma_m\lambda^l_i)g^{\alpha\beta}_{lj}+2\lambda^l_i\tilde{C}^{\alpha\beta\gamma}_{ljm}.$$

Proof. We have

$$(3.5)\qquad \begin{aligned} &\text{(a)}\quad C^{\gamma i\beta}_{m\alpha j}=\frac{1}{2}g^{is}_{\alpha\varepsilon}\partial^\beta_j\partial^\gamma_m\partial^\varepsilon_s\mathscr{L}=\frac{1}{2}g^{is}_{\alpha\varepsilon}\partial^\varepsilon_s g^{\beta\gamma}_{jm},\\ &\text{(b)}\quad \tilde{C}^{\gamma\varepsilon\beta}_{msj}=g^{\varepsilon\alpha}_{si}C^{\gamma i\beta}_{m\alpha j}=\frac{1}{2}\partial^\gamma_m g^{\alpha\beta}_{ij},\end{aligned}$$

(cf. [4]). Hence a direct calculus leads to (3.4).

Using the result of the above Proposition we shall prove that our equivalence-problem can be reduced to the Moór one ([8]), i.e. his equivalence is a special case of relation (2.5).

Theorem 3.2. *If two variational problems of multiple integrals are equivalent in the sense of Moór then the k-conformal factor* $\lambda^j_i(x, y)$ *is necessarily independent of* $y^i_\alpha$.

Proof. Differentiating (3.3) with respect to $y^l_\gamma$ we obtain

$$(3.7)\qquad \partial^\gamma_l\overset{*}{g}{}^{\alpha\beta}_{is}=(\partial^\gamma_l\lambda^j_i)g^{\alpha\beta}_{js}+\lambda^j_i(\partial^\gamma_l g^{\alpha\beta}_{js})$$

and by virtue of (3.4) we have

$$(3.8)\qquad 2\overset{*}{\tilde{C}}{}^{\alpha\beta\gamma}_{isl}=(\partial^\gamma_l\lambda^j_i)g^{\alpha\beta}_{js}+2\lambda^j_i\tilde{C}^{\alpha\beta\gamma}_{jsl}.$$

Since the *d-tensor fields* $\overset{*}{\tilde{C}}$ and $\tilde{C}$ are totally symmetric ([4]), after the cyclic permutation of the indices we get

$$(3.9) \qquad (\partial^\gamma_l \lambda^j_i) g^{\alpha\beta}_{js} = (\partial^\alpha_j \lambda^j_i) g^{\beta\gamma}_{sl} = (\partial^\beta_s \lambda^j_i) g^{\gamma\alpha}_{lj}.$$

By using the symmetric property of the metric tensor $g^{\alpha\beta}_{is}$ from (3.9) it follows that

$$(3.10) \qquad (\partial^\gamma_l \lambda^j_i) g^{\beta\alpha}_{sj} - (\partial^\beta_s \lambda^j_i) g^{\gamma\alpha}_{lj} = 0.$$

Contracting by $g^{js}_{\alpha\beta}$ the last relation we get

$$(3.11) \quad \begin{array}{ll} \text{(a)} & (\partial^\gamma_l \lambda^j_i) nk - (\partial^\beta_s \lambda^j_i) \delta^{\gamma s}_{l\beta} = 0, \\ \text{(b)} & (\partial^\gamma_l \lambda^j_i) nk - \partial^\gamma_l \lambda^j_i = 0, \end{array}$$

respectively. This means that

$$(3.12) \qquad (\partial^\gamma_l \lambda^j_i)(nk-1) = 0.$$

Because of $(nk-1) \neq 0$ the relation (3.12) holds iff

$$(3.13) \qquad \partial^\gamma_l \lambda^j_i (x, y) = 0.$$

Thus $\lambda$ is independent of $y^l_\gamma$.

Corollary. *A geometrical character of the equivalence in* (2.5) *with the k-conformal factor* $\lambda^j_i(x)$ *is that the torsion tensor field* $C^{\beta s\gamma}_{j\varepsilon m}$ *of the metrical d-connection LD is invariant.*

Proof. Suppose that (2.5) holds with $\lambda^j_i(x)$. Using the relation $C^{\beta s\gamma}_{j\varepsilon m} = g^{si}_{\varepsilon\alpha} C^{\beta\alpha\gamma}_{jim}$, from Proposition 3.1 we directly get

$$(3.14) \qquad \overset{*}{C}{}^{\beta s\gamma}_{j\varepsilon m} = \overset{*}{g}{}^{si}_{\varepsilon\alpha} \tilde{C}^{\beta\alpha\gamma}_{jim} = \tilde{\lambda}^i_t g^{st}_{\varepsilon\alpha} \lambda^l_i \tilde{C}^{\beta\alpha\gamma}_{jlm} = \delta^l_t g^{st}_{\varepsilon\alpha} \tilde{C}^{\beta\alpha\gamma}_{jlm} = g^{sl}_{\varepsilon\alpha} \tilde{C}^{\beta\alpha\gamma}_{jlm} = C^{\beta s\gamma}_{j\varepsilon m},$$

where $\tilde{\lambda}^i_t \lambda^l_i = \delta^l_t$.

**4. Transformation of the Lagrangians.** We can easily check if the Lagrangians differ by a total derivative, i.e. $\mathscr{L}^*(x, y) = \mathscr{L}(x, y) + \partial^\beta_s A(x) y^s_\beta$, then $\mathrm{E}_i(\mathscr{L}^*) \equiv \equiv \mathrm{E}_i(\mathscr{L})$. This means that two variational problems of multiple integrals are equivalent in the sense of Moór with tensor field $\delta^j_i$.

Now we examine the transformation of the Lagrangians under the equivalence relation in (2.5). First we prove

Proposition 4.1. *If the relation* (2.5) *holds and the k-conformal factor* $\lambda$ *is independent of* $y$ *then it is necessary that* $\lambda^j_i(x) = \delta^j_i \lambda(x)$.

Proof. Let us consider relation (3.3). Since the metric tensor fields $g^*$ and $g$ are symmetric in the indices $\binom{\alpha}{i}$ and $\binom{\beta}{s}$ we get

$$(4.1) \qquad \lambda^j_i g^{\alpha\beta}_{js} - \lambda^j_s g^{\beta\alpha}_{ji} = 0,$$

which can be written in the following form

$$g^{\varepsilon\gamma}_{jh}(\lambda^j_i\delta^\beta_\gamma\delta^h_s\delta^\alpha_\varepsilon-\lambda^j_s\delta^h_i\delta^\beta_\varepsilon\delta^\alpha_\gamma)=0\quad(\beta,\gamma,\varepsilon=\overline{1,k};\ i,j,s,h=\overline{1,n}).\tag{4.2}$$

We infer from the symmetry of the $d$-tensor field $g$ that the coefficients of $g^{\varepsilon\gamma}_{jh}$ in (4.2) must be skewsymmetric in $\binom{\varepsilon}{j}$ and $\binom{\gamma}{h}$. This gives for the symmetric part:

$$\lambda^j_i\delta^\beta_\gamma\delta^h_s\delta^\alpha_\varepsilon-\lambda^j_s\delta^h_i\delta^\beta_\varepsilon\delta^\alpha_\gamma+\lambda^h_i\delta^\beta_\varepsilon\delta^j_s\delta^\alpha_\gamma-\lambda^h_s\delta^j_i\delta^\beta_\gamma\delta^\alpha_\varepsilon=0.\tag{4.3}$$

Let $h=s$, $\beta=\gamma$, $\alpha=\varepsilon$, then we obtain

$$\lambda^j_i k^2 n-\lambda^j_i k+\lambda^j_i k-\lambda^h_h\delta^j_i k^2=0.\tag{4.4}$$

Now putting

$$\lambda(x)=\frac{1}{n}\lambda^h_h(x),\tag{4.5}$$

we get from (4.4)

$$k^2 n\lambda^j_i(x)-k^2 n\delta^j_i\lambda(x)=0.\tag{4.6}$$

Thus

$$\lambda^j_i(x)=\delta^j_i\lambda(x).\tag{4.7}$$

Proposition 4.2. *If relation* (2.5) *holds with* $\lambda^j_i(x)=\delta^j_i\lambda(x)$ *then the transformation beetwen the Lagrangians* $\mathscr{L}^*(x,y)$ *and* $\mathscr{L}(x,y)$ *is as follows*:

$$\mathscr{L}^*(x,y)=\lambda(x)\mathscr{L}(x,y)+A^\beta_s(x)y^s_\beta+U(x).\tag{4.8}$$

Proof. By Theorem 3.1 we obtain $\overset{*}{g}{}^{\alpha\beta}_{is}=\delta^j_i\lambda(x)^{\alpha\beta}_{js}$. In view of property of Lagrangians we get

$$\partial^\alpha_i\partial^\beta_s(\mathscr{L}^*-\lambda(x)\mathscr{L})=0.\tag{4.9}$$

Hence the function $\mathscr{L}^*-\lambda(x)\mathscr{L}$ is necessarily linear in $y^s_\beta$.

**5. Some remarks about the normal form of the Euler—Lagrange equations in $\mathbf{L}^n_k$.** It is known that in the equations of geodesics of Lagrange space the second derivatives $\ddot{x}^i$ appear explicitly and the functions $G^i(x,y)$ can be derived directly from the Lagrangians (cf. [6]). This suggests us to write the $\mathbf{E}_i(\mathscr{L}(x,y))$ in such form which is a generalization of that of geodesics. Hence we get

$$\mathbf{E}_i(\mathscr{L}(x^j,y^j_\gamma))=g^{\alpha\beta}_{is}y^s_{\alpha\beta}+G_i(x^j,y^j_\gamma)\quad\left(y^s_{\alpha\beta}:=\frac{\partial^2x^s}{\partial t^\alpha\partial t^\beta}\right)\tag{5.1}$$

where the generalized $G_i(x^j,y^j_\gamma)$ are defined by

$$G_i:=(\partial^\gamma_i\partial_s\mathscr{L})y^s_\gamma-\partial_i\mathscr{L}.$$

By means of $g^{\alpha\beta}_{ih}g^{hl}_{\beta\gamma}=\delta^{\alpha l}_{i\gamma}$ equation (5.1) can be written in the following form

$$\mathbf{E}_i(\mathscr{L}(x,y))=g^{\alpha\beta}_{is}(y^s_{\alpha\beta}+G^s_{\alpha\beta}(x,y)),\tag{5.2}$$

where the generalized $G^s_{\alpha\beta}$ are defined by

$$G^s_{\alpha\beta} := g^{is}_{\alpha\beta} G_i \quad (G_i := G^s_{\alpha\beta} g^{\alpha\beta}_{is}).$$

Finally we directly obtain

Proposition 5.1. *If two variational problems in* $\mathrm{L}^{*n}_k$ *and* $\mathrm{L}^n_k$ *are equivalent in the sense of Moór then*

$$\overset{*}{G}{}^s_{\alpha\beta} = G^s_{\alpha\beta}. \tag{5.3}$$

Indeed, from the equivalence relation (2.5) using Theorem 3.1 and relation (5.2) we obtain (5.3).

Remark. Relation (5.3) corresponds to that result which was obtained for equivalent single-integral variational problems in Lagrange spaces (cf. [3]).

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# On an integral inequality for concave functions

HORST ALZER

In 1987 A. BEZDEK and K. BEZDEK [2] proved the following interesting proposition:

Theorem A. *Let $S$ be a convex solid of revolution in $\mathbf{R}^3$ with axis of revolution $AB$. Further, let $C$ be the centroid of $S$ and let $C'$ be the centroid of the 2-dimensional domain obtained by intersecting $S$ with a plane through $AB$. Then*

$$\frac{1}{2} < \frac{|AC|}{|AC'|} < \frac{3}{2}. \tag{1}$$

As it was shown by the authors double-inequality (1) is a consequence of the following sharp integral inequalities.

Theorem B. *If $f$ is a non-negative concave function defined on* $[0, 1]$ *with* $\sup_{0 \leqq x \leqq 1} f(x) = 1$, *then*

$$\frac{2}{3} \leqq \frac{\int_0^1 f^2(t)\,dt}{\int_0^1 f(t)\,dt} \leqq 1 \tag{2}$$

*and*

$$\frac{1}{2} \leqq \frac{\int_0^1 tf^2(t)\,dt}{\int_0^1 tf(t)\,dt} \leqq 1. \tag{3}$$

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The aim of this paper is to present a short and simple proof for an integral inequality for concave functions which includes the left-hand sides of (2) and (3) as special cases.

Theorem. *Let $f$ be a non-negative, continuous, concave function on $[a, b]$ and let $g$ be a non-negative differentiable function such that the derivative $g'$ is integrable on $[a, b]$. If $\alpha$ and $\beta$ are real numbers with $\alpha \geqq 0$ and $0 < \beta \leqq 1$, then we have for all $x \in [a, b]$:*

$$(4) \qquad \frac{\alpha+\beta}{\alpha+2\beta} f^{\beta}(x) \int_a^b g(t) f^{\alpha}(t)\,dt + \frac{\beta}{\alpha+2\beta} \int_a^b (x-t) g'(t) f^{\alpha+\beta}(t)\,dt \leqq$$

$$\leqq \int_a^b g(t) f^{\alpha+\beta}(t)\,dt.$$

Proof. First we note that the function $f^{\beta}$ is concave on $[a, b]$ (see [6, p. 20]). Further, since every continuous concave function defined on a compact interval can be approximated uniformly by differentiable concave functions (see [6, p. 269]), we may assume that $f$ and $f^{\beta}$ are differentiable on $[a, b]$. Then we conclude from the mean-value theorem:

$$f^{\beta}(x) \leqq f^{\beta}(t) + \beta(x-t) f^{\beta-1}(t) f'(t) \quad \text{for all } x, t \in [a, b].$$

Multiplication by $g(t) f^{\alpha}(t)$ and integration with respect to $t$ yields:

$$(5) \quad f^{\beta}(x) \int_a^b g(t) f^{\alpha}(t)\,dt \leqq \int_a^b g(t) f^{\alpha+\beta}(t)\,dt + \frac{\beta}{\alpha+\beta} \int_a^b (x-t) g(t) (f^{\alpha+\beta}(t))'\,dt.$$

Integration by parts leads to

$$(6) \qquad \int_a^b (x-t) g(t) (f^{\alpha+\beta}(t))'\,dt = (x-b) g(b) f^{\alpha+\beta}(b) -$$

$$-(x-a) g(a) f^{\alpha+\beta}(a) + \int_a^b g(t) f^{\alpha+\beta}(t)\,dt - \int_a^b (x-t) g'(t) f^{\alpha+\beta}(t)\,dt \leqq$$

$$\leqq \int_a^b g(t) f^{\alpha+\beta}(t)\,dt - \int_a^b (x-t) g'(t) f^{\alpha+\beta}(t)\,dt,$$

and from (5) and (6) we conclude

$$f^{\beta}(x) \int_a^b g(t) f^{\alpha}(t)\,dt \leqq \frac{\alpha+2\beta}{\alpha+\beta} \int_a^b g(t) f^{\alpha+\beta}(t)\,dt - \frac{\beta}{\alpha+\beta} \int_a^b (x-t) g'(t) f^{\alpha+\beta}(t)\,dt$$

which is equivalent to inequality (4).

Remark. Inequality (4) is an extension of a result given in [3].

If we set $g(t)\equiv 1$ and $\alpha=\beta=1$, then we get the following (slightly modified) version of the left-hand side of (2):

$$\frac{2}{3}\max_{0\leqq x\leqq 1} f(x) \leqq \frac{\int_0^1 f^2(t)\,dt}{\int_0^1 f(t)\,dt}. \tag{7}$$

Since the sign of equality holds for $f(x)=x$ we conclude that the constant $2/3$ cannot be replaced by a greater number. Furthermore, setting $g(t)=t$ and $\alpha=\beta=1$ in (4) we obtain:

Corollary. *If $f(\not\equiv 0)$ is a non-negative, continuous, concave function on* $[0, 1]$, *then we have for all* $x\in[0, 1]$:

$$\frac{f(x)}{2}+\frac{x}{4}\,\frac{\int_0^1 f^2(t)\,dt}{\int_0^1 tf(t)\,dt} \leqq \frac{\int_0^1 tf^2(t)\,dt}{\int_0^1 tf(t)\,dt}. \tag{8}$$

Remarks. 1) As an immediate consequence of (8) we get the following form of the left-hand inequality of (3):

$$\frac{1}{2}\max_{0\leqq x\leqq 1} f(x) \leqq \frac{\int_0^1 tf^2(t)\,dt}{\int_0^1 tf(t)\,dt}. \tag{9}$$

Putting $f(x)=1-x$ equality holds in (9); hence the constant $1/2$ is best possible.

We note that (7) and (9) are striking companions of Favard's inequality

$$\frac{1}{2}\max_{0\leqq x\leqq 1} f(x) \leqq \int_0^1 f(t)\,dt$$

which is true for all functions $f$ which are non-negative, continuous and concave on $[0, 1]$; see [1, p. 44] and [4].

2) If $f$ is monotonic, then the two integral ratios given in (2) and (3) can be compared:

Let $f(\not\equiv 0)$ be a non-negative and decreasing function on $[0, 1]$, then

$$\frac{\int_0^1 tf^2(t)\,dt}{\int_0^1 tf(t)\,dt} \leqq \frac{\int_0^1 f^2(t)\,dt}{\int_0^1 f(t)\,dt}.$$

If $f$ is increasing, then the reversed inequality holds; see [5, pp. 302—303].

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# On the generalized strong de la Vallée Poussin approximation

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*Dedicated to Professor Béla Csákány on his 60th birthday*

**1.** Let $\{\varphi_n(x)\}$ be an orthonormal system on a finite interval $(a, b)$. In this paper we shall consider real orthogonal series

$$(1.1)\qquad \sum_{n=0}^{\infty} c_n \varphi_n(x) \quad \text{with} \quad \sum_{n=0}^{\infty} c_n^2 < \infty.$$

By the Riesz—Fischer theorem the partial sums $s_n(x)$ of any such series converge in the $L^2$ norm to a square-integrable function $f(x)$.

It is well known that there are many interesting results stating certain summability properties of series (1.1) or providing accurate rate of the approximation for special summation methods both in ordinary and strong sense. Some sample theorems for approximation can be found e.g. in the works [1], [2], [3], [5].

Analysing the theorems being in the above mentioned works we can realize that most of the results concerning any property of ordinary approximation have an analogue in strong sense. In other words, we have the same rate of approximation for strong means as for ordinary ones. But there is a lack in the case of the generalized de la Vallée Poussin summability.

The aim of the present paper is to bring this discrepansy to an end, that is, to show that the analogy also holds for this summability. Namely we shall prove that two theorems of [2] (see Theorems V and VI) can be extended to strong approximation by the same rate, too.

Now we recall the definitions of the generalized ordinary, strong and very strong de la Vallée Poussin summability methods (see [2]).

Let $\lambda := \{\lambda_n\}$ be a non-decreasing sequence of natural numbers for which $\lambda_0 = 1$ and $\lambda_{n+1} \leqq \lambda_n + 1$. Series (1.1) is $(V, \lambda)$-summable if

$$V_n(x) := V_n(\lambda;\ x) := \frac{1}{\lambda_n} \sum_{k=n-\lambda_n+1}^{n} s_k(x) \to f(x)$$

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almost everywhere (a.e.); *strongly* $(V, \lambda)$*-summable* if

$$V_n|x| := V_n|\lambda;\, x| := \left\{\frac{1}{\lambda_n}\sum_{k=n-\lambda_n+1}^{n} |s_k(x)-f(x)|^2\right\}^{1/2} \to 0$$

a.e.; and *very strongly* $(V, \lambda)$*-summable* if for any increasing sequence $\nu:=\{\nu_k\}$ of natural numbers

$$V_n^{\nu}|x| := V_n|\lambda, \nu;\, x| := \left\{\frac{1}{\lambda_n}\sum_{k=n-\lambda_n+1}^{n} |s_{\nu_k}(x)-f(x)|^2\right\}^{1/2} \to 0 \quad \text{a.e.}$$

We also note that if $\lambda_n=n$ then the $V_n(x)$-means reduce to the $(C, 1)$-means, if $\lambda_n\equiv 1$ then to the partial sums $s_n(x)$, and if $\lambda_n=\left[\frac{n}{2}\right]$ $(n\geqq 2)$, where $[\beta]$ denotes the integral part of $\beta$, then we get the classical de la Vallée Poussin means.

**2.** Now we can formulate our theorems:

Theorem 1. *Let* $\varrho:=\{\varrho_n\}$ *and* $l:=\{l_n\}$ *be monotone non-decreasing sequences. If the condition*

$$\sum_{n=0}^{\infty} c_n^2\varrho_n^2 < \infty \tag{2.1}$$

*implies the* $(V, \lambda)$*-summability of* (1.1) *for any* $\{\varphi_n(x)\}$ *and* $\{c_n\}$ *almost everywhere on a set E of positive measure, then the conditions*

$$\sum_{n=0}^{\infty} c_n^2\varrho_n^2 l_n^2 < \infty \quad \textit{and} \quad l_{\mu_{m+1}} \leqq K l_{\mu_m} \quad \textit{with} \quad 1 \leqq K < \sqrt{2}, \tag{2.2}$$

*where* $\mu_0=0$ *and* $\mu_m:=\sum_{k=0}^{m-1}\lambda_{\mu_k}$, *imply that*

$$V_n|\lambda, \nu;\, x| = o_x(l_n^{-1}) \tag{2.3}$$

*holds almost everywhere on the set E for any increasing sequence* $\nu=\{\nu_k\}$ *of positive integers.*

Theorem 2. *If a monotone non-decreasing sequence* $l=\{l_n\}$ *satisfies the conditions*

$$l_{\mu_{m+1}} \leqq K l_{\mu_m} \quad \textit{with} \quad 1 \leqq K < \sqrt{2}; \quad \textit{and} \quad \sum_{k=0}^{m} l_{\mu_k}^2 = O(l_{\mu_m}^2); \tag{2.4}$$

*then already the following condition*

$$\sum_{n=0}^{\infty} c_n^2 l_n^2 < \infty \tag{2.5}$$

*implies the validity of* (2.3) *almost everywhere in* $(a, b)$ *for any* $\{\varphi_n(x)\}$ *and* $\{\nu_n\}$.

We remind the reader of that these theorems are the strong analogues of Theorems V and VI proved in [2]. Furthermore we recall that the condition

$$\sum_{m=1}^{\infty}\Big\{\sum_{n=\mu_m+1}^{\mu_{m+1}} c_n^2\Big\}\log^2 m<\infty$$

implies the $(V,\lambda)$-summability of of (1.1) (see [2], Theorem II).

**3.** In order to prove our theorems we require some lemmas. In what follows $M$ will denote an absolute constant.

Lemma 1 ([2], Lemma II). *Let $\{p_m\}$ be an increasing sequence of positive integers, let $\{\gamma_m\}$ be a non-decreasing sequence of positive numbers so that*

$$\sum_{m=1}^{n}\gamma_{p_m}^2\leqq M\gamma_{p_n}^2,\quad n=1,2,\ldots. \tag{3.1}$$

*If*

$$\sum_{n=1}^{\infty}c_n^2\gamma_n^2<\infty, \tag{3.2}$$

*then*

$$s_{p_m}(x)-f(x)=o_x(\gamma_{p_m}^{-1}) \tag{3.3}$$

*a.e. in $(a,b)$.*

Lemma 2 ([2], Lemma III). *Let $\{p_m\}$ be an increasing sequence of positive integers, $\{u_n\}$ be an arbitrary sequence, furthermore let $\{v_n\}$ be a positive, monotone non-decreasing sequence with the property $v_{p_m+1}=\ldots=v_{p_{m+1}}$ $(m=1,2,\ldots)$. If the $p_m$-th partial sums of the series $\sum_{n=0}^{\infty}u_nv_n$ converge then the $p_m$-th partial sums of the series $\sum_{n=1}^{\infty}u_n$ also converge, furthermore if $s=\lim_{m\to\infty}s_{p_m}$, where $s_k:=\sum_{n=1}^{k}u_n$, we also have that*

$$|s_{p_m}-s|=o(v_{p_{m+1}}^{-1}).$$

Lemma 3 ([2], Theorem I). *In order that series (1.1) a.e. on a set $E$ of positive measure should be $(V,\lambda)$-summable, it is necessary and sufficient that the partial sums $s_{\mu_m}(x)$ of (1.1) ($\mu_0=1$ and $\mu_m:=\sum_{k=0}^{m-1}\lambda_{\mu_k}$) converge a.e. on $E$.*

Lemma 4 ([4], Lemma 3). *Let $\delta>0$ and $\{\delta_n\}$ be an arbitrary sequence of non-negative numbers. Suppose that for any orthonormal system $\{\varphi_n(x)\}$ the condition*

$$\sum_{n=1}^{\infty}\delta_n\Big(\sum_{k=n}^{\infty}c_k^2\Big)^{\delta}<\infty$$

*implies that the partial sums $s_n(x)$ of (1.1) possess a property $P$, then any subsequence $\{s_{\nu_n}(x)\}$ $(\nu_n<\nu_{n+1})$ of the partial sums of (1.1) also possesses property $P$.*

Finally we need to prove the following new lemma.

Lemma 5. *If a monotone non-decreasing sequence* $l=\{l_n\}$ *satisfies the conditions*

(3.4) $$l_{\mu_{m+1}} \leqq K l_{\mu_m} \quad \text{with} \quad 1 \leqq K < \sqrt{2}, \quad m=1,2,\dots;$$

*then condition* (2.5) *implies that*

(3.5) $$\Big\{\lambda_{\mu_n}^{-1} \sum_{k=\mu_n-\lambda_{\mu_n}+1}^{\mu_n} |s_k(x)-s_{\mu_n}(x)|^2\Big\}^{1/2} = o_x(l_{\mu_n}^{-1})$$

*holds a.e. in* $(a,b)$.

Proof. An elementary calculation gives that

(3.6) $$\sum_{m=1}^{\infty} \int_a^b \frac{l_{\mu_m}^2}{\lambda_{\mu_m}} \sum_{k=\mu_m-\lambda_{\mu_m}+1}^{\mu_m} |s_k(x)-s_{\mu_m}(x)|^2\,dx =$$
$$= \sum_{m=1}^{\infty} l_{\mu_m}^2 \sum_{k=\mu_m-\lambda_{\mu_m}+1}^{\mu_m} \Big(1-\frac{\mu_m+1-k}{\lambda_{\mu_m}}\Big) c_k^2 =: \sum\nolimits_1.$$

Let $\alpha^+$ denote the positive part of $\alpha$. Using this notion we can estimate $\sum_1$ as follows:

(3.7) $$\sum\nolimits_1 \leqq \sum_{m=1}^{\infty} \sum_{k=\mu_m+1}^{\mu_m} c_k^2 \sum_{n=m}^{\infty} \lambda_{\mu_n}^{-1} l_{\mu_n}^2 (\mu_m-\mu_n+\lambda_{\mu_n})^+.$$

Next we show that

(3.8) $$R_m := \sum_{n=m}^{\infty} \lambda_{\mu_n}^{-1} l_{\mu_n}^2 (\mu_m-\mu_n+\lambda_{\mu_n})^+ = O(l_{\mu_m}^2)$$

holds. On account of the definition of $\mu_m$ we have

(3.9) $$R_m = \sum_{n=m}^{\infty} l_{\mu_n}^2 \Big(1-\frac{\mu_n-\mu_m}{\lambda_{\mu_n}}\Big)^+ = l_{\mu_m}^2 + \sum_{n=m+1}^{\infty} l_{\mu_n}^2 \Big(1-\lambda_{\mu_n}^{-1} \sum_{k=m}^{n-1} \lambda_{\mu_k}\Big)^+.$$

Putting

$$\Lambda_n^{(m)} := \Big(1-\lambda_{\mu_n}^{-1} \sum_{k=m}^{n-1} \lambda_{\mu_k}\Big)^+$$

and taking into account that $\lambda_{\mu_{n+1}} \leqq 2\lambda_{\mu_n}$ always holds, thus we get for any $n>m$ that

$$\Lambda_n^{(m)} \leqq \Big(1-(2\lambda_{\mu_{n-1}})^{-1} \sum_{k=m}^{n-1} \lambda_{\mu_k}\Big)^+ = \Big(\frac{1}{2}-(2\lambda_{\mu_{n-1}})^{-1} \sum_{k=m}^{n-2} \lambda_{\mu_k}\Big)^+ \leqq$$
$$\leqq \Big(\frac{1}{2}-(4\lambda_{\mu_{n-2}})^{-1} \sum_{k=m}^{n-2} \lambda_{\mu_k}\Big)^+ = \Big(\frac{1}{4}-(4\lambda_{\mu_{n-2}})^{-1} \sum_{k=m}^{n-3} \lambda_{\mu_k}\Big) \leqq \dots \leqq \Big(\frac{1}{2}\Big)^{n-m}.$$

Hence, by (3.4) and (3.9), it follows that

$$R_m \leqq l_{\mu_m}^2 + \sum_{n=m+1}^{\infty} l_{\mu_n}^2 \left(\frac{1}{2}\right)^{n-m} \leqq l_{\mu_m}^2 \left(1 + \sum_{n=m+1}^{\infty} \left(\frac{K^2}{2}\right)^{n-m}\right) = O(l_{\mu_m}^2),$$

and this proves (3.8). Consequently, by (3.6), (3.7) and (3.8), using the Beppo Levi theorem, we get that

$$\sum_{m=1}^{\infty} \frac{l_{\mu_m}^2}{\lambda_{\mu_m}} \sum_{k=\mu_m-\lambda_{\mu_m}+1}^{\mu_m} |s_k(x) - s_{\mu_m}(x)|^2 < \infty$$

almost everywhere in $(a, b)$, whence (3.5) obviously follows.

**4.** Proof of Theorem 1. On account of Lemma 4 with $\delta=1$ and $\delta_n := \varrho_n^2 l_n^2 - \varrho_{n-1}^2 l_{n-1}^2$ it is clear that we have to carry the proof only when $v_k = k$.

On the other hand a straightforward calculation gives that if $\mu_m < n \leqq \mu_{m+1}$ holds then

$$(V_n|\lambda;\, x|)^2 \leqq (V_{\mu_m}|\lambda;\, x|)^2 + 2(V_{\mu_{m+1}}|\lambda;\, x|)^2;$$

so in order to prove (2.3) it is sufficient to verify that

$$V_{\mu_m}|\lambda;\, x| = o_x(l_{\mu_m}^{-1}) \tag{4.1}$$

holds a.e. on $E$.

Now we put $\bar{l}_k := l_{\mu_m}$ for $\mu_m < k \leqq \mu_{m+1}$, $m = 0, 1, 2, \ldots$. Then, by (2.2), the series

$$\sum_{n=1}^{\infty} c_n \bar{l}_n \varphi_n(x) \tag{4.2}$$

is $(V, \lambda)$-summable a.e. on $E$; consequently, by Lemma 3, the $\mu_m$-th partial sums of (4.2) also converge a.e. on $E$. In the next step we use Lemma 2 whence the estimations

$$s_{\mu_m}(x) - f(x) = o_x(\bar{l}_{\mu_{m+1}}^{-1}) = o_x(l_{\mu_m}^{-1}) \tag{4.3}$$

follow a.e. on $E$.

Since

$$(V_{\mu_m}|\lambda;\, x|)^2 \leqq \frac{2}{\lambda_{\mu_m}} \sum_{k=\mu_m-\lambda_{\mu_m}+1}^{\mu_m} \{|s_k(x) - s_{\mu_m}(x)|^2 + |s_{\mu_m}(x) - f(x)|^2\}, \tag{4.4}$$

so, by Lemma 5 and (4.3), we get (4.1), what completes the proof of Theorem 1.

Proof of Theorem 2. By the same token as in the proof of Theorem 1 we only have to prove estimation (4.1). Now we can use Lemma 1 with $\gamma_m := l_m$ and $p_m := \mu_m$ taking into account conditions (2.4) and (2.5), so we get that

$$s_{\mu_m}(x) - f(x) = o_x(l_{\mu_m}^{-1}) \tag{4.5}$$

holds a.e. in $(a, b)$. By (2.4) and (2.5) we can apply Lemma 5, too; therefore (3.5) and (4.5), regarding (4.4), verify (4.1). Herewith Theorem 2 is also proved.

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# An additional note on strong approximation by orthogonal series

L. LEINDLER and A. MEIR

**1.** Let $\{\varphi_n(x)\}$ be an orthogonal system on a finite interval $(a, b)$. In this note we consider real orthogonal series

$$(1.1)\qquad \sum_{n=0}^{\infty} c_n \varphi_n(x) \quad \text{with} \quad \sum_{n=0}^{\infty} c_n^2 < \infty.$$

It is well known that the partial sums $s_n(x)$ of any such series converge in the $L^2$ norm to a function $f(x) \in L^2(a, b)$.

A very general theorem we proved in [5] concerning strong approximation by orthogonal series included, as special cases, many of the results obtained previously by several authors. In addition, our theorem in [5] yielded some new results pertaining to strong approximation by certain Hausdorff and $[J, f]$-means. We refer the reader to Theorems A, B, C, D and E cited in our paper as previously known and to Theorems 2, 3, 2* and 3* as the new results obtained by means of our main theorem.

In order to recall the main theorem and to state the purpose of the present note, we need the following definitions and notations:

Let $\boldsymbol{\alpha} := \{\alpha_k(\omega)\}$, $k = 0, 1, \ldots$ denote a sequence of non-negative functions defined for $0 \leqq \omega < \infty$, satisfying

$$(1.2)\qquad \sum_{k=0}^{\infty} \alpha_k(\omega) \equiv 1.$$

We assume that the linear transformation of real sequences $\mathbf{x} := \{x_k\}$ given by

$$A_\omega(\mathbf{x}) := \sum_{k=0}^{\infty} \alpha_k(\omega) x_k, \quad \omega \to \infty,$$

is regular [1, p. 49]. Let $\gamma := \gamma(t)$ denote a non-decreasing positive function defined

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for $0\leqq t<\infty$ and $\boldsymbol{\mu}:=\{\mu_m\}$ $m=0, 1, \ldots$ an increasing sequence of integers with $\mu_0=0$ satisfying the following conditions:

There exist positive integers $N$ and $h$ so that

$$\mu_{m+1} \leqq N\mu_m, \quad \gamma(\mu_{m+1}) \leqq N\gamma(\mu_m), \quad \gamma(\mu_{m+h}) \geqq 2\gamma(\mu_m) \tag{1.3}$$

hold for all $m$.

For $r>1$, $\omega>0$ and $m=0, 1, \ldots$ we define

$$\varrho_m(\omega, r) := \left\{\frac{1}{\mu_{m+1}} \sum_{k=\mu_m}^{\mu_{m+1}-1} (\alpha_k(\omega))^r\right\}^{1/r}. \tag{1.4}$$

In terms of the quantities introduced above we can recall our result in [5]:

Theorem I. *Let $p>0$ and $g(t)$ a non-decreasing positive function on* $[0, \infty)$. *Suppose that there exist $r>1$ and a constant $K(r, \boldsymbol{\mu}, \boldsymbol{\gamma})$ such that for every $\omega>0$*

$$\sum_{m=0}^{\infty} \mu_m \varrho_m(\omega, r)\gamma(\mu_m)^{-p} \leqq K(r, \boldsymbol{\mu}, \boldsymbol{\gamma})\big(g(\omega)/\gamma(\omega)\big)^p. \tag{1.5}$$

*If*

$$\sum_{n=0}^{\infty} c_n^2 \gamma(n)^2 < \infty, \tag{1.6}$$

*then*

$$A_\omega(f, p, \mathbf{v}; x) := \Big\{\sum_{k=0}^{\infty} \alpha_k(\omega)|s_{\nu_k}(x)-f(x)|^p\Big\}^{1/p} = O_x\big(g(\omega)/\gamma(\omega)\big) \tag{1.7}$$

*almost everywhere (a.e.) in $(a, b)$ for any increasing sequence $\mathbf{v}:=\{\nu_k\}$ of positive integers.*

*If, in addition, for every fixed $m$,*

$$\varrho_m(\omega, r) = o\big((g(\omega)/\gamma(\omega))^p\big), \quad \textit{as} \quad \omega\to\infty, \tag{1.8}$$

*then the $O_x$ in* (1.7) *can be replaced by $o_x$.*

We mention that the most important special case of Theorem I is when both (1.5) and (1.8) are satisfied with $g(\omega)\equiv 1$. In this case we get that

$$A_\omega(f, p, \mathbf{v}; x) = o_x\big(\gamma(\omega)^{-1}\big) \tag{1.9}$$

holds a.e. in $(a, b)$.

Next we recall the definition of the generalized ordinary and very strong de la Vallée Poussin summability methods (see [2]) and a theorem proved in [4].

Let $\boldsymbol{\lambda}:=\{\lambda_n\}$ be a non-decreasing sequence of natural numbers for which $\lambda_0=1$ and $\lambda_{n+1}\leqq\lambda_n+1$. Series (1.1) is $(V, \lambda)$-summable if

$$V_n(\boldsymbol{\lambda}; x) := \frac{1}{\lambda_n} \sum_{k=n-\lambda_n+1}^{n} s_k(x) \to f(x) \quad \text{a.e.};$$

and very strongly $(V, \lambda)$-summable if for any increasing sequence $\mathbf{v}=\{v_k\}$ of positive integers

$$V_n|\lambda, \mathbf{v}; x| := \left\{\frac{1}{\lambda_n} \sum_{k=n-\lambda_n+1}^{n} |s_{v_k}(x)-f(x)|^2\right\}^{1/2} \to 0 \quad \text{a.e.}$$

We also note that if $\lambda_n = n$ then the $V_n(\lambda; x)$-means reduce to the $(C, 1)$-means, if $\lambda_n \equiv 1$ then to the partial sums $s_n(x)$, and if $\lambda_n = \left[\frac{n}{2}\right]$ $(n \geqq 2)$, where $[\beta]$ denotes the integral part of $\beta$, then we get the classical de la Vallée Poussin means.

In [4] the first author proved, among others, the following result:

Theorem II. *If a monotone non-decreasing sequence* $\mathbf{l} := \{l_n\}$ *satisfies the conditions*

$$l_{\mu_{m+1}} \leqq K l_{\mu_m} \quad \text{with} \quad 1 \leqq K < \sqrt{2}; \quad \text{and} \quad \sum_{k=0}^{m} l_{\mu_k}^2 = O(l_{\mu_m}^2), \tag{1.10}$$

*where* $\mu_0 = 0$ *and* $\mu_m := \sum_{k=0}^{m-1} \lambda_{\mu_k}$; *then*

$$\sum_{n=0}^{\infty} c_n^2 l_n^2 < \infty \tag{1.11}$$

*implies that*

$$V_n|\lambda, \mathbf{v}; x| = o_x(l_n^{-1}) \tag{1.12}$$

*holds a.e. in* $(a, b)$ *for any* $\{\varphi_n(x)\}$ *and* $\mathbf{v} = \{v_n\}$.

In spite of the wide applicability of Theorem I, unfortunately, in the most important special case $g(\omega) \equiv 1$, it cannot be used to estimate the approximation-rate of the partial sums $s_n(x)$ of series (1.1) because then (1.5) does not hold for any $\boldsymbol{\mu}$. Consequently Theorem I does not include the result of Theorem II in the simplest special case when $\lambda_n \equiv 1$.

The aim of the present note is to fill this gap in Theorem I for $0 < p \leqq 2$. The corresponding problem for $p > 2$ remains open at this time.

In formulating our new result we shall use the notation as above and assume hence forth that the following conditions hold:

$$\gamma(\mu_{m+1}) \leqq N\gamma(\mu_m), \quad g(\mu_{m+1}) \leqq N g(\mu_m) \tag{1.13}$$

and

$$\sum_{m=0}^{n} \gamma(\mu_m)^2 \varrho(m) \leqq N\gamma(\mu_n)^2 \tag{1.14}$$

hold for all $m$ and $n$, where $\varrho(t)$ denotes a non-increasing positive function defined on $[0, \infty)$.

Our theorem reads as follows.

Theorem III. *Suppose that there exists a natural number $q$ such that for all $k$ and $m$*

$$\alpha_k(n) \leqq N \sum_{i=-q}^{q} \alpha_k(\mu_{m+i}) \quad \textit{with} \quad \mu_m < n < \mu_{m+1} \tag{1.15}$$

*and*

$$\sum_{i=0}^{\infty} \frac{\gamma(\mu_i)^2}{g(\mu_i)^2} \sum_{j=\mu_m}^{\mu_{m+1}-1} \alpha_j(\mu_i) \leqq N\varrho(m)\gamma(\mu_m)^2 \tag{1.16}$$

*hold. Then condition* (1.6) *implies that*

$$A_n(f, p, \mathbf{v};\, x) = o_x\big(g(n)/\gamma(n)\big) \tag{1.17}$$

*a.e. in* $(a, b)$ *for every* $p$, $0<p\leqq 2$ *and for every sequence* $\mathbf{v}$.

**2.** In order to prove our Theorem we need the following lemma.

Lemma [3]. *Let* $\delta>0$ *and* $\{\delta_n\}$ *be an arbitrary sequence of non-negative numbers. Suppose that for any orthonormal system the condition*

$$\sum_{n=1}^{\infty} \delta_n \Big(\sum_{k=n}^{\infty} c_k^2\Big)^{\delta} < \infty$$

*implies that the sequence* $\{s_n(x)\}$ *possesses a property $P$, then any subsequence* $\{s_{\mu_n}(x)\}$ *also possesses property $P$.*

**3.** Proof of Theorem III. By assumptions (1.13) we have for any $\mu_m<l<$ $<\mu_{m+1}$ $(m=0, 1, \ldots)$ that

$$\frac{g(\mu_m)}{N\gamma(\mu_m)} \leqq \frac{g(l)}{\gamma(l)} \leqq \frac{Ng(\mu_{m+1})}{\gamma(\mu_{m+1})}, \tag{3.1}$$

so, on account of (1.15), it is sufficient to prove (1.17) only for the values $\mu_n$.

First we prove (1.17) in the special case $p=2$ and $\nu_k=k$; and as we have said above, only for the indices $\mu_n$, i.e. we verify that

$$A_{\mu_n}(x) := A_{\mu_n}(f, 2, \{k\};\, x) = o_x\big(g(\mu_n)/\gamma(\mu_n)\big) \tag{3.2}$$

holds a.e. in $(a, b)$.

Then

$$\int_a^b A_{\mu_n}^2(x)\,dx = \sum_{m=0}^{\infty} \sum_{k=\mu_m}^{\mu_{m+1}-1} \alpha_k(\mu_n) \int_a^b |s_k(x)-f(x)|^2\,dx \leqq \sum_{m=0}^{\infty} \sum_{k=\mu_m}^{\mu_{m+1}-1} \alpha_k(\mu_n) \sum_{i=\mu_m+1}^{\infty} c_i^2.$$

Putting

$$R_{\mu_m}^2 := \sum_{i=\mu_m+1}^{\infty} c_i^2,$$

we get, by (1.16), that

$$\sum_{n=0}^{\infty}\frac{\gamma(\mu_n)^2}{g(\mu_n)^2}\int_a^b A_{\mu_n}^2(x)\,dx \leqq \sum_{n=0}^{\infty}\frac{\gamma(\mu_n)^2}{g(\mu_n)^2}\sum_{m=0}^{\infty}\sum_{k=\mu_m}^{\mu_{m+1}-1}\alpha_k(\mu_n)R_{\mu_m}^2 = \tag{3.3}$$

$$= \sum_{m=0}^{\infty} R_{\mu_m}^2 \sum_{n=0}^{\infty}\frac{\gamma(\mu_n)^2}{g(\mu_n)^2}\sum_{k=\mu_m}^{\mu_{m+1}-1}\alpha_k(\mu_n) \leqq N\sum_{m=0}^{\infty} R_{\mu_m}^2\varrho(m)\gamma(\mu_m)^2 := \sum\nolimits_1.$$

To estimate $\sum_1$ we use assumptions (1.6), (1.13) and (1.14), and so we have

$$1/N\sum\nolimits_1 \leqq \sum_{m=0}^{\infty}\Big(\sum_{l=m}^{\infty}\sum_{k=\mu_l}^{\mu_{l+1}-1} c_k^2\Big)\varrho(m)\gamma(\mu_m)^2 = \tag{3.4}$$

$$= \sum_{l=0}^{\infty}\Big(\sum_{k=\mu_l}^{\mu_{l+1}-1} c_k^2\Big)\sum_{m=0}^{l}\varrho(m)\gamma(\mu_m)^2 \leqq N\sum_{l=0}^{\infty}\Big(\sum_{k=\mu_l}^{\mu_{l+1}-1} c_k^2\Big)\gamma(\mu_l)^2 \leqq N\sum_{n=0}^{\infty} c_n^2\gamma(n)^2 < \infty.$$

By (3.3) and (3.4), applying Beppo Levi's theorem, we get that

$$\sum_{n=0}^{\infty}\gamma(\mu_n)^2 g(\mu_n)^{-2}A_{\mu_n}^2(x) =$$

$$= \sum_{n=0}^{\infty}\gamma(\mu_n)^2 g(\mu_n)^{-2}\sum_{k=0}^{\infty}\alpha_k(\mu_n)|s_k(x)-f(x)|^2 < \infty$$

a.e. in $(a, b)$. Hence (3.2) obviously follows.

For $0<p<2$

$$A_{\mu_n}(f, p, \{k\}; x) = o_x\big(g(\mu_n)/\gamma(\mu_n)\big) \tag{3.5}$$

follows from (3.2) using Hölder's inequality and (1.2).

Now, on account of (3.1), relation (3.5) implies

$$A_n(f, p, \{k\}; x) = o_x\big(g(n)/\gamma(n)\big) \tag{3.6}$$

a.e. in $(a, b)$.

Finally, if we apply the Lemma with property $P$ characterized by (3.6), then (1.7) follows for all $p$, $0<p\leqq 2$ and all sequences $\mathbf{v}$.

**4. Application.** We show that Theorem II can be derived from Theorem III. Since in the special case $\lambda_n\equiv 1$, Theorem II represents a statement concerning the partial sums of (1.1), it follows that under the proper conditions Theorem III yields certain results for the rate of approximation achieved by the partial sums, as well.

Now we show that Theorem III in the special case when $\varrho(m)\equiv g(m)\equiv 1$, $\gamma(n)=l_n$ and

$$\alpha_k(n) := \begin{cases} 1/\lambda_n & \text{for } n-\lambda_n < k \leqq n, \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

can be applied, with $\mu_m$ defined in Theorem II, that is, then (1.6), (1.13), (1.14), (1.15) and (1.16) are fulfilled.

Condition (1.6) holds trivially, (1.13) and (1.14) follow from (1.10).

In order to prove (1.15) we put $q=1$ and $N=2$, i.e. we want to verify that for any $k$ and $\mu_m<n<\mu_{m+1}$

$$\alpha_k(n) \leqq 2[\alpha_k(\mu_{m-1})+\alpha_k(\mu_m)+\alpha(\mu_{m+1})] \tag{4.2}$$

always holds. Since $\lambda_{n+1}-\lambda_n \leqq 1$ for all $n$, therefore $\mu_m-\lambda_{\mu_m} \leqq n-\lambda_n$, whence, by (4.1),

$$\alpha_k(n) \leqq \alpha_k(\mu_m) \tag{4.3}$$

holds for any $(n-\lambda_n<)k \leqq \mu_m$.

On the other hand, taking into account that $\lambda_{\mu_{m+1}} \leqq \lambda_{\mu_m}+\mu_{m+1}-\mu_m=2\lambda_{\mu_m}$ and $\mu_{m+1}-\lambda_{\mu_{m+1}}=\mu_m+\lambda_{\mu_m}-\lambda_{\mu_{m+1}} \leqq \mu_m$, we get

$$\alpha_k(n) \leqq 2\alpha_k(\mu_{m+1}) \tag{4.4}$$

for any $\mu_m<k(\leqq n)$. Thus (4.3) and (4.4) verify (4.2), and herewith (1.15) is also proved for the entries $\alpha_k(n)$ given in (4.1).

To show (1.16) in the case given above we have to verify that

$$\sum\nolimits_2 := \sum_{i=0}^{\infty} l_{\mu_i}^2 \sum_{j=\mu_m+1}^{\mu_{m+1}} \alpha_j(\mu_i) \leqq N l_{\mu_m}^2 \tag{4.5}$$

holds for every $m$.

By (4.1) it is clear that if $j>\mu_i$ then $\alpha_j(\mu_i)=0$, therefore

$$\sum\nolimits_2 = \sum_{i=m+1}^{\infty} l_{\mu_i}^2 \sum_{j=\mu_m+1}^{\mu_{m+1}} \alpha_j(\mu_i) = \tag{4.6}$$

$$= \sum_{i=m+1}^{\infty} l_{\mu_i}^2 \lambda_{\mu_i}^{-1}(\mu_{m+1}-\max(\mu_m, (\mu_i-\lambda_{\mu_i})))^+ =: \sum\nolimits_3,$$

where $\beta^+$ denotes the positive part of $\beta$.

On account of the definition of $\mu_m$ we have that

$$\sum\nolimits_3 \leqq \sum_{i=m+1}^{\infty} l_{\mu_i}^2 \lambda_{\mu_i}^{-1}(\mu_{m+1}-\mu_i+\lambda_{\mu_i})^+ = \tag{4.7}$$

$$= \sum_{i=m+1}^{\infty} l_{\mu_i}^2 \left(1-\frac{\mu_i-\mu_{m+1}}{\lambda_{\mu_i}}\right)^+ = l_{\mu_{m+1}}^2 + \sum_{i=m+2}^{\infty} l_{\mu_i}^2 \Big(1-\lambda_{\mu_i}^{-1} \sum_{k=m+1}^{i-1} \lambda_{\mu_k}\Big)^+.$$

Setting

$$\Lambda_i^{(m)} := \Big(1-\lambda_{\mu_i}^{-1} \sum_{k=m+1}^{i-1} \lambda_{\mu_k}\Big)^+$$

and taking into account that $\lambda_{\mu_{k+1}} \leqq 2\lambda_{\mu_k}$ always holds, we have for any $i > m+1$ that

$$\Lambda_i^{(m)} \leqq \Big(1-(2\lambda_{\mu_{i-1}})^{-1} \sum_{k=m+1}^{i-1} \lambda_{\mu_k}\Big)^+ = \left(\frac{1}{2}-(2\lambda_{\mu_{i-1}})^{-1} \sum_{k=m+1}^{i-2} \lambda_{\mu_k}\right)^+ \leqq$$

$$\leqq \left(\frac{1}{2}-(4\lambda_{\mu_{i-2}})^{-1} \sum_{k=m+1}^{i-2} \lambda_{\mu_k}\right)^+ = \left(\frac{1}{4}-(4\lambda_{\mu_{i-2}})^{-1} \sum_{k=m+1}^{i-3} \lambda_{\mu_k}\right)^+ \leqq$$

$$\leqq \left(\frac{1}{4}-(8\lambda_{\mu_{i-3}})^{-1} \sum_{k=m+1}^{i-3} \lambda_{\mu_k}\right)^+ \leqq \dots \leqq \left(\frac{1}{2}\right)^{i-m-1}.$$

Hence, by (1.10), (4.6) and (4.7), we obtain that

$$\sum\nolimits_2 \leqq l_{\mu_{m+1}}^2 + 2 \sum_{i=m+2}^{\infty} l_{\mu_i}^2 \left(\frac{1}{2}\right)^{i-m} \leqq l_{\mu_{m+1}}^2 + 2 \sum_{i=m+2}^{\infty} l_{\mu_{m+1}}^2 \left(\frac{K^2}{2}\right)^{i-m} = O(l_{\mu_m}^2),$$

that is, that (4.5) holds. This proves that (1.16) is satisfied, as stated.

It follows that all of the assumptions of Theorem III are fulfilled if the parameters are chosen according to the requirements of Theorem II; therefore we have proved that Theorem III implies Theorem II.

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# On the generalized absolute summability of double series

Z. NÉMETH

*Dedicated to Professor Béla Csákány on his 60th birthday*

**1. Introduction.** As usual we denote by $\sigma_n^{(\alpha)}$ the $n$-th Cesaro means of order $\alpha$ of a series $\sum_{n=0}^{\infty} a_n$ and $\tau_n^{(\alpha)}$ the $n$-th Cesaro means of the sequence $\{na_n\}$. The following definition is due to FLETT [3]: A series $\sum_{u=0}^{\infty} a_n$ is said to be summable $|C, \alpha, u|_\lambda$, $\alpha > -1$, $u \geqq 0$, $\lambda \geqq 1$, if the series

$$\sum_{n=1}^{\alpha} n^{\lambda u+\lambda-1} |\sigma_n^{(\alpha)} - \sigma_{n-1}^{(\alpha)}|^\lambda \equiv \sum_{n=1}^{\alpha} n^{\lambda u-1} |\tau_n^{(\alpha)}|^\lambda$$

converges.

In this note we consider the following definition of the generalized absolute Cesaro summability of double series

$$\sum_{i,k=0}^{\infty} a_{i,k}. \tag{1}$$

Let us denote by $\sigma_{m,n}^{(\alpha,\beta)}$ the $(m, n)$-th Cesaro mean of order $(\alpha, \beta)$ of series (1), that is,

$$\sigma_{m,n}^{(\alpha,\beta)} = \frac{1}{A_m^{(\alpha)}} \frac{1}{A_n^{(\beta)}} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{(\alpha)} A_{n-k}^{(\beta)} a_{i,k}, \quad m, n = -1, 0, 1, \ldots, \tag{2}$$

such that in the cases $\min(m, n) = -1$ we define $\sigma_{m,n}^{(\alpha,\beta)} = 0$, where $A_m^{(\alpha)}$ denotes the Cesaro numbers, namely, $A_0^{(\alpha)} \equiv 1$ and $A_m^{(\alpha)} = \dfrac{(1+\alpha)(2+\alpha)\ldots(m+\alpha)}{m!}$, $\alpha \neq -1, -2, \ldots$.

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Considering the notations

$$z_{m,n}^{(\alpha,\beta)}=m(\sigma_{m,n}^{(\alpha,\beta)}-\sigma_{m-1,n}^{(\alpha,\beta)})=\frac{1}{A_m^{(\alpha)}A_n^{(\beta)}}\sum_{i=0}^{m}\sum_{k=0}^{n}A_{m-i}^{(\alpha-1)}A_{n-k}^{(\beta)}ia_{i,k},$$

$$t_{m,n}^{(\alpha,\beta)}=n(\sigma_{m,n}^{(\sigma,\beta)}-\sigma_{m,n-1}^{(\alpha,\beta)})=\frac{1}{A_m^{(\alpha)}A_n^{(\beta)}}\sum_{i=0}^{m}\sum_{k=0}^{n}A_{m-i}^{(\alpha)}A_{n-k}^{(\beta-1)}ka_{i,k}$$

and

$$\tau_{m,n}^{(\alpha,\beta)}=mn(\sigma_{m,n}^{(\alpha,\beta)}-\sigma_{m-1,n}^{(\alpha,\beta)}-\sigma_{m,n-1}^{(\alpha,\beta)}+\sigma_{m-1,n-1}^{(\alpha,\beta)})=$$

$$=\frac{1}{A_m^{(\alpha)}A_n^{(\beta)}}\sum_{i=0}^{m}\sum_{k=0}^{n}A_{m-i}^{(\alpha-1)}A_{n-k}^{(\beta-1)}ika_{i,k},\quad m,n=0,1,2,\dots,$$

series (1) is said to be summable $|C,(\alpha,\beta),(u,v)|_\lambda$, where $\alpha,\beta>-1$, $u,v\geqq 0$, $\lambda\geqq 1$ if

$$\sum_{i=1}^{\infty}i^{\lambda u-1}|z_{i,0}^{(\alpha,\beta)}|^\lambda<\infty, \tag{3}$$

$$\sum_{k=1}^{\infty}k^{\lambda v-1}|t_{0,k}^{(\alpha,\beta)}|^\lambda<\infty \tag{4}$$

and

$$\sum_{i,k=1}^{\infty}i^{\lambda u-1}k^{\lambda v-1}|\tau_{i,k}^{(\alpha,\beta)}|^\lambda<\infty. \tag{5}$$

The concept of summability $|C,(\alpha,\beta),(0,0)|_1$ is well known (see e.g. [1], pp. 209—214). The generalized absolute Cesaro summability of double series was investigated by MÓRICZ [7] and SZALAY [8]. The fundamental theorems of summability $|C,\alpha,u|_\lambda$ were proved by FLETT (see [3], Theorems 1, 3, 4 and 7).

**2. Main results.** The aim of this paper is to extend the fundamental theorems for the double series (1). The author would like to thank I. Szalay for pointing out this generalization and his valuable hints.

Theorem 1.* *Let $\lambda\geqq 1$, $u,v\geqq 0$, $\alpha>\lambda u-1$ and $\beta>\lambda v-1$. If $\gamma,\delta\geqq 0$ then the summability $|C,(\alpha,\beta),(u,v)|_\lambda$ of series (1) implies the summability $|C,(\alpha+\gamma,\beta+\delta),(u,v)|_\lambda$, moreover the inequalities*

$$\sum_{m=1}^{\infty}m^{\lambda u-1}|z_{m,0}^{(\alpha+\gamma,\beta+\delta)}|^\lambda\leqq K\sum_{m=1}^{\infty}m^{\lambda u-1}|z_{m,0}^{(\alpha,\beta)}|^\lambda, \tag{6}$$

$$\sum_{n=1}^{\infty}n^{\lambda v-1}|t_{0,n}^{(\alpha+\gamma,\beta+\delta)}|^\lambda\leqq K\sum_{n=1}^{\infty}n^{\lambda v-1}|t_{0,n}^{(\alpha,\beta)}|^\lambda \tag{7}$$

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*) Throughout this article $K$ denotes a positive constant, not necessarily the same at each occurrence which does not depend on addition indices and the formal sum $\sum_{i=0}^{-1}$ means 0.

*and*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\lambda u-1} n^{\lambda v-1} |\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|^{\lambda} \leqq K \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\lambda u-1} n^{\lambda v-1} |\tau_{m,n}^{(\alpha,\beta)}|^{\lambda} \tag{8}$$

*hold.*

Theorem 2. *Let* $u, v \geqq 0$, $\alpha > \lambda u - 1$ *and* $\beta > \lambda v - 1$. *If*

i) $\mu > \lambda > 1$ *and* $\delta = 1/\lambda - 1/\mu$

*or*

ii) $\mu > \lambda = 1$ *and* $\delta > 1/\lambda - 1/\mu$

*then the summability* $|C, (\alpha, \beta), (u, v)|_{\lambda}$ *of series* (1) *implies the summability* $|C, (\alpha+\gamma, \beta+\delta), (u, v)|_{\mu}$, *moreover the inequalities*

$$\Big\{\sum_{m=1}^{\infty} m^{\mu u-1} |z_{m,0}^{(\alpha+\delta,\beta+\delta)}|^{\mu}\Big\}^{1/\mu} \leqq K \Big\{\sum_{m=1}^{\infty} m^{\lambda u-1} |z_{m,0}^{(\alpha,\beta)}|^{\lambda}\Big\}^{1/\lambda}, \tag{9}$$

$$\Big\{\sum_{n=1}^{\infty} n^{\mu v-1} |t_{0,n}^{(\alpha+\delta,\beta+\delta)}|^{\mu}\Big\}^{1/\mu} \leqq K \Big\{\sum_{n=1}^{\infty} n^{\lambda v-1} |t_{0,n}^{(\alpha,\beta)}|^{\lambda}\Big\}^{1/\lambda} \tag{10}$$

*and*

$$\Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\mu u-1} n^{\mu v-1} |\tau_{m,n}^{(\alpha+\delta,\beta+\delta)}|^{\mu}\Big\}^{1/\mu} \leqq K \Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\lambda u-1} n^{\lambda v-1} |\tau_{m,n}^{(\alpha,\beta)}|^{\lambda}\Big\}^{1/\lambda} \tag{11}$$

*hold.*

We remark that part i) of Theorem 2, together with Theorem 1 is sharper than a former result of SZALAY ([8], Theorem 1).

Theorem 3. *If* $\lambda \geqq 1$, $u, v \geqq 0$, $\alpha > \lambda u - 1$, $\beta > \lambda v - 1$, $\xi \leqq u$, $\eta \leqq v$, $\gamma \geqq \xi - u$, $\delta \geqq \eta - v$, $\alpha+\gamma$, $\beta+\delta > -1$, *and series* (1) *is* $|C, (\alpha, \beta), (u, v)|_{\lambda}$ *summable, then the inequalities*

$$\sum_{m=1}^{\infty} m^{\lambda\xi-1} |z_{m,0}^{(\alpha+\gamma,\beta+\delta)}|^{\lambda} \leqq K \sum_{m=1}^{\infty} m^{\lambda u-1} |z_{m,0}^{(\alpha,\beta)}|^{\lambda}, \tag{12}$$

$$\sum_{n=1}^{\infty} n^{\lambda\eta-1} |t_{0,n}^{(\alpha+\gamma,\beta+\delta)}|^{\lambda} \leqq K \sum_{n=1}^{\infty} n^{\lambda v-1} |t_{0,n}^{(\alpha,\beta)}|^{\lambda} \tag{13}$$

*and*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\lambda\xi-1} n^{\lambda\eta-1} |\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|^{\lambda} \leqq K \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\lambda u-1} n^{\lambda v-1} |\tau_{m,n}^{(\alpha,\beta)}|^{\lambda} \tag{14}$$

*are valid.*

Using Theorem 3, in the case of parameters $u=v=0$, $-\alpha \leqq \xi \leqq 0$, $-\beta \leqq \eta \leqq 0$, $\gamma = \xi$ and $\delta = \eta$ and writing $\xi' = -\xi$, $\eta' = -\eta$ we have the following

Corollary 1. *If $\lambda\geqq 1$, $0\leqq\xi'\leqq\alpha$, $0\leqq\eta'\leqq\beta$ and series* (1) *is* $|C, (\alpha,\beta), (0,0)|_\lambda$ *summable, then the following inequalities*

$$\sum_{m=1}^{\infty} m^{-1-\lambda\xi'}|z_{m,0}^{(\alpha-\xi',\beta-\eta')}|^\lambda \leqq K\sum_{m=1}^{\infty} m^{-1}|z_{m,0}^{(\alpha,\beta)}|^\lambda,$$

$$\sum_{n=1}^{\infty} n^{-1-\lambda\eta'}|t_{0,n}^{(\alpha-\xi',\beta-\eta')}|^\lambda \leqq K\sum_{n=1}^{\infty} n^{-1}|t_{0,n}^{(\alpha,\beta)}|^\lambda$$

*and*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-1-\lambda\xi'}n^{-1-\lambda\eta'}|\tau_{m,n}^{(\alpha-\xi',\beta-\eta')}|^\lambda \leqq K\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-1}n^{-1}|\tau_{m,n}^{(\alpha,\beta)}|^\lambda$$

*hold.*

Considering the case $\alpha=\xi'$, $\beta=\eta'$ a further specialization is the

Corollary 2. *If $\lambda\geqq 1$, $\alpha,\beta\geqq 0$ and series* (1) *is* $|C,(\alpha,\beta),(0,0)|_\lambda$ *summable, then*

$$\sum_{m=1}^{\infty} m^{\lambda-1-\lambda\alpha}|a_{m,0}|^\lambda \leqq K\sum_{m=1}^{\infty} m^{-1}|z_{m,0}^{(\alpha,\beta)}|^\lambda,$$

$$\sum_{n=1}^{\infty} n^{\lambda-1-\lambda\beta}|a_{0,n}|^\lambda \leqq K\sum_{n=1}^{\infty} n^{-1}|t_{0,n}^{(\alpha,\beta)}|^\lambda$$

*and*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\lambda-1-\lambda\alpha}n^{\lambda-1-\lambda\beta}|a_{m,n}|^\lambda \leqq K\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-1}n^{-1}|\tau_{m,n}^{(\alpha,\beta)}|^\lambda$$

*are valid.*

The Corollary 2 is a useful necessary condition of the generalized absolute Cesaro summability of double series and it is an extension of results of Kogbetlianz ([5], Théoréme VI), Flett ([2], Theorem 3) and Zak and Timan ([11], § 3, Theorem 3).

Theorem 4. *If $\lambda>\mu\geqq 1$, $u,v\geqq 0$, $\alpha>\mu u-1$, $\beta>\mu v-1$, $\xi\leqq u$, $\eta\leqq v$, $\gamma>\xi-u$, $\delta>\eta-v$ and series* (1) *is* $|C,(\alpha,\beta),(u,v)|_\lambda$ *summable, then the inequalities*

$$\Big\{\sum_{m=1}^{\infty} m^{\mu\xi-1}|z_{m,0}^{(\alpha+\gamma,\beta+\delta)}|^\mu\Big\}^{1/\mu} \leqq K\Big\{\sum_{m=1}^{\infty} m^{\lambda u-1}|z_{m,0}^{(\alpha,\beta)}|^\lambda\Big\}^{1/\lambda}, \tag{15}$$

$$\Big\{\sum_{n=1}^{\infty} n^{\mu\eta-1}|t_{0,n}^{(\alpha+\gamma,\beta+\delta)}|^\mu\Big\}^{1/\mu} \leqq K\Big\{\sum_{n=1}^{\infty} n^{\lambda v-1}|t_{0,n}^{(\alpha,\beta)}|^\lambda\Big\}^{1/\lambda} \tag{16}$$

*and*

$$\Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\mu\xi-1}n^{\mu\eta-1}|\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|^\mu\Big\}^{1/\mu} \leqq K\Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\lambda u-1}n^{\lambda v-1}|\tau_{m,n}^{(\alpha,\beta)}|^\lambda\Big\}^{1/\lambda} \tag{17}$$

*hold.*

We remark that in cases $\xi, \eta \geqq 0$ Theorems 3 and 4 mean, in other words, that if the suitable conditions are satisfied, the summability $|C, (\alpha, \beta), (u, v)|_\lambda$ of series (1) implies the summability $|C, (\alpha+\gamma, \beta+\delta), (\xi, \eta)|_\lambda$ and $|C, (\alpha+\gamma, \beta+\delta), (\xi, \eta)|_\mu$, respectively.

Series (1) is said to be summable $(C, (\alpha, \beta))$, $\alpha, \beta > -1$ to $S$, if the double sequence (2) is bounded and converges to $S$ in Pringsheim's sense. Finally we have

Theorem 5. *If* $\lambda > 1, u, v > 0, \alpha > u-1, \beta > v-1, \gamma > \alpha - u - 1/\lambda > 0$ *and* $\delta > \beta - v - 1/\lambda > 0$ *then the summability* $|C, (\alpha, \beta), (u, v)|_\lambda$ *of series* (1) *implies the summability* $(C, (\gamma, \delta))$.

Part 4 of this note contains some negative results. We show that Theorem 2 is the best possible. In relation to Theorems 3 and 4 we show that the parameter $u$ (or $v$) of summability cannot be increased by no means and parameter $\lambda$ cannot be decreased if parameters $u$, $v$ are fixed.

**3. Proof of Theorems.** If $\tau_n^{(\alpha)}$ denotes the $n$-th $(C, \alpha)$ mean of the sequence $\{na_n\}$ then it is well known that if $\alpha, \beta, \alpha+\gamma, \beta+\delta \neq -1, -2, \dots$, then

(18)
$$\tau_n^{(\alpha+\delta)} = \frac{1}{A_n^{(\alpha+\delta)}} \sum_{k=0}^{n} A_{n-k}^{(\delta-1)} A_k^{(\alpha)} \tau_k^{(\alpha)},$$

(19)
$$\tau_{m,n}^{(\alpha+\gamma, \beta+\delta)} = \frac{1}{A_m^{(\alpha+\gamma)} A_n^{(\beta+\delta)}} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{(\gamma-1)} A_{n-k}^{(\delta-1)} A_i^{(\alpha)} A_k^{(\beta)} \tau_{i,k}^{(\alpha,\beta)},$$

$$A_n^{(\beta+\delta)} = \sum_{k=0}^{n} A_{n-k}^{(\delta-1)} A_k^{(\beta)} \quad (n = 0, 1, 2, \dots),$$

moreover

(20)
$$A_n^{(\alpha)}/n^\alpha \to 1/\Gamma(\alpha+1) \quad (n \to \infty).$$

In order to prove Theorems we require the following lemmas.

Lemma 1 (Szalay [9]). *If* $\alpha, \beta, \alpha+\gamma, \beta+\delta \neq -1, -2, \dots$, *then for any* $m, n = 0, 1, 2, \dots$

$$z_{m,n}^{(\alpha+\gamma, \beta+\delta)} = \frac{1}{A_m^{(\alpha+\gamma)} A_n^{(\beta+\delta)}} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{(\gamma-1)} A_{n-k}^{(\delta-1)} A_i^{(\alpha)} A_k^{(\beta)} z_{i,k}^{(\alpha,\beta)}$$

*and*

$$t_{m,n}^{(\alpha+\gamma, \beta+\delta)} = \frac{1}{A_m^{(\alpha+\gamma)} A_n^{(\beta+\delta)}} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{(\gamma-1)} A_{n-k}^{(\delta-1)} A_i^{(\alpha)} A_k^{(\beta)} t_{i,k}^{(\alpha,\beta)}.$$

Lemma 2 (Hardy—Littlewood—Pólya [6]). *Let* $\{d_i\}_{i=0}^{\infty}$ *be a non-negative se-*

*quence. If* $\mu>\lambda>1$ *and* $\delta=1/\lambda-1/\mu$, *then exists a* $K=K(\lambda,\mu)$ *constant, such that*

$$\Big\{\sum_{m=0}^{M}\Big(\sum_{i=0}^{m-1}(m-i)^{\delta-1}d_i\Big)^{\mu}\Big\}^{1/\mu}\leqq K\Big(\sum_{i=0}^{M}d_i^{\lambda}\Big)^{1/\lambda}$$

*is valid for any* $M=0,1,2,\dots$.

Lemma 3 (SZALAY [10]). *Let* $\{d_{i,k}\}_{i,k=0}^{\infty}$ *be a non-negative double sequence. If* $\mu>\lambda>1$ *and* $\delta=1/\lambda-1/\mu$, *then exists a* $K=K(\lambda,\mu)$ *constant, such that*

$$\Big\{\sum_{m=0}^{M}\sum_{n=0}^{N}\Big(\sum_{i=0}^{m-1}\sum_{k=0}^{n-1}(m-i)^{\delta-1}(n-k)^{\delta-1}d_{i,k}\Big)^{\mu}\Big\}^{1/\mu}\leqq K\Big(\sum_{i=0}^{M}\sum_{k=0}^{N}d_{i,k}^{\lambda}\Big)^{1/\lambda}$$

*is valid for any* $M,N=0,1,2,\dots$.

Lemma 4 (ZAK—TIMAN [11]). *If series* (1) *is* $|C,(\gamma,\delta),(0,0)|_1$ *summable, then it is* $(C,(\gamma,\delta))$ *summable, too.*

We remark that if $\lambda>1$ then the summability $|C,(\gamma,\delta),(0,0)|_\lambda$ does not imply the ordinary summability $(C,(\gamma,\delta))$.

Lemma 5. *If* $a_{i,k}=c_i$ *for* $k=0$ *and* $a_{i,k}=0$ *otherwise, then the* $|C,(\alpha,\beta),(u,v)|_\lambda$ *summability of series* (1) *and* $|C,\alpha,u|_\lambda$ *summability of the series* $\sum_{i=0}^{\infty}c_i$ *are equivalent. Similarly, if* $a_{i,k}=c_k$ *for* $i=0$ *and* $a_{i,k}=0$ *otherwise, then the* $|C,(\alpha,\beta),(u,v)|_\lambda$ *summability of series* (1) *and* $|C,\beta,v|_\lambda$ *summability of the series* $\sum_{k=0}^{\infty}c_k$ *are equivalent.*

Proof. A fairly trivial calculation gives that for any $n$, $\beta$

$$\sigma_{m,n}^{(\alpha,\beta)}(a_{i,k})=\frac{1}{A_m^{(\alpha)}A_n^{(\beta)}}\sum_{i=0}^{m}\sum_{k=0}^{n}A_{m-i}^{(\alpha)}A_{n-k}^{(\beta)}a_{i,k}=\frac{1}{A_m^{(\alpha)}}\sum_{i=0}^{m}A_{m-i}^{(\alpha)}a_{i,0}=\sigma_m^{(\alpha)}(c_i)$$

and

$$z_{m,n}^{(\alpha,\beta)}(a_{i,k})=\frac{1}{A_m^{(\alpha)}}\sum_{i=0}^{m}A_{m-i}^{(\alpha-1)}ic_i\equiv\tau_m^{(\alpha)}(c_i),\quad t_{m,n}^{(\alpha\beta)}(a_{i,k})=\tau_{m,n}^{(\alpha,\beta)}(a_{i,k})=0,$$

so by (3)—(5) the statement is obvious.

Proof of Theorem 1. Considering (18) and Lemma 1, it is clear, that $z_{m,0}^{(\alpha,\beta)}$, the $m$-th $\tau$-mean of order $\alpha$ of the single series $\sum_{i=0}^{\infty}a_{i,0}$, does not depend on $\beta$, hence the inequality (6) follows directly from FLETT's result ([3], Theorem 1). The proof of (7) is carried out analogously. In the case $\lambda>1$, to verify (8) we use Hölder's inequality with indices $\lambda$ and $\lambda/(\lambda-1)$. By (19) and (20) we obtain that for any

$M, N=1, 2, \ldots$

$$\sum_{m=1}^{M}\sum_{n=1}^{N} m^{\lambda u-1} n^{\lambda v-1} |\tau_{m,n}^{(\lambda+\gamma,\beta)}|^{\lambda} \leqq$$

$$\leqq \sum_{m=1}^{M}\sum_{n=1}^{N} m^{\lambda u-1} n^{\lambda v-1} \left(\frac{1}{A_m^{(\alpha+\gamma)}} \sum_{i=0}^{m} A_{m-i}^{(\gamma-1)} A_i^{(\alpha)} |\tau_{i,n}^{(\alpha,\beta)}|\right)^{\lambda} \leqq$$

$$\leqq \sum_{m=1}^{M}\sum_{n=1}^{N} m^{\lambda u-1} n^{\lambda v-1} \left(\frac{1}{A_m^{(\alpha+\gamma)}}\right)^{\lambda} \left(\sum_{i=0}^{m} A_{m-i}^{(\gamma-1)} A_i^{(\alpha)} |\tau_{i,n}^{(\alpha,\beta)}|^{\lambda}\right) \left(\sum_{i=0}^{m} A_{m-i}^{(\gamma-1)} A_i^{(\alpha)}\right)^{\lambda/\lambda'} \leqq$$

$$\leqq K \sum_{m=1}^{M}\sum_{n=1}^{N} m^{\lambda u-1} n^{\lambda v-1} m^{-\alpha-\gamma} \sum_{i=1}^{m} (m-i+1)^{\gamma-1} i^{\alpha} |\tau_{i,n}^{(\alpha,\beta)}|^{\lambda} = \tag{21}$$

$$= K \sum_{n=1}^{N} n^{\lambda v-1} \sum_{i=1}^{M} i^{\alpha} |\tau_{i,n}^{(\alpha,\beta)}|^{\lambda} \sum_{m=i}^{M} m^{\lambda u-\alpha-\gamma-1} (m-i+1)^{\gamma-1} \leqq$$

$$\leqq K \sum_{n=1}^{N}\sum_{i=1}^{M} i^{\lambda u-1} n^{\lambda v-1} |\tau_{i,n}^{(\alpha,\beta)}|^{\lambda},$$

because a routine calculation gives that if $\gamma>0$, then for any $M=1, 2, \ldots$

$$\sum_{m=i}^{M} m^{\lambda u-\alpha-\gamma-1} (m-i+1)^{\gamma-1} \leqq$$

$$\leqq \sum_{m=i}^{2i} m^{\lambda u-\alpha-\gamma-1} (m-i+1)^{\gamma-1} + \sum_{m=2i}^{\infty} m^{\lambda u-\alpha-\gamma-1} (m-i+1)^{\gamma-1} \leqq K i^{\lambda u-1-\alpha}.$$

A similar method can be used if $\delta>0$. In the case $\lambda=1$, we prove (8) in the same way, omitting the last factor in (21).

Proof of Theorem 2. Inequalities (9) and (10) follow directly from FLETT's result ([3], Theorem 1) by similar arguments to the proof of (6) and (7). Turning to the proof of (11), we denote by $S$ the sum of the series on the right side of (11).

In the case i), $\delta=1/\lambda-1/\mu$, by (19) we have

$$|\tau_{m,n}^{(\alpha+\delta,\beta+\delta)}| \leqq \frac{1}{A_m^{(\alpha+\delta)} A_n^{(\beta+\delta)}} \sum_{i=0}^{m}\sum_{k=0}^{n} A_{m-i}^{(\delta-1)} A_{n-k}^{(\delta-1)} A_i^{(\alpha)} A_k^{(\beta)} |\tau_{i,k}^{(\alpha,\beta)}| \leqq$$

$$\leqq \frac{1}{A_m^{(\alpha+\delta)}} \frac{1}{A_n^{(\beta+\delta)}} \left(\sum_{i=0}^{m/2}\sum_{k=0}^{n/2} + \sum_{i=0}^{m/2}\sum_{k=n/2}^{n} + \sum_{i=m/2}^{m}\sum_{k=0}^{n/2} + \sum_{i=m/2}^{m}\sum_{k=n/2}^{n}\right) \equiv$$

$$\equiv T_{m,n}^{(1)} + T_{m,n}^{(2)} + T_{m,n}^{(3)} + T_{m,n}^{(4)}.$$

By (20) we have

$$T_{m,n}^{(1)} = \frac{1}{A_m^{(\alpha+\delta)}} \frac{1}{A_n^{(\beta+\delta)}} \sum_{i=0}^{m/2} \sum_{k=0}^{n/2} A_{m-i}^{(\delta-1)} A_{n-k}^{(\delta-1)} A_i^{(\alpha)} A_k^{(\beta)} |\tau_{i,k}^{(\alpha,\beta)}| \leqq$$
$$\leqq K \frac{1}{A_m^{(\alpha+1)}} \frac{1}{A_n^{(\beta+1)}} \sum_{i=1}^{m} \sum_{k=1}^{n} A_i^{(\alpha)} A_k^{(\beta)} |\tau_{i,k}^{(\alpha,\beta)}|.$$

Let $\omega$ be a number such that

$$\max(-\alpha - 1/\lambda + u,\ -\beta - 1/\lambda + v) < \omega < (\lambda-1)/\lambda.$$

A routine calculation gives that

$$\Big\{\sum_{i=1}^{m} \sum_{k=1}^{n} i^{-\omega\lambda/(\lambda-1)} k^{-\omega\lambda/(\lambda-1)}\Big\}^{(\lambda-1)/\lambda} \leqq K m^{-\omega+(\lambda-1)/\lambda} n^{-\omega+(\lambda-1)/\lambda}. \tag{22}$$

Applying the Hölder inequality with indices $\mu$, $\lambda/(\lambda-1)$, $\mu\lambda/(\mu-\lambda)$, we obtain that

$$T_{m,n}^{(1)} \leqq K m^{-\alpha-1} n^{-\beta-1} \sum_{i=1}^{m} \sum_{k=1}^{n} \{i^{\alpha+\omega-(\lambda u-1)(\mu-\lambda)/\lambda\mu} k^{\beta+\omega-(\lambda v-1)(\mu-\lambda)/\lambda\mu} |\tau_{i,k}^{(\alpha,\beta)}|^{\lambda/\mu}\} \times$$
$$\times \{i^{-\omega} k^{-\omega}\} \{i^{(\lambda u-1)(\mu-\lambda)/\lambda\mu} k^{(\lambda v-1)(\mu-\lambda)/\lambda\mu} |\tau_{i,k}^{(\alpha,\beta)}|^{(\mu-\lambda)/\mu}\} \leqq$$
$$\leqq K m^{-\alpha-1} n^{-\beta-1} \Big\{\sum_{i=1}^{m} \sum_{k=1}^{n} i^{\alpha\mu+\omega\mu-(\lambda u-1)(\mu-\lambda)/\lambda} k^{\beta\mu+\omega\mu-(\lambda v-1)(\mu-\lambda)/\lambda} |\tau_{i,k}^{(\alpha,\beta)}|^{\lambda}\Big\}^{1/\lambda} \times$$
$$\times \Big\{\sum_{i=1}^{m} \sum_{k=1}^{n} i^{-\omega\lambda/(\lambda-1)} k^{-\omega\lambda/(\lambda-1)}\Big\}^{(\lambda-1)/\lambda} \Big\{\sum_{i=1}^{m} \sum_{k=1}^{n} i^{\lambda u-1} k^{\lambda v-1} |\tau_{i,k}^{(\alpha,\beta)}|^{\lambda}\Big\}^{(\mu-\lambda)/\lambda\mu} \leqq$$
$$\leqq K S^{(\mu-\lambda)/\lambda\mu} m^{-\alpha-\omega-1/\lambda} n^{-\beta-\omega-1/\lambda} \Big\{\sum_{i=1}^{m} \sum_{k=1}^{n} i^{\alpha\mu+\omega\mu-(\lambda u-1)(\mu-\lambda)/\lambda} \times$$
$$\times k^{\beta\mu+\omega\mu-(\lambda v-1)(\mu-\lambda)/\lambda} |\tau_{i,k}^{(\alpha,\beta)}|^{\lambda}\Big\}^{1/\mu},$$

whence

$$(T_{m,n}^{(1)})^{\mu} \leqq K S^{(\mu-\lambda)/\lambda} m^{-\alpha\mu-\omega\mu-\mu/\lambda} n^{-\beta\mu-\omega\mu-\mu/\lambda} \sum_{i=1}^{m} \sum_{k=1}^{n} i^{\lambda u-1} k^{\lambda v-1} \times$$
$$\times |\tau_{i,k}^{(\alpha,\beta)}|^{\lambda} i^{(\alpha\lambda\mu+\omega\lambda\mu-u\lambda\mu+\mu)/\lambda} k^{(\beta\lambda\mu+\omega\lambda\mu-v\lambda\mu+\mu)/\lambda},$$

and for any $M, N=1, 2, \ldots$

$$\sum_{m=1}^{M} \sum_{n=1}^{N} m^{\mu u-1} n^{\mu v-1} (T_{m,n}^{(1)})^{\mu} \leqq K S^{(\mu-\lambda)/\mu} \sum_{i=1}^{M} \sum_{k=1}^{N} i^{\lambda u-1} k^{\lambda v-1} |\tau_{i,k}^{(\alpha,\beta)}|^{\lambda} \times$$
$$\times i^{(\alpha\lambda\mu+\omega\lambda\mu-u\lambda\mu+\mu)/\lambda} k^{(\beta\lambda\mu+\omega\lambda\mu-v\lambda\mu+\mu)/\lambda} \times$$
$$\times \sum_{m=i}^{M} \sum_{n=k}^{N} m^{-\alpha\mu-\omega\mu-\mu/\lambda+\mu u-1} n^{-\beta\mu-\omega\mu-\mu/\lambda+\mu v-1} \leqq K S^{\mu/\lambda},$$

because, with standard computation,

$$i^{(\alpha\lambda\mu+\omega\lambda\mu-u\lambda\mu+\mu)/\lambda}\sum_{m=i}^{M} m^{-\alpha\mu-\omega\mu-\mu/\lambda+\mu u-1} \leqq K. \tag{23}$$

$$T_{m,n}^{(2)} = \frac{1}{A_m^{(\alpha+\delta)}}\frac{1}{A_n^{(\beta+\delta)}}\sum_{i=0}^{m/2}\sum_{k=n/2}^{n} A_{m-i}^{(\delta-1)}A_{n-k}^{(\delta-1)}A_i^{(\alpha)}A_k^{(\beta)}|\tau_{i,k}^{(\alpha,\beta)}| \leqq$$

$$\leqq K\frac{1}{A_m^{(\alpha+1)}}\sum_{i=0}^{m/2}\sum_{k=n/2}^{n} A_{n-k}^{(\delta-1)}A_i^{(\alpha)}k^{-\delta}|\tau_{i,k}^{(\alpha,\beta)}|$$

and

$$n^{v-1/\mu}T_{m,n}^{(2)} \leqq K\frac{1}{A_m^{(\alpha+1)}}\sum_{i=0}^{m/2}\sum_{k=n/2}^{n}(n-k+1)^{\delta-1}A_i^{(\alpha)}k^{v-1/\lambda}|\tau_{i,k}^{(\alpha,\beta)}| \leqq$$

$$\leqq K\frac{1}{A_m^{(\alpha+1)}}\sum_{i=0}^{m}\sum_{k=0}^{n-1}(n-k)^{\delta-1}A_i^{(\alpha)}(k+1)^{v-1/\lambda}|\tau_{i,k}^{(\alpha,\beta)}| +$$

$$+K\frac{1}{A_m^{(\alpha+1)}}\sum_{i=0}^{m}A_i^{(\alpha)}n^{v-1/\lambda}|\tau_{i,n}^{(\alpha,\beta)}| \equiv T_{m,n}^{(2,1)}+T_{m,n}^{(2,2)}.$$

Applying Lemma 2, with the single sequence

$$d_k^{(m)} = (k+1)^{v-1/\lambda}\sum_{i=1}^{m}A_i^{(\alpha)}|\tau_{i,k}^{(\alpha,\beta)}|,$$

we obtain that for any $N=1,2,\dots$

$$\sum_{n=1}^{N}(T_{m,n}^{(2,1)})^{\mu} \leqq Km^{-\mu\alpha-\mu}\sum_{n=0}^{N}\Big(\sum_{k=0}^{n-1}(n-k)^{\delta-1}d_k^{(m)}\Big)^{\mu} \leqq$$

$$\leqq Km^{-\mu\alpha-\mu}\Big\{\sum_{k=0}^{N}(k+1)^{\lambda v-1}\Big(\sum_{i=1}^{m}A_i^{(\alpha)}|\tau_{i,k}^{(\alpha,\beta)}|\Big)^{\lambda}\Big\}^{\mu/\lambda}.$$

Applying the Hölder inequality with indices $\lambda$ and $\lambda/(\lambda-1)$, by (22) we have

$$\sum_{i=1}^{m}A_i^{(\alpha)}|\tau_{i,k}^{(\alpha,\beta)}| \leqq K\sum_{i=0}^{m}\{i^{\alpha+\omega}|\tau_{i,k}^{(\alpha,\beta)}|\}\{i^{-\omega}\} \leqq$$

$$\leqq K\Big\{\sum_{i=1}^{m}i^{\lambda\alpha+\lambda\omega}|\tau_{i,k}^{(\alpha,\beta)}|^{\lambda}\Big\}^{1/\lambda}\Big\{\sum_{i=1}^{m}i^{-\omega\lambda/(\lambda-1)}\Big\}^{(\lambda-1)/\lambda} \leqq \tag{24}$$

$$\leqq Km^{-\omega+(\lambda-1)/\lambda}\Big\{\sum_{i=1}^{m}i^{\lambda\alpha+\lambda\omega}|\tau_{i,k}^{(\alpha,\beta)}|^{\lambda}\Big\}^{1/\lambda},$$

and, by Hölder's inequality with indices $\mu/\lambda$ and $\mu/(\mu-\lambda)$ we have

$$\sum_{n=1}^{N}(T_{m,n}^{(2,1)})^{\mu} \leqq Km^{-\mu\alpha-\mu-\omega\mu+(\lambda-1)\mu/\lambda}\Big\{\sum_{k=0}^{N}\sum_{i=1}^{m}(k+1)^{\lambda v-1}i^{\lambda\alpha+\lambda\omega}|\tau_{i,k}^{(\alpha,\beta)}|^{\lambda}\Big\}^{\mu/\lambda} =$$

$$(25) \qquad = Km^{-\mu\alpha-\mu-\omega\mu+(\lambda-1)\mu/\lambda}\Big[\sum_{k=0}^{N}\sum_{i=1}^{m}\{(k+1)^{(\lambda v-1)\lambda/\mu}i^{\lambda\alpha+\lambda\omega-(\lambda u-1)(\mu-\lambda)/\mu}\times$$

$$\times|\tau_{i,k}^{(\alpha,\beta)}|^{\lambda^2/\mu}\}\{(k+1)^{(\lambda v-1)(\mu-\lambda)/\mu}i^{(\lambda u-1)(\mu-\lambda)/\mu}|\tau_{i,k}^{(\alpha,\beta)}|^{(\mu-\lambda)\lambda/\mu}\}\Big]^{\mu/\lambda} \leqq$$

$$\leqq Km^{-\mu\alpha-\mu-\omega\mu+(\lambda-1)\mu/\lambda}S^{(\mu-\lambda)/\mu}\sum_{k=0}^{N}\sum_{i=1}^{m}(k+1)^{\lambda v-1}i^{\alpha\mu+\omega\mu-(\lambda u-1)(\mu-\lambda)/\lambda}|\tau_{i,k}^{(\alpha,\beta)}|^{\lambda}.$$

Finally, by using (23), for any $M=1, 2, \ldots$

$$\sum_{m=1}^{M}\sum_{n=1}^{N}m^{\mu u-1}(T_{m,n}^{(2,1)})^{\mu} \leqq KS^{(\mu-\lambda)/\lambda}\sum_{k=0}^{N}\sum_{i=1}^{M}i^{\lambda u-1}(k+1)^{\lambda v-1}|\tau_{i,k}^{(\alpha,\beta)}|^{\lambda}\times$$

$$\times i^{\alpha\mu+\omega\mu-u\mu+\mu/\lambda}\sum_{m=i}^{M}m^{-\alpha\mu-\omega\mu-\mu/\lambda+\mu u-1} \leqq KS^{\mu/\lambda}.$$

By (24) we obtain that for any $N=1, 2, \ldots$

$$\sum_{n=1}^{N}(T_{m,n}^{(2,2)})^{\mu} = Km^{-\mu\alpha-\mu}\sum_{n=1}^{N}n^{(v-1/\lambda)\mu}\Big(\sum_{i=1}^{m}A_i^{(\alpha)}|\tau_{i,n}^{(\alpha,\beta)}|\Big)^{\mu} \leqq$$

$$\leqq Km^{-\mu\alpha-\mu-\omega\mu+(\lambda-1)\mu/\lambda}\sum_{n=1}^{N}n^{(v-1/\lambda)\mu}\Big(\sum_{i=1}^{m}i^{\lambda\alpha+\lambda\omega}|\tau_{i,n}^{(\alpha,\beta)}|^{\lambda}\Big)^{\mu/\lambda}.$$

It is known that if $a_i \geqq 0$ and $0<p\leqq 1$ then

$$(26) \qquad (a_1+a_2+\ldots+a_k)^p \leqq a_1^p+a_2^p+\ldots+a_k^p,$$

whence

$$\sum_{n=1}^{N}(T_{m,n}^{(2,2)})^{\mu} \leqq Km^{-\mu\alpha-\mu-\omega\mu+(\lambda-1)\mu/\lambda}\Big\{\sum_{n=1}^{N}\sum_{i=1}^{m}n^{\lambda v-1}i^{\lambda\alpha+\lambda\omega}|\tau_{i,n}^{(\alpha,\beta)}|^{\lambda}\Big\}^{\mu/\lambda},$$

and we may finish the estimate as in (25). This completes the proof of

$$\sum_{m=1}^{M}\sum_{n=1}^{N}m^{\mu u-1}n^{\mu v-1}(T_{m,n}^{(2)})^{\mu} \leqq KS^{\mu/\lambda},$$

and, by similar arguments, we have that

$$\sum_{m=1}^{M}\sum_{n=1}^{N}m^{\mu u-1}n^{\mu v-1}(T_{m,n}^{(3)})^{\mu} \leqq KS^{\mu/\lambda}.$$

Now let us consider $T_{m,n}^{(4)}$.

$$T_{m,n}^{(4)} = \frac{1}{A_m^{(\alpha+\delta)} A_n^{(\beta+\delta)}} \sum_{i=m/2}^{m} \sum_{k=n/2}^{n} A_{m-i}^{(\delta-1)} A_{n-k}^{(\delta-1)} A_i^{(\alpha)} A_k^{(\beta)} |\tau_{i,k}^{(\alpha,\beta)}| \leqq$$

$$\leqq K \sum_{i=m/2}^{m} \sum_{k=n/2}^{n} i^{-\delta} k^{-\delta} A_{m-i}^{(\delta-1)} A_{n-k}^{(\delta-1)} |\tau_{i,k}^{(\alpha,\beta)}|,$$

and

$$m^{u-1/\mu} n^{v-1/\mu} T_{m,n}^{(4)} \leqq K \sum_{i=m/2}^{m-1} \sum_{k=n/2}^{n-1} (m-i)^{\delta-1} (n-k)^{\delta-1} i^{u-1/\lambda} k^{v-1/\lambda} |\tau_{i,k}^{(\alpha,\beta)}| +$$

$$+ K n^{v-1/\lambda} \sum_{i=m/2}^{m-1} (m-i)^{\delta-1} i^{u-1/\lambda} |\tau_{i,n}^{(\alpha,\beta)}| + K m^{u-1/\lambda} \sum_{k=n/2}^{n-1} (n-k)^{\delta-1} k^{v-1/\lambda} |\tau_{m,k}^{(\alpha,\beta)}| +$$

$$+ K m^{u-1/\lambda} n^{v-1/\lambda} |\tau_{m,n}^{(\alpha,\beta)}| \equiv T_{m,n}^{(4,1)} + T_{m,n}^{(4,2)} + T_{m,n}^{(4,3)} + T_{m,n}^{(4,4)}.$$

Applying Lemma 3, with the double sequence

$$d_{i,k} = (i+1)^{u-1/\lambda} (k+1)^{v-1/\lambda} |\tau_{i,k}^{(\alpha,\beta)}|$$

we obtain that for any $M, N = 1, 2, \dots$

$$\sum_{m=1}^{M} \sum_{n=1}^{N} (T_{m,n}^{(4,1)})^{\mu} \leqq K \sum_{m=1}^{M} \sum_{n=1}^{N} \Big( \sum_{i=0}^{m-1} \sum_{k=0}^{n-1} (m-i)^{\delta-1} (n-k)^{\delta-1} d_{i,k} \Big)^{\mu} \leqq$$

$$\leqq K \Big( \sum_{i=1}^{M} \sum_{k=1}^{N} (i+1)^{\lambda u-1} (k+1)^{\lambda v-1} |\tau_{i,k}^{(\alpha,\beta)}|^{\lambda} \Big)^{\mu/\lambda} \leqq K S^{\mu/\lambda}.$$

Applying Lemma 2, with the single sequence

$$d_i^{(n)} = (i+1)^{u-1/\lambda} |\tau_{i,n}^{(\alpha,\beta)}|$$

we obtain that for any $M = 1, 2, \dots$

$$\sum_{m=1}^{M} (T_{m,n}^{(4,2)})^{\mu} \leqq K n^{v\mu-\mu/\lambda} \sum_{m=1}^{M} \Big( \sum_{i=0}^{m-1} (m-i)^{\delta-1} d_i^{(n)} \Big)^{\mu} \leqq$$

$$\leqq K n^{v\mu-\mu/\lambda} \Big( \sum_{i=1}^{M} (i+1)^{\lambda u-1} |\tau_{i,n}^{(\alpha,\beta)}|^{\lambda} \Big)^{\mu/\lambda},$$

and, by (26), for any $N = 1, 2, \dots$

$$\sum_{m=1}^{M} \sum_{n=1}^{N} (T_{m,n}^{(4,2)})^{\mu} \leqq K \Big\{ \sum_{i=1}^{M} \sum_{n=1}^{N} (i+1)^{\lambda u-1} n^{\lambda v-1} |\tau_{i,n}^{(\alpha,\beta)}|^{\lambda} \Big\}^{\mu/\lambda} \leqq K S^{\mu/\lambda}.$$

By similar arguments we have

$$\sum_{m=1}^{M} \sum_{n=1}^{N} (T_{m,n}^{(4,3)})^{\mu} \leqq K S^{\mu/\lambda}$$

and, finally, using (26) again

$$\sum_{m=1}^{M}\sum_{n=1}^{N}(T_{m,n}^{(4,4)})^{\mu} \leqq K\sum_{m=1}^{M}\sum_{n=1}^{N} m^{u\mu-\mu/\lambda} n^{v\mu-\mu/\lambda}|\tau_{m,n}^{(\alpha,\beta)}|^{\mu} \leqq$$

$$\leqq K\Big(\sum_{m=1}^{M}\sum_{n=1}^{N} m^{\lambda u-1} n^{\lambda v-1}|\tau_{m,n}^{(\alpha,\beta)}|^{\lambda}\Big)^{\mu/\lambda} \leqq KS^{\mu/\lambda}.$$

Estimates for $i=1, 2, 3, 4$

$$\sum_{m=1}^{M}\sum_{n=1}^{N} m^{\mu u-1} n^{\mu v-1}(T_{m,n}^{(i)})^{\mu} \leqq KS^{\mu/\lambda}$$

complete the proof of (11) in the case i), for $\delta=1/\lambda-1/\mu$.

In the case ii) for $\delta>1/\lambda-1/\mu=1-1/\mu$, by (19), we have that for any $M, N= =1, 2, \ldots$

$$\sum_{m=1}^{M}\sum_{n=1}^{N} m^{\mu u-1} n^{\mu v-1}|\tau_{m,n}^{(\alpha+\delta,\beta+\delta)}|^{\mu} \leqq K\sum_{m=1}^{M}\sum_{n=1}^{N} m^{\mu u-1-\mu\alpha-\mu\delta} n^{\mu v-1-\mu\beta-\mu\delta}\times$$

$$\times\Big\{\sum_{i=1}^{m}\sum_{k=1}^{n}(m-i+1)^{\delta-1}(n-k+1)^{\delta-1} i^{\alpha} k^{\beta}|\tau_{i,k}^{(\alpha,\beta)}|\Big\}^{\mu}.$$

Applying Hölder's inequality with indices $\mu$ and $\mu/(\mu-1)$ we obtain that

$$\sum_{i=1}^{m}\sum_{k=1}^{n}(m-i+1)^{\delta-1}(n-k+1)^{\delta-1} i^{\alpha} k^{\beta}|\tau_{i,k}^{(\alpha,\beta)}| =$$

$$= \sum_{i=1}^{m}\sum_{k=1}^{n}\{(m-i+1)^{\delta-1}(n-k+1)^{\delta-1} i^{\alpha-(u-1)(\mu-1)/\mu} k^{\beta-(v-1)(\mu-1)/\mu}|\tau_{i,k}^{(\alpha,\beta)}|^{1/\mu}\}\times$$

$$\times\{i^{(u-1)(\mu-1)/\mu} k^{(v-1)(\mu-1)/\mu}|\tau_{i,k}^{(\alpha,\beta)}|^{(\mu-1)/\mu}\} \leqq$$

$$\leqq \Big\{\sum_{i=1}^{m}\sum_{k=1}^{n}(m-i+1)^{(\delta-1)\mu}(n-k+1)^{(\delta-1)\mu} i^{\alpha\mu-(u-1)(\mu-1)} k^{\beta\mu-(v-1)(\mu-1)}|\tau_{i,k}^{(\alpha,\beta)}|\Big\}^{1/\mu}\times$$

$$\times\Big\{\sum_{i=1}^{m}\sum_{k=1}^{n} i^{u-1} k^{v-1}|\tau_{i,k}^{(\alpha,\beta)}|\Big\}^{(\mu-1)/\mu},$$

and

$$\sum_{m=1}^{M}\sum_{n=1}^{N} m^{\mu u-1} n^{\mu v-1}|\tau_{m,n}^{(\alpha+\delta,\beta+\delta)}|^{\mu} \leqq$$

$$KS^{\mu-1}\sum_{i=1}^{M}\sum_{k=1}^{N} i^{u-1} k^{v-1}|\tau_{i,k}^{(\alpha,\beta)}|\, i^{\alpha\mu-u\mu+\mu} k^{\beta\mu-v\mu+\mu}\times$$

$$\times\sum_{m=i}^{M}\sum_{n=k}^{N}(m-i+1)^{\delta\mu-\mu}(n-k+1)^{\delta\mu-\mu} m^{\mu u-1-\mu\alpha-\mu\delta} n^{\mu v-1-\mu\beta-\mu\delta} \leqq KS^{\mu},$$

since

$$i^{\alpha\mu-u\mu+\mu}\Big(\sum_{m=i}^{2i}+\sum_{m=2i}^{\infty}\Big)(m-i+1)^{\delta\mu-\mu}m^{\mu u-1-\mu\alpha-\mu\delta}\leqq \tag{27}$$

$$\leqq Ki^{\mu-\delta\mu-1}\sum_{m=1}^{i}m^{\delta\mu-\mu}+Ki^{\alpha\mu-u\mu+\mu}\sum_{m=i}^{\infty}m^{-\mu+\mu u-1-\mu\alpha}\leqq K.$$

Proof of Theorem 3. Inequalities (12) and (13) follow directly from Flett's result ([3], Theorem 3), by similar arguments to the proof of (6) and (7). In the proof of (14), considering Theorem 1, we may assume that $\gamma=\xi-u<0$ and $\delta=\eta-v<0$.

Let $\lambda>1$. Using (19), we have that

$$m^{\gamma}n^{\delta}|\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|\leqq Km^{-\alpha}n^{-\beta}\sum_{i=1}^{m}\sum_{k=1}^{n}|A_{m-i}^{(\gamma-1)}||A_{n-k}^{(\delta-1)}|i^{\alpha}k^{\beta}|\tau_{i,k}^{(\alpha,\beta)}|.$$

Let $\omega$ be a number such that

$$\max(-\alpha-1/\lambda+u,\ -\beta-1/\lambda+v)<\omega<(\lambda-1)/\lambda.$$

Applying Hölder's inequality with indices $\lambda$, $\lambda/(\lambda-1)$ we obtain that

$$m^{\lambda\gamma}n^{\lambda\delta}|\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|^{\lambda}\leqq Km^{-\alpha\lambda}n^{-\beta\lambda}\Big\{\sum_{i=1}^{m}\sum_{k=1}^{n}|A_{m-i}^{(\gamma-1)}||A_{n-k}^{(\delta-1)}|i^{\lambda\alpha+\lambda\omega}k^{\lambda\beta+\lambda\omega}|\tau_{i,k}^{(\alpha,\beta)}|^{\lambda}\Big\}\times$$

$$\times\Big\{\sum_{i=1}^{m}\sum_{k=1}^{n}|A_{m-i}^{(\gamma-1)}||A_{n-k}^{(\delta-1)}|i^{-\omega\lambda/(\lambda-1)}k^{-\omega\lambda/(\lambda-1)}\Big\}^{(\lambda-1)}.$$

A routine calculation gives that

$$\sum_{i=1}^{m}|A_{m-i}^{(\gamma-1)}|i^{-\omega\lambda/(\lambda-1)}\leqq K\sum_{i=1}^{m}(m-i+1)^{\gamma-1}i^{-\omega\lambda/(\lambda-1)}\leqq K\Big(\sum_{i=1}^{m/2}+\sum_{i=m/2}^{m}\Big\}\leqq$$

$$\leqq Km^{\gamma-1}\sum_{i=1}^{m/2}i^{-\omega\lambda/(\lambda-1)}+Km^{-\omega\lambda/(\lambda-1)}\sum_{i=m/2}^{\infty}(m-i+1)^{\gamma-1}\leqq Km^{\gamma-\omega\lambda/(\lambda-1)},$$

whence

$$m^{\lambda\gamma}n^{\lambda\delta}|\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|^{\lambda}\leqq Km^{-\alpha\lambda-\omega\lambda}n^{-\beta\lambda-\omega\lambda}\times \tag{28}$$

$$\times\sum_{i=1}^{m}\sum_{k=1}^{n}(m-i+1)^{\gamma-1}(n-k+1)^{\delta-1}i^{\lambda\alpha+\lambda\omega}k^{\lambda\beta+\lambda\omega}|\tau_{i,k}^{(\alpha,\beta)}|^{\lambda},$$

and for any $M, N=1,2,\ldots$

$$\sum_{m=1}^{M}\sum_{n=1}^{N} m^{\lambda(u+\gamma)-1} n^{\lambda(v+\delta)-1} |\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|^{\lambda} \leqq$$

$$\leqq K \sum_{m=1}^{M}\sum_{n=1}^{N} m^{\lambda u-1-\alpha\lambda-\omega\lambda} n^{\lambda v-1-\beta\lambda-\omega\lambda} \times$$

$$\times \sum_{i=1}^{m}\sum_{k=1}^{n} (m-i+1)^{\gamma-1}(n-k+1)^{\delta-1} i^{\lambda\alpha+\lambda\omega} k^{\lambda\beta+\lambda\omega} |\tau_{i,k}^{(\alpha,\beta)}|^{\lambda} =$$

$$= K \sum_{i=1}^{M}\sum_{k=1}^{N} i^{\lambda u-1} k^{\lambda v-1} |\tau_{i,k}^{(\alpha,\beta)}|^{\lambda} i^{\lambda\alpha+\lambda\omega-\lambda u+1} k^{\lambda\beta+\lambda\omega-\lambda v+1} \times$$

$$\times \sum_{m=i}^{M}\sum_{n=k}^{N} (m-i+1)^{\gamma-1}(n-k+1)^{\delta-1} n^{\lambda u-1-\alpha\lambda-\omega\lambda} n^{\lambda v-1-\alpha\lambda-\omega\lambda} \leqq$$

$$\leqq K \sum_{i=1}^{M}\sum_{k=1}^{N} i^{\lambda u-1} k^{\lambda v-1} |\tau_{i,k}^{(\alpha,\beta)}|^{\lambda},$$

because

$$i^{\lambda\alpha+\lambda\omega-\lambda u+1}\Big(\sum_{m=i}^{2i-1}+\sum_{m=2i}^{M}\Big)\big((m-i+1)^{\gamma-1} m^{\lambda u-1-\alpha\lambda-\omega\lambda}\big) \leqq$$

$$\leqq K \sum_{m=i}^{2i-1} (m-i+1)^{\gamma-1} + K i^{\lambda\alpha+\lambda\omega-\lambda u+1} \sum_{m=2i}^{\infty} (m-i+1)^{\lambda u-2-\alpha\lambda-\omega\lambda+\gamma} \leqq \tag{29}$$

$$\leqq K \sum_{m=1}^{i} m^{\gamma-1} + K i^{\lambda\alpha+\lambda\omega-\lambda u+1} \sum_{m=i+1}^{\infty} m^{\lambda u-2-\alpha\lambda-\omega\lambda+\gamma} \leqq K.$$

In the case of $\lambda=1$ we set $\omega=0$ and the inequality (28) remains valid and we obtain (29) in this case, too, so our proof is complete.

Proof of Theorem 4. Inequalities (15) and (16) follow directly from Flett's result ([3], Theorem 4), by similar arguments to the proof of (6) and (7). In the proof of (17), thinking about Theorem 1, we may assume that $\gamma, \delta<0$. Using (14) we have that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\lambda(\gamma+u)-1} n^{\lambda(\delta+v)-1} |\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|^{\lambda} \leqq K \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\lambda u-1} n^{\lambda v-1} |\tau_{m,n}^{(\alpha,\beta)}|^{\lambda}. \tag{30}$$

Applying Hölder's inequality with indices $\lambda/\mu$ and $\lambda/(\lambda-\mu)$, we obtain that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\mu\xi-1} n^{\mu\eta-1} |\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|^{\mu} =$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \{m^{\mu(\gamma+u-1/\lambda)} n^{\mu(\delta+v-1/\lambda)} |\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|^{\mu}\} \times$$

$$\times \{m^{-1-\mu(\gamma+u-\xi-1/\lambda)} n^{-1-\mu(\delta+v-\eta-1/\lambda)}\} \leqq$$

$$\leqq \Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\lambda(\gamma+u)-1} n^{\lambda(\delta+v)-1} |\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|^{\lambda}\Big\}^{\mu/\lambda} \times$$

$$\times \Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-1-\lambda\mu(\gamma+u-\xi)/(\lambda-\mu)} n^{-1-\lambda\mu(\delta+v-\eta)/(\lambda-u)}\Big\}^{(\lambda-\mu)/\lambda}.$$

The last factor is bounded, because $\gamma+u-\xi$, $\delta+v-\eta$ are positive, and inequality (17) follows from this and (30).

Proof of Theorem 5. We can observe that if the conditions are satisfied, applying Theorem 4, we obtain that the summability $|C, (\alpha, \beta), (u, v)|_{\lambda}$ of series (1) implies the summability $|C, (\gamma, \delta), (0, 0)|_1$. Now Theorem 5 follows from Lemma 4.

**4. Negative results.** First we show that Theorem 2 is the best possible in the following sense:

a) If $\mu>\lambda>1$ and $\min(\gamma, \delta)<1/\lambda-1/\mu$, then for any $\xi, \eta\geqq 0$ the summability $|C, (\alpha, \beta), (u, v)|_{\lambda}$ does not imply the summability $|C, (\alpha+\gamma, \beta+\delta), (\xi, \eta)|_{\mu}$.

Without loss of generality, we can assume that $\gamma<1/\lambda-1/\mu$ and $0\leqq u, v\leqq 1/\lambda$. Applying Lemma 5, let $\sum_{i=0}^{\infty} c_i$ be a single series, such that $\tau_m^{(\alpha)}=m^{1/p}$ if $m=2^{\nu}$ and $\tau_m^{(\alpha)}=0$ otherwise, where $\lambda/(1-u\lambda)<p<\lambda$ and $u>0$. The series $\sum_{i=0}^{\infty} c_i$ is summable $|C, \alpha, u|_{\lambda}$, since

$$\sum_{m=1}^{\infty} m^{\lambda u-1} |\tau_m^{(\alpha)}|^{\lambda} = \sum_{\nu=0}^{\infty} 2^{\nu(u\lambda-1-\lambda/p)} < \infty,$$

but not summable $|C, \alpha+\gamma, 0|_{\mu}$, since

$$\sum_{m=1}^{\infty} m^{-1-\mu\gamma} |\tau_m^{(\alpha)}|^{\mu} = \sum_{\nu=0}^{\infty} 2^{\nu(-1-\mu\gamma+\mu/p)} = \infty,$$

and we may use Corollary 1. Thus the assertion is proved, because it is clear that the summability $|C, (\alpha+\gamma, \beta+\delta), (\xi, \eta)|_{\mu}$ implies the summability $|C, (\alpha+\gamma, \beta+\delta), (0, 0)|_{\mu}$. In the case $u=0$, the assertion was proved by Flett ([2], part 2.7).

b) If $\mu>\lambda=1$ and $\min(\gamma,\delta)\leqq 1-1/\mu$ then for any $\xi,\eta\geqq 0$ the summability $|C,(\alpha,\beta),(u,v)|_1$ does not imply the summability $|C,(\alpha,\beta),(\xi,\eta)|_\mu$.

Without loss of generality, we can assume that $\gamma=1-1/\mu$ and $\xi=0$. If $u>0$, the proof is carried out analogously to the proof of preceding assertion. In the case $u=0$, by using Lemma 5, let $\sum_{i=0}^{\infty} c_i$ be a single series such that $\tau_m^{(\alpha)}=p^{-2}l_p$ if $m=l_p=2^{2^p}$ and $\tau_m^{(\alpha)}=0$ otherwise. It is clear, that

$$\sum_{m=1}^{\infty} m^{-1}|\tau_m^{(\alpha)}| = \sum_{p=1}^{\infty} p^{-2}<\infty,$$

so the series $\sum_{i=0}^{\infty} c_i$ is summable $|C,\alpha,0|_1$. On the other hand, from (18), with the notation $n=l_p+t$, $0\leqq t\leqq l_p$

$$\tau_n^{(\alpha+\gamma)} \geqq \frac{1}{A_n^{(\alpha+\gamma)}} A_{n-l_p}^{(-1/\mu)} A_{l_p}^{(\alpha)} \tau_{l_p}^{(\alpha)} \geqq K(t+1)^{-1/\mu} l_p^{1/\mu} p^{-2}$$

and

$$\sum_{n=1}^{\infty} n^{-1}|\tau_n^{(\alpha+\gamma)}|^\mu \geqq \sum_{p=1}^{\infty}\sum_{n=l_p}^{2l_p} n^{-1}\tau_n^{(\alpha+\gamma)} \geqq K\sum_{p=1}^{\infty}\sum_{t=0}^{l_p}(l_p+t)^{-1}(t+1)^{-1}l_p p^{-2\mu}\geqq$$

$$\geqq K\sum_{p=1}^{\infty} p^{-2\mu}\sum_{t=0}^{l_p}(t+1)^{-1}\geqq K\sum_{p=1}^{\infty}2^p p^{-2\mu}=\infty,$$

and therefore this series is not summable $|C,\alpha+\gamma,0|_\mu$. We remark that this example is due to Flett [4], in connection with strong summability $[C,\alpha]_\lambda$.

Now we investigate the parameters $u$, $v$. The following result shows that the parameters $u$, $v$ cannot be increased. (It is clear that ones can be decreased.)

c) If $\lambda,\mu\geqq 1$, $u,v\geqq 0$, $\alpha>\mu-1$, $\beta>v-1$ and $\xi>u$ or $\eta>v$ then for any $\alpha_1,\beta_1(\alpha_1>\xi-1,\ \beta_1>\eta-1)$ the summability $|C,(\alpha,\beta),(u,v)|_\lambda$ does not imply the summability $|C,(\alpha_1,\beta_1),(\xi,\eta)|_\mu$.

We can assume that $\xi>u$. Applying Lemma 5, with $c_i=i^{-p}$, where $u+1<p<\xi+1$, we obtain that the series $\sum_{i=0}^{\infty} c_i$ is summable $|C,\alpha,u|_\lambda$, since

$$\sum_{m=1}^{\infty} m^{\lambda u-1}|\tau_m^{(\alpha)}|^\lambda \leqq K\sum_{m=1}^{\infty} m^{\lambda(u+1-p)-1}<\infty,$$

but is not summable $|C,\alpha_1,\xi|_\mu$, since

$$\sum_{m=1}^{\infty} m^{\lambda\xi-1}|\tau_m^{(\alpha_1)}|^\mu \geqq K\sum_{m=1}^{\infty} m^{\mu(\xi+1-p)-1}=\infty.$$

Finally we prove that the parameter $\lambda$ cannot be decreased if parameters $u$, $v$ are fixed.

d) If $\lambda>\mu\geqq 1$ and $u, v\geqq 0$, then for any $\alpha, \alpha_1, \beta, \beta_1 (\alpha, \alpha_1>u-1, \beta, \beta_1>v-1)$ the summability $|C, (\alpha, \beta), (u, v)|_\lambda$ does not imply the summability $|C, (\alpha_1, \beta_1), (u, v)|_\mu$.

We apply Lemma 5, with a single series $\sum_{i=0}^{\infty} c_i$ such that $\tau_m^{(\alpha)}=(\log m)^{-1/p} m^{-u}$, $\tau_0^{(\alpha)}=0$, where $\mu<p<\lambda$. Since

$$\sum_{m=1}^{\infty} m^{\lambda u-1}|\tau_m^{(\alpha)}|^\lambda = \sum_{m=1}^{\infty} m^{-1}(\log m)^{-\lambda/p} <\infty,$$

the series $\sum_{i=0}^{\infty} c_i$ is summable $|C, \alpha, u|_\lambda$. On the other hand, using (18), a routine calculation gives that

$$\tau_m^{(\alpha_1)} = \frac{1}{A_m^{(\alpha_1)}} A_{m-i}^{(\alpha_1-\alpha-1)} A_i^{(\alpha)} \tau_i^{(\alpha)} \geqq K m^{-\alpha_1} \sum_{i=0}^{m} (m-i+1)^{\alpha_1-\alpha-1} \times$$

$$\times i^\alpha (\log i)^{-1/p} i^{-u} = \sum_{i=1}^{m/2} + \sum_{i=m/2}^{m} \equiv T_m^{(1)} + T_m^{(2)},$$

and therefore

$$T_m^{(1)} \geqq K m^{-\alpha-1} (\log m)^{-1/p} \sum_{i=1}^{m/2} i^{\alpha-u} \geqq K m^{-u} (\log m)^{-1/p}$$

and

$$T_m^{(2)} \geqq K m^{-\alpha_1+\alpha-u} (\log m)^{-1/p} \sum_{i=1}^{m/2} i^{\alpha_1-\alpha-1} \geqq K m^{-u} (\log m)^{-1/p},$$

furthermore

$$\sum_{m=1}^{\infty} m^{\mu u-1} |\tau_m^{(\alpha_1)}|^\mu \geqq K \sum_{m=1}^{\infty} m^{-1} (\log m)^{-\mu/p} = \infty,$$

so the series $\sum_{i=0}^{\infty} c_i$ is not summable $|C, \alpha_1, u|_\mu$.

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# On an imbedding theorem

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**Introduction.** In 1968, P. L ULJANOV [13] gave a sufficient and necessary condition for the imbedding of Hölder class $H_p^\omega$ into the space $L^q$ $(1 \leqq p < q < \infty)$. The result of Uljanov was generalized later by L. LEINDLER [5], [6]. In this paper we consider an analogous problem for the case of the new modulus $\omega_{\varphi,w}(f,\delta)_p$ introduced by Z. DITZIAN and V. TOTIK [1], namely we give a necessary and sufficient condition for the imbedding of Hölder type class of functions determined by $\omega_{\varphi,w}(f,\delta)_p$ with $w(x)=(1-x)^\alpha(1+x)^\beta$, $\varphi(x)=\sqrt{1-x^2}$ $(\alpha,\beta \geqq 0, x\in(-1,1))$ into another class of functions.

**An imbedding theorem.** Let $1 \leqq p < \infty$. Let $u(x)$ be a nonnegative, integrable function on the finite interval $(a,b)$. Denote by $L_u^p(a,b)$ the Banach space of all measurable functions on $(a,b)$ with the norm

$$\|f\|_{L_u^p(a,b)} = \Big\{\int_a^b |f(x)|^p u(x)\Big\}^{1/p}.$$

In the case $u \equiv 1$ we use the notations $L^p(a,b)$, $\|f\|_{L^p(a,b)}$, respectively.

The modulus of a function $f \in L^p(a,b)$ is defined by the formula

$$\omega(f,\delta)_{L^p(a,b)} = \sup_{0<h\leqq\delta}\Big\{\int_a^{b-h} |f(x+h)-f(x)|^p\,dx\Big\}^{1/p}, \quad (0 \leqq \delta \leqq b-a).$$

Let (we shall use these notations throughout this paper)

$$w(x) = w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta, \quad (\alpha,\beta \geqq 0,\ x\in(-1,1));$$

$$\varphi(x) = \sqrt{1-x^2} \quad (x\in(-1,1)).$$

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The weighted modulus of a function $f$ for which $wf\in L^p(-1, 1)$ was introduced by Z. Ditzian and V. Totik as follows:

$$\omega_{\varphi, w}(f, \delta)_p := \sup_{0<h\le\delta} \|w\Delta_{\varphi(x)h}f(x)\|_{L^p(-1,1)}$$

where

$$\Delta_{\varphi(x)h}f(x) := \begin{cases} f(x+\varphi(x)h)-f(x) & \text{for} \quad x\colon\ x+\varphi(x)h\in(0, 1), \\ 0 & \text{elsewhere.} \end{cases}$$

Let $\omega(\delta)$ be a modulus of continuity, i.e. $\omega(\delta)$ is an onnegative, increasing continuous function on $[0, 1]$, $\omega(0)=0$ and $\omega(\delta_1+\delta_2)\le\omega(\delta_1)+\omega(\delta_2)$ $(0\le\delta_1<\delta_1+\delta_2\le 1)$. Define the Hölder type class

$$H^\omega_{\varphi, w, p} := \left\{f\colon\ wf\in L^p(-1, 1),\ \omega_{\varphi, w}(f, \delta)_p = O_f\{\omega(\delta)\}\ (\delta\to 0)\right\}.$$

We shall prove

Theorem 1. *Let* $1\le p<q<\infty$. *Let* $\omega(\delta)$ *be an arbitrary modulus of continuity. Then*

$$H^\omega_{\varphi, w, p} \subset L^q_{w^{1/q}\varphi^{(q/p)-1}}(-1, 1) \tag{3}$$

*iff*

$$\sum_{n=1}^{\infty} n^{(q/p)-2}\omega^q\left(\frac{1}{n}\right) < \infty. \tag{4}$$

For the proof of Theorem 1 we need some lemmas.

For any function $f(x)$ defined on $(-1, 1)$, let $f^*(\Theta):=f(\cos\Theta)$ $(\Theta\in(0, \pi))$. Let $P_n(\alpha, \beta, x)$ be the $n$-th orthonormal polynomials with respect to the parameters $\alpha, \beta$. Then the system

$$\Phi = \{J_n(\alpha, \beta, \theta)\} := \{P_n^*(\alpha, \beta, \theta)[w^*_{\alpha,\beta}(\theta)\varphi^*(\theta)]^{1/2}\}$$

is orthonormal on $(0, \pi)$. Denote by $\Phi_n$ the set of all $\varphi$-polynomials of degree at most $n$, i.e. the set of all functions of the form $\sum_{k=0}^{n}\lambda_k J_k(\alpha, \beta, \theta)$ ($\lambda_k$ are real numbers, $k=0, \dots, n$).

*Lemma* 1. *For any* $\varphi_n\in\Phi_n$ $(n=1, 2, \dots)$ *and* $1\le p<q<\infty$, *the inequalities*

$$\|\varphi_n'\|^*_{L^p(0,\pi)} \le cn\|\varphi_n\|_{L^p(0,\pi)} \tag{5}$$

*and*

$$\|\varphi_n\|_{L^p(0,\pi)} \le cn^{1/p-1/q}\|\varphi_n\|_{L^p(0,\pi)} \tag{6}$$

*hold.*

Proof. Combining [3, T. 4] with [8, T. 14] we get (5) and (6).

For $wf\in L^p(-1, 1)$ let

$$E_n(w, f)_p = \inf\|w(f-p_n)\|_{L^p(-1,1)}, \quad p_n\in\pi_n. \tag{7}$$

where $\pi_n$ denotes the set of all algebraic polynomials of degree at most $n$ $(n=0, 1, \ldots)$.

We define also the best approximation of a function $g\in L^p(0,\pi)$ by $\Phi$-polynomials:

(8) $$E_n^*(g)_p = \inf \|g-\varphi_n\|_{L^p(0,\pi)}, \quad \varphi_n\in\Phi_n.$$

It is clear that

(9) $$E_n(w,f)_p = E_n^*(g_f)_p,$$

where

(10) $$g_f(\theta) := f^*(\theta)w^*(\theta)\sin^{1/p}\theta.$$

Lemma 2 ([11], T. 3). *Let* $1\leqq p<\infty$. *We have for every* $wf\in L^p(-1,1)$

(11) $$E_n(w,f)_p \leqq c\omega_{\varphi,w}\left(f, \frac{1}{n}\right)_p \quad (n=1, 2, \ldots).$$

Lemma 3. *Let* $1\leqq p<\infty$. *For every* $g\in L^p(0,\pi)$ *the inequality*

(12) $$\omega\left(g, \frac{1}{n}\right)_{L^p(0,\pi)} \leqq cn^{-1}\sum_{k=0}^{n} E_k^*(g)_p$$

*holds.*

Proof. Using inequality (5) we can prove this Lemma by the same way as that of the inverse theorem for the best trigonometric approximation (see e.g. [7]).

By a result of Ditzian and Totik (see [1], T. 2.1.1.) we have that $\omega_{\varphi,w}(f,\delta)_p$ is equivalent to the $K$-functional

$$K_{\varphi,w}(f,\delta)_p := \inf_{h\in D^p_{\varphi,w}} \{\|w(f-h)\|_{L^p(-1,1)} + \delta\|w\varphi h'\|_{L^p(-1,1)}\}$$

where $D^p_{\varphi,w}$ denotes the class of all functions $g$, which are locally absolutely continuous on $(-1,1)$ and for which $wg$, $w\varphi g'\in L^p(-1,1)$.

On the other hand, the other $K$-functional defined on $L^p(0,\pi)$:

$$K^*(g,\delta)_p := \inf_{h\in D_p} \{\|w^*(\varphi^*)^{1/p}(h-g)\|_{L^p(0,\pi)} + \delta\|w^*(\varphi^*)^{1/p}h'\|_{L^p(0,\pi)}\}$$

where $D_p$ denotes the class of all locally absolutely continuous functions $h$ on $(0,\pi)$ for which $(\varphi^*)^{1/p}w^*h\in L^p(0,\pi)$, is equivalent to the following modulus of continuity

(13) $$\Omega_{A,B}(g,\delta)_p := \sup_{0<h\leqq\delta}\left\{\int_0^B |g(\theta+h)-g(\theta)|^p(w^*(\theta))^p\varphi^*(\theta)\,d\theta\right\}^{1/p} +$$

$$+\sup_{0<h\leqq\delta}\left\{\int_A^\pi |g(\theta-h)-g(\theta)|^p(w^*(\theta))^p\varphi^*(\theta)\,d\theta\right\}^{1/p},$$

$$(0<A<B<\pi;\ 0<\delta<\min(A,\pi-B)).$$

This fact was proved essentially in [11], special cases of which were proved in [9] and [10].

Summing the mentioned statements we have

Lemma 4. *Let* $1<p<\infty$, $0<A<B<\pi$. *Let* $wf\in L^p(-1,1)$ *and*

$$g_f(\theta):=f^*(\theta)w^*(\theta)\sin^{1/p}\theta.$$

*Then*

$$\omega_{\varphi,w}(f,\delta)_p\sim\Omega_{A,B}(g,\delta)_p\quad(\delta\to 0). \tag{14}$$

After these, let us turn to the

Proof of Theorem 1. a) (4)⇒(3). Let $wf\in L^p(-1,1)$. From (4) it follows by (11), that

$$\sum_{o=1}^{\infty}n^{(q/p)-2}E_n^q(w,f)_p<\infty$$

and so, we have for the function $g_f$ defined by (10)

$$\sum_{n=1}^{\infty}n^{(q/p)-2}E_n^{*q}(g_f)_p<\infty.$$

Hence, by Hardy inequality and (12) we get

$$\sum_{n=1}^{\infty}\omega^q\left(g_f,\frac{1}{n}\right)_{L^p(0,\pi)}n^{(q/p)-2}<\infty$$

which implies by *T.* 1 of [13] that $g_f\in L^q(0,\pi)$, therefore $f\in L^q_{w^q\varphi^{q/p-1}}(-1,1)$.

b) (3)⇒(4). Suppose, that (4) does not hold. Using the method applied in [13], p. 673 one can contruct a function $\varphi_0\in L^p\left[\frac{1}{4},\frac{5}{4}\right]$ satisfying the following conditions

$$\varphi_0(x)=0,\quad x\in[3/4,5/4]; \tag{15}$$

$$\int_{1/4}^{1/4+h}|\varphi_0(x)|^p\,dx\leqq c\omega^p(h); \tag{16}$$

$$\omega(\varphi_{0,\delta})_{L^p(1/4,5/4)}\leqq c\omega(\delta); \tag{17}$$

$$\varphi_0\notin L^q[1/4,5/4]. \tag{18}$$

Let now

$$g_0(\theta):=\begin{cases}\varphi_0(\theta)w^*(\theta)[\varphi^*(\theta)]^{1/p} & \text{for}\quad \theta\in[1/4,5/4],\\ 0\quad \text{for}\quad \theta\in[0,\pi]\setminus[1/4,5/4].\end{cases}$$

We estimate the modulus (13) with $A=3/2$, $B=2$ of the function $g_0$. By (15), (16) and (17) one can see that

$$\Omega_{3/2,2}(g_0, \delta)_p = O\{\omega(\delta)\} \quad (\delta \to 0).$$

Therefore by (14) we have for the function

$$f_0(x) := g_0(\arc\cos x) w^{-1}(x) \varphi^{-1/p}(x)$$

$$\omega_{\varphi,w}(f_0, \delta)_p = O\{\omega(\delta)\} \quad (\delta \to 0),$$

which means that $f_0 \in H^{\omega}_{\varphi,w,p}$.

On the other hand by (18) it follows that

$$f_0 \notin L^q_{w^q \varphi^{q/p-1}}(-1, 1).$$

Thus, the necessity of (4) is proved.

Remark 1. The part (3)⇒(4) indeed can be obtained immediately from inequality (6) and T. 1 of [12]. Besides, we have appeared the other proof, because by this method we can prove a generalization of Theorem 1, which will be stated in the following.

For a nonnegative monotonic sequence of numbers $\{\varphi_k\}$, the function

$$\Phi(x) = \sum_{k=1}^{x} k^{(\gamma/p)-2} \varphi_k \quad (\gamma, p \geqq 1)$$

was introduced by LEINDLER [6]. We denote by $M^{\gamma,p}_{w,\varphi}$ the class of measurable functions $f$ on $(-1, 1)$, for which

$$\int_0^{\pi} g_f^{q+1-(q/p)}(\theta)\, \Phi(|g_f(\theta)|)\, d\theta < \infty$$

where $g_f$ is defined by (10). Then the following theorem is true.

Theorem 2. *Let* $1 \leqq p \leqq \gamma < \infty$. *Let* $\{\varphi_k\}$ *be a nonnegative monotonic sequence of numbers satisfying* $\varphi_{k^2} \leqq c\varphi_k$ *and in the case* $\gamma > p$, *moreover let*

$$\varphi_k \leqq \varphi_{k+1} \quad (k = 1, 2, \ldots).$$

*Then*

$$H^{\omega}_{w,\varphi,p} \subset M^{\gamma,p}_{w,\varphi} \tag{19}$$

*iff*

$$\sum_{n=1}^{\infty} n^{(\gamma/p)-2} \varphi_n \omega^{\gamma}\left(\frac{1}{n}\right) < \infty. \tag{20}$$

Using Lemmas 1—4 we can prove this theorem by the same method as we used to prove Theorem 1, with the modification that the results of Uljanov applied in the proof of Theorem 1 will be replaced by the generalized results of Leindler (see Theorem 3 and its proof in [6]), while the inequality of Hardy used in the proof will be replaced by a generalized inequality (see [4], inequality (1′)).

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# Relating the normal extension and the regular unitary dilation of a subnormal tuple of contractions

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In this paper we deal with only bounded linear operators on complex infinite dimensional separable Hilbert spaces. If $S=(S_1, \ldots, S_n)$ is a tuple of operators on a Hilbert space $\mathscr{H}$, then for any $n$-tuple $k=(k_1, \ldots, k_n)$ of integers $k_i$, $S^k$ denotes $S_1^{k_1}S_2^{k_2}\ldots S_n^{k_n}$, where $S_i^{k_i}$ is to be interpreted as $S_i^{*(-k_i)}$ if $k_i$ is negative. If a Hilbert space $\mathscr{H}$ is contained in some Hilbert space $\mathscr{K}$, then $P(\mathscr{K}, \mathscr{H})$ will denote the projection of $\mathscr{K}$ onto $\mathscr{H}$. If for a tuple $S=(S_1, \ldots, S_n)$ of $n$ commuting operators on $\mathscr{H}$, there exist a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ and a tuple $M=(M_1, \ldots, M_n)$ of $n$ commuting operators on $\mathscr{K}$ such that $S^kx=P(\mathscr{K}, \mathscr{H})M^kx$ for any $x$ in $\mathscr{H}$ and any $n$-tuple $k$ of non-negative integers, then $S$ on $\mathscr{H}$ is said to *dilate to M on* $\mathscr{K}$; if moreover $\mathscr{H}$ is invariant for each $M_i$, then $S$ *on* $\mathscr{H}$ is said to *extend to M on* $\mathscr{K}$. If $S$ on $\mathscr{H}$ dilates to $M$ on $\mathscr{K}$ and each $M_i$ is unitary, then $M$ *on* $\mathscr{K}$ is said to be a *unitary dilation of S on* $\mathscr{H}$. If $S$ on $\mathscr{H}$ extends to $M$ on $\mathscr{K}$ and each $M_i$ is normal, then $M$ *on* $\mathscr{K}$ is said to be a *normal extension of S on* $\mathscr{H}$, and $S$ is said to be *subnormal.* Among all the normal extensions of a subnormal tuple $S$, there is a *minimal* one which is unique up to unitary equivalence (see [4]). In particular, if $N$ on $\mathscr{K}$ is the minimal normal extension of $S$ on $\mathscr{H}$, then $\mathscr{K}=\vee(N^k\mathscr{H}$: $k$ is a tuple of non-positive integers), where $\vee$ denotes the closed linear span in the norm $\|\cdot\|_{\mathscr{K}}$ of $\mathscr{K}$.

For our purposes, a special type of unitary dilation, known in the literature as *regular unitary dilation* (or *Sz.-Nagy—Brehmer dilation*) (see [3], [7]) is important. For any $n$-tuple $k=(k_1, \ldots, k_n)$ of integers, define $k+=(\max(k_1, 0)), \ldots, \max(k_n, 0))$ and $k-=(\min(k_1, 0), \ldots, \min(k_n, 0))$. If for a tuple $S$ of $n$ commuting operators on $\mathscr{H}$, there exist a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ and a tuple $U$ of $n$ commuting unitaries on $\mathscr{K}$ such that $S^{k-}S^{k+}x=P(\mathscr{K}, \mathscr{H})U^{k-}U^{k+}x$ for any $x$ in $\mathscr{H}$ and any $n$-tuple $k$ of integers, then $U$ *on* $\mathscr{K}$ is said to be a *regular unitary dilation* of $S$ on $\mathscr{H}$; $U$ is *minimal* if $\mathscr{K}=\vee\{U^k\mathscr{H}$: $k$ is an $n$-tuple of integers$\}$.

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In what follows, we will use the symbols $\bar{D}^n$, $T^n$ and $m_n$ to denote the closed unit polydisk in $\mathbf{C}^n$, the unit polycircle in $\mathbf{C}^n$ and the normalized product arc-length measure on $T^n$ respectively. The spectral measure of a normal or unitary tuple $M$ will be denoted by $\mu(M)$. In case $n=1$, it is well known (see [2], [5]) that if $N$ on $\mathcal{K}$ is the minimal normal extension of a contraction $S$ on $\mathcal{H}$, then $\mu(N)|T^1$ is absolutely continuous with respect to $m_1$, provided $S$ is pure; that is, there does not exist a non-trivial closed reducing subspace $\mathcal{H}'$ of $\mathcal{H}$ such that $S|\mathcal{H}'$ is normal. (An examination of the proof in [5] and Theorem 6.4 in Chapter II of [7] actually reveals that "$S|\mathcal{H}'$ is normal" can be replaced by "$S|\mathcal{H}'$ is unitary".) A contranction $S$ on a Hilbert space $\mathcal{H}$ is said to be $C_{0.}$ (see [7]) if $\|S^n h\|_{\mathcal{H}} \to 0$ as $n \to \infty$ for any $h$ in $\mathcal{H}$. It is obvious from Theorem 3.2 in Chapter I of [7] that a $C_{0.}$ contraction does not have a non-trivial unitary part. At this stage, the reader may refer to the statement of Theorem 1 below and the question raised at the end of the paper.

Lemma 1. *If $S$ is a subnormal tuple of contractions on $\mathcal{H}$, then $S$ has a regular unitary dilation.*

Proof. This follows from Theorem 4.1 of [1] and from the observation made in the proof of Corollary to Theorem 3.1 of [1].

Lemma 2. *If $U$ on $\mathcal{K}$ is a minimal regular unitary dilation of $S$ on $\mathcal{H}$ and each $S_i$ is a $C_{0.}$ contraction, then $\|\mu(U)(\cdot)x\|_{\mathcal{K}}^2$ is absolutely continuous with respect to $m_n$ for any $x$ in $\mathcal{H}$.*

Proof. Let $U$ on $\mathcal{K}$ be a minimal regular unitary dilation of $S$ on $\mathcal{H}$. Define operators $D_i$ $(i=0, 1, \ldots, n)$ from $\mathcal{H}$ to $\mathcal{K}$ as follows: $D_0=I$ $(Ix=x$ for any $x$ in $\mathcal{H})$, $D_{i+1}=D_i-U_{i+1}^* DT_{i+1}$, $(i=0, \ldots, n-1)$. Let $A$ be the closed linear span of $D_n\mathcal{H}$ in $\mathcal{K}$. It follows from Theorem 1 of [3] that $U^k A$ and $U^l A$ are orthogonal to each other with respect to the inner product $\langle .,. \rangle$ of $\mathcal{K}$ for any two distinct integer $n$-tuples $k$ and $i$, and

$$\mathcal{K} = \bigvee\{U^m A\colon\ m \text{ is an } n\text{-tuple of integers}\}.$$

Let $\xi=(\xi_1, \ldots, \xi_n)$ denote a generic point of $T^n$. For any $a$ in $A$ and any $n$-tuple $k$ of integers, we have

$$\int_{T^n} \xi_1^{k_1}\xi_2^{k_2}\ldots\xi_n^{k_n}\, d\|\mu(U)(\xi)a\|_{\mathcal{K}}^2 = \langle U^k a, a\rangle =$$

$$= \begin{cases} \|a\|_{\mathcal{K}}^2 & \text{if} \quad k_i = 0, \quad \text{for each } i, \\ 0, & \text{otherwise.} \end{cases}$$

Since the trigonometric polynomials are dense in $C(T^n)$, the space of continuous functions with the supremum norm, it follows that

$$\|\mu(U)(.)a\|_{\mathcal{K}}^2 = \|a\|_{\mathcal{K}}^2 m_n(.).$$

From our observations above and utilizing the fact that $\mu(U)$ commutes with all $U^m$, it is easy to deduce that $\|\mu(U)(.)x\|_{\mathcal{K}}^2$ is absolutely continuous with respect to $m_n$ for any $x$ in $\mathcal{K}$.

Theorem 1. *Let $S$ be a subnormal tuple of $C_{0.}$ contractions on $\mathcal{H}$. If $N$ on $\mathcal{K}$ is the minimal normal extension of $S$, then $\mu(N)/T^n$ is absolutely continuous with respect to $m_n$.*

Proof. Let $S$ on $\mathcal{H}$ be a subnormal tuple of $C_{0.}$ contractions and $N$ on $\mathcal{K}$ be its minimal normal extension. By Lemma 1, $S$ has a regular unitary dilation $U$ on some Hilbert space $\mathcal{K}'$. Define

$$\mathcal{L} = \vee\{U^k\mathcal{H}\colon k \text{ is an } n\text{-tuple of integers}\},$$

$\vee$ denoting the closed linear span in the norm of $\mathcal{K}'$, and let $W_i = U_i/\mathcal{L}$ $(i=1, \ldots, n)$. Then $W$ on $\mathcal{L}$ is a minimal regular unitary dilation of $S$ on $\mathcal{H}$.

Now for any $h$ in $\mathcal{H}$ and any $n$-variable complex polynomial $q$, we have

$$\|q(N)h\|_{\mathcal{K}}^2 = \int_{D^n} |q(y)|^2 \, d\|\mu(N)(y)h\|_{\mathcal{K}}^2$$

and

$$\|q(W)h\|_{\mathcal{L}}^2 = \int_{T^n} |q(\xi)|^2 \, d\|\mu(W)(\xi)h\|_{\mathcal{L}}^2.$$

Since

$$\|q(N)h\|_{\mathcal{K}}^2 = \|q(S)h\|_{\mathcal{H}}^2 = \|P(\mathcal{L}, \mathcal{H})q(W)h\|_{\mathcal{H}}^2 \leqq \|q(W)h\|_{\mathcal{L}}^2,$$

it follows in particular that

$$(1)\qquad \int_{T^n} |q(\xi)|^2 \, d\|\mu(N)(\xi)h\|_{\mathcal{K}}^2 \leqq \int_{T^n} |q(\xi)|^2 \, d\|\mu(W)(\xi)h\|_{\mathcal{L}}^2.$$

It is known that the unit polydisk algebra, as restricted to $T^n$, is an approximating in modulus algebra (see [6]); that is, any positive continuous function on $T^n$ can be approximated uniformaly on $T^n$ by the modulii of polynomials. It follows from (1) that if $f$ is any positive continuous function on $T^n$, then

$$(2)\qquad \int_{T^n} f(\xi) \, d\|\mu(N)(\xi)h\|_{\mathcal{K}}^2 \leqq \int_{T^n} f(\xi) \, d\|\mu(W)(\xi)h\|_{\mathcal{L}}^2.$$

It is clear from (2) that $\|(\mu(N)|T^n)(.)h\|_{\mathcal{K}}^2$ is absolutely continuous with respect to $\|\mu(W)(.)h\|_{\mathcal{L}}^2$ for any $h$ in $\mathcal{H}$. Next appeal to Lemma 2 to deduce that $\|(\mu(N)|T^n)(.)h\|_{\mathcal{K}}^2$ is absolutely continuous with respect to $m_n$ for any $h$ in $\mathcal{H}$. The desired conclusion now follows by using the minimality of $N$.

Question. If $S$ is a subnormal tuple of contractions on $\mathscr{H}$ and if there is no non-trivial closed subspace $\mathscr{H}'$ of $\mathscr{H}$ which is reducing for each $S_i$ and on which each $S_i$ is unitary, is it true that $\mu(N)|T^n$ is absolutely continuous with respect to $m_n$, where $N$ is the minimal normal extension of $S$?

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# Linear operators with a normal factorization through Hilbert space

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**Introduction.** Let $(\Omega, \mu)$ be a $\sigma$-finite measure space, and suppose that $K(x, t)$ is a kernel on $\Omega\times\Omega$ which is selfadjoint, that is, $K(x, t)=\overline{K(t, x)}$ a.e. on $\Omega\times\Omega$. Let $X$ be some Banach space of functions on $\Omega$, and assume that the integral operator

$$S(f)(x) = \int_{\Omega} K(x, t) f(t)\, d\mu(t) \quad (f\in X)$$

is a bounded linear operator on $X$. When $X=L^2(\Omega)$ and the kernel $|K|$ determines a bounded linear operator on $X$, then $S$ is a selfadjoint operator on $X$. However, in general, the operator $S$ may not have properties analogous to those of a selfadjoint operator. The purpose of this paper is to study a large class of operators which in many respects do behave like selfadjoint (or normal) operators. One motivation here is to find conditions under which selfadjoint kernels determine operators which have many of the properties of selfadjoint operators. This question is addressed implicitly in the context of the examples considered in Section 3.

There is a long history of interest in operators on a Banach space that have some properties in common with selfadjoint operators. Examples include symmetrizable operators [10], [11], the quasi-hermitian operators studied by J. DIENDONNÉ [7], and hermitian operators on Banach spaces [5], [6, Part 3]. The class of operators we study has some overlap with these classes. We consider linear operators that have a selfadjoint (or normal) factorization through a Hilbert space in the following sense.

Definition 0.1. An operator $S\in\mathscr{B}(X)$ has a *selfadjoint (normal) factorization through a Hilbert space $H$,* if there exist bounded linear maps $A$ and $T$,

$$T\colon X\to H, \quad A\colon H\to X,$$

with $S=AT$ and $TA$ selfadjoint (normal) on $H$.

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When $S=AT$ is a factorization of $S$ with $TA$ normal, then many properties of $S$ and $TA$ are closely linked. In particular, the spectral theory of the two operators is very much the same. For example, using the operational calculus of the normal operator $TA$, a rich operational calculus may be defined for $S$. This is done in Section 2. There is a large collection of examples in Section 3 which makes it clear that the theory has broad application.

Now we establish some notation. Throughout $X$ is a Banach space and $H$ is a Hilbert space. The algebra of all bounded linear operators on $X$ is denoted $\mathscr{B}(X)$. For $S\in\mathscr{B}(X)$, let $\sigma(S)$ be the spectrum of $S$. If $T$ is a linear map, then let $\mathfrak{N}(T)$ be the null space of $T$, and let $\mathscr{R}(T)$ be the range of $T$.

**1. Some preliminaries.** In this section we derive some preliminary results concerning factorizations. *We assume throughout that $S\in\mathscr{B}(X)$ has a factorization $S=AT$ where $T\colon X\to H$, $A\colon H\to X$ and $TA$ is normal on $H$.*

Definition 1.1. Let $E_0$ be the selfadjoint projection in $\mathscr{B}(H)$ with range $\mathfrak{N}(TA)$. Set $N=AE_0T$. Then $N$ is called *the nilpotent part of $S$*. Note that $NA==AE_0TA=0$, and $SN=NS=0$.

Proposition 1.2. *Let $E_0$ and $N$ be as above and set $\tilde{S}=S-N$. Then $\tilde{S}$ has a normal factorization $\tilde{S}=\tilde{A}\tilde{T}$ through a Hilbert space $\tilde{H}$ with the property that $\mathfrak{N}(\tilde{T}\tilde{A})=\{0\}$.*

Proof. Set $\tilde{H}=(I-E_0)H$, and define $\tilde{T}\colon X\to\tilde{H}$ by $\tilde{T}(x)=(I-E_0)Tx$ and $\tilde{A}\colon \tilde{H}\to X$ by $\tilde{A}(y)=Ay$. For $x\in X$, $\tilde{A}\tilde{T}(x)=A(I-E_0)Tx=ATx-AE_0Tx=Sx--Nx=\tilde{S}(x)$. For $y\in\tilde{H}$, $\tilde{T}\tilde{A}y=(I-E_0)TAy=TAy$. Since $TA$ restricted to $(I-E_0)H$ is normal, we have $\tilde{T}\tilde{A}$ is normal on $\tilde{H}$.

Next we verify that $\mathfrak{N}(\tilde{T}\tilde{A})=\{0\}$. Assume $y\in\tilde{H}$ and $\tilde{T}\tilde{A}y=0$. From the previous computation, we have $TAy=\tilde{T}\tilde{A}y=0$. Then by definition, $E_0y=y$, so $y=(I-E_0)y=0$.

Let $\tilde{S}=S-N$ as in Proposition 1.2. Then the spectral theory of $\tilde{S}$ is essentially the same as that of $S$. Now by Proposition 1.2 $\tilde{S}$ has a normal factorization with the property that $\mathfrak{N}(\tilde{T}\tilde{A})=\{0\}$. This means that from the point of view of spectral theory, we may make the following assumption without loss of generality.

$$\text{(A1)} \qquad \mathfrak{N}(TA) = \{0\}.$$

An operator $R\in\mathscr{B}(X)$ is similar to a normal operator $W\in\mathscr{B}(H)$ if $\exists\, U\colon X\to H$ such that $U$ is a bicontinuous linear isomorphism of $X$ onto $H$ with $R=U^{-1}WU$. In this situation $X$ is a Hilbert space in an equivalent renorming, and the spectral theory of $R$ is completely determined by that of the normal operator $W$.

Proposition 1.3. *If $TA$ is invertible and $\mathfrak{N}(S)=\{0\}$ then $S$ is invertible. When* (A1) *holds and $S$ is invertible, then $TA$ is invertible. Furthermore, in this case $S$ is similar to the normal operator $TA$.*

Proof. Assume (A1) holds and $S$ is invertible. The $\mathscr{R}(A)=X$ and $\mathfrak{N}(T)=\{0\}$. Also, since $\mathfrak{N}(TA)=\{0\}$, $\mathfrak{N}(A)=\{0\}$. We verify that $\mathscr{R}(T)=H$. For suppose $y\in H$. We have $A(\mathscr{R}(T))=X$, so $\exists z\in X$ with $ATz=Ay$. Then $A(Tz-y)=0$, so $Tz=y$. This proves that both $A$ and $T$ are one-to-one and onto maps. Thus, $TA$ is invertible with $(TA)^{-1}=A^{-1}T^{-1}$. Also, in this case, setting $U=T$, $S=AT=U^{-1}(TA)U$.

The proof that when $TA$ is invertible and $\mathfrak{N}(S)=\{0\}$, then $S$ is invertible, is similar to the proof above.

Suppose $S=AT$ with $TA$ invertible, but $\mathfrak{N}(S)\neq\{0\}$. We show that in this case $S$ is the direct sum of the zero operator and an operator which is similar to a normal operator. Let $R=(TA)^{-1}$, and let $P=ART$. Elementary computations show that $P^2=P$ and $SP=PS=S$. It follows that $S(I-P)(X)=\{0\}$. Also, if $Sx=ATx=0$, then since $\mathfrak{N}(A)=\{0\}$, $Tx=0$, and thus $Px=0$. This implies that $\mathfrak{N}(S)=(I-P)X$. Therefore $X=P(X)\oplus\mathfrak{N}(S)$, and $S=SP\oplus 0$. Define $U$: $P(X)\to H$ by $UPx=TPx=Tx$. Since $T(X)=H$, $U$ is onto, and when $Px\in\mathfrak{N}(U)$, then $TPx=0$, so $SPx=0$, and finally, $Px=0$. Therefore $U$ has a bounded inverse. An easy computation shows $SP=U^{-1}TAU$ on $P(X)$. Therefore $S$ is the direct sum of an operator similar to a normal operator ($SP$ on $P(X)$ and 0 on $(I-P)(X)$). In this case the spectral theory of $S$ is easily derived from that of $TA$. Thus, in studying the spectral theory of $S$, we can make the following assumption without loss:

(A2) $TA$ is not invertible.

Note that when (A1) and (A2) hold then Proposition 1.3 implies that $S$ is not invertible.

**2. Spectral theory.** *Throughout this section it is assumed that $S$ has a normal factorization, $S=AT$ with $TA$ normal.* Most of the properties of normal operators used in this paper can be found in M. Schechter's book [13].

Theorem 2.1.

(1) $\sigma(S)\cup\{0\}=\sigma(TA)\cup\{0\}$. *When* (A1) *and* (A2) *hold, then* $0\in\sigma(S)=\sigma(TA)$.

(2) *If $\lambda\neq 0$ with $\lambda\notin\sigma(TA)$, then*

$$(\lambda-S)^{-1}=\lambda^{-1}+\lambda^{-1}A(\lambda-TA)^{-1}T.$$

Proof. Assume $\lambda\notin\sigma(TA)$, $\lambda\neq 0$. The formula in (2) is verified by direct computation:

$$(\lambda-AT)\{\lambda^{-1}+\lambda^{-1}A(\lambda-TA)^{-1}T\}=$$
$$=I+A(\lambda-TA)^{-1}T-\lambda^{-1}AT-\lambda^{-1}ATA(\lambda-TA)^{-1}T.$$

Now the last term

$$-\lambda^{-1}ATA(\lambda-TA)^{-1}T = \lambda^{-1}A(\lambda-TA)(\lambda-TA)^{-1}T - A(\lambda-TA)^{-1}T,$$

and substituting the expression on the right for this term yields the result. This proves (2).

To prove (1) note that the same computation that establishes (2) shows that when $\lambda\neq 0$ is in the resolvent of $S$, then

$$(\lambda-TA)^{-1} = \lambda^{-1}+\lambda^{-1}T(\lambda-S)^{-1}A.$$

Now (1) follows from (2) and the remark following the statement of (A2).

Corollary 2.2. *Assume $S\in\mathscr{B}(X)$ has a normal factorization through Hilbert space. Then $\exists M>0$ such that when $\lambda\notin\sigma(S)$, $\lambda\neq 0$,*

$$\|(\lambda-S)^{-1}\| \leqq M|\lambda|^{-1}(1+d(\lambda)^{-1})$$

*where* $d(\lambda)=\inf\{|\lambda-\mu|:\ \mu\in\sigma(S)\}$.

Assume $\Delta$ is a compact subset of **C**. Let $\mathrm{BM}(\Delta)$ be the algebra of all bounded Borel measurable functions on $\Delta$. Define $\mathfrak{M}(\Delta)$ to be the set of all $f\in\mathrm{BM}(\Delta)$ such that $\exists g\in\mathrm{BM}(\Delta)$ with $f(\lambda)=\lambda g(\lambda)$ for all $\lambda\in\Delta$. Now assume that (A1) and (A2) hold. Set $\Delta=\sigma(S)=\sigma(TA)$. Using the fact that the normal operator $TA$ has an operational calculus $g\to g(TA)$ for all $g\in\mathrm{BM}(\Delta)$, we construct an operational calculus $f\to f(S)$ for functions $f\in\mathfrak{M}(\Delta)$.

Definition 2.3. For $f\in\mathfrak{M}(\Delta)$ with $f(\lambda)=\lambda g(\lambda)$ for all $\lambda\in\Delta$, and where $g\in\mathrm{BM}(\Delta)$, define

$$f(S) = Ag(TA)T.$$

By assumption (A2), $0\in\Delta$. This means that $g(0)$ is not uniquely determined by the requirement $f(\lambda)=\lambda g(\lambda)$ on $\Delta$. Nevertheless, $f(S)$ is well-defined. To check this it suffices to show that when $e(\lambda)=0$, $\lambda\in\Delta\setminus\{0\}$, and $e(0)=1$, then $e(TA)=0$. Since $\lambda e(\lambda)=0$ for all $\lambda\in\Delta$, we have $TAe(TA)=0$. Then by (A1), $e(TA)=0$.

Theorem 2.4. *Assume* (A1) *and* (A2) *hold. Let* $\Delta=\sigma(S)=\sigma(TA)$.

(1) *The operational calculus $f\to f(S)$ is an algebra homomorphism of $\mathfrak{M}(\Delta)$ into $\mathscr{B}(X)$.*

(2) [The Spectral Mapping Theorem.] *For* $f\in\mathfrak{M}(\Delta)$

$$\sigma(f(S)) = \sigma(f(TA)).$$

*In particular, if $f\in\mathfrak{M}(\Delta)$ and $f$ is continuous on $\Delta$, then*

$$\sigma(f(S)) = \{f(\lambda):\ \lambda\in\Delta\}.$$

(3) *Assume* $\{f_n\}$ *is a sequence in* $\mathfrak{M}(\Delta)$ *with* $f_n(\lambda)=\lambda g_n(\lambda)$, $\{g_n\}\subseteq \mathrm{BM}(\Delta)$ *and* $g_n\to g$ *uniformly on* $\Delta$. *Then* $f_n(S)\to f(S)$ *in* $\mathscr{B}(X)$.

(4) *Assume either* $\mathfrak{N}(S)=\{0\}$ *or* $\mathscr{R}(S)$ *is dense in X. If* $P\in\mathscr{B}(X)$ *with* $PS=SP$, *then for every* $f\in\mathfrak{M}(\Delta)$, $Pf(S)=f(S)P$.

Proof. Part (1) follows from the fact that $g\to g(TA)$ is an algebra homomorphism of $\mathrm{BM}(\Delta)$ into $\mathscr{B}(H)$. We check the property that when $f_1$ and $f_2$ are in $\mathfrak{M}(\Delta)$, then $f_1f_2(S)=f_1(S)f_2(S)$. Write $f_k(\lambda)=\lambda g_k(\lambda)$ on $\Delta$ for $k=1,2$. Then $f_1f_2(\lambda)=\lambda g(\lambda)$ on $\Delta$ where $g(\lambda)=g_1(\lambda)\lambda g_2(\lambda)$. Therefore $f_1f_2(S)=Ag(TA)T=Ag_1(TA)TAg_2(TA)T=f_1(S)f_2(S)$.

To prove (2), note that $f(S)$ factors through $H$ where the factors are $T\colon X\to H$ and $Ag(TA)\colon H\to X$. We have $f(S)=\big(Ag(TA)\big)T$ and $T\big(Ag(TA)\big)=f(TA)$. Therefore Theorem 2.1 implies that the nonzero spectrum of $f(S)$ and $f(TA)$ is the same. But also, by (A2) $TA$ is not invertible, so $f(TA)=TAg(TA)$ is not invertible. By Proposition 1.3 it follows that $f(S)$ is not invertible. This proves $0\in\sigma\big(f(TA)\big)$ and $0\in\sigma\big(f(S)\big)$.

The proof of (3) is elementary. Assuming the hypothesis in (3), it follows $g_n(TA)\to g(TA)$ in $\mathscr{B}(H)$. Therefore $f_n(S)=Ag_n(TA)T\to Ag(TA)T=f(S)$ in $\mathscr{B}(X)$.

Now assume $P\in\mathscr{B}(X)$ and $P(AT)=(AT)P$. Then $(TPA)(TA)=(TA)(TPA)$. Assume $f\in\mathfrak{M}(\Delta)$ with $f(\lambda)=\lambda g(\lambda)$ on $\Delta$. Then

$$(TPA)g(TA) = g(TA)(TPA). \tag{1}$$

Applying the operator $T$ on the right to the equality in (1), we have

$$TPAg(TA)T = g(TA)TPAT = g(TA)TATP = TAg(TA)TP.$$

When $\mathfrak{N}(S)=\{0\}$, then $\mathfrak{N}(T)=\{0\}$. Thus,

$$P\big(Ag(TA)T\big) = \big(Ag(TA)T\big)P$$

which proves (4) in this case. When $\mathscr{R}(S)$ is dense, apply $A$ on the left in equality (1), make a computation analogous to the one above, and use the fact that $\mathscr{R}(A)$ must be dense to arrive at the conclusion.

Corollary 2.5 *Assume that* $S\in\mathscr{B}(X)$ *has a selfadjoint factorization through Hilbert space and* (A1) *and* (A2) *hold. Then* $\exists M>0$ *such that for all* $t\in\mathbf{R}$

$$\|e^{itS}\| \leqq M|t|.$$

*Therefore, if* $f(t)$ *and* $tf(t)$ *are in* $L^1(\mathbf{R})$, *then* $\int_{-\infty}^{+\infty} f(t)e^{itS}\,dt$ *converges in* $\mathscr{B}(X)$.

Proof. Assume $S=AT$ with $TA$ selfadjoint. $\exists J>0$ such that $|w^{-1}(e^{iw}-1)|\leqq J$ for all $w\in\mathbf{R}$, $w\neq 0$. For $\lambda\in\mathbf{R}$, $\lambda\neq 0$, let $g(\lambda)=\lambda^{-1}(e^{i\lambda t}-1)$. Then $|g(\lambda)|\leqq J|t|$

on **R**. Thus

$$\|e^{itS}\| = \|Ag(TA)T\| \leqq \|A\| \|T\| J|t|.$$

Corollary 2.5 shows that when $S$ has a selfadjoint factorization through Hilbert space, then $S$ is in the class of operators studied in [2].

Corollary 2.6. *Assume* (A1) *and* (A2) *hold. Assume* $f \in \mathfrak{M}(\Delta)$ *with* $f(\lambda) = \lambda g(\lambda)$ $\lambda \in \Delta$, $g \in \mathrm{BM}(\Delta)$, *where in addition* $\lim_{\lambda \to 0} g(\lambda) = g(0) = 0$. *Then* $\exists \{f_n\}$ *a sequence of simple functions in* $\mathfrak{M}(\Delta)$ *with* $f_n(S) \to f(S)$ *in* $\mathscr{B}(X)$. *In particular such a sequence exists for* $f(S) = S^2$.

Proof. Let

$$\varepsilon_n = \sup \{|g(\lambda)| \colon \lambda \in \Delta, |\lambda| < n^{-1}\}.$$

Then by hypotheis, $\varepsilon_n \to 0$. Choose $\{t_n\}$ a sequence of simple functions such that for each $n \geqq 1$,

$$|f(\omega) - t_n(\omega)| \leqq n^{-2} \quad (\omega \in \Delta).$$

Thus,

$$\left|(f(\lambda)/\lambda) - (t_n(\lambda)/\lambda)\right| \leqq n^{-1}$$

whenever $\lambda \in \Delta$ and $|\lambda| \geqq n^{-1}$. Let $\chi_n$ be the characteristic function of the $\{\lambda \in \Delta \colon |\lambda| \geqq n^{-1}\}$. Define $f_n$ to be the simple function $f_n = \chi_n t_n$, $n \geqq 1$. Then

$$\left|g(\lambda) - (f_n(\lambda)/\lambda)\right| \leqq n^{-1} + \varepsilon_n$$

for all $\lambda \in \Delta$. Therefore $(f_n(\lambda)/\lambda) \to g(\lambda)$ uniformly on $\Delta$, so $f_n(S) \to f(S)$ in $\mathscr{B}(X)$ by Theorem 2.4 (3).

Assume $S = AT$ with $TA$ normal, and assume $0 \in \Delta = \sigma(TA)$. Let $U$ be an open set with $\Delta \subseteqq U$ and suppose $f$ is holomorphic on $U$ with $f(0) = 0$. Then $g(\lambda) = f(\lambda)/\lambda$ is holomorphic on $U$ $(g(0) = f'(0))$, thus $f \in \mathfrak{M}(\Delta)$. Let $f(S)$ be the operator in $\mathscr{B}(X)$ defined by the operational calculus constructed above. Now $f(S)$ has another meaning defined in terms of the usual holomorphic operational calculus. In fact, in this case the two possible meanings of $f(S)$ are the same. For let $\gamma$ be an appropriate curve in $U$ surrounding $\Delta$. Then using Theorem 2.1 we have

$$\frac{1}{2\pi i} \int_\gamma f(\lambda)(\lambda - S)^{-1}\, d\lambda = \frac{1}{2\pi i} \int_\gamma f(\lambda)[\lambda^{-1} + \lambda^{-1} A(\lambda - TA)^{-1} T]\, d\lambda =$$

$$= A\left\{\frac{1}{2\pi i} \int_\gamma g(\lambda)(\lambda - TA)^{-1}\, d\lambda\right\} T = Ag(TA)T = f(AT).$$

Here we have used the fact that the operational calculus determined by functions in $\mathrm{BM}(\Delta)$ and the holomorphic operational calculus coincide for normal operators.

As a consequence of the coincidence of the two operational calculi, it follows that when $\Gamma$ is an open and closed subset of $\Delta$ with $0\notin\Gamma$, then the spectral idempotent $P_\Gamma$ determined by the usual holomorphic operational calculus satisfies $P_\Gamma=\chi_\Gamma(S)$ where $\chi_\Gamma$ denotes the characteristic function of $\Gamma$.

Next we turn to some results concerning eigenvalues and eigenspaces.

Proposition 2.7. *If* $\lambda_0\in\mathbf{C}$, $\lambda_0\neq 0$, *then*

$$\mathfrak{N}((\lambda_0-S)^n)=\mathfrak{N}(\lambda_0-S)=A\{\mathfrak{N}(\lambda_0-TA)\}.$$

Proof. If $TAx=\lambda_0x$, then $ATAx=\lambda_0Ax$. Thus $A\{\mathfrak{N}(\lambda_0-TA)\}\subseteq\mathfrak{N}(\lambda_0-S)$. Conversely, if $ATy=\lambda_0y$, then $TATy=\lambda_0Ty$, so $Ty\in\mathfrak{N}(\lambda_0-TA)$. Also $y=A(\lambda_0^{-1}Ty)\in A\{\mathfrak{N}(\lambda_0-TA)\}$. This proves

$$\mathfrak{N}(\lambda_0-S)=A\{\mathfrak{N}(\lambda_0-TA)\}. \tag{1}$$

To show $\mathfrak{N}((\lambda_0-S)^n)=\mathfrak{N}(\lambda_0-S)$, it suffices to prove this for $n=2$. Suppose $x\in\mathfrak{N}((\lambda_0-S)^2)$, so $(\lambda_0-S)x\in\mathfrak{N}(\lambda_0-S)$. By (1), $\exists y\in\mathfrak{N}(\lambda_0-TA)$ with $(\lambda_0-S)x=Ay$. Then $(\lambda_0-TA)Tx=T(\lambda_0-AT)x=TAy=\lambda_0y$. Therefore $(\lambda_0-TA)^2Tx=\lambda_0(\lambda_0-TA)y=0$. Since $TA$ is normal, this implies $0=(\lambda_0-TA)Tx=T(\lambda_0-AT)x$. Then as $(\lambda_0-S)x\in\mathfrak{N}(\lambda_0-S)$, we have $0=AT(\lambda_0-AT)x=\lambda_0(\lambda_0-AT)x$. Thus, $(\lambda_0-AT)x=0$.

Proposition 2.8. *Assume* (A1) *and* (A2) *hold.*

(1) *If* $\lambda_0\neq 0$ *is an isolated point of* $\sigma(S)$, *then* $\lambda_0$ *is an eigenvalue of* $S$.

(2) *Assume* $\lambda_0\neq 0$ *is an eigenvalue of* $S$. *Let* $X_0$ *be the corresponding eigenspace. Let* $\chi_0$ *be the characteristic function of* $\{\lambda_0\}$, *so* $\chi_0\in\mathfrak{M}(\Delta)$. *Then* $P_0=\chi_0(S)$ *is a projection with* $\mathcal{R}(P_0)=X_0$ *and* $\mathcal{R}(\lambda_0-S)\subseteq\mathfrak{N}(P_0)$.

Proof. Assume $\lambda_0\neq 0$ is an isolated point of $\sigma(S)$. Then $\lambda_0$ is an isolated point of $\sigma(TA)$, and since $TA$ is normal, it follows that $\lambda_0$ is an eigenvalue of $TA$. By Proposition 2.7 $\lambda_0$ is an eigenvalue of $S$.

Now assume $\lambda_0\neq 0$ is an eigenvalue of $S$. Let $X_0$, $\chi_0$, and $P_0$ be as in (2). By Proposition 2.7 $\lambda_0$ is an eigenvalue of $TA$. Let $H_0$ be the corresponding eigenspace. Since $TA$ is normal, $Q_0=\chi_0(TA)$ is the orthogonal projection with $\mathcal{R}(Q_0)=H_0$. Now $\lambda\chi_0(\lambda)=\lambda_0\chi_0(\lambda)$ on $\Delta$, so $P_0(\lambda_0-S)=(\lambda_0-S)P_0=0$. This proves $\mathcal{R}(P_0)\subseteq X_0$ and $\mathcal{R}(\lambda_0-S)\subseteq\mathfrak{N}(P_0)$. By Proposition 2.7 $AH_0=X_0$. We prove that $AH_0\subseteq\mathcal{R}(P_0)$ to complete the proof of (2). Set $g(\lambda)=\lambda^{-1}\chi_0(\lambda)$, $\lambda\in\Delta$. Then $P_0=Ag(TA)T$, and for $x\in H_0$, $P_0Ax=Ag(TA)TAx=AQ_0x=Ax$.

A number $\lambda\in\mathbf{C}$ is a Fredholm point of $T\in B(X)$ if $\lambda-T$ is a Fredholm operator. Let $\pi_{00}(T)$ denote the set of eigenvalues of $T$ of finite multiplicity. When $T$ is a normal operator on Hilbert space, then a Fredholm point of $T$ with $\lambda\in\sigma(T)$ is an isolated point of $\sigma(T)$ and $\lambda\in\pi_{00}(T)$.

In the next theorem we prove some results concerning eigenvectors and Fredholm points of $S$.

Theorem 2.9.

(1) *Assume that* $\mathscr{R}(S)$ *is dense in* $X$. *For* $\lambda_0 \in \sigma(S)$, $\lambda_0 \neq 0$, *either* $\lambda_0$ *is an eigenvalue of* $S$, *or* $\mathscr{R}(\lambda_0 - S)$ *is dense in* $X$. *When* (A1) *and* (A2) *hold*, $\sigma(S)$ *is the union of the point spectrum and continuous spectrum of* $S$.

(2) *Assume* $\lambda \in \mathbf{C}$, $\lambda \neq 0$. *Then* $\lambda$ *is a Fredholm point of* $S$ *if and only if* $\lambda$ *is a Fredholm point of* $TA$.

(3) *When* $\lambda \in \sigma(S)$, $\lambda \neq 0$, *and* $\lambda$ *is a Fredholm point of* $S$, *then* $\lambda$ *is an isolated point of* $\sigma(S)$ *and* $\lambda \in \pi_{00}(S)$.

(4) *If* $\lambda \in \pi_{00}(S)$, $\lambda \neq 0$, *then* $\lambda$ *is a Fredholm point of* $S$ *and* $(\lambda - S)$ *has index zero.*

Proof. First we prove (1). Assume $\lambda_0 \neq 0$ and $\mathfrak{N}(\lambda_0 - S) = \{0\}$. Then by Proposition 2.7 $\mathfrak{N}(\lambda_0 - TA) = \{0\}$, and since $TA$ is normal, we have $(\lambda_0 - TA)H$ is dense in $H$. Since $\mathscr{R}(S)$ is dense in $X$, it follows that $\mathscr{R}(A)$ is dense in $X$. Therefore $A(\lambda_0 - TA)H = (\lambda_0 - AT)AH$ is dense in $X$. Thus, $\mathscr{R}(\lambda_0 - S)$ is dense in $X$.

Now we prove (2). Assume $\lambda \neq 0$ is a Fredholm point of $TA$. Then $\exists R \in \mathscr{B}(H)$ and $\exists F, G \in \mathscr{B}(H)$ with $\mathscr{R}(F)$ and $\mathscr{R}(G)$ finite dimensional so that

$$(\lambda - TA)R = I - F \quad \text{and} \quad R(\lambda - TA) = I - G.$$

Then

$$(\lambda - AT)(\lambda^{-1} + \lambda^{-1}ART) = I + ART - \lambda^{-1}AT - \lambda^{-1}ATART =$$

$$= I - \lambda^{-1}AT + \lambda^{-1}A(\lambda - TA)RT = I - \lambda^{-1}AT + \lambda^{-1}A(I - F)T = I - \lambda^{-1}AFT.$$

Similarly,

$$(\lambda^{-1} + \lambda^{-1}ART)(\lambda - AT) = I - \lambda^{-1}AGT.$$

Therefore $\lambda$ is a Fredholm point of $S = AT$. The converse is proved in exactly the same way.

Now assume as in (3) that $\lambda \in \sigma(S)$, $\lambda \neq 0$, and $\lambda$ is a Fredholm point of $S$. By (2), $\lambda$ is a Fredholm point of $TA$. Since $TA$ is normal, this implies that $\lambda$ is an isolated point of $\sigma(TA)$ and $\lambda \in \pi_{00}(TA)$. Then by Theorem 2.1 $\lambda$ is an isolated point of $\sigma(S)$. Also, by Proposition 2.7 $\mathfrak{N}(\lambda - S) = A\mathfrak{N}(\lambda - TA)$. Since $A$ is one-to-one on $\mathfrak{N}(\lambda - TA)$, $\mathfrak{N}(\lambda - S)$ has finite dimension. Therefore $\lambda \in \pi_{00}(S)$.

Assume $\lambda \in \pi_{00}(S)$, $\lambda \neq 0$. Then just as above, $\lambda \in \pi_{00}(TA)$. Since $TA$ is normal $\lambda$ is a Fredholm point of $TA$ and an isolated point of $\sigma(TA)$. Thus, by part (2), $\lambda$ is a Fredholm point of $S$ and an isolated point of $\sigma(S)$. It follows that $\lambda - S$ has index zero [13, VI, Theorem 4.5]. This proves (4).

For an operator $T \in B(X)$, let

$$W(T) = \{\lambda \in \mathbf{C}\colon \lambda - T \text{ is not a Fredholm operator with index zero}\}.$$

The set $W(T)$ is called the Weyl spectrum of $T$. When $T$ is a normal operator on Hilbert space,

$$W(T) = \sigma(T)\setminus\pi_{00}(T).$$

When this equality holds for some $T\in\mathscr{B}(X)$, then one says that Weyl's Theorem holds for $T$; see [3].

Parts (3) and (4) of Theorem 2.9 imply the following corollary.

Corollary 2.10. *When* $0\notin\pi_{00}(S)$, *then* $W(S)=\sigma(S)\setminus\pi_{00}(S)$. *Therefore in this case Weyl's Theorem holds for* $S$.

An operator $T\in\mathscr{B}(X)$ is a Riesz operator if the nonzero spectrum of $T$ consists of poles of finite rank of the resolvent of $T$. This implies that $\sigma(T)$ is either finite or a sequence converging to zero, and $\sigma(T)\setminus\{0\}\subseteq\pi_{00}(T)$. Every compact operator is a Riesz operator.

Proposition 2.11. *If* $S$ *is a Riesz operator, then* $TA$ *is compact and* $S^2$ *is compact.*

Proof. Assume $S$ is a Riesz operator. If $S$ has no nonzero eigenvalue, then $\sigma(S)=\{0\}$, which implies $\sigma(TA)=0$. In this case $TA=0$ and $S^2=A(TA)T=0$.

Now assume $S$ has a nonzero eigenvalue, and let $\{\lambda_k\}_{k\geqq 1}$ be the sequence of distinct nonzero eigenvalues of $S$ (of course, this set may be finite). For each $k$ let $X_k$ be the eigenspace of $S$ corresponding to the eigenvalue $\lambda_k$. Since $S$ is a Riesz operator, $\lambda_k\to 0$ and each $X_k$ is finite dimensional. By Proposition 2.7 $\lambda_k$ is an eigenvalue of $TA$ and $X_k=A\mathfrak{N}(\lambda_k-TA)$, $k\geqq 1$. Clearly $A$ is one-to-one on $\mathfrak{N}(\lambda_k-TA)$, so $\mathfrak{N}(\lambda_k-TA)$ is finite dimensional. Then as $TA$ is normal, $TA$ must be compact. It follows that $S^2=A(TA)T$ is compact.

Theorem 2.12. *Assume* $S$ *is a Riesz operator. Let* $\{\lambda_k\}_{k\geqq 1}$ *be the sequence of distinct nonzero eigenvalues of* $S$, *and let* $X_k$ *be the eigenspace of* $S$ *corresponding to the eigenvalue* $\lambda_k$, $k\geqq 1$. *Then there exists a sequence of projection operators,* $\{P_k\}\subseteq\mathscr{B}(X)$ *with* $P_kP_j=0$ *if* $k\neq j$, $SP_k=P_kS=\lambda_kP$, *and* $\mathscr{R}(P_k)=X_k$, $k\geqq 1$, *such that for all* $x\in X$,

$$Sx = \sum_{k\geqq 1}\lambda_k P_k x + Nx.$$

*Here* $N$ *is the nilpotent part of* $S$. *Furthermore, for* $n\geqq 2$,

$$S^n = \sum_{k\geqq 1}\lambda_k^n P_k$$

*where convergence is in the operator norm.*

Proof. By Proposition 2.11 $TA$ is compact. Let $E_k$ be the orthogonal projection with range the eigenspace of $TA$ corresponding to $\lambda_k$. Define $P_k=\lambda_k^{-1}AE_kT$,

$k \geqq 1$. Then

$$P_k P_j = \lambda_k^{-1}\lambda_j^{-1} A E_k T A E_j T = \lambda_k^{-1} A E_k E_j T = \begin{cases} 0 & \text{if } k \neq j; \\ P_k & \text{if } k = j. \end{cases}$$

Also, $SP_k = \lambda_k^{-1} ATAE_kT = AE_kT = \lambda_k P$, and similarly, $P_k S = \lambda_k P$.

Since $SP_k = \lambda_k P_k$, it follows that $\mathscr{R}(P_k) \subseteqq X_k$. Now by Proposition 2.7 $X_k = A\mathfrak{N}(\lambda_k - TA)$. If $x \in X_k$, then choose $y$ with $TAy = \lambda_k y$ and $x = Ay$. Then $P_k x = \lambda_k^{-1} AE_k Tx = \lambda_k^{-1} AE_k TAy = AE_k y = Ay = x$. This proves $\mathscr{R}(P_k) = X_k$.

Let $E_0$ be the orthogonal projection with range $\mathfrak{N}(TA)$. Since $TA$ is normal and compact, for every $y \in H$ we have

$$y = \sum_{k \geqq 1} E_k y + E_0 y.$$

Thus, for $x \in X$,

$$Tx = \sum_{k \geqq 1} E_k Tx + E_0 Tx,$$

and applying $A$,

$$Sx = ATx = \sum_{k \geqq 1} AE_k Tx + AE_0 Tx = \sum_{k \geqq 1} \lambda_k P_k x + Nx.$$

Finally for $n \geqq 2$, $(TA)^{n-1} = \sum_{k \geqq 1} \lambda_k^{n-1} E_k$, so

$$S^n = A(TA)^{n-1} T = \sum_{k \geqq 1} \lambda_k^n P_k.$$

The next result concerns the restriction of $S$ to a closed $S$-invariant subspace of $X$. It has application to the situation when $X = L^\infty(\Omega, \mu)$, where $\Omega$ is a locally compact Hausdorff space and $\mu$ is a regular Borel measure, and $S \in \mathscr{B}(X)$ leaves invariant the subspace of bounded continuous functions on $\Omega$.

Proposition 2.13. *Assume $S = AT$ where $TA$ is selfadjoint. Assume $Y$ is a closed $S$-invariant subspace of $X$. Let $\tilde{S}$ be the restriction of $S$ to $Y$, so $\tilde{S} \in \mathscr{B}(Y)$. Then $\tilde{S}$ has a selfadjoint factorization through Hilbert space. Furthermore, $\sigma(\tilde{S}) \subseteqq \sigma(S) \cup \{0\}$.*

Proof. Let $\tilde{H}$ be the closure of $T(Y)$ in $H$. Define $\tilde{T}\colon Y \to \tilde{H}$ by $\tilde{T}(y) = Ty$ for $y \in Y$. Define $\tilde{A}\colon \tilde{H} \to Y$, $\tilde{A}(z) = Az$ for $z \in \tilde{H}$. Here one notes that $A(T(Y)) \subseteqq Y$, so $A(\tilde{H}) \subseteqq Y$. Then $\tilde{S} = \tilde{A}\tilde{T}$ and $\tilde{T}\tilde{A}$ is selfadjoint in $\tilde{H}$. In fact, since $A(\tilde{H}) \subseteqq Y$, we have $TA(\tilde{H}) \subseteqq T(Y) \subseteqq \tilde{H}$. This last inclusion shows that $\tilde{H}$ is $TA$-invariant. It follows that $\sigma(\tilde{T}\tilde{A}) \subseteqq \sigma(TA) \cup \{0\}$. Therefore $\sigma(\tilde{S}) \subseteqq \sigma(S) \cup \{0\}$.

**3. Examples.** This section is devoted to examples of classes of operators on Banach spaces which have selfadjoint or normal factorizations through a Hilbert space. The specific operators involved are of the type that occur commonly in operator theory and the applications of operator theory. There are also a few examples of operators which are closely related to selfadjoint operators, but which do not have a selfadjoint factorization on Hilbert space.

Example I. Let $H$ be a Hilbert space. Assume $V, W, R \in \mathscr{B}(H)$ with $R \geqq 0$ and $WV$ selfadjoint. Then $S=VRW$ has a selfadjoint factorization through $H$. For set $T=R^{1/2}W$ and $A=VR^{1/2}$. Then $A, T \in \mathscr{B}(H)$, $S=AT$ and $TA=R^{1/2}WVR^{1/2}$ is a selfadjoint operator.

Specific examples of operators $S$ of the type considered above are well known in operator theory; see [10, p. 345].

Example II. Let $X$ be a Banach space with a bounded pre-innerproduct $(x, y)$, $x, y \in X$. This means that the form $(x, y)$ has all the properties of an innerproduct except that

$$K = \{x \in X\colon (x, y) = 0 \text{ for all } y \in X\}$$

may be nonzero. Also, that the form is bounded means $\exists C>0$ such that

$$|(x, y)| \leqq C\|x\|\,\|y\| \quad (x, y \in X).$$

The quotient space $X/K$ has an innerproduct determined in the natural way

$$(x+K, y+K) = (x, y) \quad (x, y \in X).$$

Let $H$ be the completion of $X/K$ in the norm determined by the innerproduct. Many authors study operators in $\mathscr{B}(X)$ which are selfadjoint with respect to a given bounded innerproduct on $X$; see for example [10, Chapter 9]. We consider the case where $S \in \mathscr{B}(X)$, $S$ has an adjoint $S^* \in \mathscr{B}(X)$ where $(Sx, y) = (x, S^*y)$ for all $x, y \in X$, and $\exists J>0$ with

$$(\#) \qquad \|Sx\|_X \leqq J(x, x)^{1/2} \quad (x \in X).$$

Using the special assumption (#), we will show that $S$ has a factorization through $H$. Note that (#) implies that $S(K)=\{0\}$. Then $S$ determines an operator $\tilde{S}\colon X/K \to X$ in the natural way

$$\tilde{S}(x+K) = Sx \quad (x \in X).$$

(#) implies that

$$\|\tilde{S}(x+K)\|_X \leqq J(x+K, x+K)^{1/2} \quad (x \in X),$$

and it follows that $\tilde{S}$ has an extension to a bounded linear operator $A\colon H \to X$ with $A(x+K) = Sx$ for all $x \in X$. Let $T\colon X \to H$ be given by $Tx = x+K$. The fact that the pre-innerproduct is bounded implies the continuity of $T$. Then $Sx = ATx$ for all $x \in X$, and

$$TA(x+K) = Sx+K \quad (x \in X).$$

It follows immediately that when $S=S^*$, then $TA$ is selfadjoint. When $S$ is normal further argument is necessary. First, let $W$ be the adjoint of $TA$ on $H$. Note that

$S^*(K)=\{0\}$ since $(Sx, Sx)=(S^*Sx, x)=(SS^*x, x)=(S^*x, S^*x)$ for all $x\in X$. Let $S^*$ be defined on $X/K$ in the usual way, $S^*(x+K)=S^*x+K$. For $x, y\in X$, $(x+K, W(y+K))=(Sx+K, y+K)=(x+K, S^*y+K)$. Therefore $W(y+K)= =S^*y+K$ for all $y\in X$. Thus, for $y\in X$,

$$(TA)W(y+K) = SS^*y+K = S^*Sy+K = W(TA)(y+K).$$

This proves $TA$ is normal on $H$.

Note that in the situation describe above, $TA$ is the unique extension of $S$ to an operator on $H$. By the theory in Section 2, $S$ and this extension have essentially the same spectral theory.

Now we consider a specific class of examples where this discussion applies. Let $(\Omega, \mu)$ be a $\sigma$-finite measure space, and let $X=L^2=L^2(\Omega, \mu)$. Assume $\varrho\in L^\infty(\Omega)$ with $\varrho(t)\geqq 0$ a.e. on $\Omega$. Define a pre-innerproduct on $X$ by

$$(f, g)_\varrho = \int_\Omega f(x)\overline{g(x)}\varrho(x)\, d\mu(x) \quad (f, g\in X).$$

Then

$$|(f, g)_\varrho| \leqq \|\varrho\|_\infty \|f\|\, \|g\| \quad (f, g\in X),$$

so this pre-innerproduct is bounded. Let $V\in\mathscr{B}(X)$ be selfadjoint, and define $S\in\mathscr{B}(X)$ by

$$S(f) = V(\varrho f) \quad (f\in X).$$

It is easy to verify that $S$ satisfies (♯): For $f\in X$, $\|Sf\|=\|V(\varrho f)\|\leqq\|V\|\,\|\varrho f\|\leqq \leqq\|V\|\,\|\varrho\|_\infty^{1/2}\left(\int_\Omega |f(x)|^2\varrho(x)\, d\mu(x)\right)^{1/2}$. Also, $S$ is symmetric with respect to the preinnerproduct:

$$(S(f), g)_\varrho = \int_\Omega V(\varrho f)\bar{g}\varrho\, d\mu = (V(\varrho f), \varrho g) = (\varrho f, V(\varrho g)) = (f, S(g))_\varrho.$$

In this case $H=L^2(\varrho)$, the $L^2$-space corresponding to the measure $\varrho\, d\mu$. Then $S$ has a selfadjoint factorization $S=AT$ with $TA$ the unique extension of $S$ to a bounded selfadjoint operator on $L^2(\varrho)$. As noted before, the spectral theory of $S$ on $L^2$ is essentially the same as that of the selfadjoint operator $TA$ on $L^2(\varrho)$.

Now we give an example of an operator selfadjoint with respect to an innerproduct which does not have a selfadjoint factorization through Hilbert space. Let $X$ be the disk algebra; the algebra of all continuous complex-valued functions defined on the closed unit disk $D$, and holomorphic on the interior of $D$. Define a bounded innerproduct on $X$ by

$$(f, g) = \sum_{n=1}^{\infty} f(n^{-1})\overline{g(n^{-1})}n^{-2} \quad (f, g\in X).$$

Let $S\in\mathscr{B}(X)$ be given by

$$S(f)(z) = zf(z) \quad (z\in D, f\in X).$$

Then $S$ is selfadjoint with respect to the given innerproduct. But $\sigma(S)=D$, so $S$ has no selfadjoint factorization on a Hilbert space.

There is an extension $\bar{S}$ of $S$ to a selfadjoint operator on $H$, the completion of $X$ with respect to the innerproduct. The vectors in $H$ are sequences in $l^2(n^{-2})$, and

$$\bar{S}(\{a_n\}_{n\geqq 1}) = \{n^{-1}a_n\}_{n\geqq 1}.$$

It is easy to see that $\bar{S}$ is a Hilbert—Schmidt operator on $H$. Thus, there exists a Hilbert—Schmidt operator on $H$ which when restricted to an invariant Banach subspace of $H$ no longer has the properties of a selfadjoint operator.

Example III. Let $(\Omega, \mu)$ be a measure space with $\mu$ a finite measure. We set $L^p=L^p(\Omega,\mu)$ for $1\leqq p\leqq\infty$. Assume $S\colon L^1\to L^\infty$. Then for $1\leqq p\leqq\infty$,

$$S(L^p)\subseteqq S(L^1)\subseteqq L^\infty\subseteqq L^p.$$

It is straightforward to check that $S$ is closed as an operator from $L^p$ to $L^p$. Thus for each $p$, $S$ determines a bounded linear operator $S_p\colon L^p\to L^p$. We prove that for each $p$, $S_p$ has a factorization through Hilbert space. First consider the case where $1\leqq p<2$. Then

$$S(L^p)\subseteqq L^\infty\subseteqq L^2, \quad\text{and}\quad L^2\subseteqq L^p.$$

Let $T\colon L^p\to L^2$ be determined by $S$ (again, $T$ is closed, hence continuous). Let $A$ be the continuous embedding of $L^2$ into $L^p$. Then $S_p=AT$ is a factorization of $S_p$ through $L^2$. Note that $TA(f)=S(f)$ for all $f\in L^2$, so $TA=S_2$.

Now suppose $2<p\leqq\infty$, in which case

$$S(L^2)\subseteqq L^\infty\subseteqq L^p, \quad\text{and}\quad L^p\subseteqq L^2.$$

Let $T$ be the continuous embedding of $L^p$ into $L^2$, and let $A$ be the bounded linear operator from $L^2$ into $L^p$ determined by $S$. Then $S_p=AT$ is a factorization of $S_p$ through $L^2$, and $TA=S_2$ on $L^2$. We summarize these results in a theorem.

Theorem 3.1. *Let $(\Omega,\mu)$ be a finite measure space. Assume $S\colon L^1\to L^\infty$. Then for each $p$, $1\leqq p\leqq\infty$, $S$ determines an operator $S_p\in\mathscr{B}(L^p)$, $S_p$ has a factorization $S_p=AT$ through $L^2$, and $TA=S_2$. Therefore if $S_2$ is normal then the factorization is normal.*

Now we look at two specific classes of examples where Theorem 3.1 applies.

Corollary 3.2. *Assume* $(\Omega, \mu)$ *is a finite measure space and that* $K \in L^\infty(\Omega \times \Omega)$. *Let* $S$ *be defined by*

$$S(f)(x) = \int_\Omega K(x, t) f(t)\, d\mu(t) \quad (f \in L^1).$$

*Then* $S(L^1) \subseteq L^\infty$. *If* $S_2$ *is a normal operator, then* $S_p$ *has a normal factorization through* $L^2$ *for* $1 \leqq p \leqq \infty$. *In particular, if* $K(x, t) = \overline{K(t, x)}$ *a.e. on* $\Omega \times \Omega$, *then* $S_p$ *has a selfadjoint factorization through* $L^2$ *for all* $p$.

Corollary 3.3. *Let* $\psi(t)$ *and* $\varphi(t)$ *be complex-valued measurable functions on* $(a, b)$ *with*

(i) $|\psi(t)|$ *increasing on* $(a, b)$;

(ii) $|\varphi(t)|$ *decreasing on* $(a, b)$;

(iii) $\varphi\psi \in L^\infty[a, b]$.

*Define*

$$K(x, t) = \begin{cases} \overline{\varphi(x)}\psi(t) & a \leqq t \leqq x \leqq b, \\ \varphi(t)\overline{\psi(x)} & a \leqq x \leqq t \leqq b. \end{cases}$$

*Let* $S$ *be the integral operator determined by the kernel* $K$. *Then* $S\colon L^1[a, b] \to L^\infty[a, b]$ *and* $S_2$ *is selfadjoint.*

Proof. It is straightforward to check that $K(x, t)$ is bounded.

Example IV. Let $X$ be a Banach space which is a subspace of a Hilbert space $H$ with $X$ continuously embedded in $H$. Assume $R \in \mathcal{B}(H)$ with $R$ selfadjoint (or normal) and suppose $R(H) \subseteq X$. Let $S$ be the restriction of $R$ to $X$. It is easy to check that $S$ is closed on $X$ so $S \in \mathcal{B}(X)$. Let $T\colon X \to H$ be the continuous embedding. Define $A\colon H \to X$ by $Ay = Ry$, $y \in H$. Again, $A$ is closed, hence continuous. Then $S = AT$ and $TA = R$. Examples of this type are quite common.

Here is a specific example. Let $G$ be a locally compact unimodular group with a fixed left Haar measure. Fix $k \in L^1(G) \cap L^2(G)$ such that $k(x^{-1}) = \overline{k(x)}$, $x \in G$. Then $k(xt^{-1})$ is a selfadjoint kernel, and the corresponding convolution operator

$$R(f)(x) = \int_G k(xt^{-1}) f(t)\, dt \quad (f \in L^2(G))$$

is selfadjoint on $L^2(G)$. Let $X$ be the Banach subspace of $L^2(G)$ consisting of all those $f \in L^2(G)$ which are continuous and bounded on $G$. By [9, (20.19)(iii)] $R(L^2(G)) \subseteq X$. Thus, as indicated above the operator $S \in \mathcal{B}(X)$ defined by

$$S(f)(x) = \int_G k(xt^{-1}) f(t)\, dt \quad (f \in X)$$

has a selfadjoint factorization through $L^2(G)$.

We give one more specific class of examples. Let $X=L^2[0,\infty)\cap L^p[0,\infty)$ for some $p$, $1\leqq p\leqq\infty$; or let $X$ be the set of $f\in L^2[0,\infty)$ which are bounded and continuous on $[0,\infty)$. For $\varphi,\psi\in X$, let $K(x,t)$ be the kernel

$$K(x,t)=\begin{cases}\overline{\varphi(x)}\psi(t) & 0\leqq t\leqq x,\\ \varphi(t)\overline{\psi(x)} & 0\leqq x\leqq t.\end{cases}$$

Let $R$ be the selfadjoint operator on $L^2[0,\infty)$ determined by this kernel. For $f\in L^2[0,+\infty)$,

$$R(f)(x)=\overline{\varphi(x)}\int\limits_0^x\psi(t)f(t)\,dt+\overline{\psi(x)}\int\limits_x^\infty\varphi(t)f(t)\,dt.$$

For $x\geqq 0$, $f\in L^2[0,\infty)$,

$$|R(f)(x)|\leqq|\varphi(x)|\,\|\psi\|_2\|f\|_2+|\psi(x)|\,\|\varphi\|_2\|f\|_2.$$

This inequality proves that $R(f)\in X$. Thus, as before, the integral operator $S$ on $X$ determined by the kernel $K$ has a selfadjoint factorization through $L^2[0,\infty)$.

Example V. Let $(\Omega,\mu)$ be a $\sigma$-finite measure space. We construct a class of operators on $L^\infty$ (and later on $L^1$) which have a selfadjoint factorization through $L^2$. If $f$ and $g$ are measurable functions on $\Omega$ with $\bar fg\in L^1$, then let $(f,g)=\int\limits_\Omega\bar fg\,d\mu$. Assume

$(*)$ $\quad V\colon L^1\to L^\infty$ with $\big(V(f),g\big)=\big(f,V(g)\big)$ for all $f,g\in L^1$.

Assume $k\in L^1$, $k\geqq 0$ a.e. on $\Omega$. Define $T\colon L^\infty\to L^2$ by

$$T(f)=k^{1/2}f\quad(f\in L^\infty).$$

Define $A\colon L^2\to L^\infty$ by

$$A(f)=V(k^{1/2}f)\quad(f\in L^2).$$

For $f\in L^2$,

$$\|Af\|_\infty=\|V(k^{1/2}f)\|_\infty\leqq\|V\|\,\|k^{1/2}f\|_1\leqq\|V\|\,\|k^{1/2}\|_1\|f\|_2$$

where the last inequality follows by applying the Cauchy—Schwarz Inequality. Therefore $A$ is bounded. Thus, $S=AT\colon L^\infty\to L^\infty$,

$$S(f)=V(kf)\quad(f\in L^\infty),$$

has a factorization through $L^2$. We check that $TA\colon L^2\to L^2$ is selfadjoint. For $f,g\in L^2$, $k^{1/2}f$ and $k^{1/2}g$ are in $L^1$, so using $(*)$ we have

$$\big(TA(f),g\big)=\big(k^{1/2}V(k^{1/2}f),g\big)=\big(V(k^{1/2}f),k^{1/2}g\big)=\big(k^{1/2}f,V(k^{1/2}g)\big)=\big(f,TA(g)\big).$$

Now we consider a related operator on $L^1$ that factors. Again, assume $V$ is as in $(*)$, and $k\in L^1$, $k\geqq 0$ a.e. on $\Omega$. Define $T\colon L^1\to L^2$ by

$$T(f)=k^{1/2}V(f)\quad(f\in L^1),$$

and $A\colon L^2 \to L^1$ by

$$A(f) = k^{1/2} f \quad (f \in L^2).$$

Then $R = AT\colon L^1 \to L^1$,

$$R(f) = kV(f) \quad (f \in L^1),$$

and a computation similar to that above proves that $TA$ is selfadjoint on $L^2$.

We summarize this discussion in a theorem.

Theorem 3.4. *Assume $V\colon L^1 \to L^\infty$ with $(V(f), g) = (f, V(g))$ for all $f, g \in L^1$. Assume $k \in L^1$, $k \geqq 0$ a.e. on $\Omega$. Then $S\colon L^\infty \to L^\infty$ and $R\colon L^1 \to L^1$ defined by*

$$S(f) = V(kf) \quad (f \in L^\infty),$$

$$R(f) = kV(f) \quad (f \in L^1),$$

*have selfadjoint factorizations through $L^2$.*

Next we give some examples of operators $V$ which satisfy $(*)$.

Proposition 3.5. *Let $(\Omega, \mu)$ be a $\sigma$-finite measure space and assume $K \in L^\infty(\Omega \times \Omega)$ with $K(x, t) = \overline{K(t, x)}$ a.e. Let $V$ be the corresponding integral operator*

$$V(f)(x) = \int_\Omega K(x, t) f(t)\, dt \quad (f \in L^1(\Omega)).$$

*Then $V$ satisfies $(*)$.*

The proof is elementary, so it will not be given.

Proposition 3.6. *Assume $\varphi$ and $\psi$ are* C*-valued functions on $(0, \infty)$ with*
(i) *$|\psi(t)|$ is increasing on $(0, +\infty)$;*
(ii) *$|\varphi(t)|$ is decreasing on $(0, +\infty)$;*
(iii) *$\varphi\psi \in L^\infty(0, +\infty)$.*

*Let*

$$K(x, t) = \begin{cases} \overline{\varphi(x)}\psi(t) & 0 \leqq t \leqq x, \\ \varphi(t)\overline{\psi(x)} & 0 \leqq x \leqq t. \end{cases}$$

*Let $V$ be the integral operator determined by the kernel $K$. Then $V$ satisfies $(*)$.*

The proof of this proposition is straightforward.

Now, without providing the details, we discuss two concrete situations involving kernels of the types in Propositions 3.5 and 3.6. Let $W$ be the space of all bounded C-valued continuous functions $f$ on $(0, \infty)$ such that $f'$ exists and is continuous on $(0, \infty)$, and $f''(x)$ exists for a.e. $x$ in $(0, \infty)$. Assume $\varrho(t) > 0$ on $(0, \infty)$ and $\varrho \in L^1(0, \infty)$. Fix $a > 0$.

First consider the differential operator

$$L = \varrho(t)^{-1}\left(\frac{d^2}{dt^2} + a^2\right)$$

with domain $\mathscr{D}(L)\subseteqq L^\infty(0,\infty)$ given by

$$\mathscr{D}(L)=\{f\in W\colon\ \varrho(t)^{-1}(f''(t)+a^2f(t))\in L^\infty(0,\infty)\},$$
$$\text{and}\quad L(f)=\varrho(t)^{-1}(f''(t)+a^2f(t))\quad\text{for}\quad f\in\mathscr{D}(L).$$

Let

$$K(x,t)=-a^{-1}\begin{cases}\cos(ax)\sin(at) & 0<t<x,\\ \sin(ax)\cos(at) & 0<x<t.\end{cases}$$

$K(x,t)$ is a bounded kernel. Set $J(x,t)=K(x,t)\varrho(t)$, $x,t>0$, and let $S$ be the integral operator on $L^\infty(0,\infty)$ determined by $J$. By Theorem 3.4 $S$ has a selfadjoint factorization through $L^2(0,\infty)$. Also, $S$ is a right inverse for $L$, meaning $S(L^\infty)\subseteqq\mathscr{D}(L)$ and $L(S(f))=f$ for $f\in L^\infty$. In addition, $S$ is a Fredholm inverse for $L$.

Let $W$, $\varrho$, and $a$ be as above. We consider a second differential operator

$$L=\varrho(t)^{-1}\left(\frac{d^2}{dt^2}-a^2\right)$$

with

$$\mathscr{D}(L)=\{f\in W\colon\ \varrho(t)^{-1}(f''(t)-a^2f(t))\in L^\infty(0,\infty)\}.$$

Let

$$K(x,t)=-(2a)^{-1}\begin{cases}e^{at}e^{-ax} & 0<t<x,\\ e^{ax}e^{-at} & 0<x<t.\end{cases}$$

$K$ is a kernel of the type considered in Proposition 3.6. Set $J(x,t)=K(x,t)\varrho(t)$, $x,t>0$, and let $S$ be the integral operator on $L^\infty(0,\infty)$ with kernel $J$. By Theorem 3.4 $S$ has a selfadjoint factorization on $L^2(0,\infty)$. Again in this case $S$ is a right inverse of $L$ and a Fredholm inverse for $L$.

Example VI. Let $\varrho(t)$ be the weight function on $[0,1]$ defined by $\varrho(t)=e^{1/t}$, $0<t\leqq 1$. Let $L^2(\varrho)$ be the Hilbert space of $L^2$-functions on $[0,1]$ relative to the measure $\varrho(t)\,dt$. Let $L^2=L^2[0,1]$, and note $L^2(\varrho)\subseteqq L^2$. We construct a selfadjoint Hilbert—Schmidt operator $S$ on $L^2(\varrho)$ such that $S$ has an extension $\bar S\in\mathscr{B}(L^2)$ such that $\bar S$ is not compact and $\sigma(\bar S)$ is not a subset of $\mathbf{R}$.

Let $K(x,t)$ be the kernel

$$K(x,t)=\begin{cases}x^{-1}\varrho(t)^{-1} & 0\leqq t\leqq x\leqq 1,\\ 0 & 0\leqq x<t\leqq 1.\end{cases}$$

(1) *$K$ is a Hilbert—Schmidt kernel on $L^2(\varrho)$.*

Proof. First note that

$$\int_0^x e^{-(1/t)}\,dt=\int_0^x t^2(t^{-2}e^{-(1/t)})\,dt\leqq x^2\int_0^x t^{-2}e^{-(1/t)}\,dt=x^2e^{-(1/x)}.$$

Then

$$\int_0^1 \Big(\int_0^1 K(x,t)^2 \varrho(t)\,dt\Big) \varrho(x)\,dx = \int_0^1 x^{-2}\varrho(x)\Big[\int_0^x e^{-(1/t)}\,dt\Big]\,dx \leqq$$

$$\leqq \int_0^1 x^{-2}\varrho(x)[x^2 e^{-(1/x)}]\,dx = 1.$$

Let $T$ be the Hilbert—Schmidt operator determined by the kernel $K$. The adjoint kernel of $K$, $K^*$, is given by

$$K^*(x,t) = \begin{cases} 0 & 0 \leqq t < x \leqq 1, \\ t^{-1}\varrho(x)^{-1} & 0 \leqq x \leqq t \leqq 1. \end{cases}$$

The corresponding operator is the adjoint of $T$. Let $S=T+T^*$ on $L^2(\varrho)$. $S$ is determined by the kernel $K+K^*$, so for $f \in L^2(\varrho)$,

$$S(f)(x) = \int_0^1 K(x,t) f(t)\varrho(t)\,dt + \int_0^1 K^*(x,t) f(t)\varrho(t)\,dt =$$

$$= x^{-1}\int_0^x f(t)\,dt + \varrho(x)^{-1}\int_x^1 t^{-1}\varrho(t) f(t)\,dt.$$

Let

$$J(x,t) = \begin{cases} 0 & 0 \leqq t < x \leqq 1, \\ \varrho(x)^{-1} t^{-1}\varrho(t) & 0 \leqq x \leqq t \leqq 1. \end{cases}$$

(2) *$J$ is a Hilbert—Schmidt kernel on $L^2$.*

Proof. For $x>0$

$$\int_0^1 J(x,t)^2\,dt = \varrho(x)^{-2}\int_x^1 t^{-2}e^{2/t}\,dt = \varrho(x)^{-2}\Big[-\frac{1}{2}e^2 + \frac{1}{2}e^{2/x}\Big]$$

which is a bounded continuous function of $x$ on $(0, 1]$.

Define $\bar{S}$ on $L^2$ by

$$\bar{S}(f)(x) = x^{-1}\int_0^x f(t)\,dt + \int_0^1 J(x,t) f(t)\,dt.$$

The first summand is the Cesaro operator on $L^2[0, 1]$, while the second, as verified in (2), is a Hilbert—Schmidt operator on $L^2$. The Cesaro operator is studied in [4] where it is verified that it is bounded on $L^2$. Thus, $\bar{S} \in B(L^2)$, and by definition $\bar{S}$ is an extension of $S$. Now the Cesaro operator has spectrum a disk [4], and the operator $f \to \int_0^1 J(x,t) f(t)\,dt$ is compact. These two facts imply that $\bar{S}$ is not compact, and that $\sigma(\bar{S})$ is not a subset of $\mathbf{R}$.

**4. Regularity, hyperinvariant subspaces.** Let $\mathscr{A}$ be a commutative Banach algebra with unit. We denote the Gelfand space of $\mathscr{A}$ by $\Omega_{\mathscr{A}}$ ($\Omega_{\mathscr{A}}$ is the set of all nonzero multiplicative linear functionals on $\mathscr{A}$ equipped with the relative weak-$*$topology). For $f \in \mathscr{A}$, let $\hat{f}$ denote the Gelfand transform of $f$, so $\hat{f}(\psi) = \psi(f)$ for $\psi \in \Omega_{\mathscr{A}}$. A subset $D$ of $\mathscr{A}$ strongly separates points of $\Omega_{\mathscr{A}}$ if whenever $\psi_1, \psi_2 \in \Omega_{\mathscr{A}}$, $\psi_1 \neq \psi_2$, then $\exists f \in D$ with $\hat{f}(\psi_1) \neq \hat{f}(\psi_2)$, and whenever $\psi \in \Omega_{\mathscr{A}}$, $\exists g \in D$ with $\hat{g}(\psi) \neq 0$. The algebra $\mathscr{A}$ is regular if whenever $\Gamma$ is a closed subset of $\Omega_{\mathscr{A}}$ and $\psi \in \Omega_{\mathscr{A}} \setminus \Gamma$, then $\exists f \in \mathscr{A}$ such that $\hat{f}(\Gamma) = \{0\}$ and $\hat{f}(\psi) \neq 0$. Let $\operatorname{rad}(\mathscr{A})$ denote the Jacobson radical of $\mathscr{A}$. A good reference for the theory of Banach algebras is [5].

Now we prove a general result in a Banach algebra setting which applies to operators $S$ that have selfadjoint factorizations. Some form of this result is certainly known (see [8]), but we include it since the proof is short and elementary.

Theorem 4.1. *Let $\mathscr{B}$ be a regular commutative semisimple Banach algebra with unit. Let $\mathscr{A}$ be a commutative Banach algebra with unit. Assume $\varphi\colon \mathscr{B} \to \mathscr{A}$ is a unital algebra homomorphism such that $\varphi(\mathscr{B})$ strongly separates points of $\Omega_{\mathscr{A}}$. Then*

(1) *$\mathscr{A}$ is regular;*

(2) *Assume $S \in \mathscr{B}(X)$ and $\mathscr{A}$ is a closed subalgebra of $\mathscr{B}(X)$ with $S$ and $I$ in $\mathscr{A}$ such that $\mathscr{A}$ satisfies the hypotheses of the theorem. Also assume that $\mathscr{A}$ has the property that when $R \in \mathscr{B}(X)$ and $RS = SR$, then $RT = TR$ for all $T \in \mathscr{A}$. If $\sigma(S)$ contains more than one number, then $S$ has a proper closed hyperinvariant subspace.*

Proof. Define $\tau\colon \Omega_{\mathscr{A}} \to \Omega_{\mathscr{B}}$ by $\tau(\psi) = \psi \circ \varphi$. Then $\tau$ is one-to-one and continuous. Now assume $\Gamma$ is a closed subset of $\Omega_{\mathscr{A}}$ and $\psi_1 \in \Omega_{\mathscr{A}} \setminus \Gamma$. Since $\Gamma$ is compact, $\tau(\Gamma)$ is compact, and also, $\tau(\psi_1) \notin \tau(\Gamma)$. Then $\exists f \in \mathscr{B}$ such that $\hat{f}(\tau(\Gamma)) = \{0\}$ and $\hat{f}(\tau(\psi_1)) \neq 0$. This proves $\mathscr{A}$ is regular.

Now assume $S$ and $\mathscr{A}$ are as in (2). By hypothesis $\sigma(S)$ contains at least two points. Since $\sigma(S) \subseteq \sigma_{\mathscr{A}}(S)$, $\sigma_{\mathscr{A}}(S)$ contains at least two points. Thus, $\exists \psi_1, \psi_2 \in \Omega_{\mathscr{A}}$ with $\psi_1 \neq \psi_2$, then $\tau(\psi_1) \neq \tau(\psi_2)$ so we can choose $f_1, f_2 \in \mathscr{B}$ such that $\hat{f}_k(\tau(\psi_k)) \neq 0$, $k = 1, 2$, and $f_1 f_2 = 0$. Therefore $\varphi(f_k) \neq 0$, $k = 1, 2$, and $\varphi(f_1)\varphi(f_2) = 0$. Let $W$ be the closure of $\varphi(f_2)X$ in $X$. $W$ is proper since $\varphi(f_1)W = \{0\}$. If $R \in \mathscr{B}(X)$ commutes with $S$, then $R\varphi(f_2) = \varphi(f_2)R$, so $R(W) \subseteq W$.

Theorem 4.1 applies to the situation where $S$ has a selfadjoint factorization on Hilbert space. The map $\varphi$ involved is the operational calculus. As part of the proof of this result, we prove a preliminary proposition.

Let $\Delta$ be a compact subset of $\mathbf{C}$. For $f \in \mathrm{BM}(\Delta)$, define

$$\|f\|_\Delta = \sup\{|f(\lambda)| \colon \lambda \in \Delta\}.$$

Also, let $C(\Delta)$ denote the algebra of all complex-valued continuous functions on $\Delta$.

Proposition 4.2. *Assume that $S$ has a selfadjoint factorization through Hilbert space. Also, assume that* (A1) *and* (A2) *hold. Set* $\Delta=\sigma(S)$. *Define*

$$\mathscr{I} = \{f\in C(\Delta)\colon \exists g\in C(\Delta) \text{ with } f(\lambda) = \lambda g(\lambda) \text{ on } \Delta\}.$$

*Let $\mathscr{B}$ be $\mathscr{I}$ with a unit adjoined. Let $\mathscr{A}$ be the closed subalgebra of $\mathscr{B}(X)$ generated by $S$ and $I$. Then $\mathscr{B}$ is a regular semisimple Banach algebra, and $\exists\varphi\colon \mathscr{B}\to\mathscr{A}$ with $\varphi$ a continuous unital algebra homomorphism such that $\varphi(\mathscr{B})$ separates points of $\Omega_{\mathscr{A}}$.*

Proof. One easily checks that $\mathscr{I}$ is an ideal in $C(\Delta)$ and that $\mathscr{I}$ is a Banach. algebra in the norm $\|f\|=\max(\|f\|_\Delta, \|g\|_\Delta)$ where $f\in\mathscr{I}$, $g\in C(\Delta)$ with $f(\lambda)=$ $=\lambda g(\lambda)$ on $\Delta$. It follows that $\mathscr{I}$, and hence $\mathscr{B}$, is a regular semisimple Banach algebra.

Now $\mathscr{I}\subseteqq\mathfrak{M}(\Delta)$. For $f\in\mathscr{I}$, let $\varphi(f)=f(S)$, and extend $\varphi$ to $\mathscr{B}$ by setting $\varphi(1)=I$. Note that $\varphi$ is continuous on $\mathscr{I}$ by Theorem 2.4 (3). We still must check that $\varphi(\mathscr{I})\subseteqq\mathscr{A}$. Assume $f\in\mathscr{I}$ with $g\in C(\Delta), f(\lambda)=\lambda g(\lambda)$ on $\Delta$. Choose a sequence of polynomials $\{q_n(\lambda)\}$ such that $\|q_n-g_n\|_\Delta\to 0$. Set $p_n(\lambda)=\lambda q_n(\lambda)$, so $\{p_n\}\subseteqq\mathscr{I}$. Then $\|p_n-f\|_\Delta\to 0$, so $p_n\to f$ in the norm on $\mathscr{I}$. Therefore $\{p_n(S)\}\subseteqq\mathscr{A}$ and $p_n(S)\to f(S)$ in $\mathscr{B}(X)$. Thus, $f(S)\in\mathscr{A}$. Finally, $\varphi(\mathscr{B})$ separates points of $\Omega_{\mathscr{A}}$ since $I, S\in\varphi(\mathscr{B})$.

Theorem 4.3. *Assume $S\in\mathscr{B}(X)$ has a selfadjoint factorization through Hilbert space.*

(1) *If $\sigma(S)$ contains at least two numbers, then $S$ has a proper closed hyperinvariant subspace.*

*Let $\mathscr{A}$ be the closed subalgebra of $\mathscr{B}(X)$ generated by $S$ and $I$. Assume* (A1) *and* (A2) *hold. Then*

(2) *$\mathscr{A}$ is a regular Banach algebra,* $\operatorname{rad}(\mathscr{A})^2=\{0\}$, *and $SR=RS=0$ for all* $R\in\operatorname{rad}(\mathscr{A})$;

(3) *If $\mathscr{R}(S)$ is dense in $X$ or $\mathfrak{N}(S)=\{0\}$, then $\mathscr{A}$ is semisimple.*

Proof. If $S\neq 0$ and $\mathfrak{N}(S)\neq\{0\}$, then $\mathfrak{N}(S)$ is a proper closed hyperinvariant subspace of $S$. Thus we may assume $\mathfrak{N}(S)=\{0\}$. Let $N$ be the nilpotent part of $S$. Since $SN=0$, in this case $N=0$. Then by Propositions 1.2 and 1.3 we may assume that $S$ has a factorization $S=AT$ with $TA$ selfadjoint such that (A1) and (A2) hold. If $TA$ is invertible, then by Proposition 1.3 $S$ is invertible. In this case $S$ is similar to the selfadjoint operator $TA$. It follows easily that $S$ has a proper closed hyperinvariant subspace (assuming $\sigma(S)$ has more than one point). Thus, to establish (1) we may assume (A1) and (A2) hold.

Assuming (A1) and (A2) hold, Proposition 4.2 applies. Then Theorem 4.1 proves (1) and that $\mathscr{A}$ is regular.

Now assume $R\in\operatorname{rad}(\mathscr{A})$. Choose a sequence of polynomials $\{p_n(\lambda)\}$ such that $p_n(S)\to R$ in $\mathscr{A}$. Since $\mathscr{A}$ is a closed subalgebra of $\mathscr{B}(X)$, the spectral radius of $V\in\mathscr{A}$ relative to $\mathscr{A}$ is the same as $r(V)$, the spectral radius of $V$ in $B(X)$. Now $r(p_n(S))\to r(R)=0$. Since $\sigma(p_n(S))=\{p_n(\mu)\colon \mu\in\sigma(S)\}$, it follows that $p_n(\lambda)\to 0$ uniformly on $\Delta$. Therefore by Theorem 2.4 (3) $Sp_n(S)\to 0$, so $SR=0$. Also, it now follows that $Rp_n(S)=p_n(0)R$, and since $p_n(0)\to 0$, we have $R^2=0$. This completes the proof of (2).

(3) follows easily from the fact derived in (2) that for $R\in\operatorname{rad}(\mathscr{A})$, $SR=RS=0$.

In our final result, we show that when $S$ has a selfadjoint factorization through Hilbert space, then $S$ can be approximated by operators which are similar to selfadjoint operators.

Theorem 4.4. *Assume $S\in\mathscr{B}(X)$ has a selfadjoint factorization through a Hilbert space, and that* (A1) *and* (A2) *hold. Then there exists a collection of projection operators $\{P_\varepsilon\}_{\varepsilon>0}\subseteq\mathscr{B}(X)$ such that $P_\varepsilon S=SP_\varepsilon$ for $\varepsilon>0$, and*

(i) *$P_\varepsilon S$ considered as an operator on $P_\varepsilon(X)$ is similar to a selfadjoint operator for each $\varepsilon>0$; and*

(ii) *$S$ is the strong limit as $\varepsilon\to 0^+$ of $SP_\varepsilon$ on $X$.*

Proof. Assume $S=AT$ is factorization of $S$ through $H$ with $TA$ selfadjoint and that (A1) and (A2) hold. Let $\Delta=\sigma(TA)$. For $t\in\mathbf{R}$, let $\chi_{(-\infty,t]}$ be the characteristic function of the specified interval and set $E_t=\chi_{(-\infty,t]}(TA)$. Thus, $\{E_t\}_{t\in\mathbf{R}}$ is the usual spectral resolution of the identity for $TA$. In this situation the strong limit of $E_t-E_0$ as $t\to 0^-$ is the projection on $\mathfrak{N}(TA)$ which is 0 by (A1) [11, p. 361]. Also, $E_t$ is strongly continuous from the right on $\mathbf{R}$, so the strong limit of $E_\varepsilon-E_{-\varepsilon}=0$ as $\varepsilon\to 0^+$. Let $\chi_\varepsilon$ be the characteristic function of $(-\infty,\varepsilon]\cup(\varepsilon,+\infty)$. Then $Q_\varepsilon= =\chi_\varepsilon(TA)=I-(E_\varepsilon-E_{-\varepsilon})$ has strong limit $I$ as $\varepsilon\to 0^+$. Let $P_\varepsilon=\chi_\varepsilon(S)$. Consider the operator $SP_\varepsilon$ on the space $X_\varepsilon=P_\varepsilon X$. Let $H_\varepsilon=Q_\varepsilon H$. Applying the operational calculus to the function $\lambda\chi_\varepsilon(\lambda)$, we have

$$SP_\varepsilon = AQ_\varepsilon T. \tag{1}$$

Then $TA(TP_\varepsilon A)=T(SP_\varepsilon)A=T(AQ_\varepsilon T)A$ by (1). Then $TA(TP_\varepsilon A-Q_\varepsilon TA)=0$, so

$$(TA)Q_\varepsilon = TP_\varepsilon A. \tag{2}$$

Let

$$T_\varepsilon = Q_\varepsilon TP_\varepsilon\colon X\to H_\varepsilon, \quad\text{and}\quad A_\varepsilon = P_\varepsilon AQ_\varepsilon\colon H_\varepsilon\to X_\varepsilon.$$

Then using (1) and (2) we have

$$A_\varepsilon T_\varepsilon = P_\varepsilon AQ_\varepsilon TP_\varepsilon = SP_\varepsilon, \quad\text{and}\quad T_\varepsilon A_\varepsilon = Q_\varepsilon TP_\varepsilon AQ_\varepsilon = (TA)Q_\varepsilon.$$

Now let $f(\lambda)=\lambda^{-1}\chi_\varepsilon(\lambda)$. Then $TAf(TA)=Q_\varepsilon$, while $Sf(S)=P_\varepsilon$. Therefore $TAQ_\varepsilon$

is invertible on $H_\varepsilon$ and $SP_\varepsilon$ is invertible on $X_\varepsilon$. By Proposition 1.3 $SP_\varepsilon$ as an operator on $X_\varepsilon$ is similar to the selfadjoint operator $TAQ_\varepsilon$ on $H_\varepsilon$. This proves (i).

To prove (ii), recall that we have shown that $I$ is the strong limit of $Q_\varepsilon$ as $\varepsilon \to 0^+$. Then for $x \in X$, $Q_\varepsilon Tx \to Tx$, and therefore by (1),

$$SP_\varepsilon x = AQ_\varepsilon Tx \to ATx = Sx.$$

Thus, (ii) holds.

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# Shorts of operators and some extremal problems

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## 0. Introduction

Let $\mathscr{H}$ be a Hilbert space, $\mathscr{L}$ its closed subspace, and $A$ a non-negative operator in $\mathscr{H}$. As M. G. KREIN [1] showed, the set of operators

$$\mathscr{X}(A, \mathscr{L}) = \{X: 0 \leqq X \leqq A, \mathscr{R}(X) \subset \mathscr{L}\} \tag{0.1}$$

contains a maximum element, denoted by $A_{\mathscr{L}}$ and called the *short* of $A$ to $\mathscr{L}$:

$$A_{\mathscr{L}} = \max \mathscr{X}(A, \mathscr{L}). \tag{0.2}$$

The properties of the correspondence $A \mapsto A_{\mathscr{L}}$ were studied in detail in [2] and found various applications to the theory of characteristic operator-functions [3, 4], electrical networks [5, 6], Lebesgue decomposition of nonnegative operators and positive definite operator-functions [7—9], operator means [10] and other problems. The notion of short was generalized to the case of non-closed $\mathscr{L}$, which is the range of a bounded operator, and it became clear that shorting is closely connected with the operation of parallel addition arising in the theory of electrical networks and with its inverse operation, parallel subtraction.

A recent work of S. L. ERIKSSON and LEUTWILLER [13], related to parallel addition, indicates essential connection between shorts and extreme points of some set of operators. In the present note we continue the study of this connection, and also give proofs to some assertions, announced earlier in [12, 14].

For Hilbert spaces $\mathscr{G}$, $\mathscr{H}$, let us denote by $\mathscr{B}(\mathscr{G}, \mathscr{H})$ the class of all bounded linear operators from $\mathscr{G}$ to $\mathscr{H}$. For $\mathscr{L} \subset \mathscr{H}$ and $T \in \mathscr{B}(\mathscr{G}, \mathscr{H})$, we denote by $T^{-1}\mathscr{L} (\subset \mathscr{G})$ the preimage of $\mathscr{L}$ under $T$, $\mathscr{N}(T) = T^{-1}\{0\}$ and $\mathscr{R}(T) = T\mathscr{G}$. When $\mathscr{G} = \mathscr{H}$, we shall write $\mathscr{B}(\mathscr{H}) = \mathscr{B}(\mathscr{H}, \mathscr{H})$. The class of all non-negative operators in $\mathscr{B}(\mathscr{H})$ is denoted by $\mathscr{B}_+(\mathscr{H})$.

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## 1. Shorts and complements

Let $\mathscr{G}$, $\mathscr{H}$ be Hilbert spaces, $\mathscr{F}=\mathscr{G}\oplus\mathscr{H}$, $T$ an operator $\in\mathscr{B}(\mathscr{G},\mathscr{H})$. In accordance with [11] let

$$\{\mathscr{H}\}_T = \{A\in\mathscr{B}_+(\mathscr{H}):\ \mathscr{R}(A^{1/2})\supset\mathscr{R}(T)\}.$$

The class $\{\mathscr{H}\}_T$ consists of operators $A\in\mathscr{B}_+(\mathscr{H})$ such that $\begin{bmatrix} G & T^* \\ T & A\end{bmatrix}\geqq 0$ for some $G\in\mathscr{B}_+(\mathscr{G})$. If $A\in\{\mathscr{H}\}_T$, then among those $G\in\mathscr{B}_+(\mathscr{G})$ there is a minimum, called the *complement* of $A$ relative to $T$ denoted by $A_T$. Namely, $A_T=WW^*$, where the operator $W\in\mathscr{B}(\mathscr{H},\mathscr{G})$ is uniquely determined by the condition

$$\text{(1.1)}\qquad WA^{1/2} = T^*,\quad \mathscr{N}(W)\supset\mathscr{N}(A).$$

If $A\in\mathscr{B}_+(\mathscr{H})$ is invertible, then $A\in\{\mathscr{H}\}_T$ for any $T\in\mathscr{B}(\mathscr{G},\mathscr{H})$ and $A_T=T^*A^{-1}T$. In the general case by [2] the following identity is fulfilled

$$\text{(1.2)}\qquad (A_Tg,g) = \sup_{h\in\mathscr{H}}\{(Tg,h)+(T^*h,g)-(Ah,h)\} = \sup_{h\in\mathscr{H}}\frac{|(h,Tg)|^2}{(Ah,h)}.$$

The class $\{\mathscr{G}\}_{T*}$ and the complement $G_{T*}\in\mathscr{B}_+(\mathscr{H})$ are defined correspondingly. The mappings $A\mapsto A_T$ and $G\mapsto G_{T*}$ define a Galois correspondence between the classes $\{\mathscr{H}\}_T$ and $\{\mathscr{G}\}_{T*}$ (if in these classes the order relations are defined inversely to the usual ones) [11]. This gives rise to a closure operations $A\mapsto(A_T)_{T*}\equiv\gamma_T(A)$ and $G\mapsto(G_{T*})_T\equiv\gamma_{T*}(G)$ in $\{\mathscr{H}\}_T$ and $\{\mathscr{G}\}_{T*}$ respectively. These operations are monotone in their respective classes. Thus if, for instance, $A_1$ and $A_2\in\{\mathscr{H}\}_T$ and $A_1\leqq A_2$, then $\gamma_T(A_1)\leqq\gamma_T(A_2)$. Besides, for any $A\in\{\mathscr{H}\}_T$ $(G\in\{\mathscr{G}\}_{T*})$ we have $\gamma_T(A)\leqq A\big(\gamma_{T*}(G)\leqq G\big)$. In case $A=\gamma_T(A)$ $\big(G=\gamma_{T*}(G)\big)$, then the operator $A(G)$ is called *T-closed ($T^*$-closed)*. The class of $T$-closed ($T^*$-closed) operators is denoted by $[\mathscr{H}]_T$ $([\mathscr{G}]_{T*})$.

The proof of the following theorem is found in [11].

Theorem 1.1. *In order that an operator $A\in\{\mathscr{H}\}_T$ belongs to $[\mathscr{H}]_T$ it is necessary and sufficient that the equality $\overline{A^{-1/2}\mathscr{R}(T)}=\mathscr{H}$ holds.*

Theorem 1.2. *If $\mathscr{R}(T_1)=\mathscr{R}(T_2)$, then $\{\mathscr{H}\}_{T_1}=\{\mathscr{H}\}_{T_2}$, $[\mathscr{H}]_{T_1}=[\mathscr{H}]_{T_2}$ and $\gamma_{T_1}(A)=\gamma_{T_2}(A)$ for any $A\in\{\mathscr{H}\}_{T_1}$.*

Proof. Since for any operator $T$

$$\{\mathscr{H}\}_T = \{A\in\mathscr{B}_+(\mathscr{H}):\ \mathscr{R}(A^{1/2})\supset\mathscr{R}(T)\},$$

$\{\mathscr{H}\}_{T_1}=\{\mathscr{H}\}_{T_2}$ whenever $\mathscr{R}(T_1)=\mathscr{R}(T_2)$. Now it follows easily from the preceding proposition that $[\mathscr{H}]_{T_1}=[\mathscr{H}]_{T_2}$. Then $\gamma_{T_2}(\gamma_{T_1}(A))=\gamma_{T_1}(A)$ for any $A\in\{\mathscr{H}\}_{T_1}$, hence $\gamma_{T_2}(A)\leqq A$ implies $\gamma_{T_2}(A)=\gamma_{T_1}(A)$. Exchanging the roles of $T_1$ and $T_2$, we have $\gamma_{T_2}(A) \geqq \gamma_{T_1}(A)$. Thus for any $A \in \{\mathscr{H}\}_{T_1}$ the expected equality $\gamma_{T_1}(A) = \gamma_{T_2}(A)$ is satisfied.

Corollary. *If $\mathscr{G} = \mathscr{H}$, $T \in \mathscr{B}(\mathscr{H})$ and $|T^*| = (TT^*)^{1/2}$, then $\{\mathscr{H}\}_T = \{\mathscr{H}\}_{|T^*|}$ and $\gamma_T(A) = \gamma_{|T^*|}(A)$ for any $A \in \{\mathscr{H}\}_T$.*

In many occasions it is convenient to use the following formula, established in [11].

$$\gamma_T(A) = A^{1/2} P_{\mathscr{M}} A^{1/2} \quad (A \in \{\mathscr{H}\}_T), \tag{1.3}$$

where $P_{\mathscr{M}}$ is the orthogonal projection to the subspace $\mathscr{M} = \overline{A^{-1/2}\mathscr{R}(T)}$.

Suppose that there is given an *operator range* $\mathscr{L}$, that is, the range of a bounded operator. In view of Theorem 1.2 it is possible to define the classes $\{\mathscr{H}\}_{\mathscr{L}}$, $[\mathscr{H}]_{\mathscr{L}}$ and the operation $\gamma_{\mathscr{L}}(A)$. More precisely, if $\mathscr{L} = \mathscr{R}(T)$ for $T \in \mathscr{B}(\mathscr{G}, \mathscr{H})$ with some $\mathscr{G}$, then let $\{\mathscr{H}\}_{\mathscr{L}} = \{\mathscr{H}\}_T$, $[\mathscr{H}]_{\mathscr{L}} = [\mathscr{H}]_T$, $\gamma_{\mathscr{L}}(A) = \gamma_T(A)$ for $A \in \{\mathscr{H}\}_T$.

Formula (1.3) is written in the form

$$\gamma_{\mathscr{L}}(A) = A^{1/2} P_{\mathscr{M}} A^{1/2} \quad \text{for} \quad A \in \{\mathscr{H}\}_{\mathscr{L}}, \tag{1.4}$$

where $P_{\mathscr{M}}$ is the orthogonal projection to the subspace $\mathscr{M} = \overline{A^{-1/2}\mathscr{L}}$.

Remark that an operator range $\mathscr{L}_1$ is contained in $\mathscr{L}$, $\mathscr{L}_1 \subset \mathscr{L} (\subset \mathscr{H})$, then $\{\mathscr{H}\}_{\mathscr{L}} \subset \{\mathscr{H}\}_{\mathscr{L}_1}$ and for $A \in \{\mathscr{H}\}_{\mathscr{L}}$, by (1, 4),

$$\gamma_{\mathscr{L}_1}(A) \leqq \gamma_{\mathscr{L}}(A). \tag{1.5}$$

Theorem 1.3. *Let $\mathscr{L} \subset \mathscr{H}$ be an operator range, and $A \in \{\mathscr{H}\}_{\mathscr{L}}$. Then $\gamma_{\mathscr{L}}(A)$ is a maximum element of the set*

$$\mathscr{X}_0(A, \mathscr{L}) = \{X:\ 0 \leqq X \leqq A,\ \overline{X^{-1/2}\mathscr{L}} = \mathscr{H}\}. \tag{1.6}$$

*In particular, if $\overline{\mathscr{L}} = \mathscr{L}$, then $\gamma_{\mathscr{L}}(A)$ is also a maximum element in the set* (0.1).

Proof. Remark that if $X \in \mathscr{X}_0(A, \mathscr{L})$ then $X \in \mathscr{X}_0(A, \mathscr{L}_1)$ too, where $\mathscr{L}_1 = \mathscr{R}(X^{1/2}) \cap \mathscr{L}$ is also an operator range (see [15]). Here $X \in \{\mathscr{H}\}_{\mathscr{L}_1}$ and further more $X \in [\mathscr{H}]_{\mathscr{L}_1}$, because $\overline{X^{-1/2}\mathscr{L}_1} = \mathscr{H}$. Therefore $X = \gamma_{\mathscr{L}_1}(X) \leqq \gamma_{\mathscr{L}_1}(A)$ because of the monotoneness of $\gamma_{\mathscr{L}_1}(\cdot)$, and it follows from (1.5) that $X \leqq \gamma_{\mathscr{L}}(A)$. Together with $\gamma_{\mathscr{L}}(A) (\in [\mathscr{H}_{\mathscr{L}}])$, by Theorem 1.1, $\mathscr{X}_0(A, \mathscr{L})$ contains

$$\gamma_{\mathscr{L}}(A) = \max \mathscr{X}_0(A, \mathscr{L}).$$

Suppose, in particular, that $\overline{\mathscr{L}} = \mathscr{L}$. Then we have by (1.4)

$$\mathscr{R}(\gamma_{\mathscr{L}}(A)) \subset \mathscr{R}(\gamma_{\mathscr{L}}(A)^{1/2}) = \mathscr{B}(A^{1/2} P_{\mathscr{M}}) \subset \mathscr{L},$$

hence $\gamma_{\mathscr{L}}(A) \in \mathscr{X}(A, \mathscr{L})$. Since obviously $\mathscr{X}(A, \mathscr{L}) \subset \mathscr{X}_0(A, \mathscr{L})$, $\gamma_{\mathscr{L}}(A)$ is a maximum operator in $\mathscr{X}(A, \mathscr{L})$, what is to prove.

Now consider an arbitrary operator range $\mathcal{L}\subset\mathcal{H}$ and an arbitrary operator $A\in\mathcal{B}(\mathcal{H})$ (not supposing $A\in\{\mathcal{H}\}_{\mathcal{L}}$). With $\mathcal{L}_1=\mathcal{L}\cap\mathcal{R}(A^{1/2})$, we have $A\in\{\mathcal{H}\}_{\mathcal{L}_1}$ and $\mathcal{X}_0(A,\mathcal{L})=\mathcal{X}_0(A,\mathcal{L}_1)$. Therefore according to Theorem 1.3 the set $\mathcal{X}_0(A,\mathcal{L})$ has a maximum element $\max\mathcal{X}_0(A,\mathcal{L})=\gamma_{\mathcal{L}_1}(A)$. As in Theorem 1.3 it is not difficult to see that if $\overline{\mathcal{L}}=\mathcal{L}$ then $\max\mathcal{X}_0(A,\mathcal{L})=\max\mathcal{X}(A,\mathcal{L})$.

In accordance with (0.2) let us introduce

Definition [12]. The *short* of an operator $A\in\mathcal{B}_+(\mathcal{H})$ to an operator range $\mathcal{L}\subset\mathcal{H}$ is the operator, defined by the relation

$$A_{\mathcal{L}} = \max\mathcal{X}_0(A,\mathcal{L}). \tag{1.7}$$

Since $A_{\mathcal{L}}=\gamma_{\mathcal{L}_1}(A)$ with $\mathcal{L}_1=\mathcal{L}\cap\mathcal{R}(A^{1/2})$, by (1.4) we have immediatley the following representation:

Theorem 1.4. *If $A\in\mathcal{B}_+(\mathcal{H})$ and $\mathcal{L}$ is an operator range $\subset\mathcal{H}$, then*

$$A_{\mathcal{L}} = A^{1/2}P_{\mathcal{M}}A^{1/2} \tag{1.8}$$

*where $P_{\mathcal{M}}$ is the orthogonal projection to the subspace $\mathcal{M}=\overline{A^{-1/2}\mathcal{L}}$.*

Corollary 1. $(A^2)_{\mathcal{L}}\leqq(A_{\mathcal{L}})^2$.

In fact, let $\mathcal{M}=\overline{A^{-1/2}\mathcal{L}}$, $\mathcal{M}_1=\overline{A^{-1}\mathcal{L}}$, and $P_{\mathcal{M}}$ and $P_{\mathcal{M}_1}$ the orthogonal projections to $\mathcal{M}$ and $\mathcal{M}_1$, respectively. If $g\in A^{-1}\mathcal{L}$ then $Ag\in\mathcal{L}$, and $A^{1/2}g\in A^{-1/2}\mathcal{L}\subset\mathcal{M}$. Therefore $P_{\mathcal{M}_1}A^{1/2}(I-P_{\mathcal{M}})=0$ and for any $h\in\mathcal{H}$ we have the inequality $\|P_{\mathcal{M}_1}A^{1/2}h\|=$ $=\|P_{\mathcal{M}_1}A^{1/2}(I-P_{\mathcal{M}})h\|+\|P_{\mathcal{M}_1}A^{1/2}P_{\mathcal{M}}h\|\leqq\|A^{1/2}P_{\mathcal{M}}h\|$. This implies that $A^{1/2}P_{\mathcal{M}_1}A^{1/2}\leqq$ $\leqq P_{\mathcal{M}}AP_{\mathcal{M}}$, and hence $AP_{\mathcal{M}_1}A\leqq A^{1/2}P_{\mathcal{M}}AP_{\mathcal{M}}A^{1/2}$. The last inequality means, by Theorem 1.4, that $(A^2)_{\mathcal{L}}\leqq(A_{\mathcal{L}})^2$.

Corollary 2. *$A_{\mathcal{L}}=A_{\mathcal{L}_1}$ if and only if $\overline{A^{-1/2}\mathcal{L}}=\overline{A^{-1/2}\mathcal{L}_1}$. In particular, $A_{\mathcal{L}}=A_{\overline{\mathcal{L}}}$ if and only if $\overline{A^{-1/2}\mathcal{L}}=\overline{A^{-1/2}\overline{\mathcal{L}}}$.*

It is easy to construct an example in which the last equality does not take place In fact, if $\mathcal{R}(A)\neq\overline{\mathcal{R}(A)}=\mathcal{H}$ and $h_0\notin\mathcal{R}(A^{1/2})$, let $\mathcal{L}=\{A^{1/2}h\colon h\in\mathcal{H}, h\perp h_0\}$ to get $h_0\perp\overline{A^{-1/2}\mathcal{L}}$. Then as $\overline{\mathcal{L}}=\mathcal{H}$, $A^{-1/2}\overline{\mathcal{L}}=\mathcal{H}$.

Clearly if $\mathcal{L}=\overline{\mathcal{L}}(\subset\mathcal{H})$, $A\in\mathcal{B}_+(\mathcal{H})$ and $\mathcal{L}'=\mathcal{H}\ominus\mathcal{L}$, then with respect to the orthogonal decomposition $\mathcal{H}=\mathcal{L}\oplus\mathcal{L}'$, we have a matrix representation of operators

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix},\quad A_{\mathcal{L}} = \begin{bmatrix} A_0 & 0 \\ 0 & 0\end{bmatrix}$$

where $A_0=A_{11}-(A_{22})_{A_{12}}$, hence by (1.2)

$$(A_{\mathcal{L}}h,h) = \inf_{g\perp\mathcal{L}}(A(h+g),h+g)\quad (h\in\mathcal{H}). \tag{1.9}$$

In concluding this section, remark that in case $\mathscr{L}=\overline{\mathscr{L}}$. the operator $A_{\mathscr{L}}(=\max \mathscr{X}(A, \mathscr{L})=\max \mathscr{X}_0(A, \mathscr{L}))$ becomes, obviously, the maximum in the set

$$\mathscr{X}'(A, \mathscr{L}) = \{X\colon 0 \leqq X \leqq A, \mathscr{R}(X^{1/2}) \subset \mathscr{L}\} \tag{1.10}$$

(because $\mathscr{X}'(A, \mathscr{L})=\mathscr{X}(A, \mathscr{L})$ in case $\mathscr{L}=\overline{\mathscr{L}}$.) If $\mathscr{L}\neq\overline{\mathscr{L}}$, as shown in the following section, the sets (0.1) and (1.10) do not contain maximum elements in general.

## 2. Short, parallel addition and parallel subtraction

In the theory of electrical network the parallel sum of invertible operators (matrices) $A$ and $B$ corresponding to the impedances of branches of the network, is the operator $A\colon B=(A^{-1}+B^{-1})^{-1}$, which becomes the impedance of the parallel connection of those branches. When $A$ and $B$ are non-negative, their parallel sum is suitably defined [16, 5, 11]*).

Let $\mathscr{H}$ be a Hilbert space, and $A, B\in\mathscr{B}_+(\mathscr{H})$. The operator

$$\mathbf{A} = \begin{bmatrix} A & A \\ A & A+B \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} I & I \\ 0 & I \end{bmatrix}, \tag{2.1}$$

acting on the space $\mathscr{F}=\mathscr{H}\oplus\mathscr{H}$, belongs to $\mathscr{B}_+(\mathscr{F})$, so that $A+B\in\{\mathscr{H}\}_A$ and $A\geqq(A+B)_A$.

Definition [5, 11]. The operator

$$A\colon B = A-(A+B)_A \quad (\geqq 0) \tag{2.2}$$

is called the *parallel sum* of $A$ and $B$.

It is clear that if $A$ and $B$ are invertible then

$$A\colon B = A-A(A+B)^{-1}A = A(A+B)^{-1}B = (A^{-1}+B^{-1})^{-1}.$$

Since the short of the operator $\mathbf{A}$ in (2.1) to the first component $\mathscr{H}$ coincides with $\begin{bmatrix} A\colon B & 0 \\ 0 & 0 \end{bmatrix}$, we have from (1.9)

$$((A\colon B)f,f) = \inf_{\substack{g,h\in\mathscr{H} \\ g+h=f}} \{(Ag, g)+(Bh, h)\}. \tag{2.3}$$

*) Remark that in a number of papers the operation of parallel addition is extended to a wider class of operators, in particular, to a class of non-linear operators [17, 18, 19].

The following properties of parallel sum follow easily from (2.2) and (2.3) (see, for instance [5, 6], and also [11]):

$$A:B = B:A, \quad (A:B):C = A:(B:C) \tag{2.4}$$

$$C(A:B)C \leqq (CAC):(CBC) \tag{2.5}$$

$$(A+B):(C+D) \geqq A:C+B:C \tag{2.6}$$

$$A_n \downarrow A, \quad B_n \downarrow B \Rightarrow A_n:B_n \downarrow A:B \tag{2.7}$$

$$\mathscr{R}\big((A:B)^{1/2}\big) = \mathscr{R}(A^{1/2}) \cap \mathscr{R}(B^{1/2}). \tag{2.8}$$

All operators in (2.4)—(2.8) are assumed non-negative. Notice that without any additional condition in (2.5) equality sign may not occur even in the two dimensional case.

Example. Let $A=\begin{bmatrix}1 & \alpha\\ \alpha & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & \beta\\ \beta & 1\end{bmatrix}$ where $0<\alpha, \beta<1$, $\alpha\neq\beta$ and $C=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$. Then equality sign does not occur in (2.5), because

$$(CAC):(CBC) = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad C(A:B)C = \frac{1}{\Delta}\begin{pmatrix}2-(\alpha^2+\beta^2) & 0\\ 0 & 0\end{pmatrix}$$

where $\Delta = \det(A+B) = 4-(\alpha+\beta)^2$.

Lemma 1. *If* $A, B, C \in \mathscr{B}_+(\mathscr{H})$, *and* $\overline{\mathscr{R}(A)}+\overline{\mathscr{R}(B)} \subset \overline{\mathscr{R}(C)}$, *then*

$$C(A:B)C = (CAC):(CBC).$$

Proof. With $\mathscr{L}=\mathscr{R}(C)$ and $h'=Ch$, using formula (2.3) we have

$$\big((CAC):(CBC)f,f\big) = \inf_{h\in\mathscr{H}}\big\{(AC(f-h), C(f-h))+(BCh, Ch)\big\} =$$
$$= \inf_{h'\in\mathscr{L}}\big\{(A(Cf-h'), Cf-h')+(Bh', h')\big\}.$$

The infimum does not change even if $\mathscr{L}$ is replaced by $\overline{\mathscr{L}}$. On the other hand, putting $g=h'+h''$, where $h'$ runs over $\overline{\mathscr{L}}$ and $h''$ over $\mathscr{H}\ominus\mathscr{L}$, since $Ah''=Bh''=0$, we have

$$\big(C(A:B)Cf,f\big) = \inf_{g\in\mathscr{H}}\big\{(A(Cf-g), Cf-g)+(Bg, g)\big\} =$$
$$= \inf_{h'\in\overline{\mathscr{L}}}\big\{(A(Cf-h'), Cf-h')+(Bh', h')\big\}.$$

This completes the proof.

Remark that all operators in (2.7) are invertible then convergence from above can be changed to convergence from below, though this is not the case in general. At the same time parallel sum becomes a non-decreasing function of "summands", and $0\leqq A:B\leqq A, B$. Hence for any non-decreasing sequences $\{A_n\}$, $\{B_n\}\subset\mathscr{B}_+(\mathscr{H})$ the sequence $\{A_n:B_n\}$ is also non-decreasing, and if one of the given sequences is bounded, then a limit $\lim_{n\to 0} A_n:B_n$ exists. Supposing that $A_n$, $B_n$ $(n=1, 2, \ldots)$ are invertible, let $A_\infty = \lim_{n\to\infty} A_n^{-1}$, $B_\infty = \lim_{n\to\infty} B_n^{-1}$. Then it is clear that

$$(A_n+B_n)^{-1} = A_n^{-1}:B_n^{-1}\downarrow A_\infty:B_\infty.$$

Let the sequence $\{A_n\}$ be bounded and $A=\lim_{n\to\infty} A_n(=A_\infty^{-1})$. Then

$$A_n:B_n = A_n - A_n(A_n+B_n)^{-1}A_n \xrightarrow[n\to\infty]{} A-A(A_\infty:B_\infty)A$$

and by Lemma 2.1

$$\lim_{n\to\infty} A_n:B_n = A-A:(AB_\infty A).$$

It follows, in particular, that with $A_n=A$ and $B_n=nB$

$$\lim_{n\to\infty} A:nB=A$$

for any invertible $B\in\mathscr{B}_+(\mathscr{H})$. Here the last equality is obvioualy satisfied for any $A\in\mathscr{B}_+(\mathscr{H})$, because

$$A:nB = A-A(A+nB)^{-1}A \quad \text{and} \quad (A+nB)^{-1}\downarrow 0.$$

For a non-invertible operator $B\in\mathscr{B}_+(\mathscr{H})$ the following assertion is valid (see [12, 21]):

Theorem 2.2. *If $A, B\in\mathscr{B}_+(\mathscr{H})$, then $C=\lim_{n\to\infty} A:nB$ is a minimum non-negative solution of the equation $X:B=A:B$.*

Recall that if $B$, $S\in\mathscr{B}_+(\mathscr{H})$, then the equation $X:B=S$ with unknown $X\in\mathscr{B}_+(\mathscr{H})$ has a solution exactly when $B-S\in[\mathscr{H}]_B$, and in this case there exists a minimum solution.

Definition [11]. The minimum solution of the equation $X:B=S$ is called the *parallel difference* of operators $S$ and $B$, and is denoted by $S\div B$.

We can prove that

$$S\div B = (B-S)_B - B = (B-S)_S+S, \tag{2.9}$$

hence by (1.2) we have

$$((S\div B)f,f) = \sup_{g\in\mathscr{H}}\{(S(f+g),f+g)-(Bg,g)\} \quad (\forall f\in\mathscr{H}). \tag{2.10}$$

It is clear that if operator $B-S$ is continuously invertible then Theorem 1.1 implies that the parallel difference $S\div B$ exists, and by (2.9)

$$S\div B = B(B-S)^{-1}B-B = B(B-S)^{-1}S.$$

Let us cite, for completeness, some properties of parallel subtraction, proved in [11] (provided that parallel subtraction in question exists):

$$S_1 \geqq S_2, \quad B_1 \leqq B_2 \Rightarrow S_1 \div B_1 \geqq S_2 \div B_2$$

$$S = A:B \Rightarrow S \div (S \div B) = S \div A$$

$$S = A:B \Rightarrow (S \div B):(S \div A) = S$$

$$S_n \uparrow S, \quad A_n \downarrow A \Rightarrow S_n \div A_n \uparrow S \div A$$

Given $S \in \mathscr{B}_+(\mathscr{H})$, let $\mathscr{M}(S)$ denote the class of those $A \in \mathscr{B}_+(\mathscr{H})$, for which the equation $A:X=S$ is solvable. When twice applied, the mapping $A \mapsto S \div A$ defines a Galois correspondence between $\mathscr{M}(S)$ and $\mathscr{M}(S)$: $A \mapsto S \div (S \div A) \equiv \varrho_S(A)$. Since $B-S=(A+B)_B$ for $A:B=S$, by (2.9) we have

$$\varrho_S(A) = \gamma_B(A+B)-B. \tag{2.11}$$

Remark that if $A \in \mathscr{M}(S)$, then $A:B=S$ for some $B \in \mathscr{B}_+(\mathscr{H})$, hence by Theorem 2.2

$$\varrho_S(A) = S+B = \lim_{n\to\infty} A:nB.$$

On the other hand

$$A:nB = \frac{n}{n-1}\{A:(n-1)A:(n-1)B\} = \frac{n}{n-1}\{A:(n-1)S\},$$

and letting $n \to \infty$ we have

$$\varrho_S(A) = \lim_{n\to\infty} A:nS.$$

An operator $A \in \mathscr{M}(S)$ is called *$\varrho_S$-closed* if $\varrho_S(A)=A$; the set of all $\varrho_S$-closed operators will be denoted by $\mathscr{M}[S]$. For an operator $A \in \mathscr{M}(S)$ the following three conditions are equivalent [11];

$$A \in \mathscr{M}[S] \Leftrightarrow A-S \in [\mathscr{H}]_S \Leftrightarrow \overline{A^{-1/2}\mathscr{R}(S^{1/2})} = \mathscr{H}. \tag{2.12}$$

Notice further that the following conditions are equivalent;

$$A \in \mathscr{M}[S] \Leftrightarrow A+B \in [\mathscr{H}]_B \Leftrightarrow A+B \in [\mathscr{H}]_{B^{1/2}}. \tag{2.13}$$

Now let us cite an assertion, proved in [12].

Theorem 2.3. *Given $A \in \mathscr{B}_+(\mathscr{H})$, let $A_{\mathscr{L}}$ be its short to an operator range $\mathscr{L} \subset \mathscr{H}$. Then $A_{\mathscr{L}}=(A:L)\div L$ for any $L \in \mathscr{B}_+(\mathscr{H})$ such that $\mathscr{R}(L^{1/2})=\mathscr{L}$.*

Applying to this representation of a short the expression (2.10) for parallel subtraction and (2.3) for parallel addition, we obtain

Corollary. *Under the assumptions of the theorem*

$$(A_{\mathscr{L}}f, f) = \sup_g \inf_h \{(Lh, h)-(Lg, g)+\big(A(f+g+h), f+g+h\big)\} \quad (\forall f \in \mathscr{H}).$$

Comparison of Theorems 2.2 and 2.3 leads to the identity, first proved in [7]

$$A_{\mathscr{L}} = \lim_{n\to\infty} A:nL \quad (\mathscr{L} = \mathscr{R}(L^{1/2})). \tag{2.14}$$

If $S=A:L$, then clearly

$$A_{\mathscr{L}} = \varrho_S(A) = \lim_{n\to\infty} A:nS. \tag{2.15}$$

The following properties of short operation to an operator range follows immediately from definition (1.7) and relation (2.14):

$$1.\ A_{\mathscr{L}} \leqq A; \qquad 2.\ (\alpha A)_{\mathscr{L}} = \alpha A_{\mathscr{L}}; \qquad 3.\ (A_{\mathscr{L}})_{\mathscr{L}} = A_{\mathscr{L}};$$

$$4.\ (A+B)_{\mathscr{L}} \geqq A_{\mathscr{L}}+B_{\mathscr{L}}; \qquad 5.\ (A:B)_{\mathscr{L}} \leqq A_{\mathscr{L}}:B_{\mathscr{L}} \leqq A_{\mathscr{L}}:B,\ A:B_{\mathscr{L}}.$$

In view of [5] if $\overline{\mathscr{L}}=\mathscr{L}$ equality sign occurs in the last part. But if $\overline{\mathscr{L}}\neq\mathscr{L}$, equality sign may break down. In fact, let $\mathscr{L}\neq\overline{\mathscr{L}}=\mathscr{H}$, $A=I$, $B\neq 0$ and $\mathscr{R}(B^{1/2})\cap\mathscr{L}=\{0\}$. Then $(A:B)_{\mathscr{L}}=A_{\mathscr{L}}:B_{\mathscr{L}}=A:B_{\mathscr{L}}=0$, but $A_{\mathscr{L}}:B=B(I+B)^{-1}\neq 0$.

It is known [2] that when $\mathscr{L}$ is a closed subspace then $A_n\downarrow A$ implies $(A_n)_{\mathscr{L}}\downarrow A_{\mathscr{L}}$. This property breaks down when $\mathscr{L}$ is a non-closed operator range. We can only assert the following

$$A_n\downarrow A \Rightarrow A_{\mathscr{L}} \leqq \lim_{n\to\infty} (A_n)_{\mathscr{L}} \leqq A_{\overline{\mathscr{L}}}.$$

In fact, since clearly $(A_n)_{\mathscr{L}}\leqq(A_n)_{\overline{\mathscr{L}}}$, letting $\alpha\to\infty$ in the inequalities

$$A:\alpha L \leqq A_n:\alpha L \leqq (A_n)_{\mathscr{L}} \leqq (A_n)_{\overline{\mathscr{L}}}$$

where $L\in\mathscr{B}_+(\mathscr{H})$ such that $\mathscr{R}(L^{1/2})=\mathscr{L}$, we have

$$A_{\mathscr{L}} \leqq (A_n)_{\mathscr{L}} \leqq (A_n)_{\overline{\mathscr{L}}}.$$

It remains to remark that the sequence $\{(A_n)_{\mathscr{L}}\}$ monotonely decreases and $\lim\limits_{n\to\infty}(A_n)_{\overline{\mathscr{L}}}= =A_{\overline{\mathscr{L}}}$.

If $A_{\mathscr{L}}\neq A_{\overline{\mathscr{L}}}$ (as in the example for Corollary 2 to Theorem 1.4), then putting $A_n=A+\frac{1}{n}I$ $(n=1, 2, \ldots)$ we have

$$A_n\downarrow A, \quad (A_n)_{\mathscr{L}} = \left(A+\frac{1}{n}I\right)_{\mathscr{L}} = \left(A+\frac{1}{n}I\right)_{\overline{\mathscr{L}}} \geqq A_{\overline{\mathscr{L}}}$$

(shorts of an invertible operator to $\mathscr{L}$ and $\overline{\mathscr{L}}$ coincide because of formule (1.8)). Therefore, in the present case $\lim\limits_{n\to\infty}(A_n)_{\mathscr{L}}=A_{\overline{\mathscr{L}}}\neq A_{\mathscr{L}}$.

Remark that if $\mathscr{L}_1\supset\mathscr{L}_2$ then $A_{\mathscr{L}_1}\geqq A_{\mathscr{L}_2}$ and $A_{\mathscr{L}_1\cap\mathscr{L}_2}=(A_{\mathscr{L}_1})_{\mathscr{L}_2}$. It follows easily from this that $\{\lim\limits_{n\to\infty}(A_n)_{\mathscr{L}}\}_{\mathscr{L}}=A_{\mathscr{L}}$.

In view of [5], for any closed subspaces $\mathscr{L}_1$ and $\mathscr{L}_2$ the following identity holds

$$A_{\mathscr{L}_1\cap\mathscr{L}_2} = (A_{\mathscr{L}_1})_{\mathscr{L}_2}.$$

For non-closed $\mathscr{L}_1$, $\mathscr{L}_2$ this may break down. For instance, if $\mathscr{L}_1 \cap \mathscr{L}_2 = \{0\}$, $\overline{\mathscr{L}_1} = \overline{\mathscr{L}_2} = \mathscr{H}$ (examples of such operator ranges can be seen in [15]) and $A=I$, then $A_{\mathscr{L}_1 \cap \mathscr{L}_2} = 0$ but $(A_{\mathscr{L}_1})_{\mathscr{L}_2} = I$.

In the case of arbitrary operator ranges $\mathscr{L}_i = \mathscr{R}(L_i^{1/2})$ $(i=1,2)$ it follows from the inequalities

$$A:n(L_1:L_2) = (A:nL_1):nL_2 \leqq A_{\mathscr{L}_1}:nL_2 \leqq (A_{\mathscr{L}_1})_{\mathscr{L}_2}$$

that $A_{\mathscr{L}_1 \cap \mathscr{L}_2} \leqq (A_{\mathscr{L}_1})_{\mathscr{L}_2}$. Further we have

$$A_{\mathscr{L}_1 \cap \mathscr{L}_2} \leqq 2(A_{\mathscr{L}_1}:A_{\mathscr{L}_2}). \tag{2.16}$$

In fact, denote by $P_1$, $P_2$ and $P$ the orthoprojections to the subspaces $\overline{A^{-1/2}\mathscr{L}_1}$, $\overline{A^{-1/2}\mathscr{L}_2}$ and $\overline{A^{-1/2}(\mathscr{L}_1 \cap \mathscr{L}_2)}$ respectively. Since $2(P_1:P_2)$ becomes the orthoprojection to the intersection of the subspaces $\overline{A^{-1/2}\mathscr{L}_1}$ and $\overline{A^{-1/2}\mathscr{L}_2}$ and $\overline{A^{-1/2}(\mathscr{L}_1 \cap \mathscr{L}_2)} \subset \overline{A^{-1/2}\mathscr{L}_1} \cap \overline{A^{-1/2}\mathscr{L}_2}$, we have

$$P \leqq 2(P_1:P_2).$$

Multiplying both sides of this inequality by $A^{1/2}$ from left and right, we have inequality (2.16) by Theorem 1.4 and Lemma 2.1. Here in (2.16) equality sign can occur exactly when $\overline{A^{-1/2}(\mathscr{L}_1 \cap \mathscr{L}_2)} = \overline{A^{-1/2}\mathscr{L}_1} \cap \overline{A^{-1/2}\mathscr{L}_2}$. In particular, it is the case when $\mathscr{L}_1$ and $\mathscr{L}_2$ are closed.

On the other hand, it is easy to construct an example in which one of operator ranges $\mathscr{L}_1$ and $\mathscr{L}_2$ is non-closed and $A_{\mathscr{L}_1 \cap \mathscr{L}_2} \neq 2(A_{\mathscr{L}_1}:A_{\mathscr{L}_2})$. In fact, if $A \in \mathscr{B}_+(\mathscr{H})$ and $\mathscr{R}(A) \neq \overline{\mathscr{R}(A)} = \mathscr{H}$, then with a dense operator range $\mathscr{M}_1 \neq \mathscr{H}$ and any operator range $\mathscr{M}_1 \neq \{0\}$ such that $\mathscr{M}_1 \cap \mathscr{M}_2 = \{0\}$, let $\mathscr{L}_i = A^{1/2}\mathscr{M}_i$ $(i=1,2)$. Then clearly $P=0$ but $2(P_1:P_2)$ is the orthoprojection to $\overline{\mathscr{M}_2} \neq \{0\}$.

In concluding this section let us show that if an operator range $\mathscr{L}$ is non-closed then the sets (0.1) and (1.10) cannot posses maximum elements. To this end, let us consider the example, used earlier in [11]].

Let $A$ be an operator on $\mathscr{H}$, $0 \leqq A \leqq I$, for which 1 is a continuous spectrum, and $B = I - A$. Then with $S = A:B = A(I-A)$, by (2.13) $A \in \mathscr{M}[S]$. With $\mathscr{L} = \mathscr{R}(B^{1/2})$, by (2.12) we see that $A = A_{\mathscr{L}} = \lim_{n\to\infty} A:nB$. According to the definition of the sets (0.1) and (1.10) we have for all $n = 1, 2, \ldots$

$$A:nB \in \mathscr{X}(A, \mathscr{L}) \cap \mathscr{X}'(A, \mathscr{L}),$$

so that $\mathscr{R}(A:nB) \subset \mathscr{R}((A:nB)^{1/2}) \subset \mathscr{L}$. Therefore if $C$ (resp. $C'$) is the maximum operator in $\mathscr{X}(A, \mathscr{L})$ (resp. $\mathscr{X}'(A, \mathscr{L})$), then $A = \lim_{n\to\infty} A:nB \leqq C$ (resp. $C'$), hence $A = C$ (resp. $C'$). But the equality $A = C$ implies that $\mathscr{R}(A) \subset \mathscr{R}(B^{1/2}) \subset \mathscr{R}(S^{1/2})$, that is, for some $\lambda > 0$, $A^2 \leqq \lambda A(I-A)$. Thus $I - A \geqq (1+\lambda)^{-1} I$, which contradicts that 1 is a continuous spectrum of $A$. Impossibility of the equality $A = C'$ is proved in an analogous way.

### 3. Short and extreme points

In the theory of characteristic operator functions, on studying completely non-regular factorizations the description of solutions of the following equation plays a basic role;

(3.1) $$(A-X):(B+X) = 0 \quad (A, B\in\mathscr{B}_+(\mathscr{H}), 0 \leqq X \leqq A).$$

Formulas for this problem were announced in [14, 12]. In this section, together with the proofs of those formulas, we shall present, as a consequence, some propositions, related to extreme point of some convex sets of non-negative operators in a Hilbert space. In particular, short to an operator range is interpreted as an extreme point.

In analogy to the scalar case an *operator interval* $[B, C]$ $(0\leqq B\leqq C)$ will mean the set

$$[B, C] = \{X\in\mathscr{B}_+(\mathscr{H});\ B \leqq X \leqq C\}.$$

The class of all solutions of equation (3.1) will be denoted by $\mathscr{E}(A, B)(\subset[0\ \ A])$.

Theorem 3.1. *Let* $A, B\in\mathscr{B}_+(\mathscr{H})$ *and* $S=A:B$. *Then*

1) *an operator* $X\in[0, A]$ *becomes a solution of equation* (3.1) *exactly when it admits a representation*

(3.2) $$X = (A+B)^{1/2}P(A+B)^{1/2}-B,$$

*where $P$ is an orthoprojection to a subspace of* $\overline{\mathscr{R}(A+B)}$;

2) *the operator* $X_0=\varrho_S(A)$ *is a minimum solution of equation* (3.1).

Proof. 1) In fact, if $X\in[0, A]$, by definition of parallel sum (2.2)

$$(A-X):(B+X) = B+X-(A+B)_{B+X}.$$

Therefore if $X\in\mathscr{E}(A, B)$, then

(3.3) $$B+X = \omega\omega^*,$$

where according to (1.1) an operator $\omega$ is uniquely defined by the conditions $B+X=$ $=\omega(A+B)^{1/2}$, $\mathscr{N}(\omega)\supset\mathscr{N}(A+B)$.

Let $\omega^*=W(\omega\omega^*)^{1/2}$ be the polar representation of $\omega^*$; here $W$ is a partial isometry with initial space $\overline{\mathscr{R}(\omega)}$ and final space $\mathscr{R}(W)=\overline{\mathscr{R}(\omega^*)}(\subset\overline{\mathscr{R}(A+B)})$. It follows from the equality $\omega\omega^*=\omega(A+B)^{1/2}$ that

$$\omega\omega^* = (\omega\omega^*)^{1/2}W^*(A+B)^{1/2},$$

and since $\mathscr{R}(\omega^*)\subset\overline{\mathscr{R}(\omega)}$, $(\omega\omega^*)^{1/2}=W^*(A+B)^{1/2}$. Hence $\omega^*=W(\omega\omega^*)^{1/2}=$

$=WW^*(A+B)^{1/2}$, and consequently

$$X = \omega\omega^* - B = (A+B)^{1/2}P(A+B)^{1/2} - B,$$

where $P$ is the orthoprojection to $\mathscr{R}(W)$.

Conversely, if $X \in \mathscr{B}_+(\mathscr{H})$ is represented in the form (3.2), then clearly $X \leqq A$, and since

$$B+X = (A+B)^{1/2}P(A+B)^{1/2}, \quad \mathscr{N}((A+B)^{1/2}P) \subset \mathscr{N}(A+B),$$

by (1.1) we have $(A+B)_{B+X} = B+X$. Thus $(A-X):(B+X)=0$.

2) It remains to prove that $X_0 = \varrho_S(A)$ is a minimum solution of equation (3.1). For this, remark first that if $X \in \mathscr{E}(A, B)$ then by (3.3) $B \leqq \omega\omega^*$, where, as clarified in the first part of the proof, $\omega = (A+B)^{1/2}P$. Then by [20] there is a contraction $V \in \mathscr{B}(\mathscr{H})$, for which

$$B^{1/2} = V\omega^* = VP(A+B)^{1/2}.$$

Since here $\mathscr{N}(B^{1/2}VP) \supset \mathscr{N}((A+B)^{1/2})$, we have

$$(A+B)_B = B^{1/2}VPV^*B^{1/2}. \tag{3.4}$$

On the other hand, by (3.3) again

$$B = B^{1/2}V\omega^* = B^{1/2}VW(B+X)^{1/2},$$

and since $\mathscr{N}(W) = \mathscr{N}(\omega\omega^*) = \mathscr{N}(B+X)$, we have

$$(X+B)_B = B^{1/2}VWW^*V^*B^{1/2} = B^{1/2}VPV^*B^{1/2}. \tag{3.5}$$

Now we obtain from (3.4) and (3.5)

$$X:B = B-(B+X)_B = B-(A+B)_B = A:B.$$

This implies that if $X \in \mathscr{E}(A, B)$ then $X \leqq \varrho_S(A)$. But by (2.1) and (1.3) we have

$$\varrho_S(A) = \gamma_B(A+B) - B = (A+B)^{1/2}P_{\mathscr{M}}(A+B)^{1/2} - B,$$

where $P_{\mathscr{M}}$ is the orthoprojection to the subspace $\mathscr{M} = \overline{(A+B)^{-1/2}\mathscr{R}(B)} \cap \overline{\mathscr{R}(A+B)}$. Thus $\varrho_S(A)$ admits a representation of the form (3.2), hence by 1) $\varrho_S(A) \in \mathscr{E}(A, B)$. Thus $\varrho_S(A) = \min \mathscr{E}(A, B)$, which is to prove.

Corollary 1. *If* $X = \alpha X_1 + (1-\alpha)X_2 \in \mathscr{E}(A, B)$, *where* $X_1, X_2 \in \mathscr{B}_+(\mathscr{H})$, $0<\alpha<1$, *then* $X_1 = X_2 \in \mathscr{E}(A, B)$.

If fact, according to inequality (2.6)

$$(A-X):(B+X) = \{\alpha(A-X_1)+(1-\alpha)(A-X_2)\}:\{\alpha(B+X_1)+(1-\alpha)(B+X_2)\} \geqq$$
$$\geqq \alpha(A-X_1):(B+X_1)+(1-\alpha)(A-X_2):(B+X_2).$$

Therefore if $X \in \mathscr{E}(A, B)$, then $X_1, X_2 \in \mathscr{E}(A, B)$. Denoting by $P$, $P_1$ and $P_2$ the orthoprojections, corresponding to $X$, $X_1$ and $X_2$ in (3.2) respectively, we have

$$P = \alpha P_1 + (1-\alpha) P_2,$$

clearly this last equality is possible only when $P=P_1=P_2$, hence $X=X_1=X_2$.

Corollary 2. *If $\mathcal{P}_A$ denotes the class of all orthoprojections to subspaces of $\overline{\mathcal{R}(A)}$, then*

$$\mathcal{E}(A, 0) = \{X \in [0, A]: X = A^{1/2} P A^{1/2},\ P \in \mathcal{P}_A\}. \tag{3.6}$$

Recall that $X$ is called an *extreme point* of a convex set $\mathcal{S}$, if the relation

$$X = \alpha X_1 + (1-\alpha) X_2$$

where $X_1, X_2 \in \gamma$, $0<\alpha<1$, is possible only when $X_1=X_2=X$. The class of all extreme points a set $\mathcal{S}$ is denoted by $\operatorname{ex}\mathcal{S}$.

Corollary 3. $\mathcal{E}(A, 0) = \operatorname{ex}[0, A]$.

In fact, if $X \in \mathcal{E}(A, 0)$ and $X=\alpha X_1+(1-\alpha)X_2$, where $X_1, X_2 \in [0, A]$, $0<\alpha<1$, then by Corollary 1 $X_1=X_2=X$, that is, $X \in \operatorname{ex}[0, A]$. If $X \notin \mathcal{E}(A, 0)$, then $X \neq X_1 \equiv \equiv X+(A-X):X \in [0, A]$, $X \neq X_2 \equiv X-(A-X):X \in [0, A]$ and $X=\frac{1}{2}(X_1+X_2)$, so that $X \notin \operatorname{ex}[0, A]$.

Remark. The relation $\mathcal{E}(A, 0) = \operatorname{ex}[0, A]$ is found in [13], where it is proved in essence that if $X_1, X_2 \in \operatorname{ex}[0, A]$ then $2(X_1:X_2) \in \operatorname{ex}[0, A]$. This follows also from Corollary 2 by Lemma 2.1, because

$$2(X_1:X_2) = 2((A^{1/2}P_1A^{1/2}):(A^{1/2}P_2A^{1/2})) = A^{1/2}(2(P_1:P_2))A^{1/2} = A^{1/2}P_{\mathcal{M}}A^{1/2},$$

where $P_1$ and $P_2$ are the orthoprojections corresponding to $X_1$ and $X_2$ through (3.6), while $P_{\mathcal{M}}$ is the orthoprojection to the subspace $\mathcal{M}=\mathcal{R}(P_1)\cap\mathcal{R}(P_2)$.

Let $A, B \in \mathcal{B}_+(\mathcal{H})$ and $C=A+B$. Remark that $X \in \operatorname{ex}[B, C]$ exactly then $X-B \in \operatorname{ex}[0, A]$ thus by Corollaries 2 and 3 to Theorem 3.1

$$\operatorname{ex}[0, A] = \{X \in \mathcal{B}_+(\mathcal{H}): X = A^{1/2}PA^{1/2},\ P \in \mathcal{P}_A\} \tag{3.7}$$

hence

$$\operatorname{ex}[B, C] = \{X \in \mathcal{B}_+(\mathcal{H}): X = A^{1/2}PA^{1/2}+B,\ P \in \mathcal{P}_A\}. \tag{3.8}$$

Obviously $\operatorname{ex}[B, C]$ contains all points of $\operatorname{ex}[0, C]$ in $[B, C]$. The converse is not true, if $B \notin \operatorname{ex}[0, C]$. Relation (3.8) implies that in this case $\operatorname{ex}[B, C]$ contains, together with $B$, other points not belonging to $\operatorname{ex}[0, C]$.

Indeed, in the contrary case, taking a sequence $\{P_n\} \subset \mathcal{P}_A$ such that $0 \neq P_n \to 0$ $(n \to \infty)$, we have a sequence $\{Q_n\} \subset \mathcal{P}_C$ for which

$$A^{1/2}P_nA^{1/2}+B = C^{1/2}Q_nC^{1/2} \quad (n = 1, 2, \ldots).$$

Since there exists a limit $Q=\lim_{n\to\infty} Q_n$, and $B=C^{1/2}QC^{1/2}$ $(Q\in\mathscr{P}_C)$, hence $B\in\operatorname{ex}[0, C]$, contradicting the assumption.

Theorem 3.2. *Let* $B\in[0, C]$ $(C\in\mathscr{B}_+(\mathscr{H}))$. *Then* $\operatorname{ex}[B, C]\subset\operatorname{ex}[0, C]$ *exactly when* $B\in\operatorname{ex}[0, C]$.

Proof. Since always $B\in\operatorname{ex}[B, C]$, it suffices to consider the case $B\in\operatorname{ex}[0, C]$. With $A=C-B$, remark that since $\operatorname{ex}[0, C]=\mathscr{E}(C, 0)$ we have $A:B=(C-B):B=0$. Thus by (2.8)

$$\mathscr{R}(A^{1/2})\cap\mathscr{R}(B^{1/2}) = \mathscr{R}((A:B)^{1/2}) = \{0\}.$$

Then, with $\mathscr{L}=\mathscr{R}(B^{1/2})$, for any $P\in\mathscr{P}_A$

$$\mathscr{R}((A-A^{1/2}PA^{1/2})^{1/2})\cap\mathscr{R}((A^{1/2}PA^{1/2}+B)^{1/2}) =$$

$$= \mathscr{R}(A^{1/2}(1-P))\cap(\mathscr{R}(A^{1/2}P)+\mathscr{L}) = \{0\},$$

hence

$$(A-A^{1/2}PA^{1/2}):(B+A^{1/2}PA^{1/2}) = 0.$$

Since this relation means that each operator $X=A^{1/2}PA^{1/2}+B$ is contained in $\mathscr{E}(C, 0)(P\in\mathscr{P}_A)$. Therefore in view of (3.8) we can conclude that $\operatorname{ex}[B, C]\subset\operatorname{ex}[0, C]$, what is to prove.

Remark. Let $A, B\in\mathscr{B}_+(\mathscr{H})$, and $C=A+B$. Then $B\in\operatorname{ex}[0, C]$ if and only if $A\in\operatorname{ex}[0, C]$. In this case it follows from the already proved fact and relation (3.8) that $\operatorname{ex}[0, A]\subset\operatorname{ex}[0, C]$. In other words, if $A=C^{1/2}PC^{1/2}$ $(P\in\mathscr{P}_C)$ and $X=A^{1/2}XA^{1/2}=A^{1/2}QA^{1/2}$ $(Q\in\mathscr{P}_A)$, then $X=C^{1/2}RC^{1/2}$ for some $R\in\mathscr{P}_C$.

Theorem 3.3. *Suppose that there are given an operator* $A\in\mathscr{B}_+(\mathscr{H})$ *and an operator range* $\mathscr{L}\subset\mathscr{H}$. *If* $\mathscr{L}=\mathscr{R}(B^{1/2})$ $(B\in\mathscr{B}_+(\mathscr{H}))$, *then*

$$A_{\mathscr{L}} = \min\{\operatorname{ex}[0, A+B]\cap\operatorname{ex}[B, A+B]\}-B.$$

Proof. In view of (3.7) and (3.8)

$$X\in\operatorname{ex}[0, A+B]\cap\operatorname{ex}[B, A+B]$$

if and only if there are representations

$$X = (A+B)^{1/2}P(A+B)^{1/2} = A^{1/2}P'A^{1/2}+B,$$

where $P\in\mathscr{P}_{A+B}$ and $P'\in\mathscr{P}_A$. Here $P$, which runs over all admissible elements of $\mathscr{P}_{A+B}$, can attain a minimum value for the operator

$$X-B = (A+B)^{1/2}P(A+B)^{1/2}-B.$$

In view of Theorem 3.1 this minimum value exists and coincides with $\varrho_S(A)$ $(S=A:B)$

$$X_{min}-B = \varrho_S(A).$$

On the other hand, by (2.15) $A_{\mathscr{L}}=\varrho_S(A)$. This proves the theorem.

It is easy to see that if $A\in\mathscr{B}_+(\mathscr{H})$ and $\mathscr{L}=\mathscr{R}(B^{1/2})=\mathscr{R}(B_1^{1/2})$, where $B, B_1\in\mathscr{B}_+(\mathscr{H})$, then equations $(A-X):(B+X)=0$ and $(A-X):(B_1+X)=0$ are equivalent. Thus their solutions, determined by $A$ and $\mathscr{L}$, will be denoted by $\mathscr{E}(A, \mathscr{L})(=\mathscr{E}(A, B)=\mathscr{E}(A, B_1))$.

Theorem 3.4. *For an operator $A\in\mathscr{B}_+(\mathscr{H})$ and an operator range $\mathscr{L}\subset\mathscr{H}$ the following relation holds*

$$\mathscr{E}(A, \mathscr{L}) = \operatorname{ex}[A_{\mathscr{L}}, A].$$

Proof. Indeed, if $X\in\mathscr{E}(A, \mathscr{L})$ and $\mathscr{L}=\mathscr{R}(B^{1/2})\ (B\in\mathscr{B}_+(\mathscr{H}))$, then

$$(A-X):X \leqq (A-x):(B+X) = 0,$$

hence $X\in\operatorname{ex}[0, A]$. But by Theorem 3.1 we have $X\in[A_{\mathscr{L}}, A]$, hence $X\in\operatorname{ex}[A_{\mathscr{L}}, A]$.

Suppose, conversely, that $X\in\operatorname{ex}[A_{\mathscr{L}}, A]$. Since by (1.8) $A_{\mathscr{L}}=A^{1/2}P_{\mathscr{M}}A^{1/2}$, where $P_{\mathscr{M}}$ is the orthoprojection to the subspace $\mathscr{M}=\overline{A^{-1/2}\mathscr{L}}$, we have $A_{\mathscr{L}}\in\operatorname{ex}[0, A]$ and by Theorem 3.2 $\operatorname{ex}[A_{\mathscr{L}}, A]\subset\operatorname{ex}[0, A]$. Thus $X\in\operatorname{ex}[0, A]$, that is, $(A-X):X=0$. But by (2.8) this relation means that

$$\mathscr{R}\big((A-X)^{1/2}\big)\cap\mathscr{R}(X^{1/2}) = \{0\}. \tag{3.9}$$

Further, it follows from the relations

$$A:B \leqq A_{\mathscr{L}} \leqq X \leqq A\big(B\in\mathscr{B}_+(\mathscr{H}),\ \mathscr{R}(B^{1/2}) = \mathscr{L}\big)$$

that

$$\mathscr{R}(A^{1/2})\cap\mathscr{R}(B^{1/2}) \subset \mathscr{R}(A_{\mathscr{L}}^{1/2}) \subset \mathscr{R}(X^{1/2}) \subset \mathscr{R}(A^{1/2}).$$

Now it is clear that $(A-X):(B+X)=0$, since

$$\mathscr{R}\big((A-X)^{1/2}\big)\cap\mathscr{R}\big((B+X)^{1/2}\big) = \{0\}.$$

In fact, if for some $f, g, h\in\mathscr{H}$

$$(A-X)^{1/2}f = B^{1/2}g+X^{1/2}h,$$

then $B^{1/2}g\in\mathscr{R}(A^{1/2})\cap\mathscr{R}(B^{1/2})\subset\mathscr{R}(X^{1/2})$ and by (3.9) $(A-X)^{1/2}f=0$. This completes the proof.

Remark. If $X\in[0, A]$, then $\mathscr{R}(X)\subset\mathscr{R}(X^{1/2})\subset\mathscr{R}(A^{1/2})$, that is, $A\in\{\mathscr{H}\}_X$ and by definition of parallel sum (2.2) $(A-X):X=X-A_X$. Consequently, $X\in\operatorname{ex}[0, A]$ exactly when $X=A_X$. For invertible $A$ this condition $(X=XA^{-1}X)$ is found in

[13], in which there is proved that

$$X \in \operatorname{ex}[0, A] \Leftrightarrow D_A X = X,$$

where by definition

$$D_A X = \sup_{\lambda > 0} \frac{X - Q_\lambda^A X}{\lambda} \quad \text{and} \quad Q_\lambda^A X = \sup_\lambda X : \frac{1}{\lambda} A.$$

Remark, in this connection, that there follows from Proposition 6.2 of [13] and formula (1.2) the relation $D_A X = A_X$, valid for all $X$ in the domain of definition of $D_A$, namely under the condition $A \in \{\mathscr{H}\}_X$.

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# A note on compact weighted composition operators on $L^p(\mu)$

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## 1. Introduction

Let $(X, \Sigma, \mu)$ be a complete $\sigma$-finite measure space and let $T$ be a measurable transformation from $X$ into itself, by which we mean $T^{-1}A \in \Sigma$ for every $A \in \Sigma$. Define the composition operator $C_T$ by $C_T = f \circ T$ for every $\Sigma$-measurable function $f$ on $X$. In order that functions which agree almost everywhere are mapped to functions with the same property, we require the measure $\mu \circ T^{-1}$ to be absolutely continuous with respect to $\mu$. For a fixed $\Sigma$-measurable function $\Theta$ on $X$ define the multiplication operator by $M_\Theta f = \Theta \cdot f$. The composite $M_\Theta \circ C_T$ is called a weighted composition operator. In this note we shall give a necessary and sufficient condition under which $M_\Theta \circ C_T$ is a compact operator on $L^p(\mu)$ $(1 \leqq p < \infty)$. The case $p=2$ has been investigated by SINGH and DHARMADHIKARI in [3] under the assumption that $C_T$ is a bounded operator on $L^2(\mu)$ with dense range. But as pointed out by CAMPBELL and JAMISON [1], one of the interesting features of a weighted composition operator is that the composition operator alone may not define a bounded operator on $L^p(\mu)$. To quote their example, let $T$ be the map $T(x)=x^2$ on $[0, 1]$. Then $C_T$ does not define a mapping on $L^1[0, 1]$. However with $\Theta(x) \equiv x$, $M_\Theta \circ C_T$ is bounded operator on $L^1[0, 1]$. A necessary and sufficient condition on $\Theta$ and $T$ under which $M_\Theta \circ C_T$ is a bounded operator on $L^p(\mu)$ is given in [2]. Before stating their result, some preliminaries are in order.

Let $T^{-1}\Sigma$ denote the relative completion of the $\sigma$-algebra generated by $\{T^{-1}A \colon A \in \Sigma\}$. If $f$ is a non-negative $\Sigma$-measurable function on $X$, then there exists a unique (a.e.) $T^{-1}\Sigma$-measurable function $E(f)$, called the conditional expectation of $f$ with respect to $T^{-1}\Sigma$, such that

$$\int_A f \, d\mu = \int_A E(f) \, d\mu \quad \text{for all} \quad A \in T^{-1}\Sigma.$$

We shall also need the following facts:

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(1) $f$ is $T^{-1}\Sigma$-measurable if and only if $f \equiv g \circ T$ for some $\Sigma$-measurable function $g$.

(2) $E(f \cdot g \circ T) = E(f) \cdot g \circ T$ whenever the conditional expectations are well-defined.

In view of (1) above we write $E(f) \circ T^{-1}$ to denote a function $g$ for which $E(f) \equiv g \circ T$.

Now let $h$ be the Radon—Nikodym derivative $\dfrac{d\mu \circ T^{-1}}{d\mu}$. Then $M_\Theta \circ C_T$ is a bounded operator on $L^p(\mu)$ if and only if $h \cdot E(|\Theta|^p) \circ T^{-1}$ is $\mu$-essentially bounded and in this case the operator norm of $M_\Theta \circ C_T$ equals $\|h \cdot E(|\Theta|^p) \circ T^{-1}\|_\infty^{1/p}$ [2, Proposition 2.1]. We shall include the proof for easy reference. For any $f \in L^p(\mu)$, we have

$$\int_X |\theta \cdot f \circ T|^p \, d\mu = \int_X |\theta|^p \cdot |f|^p \circ T \, d\mu = \int_X E(|\theta|^p \cdot |f|^p \circ T) \, d\mu =$$

$$= \int_X E(|\theta|^p) \cdot |f|^p \circ T \, d\mu = \int_X E(|\theta|^p) \circ T^{-1} \cdot |f|^p \, d\mu \circ T^{-1} =$$

$$= \int_X E(|\theta|^p) \circ T^{-1} \cdot |f|^p \cdot h \, d\mu = \int_X (h \cdot E(|\theta|^p) \circ T^{-1}) |f|^p \, d\mu.$$

The assertion follows immediately from the equations. We also note that as far as the condition is concerned, $E(|\Theta|^p) \circ T^{-1}$ does not depend on any particular choice of $g$ for which $E(|\Theta|^p) \equiv g \circ T$. For a thorough discussion of what appeared above, please consult [1] and [2].

## 2. The results

Theorem 2.1. *The weighted composition operator $M_\Theta \circ C_T$ is a compact operator on $L^p(\mu)$ if and only if for any $\varepsilon > 0$, the set $\{h \cdot E(|\Theta|^p) \circ T^{-1} \geqq \varepsilon\}$ consists of finitely many atoms.*

Proof. ($\Rightarrow$) Assume the contrary. Then for some $\varepsilon > 0$, the set

$$\{h \cdot E(|\Theta|^p) \circ T^{-1} \geqq \varepsilon\}$$

either contains a nonatomic subset or has infinitely many atoms. In both cases we can find a sequence of pairwise disjoint measurable subsets $\{A_n\}$ with $0 < \mu(A_n) < \infty$ for every $n$. Define $f_n = \mu(A_n)^{-1/p} \chi_{A_n}$. Then $\|f_n\| = 1$ and $\|M_\Theta \circ C_T f_n\|^p = \mu(A_n)^{-1} \int_X h \cdot E(|\theta|^p) \circ T^{-1} \cdot \chi_{A_n} \, d\mu \geqq \varepsilon$. When $n \neq m$, the functions $M_\Theta \circ C_T f_n$ and

$M_\Theta \circ C_T f_m$ have disjoint supports and hence $\|M_\Theta \circ C_T f_n - M_\Theta \circ C_T f_n\|^p > 2\varepsilon$. Therefore $M_\Theta \circ C_T$ is not compact.

($\Leftarrow$) Let $\varepsilon > 0$ and let $A = \{h \cdot E(|\theta|^p) \circ T^{-1} \geqq \varepsilon\}$. Put $\Theta' = \Theta \chi_{T^{-1}A}$. Then under the hypothesis that $A$ consists finitely many atoms, $M_{\Theta'} \circ C_T$ is a finite rank operator. For every $f \in L^p(\mu)$,

$$\|M_\theta \circ C_T(f) - M_{\theta'} \circ C_T(f)\|^p = \int_X |\theta \chi_{X \setminus T^{-1}A}|^p \cdot |f|^p \circ T \, d\mu =$$

$$= \int_X |\theta|^p \cdot (\chi_{X \setminus A} \circ T) \cdot (|f|^p \circ T) \, d\mu = \int_X hE(|\theta|^p) \circ T^{-1} \cdot |f|^p \chi_{X \setminus A} \, d\mu \leqq \varepsilon \|f\|^p.$$

So $M_\Theta \circ C_T$ is the limit of some finite rank operators and is therefore compact.

Corollary 2.2. *If $X$ is nonatomic, then a weight composition operator is not compact unless it is the zero operator.*

In [3, Theorem 3.6] Singh and Dharmadhikari assert that if $C_T$ is a bounded operator on $L^2(\mu)$ with dense range, then $M_\Theta \circ C_T$ is compact if and only if $\Theta \equiv 0$ on the set $\{h \circ T \neq 0\}$. A closer look at their proof reveals that the latter condition is equivalent to $|\Theta|^2 \cdot h \circ T \equiv 0$ a.e. But then $(M_\Theta \circ G) \circ (M_\Theta \circ G)^* \equiv M_{|\Theta|^2 \cdot h \circ T} \equiv 0$ gives $M_\Theta \circ C_T \equiv 0$.

If $X$ is $\mathbf{N}$, the set of natural numbers and if $\mu$ is the counting measure on $\mathbf{N}$, we denote as usual $L^p(\mu)$ by $l^p$. Suppose that $C_T$ does not define a bounded operator on $l^p$, then in contrast to [3, Theorem 3.3], the condition $\lim_{n \to \infty} \Theta(n) \equiv 0$ does not imply that $M_\Theta \circ G$ is compact.

Example 2.3. On $l^1$, consider the mapping

$$(x_1, x_2, x_3, \ldots) \mapsto \left(x_1, \frac{1}{2} x_2, \frac{1}{2} x_2, \frac{1}{3} x_3, \frac{1}{3} x_3, \frac{1}{3} x_3, \ldots\right).$$

This mapping can be realized as a weighted composition operator with $T(n) \equiv k$ and $\Theta(n) \equiv \frac{1}{k}$ whenever $\frac{(k-1)k}{2} < n \leqq \frac{k(k+1)}{2}$. A simple computation shows $h(n) \equiv n$ and $E(\Theta) \circ T^{-1}(n) \equiv \frac{1}{n}$. An appeal to Theorem 2.1 yields $M_\Theta \circ C_T$ is not compact. Actually this fact can be established by a direct argument. Let $\{e_n\}$ be the canonical basis in $l^1$. Then clearly $\{M_\Theta \circ C_T(e_n)\}$ does not have any norm-convergent subsequence.

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# Strongly dense simultaneous similarity orbits of operators

JOSÉ BARRÍA

## Introduction

Let $X$ be a (real or complex) Banach space and let $B(X)$ denote the algebra of all bounded linear operators on $X$. Let $B^{(n)}(X)$ denote the product $B(X)\times\ldots\times B(X)$ of $n$ copies of $B(X)$. The group of invertible operators in $B(X)$ acts on $B^{(n)}(X)$ by conjugation $A^{-1}(T_1, \ldots, T_n)A=(A^{-1}T_1A, \ldots, A^{-1}T_nA)$. For $(T_1, \ldots, T_n)$ in $B^{(n)}(X)$ denote by $S(T_1, \ldots, T_n)$ the orbit of $(T_1, \ldots, T_n)$ in $B^{(n)}(X)$,

$$S(T_1, \ldots, T_n) =$$
$$= \{A^{-1}(T_1, \ldots, T_n)A = (A^{-1}T_1A, \ldots, A^{-1}T_nA)\colon A \text{ is invertible in } B(X)\}.$$

The purpose of this paper is to describe those orbits $S(T_1, \ldots, T_n)$ which are strongly dense in $B^{(n)}(X)$. Recall that a net $\{S_\lambda\}$ in $B(X)$ converges strongly to an operator $S$ in $B(X)$ if and only if $\lim_\lambda S_\lambda f=Sf$ for all $f$ in $X$. If $X$ is finite-dimensional then the strong topology coincides with the norm topology, and therefore $S(T_1, \ldots, T_n)$ is never dense in $B^{(n)}(X)$. If $X$ is infinite-dimensional (and $n=1$), then $S(T)$ is strongly dense in $B(X)$ for a very large set of $T$'s. More precisely, in [2] it was shown that $S(T)$ is strongly dense if and only if $T$ is in the complement of the set $\{\lambda I+F\colon \lambda\in\mathbf{K},\ F \text{ has finite rank}\}$ ($\mathbf{K}$ is the field of scalars and $I$ is the identity operator on $X$). Observe that an operator $T$ is not a scalar plus a finite rank operator if and only if $\alpha_0I+\alpha_1T$ has infinite rank for all nonzero $(\alpha_0, \alpha_1)$ in $\mathbf{K}^2$. This suggests to consider those $n$-tuples $(T_1, \ldots, T_n)$ such that $\alpha_0I+\alpha_1T_1+\ldots+\alpha_nT_n$ has infinite rank for all nonzero $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ in $\mathbf{K}^{n+1}$. In this paper we show that this condition on $(T_1, \ldots, T_n)$ characterizes the strong density of $S(T_1, \ldots, T_n)$ in $B^{(n)}(X)$. Another result from [2] states that $S(T)$ is strongly dense if and only if $S(T)$ is weakly dense. The corresponding generalization to $n$-tuples is also true. From [1] it follows that the strong density of $S(T)$ can be described in terms of compressions. If $P$ is an idempotent in $B(X)$ with range $X_0$, then the

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compression of $S(T_1, \dots, T_n)$ to $X_0$ is defined as the restriction of $PS(T_1, \dots, P_n)P$ to $X_0$. Then for $n$-tuples the density of $S(T_1, \dots, T_n)$ is characterized by the condition that the compression of $S(T_1, \dots, T_n)$ to any finite-dimensional subspace $X_0(\subseteqq X)$ is equal to the full algebra $B^{(n)}(X_0)$.

## Preliminaries

Lemma 1. *Let $n$ be a fixed positive integer. For $1 \leqq i \leqq n$ and $m \geqq 1$, let $f_m^{(i)}$, $f^{(i)}$ be vectors in $X$ such that $f_m^{(i)} \to f^{(i)}$ $(m \to \infty)$. Let $g_m = \alpha_m^{(1)} f_m^{(1)} + \dots + \alpha_m^{(n)} f_m^{(n)}$, with $\alpha_m^{(i)} \in \mathbf{K}$. If $f^{(1)}, \dots, f^{(n)}$ are linearly independent and if the sequence $\{g_m\}_{m=1}^{\infty}$ converges, then there are scalars $\alpha^{(1)}, \dots, \alpha^{(n)}$ such that $\alpha_m^{(i)} \to \alpha^{(i)}$ $(m \to \infty)$ for $i = 1, \dots, n$.*

Proof. If $n=1$ we choose a bounded linear functional $\Phi$ on $X$ such that $\Phi(f_1)=1$, then $g_m = \alpha_m^{(1)} f_m^{(1)}$ implies that $\lim_{m\to\infty} \alpha_m^{(1)} = \lim_{m\to\infty} \Phi(g_m)$. Now we assume that $n \geqq 2$. The next step is to show that $\{|\alpha_m^{(1)}|\}_{m=1}^{\infty}$ cannot converge to infinity. Indeed, if $|\alpha_m^{(1)}| \to \infty$ $(m \to \infty)$, then the left hand side of

$$\frac{g_m}{\alpha_m^{(1)}} - f_m^{(1)} = \frac{\alpha_m^{(2)}}{\alpha_m^{(1)}} f_m^{(2)} + \dots + \frac{\alpha_m^{(n)}}{\alpha_m^{(1)}} f_m^{(n)}$$

converges to $-f^{(1)}$ and then the induction hypothesis can be applied to $f_m^{(2)}, \dots, f_m^{(n)}$ to conclude that there are scalars $\beta^{(2)}, \dots, \beta^{(n)}$ such that $-f^{(1)} = \beta^{(2)} f^{(2)} + \dots + \beta^{(n)} f^{(n)}$. This contradicts the fact that $f^{(1)}, \dots, f^{(n)}$ are linearly independent. The same reasoning applies to any subsequence of $\{|\alpha_m^{(1)}|\}_{m=1}^{\infty}$, therefore $\{\alpha_m^{(1)}\}_{m=1}^{\infty}$ is bounded. Next, let $\{m_k\}_{k=1}^{\infty}$ be an increasing sequence of positive integers such that $\alpha_{m_k}^{(1)} \to \alpha^{(1)}$ $(k \to \infty)$ for some scalar $\alpha^{(1)}$. Then from the induction hypothesis it follows that there are scalars $\alpha^{(2)}, \dots, \alpha^{(n)}$ such that $\alpha_{m_k}^{(i)} \to \alpha^{(i)}$ $(k \to \infty)$ for $i = 1, \dots, n$. Since $f^{(1)}, \dots, f^{(n)}$ are linearly independent, the scalars $\alpha^{(1)}, \dots, \alpha^{(n)}$ are independent of the sequence $\{m_k\}_{k=1}^{\infty}$. Then it follows that $\alpha_m^{(i)} \to \alpha^{(i)}$ $(m \to \infty)$ for $i = 1, \dots, n$.

Lemma 2. *Let $T_1, T_2, \dots, T_n \in B(X)$. Assume that for every vector $f$ in $X$ the set $\{T_1 f, T_2 f, \dots, T_n f\}$ is linearly dependent. Then there is a nonzero $n$-tuple $(\alpha_1, \dots, \alpha_n)$ in $\mathbf{K}^n$ such that $\alpha_1 T_1 + \dots + \alpha_n T_n$ has rank less than or equal to $n-1$.*

Proof. If $n=1$ then the hypothesis reduces to $T_1 f = 0$ for all $f$ in $X$, and the conclusion holds. Assume that $n \geqq 2$. Let $D$ be the set of all vectors $f$ in $X$ such that $\{T_1 f, \dots, T_{n-1} f\}$ is linearly dependent. If $D = X$ then the conclusion follows by induction. Assume that $D \neq X$. An easy compactness argument in $\mathbf{K}^n$ implies that $D$ is a closed set. For every vector $h$ in $X \setminus D$ (the complement of $D$) the set $\{T_1 h, \dots, T_{n-1} h\}$ is linearly independent; then from the linear dependence of $\{T_1 h, \dots, T_{n-1} h, T_n h\}$ it follows that there are functions $\alpha_1, \dots, \alpha_{n-1}$ from $X \setminus D$

to $\mathbf{K}$ such that

(1) $$\alpha_1(h)T_1h+\ldots+\alpha_{n-1}(h)T_{n-1}h+T_nh=0 \quad \text{for all } h \text{ in} X\setminus D.$$

Let $f$ be a fixed vector in $X\setminus D$, and let $M$ be the subspace spanned by $\{T_1f, \ldots, T_{n-1}f\}$. The proof will be completed by showing that the range of $\alpha_1(f)T_1+\ldots+\alpha_{n-1}(f)T_{n-1}+T_n$ is contained in $M$. Let $g$ be an arbitrary vector in $X$. Since $X\setminus D$ is open, there is a positive $\delta$ such that $f+\lambda g\in X\setminus D$ for $|\lambda|<\delta$. If $|\lambda|<\delta$, from (1) we obtain

(2) $$\alpha_1(f+\lambda g)T_1(f+\lambda g)+\ldots+\alpha_{n-1}(f+\lambda g)T_{n-1}(f+\lambda g)+T_n(f+\lambda g)=0,$$

and (with $\lambda=0$)

(3) $$\alpha_1(f)T_1f+\ldots+\alpha_{n-1}(f)T_{n-1}f+T_nf=0.$$

Subtracting (3) from (2) we get

$$\lambda[\alpha_1(f+\lambda g)T_1g+\ldots+\alpha_{n-1}(f+\lambda g)T_{n-1}g+T_ng]=$$
$$=[\alpha_1(f)-\alpha_1(f+\lambda g)]T_1f+\ldots+[\alpha_{n-1}(f)-\alpha_{n-1}(f+\lambda g)]T_{n-1}f$$

which implies that

(4) $$\alpha_1(f+\lambda g)T_1g+\ldots+\alpha_{n-1}(f+\lambda g)T_{n-1}g+T_ng\in M \quad \text{for} \quad 0<|\lambda|<\delta.$$

Let $\{\lambda_m\}_{m=1}^{\infty}$ be a sequence of scalars such that $\lambda_m\to 0$ $(m\to\infty)$. If we define $f_m^{(i)}=$ $=T_i(f+\lambda_m g)$ $(1\leqq i\leqq n-1)$, then $f_m^{(i)}\to T_if$ $(m\to\infty)$, and $T_1f, \ldots, T_{n-1}f$ are linearly independent. Then, using (2), we can apply Lemma 1, with $g_m=-T_n(f+\lambda_m g)$, to conclude that $\alpha_i(f+\lambda_m g)\to\alpha^{(i)}$ $(m\to\infty)$ for $i=1, \ldots, n-1$. Then, from (2) again, $\alpha^{(1)}T_1f+\ldots+\alpha^{(n-1)}T_{n-1}f+T_nf=0$, and comparing with (3) it follows that $\alpha^{(i)}=\alpha_i(f)$ for $i=1, \ldots, \ldots, n-1$. This shows that the functions $\lambda\to\alpha_i(f+\lambda g)(|\lambda|<\delta)$ are continuous at $\lambda=0$ in every direction. Since $M$ is a closed subspace, from (4) we conclude that $\alpha_1(f)T_1g+\ldots+\alpha_{n-1}(f)T_{n-1}g+T_ng\in M$. Since $g$ is an arbitrary vector, then the range of $\alpha_1(f)T_1+\ldots+\alpha_{n-1}(f)T_{n-1}+T_n$ is contained in $M$.

Lemma 3. *Let $T_1, T_2, \ldots, T_n\in B(X)$. Assume that every nontrivial linear combination of $T_1, \ldots, T_n$ has infinite rank. Then given a positive integer $m$ there are vectors $f_1, \ldots, f_m$ in $X$ such that $\{T_if_j\colon 1\leqq i\leqq n, 1\leqq j\leqq m\}$ is a linearly independent set.*

Proof. If $f_1, \ldots, f_m$ are vectors in $X$ then we denote by $L(f_1, \ldots, f_m)$ the set $\{T_if_j\colon 1\leqq i\leqq n, 1\leqq j\leqq m\}$. If $m=1$, then what is wanted is a vector $f$ in $X$ such that $T_1f, \ldots, T_nf$ are linearly independent. If this is not true then Lemma 2 implies that some nontrivial linear combination of $T_1, \ldots, T_n$ has finite rank. Since this contradicts the hypothesis, the lemma holds for $m=1$. Now we assume that $L(f_1, \ldots, f_m)$ is a linearly independent set for some vectors $f_1, \ldots, f_m$. Let $M$ be the subspace spanned by $L(f_1, \ldots, f_m)$ and let $N$ be a closed subspace which is a complement of

$M$ (i.e., $X=M+N$ and $M\cap N=(0)$). Let $P$ be the idempotent in $B(X)$ with range $N$ and null space $M$. Since $T_i=(I-P)T_i+PT_i(I-P)+PT_iP$, and since $I-P$ has finite rank, then every nontrivial linear combination of $PT_1P, \ldots, PT_nP$ has infinite rank. Now from the first part of the proof it follows that there is a vector $g$ in $N$ such that $PT_1g, \ldots, PT_ng$ are linearly independent. If we define $f_{m+1}=g$, then $L(f_1, \ldots, f_m, f_{m+1})$ is linearly independent. Indeed, if $\sum_{i=1}^{n}\sum_{j=1}^{m+1} \alpha_{ij}T_if_j=0$, and since $P$ annihilates $L(f_1, \ldots, f_m)$, it follows that $\sum_{i=1}^{n} \alpha_{i,m+1}PT_ig=0$, and therefore $\alpha_{i,m+1}=0$ for $i=1, \ldots, n$; finally, since $L(f_1, \ldots, f_m)$ is linearly independent we conclude that $\alpha_{ij}=0$ for all $i$ and $j$.

## Density

Theorem 4. *Let $T_1, T_2, \ldots, T_n \in B(X)$. Assume that every nontrivial linear combination of $I$, $T_1, \ldots, T_n$ has infinite rank. Then the similarity orbit $S(T_1, \ldots, T_n)$ is strongly dense in $B^{(n)}(X)$.*

Proof. Let $\tilde{S}=(S_1, \ldots, S_n)\in B^{(n)}(X)$ and let $U$ be a strong neighborhood of $\tilde{S}$. Then there are linearly independent vectors $e_1, \ldots, e_m$ in $X$ and a positive number $\varepsilon$ such that $U$ contains

$$\{(A_1, \ldots, A_n)\in B^{(n)}(X)\colon \|(A_i-S_i)e_j\| < \varepsilon,\ 1 \leqq i \leqq n,\ 1 \leqq j \leqq m\}.$$

Let $M$ be the span of $\{e_1, \ldots, e_m\}$. Let $N$ be a complement of the subspace $M+S_1M+\ldots+S_nM$. Since $N$ is infinite-dimensional, we can choose in $N$ a set $\{h_{ij}\colon 1\leqq i\leqq n, 1\leqq j\leqq m\}$ of linearly independent vectors such that $\|h_{ij}\|<\varepsilon$ for all $i, j$. Let $f_{ij}=S_ie_j+h_{ij}$. Then the set $\{e_i, f_{ij}\colon 1\leqq i\leqq n, 1\leqq j\leqq m\}$ is linearly independent and $\|S_ie_j-f_{ij}\|<\varepsilon$ for all $i$ and $j$. We apply Lemma 3 to $I$, $T_1, \ldots, T_n$ to find vectors $f_1, \ldots, f_m$ in $X$ such that $\{f_j, T_if_j\colon 1\leqq i\leqq n, 1\leqq j\leqq m\}$ is a linearly independent set. If $A$ is an invertible operator on $X$ such that $Ae_j=f_j$ and $Af_{ij}=T_if_j$ for $1\leqq i\leqq n$ and $1\leqq j\leqq m$, then

$$\|(A^{-1}T_iA-S_i)e_j\| = \|A^{-1}T_if_j-S_ie_j\| = \|A^{-1}Af_{ij}-S_ie_j\| = \|f_{ij}-S_ie_j\| < \varepsilon$$

for all $i$ and $j$. Therefore $(A^{-1}T_1A, \ldots, A^{-1}T_nA)\in U$, and $S(T_1, \ldots, T_n)$ is strongly dense in $B^{(n)}(X)$.

Theorem 5. *Let $T_1, T_2, \ldots, T_n\in B(X)$. Assume that every nontrivial linear combination of $I$, $T_1, \ldots, T_n$ has infinite rank. Then the compression of $S(T_1, \ldots, T_n)$ to a given finite-dimensional subspace $M$ is equal to $B^{(n)}(M)$. More precisely, if $P$ is an idempotent in $B(X)$ with range $M$, then the restriction of $PS(T_1, \ldots, T_n)P$ to $M$ is $B^{(n)}(M)$.*

Proof. Let $P$ be a fixed idempotent in $B(X)$ with range $M$. Let $(F_1, \ldots, F_n)$ be arbitrary in $B^{(n)}(M)$. Let $T_0=I$ and $m=\dim M$. By Lemma 3 there are vectors $f_1, \ldots, f_m$ such that $\{T_i f_j\colon 0\leqq i\leqq n,\ 1\leqq j\leqq m\}$ is a linearly independent set. For $0\leqq i\leqq n$ let $N_i$ be the subspace spanned by $\{T_i f_1, \ldots, T_i f_m\}$. We choose linearly independent subspaces $M_0, M_1, \ldots, M_n$ (i.e., $g_i\in M_i$ and $g_0+g_1+\ldots+g_n=0$ imply that $g_i=0$ for all $i$) satisfying the following conditions: $M_0=M$, $M_i\subset\ker P$ for $1\leqq i\leqq n$, and $\dim M_i=m$ for all $i$. Let $B\in B(X)$ be an invertible operator such that $BM_i=N_i$ for $0\leqq i\leqq n$. Let $S_i=B^{-1}T_iB$ $(1\leqq i\leqq n)$. Then

$$BS_i(M) = T_iBM_0 = T_iN_0 = N_i = BM_i$$

and therefore $S_iM=M_i$. In particular, $S_i$ is injective on $M$, and we can find $C_i\in B(M_i, M)$ such that $C_iS_if=-F_if$ for all $f$ in $M$. Let $M_{n+1}$ be a subspace of $\ker P$ which is a complement (in $\ker P$) of the subspace $M_1+M_2+\ldots+M_n$. Then $X=M_0+M_1+\ldots+M_{n+1}$, and we use this decomposition of $X$ to define the operator $C$ on $X$ given by the $(n+2)\times(n+2)$ operator matrix,

$$C = \begin{bmatrix} I & C_1 & C_2 \ldots & C_n & 0 \\ 0 & I & 0 \ldots & 0 & 0 \\ & & \ddots & \vdots & \vdots \\ & 0 & & I & 0 \\ & & & 0 & I \end{bmatrix}.$$

Then $C$ is invertible, and $C^{-1}$ is the operator matrix whose first row is $[I, -C_1, -C_2, \ldots, -C_n, 0]$, and the other rows are identical to the corresponding rows of $C$. Now for $f\in M$ and $1\leqq i\leqq n$ we have (denoting the $(i+1)$-th component of the vector $f$ by $S_if$)

$$C^{-1}S_iCf = C^{-1}S_iC\langle f, 0, \ldots, 0\rangle = C^{-1}S_i\langle f, 0, \ldots, 0\rangle =$$
$$= C^{-1}\langle 0, \ldots, 0, S_if, 0, \ldots, 0\rangle = \langle -C_iS_if, *, \ldots, *\rangle$$

(the third equality follows from $S_iM=M_i$), and therefore $PC^{-1}S_iCf=-C_iS_if=$ $=F_if$. Finally, with $A=BC$, the restriction of $PA^{-1}T_iA$ to $M$ is $F_i$ for $i=1, \ldots, n$.

Corollary 6. *Let $T_1, T_2, \ldots, T_n\in B(X)$. The following statements are equivalent:*

(1) *$S(T_1, \ldots, T_n)$ is strongly dense in $B^{(n)}(X)$.*

(2) *$S(T_1, \ldots, T_n)$ is weakly dense in $B^{(n)}(X)$.*

(3) *Every nontrivial linear combination of $I$, $T_1, \ldots, T_n$ has infinite rank.*

(4) *For every finite-dimensional subspace $M$ of $X$ the compression of $S(T_1, \ldots, T_n)$ to $M$ is equal to $B^{(n)}(M)$.*

Proof. Since the strong topology is finer than the weak topology, then (1) implies (2). Next we assume that some linear combination $\alpha_0I+\alpha_1T_1+\ldots+\alpha_nT_n=F$

has finite rank and $(\alpha_0, \alpha_1, \ldots, \alpha_n) \neq 0$. Let $(S_1, \ldots, S_n) \in S(T_1, \ldots, T_n)$. Then there is an invertible operator $A$ on $X$ such that $S_i = A^{-1} T_i A$ for $1 \leqq i \leqq n$. Therefore $\alpha_0 I + \alpha_1 S_1 + \ldots + \alpha_n S_n = A^{-1} F A$ and $\operatorname{rank}(\alpha_0 I + \alpha_1 S_1 + \ldots + \alpha_n S_n) = \operatorname{rank} F < \infty$. Since the set $\{S \in B(X)\colon \operatorname{rank} S \leqq \operatorname{rank} F\}$ is weakly closed, it follows that the weak closure of $S(T_1, \ldots, T_n)$ is contained in the set

$$\{(S_1, \ldots, S_n) \in B^{(n)}(X)\colon \operatorname{rank}(\alpha_0 I + \alpha_1 S_1 + \ldots + \alpha_n S_n) \leqq \operatorname{rank} F\},$$

and this set is smaller than $B^{(n)}(X)$. Hence (2) implies (3). Now by Theorem 4 we conclude that (1), (2), and (3) are equivalent. By Theorem 5, (3) implies (4). Now we assume that (4) holds. Let $(\alpha_0, \alpha_1, \ldots, \alpha_n) \neq 0$. Let $M$ be an arbitrary finite-dimensional subspace of $X$. Choose $(F_1, \ldots, F_n)$ in $B^{(n)}(M)$ such that $\alpha_0 I + \alpha_1 F_1 + \ldots + \alpha_n F_n = I$ (the identity on $M$). By (4), there is an invertible operator $A$ on $X$ such that the compression of $A^{-1} T_i A$ to $M$ is $F_i$ $(1 \leqq i \leqq n)$. Then

$$\operatorname{rank}(\alpha_0 I + \alpha_1 T_1 + \ldots + \alpha_n T_n) = \operatorname{rank} A^{-1}(\alpha_0 I + \alpha_1 T_1 + \ldots + \alpha_n T_n) A \geqq$$
$$\geqq \operatorname{rank}(\alpha_0 I + \alpha_1 F_1 + \ldots + \alpha_n F_n) = \dim M.$$

Since $M$ is arbitrary, we conclude that $\alpha_0 I + \alpha_1 T_1 + \ldots + \alpha_n T_n$ has infinite rank. This shows that (4) implies (3).

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# $\varrho$-dilations and hypo-Dirichlet algebras

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**1. Introduction.** Let $X$ be a compact Hausdorff space, let $C(X)$ be the algebra of complex-valued continuous functions on $X$, and let $A$ be a uniform algebra on $X$. Let $H$ be a complex Hilbert space and $L(H)$ the algebra of all bounded linear operators on $H$. $I=I_H$ is the identity operator in $H$. An algebra homomorphism $f\to T_f$ of $A$ in $L(H)$, which satisfies

$$T_1 = I \quad \text{and} \quad \|T_f\| \leqq \|f\|$$

is called a representation of $A$ on $H$. A representation $\varphi\to U_\varphi$ of $C(X)$ on a Hilbert space $K$ is called a $\varrho$-dilation of the representation $f\to T_f$ of $A$ if $H$ is a Hilbert subspace of $K$ and

$$T_f = \varrho PU_f|H \quad (f\in A_\tau)$$

where $P$ is the orthogonal projection of $K$ onto $H$, $A_\tau$ is the kernel of a nonzero complex homomorphism $\tau$ of $A$, and $0<\varrho<\infty$.

If the uniform closure of $A+\bar{A}$, that is, $[A+\bar{A}]$ has finite codimension in $C(X)$ then $A$ is called a hypo-Dirichlet algebra and it is called a Dirichlet algebra when $[A+\bar{A}]=C(X)$. If $A$ is a Dirichlet algebra on $X$ and $f\to T_f$ a representation of $A$ on $H$, then there exists a 1-dilation (cf. [7], [5]). It is known that only two hypo-Dirichlet (non-Dirichlet) algebras have 1-dilations [1], [9]. R. G. DOUGLAS and V. I. PAULSEN [4, Corollary 2.3] showed that an operator representation of a hypo-Dirichlet algebra is similar to an operator representation which has a 1-dilation.

In this paper, using their method we show that many natural hypo-Dirichlet algebras have $\varrho$-dilations. Then it follows that their representations are similar to those which have 1-dilations. A well known theorem of T. ANDO [2] shows that the

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bidisc algebra has 1-dilation. The theory of spectral sets is concerned with determing when a particular set $Y$ in $\mathbf{C}$ is spectral for an operator $T$ and if it is, deciding whether or not $T$ possesses a normal 1-dilation whose spectrum is contained in $\partial Y$. Our main theorem shows that $T$ possesses a normal $\varrho$-dilation. The other applications are related with $\varrho$-contractions (cf. [8]).

**2. Main theorem.** Let $A$ be a hypo-Dirichlet algebra with $\dim C(X)/[A+\bar{A}]=$ $=n<\infty$. Fix $\tau$ a non-zero complex homomorphism of $A$ and let $N_\tau$ be the set of all representing measures of $\tau$. Then $\dim N_\tau=n$ and there exists a core measure $m$ of $N_\tau$ (cf. [6, p. 106]). Hence if $v\in N_\tau$ then $v=hdm$ and $h\in L^\infty(m)$ where $L^\infty(m)$ denotes the usual Lebesgue space. Thus $N_\tau$ can be considered as a subset of $L^\infty(m)$. In this paper we put a natural condition on $N_\tau$: $N_\tau\subset C(X)$. Many important hypo-Dirichlet algebras satisfy it.

Theorem. *Let $A$ be a hypo-Dirichlet algebra and let $\tau$ be a nonzero complex homomorphism with $N_\tau\subset C(X)$. Then a representation of $A$ has a $\varrho$-dilation with respect to $\tau$.*

Proof. Let $\{u_j\}_{j=1}^n$ be a normalized orthogonal basis in the real linear span of $(N_\tau-N_\tau)$ (r.l.s. of $(N_\tau-N_\tau)$) with respect to the inner product in $L^2(m)$ where $m$ is a core measure of $N_\tau$. Then for $1\leqq j\leqq n$

$$u_j = \sum_{l=1}^{n} \alpha_l^{(j)}(h_l^{(j)}-k_l^{(j)})$$

where each $\alpha_l^{(j)}$ is a real constant and $h_l^{(j)}, k_l^{(j)}\in N_\tau$. Put for $v\in C(X)$

$$\Phi(v) = v-\sum_{j=1}^{n}\Big(\int vu_j\,dm\Big)u_j+s(v)$$

where

$$s(v) = \sum_{j=1}^{n}\Big(\sum_{l=1}^{n}|\alpha_l^{(j)}|\int v(h_l^{(j)}+k_l^{(j)})\,dm\Big)\|u_j\|_\infty.$$

Then $\Phi$ is a positive map from $C(X)$ to $[A+\bar{A}]$, $\Phi(1)=1+s(1)$ and

$$s(1) = 2\sum_{j=1}^{n}\Big(\sum_{l=1}^{n}|\alpha_l^{(j)}|\Big)\|u_j\|_\infty<\infty. \tag{1}$$

In fact, since $N_\tau\subset C(X)$,

$$[A+\bar{A}]\oplus[N_\tau-N_\tau] = C(X)$$

where $\oplus$ denotes the orthogonal direct sum of $L^2(m)$. Hence if $v\in C(X)$ then

$$v = v_1+v_2$$

where $v_1 \in [A+\bar{A}]$ and $v_2 \in [N_\tau - N_\tau]$, consequently

$$\Phi(v) = v_1 + s(v)$$

and therefore $\Phi(v) \in [A+\bar{A}]$, and the positivity and the finiteness of $\Phi(1)$ are clear. If $f \in A_\tau$ then

$$\Phi(f) = f \tag{2}$$

because $s(f)=0$. This is different from Lemma 2.1 in [4].

If we extend $T$ to $\tilde{T}$: $[A+\bar{A}] \to L(H)$ by $\tilde{T}_{f+\bar{g}} = T_f + T_g^*$, then $\tilde{T}$ is positive by [3, p. 152—153]. Thus $\Phi(1)^{-1}\tilde{T}\circ\Phi$: $C(X) \to L(H)$ is positive and $\Phi(1)^{-1}\tilde{T}\circ\Phi(1) = I_H$. By the dilation theorem of M. A. Naimark (cf. [10, Theorem 7.5]) there exists a Hilbert space $K$, an orthogonal projection $P$: $H \to K$ and a multiplicative linear map $\varphi \to U_\varphi$ of $C(X)$ in $L(K)$, which satisfies $U_1 = I_K$, $\|U_\varphi\| \leqq \|\varphi\|$, $\varphi \in C(X)$ and

$$\tilde{T}\circ\Phi(\varphi) = \Phi(1) P U_\varphi | H.$$

By (2), if $f \in A_\tau$,

$$T_f = \Phi(1) P U_f | H.$$

Corollary. *Suppose* $\dim N_\tau = 1$ *in Theorem, then*

$$\varrho = \inf \left\{ \frac{2\|h-k\|_\infty}{\int |h-k|^2\, dm} : h, k \in N_\tau \right\} + 1.$$

Proof. By (1) in the proof of Theorem with $n=1$

$$\Phi(1) = 2|\alpha_1^{(1)}|\, \|u_1\|_\infty + 1$$

where

$$u_1 = \alpha_1^{(1)}(h_1^{(1)} - k_1^{(1)}), \quad h_1^{(1)}, k_1^{(1)} \in N_\tau$$

and

$$|\alpha_1^{(1)}|^2 \int |h_1^{(1)} - k_1^{(1)}|^2\, dm = 1.$$

This implies the corollary.

We concentrated in unital contractive homomorphism but our technique can be used for unital contractions.

**3. Concrete examples.** In this section we will calculate $\varrho$ of $\varrho$-dilation in few concrete examples or apply Theorem to them.

(1) Let $n$ be a positive integer and $Y_i$ $(1 \leqq i \leqq n)$ disjoint compact subsets of $\mathbf{C}$ with non-empty interior $Y_i^0$. Suppose $R(Y_i)|X_i$ is a Dirichlet algebra on $X_i$ where $R(Y_i)$ denotes the uniform closure of the set of the rational functions with poles off $Y_i$ and $X_i$ is the boundary of $Y_i$. Put $X = \bigcup_{i=1}^{n} X_i$ and $Y = \bigcup_{i=1}^{n} Y_i$, then $X$ is the boundary of $Y$ and $R(Y)|X$ is a Dirichlet algebra on $X$. Put

$$A = \{f \in R(Y)|X : f(x_i) = f(x_1) \text{ for } i > 1\}$$

where $x_i \in Y_i^0$ $(1 \leqq i \leqq n)$. $A$ is a uniform algebra on $X$ and if $n>1$ then $A$ is not a Dirichlet algebra but a hypo-Dirichlet algebra.

A representation of $A$ has a $\varrho$-dilation with $\varrho = n$.

Proof. Let $\tau(f) = f(x_1)$ then $\tau$ is a nonzero complex homomorphism. Put $u_i$ be a characteristic function of $X_i$ $(1 \leqq i \leqq n)$ and let $D$ be the commutative $C^*$-algebra generated by $\{u_i\colon 1 \leqq i \leqq n\}$. Then $A_\tau D \subset A_\tau$, $A_\tau + \bar{A}_\tau + D$ is uniformly dense in $C(X)$ and $\dim D = n$. Let $m_i$ be a harmonic measure of $x_i$ $(1 \leqq i \leqq n)$ and $m = \sum_{i=1}^{n} m_i/n$ then $m$ is a representing measure of $\tau$. In the proof of Theorem, put

$$\Phi(v) = v - \sum_{j=1}^{n} \frac{1}{m(X_j)} \int_{X_j} v \, dm \, u_j + s(v) \quad (v \in C(X))$$

and

$$s(v) = \sum_{j=1}^{n} \frac{1}{m(X_j)} \int_{X_j} v \, dm.$$

Then $\Phi$ is a positive map from $C(X)$ to $[A+\bar{A}]$, and if $f \in A_\tau$ then $\Phi(f) = f$ and $\Phi(1) = n$. This can be shown as in the proof of Theorem because

$$[A_\tau + \bar{A}_\tau] \oplus D = C(X)$$

and $DA_\tau \subset A_\tau$. Thus a representation of $A$ has a $\varrho$-dilation with $\varrho = \Phi(1) = n$.

If $\dim D = n$ then $A$ is one kind of hypo-Dirichlet algebras of finite codimension $n-1$. By a theorem of R. G. Douglas and V. I. Paulsen [4] the completely bounded norm of the representation $T$ of $A$, $\|T\|_{cb} \leqq 2n-1$ but our result implies $\|T\|_{cb} \leqq$ $\leqq n-1$.

(2) Let $\mathscr{A}$ be the disc algebra on the circle $\Gamma$ and

$$A = \{f \in \mathscr{A}\colon f'(0) = \ldots = f^{(n)}(0)\}$$

where $f^{(j)}(0)$ denotes the $j$-derivative at the origin. Then $A$ is a hypo-Dirichlet algebra on $X = \Gamma$ and $\dim C(\Gamma)/[A+\bar{A}] = 2n$.

A representation of $A$ has a $\varrho$-dilation with $\varrho = 8n+1$.

Proof. $d\theta/2\pi$ is the core measure of $N_\tau$ where $\tau(f) = f(0)$. Then

$$\text{r.l.s.}(N_\tau - N_\tau) = \text{r.l.s.}(\cos\theta, \cos 2\theta, \ldots, \cos n\theta; \sin\theta, \sin 2\theta, \ldots, \sin n\theta).$$

In the proof of Theorem, put for $v \in C(\Gamma)$

$$\Phi(v) = v - 2\sum_{j=1}^{n} \left\{ \left( \frac{1}{2\pi} \int v \sin j\theta \, d\theta \right) \sin j\theta + \left( \frac{1}{2\pi} \int v \cos j\theta \, d\theta \right) \cos j\theta \right\} + s(v)$$

and

$$s(v) = 2\sum_{j=1}^{n} \left\{ \frac{1}{2\pi} \int v(2 - \sin j\theta) \, dm + \frac{1}{2\pi} \int v(2 - \cos j\theta) \, d\theta \right\}.$$

Then $\Phi$ is a positive map from $C(\Gamma)$ to $[A+\bar{A}]$, if $f\in A_\tau$ then $\Phi(f)=f$ and $\Phi(1)= =8n+1$. In fact, since

$$\Phi(v) = v+2\sum_{j=1}^{n}\left\{\left(\frac{1}{2\pi}\int v\,d\theta\right)(1-\sin j\theta)+\left(\frac{1}{2\pi}\int v(1-\sin j\theta)\,d\theta\right)(1+\sin j\theta)+\right.$$

$$\left.+\left(\frac{1}{2\pi}\int v\,d\theta\right)(1-\cos j\theta)+\left(\frac{1}{2\pi}\int v(1-\cos j\theta)\,d\theta\right)(1+\cos j\theta)\right\},$$

$\Phi$ is positive. The other statements are clear. Thus a representation of $A$ has a $\varrho$-dilation with $\varrho=\Phi(1)=8n+1$.

(3) Let $a_1, \ldots, a_n$ be distinct points in the open unit disc and

$$A = \{f\in\mathscr{A}: f(a_j) = f(0),\ j = 1, \ldots, n\}.$$

Then $A$ is a hypo-Dirichlet algebra on $X=\Gamma$ and $\dim C(\Gamma)/[A+\bar{A}]=2n$. $d\theta/2\pi$ is the core measure of $N_\tau$ where $\tau(f)=f(0)$. Then $N_\tau\subset C(\Gamma)$ and hence we can apply Theorem to this hypo-Dirichlet algebra.

(4) Let $Y$ be a compact subset of $\mathbf{C}$ and let $R(Y)$ be the uniform closure of the set of rational functions in $C(Y)$. Suppose the complement $Y^C$ of $Y$ has a finite number $n$ of components and $Y^0$ is a nonempty connected set. Let $A=R(Y)|X$ where $X$ is the boundary of $Y$ and $\tau$ a nonzero complex homomorphism defined by the evaluation at a point $t$ in $Y^0$. Then $A$ is a hypo-Dirichlet algebra on $X$ and $\dim C(X)/[A+\bar{A}]=n$. If $m$ is a harmonic measure for $t$ then $m$ is a core measure in $N_\tau$ and $N_\tau\subset C(X)$. Hence we apply Theorem to this hypo-Dirichlet algebra and hence a representation of $A$ has a $\varrho$-dilation.

In the four examples we concentrated in unital contractive homomorphisms our technique can be used for unital contractions.

(5) Let

$$A = \{f\in\mathscr{A}: f(0) = f(1)\}.$$

Then $A$ is a hypo-Dirichlet algebra on $X=\Gamma$ and $\dim C(\Gamma)/[A+\bar{A}]=1$. $(d\theta/2\pi+d\delta_1)/2$ is the core measure of $N_\tau$ where $\tau(f)=f(0)=f(1)$ and $\delta_1$ is a dirac measure at 1. Then $N_\tau$ can not be embedded in $C(\Gamma)$ and hence we can not Theorem to this hypo-Dirichlet algebra. However the author [9] showed previously by the different method that a representation of $A$ has a 1-dilation.

**4. Normal $\varrho$-dilation.** Results in this section are corollaries of Theorem and Examples (2)—(4).

Corollary 1. *If $T\in L(H)$ and*

$$\|f(T)\| \leqq \sup_{|z|\leqq 1}|f(z)|$$

*for all analytic polynomials* $f$ *with* $f'(0)=\dots=f^{(n)}(0)$, *then there exists a Hilbert space* $K\supseteq H$ *and a unitary operator* $U$ *on* $K$ *such that*

$$f(T) = (8n+1)Pf(U)|K$$

*for all analytic polynomials with* $f(0)=f'(0)=\dots=f^{(n)}(0)=0$, *where* $P$ *is the orthogonal projection from* $K$ *to* $H$.

Proof. Put $T_f=f(T)$ for each analytic polynomials $f$ with $f'(0)=\dots=f^{(n)}(0)$, then $f\to T_f$ extends to a representation of $A$ in Example 2. Thus the representation of $A$ has a $\varrho$-dilation with $\varrho=8n+1$ and the corollary follows.

Corollary 2. *Let* $\{a_j\}_{j=1}^n$ *be in the open unit disc. If* $T\in L(H)$ *and*

$$\|f(T)\| \leqq \sup_{|z|\leqq 1} |f(z)|$$

*for all analytic polynomials* $f$ *with* $f(0)=f(a_1)=\dots=f(a_n)$, *then there exists a Hilbert space* $K\supseteq H$ *and a unitary operator* $U$ *on* $K$ *such that*

$$f(T) = \varrho Pf(U)|K$$

*for all analytic polynomials with* $f(0)=f(a_1)=\dots=f(a_n)=0$, *where* $P$ *is the orthogonal projection from* $K$ *to* $H$.

Proof. It can be shown that this is a corollary of Example 3 as in the proof of Corollary 1.

Corollary 3. *Let* $Y$ *be a compact subset of* $\mathbf{C}$ *in Example* 4. *If* $Y$ *contains the spectrum* $\sigma(T)$ *of* $T\in L(H)$ *and*

$$\|f(T)\| \leqq \sup_{z\in Y} |f(z)|$$

*for all* $f$ *in* $R(Y)$ *then there exists a Hilbert space* $K\supseteq H$ *and a normal operator* $N$ *on* $K$ *with* $\sigma(N)\subseteq\partial Y$ *such that*

$$f(T) = \varrho Pf(N)|H$$

*for all* $f$ *in* $R(Y)$ *with* $\tau(f)=0$, *where* $P$ *is the orthogonal projection from* $K$ *to* $H$.

Proof. It can be shown that this is a corollary of Example 4 as in the proof of Corollary 1.

J. Agler [1] proved $\varrho=1$ when $n=1$. R. G. Douglas and V. I. Paulsen [4] showed that there exists an invertible operator $S$ on $H$ such that $S^{-1}TS$ has a normal dilation.

Corollary 4. *If* $T\in L(H)$ *and*

$$\|f(T)\| \leqq \sup_{|z|\leqq 1} |f(z)|$$

*for all analytic polynomials f with* $f(0)=f(1)$, *then there exists a Hilbert space* $K \supseteqq H$ *and a unitary operator U on K such that*

$$f(T) = Pf(U)|H$$

*for all analytic polynomials f with* $f(0)=f(1)$, *where P is the orthogonal projection from K to H. In particular, for all* $n \geqq 1$

$$T^n - PU^n H = T - PU|H.$$

Proof. It can be shown that this is a corollary of Example 5 as in the proof of Corollary 1.

In Corollary 4, $T$ is a polynomially bounded operator. We could not answer the following question which is a special case of Problem 6 of Halmos: Is $T$ similar to a contraction?

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# Bibliographie

**Algebraic Logic and Universal Algebra in Computer Science,** Edited by C. H. Bergman, R. D. Maddux and D. L. Pigozzi (Lecture Notes in Computer Science, 425), XI+292 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1990.

The conference "Algebraic Logic and Universal Algebra in Computer Science" was held in Ames, Iowa in June 1988. The aim of the conference was to bring together researchers from computer science and mathematicians working in universal algebra or algebraic logic. The LNCS volume contains the text of 6 invited papers and 10 contributed papers.

Two questions concerning finitely generated free algebras in a nontrivial variety of relation algebras are of particular interest in the paper *"Relatively Free Relation Algebras"* by H. Andréka, B. Jónsson and I. Németi. The first one is whether an $n$-generated free algebra contains as a subalgebra a free algebra on $n+1$ generators. The second question is if a free algebra on $n$ generators can be nonfreely generated by some $n$ element subset. The results on the first question are derived from general facts such as the congruence extension property and the existence of a nontrivial absolute retract. On the other hand, the results on the second question make use of arguments specific for relation algebras. It is shown that in general the two questions are completely independent of one another.

The informal paper *"The Value of Free Algebras"* by J. Berman exhibits through a series of examples how free algebras occur in computer science and how these free algebras are useful in solving problems in computer science. The examples include non-classical logics, one-pass algebras and data bases.

The paper *"Dynamic Algebras as a Well-Behaved Fragment of Relation Algebras"* by V. Pratt is devoted to the comparison of the merits of relation and dynamic algebras with converse and sometimes with star. Tarski proved in the 1940's that the equational theory of representable relation algebras is undecidable and not finitely based. On the other hand, the equational theory of dynamic algebras is both decidable and finitely based. Pratt attributes these advantages to the "maintenance of a suitable distance between the Boolean and monoidal sorts". One more argument justifying his opinion would be a proof that the equational theory of representable relation algebras with disjunction, relative product, converse, star and constants 0 and $1'$ is decidable, so that after dropping one part of the Boolean structure there results a decidable equational theory.

Conditional logic, studied in the paper *"The Implications in Conditional Logic"* by F. Guzmán, is a 3-valued logic which is a regular extension of Boolean logic. Because disjunction is not commutative, it is possible to define two kinds of implications. The main result is a complete equational axiomatization for these implications.

The contribution *"Other Logics for (Equational) Theories"* by G. C. Nelson consists of two parts. In the first part a complete proof system is described suitable for deriving all positive sentences that are logical consequences of a set of equational axioms. The proof system is extended to the case that the axioms are universal Horn sentences. Some computer science applications are mentioned. The second part is concerned with proving equations true in finite algebras. All facts and ideas exploited in the paper are well known in some form. The way how these facts are arranged accounts for the value of the paper.

Mal'cev algebras have been intended to serve as a variable free and signature independent formal treatment of (function) composition and term substitution in universal algebra. These algebras are the subject of the paper *"Mal'cev Algebras for Universal Algebra Terms"* by I. G. Rosenberg. After a definition of various Mal'cev algebras, it is shown how these algebras are related to varieties. Of course, the connection is the same as that between varieties and Lawvere theories.

The volume contains really good papers and covers a wide range. Everybody working in algebraic or logical aspects of computer science may find some papers of particular interest. The volume is dedicated to the memory of Evelyn M. Nelson.

*Z. Ésik* (Szeged)

**Analysis III,** Spaces of Differentiable Functions. Edited by S. M. Nikol'skiĭ (Encyclopaedia of Mathematical Sciences, 26), 221 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong—Barcelona, 1990.

In this volume the theory of differentiable functions in several variables is treated in detail. The book consists of two parts. Part I: Spaces of differentiable Functions of Several Variables and Imbedding Theorems (by L. D. Kudryavtsev and S. M. Nikol'skiĭ); Part II: Classes of Domains, Measures and Capacities in the Theory of Differentiable Functions (by V. G. Maz'ya). The aim of the authors of Part I is laid in the Introduction as follows: "... the authors undertake to give a presentation of the historical development of the theory of imbedding of function spaces, of the internal as well as the external motives which have stimulated it and of the current state of art in the field, in particular, what regards the methods employed today." The reader can be convicted that this aim is overfulfilled in many senses even if she or he only takes a look at the chapter headings, most significant ones of which are: Sobolev-Spaces; The Imbedding Theorems of Nikol'skiĭ; Sobolev—Liouville Spaces; Weighted Function Spaces; Orlicz and Orlicz—Sobolev Spaces. The main topic of Part II is justified by the author as follows: "An adequate description of the properties of function spaces has made it necessary to introduce new classes of domains of definition for the functions, or classes of measures entering in the norms. In this connection the universal importance of the notion of capacity of a set became manifest".

The principal questions treated in this Part are: The Influence of the Geometry of the Domain on the Properties of Sobolev Spaces; Inequalities for Potentials and Their Applications to the Theory of Spaces of Differentiable Functions; Imbedding Theorems for Spaces of Functions Satisfying Homogeneous Boundary Conditions.

In my opinion this book is very clearly and well written and it is warmly recommended both to researchers and to graduate students.

*J. Németh* (Szeged)

**Béla Andrásfai, Graph Theory: Flows, Matrices,** x+280 pages, Akadémiai Kiadó, Budapest, Hungary, 1991.

This book is the English translation (and revised version) of Béla Andrásfai's book Graph Theory: Flows, Matrices (in Hungarian, Akadémiai Kiadó, Budapest). The book includes various topics from graph theory and their applications to physical sciences, operation research and economics. The author also covers the algorithmic aspects of the topics discussed in the book.

The first chapter contains the basic results on connectivity, blocks and strongly connected digraphs. The second chapter includes results on bipartite graph matching, the Hungarian method, the max flow — min cut theorem and different flow problems. The final (third) chapter deals with some matrices related to graphs. Spectrum of graphs and planar graphs are also considered. The theory of linear electrical networks is discussed as an application of the matrix method.

At the end of each section there are several exercises with solutions (91 altogether). Solving these exercises gives a good practice for the methods.

Students and lecturers will enjoy this book. It can be also used as a textbook for classes in different fields where graph theoretical methods are used.

*Péter Hajnal* (Szeged)

**D. K. Arrowsmith—C. M. Place, An Introduction to Dynamical Systems,** VIII+423 pages, Cambridge University Press, Cambridge—New York—Port Chester—Melbourne—Sidney, 1990.

In the classical sense, a dynamical system is a system of ordinary differential equations. The solutions of such a system defines a flow in a space. Similarly, if $f$ is a diffeomorphism, then the iteration $x_{t+1}=f(x_t)$, where $t$ is a natural number, also gives a dynamical system. Besides their great natural beauty, there are two reasons for studying these "discrete" dynamical systems: on the one hand there are tight connections between time-periodic vector fields and diffeomorphism problems; on the other hand, the same phenomena and problems of the qualitative theory of ordinary differential equations are present in their simplest form in the theory of discrete dynamical systems. In recent years there has been a marked increase of research interest in dynamical systems both continuous and discrete, and a number of good postgraduate texts have been published. The present book is specially aimed at the interface between undergraduate and postgraduate studies. The reader is assumed to be familiar with courses in analysis and linear algebra to second-year undergraduate standard.

The first chapter (Diffeomorphisms and flows) contains the basic definitions. In the second chapter (Local properties of flows and diffeomorphisms) the topological behaviour of diffeomorphisms and flows in the neighbourhood of an isolated fixed point is considered. The third chapter (Structural stability, hyperbolicity and homoclinic points) gives a description of the flows on two-dimensional manifolds, of the Anosov diffeomorphisms, and a very nice presentation of the horseshoe diffeomorphisms. The fourth and fifth chapters are devoted to the local bifurcations. The last Chapter 6 (Area-preserving maps and their perturbations) is directed at first-year postgraduate students. It contains current research topics arisen from the interaction of the theories of area-preserving and non-area-preserving maps.

The whole book is excellent, but its main value is its extensive set of exercises; more than 300 in all. They are companied by model solutions and hints to their construction.

We warmly recommend this book to both senior undergraduates and postgraduate students in mathematics, physics engineering, to the instructors and researchers interested in qualitative theory of nonlinear systems.

*L. Hatvani* (Szeged)

**Bernard Aupetit, A Primer on Spectral Theory,** (Universitext) x+193 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong—Barcelona, 1991.

The text is divided into seven chapters and an appendix. The first two chapters give a list of basic results in functional analysis without proofs and introduce the reader to the theory of operators on Banach spaces and Hilbert spaces. Some special types of operators are also examined. The third chapter introduces the notion of Banach algebras, gives some examples of commutative and non-commutative Banach algebras and develops the basic spectral theory of Banach algebras. In the fourth chapter the Gelfand representation theory of commutative Banach algebras and the representation theory of non-commutative Banach algebras are presented. The fifth chapter is devoted to some applications of subharmonicity. Here the spectral characterisations of commutative Banach algebras and finite-dimensional Banach algebras and the spectral characterizations of the radical are also discussed. The sixth chapter deals with special Banach algebras in which a continuous involution is given. Proving the basic theorem for the Gelfand representation of $C^*$ algebras, as an application develops the spectral representation theory for selfadjoint and normal operators in a Hilbert space. The seventh chapter is an introduction to the theory of analytic multifunctions which has very important applications for instance to the distribution of spectral values in the plane. The appendix is essentially a list of results without proofs concerning subharmonic functions and functions of several complex variables. Each chapter ends with a collection of problems.

*L. Gehér* (Szeged)

**Joseph A. Ball—Israel Gohberg——Leiba Rodman, Interpolation of rational matrix functions** (Operator Theory: Advances and Applications, 45), XII+605 pages, Birkhäuser, Basel—Boston—Berlin, 1990.

The development over close to 100 years of the interpolation theory reached a considerable phase 40 years ago. Namely, since the early 1950's interpolation problems have been considered for matrix-valued functions, too.

A scalar interpolation problem admits often several generalizations for matrix case. For example: $\lambda_0$ can be a zero of the matrix-valued function $P(\lambda)$ in the sense that 1) $P(\lambda_0)$ is zero matrix, 2) $P(\lambda_0)\mathbf{x}=\mathbf{0}$, 3) $\mathbf{y}P(\lambda_0)=\mathbf{0}$, 4) $\mathbf{u}P(\lambda_0)\mathbf{v}=\mathbf{0}$ (with appropriate column or/and row vectors).

This book presents the interpolation theory for rational matrix functions. It would be difficult to list its content, it is much more informative — but not exhaustive — to say that classical results are generalized to this case. The presented theory admits applications to control and system theory; the last part of the book is devoted to such applications. In fact, the objects of this part are sensitivity minimization, model reduction and robust stabilization included their engineering motivations. An Appendix dealing with Sylvester, Lyapunov and Stein matrix-equations completes the main text, and more than two hundred items are listed as references.

The mean feature of this systematic and self-contained treatment is the realization approach. This is based on the fact that every proper rational matrix function can be expressed in the form

$$W(\lambda) = D+C(\lambda I-A)^{-1}B,$$

which allows to reduce the interpolation problems to problems in matrix theory.

This book certainly meets the interest of a great number of mathematicians and ingeneers as well as advanced students.

*E. Durszt* (Szeged)

**Bifurcation and Chaos: Analysis, Algorithms, Applications,** Edited by R. Seydel, F. W. Schneider, T. Küpper and H. Troger (International Series of Numerical Mathematics, 97), X+388 pages, Birkhäuser Verlag, Basel—Boston—Berlin, 1991.

This volume is the proceedings of a conference held in Würzburg, August 20—24, 1990. The main topics discussed in the papers are the following: symmetry, applications of manifolds, Takens—Bogdanov bifurcation, homoclinic orbits, oscillators, controllability, characterization of dynamical systems, general numerical procedures and specific algorithmic topics. The connection with applications is also strongly felt in many papers including chemical oscillations convection problems, climate modeling, economy, robot control, rolling motion of ships, motion of a moored pontoon, galvanostatic oscillation, excitable systems, dry friction, rotating shafts, an elastic model with continuous spectrum, rings under hydrostatic pressure, combustion, Turing structures, and a spinning satellite.

The volume gives the reader a good opportunity for getting an overview of the actual problems and results of the world of nonlinear phenomena.

*L. Hatvani* (Szeged)

**Böhme, Analysis 1** (Anwendungsorientierte Mathematik, Funktionen, Differentialrechnung, 6. Auflage), XI+492 pages, Springer-Verlag, Berlin—Heidelberg—New York—London— Paris—Tokyo—Hong Kong—Barcelona, 1990.

The book essentially contains the material of the first semester. The discussion attaches great importance to applications. The text is divided into four parts. The first part is devoted to elementary functions of one real variable. The second part is a short glimpse into functions of one complex variable. The third part develops the differentiation of real function and gives the differentiation rules. The last part deals with the differentiation of functions of two real variables. To make easier the understanding lots of exercises are given, the solutions of which at the end of the book can be found.

*L. Gehér* (Szeged)

**P. Concus—R. Finn—D. A. Hoffman, Geometric Analysis and Computer Graphics,** Proceedings of a Workshop held May 23—25, 1988 (Mathematical Sciences Research Institute Publications, 17), IX+203 pages, 60 illustrations — 30 in full color, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong—Barcelona, 1991.

The unexpected title of this book comes from a workshop on differential geometry, calculus of variations, and computer graphics held at the Mathematical Sciences Research Institute in Berkley, May 23—25, 1988. Although nobody could imagine a meeting on such a divergent background in the past, now this book proves the successs. Reading the gathered papers in this book, it comes to light that scientific and technological frontiers being crossed with impressive speed and so the title gets a deeper meaning. Maybe this is the way of the future.

One reads about the multi-functions, Monge-Ampère equation, rendering algebraic surfaces, minimal surfaces, capillary surfaces, tories and so on from the papers by Almgren, Baldes, Wohlrab, Banchoff, Callahan, Concus, Finn, Sterling and others.

We recommend the book mainly to those who want to know the interaction between the two subjects in the title, but also to anyone interested in any of these subjects alone.

*Á. Kurusa* (Szeged)

**M. Coornaert—T. Delzant—A. Papadopoulos, Géométrie et théorie des groupes** (Lecture Notes in Mathematics, 1441), X+169 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong—Barcelona, 1990.

The main purpose of this book is to give a detailed treatment of the Gromov theory of hyperbolic groups. The material is based on the lectures which were held by the three authors at the University of Strasbourg. The text is divided into 12 chapters. In the first four chapters the basic concepts of Gromov product, hyperbolicity of metric spaces, Gromov boundary and hyperbolic groups are introduced, and hyperbolicity of the $n$-dimensional simply-connected Riemannian space with constant curvature $-1$ and more generally the hyperbolicity of simply connected Riemannian spaces the sectional curvatures of which is bounded from above by a strictly negative constant are investigated. In Chapter 5 for a given hyperbolic group a contractible locally finite and finite dimensional simplicial complex is constructed. Chapters 6, 7 examine linear isoperimetric inequalities in hyperbolic spaces and give isoperimetric characterisation of hyperbolic groups. Chapters 8, 9, 10 deal with approximations, isometries and quasi convexity. Chapter 11 is devoted to the investigation of the boundary of hyperbolic groups and the theory of automata.

*L. Gehér* (Szeged)

**C. Corduneanu: Integral equations and applications**, IX+366 pages, Cambridge University Press, Cambridge—New York—Port Chester—Melbourne—Sidney, 1991.

Since it is so classical subject of the analysis it is a very natural question for a nonspecialist mathematician that "What new about the integral equations can a book write?". The book of Corduneanu gives a very striking answer. I recommend to all to read the excellent "Introduction" of the book. It is very well written, and contains not only a detailed description of the book's contents, but also some interesting historical considerations as well as some important notes of the author about the built up of the theory.

I think the author successfully reached his aim to write the book for three purposes. It is good for graduate textbook and for reference book as well as for young researchers to become acquainted with this field.

The book is based on the integral and the abstract Volterra equation as a unified starting point. It deals with the Fredholm theory of the linear integral equations with the Hammerstein equations and some of their generalizations to the Banach spaces. Applications of these integral equations are discussed in the last chapter. A very valuable part of the book is its big list of references, that contains more than 500 entries.

I recommend this very well written book to everybody who get in touch with the integral equations even in teaching, learning or in research.

*Á. Kurusa* (Szeged)

**CSL'89**, Edited by E. Börger, H. Kleine Büning and M. M. Richter (Lecture Notes in Computer Science, 440), VI+437 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1990.

These are the proceedings of the 3rd Workshop on Computer Science Logic held in Kaiserslautern, Germany, in October 1989. Altogether 45 talks were presented at the workshop, 28 of which have been collected in the volume. The authors of the papers are: *K. Ambos-Spies and D. Yang;*

*G. Antoniou and V. Sperscheider; E. Börger; D. Cantone, V. Cutello and A. Policriti; E. Dahlhaus; B. I. Dahn; H. Decker and L. Cavedon; M. Droste and R. Göbel; A. Goerdt; E. Grädel; Y. Gurevich and L. S. Moss; J. Krajicek and P. Pudlák; H. Leiss; A. Leitsch; C. Meinel; D. Mey; D. Mundici; H.-J. Ohlbach; M. Parigot; A. Pasztor and I. Sain; W. Penczek; L. Priese and D. Note; E. Speckenmeyer and R. Kemp; R. F. Stärk* (two papers), *O. Stepankova and P. Stepanek; H. Volger; E. Wette.*

The volume can be recommended to those interested in logical aspects of theoretical computer science.

*Z. Ésik* (Szeged)

**Effective Methods in Algebraic Geometry** (Progress in Mathematics, 94), Edited by Teo Mora and Carlo Traverso, XIV+500 pages, Birkhäuser, Boston—Basel—Berlin, 1991.

The development of computers has made it possible to complete calculations which previously were not feasible, thus the formulation of effective methods is now an important part of many areas of mathematics.

This book contains the proceedings of the symposium "MEGA—90 — Effective Methods in Algebraic Geometry", Castiglioncello, April 17—21, 1991. Two main areas were addressed at the symposium, that of effective methods and complexity issues in algebraic geometry and related areas (such as commutative algebra and algebraic number theory) and the use of algebraic geometry in algebraic computing. The book contains 33 papers, treating the resolution of singularities, codes and elliptic curves, algebraic differential equations, membership problems and other topics in algebraic geometry and algebra.

The book is recommended to those interested in the algorithmic aspects of algebraic geometry at graduate level and beyond.

*G. Megyesi* (Szeged)

**A. Simovici—Peter A. Fejer—Peter Dan, Mathematical Foundations of Computer Science,** Vol. 1, X+425 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1990.

The volume presents basic discrete mathematics relevant to computer science courses. The five chapters collected in the first volume are *Elementary set theory, Relations and functions, Partially ordered sets, Induction* and *Enumerability and diagonalization.* The computer science orientation can be witnessed by a thorough treatment of induction and diagonalization and topics such as databases, complete partially ordered sets, grammars, primitive recursive and partial recursive functions. The book is written in a rigorous style. New concepts are usually introduced through a series of examples and a number of applications are given for most theorems. In addition, each chapter contains a large number of exercises. Many of them are related to various fields of computer science or provide background information. Care is taken that general results are preceeded by a treatment of some particular instances. All these make the volume available for a large audience including undergraduate students. The second volume will cover topics of logical nature.

*Z. Ésik* (Szeged)

**C. A. Floudas—P. M. Pardalos, A Collection of Test Problems for Constrained Global Optimization Algorithms** (Lecture Notes in Computer Science, 455), XIV+180 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong—Barcelona, 1990.

Global optimization has been extensively studied in recent years, and numerous new theoretical, algorithmic and computational results have been achieved. In spite of these contributions, there has been still a lack of nonconvex test problems for comparing constrained global optimization algorithms.

The book of the authors contains a systematic collection of over 50 test problems for evaluating and testing constrained global optimization methods. For each test problem, the problem formulation, data problem statistics (like number of variables, linear and nonlinear constraints) and global or best known solutions are given. The test problems collected reflect a wide range of practical applications: e.g. distillation column sequencing, pooling, blending, heat exchanger network synthesis, reactor-separator-recycle system design etc.
An extensive bibliography of more than 250 references completes the book. The volume can be recommended to those working in the field of nonlinear constrained optimization and to engineers who want to test the numerical effectivity, efficiency and reliability of optimization algorithms.

*T. Csendes* (Szeged)

**Functional-Analytic Methods for Partial Differential Equations,** Proceedings of a Conference and a Symposium held in Tokyo, July 3—9, 1989. Edited by H. Fujita, T. Ikebe, and S. T. Kuroda, (Lecture Notes in Mathematics, 1450) VII+251 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong—Barcelona, 1990.

An "International Conference on Functional Analysis and its Application in Honor of professor Tosio Kato" was held on July 3 through 6, 1989 at University of Tokyo, which was followed by a "Symposium on Spectral and Scattering Theory" held on July 7 through 9 at Gakushuin University. In these meetings the study of Schrödinger operators and functional analytic study of nonlinear PDEs were the major subjects. The connection with applications is also strongly discussed in many papers.

*L. Hatvani* (Szeged)

**I. M. Gel'fand—E. G. Glagoleva—E. E. Shnol, Functions and Graphs,** IX+105 pages, Birkhäuser, Boston—Basel—Berlin, 1990.

The book is dealing with transferring of formulae and data into geometrical form by sketching the graphs of several functions without calculus.

It is very important to show the way how to "see" functions, formulae and how to observe the ways in which these functions change. To see simultaneously the formula of a given function and its geometrical representation and to draw the graph of a function is very useful not only in studying mathematics but in studying any subject, because the graphs are widely used not only in mathematics but in economy, medicine, engineering, physics, biology, business and so on.

The chapter headings of the book are: Examples; The Linear Function; The Function $y=|x|$; The Quadratic Trinomial; The Linear Fractional Function; Power Functions; Rational Function; Problems for Independent Solution; Answers and Hints to Problems and Exercises.

The book is very useful for high school teachers helping them in presenting basic mathematics in a clear and simple form.

*J. Németh* (Szeged)

**B. Golubov—A. Efimov—V. Skvortsov, Walsh Series and Transforms.** Theory and Applications (Mathematics and its Applications, 64), XIII+368 pages, Kluwer Academic Publishers, Dordrecht—Boston—London, 1991.

This book is the translation of the work published in Russian in 1987. This volume is a very good and useful introduction to Walsh—Fourier analysis with applications of the theory. Chapters 1 and 2 give the definitions of Walsh system and examine the basic properties of Walsh—Fourier series. Chapters 3—5 deal with the uniqueness of representation of functions by Walsh series, summation of Walsh series by the method of arithmetic means and convergence in $L^p$ of Walsh—Fourier series. The main topic of Chapter 6 is the theory of generalized multiplicative transforms. In Chapters 7 and 8 Walsh series with monotone decreasing coefficients and lacunary subsystems of the Walsh system are considered.

Chapter 9 is dealing with divergence, almost everywhere convergence of Walsh—Fourier series of $L^2$ functions. Chapter 10 is devoted to the question of approximation by Walsh and Haar polynomials.

The last chapters (11 and 12) contain the methods for applying the Walsh system and its generalizations to digital information processing, to construct special computational devices to digital filtering, and to digital holograms. The appendices at the end of the book contain background information relating to more advanced material (group theory, measure theory, the Lebesgue integral, functional analysis). The appendices are followed by commentary including some remarks of historical nature and information about the latest developments in the area.

The book ends with a very rich, valuable "References" containing more than 150 items (30 of them are books). The volume is clearly and very well written. It will certainly be very useful book for engineers, technical specialists, graduate students of applied mathematics, and for everybody interested in Fourier analysis and its application.

*J. Németh* (Szeged)

**The Grothendieck Festschrift,** A collection of Articles Written in Honor of the 60th Birthday of Alexander Grothendieck (Progress in Mathematics, 86—88), Edited by P. Cartier, L. Illusie, N. M. Katz, G. Laumon, Y. Manin and K. A. Ribet, 3 volumes, Volume I XX+498 pages, Volume II VIII+563 pages, Volume III VII+495 pages, Birkhäuser, Boston—Basel—Berlin, 1991.

This book contains 35 papers by leading mathematicians from around the world. Most of the contributions are on various areas of algebraic geometry, but there are also several on algebraic number theory, topology and other areas of geometry. This variety of topics reflects the vast area on which Grothendieck worked and the book is a worthy tribute for his 60th birthday.

The diversity of the topics in this book, and also its price, mean that this book is less suitable for the individual, but it would be a good addition to any mathematical library.

*G. Megyesi* (Szeged)

**Martin C. Gutzwiller, Chaos in Classical and Quantum Mechanics** (Interdisciplinary Applied Mathematics, 1), xiv+432 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong—Barcelona, 1990.

Simple elementary and deterministic mechanical systems can have very complicated motions. Their behavior is exceedingly sensitive to the precise starting conditions and they do not follow simple, regular and predictable patterns, but run along a seemingly random, yet well-defined, trajectory. The name for this phenomenon is chaos.

"Chaos ... will challenge many of our assumptions about the typical behavior of dynamical systems. Since mechanics underlies our view of nature, we will probably have to modify some of our ideas concerning the harmony and beauty of the universe. As a first step, we will have to study entirely different basic examples in order to re-form our intuition. We must become familiar with certain novel specimens of simple mechanical systems based on chaotic rather than regular behavior."

This book offers a collection of instructive examples, which are chaotic, yet simple enough to be understood thoroughly. The central theme is the connection between classical and quantum mechanics: classical chaos should be the limit of quantum chaos as Planck's quantum becomes small. The style of the book is informal. The arguments based on elementary rather than algebraic manipulations. In order to gain a better perspective on the more important results, the historical and cultural background is mentioned and related disciplines are connected. The comments on the motivation behind certain results and on possible future developments provide the reader with a new perspective and prepare her/him to attack new problems.

Reading the book requires a knowledge of both classical and quantum mechanics at the level of beginning graduate students. This excellent book will certainly appeal to people working on this very active area of physics and its closest relatives: mathematics, astronomy and chemistry.

*T. Krisztin* (Szeged)

**Werner Heise—Pasquale Quattrocchi, Informations- und Codierungstheorie,** (Studienreihe Informatik), XII+392 pages, Springer-Verlag, Berlin—Heidelberg—New York—Paris—Tokyo, 1989.

The aim of the German and Italian authors is to provide a tutorial for those who are interested in the mathematical theory of communication. According to the preface of the first edition, the assumed readers are students majored in informatics. However, the authors present the applied mathematical background, so no specific knowledge is required to understand the book.

In the first part we can get an introduction to the theory of message transmission. The first two chapters introduce the basic concepts, such as code, source, channel, and give the definition of some special classes of channel. Chapter 3 summarizes the classical results of information theory, Chapters 4 and 5 discuss source and channel encoding.

The second part of the book is devoted to the theory of error detecting and correcting codes. After discussing the best known combinatorial bounds for these codes in Chapter 6, in Chapter 7 the authors present those algebraic concepts and theorems which are necessary to understand the theory of linear codes. The detailed description of these codes can be found in Chapter 8. The sections of this chapter deal with the basic concepts of linear codes (such as generator and parity check matrix, syndrome decoding, etc.), the modifications which preserve the linearity and the Reed—Muller codes. A separate chapter (Chapter 9) is devoted to the special class of linear codes, the cyclic codes, paying particular attention to the *BCH* and quadratic residue (*QR*) codes.

A lot of example help to understand the theoretical material of this clearly written book, besides funny pictures make the reading more enjoyable. We can warmly recommend this work both to students and teachers.

*T. Gaizer* (Szeged)

**J. H. Hubbard—B. H. West, Differential Equations.** A Dynamical Systems Approach, Part I: Ordinary Differential Equations, (Texts in Applied Mathematics, 5), XIX+348 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong—Barcelona, 1991.

This is an introductory textbook, which essentially differs from the traditional courses on differential equations. According to the fact that most of the differential equations do not admit solutions which can be written in elementary terms, it takes the view that a differential equation defines functions, and the object of the theory is to understand the behaviour of these functions. To this end it uses numerical and qualitative methods. While numerical methods approximate a single solution as closely as one wishes qualitative methods involve graphing the field of slopes, which enables one to draw approximate solutions following the slopes, and to study these solutions all at once. These method, are companied with a software, MacMath, which brings the notions to life and yields the majority of the 144 illustrations.

Not only the approach is new but the basic terminology as well. The authors introduce the terms "fence" "funnel" and "antifunnel". A fence is a curve on the $(t, x)$ plane that channels the solutions in the direction of the slope field. A lower fence pushes solutions up, an upper fence pushes solutions down. A set bounded above by an upper fence and below by a lower fence is called a funnel. A set bounded above by a lower fence and below by an upper fence is called an antifunnel. It is interesting that these concepts give simple, noniterative proofs of the important theorems, e.g. the Sturm comparison theorem.

The book is ended by a chapter on iteration, which is also unusual in a text on differential equations. The reason of the appearance is that the iteration is another type of dynamical systems playing an important role in the theory of continuous dynamical systems generated by differential equations.

This excellent book will be very useful for instructors and students of undergraduate courses in differential equations and their applications.

*L. Hatvani* (Szeged)

**J. E. Humpreys, Reflection Groups and Coxeter Groups,** (Cambridge studies in advanced mathematics, 29), XII+204 pages, Cambridge University Press, Cambridge—New York—Port Chester—Melbourne—Sydney, 1990.

This is an easy-to-follow introductory graduate text on the theory of Coxeter groups.

The book consists of two parts. Part I describes the classical examples of Coxeter groups and provides the motivation for Part II, which is devoted to the general study of Coxeter groups. The first two chapters introduce the basic notions, such as for example roots, Coxeter graphs and Coxeter systems of generators, on the example of finite reflection groups, and give the classification of such groups. The next chapter describes in detail the theory of polynomial invariants of finite reflection groups. In particular, it presents the interesting relationships between the properties of the Coxeter elements and the orders of the fundamental invariants. In this part special attention is payed to the important examples of finite reflection groups provided by the Weyl groups of semisimple Lie algebras and to the related affine Weyl groups.

The first chapter in Part II develops the theory of Coxeter groups in general. For example, the geometric representation of general Coxeter groups and the properties of the Bruhat ordering are among the topics treated here. The following chapter deals with special cases: finite, affine, crystallographic and hyperbolic Coxeter groups. Then the author gives an introduction to the theory of Hecke algebras associated to Coxeter groups. The last chapter provides the reader with a guide to additional topics related to the subject-matter treated in the book, which can be further studied by using the extensive bibliography.

The book is clearly written and is self-contained. It can be profitably used by everybody interested in the general theory of Coxeter groups and its applications, or in the special Coxeter groups featuring so prominently in Lie theory.

*László Fehér* (Szeged)

**A. E. Ingham, The Distribution of Prime Numbers** (Cambridge Mathematical Library), XVII + 114 pages, Cambridge University Press, Cambridge—New York—Port Chester—Melbourne—Sidney, 1990.

This book was first published in 1932. Number theory is a very strange part of mathematics. One of my professors told us that number theory is beautiful and good for nothing. But this was more than forty years ago. Nowadays number theory plays a more and more important role in real applications, too. The "fairly tales" become reality.

One of the most interesting parts of number theory deals with prime numbers. The subject of this book is the discussion of the theory of distribution of prime numbers in the series of natural numbers. After an introduction which contains the history of the problem and elementary facts too, the discussion depends on the theory of zeta-function. Chapter headings are: Foreword, Preface, Introduction, Elementary theorems, The prime number theorem, Further theory of $(s)$, Applications, Explicit formulae, Irregularities of distribution.

An important part of this book is the Foreword written by R. C. Vaughan containing up to date results, comments and references.

This work is warmly recommended to teachers and students as well.

Finally we cite two interesting things. The last two sentences of the author's preface are: "The proof-sheets have been read by Prof. H. Bohr and Proof. J. E. Littlewood and also by Prof. G. H. Hardy, Dr. A. Zygmund ..., To Prof. N. Wiener I am indebted for some valuable comments ...". What a list of names!

One of the first reviews of the book from Zentralblatt für Mathematik (1933) was written by F. Bohnenblust.

A sentence from the review: Von vielen Sätzen werden verschiedene Beweisvarianten manchmal vollständig ausgeführt, manchmal nur skizziert, so dass der Leser-neben einer durchsichtigen systematischen Darstellung — eine klare Übersicht über die inneren Zusammenhänge der Theorie gewinnen kann.

*L. Pintér* (Szeged)

**Bernd Jähne, Digitale Bildverarbeitung,** XII + 331 pages, 144 pictures, Springer-Verlag, Berlin—Heidelberg—New York—Paris—Tokyo, 1989.

To analyze and understand pictures is a simple task for us, humans, but — at least for the first sight — hardly tractable for computers. Even the most obvious operations: storing pictures, making simple corrections or detecting some simple patterns can require strong hardware background and involved algorithms.

Jähne's book provides a good overview of the present state of image processing. The outline of the book was the author's two-semester course had been held at University of Heidelberg. In spite of this the book is far more than a tutorial, it can also be a helpful guide for those ingeneers and researchers who want to use image processing in their work.

The chapters of the book cover the phases of image processing. The first two chapters contain some general introduction and concentrate on the problem of making and digitalizing pictures. The next two chapters are devoted to the mathematical background applied in image processing: Chapter 3 to the unitary transformations, Chapter 4 to the basic concepts of one and two variable statistics. In Chapters 5 to 10 we can read about the techniques that are used for modifying and analyzing digitalized pictures: for example, filtering and clustering procedures are discussed here. Chapter 11 is devoted to an area which has a lot of use e.g. in medical applications: the problem of reconstructing a picture from its projections. The last four chapter focus on analyzing and processing a sequence of pictures that were taken of moving objects or by moving camera.

The importance of Fourier transformation in image processing is emphasized by the fact that besides the *DFT* algorithm is described in Chapter 3, in Appendix A the author provides a summary of the one and two dimensional Fourier transformation. Finally, Appendix B contains a complete description of a *PC*-based digital image processing system.

The book is well illustrated with experimental results: 144 pictures help to demonstrate the effects of the studied procedures. We can warmly recommend this work both to those who just wish to get familiar with image processing and to those who want to apply it in the practice.

*T. Gaizer* (Szeged)

**N. Korneichuk, Exact Constants in Approximation Theory** (Encyclopedia of Mathematics and its Applications, 38), XII+452 pages, Cambridge University Press, Cambridge—New York—Port Chester—Malbourne—Sidney, 1991.

This book is very useful for non-specialists as a self-contained introduction to the important and widely applied area of approximation theory that is dealing with exact constants and for experts as a rich reference book to this topic (28 monographs, 17 books and more than 300 articles are cited in the References). The results are concerning extremal problems in approximation theory and are tightly related to numerical analysis and optimization.

Chapter 1 (Best approximation and duality in extremal problems) and Chapter 3 (Comparison theorems and inequalities for the norms of functions and their derivatives) contain the deep theorems of analysis and function theory on which the exact constant results are based. Chapter 2 (Polynomials and spline functions as approximating tools) gives an introduction to polynomial and spline approximation. Chapters 4 to 7 (Polynomial approximation of classes of functions with bounded $r$th derivative in $L_p$; Spline approximation of classes of functions with a bounded $r$th derivative; Exact constants in Jackson inequalities; Approximation of classes of functions determined by modulus of continuity) are devoted to approximation by polynomials (trigonometric or algebraic) and by polynomial splines. Chapter 8 ($N$-widths of functional classes and closely related extremal problems) deals with $n$-widths and generalizes some of the ideas of the earlier chapters.

Each chapter ends with valuable commentary and exercises.

The former contains references to the authors and their works related to the results included in the chapter in question and the latter contains in many cases the extensions of the corresponding results.

Since many of the results collected in this book have not been gathered together in book form before, this excellently written book of high level is warmly recommended to everybody who searches, teaches or applies the approximation theory.

*J. Németh* (Szeged)

**Logic and Computer Science**, Edited by P. Odifreddi (Lecture Notes in Mathematics, 1429), V+162 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1990.

This volume contains the lecture notes of the C.I.M.E. meeting on Logic and Computer Science held in June 1988 in Monteatini, Italy.

*Table of Contents:*

S. Homer, *The Isomorphism Conjecture and its Generalizations*

A. Nerode, *qome Lectures on Intuitionistic Logic*

R. A. Platek, *Making Computers Safe for the World. An Introduction to Proofs of Programs.* Part I

G. E. Sacks, *Prolog Programming*

A. Scedrov, *A Guide to Polymorphic Types*

It has been conjectured by L. Berman and J. Hartmanis that all *NP*-complete problems are are polynomial time isomorphic. This conjecture and its generalizations are discussed in the paper by S. Homer. The second paper, written by A. Nerode, is an exposition of one part of the undergraduate course on Intuitionistic Logic at Cornell. It focuses on Kripke's frame semantics for intuitionistic predicate logic (without function symbols) and on the correctness and completeness of a variant of Hughes and Cresswell's, or Fitting's prefixed tableaux. The paper by R. A. Platek develops flowchart semantics and Floyd's inductive assertion method on the basis of inductive definability. The fourth paper is written in a rather technical style. It fails to explain the aim and scope of PROLOG programming and its scientific level is well below the level of the other contributions. The last paper provides a highlight of (second order) polymorphic lambda calculus and the semantics of polymorphism. Complete proofs of the confluence theorem and the strong minimalization theorem are given.

*Z. Ésik* (Szeged)

**Mathematical Foundations of Programming Semantics**, Edited by M. Main, A. Melton and M. Mislove (Lecture Notes in Computer Science, 442), VI+pages, Springer-Verlag, Berlin—Heidelberg—New York, 1990.

The volume contains the papers presented at the Fifth International Conference on the Mathematical Foundations of Programming Semantics held at Tulane University, New Orleans, Louisiana' from March 29 to April 1, 1989. The contributions address concurrency, domain theory, type theory and lambda calculus, categorial semantics and program correctness. The authors of the papers are: *S. Abramsky; L. Cardelli and J. C. Mitchell; E. W. Stark; G. M. Reed; J. Davies and S. Schneider; A. W. Roscoe and G. Barrett; G. Barrett; F. Pfenning and C. Paulin-Mohring; K. Malmkjaer; M. G. Main and D. L. Black; A. Stoughton; L. S. Moss and S. R. Thatte; L. Aceto and M. Henessey; P. Panangaden and J. R. Russell; J. M. E. Hyland, E. P. Robinson and G. Rosolini; E. L. Gunter; R. Jagadeesan; A. Pasztor; A. J. Power; H. Jifeng and C. A. R. Hoare; J. W. Gray.*

The volume can be recommended to those interested in recent research in semantics.

*Z. Ésik* (Szeged)

**Mappings of Operator Algebras.** Proceedings of the Japan—U.S. Joint Seminar, University of Pennsylvania, 1988. Edited by Huzihiro Araki and Richard V. Kadison (Progress in Mathematics, 84), X+307 pages, Birkhäuser, Boston—Basel—Berlin, 1991.

This volume is dedicated to Professor Shôichirô Sakai and it is the proceedings of the fourth Japan — U.S. Joint Seminar on Operator Algebras held in honor of his 60th birthday. The content is (of course) adequate to this occasion. Index theory is a frequented topic in the papers, and — among others — derivation of operator algebras, actions of groups on $C^*$-algebras and completely bounded mappings are discussed. The content is connected also with quantum physics and ergogic theory.

The 19 articles of several length form a really good proceedings. The reader can find some new results, but mainly expositions of recent results and effective methods. Open problems and conjectures presented with hints stimulate to make attempt at solving some of them. Thus, this book provides a useful reference for researchers and graduate students working in the field of operator algebras.

*E. Durszt* (Szeged)

**Mircea Martin—Mihai Putinar, Lectures on Hyponormal Operators,** (Operator Theory, 39), 304 pages, Birkhäuser Verlag, Basel—Boston—Berlin, 1989.

The Hilbert space operator $T$ is called hyponormal if its selfcommutator $T^*T-TT^*$ is a positive operator. An important subclass of hyponormal operators is formed by the subnormal operators, which are restrictions of normal operators to invariant subspaces. The significant progress having reached in the study of subnormal operators up to 1981 was summarized in a monograph by J. Conway. There are known however hyponormal operators which are not subnormal, even more such operators naturally arise in many applications, e.g. in the theory of singular integral operators.

This book collects the various results achieved in the study of hyponormal operators in the last decades, including the basic inequalities, the invariant subspace theorems, the functional models and the role of the principal function. A number of examples and exercises make the treatment more colourful.

This volume can be recommended to graduate students as an introduction to this rapidly developing, fruitful field of mathematics. At the same time it will surely serve as an indispensable reference for the specialists.

*L. Kérchy* (Szeged)

**Jean Mawhin—Michel Willem, Critical Point Theory and Hamiltonian Systems** (Applied Mathematical Sciences, 74), xiv+277 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1989.

The development of a general theory of periodic solutions of Hamiltonian systems is a fundamental step in understanding the structure of their solution set. The main difficulty in applying the naive idea of finding the periodic solutions of a general Hamiltonian system through the critical points of its Hamiltonian action on a suitable space of periodic functions lies in the fact that this action is unbounded from below and from above. Therefore, the direct method of the calculus of variations (which deals with absolute minima) cannot be applied in a straightforward way and more sophisticated approaches like minimax methods and dual least action principles have to be used.

The aim of this interesting survey is to initiate the reader to the fundamental techniques of critical point theory which have been used recently in the framework of periodic solutions of Hamiltonian systems. The main subjects are the dual least action principle developed by Clarke and Ekeland, minimax approaches such as the Lusternik—Schnirelman theory and the mountain pass theorem of Ambrosetti and Rabinowitz, the Morse theory and some local and global aspects of the theory of nondegenerate critical manifolds. Various important problems concerning Hamiltonian systems are considered as applications of the techniques.

The book consists of ten chapters. The titles of the chapters are the following: the direct method of the calculus of variation, the Fenchel transform and duality, minimization of the dual action, minimax theorems for indefinite functionals, a Borsuk—Ulam theorem and the index theories, Lusternik—Schnirelman theory and multiple periodic solutions with fixed energy, Morse—Ekeland index and multiple periodic solutions with fixed energy, Morse theory, application of Morse theory to second order systems, nondegenerate critical manifolds. Some exercises are provided at the end of each chapter and a very extensive bibliography is presented.

The excellent style of the presentation of the book may help to make critical point theory more popular among people working and trained in ordinary differential equations.

*T. Krisztin* (Szeged)

**Nonlinear Analysis and Applications.** Edited by V. Lakshmikantham (Lecture Notes in Pure and Applied Mathematics, *109*), XIX+649 pages, Marcel Dekker, Inc., New York and Basel, 1987.

The 7th International Conference on Nonlinear Analysis and Applications held at the University of Texas at Arlington, July 28—August 1, 1986 was in some sense a festive occasion because the main organizer, the moving spirit of these conferences V. Lakshmikantham became sixty years old. In this volume one finds the proceedings of this conference. Nowadays nonlinear analysis is a very broad part of mathematics both in theory and applications. In this book you have more than eighty papers. To enumerate the various problems is a nearly impossible task and to cite only a few of the talks could be misleading. (To enumerate all the titles is too long.) But let us emphasize an important feature of these talks. Everyone knows that sometimes (perhaps fairly often) the talks on conferences after the first five minutes are interesting for a few specialist only. We get some results but not ideas and important problems. Im my opinion in this collection the reader will find relatively many well-written, inspiring paper. Perhaps this is the best recommendation.

*L. Pintér* (Szeged)

**Wlodzimierz Odyniec—Grzegorz Lewicki, Minimal Projections in Banach Spaces,** (Lecture Notes in Mathematics, 1449), VIII+168 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong—Barcelona, 1990.

It is a well-known problem to find the best approximation of an element $x$ in a Banach space $X$ with elements of a subspace $D$ of $X$. If the subspace $D$ is complemented in $X$, i.e. if there exists a projection of $X$ onto $D$, then it is of special interest to find a projection onto $D$ (provided that there exists) the norm of which is the distance $\varrho(x, D)$ between $x$ and $D$; such projections are called minimal projections. (The relationship between the two above mentioned problems is not apparent.) The text consists of four chapters. The first chapter is devoted to the problem of uniqueness of minimal projections. The second chapter deals with the connection between the problem of uniqueness of a minimal projection for subspaces with finite codimension of infinite dimensional spaces, and certain linear programming problems in $n$-dimensional euklidean spaces. In Chapter 3 lots of Kolmogorov type characterizations of minimal projections are presented. Chapter 4 studies isometries of a Banach space onto itself and gives characterisations of Hilbert spaces in the class of uniformly smooth strictly normed Banach spaces with the aid of minimal projections.

*L. Gehér* (Szeged)

**Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory** (Actes du colloque en l'honneur de Jacques Dixmier), Edited by A. Connes, M. Duflo, A. Joseph and R. Rentschler (Progress in Math., 92), XVI+579 pages, Birkhäuser Verlag, Boston—Basel—Berlin, 1990.

This volume is the proceedings of the Colloquium held in Paris in 1989, celebrating the 65th anniversary of Professor Jacques Dixmier. The expository and research articles presented by the 22 invited speakers cover the four great areas of research, listed in the title, where Jacques Dixmier achieved significant progress. The first chapter deals with "$C^*$-algebras" and contains papers by E. Stormer, M. Takesaki and D. Voiculescu. The second chapter is devoted to "Lie groups and Lie algebras" and includes papers by L. Pukanszky, A. A. Kirillov, B. Kostant, D. Kazhdan, M. Kashiwara—T. Tanisaki, V. Lakshmibai, P. Littelmann—C. Procesi, W. Rossmann, R. K. Brylinski, D. Barbasch, D. A. Vorgan, Jr., W. M. McGovern, J. Bernstein, J.-E. Björk—E. K. Ekström, T. Levasseur, C. De Concini—V. G. Kac. Finally papers by M. Brian—C. Procesi, V. L. Popov and H. Kraft constitute the third "Invariant Theory" chapter.

This book can be recommended first of all to the specialist, but beyond this any interested reader will find it useful who wants to get an insight into these areas of mathematics inspired by Jacques Dixmier.

*L. Kérchy* (Szeged)

**Paradoxa Klassische und neue Überraschungen aus Wahrscheinlichkeitsrechnung und mathematischer Statistik,** 240 pages, Akadémiai Kiadó, Budapest, 1990.

This book illustrates excellently that probability theory is not only a chapter of measure theory.

The book has a historical framework. Chapter 1 is devoted to the oldest and most classical paradoxes of probability theory connected to problems of chance like card-playing, lottery, horse kickings and misprints ... to name a few.

Chapter 2 presents paradoxes in mathematical statistics. The explanations of these paradoxes help the reader to see through statistical absurdities and understand the useful and essential conclusions of statistics.

In Chapter 3 the reader can find paradoxes of random processes. Most of these paradoxes arose in the second half of the last century when the results of classical deterministic mechanics proved to be insufficient in different fundamental branches of science.

Chapter 4 — the most interesting for specialists of probability theory — presents paradoxes in the foundations of probability theory. These paradoxes are closely related to the development of Kolmogorov's fundamental theory.

Each paradox is discussed in five parts: the history, formulation, explanation of the paradox, remarks and references. Each chapter finishes with quickes. These are not discussed in detail, not because they are of less importance or interest, but because they do not fit into the main line of the book.

The book is recommended to probability specialists and nonspecialists as well.

*L. Viharos* (Szeged)

**H.-O. Peitgen—E. Maletsky—H. Jürgens—T. Perciante—D. Saupe—L. Yunker, Fractals for Classroom: Strategic Activities** Volume One, XII+128 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong—Barcelona, 1991.

The subject of fractals is nowadays a rapidly increasing area of the mathematics. At the same time it is one of the most suitable territory of recent mathematics to introduce in a classroom, because also its most abstract theories keep the freshness of the basic experiences. A story of a young lady, who determined the dimensions of fractals generated by Pascal's triangle, in the foreword of Benoit Mandelbrot justifies also this establishment.

As an introduction of the fractals, this book tries to drive students along a sequence of experiments. The activities need the reader to construct, count, compute and measure. The fractal theory seems from this point of view an experimental science like physics, and this makes easier to understand the underlying mathematical principles for the students. I think this approach can be made complete in teaching thanks to the modern computers. While these experiments are very interesting they make always good opportunity for the authors to call the student's attention to the most interesting experiences. The concept the book is built on is the self-similarity, the chaos game and complexity.

It is worth noting, that all the sheets of the book are perforated. This makes possible to use the sheets as separated exercise-forms. Other interest of the book is the enclosed slide package. This contains nine very good quality slide about fractal images.

In sum, we warmly recommend this book to the teachers, who want to bring mathematics out of past history for their students, to the students, who want to know in the visual sense the most color and beauty geometric structures of mathematics and want to discover new exciting territories. I am sure that fractals, and also this publication, open new ways in teaching and learning mathematics.

*Á. Kurusa* (Szeged)

**A. M. Perelomov, Integrable Systems of Classical Mechanics and Lie Algebras,** Volume I, X+307 pages, Birkhäuser Verlag, Basel—Boston—Berlin, 1990.

This book gives a systematic, up-to-date account of the rapidly developing theory of integrable) classical mechanical systems with finitely many degrees of freedom. The reader is assumed to be familiar with the fundamentals of classical mechanics, the theory of differentiable manifolds and Lie groups, but apart from these prerequisities the vook is self-contained.

The study of such well-known integrable systems as for example the motion of a point mass in a central potential or various special cases of the motion of a rigid body about a fixed point, played an important role both in the development of the mathematical formalism of classical mechanics and in its applications in describing physical phenomena. However, until quite recently only a rather small number of nontrivial examples of integrable systems was known. During the last twenty years or so the situation changed dramatically, the complete integrability of a large number of mechanical systems has been proven, mainly by applying the isospectral deformation (inverse scattering) method to classical mechanics. Practically all know integrable systems are related to Lie algebras in some way or other, for example quite a few of them live on coadjoint orbits of Lie groups, or can be obtained by reduction of some sort from some higher dimensional system with a large, manifest symmetry group, which underlies the integrability.

The main subject of the present volume is the isospectral deformation method and its combination with various Lie algebraic techniques. The author also gives an exposition of the classical methods and results of the theory of integrable Hamiltonian systems. The book contains a detailed survey of important classes of the known integrable systems and a good bibliography as well, and thus it can serve as a standard reference on its subject.

The first chapter contains a clear presentation of the general theory, including the isospectral deformation method, the description of Hamiltonian systems with symmetry, and the closely related questions of symmetry reduction and the so called projection method. Chapter 2 deals with the simplest, classic examples of integrable systems. Chapter 3 offers a survey of many-body problems of generalized Calogero—Moser type. The subject of Chapter 4 is the non-periodic Toda lattice and its various generalizations, described here both from the point of view of coadjoint orbits and Lax pairs and also as reductions of the geodesic motion on certain symmetric spaces. The last chapter deals with additional questions of many-body problems. Throughout the book, various aspects of the theory of semisimple Lie algebras are used, and the basic facts of this theory are summarized in an appendix. There are also three further appendices, e.g. one on symmetric spaces.

The book is clearly written and is very redable. It will be useful to students and lectures in theoretical physics and mathematics as well as for researchers on related areas.

*László Fehér* (Szeged)

**M. H. Protter—C. B. Morrey, A First Course Calculus in Real Analysis** (Undergraduate Texts in Mathematics), XVIII+534 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong—Barcelona, 1991.

In this second edition of the successful book there are a lot of change and improvements. Many new problems and noticable clarifications of many proofs are added improving the readability of the book.

Since the first course in real analysis follows the elementary calculus where the emphasis is on problem-solving and the development of manipulative skills, the book like this has to show the students that the higher mathematics is not simply manipulative but the rigorous proofs have great importance from the point of view of advanced mathematics. This problem has been excellently solved in this book since precise proofs of the theorem are given but the way leading from the intuitive ideas to the end of the proofs is not exhausting. At the same time the book makes clear to the students that the proofs of the basic statements are necessary to their further study in mathematics.

The main topics of the book are: The Real Number System; Basic Properties of Functions on $R^1$; Elementary Theory of Differentiation and Integration in $R^1$ and in $R^N$; Infinite Sequences and Infinite Series; Fourier Series; Functions of Bounded Variation; Riemann—Stieltjes integral; Implicit Function Theory; Approximation Theorems; Vector Field Theory.

The great number of exercises (more than one thousand) help the students in understanding the material.

The book is excellently written, it is recommended to all instructors and students who want to teach or to learn first course in real analysis using an outstanding textbook.

*J. Németh* (Szeged)

**László Rédei, Endliche p-Gruppen (Finite p-Groups),** 304 pages, Akadémiai Kiadó, Budapest, 1989.

The large number and variety of finite $p$-groups have caused the demand for a classification or at least for the creation of a comprehensive theory. Successful efforts in this direction were made only within special (although sufficiently wide) families, such as regular $p$-groups and $p$-groups of maximal class. All these (and other) treatments mainly happen by means of the subgroups, as well as throughout in (finite) group theory.

In his last work Rédei had given a new classification method based on handling the group elements themselves. The initial idea is very simple and natural: Using a (strictly) normal series of the group with cyclic factors, one can choose for each factor an element to cover. The chosen elements form a special generating set (a *basis*) such that any group element possesses a presentation as the product of powers of the generators in fixed order, determined by the order of the factors in the series. With the aid of a basis one easily obtains a set of defining relations for the group. However, this set of relations heavily depends on the particular choice of basis and that makes the classification problem so difficult.

Since powers in finite $p$-groups are more conveniently thought of having $p$-adic integers in the exponents, Chapter 1 is completely devoted to the ring of $p$-adic integers $\mathscr{I}_p$ and the $p$-adic number field $\mathscr{K}_p$. Paragraphs 1—14 provide a practical and elementary introduction into the structure of these objects, and give the very basic terms of $p$-adic analysis. In paragraph 15 a new concept of generalized sum is defined, with a $p$-adic integer as the number of terms.

In Chapter 2 the general theory is developed; the starting point is the following.

Let $G=N_0\supset N_1\supset\ldots\supset N_l=1$ be a normal series for $G$ and $a_1, \ldots, a_l$ a corresponding basis such that $N_{i-1}=\langle a_i, a_{j+1}, a\rangle$ holds for $i=1, \ldots, l$. If $q_j$ denotes the order of $a_j$ mod $N_j$ then the group is defined by the following relations:

$$a_j^{q_j} = \prod_{k=j+1}^{l} a_k^{r_{jk}} \quad (1 \leq j \leq l), \qquad a_j^{a_i} = \prod_{k=j}^{l} a_k^{s_{ijk}} \quad (1 \leq i \leq j \leq l).$$

Let

$$\mathscr{D} = \{q_1, q_2, \ldots; r_{12}, r_{13}, \ldots; s_{122}, s_{123}, \ldots\},$$

be considered as a set of symbols, and for any integer $l$ the *l-th segment* $\mathscr{D}_l$ denotes the subset of $\mathscr{D}$ with all elements, whose indices do not exceed $l$. Thus a presentation $\mathscr{D}_l(G)$ of a group $G$ by means of an $l$ element basis can be thought of as a place (a special value) of $\mathscr{D}_l$. Places generally do not lead to desired ($p$-)groups, therefore the text goes on with determining the conditions for that. All these conditions are of the form that the structure constants be zeros of some continuous $p$-adic function.

(It is worth mentioning that *not* all the functions in question are polynomials.)

The first step towards a classification is the concept of *natural classes;* a group belongs to the $k$-th natural class $\mathscr{C}_k$ iff $k$ is the minimal length of its bases. Then natural classes split into several *parameter classes* in the following way: With given $n$, $l$ and continuous $p$-adic functions

$$q_k \ (1 \leq k \leq l), \quad r_{jk} \ (1 \leq j < k \leq l), \quad s_{ijk} \ (1 \leq i < j \leq k \leq l)$$

whose variables are $t_1, \ldots, t_n$, let $\mathscr{T}_1, \ldots, \mathscr{T}_n$ be at least two element subsets of $\mathscr{I}_p$. Suppose that the values of the above functions at all $t_h \in \mathscr{T}_h$ $(h=1, \ldots, n)$ result in presentations of pairwise nonisomorphic groups in $\mathscr{C}_l$; then the set of these groups is called a *parameter class of degree n.* To find parameter classes (for fixed $l$ and $n$) is not generally easy, and a partition of $\mathscr{C}_l$ into parameter classes of degree $n$ is far from being uniquely determined. The aim is always to get $\mathscr{C}_l$ as the disjoint union of minimal (possibly finite) number of parameter classes of lowest degree. Concerning this problem, the following conjecture is set: *$\mathscr{C}_l$ can be splitted into the disjoint union of finitely many parameter classes of degree* $\leqq \binom{l+2}{3}$.

Going through general theory one will recognize the difficulty of application to concrete classes. However, Chapter 3 is an evidence to the fact that classifying certain classes is not hopeless: Rédei succeeded in dividing $\mathscr{C}_2$ into 3 (for odd $p$) or 9 (for $p=2$) parameter classes of degree at most 4 *(6.1 Satz)*. Even in this simplest case a good deal of extra calculation was needed to get the final result, which does not give rise to much optimism concerning $\mathscr{C}_l$ in general. On the other hand, the theory yields the following beautiful "classical type" theorem:

*For any group G in $\mathscr{C}_2$ (i.e., for any metacyclic p-group)* $|G|$, $\exp(G)$, $|G/G'|$, $\exp(G/G')$, $|Z(G)|$, $\exp(Z(G))$, $|\{x^p\colon x\in G\}|$ *and* $|\{x\in G\colon x^p=e\}|$ *form a complete system of invariants.*

It seems to be obvious that similar theorems will not occur very often. Within the scope of Rédei's new method, instead, there must be more possibilities for anyone not waiting for an easy success.

*Péter Z. Hermann* (Budapest)

**Arto Salomaa, Public-Key Cryptography** (EATCS Monographs in Theoretical Computer Science, 23), X+245 pages, with 18 figures, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong—Barcelona, 1990.

Cryptography, secret writing, is probably as old as writing. This old activity, i.e., to send secret messages, has become the object of scientific research only recently. It is partly due to the need to guarantee the security of data bases, but the military aims are also important.

The first chapter is an outline of the classical two-way cryptography. All the other chapters are devoted to the public-key systems.

In the "classical" cryptosystems both keys, the encrypting and the decrypting keys, are supposed to be secret, in the public-key systems the encrypting keys can be published (like a telephon directory), but the decrypting keys are secret.

In Chapter 1 several classical systems are considered and analysed in the cryptoanalist's (a person, who wants to decypher the secret message without knowing the key) point of view.

In the subsequent five chapters we can read a systematic treatment of the public-key systems (the main point is the RSA system). These systems appeared in the middle of the 70's only. The security of this type of systems are based on results of the complexity theory. The fundamental idea of them is closely related with the following: given an argument value $x$, it is easy to compute the function value $f(x)$, whereas it is intractable (in the sense of the complexity theory) to compute $x$ from $f(x)$.

To read this very interesting book the knowledge of the basic notions and results of the complexity theory (e.g. time complexity, Turing machine, the classes $P$ and $NP$, etc.) and some results from the classical number theory (e.g. congruences, Euler's theorem, quadratic residues, etc.) are also supposed to be known. To help the reader in these fields there are two tutorials as appendices.

*Lajos Klukovits* (Szeged)

# INFORMATIONS FOR AUTHORS

Acta Scientiarum Mathematicarum will publish carefully selected pure mathematical papers. Articles submitted (in duplicate) for publication should be typed double spaced on one side of each sheet (approximately 25 rows and 50 letters in each row). Authors should take the greatest possible care in preparing the manuscript. Hand-written symbols are satisfactory if clearly done; special instructions, when necessary, should be included on a separate sheet. Manuscripts longer than 30 pages will generally not be considered. Authors should consult recent issues of our Acta for general style conventions.

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Dear Colleagues:

With the consent of the Editorial Board I pass the editorial position of our Acta by January 1st, 1993, to Professor László Kérchy.

On this accasion I wish to express my heartfelt thanks to Everybody for helping me in this honouring and delightful, but sometimes also rather tiresome position.

I also sincerely wish our Acta and the new Editor much success.

Szeged, June 1992.

László Leindler

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# INDEX — TARTALOM



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