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A JÓZSEF ATTILA TUDOMÁNYEGYETEM KÖZLEMÉNYEI

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JÓZSEF ATTILA TUDOMÁNYEGYETEM BOLYAI INTÉZETE

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Congruence lattices on a regular semigroup associated with certain operators

FRANCIS PASTIJN and MARIO PETRICH

1. Introduction and summary

To each congruence ϱ on a regular semigroup S we may associate a number of congruences on S according to the following scheme. If Φ is a complete $\cap$-congruence on the congruence lattice $\mathscr{C}(S)$ of S, then the Φ-class of ϱ has a least element ϱ_Φ and we may consider an operator on $\mathscr{C}(S)$ whose effect is: $\varrho \to \varrho_\Phi$. For Φ we may take the congruences T_l, T_r, T, U, V and some of their variants, and the $\cap$-congruence K, studied in the authors' papers [7] and [9]. Recall, for example that T_l stands for having the same left trace, T for having the same trace, K for having the same kernel; the congruences U and V have similar interpretations. We call the sublattice of $\mathscr{C}(S)$ generated by the set $\{\varrho_{T_l}, \varrho_{T_r}, \varrho_K, \varrho_U\}$ the lattice *associated* with ϱ. As we shall see, this lattice is always finite. We shall determine the lattice associated with the congruences ω, σ, ν, γ and η on a regular semigroup S. Here ω denotes the universal congruence and σ, ν, γ and η the least group, Clifford, inverse and semilattice congruences, respectively.

In the case of an inverse semigroup S, the sublattice of $\mathscr{C}(S)$ obtained from ω by successively applying the operators sub T and sub K was investigated and dubbed the min-network of S by PETRICH and REILLY in [13]. In contradistinction to this procedure, we apply our operators only once and then form the sublattice of $\mathscr{C}(S)$ generated by the congruences so obtained. In this sense, our scope is narrower than that in [13]. However, it is also wider in two different directions: we start with various congruences, not just with ω, and study the lattice generated by ϱ_{T_l}, ϱ_{T_r}, ϱ_K and ϱ_U, not just by ϱ_T and ϱ_K. Note that the notation $\varrho_{\min}$ and $\varrho^{\min}$ is used in [13] for ϱ_T and ϱ_K, respectively.

We now summarize the contents of the various sections of the paper. Section 2 contains only some terminology and notation, the rest being relegated to the per-

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tinent literature, as well as some preliminary results. The result in Section 3 relates the effect of applying some of our operators to minimal congruences in terms of Malcev products. The lattice associated with a congruence on a regular semigroup is described in Section 4 in terms of an $\cap$-sublattice of the lattice of congruences. The description of the lattices associated with ω and σ forms the content of Section 5, that of ν and γ of Section 6 and that of η of Section 7.

2. Preliminaries

Throughout the paper, S stands for an arbitrary regular semigroup and E for its set of idempotents, unless stated otherwise.

We use the following notation on S:

ω — the universal relation,
σ — the least group congruence,
ν — the least Clifford congruence,
γ — the least inverse semigroup congruence,
η — the least semilattice congruence,
ε — the equality relation (also denoted by I).

For any semigroup T, we denote by $E(T)$ the set of its idempotents, and for $a \in T$, by $V(a)$ the set of inverses of a. For any relation θ on S, θ^* is the congruence generated by θ.

We shall consider classes $\mathscr{C}$ of regular semigroups which satisfy the conditions

(i) all isomorphic copies of members of $\mathscr{C}$ belong to $\mathscr{C}$,

(ii) $\mathscr{C}$ is closed for the formation of subdirect products within the class of regular semigroups.

Remark that a class $\mathscr{C}$ which satisfies the conditions (i) and (ii) is never empty because it contains the trivial semigroup, which is the direct product of the empty system of semigroups from $\mathscr{C}$. The classes $\mathscr{C}$ which satisfy the conditions (i) and (ii) form a lattice $\mathbf{L}$ under inclusion. If $\mathscr{A}$ and $\mathscr{B}$ are classes of regular semigroups satisfying the above conditions (i) and (ii), then the meet of $\mathscr{A}$ and $\mathscr{B}$ in $\mathbf{L}$ is simply $\mathscr{A} \cap \mathscr{B}$ whereas the join $\mathscr{A} \vee \mathscr{B}$ of $\mathscr{A}$ and $\mathscr{B}$ in $\mathbf{L}$ consists of all isomorphic copies of regular semigroups which are subdirect product of members of $\mathscr{A}$ and of members of $\mathscr{B}$.

Let $\mathscr{C}$ be a class of regular semigroups satisfying the above conditions (i) and (ii). Then there exists a least congruence ϱ on S such that $S/\varrho \in \mathscr{C}$. (See [1], exercise 2 of § 11.6.) This congruence will be denoted by $\theta_{\mathscr{C}}$. In order to simplify our statements, when we write $\varrho = \theta_{\mathscr{C}}$, we tacitly imply that $\mathscr{C} \in \mathbf{L}$. The mapping

$$\mathbf{L} \to \mathscr{C}(S), \quad \mathscr{C} \to \theta_{\mathscr{C}}$$

is an antitone mapping of $\mathbf{L}$ into $\mathscr{C}(S)$ such that for $\mathscr{A}, \mathscr{B} \in \mathbf{L}$,

$$\theta_{\mathscr{A} \vee \mathscr{B}} = \theta_{\mathscr{A}} \cap \theta_{\mathscr{B}}, \ \theta_{\mathscr{A} \cap \mathscr{B}} \supseteqq \theta_{\mathscr{A}} \vee \theta_{\mathscr{B}}.$$

If $\mathscr{A}, \mathscr{B} \in \mathbf{L}$ and if $\mathscr{A}$ and $\mathscr{B}$ are closed for taking homomorphic images, then $\theta_{\mathscr{A} \cap \mathscr{B}} = \theta_{\mathscr{A}} \vee \theta_{\mathscr{B}}$. We shall apply these results without further notice.

We now list some of the classes of regular semigroups which belong to $\mathbf{L}$. The abbreviations we introduce here will be used freely throughout the paper. For some of them the defining identities can be found in [12].

$\mathscr{T}$ — trivial semigroups,
$\mathscr{L}\mathscr{Z}$ — left zero semigroups,
$\mathscr{R}\mathscr{Z}$ — right zero semigroups,
$\mathscr{R}e\mathscr{B}$ — rectangular bands,
$\mathscr{L}\mathscr{R}\mathscr{B}$ — left regular bands,
$\mathscr{R}\mathscr{R}\mathscr{B}$ — right regular bands,
$\mathscr{R}\mathscr{B}$ — regular bands,
$\mathscr{B}$ — bands,
$\mathscr{G}$ — groups,
$\mathscr{L}\mathscr{G}$ — left groups,
$\mathscr{R}\mathscr{G}$ — right groups,
$\mathscr{R}e\mathscr{G}$ — rectangular groups,
$\mathscr{S}\mathscr{G}$ — Clifford semigroups,
$\mathscr{L}\mathscr{R}\mathscr{B}\mathscr{G}$ — left regular bands of groups,
$\mathscr{R}\mathscr{R}\mathscr{B}\mathscr{G}$ – right regular bands of groups,
$\mathscr{R}\mathscr{O}\mathscr{B}\mathscr{G}$ — regular orthodox bands of groups,
$\mathscr{U}\mathscr{B}\mathscr{G}$ – E-unitary bands of groups,
$\mathscr{O}\mathscr{B}\mathscr{G}$ — orthodox bands of groups,
$\mathscr{C}\mathscr{S}$ — completely simple semigroups,
$\mathscr{L}\mathscr{U}\mathscr{B}\mathscr{G}$ — locally E-unitary bands of groups,
$\mathscr{B}\mathscr{G}$ — bands of groups,
$\mathscr{L}\mathscr{R}\mathscr{O}\mathscr{G}$ — left regular orthogroups,
$\mathscr{R}\mathscr{R}\mathscr{O}\mathscr{G}$ — right regular orthogroups,
$\mathscr{L}\mathscr{C}\mathscr{R}\mathscr{O}\mathscr{G}$ — left compatible regular orthogroups,
$\mathscr{R}\mathscr{C}\mathscr{R}\mathscr{O}\mathscr{G}$ — right compatible regular orthogroups,
$\mathscr{R}\mathscr{O}\mathscr{G}$ — regular orthogroups,
$\mathscr{O}\mathscr{G}$ — orthogroups,
$\mathscr{C}\mathscr{R}$ — completely regular semigroups,
$\mathscr{I}$ — inverse semigroups,
$\mathscr{L}\mathscr{R}\mathscr{O}$ — left regular orthodox semigroups,
$\mathscr{R}\mathscr{R}\mathscr{O}$ — right regular orthodox semigroups,

$\mathscr{RO}$ — regular orthodox semigroups,
$\mathscr{O}$ — orthodox semigroups,
$\mathscr{QO}$ — quasiorthodox semigroups,
$\mathscr{U}$ — E-unitary regular semigroups,
$\mathscr{R}$ — E-reflexive regular semigroups.

In lieu of a complete explanation of these terms, we offer here only a few basic hints; for the rest we refer to the literature on regular semigroups.

"Left regular" refers to idempotents forming a left regular band (i.e. satisfying the identity $ax=axa$); "right regular" has the corresponding meaning; "regular" means that the idempotents form a regular band (i.e. satisfy the identity $axya=axaya$). "E-unitary" means that idempotents form a unitary subset. "Locally $\mathscr{P}$" denotes that all subsemigroups of the form eSe, where $e \in E$, have property $\mathscr{P}$. "Left compatible" stands for $\mathscr{L}$ being a congruence; "right compatible" for $\mathscr{R}$ being a congruence. "Orthodox" refers to idempotents forming a subsemigroup; if also the semigroup is completely regular, it is an "orthogroup". Finally "quasiorthodox" stands for the semigroup generated by the idempotents being completely regular. "E-reflexive regular" means a semilattice of E-unitary regular semigroups.

We now establish some auxiliary statements leading to the lattice of certain quasivarieties of completely regular semigroups which will be useful for later considerations.

Lemma 1. *A regular semigroup S is in $\mathscr{UBG}$ if and only if S is a subdirect product of a band and a group.*

Proof. Let $S \in \mathscr{UBG}$. By ([6], Corollary 6.40), S is a subdirect product of a fundamental regular semigroup T and a group G. Since T is a homomorphic image of S, it must be a band of groups and hence a band.

The converse follows immediately.

Lemma 2. *A regular semigroup S is in $\mathscr{LUBG}$ if and only if S is a subdirect product of a band and a completely simple semigroup.*

Proof. Let $S \in \mathscr{LUBG}$. By ([2], Corollary 5.5(ii)), S is a subdirect product of a band B and a normal band of groups N. According to ([11], IV.4.3), N is a strong semilattice of completely simple semigroups, in notation $N=[Y; S_\alpha, \varphi_{\alpha,\beta}]$. Define a relation ϱ on N by: for $a \in S_\alpha$, $b \in S_\beta$,

$$a \varrho b \Leftrightarrow a\varphi_{\alpha,\gamma} = b\varphi_{\beta,\gamma} \quad \text{for some } \gamma \leqq \alpha\beta.$$

Straightforward verification shows that ϱ is a congruence and that S/ϱ is completely simple.

Now let $a\mathscr{H}\cap\varrho b$. Then $a, b\in S_\alpha$ for some $\alpha\in Y$ and $a\varphi_{\alpha,\gamma}=b\varphi_{\alpha,\gamma}$ for some $\gamma\leqq\alpha$. Letting $e\in E(H_a)$ and $f=e\varphi_{\alpha,\gamma}$, we get

$$(ab^{-1})f=(ab^{-1})\varphi_{\alpha,\gamma}=(a\varphi_{\alpha,\gamma})(b\varphi_{\alpha,\gamma})^{-1}=f.$$

There exits $u, v\in B$ such that $(u, ab^{-1}), (v, f)\in S$. It is easy to see that $(u, ab^{-1})\times\times(u, ab^{-1})^{-1}=(u, e)$ in the band of groups S. Hence $(uvu, f)=(u, e)(v, f)(u, e)$ and (u, ab^{-1}) both belong to $(u, e)S(u, e)$ and

$$(u, ab^{-1})(uvu, f)=(uvu, f).$$

Since $(u, e)S(u, e)$ is E-unitary, the above implies that (u, ab^{-1}) is an idempotent. Consequently $ab^{-1}=e$ and $a=b$. Therefore N is a subdirect product of a normal band and a completely simple semigroup. Thus S itself is a subdirect product of a band and a completely simple semigroup.

Any band and any completely simple semigroup is in $\mathscr{L}\mathscr{U}\mathscr{B}\mathscr{G}$ and thus so is any subdirect product of these since a quasivariety is closed under direct products and subalgebras.

Lemma 3. *Diagram 1 with vertices labelled with script letters depicts the lattice of quasivarieties of completely regular semigroups generated by the set* $\{\mathscr{L}\mathscr{G}, \mathscr{R}\mathscr{G}, \mathscr{B}, \mathscr{C}\mathscr{S}\}$.

Proof. That the meets of any two of these quasivarieties agree with those in the diagram is obvious. The joins $\mathscr{B}\vee\mathscr{G}=\mathscr{U}\mathscr{B}\mathscr{G}$ and $\mathscr{B}\vee\mathscr{C}\mathscr{S}=\mathscr{L}\mathscr{U}\mathscr{B}\mathscr{G}$ follow from Lemma 1 and Lemma 2, respectively; the remaining joins are consequences of well-known properties of these semigroups. The assertion of the lemma now follows by simple inspection.

If $\mathscr{C}\in\mathbf{L}$, then we say that the congruence ϱ on S is *over* $\mathscr{C}$ if the idempotent ϱ-classes belong to $\mathscr{C}$.

We can introduce the relations T_l, T_r, U, K on the congruence lattice $\mathscr{C}(S)$ in the following way. For $\varrho_1, \varrho_2\in\mathscr{C}(S)$ we say that ϱ_1 and ϱ_2 are $T_l-[T_r-, U-, K-]$ related if $\varrho_1/(\varrho_1\cap\varrho_2)$ and $\varrho_2/(\varrho_1\cap\varrho_2)$ are over $\mathscr{L}\mathscr{G}[\mathscr{R}\mathscr{G}, \mathscr{C}\mathscr{S}, \mathscr{B}]$. If we put

$$T=T_l\cap T_r,\ V=U\cap K,\ K_l=T_l\cap K,\ K_r=T_r\cap K,$$

then we obviously have that ϱ_1 and ϱ_2 are $T-[V-, K_l-, K_r-]$ related if and only if $\varrho_1/(\varrho_1\cap\varrho_2)$ and $\varrho_2/(\varrho_1\cap\varrho_2)$ are over $\mathscr{G}[\mathscr{R}e\mathscr{B}, \mathscr{L}\mathscr{Z}, \mathscr{R}\mathscr{Z}]$. We also see that $I=T\cap K$ is the equality relation on $\mathscr{C}(S)$. These relations

$$(1)\qquad T_l, T_r, U, K, T, V, K_l, K_r, I$$

were introduced and investigated in [7] and [9]. A survey of the principal results can be found in [5].

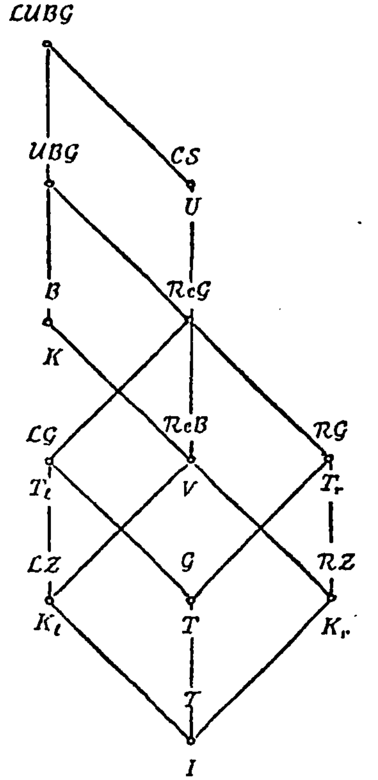

Diagram 1

In [7] and [9] it is proved that all of the relations in (1) except K are complete congruences on $\mathscr{C}(S)$. The relation K is a complete $\cap$-congruence but not necessarily a $\vee$-congruence. If $\varrho\in\mathscr{C}(S)$ and Φ is any of the relations in (1), then the Φ-class of ϱ contains a smallest element which we denote by ϱ_{Φ}. The sublattice of $\mathscr{C}(S)$ generated by the set $\{\varrho_{T_l}, \varrho_{T_r}, \varrho_U, \varrho_K\}$ will be called the lattice *associated with* ϱ.

We shall frequently use the following elementary result, the proof of which will be omitted.

Lemma 4. *Let* C *be a complete lattice and* Φ_1, Φ_2 *complete congruences over* C. *For any* $x\in$ C *denote by* $x_{\Phi_1}[x_{\Phi_2}, x_{(\Phi_1\cap\Phi_2)}, x_{(\Phi_1\vee\Phi_2)}]$ *the least element in the* $\Phi_1-[\Phi_2-, (\Phi_1\cap\Phi_2)-, (\Phi_1\vee\Phi_2)-]$ *class of* x. *Then*

$$x_{(\Phi_1\cap\Phi_2)} = x_{\Phi_1}\vee x_{\Phi_2},\quad x_{(\Phi_1\vee\Phi_2)}(\Phi_1\vee\Phi_2)\, x_{\Phi_1}\cap x_{\Phi_2}.$$

We remark that

$$I \subseteq K_l \subseteq T_l \subseteq U,\ I \subseteq T \subseteq T_l \subseteq U,\ I \subseteq K_l \subseteq V \subseteq K$$

and their (left-right) duals hold. From this we already have that

$$\varrho_U \subseteq \varrho_V \subseteq \varrho_{K_l} \subseteq \varrho_I = \varrho,\ \varrho_U \subseteq \varrho_{T_l} \subseteq \varrho_T \subseteq \varrho_I = \varrho,\ \varrho_{T_l} \subseteq \varrho_{K_l},$$

$$\varrho_K \subseteq \varrho_V \subseteq \varrho_{K_l} \subseteq \varrho_I = \varrho$$

and their duals hold. From this we find that $\varrho_V\, U\, \varrho_{K_l}$, $\varrho_T\, T_l\, \varrho$, $\varrho_{K_l}\, T_l\, \varrho$ and so on. Moreover,

Lemma 5. *Let* $\varrho \in \mathscr{C}(S)$ *and let* Φ_1, Φ_2 *be any two of the relations in* (1). *Then* $\varrho_{(\Phi_1 \cap \Phi_2)} = \varrho_{\Phi_1} \vee \varrho_{\Phi_2}$.

Proof. If neither Φ_1 nor Φ_2 is K, then we can apply Lemma 4. Let us now consider the case where one of the Φ_i equals K.

We have $\varrho_{U \cap K} = \varrho_V \supseteq \varrho_U \vee \varrho_K$. Since $\varrho_U \subseteq \varrho_U \vee \varrho_K \subseteq \varrho$, we have that $\varrho/(\varrho_U \vee \varrho_K)$ is over $\mathscr{CS}$ and since $\varrho_K \subseteq \varrho_U \vee \varrho_K \subseteq \varrho$, we have that $\varrho/(\varrho_U \vee \varrho_K)$ is over $\mathscr{B}$. Therefore $\varrho/(\varrho_U \vee \varrho_K)$ is over $\mathscr{R}e\mathscr{B}$ and we have $\varrho_V \subseteq \varrho_U \vee \varrho_K$. Consequently the equality $\varrho_V = \varrho_U \vee \varrho_K$ prevails.

The remaining cases involving K can be resolved in a similar way.

From the above we have $K_l = T_l \cap K = T_l \cap V$ and thus $\varrho_{K_l} = \varrho_{T_l} \vee \varrho_K = \varrho_{T_l} \vee \varrho_V$ for every $\varrho \in \mathscr{C}(S)$. Also

$$I = T \cap K = T_l \cap T_r \cap K = T_l \cap T_r \cap V = K_l \cap K_r$$

gives

$$\varrho = \varrho_T \vee \varrho_K = \varrho_{T_l} \vee \varrho_{T_r} \vee \varrho_K = \varrho_{T_l} \vee \varrho_{T_r} \vee \varrho_V = \varrho_{K_l} \vee \varrho_{K_r}$$

for every $\varrho \in \mathscr{C}(S)$.

The results concerning the relations (1) mentioned here will be used without further ado.

3. Malcev products

A class of semigroups is an *isomorphism class* if it is closed for taking isomorphic images. Let $\mathscr{X}$ and $\mathscr{Y}$ be isomorphism classes of regular semigroups. The *Malcev product of* $\mathscr{X}$ *and* $\mathscr{Y}$ *(within the class of all regular semigroups)* is the class of regular semigroups

$$\mathscr{X} \circ \mathscr{Y} = \{S \mid \text{there is a congruence } \varrho \text{ on } S \text{ over } \mathscr{X} \text{ such that } S/\varrho \in \mathscr{Y}\}.$$

We are interested here in the case where $\mathscr{X}$ is a variety of completely simple semigroups or a variety of bands and $\mathscr{Y} \in \mathbf{L}$.

For the notation and conventions incorporated in the next result, consult the preceding section.

We now define the following mapping

$$\chi = \begin{pmatrix} \mathcal{T} & \mathcal{LZ} & \mathcal{RZ} & \mathcal{R}e\mathcal{B} & \mathcal{G} & \mathcal{LG} & \mathcal{RG} & \mathcal{B} & \mathcal{CS} \\ I & K_l & K_r & V & T & T_l & T_r & K & U \end{pmatrix}. \tag{2}$$

Note that χ follows the labelling in Diagram 1. Let

$$\Gamma = \{\mathcal{T}, \mathcal{LZ}, \mathcal{RZ}, \mathcal{R}e\mathcal{B}, \mathcal{G}, \mathcal{LG}, \mathcal{RG}, \mathcal{B}, \mathcal{CS}\}, \tag{3}$$

$$\Delta = \{I, K_l, K_r, V, T, T_l, T_r, K, U\}, \tag{4}$$

both ordered by inclusion. Using the information concerning the elements of Δ listed in the preceding section, we see that Δ is an $\cap$-semilattice. Obviously Γ is also an $\cap$-semilattice and χ is an $\cap$-isomorphism of Γ onto Δ.

Theorem 1. *If* $\mathcal{C}\in\mathbf{L}$ *and* $\mathcal{P}\in\Gamma$, *then* $\mathcal{P}\circ\mathcal{C}\in\mathbf{L}$ *and* $(\theta_{\mathcal{C}})_{\mathcal{P}\chi}=\theta_{\mathcal{P}\circ\mathcal{C}}$.

Proof. If $\mathcal{X}, \mathcal{Y}\in\mathbf{L}$, then routine verification shows that $\mathcal{X}\circ\mathcal{Y}\in\mathbf{L}$. In particular, if $\mathcal{P}\in\Gamma$ and $\mathcal{C}\in\mathbf{L}$ we have that $\mathcal{P}\circ\mathcal{C}\in\mathbf{L}$ and $\theta_{\mathcal{P}\circ\mathcal{C}}$ exists.

For $\mathcal{P}=\mathcal{T}$ we have $(\theta_{\mathcal{C}})_I=\theta_{\mathcal{C}}=\theta_{\mathcal{T}\circ\mathcal{C}}$ and the formula holds.

We consider next the case $\mathcal{P}=\mathcal{G}$. We must show that $\theta_T=\theta_{\mathcal{G}\circ\mathcal{G}}$ where $\theta=\theta_{\mathcal{C}}$. To prove that $\theta_T\supseteq\theta_{\mathcal{G}\circ\mathcal{C}}$, we must show that θ_T is a $\mathcal{G}\circ\mathcal{C}$-congruence. By the definition of T we know that θ/θ_T is over $\mathcal{G}$. Further, $(S/\theta_T)/(\theta/\theta_T)\cong S/\theta\in\mathcal{C}$ and therefore $S/\theta_T\in\mathcal{G}\circ\mathcal{C}$. Thus indeed $\theta_{\mathcal{G}\circ\mathcal{C}}\subseteq\theta_T$. In order to establish the opposite inclusion, we consider an arbitrary $\mathcal{G}\circ\mathcal{C}$-congruence ϱ on S. There exists a congruence λ on S/ϱ such that $(S/\varrho)/\lambda\in\mathcal{C}$ and such that all idempotent λ-classes are groups. Lifting λ to S we obtain a congruence τ for which $\tau/\varrho=\lambda$. Since τ/ϱ is over $\mathcal{G}$, we have that $\varrho T\tau$. Since $S/\tau\cong(S/\varrho)/(\tau/\varrho)\in\mathcal{C}$ we have that $\theta\subseteq\tau$. Hence $\theta_T\subseteq\tau_T=\varrho_T\subseteq\varrho$ and we conclude that $\theta_T\subseteq\theta_{\mathcal{G}\circ\mathcal{C}}$.

We have proved that the above formula holds for $\mathcal{P}=\mathcal{G}$. For the remaining cases we may follow the above argument step by step.

Corollary. *For any* $\mathcal{P}, \mathcal{Q}\in\Gamma$ *and* $\mathcal{C}\in\mathbf{L}$, *we have*

$$\theta_{(\mathcal{P}\cap\mathcal{Q})\circ\mathcal{C}} = (\theta_{\mathcal{C}})_{\mathcal{P}\chi\cap\mathcal{Q}\chi} \supseteq (\theta_{\mathcal{C}})_{\mathcal{P}\chi}\vee(\theta_{\mathcal{C}})_{\mathcal{Q}\chi} = \theta_{\mathcal{P}\circ\mathcal{C}}\vee\theta_{\mathcal{Q}\circ\mathcal{C}}.$$

Proof. This follows immediately from the above theorem using the fact that χ is an $\cap$-isomorphism.

4. The lattice associated with a congruence

The main result here describes the lattice associated with any congruence ϱ on a regular semigroup S as the finite $\cap$-sublattice of the congruence lattice of S generated by eight congruences derived from ϱ by the operations introduced earlier.

In the proof of the following theorem we freely use the fact that the relations in (1) different from K are congruences on $\mathscr{C}(S)$, and that for every $\varrho \in \mathscr{C}(S)$,

(5) $$\varrho_V V \varrho_{K_l} V \varrho_{K_r} V \varrho,$$

and

(6) $$\varrho_{T_l} T_l \varrho_{K_l} T_l \varrho_T T_l \varrho$$

hold. The validity of (5) and (6) follows immediately from the definitions of the relations (1).

A glance at Diagram 2 may help visualize the heuristics behind the proof of the following theorem.

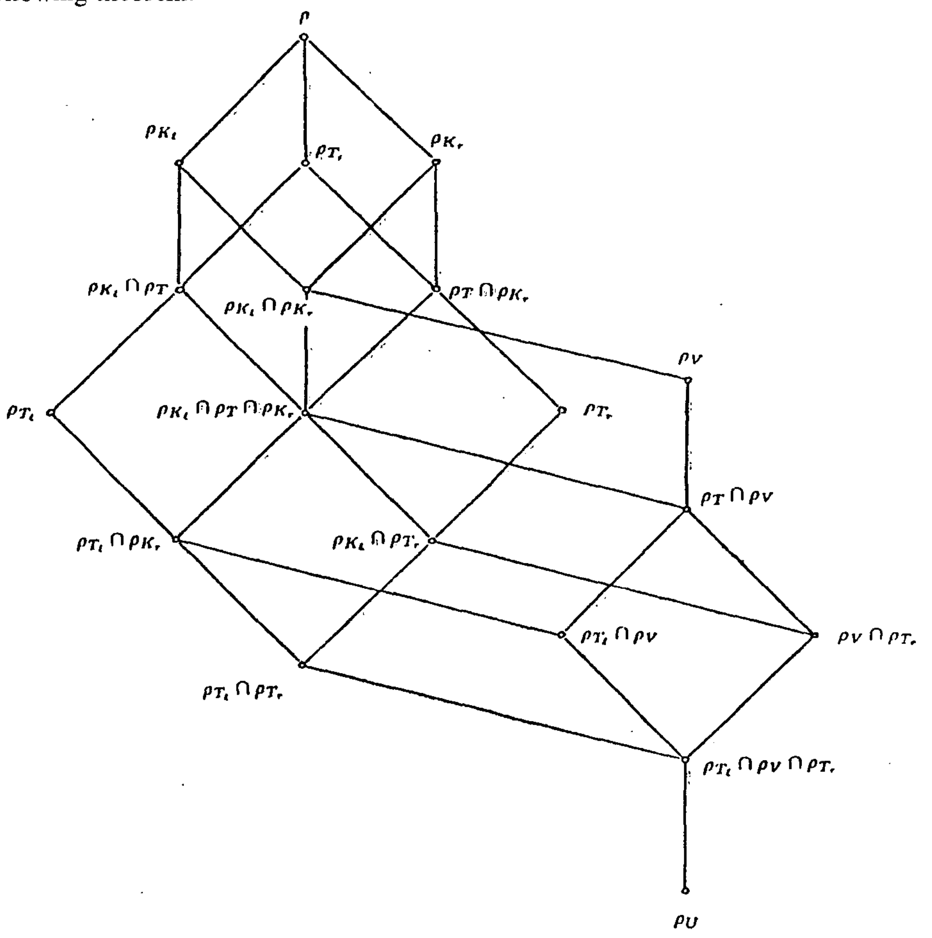

Diagram 2

Theorem 2. *Let ϱ be any congruence on a regular semigroup S. Then the sublattice of the congruence lattice $\mathscr{C}(S)$ generated by $\{\varrho_{T_l}, \varrho_{T_r}, \varrho_V, \varrho_U\}$ is the finite $\cap$-subsemilattice of $\mathscr{C}(S)$ generated by the set*

$$\{\varrho, \varrho_{K_l}, \varrho_{K_r}, \varrho_T, \varrho_{T_l}, \varrho_{T_r}, \varrho_V, \varrho_U\}. \tag{7}$$

Proof. Taking into account that

$$\varrho_U \subseteqq \varrho_{T_l} \subseteqq \varrho_{K_l} \subseteqq \varrho, \ \varrho_{T_l} \subseteqq \varrho_T \subseteqq \varrho, \ \varrho_U \subseteqq \varrho_V \subseteqq \varrho_{K_l}$$

and their duals hold, it is easy to see that Diagram 2 gives the $\cap$-semilattice L generated by the set (1). We shall now verify that L is a sublattice of $\mathscr{C}(S)$.

Obviously

$$\varrho_{K_l} \vee \varrho_{K_r} = \varrho, \ \varrho_{T_l} \vee \varrho_{T_r} = \varrho_T, \ \varrho_{T_l} \vee \varrho_V = \varrho_{K_l}, \ \varrho_{T_r} \vee \varrho_V = \varrho_{K_r}$$

in $\mathscr{C}(S)$ and therefore also

$$\varrho_T = \varrho_{T_l} \vee (\varrho_T \cap \varrho_{K_r}) = \varrho_{T_r} \vee (\varrho_{K_l} \cap \varrho_T) = (\varrho_{K_l} \cap \varrho_T) \vee (\varrho_T \cap \varrho_{K_r}). \tag{8}$$

In the following we use the fact that T_l, T_r and V are congruences, that $T_l \cap V \cap T_r$ is the equality on $\mathscr{C}(S)$, and that (5), (6) and the dual of (6) hold. From

$$(\varrho_{K_l} \cap \varrho_T) \vee (\varrho_{K_l} \cap \varrho_{K_r}) T (\varrho_{K_l} \cap \varrho) \vee (\varrho_{K_l} \cap \varrho_{K_r}) = \varrho_{K_l}$$

and

$$(\varrho_{K_l} \cap \varrho_T) \vee (\varrho_{K_l} \cap \varrho_{K_r}) V (\varrho \cap \varrho_T) \vee \varrho = \varrho V \varrho_{K_l}$$

it follows that

$$(\varrho_{K_l} \cap \varrho_T) \vee (\varrho_{K_l} \cap \varrho_{K_r}) = \varrho_{K_l}. \tag{9}$$

From

$$\varrho_{T_l} \vee (\varrho_{K_l} \cap \varrho_T \cap \varrho_{K_r}) T_l \varrho_T \cap \varrho_{K_l},$$

$$\varrho_{T_l} \vee (\varrho_{K_l} \cap \varrho_T \cap \varrho_{K_r}) T_r \varrho_{T_l} \vee (\varrho_{K_l} \cap \varrho_T \cap \varrho) = \varrho_{K_l} \cap \varrho_T$$

and

$$\varrho_{T_l} \vee (\varrho_{K_l} \cap \varrho_T \cap \varrho_{K_r}) V \varrho_{T_l} \vee (\varrho \cap \varrho_T) = \varrho \cap \varrho_T V \varrho_{K_l} \cap \varrho_T,$$

it follows that

$$\varrho_{T_l} \vee (\varrho_{K_l} \cap \varrho_T \cap \varrho_{K_r}) = \varrho_{K_l} \cap \varrho_T. \tag{10}$$

From

$$(\varrho_{T_l} \cap \varrho_{K_r}) \vee (\varrho_{K_l} \cap \varrho_{T_r}) T_l (\varrho \cap \varrho_{K_r}) \vee (\varrho \cap \varrho_{T_r}) = \\ = \varrho_{K_r} = \varrho \cap \varrho \cap \varrho_{K_r} T_l \varrho_{K_l} \cap \varrho_T \cap \varrho_{K_r},$$

the dual

$$(\varrho_{T_l} \cap \varrho_{K_r}) \vee (\varrho_{K_l} \cap \varrho_{T_r}) T_r \varrho_{K_l} \cap \varrho_T \cap \varrho_{K_r}$$

and

$$(\varrho_{T_l} \cap \varrho_{K_r}) \vee (\varrho_{K_l} \cap \varrho_{T_r}) V (\varrho_{T_l} \cap \varrho) \vee (\varrho \cap \varrho_{T_r}) = \\ = \varrho_T = \varrho \cap \varrho_T \cap \varrho V \varrho_{K_l} \cap \varrho_T \cap \varrho_{K_r},$$

it follows that

$$(\varrho_{T_l} \cap \varrho_{K_r}) \vee (\varrho_{K_l} \cap \varrho_{T_r}) = \varrho_{K_l} \cap \varrho_T \cap \varrho_{K_r}. \tag{11}$$

From

$$(\varrho_{T_l}\cap\varrho_V)\vee(\varrho_V\cap\varrho_{T_r})T_l(\varrho\cap\varrho_V)\vee(\varrho_V\cap\varrho_{T_r}) = \varrho_V = \varrho\cap\varrho_V T_l\varrho_T\cap\varrho_V,$$

the dual

$$(\varrho_{T_l}\cap\varrho_V)\vee(\varrho_V\cap\varrho_{T_r})T_r\varrho_T\cap\varrho_V$$

and

$$(\varrho_{T_l}\cap\varrho_V)\vee(\varrho_V\cap\varrho_{T_r})V(\varrho_{T_l}\cap\varrho)\vee(\varrho\cap\varrho_{T_r}) = \varrho_T = \varrho_T\cap\varrho V\varrho_T\cap\varrho_V,$$

it follows that

(12) $$(\varrho_{T_l}\cap\varrho_V)\vee(\varrho_V\cap\varrho_{T_r}) = \varrho_T\cap\varrho_V.$$

From

$$(\varrho_{T_l}\cap\varrho_{T_r})\vee(\varrho_{T_l}\cap\varrho_V)T_l(\varrho\cap\varrho_{T_r})\vee(\varrho\cap\varrho_V) = \varrho_{K_r} = \varrho\cap\varrho_{K_r}T_l\varrho_{T_l}\cap\varrho_{K_r},$$

$$(\varrho_{T_l}\cap\varrho_{T_r})\vee(\varrho_{T_l}\cap\varrho_V)T_r(\varrho_{T_l}\cap\varrho)\vee(\varrho_{T_l}\cap\varrho_V) = \varrho_{T_l} = \varrho_{T_l}\cap\varrho T_r\varrho_{T_l}\cap\varrho_{K_r}$$

and

$$(\varrho_{T_l}\cap\varrho_{T_r})\vee(\varrho_{T_l}\cap\varrho_V)V(\varrho_{T_l}\cap\varrho_{T_r})\vee(\varrho_{T_l}\cap\varrho) = \varrho_{T_l} = \varrho_{T_l}\cap\varrho V\varrho_{T_l}\cap\varrho_{K_r},$$

it follows that

(13) $$(\varrho_{T_l}\cap\varrho_{T_r})\vee(\varrho_{T_l}\cap\varrho_V) = \varrho_{T_l}\cap\varrho_{K_r}.$$

From

$$(\varrho_{T_l}\cap\varrho_{T_r})\vee\varrho_V T_l(\varrho\cap\varrho_{T_r})\vee\varrho_V = \varrho_{K_r} = \varrho\cap\varrho_{K_r}T_l\varrho_{K_l}\cap\varrho_{K_r},$$

the dual

$$(\varrho_{T_l}\cap\varrho_{T_r})\vee\varrho_V T_r\varrho_{K_l}\cap\varrho_{K_r},$$

and

$$(\varrho_{T_l}\cap\varrho_{T_r})\vee\varrho_V V(\varrho_{T_l}\cap\varrho_{T_r})\vee\varrho = \varrho V\varrho_{K_l}\cap\varrho_{K_r}$$

it follows that

(14) $$(\varrho_{T_l}\cap\varrho_{T_r})\vee\varrho_V = \varrho_{K_l}\cap\varrho_{K_r}.$$

The remaining cases now follow easily from the above equalities (8)—(14) and their duals.

Depending on the special nature of S and ϱ, some of the elements of the lattice L occurring in Diagram 2 may coincide. Therefore, we have the following result.

Corollary. *Let ϱ be a congruence on the regular semigroup S. Then the sublattice of $\mathscr{C}(S)$ generated by $\{\varrho_{T_l}, \varrho_{T_r}, \varrho_V, \varrho_U\}$ is a homomorphic image of the lattice of Diagram 2.*

In the following we shall show that the lattice in Diagram 2 *can be* the lattice generated by $\{\varrho_{T_l}, \varrho_{T_l}, \varrho_V, \varrho_U\}$ for a suitable ϱ. For this we shall consider $\varrho=\eta$. Therefore we can say that, *in general,* Diagram 2 depicts the lattice generated by $\{\varrho_{T_l}, \varrho_{T_r}, \varrho_V, \varrho_U\}$.

5. The lattices associated with ω and σ

As we shall see, the situation here is very simple.

Theorem 3. *Diagrams* 3a *and* 3b *depict the lattices associated with the universal relation ω and the least group congruence σ, respectively.*

Proof. All the equalities in Diagram 3a follow from Theorem 1 since for any class $\mathscr{P}$ of regular semigroups we have $\mathscr{P}\circ\mathscr{T}=\mathscr{P}$ and $\mathscr{T}\chi=I$. As in the proof of Lemma 3, the only assertions about meets and joins which are not well-known are $\theta_{\mathscr{G}}\cap\theta_{\mathscr{B}}=\theta_{\mathscr{U}\mathscr{B}\mathscr{G}}$ and $\theta_{\mathscr{C}\mathscr{S}}\cap\theta_{\mathscr{B}}=\theta_{\mathscr{L}\mathscr{U}\mathscr{B}\mathscr{G}}$. These follow directly from Lemma 1 and Lemma 2, respectively.

The equalities at the vertices of Diagram 3b follow from Theorem 1 in view of the equalities

$$\mathscr{G}\circ\mathscr{G}=\mathscr{G},\quad \mathscr{L}\mathscr{G}\circ\mathscr{G}=\mathscr{L}\mathscr{Z}\circ\mathscr{G}=\mathscr{L}\mathscr{G},$$

$$\mathscr{R}\mathscr{G}\circ\mathscr{G}=\mathscr{R}\mathscr{Z}\circ\mathscr{G}=\mathscr{R}\mathscr{G},\quad \mathscr{B}\circ\mathscr{G}=\mathscr{U},\quad \mathscr{C}\mathscr{S}\circ\mathscr{G}=\mathscr{C}\mathscr{S}.$$

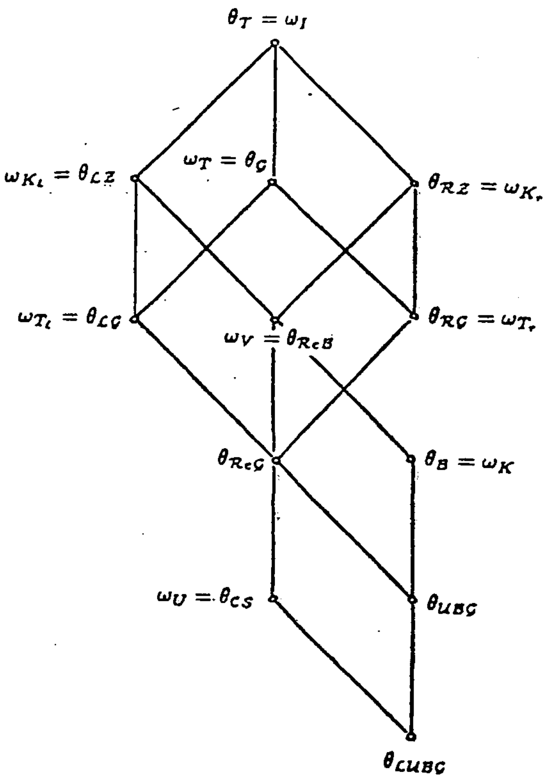

Diagram 3a

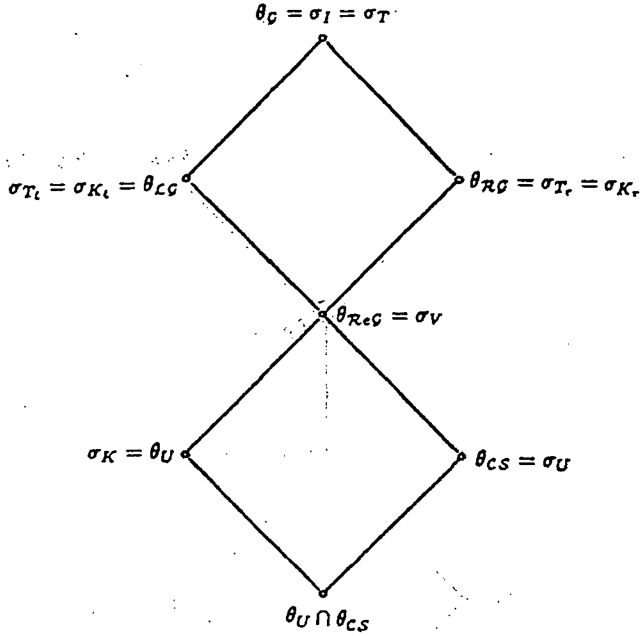

Diagram 3b

Here $\mathcal{B} \circ \mathcal{G} = \mathcal{U}$ follows from ([6], Theorem 6.37); the remaining equalities can be easily verified. Joins and meets follow from well-known characterizations of semigroups in the respective classes.

The above proof also indicates that Diagram 3a is just the inverted Diagram 1. To see that the meets in Diagram 3a are the correct ones, we observe that the joins in Diagram 1 for quasivarieties amount to taking subdirect products which corresponds to taking intersection of minimal congruences $\theta_{\mathcal{C}}$ in Diagram 1. We leave the structural description of $\mathcal{U} \vee \mathcal{CS}$ open.

6. The lattices associated with ν and γ

Recall that Clifford semigroup is a synonym for semilattice of groups. The situation here is somewhat more complex.

Theorem 4. *Diagrams* 4a *and* 4b *depict the lattices associated with the least Clifford congruence* ν *and the least inverse congruence* γ*, respectively.*

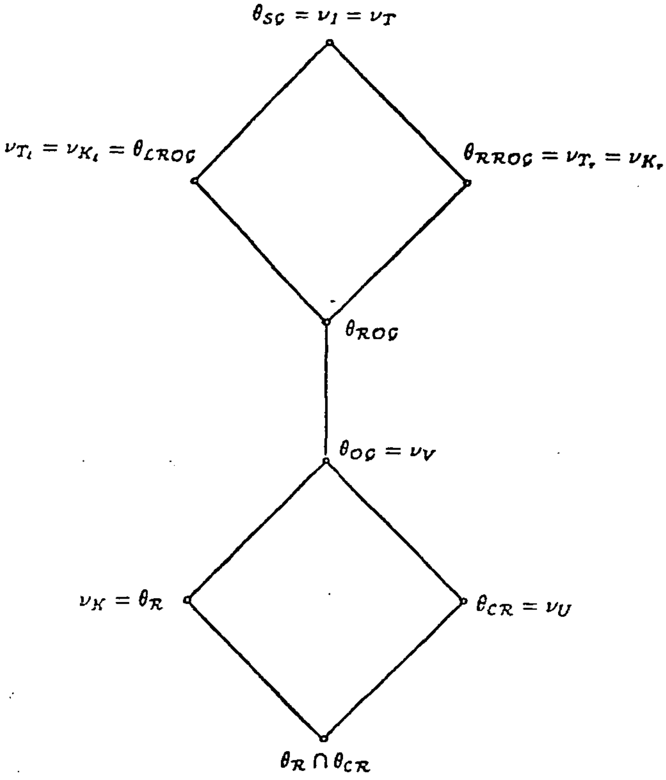

Diagram 4a

Proof. The equalities at the vertices of Diagrams 4a and 4b will follow from Theorem 1 if we establish the corresponding statements about Malcev products, which we proceed to do.

1. $\mathcal{G} \circ \mathcal{SG} = \mathcal{SG}$. Let $S \in \mathcal{G} \circ \mathcal{SG}$ so that there exists a congruence θ on S such that θ is over $\mathcal{G}$ and $S/\theta \in \mathcal{SG}$. Hence S/θ is a semilattice Y of groups G_α. For each $\alpha \in Y$, let $S_\alpha = G_\alpha \theta$ so that S is a semilattice Y of semigroups S_α. But $S_\alpha \in \mathcal{G} \circ \mathcal{G}$ whence $S_\alpha \in \mathcal{G}$. Thus S is a semilattice of groups S_α whence $S \in \mathcal{SG}$. Therefore $\mathcal{G} \circ \mathcal{SG} \subseteq \mathcal{SG}$; the oppositive inclusion is trivial.

2. $\mathcal{LZ} \circ \mathcal{SG} = \mathcal{LG} \circ \mathcal{SG} = \mathcal{LROG}$. Trivially $\mathcal{LZ} \circ \mathcal{SG} \subseteq \mathcal{LG} \circ \mathcal{SG}$. Next let $S \in \mathcal{LG} \circ \mathcal{SG}$ so that there exists a congruence θ on S such that θ is over $\mathcal{LG}$ and $S/\theta \in \mathcal{SG}$. Let $a \in S$ and $b \in V(a)$. Then $b\theta \in V(a\theta)$ so $(a\theta)(b\theta) = (b\theta)(a\theta)$ since $S/\theta \in \mathcal{SG}$. Thus $ab\,\theta\,ba$ and $(ab)\theta \in \mathcal{LG}$ so there exists $x \in S$ such that $ab = xba$ whence $a = aba = xba^2$. It follows that $a \in aSa^2$ for every $a \in S$ and thus $S \in \mathcal{CR}$ by ([11], IV. 1.6). Now let $e, f \in E$. Then $ef\,\theta\,fe$ and $(ef)\theta = (e\theta)(f\theta) \in E(S/\theta)$ since

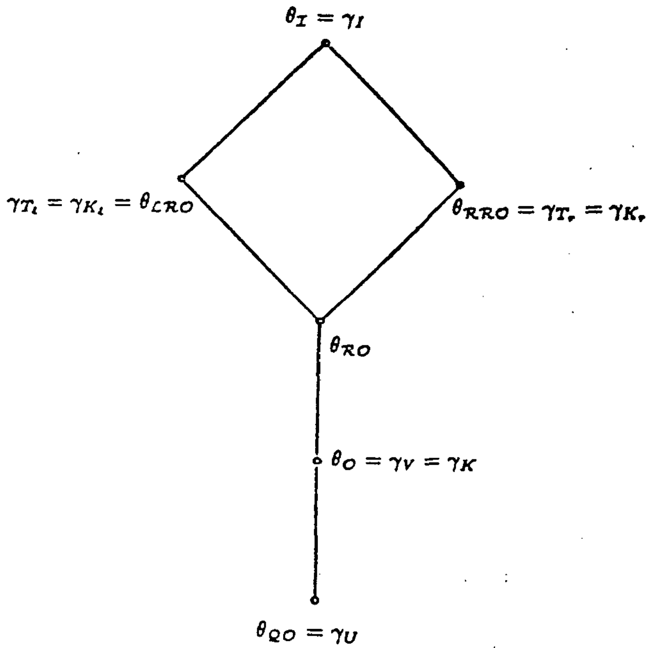

Diagram 4b

$S/\theta \in \mathcal{S}\mathcal{G}$. Hence $(ef)\theta \in \mathcal{L}\mathcal{G}$ and there exists $x \in (ef)\theta$ such that $ef = xfe$. It follows that $efe = ef$ which proves that $S \in \mathcal{L}\mathcal{R}\mathcal{O}\mathcal{G}$.

We now let $S \in \mathcal{L}\mathcal{R}\mathcal{O}\mathcal{G}$. Then ([6], Theorem 6.20) gives the structure of S in terms of $L = (Y; L_\alpha) \in \mathcal{L}\mathcal{R}\mathcal{B}$, $T = (Y; G_\alpha) \in \mathcal{S}\mathcal{G}$ and $R = Y \in \mathcal{R}\mathcal{R}\mathcal{B}$. We identify S with the construction in the above reference. Let $\varphi\colon S \to T$ be the homomorphism $(i, g, \lambda) \to g$. Let $e, f \in E(T)$ be such that $(i, e, \lambda)\varphi = (j, f, \mu)\varphi$. Then $e = f$ and thus $\lambda = \mu$ which implies that $(i, e, \lambda) = (i, e, \lambda)(j, e, \lambda)$. Therefore θ is over $\mathcal{L}\mathcal{Z}$ and $S/\theta \in \mathcal{S}\mathcal{G}$ and thus $S \in \mathcal{L}\mathcal{Z} \circ \mathcal{S}\mathcal{G}$.

3. $\mathcal{R}e\mathcal{B} \circ \mathcal{S}\mathcal{G} = \mathcal{O}\mathcal{G}$. Let $S \in \mathcal{R}e\mathcal{B} \circ \mathcal{S}\mathcal{G}$ so that there exists a congruence θ on S such that θ is over $\mathcal{R}e\mathcal{B}$ and $S/\theta \in \mathcal{S}\mathcal{G}$. Let $a \in S$ and $b \in V(a)$. Then $(ab)\theta = (ba)\theta \in E(S/\theta)$ and so $ab = abbaab$ since θ is over $\mathcal{R}e\mathcal{B}$. Therefore

$$a = aba = abbaaba = abba^2 \in aSa^2$$

which in view of ([11], IV. 1.6) gives that $S \in \mathcal{C}\mathcal{R}$. Next let $e, f \in E$. We have $(ef)\theta = (e\theta)(f\theta) \in E(S/\theta)$ since $S/\theta \in \mathcal{S}\mathcal{G}$. Since θ is over $\mathcal{R}e\mathcal{B}$, it follows that $ef \in E$. Therefore $S \in \mathcal{O}\mathcal{G}$.

Conversely, let $S \in \mathcal{O}\mathcal{G}$. Then the relation γ defined by

$$a\gamma b \Leftrightarrow V(a) = V(b)$$

is the least inverse congruence on S, see ([3], VI. 1.12). Hence $S/\gamma\in\mathscr{SG}$. Let $e\in E$ and $a\,\gamma\, e$. Then $e\in V(e)=V(a)$ so that $a\in V(e)$. By ([14], Lemma 1.3), we must have $a\in E$. In addition $a=aea$ and since e and a are arbitrary γ-related elements, we conclude that γ is over $\mathscr{R}e\mathscr{B}$. Consequently $S\in\mathscr{R}e\mathscr{B}\circ\mathscr{SG}$.

4. $\mathscr{B}\circ\mathscr{SG}=\mathscr{R}$. This forms a part of ([4], Theorem 2 and [6], Theorem 6.43).

5. $\mathscr{CS}\circ\mathscr{SG}=\mathscr{CR}$. Let $S\in\mathscr{CS}\circ\mathscr{SG}$ so that there exists a congruence θ on S such that θ is over $\mathscr{CS}$ and $S/\theta\in\mathscr{SG}$. Then S/θ is a semilattice Y of groups G_α, say. Letting $S_\alpha=G_\alpha\theta$ for every $\alpha\in Y$, we get that S is a semilattice Y of semigroups S_α, where $S_\alpha\in\mathscr{CS}\circ\mathscr{G}$. It follows easily that $\mathscr{CS}\circ\mathscr{G}=\mathscr{CS}$. Hence S is a semilattice of completely simple semigroups and therefore $S\in\mathscr{CR}$. Thus $\mathscr{CS}\circ\mathscr{SG}\subseteq\mathscr{CR}$; the opposite inclusion follows from $\mathscr{CS}\circ\mathscr{S}=\mathscr{CR}$.

That $\theta_{\mathscr{LROG}}\cap{}_{\mathscr{RROG}}=\theta_{\mathscr{ROG}}$ follows directly from ([15], Theorem 3).

6. $\mathscr{LZ}\circ\mathscr{I}=\mathscr{LG}\circ\mathscr{I}=\mathscr{LRO}$. The argument here amounts to a simplification of that in part 2 above.

7. $\mathscr{R}e\mathscr{B}\circ\mathscr{I}=\mathscr{B}\circ\mathscr{I}=\mathscr{O}$. Trivially $\mathscr{R}e\mathscr{B}\circ\mathscr{I}\subseteq\mathscr{B}\circ\mathscr{I}$. Let $S\in\mathscr{B}\circ\mathscr{I}$ so that there exists a congruence θ on S such that θ is over $\mathscr{B}$ and $S/\theta\in\mathscr{I}$. Let $e,f\in E$. Then $ef\,\theta\, fe$ since $S/\theta\in\mathscr{I}$ and hence $ef\,\theta\, fe\,\theta\, efef$. It follows that $(ef)\theta$ is an idempotent θ-class so that $(ef)\theta\in\mathscr{B}$. But then $ef\in E$ which proves that $S\in\mathscr{O}$. The argument for $\mathscr{O}\subseteq\mathscr{R}e\mathscr{B}\circ\mathscr{I}$ is virtually identical to the proof of the converse of Part 2 above.

8. $\mathscr{CS}\circ\mathscr{I}=\mathscr{QO}$. This is the content of ([17], Theorem 7.1).

The relation $\theta_{\mathscr{LRO}}\cap\theta_{\mathscr{RRO}}=\theta_{\mathscr{RO}}$ follows directly from ([15], Theorem 3).

7. The lattice associated with η

In order to treat this case, we need some preparation.

Lemma 6. *A regular semigroup is in $\mathscr{LRBG}$ if and only if it is a subdirect product of a Clifford semigroup and a left regular band.*

Proof. Let $S\in\mathscr{LRBG}$. By ([10], Theorem 3.2), S is a subdirect product of $S/\theta_{\mathscr{SG}}$ and $S/\theta_{\mathscr{B}}$. Since $S\in\mathscr{LRBG}$, we have that $\theta_{\mathscr{B}}=\theta_{\mathscr{LRB}}$ so that $\theta_{\mathscr{SG}}\cap\cap\theta_{\mathscr{LRB}}=\varepsilon$ and the assertion follows. The converse is trivial.

Lemma 7. *A regular semigroup is in $\mathscr{LCROG}$ if and only if it is a subdirect product of a left regular orthogroup and a right regular band.*

Proof. Let S be in $\mathscr{LCROG}$. By ([16], Theorem 2), we have $\gamma=(\mathscr{L}|_E)^*\vee(\mathscr{R}|_E)^*$, the least inverse congruence. The argument in Part 3 of the proof of Theorem 5 shows that γ is over $\mathscr{RB}$. Since $(\mathscr{R}|_E)^*\subseteq\gamma$, it follows that $\ker(\mathscr{R}|_E)^*\subseteq\ker\gamma=E$

and equality prevails so that

$$\ker((\mathscr{R}|_E)^* \cap \mathscr{L}^*) = \ker(\mathscr{R}|_E)^* \cap \ker \mathscr{L}^* = E. \tag{1}$$

By ([16], Theorem 2), $(\mathscr{L}|_E)^* \cap (\mathscr{R}|_E)^* = \theta_{\mathscr{RO}}$ and since $S \in \mathscr{LCROG}$, we get $(\mathscr{L}|_E)^* \cap (\mathscr{R}|_E)^* = \varepsilon$, the equality relation on S. Hence no distinct $\mathscr{L}$-related idempotents of S can be $(\mathscr{R}|_E)^*$-related and we conclude that $\operatorname{tr}(\mathscr{R}|_E)^* = \mathscr{R}|_E$. Since $S \in \mathscr{LCROG}$, we also have that $\mathscr{L} = \mathscr{L}^*$ so that $\operatorname{tr} \mathscr{L}^* = \mathscr{L}|_E$ which gives

$$tr((\mathscr{R}|_E)^* \cap \mathscr{L}^*) = tr(\mathscr{R}|_E)^* \cap tr\mathscr{L}^* = \mathscr{R}|_E \cap \mathscr{L}|_E = \varepsilon, \tag{2}$$

the equality relation on E. It is well-known that relations (1) and (2) imply that $(\mathscr{R}|_E)^* \cap \mathscr{L}^* = \varepsilon$. It now follows from ([16], Theorem 2) and ([8], Theorem 1(i)), that $\theta_{\mathscr{LRO}} \cap \theta_{\mathscr{RRB}} = \varepsilon$ whence $\theta_{\mathscr{LROG}} \cap \theta_{\mathscr{RRB}} = \varepsilon$ and thus S is a subdirect product of a left regular orthogroup and a right regular band.

Conversely, let S be a subdirect product of a left regular orthogroup T and a right regular band B. Then S is a regular orthogroup since $\mathscr{ROG}$ is closed under direct products and regular subsemigroups. Since $\mathscr{L}$ is a congruence in both T and B, it follows easily that the same holds for $T \times B$ and hence also for S. Therefore $S \in \mathscr{LCROG}$.

Theorem 5. *Diagram 5 depicts the lattice associated with the least semilattice congruence η.*

Proof. Equalities at the vertices of Diagram 5 follow directly from Theorem 1 in view of the well-known equalities:

$$\mathscr{LZ} \circ \mathscr{S} = \mathscr{LRB},\ \mathscr{G} \circ \mathscr{S} = \mathscr{SG},\ \mathscr{LG} \circ \mathscr{S} = \mathscr{LROG},\ \mathscr{CS} \circ \mathscr{S} = \mathscr{CR},$$

$$\mathscr{RB} \circ \mathscr{S} = \mathscr{B} \circ \mathscr{S} = \mathscr{B}$$

and their duals.

The relation $\theta_{\mathscr{LRB}} \cap \theta_{\mathscr{SG}} = \theta_{\mathscr{LRBG}}$ follows from Lemma 6 and $\theta_{\mathscr{LROG}} \cap \theta_{\mathscr{RRB}} = \theta_{\mathscr{LCROG}}$ from Lemma 7. The relations

$$\theta_{\mathscr{LRB}} \cap \theta_{\mathscr{RRB}} = \theta_{\mathscr{RB}},\ \theta_{\mathscr{LRBG}} \cap \theta_{\mathscr{RRBG}} = \theta_{\mathscr{ROBG}},\ \theta_{\mathscr{LROG}} \cap \theta_{\mathscr{RROG}} = \theta_{\mathscr{ROG}}$$

follow easily from ([15], Theorem 3). Also, the relation $\theta_{\mathscr{SG}} \cap \theta_{\mathscr{B}} = \theta_{\mathscr{OBG}}$ follows from ([10], Theorem 3.4).

One can convine oneself on examples of regular semigroups that the classes $\mathscr{OBG}$, $\mathscr{LROG} \vee \mathscr{B}$, $\mathscr{RROG} \vee \mathscr{B}$, $\mathscr{ROG} \vee \mathscr{B}$ and $\mathscr{CR}$ are distinct. Hence the lower right part of Diagram 5 does not collapse in general. The assertion of the theorem now follows by Theorem 2, see Diagram 2.

We leave the structural description of the semigroups in $\mathscr{LROG} \vee \mathscr{B}$, $\mathscr{RROG} \vee \mathscr{B}$ and $\mathscr{ROG} \vee \mathscr{B}$ open.

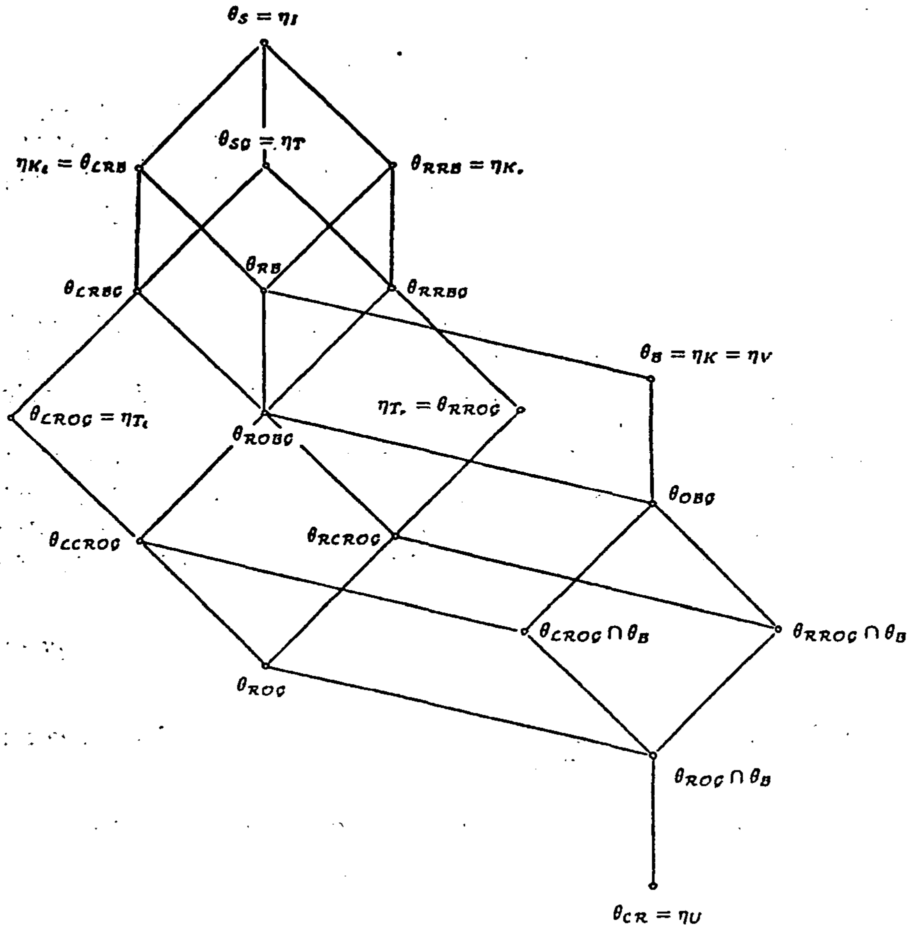

Diagram 5

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(F. P.)
DEPARTMENT OF MATHEMATICS
MARQUETTE UNIVERSITY
MILWAUKEE
WISCONSIN, WI 53233
U.S.A.

(M. P.)
SIMON FRASER UNIVERSITY
BURNABY, BC V5A 1S6
CANADA

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Arithmetical functions satisfying some relations

IMRE KÁTAI*)

1. Let $A(A^*)$ be the set of additive (completely additive) functions, $M(M^*)$ be the set of multiplicative (completely multiplicative) functions. $\|x\| = \min_{k\in\mathbf{Z}} |x-k|$.

Let $L_f(n):=f_0(n)+f_1(n+a_1)+\ldots+f_k(n+a_k)$, where $f_j\in A^*$ and $a_1, \ldots, a_k$ are mutually distinct natural numbers. It is probable that $\|L_f(n)\|\to 0$ $(n\to\infty)$ implies that $f_j(n)\equiv\tau_j\log n+u_j(n)$ (mod 1), with some $\tau_j\in\mathbf{R}$ such that $\tau_0+\ldots+\tau_k=0$ and $L_u(n):=u_0(n)+u_1(n+1)+\ldots+u_k(n+a_k)$ satisfies $L_u(n):\equiv 0$ (mod 1) for every $n\geqq 1$. This question was raised by the author and solved by E. Wirsing in the special case $k=1$.

Furthermore we guess that

$$L_u(n) \equiv 0 \pmod 1 \quad (n = 1, 2 \ldots) \tag{1.1}$$

implies that $u_j(n)\equiv 0$ (mod 1) for every $n\in\mathbf{N}$ and for every j. This was proved for $k=3$, $a_1=1$, $a_2=2$, $a_3=3$ in [2]. Marijke van Rossum investigated the solutions of the relation

$$g_0(\alpha)+g_1(\alpha+1)+g_2(\alpha+2)+g_3(\alpha+3) \equiv 0 \pmod 1 \quad (\forall\alpha\in G), \tag{1.2}$$

where $g_0, \ldots, g_3$ are completely additive functions defined on the set of $\mathbf{G}$ of Gaussian integers. She found that (1.2) has only trivial solutions.

The simple idea to prove that a recursion

$$L_f(n) = f_0(n)+f_1(n+1)+\ldots+f_k(n+k), \quad L_f(n) \equiv 0 \pmod 1 \tag{1.3}$$

has only trivial solution, is the following one:

1) *Initial step:* by taking $L_f(n)\equiv 0$ (mod 1) for $n=1, 2, \ldots, N$ with a large N, solving a linear equation system without multiplication and divisions, one conclude that $f_j(n)\equiv 0$ (mod 1) holds true for all n up to N_0.

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2) *Induction step:* If (1.3) holds and $f_j(n)\equiv 0 \pmod 1$ holds for $k=1, 2, \ldots, n$, then it is true for $k=n+1$ as well, assuming that $n\geqq N_1$, where $N_1\leqq N_0$.
The initial step can be handled by using computer for a moderate size of k. The induction could be deduced simply from the following.

Conjecture. For every integer $k\geqq 1$ there exists a constant $C_0(k)$ such that

$$\min_{P(j)<Q}\max_{l=1,\ldots,k}\max\{P(jQ+l),\, P(jQ-l)\} < Q$$

hold for every prime $Q>C_0(k)$. Here $P(n)$ denotes the largest prime divisor of n.

This is clearly true, if $k=1$, by choosing $j=1$. The conjecture is open for $k\geqq 2$, and even in the case $k=1$ if we exclude $j=1$.

In Section 2 we shall prove the following

Theorem 1. *Let a, δ be positive integers, $f_1, f_2, f_3\in A^*$ such that $L(n):\equiv f_1(n-a)+f_2(n)+f_3(n+\delta)$ satisfies the relation*

$$L(n)\equiv 0 \pmod 1, \tag{1.4}$$

for every integer $n\geqq a+1$. Assume furthermore that $f_j(n)\equiv 0 \pmod 1$ for $j=1, 2, 3$ and for all $n\leqq \max(3, a+\delta)$. Then $f_j(n)\equiv 0 \pmod 1$ $(j=1, 2, 3)$ for all $n\in\mathrm{N}$ and $j=1, 2, 3$.

Hence immediately follows

Theorem 2. *If $f_1, f_2, f_3\in A^*$ and*

$$f_1(n-a)+f_2(n)+f_3(n+b) = 0 \tag{1.5}$$

holds for all $n\geqq a+1$, then for every prime $p>\max(3, a+b)$ the values $f_1(p), f_2(p), f_3(p)$ are determined by the collection of the values $f_1(q), f_2(q), f_3(q)$ taken on at primes $q\leqq\max(3, a+b)$. Thus the set of solutions (f_1, f_2, f_3) of (1.5) forms a finite dimensional space.

Let E denote the operator $Ex_n=x_{n+1}$ in the linear space of infinite sequences, and for an arbitrary polynomial $P(z)=a_0+a_1z+\ldots+a_kz^k$ let $P(E)x_n=a_0x_n+a_1x_{n+1}+\ldots+a_kx_{n+k}$. A. SÁRKÖZY [4] determined all $f\in M$ which satisfy a linear recurrence. From his theorem one can deduce immediately the following

Lemma 1. *Let $B\geqq 1$ be an integer, $f\in M$ for which $f(n+B)=f(n)$ $(n=1, 2, \ldots)$ holds. Then either $f(n)=0$ for all $n\in\mathrm{N}$, or $f(n)=\chi_B(n)$ for all n coprime to B, where $\chi_B(n)$ is a character $\bmod B$. Let $B=B_1B_2$, $(B_1, B_2)=1$, $B_1=p_1^{\alpha_1}\ldots p_r^{\alpha_r}$, where $f(p_j^{\alpha_j})\neq 0$ $(j=1, \ldots, r)$, $B_2=q_1^{\beta_1}\ldots q_s^{\beta_s}$, where $f(q_l^{\beta_l})=0$. The cases $B_1=1$ or $B_2=1$ are included. Let δ_l be the largest exponent $(\delta_l\geqq 0)$ for which $f(q_l^{\delta_l})\neq 0$. Then $0\leqq\delta_l<\beta_l$ $(l=1, \ldots, S)$. Let $D=q^{\beta_1-\delta_1}\ldots q^{\beta_s-\delta_s}$. Then $\chi_B(n)=\chi_D(n)$ for $(n, B)=1$, χ_D is a character $\bmod D$. Furthermore $f(p^\gamma)=f(p^\alpha)\,\chi_E(p^{\gamma\alpha})$ holds for all $p^\alpha\| B$ and $\gamma>\alpha$.*

All the functions with the above conditions are periodic mod B.

In Section 3 we give all the solutions of $V(n+k)=U(n)$ $(n=1, 2, \ldots)$ for $U, V \in M$ under the condition $U(n)\neq 0$ if $(n, k)=1$. This equation for completely multiplicative functions was solved earlier in [1]. We present it now as

Lemma 2. *Let* $G(n+k)=F(n)$ *hold for all* $n\in N$, $F, G\in M^*$, $F(n)$ *be non-identically zero,* $F(n)=0$ *if* $(n, k)>1$. *Then*

a) $F(n)=G(n)=\chi_k(n)$ *is a solution for an arbitrary multiplicative character* χ_k (mod K),

b) *there is no other solution if* $4|K$ *or if* $(2, K)=1$,

c) *if* $K=2R$, $(R, 2)=1$, *then all further solutions have the form*

$$F(n) = \chi(n, 8)\psi_R(n), \quad G(n) = \chi(n, 4)F(n),$$

where $\psi_R(n)$ *is an arbitrary character* mod R, $\chi(n, 4)$ *is the nonprincipal character* mod 4, *and* $\chi(n, 8)$ *is the character* mod 8 *defined by the relations.*

$$\chi_8(n) = \begin{cases} 1 & n \equiv \pm 1 \pmod 8 \\ -1 & n \equiv \pm 3 \pmod 8 \end{cases} \quad \text{if} \quad R \equiv 1 \pmod 4,$$

$$\chi_8(n) = \begin{cases} 1 & n \equiv \pm 3 \pmod 8 \\ -1 & n \equiv 5, 7 \pmod 8 \end{cases} \quad \text{if} \quad R \equiv -1 \pmod 4.$$

The equation $G(n+k)=F(n)$, $F(1)\neq 0$ implies that $F(n)G(n)\neq 0$ for $(n, k)=1$, assuming that F and G are completely multiplicative. This is not true if we assume only that $F, G\in M$.

In Section 4 we solve the equation $G(n+1)=F(n)$ for $F, G\in M$ without any additional conditions.

2. Proof of Theorem 1. The case $a=b=1$ has been proved in [2]. We may assume that $(a, b)=1$. Indeed, by substituting $n\delta$ into the place of n, observing that $f_j(\delta)\equiv 0 \pmod 1$, we have

$$f_1(n-a_1)+f_2(n)+f_3(n+a_1) \equiv 0 \pmod n \quad (\forall n),$$

and $f_j(n)\equiv 0 \pmod 1$ $(j=1, 2, 3)$ for every $n\leqq \max(3, a+b)$, $a=\delta a_1$, $b=\delta b_1$.

Let A_n denote the event that $f_j(n)\not\equiv 0 \pmod 1$ holds for at least one j. We shall prove that under the condition of the theorem there exists no such an integer. If such an n exists, then $n\geqq k+1$, furthermore the smallest n for which A_n is true has to be a prime number P.

Now we distinguish three cases according to the parity of a and b. Let $k=a+b$.

Case I: a and b are odd numbers. Since P is the smallest integer n for which A_n is true, therefore $f_3(P)\equiv 0 \pmod 1$ cannot occur, since $f_2(P-a)\equiv 0 \pmod 1$,

$f_1(P-k)\equiv 0 \pmod 1$. Similarly, $f_2(P)\equiv 0 \pmod 1$, since $2|P+b$, and $\frac{P+b}{2}<P$. Thus $f_1(P)\equiv\alpha\ (\not\equiv 0) \pmod 1$. Since $L(P+a)\equiv 0 \pmod 1$, and $2|P+a$, $\frac{P+a}{2}<P$, $f_2(P+a)\equiv 0 \pmod 1$, therefore $f_3(P+k)\equiv -\alpha \pmod 1$.

Let now $\delta|k$, $\delta>1$. Since $L(P+a)\equiv 0 \pmod 1$, $L\left(P+\frac{k}{\beta}-b\right)\equiv 0 \pmod 1$, therefore

(2.1) $$f_1(\delta P)+f_2(\delta P+a)+f_3(\delta P+k)\equiv 0 \pmod 1$$

(2.2) $$f_1\left(P+\frac{k}{\delta}+k\right)+f_2\left(P+\frac{k}{\delta}-b\right)+f_3\left(P+\frac{k}{\delta}\right)\equiv 0 \text{ mod } 1),$$

$f_1\left(P+\frac{k}{\delta}-k\right)\equiv 0 \pmod 1$. If $f_3(P+k/\delta)\equiv\beta\not\equiv 0 \pmod 1$, then k/δ is an even number, since in the opposite case $2|P+k/\delta$, and from $\frac{1}{2}(P+k/\delta)<P$ it would follow $f_3(\cdot)\equiv 0 \pmod 1$. But then $f_2(P+k/\delta-b)\equiv-\beta\not\equiv 0 \pmod 1$, $P+\frac{k}{\delta}-b$ is an even number and $\frac{1}{2}\left(P+\frac{k}{\delta}-b\right)<P$. This cannot be occur. Thus $f_3(\delta P+k)\equiv$ $\equiv f_3(\delta)+f_3\left(P+\frac{k}{\delta}\right)\equiv 0 \pmod 1$. So we have

(2.3) $$f_2(\delta P+a)\equiv -\alpha \pmod 1 \text{ whenever } \delta|k,\ \delta>1.$$

Assume first that $3|k$. Then, from (2.3) we have $f_2(3P+a)\equiv-\alpha \pmod 1$. Since $2|3P+a$, therefore $3P+a=2Q$, where Q is a prime number, $P<Q<2P$. Since $f_1(Q-a)+f_2(Q)+f_3(Q+b)\equiv 0 \pmod 1$, $2|Q-a$, $2|Q+b$, $Q-a<2P$, $Q+b<$ $<2(P+k)$, therefore $f_1(Q-a)\equiv 0 \pmod 1$, $f_3(Q+b)\equiv 0 \pmod 1$, and so $f_2(Q)\equiv$ $\equiv 0 \pmod 1$, $\alpha\equiv 0 \pmod 1$. It remains the case $3\nmid k$. Since $f_3(P+k)\not\equiv 0 \pmod 1$, and from (2.3), $f_2(2P+a)\not\equiv 0 \pmod 1$, thus P, $P+k$, $2P+a$ are prime numbers.

Assume first that $3\nmid a$. Since $P>3$, therefore either $3|2P+a$ or $3|4P+a$. Since $f_2(2P+a)\not\equiv 0 \pmod 1$, therefore $3\nmid 2P+a$, so $3|4P+a$. Let us consider now

(2.4) $$f_1(4P)+f_2(4P+a)+f_3(4P+k)\equiv 0 \pmod 1.$$

We shall prove that $f_2(4P+a)\equiv 0 \pmod 1$. Since $4P+a=3Q$, it is true, if Q is a composite number. If it is a prime, then we may consider

$$f_1(Q-a)+f_2(Q)+f_3(Q+b)\equiv 0 \pmod 1,$$

which by $2|Q+b$, $2|Q-a$, $Q<2P$ gives that $f_2(Q)\equiv 0 \pmod 1$. So, from (2.4) we infer $f_3(4P+k)\equiv-\alpha \pmod 1$. If $4|k$, then it cannot be occur, since $P+k$ is the smallest integer n for which $f_3(n)\not\equiv 0 \pmod 1$. If $k=2l$, $(l,2)=1$, then

$f_3(2P+l) \equiv -\alpha \pmod 1$. If $k=2l$, $(l,2)=1$, then $f_3(2P+l) \equiv -\alpha \pmod 1$. But

$$f_1(2P-l)+f_2(2P-l+a)+f_3(2P+l) \equiv 0 \pmod 1. \tag{2.5}$$

Since $2|a-l$, $2|2P-l+a<2P+a$, therefore $f_2(2P-l+a) \equiv 0 \pmod 1$, and so $f_1(2P-l) \equiv \alpha \pmod 1$.

Since $2P-l$, $(2P-l)+l=2P$, $2P+l$ cover all the residue classes mod 3, $3 \nmid 2P$, thus $3|2P+l$ or $3|2-l$. Both of these cases imply that $\alpha \equiv 0 \pmod 1$.

It remains the case $3|a$ and $3 \nmid k$. Then $k \equiv b \pmod 3$. Let $Q:=P+k$. Then $f_3(Q) \equiv -\alpha \pmod 1$. Let us consider $f_1(2Q-k)+f_2(2Q-b)+f_3(2Q) \equiv 0 \pmod 1$. Since $2Q-k \equiv 2Q-b \pmod 3$, $3|2Q-b$, and $2Q-b<3(P+a)$, would imply $f_2(2Q-b) \equiv 0 \pmod 1$, $f_1(2Q-k) \equiv 0 \pmod 1$, thus we may assume that $3 \nmid 2Q-b$. But then $P, P+k, 2P+k$, are coprime to 3. Since $3 \nmid k$, $3 \nmid P$, therefore either $P \equiv k \pmod 3$ or $P \equiv -k \pmod 3$. In both cases, at least one of $P, P+k, 2P+k$ is a multiple of 3. This is a contradiction.

By this the proof of Case I is completed.

Case II: a is odd, b is even. Let $n=P$ be the smallest integer for which A_n holds true. Then n is a prime, $P>3$, $P>k$. We can see, similarly as earlier, that $f_2(P) \equiv \alpha \not\equiv 0 \pmod 1$ with some α, $f_1(P) \equiv 0$, $f_3(P) \equiv 0 \pmod 1$. Observe that $f_3(n) \equiv 0 \pmod 1$ if $n<P+b$, and that $f_3(P+b) \equiv -\alpha \pmod 1$, which immediately follows from $L(P) \equiv 0 \pmod 1$. Furthermore, we can get that $f_1(n) \equiv 0 \pmod 1$, if $n<2P-a$. It is enough to prove this for odd, even for prime number integer $n=Q$. Since $L(Q+a) \equiv 0 \pmod 1$, $2|Q+a$, $2|Q+k$, $Q+a<2P$, therefore $f_2(Q+a) \equiv$ $\equiv 0 \pmod 1$, $f_3(Q+k) \equiv 0 \pmod 1$, and so $f_1(Q) \equiv 0 \pmod 1$ as well. Then, for $\delta|b$, $\delta>1$, we get that $f_3(\delta P+b) \equiv 0 \pmod 1$, and by $L(\delta P) \equiv 0 \pmod 1$, that

$$f_1(\delta P-a) \equiv -\alpha \pmod 1 \quad \text{if} \quad \delta|b \quad \text{and} \quad \delta>1. \tag{2.6}$$

Let us consider the equation $L(3P) \equiv 0 \pmod 1$.

Since $2|3P-a$, $3P-a=2Q$, $Q<2P-a$, therefore $f_1(3P-a) \equiv 0 \pmod 1$. This implies that either $\alpha \equiv 0 \pmod 1$, or $3 \nmid b$, furthermore in the second case that $f_3(3P+b) \equiv -\alpha \pmod 1$. Thus $3P+b$ is a prime number since if it would be composite then its prime factors would be smaller than $P+b$. So P, $P+b$, $3P+b$ are prime numbers greater than 3, thus $P \equiv b \pmod 3$.

Since $2|b$, thus from (2.6) it follows that $2P-a$ is a prime, and so that $3 \nmid 2b-a$. If $4|b$, then by (2.6) we get that $4P-a$ is a prime, and $f_1(4P-a) \equiv -\alpha \pmod 1$. Assume that $2 \| b$, $b=2b_1$. Since $P \equiv b \pmod 3$, $P \equiv 2b_1 \pmod 3$, from $L(2P+b_1-b) \equiv$ $\equiv 0 \pmod 1$, by $2|2P+b_1-k<P$, $3|2P+b_1-b$ we deduce that $f_1(2P+b_1-k) \equiv$ $\equiv 0 \pmod 1$, $f_2(2P+b_1-b) \equiv 0 \pmod 1$, and so that $f_3(2P+b_1) \equiv 0 \pmod 1$. But then, from $L(4P) \equiv 0 \pmod 1$ we have

$$f_1(4P-a)+f_2(4P)+f_3(2(2P+b_1)) \equiv 0 \pmod 1,$$

and so that $f_1(4P-a)\equiv-\alpha$ (mod 1). Thus $4P-a$ is a prime, since in the case $4P-a=3Q$, $Q<2P-a$ would imply $f_1(4P-a)\equiv 0$ (mod 1). So $P, P+b, 2P-a$, $4P-a$ are all prime numbers which can be occur only if $3|a$.

It remained to consider the case $3|a$, $P\equiv b$ (mod 3). Furthermore $f_1(4P-a)\equiv$ $\equiv-\alpha$ (mod 1). Since $3|2(P+b)-b$, $3|2(P+b)-b-a$, and $L(2(P+b)-b)\equiv 0$ (mod 1), therefore $f_1(2(P+b)-b)\equiv 0$ (mod 1), $f_2(2(P+b)-b-a)\equiv 0$ (mod 1), consequently $f_3(2(P+b))\equiv 0$ (mod 1), which implies $\alpha\equiv 0$ (mod 1).

The proof of Case II is completed.

Case III: a is even, b is odd. Then we have $f_1(P)\equiv\alpha(\not\equiv 0)$ (mod 1), $f_2(P+a)\equiv$ $\equiv-\alpha$ (mod 1, $P+a$ is a prime number. Furthermore, $f_2(n)\equiv 0$ (mod 1) if $n<P+a$. Now we observe that $f_3(n)\equiv 0$ (mod 1) for all $n<2P+k$. Since $f_3(2)\equiv 0$ (mod 1), therefore enough to prove this for odd prime Q. Let $Q<2P+k$. If $f_3(Q)\not\equiv 0$ (mod 1), then by $L(Q-b)\equiv 0$ (mod 1) we have that $f_1(Q-k)+f_2(Q-b)\not\equiv 0$ (mod 1). But $2|Q-b$, $2|Q-k$, and $Q-k<2P$, $Q-b<2(P+a)$. Consequently $f_3(Q)\equiv 0$ (mod 1).

Let $\delta|a$ and $\delta>1$. By $f_2(P+a/\delta)\equiv 0$ (mod 1), and $L(\delta P+a)\equiv 0$ (mod 1) we deduce that

$$f_3(\delta P+k)\equiv-\alpha \pmod 1 \quad \text{if} \quad \delta>1 \quad \text{and} \quad \delta/a. \tag{2.7}$$

Let $\mu|k$. Since $L(\mu P+a)\equiv 0$ (mod 1) and $f_3\left(\mu P+\mu\cdot\frac{k}{\mu}\right)\equiv 0$ (mod 1), therefore

$$f_2(\mu P+a)\equiv-\alpha \pmod 1 \quad \text{if} \quad \mu|k. \tag{2.8}$$

Assume now that $\mu>1$. Then $L(2\mu P+a)\equiv 0$ (mod 1), $2\mu P+k=(\mu 2P+k/\mu)$, $2P+k/\mu<2P+k$, $f_3(2\mu P+k)\equiv 0$ (mod 1), and so

$$f_2(2\mu P+a)\equiv-\alpha \pmod 1 \quad \text{if} \quad \mu|k \quad \text{and} \quad \mu>1. \tag{2.9}$$

So $P, P+a, 2P+k$ are prime numbers.

Since $2|3P+k$, $\frac{3P+k}{2}<2P+k$, therefore $f_3(3P+k)\equiv 0$ (mod 1), and so, by $L(3P+a)\equiv 0$ (mod 1) we have $f_2(3P+a)\equiv-\alpha$ (mod 1). This implies that either $\alpha\equiv 0$ (mod 1) or $3\nmid a$. Assume that $3\nmid a$. Since $P, P+a$ are primes larger than 3, therefore $P\equiv a$ (mod 3). If $4|a$, then $f_3(4P+k)\equiv-\alpha$ (mod 3) and 3 cannot be a divisor of $4P+k$ if $\alpha\not\equiv 0$ (mod 3), consequently $4P+k$ is a prime number. If $2\|a$, $a=2a_1$, then by

$$f_1(4P)+f_2(2(2P+a_1))+f_3(4P+k)\equiv 0 \pmod 1$$

$$f_1(2P-a_1)+f_2(2P+a_1)+f_3(2P+a_1+b)\equiv 0 \pmod 1$$

and by taking into account that $3|2P-a_1$, $2|a_1+b$, first we deduce that $f_1(2P-a_1)\equiv$ $\equiv 0$ (mod 1), $f_3(2P+a_1+b)\equiv 0$ (mod 1) and so that $f_2(2P+a_1)\equiv 0$ (mod 1), we have $f_3(4P+k)\equiv-\alpha \pmod 1$. This implies that $4P+k$ is a prime number. Since $0, 2P, 2\cdot 2P$ are incongruent residues $\bmod 3$, therefore so are $k, 2P+k, 4P+k$, consequently one of them is a multiple of 3. Since $2P+k$, $4P+k$ are primes larger than 3, only the case $3|k$ can be occur. Assume that $3|k$. Then $a\equiv-b \pmod 3$. From

$$f_1(2P+a)+f_2(2P+2a)+f_3(2P+2a+b)\equiv 0 \pmod 1$$

we have $3|2P+a$, $3|2P+2a+b$, which implies that $f_1(2P+a)\equiv 0 \pmod 1$, $f_3(2P+2a+b)\equiv 0 \pmod 1$, and so that $f_2(P+a)\equiv 0 \pmod 1$, which can be occur only if $\alpha\equiv 0 \pmod 1$.

This completes the proof of Case III. The theorem is proved.

3. Let us consider now the equation

$$V(n+K)=U(n) \quad (n=1,2,\ldots), \tag{3.1}$$

where U, V are multiplicative functions, K is a fixed positive integer. We are interested in to give all the solutions under the condition

$$U(n)\neq 0 \quad \text{whenever} \quad (n,K)=1. \tag{3.2}$$

The same equation for completely multiplicative functions was considered in our earlier paper [1]. We solved (3.1) for $K=1$ assuming (3.2) in [1]. The case $K>1$ is more complicated. Assume that (3.1) and (3.2) hold.

Let

$$H(n):=\frac{V(n)}{U(n)} \tag{3.3}$$

be defined on the set of integers n, coprime to K. Let furthermore

$$\delta_p(m):=H(p)H(m)H(m+k)\ldots H(m+(p-2)K). \tag{3.4}$$

If $(p,n(n+K))=1$, then

$$H(p)=\frac{V(p(n+k))}{U(pn)}=\frac{1}{H(pn+K)\ldots H(pn+(p-1)K)}, \tag{3.5}$$

i.e.

$$\delta_p(pn+K)=1 \quad \text{if} \quad (p,n(n+K))=1. \tag{3.6}$$

Let $p>q$, $r=p-q+1$. Then

$$\begin{aligned}\delta_p(m)&=H(p)[H(m)H(m+K)\ldots H(m+(q-2)K)]\times\\&\quad\times[H(m+(q-1)K)\ldots H(m+(p-2)K)]=\\&=H(p)\frac{\delta_q(m)}{H(q)}\cdot\frac{\delta_r(m+(q-1)K)}{H(r)},\end{aligned}$$

and so

$$\frac{H(p)}{H(q)H(r)} = \frac{\delta_p(m)}{\delta_q(m)\cdot\delta_r(m+(q-1)K)}. \tag{3.7}$$

We should like to give some conditions which imply that the right hand side equals 1. This holds true if all the next relations are satisfied, with a suitable integer m:

$$m \equiv K(\mathrm{mod}\, p);\ m \equiv K(\mathrm{mod}\, q);\ m+(q-2)K \equiv 0 \quad (\mathrm{mod}\, r), \tag{3.8}$$

$$\left(\frac{m-K}{p}\cdot\frac{m+(p-1)K}{p}, p\right) = 1;\ \left(\frac{m-K}{q}\cdot\frac{m+(q-1)K}{q}, q\right) = 1, \tag{3.9}$$

$$\left(\frac{m+(q-2)K}{r}\cdot\frac{m+(q-1)K-K+rK}{r}, r\right) = 1;\quad (pqr, K) = 1. \tag{3.10}$$

Let

$$K^* = \begin{cases} K & \text{if } K \text{ is even,} \\ 2K & \text{if } K \text{ is odd.}\end{cases}$$

Assume that r is given, $(r, K)=1$. Let λ be an integer which will be chosen later, $\eta:=\lambda K^*$. Let p and q by defined by

$$p = (1+\eta)r,\ q = \eta r+1.$$

If (3.8), (3.9), (3.10) hold with some m, then

$$H(p) = H(1+\lambda K^*)H(r) \tag{3.11}$$

is valid.

We shall search m in the form $m=pq\,v+K$. The conditions $m\equiv K\ (\mathrm{mod}\, p)$, $m\equiv K\ (\mathrm{mod}\, q)$, $m+(q-2)K\equiv pq\, v+(q-1)K\equiv 0\ (\mathrm{mod}\, r)$ are satisfied clearly, the condition $(pqr, K)=1$ is equivalent to $(r(1+\eta((\eta r+1), K)=1$ which is true since $(r, K)=1$ was assumed.

We have

$$\frac{m-K}{p}\cdot\frac{m+(p-1)K}{p} = qv(qv+K),\ \frac{m-K}{q}\cdot\frac{m+(q-1)K}{q} = pv(pv+K),$$

$$m+(q-2)K = pqv+(q-1)K = [(1+\eta)qv+\eta K]r,$$

$$m+(q-2)K+rK = [(1+\eta)qv+(\eta+1)K]r = (1+\eta)r(qv+K).$$

So, to satisfy (3.9), (3.10) we have to find such v, for which

$$\left(qv(qv+K), p\right) = 1,\quad \left(pv(pv+K), q\right) = 1 \tag{3.12}$$

$$\left(((1+\eta)qv+\eta K)\cdot(1+\eta)(qv+K), r\right) = 1 \tag{3.13}$$

simultaneously hold.

The condition $(p, q)=1$ will be guaranteed by restricting r to satisfy the relation

$$(r(r-1), 1+\eta) = 1. \tag{3.14}$$

Since η is an even number, there exists such an r. Now we prove that (3.14) implies that $(p, q)=1$. Assume the contrary. Let $\delta|(p, q)$, δ be a prime number. Since $p=(1+\eta)r$, $q=\eta r+1$, therefore $\delta\nmid r$, and so $\delta|1+\eta$. But $q=(\eta+1)r+ +(1-r)$, whence $\delta|1-r$. This case was excluded by (3.14).

Now our conditions can be rewritten in the form

$$(v(pv+K), q) = 1 \tag{1}$$

$$(v(qv+K), p) = 1 \tag{2}$$

$$((1+\eta)qv+\eta K, r) = 1 \tag{3}$$

$$(qv+K, r) = 1. \tag{4}$$

Since (2) implies (4), therefore (4) can be omitted. Since $p=(1+\eta)r$, then we may substitute them with

$$(v(pv+K), q) = 1 \tag{A}$$

$$(v(qv+K), r) = 1 \tag{B}$$

$$(v(qv+K), (1+\eta)) = 1 \tag{C}$$

$$((1+\eta)qv+\eta K, r) = 1. \tag{D}$$

Since $(p, q)=1$, therefore $(q, r)=1$, consequently q, r, $1+\eta$ are pairwise coprime integers. To prove that (A), (B), (C), (D) hold simultaneously with a suitable v, it is enough to show that there is a solution of (B) and (D), furthermore that of (A), and of (C).

Since q and $1+\eta$ are both odd numbers, therefore (A) and (C) can be solved.

Assume that there exist no v for which (B) and (D) would hold simultaneously. Then there exists a prime divisor Q of r such that for every integer v, either $(v(qv+K), Q)=Q$ or $((1+\eta)qv+\eta K, Q)=Q$. Let us observe that it can be occur only if $Q=3$, i.e. if $3|r$.

If $3|r$, then $3\nmid K$, $q\equiv 1 \pmod 3$, thus we have $v(qv+K)\equiv v(v+K) \pmod 3$, $(1+\eta)qv+\eta K\equiv(1+\eta)v+\eta K \pmod 3$. If $3|\eta$, then the last congruence can be reduced to $\equiv v \pmod 3$. In this case (B) and (D) can be solved as well.

We shall exclude the case when $3|r$ and $3\nmid\eta$, i.e. the case: $3|r$ and $\eta\equiv 1$ (mod3.) Since $H(p)=H(q)H(r)$, by (3.9) we have

$$H(1+\lambda K^*) = H(1+\lambda r K^*) \tag{3.15}$$

if

$$(r(r-1), 1+\lambda K^*) = 1 \quad (r, K) = 1 \tag{3.16}$$

and in the case $3|r$, the relation $\eta \not\equiv 1 (3)$ holds.

Lemma 3. *If* $(\lambda, K)=1$, $(\mu, K)=1$ *and in the case* $3 \nmid K$, $\lambda K^* \not\equiv 1 \pmod 3$, $\mu K^* \not\equiv 1 \pmod 3$, *then*

$$H(1+\lambda K^*) = H(1+\mu K^*) \tag{3.17}$$

Proof. We can find positive integers r and s such that

$$r\lambda = s\mu \tag{3.18}$$

and

$$(r(r-1), 1+\lambda K^*) = 1 \tag{3.19}$$

$$(s(s-1), 1+\mu K^*) = 1. \tag{3.20}$$

Indeed, if $\delta=(\lambda, \mu)$, $\lambda=\delta\lambda_1$, $\mu=\delta\mu_1$, then $r=\mu_1 t$, $s=\lambda_1 t$ is a solution of (3.18) for every positive integer t. Assume that $(t, K)=1$. Then $(r, K)=(s, K)=1$ holds true. Since K is coprime to both of the integers $1+\lambda K^*$, $1+\mu K^*$, we have to consider only the solvability of (3.19) and that of (3.20). Both of them have solutions.

Assume that there exists no t for which (3.19) and (3.20) would be satisfied. Then there would exist a prime divisor Q of $(1+\lambda K^*, 1+\mu K^*)$ such that $\mu_1 t(\mu_1 t-1)\cdot\lambda_1 t\cdot(\lambda_1 t-1)\equiv 0 \pmod Q$ holds for every integer t.

We have $(\lambda_1\mu_1, Q)=1$. Furthermore $Q|(\lambda-\mu)K^*$, $(Q, K^*)=1$, therefore $Q|\delta(\lambda_1-\mu_1)$. $Q|\delta$ cannot be occur, thus $\lambda_1-\mu_1\equiv 0 \pmod Q$. Consequently our congruence can be reduced to the form $t(\lambda_1 t-1)\equiv 0 \pmod Q$. But it has at most two solutions mod Q, consequently there is a t for which both of (3.19), (3.20) holds. By this we proved our Lemma 3.

Lemma 4. *If* $A\equiv B \pmod{K^*K}$ *and* $(A, K^*)=1$, *then*

$$H(A) = H(B). \tag{3.21}$$

Proof. Let $3\nmid K$. Assume first that $3\nmid A$ and $3\nmid B$ or $3|(A, B)$. In the former case let $A_1=3A$, $B_1=3B$, in the second case $A=A_1$, $B=B_1$. In both cases $A_1=B_1 \pmod 3$.

If Θ is such an integer for which $A_1\Theta\equiv 1+K^* \pmod{K^*K}$ holds, then $B_1\Theta\equiv 1+K^* \pmod{K^*K}$ is satisfied as well. Writing $A_1\Theta=1+\lambda K^*$, $B_1\Theta=1+{}+\mu K^*$, $\lambda K^*\not\equiv 1$, $\mu K^*\not\equiv 1$ obviously hold. Since the solutions Θ give a whole residue class $(\bmod K^*K)$, which is reduced to the module, we can choose Θ to be a large prime. By Lemma 3 we have $H(A_1\Theta)=H(B_1\Theta)$, which implies that $H(A_1)=H(B_1)$, and so that $H(A)=H(B)$.

If $3|A$, $3\nmid B$, then the general solution of the congruence $B\Theta\equiv 1+K^*$ (mod K^*K) can be written as $\Theta=\Theta^*+hK^*K$ ($h=0, 1, 2, \dots$) where Θ^* is a particular solution. Since $B\Theta\equiv B\Theta^*+hBK^*K$ (mod 3), $3\nmid BK^*K$, therefore $B\Theta\equiv 1$ (mod 3) holds if h is falling into the appropriate residue class mod 3. Then $A\Theta\equiv 0$ (mod 3). We may choose Θ to be a large prime, and by Lemma 2, $H(A\Theta)=H(B\Theta)$ we conclude that $H(A)=H(B)$.

In the case $3|K$ we get the lemma similarly, but without taking care of the requirements $\lambda K^*\not\equiv 1$, $\mu K^*\not\equiv 1$ (mod 3).

Let χ_0 be the principal character mod K^*K. Since the conditions of Lemma 1 are satisfied for the function $f(n):=\chi_0(n)H(n)$, $B=K^*K$, therefore there exists a character χ_{K^*K} such that

$$H(n) = \chi_{K^*K}(n) \quad \text{whenever} \quad (n, K^*K) = 1. \tag{3.22}$$

We distinguish two cases according to the parity of k.

Case K=even. For every m, n integers coprime to K, let

$$A(m, n) := \frac{U(mn)}{U(m)U(n)},$$

$$S(m, n) := \frac{1}{\chi(n+K)} \prod_{l=1}^{m} \chi(mn+lK),$$

where χ is the character given in (3.22). Since χ is periodic mod K^2, therefore $S(m, n)$ is periodic mod K^2 in both of its variables m and n. Furthermore, $A(m, n)=1$ if m and n are coprimes.

Since

$$U(n) = V(n+K) = H(n+K)U(n+K) = \chi(n+K)U(n+K),$$

consequently

$$U(nm) = \chi(mn+K)\chi(mn+2K)\dots\chi(mn+mK)U\big(m(n+K)\big) = S(m, n)U\big(m(n+K)\big),$$

i.e.

$$A(m, n) = S(m, n) \tag{3.23}$$

holds under the condition $(mn, K)=1$, $(m, n+K)=1$.

Let p be an arbitrary prime, $(p, K)=1$. Then p is an odd integer. Take $m=p^\alpha$, $n=pv$, where $(v, p)=1$. Then $A(p^\alpha, pv)=\dfrac{U(p^{\alpha+1})}{U(p)U(p^\alpha)}$. Since $(n+K, m)=1$ clearly holds, therefore

$$A(p^\alpha, pv) = S(p^\alpha, pv).$$

Since $S(p^\alpha, pv)=S(p^\alpha, pv+K^2)=A(p^\alpha, pv+K^2)=1$, we deduced that

$$U(p^{\alpha+1}) = U(p^\alpha)U(p)$$

valid for all prime pover p^α coprime to K. This shows that U is completely multiplicative on the set $(n, K)=1$. Since $V=H\cdot U$, and H is completely multiplicative on the set $(n, K)=1$, so is V. Therefore, we may apply Lemma 2 for the characterization of the solution (U, V) at least on the set $(n, K)=1$.

Case $K=odd$. Let $n=2^\gamma v$, $\gamma\geqq 1$ and $(v, K)=1$. Then

$$1=\frac{V(2^\gamma v+K)}{U(2^\gamma v)}=\frac{V(2^{\gamma+1}v+2K)}{U(v(U(2^\gamma)}\cdot\frac{U(2^{\gamma+1})}{V(2)U(2^{\gamma+1})}=\frac{U(2^{\gamma+1})}{V(2)U(2^\gamma)}H(2^{\gamma+1}\gamma+K).$$

Thus we proved that

$$H(2^{\gamma+1}v+K)=D_\gamma,\quad\text{for every}\quad(v, 2K)=1, \tag{3.24}$$

where

$$D_\gamma=\frac{U(2^{\gamma+1})}{V(2)U(2^\gamma)},\quad(\gamma\geqq 1) \tag{3.25}$$

Similarly, we can prove that

$$H(2^{\gamma+1}v-K)=E_\gamma\quad\text{for every}\quad(v, 2K)=1, \tag{3.26}$$

$$E_\gamma=\frac{U(2)V(2^\gamma)}{V(2^{\gamma+1})}\quad\gamma\geqq 1 \tag{3.27}$$

From (3.22) we know that $H(n)=\chi(n)$ for $(n, 2K)=1$, where χ is a character mod $2K^2$ For odd K we can prove more, namely that H is periodic mod $2K$. The worst case is the case $3\nmid K$. Assume that $3\nmid K$.

If $K^*\equiv 1 \pmod 3$, then, by Lemma 2,

$$H(1+3K^*)=H(1+4K^*);\quad(\lambda=3, \mu=4),$$

if $K^*\equiv -1 \pmod 3$, then

$$H(1+2K^*)=H(1+3K^*)\quad(\lambda=2, \mu=3),$$

consequently, by

$$H(1+vK^*)=\chi_{2K^2}(1+vK^*)=\chi_{2K^2}(1+K^*)^v=H(1+K^*)^v,$$

we get that $H(1+K^*)=\chi_{2K^2}(1+K)=1$. If $3|K$, then we have $H(1+K^*)=H(1+2K^*)$, and conclude to the same result. But then $H(1+vK^*)=\chi_{2K^2}(1+vK^*)=1$ holds for every integer v. If $A\equiv B \pmod{K^*}$ such that $(A, K^*)=1$, then one can choose a large prime Θ such that $A\Theta\equiv 1 \pmod{K^*}$, which implies that $B\Theta\equiv 1 \pmod{K^*}$, and $H(A\Theta)=H(B\Theta)$, whence by $(A, \Theta)=(B, \Theta)=1$, $H(\Theta)\neq 0$, we infer $H(A)=H(B)$. So we proved that H is periodic mod $2K$; consequently, by Lemma 1,

$$H(n)=\chi_{2K}(n)\quad\text{if}\quad(n, 2K)=1. \tag{3.28}$$

Let us consider now (3.24). Observe that if $v_1, v_2, \ldots, v_s$, $S=\varphi(2K)$ is a complete reduced residue system mod $2K$, then so is $2^{\gamma+1}v_j+K$ $(j=1, \ldots, S)$. Indeed, these numbers are coprime to $2K$, and if $2^{\gamma+1}v_i+K\equiv 2^{\gamma+1}v_j+K \pmod{2K}$, for some suitable $i\neq j$, then $K|(v_i-v_j)$. Since v_i, v_j are odd numbers, therefore $2|(v_i-v_j)$, so $v_i\equiv v_j \pmod{2K}$, which cannot be occur. It implies that the left-hand side does not change its value if v run over a reduced residue set, whence we have that $H(n)=1$ for every $(n, 2K)=1$, furthermore that $D_\gamma=1$ and similarly that $E_\gamma=1$ for every $\gamma\geqq 1$. From the relation $D_\gamma E_\gamma=1$ we obtain that

$$H(2^{\gamma+1}) = \frac{H(2^\gamma)}{H(2)} \quad (\gamma \geqq 1),$$

which implies that $H(2^2)=1$. We shall show that there exists such an integer Γ for which $H(2^\Gamma)=H(2^{\Gamma+1})$, which will imply that $H(2)=1$, and so that $H(2^\gamma)=1$ for every $\gamma\geqq 1$.

To do this, let us consider the product

$$\Delta(s, n) = \prod_{l=1}^{s-1} H(sn+lK)$$

defined for positive integers s, n such that $(sn, K)=1$. Observing that for $(s, n+K)=1$ we have

$$U(sn) = H(sn+K)\ldots H(sn+sK)\,U\big(s(n+K)\big) = \Delta(s, n)\,U(s)\,U(n),$$

consequently, if additionally $(s, n)=1$, then

$$\Delta(s, n) = 1.$$

Assume that the conditions

$$(3.29) \qquad (s, n) = 1, \quad (s, n+K) = 1, \quad (s, K) = (n, K) = 1$$

hold for some pairs of integers s, n. They imply that $\Delta(s, n)=1$. Let us change n by $N=n+RsK$, where R is an arbitrary positive integer. Since the conditions (3.29) will be held replacing n by N, therefore $\Delta(s, N)=1$ holds for all $R\geqq 1$. Let $A_l=sn+lK$, then $A_1<A_2<\ldots<A_{s-1}$. Let Γ_0 be so large that $A_{s-1}-A_1<2^{\Gamma_0}$. Let us choose $R=R_1$ such that $2^\Gamma\|A_0+s^2R_1K$. Let $b_2, \ldots, b_{s-1}$ be defined as the exponents of 2, such that $2^{b_j}\|A_j+s^2R_1K$ $(j=2, \ldots, s-1)$. It is clear that $\max b_j<\Gamma_0$. Now we choose an R_2 such that $2^{\Gamma+1}\|A_0+s^2R_2K$. For this choice of R the exponents of 2 in $A_j+s^2R_2k$ $(j=2, \ldots, s-1)$ are unchanged, $2^{b_j}\|A_j+s^2R_2K$. Thus we have

$$1 = \Delta(s, n+R_1sK) = H(2^\Gamma)\prod_{l=2}^{s-1} H(2^{b_l}) = H(2^{\Gamma+1})\prod_{l=2}^{s-1} H(2^{b_l}) = \Delta(s, n+R_2sK).$$

whence we have $H(2^\Gamma)=H(2^{\Gamma+1})$.

So we proved that $U(n)=V(n)$ on the set $(n, K)=1$. By taking $f(n)=$ $=\chi_0(n)U(n)$, where $\chi_0(n)$ is the principal character $\bmod K$, we have $f(n+K)=$ $=f(n)$ for all $(n,K)=1$. From Lemma 1 we get that $U(n)=V(n)=\chi_K(n)$ on the set $(n,K)=1$. Hence, by Lemma 3, after a simple discussion we shall deduce our

Theorem 3. *Let* $K\geqq 1$ *be an integer,* $F, G\in M$ *such that* $G(n+K)=F(n)$ *holds for every* $n\in\mathbf{N}$, *furthermore that* $F(n)\neq 0$ *if* $(n,K)=1$. *Then the following assertions hold:*

(A) $F(n)=G(n)=\chi(n;\,K)$ *on the set* $n, (n,K)=1$,

or

(B) *in the case* $K=2R$, $(R,2)=1$,

$$G(n) = \chi(n;\,4)F(n); \quad F(n) = \chi(n;\,8)\chi(n;\,R),$$

for every n, $(n,K)=1$, *where* $\chi(n;4)$ *is the nonprincipal character* mod 4; *by*

$$\chi(n;\,8) = \begin{cases} -1 & n\equiv \pm 1 \pmod 8 \\ \ \ 1 & n\equiv \pm 3 \pmod 8 \end{cases} \quad \text{if} \quad R\equiv 1 \pmod 4,$$

$$\chi(n;\,8) = \begin{cases} \ \ 1 & n\equiv 1,3 \pmod 8 \\ -1 & n\equiv 5,7 \pmod 8 \end{cases} \quad \text{if} \quad R\equiv -1 \pmod 4.$$

(C) *If* $\delta\geqq 1$ *and* $p^{\delta+1}|K$, *then* $F(p^\delta)=0$ *holds if and only if* $G(p^\delta)=0$ *is satisfied. In the case* (A), *if* p *is odd and* $F(p^\delta)\neq 0$ *then* $G(p^\delta)=F(p^\delta)$ *and* $\chi(n;K)$ *is periodic with the period* K/p^δ. *In the case* (A), *if* $p=2$ *and* $F(p^\delta)\neq 0$, *then* $\chi(n;K)$ *is periodic with the period* $K/2^{\delta-1}$ *and* $G(p^\delta)=\chi\big(1+(K/2^\delta);K\big)F(p^\delta)$. *In the case* (B), *if* p *is odd, then* $G(p^\delta)=\chi(p^\delta;4)F(p^\delta)$, *and* $F(p^\delta)\neq 0$ *implies that* $\chi(n;R)$ *is periodic* mod R/p^δ.

(D) *In the case* (B), $F(2^\gamma)=G(2^\gamma)=0$ *for every* $\gamma\geqq 1$.

(E) *If* $p^\alpha\|K$, *then* $F(p^\alpha)=0$ *is true if and only if* $G(p^\gamma)=0$ *for every* $\gamma>\alpha$ *furthermore* $G(p^\alpha)=0$ *if and only if* $F(p^\gamma)=0$ *is satisfied for every* $\gamma>\alpha$. *If* $p>2$, *then the statement* $G(p)=0$, $F(p)=0$ *are equivalent.*

(F) *If* $p^\alpha\|K$ *and* $F(p^\alpha)\neq 0$ *or* $G(p^\alpha)\neq 0$, *then* $\chi(n;K)$ *is induced by* $\chi(n;K_1)$, $K=p^\alpha K_1$ *in case* (A), *and* $\chi(n;R)$ *is induced by* $\chi(n;R_1)$, $R=p^\alpha R_1$ *in case* (B).

(G) *In case* (A) *let* $K=B_1B_2$, $(B_1,B_2)=1$, *where* B_1 *is the product of those prime powers* p^α, $p^\alpha\|K$, *for which at least one of* $G(p^\alpha)\neq 0$, $F(p^\alpha)\neq 0$ *holds. Then* $\chi(n;K)$ *is induced by some character* $\chi(n;B_2)$, *and*

$$\frac{G(p^\alpha)}{\chi(p^\alpha;\,B_2)} = \frac{F(p^\gamma)}{\chi(p^\gamma;\,B_2)} \quad (\text{for every } \gamma>\alpha)$$

$$\frac{F(p^\alpha)}{\chi(p^\alpha;\,B_2)} = \frac{G(p^\gamma)}{\chi(p^\gamma;\,B_2)} \quad (\text{for every } \gamma>\alpha),$$

moreover for $p\neq 2$,

$$F(p^\alpha) = G(p^\alpha)$$

hold.

(H) *In the case* (B) *let* $R=D_1\cdot D_2$, $(D_1, D_2)=1$, *where* D_1 *is the product of the prime powers* p^α, $p^\alpha\|R$, *for which* $F(p^\alpha)\neq 0$, *then* $\chi(n; R)$ *is induced by a character* $\chi(n; D_2)$. *Then*

$$a(p;\gamma) := \frac{F(p^\gamma)}{\chi(p^\gamma; 8)\chi(p^\gamma; D_2)} = a(p;\alpha)$$

$$b(p;\gamma) := \frac{G(p^\gamma)}{\chi(p^\gamma; 8)\chi(p^\gamma; D_2)} = b(p;\gamma)$$

hold for every $\gamma>\alpha$, *furthermore*

$$G(p^\gamma) = X(p^\alpha; 4)F(p^\gamma)$$

for every $\gamma\geqq\alpha$.

If F and G is such a pair of functions for which the above conditions hold, then the relation $G(n+K)=F(n)$ $(n\in N)$ *is satisfied.*

Proof. We shall prove only the necessity of the conditions, the sufficiency part can be verified easily. (A) and (B) were proved earlier. To prove (E) take $n=p^\gamma v$, where $\gamma>\alpha$, $(v, K)=1$, and consider only the equations $G(p^\gamma v+K)=F(p^\gamma v)$, $F(p^\gamma v-K)=G(p^\gamma v)$. Since $p^\gamma v\pm K=p^\alpha(p^{\gamma-\alpha}v\pm K_1)$, $K=p^\alpha K_1$, and $(p^{\gamma-\alpha}v\pm K_1, K)=1$, $F(p^{\gamma-\alpha}v-K_1)\neq 0$, $G(p^{\gamma-\alpha}v+K_1)\neq 0$, and since the same is true if $\gamma=\alpha$, $p>2$, for v, $(v(v-K_1), K)=1$, we obtain (E).

Now we prove (C). The assertion that $F(p^\delta)=0$ iff $G(p^\delta)=0$ is clear. Consider first the case (A). Assume that $F(p^\delta)\neq 0$. Let $n=p^\delta v$, $K=p^\alpha K_1$, $p^\alpha\|K$, $\delta<\alpha$. Then $G(p^\delta)G(v+p^{\alpha-\delta}K_1)=F(p^\delta)F(v)$, whence

$$a := \frac{G(p^\delta)}{F(p^\delta)} = \frac{\chi(v; K)}{\chi(v+p^{\alpha-\delta}K_1; K)} \quad \text{if} \quad (v, K) = 1. \tag{3.30}$$

If we write this equation replacing v by $v+s\,p^{\alpha-1}K_1$, and multiply the equations for $s=0, \dots, v-1$, we get that

$$a^v = \frac{\chi(v; K)}{\chi(v+vp^{\alpha-\delta}K_1; K)},$$

whence we obtain, that

$$a^v = \frac{1}{\chi(1+p^{\alpha-\delta}K_1, K)}$$

is true for every v, $(v, K)=1$. The right hand side does not depend on v. If $2\nmid K$ we can choose $v=1$, $v=2$ and conclude that $a=1$. If $2|K$, then we take $v=K-1$,

$v=K+1$, and deduce that $a^2=1$. In both cases we have

$$\chi(v+2p^{\alpha-\delta}K_1, K) = \chi(v; K) \quad \text{if} \quad (v, K) = 1,$$

which implies that $\chi(v, K)$ is periodic with period $2p^{\alpha-\delta}K_1$, and so it is periodic with $(2p^{\alpha-\delta}K_1, K)$. This implies condition (C) for the case (A).

Now we shall consider case (B). Observe that for the characters given in (B), the product

$$T_R(\mu) := \frac{\chi(\mu; 8)}{\chi(\mu+2R; 8)\chi(\mu; 4)} = -1 \tag{3.31}$$

for every odd μ and for $R\equiv\pm 1 \pmod 4$.

Assume that $p\neq 2$, $p^\delta|R$, $R=p^\alpha R_1$, $p^\alpha\|R$, $\delta<\alpha$. By choosing $n=p^\delta\alpha$, starting from the relation $G(p^\delta)G(v+2p^{\alpha-\delta}R_1)=F(p^\delta)F(v)$, substituting the values for $F(v)$ and $G(v+2p^{\alpha-\delta}R_1)$ given in (B), after some calculation we obtain

$$\frac{G(p^\delta)}{F(p^\delta)} = -\chi(p^\delta; 4)T_R(p^\delta v)\frac{\chi(v; R)}{\chi(v+2p^{\alpha-\delta}R_1; R)},$$

whence, by (3.32) we have that

$$b := \frac{G(p^\delta)}{(p^\delta; 4)F(p^\delta)} = \frac{\chi(v; R)}{\chi(v+2p^{\alpha-\delta}R_1; R)},$$

for every v, $(v, 2R)=1$. Arguing as at the former case we deduce that $b^2=1$, and so that $\chi(\cdot, R)$ is periodic $\bmod 4p^{\alpha-\delta}R_1$, and so $\bmod (4p^{\alpha-\delta}R_1, R)=p^{\alpha-\delta}R_1$. But then $b=1$, $G(p^\delta)=\chi(p^\delta; 4)F(p^\delta)$. This proves condition (C).

The next step is to prove (D). Assume that $G(2)\neq 0$, choose $n=2^\gamma v$, $\gamma\geqq 2$. Then

$$G(2)G(2^{\gamma-1}v+R) = F(2^\gamma)F(v)$$

and by using the explicit form of F and G, after some cancellation, we have

$$G(2)\chi(2^{\gamma-1}v+R; 4)\chi(2^{\gamma-1}v+R; 8)\chi(2^{\gamma-1}; R) = F(2^\gamma)\chi(v; 8). \tag{3.32}$$

If $\gamma\geqq 4$, then the left-hand side does not depend on v, while $\chi(v; 8)$ does. It implies that $F(2^\gamma)=0$ for $\gamma\geqq 4$, and so $G(2)=0$. We can prove impossibility of the case $F(2)\neq 0$ similarly. By (C) the proof of (D) is completed.

Let us prove now (G). By choosing $n=p^\gamma v$, $(v, K)=1$, $\gamma>\alpha$, $p^\alpha\|K$, $K=K_1p^\alpha$, under conditions (A), we have

$$\frac{G(p^\alpha)}{F(p^\gamma)} = \frac{\chi(v; K)}{\chi(p^{\gamma-\alpha}v+K_1; K)}, \tag{3.33}$$

which is valid if $G(p^\alpha)\neq 0$. Assume $G(p^\alpha)\neq 0$. Then $F(p^\gamma)\neq 0$ holds for $\gamma>\alpha$, and the right-hand side does not depend on v. Let $\gamma\geqq 2\alpha$. Then the denominator is periodic $\bmod K_1$, which implies that $\chi(\nu+K_1; K)=\chi(\nu; K)$, consequently $\chi(\nu; K)=\chi(\nu; K_1)$ with some character $\bmod K_1$, and so the right-hand side is $\chi(p^{\gamma-\alpha}; K_1)$. This assertion hold for every $\gamma>\alpha$, and in the case $p\neq 2$ even for $\gamma=\alpha$. The case $F(p^\alpha)\neq 0$ is similar. Doing this for all p^α, $p^\alpha \| B_1$, we get that $\chi(n; K)$ is periodic $\bmod B_2$, and this leads to the equations given in (G). We proved the first part of (F), as well.

Let us finally consider (H). Let $G(p^\alpha)\neq 0$. Let $R=p^\alpha R_1, \gamma>\alpha, n=p^\gamma \nu, (\nu, K)=1$. Then $p>2$. From $G(p^\alpha)G(p^{\gamma-\alpha}\nu+2R_1)=F(p^\gamma)F(\nu)$, we deduce that

$$\frac{G(p^\alpha)}{F(p^\gamma)}\cdot\frac{\chi(p^{\gamma-\alpha}; 8)}{\chi(p^\alpha; 4)} = -\frac{\chi(\nu, R)}{\chi(p^{\gamma-\alpha}\nu+2R_1, R)}\cdot T_R(p^\gamma \nu),$$

which by (3.31) and by choosing $\gamma\geqq 2\alpha$, gives that $X(\cdot, R)$ is periodic $\bmod 2R$ and so $\bmod R$. Furthermore the right-hand side equals $\chi(p^{\gamma-\alpha}; R_1)$, for every $\gamma\geqq\alpha$. We can deduce a similar formula assuming $F(p^\alpha)\neq 0$. Doing this for every p^α, $p^\alpha \| D_1$, we can finish the proof rapidly.

By this the proof of our theorem is completed.

4. Let $A, G\in M$ be connected by the equation $G(n+1)=F(n)$. This was solved in Section 3 under the additional condition $F(n)\neq 0$ $(n=1, 2, \ldots)$. It was found that $F(n)=G(n)=1$ identically.

Let now α be such an exponent for which $2^\alpha-1=P$, where P is a prime power, $P=Q^\beta$, allowing the case $\beta=1$. Let $G_\alpha, F_\alpha\in M$ as follow: $F(1)=G(1)=1$, $G_\alpha(2)=1$, $G_\alpha(2^\alpha)=F_\alpha(P)=$arbitrary nonzero value, $F_\alpha(n)=0$ if $n\neq 1, P$; $G_\alpha(n)=0$ if $n\neq 1, 2, 2^\alpha$. It is clear that F_α and G_α will be multiplicative functions, and the equation $G_\alpha(n+1)=F_\alpha(n)$ $(n=1, 2, \ldots)$ will be true.

It is an open question, whether $2^\alpha-1$ can be a prime power for infinitely many α or not. The list of $\alpha=2, 3, 5$ shows that such α values exist.

We shall prove the next

Theorem 4. *If $F, G\in M$ and $G(n+1)=F(n)$ holds for every $n\in N$, then either $F(n)=G(n)$ are identically zero, or identically one, or there exists an integer $\alpha\geqq 2$ such that $2^\alpha-1=$prime power$=P$, such that $G(2)=F(1)=G(1)=1$, $G(2^\alpha)=$ $=F(P)$ and $F(n)=0$, $G(n)=0$ holds for all other $n\in N$.*

Proof. Let $\mathcal{P}$ be the set of those prime powers P for which $F(P)\neq 0$, and $\mathcal{R}$ be the set of those powers Q for which $G(Q)\neq 0$. Let $\overline{\mathcal{P}}$, $\overline{\mathcal{R}}$ denote the complement sets with respect to the whole set of prime powers. If $\mathcal{P}$ or $\overline{\mathcal{P}}$ are empty sets, then so are $\mathcal{R}$ and $\overline{\mathcal{R}}$, and these lead to the equation $F(n)=G(n)$ as it was proved in Section 3. Thus, we may assume that $\mathcal{P}$ and $\mathcal{R}$ are non-empty proper subsets of the whole set of the prime powers.

It is well known that all solutions of the Diophantine equation $3^x-2^y=1$ are $x=y=1$, and $x=2, y=3$ while $2^x-3^y=1$ implies that $x=2, y=1$.

Lemma 5. *Let P be the smallest integer n, for which $F(n)G(n)=0$. Then P=prime power, furthermore $P=2, 4$ or 8; $F(P)=0$ and $G(P)\neq 0$.*

Proof. It is clear that the smallest integer n for which $F(n)G(n)=0$ holds, has to be a prime power P, and $G(n)=F(n-1)\neq 0$. Thus $F(P)=0$.

Assume first that P is even, and $P>2$. Then $P=2^a$. We have $G(P+1)=0$, $G(2P+2)=F(2P+1)=0$. From the minimality of P we have that both of $P+1$ and $2P+1$ are prime powers. Since at least one of them is a multiple of 3, therefore either $2^a+1=3^b$ or $2^{a+1}+1=3^b$, which implies that $P=4$ or $P=8$.

Assume that P is an odd number. Then $G(P+1)=0$, and we can get rapidly that $P+1=2^s$. If $3|P$, then $P=3^a$, $2^s-3^a=1$, whence $s=2$, $a=1$, i.e. $P=3$ follows. In this case $F(2)\neq 0$, $F(3)=0$. But $F(2)\neq 0$, $\Rightarrow G(3)\neq 0$, $G(6)=F(5)\neq 0$, $F(10)\neq 0$, $G(11)\neq 0$, $\Rightarrow G(22)\neq 0$, $\Rightarrow F(21)=F(3)F(7)\neq 0$, $\Rightarrow F(3)\neq 0$. This leads to a contradiction. If $3|P$, the $2P+1\equiv 0(3)$, $G(2P+1)=0$, and we deduce that $2P+1=3^b$, whence $2^{s+1}-3^b=1$, and so $s=1$, $P=1$ follows. This cannot occur. We finished the proof of our lemma.

Lemma 6. *In the notations of Lemma 5, $P=4$ or $P=8$ cannot be occur.*

Proof. I. The case $P=8$. Then $\{2, 3, 2^2, 5, 7\}\in\mathcal{P}$, $\{2, 3, 2^2, 5, 7, 2^3\}\in\mathcal{R}$, whence $G(5\cdot 3\cdot 7)=G(105)\neq 0$, $F(104)=F(2^3)\cdot(13)\neq 0$, $F(2^3)\neq 0$, and this is a contradiction.

II. The case $P=4$. Then $\{2, 3\}\in\mathcal{P}$, $\{2, 3, 2^2\}\in\mathcal{R}$, and so

$$7 = 2\cdot 3+1\in\mathcal{R},\ G(7\cdot 3) = G(21) = F(20) = F(5\cdot 4) \neq 0, \quad \text{i.e.} \quad F(4) \neq 0,$$

contrary to our assumption.

Lemma 7. *If $\mathcal{R}$ contains at least one odd prime powers, then $F(n)$ and $G(n)$ are nowhere zero.*

Proof. Assume that $\varkappa$ is the smallest odd prime power in $\mathcal{R}$. $\varkappa=3$ would imply that $F(2)\neq 0$, and this case was treated earlier. Assume that $\varkappa>3$. Then $\varkappa-1$ is a power of 2, since in the opposite case, $\varkappa-1=2^sA$, $A>1$ would imply that $F(2^s)\neq 0$, $G(2^s+1)\neq 0$, and $2^s+1<\varkappa$. Thus $\varkappa-1=2^s\in\mathcal{P}$. Since $G(2)\neq 0$, therefore $0\neq G(2\varkappa)=F(2\varkappa-1)$. If $3|\varkappa$, then $\varkappa=3^b$, and from the equation $3^b-2^s=1$ we deduce that either $\varkappa=3$ $(b=1)$, or $\varkappa=3^2$ $(b=2)$. If $\varkappa=3$ then $2\in\mathcal{P}$, which was considered earlier. If $\varkappa=3^2$, then

$$\{2,3^2\}\in\mathcal{R},\ 2^3\in\mathcal{P},\ G(18) \neq 0,\ 17\in\mathcal{P},\ F(136) = F(8\cdot 17) \neq 0,$$

$$137\in\mathcal{R},\ G(1233) = G(9\cdot 137) \neq 0,\ F(1232) = F(2^4\cdot 7\cdot 11) \neq 0,\ 11\in\mathcal{R},$$

$G(12)=G(4\cdot 3)\neq 0, 3\in\mathcal{R}$, which is a contradiction. Assume that $3\nmid \varkappa$. Then $3|2\varkappa-1$, $F(2\varkappa-1)\neq 0$. If $2\varkappa-1$ is not a power of 3, then $2\varkappa-1=3^b B$, where $B>1$, $3\nmid B$, consequently $B\geqq 5$, $3^b\in P$, $F(B)=G(B+1)\neq 0$, and the odd parts of both of 3^b+1, $B+1$ have to be 1, taking into account the minimality of $\varkappa$. But then $3^b+1=2^l$, whence $b=1$, $B=2^d-1$, $d\geqq 3$, and $2(2^s+1)-1=3\cdot(2^d-1)$, i.e. $2^{s+1}-3\cdot 2^d=-4$, which is impossible, since $s+1\geqq d\geqq 3$.

We finished the proof of our lemma.

Lemma 8. *If $\mathcal{P}$ contains at least two distinct odd prime powers, then $\mathcal{R}$ contains at least one odd number.*

Proof. Let $Q_1, Q_2\in\mathcal{P}$ be odd numbers. Assume first that $(Q_1, Q_2)=1$. If the lemma fails to hold, then $G(Q_1+1)\neq 0$, $G(Q_2+1)\neq 0$, $G(Q_1Q_2+1)\neq 0$, and so $Q_1+1=2^a$, $Q_2+1=2^b$, $Q_1Q_2+1=2^c$, $a>b\geqq 2$. Then $(2^c-1)=(2^a-1)(2^b-1)$ and the two sides of this equation are incongruent mod 2^b.

It remains the case when $Q_1=Q^u$, $Q_2=Q^v$ with some odd prime Q. Let $Q_1+1= =2^a$, $Q_2+1=2^b$, $a>b\geqq 2$. Hence we get that $Q_j\equiv -1 \pmod 4$, i.e. that $Q\equiv - -1 \pmod 4$, u, v are both odd numbers. First we observe that $Q^v+1|Q^u+1$. But then $v|u$, which can be proved easily. Assume that $u=kv+r$, where $0\leqq r<v$. If k is an even number, $k=2h$,

$$Q^u+1 = Q^r(Q^{2hv}-1)+Q^r+1,$$

which by $Q^v+1|Q^{2hv}-1$ implies that $Q^v+1|Q^r+1$, and this cannot occur. If k is an odd number, then $Q^u+1=Q^r(Q^{kv}+1)+(1-Q^r)$, and by $Q^v+1|Q^{kv}+1$, $Q^v+1|Q^r-1$, which implies that $r=0$.

So we have, $u=kv$, k is odd. In the same way, starting from $Q^v|Q^u$, we deduce that $b|a$, $a=bt$. So we have

$$Q^v+1 = 2^b,\ Q^{kv}+1 = 2^{bt},\ t\geqq 2.$$

Then

$$2^{bt}-1 = (2^b-1)^k \equiv -1+(1^k)\cdot 2^b \pmod{2^{b+1}}$$

which is impossible for odd k.

The proof of our lemma is completed. By this we proved our theorem.

Remark. The general case $G(n+K)=F(n)$ can be treated similarly, at least for small fixed values of K, but it involves the knowledge of all solutions of Diophantine equations like $a^x-b^y=h$ for some values of a, b, h.

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EÖTVÖS LORÁND UNIVERSITY
COMPUTER CENTER
BUDAPEST, H-1117
BOGDÁNFY ÚT 10/B

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A problem of Kátai on sums of additive functions

ROBERT STYER

1. Introduction

KÁTAI [4] has shown the following result about completely additive functions:

Theorem. *Let F_1, F_2, F_3, F_4 be completely additive functions on the positive integers. Assume that*

$$F_1(n)+F_2(n+1)+F_3(n+2)+F_4(n+3)$$

is an integer for every positive integer n. Then $F_j(n)$, $j=1, 2, 3, 4$, is an integer for every positive integer n.

The theorem can be extended to Gaussian integers, as was done by VAN ROSSUM-WIJSMULLER [9] for four functions and recently has been extended to six functions by KÁTAI and VAN ROSSUM-WIJSMULLER [6].

KÁTAI [5] has shown the analogy of his theorem holds for two *additive* functions by using properties of multiplicative functions. This reference to Kátai's paper may not seem relevant at first glance. But if F and G are additive functions, then $f(n)=\exp(2\pi i F(n))$ and $g(n)=\exp(2\pi i G(n))$ are multiplicative functions; now [5, II, Theorem 2, p. 105] gives the explicit form for f and g and one can then deduce the result.

We wish to extend this to three additive functions. Of course Kátai's theorem as stated is not true for three additive functions. For instance, one can let $F_1(2)=r$, $F_2(2^b)=s$ for all $b\geqq 1$, $F_3(2)=t$, $F_1(2^b)=s-t$, for all $b>1$, $F_3(2^b)=s-r$, for all $b>1$, $F_j(3^b)=-s$, $j=1, 2, 3$, for all $b\geqq 1$, and $F_j(q)=0$, $j=1, 2, 3$, for all prime powers q relatively prime to 6. No matter what real numbers r, s, t one chooses, $F_1(n)+F_2(n+1)+F_3(n+2)=0$. We will show, however, that this counterexample is the only way that a sum of three additive functions can be integral without the functions being integral.

More generally, Kátai (personal communication) believes that the following might be true.

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Conjecture. Let $F_0, F_1, ..., F_{k-1}$ be k additive functions. Assume that

$$(*) \qquad F_0(n)+F_1(n+1)+ ... +F_{k-1}(n+k-1) \equiv 0 \pmod 1$$

for all $n>1$. Then each F_j, $j=0, 1, ..., k-1$, has finite support.

Here we will say $F_j(n)\equiv 0 \pmod 1$ whenever $F_j(n)$ is an integer. The hypothesis $(*)$ probably need only hold for n sufficiently large. We define finite support to mean.

Definition. An additive function F is of finite support mod 1 if $F(p^a)\equiv \equiv 0 \pmod 1$, $a=1, 2, 3, ...$, is true for all but finitely many primes p.

This paper has two parts. In the first part we assume Kátai's conjecture and then investigate which primes are within the finite support of the F_j for a fixed arbitrary number of additive functions. The proof is essentially the Chinese remainder theorem. We will see that for k additive functions, only primes p with $p\leqq k$ are in the set of finite support. Indeed, we will explicitly give all the relationships between the nonzero values of the additive functions at these exceptional primes.

The second half of this paper will prove Kátai's conjecture when we have three additive functions. This proof follows closely the proof of Kátai's theorem in [4]. We will, however, find several exponential Diophantine equations arising in our modification of his proof.

2. Primes in the set of finite support

We now begin to investigate the structure of the primes in the set of finite support, assuming Kátai's conjecture. To prepare for this, let k be the number of additive functions. For a prime p, define $\alpha=\alpha(p)$ to be the integer such that $p^\alpha > > k\geqq p^{\alpha-1}$.

First Main Theorem. *Let $F_0, F_1, ..., F_{k-1}$ be k additive functions on the positive integers. Assume that*

$$(*) \qquad F_0(n)+F_1(n+1)+ ... +F_{k-1}(n+k-1) \equiv 0 \pmod 1$$

for all $n>N$, some integer N. Also assume that each F_j is of finite support mod 1. *Then* $F_j(q)\equiv 0 \pmod 1$, $j=0, 1, ..., k-1$, *for every prime power $q=p^b$ with drime $p>k$.*

Now consider only prime powers $q=p^b$ for any prime $p\leqq k$. The number of $F_j(q)$ which may be assigned arbitrary real values is

$$-1+\sum_{\text{primes } p\leqq k} d(p)$$

where

$$d(p)=\begin{cases}(\alpha-1)k-p^{\alpha-1}+1 & \text{if } p^\alpha-k>p^{\alpha-1};\\ \alpha k-p^\alpha+1 & \text{if } p^\alpha-k\leqq p^{\alpha-1}.\end{cases}$$

One can explicitly find the relationship of the remaining $F_j(q)$ in terms of the ones assigned arbitrary real values.

Proof. We will establish a series of lemmas: the first will remove from consideration all prime powers where the prime exceeds k, the second will show the relationship of $F_j(p^b)$ and $F_j(p^\alpha)$ for $b>\alpha$. Finally, we will see that the rest of the small prime powers lead to a simple linear algebra problem. The proof of each lemma will depend on an application of the Chinese remainder theorem.

The author wishes to thank Professor Kátai for suggesting this problem, and also notes that Professor Kátai independently proved this result.

Lemma 1. *Assume that $F_0, F_1, \ldots, F_{k-1}$ are additive functions of finite support, satisfying $(*)$. Let p be a prime with $p>k$. Then $F_j(q)\equiv 0 \pmod 1$, $j=0, 1, \ldots, k-1$ for all prime powers $q=p^b$.*

Notation. Number the primes $p_1<p_2<p_3<\ldots<p_s$ where p_s is the largest prime within the finite support. Number the prime powers of these primes by $q_1<<q_2<\ldots$. We say that a prime power $q\|n$ if $q=p^b$ and $p^b|n$ but $p^{b+1}\nmid n$.

Define F to be the infinite vector

$$F=(F_0(q_1), F_1(q_1), \ldots, F_{k-1}(q_1), F_0(q_2), \ldots, F_{k-1}(q_2), F_0(q_3), \ldots).$$

For a positive integer n, define $R(n)$ by

$$R(n)=(\delta_{0,1}, \delta_{1,1}, \ldots, \delta_{k-1,1}, \delta_{0,2}, \delta_{1,2}, \ldots, \delta_{k-1,2}, \delta_{0,3}, \ldots),$$

where

$$\delta_{i,j}=\begin{cases}1 & \text{if } q_i\|n+j;\\ 0 & \text{otherwise.}\end{cases}$$

$R(n)$ is an infinite sequence of 0 or 1 values. We note that the inner product

$$R(n)\cdot F=F_0(n)+F_1(n+1)+\ldots+F_{k-1}(n+k-1).$$

Thus, the assumption $(*)$ can be written $R(n)\cdot F\equiv 0 \pmod 1$ for all $n>N$.

Proof of Lemma 1. Fix j. We let p be any prime with $p_s\geqq p>k$. Let $q=p^b$. Recall that we defined α_i for prime p_i by the condition $p_i^{\alpha_i}>k\geqq p_i^{\alpha_i-1}$. By the Chinese remainder theorem, we may choose n_1 and n_2 greater than N such that

$$n_1\equiv 1 \pmod{p_i^{\alpha_i}}, \quad i=1, 2, \ldots, s$$

and

$$n_2 \equiv 1 \pmod{p_i^{\alpha_i}}, \quad i = 1, 2, \ldots, s, \; p_i \neq p;$$
$$n_2 \equiv -j \pmod{p^b};$$
$$n_2 \not\equiv -j \pmod{p^{b+1}}.$$

In other words, n_1 and n_2 (and the next $k-1$ pairs of values) are the same modulo p_i for all the p_i except p. In fact, one can see that

$$[R(n_2)-R(n_1)]\cdot F = F_j(p^b)$$

and so by $(*)$, $F_j(p^b)\equiv 0 \pmod 1$. This proves this lemma.

We now may assume without loss of generality that the primes in the finite support of our additive functions all satisfy $p \leqq k$.

Lemma 2. *Assume that $F_0, F_1, \ldots, F_{k-1}$ are additive functions with finite support satisfying $(*)$. Let p be a prime, and α defined as above. Then $F_j(p^{\alpha+b}) - F_j(p^\alpha)\equiv 0 \pmod 1$, $j=0, 1, \ldots, k-1$, for all integers $b \geqq 1$.*

Proof. Again we merely apply the Chinese remainder theorem. Fix p, b and j. Choose n_1 and n_2 greater than N such that

$$n_1 \equiv 1 \pmod{p_i^{\alpha_i}}, \quad i = 1, 2, \ldots, s, \; p_i \neq p;$$
$$n_1 \equiv -j \pmod{p^\alpha};$$
$$n_1 \not\equiv -j \pmod{p^{\alpha+1}},$$

and

$$n_2 \equiv 1 \pmod{p_i^{\alpha_i}}, \quad i = 1, 2, \ldots, s, \; p_i \neq p;$$
$$n_2 \equiv -j \pmod{p^{\alpha+b}};$$
$$n_2 \not\equiv -j \pmod{p^{\alpha+b+1}}.$$

Then one can see that

$$[R(n_2)-R(n_1)]\cdot F = F_j(p^{\alpha+b}) - F_j(p^\alpha).$$

This proves the lemma.

We now only have a finite number of prime powers to consider, since any large power will give the same values as a power "close to k". Fix a prime $p \leqq k$, and let r be chosen so that $r+k\equiv 0 \pmod{p^{\alpha-1}}$. The r simply shifts some columns so that we will get an upper triangular matrix.

Define a vector

$$F(p) = \big(F_0(p^\alpha), F_1(p^\alpha), \ldots, F_{k-1}(p^\alpha),$$
$$F_r(p^{\alpha-1}), F_{r+1}(p^{\alpha-1}), \ldots, F_{k-1}(p^{\alpha-1}), F_0(p^{\alpha-1}), F_1(p^{\alpha-1}), F_{r-1}(p^{\alpha-1}),$$
$$F_0(p^{\alpha-2}), F_1(p^{\alpha-2}), \ldots, F_{k-1}(p^{\alpha-2}), \ldots, F_0(p), F_1(p), \ldots, F_{k-1}(p)\big).$$

Also define

$$R(n,p) = \delta_{\alpha,0}, \delta_{\alpha,1}, \ldots, \delta_{\alpha,k-1},$$
$$\delta_{\alpha-1,r}, \delta_{\alpha-1,r+1}, \ldots, \delta_{\alpha-1,k-1}, \delta_{\alpha-1,0}, \delta_{\alpha-1,1}, \ldots, \delta_{\alpha-1,r-1},$$
$$\delta_{\alpha-2,0}, \delta_{\alpha-2,1}, \ldots, \delta_{\alpha-2,k-1}, \ldots, \delta_{1,0}, \delta_{1,1}, \ldots, \delta_{1,k-1}$$

where

$$\delta_{a,j} = \begin{cases} 1 & \text{if } \ a<\alpha \text{ and } p^a \| n+j; \\ 1 & \text{if } \ a=\alpha \text{ and } p^a | n+j; \\ 0 & \text{otherwise.} \end{cases}$$

We note that $F(p)$ and $R(n,p)$ are vectors of length αk. We also note that $R(n)\cdot F = \sum_{i=1}^{s} R(n,p_i)\cdot F(p_i)$ and that $R(n_1,p)=R(n_2,p)$ whenever $n_1\equiv n_2 \pmod{p^\alpha}$.

Lemma 3. *Assume that $F_0, F_1, \ldots, F_{k-1}$ are additive functions of finite support satisfying $(*)$. Let p be any prime with $p\leqq k$. Then*

$$R(n_2,p)\cdot F(p) \equiv R(n_1,p)\cdot F(p) \pmod 1$$

for any positive integers n_1 and n_2.

Proof. Again we use the Chinese remainder theorem. Choose integers n_3 and n_4 greater than N such that

$$n_3 \equiv n_1 \pmod{p^\alpha};$$
$$n_3 \equiv 1 \pmod{p_i^{\alpha_i}}, \quad i = 1, 2, \ldots, s, \ p_i \neq p,$$

and

$$n_4 \equiv n_2 \pmod{p^\alpha};$$
$$n_4 \equiv 1 \pmod{p_i^{\alpha_i}}, \quad i = 1, 2, \ldots, s, \ p_i \neq p.$$

Then one can see that $R(n_2,p)\cdot F(p)-R(n_1,p)\cdot F(p)=R(n_4,p)\cdot F(p)-R(n_3,p)\times \times F(p)=R(n_4)\cdot F-R(n_3)\cdot F\equiv 0 \pmod 1$. This proves the lemma.

We now prove the first main theorem.

By Lemma 3, we know that for every prime p there is some real number b such that for every n we have $R(n,p)\cdot F(p)\equiv b \pmod 1$. For each prime $p\leqq k$, choose an arbitrary real number $b=b(p)$. Fix a prime p and choose any n with $n\equiv 1 \pmod{p^{\alpha+1}}$. Now define a matrix with p^α rows and αk columns by

$$A = \begin{pmatrix} R(n+p^\alpha, p) \\ R(n+p^\alpha-1, p) \\ \vdots \\ R(n+1,p) \\ R(n,p) \end{pmatrix}.$$

One can verify (because of the way we chose r) that if $p^\alpha - k \leqq p^{\alpha-1}$ then A is of the form

$$A = \begin{pmatrix} I_k & * & * & * \\ 0 & I_{p^\alpha - k} & * & * \end{pmatrix}.$$

If $p^\alpha - k \geqq p^{\alpha-1}$ then A is of the form

$$A = \begin{pmatrix} I_k & * & * & * \\ 0 & I_{p^{\alpha-1}} & * & * \\ 0 & * & * & * \end{pmatrix}.$$

Here I_m is the identity matrix of size m. If we consider this last matrix as having three divisions of the rows, then one can see that every row of the third division is identical to one of the rows of the second division.

Now we note that the matrix equation $AF(p)^t \equiv b(p)(1, 1, ..., 1)^t \pmod 1$ has either $\alpha k - p^\alpha$ or $\alpha k - k - p^{\alpha-1}$ free variables. We also have the free variable $b(p)$ and so this gives us the expression for $d(p)$ stated in the theorem. But now we note that the $b(p)$ are not really free—indeed, since $\sum_{i=1}^{s} b(p) = \sum_{i=1}^{s} R(n, p) \cdot F(p) = R(n) \cdot F \equiv 0 \pmod 1$, we have one linear relation among the $b(p)$. This explains the -1 in the theorem. (The Chinese remainder theorem again implies that the $b(p)$ have no other relations.) One also sees explicitly in the matrix A the relations between the $F_j(p^b)$ for any given prime $p \leqq k$. This proves the first main theorem.

3. Sums of three additive functions

We now will embark on a proof that when $k=3$, Kátai's conjecture about finite support is indeed true. We will follow the broad outlines of the proof of his theorem quoted at the beginning of this paper. His proof begins by showing that the theorem holds for small prime n, and then he uses induction (with many subcases) to complete the proof. When we attempt to modify his proof, however, we will encounter dozens of exponential Diophantine equations. Fortunately, most of these equations have been studied previously.

Theorem. *Let F_1, F_2, F_3 be additive functions. Assume that*

$$(*) \qquad F_1(n) + F_2(n+1) + F_3(n+2) \equiv 0 \pmod 1, \quad n > 1.$$

Then F_1, F_2 and F_3 have finite support.

Indeed, if r, s and t are arbitrary real numbers, and if $F_1(2) \equiv r$, $F_2(2) \equiv s$ and $F_3(2) \equiv t \pmod 1$, then $F_1(2^b) \equiv s - t$, for all $b > 1$, $F_2(2^b) \equiv s$ for all $b > 1$, $F_3(2^b) \equiv$

$\equiv s-r$, for all $b>1$, $F_j(3^c)\equiv -s$, $j=1, 2, 3$, for all b, and $F_j(q)\equiv 0 \pmod 1$, $j=1, 2, 3$, for all prime powers q relatively prime to 2 and 3.

By our work above, we already know the structure of the nonzero solutions must be the ones stated in the second half of this theorem. Because we could subtract two solutions with $F_1(2)\equiv r$, $F_2(2)\equiv s$ and $F_3(2)\equiv t \pmod 1$, we may assume these values are all zero mod 1. We are then proving

Second Main Theorem. *Let F_1, F_2 and F_3 be additive functions on the positive integers. Assume that*

$$(*) \qquad F_1(n)+F_2(n+1)+F_3(n+2) \equiv 0 \pmod 1$$

for all $n>1$. Also assume that $F_1(2)$, $F_2(2)$ and $F_3(2)$ are $\equiv 0 \pmod 1$. Then $F_j(n)\equiv 0 \pmod 1$ for every n, $j=1, 2, 3$.

Proof. We first show that our theorem's conclusion holds for small prime powers n, then that it holds for *all* powers of a few small primes, and finally use induction to show the theorem for general n. As in Kátai's proof, we will have many cases depending on the prime power mod low primes. Unlike Kátai's case, however, we find a multitude of exponential Diophantine equations arising.

We first show that the $F_j(n)\equiv 0 \pmod 1$ for small n.

Lemma 4. *Assume that F_1, F_2, and F_3 are additive functions of the positive integers. Assume that*

$$(*) \qquad F_1(n)+F_2(n+1)+F_3(n+2) \equiv 0 \pmod 1$$

for all $n>1$. Then $F_j(n)\equiv 0 \pmod 1$ for all $n<38, j=1, 2, 3$.

Before proving the case of three additive functions, we will illustrate the idea with the case of two additive functions satisfying the analog of $(*)$, namely, $F_1(n)+$ $+F_2(n+1)\equiv 0 \pmod 1$. Consider the set of prime powers $\{2, 3, 4, 5, 7, 8, 9, 11\}$. Consider the sixteen values $n=2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14, 20, 21, 35, 44, 55$. These sixteen n give rise to sixteen equations $F_1(n)+F_2(n+1)\equiv 0 \pmod 1$ which can be expressed in terms of the prime powers in $\{2, 3, 4, 5, 7, 8, 9, 11\}$. For instance, $n=55$ gives rise to the equation $F_1(5)+F_1(11)+F_2(7)+F_2(8)\equiv 0 \pmod 1$. We therefore have 16 equations in the 16 variables $F_j(q)$ with $j=1, 2$ and $q\in \{2, 3, 4, 5, 7, 8, 9, 11\}$. One may set up a matrix equation to represent these, say $AF\equiv 0 \pmod 1$, where

$$F^t = (F_1(2),\ F_2(2),\ F_1(3),\ F_2(3),\ \ldots,\ F_1(11),\ F_2(11)),$$

and

$$A=\begin{pmatrix}
1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0\\
0&1&0&1&0&0&1&0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0\\
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1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1\\
0&0&0&1&0&1&0&0&0&0&0&0&0&0&1&0\\
1&0&0&1&0&0&0&1&1&0&0&0&0&0&0&0\\
0&0&0&1&1&0&1&0&0&1&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&1&0&0&0&0&0&0&1\\
0&0&0&0&0&1&1&0&1&0&0&0&0&1&0&0\\
0&0&0&0&1&0&0&1&0&0&0&0&0&1&1&0\\
0&0&0&0&0&0&1&0&0&1&0&1&0&0&1&0
\end{pmatrix}.$$

Amazingly, this matrix A has determinant -1. Thus we may conclude that it has an inverse with integer entries and therefore that the vector F must have each component $\equiv 0 \pmod 1$. In other words, $F_1(n)$ and $F_2(n)$ are integers for $n=2, 3, 4, 5, 7, 8, 9, 11$.

Indeed, we need not assume that the above matrix A is square; even if A were overdetermined one would "row reduce" with the proviso that one may not divide by integer factors. If one only switches rows or adds integral multiples of one row to another, then one hopes to reduce the matrix A to a diagonal matrix with diagonal entries equal to 1 or -1. If this is possible, then every variable $F_j(q) \equiv 0 \pmod 1$.

We will follow the same ideas when we have three additive functions. One sets up the matrix equation $AF \equiv 0 \pmod 1$ where the vector F contains the variables $F_j(q)$, $j=1, 2, 3$, for the nineteen prime powers q equal to

$$2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, 29, 31, 32, 37.$$

Recall that we have hypothesized that $F_j(2) \equiv 0 \pmod 1$, $j=1, 2, 3$. This hypothesis eliminates three variables, so we actually have 54 variables. The rows of A come from expanding $F_1(n)+F_2(n+1)+F_3(n+2)$ for the fifty-four values of n

$$2, 3, \ldots, 36, 37, 38, 44, 50, 54, 55, 56, 68, 74, 75, 76, 90, 91, 110, 115, 143, 152, 154, 713.$$

One can verify that the prime factorizations of these fifty-four triples n, $n+1$ and $n+2$ have only prime powers in the set of nineteen powers listed above. Therefore, we get the matrix A to be a 54 by 54 matrix of zeros and ones.

Amazingly, the determinant of A is ± 1 (depending on the ordering of the columns). We conclude that A can be inverted with integer entries and therefore each $F_j(q) \equiv 0 \pmod 1$ for the q listed.

One can change the hypothesis. Instead of $F_j(2) \equiv 0 \pmod 1$, $j=1, 2, 3$, one may assume $F_j(4) \equiv 0 \pmod 1$, $j=1, 2, 3$, or $F_j(8) \equiv 0 \pmod 1$, $j=1, 2, 3$, or $F_1(2) \equiv 0 \pmod 1$, $F_1(3) \equiv 0 \pmod 1$, $F_1(4) \equiv 0 \pmod 1$ or any combination that would lead to $r \equiv s \equiv t \equiv 0 \pmod 1$ in our counterexample.

Also, one need not start the hypothesis on $F_1(n)+F_2(n+1)+F_3(n+2)$ with $n=2$. It seems that one might be able to start at any value of n as long as one has enough rows. For instance, if we begin with $n=17$, (adding 15 new values n to replace the ones we have eliminated) we get a matrix which row reduces to give all the $F_j(q) \equiv 0 \pmod 1$.

At any rate, we have taken care of small values of prime powers q. We must now take care of the case when q is an arbitrary power of 2 or 3. So suppose q is a power of 2.

Lemma 2. *Let $a>5$. Assume that $F_j(n) \equiv 0 \pmod 1$ for all n less than 2^a-3, $j=1, 2, 3$. Then $F_1(2^a)$, $F_2(2^a+1)$, $F_1(2^a-1)$, $F_2(2^a)$, $F_3(2^a+1)$, $F_2(2^a-1)$, and $F_3(2^a)$ are all $\equiv 0 \pmod 1$.*

Remark. The condition $n<2^a-3$ could be replaced by $n<7 \cdot 2^{a-3}$ but we only need 2^a-3 (in Case 18).

Proof. We give a case by case analysis depending on what the power a is modulo 12. Each case will state the result obtained, the assumption on a, the exceptions to the proof (invariably Diophantine equations which will be dealt with later), and the synopsis of the proof for the case.

Case 1. $F_1(2^a) \equiv 0 \pmod 1$ for a odd,

unless: $2^a+1=3^b$ for some positive integer b.

$$2^a;\ 3\frac{2^a+1}{3},\ 2(2^{a-1}+1).$$

This last line will be our abbreviated notation for

$$F_1(2^a)+F_2(2^a+1)+F_3(2^a+2) \equiv 0 \pmod 1$$

and the fact that some power of 3 divides 2^a+1 as well as that 2 divides 2^a+2.

Using the fact that 3 divides 2^a+1 when a is odd, we have

$$F_1(2^a)+F_2(3^c)+F_2((2^a+1)/3^c)+F_3(2)+F_3(2^{a-1}+1) \equiv 0 \pmod 1$$

for some positive integer c such that 3^c divides 2^a+1 but 3^{c+1} does not. We exclude the case when $2^a+1=3^b$ for some positive integer b so we may assume that $3^c<2^a-2$. (Fortunately, we will see that this exponential Diophantine equation has no solutions with $a>5$.) By our inductive hypothesis, $F_2(3^c)$, $F_2((2^a+1)/3^c)$, $F_3(2)$, and $F_3(2^{a-1}+1)$ are all $\equiv 0 \pmod 1$. We therefore conclude that $F_1(2^a)\equiv 0 \pmod 1$.

Case 2. $F_1(2^a)\equiv 0 \pmod 1$ and $F_2(2^a+1)\equiv 0 \pmod 1$ for $a\equiv 0 \pmod 4$,

unless: $2^{a+1}+1=3^b$ for some positive integer b, or $2^{a+1}+3=5^b$ for some positive integer b.

$$2^a;\ 2^a+1;\ 2(2^{a-1}+1),$$

$$3\frac{2^{a+1}+1}{3};\ 2(2^a+1);\ 5\frac{2^{a+1}+3}{5}.$$

These lists are shortland for the following argument: starting with the last line of our proof list,

$$F_1\left(3\frac{2^{a+1}+1}{3}\right)+F_2(2(2^a+1))+F_3\left(5\frac{2^{a+1}+3}{5}\right)\equiv 0 \pmod 1$$

or thus

$$F_1(3^c)+F_1\left(\frac{2^{a+1}+1}{3^c}\right)+F_2(2)+F_2(2^a+1)+F_3(5^d)+F_3\left(\frac{2^{a+1}+3}{5^d}\right)\equiv 0 \pmod 1$$

for some c and d with 3^c the highest power dividing $2^{a+1}+1$ and 5^d the highest power dividing $2^{a+1}+3$.

With the inductive hypothesis, noting our exceptions, we have that $F_1(3^c)$, $F_1\left(\frac{2^{a+1}+1}{3^c}\right)$, $F_2(2)$, $F_3(5^d)$, $F_3\left(\frac{2^{a+1}+3}{5^d}\right)$ are all $\equiv 0 \pmod 1$. Thus, $F_2(2^a+1)\equiv$ $\equiv 0 \pmod 1$.

The first line of our proof list says

$$F_1(2^a)+F_2(2^a+1)+F_3(2(2^{a-1}+1))\equiv 0 \pmod 1$$

which says

$$F_1(2^a)+F_2(2^a+1)+F_3(2)+F_3(2^{a-1}+1)\equiv 0 \pmod 1.$$

Using the inductive hypothesis, we have $F_1(2^a)+F_2(2^a+1)\equiv 0 \pmod 1$. Then $F_1(2^a)\equiv 0 \pmod 1$.

As before, the exceptions are exponential Diophantine equations which fortunately will have no solutions with $a>5$.

We will now only give the results without filling in the details.

Case 3. $F_1(2^a)\equiv 0 \pmod 1$ for $a\equiv 2 \pmod 4$,

unless: $2^a+1=5^b$ for some positive integer b.

$$2^a;\ 5\frac{2^a+1}{5};\ 2(2^{a-1}+1).$$

Case 4. $F_2(2^a)\equiv 0 \pmod 1$ and $F_3(2^a+1)\equiv 0 \pmod 1$ for $a\equiv 0 \pmod{12}$,

unless: $2^a-1=3^b$ for some positive integer b, or $2^{a+1}+1=3^b$ for some positive integer b, or $2^{a+2}+3=7^b$ for some positive integer b.

$$3\frac{2^a-1}{3};\ 2^a;\ 2^a+1,$$

$$6\frac{2^{a+1}+1}{3};\ 7\frac{2^{a+2}+3}{7};\ 4(2^a+1).$$

Case 5. $F_1(2^a-1)\equiv 0 \pmod 1$ and $F_2(2^a)\equiv 0 \pmod 1$ for $a\equiv 1 \pmod 4$,

unless: $2^a+1=3^b$ for some positive integer b, or $2^{a+1}-1=3^b$ for some positive integer b, or $2^{a+2}-3=5^b$ for some positive integer b.

$$2^a-1;\ 2^a;\ 3\frac{2^a+1}{3},$$

$$4(2^a-1);\ 5\frac{2^{a+2}-3}{5},\ 6\frac{2^{a+1}-1}{3}.$$

A minor note: the Diophantine equation $2^\alpha-3=5^\beta$ has a rather large solution, namely $2^7-3=5^3$. We are fortunate that $\alpha=7$ corresponds to $a=5$.

Case 6.. $F_2(2^b)\equiv 0 \pmod 1$ for $a\equiv 2 \pmod 4$,

unless: $2^a-1=3^b$ for some positive integer b, or $2^a+1=5^b$ for some positive integer b.

$$3\frac{2^a-1}{3};\ 2^a;\ 5\frac{2^a+1}{5}.$$

Case 7. $F_1(2^a-1)\equiv 0 \pmod 1$ and $F_2(2^a)\equiv 0 \pmod 1$ for $a\equiv 3 \pmod 4$,

unless: $2^a+1=3^b$ for some positive integer b, or $2^a-1=7^b$ for some positive integer b, or $7\cdot 2^{a-1}-3=5^b$ for some positive integer b, or $7\cdot 2^a-5=3^b$ for some positive integer b, or $7\cdot 2^a-11=5\cdot 3^b$ for some positive integer b, or $7\cdot 2^a-11=$ $=3\cdot 5^b$ for some positive integer b.

$$2^a-1;\ 2^a;\ 3\frac{2^a+1}{3},$$

$$7(2^a-1);\ 10\frac{7\cdot 2^{a-1}-3}{5};\ 3\frac{7\cdot 2^a-5}{3},$$

$$5\frac{7\cdot 2^a-11}{15};\ 8\frac{7\cdot 2^{a-3}-1}{3};\ \frac{7\cdot 2^a-5}{3}.$$

Case 8. $F_2(2^a)\equiv 0 \pmod 1$ and $F_3(2^a+1)\equiv 0 \pmod 1$ for $a\equiv 4 \pmod{12}$,

unless: $2^a-1=3^b$ for some positive integer b, or $3\cdot 2^a+1=7^b$ for some positive integer b, or $3\cdot 2^{a-1}+1=5^b$ for some positive integer b.

$$3\,\frac{2^a-1}{3};\ 2^a;\ 2^a+1,$$

$$7\,\frac{3\cdot 2^a+1}{7};\ 10\,\frac{3\cdot 2^{a-1}+1}{5};\ 3(2^a+1).$$

Case 9. $F_2(2^a)\equiv 0 \pmod 1$ and $F_3(2^a+1)\equiv 0 \pmod 1$ for $a\equiv 8 \pmod{12}$,

unless: $2^a-1=3^b$ for some positive integer b, or $13\cdot 2^a+11=3\cdot 7^b$ for some positive integer b, or $13\cdot 2^a+11=7\cdot 3^b$ for some positive integer b, or $13\cdot 2^{a-2}+3=5^b$ for some positive integer b, or $2^a+1=13^b$ for some positive integer b.

$$3\,\frac{2^a-1}{3};\ 2^a;\ 2^a+1,$$

$$21\,\frac{13\cdot 2^a+11}{21};\ 20\,\frac{13\cdot 2^{a-2}+3}{5};\ 13(2^a+1).$$

Case 10. $F_3(2^a)\equiv 0 \pmod 1$ for a even,

unless: $2^a-1=3^b$ for some positive integer b.

$$2(2^a-1);\ 3\,\frac{2^a-1}{3};\ 2^a.$$

Case 11. $F_2(2^a-1)\equiv 0 \pmod 1$ and $F_3(2^a)\equiv 0 \pmod 1$ for $a\equiv 1 \pmod 4$,

unless: $11\cdot 2^{a-2}-3=5^b$ for some positive integer b, or $2^a-1=11^b$ for some positive integer b, or $11\cdot 2^{a-1}-5=3^b$ for some positive integer b, or $2^{a-1}-1=3\cdot 11^b$ for some positive integer b.

$$2(2^{a-1}-1);\ 2^a-1;\ 2^a,$$

$$20\,\frac{11\cdot 2^{a-2}-3}{5};\ 11(2^a-1);\ 6\,\frac{11\cdot 2^{a-1}-5}{3},$$

$$11\,\frac{2^{a-1}-1}{3};\ 8\,\frac{11\cdot 2^{a-4}-1}{3};\ \frac{11\cdot 2^{a-1}-5}{3}.$$

Case 12. $F_2(2^a-1)\equiv 0 \pmod 1$ and $F_3(2^a)\equiv 0 \pmod 1$ for $a\equiv 3 \pmod 4$, *unless:* $2^a-1=7^b$ for some positive integer b, or $7\cdot 2^{a-1}-3=5^b$ for some positive integer b.

$$2(2^{a-1}-1);\ 2^a-1;\ 2^a,$$

$$8(7\cdot 2^{a-3}-1);\ 7(2^a-1);\ 10\frac{7\cdot 2^{a-1}-3}{5}.$$

We also need to consider all powers of three. Fortunately, the powers of 3 are much easier.

Lemma 6. *Let* $a>3$. *Assume that* $F_j(n)\equiv 0 \pmod 1$ *for all n less than* 3^a-2, $j=1,2,3$. *Then* $F_1(3^a)$, $F_3(3^a+2)$, $F_2(3^a)$, $F_1(3^a-2)$, *and* $F_3(3^a)$ *are* all $\equiv 0 \pmod 1$.

Remark. The condition $n<3^a-2$ could be replaced by $n<2(3^{a-1}+1)$ but we only need the stated condition (for Case 16).

Proof. As with the powers of 2, we will do a case analysis, only this time each case will have arbitrary powers a. We will again find several exponential Diophantine equations which we deal with in a later section.

Case 13. $F_1(3^a)\equiv 0 \pmod 1$ and $F_3(3^a+2)\equiv 0 \pmod 1$,

unless: $3^a+1=2^b$ for some positive integer b.

$$3^a;\ 2\frac{3^a+1}{2};\ 3^a+2,$$

$$4\frac{3^a+1}{2};\ 3(2\cdot 3^{a-1}+1);\ 2(3^a+2).$$

Case 14. $F_2(3^a)\equiv 0 \pmod 1$,

unless: $3^a+1=2^b$ for some positive integer b, or $3^a-1=2^b$ for some positive integer b.

$$2\frac{3^a-1}{2};\ 3^a;\ 2\frac{3^a+1}{2}.$$

Case 15. $F_1(3^a-2)\equiv 0 \pmod 1$ and $F_3(3^a)\equiv 0 \pmod 1$,

unless: $3^a-1=2^b$ for some positive integer b.

$$3^a-2;\ 2\frac{3^a-1}{2};\ 3^a,$$

$$2(3^a-2);\ 3(2\cdot 3^{a-1}-1);\ 4\frac{3^a-1}{2}.$$

Now we can do the general prime power q case.

Lemma 7. *Let* $q>37$ *be a prime power. Assume that* $F_j(n)\equiv 0 \pmod 1$ *for all n less than* q, $j=1,2,3$. *Then* $F_1(q)$, $F_2(q)$, *and* $F_3(q)$ *are all* $\equiv 0 \pmod 1$.

Proof. Suppose q is even; then q is a power of 2 which we have already completed above in the fifth lemma. If q is divisible by 3, we see that the sixth lemma completed the proof. Therefore, we may assume q is not divisible by 2 or 3.

Since $F_1(q-2)+F_2(q-1)+F_3(q)\equiv 0 \pmod 1$, the induction hypothesis immediately gives that $F_3(q)\equiv 0 \pmod 1$.

Case 16. $F_1(q)\equiv 0 \pmod 1$ for $q\equiv 1 \pmod 3$,

unless: $q+1=2^b$ for some positive integer b, or $q+2=3^b$ for some positive integer b.

$$q;\ 2\frac{q+1}{2};\ 3\frac{q+2}{3}.$$

Fortunately, if $q=2^b-1$, then we have already shown that $F_1(q)\equiv 0 \pmod 1$ (Cases 5 and 7 above). If $q=3^b-2$ then $F_1(q)\equiv 0 \pmod 1$ from Case 15.

When $q\equiv 2 \pmod 3$, we will give two different ways to achieve the desired result.

Case 17. $F_1(q)\equiv 0 \pmod 1$ for $q\equiv 2 \pmod 3$,

unless: $4q+1=3^b$ for some positive integer b, or $2q-1=3^b$ for some positive integer b, or $q+1=2^b$ for some integer b.

$$4q;\ 3\frac{4q+1}{3};\ 2(2q+1),$$

$$2\frac{2q-1}{3};\ \frac{4q+1}{3};\ 4\frac{q+1}{3},$$

$$3\frac{2q-1}{3};\ 2q;\ 2q+1,$$

$$2\frac{q-1}{2};\ q;\ 2\frac{q+1}{2}.$$

Fortunately, $q+1=2^b$ has already been covered by Lemma 5.

Case 18. $F_1(q)\equiv 0 \pmod 1$ for $q\equiv 2 \pmod 3$,

unless: $q+1=2^b$ for some positive integer b,

$4q+7=3^b$ for some positive integer b, or $2q+5=3^b$ for some positive integer b, or $q+3=2^b$ for some positive integer b.

$$q;\ 2\frac{q+1}{2};\ q+2,$$

$$2(2q+3);\ 3\frac{4q+7}{3};\ 4(q+2),$$

$$2q+3;\ 2(q+2);\ 3\frac{2q+5}{3},$$

$$2\frac{q+1}{2};\ q+2;\ 2\frac{q+3}{2},$$

$$4\frac{q+1}{3};\ \frac{4q+7}{2};\ 2\frac{2q+5}{3}.$$

Fortunately, when $q=2^b-3$, Lemma 5 tells us that $F_2(2^b-1)\equiv 0 \pmod 1$ and. $F_3(2^b)\equiv 0 \pmod 1$ so the fourth line of this proof list is still valid even when $q+3=2^b$ We therefore only have two exceptions to consider.

Cases 17 and 18 give us a choice; we will choose the one which avoids the exceptions listed whenever possible. In particular, we can avoid the exceptions listed unless we have one of the following:

$4q+1=3^b$ for some positive integer b and $4q+7=3^c$ for some positive integer c,
$4q+1=3^b$ for some positive integer b and $2q+5=3^c$ for some positive integer c,
$2q-1=3^b$ for some positive integer b and $4q+7=3^c$ for some positive integer c,
$2q-1=3^b$ for some positive integer b and $2q+5=3^c$ for some positive integer c.

These give rise to the exponential Diophantine equations:

$$6 = 3^c-3^d,\quad 9 = 2\cdot 3^c-3^b,\quad 9 = 3^c-2\cdot 3^b,\quad \text{and}\quad 6 = 3^c-3^b.$$

Of course, these are rather trivial and one sees that these have no solutions with c or d exceeding 3.

Putting Lemmas 4—7 together, we find an inductive proof for our main theorem provided that we can remove the exceptions from each case. In other words, we have reduced the entire problem to solving several two variable exponential Diophantine equations. Most of these have been solved (in much greater generality) by TRYGVE NAGELL [8] and later TOSHIRO HADANO [3]. Nagell solved all equations of the form $a^x+b^y=c^z$ for distinct a, b and c primes less than or equal to seven. Hadano extended this to a, b and c primes up to seventeen. In particular, their results take care of

$$2^a+1 = 3^b,$$
$$2^{a+1}+1 = 3^b,$$
$$2^{a+1}+3 = 5^b,$$
$$2^a+1 = 5^b,$$
$$2^a-1 = 3^b,$$
$$2^{a+2}+3 = 7^b,$$
$$2^{a+1}-1 = 3^b,$$
$$2^{a+2}-3 = 5^b,$$
$$2^a-1 = 7^b,$$
$$2^a+1 = 13^b,$$
$$2^a-1 = 11^b.$$

D. H. LEHMER [7] solved a host of exponential Diophantine equations of the form $S+1=T$ where S and T have prime factors in some small set. His calculations take care of our equations

$$3\cdot 2^a+1 = 7^b,$$
$$3\cdot 2^{a-1}+1 = 5^b,$$
$$2^{a-1}-1 = 3\cdot 11^b.$$

LEO ALEX [1], when looking at possible indices for simple groups, has solved equations of the form $x+y=z$ where x, y, and z are of the form $2^r 3^s 5^t 7^u$. His work takes care of the equations

$$7\cdot 2^{a-1}-3 = 5^b,$$
$$7\cdot 2^a-5 = 3^b.$$

The rest of the exponential Diophantine equations are

$$7\cdot 2^a-11 = 5\cdot 3^b,$$
$$7\cdot 2^a-11 = 3\cdot 5^b,$$
$$13\cdot 2^a+11 = 3\cdot 7^b,$$
$$13\cdot 2^a+11 = 7\cdot 3^b,$$
$$13\cdot 2^{a-2}+3 = 5^b,$$
$$11\cdot 2^{a-2}-3 = 5^b,$$
$$11\cdot 2^{a-1}-5 = 3^b.$$

We are only interested in solutions when $a>5$; indeed, one can easily compute that these have no solutions for $a=6$ so we may certainly view these equations modulo 16. Fortunately, the equations $7\cdot 2^a-11=3\cdot 5^b$, $13\cdot 2^a+11=3\cdot 7^b$, $13\cdot 2^a+11=7\cdot 3^b$, and $13\cdot 2^{a-2}+3=5^b$ are all impossible modulo 16.

$11\cdot 2^{a-2}-3=5^b$ and $11\cdot 2^{a-1}-5=3^b$ are impossible modulo 11.

This leaves $7\cdot 2^a-11=5\cdot 3^b$ which has a solution $7\cdot 2^3-11=5\cdot 3^2$. Then $7\cdot 2^3(2^\alpha-1)=5\cdot 3^2(3^\beta-1)$. Viewing this modulo 16 gives $\beta=2\gamma$ with γ odd, unless $\beta=0$. Now 3^2 divides $2^\alpha-1$ and one can verify that this implies $\alpha\equiv 0 \pmod 6$ Then $7=2^3-1$ divides $2^\alpha-1$, so 7^2 divides $3^\beta-1$. One verifies that this gives $\beta\equiv 0 \pmod{42}$. Then 1093 divides 3^7-1 which divides $3^\beta-1$, so 1093 must divide $2^\alpha-1$. One can verify that this implies $\alpha\equiv 0 \pmod{364}$. Then 113 divides $2^{14}+1$ which divides $2^\alpha-1$, so 113 divides $3^\beta-1$. One verifies that this implies $\beta\equiv 0 \pmod{112}$. But then 4 divides β, a contradiction, unless $\beta=0$, that is, unless $7\cdot 2^3-11=5\cdot 3^2$ is the largest solution to this exponential Diophantine equation.

The procedure used to solve this last equation is exactly the same that Guy, Lacampagne, and Selfridge [2] use to solve equations such as $5=2^a-3^b$.

This finishes the solution to all of the Diophantine equations, which removes the exceptions from the cases analyzed above, and so one can now use the lemmas to prove the main theorem by induction.

Similar ideas surely work when one considers the analogy of $(*)$ with four additive functions. One can easily find a matrix A involving all prime powers up to 89 which will give the analogy of Lemma 1 for the small prime powers. Instead of dealing with powers of 2 and 3, one must now deal with all powers of 2, 3, 5, 7, and 13. One can then find the necessary cases to deal with the general prime power. But by now one has over a hundred cases, each with many exceptions. Even the task of listing all of the relevant Diophantine equations would be formidable. To attempt this approach with five additive functions seems untenable. Our method is clearly not appropriate for large numbers of additive functions, and we hope that someone will find a better approach which proves the problem in its deserved generality.

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DEPARTMENT OF MATHEMATICAL SCIENCES
VILLANOVA UNIVERSITY
VILLANOVA, PA 19085 USA

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Number systems in integral domains, especially in orders of algebraic number fields

B. KOVÁCS[1]) and A. PETHŐ[2])

1. Introduction

Let $\mathbf{R}$ be an integral domain, $\alpha\in\mathbf{R}$, $\mathcal{N}=\{n_1, n_2, \ldots, n_m\}\subset\mathbf{Z}$, where $\mathbf{Z}$ denotes the ring of integers. $\{\alpha, \mathcal{N}\}$ is called a number system in $\mathbf{R}$ if any $\gamma\in\mathbf{R}$ has a unique representation

$$(1.1)\quad \gamma = c_0+c_1\alpha+\ldots+c_h\alpha^h;\; c_j\in\mathcal{N}\;\; (i = 0, 1, \ldots, h),\; c_h \neq 0, \;\text{ if }\; h \neq 0.$$

If $\mathcal{N}=\mathcal{N}_0=\{0, 1, \ldots, m\}$ for some $m\geqq 1$, then $\{\alpha, \mathcal{N}\}$ is called canonical number system. In the sequel α will be called the base and $\mathcal{N}$ the set of digits of the number system.

If the characteristic of $\mathbf{R}$ is p, then we may identify any $n\in\mathbf{Z}$ with $n_1\in\mathbf{R}$, where $0\leqq n_1<p$ and 1 is the identity element of $\mathbf{R}$. Hence, in this case we may assume without loss of generality that $\mathcal{N}\subseteqq\{0, \ldots, p-1\}$.

This concept is a natural generalization of negative base number systems in $\mathbf{Z}$ considered by several authors. For an extensive literature we refer to KNUTH [10, 4.1]. The canonical number systems in the ring of integers of quadratic number fields were completely described by KÁTAI and SZABÓ [7], KÁTAI and KOVÁCS [5], [6].

Kovács [8] gave a necessary and sufficient condition for the existence of canonical number systems in $\mathbf{R}$. In [9] we proved that for any $q\in\mathbf{Z}$, $q<-1$ there exist infinitely many $\mathcal{N}\subset\mathbf{Z}$ such that $\{q, \mathcal{N}\}$ is a number system.

In this paper we first characterize all those integral domains which have number systems. If the characteristic of $\mathbf{R}$ is a prime, then we are able to establish all number systems in $\mathbf{R}$. This problem is more difficult if the characteristic of $\mathbf{R}$ is 0.

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It is considered for orders $\mathcal{O}$ of algebraic number fields. In Theorem 3 and 4 we give necessary and sufficient conditions for $\{\alpha, \mathcal{N}\}$ to be a number system in $\mathcal{O}$. Theorem 5 effectively characterizes the bases of all canonical number systems of $\mathcal{O}$. This solves a problem of GILBERT [3]. Combining results of GAÁL and SHULTE [2]. and the enumeration technique of FINCKE and POHST [1] with our Theorems we computed the representatives of all but one classes of basis of canonical number systems in rings of integers of totally real cubic fields with discriminant $\leqq 564$.

2. Results

In the sequel $\mathbf{R}$ will denote an integral domain, $\mathbf{Z}$ the ring of integers, $\mathbf{Q}$ the field of rational numbers, $\mathbf{K}$ an algebraic number field of degree n, with ring of integers $\mathbf{Z}_K$. If α is algebraic over $\mathbf{Q}$, $\mathbf{Z}[\alpha]$ denotes the smallest ring of $\mathbf{Q}(\alpha)$ containing $\mathbf{Z}$ and α. Finally $\mathbf{F}_p$ denotes the finite field with p elements, where p is a prime. With this notations we have

Theorem 1. *There exists a number system in* $\mathbf{R}$ *if and only if*

(i) $\mathbf{R}=\mathbf{Z}[\alpha]$ *for an* α, *algebraic over* $\mathbf{Q}$, *if* $\operatorname{char}\mathbf{R}=0$,

(ii) $\mathbf{R}=\mathbf{F}_p[x]$, *where* x *is transcendental over* $\mathbf{F}_p$, *if* $\operatorname{char}\mathbf{R}=p$, p *is a prime.*

This theorem generalizes a result of KOVÁCS [8], where integral domains with canonical number systems were characterized.

If $\operatorname{char}\mathbf{R}=p$, then $\mathbf{R}=\mathbf{F}_p[x]$ and we can describe all number systems.

Theorem 2. $\{\alpha, \mathcal{N}\}$ *is a number system in* $\mathbf{F}_p[x]$ *if and only if* $\alpha=a_0+a_1x$, *where* $a_0, a_1\in\mathbf{F}_p, a_1\neq 0$ *and* $\mathcal{N}=\mathcal{N}_0=\{0, 1, \ldots, p-1\}$.

From now on we are dealing with integral domains $\mathbf{R}$ with $\operatorname{char}\mathbf{R}=0$. If $\mathbf{R}$ has a number system, then there exists an $\alpha\in\mathbf{R}$, algebraic over $\mathbf{Q}$, such that $\mathbf{R}=\mathbf{Z}[\alpha]$. Let $\mathbf{K}=\mathbf{Q}(\alpha)$ be of degree n, and denote by $\gamma=\gamma^{(1)}, \ldots, \gamma^{(n)}$ the conjugates of a $\gamma\in\mathbf{K}$. If $\{\beta, \mathcal{N}\}$ is a number system in $\mathbf{Z}[\alpha]$, then $\mathbf{Q}(\alpha)=\mathbf{Q}(\beta)$, hence the discriminant of β, $\mathbf{D}(\beta)\neq 0$. In the following two theorems we give necessary and sufficient conditions for $\{\beta, \mathcal{N}\}$ to be a number system in $\mathbf{Z}[\alpha]$, where α is an algebraic integer over $\mathbf{Q}$.

Theorem 3. *Let* α *be an algebraic integer over* $\mathbf{Q}$. *Let* $\beta\in\mathbf{Z}[\alpha]$, $\mathcal{N}\subset\mathbf{Z}$ *and put* $A=\max\limits_{a\in\mathcal{N}}|a|$. $\{\beta, \mathcal{N}\}$ *is a number system in* $\mathbf{Z}[\alpha]$ *if and only if*

(i) $|\beta^{(j)}|>1$ *for* $j=1, 2, \ldots, n$,

(ii) $\mathcal{N}$ *is a complete residue system* mod $|N_{\mathbf{K}/\mathbf{Q}}(\beta)|$ *containing* 0,

(iii) $\alpha\in\mathbf{Z}[\beta]$,

(iv) *all* $\gamma\in\mathbf{Z}[\alpha]$ *with*

$$|\gamma^{(j)}| \leqq \frac{A}{|\beta^{(j)}|-1}, \quad (j = 1, \dots, n) \tag{2.1}$$

have a representation (1.1) *in* $\{\beta, \mathcal{N}\}$.

This theorem is well applicable in practice, because there exist only finitely many $\gamma\in\mathbf{Z}[\alpha]$ with (2.1). The disadvantage of condition (iv) is that it is not clear, if the representability of $\gamma\in\mathbf{Z}[\alpha]$ can be decided in finitely many steps. Therefore we give another characterization.

Theorem 4. *Let the notation be the same as in Theorem* 3. $\{\beta, \mathcal{N}\}$ *is a number system in* $\mathbf{Z}[\alpha]$ *if and only if* (i), (ii), (iii) *and*

$$\frac{\sum_{i=0}^{k-1} a_i\beta^i}{(\beta^k-1)} \notin \mathbf{Z}[\beta] \tag{v}$$

hold for any $a_i\in\mathcal{N}$, $(i=0, \dots, k-1)$, $a_j\neq 0$ *for at least one* $0\leqq j\leqq k-1$ *and*

$$0 < k \leqq c = \left(\frac{2^{t+1}(A+1)}{D(\beta)^{1/2}}\sqrt{\sum_{j=1}^{n}\left(\frac{1}{|\beta^{(j)}|-1}\right)^2}\,(n|\beta|^n)^{(n-1)/2}\right)^n \max_{1\leqq j\leqq n}\frac{\log(A+1)}{\log(|\beta^{(j)}|)},$$

where t denotes the number of non-real conjugates of $\mathbf{K}$, *and*

$$|\beta| = \max_{1\leqq j\leqq n}|\beta^{(j)}|.$$

For an algebraic integer α let $\mathcal{N}_0(\alpha)=\{0, 1, \dots, |N_{\mathbf{K}/\mathbf{Q}}(\alpha)|-1\}$.

Theorem 5. *Let* $\mathcal{O}$ *be an order in the algebraic number field* $\mathbf{K}$. *There exist* $\alpha_1, \dots, \alpha_t\in\mathcal{O}$; $n_1, \dots, n_t\in\mathbf{Z}$, $N_1, \dots, N_t$ *finite subsets of* $\mathbf{Z}$, *which are all effectively computable, such that* $\{\alpha, \mathcal{N}_0(\alpha)\}$ *is a canonical number system in* $\mathcal{O}$, *if and only if* $\alpha=\alpha_i-h$ *for some integers i, h with* $1\leqq i\leqq t$ *and either* $h\geqq n_i$ *or* $h\in N_i$.

3. Number systems in integral domains

To prove Theorem 1 we need two Lemmas.

Lemma 1. *If* $\{\alpha, \mathcal{N}\}$ *is a number system in the integral domain* $\mathbf{R}$, *then* $0\in\mathcal{N}$.

Proof. Assume that $0\notin\mathcal{N}$. Then there exist $b_i\in\mathcal{N}$, $(i=0, \dots, k)$, such that

$$0 = b_0+b_1\alpha+\dots+b_k\alpha^k, \quad b_k\neq 0. \tag{3.1}$$

Let $0\neq\gamma\in\mathbf{R}$, then there exist $c_i\in\mathcal{N}$, $(i=0, \dots, h)$ with

$$\gamma = c_0+c_1\alpha+\dots+c_h\alpha^h, \quad c_h\neq 0. \tag{3.2}$$

From (3.1) and (3.2) it follows easily that $0\neq\gamma\alpha^{k+1}\in\mathbf{R}$ has at least two different representations. Thus Lemma 1 is proved.

Lemma 2. *Let* $\{\alpha, \mathcal{N}\}$ *be a number system in* $\mathbf{R}$ *with* $\operatorname{char}\mathbf{R}=p$. *Then* $\mathcal{N}=\mathcal{N}_0(p)=\{0, 1, ..., p-1\}$.

Proof. We may assume by $\operatorname{char}\mathbf{R}=p$, that $0\leqq a<p$ holds for all $a\in\mathcal{N}$. Obviously $0\in\mathcal{N}$ by Lemma 1. Assume now that there exists an $0<a<p$ with $a\notin\mathcal{N}$. Then there exist $c_i\in\mathcal{N}$, $i=0, ..., k$, $c_k\neq 0$ with

$$a = c_0+c_1\alpha+...+c_k\alpha^k. \tag{3.3}$$

This implies that α is algebraic over $\mathbf{F}_p$. Hence $\mathbf{R}\subset\mathbf{F}_p[\alpha]$ is finite. But the number of different representations (1.1) in $\{\alpha, \mathcal{N}\}$ is infinite. Hence there exists $\gamma\in\mathbf{R}$ with infinitely many different representations. This contradiction proves Lemma 2.

Proof of Theorem 1. First let $\operatorname{char}\mathbf{R}=0$. Assume that there exists a number system $\{\alpha, \mathcal{N}\}$ in $\mathbf{R}$. Let $N=\max_{a\in\mathcal{N}}|a|+1$. Then $N\geqq 1$, because $\mathbf{R}\neq\{0\}$. Since $N\in\mathbf{R}$, there exist $k\geqq 0$, $c_i\in\mathcal{N}$, $i=0, ..., k$ with $N=c_0+c_1\alpha+...+c_k\alpha^k$. We have $k>0$ because $(N-c_0)\neq 0$. Therefore α is algebraic over $\mathbf{Q}$. All $\gamma\in\mathbf{R}$ have representations (1.1), whence $\mathbf{R}=\mathbf{Z}[\alpha]$.

On the other hand, by [8, Theorem 1] there exists a canonical number system in $\mathbf{Z}[\alpha]$, which proves the first assertion of Theorem 1.

Let now $\operatorname{char}\mathbf{R}=p$, where p is a prime, and let $\{\alpha, \mathcal{N}\}$ be a number system in $\mathbf{R}$. Then by Lemma 2, $\mathcal{N}=\mathcal{N}_0$, i.e. $\{\alpha, \mathcal{N}\}$ is a canonical number system in $\mathbf{R}$. This implies by [8, Theorem 2] that $\mathbf{R}=\mathbf{F}_p[x]$. On the other hand there exists a number system in this ring.

Proof of Theorem 2. Let $\{\alpha, \mathcal{N}\}$ be a number system in $\mathbf{F}_p[x]$. Then by Lemma 2, $\mathcal{N}=\{0, 1, ..., p-1\}$. Let $\alpha=P(x)\in\mathbf{F}_p[x]$, then the degree of P in x is at least 1. On the other hand there exist $k\geqq 1$, $a_i\in\mathcal{N}$, $0\leqq i\leqq k$, $a_k\neq 0$ with $x=a_0+$ $+a_1(P(x))+...+a_k(P(x))^k$. This implies that $P(x)|(x-a_0)$, hence $\deg P(x)\leqq 1$. Combining the inequalities for $\deg P(x)$ we conclude $\alpha=a_0+a_1x$ with $a_1\neq 0$. Thus the condition is necessary.

Let now $\alpha=a_0+a_1x$, $a_1\neq 0$. From $x=a_1^{-1}(\alpha-a_0)$ it follows that all elements of $\mathbf{F}_p[x]$ is representable in $\{\alpha, \mathcal{N}\}$. Theorem 2 is proved.

4. Number systems in $\mathbf{Z}[\alpha]$

The main purpose of this section is to prove Theorems 3, and 4. We shall use the notation introduced in Section 2.

Lemma 3. *Let α be algebraic over $\mathbf{Q}$, of degree n. If $\{\beta, \mathcal{N}\}$ is a number system in $\mathbf{Z}[\alpha]$, then $|\beta^{(j)}|\geqq 1$ for all $j=1, \dots, n$.*

Proof. Assume that there exists a j, $1\leqq j\leqq n$ with $|\beta^{(j)}|<1$. Suppose that $\gamma\in\mathbf{Z}[\alpha]$ has the representation $\gamma=a_0+a_1\beta+\dots+a_h\beta^h$ in $\{\beta, \mathcal{N}\}$. Then

$$|\gamma^{(j)}| < A\frac{1}{1-|\beta^{(j)}|},$$

where $A=\max_{a\in\mathcal{N}}|a|$. But this is impossible because $\mathbf{Z}[\alpha^{(j)}]$ has elements with absolute value larger than $\frac{A}{1-|\beta^{(j)}|}$. Lemma 3 is proved.

From now on α will denote an algebraic integer of degree n over $\mathbf{Q}$. Let $\mathbf{K}=\mathbf{Q}(\alpha)$ and denote $\mathbf{Z_K}$ its ring of integers.

Lemma 4. *Let $\beta\in\mathbf{Z_K}$ be of degree n, such that $|\beta^{(j)}|>1$, $j=1, \dots, n$; and $\mathcal{N}\subset\mathbf{Z}$ a complete residue system $\bmod |N_{\mathbf{K}/\mathbf{Q}}(\beta)|$. Put $A=\max_{a\in\mathcal{N}}|a|$. Then for any $\gamma\in\mathbf{Z}[\beta]$ and $k\in\mathbf{Z}$, $k\geqq 1$ there exist $a_0, \dots, a_{k-1}\in\mathcal{N}$ and $\gamma'\in\mathbf{Z}[\beta]$ such that*

$$\gamma = \sum_{i=0}^{k-1} a_i\beta^i+\gamma'\beta^k \tag{4.1}$$

and

$$|\gamma'^{(j)}| < \frac{|\gamma^{(j)}|}{|\beta^{(j)}|^k}+\frac{A}{|\beta^{(j)}|-1}, \quad (j = 1, \dots, n). \tag{4.2}$$

Proof. Let $x^n+b_{n-1}x^{n-1}+\dots+b_0$ be the defining polynomial of β. Then $|b_0|=|N_{\mathbf{K}/\mathbf{Q}}(\beta)|$. Let $\gamma\in\mathbf{Z}[\beta]$. The assertion is trivially true for $k=1$. Assume that it holds for a $k\geqq 1$, i.e.

$$\gamma = \sum_{i=0}^{k-1} a_i\beta^i+\gamma_k\beta^k, \tag{4.3}$$

where $a_i\in\mathcal{N}$, $i=0, 1, \dots, k-1$ and $\gamma_k\in\mathbf{Z}[\beta]$. $\mathbf{Z}[\beta]$ is an order in $\mathbf{K}$, hence there exist $c_0, \dots, c_{n-1}\in\mathbf{Z}$ with

$$\gamma_k = c_0+c_1\beta+\dots+c_{n-1}\beta^{n-1}.$$

Let $a\in\mathcal{N}$ with $c_0\equiv a \pmod{|b_0|}$ and $h=(c_0-a)/b_0$. Then

$$\gamma_k = \gamma_k-h(b_0+b_1\beta+\dots+b_{n-1}\beta^{n-1}+\beta^n) =$$
$$= a+(c_1-hb_1)\beta+\dots+(c_{n-1}-hb_{n-1})\beta^{n-1}-h\beta^n = a+\beta\gamma_{k+1}.$$

Inserting this into (4.3), we get (4.1) for $k+1$, which proves (4.1) for any $\gamma\in\mathbf{Z}[\beta]$ and $k\geqq 0$.

Taking conjugates in (4.1) we obtain

$$\gamma^{(j)} = \sum_{i=0}^{k-1} a_i(\beta^{(j)})^i+\gamma'^{(j)}(\beta^{(j)})^k$$

for any $j=1, \ldots, n$. This implies

$$|\gamma'^{(j)}| \leqq \frac{|\gamma^{(j)}|}{|\beta^{(j)}|^k}+\frac{1}{|\beta^{(j)}|^k}\sum_{i=0}^{k-1} |a_i|\,|\beta^{(j)}|^i,$$

from which (4.2) follows immediately. Lemma 4 is proved.

Proof of Theorem 3. First we prove the necessity of the conditions. Let $\{\beta, \mathcal{N}\}$ be a number system in $\mathbf{Z}[\alpha]$. Then $\beta\in\mathbf{Z}[\alpha]$ and so $\beta\in\mathbf{Z}_{\mathbf{K}}$. By Lemma 1, $0\in\mathcal{N}$, and by [3], $\mathcal{N}$ is a complete residue system mod $|N_{\mathbf{K}/\mathbf{Q}}(\beta)|$. This proves (ii).

By Lemma 3 we have $|\beta^{(j)}|\geqq 1$, $j=1, \ldots, n$. $|\beta^{(j)}|=1$, $j=1, \ldots, n$ is not possible, because in this case $|N_{\mathbf{K}/\mathbf{Q}}(\beta)|=1$ and so $\mathcal{N}$ may contain only one integer. Hence there exists $1\leqq j\leqq n$ with $|\beta^{(j)}|>1$. If for an ℓ $(1\leqq\ell\leqq n)$ we have $|\beta^{(\ell)}|=1$, then $\beta^{(\ell)}$ is not real. Taking $\mathbf{L}=\mathbf{Q}(\beta^l+\bar{\beta}^l)$, then $\mathbf{L}$ is real and we have $[\mathbf{K}^{(\ell)}: \mathbf{L}]=2$, hence $\beta^{(\ell)}$ is a relative unit in $\mathbf{K}^{(\ell)}$, but then β is a unit and so there exists a h $(1\leqq h\leqq n)$ with $|\beta^{(h)}|<1$, which is impossible by Lemma 3.

(iii) and (iv) are obviously necessary for $\{\beta, \mathcal{N}\}$ to be a number system in $\mathbf{Z}[\alpha]$.

We proceed now to the proof of sufficiency. Let $\gamma\in\mathbf{Z}[\alpha]$. By (iii) $\mathbf{Z}[\alpha]\subset\mathbf{Z}[\beta]$ and so $\gamma\in\mathbf{Z}[\beta]$. There exists by (i) for any $\varepsilon>0$ an integer $k=k(\varepsilon)$ with

$$|\gamma^{(j)}| < \varepsilon|\beta^{(j)}|^k, \quad j = 1, \ldots, n.$$

It is possible to find by Lemma 4 $a_i\in\mathcal{N}$, $i=0, \ldots, k-1$ and $\gamma_k\in\mathbf{Z}[\beta]$ such that

$$\gamma = \sum_{i=0}^{k-1} a_i\beta^i+\gamma_k\beta^k \tag{4.4}$$

and

$$|\gamma_k^{(j)}| < \frac{|\gamma^{(j)}|}{|\beta^{(j)}|^k}+\frac{A}{|\beta^{(j)}|-1} < \varepsilon+\frac{A}{|\beta^{(j)}|-1}, \quad j = 1, \ldots, n.$$

This inequality has only finitely many solutions for $\varepsilon=1$. This means, that we can choose ε such that for the corresponding k (2.1) holds. By (iv) and (4.4) we get the desired representation of γ. Theorem 3 is proved.

Proof of Theorem 4. In the proof of Theorem 3 we have seen that (i), (ii) and (iii) are necessary conditions for $\{\beta, \mathcal{N}\}$ to be a number system in $\mathbf{Z}[\alpha]$. Assume now that there exist a $0<k$ and $a_i\in\mathcal{N}$, $i=0, \ldots, k-1$ such that

$$0 \neq -\gamma = \frac{\sum_{i=0}^{k-1} a_i\beta^i}{(\beta^k-1)} \in \mathbf{Z}[\beta],$$

then

$$\gamma = \sum_{i=0}^{k-1} a_i\beta^i + \gamma\beta^k. \tag{4.5}$$

But $\gamma\in\mathbf{Z}[\beta]$ implies the representability of γ in the form

$$\gamma = c_0 + c_1\beta + \ldots + c_h\beta^h, \quad c_i\in\mathcal{N}, \ 1 \leqq i \leqq h. \tag{4.6}$$

Inserting (4.6) into the right-hand side of (4.5) we get a second finite representation of γ in $\{\beta, \mathcal{N}\}$ which is not allowed. Hence assumption (v) is necessary.

To prove the sufficiency of (v), it is enough to show that any $\gamma\in\mathbf{Z}[\alpha]$ with

$$|\gamma^{(j)}| \leqq \frac{A+1}{|\beta^{(j)}|-1}, \quad j = 1, \ldots, n \tag{4.7}$$

have a representation (1.1) in $\{\beta, \mathcal{N}\}$.

Let $\mathbf{K}^{(1)}, \ldots, \mathbf{K}^{(s)}$ be the real, $\mathbf{K}^{(s+1)}, \ldots, \mathbf{K}^{(s+2t)}$ the non-real conjugates of $\mathbf{K}$; $s+2t=n$. Then (4.7) implies

$$\begin{gathered} |\gamma^{(j)}| \leqq \frac{A+1}{|\beta^{(j)}|-1}, \quad j = 1, \ldots, s, \\ |\mathrm{Re}\,\gamma^{(s+j)}|, \ |\mathrm{Im}\,\gamma^{(s+j)}| \leqq \frac{A+1}{|\beta^{(j)}|-1} \quad j = 1, \ldots, t. \end{gathered} \tag{4.8}$$

Write $\gamma=c_0+c_1\beta+\ldots+c_{n-1}\beta^{n-1}$ with $c_i\in\mathbf{Z}$, $i=0, \ldots, n-1$. The number of solutions of (4.8) in $c_0, \ldots, c_{n-1}$, and so, the number of $\gamma\in\mathbf{Z}[\alpha]$ satisfying (4.7) is bounded above by

$$\left(\frac{2^{t+1}(A+1)}{D(\beta)^{1/2}} \sqrt{\sum_{j=1}^{n}\left(\frac{1}{|\beta^{(j)}|-1}\right)^2} (n|\beta|^n)^{(n-1)/2}\right)^n.$$

Let $\gamma\in\mathbf{Z}[\alpha]$ satisfying (4.7). Choose k so that

$$\frac{|\gamma^{(j)}|}{|\beta^{(j)}|^k} \leqq \frac{A+1}{|\beta^{(j)}|^k(|\beta^{(j)}|-1)} \leqq \frac{1}{|\beta^{(j)}|-1}$$

holds for any $j=1, \ldots, n$, i.e. let

$$k = \max_{1\leqq j\leqq n} \frac{\log(A+1)}{\log|\beta^{(j)}|}.$$

Then by Lemma 4, there exist $a_0, ..., a_{k-1} \in \mathcal{N}$ and $\gamma_1 \in \mathbf{Z}[\alpha]$ such that

$$\gamma = \sum_{i=0}^{k-1} a_i \beta^i + \gamma_1 \beta^k.$$

and γ_1 satisfies (4.7). Repeating the application of Lemma 4 to γ_1 instead of γ we get a sequence $\gamma, \gamma_1, \gamma_2, ...$ of elements of $\mathbf{Z}[\alpha]$ with (4.7). This procedure either terminates with $\gamma_i = 0$ or will be periodic. If it is periodic, then we may assume that it is purely periodic, i.e.

$$\gamma = a_0 + a_1 \beta + ... + a_{h-1} \beta^{h-1} + \gamma \beta^h \tag{4.9}$$

holds with $a_i \in \mathcal{N}$ and $h \leqq c$. At least one of $a_i \neq 0$, because otherwise β would be a root of unity. (4.9) implies that

$$-\gamma = (a_0 + a_1 \beta + ... + a_{h-1} \beta^{h-1})/(\beta^h - 1) \in \mathbf{Z}[\alpha],$$

which contradicts the assumption. Theorem 4 is proved.

5. Canonical number systems in orders of algebraic number fields

In the sequel we set $\mathcal{N}_0(\alpha) = \{0, 1, ..., |a_0| - 1\}$ for an algebraic number α. Let the defining polynomial of α in $\mathbf{Z}[x]$ be $a_n x^n + ... + a_1 x + a_0$.

Theorem 6. *Let α and β be algebraic integers over* $\mathbf{Q}$ *such that* $\mathbf{Z}[\alpha] = \mathbf{Z}[\beta]$. *Assume that the coefficients of the defining polynomial* $x^n + ... + b_1 x + b_0 \in \mathbf{Z}[x]$ *of* β *satisfy*

$$0 < b_{n-1} \leqq ... \leqq b_0, \quad b_0 \geqq 2. \tag{5.1}$$

Then $\{\beta, \mathcal{N}_0(\beta)\}$ *is a canonical number system in* $\mathbf{Z}[\alpha]$.

Proof. See the proof of Theorem 1 in [8].

Corollary. *Let α be an algebraic integer over* $\mathbf{Q}$. *There exists an* $N_0 \in \mathbf{Z}$ *such that* $\{\alpha - N, \mathcal{N}_0(\alpha - N)\}$ *is a canonical number system in* $\mathbf{Z}[\alpha]$ *for all* $N \geqq N_0$.

Proof. Let the defining polynomial of α over $\mathbf{Z}[x]$ be $P(x) = a_n x^n + ... + a_1 x + a_0$. We may assume that $a_n > 0$. Let $N > 0$ and $P(x+N) = b_n(N) x^n + ... + b_1(N) x + b_0(N)$, then $b_i(N)$'s $(i = 0, 1, ..., n)$ are polynomials of degree $n - i$ in N with positive leading coefficients. Hence for all sufficiently large N, the $b_i(N)$ satisfy (5.1). Therefore by Theorem 6 $\{\alpha - N, \mathcal{N}_0(\alpha - N)\}$ are canonical number systems in $\mathbf{Z}[\alpha]$.

Lemma 5. *Let α be an algebraic integer over* $\mathbf{Q}$. *There exists an* $M_0 \in \mathbf{Z}$ *such that* $\{\alpha + M, \mathcal{N}_0(\alpha + M)\}$ *is not a canonical number system in* $\mathbf{Z}[\alpha]$ *for all* $M \geqq M_0$.

Proof. Let $P(x)$ be as in the proof of the Corollary. Let $M>0$ and $P(x-M)=$ $=c_n(M)x^n+\ldots+c_1(M)x+c_0(M)$. Then $c_0(M)=P(-M)$, hence there exists an $M_0\in\mathbf{Z}$ such that $c_0(M)$ is strictly decreasing (strictly increasing if n is even) for $M>M_0$. This means that $|c_0(M)|\in\mathcal{N}_0(\alpha+M+1)$. We have further

$$\frac{|c_0(M)|}{(\alpha+M+1)-1}=\frac{|c_0(M)|}{\alpha+M}\in\mathbf{Z}[\alpha],$$

and so $\{\alpha+M+1, \mathcal{N}_0(\alpha+M+1)\}$ is not a number system in $\mathbf{Z}[\alpha]$ by Theorem 4.

Lemma 6. *Let α be an algebraic integer over* $\mathbf{Q}$. *If $\alpha^{(i)}\geqq -1$ holds for some real conjugate of α, then $\{\alpha, \mathcal{N}_0(\alpha)\}$ is not a canonical number system in* $\mathbf{Z}[\alpha]$.

Proof. Let $\alpha^{(i)}$ be a real conjugate of α. If $\{\alpha, \mathcal{N}_0(\alpha)\}$ is a number system, then we have $|\alpha^{(i)}|\geqq 1$ by Lemma 3. $\alpha^{(i)}=-1$ is obviously impossible. If $\alpha^{(i)}\geqq 1$ and $a_j\in\mathcal{N}_0(\alpha)$, then $a_0+a_1\alpha^{(i)}+\ldots+a_l(\alpha^{(i)})^l\geqq 0$, i.e. the negative integers are not representable in $\{\alpha^{(i)}, \mathcal{N}_0(\alpha^{(i)})\}$. Lemma 6 is proved.

Proof of Theorem 5. By the assumption $\mathcal{O}$ is an integral domain of characteristic 0, so if there exists a canonical number system $\{\alpha, \mathcal{N}_0(\alpha)\}$ in $\mathcal{O}$, then $\mathcal{O}=\mathbf{Z}[\alpha]$, i.e. $1, \alpha, \ldots, \alpha^{n-1}$ is a power basis in $\mathcal{O}$, by Theorem 1 GYŐRY [4] proved that there exist finitely many effectively computable element $\beta_1, \beta_2, \ldots, \beta_t$ in $\mathcal{O}$ such that $1, \alpha, \ldots, \alpha^{n-1}$ is a power basis in $\mathcal{O}$, if and only if $\alpha=\beta_i+H$, for some integers H, $1\leqq i\leqq t$.

Let $1\leqq i\leqq t$ be fixed. By Lemma 5, one can find an integer M_i such that $\{\beta_i+M, \mathcal{N}_0(\beta_i+M)\}$ is not a number system in $\mathcal{O}$ for all $M>M_i$. On the other hand, by the Corollary there exists an $m_i\in\mathbf{Z}$ such that $\{\beta_i+m, \mathcal{N}_0(\beta_i+m)\}$ is a number system in $\mathcal{O}$, for all $m\leqq m_i$. Finally by Theorem 4 it is possible to decide for every $m_i<m\leqq M_i$ whether $\{\beta_i+m, \mathcal{N}_0(\beta_i+m)\}$ is a number system in $\mathcal{O}$. Taking

$$N_i=\{m|m_i<m\leqq M_i,\ \{\beta_i+m, \mathcal{N}_0(\beta_i+m)\}\text{ is number system in }\mathcal{O}\}$$

and $n_i=-m_i$, they satisfy the assertion of Theorem 5, which completes the proof.

6. Computational results

Let $\mathbf{K}$ be an algebraic number field of degree n. Let $\mathbf{K}^{(1)}, \ldots, \mathbf{K}^{(s)}$ the real and $\mathbf{K}^{(s+1)}, \ldots, \mathbf{K}^{(s+t)}, \overline{\mathbf{K}^{(s+1)}}, \ldots, \overline{\mathbf{K}^{(s+t)}}$ the non-real conjugates of $\mathbf{K}$, $n=s+2t$. Let $\mathcal{O}$ be an order in $\mathbf{K}$. For the maximal orders of $\mathbf{Q}$ and for the quadratic extensions of $\mathbf{Q}$ all canonical number systems are known of [10], [5], [6]. For higher degree fields the problem is more difficult.

Based on Theorem 5 we can give the following algorithm to determine the canonical number systems in $\mathcal{O}$:

1. Compute $\alpha_1, \ldots, \alpha_h \in \mathcal{O}$ such that $1, \alpha, \ldots, \alpha^{n-1}$ is a power basis in $\mathcal{O}$, if and only if $\alpha = \alpha_i + H$ for some $1 \leqq i \leqq h$ and $H \in \mathbf{Z}$.

2. If $s > 0$, then find the minimal n_i, $(i = 1, \ldots, h)$ such that for any $m \geqq n_i$

$$\alpha_i^{(j)} - m < -1 \quad (j = 1, \ldots, s) \quad \text{and} \quad |\alpha_i^{(s+j)} - m| > 1, \quad j = 1, \ldots, t.$$

Otherwise, compute the minimal n_i such that $P_i(-x)$ is strictly increasing for $x \geqq n_i$, where $P_i(x)$ denotes the defining polynomial of α_i over $\mathbf{Z}$.

3. Calculate M_i $(i = 1, \ldots, h)$ such that for all $m > M_i$ the coefficients of the defining polynomials of $\alpha_i - m$ satisfy (5.1).

4. Decide for every m with $n_i < m \leqq M_i$ whether $\{\alpha_i - m, \mathcal{N}_0(\alpha_i - m)\}$ is number system in $\mathcal{O}$.

The hardest problem in this algorithm is step 1. Győry [4] proved that $\alpha_1, \ldots, \alpha_h$ are effectively computable by giving explicit upper bounds for their heights. His result is based on A. Baker's theorem on linear forms in the logarithms of algebraic numbers, hence in practice it is not applicable at this time. For totally real cubic fields with discriminant $\leqq 3137$ Gaál and Schulte [2] computed such complete systems, using the Baker—Davenport reduction method.

Using their results we computed — in the sense of Theorem 5 — all but one canonical number systems in the maximal orders of totally real cubic fields with discriminant $\leqq 564$.

Steps 2 and 3 are easy to perform. For the computation of M_i we remark that it is the smallest value of $m \in \mathbf{Z}$ such that the coefficients of the defining polynomial of $\alpha_i - m$ satisfy (5.1). Of course assume that

$$(6.1) \qquad 1 \leqq a_1 \leqq a_2 \leqq a_3$$

and the roots $\beta_1, \beta_2, \beta_3$ of the polynomial $P(x) = x^3 + a_1 x^2 + a_2 x + a_3$ are real with $\beta_i < -1$ $(i = 1, 2, 3)$. This implies $a_1 \geqq 4$. Since both roots of $P'(x) = 3x^2 + 2a_1 x + a_2$ are real and are less then -1 we get

$$(6.2) \qquad a_2 \geqq 2a_1 - 3 \geqq a_1 + 2.$$

On the other hand $P(x+1) = x^3 + (a_1+3)x^2 + (a_2 + 2a_1 + 3)x + (a_3 + a_2 + a_1 + 1)$. Using (6.1) and (6.2) we get

$$a_3 + a_2 + a_1 + 1 \geqq 2a_1 + a_2 + 3,$$

hence the coefficients of x in $P(x+1)$ satisfy (5.1) too.

To perform Step 4 we have to enumerate all $\gamma \in \mathbf{Z}_K$ with (2.1) and then to check whether they are representable in the corresponding number system. For the enumeration we used the method of Fincke and Pohst [1].

In the table we listed the discriminants D of all totally real cubic fields $\mathbf{K}$ with $D \leqq 564$, which have power basis. In the column (x, y) we displayed the solutions — computed by GAÁL and SCHULTE [2] — of the index form equation of $\mathbf{K}$, corresponding to an integral basis $1, \omega_1, \omega_2$ of $\mathbf{Z}_\mathbf{K}$. Then in the columns $P_+(x)$, $(P_-(x))$ you find the coefficients — starting with the leading coefficient 1 — of the defining polynomial of $\beta = a + x\omega_1 + y\omega_2$, $(\beta = b - x\omega_1 - y\omega_2)$ $(a, b \in \mathbf{Z})$ such that $\{\alpha, \mathcal{N}_0(\alpha)\}$ is a number system in $\mathbf{Z}_\mathbf{K}$ if and only if $\alpha = \beta - h$ with some integer $h \geqq 0$. We did not find sporadic cases, i.e. the finite sets N_i defined in Theorem 5 were always empty.

The computer program was developed in FORTRAN and was executed on an IBM PC—AT compatible computer. If the sequence of the coefficients of $P_+(x)$ $(P_-(x))$ is not monotonic, then the execution time depends on the number of solutions of (2.1), which was between 600 and 18 000. The computer tested about 40 solutions of (2.1)/seconds.

For the field with $D = 229$; $(x, y) = (508, 273)$ we were not able to compute all solutions of (2.1) because of the large number of solutions.

Let $1, \alpha, \alpha^2$ be a power integral basis of a totally real cubic field. Our computation suggests that $\alpha^{(i)} < -1$ $(i = 1, 2, 3)$ is a sufficient condition for $\{\alpha, \mathcal{N}_0(\alpha)\}$ to be a number system in $\mathbf{Z}_\mathbf{K}$.

D	(x, y)	$P_+(x)$				$P_-(x)$			
49	$(-1, -1)$, $(0, 1)$, $(1, 0)$	1	10	31	29	1	8	19	13
	$(-2, -1)$, $(1, -1)$, $(1, 2)$	1	09	20	13	1	15	68	83
	$(-5, -9)$, $(-4, 5)$, $(9, 4)$	1	46	563	769	1	26	83	71
81	$(-3, -2)$, $(1, 3)$, $(2, -1)$	1	12	27	17	1	21	126	159
	$(-1, -1)$, $(0, 1)$, $(1, 0)$	1	09	24	19	1	9	24	17
148	$(-31, 14)$	1	305	23 515	39 349	1	154	412	278
	$(-5, -3)$	1	18	50	38	1	30	242	250
	$(-1, -1)$	1	11	37	37	1	10	30	26
	$(1, 0)$	1	9	23	17	1	12	44	46
	$(1, 2)$	1	11	27	19	1	16	72	62
169	$(-2, -1)$, $(1, 0)$, $(1, 1)$	1	10	29	25	1	11	36	31

229	(−2, 1)	1	22	134	139	1	14	38	29
	(0, 1)	1	10	28	23	1	11	35	26
	(1, 0)	1	9	23	16	1	12	44	47
	(1, 4)	1	19	43	26	1	35	331	424
	(2, 1)	1	19	105	134	1	11	25	16
	(508, 273)	1	3492	3 050 996	4 329 199	(1	1749	5975	5108)?
257	(−11, −6)	1	36	121	107	1	66	1141	1695
	(−1, −1)	1	10	29	21	1	11	36	35
	(1, 0)	1	09	22	15	1	12	43	41
	(5, 2)	1	32	93	71	1	58	873	919
	(−2, −3)	1	27	202	259	1	15	34	21
	(2, 1)	1	17	86	111	1	10	23	15
316	(1, 0)	1	10	29	22	1	11	36	34
	(1, 2)	1	13	32	22	1	23	152	218
324	(1, 0)	1	10	29	23	1	11	36	33
	(−1, −1)	1	14	59	67	1	10	27	21
364	(−1, 1), (0, −1), (1, 0)	1	13	50	49	1	11	34	31
	(−7, −2), (−2, 9), (9, −7)	1	40	109	77	1	77	1552	2653
404	(1, 0)	1	10	28	22	1	11	35	27
	(1, 1)	1	11	33	29	1	13	49	43
469	(1, 0)	1	10	26	19	1	14	58	61
	(−2, −1)	1	13	51	56	1	11	35	32
473	(−2, −1)	1	13	34	25	1	20	111	107
	(0, 1)	1	11	32	27	1	13	48	37
	(1, 5)	1	28	63	37	1	53	738	935
	(7, −3)	1	39	124	103	1	72	1345	1747
	(1, 0)	1	12	43	45	1	12	43	43
564	(−3, −7)	1	77	1 541	2 239	1	40	98	62
	(−3, −1)	1	17	49	39	1	28	214	246
	(−3, 2)	1	41	455	697	1	22	56	38
	(1, 0)	1	13	51	57	1	11	35	31

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KOSSUTH LAJOS UNIVERSITY
MATHEMATICAL INSTITUTE
4010 DEBRECEN, P.O. BOX 12
HUNGARY

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Note on multiplicative functions satisfying a congruence property

I. JOÓ

1. An arithmetical function $f(n)\not\equiv 0$ is said to be multiplicative if $(n, m)=1$ implies

$$f(nm)=f(n)f(m)$$

and it is called completely multiplicative if the above equation holds for all pairs of positive integers n and m. In the following let $\mathcal{M}$ and $\mathcal{M}^*$ denote the set of integer-valued multiplicative and completely multiplicative functions, respectively.

In 1966 M. V. SUBBARAO [3] proved that if $f\in\mathcal{M}$ and f satisfies the relation

$$f(n+m)\equiv f(m) \pmod n \tag{1}$$

for every positive integers n and m, then $f(n)$ is a power of n with non-negative integer exponent. In 1972 A. IVÁNYI [1] showed that if $f\in\mathcal{M}^*$ and (1) holds for a fixed m and every n, then $f(n)$ also has the same form. Recently, B. M. PHONG and J. FEHÉR [2] extended the results of Subbarao and Iványi mentioned above, proving that if $f\in\mathcal{M}$ and (1) holds for a fixed m with $f(m)\neq 0$ and for every positive integer n, then there is a non-negative integer a such that

$$f(n)=n^a \quad (n=1, 2, \ldots).$$

In this paper we shall give a characterization of those elements $f\in\mathcal{M}$ which satisfy

$$f(pn+M)\equiv f(M) \pmod n$$

for every positive integer n, where p is a fixed prime, M is a fixed positive integer with the conditions $(p, M)=1$ and $f(M)\neq 0$.

We prove the following

Theorem. *Let p be a prime, M be a positive integer for which $(p, M)=1$. Moreover let $f\in\mathcal{M}$ with $f(M)\neq 0$. If f satisfies the relation*

$$f(pn+M)\equiv f(M) \pmod n \tag{2}$$

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for every positive integer n, then either

$$f(n) = n^a \tag{3}$$

or

$$f(n) = \left(\frac{n}{p}\right) \cdot n^a \tag{4}$$

for all positive integers n which are prime to p, where $a \geqq 0$ *is an integer and* $\left(\frac{n}{p}\right)$ *denotes the Legendre symbol.*

Example. All solutions $f \in \mathcal{M}$ of the following congruence

$$f(5n+1) \equiv 1 \pmod{n} \quad (n = 1, 2, \ldots)$$

are

$$f(n) = n^a \quad \text{for all } n \text{ prime to } 5$$

or

$$f(n) = \left(\frac{n}{5}\right) n^a = \begin{cases} n^a & \text{if } n \equiv \pm 1 \pmod 5 \\ -n^a & \text{if } n \equiv \pm 2 \pmod 5, \end{cases}$$

where a is a non-negative integer.

2. Lemmas

Lemma 1. *Assume that p, M and f satisfy the conditions of Theorem and* (2) *holds for every positive integer n. If Q is a prime for which* $(Q, pM) = 1$, *then*

$$f(Q^k) = f(Q)^k \quad (k = 1, 2, \ldots). \tag{5}$$

Proof. Let Q be a prime with $(Q, pM) = 1$. We prove (5) by induction on k. It is obvious that (5) holds for $k = 1$. Assume that (5) is true for k and prove it for $k+1$, and (5) will be proved.

Let q be a prime for which

$$q > QM|f(M)|. \tag{6}$$

Then there exist positive integers x and y such that

$$Q^k x = 1 + pqy \quad \text{and} \quad (x, QM) = 1.$$

Applying (2) with $n = qyM$, we get

$$f(Q^k) f(x) f(M) = f(Q^k x M) = f(M + pqyM) \equiv f(M) \pmod{q},$$

which with (6) implies

$$f(Q^k) f(x) \equiv 1 \pmod{q}. \tag{7}$$

On the other hand, using the fact $(QM, pxq)=1$ we can choose positive integers u and v such that

$$Qu = M+pxqv \quad \text{and} \quad (u, Q) = 1.$$

Then, we have

$$f(Q^{k+1})f(xu) = f(Q^{k+1}xu) = f[Q^k x)(Qu)] =$$
$$= f[Q^k x(M+pxqv)] = f[MQ^k x+px^2qvQ^k] =$$
$$= f(M+pq(My+x^2vQ^k)) \equiv f(M) \pmod q$$

and

$$f(Q)f(xu) = f(Qxu) = f[x(M+pxqv)] = f(x)f(M+pxqv) \equiv fx)f(M) \pmod q.$$

These give

$$f(Q^{k+1})f(x)f(M) \equiv f(Q)f(M) \pmod q$$

which, using (6), implies

(8) $$f(Q^{k+1})f(x) \equiv f(Q) \pmod q.$$

From (7) we get that $f(x)\not\equiv 0 \pmod q$, and so (7) and (8) imply that

$$f(Q^{k+1}) \equiv f(Q^k)f(Q) \pmod q.$$

This shows that

$$f(Q^{k+1}) = f(Q^k)f(Q) = f(Q)^{k+1},$$

since there are infinitely many primes q satisfying (6). Thus (5) is proved for $k+1$. Lemma 1 is proved.

Lemma 2. *Assume that p, M and f satisfy the conditions of Theorem and* (2) *holds for every positive integer n. Then there exists a non-negative integer a such that*

(9) $$|f(n)| = n^a$$

for all positive integers n which are prime to p.

Proof. We first prove that there exists a non-negative integer a such that

(10) $$|f(n)| = n^a \quad \text{if} \quad (n, pM) = 1.$$

In order to prove (10) it is enough to show that

(11) $$f(Q) = \pm Q^{a(Q)}$$

for each prime Q coprime to pM, where $a(Q)\geqq 0$ is an integer, furthermore if P, Q are distinct primes with $(PQ, pM)=1$, then

(12) $$a(P) = a(Q).$$

Let Q be a prime for which $(Q, pM)=1$. Assume that there is a prime $q\neq Q$ and $q|f(Q)$. Then, by Lemma 1, we have

(13) $$q^s|f(Q)^s = f(Q^s) \quad (s = 1, 2, \ldots).$$

For each positive integers s there are positive integers $t=t(s)$ and $h=h(s)$ such that

$$Q^s t = M+pq^s h, \quad (Q, t) = 1.$$

Then we get from (2) and (13) that

$$0 \equiv f(Q^s)f(t) = f(Q^s t) = f(M+pq^s h) \equiv f(M) \pmod{q^s},$$

holds for every s, which implies $f(M)=0$. This is a contradiction and so (11) holds.

Now let P, Q be distinct primes for which $(PQ, pM)=1$. Then, by using (11) we have

$$f(P) = \pm P^{a(P)}, \quad f(Q) = \pm Q^{a(Q)}.$$

Assume that $a(P)\geqq a(Q)$ and let $d=a(P)-a(Q)$. Since p is a prime and $(PQ, p)=1$, we have

$$(PQ^s)^{2(p-1)} \equiv 1 \pmod p \quad (s = 1, 2, \ldots)$$

and so we get from (2) that

$$f(M) \equiv f[(PQ^s)^{2(p-1)}M] = f(P)^{2(p-1)}f(Q)^{2s(p-1)}f(M) =$$

$$= P^{2d(p-1)}(PQ^s)^{2(p-1)a(Q)}f(M) \equiv P^{2d(p-1)}f(M) \mod \left(\frac{(PQ^s)^{2(p-1)}-1}{p}\right)$$

holds for every positive integer s, consequently

$$P^{2d(p-1)}f(M) = f(M)$$

This shows that $d=a(P)-a(Q)=0$, which implies (12). From (11) and (12) it follows that (10) holds.

Now we prove (9).

By using (10), in order to prove (9) it is enough to show that

$$|f(q^k)| = q^{ka} \quad (k = 1, 2, \ldots) \tag{14}$$

holds for all prime divisors q of M, where a is a non-negative integer determined in (10).

Let m be a positive integer for which

$$(m, pM) = 1. \tag{15}$$

Then we have $(pm+M, pM)=1$ and so from (2) and (10) we get

$$f(M) \equiv f(M+pm) = \pm(M+pm)^a \equiv \pm M^a \pmod m,$$

which, as $m\to\infty$ with $(m, M)=1$, implies that

$$|f(M)| = M^a, \tag{16}$$

where a is an integer given in (10).

Let q be a prime divisor of M and $q^{k_0}\|M$. Let $k\leqq k_0$. Then there exist infinitely many positive integers m such that

$$\left(pm+\frac{M}{q^k},\, pM\right)=1.$$

For these m using (2) and (10), we have

$$f(M)\equiv f(pq^k m+M)=f(q^k)f\left(pm+\frac{M}{q^k}\right)=\pm f(q^k)\left(pm+\frac{M}{q^k}\right)^a\equiv$$
$$\equiv \pm f(q^k)\left(\frac{M}{q^k}\right)^a \pmod{m},$$

which implies

(17) $$f(M)=\pm f(q^k)\frac{M^a}{q^{ka}}.$$

Thus, by (16) and (17), it follows that (14) holds for $k\leqq k_0$.

Now let $k>k_0$. Then there exists a prime $Q_0=Q_0(k)$ such that

(18) $$q^{k-k_0}Q_0\equiv 1 \pmod{p},\quad (Q_0, pM)=1.$$

From (2), (10) and (18) we get that

$$f(M)\equiv f[q^{k-k_0}Q_0^{1+(p-1)t}M]=f(Q_0)^{1+(p-1)t}f(q^{k-k_0}M)=$$
$$=\pm Q_0^{a(1+(p-1)t)}f(q^{k-k_0}M)\ \left(\operatorname{mod}\frac{q^{k-k_0}Q_0^{1+(p-1)t}-1}{p}\right)$$

holds for every positive integer t. Thus, we have

$$f(q^{k-k_3}M)=\pm q^{a(k-k_3)}f(M),$$

which, using the fact that (14) holds for every positive integer $k\leqq k_0$, implies that

$$f(q^k)=\pm q^{a(k-k_0)}f(q^{k_0})=\pm q^{ak}.$$

It follows that (14) holds for every positive integer $k>k_0$, and this completes the proof of Lemma 2.

Lemma 3. *Assume p, M and f satisfy the conditions of Theorem and* (2) *holds for every positive integer n. Then we have*

$$f(nM)=n^a f(M)$$

for each quadratic residue n (mod p), i.e. for $\left(\frac{n}{p}\right)=1$, where a is the same integer as in Lemma 2.

Proof. Assume that $(n,p)=1$ and $\left(\frac{n}{p}\right)=1$, i.e. the quadratic congruence

$$z^2 \equiv n \pmod{p}$$

is solvable. It is clear that there exists a prime $Q_1=Q_1(n)$ such that

$$nQ_1^2 \equiv 1 \pmod{p} \quad \text{and} \quad (Q_1, pM) = 1. \tag{19}$$

Let $s(t)=1+(p-1)t$. Then, from (2) and (19) we get that

$$f(nQ_1^{2s(t)}M) \equiv f(M) \left(\mathrm{mod}\, \frac{nQ_1^{2s(t)}-1}{p}\right) \tag{20}$$

holds for every positive integer t. Since $(Q_1, pM)=1$, from Lemmas 1 and 2 we have

$$f(nQ_1^{2s(t)}M) = f(Q_1)^{2s(t)}f(nM) = Q_1^{2as(t)}f(nM)$$

which with (19) implies that

$$n^a f(M) \equiv n^a Q_1^{2as(t)} f(nM) \left(\mathrm{mod}\, \frac{nQ_1^{2s(t)}-1}{p}\right)$$

holds for every positive integer t. The last congruence shows that

$$f(nM) = n^a f(M)$$

since $nQ_1^{2s(t)}-1 \to \infty$ as $t \to \infty$. Thus, Lemma 3 is proved.

3. Proof of Theorem

Assume that p, M and f satisfy the conditions of Theorem and (2) holds for every positive integer n. At first we obtain from Lemma 2 that

$$f(n) = \pm n^a \quad \text{if} \quad (n, p) = 1, \tag{21}$$

where $a \geqq 0$ is an integer and from Lemma 3 that

$$f(n) = n^a \quad \text{if} \quad (n, pM) = 1, \left(\frac{n}{p}\right) = 1. \tag{22}$$

First we shall prove that our theorem holds for all n coprime to pM. Assume that $f(n) \not\equiv n^a$ on the set of integers n with $(n, pM)=1$. We prove that

$$f(n) = \left(\frac{n}{p}\right) n^a \quad \text{if} \quad (n, pM) = 1. \tag{23}$$

It is obvious that (23) follows from (22) in the case $\left(\frac{n}{p}\right)=1$. Since $f(n) \not\equiv n^a$ on the set of integers n coprime to pM, hence there exists a positive integer n_0 such that

(24) $$f(n_0) = -n_0^a \quad \text{and} \quad (n_0, pM) = 1.$$

It follows from (22) and (24) that $\left(\frac{n_0}{p}\right) = -1$.

If $(n, pM)=1$ and $\left(\frac{n}{p}\right) = -1$, then $\left(\frac{n_0 n}{p}\right) = 1$ and so from (22), (24), and Lemma 1 we obtain

$$-n_0^a f(n) = f(n_0) f(n) = f(nn_0) = (nn_0)^a.$$

This shows that

$$f(n) = -n^a = \left(\frac{n}{p}\right) n^a.$$

Thus, (23) is proved.

Using (23) and the method which was used in the proof of Lemma 2 (see the proof of (14)), one can deduce that if $q|M$ then

$$f(q^k) = q^{ka} \quad (k = 1, 2, \ldots)$$

in the case when $f(n) = n^a$ for all n coprime to pM and

$$f(q^k) = \left(\frac{q^k}{p}\right) q^{ka} \quad (k = 1, 2, \ldots)$$

in the case when $f(n) = \left(\frac{n}{p}\right) n^a$ for all n coprime to pM.

The theorem is proved.

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MATHEMATICAL INSTITUTE OF THE
HUNGARIAN ACADEMY OF SCIENCES
REÁLTANODA U. 13—15
H-1053 BUDAPEST
HUNGARY

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Strong limit theorems for quasi-orthogonal random fields. II

F. MÓRICZ

1. Introduction. Let $\{X_{ik}: i, k \geqq 1\}$ be a random field (in abbreviation: r.f.). We say that $\{X_{ik}\}$ is quasi-orthogonal if

$$EX_{ik}^2 = \sigma_{ik}^2 < \infty \tag{1.1}$$

and there exists a double sequence $\{\varrho(m, n): m, n \geqq 0\}$ of nonnegative numbers such that

$$|EX_{ik}X_{jl}| \leqq \varrho(|i-j|, |k-l|)\sigma_{ik}\sigma_{jl} \quad (i, j, k, l \geqq 1) \tag{1.2}$$

and

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varrho(m, n) < \infty. \tag{1.3}$$

In the special case when $\varrho(m, n)=0$ except $m=n=0$, we say that $\{X_{ik}\}$ is an orthogonal r.f.

2. Main results. We will study the almost sure (in abbreviation: a.s.) behavior of the Cesàro type means

$$\zeta_{mn} = \frac{1}{mn} \sum_{i=1}^{m} \sum_{k=1}^{n} \left(1-\frac{i-1}{m}\right)\left(1-\frac{k-1}{n}\right) X_{ik} \quad (m, n \geqq 1) \tag{2.1}$$

as $m+n \to \infty$.

Theorem 1. *If $\{X_{ik}\}$ is a quasi-orthogonal r.f. and*

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \frac{\sigma_{ik}^2}{i^2 k^2} < \infty, \tag{2.2}$$

then

$$\lim_{m+n \to \infty} \zeta_{mn} = 0 \quad a.s. \tag{2.3}$$

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It is instructive to compare Theorem 1 with the corresponding result in [4, Theorem 1] according to which

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{\sigma_{ik}^2}{i^2k^2}[\log(i+1)]^2[\log(k+1)]^2<\infty \tag{2.4}$$

is a sufficient (and in the monotonic case, necessary) condition for the following strong law of large numbers:

$$\lim_{m+n\to\infty}\frac{1}{mn}\sum_{i=1}^{m}\sum_{k=1}^{n}X_{ik}=0\quad \text{a.s.} \tag{2.5}$$

The surprising fact is that the logarithmic factors are missing in condition (2.2). We note that the logarithms are to the base 2 in this paper.

We will prove Theorem 1 in a more general setting which provides information on the rate of convergence in (2.3). In the sequel, p and q denote nonnegative integers.

Proposition 1. *If the conditions of Theorem* 1 *are satisfied and* $\varepsilon>0$, *then*

$$P[\sup_{m\ge 2^p}\sup_{n\ge 2^q}|\zeta_{mn}|>\varepsilon]=O(1)\left\{\frac{1}{2^{2p}2^{2q}}\sum_{i=1}^{2^p}\sum_{k=1}^{2^q}\sigma_{ik}^2+\frac{1}{2^{2p}}\sum_{i=1}^{2^p}\sum_{k=2^q+1}^{\infty}\frac{\sigma_{ik}^2}{k^2}+\right.$$
$$\left.+\frac{1}{2^{2q}}\sum_{i=2^p+1}^{\infty}\sum_{k=1}^{2^q}\frac{\sigma_{ik}^2}{i^2}+\sum_{i=2^p+1}^{\infty}\sum_{k=2^q+1}^{\infty}\frac{\sigma_{ik}^2}{i^2k^2}\right\}. \tag{2.6}$$

Applying the well-known Kronecker lemma (see, e.g. [5, p. 35]), Proposition 1 implies Theorem 1.

We note that a result analogous to Proposition 1 was proved in [3, Theorem 4] for sequences of random variables (in abbreviation: r.v.'s).

We also consider other Cesàro type means defined by

$$\tau_{nm}=\frac{1}{mn}\sum_{i=1}^{m}\sum_{k=1}^{n}\left(1-\frac{i-1}{m}\right)X_{ik}\quad (m,n\ge 1). \tag{2.7}$$

Clearly, the τ_{mn} are intermediate between the rectangular arithmetic means occurring in (2.5) and the means (2.1).

Theorem 2. *If* $\{X_{ik}\}$ *is a quasi-orthogonal r.f. and*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{\sigma_{ik}^2}{i^2k^2}[\log(k+1)]^2<\infty, \tag{2.8}$$

then

$$\lim_{m+n\to\infty}\tau_{mn}=0\quad \textit{a.s.} \tag{2.9}$$

A more general statement giving information on the convergence rate in (2.9) reads as follows.

Proposition 2. *If the conditions of Theorem 2 are satisfied and $\varepsilon>0$, then*

(2.10)
$$P[\sup_{m\geqq 2^p}\sup_{n\geqq 2^q}|\tau_{mn}|>\varepsilon]=O(1)\Big\{\frac{1}{2^{2p}2^{2q}}\sum_{i=1}^{2^p}\sum_{k=1}^{2^q}\sigma_{ik}^2+$$
$$+\frac{1}{2^{2p}}\sum_{i=1}^{2^p}\sum_{k=2^q+1}^{\infty}\frac{\sigma_{ik}^2}{k^2}[\log(k+1)]^2+\frac{1}{2^{2q}}\sum_{i=2^p+1}^{\infty}\sum_{k=1}^{2^q}\frac{\sigma_{ik}^2}{i^2}+$$
$$+\sum_{i=2^p+1}^{\infty}\sum_{k=2^q+1}^{\infty}\frac{\sigma_{ik}^2}{i^2k^2}[\log(k+1)]^2\Big\}.$$

Condition (2.8) lies between (2.2) and (2.4) (cf. conclusions (2.3), (2.5), and (2.9)).

We guess that the logarithmic factor in condition (2.8) is exact.

Conjecture. If $\{\sigma_{ik}\geqq 0\}$ is a double sequence such that

$$\frac{\sigma_{ik}}{k}\geqq\frac{\sigma_{i,k+1}}{k+1}\quad(i,k\geqq 1)$$

and

(2.11)
$$\sum_{i=r}^{\infty}\sum_{k=r}^{\infty}\frac{\sigma_{ik}^2}{i^2k^2}[\log(k+1)]^2=\infty$$

with $r=1$, then there exists an orthogonal r.f. $\{X_{ik}\}$ such that

$$EX_{ik}=0,\ EX_{ik}^2\leqq\sigma_{ik}^2\quad(i,k\geqq 1)$$

and

$$\limsup_{m+n\to\infty}|\tau_{mn}|=\infty\quad\text{a.s.}$$

If condition (2.11) is satisfied with any $r\geqq 1$, then we can state

$$\limsup_{m,n\to\infty}|\tau_{mn}|=\infty\quad\text{a.s.}$$

3. Proof of Proposition 1. We begin with a known result [2].

Lemma 1. *If $\{X_{ik}\}$ satisfies conditions* (1.1)—(1.3), *and $\{a_{ik}\}$ is any sequence of numbers, then*

(3.1)
$$E[\sum_{i=a+1}^{a+m}\sum_{k=b+1}^{b+n}a_{ik}X_{ik}]^2=O(1)\sum_{i=a+1}^{a+m}\sum_{k=b+1}^{b+n}a_{ik}^2\sigma_{ik}^2\quad(a,b\geqq 0;\ m,n\geqq 1).$$

We emphasize that in the proofs of Propositions 1 and 2 the condition that $\{X_{ik}\}$ is a quasi-orthogonal r.f. is used only to the extent that this implies the moment inequality (3.1).

Now we turn to the proof of Proposition 1. We start with the inequality

$$(3.2)\qquad P[\sup_{m\geqq 2^p}\sup_{n\geqq 2^q}|\zeta_{mn}|>\varepsilon]\leqq\sum_{r=p}^{\infty}\sum_{s=q}^{\infty}P[\max_{2^r\leqq m\leqq 2^{r+1}}\max_{2^s\leqq n\leqq 2^{s+1}}|\zeta_{mn}|>\varepsilon].$$

Let $2^r\leqq m\leqq 2^{r+1}$ and $2^s\leqq n\leqq 2^{s+1}$. Since

$$(3.3)\qquad \zeta_{mn}=\zeta_{2^r,2^s}+(\zeta_{m,2^s}-\zeta_{2^r,2^s})+(\zeta_{2^r,n}-\zeta_{2^r,2^s})+(\zeta_{mn}-\zeta_{m,2^s}-\zeta_{2^r,n}+\zeta_{2^r,2^s})$$

we can estimate as follows

$$(3.4)\qquad P[\max_{2^r\leqq m\leqq 2^{r+1}}\max_{2^s\leqq n\leqq 2^{s+1}}|\zeta_{mn}|>\varepsilon]\leqq P\left[[|\zeta_{2^r,2^s}|>\frac{\varepsilon}{4}\right]+\sum_{j=1}^{3}P_{rs}^{(j)},$$

where

$$P_{rs}^{(1)}=P\left[\max_{2^r<m\leqq 2^{r+1}}|\zeta_{m,2^s}-\zeta_{2^r,2^s}|>\frac{\varepsilon}{4}\right],$$

$$P_{rs}^{(2)}=P\left[\max_{2^s<n\leqq 2^{s+1}}|\zeta_{2^r,n}-\zeta_{2^r,2^s}|>\frac{\varepsilon}{4}\right],$$

$$P_{rs}^{(3)}=P\left[\max_{2^r<m\leqq 2^{r+1}}\max_{2^s<n\leqq 2^{s+1}}|\zeta_{mn}-\zeta_{m,2^s}-\zeta_{2^r,n}+\zeta_{2^r,2^s}|>\frac{\varepsilon}{4}\right].$$

By the Chebyshev inequality and (3.1),

$$(3.5)\qquad P\left[|\zeta_{2^r,2^s}|>\frac{\varepsilon}{4}\right]\leqq\frac{16}{\varepsilon^2}E\zeta^2_{2^r,2^s}=\frac{O(1)}{\varepsilon^2}\sum_{i=1}^{2^r}\sum_{k=1}^{2^s}\sigma_{ik}^2.$$

By the Cauchy inequality,

$$(3.6)\qquad [\max_{2^r<m\leqq 2^{r+1}}|\zeta_{m,2^s}-\zeta_{2^r,2^s}|]^2\leqq\sum_{m=2^r+1}^{2^{r+1}}m[\zeta_{m,2^s}-\zeta_{m-1,2^s}[^2.$$

An elementary calculation shows that

$$\zeta_{m,2^s}-\zeta_{m-1,2^s}=\sum_{i=1}^{m}\sum_{k=1}^{2^s}a_{ik}(m,s)X_{ik}$$

where

$$a_{ik}(m,s)=\frac{1}{2^s}\left(1-\frac{k-1}{2^s}\right)\left[\frac{(i-1)(2m-1)}{m^2(m-1)^2}-\frac{1}{m(m-1)}\right].$$

Clearly,

$$|a_{ik}(m,s)|\leqq\frac{1}{m(m-1)2^s}.$$

Hence, by the Chebyshev inequality and (3.1),

$$(3.7)\qquad P_{rs}^{(1)}\leqq\frac{16}{\varepsilon^2}\sum_{m=2^r+1}^{2^{r+1}}mE[\zeta_{m,2^s}-\zeta_{m-1,2^s}]^2=\frac{O(1)}{\varepsilon^2}\sum_{m=2^r+1}^{2^{r+1}}\sum_{i=1}^{m}\sum_{k=1}^{2^s}\frac{\sigma_{ik}^2}{m(m-1)^2 2^{2s}}.$$

The symmetric counterpart of (3.7) is

$$P_{rs}^{(2)} = \frac{O(1)}{\varepsilon^2} \sum_{n=2^s+1}^{2^{s+1}} \sum_{i=1}^{2^r} \sum_{k=1}^{n} \frac{\sigma_{ik}^2}{n(n-1)^2 2^{2r}}. \tag{3.8}$$

Finally, by the Cauchy inequality,

$$[\max_{2^r<m\leqq 2^{r+1}} \max_{2^s<n\leqq 2^{s+1}} |\zeta_{mn}-\zeta_{m,2^s}-\zeta_{2^r,n}+\zeta_{2^r,2^s}|]^2 \leqq$$

$$\leqq \sum_{m=2^r+1}^{2^{r+1}} \sum_{n=2^s+1}^{2^{s+1}} mn[\zeta_{mn}-\zeta_{m-1,n}-\zeta_{m,n-1}+\zeta_{m-1,n-1}]^2$$

and by an elementary calculation,

$$\zeta_{mn}-\zeta_{m-1,n}-\zeta_{m,n-1}+\zeta_{m-1\,n-1} = \sum_{i=1}^{m} \sum_{k=1}^{n} b_{ik}(m,n) X_{ik}$$

where

$$b_{ik}(m,n) = \left[\frac{(i-1)(2m-1)}{m^2(m-1)^2} - \frac{1}{m(m-1)}\right]\left[\frac{(k-1)(2n-1)}{n^2(n-1)^2} - \frac{1}{n(n-1)}\right].$$

Clearly,

$$|b_{ik}(m,n)| \leqq \frac{1}{m(m-1)n(n-1)}.$$

Hence, by the Cauchy inequality and (3.1),

$$P_{rs}^{(3)} \leqq \frac{16}{\varepsilon^2} \sum_{m=2^r+1}^{2^{r+1}} \sum_{n=2^s+1}^{2^{s+1}} mnE[\zeta_{mn}-\zeta_{m-1,n}-\zeta_{m,n-1}+\zeta_{m-1,n-1}]^2 = \tag{3.9}$$

$$= \frac{O(1)}{\varepsilon^2} \sum_{m=2^r+1}^{2^{r+1}} \sum_{n=2^s+1}^{2^{s+1}} \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{\sigma_{ik}^2}{m(m-1)^2 n(n-1)^2}.$$

Next, we combine the above estimates in four parts.

Part 1. By (3.2) and (3.5), while decomposing the inner double sum and interchanging the order of summations, we get that

$$\sum_{r=p}^{\infty} \sum_{s=q}^{\infty} P\left[|\zeta_{2^r,2^s}| > \frac{\varepsilon}{4}\right] = O(1) \sum_{r=p}^{\infty} \sum_{s=q}^{\infty} \frac{1}{2^{2r}2^{2s}} \times \tag{3.10}$$

$$\times \Big\{\sum_{i=1}^{2^p} \sum_{k=1}^{2^q} + \sum_{i=1}^{2^p} \sum_{k=2^q+1}^{2^s} + \sum_{i=2^p+1}^{2^r} \sum_{k=1}^{2^q} + \sum_{i=2^p+1}^{2^r} \sum_{k=2^q+1}^{2^s}\Big\} \sigma_{ik}^2 =$$

$$= O(1)\left\{\frac{1}{2^{2p}2^{2q}} \sum_{i=1}^{2^p} \sum_{k=1}^{2^q} \sigma_{ik}^2 + \frac{1}{2^{2p}} \sum_{i=1}^{2^p} \sum_{k=2^q+1}^{\infty} \frac{\sigma_{ik}^2}{k^2} + \right.$$

$$\left. + \frac{1}{2^{2q}} \sum_{i=2^p+1}^{\infty} \sum_{k=1}^{2^q} \frac{\sigma_{ik}^2}{i^2} + \sum_{i=2^p+1}^{\infty} \sum_{k=2^q+1}^{\infty} \frac{\sigma_{ik}^2}{i^2k^2}\right\}.$$

Part 2. By (3.2) and (3.7), we obtain in a similar way that

$$\sum_{r=p}^{\infty}\sum_{s=q}^{\infty} P_{rs}^{(1)} = O(1) \sum_{m=2^p+1}^{\infty} \sum_{s=q}^{\infty}\sum_{i=1}^{m}\Big\{\sum_{k=1}^{2^q}+\sum_{k=2^q+1}^{2^s}\Big\}\frac{\sigma_{ik}^2}{m^3 2^{2s}} = \tag{3.11}$$

$$= O(1)\Big\{\frac{1}{2^{2q}}\sum_{i=2^p+1}^{\infty}\sum_{k=1}^{2^q}\frac{\sigma_{ik}^2}{k^2}+\sum_{i=2^p+1}^{\infty}\sum_{k=2^q+1}^{\infty}\frac{\sigma_{ik}^2}{i^2k^2}\Big\}.$$

Part 3. By (3.2) and (3.8),

$$\sum_{r=p}^{\infty}\sum_{s=q}^{\infty} P_{rs}^{(2)} = O(1)\Big\{\frac{1}{2^{2p}}\sum_{i=1}^{2^p}\sum_{k=2^q+1}^{\infty}\frac{\sigma_{ik}^2}{i^2}+\sum_{i=2^p+1}^{\infty}\sum_{k=2^q+1}^{\infty}\frac{\sigma_{ik}^2}{i^2k^2}\Big\}. \tag{3.12}$$

Part 4. By (3.2) and (3.9),

$$\sum_{r=p}^{\infty}\sum_{s=q}^{\infty} P_{rs}^{(3)} = O(1)\sum_{m=2^p+1}^{\infty}\sum_{n=2^q+1}^{\infty}\Big\{\sum_{i=1}^{2^p}\sum_{k=1}^{2^q}+\sum_{i=1}^{2^p}\sum_{k=2^q+1}^{n}+\sum_{i=2^p+1}^{m}\sum_{k=1}^{2^q}+ \tag{3.13}$$

$$+\sum_{i=2^p+1}^{m}\sum_{k=2^q+1}^{n}\Big\}\frac{\sigma_{ik}^2}{m^3n^3} = O(1)\Big\{\frac{1}{2^{2p}2^{2q}}\sum_{i=1}^{2^p}\sum_{k=1}^{2^q}\sigma_{ik}^2+\frac{1}{2^{2p}}\sum_{i=1}^{2^p}\sum_{k=2^q+1}^{\infty}\frac{\sigma_{ik}^2}{k^2}+$$

$$+\frac{1}{2^{2q}}\sum_{i=2^p+1}^{\infty}\sum_{k=1}^{2^q}\frac{\sigma_{ik}^2}{i^2}+\sum_{i=2^p+1}^{\infty}\sum_{k=2^q+1}^{\infty}\frac{\sigma_{ik}^2}{i^2k^2}\Big\}.$$

Collecting (3.2) and (3.10)—(3.13) yields (2.6) to be proved.

4. Proof of Proposition 2. This proof is essentially a combination of the techniques of Section 3 and the proof of [4, Proposition 1]. Therefore, we do not go into full details.

The next lemma is a version of the well-known Rademacher—Menshov inequality (see, e.g. [1, Theorem 2]).

Lemma 2. *If* $\{X_{ik}\}$ *satisfies conditions* (1.1)—(1.3), *and* $\{a_{ik}\}$ *is any sequence of numbers, then*

$$E\Big[\max_{1\le l\le n}\Big|\sum_{i=a+1}^{a+m}\sum_{k=b+1}^{b+l} a_{ik}X_{ik}\Big|\Big]^2 = O(1)[\log 2n]^2\sum_{i=a+1}^{a+m}\sum_{k=b+1}^{b+n} a_{ik}^2\sigma_{ik}^2 \tag{4.1}$$

$$(a, b \geqq 0;\ m, n \geqq 1).$$

To start the proof of Proposition 2, assume that $2^r \leqq m \leqq 2^{r+1}$ and $2^s \leqq n \leqq 2^{s+1}$ with nonnegative integers r and s. Obviously, it is enough to prove (2.10) for the slightly modified means

$$\tau_{mn}^* = \frac{1}{m2^s}\sum_{i=1}^{m}\sum_{k=1}^{n}\Big(1-\frac{i-1}{m}\Big)X_{ik}$$

in the place of τ_{mn}. We use a decomposition analogous to (3.3), according to which we can write

$$P\Big[\max_{2^r\leq m\leq 2^{r+1}}\max_{2^s\leq n\leq 2^{s+1}}|\tau^*_{mn}|>\varepsilon\Big]\leq P\Big[|\tau^*_{2^r,2^s}|>\frac{\varepsilon}{4}\Big]+\sum_{j=1}^{3}Q^{(j)}_{rs} \tag{4.2}$$

(cf. (3.4)), where

$$Q^{(1)}_{rs}=P\Big[\max_{2^r<m\leq 2^{r+1}}|\tau^*_{m,2^s}-\tau^*_{2^r,2^s}|>\frac{\varepsilon}{4}\Big],$$

$$Q^{(2)}_{rs}=P\Big[\max_{2^s<n\leq 2^{s+1}}|\tau^*_{2^r,n}-\tau^*_{2^r,2^s}|>\frac{\varepsilon}{4}\Big],$$

$$Q^{(3)}_{rs}=P\Big[\max_{2^r<m\leq 2^{r+1}}\max_{2^s<n\leq 2^{s+1}}|\tau^*_{mn}-\tau^*_{m,2^s}-\tau^*_{2^r,n}+\tau^*_{2^r,2^s}|>\frac{\varepsilon}{4}\Big].$$

Imitating the corresponding steps in the proof of Proposition 1, it is easy to verify that

$$P\Big[|\tau^*_{2^r,2^s}|>\frac{\varepsilon}{4}\Big]=\frac{O(1)}{\varepsilon^2}\sum_{i=1}^{2^r}\sum_{k=1}^{2^s}\sigma^2_{ik} \tag{4.3}$$

and

$$Q^{(1)}_{rs}=\frac{O(1)}{\varepsilon^2}\sum_{m=2^r+1}^{2^{r+1}}\sum_{i=1}^{m}\sum_{k=1}^{2^s}\frac{\sigma^2_{ik}}{m(m-1)^2 2^{2s}} \tag{4.4}$$

(cf. (3.5) and (3.7), respectively).

The following two estimates are different from (3.8) and (3.9). By the Chebyshev inequality and (4.1),

$$Q^{(2)}_{rs}=\frac{O(1)}{\varepsilon^2}\frac{[\log 2^{s+1}]^2}{2^{2r}2^{2s}}\sum_{i=1}^{2^r}\sum_{k=2^s+1}^{2^{s+1}}\Big(1-\frac{i-1}{m}\Big)^2\sigma^2_{ik}= \tag{4.5}$$

$$=\frac{O(1)}{\varepsilon^2}\frac{1}{2^{2r}}\sum_{i=1}^{2^r}\sum_{k=2^s+1}^{2^{s+1}}\frac{\sigma^2_{ik}}{k^2}[\log 2k]^2.$$

To estimate $Q^{(3)}_{rs}$, we set $\eta_{mn}=\tau^*_{mn}-\tau^*_{m,2^s}$. Then

$$\eta_{mn}-\eta_{2^r,n}=\tau^*_{mn}-\tau^*_{m,2^s}-\tau^*_{2^r,n}+\tau^*_{2^r,2^s}.$$

Similarly to the reasoning in (3.6) we estimate as follows

$$\Big[\max_{2^r\leq m\leq 2^{r+1}}\max_{2^s\leq n\leq 2^{s+1}}|\tau^*_{mn}-\tau^*_{m,2^s}-\tau^*_{2^r,n}+\tau^*_{2^r,2^s}|\Big]^2\leq$$

$$\leq\sum_{m=2^r+1}^{2^{r+1}}m\Big[\max_{2^s<n\leq 2^{s+1}}|\eta_{mn}-\eta_{m-1,n}|\Big]^2.$$

A simple computation shows that

$$\eta_{mn}-\eta_{m-1,n}=\sum_{i=1}^{m}\sum_{k=2^s+1}^{n}c_{ik}(m,n)X_{ik},$$

where

$$c_{ik}(m, n) = \frac{1}{2^s}\left[\frac{(i-1)(2m-1)}{m^2(m-1)^2} - \frac{1}{m(m-1)}\right].$$

Clearly,

$$|c_{ik}(m, n)| \leqq \frac{1}{m(m-1)2^s}.$$

Thus, by the Chebyshev inequality and (4.1),

$$(4.6) \qquad Q_{rs}^{(3)} = \frac{O(1)}{\varepsilon^2} \sum_{m=2^r+1}^{2^{r+1}} m[\log 2^{s+1}]^2 \sum_{i=1}^{m} \sum_{k=2^s+1}^{2^{s+1}} c_{ik}^2(m, n)\sigma_{ik}^2 =$$

$$= \frac{O(1)}{\varepsilon^2} \sum_{m=2^r+1}^{2^{r+1}} \sum_{i=1}^{m} \sum_{k=2^s+1}^{2^{s+1}} \frac{\sigma_{ik}^2}{m(m-1)^2 2^{2s}} [\log 2k]^2 = \frac{O(1)}{\varepsilon^2} \sum_{i=1}^{2^{r+1}} \sum_{k=2^s+1}^{2^{s+1}} \frac{\sigma_{ik}^2}{2^{2r} k^2} [\log 2k]^2.$$

Now to complete the proof on the basis of (4.2)—(4.6) we have to go along the same lines as in the proof of Proposition 1 (cf. Parts 1—4 there).

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UNIVERSITY OF SZEGED
BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

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General results on strong approximation by orthogonal series

L. LEINDLER and A. MEIR

1. Introduction. Let $\{\varphi_n(x)\}$ denote an orthonormal system on a finite interval (a, b). In this paper we shall consider real orthogonal series

$$(1.1) \qquad \sum_{n=0}^{\infty} c_n \varphi_n(x) \quad \text{with} \quad \sum_{n=0}^{\infty} c_n^2 < \infty.$$

It is well known that the partial sums $s_n(x)$ of any such series converge in the L^2 norm to a function $f(x) \in L^2(a, b)$.

The following theorem, proved in [6], provides a quantitative estimate for the pontwise approximation of $f(x)$ by the arithmetic means of $s_n(x)$:

Let $0<\gamma<1$. *If*

$$(1.2) \qquad \sum_{n=0}^{\infty} c_n^2 n^{2\gamma} < \infty,$$

then

$$\frac{1}{n+1} \sum_{k=0}^{n} s_k(s) - f(x) = o_x(n^{-\gamma})$$

almost everywhere (a.e.) in (a, b).

This result was extended by G. Sunouchi [17] to strong approximation. Earlier G. Alexits, who was first to propose the problem of strong approximation, in cooperation with his coauthors established various results pertaining to Fourier series [2], [3]. As far as we know it was Sunouchi's result the first to deal with strong approximation by general orthogonal series. His result reads as follows:

Let $0<\gamma<1$ *and* $\varkappa>0$. *If* (1.2) *holds and* $0<p\gamma<1$, *then*

$$\left\{\frac{1}{A_n^{\varkappa}} \sum_{k=0}^{n} A_{n-k}^{\varkappa-1} |s_k(x) - f(x)|^p\right\}^{1/p} = o_x(n^{-\gamma})$$

a.e. in (a, b), where $A_n^{\varkappa} = \binom{n+\varkappa}{n}$.

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After several articles of the first author have dealt with strong approximation [9], [10], [11], the following two general results (the first for Cesàro means, the second for Riesz means) were established by the first author and H. SCHWINN [14]:

Theorem A. *Let* $\gamma>0$, $\varkappa>0$. *If* (1.2) *holds and* $0<p\gamma<1$, *then*

$$C_n(f,\varkappa,p,v;x):=\left\{\frac{1}{A_n^{\varkappa}}\sum_{k=0}^{n}A_{n-k}^{\varkappa-1}|s_{v_k}(x)-f(x)|^p\right\}^{1/p}=o_x(n^{-\gamma}) \tag{1.3}$$

a.e. in (a,b) *for any increasing sequence* $v:=\{v_k\}$ *of positive integers.*

Theorem B. *Let* $\gamma>0$, $\beta>0$. *If* (1.2) *holds and* $0<p\gamma<\beta$, *then*

$$R_n(f,\beta,p,v;\ x):=\left\{(n+1)^{-\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_{v_k}(x)-f(x)|^p\right\}^{1/p}=o_x(n^{-\gamma}) \tag{1.4}$$

a.e. in (a,b) *for any increasing sequence* $v:=\{v_k\}$ *of positive integers.*

L. REMPULSKA [15] investigated the approximation properties of generalized Abel means of orthogonal series. One of her results, relevant to our present interest, is as follows:

Let q *be a non-negative integer and* $\gamma>0$. *If* (1.2) *holds, then*

$$(1-t)^{q+1}\sum_{k=0}^{\infty}\binom{q+k}{k}t^k s_k(x)-f(x)=\begin{cases}o_x((1-t)^{\gamma}) & \text{if } q+1>\gamma,\\ o_x((1-t)^{\gamma}|\log(1-t)|) & \text{if } q+1=\gamma,\\ O_x((1-t)^{q+1}) & \text{if } q+1<\gamma,\end{cases}$$

a.e. in (a,b), *as* $t\to 1^-$.

This result was extended to strong Abel means, by the first author, in [8]:

Theorem C. *Let* q *be a non-negative integer and* $\gamma>0$. *If* (1.2) *holds and* $0<p\gamma<1$, *then*

$$Q(f,q,p,v;\ t):=\left\{(1-t)^{q+1}\sum_{k=0}^{\infty}\binom{q+k}{k}t^k|s_{v_k}(x)-f(x)|^p\right\}^{1/p}=o_x((1-t)^{\gamma});$$

furthermore if $p\gamma\geqq 1$ *and* $p\leqq 2$, *then*

$$Q(f,q,p,v;\ t)=\begin{cases}o_x((1-t)^{\gamma}) & \text{if } q+1>p\gamma,\\ o_x((1-t)^{\gamma}|\log(1-t)|^{1/p}) & \text{if } q+1=p\gamma,\\ O_x((1-t)^{(q+1)/p}) & \text{if } q+1<p\gamma,\end{cases}$$

hold a.e. in (a,b), *as* $t\to 1^-$, *for any increasing sequence* $v:=\{v_k\}$ *of positive integers.*

An investigation, pertaining to the Riesz means dealing with a question similar to the special case when $q+1=p\gamma$ in Theorem C, was started in [10]. These results are often referred to as "limit case" theorems, since the restrictions, concerning the parameters, $p\gamma<1$ and $p\gamma<\beta$ are replaced by $p\gamma=1$ and $p\gamma=\beta$, respectively.

Theorem D. *Let $\varkappa$ and p be positive numbers. If*

$$\sum_{n=1}^{\infty} c_n^2 n^{2/p} < \infty,$$

then

$$C_n(f, \varkappa, p, \mathbf{v};\ x) = o_x\big(n^{-1/p}(\log n)^{1/p}\big)$$

a.e. in (a, b) for any increasing sequence $\mathbf{v}:=\{v_k\}$ of positive integers.

This corresponds to the case $\gamma=1/p$.

Theorem E. *Let β and p be positive numbers. If*

$$\sum_{n=1}^{\infty} c_n^2 n^{2\beta/p} < \infty,$$

then

$$R_n(f, \beta, p, \mathbf{v};\ x) = o_x\big(n^{-\beta/p}(\log n)^{1/p}\big)$$

a.e. in (a, b) for any increasing sequence $\mathbf{v}:=\{v_k\}$ of positive integers.

This corresponds to the case $\gamma=\beta/p$.

The aim of our present paper is to extend these results of strong approximation to certain more general classes of strong summation methods. These methods will include, as we shall show, a large family of Hausdorff transformations and $[J,f]$-transformations. We hope that the forthcoming result will throw additional light on the common kernel of the previously established results.

2. The main result. Let $\boldsymbol{\alpha}:=\{\alpha_k(\omega)\}$, $k=0, 1, \ldots$ denote a sequence of non-negative functions defined for $0\leqq\omega<\infty$, satisfying

$$\sum_{k=0}^{\infty} \alpha_k(\omega) \equiv 1.$$

We shall assume that the linear transformation of real sequences $\mathbf{x}:=\{x_k\}$ given by

$$A_\omega(\mathbf{x}) := \sum_{k=0}^{\infty} \alpha_k(\omega) x_k, \quad \omega\to\infty$$

is regular [4; p. 49]. Let $\gamma:=\gamma(t)$ and $g(t)$ denote non-decreasing positive functions defined for $0\leqq t<\infty$, furthermore let $\boldsymbol{\mu}:=\{\mu_m\}$, $m=0, 1, \ldots$ denote a fixed, increasing sequence of integers with $\mu_0=0$. We shall assume throughout this paper that the following conditions are satisfied:

There exist positive integers N and h so that

$$\mu_{m+1} \leqq N \cdot \mu_m, \quad m = 1, 2, \ldots \tag{2.1}$$

$$\gamma(\mu_{m+1}) \leqq N \cdot \gamma(\mu_m), \quad m = 1, 2, \ldots \tag{2.2}$$

$$\gamma(\mu_{m+h}) \geqq 2\gamma(\mu_m), \quad m = 1, 2, \ldots . \tag{2.3}$$

For $r>1$, $\omega>0$ and $m=0, 1, \ldots$ we define

$$\varrho_m(\omega, r) := \left\{ \frac{1}{\mu_{m+1}} \sum_{k=\mu_m}^{\overline{\mu_{m+1}-1}} (\alpha_k(\omega))^r \right\}^{1/r}. \tag{2.4}$$

In terms of the quantities introduced above we are ready to state our main result.

Theorem 1. *Let $p>0$. Suppose that there exist $r>1$ and a constant $K(r, \mu, \gamma)$ such that for any $\omega>0$*

$$\sum_{m=0}^{\infty} \mu_m \varrho_m(\omega, r) \gamma(\mu_m)^{-p} \leqq K(r, \mu, \gamma)\big(g(\omega)/\gamma(\omega)\big)^p. \tag{2.5}$$

If

$$\sum_{n=1}^{\infty} c_n^2 \gamma(n)^2 < \infty, \tag{2.6}$$

then

$$A_\omega(f, p, \nu;\ x) := \Big\{ \sum_{k=0}^{\infty} \alpha_k(\omega) \, |s_{\nu_k}(x) - f(x)|^p \Big\}^{1/p} = O_x\big(g(\omega)/\gamma(\omega)\big) \tag{2.7}$$

a.e. in (a, b) for any increasing sequence $\nu := \{\nu_k\}$ of positive integers.

If, in addition, for every fixed m,

$$\varrho_m(\omega, r) = o\big((g(\omega)/\gamma(\omega))^p\big), \quad \text{as} \quad \omega \to \infty, \tag{2.8}$$

then the O_x in (2.7) can be replaced by o_x.

We mention that the most important special case of Theorem 1 is when both (2.5) and (2.8) are satisfied with $g(\omega)\equiv 1$. In this case we get that

$$A_\omega(f, p, \nu;\ x) = o_x\big(\gamma(\omega)^{-1}\big) \tag{2.9}$$

holds a.e. in (a, b).

3. Lemmas. In order to prove Theorem 1 and for its applications we need a number of results; some were proved earlier, others will be proven here. In what follows K will denote absolute constants and $K(.)$ constants depending only on those parameters as indicated.

Lemma 1. [13]. *If $\{a_m\}$ is a sequence of non-negative numbers, then*

$$\sum_{m=1}^{n} a_m \leqq K a_n, \quad n = 1, 2, \ldots$$

hold if and only if there exist positive integers N and s so that

$$a_{m+1} \leqq N a_m \quad \text{and} \quad a_{m+s} \geqq 2a_m, \quad m = 1, 2, \ldots .$$

Lemma 2. [7]. *Let $\{\lambda_m\}$ be an increasing sequence of positive integers, let $\{\gamma_m\}$ be a non-decreasing sequence of positive numbers so that*

$$\sum_{m=1}^{n} \gamma_{\lambda_m}^2 \leqq K \gamma_{\lambda_n}^2, \quad n = 1, 2, \ldots . \tag{3.1}$$

If

$$\sum_{n=1}^{\infty} c_n^2 \gamma_n^2 < \infty, \tag{3.2}$$

then

$$s_{\lambda_n}(x) - f(x) = o_x(\gamma_{\lambda_n}^{-1}) \tag{3.3}$$

a.e. in (a, b).

Lemma 3. [10]. *Let $\delta > 0$ and $\{\delta_n\}$ an arbitrary sequence of positive numbers. Suppose that for any orthonormal system the condition*

$$\sum_{n=1}^{\infty} \delta_n \Big(\sum_{k=n}^{\infty} c_k^2 \Big)^{\delta} < \infty$$

implies that the sequence $\{s_n(x)\}$ possesses a property P, then any subsequence $\{s_{\nu_n}(x)\}$ also possesses property P.

Lemma 4. *Let $\sigma_k(x) := (k+1)^{-1} \sum_{i=0}^{k} s_i(x)$, $k = 0, 1, \ldots$. If*

$$\sum_{n=0}^{\infty} c_n^2 < \infty,$$

then

$$\sum_{n=1}^{\infty} n \int_a^b (\sigma_n(x) - \sigma_{n-1}(x))^2 \, dx \leqq K \sum_{n=0}^{\infty} c_n^2; \tag{3.4}$$

and for every $p > 0$

$$\int_a^b \Big\{ \sup_{n \geqq 0} \Big((n+1)^{-1} \sum_{k=0}^{n} |s_k(x) - \sigma_k(x)|^p \Big)^{1/p} \Big\}^2 dx \leqq K(p) \sum_{n=0}^{\infty} c_n^2. \tag{3.5}$$

Inequality (3.4) can be found in [1] and (3.5) was proved in [16].

Lemma 5. *Let $p > 0$ and $M < N$ positive integers. Let*

$$\bar{\sigma}_n(x) = \begin{cases} 0, & \text{if } n \leqq M, \\ (n+1)^{-1} \sum_{k=M+1}^{n} (s_k(x) - s_M(x)), & \text{if } M < n \leqq N, \\ (n+1)^{-1} \sum_{k=M+1}^{N} (s_k(x) - s_M(x)), & \text{if } n > N. \end{cases} \tag{3.6}$$

Then

$$\sum_{n=M+1}^{N} \int_a^b n(\bar{\sigma}_n(x)-\bar{\sigma}_{n-1}(x))^2\,dx \leqq K \sum_{n=M+1}^{N} c_n^2, \tag{3.7}$$

and

$$\int_a^b \left\{\frac{1}{N+1} \sum_{n=M+1}^{N} |s_n(x)-s_M(x)-\bar{\sigma}_n(x)|^p\right\}^{2/p} \leqq K(p) \sum_{n=M+1}^{N} c_n^2. \tag{3.8}$$

Proof. Let

$$\bar{c}_k = \begin{cases} c_k, & \text{if } M < k \leqq N, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that for the corresponding partial sums $\bar{s}_n(x)$ of (1.1) we have

$$\bar{s}_n(x) = \begin{cases} 0, & \text{if } n \leqq M, \\ s_n(x)-s_M(x), & \text{if } M < n \leqq N, \\ s_N(x)-s_M(x), & \text{if } N < n, \end{cases}$$

and therefore the $(C, 1)$-means $\bar{\sigma}_n(x)$ of $\{\bar{s}_n(x)\}$ are given by (3.6). The application of (3.4) to $\{\bar{c}_k\}$ clearly implies (3.7), the application of (3.5) to $\{\bar{c}_k\}$ implies (3.8).

Lemma 6. *Let $p>0$ and let $\sigma_n^*(x)$ be defined by*

$$\sigma_n^*(x) := \sigma_n^*(\mu;\, x) := \frac{1}{n+1} \sum_{k=\mu_m}^{n} (s_k(x)-s_{\mu_m}(x)) \tag{3.9}$$

for $\mu_m \leqq n < \mu_{m+1}$, $m=0, 1, \ldots$. If (2.6) holds, then

$$\Delta_m(x) := \left\{\frac{1}{\mu_{m+1}} \sum_{k=\mu_m}^{\mu_{m+1}-1} |s_k(x)-s_{\mu_m}(x)-\sigma_k^*(x)|^p\right\}^{1/p} = o_x(\gamma(\mu_m)^{-1}) \tag{3.10}$$

a.e. in (a, b).

Proof. We set $M=\mu_m$ and $N=\mu_{m+1}-1$ with $m=0, 1, \ldots$ successively into (3.8) and observe that for $\mu_m \leqq n < \mu_{m+1}$, $\sigma_n^*(x)$ equals $\bar{\sigma}_n(x)$ of (3.6). Multiplying by $\gamma(\mu_m)^2$ and summing over m, we get

$$\sum_{m=0}^{\infty} \int_a^b \gamma(\mu_m)^2 \Delta_m^2(x)\,dx \leqq K(p) \sum_{m=0}^{\infty} \gamma(\mu_m)^2 \sum_{k=\mu_m}^{\mu_{m+1}-1} c_k^2.$$

The sum on the right-hand side is finite on account of (2.2) and (2.6). This implies the required result.

Lemma 7. *Let $\sigma_n^*(x)$ be as defined by (3.9). If (2.6) holds, then*

$$\sigma_n^*(x) = o_x(\gamma(n)^{-1})$$

a.e. in (a, b).

Proof. Since $\sigma^*_{\mu_m}(x)=0$ for every m, we have

$$(3.11)\qquad \max_{\mu_n\leqq k<\mu_{m+1}} |\sigma^*_k(x)|^2 \leqq \max_{\mu_m\leqq k<\mu_{m+1}} \Big(\sum_{j=\mu_m+1}^{k} |\sigma^*_j(x)-\sigma^*_{j-1}(x)|\Big)^2 \leqq$$

$$\leqq \Big(\sum_{j=\mu_m+1}^{\mu_{m+1}-1} |\sigma^*_j(x)-\sigma^*_{j-1}(x)|\Big)^2 \leqq K \sum_{j=\mu_m+1}^{\mu_{m+1}-1} j\,|\sigma^*_j(x)-\sigma^*_{j-1}(x)|^2,$$

where the last inequality is the consequence of the Schwarz inequality and (2.1). To the last expression we may apply (3.7) with $M=\mu_m$ and $N=\mu_{m+1}-1$, since in the required range $\sigma^*_j(x)=\bar\sigma_j(x)$. Thus we obtain from (3.11)

$$(3.12)\qquad \int_a^b \max_{\mu_m\leqq k<\mu_{m+1}} |\sigma^*_k(x)|^2\,dx \leqq K \sum_{j=\mu_m+1}^{\mu_{m+1}-1} c_j^2.$$

It follows now from (3.12) that

$$\sum_{m=0}^{\infty} \gamma(\mu_m)^2 \int_a^b \max_{\mu_m\leqq k<\mu_{m+1}} |\sigma^*_k(x)|^2\,dx \leqq K \sum_{m=0}^{\infty} \gamma(\mu_m)^2 \sum_{j=\mu_m+1}^{\mu_{m+1}-1} c_j^2 \leqq K \sum_{k=0}^{\infty} \gamma(k)^2 c_k^2 < \infty,$$

on account of (2.2) and (2.6). The last inequality implies the required result using (2.2) once again.

Lemma 8. [4]. *Let $\{\alpha_k(n)\}$, the coefficients of a regular Hausdorff transformation, be given by*

$$(3.13)\qquad \alpha_k(n) = \int_0^1 \binom{n}{k} t^k(1-t)^{n-k}\,\phi(t)\,dt,$$

where $\phi(t)\in L^r(0,1)$ for some $r>1$. Then

$$(3.14)\qquad \sum_{k=0}^{n} |\alpha_k(n)|^r \leqq K(r)(n+1)^{1-r}.$$

Lemma 9. *Let $\{\alpha_k(\omega)\}$, the coefficients of a regular $[J,f]$-transformation, be given by*

$$(3.15)\qquad \alpha_k(\omega) = \frac{\omega^k}{k!} \int_0^1 t^{\omega}(\log(1/t))^k\,\phi(t)\,dt,$$

where $\phi(t)\in L^r(0,1)$ for some $r>1$. Then for $l=0,1,\dots$

$$(3.16)\qquad \sum_{k=l}^{\infty} |\alpha_k(\omega)|^r \leqq K(r)\big((1+\omega)^{-1}e^{-l/1+\omega}\big)^{r-1}.$$

Proof. Denote $\lambda_k(\omega, t)=(k!)^{-1}(\log(1/t))^k t^\omega$ and let $r^{-1}+s^{-1}=1$. By Hölder's inequality we get from (3.15)

$$(3.17)\qquad |\alpha_k(\omega)|^r \leqq \Big(\int_0^1 \lambda_k(\omega, t)\,dt\Big)^{r-1}\cdot \int_0^1 \lambda_k(\omega, t)\,|\phi(t)|^r\,dt.$$

Now, we find by an easy calculation that for $k=0, 1, \ldots$

$$(3.18)\qquad \int_0^1 \lambda_k(\omega, t)\,dt = \omega^k(1+\omega)^{-k-1} \leqq (1+\omega)^{-1}e^{-k/1+\omega}$$

and that

$$\sum_{k=0}^{\infty} \lambda_k(\omega, t) \equiv 1.$$

Inequality (3.16) is therefore a consequence of (3.17) and (3.18).

4. Proof of Theorem 1. First we carry out the proof when $\nu_k=k$. Using elementary considerations we see that

$$(4.1)\qquad \Big\{\sum_{k=0}^{\infty}\alpha_k(\omega)\,|s_k(x)-f(x)|^p\Big\}^{1/p} \leqq K(p)\big(\textstyle\sum_1+\sum_2+\sum_3\big),$$

where

$$\textstyle\sum_1 = \Big\{\displaystyle\sum_{m=0}^{\infty}\sum_{k=\mu_m}^{\mu_{m+1}-1} \alpha_k(\omega)\,|s_k(x)-s_{\mu_m}(x)-\sigma_k^*(x)|^p\Big\}^{1/p},$$

$$(4.2)\qquad \textstyle\sum_2 = \Big\{\displaystyle\sum_{m=0}^{\infty}\sum_{k=\mu_m}^{\mu_{m+1}-1} \alpha_k(\omega)\,|s_{\mu_m}(x)-f(x)|^p\Big\}^{1/p},$$

$$\textstyle\sum_3 = \Big\{\displaystyle\sum_{m=0}^{\infty}\sum_{k=\mu_m}^{\mu_{m+1}-1} \alpha_k(\omega)\,|\sigma_k^*(x)|^p\Big\}^{1/p}.$$

Let $r^{-1}+s^{-1}=1$. By Hölder's inequality, using (2.2) and (3.10) with ps in place of p, we get

$$(4.3)\qquad \textstyle\sum_1^p \leqq \displaystyle\sum_{m=0}^{\infty}\Big\{\sum_{k=\mu_m}^{\mu_{m+1}-1}\alpha_k(\omega)^r\Big\}^{1/r}\cdot\Big\{\sum_{k=\mu_m}^{\mu_{m+1}-1}|s_k(x)-s_{\mu_m}(x)-\sigma_k^*(x)|^{ps}\Big\}^{1/s} \leqq$$

$$\leqq K\sum_{m=0}^{\infty}\mu_m\varrho_m(\omega, r)\cdot\Big\{\frac{1}{\mu_{m+1}}\sum_{k=\mu_m}^{\mu_{m+1}-1}|s_k(x)-s_{\mu_m}(x)-\sigma_k^*(x)|^{ps}\Big\}^{1/s} \leqq$$

$$\leqq K\sum_{m=0}^{\infty}\mu_m\varrho_m(\omega, r)\cdot o_x\big(\gamma(\mu_m)^{-p}\big),$$

a.e. in (a, b).

In order to estimate $\sum_2$ we use (3.3) with $\lambda_n=\mu_n$ and $\gamma_n:=\gamma(n)$, observing that (3.1) is satisfied due to our assumptions (2.2) and (2.3) and Lemma 1. By Hölder's inequality

$$\Sigma_2^p \leqq \sum_{m=0}^{\infty} \Big\{ \sum_{k=\mu_m}^{\mu_{m+1}-1} \alpha_k(\omega)^r \Big\}^{1/r} \cdot \Big\{ \sum_{k=\mu_m}^{\mu_{m+1}-1} |s_{\mu_m}(x)-f(x)|^{ps} \Big\}^{1/s} \leqq \tag{4.4}$$

$$\leqq K \sum_{m=0}^{\infty} \mu_m \varrho_m(\omega, r) o_x\big(\gamma(\mu_m)^{-p}\big),$$

taking (2.1) and (2.2) into account.

For estimating Σ_3 we use Lemma 7. By Hölder's inequality

(4.5)

$$\Sigma_3^p \leqq \sum_{m=0}^{\infty} \mu_m \varrho_m(\omega, r) \cdot \Big\{ \frac{1}{\mu_m} \sum_{k=\mu_m}^{\mu_{m+1}-1} |\sigma_k^*(x)|^{ps} \Big\}^{1/s} \leqq K \sum_{m=0}^{\infty} \mu_m \varrho_m(\omega, r) o_x\big(\gamma(\mu_m)^{-p}\big).$$

Collecting these estimations and taking account of assumption (2.5) we immediately get the required result (2.7) when $v_k = k$.

If (2.8) is also satisfied, then the proof runs as follows. By (4.1)—(4.5) we have, when $v_k = k$, that

$$A_\omega(f, p, v;\ x)^p \leqq K \sum_{m=0}^{\infty} \mu_m \varrho_m(\omega, r) o_x\big(\gamma(\mu_m)^{-p}\big) \tag{4.6}$$

holds a.e. in (a, b).

Let now $\varepsilon > 0$ be given. If x is a point where (4.6) holds, then let $M(x)$ be a positive integer such that for $m > M(x)$ the inequality $o_x\big(\gamma(\mu_m)^{-p}\big) < \varepsilon^p \gamma(\mu_m)^{-p}$ is valid. For such x we get from (4.6) that

$$\big(\gamma(\omega)/g(\omega)\big)^p A_\omega(f, p, v;\ x)^p \leqq K(x) \Big\{ \sum_{m=0}^{M(x)} \mu_m \varrho_m(\omega, r) \gamma(\mu_m)^{-p} \Big\} \big(\gamma(\omega)/g(\omega)\big)^p +$$

$$+ K\varepsilon^p \big(\gamma(\omega)/g(\omega)\big)^p \sum_{m=M(x)+1}^{\infty} \mu_m \varrho_m(\omega, r) \gamma(\mu_m)^{-p}.$$

When $\omega \to \infty$, the first sum on the right converges to zero by (2.8); and the second sum remains $O\big((g(\omega)/\gamma(\omega))^p\big)$, by (2.5).

Hence, for $v_k = k$,

$$A_\omega(f, p, v;\ x) = o_x\big(g(\omega)/\gamma(\omega)\big), \quad \text{as} \quad \omega \to \infty \tag{4.7}$$

clearly follows. Since (4.6) holds a.e. in (a, b), it follows that (4.7) also holds a.e. in (a, b). This completes the proof when $v_k = k$.

The statements of Theorem 1 in their generality — for arbitrary $v := \{v_k\}$ — follow from the results just proved and (2.6) using Lemma 3 with $\delta = 1$ and $\delta_n := \gamma(n)^2 - \gamma(n-1)^2$.

5. Applications. First we treat those results which can be derived from Theorem 1 in the special case when $g(\omega) \equiv 1$ and both (2.5) and (2.8) are satisfied.

(i) If

$$p_{nk}(t) = \binom{n}{k} t^k (1-t)^{n-k}, \quad k = 0, 1, \ldots, n; \; n = 1, 2, \ldots \tag{5.1}$$

and $\phi(t) \in L^1(0,1)$ is a non-negative function with $\|\phi\|_1 = 1$, then the matrix $\{\alpha_k(n)\}$ defined by

$$\alpha_k(n) = \int_0^1 p_{nk}(t)\phi(t)\,dt, \quad k = 0, 1, \ldots, n; \; n = 1, 2, \ldots \tag{5.2}$$

yields the coefficients of a regular Hausdorff transformation. For these transformations we have the following result.

Theorem 2. *Let $\gamma > 0$. Suppose that $\{\alpha_k(n)\}$ is given by* (5.2), *where $\phi(t) \in L^r(0,1)$ with some $r > 1$. If* (1.2) *holds and*

$$0 < p\gamma < 1 - r^{-1}, \tag{5.3}$$

then

$$\Big\{\sum_{k=0}^{n} \alpha_k(n) |s_{\nu_k}(x) - f(x)|^p\Big\}^{1/p} = o_x(n^{-\gamma}) \tag{5.4}$$

a.e. in (a, b) for any increasing sequence $\{\nu_k\}$ of positive integers.

Corollary 2.1. *If $\{\alpha_k(n)\}$ is the matrix of a Cesàro $(C, \varkappa)$ or a Hölder $(H, \varkappa)$ transformation, then* (5.4) *holds whenever $0 < p\gamma < \min(1, \varkappa)$.*

Remark. Although Theorem 2 does not include Theorem A for arbitrary $\varkappa > 0$, if we take into acoount the special properties of the $(C, \varkappa)$ transformation matrix, we find easily that Theorem 1 is applicable. For, in this case, (2.5) with $g(t) \equiv 1$ and $\gamma(t) = t^\gamma$ will be satisfied if we choose $r(>1)$ so that $\varkappa > 1 - r^{-1}$.

Proof of Theorem 2. We wish to show that conditions (2.5) and (2.8) of Theorem 1 are satisfied if $[\omega] = n$, $\gamma(t) = t^\gamma$ and $g(\omega) \equiv 1$. From (3.14) we get

$$\varrho_m(\omega, r)\gamma(\omega)^p \leqq K(r)\mu_m^{-1/r}\omega^{(1/r)-1+p\gamma},$$

whence (2.8) follows by (5.3). Now we observe that in this case $\varrho_m(\omega, r) = 0$ if $\mu_m > \omega$. Hence, again from (3.14),

$$\sum_{m=0}^{\infty} \mu_m \varrho_m(\omega, r)\gamma(\mu_m)^{-p} \leqq K\omega^{(1/r)-1} \sum \mu_m^{1-(1/r)-p\gamma},$$

where the summation on the right is for $\mu_m \leqq \omega$. Because of the assumptions made on the sequence $\{\mu_m\}$, this last sum is $O(\omega^{1-(1/r)-p\gamma})$. This proves (2.5). The conclusion of Theorem 2 now follows from Theorem 1.

Proof of Corollary 2.1. Both the $(C, \varkappa)$ and $(H, \varkappa)$ transforms are Hausdorff transforms with $\phi_1(t)=\varkappa(1-t)^{\varkappa-1}$ and $\phi_2(t)=\Gamma(\varkappa)^{-1}(\log 1/t)^{\varkappa-1}$, respectively. If $\varkappa \geqq 1$, then $\phi_i(t) \in L^r(0, 1)$, for arbitrary large r, hence (5.3) will hold whenever $0<p\gamma<1$ and r is large enough. If $0<\varkappa<1$, then $\phi_i(t) \in L^r(0, 1)$ if $r^{-1}>1-\varkappa$, hence in this case (5.3) holds whenever $0<p\gamma<1-\frac{1}{r}<\varkappa$.

(ii) If

$$\lambda_k(\omega, t) = \frac{(\omega \log(1/t))^k}{k!} t^\omega, \quad k = 0, 1, \ldots \tag{5.5}$$

and $\phi(t) \in L^1(0, 1)$ is a non-negative function with $\|\phi\|_1=1$, then the function-sequence $\{\alpha_k(\omega)\}$ defined by

$$\alpha_k(\omega) = \int_0^1 \lambda_k(\omega, t)\phi(t)\, dt, \quad k = 0, 1, \ldots \tag{5.6}$$

yields the coefficients of a regular $[J, f]$-transformation. For this transformation we have the following result.

Theorem 3. *Let $\gamma>0$. Suppose that $\{\alpha_k(\omega)\}$ is given by* (5.6), *where $\phi(t) \in L^r(0, 1)$ with some $r>1$. If* (1.2) *holds and*

$$0 < p\gamma < 1-r^{-1}, \tag{5.7}$$

then

$$\Big\{\sum_{k=0}^{\infty} \alpha_k(\omega)|s_{\nu_k}(x)-f(x)|^p\Big\}^{1/p} = o_x(\omega^{-\gamma}) \tag{5.8}$$

a.e. in (a, b) for any increasing sequence $\{\nu_k\}$ of positive integers.

Corollary 3.1. *If $\{\alpha_k(\omega)\}$ is the coefficient-sequence of the Abel transformation, then* (5.8) *holds whenever $0<p\gamma<1$.*

Proof of Theorem 3. We shall show that the conditions (2.5) and (2.8) of Theorem 1 are satisfied in this case with $\gamma(t)=t^\gamma$ and $g(\omega)\equiv 1$. From (3.16) we obtain

$$\varrho_m(\omega, r)\gamma(\omega)^p \leqq K(r)\mu_m^{-1/r}\omega^{(1/r)-1+p\gamma},$$

whence (2.8) follows by (5.7). Also from (3.16)

$$\sum_{\mu_m \leqq \omega} \mu_m \varrho_m(\omega, r)\mu_m^{-p\gamma} \leqq \frac{K(r)}{(1+\omega)^{1-1/r}} \cdot \sum_{\mu_m \leqq \omega} \mu_m^{1-(1/r)-p\gamma} \leqq K(r)\omega^{-p\gamma}, \tag{5.9}$$

due to the assumptions concerning $\{\mu_m\}$. Finally, again from (3.16),

$$\sum_{\mu_m>\omega}\mu_m\varrho_m(\omega,r)\mu_m^{-p\gamma}\leqq K(r)\cdot\sum_{\mu_m>\omega}\left(\frac{\mu_m}{1+\omega}\right)^{1-1/r}e^{-\frac{\mu_m}{1+\omega}\left(1-\frac{1}{r}\right)}\mu_m^{-p\gamma}\leqq \tag{5.10}$$

$$\leqq K(r)\sum_{\mu_m>\omega}\mu_m^{-p\gamma}\leqq K(r)\omega^{-p\gamma},$$

due to the fact that $xe^{-x}<1$ for $x>0$ and the properties of $\{\mu_m\}$.

Inequalities (5.9) and (5.10) prove (2.5), hence Theorem 3 is a consequence of Theorem 1.

Proof of Corollary 3.1. If $\phi(t)\equiv 1$, then $\alpha_k(\omega)=\omega^k/(1+\omega)^{k+1}$ for $k=0,1,\dots$, which yield the classical Abel transformation. In this case, clearly, $\phi(t)\in L^r(0,1)$ for any $r>0$, hence the result follows from Theorem 3.

(iii) If the function $\phi(t)$ in (5.2) satisfies

$$0\leqq\phi(t)\leqq K(\beta)t^{\beta-1},$$

with $\beta>0$, then it is easy to see that

$$\alpha_k(n)\leqq K(\beta)\frac{(k+1)^{\beta-1}}{(n+1)^{\beta}} \tag{5.11}$$

for $0\leqq k\leqq n$, $n=1,2,\dots$. Using (5.11) one can establish by easy estimations that in these cases (2.5) and (2.8) hold whenever $\gamma(t)=t^\gamma$, $g(t)\equiv 1$ and $0<p\gamma<\beta$. For example if $\phi(t)=\beta t^{\beta-1}$, then

$$\alpha_k(n)=\beta\frac{n!}{\Gamma(n+\beta+1)}\frac{\Gamma(k+\beta)}{k!},\quad k=0,1,\dots,n,$$

which yield, essentially, the Riesz transformation of order β. Hence Theorem B follows from Theorem 1.

(iv) If the function $\phi(t)$ in (5.6) satisfies

$$0\leqq\phi(t)\leqq K(q)\left(\log\frac{1}{t}\right)^q$$

with $q\geqq 0$, then easy calculations yield that

$$\alpha_k(\omega)\leqq K(q)\frac{(k+1)^q}{(\omega+1)^{q+1}}\left(\frac{\omega}{\omega+1}\right)^k \tag{5.12}$$

for $k=0,1,\dots$. Using (5.12) it is not difficult to show that in these cases (2.5) and (2.8) hold whenever $0<p\gamma<q+1$. For example, if $\phi(t)=\dfrac{1}{\Gamma(q+1)}\left(\log\dfrac{1}{t}\right)^q$,

$q \geqq 0$, then

$$\alpha_k(\omega) = (\omega+1)^{-q-1}\binom{k+q}{k}\cdot\left(\frac{\omega}{\omega+1}\right)^k, \quad k = 0, 1, \dots$$

which yields the generalized Abel transforms of order $q+1$, $q \geqq 0$. Hence the first statement of Theorem C with the relaxed condition $0 < p\gamma < q+1$ follows from Theorem 1 for all $q \geqq 0$.

It seems worthwhile mentioning that Theorem 1 with suitable choices of $\gamma(t)$ and $\{\mu_m\}$ can also be applied to strong approximation by certain Nörlund and Riesz means having the form

$$N_n^\lambda(f, p;\, x) := \left\{\frac{1}{\lambda(n)}\sum_{k=0}^{n-1}(\lambda(n-k)-\lambda(n-k-1))\,|s_k(x)-f(x)|^p\right\}^{1/p}$$

and

$$R_n^\lambda(f, p;\, x) := \left\{\frac{1}{\lambda(n)}\sum_{k=0}^{n-1}(\lambda(k+1)-\lambda(k))\,|s_k(x)-f(x)|^p\right\}^{1/p}$$

where $\lambda = \{\lambda(n)\}$ denotes an increasing unbounded sequence of positive numbers satisfying

$$\lambda(n) \geqq cn^\varepsilon \quad \text{or} \quad \lambda(n)-\lambda(n-1) \leqq \lambda(n)n^{-\varepsilon}$$

with $c > 0$ and $\varepsilon > 0$, respectively.

Furthermore, the function $\gamma(t)$ chosen as t^γ in Theorems 2 and 3 could be replaced by functions of the form $\gamma(t) = t^\gamma(\log t)^\beta$.

Next, without proof, we mention some further applications of Theorem 1 with $g(\omega) := (\log(1+\omega))^{1/p}$. The proofs would run as in the previous cases. These special cases of Theorem 1 include certain parts of the so-called limit-case theorems. For example Theorems D and E, moreover the second part of Theorem C quoted in this paper, belong to these cases.

(v) Let $\{\alpha_k(n)\}$ denote the coefficient matrix of a regular Hausdorff transformation with $\phi(t) \in L^r(0, 1)$ for some $r > 1$.

Theorem 2*. *If* (1.2) *holds and* $p\gamma = 1 - r^{-1}$, *then*

$$\left\{\sum_{k=0}^{n}\alpha_k(n)|s_{\nu_k}(x)-f(x)|^p\right\}^{1/p} = o_x((\log n)^{1/p}n^{-\gamma})$$

a.e. in (a, b) *for any increasing sequence* $\{\nu_k\}$ *of positive integers.*

This result $(r = \infty)$ includes the special case $\varkappa = 1$ of Theorem D. Similarly it includes the special case $\beta \leqq 1$ of Theorem E.

(vi) Let $\{\alpha_k(\omega)\}$ denote the function-sequence of coefficients of a regular $[J, f]$-transformation with $\phi(t) \in L^r(0, 1)$ for some $r > 1$.

Theorem 3*. *If* (1.2) *holds and* $p\gamma=1-r^{-1}$, *then*

$$\Big\{\sum_{k=0}^{\infty}\alpha_k(\omega)\,|s_{\nu_k}(x)-f(x)|^p\Big\}^{1/p}=o_x((\log(1+\omega))^{1/p}\omega^{-\gamma})$$

a.e. in (a, b) *for any increasing sequence* $\{\nu_k\}$ *of positive integers.*

Theorems 2* and 3*, because of their generality, do not yield the limit-cases included in Theorems C and E. However, if we take into account the special properties of the coefficients of the Riesz and the generalized Abel summation methods, as appear under (5.11) and (5.12), then our main result, Theorem 1, yields the results for the above mentioned cases as well.

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(L.L.)
BOLYAI INSTITUTE
JÓZSEF ATTILA UNIVERSITY
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

(A.M.)
DEPT. OF MATHEMATICS
UNIVERSITY OF ALBERTA
EDMONTON, CANADA
T6G 2G1

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On the central limit theorem for series with respect to periodical multiplicative systems. I

S. V. LEVIZOV*)

Introduction. It is well known that many important properties of independent random variables are transfered on broad classes of various orthonormal systems. The questions concerning the statistical properties of lacunary subsystems of orthonormal systems have been studied by many authors. For the trigonometric systems the first result in this direction is due to SALEM and ZYGMUND.

Theorem ([13]). *Let* $S_N(t)=\sum_{k=1}^{N} a_k \cos 2\pi n_k(t+\alpha_k)$, *where* $\{n_k\}$ *is an infinite sequence of positive integers satisfying the condition* $\frac{n_{k+1}}{n_k} \geqq \lambda$ *for certain* $\lambda>1$ *(so-called Hadamard's lacunarity); furthermore let* $\{a_k\}$ *be a sequence of real numbers such that*

$$A_N = \{\sum_{k=1}^{N} a_k^2\}^{1/2} \to \infty, \quad a_N = o(A_N) \quad as \quad N \to \infty,$$

and $\{\alpha_k\}$ *be an arbitrary sequence of real numbers. Then for any set* $E\subset[0, 1]$ *of positive measure and for any* $x\in\mathbf{R}$ *we have*

$$\lim_{N\to\infty} \frac{1}{|E|} |\{t\colon t\in E, S_N(t) \leqq x\cdot A_N\}| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{z^2}{2}\right) dz,$$

where $|E|$ *denotes the Lebesgue measure of* E.

This result is called central limit theorem (abbrev. CLT for lacunary trigonometric series and it has been generalized by many ways ([1], [15]—[16]).

For Walsh—Paley's system $\{W_n(x)\}$ the first analogous result was achieved in [12] and afterwards it was extended in [2]—[3], [7].

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Theorem ([3]). *Let us assume that a sequence* $\{n_k\}$ *satisfies the conditions*

(1) $$\frac{n_{k+1}}{n_k} \geqq 1+\frac{c}{k^\alpha}, \quad c>0, \quad 0 \leqq \alpha \leqq \frac{1}{2}, \quad k=1,2,\dots;$$

and $\{a_k\}$ *has the properties*

(2) $$A_N = \{\sum_{k=1}^{N} a_k^2\}^{1/2} \to \infty, \quad a_N = o(A_N \cdot N^{-\alpha}).$$

Then for any $x \in \mathbf{R}$ *we have*

(3) $$\lim_{N\to\infty} \left|\{t\colon\ t\in[0,1],\ \sum_{k=1}^{N} a_k W_{n_k}(t) \leqq x\cdot A_N\}\right| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{z^2}{2}\right) dz.$$

In [7] it was remarked that under hypothesis (1) the second condition of (2) is necessary for the validity of (3).

The purpose of the present work is to study the CLT for weakly lacunary series with respect to the generalized Walsh's functions, i.e. for so-called periodical multiplicative orthonormal systems (abbrev. PMONS).

We recall the definition of PMONS following the survey paper [6].

A sequence of functions $X=\{\chi_k(x)\}_{k=0}^{\infty}$ is called multiplicative system if the following conditions are fulfilled:

a) if $\chi_k(x)$, $\chi_l(x)\in X$ then the product $\chi_k(x)\cdot\chi_l(x)=\chi(k,l,x)$ also belongs to X;

b) if $\chi_k(x)\in X$ then $\{\chi_k(x)\}^{-1}$ belongs to X, too.

The system X is called periodical if for every $n=0,1,\dots$ there exists an integer k_n such that $\{\chi_n(x)\}^{k_n}\equiv 1$.

We shall define a periodical, multiplicative and orthonormal system X which will be considered later on the interval $[0,1]$. This system can be numerated in the following way (see [6]): there exist integers

$$0=m_{-1}<1=m_0<m_1<m_2\dots$$

and functions $\chi_0(x)\equiv 1$, $\chi_{m_0}(x)$, $\chi_{m_1}(x)$, ... such that the quotients $\frac{m_{n+1}}{m_n}=p_{n+1}$ are prime numbers *) and every functions $\chi_k(x)$ of the system X has the representation

$$\chi_k(x) = \prod_{j=0}^{n} \{\chi_{m_j}(x)\}^{\alpha_j},$$

*) We remark that p_n has not to be a prime number necessarily.

provided that k is expressed in the from

$$k = \sum_{j=0}^{n} \alpha_j \cdot m_j, \quad \text{where} \quad 0 \leqq \alpha_j < p_{j+1}, \quad k > 0.$$

The choice of $\{\chi_{m_n}(x)\}$ may be also ambiguous, but we suppose that it is made by certain determined manner.

We shall study the properties of the series having the form $\sum_{k=1}^{\infty} a_k \chi_{n_k}(x)$, where $\{n_k\}$ is a sequence of positive integers such that

$$\frac{n_{k+1}}{n_k} \geqq 1+\omega(k) \quad \text{for} \quad k = 1, 2, \ldots, \tag{4}$$

and $\{\omega(k)\}$ is a non-negative, non-increasing sequence such that

$$k^{\alpha} \cdot \omega(k) \uparrow \infty \quad \text{for some} \quad \alpha, 0 < \alpha < 1. \tag{5}$$

Finally we assume that the sequence of the coefficients $\{a_k\}$ satisfies the condition

$$A_N = \Big\{\sum_{k=1}^{N} a_k^2\Big\}^{1/2} \to \infty. \tag{6}$$

We shall consider the following sum

$$T_N(x) := \frac{1}{A_N} \sum_{k=1}^{N} a_k \chi_{n_k}(x). \tag{7}$$

Further on the sequence of the complex-valued functions $T_N(x)$ will be understood as a sequence of two-dimensional random vectors. These vectors are defined on the probability space $(\Omega, \mathscr{F}, P)$, where Ω is the square $[0, 1]\times[0, 1]$, $\mathscr{F}$ is the σ-field of all Borel-measurable sets on Ω and P is the Lebesgue measure on $\mathscr{F}$. The components of the vector $T_N(x)$ are the real part and the imaginary part of the function $T_N(x)$. If it will be necessary, we shall represent the vector $T_N(x)$ in the form

$$T_N(x) = \big(\xi_N^1(x), \xi_N^2(x)\big),$$

where

$$\xi_N^1(x) = \operatorname{Re}\{T_N(x)\}, \quad \xi_N^2(x) = \operatorname{Im}\{T_N(x)\}.$$

In the case of the trigonometric system (or the Walsh's system) the CLT was proved by a direct proof showing the convergence of the sequence $\{T_N(x)\}$ to the normal distributions. But in our case the corresponding distributions have two-dimensional character and it requires a special approach.

We shall require some informations from the theory of probability. The terminology and the facts are taken from [14].

Definition 1. A random vector $\xi=(\xi_1, \xi_2, \ldots, \xi_n)$ is called *normally distributed (Gaussian)* if its characteristic function $\varphi_\xi(t)$ has the form

$$\varphi_\xi(t) = \exp\left\{i\cdot\langle t, m\rangle - \frac{1}{2}\langle \mathbf{R}t, t\rangle\right\},$$

where $m=(m_1, m_2, \ldots, m_k)$, $|m_k|<\infty$; $\mathbf{R}=\|r_{kl}\|$ is a symmetrical, positive semidefinite matrix, the dimension of which is equal to $n\times n$; $\langle .,.\rangle$ denotes a scalar product. For brevity we shall use the notation $\xi\sim\mathcal{N}(m, \mathbf{R})$.

In this connection m is a vector of mean value, i.e.

$$m_k = M\xi_k \quad \text{for} \quad k = 1, 2, \ldots, n;$$

and $\mathbf{R}$ is a covariance matrix, i.e.

$$r_{kl} = M\{(\xi_k - M\xi_k)\cdot(\xi_l - M\xi_l)\} = \operatorname{cov}(\xi_k, \xi_l); \quad k, l = 1, 2, \ldots, n.$$

Here the symbol $M\xi$ denotes the mathematical expectation of random variable ξ and r_{kl} are the elements of $\mathbf{R}$.

Definition 2. If there exists a two-dimensional Gaussian vector $T(x)= =(\xi^1(x), \xi^2(x))$ such that the sequence of random vectors $T_N(x)$ weakly converges to $T(x)$ as $N\to\infty$ (in distribution) then the subsystem $\{a_k\chi_{n_k}(x)\}$ is called *the subject to CLT*. We denote these facts as follows:

$$T_N(x)\xrightarrow{d} T(x) \quad \text{and} \quad \{a_k\chi_{n_k}(x)\}\subset CLT,$$

where the symbol $\xrightarrow{d}$ means the weak convergence.

In other words, there exist a vector $m=(m_1, m_2)$ and a covariance matrix $\mathbf{R}=\|r_{kl}\|$; $k, l=1, 2$; such that

$$T_N(x)\xrightarrow{d}\mathcal{N}(m, \mathbf{R}) \quad \text{as} \quad N\to\infty.$$

1. The main theorem. Let the PMONS $X=\{\chi_n(x)\}_{n=0}^{\infty}$ be defined by means of the sequence $\{p_n\}$. As earlier, we assume that $m_0=1$, $m_{n+1}=m_n\cdot p_{n+1}$; $n=1, 2, \ldots$. The functions $\chi_{m_n}(x)$ are used as "basis" elements in the system X. The set of the functions $\chi_k(x)$ having the index-number from m_n to $m_{n+1}-1$ (inclusively) will be called the "n-th block of X" and denoted by $[m_n, m_{n+1})$. Also let us define the operations of addition and subtraction on the group of non-negative integers according to the following rules:

$$m = k\dot{+}l, \quad \text{if} \quad \chi_m(x) = \chi_k(x)\cdot\chi_l(x);$$

$$m = k\dot{-}l, \quad \text{if} \quad \chi_m(x) = \chi_k(x)\cdot\overline{\chi_l(x)},$$

where $\overline{\chi_l(x)}=\{\chi_l(x)\}^{-1}$ denotes the complex conjugate function of $\chi_l(x)$.

To formulate the further results we shall introduce some additional concepts. Let $\chi_k(x)\in X$. The number s is called conjugate to the number k, if $s+k=0$ (i.e. $\chi_s(x)=\overline{\chi_k(x)}$). The coefficients at the conjugate functions $\chi_{n_k}(x)$ and $\overline{\chi_{n_k}(x)}$ (if such pair there will be in our subsystem $\{\chi_{n_k}(x)\}$) will be denote by a_k and $\hat{a}_k$, respectively.

Furthermore, let the numbers q, r be given such that $m_n\leqq q,\ r<m_{n+1}$ for some n. Suppose that $q+r\neq 0$ and let $l=\min\{i\colon 0<q+r<m_{i+1}\}$ (l can be equal to $0, 1, \ldots, n$). In this case we shall call the numbers q and r (l, n)-adjoint.

If a sequence $\{n_k\}$ is given, then, in general, there exist both conjugate and (l, n)-adjoint numbers in $\{n_k\}$. The quantity of the conjugate pairs (n_q, n_r), where $m_n\leqq q,\ r<m_{n+1}$ will be denoted by λ_n (in addition, we suppose that the pairs (n_q, n_r) and (n_r, n_q) are distinct if $q\neq r$). The value $\lambda_n^l(q)$ will be defined as quantity of the numbers n_r being (l, n)-adjoint with n_q for a fixed q.

Finally, for given sequences $\{n_k\}$ and $\{a_k\}$ we put

$$f(0) = 0,\quad f(k) = \max\{i\colon n_i < m_k\},\quad k = 1, 2, \ldots$$

$$\Delta_k(x) = \sum_{i=f(k)+1}^{f(k+1)} a_i\chi_{n_i}(x);\quad B_k = A_{f(k+1)};\quad k = 0, 1, \ldots \tag{1.1}$$

$$b_k = \max\{|a_j|\colon f(k)+1 \leqq j < f(k+1)\};\quad \delta_k = f(k+1)-f(k).$$

Remark 1.1. If for some k $f(k)=f(k+1)$, then we assume that $\Delta_k(x)\equiv 0$; $B_k=B_{k-1}$; $\delta_k=b_k=0$.

Now we can formulate the main statement of our work.

Theorem A. *Suppose that for a given system X the corresponding sequence $\{p_n\}$ is bounded. We also assume that the sequences $\{n_k\}$, $\{\omega(k)\}$ and $\{a_k\}$ satisfy conditions* (4)—(6), *respectively. Additionally if*

a)

$$a_k = o(A_k\omega(k)); \tag{1.2}$$

b) *there exists a real number η, $0\leqq\eta\leqq 1$ such that*

$$\lim_{N\to\infty}\frac{1}{B_N^2}\sum_{j}^{f(N+1)}(a_j\cdot\hat{a}_j) = \eta, \tag{1.3}$$

where the summation is taken for all conjugate numbers being not greater than $f(N+1)$;

c) *there exists a constant $C>1$ (independent of q, j) such that for any fixed q and for any j, $0\leqq j\leqq k-1$*

$$\lambda_k^j(q)\cdot\omega(f(k)) = O(C^{j-k})\quad \text{as}\quad k\to\infty \tag{1.4}$$

holds, then the subsystem $\{a_k\chi_{n_k}(x)\}$ is the subject to CLT.

It is easy to see that Theorem A will be proved if we can show the existence of a vector $m=(m_1, m_2)$ and a matrix $\mathbf{R}=\|r_{kl}\|$; $k, l=1, 2$; such that

$$T_N(x) \xrightarrow{d} \mathcal{N}(m, \mathbf{R}) \quad \text{as} \quad N \to \infty. \tag{1.5}$$

2. Lemmas. First we shall recall some further facts of the probability.

Definition 3 ([14], pp. 467—474). Let $\{l_n\}$ be a certain sequence of indices and $\{X_{n,i};\ n=0, 1, \ldots;\ 0 \leqq i \leqq l_n\}$ be an array of random variables on the probability space $(\Omega, \mathcal{F}, P)$. Let $\{\mathcal{F}_{n,i};\ n=0, 1, \ldots;\ 0 \leqq i \leqq l_n\}$ be any triangular array of sub σ-fields of $\mathcal{F}$ such that

$$\mathcal{F}_{n,i-1} \subset \mathcal{F}_{n,i} \quad \text{for all} \quad n = 0, 1, \ldots;\ 1 \leqq i \leqq l_n.$$

Then we shall call the array $\{X_{n,i}\}$ a *martingale difference array* (briefly MDA) with respect to $\{\mathcal{F}_{n,i}\}$ if $X_{n,i}$ is $\mathcal{F}_{n,i}$-measurable and $M\{|X_{n,i}|\}<\infty$, $M\{X_{n,i}|\mathcal{F}_{n,i-1}\}=0$ almost everywhere (a.e.) for all n and $i \geqq 1$ (the definition of the conditional expectation with respect to σ-field can be found in [14], p. 227).

Definition 4 ([14], p. 204). The class of random variables is called *uniformly integrable* if

$$\sup_n M\{|\xi_n| \cdot I_{[|\xi_n|>c]}\} \to 0 \quad \text{as} \quad c \to \infty.$$

Now we shall prove some auxiliary assertions.

Lemma 2.1. *On the probability space* $(\Omega, \mathcal{F}, P)$ *let the sequences* $\{T_n\}$ *and* $\{\xi_n\}$ *of random variables be given such that*

a) $\{T_n\}$ *is uniformly integrable;*

b) $\xi_n \xrightarrow{P} 0$ *as* $n \to \infty$;

c) $\{T_n \cdot \xi_n\}$ *is uniformly integrable.*

Then $T_n \cdot \xi_n \xrightarrow{L_1} 0$ *(here* $\xrightarrow{P}$ *and* $\xrightarrow{L_1}$ *denote the convergence with respect to probability and* L_1*-metric, respectively).*

Proof. Let $\varepsilon>0$ be fixed. By virtue of condition c) there exists $\delta(\varepsilon)>0$ such that for any $n \in \mathrm{N}$ and $A \subset \mathcal{F}$ we have

$$\int_A |T_n \cdot \xi_n| \, dP \leqq \varepsilon \tag{2.1}$$

if $P(A)<\delta(\varepsilon)$.

Furthermore, by condition b) there exists N such that for all $n>N$

$$P\{|\xi_n| > \varepsilon\} \leqq \delta(\varepsilon). \tag{2.2}$$

From (2.1) and (2.2) we conclude that for $n>N$

$$\int_{\{|\xi_n|>\varepsilon\}} |T_n \xi_n|\, dP \leqq \varepsilon.$$

Therefore by $n>N$

(2.3)

$$\int_\Omega |T_n \xi_n|\, dP = \int_{\{|\xi_n|>\varepsilon\}} + \int_{\{|\xi_n|\leqq\varepsilon\}} \leqq \int_{\{|\xi_n|>\varepsilon\}} |T_n \xi_n|\, dP + \varepsilon \cdot \int_\Omega |T_n|\, dP < \varepsilon + \varepsilon \int_\Omega |T_n|\, dP$$

Since $\{T_n\}$ is uniformly integrable, therefore

$$\sup_n \int_\Omega |T_n|\, dP < \infty$$

(see [14], p. 206). Hence, taking into account (2.3), we obtain the assertion of Lemma 2.1.

Now let $X_{n,j} = (\mu_{n,j}; \nu_{n,j})$; $n=0, 1, \ldots$; $0 \leqq j \leqq n$ be the set of random vectors on the probability space $(\Omega, \mathscr{F}, P)$; $\mathscr{F}_{n,j}$ be the set of sub σ-fields of $\mathscr{F}$ such that for all n, j $(n=0, 1, \ldots; 0 \leqq j \leqq n)$ the variables $X_{n,j}$ are $\mathscr{F}_{n,j}$-measurable and $\mathscr{F}_{n,j-1} \subset \subset \mathscr{F}_{n,j}$.

Put

$$T_n := \prod_{j=0}^{n} (1 + i\langle t, X_{n,j}\rangle), \tag{2.4}$$

where symbol i denotes the imaginary unit, $t=(t_1, t_2)$ is any vector, and $\langle \cdot, \cdot \rangle$ denotes scalar product.

Lemma 2.2. *Let the sequence $\{X_{n,j}\}$ satisfy the following conditions:*

a) $\max_{j \leqq n} |X_{n,j}| \xrightarrow{P} 0$ *as* $n \to \infty$;

b) *there exist constants μ, ν, ξ, such that*

$$\sum_{j=0}^{n} (\mu_{n,j})^2 \xrightarrow{P} \mu; \quad \sum_{j=0}^{n} (\nu_{n,j})^2 \xrightarrow{P} \nu;$$

$$\sum_{j=0}^{n} (\mu_{n,j} \cdot \nu_{n,j}) \xrightarrow{P} \xi;$$

c) *for any vector $t=(t_1, t_2)$ the sequence $\{T_n\}$ is uniformly integrable and* $M\{T_n\} \to 1$.

Then $S_n = \sum_{j=0}^{n} X_{n,j} \xrightarrow{d} \mathscr{N}(0, \mathbf{R})$, *where* $\mathbf{R} = \|r_{kl}\| = \begin{pmatrix} \mu & \xi \\ \xi & \nu \end{pmatrix}$.

Proof. We use the relation

$$\exp(ix) = (1+ix)\left(\exp\left\{-\frac{x^2}{2}+r(x)\right\}\right), \quad \text{where} \quad |r(x)| \leqq |x|^3$$

for all $|x|<1$.

Let

$$V_n := \exp\{i\cdot\langle t, S_n\rangle\}$$

and

$$U_n := \exp\left\{-\frac{1}{2}\sum_{j=0}^{n}(\langle t, X'_{n,j}\rangle)^2 + \sum_{j=0}^{n} r(\langle t, X_{n,j}\rangle)\right\}.$$

We have

$$V_n = \exp\{i\cdot\langle t, \sum_{j=0}^{n} X_{n,j}\rangle\} = \exp\{i\cdot\sum_{j=0}^{n}\langle t, X_{n,j}\rangle\} =$$

$$= T_n\cdot\exp\left\{-\frac{1}{2}\sum_{j=0}^{n}(\langle t, X_{n,j}\rangle)^2 + \sum_{j=0}^{n} r(\langle t, X_{n,j}\rangle)\right\} =$$

$$= T_n\cdot\exp\left\{-\frac{1}{2}\langle t, \mathbf{R}t\rangle\right\} + T_n\left(U_n - \exp\left\{-\frac{1}{2}\langle t, \mathbf{R}t\rangle\right\}\right).$$

By virtue of a theorem about the connection between the pointwise convergence and the convergence of corresponding distributions (see [14], p. 343) for the proof of Lemma 2.2 it will be sufficient to show that for any $t=(t_1, t_2)$

$$M\{|V_n|\} \to \exp\left\{-\frac{1}{2}\langle t, \mathbf{R}t\rangle\right\}. \tag{2.5}$$

Since $M\{T_n\}\to 1$ thus we have to verify only that

$$T_n\left(U_n - \exp\left\{-\frac{1}{2}\langle t, \mathbf{R}t\rangle\right\}\right) \xrightarrow{L_1} 0. \tag{2.6}$$

First we show that

$$U_n - \exp\left\{-\frac{1}{2}\langle t, \mathbf{R}t\rangle\right\} \xrightarrow{p} 0. \tag{2.7}$$

According to b)

$$\sum_{j=0}^{n}(\langle t, X_{n,j}\rangle)^2 \xrightarrow{p} \langle t, \mathbf{R}t\rangle.$$

Furthermore

$$\left|\sum_{j=0}^{n} r(\langle t, X_{n,j}\rangle)\right| \leqq |t|^3\cdot\sum_{j=0}^{n}|X_{n,j}|^3 \leqq |t|^3\cdot\max_{j\leqq n}|X_{n,j}|\cdot\sum_{j=0}^{n}|X_{n,j}|^2 \xrightarrow{p} 0,$$

so long as

$$\sum_{j=0}^{n}|X_{n,j}|^2 = \sum_{j=0}^{n}(\mu_{n,j}^2+\nu_{n,j}^2) \xrightarrow{p} \mu+\nu$$

and $\max_{j \geqq n} |X_{n,j}| \xrightarrow{P} 0$ by condition a). This implies that

$$U_n \xrightarrow{P} \exp\left\{-\frac{1}{2}\langle t, \mathbf{R}t\rangle\right\},$$

i.e. (2.7) is valid.

Since $\{T_n\}$ and $\{V_n\}$ are uniformly integrable (the uniform integrability of $\{V_n\}$ follows from $M\{|V_n|^2\}=1$), thus the sequence of values

$$\eta_n := V_n - T_n \cdot \exp\left\{-\frac{1}{2}\langle t, \mathbf{R}t\rangle\right\} = T_n\left(U_n - \exp\left\{-\ \langle t, \mathbf{R}t\rangle\right\}\right)$$

is also uniformly integrable as a convex set of uniformly integrable sequences (e.g. see [illegible]. 27).

By condition c), relation (2.7) and the uniform integrability of $\{\eta_n\}$ we can see that for the sequences $\{T_n\}$ and $\left\{U_n - \exp\left\{-\frac{1}{2}\langle t, \mathbf{R}t\rangle\right\}\right\}$ all of the conditions of Lemma 2.1 are fulfilled. $\left(\text{For it is sufficient to put } \xi_n = U_n - \exp\left\{-\frac{1}{2}\langle t, \mathbf{R}t\rangle\right\}.\right)$ Applying Lemma 2.1 we obtain (2.6) and moreover (2.5). Consequently the proof is complete.

The next lemma is basic for the proof of Theorem A.

Lemma 2.3. *Let* $\{X_{n,j}; \mathscr{F}_{n,j}\}$ *be an* MDA *satisfying the conditions:*

a) $\max_{j \leqq n} |X_{n,j}|$ *is uniformly bounded (in* L_2*-norm);*

b) $\max_{j \leqq n} |X_{n,j}| \xrightarrow{P} 0$;

c) *there exist constants* μ, ν, ξ *such that*

$$\sum_{j=0}^{n} (\mu_{n,j})^2 \xrightarrow{P} \mu; \quad \sum_{j=0}^{n} (\nu_{n,j})^2 \xrightarrow{P} \nu; \quad \sum_{j=0}^{n} (\mu_{n,j} \cdot \nu_{n,j}) \xrightarrow{P} \xi,$$

where $\mu_{n,j}; \nu_{n,j}$ *are the components of random vector* X_n. *Then*

$$S_n = \sum_{j=0}^{n} X_{n,j} \xrightarrow{d} \mathscr{N}(0, \mathbf{R}) \quad \text{where} \quad \mathbf{R} = \begin{pmatrix} \mu & \xi \\ \xi & \eta \end{pmatrix}.$$

Proof. Let us define the sequence $\{Z_{n,j}\}$ in the following way:

$$Z_{n,j} := X_{n,j} \cdot I\left(\sum_{k=0}^{j-1} |X_{n,k}|^2 \leqq 2(\mu+\nu)\right),$$

where $I(A)$ denotes the characteristic function of A.

It is clear that $\{Z_{n,j}; \mathscr{F}_{n,j}\}$ also represents an MDA and

$$P\{Z_{n,j} \neq X_{n,j} \text{ for some } j \leqq n\} \leqq P\Big\{\sum_{j=0}^{n} |X_{n,j}|^2 > 2(\mu+\nu)\Big\} \to 0, \tag{2.8}$$

since $|X_{n,j}|^2 = (\mu_{n,j})^2 + (\nu_{n,j})^2$, and according to c)

$$\sum_{j=0}^{n} |X_{n,j}|^2 \xrightarrow{P} \mu+\nu.$$

Therefore $\{Z_{n,j}\}$ also satisfies the conditions a), b), c) of Lemma 2.3. Now for any $t=(t_1, t_2)$ we put

$$T_n := \prod_{j=0}^{n} (1+i\cdot\langle t, Z_{n,j}\rangle).$$

Then $M\{T_n\}=1$ for all n, because $\{Z_{n,j}; \mathscr{F}_{n,j}\}$ is an MDA.

Put

$$J_n := \begin{cases} \min\Big\{j \leqq n\colon \sum_{k=0}^{j} |X_{n,k}|^2 > 2(\mu+\nu)\Big\}, & \text{if } \sum_{k=0}^{n} |X_{n,k}|^2 > 2(\mu+\nu); \\ n, & \text{otherwise.} \end{cases}$$

We have

$$M\{|T_n|^2\} = M\Big\{\prod_{j=0}^{n} (1+(\langle t, Z_{n,j}\rangle)^2)\Big\} \leqq M\Big\{\exp\Big[|t|^2\cdot \sum_{j=0}^{J_n-1} |X_{n,j}|^2\Big]\times$$
$$\times [1+(\langle t, X_{n,J_n}\rangle)^2]\Big\} \leqq \exp\{2|t|^2\cdot(\mu+\nu)\cdot[1+|t|^2\cdot M\{|X_{n,J_n}|^2\}]\}.$$

The right side of the last inequality is uniformly bounded (by condition a)). Therefore the set $\{T_n\}$ is uniformly integrable (see [14], p. 207). Taking into account b) and c), we can see that for $\{Z_{n,j}\}$ all of the conditions of Lemma 2.2 are fulfilled. Therefore

$$\sum_{j=0}^{n} Z_{n,j} \xrightarrow{d} \mathscr{N}(0, \mathbf{R}),$$

whence, by means of (2.8), we obtain

$$S_n = \sum_{j=0}^{n} X_{n,j} \xrightarrow{d} \mathscr{N}(0, \mathbf{R}),$$

which completes our proof.

Remark 2.1 (see [10]). Lemma 2.3 is the two-dimensional extension of Mc. Leish's theorem.

3. Preparation to the proof of Theorem A. We shall suppose that the sequence $\{p_n\}$ is bounded. Using notations (1.1), we can select for any N a number k such that

$$f(k) < N \leqq f(k+1). \tag{3.1}$$

Then

$$T_n(x) = \frac{1}{A_N}\sum_{m=1}^{N} a_m\chi_{n_m}(x) = \frac{B_{k-1}}{A_N}\cdot\frac{1}{B_{k-1}}\sum_{i=0}^{k-1}\Delta_i(x)+\frac{1}{A_N}\sum_{m=f(k)+1}^{N} a_m\chi_{n_m}(x).$$

Put

(3.2) $$X_{k,i}:=\frac{\Delta_i(x)}{B_k},\quad S_k:=\sum_{i=0}^{k}X_{k,i};\quad k=0,1,\ldots;\ i=0,1,\ldots,k.$$

We rewrite $T_N(x)$ in the following form:

$$T_N(x) = \frac{B_{k-1}}{A_N}\sum_{i=0}^{k-1}X_{k-1,i}+\frac{1}{A_N}\sum_{m=f(k)+1}^{N} a_m\chi_{n_m}(x) =$$

$$= \frac{B_{k-1}}{A_N}\cdot S_{k-1}+\frac{1}{A_N}\sum_{m=f(k)+1}^{N} a_m\chi_{n_m}(x).$$

Hence, in order to prove Theorem A, it is enough to show that

(3.3) there exist a vector $m=(m_1, m_2)$ and a symmetrical positive semi-definite matrix $\mathbf{R}=\|r_{kl}\|$; $k, l=1, 2$; such that

$$S_k \xrightarrow{d} \mathcal{N}(m, \mathbf{R});$$

(3.4) $$\frac{B_{k-1}}{A_N}\to 1;$$

(3.5) $$\frac{1}{A_N}\sum_{m=f(k)+1}^{N} a_m\chi_{n_m}(x)\to 0.$$

as $N\to\infty$.

Assertions (3.4) and (3.5) follow from conditions (4)—(6) and (1.2). Now we show our assertions.

Lemma 3.1 ([8]). *Let sequences $\{n_k\}$ and $\{\omega(k)\}$ satisfy the conditions* (4) *and* (5), *respectively. Then*

(3.6) $$\delta_k = O\left\{\frac{1}{\omega(f(k))}\right\},$$

(3.7) $$f(k+1)\sim f(k),$$

(3.8) $$\omega(f(k)) = O\{\omega(k+1)\}.$$

as $k\to\infty$.

Further, using (3.1), we have

$$1 \geqq \frac{B_{k-1}}{A_N}\geqq\frac{B_{k-1}}{B_k}\geqq 0.$$

In order to verify (3.4) it is sufficient to prove that

(3.9) $$\frac{B_{k-1}}{B_k}\to 1.$$

By (1.2) and (3.3) we get

$$1-\left(\frac{B_{k-1}}{B_k}\right)^2=\frac{1}{B_k^2}(B_k^2-B_{k-1}^2)=\frac{1}{B_k^2}\sum_{m=f(k)+1}^{f(k+1)}a_m^2\leqq\frac{1}{B_k^2}\left\{\max_{f(k)<m\leqq f(k+1)}|a_m|^2\cdot\delta_k\right\}=$$

$$=\frac{1}{B_k^2}\cdot o\left\{\frac{B_k^2\cdot(\omega(f(k)))^2}{\omega(f(k))}\right\}=o\{\omega(f(k))\}=o(1),$$

consequently (3.9) and (3.4) are proved.

Furthermore (1.2) and (3.9) imply that

$$\frac{1}{A_N}\cdot\left|\sum_{m=f(k)+1}^{N}a_m\chi_{n_m}(x)\right|\leqq\frac{1}{B_{k-1}}\cdot\sum_{m=f(k)+1}^{f(k+1)}|a_m|\leqq\frac{1}{B_{k-1}}\cdot\{b_k\cdot[f(k+1)-f(k)]\}=$$

$$=\frac{1}{B_{k-1}}\cdot o\{B_k\cdot\omega(f(k))\}\cdot O\left\{\frac{1}{\omega(f(k))}\right\}=\frac{1}{B_{k-1}}\cdot o(B_k)=o(1),\quad\text{as}\quad k\to\infty,$$

and by the previous reason (3.5) is proved.

Since the functions $\chi_n(x)$ are two-dimensional random variables, defined on $(\Omega,\mathscr{F},P)$, we shall denote by $\mathscr{F}_{k,i}$ $(k=0,1,\ldots;\ 0\leqq i\leqq k)$ the sub σ-field of $\mathscr{F}$ generated by random variables $\{\chi_{m_s}(x)\colon 0\leqq s\leqq i\}$. In this case the values $X_{k,i}(x)$, defined by (3.2), are $\mathscr{F}_{k,i}$-measurable, $\mathscr{F}_{k,i-1}\subset\mathscr{F}_{k,i}$ and $M\{X_{k,i}|\mathscr{F}_{k,i-1}\}=0$ a.e. for all k,i $(1\leqq i\leqq k)$. These evidently follow from the properties of our system.

In addition, we remark that

$$M\{|X_{k,i}|\}=\int_0^1|X_{k,i}|\,dx=\frac{1}{B_k}\int_0^1|\Delta_i(x)|\,dx\leqq\frac{1}{B_k}\int_0^1\sum_{j=f(i)+1}^{f(i+1)}|a_j|\,dx\leqq$$

$$\leqq\frac{1}{B_k}\cdot b_i\cdot\delta_i=\frac{1}{B_k}\cdot o\{B_i\cdot\omega(f(i))\}\cdot o\left\{\frac{1}{\omega(f(i))}\right\}=O\left(\frac{B_i}{B_k}\right)=o(1)\quad\text{as}\quad k\to\infty.$$

Therefore $M\{|X_{k,i}|\}<\infty$ for all k, i. Thus, the sequence $\{X_{k,i};\mathscr{F}_{k,i}\}$ represents an MDA and in order to prove (3.4) it is sufficient to verify the validity of the conditions of Lemma 2.3 for the sequence $\{X_{k,i}\}$.

Using the multiplicative property and the orthogonality of the system X, we get

$$M\{\max_{0\leqq i\leqq k}|X_{k,i}|^2\}=\frac{1}{B_k^2}\int_0^1\max_{0\leqq i\leqq k}|\Delta_i(x)|^2\,dx\leqq$$

$$\leqq\frac{1}{B_k^2}\int_0^1\sum_{i=0}^{k}|\Delta_i(x)|^2\,dx=\frac{1}{B_k^2}\sum_{i=0}^{k}\int_0^1|\Delta_i(x)|^2\,dx=\frac{B_k^2}{B_k^2}=1,$$

which means that condition a) is satisfied.

Furthermore,

$$\max_{0\le i\le k}|X_{k,i}| = \frac{1}{B_k}\cdot\max_{0\le i\le k}|\Delta_i(x)| \le \frac{1}{B_k}\cdot\max_{0\le i\le k}\sup_x|\Delta_i(x)| \le$$

$$\le \frac{1}{B_k}\cdot\max_{0\le i\le k}\sum_{m=f(i)+1}^{f(i+1)}|a_m| \le \frac{1}{B_k}\cdot\max_{0\le i\le k}\Big\{\max_{f(i)<m\le f(i+1)}|a_m|\cdot\delta_i\Big\} =$$

$$= \frac{1}{B_k}\cdot\max_{0\le i\le k}\left\{o\big(B_i\cdot\omega(f(i))\big)\cdot O\left(\frac{1}{\omega(f(i))}\right)\right\} = \frac{1}{B_k}\cdot\max_{0\le i\le k}\{o(B_i)\} = o(1),\quad \text{as}\quad k\to\infty,$$

and this proves b).

For the direct proof of condition c) we require some lemmas.

Lemma 3.2. *Let the sequences* $\{n_k\}$, $\{\omega(k)\}$ *and* $\{a_k\}$ *satisfy conditions* (4)—(6), (1.2), *respectively. Then*

$$\int_0^1\left|\frac{1}{B_N^2}\sum_{k=0}^{N}|\Delta_k(x)|^2-1\right|^2dx = o(1)\quad as\quad N\to\infty.$$

The proof of this lemma can be found in [8] (replacing only the symbol O by o),

Lemma 3.3. *Let the sequences* $\{n_k\}$, $\{\omega(k)\}$ *and* $\{a_k\}$ *fulfil conditions* (4)—(6) (1.2)—(1.4), *respectively. Then*

$$\int_0^1\left|\frac{1}{B_N^2}\sum_{k=0}^{N}\big((\Delta_k(x))^2-\eta\big)\right|^2dx = o(1)\quad as\quad N\to\infty.$$

Proof. The next equalities are evident

(3.10)

$$\int_0^1\left|\frac{1}{B_N^2}\sum_{k=0}^{N}(\Delta_k(x))^2-\eta\right|^2dx = \int_0^1\frac{1}{B_N^4}\cdot\Big\{\sum_{k=0}^{N}(\Delta_k)^2-\eta B_N^2\Big\}\cdot\Big\{\sum_{k=0}^{N}(\bar\Delta_k)^2-\eta B_N^2\Big\}dx =$$

$$= \int_0^1\left\{\frac{1}{B_N^4}\Big[\sum_{k=0}^{N}(\Delta_k)^2\cdot\sum_{k=0}^{N}(\bar\Delta_k)^2-\eta B_N^2\cdot\Big(\sum_{k=0}^{N}(\Delta_k)^2+\sum_{k=0}^{N}(\bar\Delta_k)^2\Big)+\eta^2B_N^4\Big]\right\}dx =$$

$$= \frac{1}{B_N^4}\left\{\int_0^1\Big|\sum_{k=0}^{N}(\Delta_k)^2\Big|^2dx-\eta B_N^2\int_0^1\sum_{k=0}^{N}[(\Delta_k)^2+(\bar\Delta_k)^2]\,dx+\eta^2B_N^4\right\},$$

since

$$\overline{\sum_{k=0}^{N}(\Delta_k)^2} = \sum_{k=0}^{N}(\bar\Delta_k)^2\quad\text{and}\quad\bar\eta=\eta.$$

Let us evaluate the values in the brace. We have

$$(3.11)\qquad \int_0^1 \Big|\sum_{k=0}^N (\Delta_k)^2\Big|^2 dx = \sum_{k=0}^N \sum_{j=0}^N \int_0^1 (\Delta_k^2 \cdot \bar{\Delta}_j^2)\, dx.$$

The terms of the type $\int_0^1 (\Delta_k^2 \cdot \bar{\Delta}_j^2)\, dx$, in turn, consist of the summands containing the expressions of the species

$$(3.12)\qquad \int_0^1 (\chi_{n_q} \cdot \chi_{n_r} \cdot \bar{\chi}_{n_h} \cdot \bar{\chi}_{n_i})\, dx$$

(with corresponding coefficients), where

$$f(k) < q, r \leqq f(k+1);\ f(j) < h, i \leqq f(j+1);\ 0 \leqq k, j \leqq N.$$

Each of the integrals is equal to zero or one (by virtue of the multiplicative property and the orthogonality of the system X). We have to estimate the quantity of the non-zero summands. Let $k > j$ (the case $k < j$ can be treated similarly). Arguing the same way as in the proof of Lemma 2.4 in [8], we conclude that the functions $\chi_p(x) = \chi_{n_q}(x) \cdot \chi_{n_r}(x)$ belong to a block, number of which is not larger than j (otherwise the integral (3.12) will become zero). Therefore the numbers n_q and n_r have to be conjugate or (l, k)-adjoint (moreover $0 \leqq l \leqq j$).

Now we rewrite the previous equality in the form

$$\int_0^1 \Big|\sum_{k=0}^N (\Delta_k)^2\Big|^2 dx = \sum_{k,j=0}^{N}{}' \int_0^1 (\Delta_k^2 \cdot \bar{\Delta}_j^2)\, dx + \sum_{k,j=0}^{N}{}'' \int_0^1 (\Delta_k^2 \cdot \bar{\Delta}_j^2)\, dx,$$

where the symbol $\sum'$ denotes the set of that summands, for which the numbers n_q and n_r are conjugate in the fourfold product $\chi_{n_q} \cdot \chi_{n_r} \cdot \bar{\chi}_{n_h} \cdot \bar{\chi}_{n_i}$ and the symbol $\sum''$ denotes the set of all other summands.

In the sum $\sum'$ we have to consider only the summands, for which $n_q \dotplus n_r = 0$ and $n_h \dotplus n_i = 0$ simultaneously; other summands, are equal to zero, because for them the equality

$$\chi_{n_q} \cdot \chi_{n_r} \cdot \bar{\chi}_{n_h} \cdot \bar{\chi}_{n_i} \equiv 1$$

does not fulfil.

Therefore,

$$\sum_{k,j=0}^{N}{}' \int_0^1 (\Delta_k^2 \cdot \bar{\Delta}_j^2)\, dx = \sum_{k}^{f(N+1)} \sum_{j}^{f(N+1)} (a_k \cdot \hat{a}_k) \cdot (a_j \cdot \hat{a}_j) = \Big\{\sum_{k}^{f(N+1)} (a_k \cdot \hat{a}_k)\Big\}^2.$$

Using (1.3) we get

$$(3.13)\qquad \sum_{k,j=0}^{N}{}' \int_0^1 (\Delta_k^2 \cdot \bar{\Delta}_j^2)\, dx = \eta^2 \cdot B_N^4 + o(B_N^4) \quad \text{as} \quad N \to \infty.$$

In the sum $\sum''$ we select the summands for the cases $k=j$, $k>j$ and $k<j$:

$$\sum_{k,j=0}^{N}{}'' \int_0^1 (\Delta_k^2 \cdot \bar{\Delta}_j^2)\,dx = \sum_{k=0}^{N}{}'' \int_0^1 (\Delta_k^2 \cdot \bar{\Delta}_k^2)\,dx + \sum_{k=0}^{N}{}'' \sum_{j=0}^{k-1} \int_0^1 (\Delta_k^2 \cdot \bar{\Delta}_j^2)\,dx + \tag{3.14}$$

$$+ \sum_{j=1}^{N}{}'' \sum_{k=0}^{j-1} \int_0^1 (\bar{\Delta}_j^2 \cdot \Delta_k^2)\,dx = L_N^{(1)} + L_N^{(2)} + L_N^{(3)}.$$

Now we have

$$L_N^{(1)} = \sum_{k=0}^{N}{}'' \int_0^1 (\Delta_k^2 \cdot \bar{\Delta}_k^2)\,dx \leqq \sum_{k=0}^{N} \int_0^1 |\Delta_k^4|\,dx = \sum_{k=0}^{N} \int_0^1 (|\Delta_k|^2)^2\,dx. \tag{3.15}$$

Since

$$|\Delta_k|^2 = \sum_{q=f(k)+1}^{f(k+1)} a_q \chi_{n_q} \cdot \sum_{r=f(k)+1}^{f(k+1)} a_r \bar{\chi}_{n_r} = \sum_{q=f(k)+1}^{f(k+1)} a_q \Big\{ \sum_{r=f(k)+1}^{f(k+1)} a_r \chi_{n_q} \cdot \bar{\chi}_{n_r} \Big\},$$

therefore, applying Minkowski's inequality, we obtain

$$\Big\{\int_0^1 (|\Delta_k|^2)^2\,dx\Big\}^{1/2} \leqq \sum_{q=f(k)+1}^{f(k+1)} |a_q| \cdot \Big\{\int_0^1 \Big|\sum_{r=f(k)+1}^{f(k+1)} a_r \chi_{n_q} \cdot \bar{\chi}_{n_r}\Big|^2 dx\Big\}^{1/2} \leqq$$

$$\leqq \sum_{q=f(k)+1}^{f(k+1)} |a_q| \cdot \Big\{\int_0^1 \Big|\sum_{r=f(k)+1}^{f(k+1)} a_r \bar{\chi}_{n_r}\Big|^2 dx\Big\}^{1/2} = \sum_{q=f(k+1)}^{f(k+1)} |a_q| \cdot \Big\{\sum_{r=f(k)+1}^{f(k+1)} a_r^2\Big\}^{1/2}.$$

Hence,

(3.16)

$$\sum_{k=0}^{N} \int_0^1 (|\Delta_k|^2)^2\,dx = \sum_{k=0}^{N} \Big\{\sum_{q=f(k)+1}^{f(k+1)} |a_q|\Big\}^2 \cdot \sum_{r=f(k)+1}^{f(k+1)} a_r^2 \leqq \sum_{k=0}^{N} (b_k \delta_k)^2 \times \sum_{r=f(k)+1}^{f(k+1)} a_r^2 =$$

$$= \sum_{k=0}^{N} \Big\{ o\big(B_k \cdot \omega(f(k))\big) \cdot \frac{1}{\omega\big(f(k)\big)} \Big\}^2 \cdot \sum_{r=f(k)+1}^{f(k+1)} a_r^2 = \sum_{k=0}^{N} o(B_k^2) \cdot \sum_{r=f(k+1)}^{f(k+1)} a_r^2 =$$

$$= o(B_N^2) \cdot \sum_{k=0}^{N} \sum_{r=f(k)+1}^{f(k+1)} a_r^2 = o(B_N^2) \cdot B_N^2 = o(B_N^4) \quad \text{as} \quad N \to \infty$$

(we used relations (1.2) and (3.3)).

Passing on to the estimation of $L_N^{(2)}$ in (3.14), we remark that

$$\int_0^1 (\Delta_k^2 \cdot \bar{\Delta}_j^2)\,dx = \int_0^1 \Big(\sum_{q=f(k)+1}^{f(k+1)} a_q \chi_{n_q}\Big)^2 \cdot \Big(\sum_{h=f(j)+1}^{f(j+1)} a_h \bar{\chi}_{n_h}\Big)^2 dx =$$

$$= \int_0^1 \Big(\sum_{q=f(k)+1}^{f(k+1)} a_q \chi_{n_q} \cdot \sum_{r=f(k)+1}^{f(k+1)} a_r \chi_{n_r}\Big) \cdot \Big(\sum_{h=f(j)+1}^{f(j+1)} a_h \bar{\chi}_{n_h} \cdot \sum_{i=f(j)+1}^{f(j+1)} a_i \bar{\chi}_{n_i}\Big)\,dx = \tag{3.17}$$

$$= \sum_{q=f(k)+1}^{f(k+1)} a_q \sum_{h=f(j)+1}^{f(j+1)} a_h \sum_{r=f(k)+1}^{f(k+1)} a_r \sum_{i=f(j)+1}^{f(j+1)} a_i \int_0^1 (\chi_{n_q} \chi_{n_r} \bar{\chi}_{n_h} \bar{\chi}_{n_i})\,dx.$$

As it was mentioned above, for any non-zero term of $\sum''$ there should exist an (l, k)-adjoint of the numbers n_q and n_r. Therefore the total quantity of the appropriate pairs (n_q, n_r) is not more than $\lambda_k^j(q)$.

Under fixed indices q, h and for any selected number n_r there exists not more than one number n_i such that

$$\chi_{n_q}\cdot\chi_{n_r}\cdot\bar{\chi}_{n_h}\cdot\bar{\chi}_{n_i}\equiv 1.$$

Thus, by (3.17), we get the estimation

$$\int_0^1 (\Delta_k^2\cdot\bar{\Delta}_j^2)\,dx \leq b_k\cdot\sum_{q=f(k)+1}^{f(k+1)}|a_q|\cdot\lambda_k^j(q)\cdot b_j\sum_{h=f(j)+1}^{f(j+1)}|a_h|. \tag{3.18}$$

By Cauchy—Bunjakowski's inequality and by (1.2) and (3.3):

$$b_k\cdot\sum_{q=f(k)+1}^{f(k+1)}|a_q| = o\big(B_k\cdot\omega(f(k))\big)\cdot\Big\{\sum_{q=f(k)+1}^{f(k+1)}a_q^2\Big\}^{1/2}\cdot(\delta_k)^{1/2} =$$

$$= o\big(B_k\cdot\omega(f(k))\big)\cdot O\left\{\frac{1}{\sqrt{\omega(f(k))}}\right\}\cdot\Big\{\int_0^1|\Delta_k(x)|^2\,dx\Big\}^{1/2} =$$

$$= o\big(B_k\cdot\sqrt{\omega(f(k))}\big)\cdot\Big\{\int_0^1|\Delta_k(x)|^2\,dx\Big\}^{1/2}\quad\text{as}\quad k\to\infty.$$

This implies that under the realization of (1.4) the next relation holds

$$\int_0^1(\Delta_k^2\cdot\bar{\Delta}_j^2)\,dx = o\big(B_k\cdot\sqrt{\omega(f(k))}\big)\cdot\Big\{\int_0^1|\Delta_k(x)|^2\,dx\Big\}^{1/2}\times$$

$$\times O\left(\frac{C^{j-k}}{\omega(f(k))}\right)\cdot o\big(B_j\cdot\sqrt{\omega(f(j))}\big)\cdot\Big\{\int_0^1|\Delta_j|^2\,dx\Big\}^{1/2}\quad\text{as}\quad k\to\infty.$$

Hence,

$$L_N^{(2)} = \sum_{k=0}^{N}\sum_{j=0}^{k-1} o\big(B_k\cdot\sqrt{\omega(f(k))}\big)\cdot O\left(\frac{C^{j-k}}{\omega(f(k))}\right)\cdot o\big(B_j\cdot\sqrt{\omega(f(j))}\big)\times$$

$$\times\Big\{\int_0^1|\Delta_k^2|\,dx\cdot\int_0^1|\Delta_j|^2\,dx\Big\}^{1/2} = o(B_N^2)\cdot\sum_{k=1}^{N}\sum_{j=0}^{k-1}C^{j-k}\cdot\frac{1}{\sqrt{\omega(f(k))}}\times$$

$$\times\Big\{\int_0^1|\Delta_k|^2\,dx\cdot\int_0^1|\Delta_j|^2\,dx\cdot\omega(f(j))\Big\}^{1/2}\quad\text{as}\quad N\to\infty.$$

Let us show that $L_N^{(2)}=o(B_N^4)$ as $N\to\infty$. It is sufficient to show that

$$\sum_{k=1}^{N}\sum_{j=0}^{k-1}C^{j-k}\cdot\frac{1}{\sqrt{\omega(f(k))}}\cdot\Big\{\int_0^1|\Delta_k|^2\,dx\cdot\int_0^1|\Delta_j|^2\,dx\cdot\omega(f(j))\Big\}^{1/2} = O(B_N^2).$$

Consider the sum $\sum_{j=0}^{k-1} C^{j-k}\cdot\omega(f(j))$. Since $k\cdot\omega(k)\uparrow\infty$, thus for any natural $\tau\geqq 2$ we have

$$k\cdot\omega(k)\leqq\left[\frac{\tau}{2}k\right]\cdot\omega\left(\left[\frac{\tau}{2}k\right]\right)\leqq\frac{\tau}{2}k\cdot\omega\left(\left[\frac{\tau}{2}k\right]\right),$$

from which

(3.19) $$\omega\left(\left[\frac{\tau k}{2}\right]\right)\leqq\frac{2\omega(k)}{\tau},\quad k=1,2,\dots,$$

(the symbol $[x]$ denotes the integral part of x).

By (3.7) there exists a number M such that if $k>M$ then

(3.20) $$f(k+1)<\left[\frac{[C+1]}{2}f(k)\right].$$

Relations (3.19) and (3.20) imply that if $\tau=[C+1]\geqq 2$ and $k>M$, then

$$\omega(f(k+1))>\omega\left(\left[\frac{[C+1]}{2}\cdot f(k)\right]\right)\geqq\frac{2\omega(f(k))}{[C+1]}\geqq\frac{2\omega(f(k))}{C+1},$$

i.e. $\omega(f(k))<\frac{C+1}{2}\cdot\omega((f+1))$ as $k>M$.

Therefore

$$\omega(f(j))<\left(\frac{C+1}{2}\right)^{k-j}\cdot\omega(f(k))\quad\text{if}\quad k>j>M,$$

and thus

(3.21)

$$\sum_{j=M+1}^{k-1} C^{j-k}\cdot\omega(f(j))<\sum_{j=M+1}^{k-1}\left(\frac{C+1}{2C}\right)^{k-j}\cdot\omega(f(k))=O\{\omega(f(k))\}\quad\text{as}\quad k\to\infty.$$

On the other hand

$$\sum_{j=0}^{M} C^{j-k}\cdot\omega(f(j))=O(C^{-k})\quad\text{as}\quad k\to\infty.$$

At the same time, by (3.20), we have

$$f(k)<C\cdot f(k-1)<\dots<C^{k-M-1}\cdot f(M+1)\quad\text{if}\quad k>M+1,$$

hence

$$C^{-k}\leqq\frac{f(M+1)}{C^{M+1}}\cdot\frac{1}{f(k)}=O\left\{\frac{1}{f(k)}\right\}=O\{\omega(f(k))\}$$

since $f(k)\cdot\omega(f(k))\uparrow\infty$ as $k\to\infty$.

So

$$\sum_{j=0}^{M} C^{j-k}\cdot\omega(f(j)) = O\{\omega(f(k))\} \quad \text{as} \quad k\to\infty. \tag{3.22}$$

By (3.21) and (3.22) we obtain that

$$\sum_{j=0}^{k-1} C^{j-k}\cdot\omega(f(j)) = O\{\omega(f(k))\} \quad \text{as} \quad k\to\infty. \tag{3.23}$$

Applying Cauchy—Bunjakowski's inequality, by (3.23),

$$\sum_{k=1}^{N}\sum_{j=0}^{k-1} C^{j-k}\cdot\frac{1}{\sqrt{\omega(f(k))}}\cdot\Big\{\int_0^1 |\Delta_k|^2\,dx\cdot\int_0^1 |\Delta_j|^2\,dx\cdot\omega(f(j))\Big\}^{1/2} \leqq$$

$$\leqq \sum_{k=1}^{N}\{\omega(f(k))\}^{-1/2}\cdot\Big\{\int_0^1 |\Delta_k|^2\,dx\Big\}^{1/2}\cdot\Big(\sum_{j=1}^{k=1} C^{j-k}\cdot\omega(f(j))\Big)^{1/2}\times$$

$$\times\Big\{\sum_{j=0}^{k-1} C^{j-k}\cdot\int_0^1 |\Delta_j|^2\,dx\Big\}^{1/2} = O\Big\{\sum_{k=1}^{N}\Big(\int_0^1 |\Delta_k|^2\,dx\Big)^{1/2}\cdot\Big(\sum_{j=0}^{k+1} C^{j-k}\cdot\int_0^1 |\Delta_j|^2\,dx\Big)^{1/2}\Big\} =$$

$$= O\Big(\Big(\sum_{k=0}^{N}\int_0^1 |\Delta_k|^2\,dx\Big)^{1/2}\cdot\Big(\sum_{k=1}^{N}\sum_{j=0}^{k-1} C^{j-k}\cdot\int_0^1 |\Delta_j|^2\,dx\Big)^{1/2}\Big) =$$

$$= O\Big((B_N^2)^{1/2}\cdot\Big(\sum_{k=0}^{N-1}\int_0^1 |\Delta_k|^2\,dx\Big)^{1/2}\Big) = O(B_N^2).$$

Thus, the relation $L_N^{(2)} = O(B_N^4)$ is proved. The proof of $L_N^{(3)} = O(B_N^4)$ runs similarly.

Using these relations, by (3.15) and (3.16), we get

$$\sum_{k,j=0}^{N}{}'' \int_0^1 (\Delta_k^2\cdot\bar{\Delta}_j^2)\,dx = o(B_N^4) \quad \text{as} \quad N\to\infty. \tag{3.24}$$

By (3.13) and (3.24) we have

$$\int_0^1 \Big\{\Big|\sum_{k=0}^{N}(\Delta_k)^2\Big|^2\Big\}\,dx = \eta B_N^4 + o(B_N^4) \quad \text{as} \quad N\to\infty. \tag{3.25}$$

Now let us consider the value

$$\int_0^1 \Big\{\sum_{k=0}^{N}(\Delta_k^2+\bar{\Delta}_k^2)\Big\}\,dx = \sum_{k=0}^{N}\Big\{\int_0^1 (\Delta_k)^2\,dx + \int_0^1 (\bar{\Delta}_k^2)\,dx\Big\}.$$

The terms of the type $\int_0^1 (\Delta_k)^2 dx$ consist of the summands of the species

$$\int_0^1 (a_q a_r \chi_{n_q}\chi_{n_r})\,dx,$$

where the functions χ_{n_r} and χ_{n_q} belong to the k-th block. The quantity of non-zero summands of this species depends on the number of conjugate pairs (n_q, n_r) in the k-th block. Therefore, using (1.3), we have

$$\sum_{k=0}^{N}\int_0^1 (\Delta_k)^2\,dx = \sum_{k}^{f(N+1)} (a_k\cdot\hat{a}_k) = \eta B_N^2+o(B_N^2) \quad \text{as} \quad N\to\infty. \tag{3.26}$$

Analogously

$$\sum_{k=0}^{N}\int_0^1 (\bar{\Delta}_k^2)\,dx = \eta B_N^2+o(B_N^2) \quad \text{as} \quad N\to\infty. \tag{3.27}$$

Finally, substituting estimations (3.25)—(3.27) into (3.10), we receive that

$$\int_0^1 \left|\frac{1}{B_N^2}\sum_{k=0}^{N}(\Delta_k)^2-\eta\right|^2 dx = \frac{1}{B_N^4}\{\eta^2\cdot B_N^4+o(B_N^4)-\eta B_N^2\cdot(2\eta B_N^2+o(B_N^2))+B_N^4\cdot\eta^2\} =$$

$$= \frac{1}{B_N^4}\{\eta^2 B_N^4-2\eta^2 B_N^4+\eta^2 B_N^4+o(B_N^4)\} = o(1) \quad \text{as} \quad N\to\infty.$$

Thus Lemma 3.3 is proved.

4. The proof of Theorem A. Lemmas 3.2 and 3.3 imply that if the conditions of Theorem A are fulfilled then

$$\frac{1}{B_k^2}\sum_{j=0}^{k}|\Delta_j|^2-1 \xrightarrow{p} 0 \quad \text{and} \quad \frac{1}{B_k^2}\sum_{j=0}^{k}(\Delta_j)^2-\eta \xrightarrow{p} 0.$$

Regarding the definitions (3.2), we obtain

$$\sum_{j=0}^{k}|X_{k,j}|^2 \xrightarrow{p} 1 \quad \text{and} \quad \sum_{j=0}^{k}(X_{k,j})^2 \xrightarrow{p} \eta. \tag{4.1}$$

Now we show that (4.1) implies the realization of condition c) of Lemma 2.3 (for the sequence $\{x_{k,j};\ \mathscr{F}_{k,j}\}$).

Let $X_{k,j}=(\mu_{k,j};\ \nu_{k,j})$; $k=0, 1, \ldots;\ 0\leqq j\leqq k$, where

$$\mu_{k,j} = \mathrm{Re}\,\{X_{k,j}\};\ \nu_{k,j} = \mathrm{Im}\,\{X_{k,j}\}.$$

Then

$$|X_{k,j}|^2 = (\mu_{k,j})^2+(\nu_{k,j})^2,$$

$$(X_{k,j})^2 = \{(\mu_{k,j})^2-(\nu_{k,j})^2\}+2i\cdot(\mu_{k,j}\cdot\nu_{k,j})$$

(here i denotes the imaginary unit).

Substituting it into (4.1), we get

$$\sum_{j=0}^{k}\{(\mu_{k,j})^2+(\nu_{k,j})^2\} \xrightarrow{p} 1 \tag{4.2}$$

$$\sum_{j=0}^{k}\{(\mu_{k,j})^2-(\nu_{k,j})^2\}+2i(\mu_{k,j}\cdot\nu_{k,j}) \xrightarrow{p} \eta. \tag{4.3}$$

Adding and subtracting equalities (4.2) and (4.3) we conclude that

$$\text{(4.4)}\qquad \begin{cases} \sum_{j=0}^{k} (\mu_{k,j})^2 \xrightarrow{P} \frac{1+\eta}{2}, \\ \sum_{j=0}^{k} (v_{k,j})^2 \xrightarrow{P} \frac{1-\eta}{2}, \\ \sum_{j=0}^{k} (\mu_{k,j}\cdot v_{k,j}) \xrightarrow{P} 0. \end{cases}$$

Relations (4.4) show that Lemma 2.3 is applicable for the sequence $\{X_{k,j};\ \mathscr{F}_{k,j}\}$. It implies the validity of (3.6). Thus we have

$$S_k = \sum_{j=0}^{k} X_{k,j} \xrightarrow{d} \mathscr{N}(m, \mathbf{R}),$$

where $m=(0,0)$, $\mathbf{R}=\frac{1}{2}\begin{pmatrix}1+\eta & 0\\ 0 & 1-\eta\end{pmatrix}$.

Finally, taking into account the definition of the value $T_N(x)$ and relations (3.4)—(3.6), we obtain that

$$T_N(x) \xrightarrow{d} \mathscr{N}(m, \mathbf{R}),$$

where $m=(0,0)$, $\mathbf{R}=\|r_{kl}\|$, $r_{11}=\frac{1}{2}(1+\eta)$, $r_{12}=r_{21}=0$, $r_{22}=\frac{1}{2}(1-\eta)$.

Herewith Theorem A is proved completely.

Remark 4.1. The foregoing proof implies that if our system $X=\{\chi_n(x)\}_{n=0}^{\infty}$ is real-valued then $\overline{\Delta_k(x)}=\Delta_k(x)$ and the assertions of Lemmas 3.2 and 3.3 coincide, therefore we have $\eta=1$ (because $\overline{\chi_n(x)}=\chi_n(x)$ for all n). Then, in the case of Walsh—Paley's system, the realization of condition (1.2) already is sufficient. Condition (1.3) is fulfilled automatically ($\eta=1$), and condition (1.4) is furnished by conditions (4)—(6) and (1.2) (see, e.g., the proof of Lemma 2.4 in [8]). The covariant matrix in this case is the following:

$$\mathbf{R} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

(it conforms to the normal distribution of a vector such that one of its components — the imaginary part in our case — equals zero identically).

Remark 4.2. Since the divergence of the series $\sum_{k=1}^{\infty} a_k^2$ implies the divergence of the series $\sum_{k=1}^{\infty}\frac{a_k^2}{A_k^2}$, thus the sequence $\{\omega(k)\}$ in Theorem A must satisfy the condition: $\sum_{k=1}^{\infty}(\omega(k))^2=\infty$.

As a sample example realizing the conditions of Theorem A for complex-valued PMONS we bring the next one.

Let us consider the Chrestenson—Levy's system generated by $p_k\equiv 3$. Then $m_0=1, m_1=3, \dots, m_k=3^k$; $k=1, 2, \dots$; the functions $\chi_{m_k}(x)$ are "basis" in the blocks $[m_k, m_{k+1})$. Put $n_1=m_1$, $n_2=2m_1, \dots, n_{2i-1}=m_i$, $n_{2i}=2m_i$; $i=1, 2, \dots$. Let $a_k=1$ for all k. So, for our sequence $\{n_k\}$ $\frac{n_{k+1}}{n_k}\geqq\frac{3}{2}$, $k=1, 2, \dots$ hold, and we can put $\omega(k)\equiv\frac{1}{2}$.

Then the conditions of Theorem 1 are fulfilled trivially. Indeed, $A_k=\sqrt{k}$ for all k. It is also clear that $\overline{\chi_{m_k}(x)}=(\chi_{m_k}(x))^2=\chi_{2m_k}(x)$. Consequently the quantity of the conjugate pairs is equal to 2 in each block (we remind that the pairs (n_q, n_r) and (n_r, n_q) are considered as distinct if $q\neq r$).

At the same time $B_k^2=f(k+1)=2k$. Thus,

$$\eta = \lim_{N\to\infty}\frac{1}{B_N^2}\sum_{j}^{f(N+1)}(a_j\cdot\hat{a}_j) = \lim_{N\to\infty}\frac{2N}{2N} = 1.$$

The validity of (1.4) follows from the fact that there are no (l, k)-adjoint numbers in our sequence $\{n_k\}$, i.e. $\lambda_k^l(q)=0$ for all k, l, q $(0\leqq l\leqq k)$. Therefore, the constructed subsystem $\chi_{n_k}(x)$ is a subject to CLT with the covariant matrix $\mathbf{R}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$.

We also note that taking the subsystem $\{\chi_{n_k}(x)\}$ such that $n_k=m_k$ (i.e. $\{\chi_{n_k}(x)\}$ consists of the "basis" functions), then all of the conditions of Theorem A are fulfilled and we evidently have $\eta=0$ in this case. So the covariant matrix is $\mathbf{R}=\begin{bmatrix}1/2 & 0\\ 0 & 1/2\end{bmatrix}$ (as before we put $a_k=1$ for all k).

5. The sharpness of conditions of Theorem A. The following theorem shows that conditions (1.2)—(1.4) in Theorem A cannot be weakened, generally.

Theorem B. *There exist sequences* $\{a_k\}$, $\{n_k\}$ *and* $\{\omega(k)\}$, *satisfying conditions* (4)—(6) *and there exists a* PMONS $\{\chi_{n_k}(x)\}$ *such that if even any of conditions* (1.2)—(1.4) *is broken then the subsystem* $\{a_k\chi_{n_k}(x)\}$ *is not the subject to* CLT.

In the proof of Theorem B we shall use the following fact (see e.g. [9], pp. 195—198):

Let the sequence $\{\xi_n\}$ of random vectors be given, where $\xi_n=(\xi_n^1, \xi_n^2, \dots, \xi_n^k)$ weakly converges to some random vector $\xi=(\xi^1, \xi^2, \dots, \xi^k)$. Also let $\mu_n^{(r)}$ and $m_n^{(\nu)}$ denote the r-th absolute moment and the ν-th $(\nu=(\nu_1, \nu_2, \dots, \nu_k))$ mixed moment of random variable ξ_n, respectively; i.e.

$$M_n^{(r)} = M|\xi_n|^r \quad \text{and} \quad m_n^{(\nu)} = M\{(\xi_n^1)^{\nu_1}\cdot(\xi_n^2)^{\nu_2}\dots(\xi_n^k)^{\nu_k}\},$$

where $\nu_i\geqq 0$, $i=1, 2, \dots, k$.

In this case, if the sequence $\{\mu_n^{(r_0+\delta)}\}$ is bounded for some $\delta>0$, then the sequences $\{\mu_n^{(\nu)}\}$ and $\{m_n^{(\nu)}\}$ of the moments converge to the corresponding moments of the distribution of vector ξ for all r, $|\nu|\leqq r_0$ (here $|\nu|=\sum_{i=1}^{k}\nu_i$), i.e.

$$(5.1) \qquad \mu_n^{(r)} \to \mu^{(r)}, \quad m_n^{(\nu)} = m_n^{(\nu_1, \nu_2, \dots, \nu_k)} \to m^{(\nu)} \quad \text{as} \quad n\to\infty.$$

Moreover, the mentioned limits are finite.

The direct proof of Theorem B requires constructions of counterexamples, that are showing the necessity of conditions (1.2)—(1.4), by turns. First we notice that the necessity of (1.2) was shown in [7].

6. Counterexamples. Passing to the proof of the necessity of (1.3), let us choose the Chrestenson—Levy's system generated by $p_k\equiv 3$. Put $a_k=1$ for all k. The sequence $\{n_k\}$ is constructed in the following way:

$$n_1 = m_1, \quad n_2 = 2m_1, \quad n_{2k-1} = m_k \quad (m_k = 3^k, k = 1, 2, \dots)$$

$$(6.1) \qquad n_{2k} = \begin{cases} 2m_k - 1, & \text{if} \quad 10^{2l} < k \leqq 10^{2l+1}; \\ 2m_k & \text{if} \quad 10^{2l+1} < k \leqq 10^{2l+2}; \end{cases} \quad l = 0, 1, \dots$$

Thus there exists one pair of the terms of $\{\chi_{n_k}(x)\}$ in every block $[m_k, m_{k+1})$: The terms of n_k can be conjugate if $n_{2k}=2n_{2k-1}$. Let us check the fulfilment of the conditions of Theorem A for $\{n_k\}$.

Since $n_{k+1}\geqq\frac{3}{2}n_k$ for all k, it is clear that we can put $\omega(k)\equiv\frac{1}{2}$. Further, $A_k=\sqrt{k}\to\infty$ and $a_k=1=o(A_k\omega(k))$ as $k\to\infty$. For these reasons conditions (4)—(6) and (1.2) are fulfilled.

The verification of condition (1.4) is trivial, because $\lambda_k^j(q)=0$ for all q, k and $0\leqq j\leqq k-1$ and $\lambda_k^k(q)=1$ for $q=2k$, $10^{2l}+1\leqq k\leqq 10^{2l+1}$.

In the same time the value

$$C_N = \frac{1}{B_N^2}\sum_{j}^{f(N+1)}(a_j\cdot\hat{a}_j)$$

has no limit, since

$$C_{10^{2l+1}} < \frac{2\cdot 10^{2l}}{2\cdot 10^{2l+1}} = \frac{1}{10} \quad \text{for} \quad l = 0, 1, \dots,$$

but

$$C_{10^{2l+2}} \geqq \frac{2\cdot 9\cdot 10^{2l+1}}{2\cdot 10^{2l+2}} = \frac{9}{10} \quad \text{for} \quad l = 0, 1, \ldots.$$

So, (1.3) is failed. Now we shall show that CLT for $\{\chi_{n_k}(x)\}$ does not hold. Let us estimate the absolute moment of the 4-th power of the random variable

$$T_N(x) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \chi_{n_k}(x).$$

We have

$$\mu_N^{(4)} = \int_0^1 |T_N(x)|^2\, dx = \frac{1}{N^2} \sum_{1\leqq i,j,p,q\leqq N} \int_0^1 (\chi_{n_q}\chi_{n_p}\bar{\chi}_{n_j}\bar{\chi}_{n_i})\, dx. \tag{6.2}$$

The summands in the right side of (6.2) are distinct from zero if and only if

$$\chi_{n_p}\cdot\chi_{n_q}\cdot\bar{\chi}_{n_i}\cdot\bar{\chi}_{n_j} = 1. \tag{6.3}$$

Arguing the same way as in the proof of Lemma 2.4 in [8], we conclude that (6.3) holds only if the fourfold product $\chi_{n_p}\chi_{n_q}\bar{\chi}_{n_i}\bar{\chi}_{n_j}$ has a decomposition of two pairs such that each of the pairs belongs to certain block $[m_k, m_{k+1})$, perhaps, to the same. Since the number of the blocks is not more than $\left[\frac{N}{2}\right]+1$, thus the number of the non-zero summands in the right side of (6.2) does not exceed the value $4!\left(\left[\frac{N}{2}\right]+1\right)^2$. Consequently,

$$\mu_N^{(4)} \leqq \frac{1}{N^2}4!\left(\left[\frac{N}{2}\right]+1\right)^2 = O(1).$$

Thus, the sequence $\{\mu_N^{(4)}\}$ is bounded.

Now let us use relations (5.1). If the sequence $\{T_N(x)\}$ weakly converged to some (Gaussian) random variable $T(x)=(\xi^1(x), \xi^2(x))$, then the following limits would exist:

$$\lim_{N\to\infty} m_N^{(2,0)} = \lim_{N\to\infty} M\{(\xi_N^1(x))^2\} = \lim_{N\to\infty} M\{(\mathrm{Re}\,(T_N(x)))^2\},$$

$$\lim_{N\to\infty} m_N^{(0,2)} = \lim_{N\to\infty} M\{(\xi_N^2(x))^2\} = \lim_{N\to\infty} M\{(\mathrm{Im}(T_N(x)))^2\},$$

$$\lim_{N\to\infty} m_N^{(1,1)} = \lim_{N\to\infty} m\{(\xi_N^1(x))\cdot(\xi_N^2(x))\} = \lim_{N\to\infty} M\{(\mathrm{Re}(T_N(x)))(\mathrm{Im}(T_N(x)))\}.$$

These relations imply that under the assumption $T_N(x)\xrightarrow{d} T(x)$ there exists a finite limit of the value

$$M\{\mathrm{Re}^2(T_N(x)) - \mathrm{Im}^2(T_N(x)) + 2i\cdot\mathrm{Re}\,(T_N(x))\cdot\mathrm{Im}\,(T_N(x))\}$$

(where i denotes the imaginary unit).

In other words, the limit ought to exist

$$(6.4)\qquad \lim_{N\to\infty} M\{(T_N(x))^2\} = \lim_{N\to\infty}\int_0^1 (T_N(x))^2\,dx = \lim_{N\to\infty}\frac{1}{N}\sum_{1\le i,j\le N}\int_0^1 (\chi_{n_i}\cdot\chi_{n_j})\,dx.$$

The quantity of the non-zero summands of $\int_0^1 (\chi_{n_i}\chi_{n_j})dx$ is given by the number of the conjugate pairs (n_i, n_j) in our sequence $\{n_k\}$. By (6.1) it is easy to see that

$$\int_0^1 (T_{20}(x))^2\,dx = \frac{2}{20} = \frac{1}{10},$$

$$\int_0^1 (T_{200}(x))^2\,dx = \frac{182}{200} > \frac{9}{10};$$

and so on, generally,

$$(6.5)\qquad \int_0^1 (T_{2\cdot 10^{2l-1}}(x))^2\,dx \leqq \frac{1}{10},\quad \int_0^1 (T_{2\cdot 10^{2l}}(x))^2\,dx > \frac{9}{10}\quad \text{for all}\quad l = 1, 2, \dots.$$

Inequalities (6.5) show that limit (6.4) for the subsystem $\{\chi_{n_k}(x)\}$ does not exist. Therefore, the sequence $\{T_N(x)\}$ cannot converge (with respect to distribution) to any random variable $T(x)$. This contradiction proves that CLT does not hold for the subsystem $\{\chi_{n_k}(x)\}$, and this completes the proof.

Furthermore, for the proof of the necessity of (1.4), let us take the Chrestenson—Levy's system generated by $p_k\equiv 5$. As before we put $a_k=1$ for all k. The sequence $\{n_k\}$ will be defined in the following way:

$$n_1 = 2m_1,\quad n_2 = 4m_1,\quad n_{2k-1} = 2m_k\quad (m_k = 5^k, k = 1, 2, \dots)$$

$$(6.6)\qquad n_{2k} = \begin{cases} 3m_k+1, & 10^{2l} < k \leqq 10^{2l+1}; \\ 4m_k, & 10^{2l+1} < k \leqq 10^{2l+2} \end{cases}\quad l = 0, 1, \dots.$$

Let us verify the fulfilment of Theorem A. We have $n_{k+1}\geqq \frac{5}{4} n_k$ for all k, consequently we can put $\omega(k)\equiv 1/4$. Conditions (4)—(6) and (1.2) in this case are also fulfilled evidently. Condition (1.3) is fulfilled because there are no conjugate numbers in our sequence $\{n_k\}$ and by the same reason the limit in (1.3) is equal to zero.

But condition (1.4) does not fulfil. Indeed, the numbers $2m_k$ and $3m_k+1$ are $(0, k)$-adjoint, therefore

$$\lambda_k^0(q) = 1\quad\text{for}\quad q = 2k\quad\text{and}\quad 10^{2l}+1 \leqq k \leqq 10^{2l+1},$$

hence

$$\forall C > 1\quad \lambda_k^0(2k)\cdot C^k \neq O(1) = O\left\{\frac{1}{\omega(f(k))}\right\}\quad\text{as}\quad k\to\infty.$$

Let us show that CLT for $\{\chi_{n_k}(x)\}$ also does not hold. Now we consider the 6-th absolute moment of the random variable $T_N(x)$. We have

$$\mu_N^{(6)} = \int_0^1 |T_N(x)|^6\,dx = \frac{1}{N^3}\sum_{1\le i,j,h,p,q,r\le N}\int_0^1 (\chi_{n_p}\chi_{n_q}\chi_{n_r}\bar{\chi}_{n_h}\bar{\chi}_{n_i}\bar{\chi}_{n_j})\,dx.$$

Arguing as in the proof of the previous counterexample of the boundedness of $\mu_N^{(4)}$, we can see that the sequence $\{\mu_N^{(6)}\}$ is bounded. Now if we assume that $\{T_N(x)\}$ weakly converges to some Gaussian random variable $T(x)$ then (5.1) implies that the limit of the sequence $\{\mu_N^{(4)}\}$ exists. So we get

$$\mu_N^{(4)} = \int_0^1 |T_N(x)|^4\,dx = \frac{1}{N^2}\sum_{1\le i,j,p,q\le N}\int_0^1 (\chi_{n_p}\chi_{n_q}\bar{\chi}_{n_i}\bar{\chi}_{n_j})\,dx. \tag{6.7}$$

The definition of $\{n_k\}$ (see (6.6)) shows that the summands of the type $\int_0^1 (\chi_{n_p}\chi_{n_q}\bar{\chi}_{n_i}\bar{\chi}_{n_j})dx$ differ from zero if the fourfold product $\chi_{n_p}\chi_{n_q}\bar{\chi}_{n_i}\bar{\chi}_{n_j}$ either consists of the factors belonging to the same block or this product decomposes into two pairs of the factors such that each of the pairs belong to different block. Therefore, if $N=2M$ then we can rewrite (6.7) in the following form:

$$\mu_N^{(4)} = \frac{1}{4M^2}\int_0^1 \{|\sum_{k=0}^{M}\Delta_k|^2\}^2\,dx = \frac{1}{4M^2}\int_0^1 \{\sum_{k=1}^{M}\Delta_k\cdot\sum_{j=1}^{M}\bar{\Delta}_j\}^2\,dx = \tag{6.8}$$

$$= \frac{1}{4M^2}\sum_{k=1}^{M}\sum_{j=1}^{M}\left\{\int_0^1 (|\Delta_k|^2\cdot|\bar{\Delta}_j|^2+(\Delta_k)^2\cdot(\bar{\Delta}_j)^2)\,dx\right\} =$$

$$= \frac{1}{4M^2}\sum_{k=1}^{M}\sum_{j=1}^{M}\int_0^1 (|\Delta_k|^2\cdot|\bar{\Delta}_j|^2)\,dx + \frac{1}{4M^2}\sum_{k=1}^{M}\sum_{j=1}^{M}\int_0^1 (\Delta_k^2\cdot\bar{\Delta}_j^2)\,dx = L_M^{(1)}+L_M^{(2)}.$$

By a direct calculation it is possible to see that

$$\int_0^1 |\Delta_k|^2\cdot|\bar{\Delta}_j|^2\,dx = \begin{cases}4, & \text{if } k\ne j,\\ 6, & \text{if } k=j.\end{cases}$$

Hence

$$L_M^{(1)} = \frac{1}{4M^2}\left\{\sum_{k=1}^{M}\int_0^1 |\Delta_k|^2\cdot|\bar{\Delta}_k|^2\,dx + \sum_{k=2}^{M}\sum_{j=1}^{k-1}\int_0^1 |\Delta_k|^2\cdot|\bar{\Delta}_j|^2\,dx\right\} = \tag{6.9}$$

$$= \frac{1}{4M^2}\left(6M+4\cdot\frac{M(M-1)}{2}\right) = \frac{1}{2}+\frac{1}{M}.$$

So, if $M\to\infty$ (i.e. as $N\to\infty$)

$$\lim_{M\to\infty} L_M^{(1)} = \frac{1}{2}. \tag{6.9}$$

At the same time we have

(6.10)
$$\int_0^1 (\Delta_k)^2\cdot(\Delta_j)^2\,dx = \begin{cases} 6, & \text{if } k=j; \\ 4, & \text{if } \begin{cases} 10^{2l} < k \leqq 10^{2l+1}; \\ 10^{2m} < j \leqq 10^{2m+1}; \end{cases} \quad l, m = 0, 1, \dots;\ l \neq m: \\ 0, & \text{othervise} \end{cases}$$

Therefore

$$L_{10}^{(2)} \geqq \frac{1}{400}\sum_{k=2}^{10}\sum_{j=2}^{10}\int_0^1 (\Delta_k^2\cdot\bar{\Delta}_j^2)\,dx = \frac{4\cdot 9\cdot 9}{400} = \frac{81}{100},$$

$$L_{100}^{(2)} = \frac{1}{40\,000}(6\cdot 100+4\cdot 9\cdot 9) = \frac{231}{10\,000} < \frac{3}{100},$$

and so on, generally,

(6.11)
$$L_{10^{2l+1}}^{(2)} \geqq (4\cdot 10^{4l+2})^{-1}\cdot\sum_{k=10^{2l}+1}^{10^{2l+1}}\sum_{j=10^{2l}+1}^{10^{2l+1}}\int_0^1 (\Delta_k^2\cdot\bar{\Delta}_j^2)\,dx = \frac{4\cdot(9\cdot 10^{2l})^2}{4\cdot 10^{4l+2}} = \frac{81}{100};$$

(6.12) $$L_{10^{2l+2}}^{(2)} \leqq \frac{1}{4\cdot 10^{4l+4}}\cdot\Big(6\cdot\sum_{k=10^{2l}+1}^{10^{2l+1}}\sum_{j=10^{2l}+1}^{10^{2l+1}}\int_0^1 (\Delta_k^2\cdot\bar{\Delta}_j^2)\,dx+6\cdot 10^{2l+2}\Big) =$$

$$= \frac{6\cdot(10^{2l+2}+10^{4l+2})}{4\cdot 10^{4l+4}} \leqq \frac{12\cdot 10^{4l+2}}{4\cdot 10^{4l+4}} = \frac{3}{100}$$

for all $l=0, 1, 2, \dots$.

Inequalities (6.11)—(6.12) show that the value $L_M^{(2)}$ has no limit if $M\to\infty$, and so if $N\to\infty$. Comparing it with (6.9) and (6.10) we can conclude that the sequence $\{\mu_N^{(4)}\}$ also diverges as $N\to\infty$. The obtained contradiction (with assumption about the weak convergence of $\{T_N(x)\}$) implies that the subsystem $\{\chi_{n_k}(x)\}$ is not subjected to CLT as desired.

Theorem B is proved completely.

Remark 6.1. Theorems A and B demonstrate that the known results on CLT with respect to real-valued orthonormal systems (for example, trigonometric system or Walsh's system) have no direct analogues in the case of general PMONS. Namely, in order to prove the validity of CLT in our case, it is not sufficient to know the ratio of the lacunarity of $\{n_k\}$ and the magnitude of the coefficients $\{a_k\}$ but we have to know certain facts about the existence and the regularity of the conjugate and the (l, k)-adjoint numbers in the sequence $\{n_k\}$. We also mention that in our case it can occur, despite a very good lacunarity of $\{n_k\}$, that conditions (1.3) and (1.4) are not fulfilled independently of each other.

It should be noted that some problems, closely connected with them here, were studied in [4]—[5], but they were formulated in a different way; in addition, for the sequences $\{n_k\}$ there were assumed certain "arithmetical" conditions.

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USSR 600005, VLADIMIR 5
PROSPECT STROITELEY 85
POLYTECHNICAL INSTITUTE
CHAIR OF HIGHER MATHEMATICS

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Замыкание в операторной области

Э. Л. ПЕКАРЕВ

Пусть $\mathfrak{H}$ — сепарабельное гильбертово пространство, $\mathscr{B}=\mathscr{B}(\mathfrak{H})$ — совокупность всех линейных ограниченных операторов, действующих в $\mathfrak{H}$, $\mathscr{B}_+=\mathscr{B}_+(\mathfrak{H})$ — подмножество $\mathscr{B}(\mathfrak{H})$, состоящее из неотрицательных операторов. По аналогии со скалярным случаем операторным сегментом $[O, R]$, где $R\in\mathscr{B}_+$ назовем множество, определяемое равенством

$$[O, R] = \{X\in\mathscr{B} \mid O \leqq X \leqq R\},$$

а совокупность его крайних точек обозначим через $\operatorname{ex}[O, R]$.

В настоящей заметке рассматривается топология множества $\mathfrak{R}=\operatorname{ran} R^{1/2}$, в которой система замкнутых подпространств есть $\{\operatorname{ran} X^{1/2} \mid X\in\operatorname{ex}[O, R]\}$. Охарактеризован класс $\{\gamma_p\mathfrak{H}\}$ областей значений операторов, принадлежащих счетно-нормированному идеалу Шэттена γ_p $(1\leqq p<\infty)$. Описана структура замкнутых операторов, определенных на областях из $\{\gamma_p\mathfrak{H}\}$ $(1\leqq p<\infty)$.

Напомним, что если $\mathscr{L}(\subset\mathfrak{H})$ — операторная область (то есть область значений оператора из $\mathscr{B}$), то свертка $R_{\mathscr{L}}$ оператора $R\in\mathscr{B}_+(\mathfrak{H})$ на $\mathscr{L}$ — это оператор

(1) $$R_{\mathscr{L}} = R^{1/2}P_M R^{1/2},$$

где P_M — ортопроектор на подпространство $M=(R^{-1/2}\mathscr{L})^-$. Описание множества $\operatorname{ex}[O, R]$ и свойств свертки можно найти в статьях [3, 8] и цитированной в них литературе. В частности, отметим используемое в дальнейшем равенство

(2) $$\operatorname{ex}[O, R] = \{R_{\mathscr{L}} \mid \mathscr{L} \text{ — операторная область из } \operatorname{ran} R^{1/2}\}.$$

Через $\mathscr{C}(\mathfrak{H}_1, \mathfrak{H}_2)$ обозначается совокупность всех линейных замкнутых операторов, действующих из $\mathfrak{H}_1$ в $\mathfrak{H}_2$, и при $\mathfrak{H}_1=\mathfrak{H}_2=\mathfrak{H}$ полагается $\mathscr{C}(\mathfrak{H})=\mathscr{C}(\mathfrak{H}, \mathfrak{H})$.

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1. Пусть $R\in\mathscr{B}_+$ и $\mathfrak{R}=\operatorname{ran}R^{1/2}$. Согласно [9] на $\mathfrak{R}$ можно определить новую норму $\|\cdot\|'$ так, чтобы $\mathfrak{R}'=(\mathfrak{R},\|\cdot\|')$ являлось гильбертовым пространством и для исходной нормы $\|\cdot\|$ выполнялось соотношение

$$(3)\qquad \|f\|\leqq\|f\|'\quad(\forall f\in\mathfrak{R}).$$

Обратно, если на некотором линеале $\mathfrak{R}\subset\mathfrak{H}$ определена такая норма $\|\cdot\|'$, что $\mathfrak{R}'=(\mathfrak{R},\|\cdot\|')$ — гильбертово пространство и справедливо (3), то, следуя [1], рассмотрим инъекцию $S\colon\mathfrak{R}'\to\mathfrak{H}$

$$Sf=f\quad(\forall f\in\mathfrak{R})$$

и ее сопряженный оператор $S^+\colon\mathfrak{H}\to\mathfrak{R}'$. Положив $R=SS^+$, так что $0\leqq R\leqq I$, получим:

$$\operatorname{ran}R^{1/2}=\operatorname{ran}(SS^+)^{1/2}=\operatorname{ran}S=\mathfrak{R}.$$

При этом из равенства $(Ru,v)=(S^+u,S^+v)$ $(u,v\in\mathfrak{H})$ легко вытекает представление

$$(4)\qquad (f,g)'=(R^{-1/2}f,R^{-1/2}g)\quad(\forall f,g\in\mathfrak{R}),$$

где (однозначный) оператор $R^{-1/2}$ действует из $\mathfrak{R}$ в $\mathfrak{R}^-$.

Ясно, что оператор $R\in[O,I]$, $\operatorname{ran}R^{1/2}=\mathfrak{R}$, условием (4) определяется единственным образом. Этот оператор назовем метрическим для гильбертова пространства $\mathfrak{R}'=(\mathfrak{R},\|\cdot\|')$. Очевидно, множество $\mathscr{M}(\mathfrak{R})=\{R\in\mathscr{B}_+|\operatorname{ran}R^{1/2}=\mathfrak{R}\}$ выпукло, $\operatorname{ex}\mathscr{M}(\mathfrak{R})=\emptyset$.

Из теоремы о замкнутом графике следует, что если на $\mathfrak{R}$ заданы две нормы $\|\cdot\|'(\geqq\|\cdot\|)$ и $\|\cdot\|''(\geqq\|\cdot\|)$, относительно которых $\mathfrak{R}$ становится гильбертовым пространством, то эти нормы эквивалентны. Значит, для любого множества $\mathscr{L}\subset\mathfrak{R}$ его замыкания в пространствах $\mathfrak{R}'=(\mathfrak{R},\|\cdot\|')$ и $\mathfrak{R}''=(\mathfrak{R},\|\cdot\|'')$ совпадают; это замыкание обозначается через $[\mathscr{L}]_{\mathfrak{R}}$.

Определение. $\mathfrak{R}$-замыканием множества $\mathscr{L}(\subset\mathfrak{R})$ в $\mathfrak{R}$ называется множество $[\mathscr{L}]_{\mathfrak{R}}$.

Отметим простейшие свойства $\mathfrak{R}$-замыкания линеалов $\mathscr{L}\subset\mathfrak{R}$ [7]:

1) $\mathscr{L}\subset[\mathscr{L}]_{\mathfrak{R}}\subset\mathscr{L}^-$; 2) $[\mathscr{L}]_{\mathfrak{R}}=\mathscr{L}^-\Leftrightarrow\mathscr{L}^-\subset\mathfrak{R}$;

3) $\mathfrak{R}^-=\mathfrak{R}\Rightarrow[\mathscr{L}]_{\mathfrak{R}}=\mathscr{L}^-$.

Легко видеть, что если $R\in\mathscr{B}_+$, $\operatorname{ran}R^{1/2}=\mathfrak{R}$, то оператор $R^{1/2}$ взаимно однозначно и непрерывно отображает $\mathfrak{R}^-$ на $\mathfrak{R}'$, и, следовательно,

$$(5)\qquad [\mathscr{L}]_{\mathfrak{R}}=R^{1/2}(R^{-1/2}\mathscr{L})^-.$$

Кроме того, ясно, что линеал $\mathscr{L}\subset\mathfrak{R}$ является операторной областью в $\mathfrak{H}$ точно тогда, когда $\mathscr{L}$ — операторная область в $\mathfrak{R}'$.

Лемма 1 ([7]). *Если* $R\in\mathscr{B}_+$ *и* $\operatorname{ran} R^{1/2}=\mathfrak{R}$, *то для любой операторной области* $\mathscr{L}\subset\mathfrak{R}$ *справедливо равенство* $\operatorname{ran} R_{\mathscr{L}}^{1/2}=[\mathscr{L}]_{\mathfrak{R}}$.

Доказательство вытекает непосредственно из (1) и (5):

$$\operatorname{ran} R_{\mathscr{L}}^{1/2} = \operatorname{ran}(R^{1/2}P_M R^{1/2})^{1/2} = \operatorname{ran} R^{1/2}P_M = R^{1/2}(R^{-1/2}\mathscr{L})^- = [\mathscr{L}]_{\mathfrak{R}}.$$

Замечание. Из (1) и (5) вытекает, очевидно, что $R_{\mathscr{L}}=R_{[\mathscr{L}]_{\mathfrak{R}}}$, хотя как было отмечено в [6, 8], вообще говоря $R_{\mathscr{L}}\neq R_{\mathscr{L}^-}$.

Следствие. *Если* R — *метрический оператор для* $\mathfrak{R}'$, *то для любой операторной области* $\mathscr{L}\subset\mathfrak{R}$ *ее ортогональное дополнение в* $\mathfrak{R}'$ *совпадает с* $\operatorname{ran}(R-R_{\mathscr{L}})^{1/2}$.

Действительно, считая без ограничения общности $\mathscr{L}=[\mathscr{L}]_{\mathfrak{R}}$, рассмотрим вытекающие из (1) представления

$$R_{\mathscr{L}} = R^{1/2}PR^{1/2}, \quad R-R_{\mathscr{L}} = R^{1/2}QR^{1/2},$$

где P и Q — ортопроекторы на $(R^{-1/2}\mathscr{L})^-$ и $\mathfrak{R}^-\ominus(R^{-1/2}\mathscr{L})^-$ соответственно. Обозначив $\mathfrak{M}=\operatorname{ran}(R-R_{\mathscr{L}})^{1/2}$, получим, очевидно, что

$$\mathscr{L}\dotplus\mathfrak{M} = \mathfrak{R},$$

и если $f\in\mathscr{L}$, $g\in\mathfrak{M}$, то $f=R^{1/2}Pu$, $g=R^{1/2}Qv$ при некоторых $u,v\in\mathfrak{R}^-$, так что

$$(f,g)' = (R^{-1/2}f, R^{-1/2}g) = (Pu, Qv) = 0.$$

Пусть R — метрический оператор для гильбертова пространства $\mathfrak{R}'=(\mathfrak{R}, \|\cdot\|')$, удовлетворяющего условию (3), и $\mathscr{L}$ — операторная область из $\mathfrak{R}$. Если $\mathscr{L}'=(\mathscr{L}, \|\cdot\|')$ является гильбертовым пространством, то есть $\mathscr{L}=[\mathscr{L}]_{\mathfrak{R}}$, то его метрический оператор обозначим через $X(\mathscr{L}')$; совокупность всех таких метрических операторов обозначим через $\mathscr{X}(\mathfrak{R}')(=\{X(\mathscr{L}')|\mathscr{L}=[\mathscr{L}]_{\mathfrak{R}}\})$.

Теорема 1. $\mathscr{X}(\mathfrak{R}')=\operatorname{ex}[O, R]$.

Доказательство. Согласно (2), $\operatorname{ex}[O,R]$ состоит из сверток оператора R на всевозможные операторные области $\mathscr{L}\subset\mathfrak{R}$, так что ввиду замечания к лемме 1 имеем:

$$\operatorname{ex}[O, R] = \{R_{\mathscr{L}}|\mathscr{L}=[\mathscr{L}]_{\mathfrak{R}}\}.$$

С другой стороны, если $X=X(\mathscr{L}')$ — метрический оператор для $\mathscr{L}'$ $(\mathscr{L}=[\mathscr{L}]_{\mathfrak{R}})$, то

$$(X^{-1/2}f, X^{-1/2}g) = (R^{-1/2}f, R^{-1/2}g) \quad (\forall f,g\in\mathscr{L}).$$

Значит, для любых $u,v\in\mathscr{L}^-$,

$$(u,v) = (R^{-1/2}X^{1/2}u, R^{-1/2}X^{1/2}v),$$

и оператор $\omega = R^{-1/2}X^{1/2} (\in \mathscr{B})$ изометрически отображает $\mathscr{L}^{-} \to R^{-1/2}\mathscr{L}$, а $\omega(\mathfrak{H}\ominus\mathscr{L}) = \{0\}$. Поэтому $P = \omega\omega^*$ — ортопроектор на подпространство $R^{-1/2}\mathscr{L}$ и, следовательно,

$$X = R^{-1/2}\omega\omega^* R^{1/2} = R^{1/2}PR^{1/2} = R_{\mathscr{L}}.$$

Доказательство законечно.

Следствие 1. *Если R_1 и R_2 — метрические операторы для гильбертовых пространств $\mathfrak{R}_1'$ и $\mathfrak{R}_2''$ соответственно, причем $R_1 \leqq R_2$, то $\mathscr{X}(\mathfrak{R}_1') \subset \mathscr{X}(\mathfrak{R}_2'')$ тогда и только тогда, когда*

$$\operatorname{ran}(R_2 - R_1)^{1/2} \cap \operatorname{ran} R_1^{1/2} = \{0\}. \tag{6}$$

В самом деле, в силу доказанной теоремы нужно убедиться, что $\operatorname{ex}[O, R_1] \subset \subset \operatorname{ex}[O, R_2]$ точно тогда, когда имеет место равенство (6). Но согласно [3, 8] выполнение (6) равносильно тому, что $R_1 \in \operatorname{ex}[O, R_2]$, а это в свою очередь эквивалентно включению $\operatorname{ex}[O, R_1] \subset \operatorname{ex}[O, R_2]$ (см. [8], замечание к теореме 3.2).

Следствие 2. *В условиях следствия* 1, *$\mathfrak{R}_1'$ является подпространством $\mathfrak{R}_2''$ точно тогда, когда выполняется равенство* (6).

Замечание. Если $R \in \mathscr{B}_+$, $\mathfrak{R} = \operatorname{ran} R^{1/2}$, то

$$\operatorname{ex}([O, R] \cap \mathscr{M}(\mathfrak{R})) = \{R\}.$$

Действительно, считая без ограничения общности, что $\mathfrak{R}^- = \mathfrak{H}$, получим представление

$$[O, R] \cap \mathscr{M}(\mathfrak{R}) = \bigcup_{0<\delta<1} R^{1/2}[\delta I, I] R^{1/2},$$

в силу которого для каждого оператора $R^{1/2}K_0R^{1/2} \in \operatorname{ex}([O, R] \cap \mathscr{M}(\mathfrak{R}))$ $(\delta_0 I \leqq K_0 \leqq I)$ имеет место включение $K_0 \in \operatorname{ex}[\delta I, I]$ $(0 < \delta \leqq \delta_0)$. Но согласно [8] (формула (3.8))

$$\operatorname{ex}[\delta I, I] = \{K \in \mathscr{B}_+ | K = (1-\delta)P + \delta I,\ P^* = P^2 = P\},$$

откуда легко вытекает, что $\bigcap_{0<\delta\leqq 0} \operatorname{ex}[\delta I, I] = \{I\}$. Таким образом, $\operatorname{ex}([O, R] \cap \cap \mathscr{M}(\mathfrak{R})) \subset \{R\}$; противоположное включение очевидно.

Воспользуемся еще одной характеристикой операторной области, приведенной в [9]: Линеал $\mathfrak{R}$ является операторной областью тогда и только тогда, когда существует последовательность $\{\mathfrak{Q}_n\}_{n\geqq 0}$ взаимно ортогональных подпространства $\mathfrak{H}$ и убывающая числовая последовательность $\{\mu_n\}_{n\geqq 0}$ $(\mu_n \downarrow 0)$ такие, что

$$\mathfrak{R} = \Big\{ \sum_{n\geqq 0} x_n | x_n \in \mathfrak{Q}_n,\ \sum_{n\geqq 0} (\mu_n^{-1} \|x_n\|)^2 < \infty \Big\}. \tag{7}$$

В этом случае, если $R=\sum_{n\geqq 0}\mu_n Q_n$, где Q_n — ортопроектор на $\mathfrak{Q}_n$ $(n\geqq 0)$, то $\operatorname{ran} R^{1/2}=\mathfrak{R}$.

Рассмотрим две последовательности ортопроекторов $\{P_n\}_{n\geqq 0}$ и $\{Q_n\}_{n\geqq 0}$ такие, что $P_iP_j=O$, $Q_iQ_j=O$ $(i\neq j)$ и положим

(8) $$L=\sum_{n\geqq 0}\lambda_n^2 P_n,\quad R=\sum_{n\geqq 0}\mu_n^2 Q_n,$$

где числовые последовательности $\{\lambda_n\}_{n\geqq 0}$ и $\{\mu_n\}_{n\geqq 0}$ монотонно убывая стремятся к нулю. Обозначив $\mathscr{L}=\operatorname{ran} L^{1/2}$, $\mathfrak{R}=\operatorname{ran} R^{1/2}$ заметим [2], что если $L\leqq R$, то $\mathscr{L}\subset\mathfrak{R}$.

Теорема 2. *Пусть операторы L и R из* (8) *— метрические для гильбертовых пространств $\mathscr{L}'$ и $\mathfrak{R}''$ соответственно, причем $L\leqq R$. Тогда $\mathscr{L}'$ — подпространство $\mathfrak{R}''$ точно в том случае, если существует последовательность ортопроекторов $\{\pi_j\}_{j\geqq 0}$ удовлетворяющая условиям*

(9) $$\operatorname{ran}\pi_j\subset\mathfrak{R}^-\quad (j\geqq 0),\ \pi_j\pi_k=O\quad (j\neq k),$$
$$\mu_i\mu_k Q_i\pi_j Q_k=\lambda_j^2 Q_iP_jQ_k\quad (i,j,k\geqq 0),$$

Доказательство. Допустим сперва, что $\mathscr{L}'$ — подпространство $\mathfrak{R}''$. Тогда, ввиду теоремы 1 и ее следствий, если $\mathfrak{M}\subset\mathscr{L}$ и $[\mathfrak{M}]_{\mathscr{L}}=\mathfrak{M}$, то $L_{\mathfrak{M}}=R_{\mathfrak{M}}$. В частности, при $\mathfrak{M}_j=P_j\mathfrak{H}$ $(j\geqq 0)$ получаем:

(10) $$L_{\mathfrak{M}_j}=R_{\mathfrak{M}_j}=R^{1/2}\pi_j R^{1/2},$$

где в соответствии с (1) π_j — ортопроектор на подпространство $R^{-1/2}\mathfrak{M}_j\subset\mathfrak{R}^-$. Покажем, что последовательность $\{\pi_j\}_{j\geqq 0}$ удовлетворяет условиям (9). Действительно, из (8) и (1) вытекает, что

$$L_{\mathfrak{M}_j}=\lambda_j^2P_j,\ L_{\mathfrak{M}_j+\mathfrak{M}_k}=\lambda_j^2P_j+\lambda_k^2P_k\quad (k\neq j)$$

так что согласно (10)

(11) $$\lambda_j^2P_j=R^{1/2}\pi_jR^{1/2},\ \lambda_j^2P_j+\lambda_k^2P_k=R^{1/2}(\pi_j+\pi_k)R^{1/2}\quad (k\neq j).$$

Но поскольку $[\mathfrak{M}_j+\mathfrak{M}_k]_{\mathscr{L}}=\mathfrak{M}_j+\mathfrak{M}_k$ и $\operatorname{ran}(\pi_j+\pi_k)\subset\mathfrak{R}^-$, то $\pi_j+\pi_k$ — ортопроектор, а это возможно только если $\pi_j\pi_k=O$ $(j\neq k)$. Наконец, домножив обе части первого из равенств (11) слева на Q_i и справа на Q_k, получим с учетом (8), что

$$\lambda_j^2Q_iP_jQ_k=\mu_i\mu_kQ_i\pi_jQ_k\quad (i,j,k\geqq 0).$$

Таким образом, выполнены все условия (9).

Обратно, из (9) вытекают равенства

$$Q_iLQ_k=\mu_i\mu_kQ_iPQ_k\quad (i,k\geqq 0),$$

где $P=\sum\limits_{j\geqq 0}\pi_j$ — ортопроектор, $P\mathfrak{H}\subset\mathfrak{R}^-$. Отсюда, учитывая включения $\operatorname{ran} L\subset\subset\mathscr{L}\subset\mathfrak{R}^-=\operatorname{ran}\sum\limits_{i\geqq 0}Q_i$, легко получить равенство $L=R_{\mathscr{L}}$, означающее, что $\mathscr{L}'$ — подпространство $\mathfrak{R}''$.

Замечание. Вообще говоря, не всякий оператор $R\in\mathscr{M}(\mathfrak{R})$ представим в виде (8). Однако, если $\mathfrak{R}$ — область значений некоророго вполне непрерывного оператора, то R также вполне непрерывен и, следовательно, допускает представление (8), причем $\dim Q_n\mathfrak{H}<\infty$ $(n\geqq 0)$. Ясно, что тогда для любого оператора $L\in[O,R]$ справедливо разложение (8), где $\dim P_n\mathfrak{H}<\infty$ $(n\geqq 0)$.

2. Напомним [9], что линеал $\mathfrak{R}\subset\mathfrak{H}$ является операторной областью тогда и только тогда, когда существует оператор $T\in\mathscr{C}(\mathfrak{H})$ с областью определения $\vartheta(T)=\mathfrak{R}$. Введем гильбертово пространство $(\mathfrak{R},\|\cdot\|_T)$, где

$$\|f\|_T^2=\|f\|^2+\|Tf\|^2 \quad (f\in\mathfrak{R}).$$

и обозначим через $R(T)$ соответствующий метрический оператор.

Если $\vartheta(T)^-\neq\mathfrak{H}$, то рассматривая оператор T как элемент множества $\mathscr{C}(\vartheta(T)^-,\mathfrak{H})$, обозначим его сопряженный через $T^*(\in\mathscr{C}(\mathfrak{H},\vartheta(T)^-))$, а абсолютную величину — через $|T|=(T^*T)^{1/2}$ [7].

Лемма 2. *Если $T\in\mathscr{C}(\mathfrak{H})$ и $\vartheta(T)=\mathfrak{R}$, то*

$$|T|=(I-R(T))^{1/2}R(T)^{-1/2},\quad R(T)=(I+T^*T)^{-1}I_{\mathfrak{R}}. \tag{12}$$

Доказательство. Поскольку $(f,g\in\mathfrak{R})$

$$(f,g)+(Tf,Tg)=(R(T)^{-1/2}f,R(T)^{-1/2}g),$$

то

$$(Tf,Tg)=((I-R(T))^{1/2}R(T)^{-1/2}f,\ (I-R(T))^{1/2}R(T)^{-1/2}g),$$

и первое из равенств (12) справедливо, так как $(I-R(T))^{1/2}R(T)^{-1/2}$ — самосопряженный неотрицательный оператор в $\mathfrak{R}^-$. Справедливость второго равенства столь же очевидна.

Следствие 1. $\mathscr{M}(\mathfrak{R})=\{R(T)|T\in\mathscr{C}(\mathfrak{H}),\vartheta(T)=\mathfrak{R}\}$.

Следствие 2. *Если T — самосопряженный неотрицательный оператор в $\mathfrak{H}$, то*

$$T=(I-R(T))^{1/2}R(T)^{-1/2},\quad R(T)=(I+T^2)^{-1}.$$

Следствие 3. *Если $T\in\mathscr{C}(\mathfrak{H})$ и* $\operatorname{ran}T^*\subset\vartheta(T)$, *то* $\vartheta(T)=\vartheta(T)^-$ *и T ограничен.*

Действительно, в этом случае согласно (12) $(R=R(T))$

$$(I-R)^{1/2}\mathfrak{R}^-=\operatorname{ran}T^*\subset\mathfrak{R}=R^{1/2}\mathfrak{R}^-,$$

так что $(I-R)\mathfrak{R}^{-}\subset R\mathfrak{R}^{-}$ и, следовательно, $\mathfrak{R}^{-}=\mathfrak{R}$, что и требовалось доказать.

Замечание. По существу, предыдущее утверждение, вполне элементарное, было отмечено ранее в [5].

Рассмотрим оператор $T\in\mathscr{C}(\mathfrak{H})$ с областью определения $\vartheta(T)=\mathfrak{R}$ и произвольный линеал $\mathscr{L}\subset\mathfrak{R}$. Ясно, что сужение $T\restriction\mathscr{L}$ — замыкаемый оператор, $T\restriction[\mathscr{L}]_{\mathfrak{R}}$ — его замыкание и, значит, $T\restriction\mathscr{L}\in\mathscr{C}(\mathfrak{H})$ точно тогда, когда $\mathscr{L}=[\mathscr{L}]_{\mathfrak{R}}$. Отметим, что согласно теореме 1 для любой $\mathfrak{R}$-замкнутой операторной области $\mathscr{L}$ соответствующий метрический оператор $R(T\restriction\mathscr{L})$ содержится во множестве ex $[O, R]$. Значит, в силу (2) и (1) справедливо равенство

$$R(T\restriction\mathscr{L}) = R(T)_{\mathscr{L}}. \tag{13}$$

Из (12) и (13) вытекает, что

$$|T_1| = (I-R_{\mathscr{L}})^{1/2}R_{\mathscr{L}}^{-1/2}, \quad R_{\mathscr{L}} = (I+T_1^*T_1)^{-1}I_{\mathscr{L}}, \tag{14}$$

где положено $R=R(T)$, $T_1=T\restriction\mathscr{L}$, $|T_1|$ — абсолютная величина оператора T_1 как элемента $\mathscr{C}(\mathscr{L}^{-}, \mathfrak{H})$.

Очевидно, если $T\in\mathscr{B}_{+}(\mathfrak{H})$, то из (12) вытекает, что T и R имеют общие инвариантные подпространства. В случае самосопряженного $T\in\mathscr{C}(\mathfrak{H})$ справедливо следующее утверждение.

Теорема 3. *Пусть T — самосопряженный неотрицательный оператор в $\mathfrak{H}$, $\vartheta(T)=\mathfrak{R}$, а $\mathscr{L}(\subset\mathfrak{R})$ — $\mathfrak{R}$-замкнутая операторная область. Тогда подпространство $\mathscr{L}^{-}$ инвариантно относительно T и оператор $T_1=T\restriction\mathscr{L}$ самосопряжен в $\mathscr{L}^{-}$ точно в том случае, если $\mathscr{L}$ инвариантно относительно $R=R(T)$ $(R\mathscr{L}\subset\mathscr{L})$.*

Доказательство. Очевидно, $R\mathscr{L}\subset\mathscr{L}$ точно тогда, когда подпространство $\mathfrak{P}=R^{-1/2}\mathscr{L}(\subset\mathfrak{R}^{-})$ инвариантно относительно R, и так как $R_{\mathscr{L}}=R^{1/2}PR^{1/2}$, где P — ортопроектор на $\mathfrak{P}$, то

$$R_{\mathscr{L}}f = R^{1/2}PR^{1/2}f = Rf \quad (\forall f\in\mathscr{L}).$$

Отсюда на основании первых равенств в (12′) и (14) заключаем, что если $\mathscr{L}$ инвариантно относительно R, то

$$Tf = |T_1|f \quad (\forall f\in\mathscr{L}).$$

Остается заметить, что $|T_1|$ самосопряжен в $\mathscr{L}^{-}$ и $\mathscr{L}=\vartheta(T)\cap\mathscr{L}^{-}$.

Обратно, пусть $\mathscr{L}^{-}$ инвариантно относительно T и T_1 самосопряжен в $\mathscr{L}^{-}$. Обозначив $\mathscr{L}_0=\operatorname{ran} R_{\mathscr{L}}$, получим ввиду второго равенства в (14), что

$$(I+T^2)\mathscr{L}_0 = (I+T_1^2)\mathscr{L}_0 = \mathscr{L}^{-}.$$

Следовательно, в силу второго равенства в (12), $R\mathscr{L}\subset R\mathscr{L}^-\subset\mathscr{L}^-$, и так как $\mathscr{L}^-\cap\mathfrak{R}=\mathscr{L}^-\cap\vartheta(T)=\mathscr{L}$, то $R\mathscr{L}\subset\mathscr{L}$, что и требовалось доказать.

Замечание. Если в условиях теоремы $\mathfrak{R}\neq\mathfrak{H}$, то легко привести пример плотной в $\mathfrak{H}$ $\mathfrak{R}$-замкнутой операторной области $\mathscr{L}(\subset\mathfrak{R})$, не инвариантной относительно R; именно $\mathscr{L}=R^{1/2}(\mathfrak{H}\ominus\{e\})$, где $e\notin\mathfrak{R}$.

3. Рассмотрим множество операторных областей гильбертова пространства $\mathfrak{H}$, определяемое следующим образом:

$$\{\gamma\mathfrak{H}\} = \{\mathfrak{R}|\mathfrak{R} = \operatorname{ran} A,\ A\in\gamma\},$$

где γ — некоторый двусторонний идеал в $\mathscr{B}$. Отметим, что если $\operatorname{ran} A=\operatorname{ran} B$, где $A\in\gamma, B\in\mathscr{B}$, то согласно [2] $B=AC$ при некотором $C\in\mathscr{B}$ и, следовательно, $B\in\gamma$. В частности, если $R\in[O, I]$ и $\mathfrak{R}=\operatorname{ran} R^{1/2}$, то $\mathfrak{R}\in\{\gamma_p\mathfrak{H}\}$ точно тогда, когда $R^{1/2}\in\gamma_p$ $(1\leqq p<\infty)$, Отсюда вытекает, что если $\mathfrak{R}\in\{\gamma_p\mathfrak{H}\}$ $(1\leqq p<\infty)$, то и любая операторная область $\mathscr{L}(\subset\mathfrak{R})$ принадлежит $\{\gamma_p\mathfrak{H}\}$. Очевидно также, что включение $\mathfrak{R}\in\{\gamma_p\mathfrak{H}\}$ $(1\leqq p<\infty)$ равносильно существованию представления (7), в котором

$$\dim\mathfrak{Q}_n<\infty \quad (n\geqq 0),\quad \sum_{n\geqq 0}\mu_n^p\dim\mathfrak{Q}_n<\infty.$$

Выбрав в $\mathfrak{H}$ ортонормированную систему векторов $\varepsilon=\{e_n\}_{n=0}^{\omega}$ и невозрастающую последовательность положительных чисел $M=\{\mu_n\}_{n=0}^{\omega}$ $(\omega\leqq\infty)$, определим операторную область $\mathfrak{R}(\varepsilon, M)$ равенством

$$\mathfrak{R}(\varepsilon, M) = \left\{\sum_{n=0}^{\omega}\alpha_n e_n \,\Big|\, \sum_{n=0}^{\omega}\left|\frac{\alpha_n}{\mu_n}\right|^2<\infty\right\}. \tag{15}$$

Ясно, что $\mathfrak{R}(\varepsilon, M)\in\{\gamma_\infty\mathfrak{H}\}$, причем $\mathfrak{R}(\varepsilon, M)\in\{\gamma_p\mathfrak{H}\}$ $(1\leqq p<\infty)$ тогда и только тогда, когда $M=\{\mu_n\}_{n\geqq 0}\in l_p$ (если $\omega<\infty$, то при $n>\omega$ полагаем $\mu_n=0$).

Из предыдущих рассуждений легко вытекает следующая

Лемма 3. *Для того чтобы* $\mathfrak{R}\in\{\gamma_p\mathfrak{H}\}$ $(1\leqq p<\infty)$, *необходимо чтобы для любой и достаточно, чтобы для какой-либо ортонормированной системы* $\varepsilon\subset\mathfrak{R}$ *существовала невозрастающая последовательность неотрицательных чисел* $M\in\ell_p$ *такая, что* $\mathfrak{R}=\mathfrak{R}(\varepsilon, M)$.

Замечание. Непосредственно из определения (15) видно, что если $M=\{\mu_n\}_{n=0}^{\infty}$ и $\Lambda=\{\lambda_n\}_{n=0}^{\infty}$, то $\mathfrak{R}(\varepsilon, \Lambda)\subset\mathfrak{R}(\varepsilon, M)$ точно тогда, когда $\{\lambda_n\mu_n^{-1}\}_{n=0}^{\infty}\in\ell_\infty$. В часности, равенство $\mathfrak{R}(\varepsilon, M)=\mathfrak{R}(\varepsilon, \Lambda)$ имеет место точно в том случае, если $\lambda_n\asymp\mu_n$ (то есть $\{\lambda_n\mu_n^{-1}\}$ и $\{\lambda_n^{-1}\mu_n\}_{n=0}^{\infty}\in\ell_\infty$).

Теорема 4. *Пусть T — самосопряженный неотрицательный оператор в* $\mathfrak{H}$ *с областью определения* $\vartheta(T)$ *и* $\operatorname{Ker} T=\{0\}$. *Тогда* $\vartheta(T)\in\{\gamma_p\mathfrak{H}\}$ $(1\leqq p<\infty)$

в том и только том случае, если $\mathfrak{D}(T)=\mathfrak{R}(\varepsilon, M)$, *где* $\varepsilon=\{e_n\}_{n=0}^{\infty}$ *— ортонормированный базис пространства* $\mathfrak{H}$, *состоящий из собственных векторов оператора* T, *причем соответствующие собственные числа* t_n $(n\geqq 0)$ *таковы, что* $\{(1+t_n^2)^{-1/2}\}\in\ell_p$.

Доказательство. Действительно, если $\mathfrak{D}(T)\in\{\gamma_p\mathfrak{H}\}$, $\operatorname{Ker} T=\{0\}$, то оператор $R=R(T)$ представим в виде (8), где

$$\sum_{n\geqq 0} Q_n = I, \ \dim Q_n = 1 \quad (n \geqq 0), \ \{\mu_n\}_{n\geqq 0}\in\ell_p.$$

Если $e_n\in Q_n\mathfrak{H}$, $\|e_n\|=1$ $(n\geqq 0)$, то ввиду теоремы 3 $Te_n=t_ne_n$ $(n\geqq 0)$, причем согласно (12) $\mu_n=(1+t_n^2)^{-1/2}$ $(n\geqq 0)$. Обратное утверждение также очевидно.

Следствие 1. *Если* $T_i=T_i^*\geqq O$, $\operatorname{Ker} T_i=\{0\}$ *и* $T_ie_n=t_{i,n}e_n$ $(i=1,2;\ n\geqq 0)$, *где* $\{e_n\}_{n\geqq 0}$ *— ортонормированный базис в* $\mathfrak{H}$, $\{(1+t_{i,n}^2)^{-1/2}\}\in\ell_p$, *то* $\mathfrak{D}(T_1)=\mathfrak{D}(T_2)$ *тогда и только тогда, когда* $t_{1,n}\asymp t_{2,n}$.

Следствие 2. *Если* $T\in\mathscr{C}(\mathfrak{H})$ *и* $\mathfrak{D}(T)\in\{\gamma_p\mathfrak{H}\}$, *то существуют такие ортонормированные системы* $\{e_n\}_{n=0}^{\omega}$ *и* $\{g_n\}_{n=0}^{\omega}$ $(\omega\leqq\infty)$, *полные в пространствах* $\mathfrak{D}(T)^-$ *и* $(\operatorname{ran} T)^-=\operatorname{ran} T$ *соответственно, что* $Te_n=t_ng_n$ $(0\leqq n\leqq\omega)$, *причем* $\{(1+t_n^2)^{-1/2}\}_{n\geqq 0}\in\ell_p$.

В самом деле, поскольку $\mathfrak{D}(|T|)=\mathfrak{D}(T)$, то применив доказанную теорему к оператору $|T|$ в пространстве $\mathfrak{D}(T)^-$, найдем полную в $\mathfrak{D}(T)^-$ ортонормированную систему векторов $\varepsilon=\{e_n\}_{n=0}^{\omega}$ $(\omega\leqq\infty)$, для которой $|T|e_n=t_ne_n$ $(0\leqq n\leqq\omega)$, $\{(1+t_n^2)^{-1/2}\}_{n\geqq 0}\in\ell_p$. Если же $T=U|T|$ — полярное представление оператора T, то положив $g_n=Ue_n$ $(0\leqq n\leqq\infty)$, получим:

$$Te_n = t_ng_n \quad (0 \leqq n \leqq \omega).$$

При этом согласно (15)

$$\mathfrak{D}(T) = \Big\{\sum_{n=0}^{\omega} \alpha_n e_n \Big| \sum_{n=0}^{\omega} (1+t_n^2)|\alpha_n|^2 < \infty\Big\}$$

и, следовательно,

$$\operatorname{ran} T = \Big\{\sum_{n=0}^{\omega} \alpha_n t_n g_n \Big| \sum_{n=0}^{\omega} (1+t_n^2)|\alpha_n|^2 < \infty\Big\}.$$

В силу последнего равенства, очевидно, $(\operatorname{ran} T)^-=\operatorname{ran} T$.

Замечание. Поскольку операторная область класса $\{\gamma_\infty\mathfrak{H}\}$ не содержит замкнутых бесконечномерных подпространств [9] (Теорема 2.5), то при $\dim\mathfrak{D}(T)^-=\infty$ в условиях предыдущего следствия $\operatorname{ran} T\not\subset\mathfrak{D}(T)$. Однако, если $\mathfrak{D}(T)=\mathfrak{D}_1\oplus\mathfrak{G}$, где $\mathfrak{D}_1\in\{\gamma_\infty\mathfrak{H}\}$, $\mathfrak{G}=\mathfrak{H}\ominus\mathfrak{D}_1$, а $T\mathfrak{D}_1\subset\mathfrak{G}$ и $T\mathfrak{G}=\{0\}$, то, очевидно, имеет место включение $\operatorname{ran} T\subset\mathfrak{D}(T)$ (ср. [5], пример 3.1).

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Models for operators with trivial residual space

BRIAN W. McENNIS

1. Introduction

As part of their study of contractions [12], Sz.-NAGY and FOIAŞ derive a functional model for an arbitrary completely non-unitary contraction T, in terms of its characteristic function Θ_T. Also, given an arbitrary purely contractive analytic operator-valued function $\Theta(\lambda)$: $\mathscr{D}\to\mathscr{D}_*$, where $\mathscr{D}$ and $\mathscr{D}_*$ are Hilbert spaces, they are able to construct a model for a completely non-unitary operator whose characteristic function coincides with Θ. The Sz.-Nagy and Foiaş model provides, in fact, a model for the unitary dilation of the contraction, with the model for the contraction itself being obtained by a compression.

In an extension of the dilation theory of Sz.-Nagy and Foiaş, DAVIS [4] has constructed a unitary dilation of an arbitrary operator, with the dilation space being a Krein space, and the dilation preserving the indefinite inner product. In a subsequent paper, DAVIS and FOIAŞ [5] showed how the characteristic function of a noncontraction could be given a geometric interpretation on the dilation space analogous to that used by Sz.-Nagy and Foiaş in their modelling of contractions.

Models have been developed for noncontractions ([1], [3], [8]), which are given in terms of their characteristic functions, but which are along the lines of the DE BRANGES—ROVNYAK model for a contraction [6], providing no model for the dilation. In [10], a model theory is given which does model the dilation space and uses the geometric interpretation of the characteristic function, in a manner analogous to the theory of Sz.-Nagy and Foiaş. As in [5], however, it is necessary in [10] to assume the boundedness of the characteristic function in order to be able to construct this model.

The boundedness of the characteristic function is used in [5] and [10] to ensure the boundedness of the Fourier representations, which map certain subspaces of the dilation space onto L^2 spaces, and to ensure that the characteristic func-

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tion acts as a bounded operator between these spaces. In this paper, we adopt the approach that L^2 spaces are not necessarily the natural ones to be used in modelling noncontractions, and design the function spaces to fit the operator and its characteristic function. Under these circumstances, it is not necessary to assume the boundedness of the characteristic function. Although the model obtained is based on the Sz.-Nagy and Foiaş model of a contraction, the function spaces involved can be defined from the characteristic function in terms of reproducing kernels, and are similar to those considered by de Branges and Rovnyak.

Let T be a bounded operator on a Hilbert space $\mathscr{H}$. Following [12], we write $T\in C_{\cdot 0}$ if $T^{*n}h\to 0$ for all $h\in\mathscr{H}$. When T has bounded characteristic function this condition is equivalent to a condition on the geometry of the dilation space, namely that the residual space be trivial (see [12], [9]). When the characteristic function is not bounded, the condition $T\in C_{\cdot 0}$ implies that the residual space is trivial, but it is possible for an operator not in $C_{\cdot 0}$ to have a trivial residual space.

In this paper, we concentrate on operators with trivial residual space, as the description of the function spaces needed for the model is simplest in this case. The model is also described in terms of an operator valued analytic function Θ, for which we have assumed properties that guarantee that it is the characteristic function of an operator with trivial residual space. The properties assumed for Θ are valid for the characteristic function of an arbitrary $C_{\cdot 0}$ operator; it is not known if only $C_{\cdot 0}$ operators have characteristic functions with these properties.

2. The dilation

In this section, we give a brief description of the Davis dilation of a bounded operator [4].

Using the selfadjoint functional calculus, we can define the operators

$$J_T = \operatorname{sgn}(I-T^*T), \quad Q_T = |I-T^*T|^{1/2},$$
$$J_{T^*} = \operatorname{sgn}(I-TT^*), \quad Q_{T^*} = |I-TT^*|^{1/2}.$$

We have (see [4])

$$J_T Q_T^2 = I-T^*T,\ J_{T^*}Q_{T^*}^2 = I-TT^*,$$
$$TJ_T = J_{T^*}T,\ TQ_T = Q_{T^*}T,\ T^*J_{T^*} = J_T T^*,\ T^*Q_{T^*} = Q_T T^*.$$

We equip the spaces

$$\mathscr{D}_T = J_T\mathscr{H} \quad \text{and} \quad \mathscr{D}_{T^*} = J_{T^*}\mathscr{H}$$

with the respective indefinite inner products

$$[x,y] = (J_T x, y) \quad x, y\in\mathscr{D}_T$$

and

$$[x, y] = (J_{T^*}x, y) \quad x, y \in \mathscr{D}_{T^*},$$

where $(.,.)$ denotes the inner product on $\mathscr{H}$. Then, with the topology inherited from $\mathscr{H}$, $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$ become Krein spaces, with fundamental symmetries J_T and J_{T^*} (see [2]).

The Davis dilation of T is a bounded operator U, which acts on a Krein space $\mathscr{K} \supseteqq \mathscr{H}$, with Hilbert space inner product $(.,.)$ and indefinite inner product $[.,.]$ linked by a fundamental symmetry J:

$$(x, y) = [Jx, y], \quad [x, y] = (Jx, y) \quad \text{for all } x, y \in \mathscr{K}.$$

U is boundedly invertible, and the following properties hold:

(i) $(U^n x, y) = (T^n x, y)$ for all $x, y \in \mathscr{H}$;

(ii) $[Ux, Uy] = [x, y]$ for all $x, y \in \mathscr{K}$;

(iii) $Jx = x$ for all $x \in \mathscr{H}$;

(iv) $\bigvee \{U^n \mathscr{H}: -\infty < n < \infty\} = \mathscr{K}$.

Consider the subspaces

$$\mathscr{L} = \overline{(U-T)\mathscr{H}}, \ \mathscr{L}^* = \overline{(U^* - T^*)\mathscr{H}}, \ \mathscr{L}_* = U\mathscr{L}^* = \overline{(I-UT^*)\mathscr{H}}.$$

(Here, and in the sequel, adjoints are assumed to be taken in the indefinite inner product $[.,.]$ of $\mathscr{K}$.) In the Davis dilation, $\mathscr{L}$ and $\mathscr{L}^*$ are isomorphic to $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$, respectively, in the sense of both the Hilbert space and Krein space structures. There is an operator φ, mapping $\mathscr{L}$ onto $\mathscr{D}_T$, and preserving both the Hilbert space and indefinite inner products, such that

$$\varphi(U-T)h = Q_T h \quad \text{for every } h \in \mathscr{H}. \tag{2.1}$$

As in [12], it is more convenient to work with $\mathscr{L}_*$ than with $\mathscr{L}^*$. We consider an operator φ_*, mapping $\mathscr{L}_*$ onto $\mathscr{D}_{T^*}$, such that $U^*\varphi_*$ preserves both the Hilbert space and indefinite inner products, and such that

$$\varphi_*(I-UT^*)h = J_{T^*}Q_{T^*}h \quad \text{for every } h \in \mathscr{H}.$$

Because of property (ii) above, φ_* also preserves the indefinite inner product.

As in [12], $\mathscr{L}$ and $\mathscr{L}^*$ are each orthogonal to $\mathscr{H}$, with respect to both the Hilbert space and the indefinite inner product on $\mathscr{K}$. Consequently, $\mathscr{L}_* = U\mathscr{L}^* \perp U\mathscr{H}$, where we are using "$\perp$" here, and in the sequel, to denote orthogonality with respect to the indefinite inner product. Also, both $\mathscr{L}$ and $\mathscr{L}_*$ are wandering for U, i.e. $U^m \mathscr{L} \perp U^n \mathscr{L}$ and $U^m \mathscr{L}_* \perp U^n \mathscr{L}_*$ for $m \neq n$ (see [7]).

We define

$$\mathcal{K}_+ = \bigvee\{U^n\mathcal{H}: n \geqq 0\},$$
$$M_+(\mathcal{L}) = \bigvee\{U^n\mathcal{L}: n \geqq 0\},$$
$$M_+(\mathcal{L}_*) = \bigvee\{U^n\mathcal{L}_*: n \geqq 0\}.$$

In the Davis dilation, the spaces $\mathcal{H}$ and $M_+(\mathcal{L})$ are mutually orthogonal, in both the Hilbert space and indefinite inner products, and we have

$$\mathcal{K}_+ = \mathcal{H} \oplus M_+(\mathcal{L}).$$

We also have $\mathcal{L}^* \perp \mathcal{K}_+$, and thus

$$\mathcal{L}_* \perp UM_+(\mathcal{L}). \tag{2.2}$$

The *residual space* $\mathcal{R}$ is defined as the space of all vectors in $\mathcal{K}_+$ which are orthogonal to $M_+(\mathcal{L}_*)$ in the indefinite inner product:

$$\mathcal{R} = K_+ \cap M_+(\mathcal{L}_*)^\perp.$$

Theorem 2.1. *If* $T \in C_{\cdot 0}$, *then* $\mathcal{R} = \{0\}$, *i.e.,* $M_+(\mathcal{L}_*) = \mathcal{K}_+$.

Proof. See [9], Theorem 5.5.

Let Q denote the orthogonal projection onto $\mathcal{L}$; in the Davis dilation, this projection is selfadjoint in both inner products. For any $k \in M_+(\mathcal{L})$, the *Fourier coefficients of k in* $M_+(\mathcal{L})$ are defined by

$$l_n = QU^{*n}k, \quad n \geqq 0.$$

The vector k is uniquely determined by its sequence of Fourier coefficients in $M_+(\mathcal{L})$. (See [7].) In the Davis dilation we have, for $k \notin M_+(\mathcal{L})$,

$$\|k\|^2 = \sum_{n=0}^{\infty} \|\varphi l_n\|^2 \quad \text{and} \quad [k, k] = \sum_{n=0}^{\infty} [\varphi l_n, \varphi l_n]. \tag{2.3}$$

The *Fourier representation of k in* $M_+(\mathcal{L})$ is the operator Φ mapping $k \in M_+(\mathcal{L})$ to the function $\Phi k = u$, where

$$u(\lambda) = \sum_{n=0}^{\infty} \lambda^n \varphi l_n$$

and $\{l_n\}$ is the sequence of Fourier coefficients of k in $M_+(\mathcal{L})$. The function u takes its values in $\mathcal{D}_T$ and, because of (2.3), is defined in a neighborhood of zero which includes the open unit disc.

Similarly, for any $k \in M_+(\mathcal{L}_*)$, the *Fourier coefficients of k in* $M_+(\mathcal{L}_*)$ are defined by

$$l_{*n} = PU^{*n}k, \quad n \geqq 0,$$

where P is the orthogonal projection onto $\mathcal{L}_*$: P is selfadjoint in the indefinite inner product, and U^*PU (the orthogonal projection onto $\mathscr{L}^*$) is selfadjoint in both inner products. The structure of $M_+(\mathscr{L}_*)$ can be much more complicated than that described by (2.3) for $M_+(\mathscr{L})$; this will be investigated in subsequent sections. In particular, it is possible to have a vector $k\in M_+(\mathscr{L}_*)$ with the property that $[k, m]=0$ for all $m\in M_+(\mathscr{L}_*)$ (we then call $M_+(\mathscr{L}_*)$ *degenerate*); such a vector k has all of its Fourier coefficients in $M_+(\mathscr{L}_*)$ equal to zero. We will, however, be considering only the case where $M_+(\mathscr{L}_*)$ is nondegenerate, and then any vector in $M_+(\mathscr{L}_*)$ is uniquely determined by its Fourier coefficients. (See [7].)

The *Fourier representation of k in* $M_+(\mathscr{L}_*)$ is the operator Φ_* mapping $k\in M_+(\mathscr{L}_*)$ to the function $\Phi_* k=v$, where

$$v(\lambda) = \sum_{n=0}^{\infty} \lambda^n \varphi_* l_{*n}$$

and $\{l_{*n}\}$ is the sequence of Fourier coefficients of k in $M_+(\mathscr{L}_*)$. The function v takes its values in $\mathscr{D}_{T^*}$ and is defined in some neighborhood of zero.

Note that, when $M_+(\mathscr{L}_*)$ is nondegenerate, the Fourier representations Φ and Φ_* are injective, since the Fourier coefficients of a vector k in $M_+(\mathscr{L})$ or in $M_+(\mathscr{L}_*)$ uniquely determine k.

3. The space $H(T)$

Throughout this section we will be assuming that T has spectrum in the closed unit disc. By an application of the principle of uniform boundedness and the spectral radius formula, it follows that the results of this section apply to operators in $C_{\cdot 0}$.

Let us first condider the Kerin space $H^2(\mathscr{D}_T)$ of functions analytic in the open unit disc, with values in $\mathscr{D}_T$, and with square summable Taylor coefficients. If

$$u(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n \quad \text{and} \quad v(\lambda) = \sum_{n=0}^{\infty} \lambda^n v_n$$

are two functions in $H^2(\mathscr{D}_T)$, then their indefinite and Hilbert space inner products are given by

$$[u, v] = \sum_{n=0}^{\infty} [u_n, v_n] \quad \text{and} \quad (u, v) = \sum_{n=0}^{\infty} (u_n, v_n).$$

A fundamental symmetry J on $H^2(\mathscr{D}_T)$ is given by the formula

$$(Ju)(\lambda) = J_T u(\lambda).$$

The space $H^2(\mathscr{D}_{T^*})$ can be defined in a similar manner.

It follows immediately from (2.3) that the Fourier representation Φ is a unitary operator from $M_+(\mathscr{L})$ onto $H^2(\mathscr{D}_T)$, preserving both the indefinite and the Hilbert space inner products.

When T has bounded characteristic function, every $k \in M_+(\mathscr{L}_*)$ has square summable Fourier coefficients in $M_+(\mathscr{L}_*)$ [5], but for an arbitrary operator, this is not always the case. (See Example 3.1 below, which demonstrates this for a $C_{\cdot 0}$ operator.) Consequently, the Fourier representation Φ_* does not necessarily have its range in the space $H^2(\mathscr{D}_{T^*})$. We will use the notation $H(T)$ to describe the range of Φ_*, i.e.,

$$H(T) = \Phi_* M_+(\mathscr{L}_*).$$

We will assume, for the remainder of this section, that $\mathscr{R}=\{0\}$. We will describe $H(T)$ for such an operator; by Theorem 2.1, we are including the case $T \in C_{\cdot 0}$. The assumption $\mathscr{R}=\{0\}$ is equivalent to $M_+(\mathscr{L}_*)=\mathscr{K}_+$; since $\mathscr{K}_+$ is nondegenerate, it follows that the Fourier representation Φ_* is defined and injective on $\mathscr{K}_+$. Every $k \in \mathscr{K}_+$ has a unique representation of the form $k=h+m$, where $h \in \mathscr{H}$ and $m \in M_+(\mathscr{L})$; thus, since Φ_* is injective, every function in $H(T)$ has a unique representation of the form

$$\Phi_* k = \Phi_* h + \Phi_* m,$$

where $h \in \mathscr{H}$ and $m \in M_+(\mathscr{L})$.

We can define a Krein space structure on $H(T)$ by requiring that Φ_* be unitary with respect to both inner products; we define, for $k, k' \in M_+(\mathscr{L}_*)$,

$$[\Phi_* k, \Phi_* k'] = [k, k'] \tag{3.1}$$

and

$$(\Phi_* k, \Phi_* k') = (k, k'). \tag{3.2}$$

$H(T)$ is then a Krein space, with a fundamental symmetry (also denoted by J) given by

$$J\Phi_* k = \Phi_* Jk.$$

We begin our study of the structure of $H(T)$ by considering first the subspace $\Phi_* \mathscr{H}$. For $h \in \mathscr{H}$, define a function Fh by

$$[Fh](\lambda) = \sum_{n=0}^{\infty} \lambda^n J_{T^*} Q_{T^*} T^{*n} h = J_{T^*} Q_{T^*} (I-\lambda T^*)^{-1} h. \tag{3.3}$$

If we apply [9], Corollary 8.2, to the our situation, in which $\mathscr{R}=\{0\}$, we obtain

$$\Phi_* h = Fh$$

for all $h \in \mathscr{H}$. Fh is not necessarily in $H^2(\mathscr{D}_{T^*})$, since the sequence $\{J_{T^*} Q_{T^*} T^{*n} h\}_{n \geq 0}$ need not be square summable, even for $T \in C_{\cdot 0}$:

Example 3.1. Let $\{a_m\}_{m \geq 1}$ be the sequence of positive numbers given by $a_m^2 = 1-1/m^2$, and let T_m be an operator on a two dimensional Hilbert space $\mathscr{H}_m$, given by the matrix

$$T_m = \begin{pmatrix} a_m & 1 \\ 0 & 0 \end{pmatrix}.$$

Let $\mathscr{H}$ be the Hilbert space of square summable sequences $x=\{x_m\}_{m\geqq 1}$, with $x_m\in\mathscr{H}_m$ for $m\geqq 1$, and define T on $\mathscr{H}$ by $Tx=\{T_m x_m\}_{m\geqq 1}$. Since $0\leqq a_m<1$, for each $m\geqq 1$, it is easily verified that both T and its adjoint are $C_{\cdot 0}$ operators.

We have $J_{T^*}Q_{T^*}T^{*n}x=\{y_{nm}\}_{m\geqq 1}$, where

$$y_{nm} = \begin{pmatrix} -a_m^{n+1} & 0 \\ a_m^{n-1} & 0 \end{pmatrix} x_m \quad \text{when} \quad n>0$$

and

$$y_{0m} = \begin{pmatrix} -a_m & 0 \\ 0 & 1 \end{pmatrix} x_m.$$

If

$$x_m = \begin{pmatrix} 1/m \\ 0 \end{pmatrix}$$

then we have, for every $M>0$,

$$\sum_{n=0}^{\infty} \|J_{T^*}Q_{T^*}T^{*n}x\|^2 \geqq \sum_{m=1}^{M} m^{-2}\Big(a_m^2+\sum_{n=1}^{\infty}(a_m^{2n+2}+a_m^{2n-2})\Big) = \tag{3.4}$$

$$= \sum_{m=1}^{M} m^{-2}(1-a_m^2)^{-1}(1+a_m^2) = \sum_{m=1}^{M}(1+a_m^2) \to \infty$$

as $M\to\infty$. Therefore, the sequence $\{J_{T^*}Q_{T^*}T^{*n}x\}_{n\geqq 0}$ is not square summable. It also follows, from the observation made earlier, that T does not have a bounded characteristic function.

As in [12], [5], [9], and [10], the space $\Phi_* M_+(\mathscr{L})$ can be studied by introducing the characteristic function Θ_T of T, defined by

$$\Theta_T(\lambda) = [-TJ_T+\lambda J_{T^*}Q_{T^*}(I-\lambda T^*)^{-1}J_T Q_T]\,|\,\mathscr{D}_T. \tag{3.5}$$

$\Theta_T(\lambda)$ is defined for those complex numbers λ for which $I-\lambda T^*$ is boundedly invertible, and takes values which are bounded operators from $\mathscr{D}_T$ to $\mathscr{D}_{T^*}$. Since the spectrum of T is in the closed unit disc, it follows that $\Theta_T(\lambda)$ is defined for λ in the open unit disc. We can write, for $|\lambda|<1$,

$$\Theta_T(\lambda) = \sum_{n=0}^{\infty}\lambda^n\Theta_n, \tag{3.6}$$

where

$$\Theta_0 = -TJ_T \tag{3.7}$$

and

$$\Theta_n = J_{T^*}Q_{T^*}T^{*n-1}J_T Q_T \tag{3.8}$$

for $n\geqq 1$.

The characteristic function Θ_T is purely contractive, i.e., if $|\lambda|<1$, then

$$[\Theta_T(\lambda)a, \Theta_T(\lambda)a] < [a, a] \quad \text{for} \quad a\in\mathscr{D}_T,\ a\neq 0,$$

and

$$[\Theta_T(\lambda)^* b, \Theta_T(\lambda)^* b] < [b, b] \quad \text{for} \quad b\in\mathscr{D}_{T^*},\ \ b\neq 0.$$

As usual, we are using $[.,.]$ to denote the indefinite inner products on $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$, and $\Theta_T(\lambda)^*$ denotes the adjoint of $\Theta_T(\lambda)$ with respect to these inner products.

When $T\in C_{\cdot 0}$, we also have, in the strong operator topology, the telescoping series

$$\text{(3.9)} \qquad \sum_{n=0}^{\infty}\Theta_n^*\Theta_n = J_T T^* T J_T + \sum_{n=1}^{\infty} Q_T T^{n-1}(I-TT^*)T^{*n-1}J_T Q_T =$$

$$= T^*T + Q_T(I-\lim_{n\to\infty} T^n T^{*n})J_T Q_T = I.$$

Suppose $u\in H^2(\mathscr{D}_T)$, and consider the function $\Theta_T u$, defined for $|\lambda|<1$ by $[\Theta_T u](\lambda)=\Theta_T(\lambda)u(\lambda)$. If Θ_T is uniformly bounded on the open unit disc, then $\Theta_T u\in H^2(\mathscr{D}_{T^*})$ and the Fourier representation Φ_* maps $M_+(\mathscr{L}_*)$ onto $H^2(D_{T^*})$; if, in addition, $\mathscr{R}=\{0\}$, then we have, for $m\in M_+(\mathscr{L})$,

$$\text{(3.10)} \qquad \Theta_T\Phi m = \Phi_* m$$

(see [5]).

We have already noted that Example 3.1 gives an example of a $C_{\cdot 0}$ operator whose characteristic function is not bounded. Indeed $\Theta_T u\notin H^2(\mathscr{D}_{T^*})$ for the operator T of Example 3.1 and for $u(\lambda)$ equal to the constant function whose range is the vector x of Example 3.1. We can still, nevertheless, generalise (3.10) to an arbitrary operator having $\mathscr{R}=\{0\}$:

Theorem 3.1. *If* $\mathscr{R}=\{0\}$, *then* $\Phi_* m=\Theta_T\Phi m$ *for all* $m\in M_+(\mathscr{L})$.

Proof. Since the proof in [5] relies on Θ_T being a bounded operator into $H^2(\mathscr{D}_{T^*})$, it can not be used here. The proof given here does not require that the dilation be the one constructed by Davis, but only that the operators $\varphi\colon \mathscr{L}\to\mathscr{D}_T$ and $\varphi_*\colon \mathscr{L}_*\to\mathscr{D}_{T^*}$, defined in Section 2 above, be bounded.

Let us denote by P the projection onto $\mathscr{L}_*$ which is selfadjoint with respect to the indefinite inner product, and, as usual, let "$\perp$" denote orthogonality with respect to the indefinite inner product.

Suppose $h\in\mathscr{H}$, and let $m=(U-T)h\in\mathscr{L}$, so that $\varphi m=Q_T h$. Since $\mathscr{L}_*\perp U\mathscr{H}$, and since we have

$$m = (U-T)h = -(I-UT^*)Th + U(I-T^*T)h\in\mathscr{L}_* + U\mathscr{H},$$

we can conclude that

$$Pm = -(I-UT^*)Th.$$

Consequently,

(3.11) $$\varphi_* Pm = -J_{T^*}Q_{T^*}Th = -TJ_TQ_Th = -TJ_T\varphi m$$

for a set of vectors m which are dense in $\mathscr{L}$. (3.11) extends by continuity to be valid for all $m\in\mathscr{L}$.

We also have, for all $n>0$ and for $m=(U-T)h$ $(h\in\mathscr{H})$, the telescoping series

(3.12) $$U^{*n}m = U^{*n}(U-T)h =$$

$$= -U^{*n}(I-UT^*)Th+\sum_{k=0}^{n-1} U^{*n-1-k}(I-UT^*)T^{*k}(I-T^*T)h+UT^{*n}(I-T^*T)h.$$

Since $\mathscr{L}_*$ is wandering for U, and $U\mathscr{H}\perp\mathscr{L}_*$, all except one of the terms in (3.12) are orthogonal to $\mathscr{L}_*$ (in the indefinite inner product), and we can conclude that

$$PU^{*n}m = (I-UT^*)T^{*n-1}(I-T^*T)h.$$

Consequently,

(3.13) $$\varphi_* PU^{*n}m = J_{T^*}Q_{T^*}T^{*n-1}(I-T^*T)h = J_{T^*}Q_{T^*}T^{*n-1}J_TQ_T\varphi m$$

for all $n>0$ and for a set of vectors m which are dense in $\mathscr{L}$. Again, (3.13) extends by continuity to be valid for all $m\in\mathscr{L}$.

If we use the representation (3.6) of $\Theta_T(\lambda)$, then (3.11) and (3.13) can be rewritten

(3.14) $$\varphi_* PU^{*n}m = \Theta_n\varphi m,$$

for all $m\in\mathscr{L}$ and $n\geqq 0$.

Now suppose $m\in M_+(\mathscr{L})$, and let $\{l_n\}_{n\geqq 0}$ be the sequence of Fourier coefficients of m in $M_+(\mathscr{L})$. Then we have, for each $n\geqq 0$,

$$m-\sum_{k=0}^{n} U^k l_k\in U^{n+1}M_+(\mathscr{L})$$

(see [7]), and thus

(3.15) $$U^{*n}m-\sum_{k=0}^{n} U^{*n-k}l_k\in UM_+(\mathscr{L}).$$

We have $UM_+(\mathscr{L})\perp\mathscr{L}_*$ (see (2.2)), and thus (3.15) and (3.14) imply that

(3.16) $$\varphi_* PU^{*n}m = \sum_{k=0}^{n}\varphi_* PU^{*n-k}l_k = \sum_{k=0}^{n}\Theta_{n-k}\varphi l_k.$$

Since we are assuming that $\mathscr{R}=\{0\}$, the vector m is in $M_+(\mathscr{L}_*)$, and the left side of (3.16) is the coefficient of λ^n in the Taylor expansion of $\Phi_* m$. The right side of (3.16) is the coefficient of λ^n in the Taylor expansion of $\Theta_T \Phi m$. Therefore we have $\Phi_* m = \Theta_T \Phi m$, and the theorem is proved.

Corollary 3.3. *If* $\mathscr{R}=\{0\}$, *then* $\Theta_T u \in H(T)$ *for all* $u \in H^2(\mathscr{D}_T)$.

4. Some properties of $H(T)$

In this section, we derive some properties of $H(T)$ that will be useful in constructing a model for T in terms of its characteristic function. As before, we will be assuming throughout this section that T has spectrum in the closed unit disc and that $\mathscr{R}=\{0\}$, but some results will be proved only for $C_{\cdot 0}$ operators.

It follows from the results of the preceding section that, if $\mathscr{R}=\{0\}$, then the range of the Fourier representation Φ_* is of the form

$$H(T) = F\mathscr{H} + \Theta_T H^2(\mathscr{D}_T).$$

If a vector $k \in \mathscr{K}_+$ is written in the form $k=h+m$, with $h \in \mathscr{H}$ and $m \in M_+(\mathscr{L})$, then we have

$$\Phi_* k = Fh + \Theta_T u, \tag{4.1}$$

where $u=\Phi m$. The representation (4.1) of a function in $H(T)$ is unique, since Φ and Φ_* are injective. The inner products on $H(T)$ have a simple formulation in terms of the representation (4.1):

Proposition 4.1. *If* $v=Fh+\Theta_T u$ *and* $v'=Fh'+\Theta_T u'$ *are two functions in* $H(T)$, *then*

$$[v, v'] = (h, h') + [u, u'] \tag{4.2}$$

and

$$(v, v') = (h, h') + (u, u'). \tag{4.3}$$

Proof. Let $v=\Phi_* k$ and $v'=\Phi_* k'$, where $k=h+m$ and $k'=h'+m'$. It follows immediately from (4.1) and the definitions of the inner products (3.1) and (3.2) on $H(T)$ and on $\mathscr{K}$ (see [4]) that

$$[v, v'] = [k, k'] = (h, h') + [m, m'] = (h, h') + [u, u'],$$

since Φ is a unitary operator. The formula for (v, v') is proved similarly.

It is important for a later application to note that uniqueness of the representation (4.1) in fact implies the condition $\mathscr{R}=\{0\}$.

Theorem 4.2. *Suppose* T *is an operator with spectrum in the closed unit disc. Define the operator-valued functions* F *and* Θ_T *by* (3.3) *and* (3.5), *respectively. If*

$\mathscr{R}\neq\{0\}$, then there exists a vector $h\in\mathscr{H}$ and a function $u\in H^2(\mathscr{D}_T)$, not both zero, such that $Fh+\Theta_T u=0$.

Proof. If $\mathscr{R}\neq\{0\}$, then there is a nonzero vector $k\in\mathscr{R}$, i.e., $k\in\mathscr{K}_+$ and $k\perp M_+(\mathscr{L}_*)$. We can write k in the form $k=h+m$, where $h\in\mathscr{H}$ and $m\in M_+(\mathscr{L})$, and we will take $u=\Phi m\in H^2(\mathscr{D}_T)$. Since $k\neq 0$, h and u are not both zero. If $\{l_n\}_{n\geqq 0}$ is the sequence of Fourier coefficients of m in $M_+(\mathscr{L})$, then we have, for the nth coefficient in the Taylor series expansion of u,

$$u_n = \varphi l_n. \tag{4.4}$$

Since $k\in\mathscr{R}$, we can apply [9], Theorem 4.2, and assert the existence of a sequence $\{h_n\}_{n\geqq 0}$ of vectors in $\mathscr{H}$ such that

$$h_0 = h, \tag{4.5}$$

$$Th_{n+1} = h_n \quad \text{for all} \quad n \geqq 0, \quad \text{and} \tag{4.6}$$

$$l_n = (U-T)h_{n+1} \quad \text{for all} \quad n \geqq 0. \tag{4.7}$$

Combining (4.4) with (4.7) and (2.1) gives us

$$u_n = Q_T h_{n+1} \quad \text{for all} \quad n \geqq 0. \tag{4.8}$$

The nth coefficient v_n in the Taylor series expansion of $v=Fh+\Theta_T u$ can now be calculated. From the definitions of F and Θ_T we obtain

$$v_n = J_{T^*}Q_{T^*}T^{*n}h+\sum_{m=0}^{n}\Theta_m u_{n-m} =$$

$$= J_{T^*}Q_{T^*}T^{*n}h_0 - TJ_TQ_Th_{n+1}+\sum_{m=1}^{n}J_{T^*}Q_{T^*}T^{*m-1}(I-T^*T)h_{n-m+1}$$

by (4.5) and (4.8). The second term of the above line can be written as $-J_{T^*}Q_{T^*}Th_{n+1}$, and iterating (4.6) gives us $h_{n-m+1}=T^m h_{n+1}$ $(1\leqq m\leqq n+1)$. Thus we have

$$v_n = J_{T^*}Q_{T^*}\Big(T^{*n}T^{n+1}-T+\sum_{m=1}^{n}T^{*m-1}(I-T^*T)T^m\Big)h_{n+1} = 0,$$

since the series telescopes. Thus $v=0$, and the theorem is proved.

We will denote by $\mathbf{U}$ the operator of multiplication by the independent variable on $H(T)$ or $H^2(\mathscr{D}_T)$:

$$[\mathbf{U}u](\lambda) = \lambda u(\lambda)$$

for $u\in H(T)$ or $u\in H^2(\mathscr{D}_T)$. It is obvious that $H^2(\mathscr{D}_T)$ is invariant for $\mathbf{U}$, and that $\mathbf{U}$ preserves both the Hilbert space and the indefinite inner products of $H^2(\mathscr{D}_T)$. The adjoint $\mathbf{U}^*$ of $\mathbf{U}$, in both inner products of $H^2(\mathscr{D}_T)$, is given by the formula

$$[\mathbf{U}^*u](\lambda) = \lambda^{-1}\big(u(\lambda)-u(0)\big). \tag{4.9}$$

$H(T)$ is the range of the Fourier representation Φ_*, and the inner products on $H(T)$ were defined so as to make Φ_* unitary. It follows easily from the definition of Φ_* that

$$\Phi_* U = \mathbf{U}\Phi_*,$$

where U is the Davis dilation of T, and so $H(T)$ is invariant for $\mathbf{U}$. Since U is bounded and preserves the indefinite inner product of $\mathcal{K}$, we can conclude that $\mathbf{U}$ is bounded and preserves the indefinite inner product of $H(T)$. The formula (4.9) for the adjoint $\mathbf{U}^*$ of $\mathbf{U}$, in the indefinite inner product, is also valid in $H(T)$, since $\mathbf{U}^*$ acts as the backward shift on the Fourier coefficients of a vector in $M_+(\mathcal{L}_*)$. It is possible to give explicitly the action of $\mathbf{U}$ and $\mathbf{U}^*$ on $H(T)$ in terms of the representation (4.1):

Proposition 4.3. *Suppose $v\in H(T)$, with $v=Fh+\Theta_T u$ for some $h\in\mathcal{H}$ and $u\in H^2(\mathcal{D}_T)$. Then*

$$\mathbf{U}v = FTh+\Theta_T(Q_T h+\mathbf{U}u) \tag{4.10}$$

and

$$\mathbf{U}^* v = F(T^* h+J_T Q_T u(0))+\Theta_T(\mathbf{U}^* u). \tag{4.11}$$

Proof. It follows immediately from (3.5) that

$$\Theta_T(\lambda) Q_T = J_{T^*} Q_{T^*}(I-\lambda T^*)^{-1}(\lambda-T)$$

(cf. [12], p. 237). Therefore, if $v=Fh+\Theta_T u$, we have

$$\lambda v(\lambda) = \lambda J_{T^*} Q_{T^*}(I-\lambda T^*)^{-1} h+\lambda\Theta_T(\lambda) u(\lambda) =$$
$$= J_{T^*} Q_{T^*}(I-\lambda T^*)^{-1} Th+\Theta_T(\lambda) Q_T h+\lambda\Theta_T(\lambda) u(\lambda).$$

Formula (4.10) follows.

If $v'=Fh'+\Theta_T u'$, for some $h'\in\mathcal{H}$ and $u'\in H^2(\mathcal{D}_T)$, then, using (4.10) and Proposition 4.1, we have

$$[\mathbf{U}^* v, v'] = [v, \mathbf{U}v'] = (h, Th')+[u, Q_T h'+Uu'] = \tag{4.12}$$
$$= (T^* h, h')+[u(0), Q_T h']+[\mathbf{U}^* u, u'].$$

Note that

$$[u(0), Q_T h'] = (J_T u(0), Q_T h') = (J_T Q_T u(0), h'),$$

so that (4.11) follows from (4.12) and Proposition 4.1.

The fact that $\mathbf{U}$ preserves the indefinite inner product of $H(T)$ can be verified directly by observing that (4.10) and (4.11) imply that $\mathbf{U}^*\mathbf{U}=I$.

The inner products (3.1) and (3.2) have been defined on the function space $H(T)$ by making reference to the underlying Krein space $\mathcal{K}$. The indefinite inner product (3.1) can also be given, on a dense subset of $H(T)$, directly in terms of the functions involved. In this section, we will prove that, for $v \in H(T)$ and for u belonging to a dense subset of $H(T)$, with

$$u(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n \quad \text{and} \quad v(\lambda) = \sum_{n=0}^{\infty} \lambda^n v_n,$$

the indefinite inner product (3.1) is also given by

$$[u, v]_\Sigma = \sum_{n=0}^{\infty} [u_n, v_n]. \tag{4.13}$$

The formula (4.13) is identical to the one that applies in $H^2(\mathscr{D}_{T^*})$, but the Hilbert space structure on these two spaces can be quite different.

The dense subset of $H(T)$ on which (4.13) is valid includes the polynomials and a space of functions obtained from a reproducing kernel for the indefinite inner product. When $T \in C_{.0}$, (4.13) is also valid for any functions u and v which are finite linear combinations of functions of the form $\mathbf{U}^n Fh$ $(h \in \mathscr{H}, n \geqq 0)$, and for any u and v of the form $Fh + \Theta_T p$, where $h \in \mathscr{H}$ and p is a polynomial in $H^2(\mathscr{D}_T)$.

Example 5.9 of [9] shows that (4.13) can not be expected, in general, to provide the indefinite inner product on all of $H(T)$. In this example, we have $\mathscr{R} = \{0\}$, but $T \notin C_{.0}$. It is a consequence of Proposition 4.6 below that $[Fh, Fh']_\Sigma \neq (h, h')$ for some $h, h' \in \mathscr{H}$, and thus (4.13) does not give the inner product on all of $H(T)$. It is not known whether or not (4.13) is valid on all of $H(T)$ when $T \in C_{.0}$.

The space $H(T)$ contains the constant functions with values in $\mathscr{D}_{T^*}$ (they are the functions of the form $\Phi_* m$ for $m \in \mathscr{L}_*$), and hence contains all polynomials with values in $\mathscr{D}_{T^*}$. The operator mapping a vector in $\mathscr{D}_{T^*}$ to the corresponding constant function is continuous and preserves the indefinite inner product, because of the corresponding properties of the operator φ_* considered in Section 2. Moreover, the definition of $M_+(\mathscr{L}_*)$ and the wandering property of $\mathscr{L}_*$ imply that the polynomials are dense in $H(T)$ and that the indefinite inner product of a polynomial with an arbitrary function in $H(T)$ is given by the formula (4.13). These properties of the polynomials can be verified directly in terms of the representation (4.1) of functions in $H(T)$, given for polynomials in the following proposition. The operators Θ_k in (4.15) are those given by (3.7) and (3.8), and their adjoints are taken in the indefinite inner product.

Proposition 4.4. *If $a \in \mathscr{D}_{T^*}$, then the constant function with range a in $H(T)$ is of the form*

$$a = F(Q_{T^*} a) + \Theta_T(-J_T T^* a). \tag{4.14}$$

We also have, for all $n \geqq 0$,

(4.15) $$\mathbf{U}^n a = F(T^n Q_{T^*} a) + \Theta_T\Big(\sum_{m=0}^{n} \mathbf{U}^m \Theta_{n-m}^* a\Big).$$

If $v \in H(T)$ *and if p is a polynomial in* $H(T)$, *then*

(4.16) $$[p, v] = [p, v]_\Sigma .$$

Proof. From the definitions of F and Θ_T we obtain

$$[F(Q_{T^*} a) + \Theta_T(-J_T T^* a)](\lambda) =$$
$$= J_{T^*} Q_{T^*}(I-\lambda T^*)^{-1} Q_{T^*} a + TT^* a - \lambda J_{T^*} Q_{T^*}(I-\lambda T^*)^{-1} T^* Q_{T^*} a =$$
$$= J_{T^*} Q_{T^*}(I-\lambda T^*)^{-1}(I-\lambda T^*) Q_{T^*} a + TT^* a = a,$$

since $J_{T^*} Q_{T^*}^2 = I - TT^*$. Thus, (4.14) is proved, and (4.15) follows by iterating (4.10) and using the definitions (3.7) and (3.8) of Θ_k.

For an arbitrary $v = Fh + \Theta_T u \in H(T)$ we have, from (4.15) and Proposition 4.1,

(4.17) $$[\mathbf{U}^n a, v] = (T^n Q_{T^*} a, h) + \sum_{m=0}^{n} [\Theta_{n-m}^* a, u_m],$$

where u_m is the mth coefficient in the Taylor series expansion of u. Rewriting each of the terms on the right side of (4.17) in the indefinite inner product of $\mathscr{D}_{T^*}$ gives us

$$[\mathbf{U}^n a, v] = [a, J_{T^*} Q_{T^*} T^{*n} h + \sum_{m=0}^{n} \Theta_{n-m} u_m] = [a, v_n],$$

where v_n is the nth coefficient in the Taylor series expansion of v. Formula (4.16) then follows from the definition (4.13) of $[.,.]_\Sigma$ and the linearity of the inner product.

Consider the function

$$k(\mu, \lambda) = (1 - \lambda\bar{\mu})^{-1},$$

defined for λ and μ in the open unit disc, and the associated family $\{k_\mu\}$ of functions of a single variable, defined by

$$k_\mu(\lambda) = k(\mu, \lambda).$$

For any $a \in \mathscr{D}_T$ and $|\mu| < 1$, the function $k_\mu a$ is in $H^2(\mathscr{D}_T)$ and has the reproducing properties:

(4.18) $$[u, k_\mu a] = [u(\mu), a]$$

and

(4.19) $$(u, k_\mu a) = (u(\mu), a)$$

for every $u \in H^2(\mathscr{D}_T)$. The inner products on the left sides of (4.18) and (4.19) are the indefinite and Hilbert space inner products, respectively, of $H^2(\mathscr{D}_T)$, whereas the inner products on the right sides are the respective inner products of $\mathscr{D}_T$. We say that $k(\mu, \lambda)$ is a *reproducing kernel* for each of the inner products on $H^2(\mathscr{D}_T)$.

It is not obvious that the functions $k_\mu a$ $(a \in \mathscr{D}_{T^*}, |\mu| < 1)$ are in $H(T)$; we show in Theorem 4.5 that this is in fact the case. Moreover, $k(\mu, \lambda)$ is a reproducing kernel for the indefinite inner product of $H(T)$:

$$[v, k_\mu a] = [v(\mu), a] \tag{4.20}$$

for every $v \in H(T)$, $a \in \mathscr{D}_{T^*}$, and for $|\mu| < 1$. It is a consequence of (4.20) that the space of all finite linear combinations of functions of the form $k_\mu a$ $(a \in \mathscr{D}_{T^*}, |\mu| < 1)$ is dense in $H(T)$. We also show in Theorem 4.5 that the inner product in $H(T)$ on the left side of (4.20) coincides with the inner product $[.,.]_\Sigma$.

In Section 5, we will find a reproducing kernel for the Hilbert space inner product of $H(T)$, i.e., a kernel k' such that

$$(v, k'_\mu a) = (v(\mu), a)$$

for every $v \in H(T)$, $a \in \mathscr{D}_{T^*}$; and for $|\mu| < 1$.

Theorem 4.5. *The function* $k(\mu, \lambda) = (1 - \lambda\bar{\mu})^{-1}$ *is a reproducing kernel for the indefinite inner product on* $H(T)$. *If* $a \in \mathscr{D}_{T^*}$ *and* $|\mu| < 1$, *then we have*

$$k_\mu a = F(I - \bar{\mu}T)^{-1} Q_{T^*} a + \Theta_T k_\mu \Theta_T(\mu)^* a. \tag{4.21}$$

If u *is a finite linear combination of functions of the form* $k_\mu a$, *where* $a \in \mathscr{D}_{T^*}$ *and* $|\mu| < 1$, *then*

$$[v, u] = [v, u]_\Sigma, \tag{4.22}$$

for all $v \in H(T)$.

Proof. From (3.5) we can derive the formula

$$I - \Theta_T(\lambda)\Theta_T(\mu)^* = (1 - \lambda\bar{\mu}) J_{T^*} Q_{T^*} (I - \lambda T^*)^{-1} (I - \bar{\mu}T)^{-1} Q_{T^*},$$

where λ and μ are in the open unit disc, and the adjoint $\Theta_T(\mu)^*$ is computed in the indefinite inner products of $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$ (cf. [12], p. 238, and [8], Sec 4). Thus, for all $a \in \mathscr{D}_{T^*}$, we have

$$(1 - \lambda\bar{\mu})^{-1} a = J_{T^*} Q_{T^*} (I - \lambda T^*)^{-1} (I - \bar{\mu}T)^{-1} Q_{T^*} a + \Theta_T(\lambda)(1 - \lambda\bar{\mu})^{-1} \Theta_T(\mu)^* a,$$

and thus (4.21) is verified. Since $k_\mu \Theta_T(\mu)^* a \in H^2(\mathscr{D}_T)$, we have $k_\mu a \in H(T)$.

If we take $v = Fh + \Theta_T u \in H(T)$, then we obtain, using (4.21) and (4.2),

$$[v, k_\mu a] = \big(h, (I - \bar{\mu}T)^{-1} Q_{T^*} a\big) + [u, k_\mu \Theta_T(\mu)^* a] =$$

$$= \big(Q_{T^*}(I - \mu T^*)^{-1} h, a\big) + [u(\mu), \Theta_T(\mu)^* a],$$

using the reproducing property (4.18) in $H^2(\mathscr{D}_T)$. Rewriting the inner products in terms of the indefinite inner product of $\mathscr{D}_{T^*}$, we obtain

$$[v, k_\mu a] = [J_{T^*}Q_{T^*}(I-\mu T^*)^{-1}h + \Theta_T(\mu)u(\mu), a] = [v(\mu), a],$$

proving the reproducing property (4.20) for $H(T)$.

Finally, we note that, if

$$v(\lambda) = \sum_{n=0}^{\infty} \lambda^n v_n,$$

then

$$[v, k_\mu a]_\Sigma = \sum_{n=0}^{\infty} [v_n, \bar{\mu}^n a] = \sum_{n=0}^{\infty} [\mu^n v_n, a] = [v(\mu), a] = [v, k_\mu a].$$

Equation (4.22) then follows by linearity.

Although the indefinite inner product is given by $[.,.]_\Sigma$ on the dense subsets of $H(T)$ identified in Proposition 4.4 and Theorem 4.5, it does not necessarily apply on the whole space. As was noted above, it is possible to have $\mathscr{R}=\{0\}$ with $T\notin C_{.0}$; the following proposition shows that, in such a case, the inner product is not given by $[.,.]_\Sigma$ on the subspace $F\mathscr{H}$ of $H(T)$.

Proposition 4.6. *On the subspace $F\mathscr{H}$ of $H(T)$, the indefinite inner products $[.,.]$ and $[.,.]_\Sigma$ coincide if and only if $T\in C_{.0}$.*

Proof. Using (3.3) and (4.13), and the property $J_{T^*}Q_{T^*}^2 = I-TT^*$, we obtain

$$[Fh, Fk]_\Sigma = \sum_{n=0}^{\infty} [J_{T^*}Q_{T^*}T^{*n}h, J_{T^*}Q_{T^*}T^{*n}k] = \tag{4.23}$$

$$= \sum_{n=0}^{\infty} (T^n(I-TT^*)T^{*n}h, k) = (h, k) - \lim_{n\to\infty} (T^{*n}h, T^{*n}k)$$

whenever the limit exists. On the other hand, Proposition 4.1 gives $[Fh, Fk]=(h, k)$ for the inner product on $H(T)$. Thus, if $T\in C_{.0}$, the two inner products coincide. Conversely, if the inner products coincide, then, by putting $k=h$ in (4.23), we obtain $T^{*n}h\to 0$ for every $h\in\mathscr{H}$, i.e., $T\in C_{.0}$.

If we restrict ourselves to functions of the form $Fh+\Theta_T u$, where $h\in\mathscr{H}$ and u is a polynomial with values in $\mathscr{D}_T$, then we can show that, for $T\in C_{.0}$, the two indefinite inner products, given by (4.2) and (4.13), coincide. It follows immediately from the definition of $H(T)$ that the linear manifold of such functions is dense in $H(T)$.

Theorem 4.7. *Suppose $T\in C_{.0}$. Then the indefinite inner products $[.,.]$ and $[.,.]_\Sigma$ coincide on the dense linear manifold of $H(T)$ consisting of functions of the form $Fh+\Theta_T u$, with $h\in\mathscr{H}$ and u a polynomial with values in $\mathscr{D}_T$.*

Proof. Consider a function w of the form $w=Fh+\Theta_T u$, where $h\in\mathscr{H}$ and

$$u(\lambda)=\sum_{n=0}^{N}\lambda^n u_n,$$

for some $N\geqq 0$ and $u_n\in\mathscr{D}_T$ $(0\leqq n\leqq N)$. By the polarization identity, we can establish equality for the two inner products by showing that $[w,w]_\Sigma=[w,w]$, i.e., by showing that

$$[w,w]_\Sigma=\|h\|^2+[u,u]=\|h\|^2+\sum_{n=0}^{N}[u_n,u_n]. \tag{4.24}$$

We have already shown, in Proposition 4.6, that (4.24) is valid when $u=0$. Thus, to establish (4.24) it suffices to show that, if $h\in\mathscr{H}$, and if $u(\lambda)=\lambda^n a$ and $v(\lambda)=\lambda^m b$, for $a,b\in\mathscr{D}_T$ and $0\leqq m<n$, then

$$[Fh,\Theta_T u]_\Sigma=0, \tag{4.25}$$

$$[\Theta_T u,\Theta_T v]_\Sigma=0, \tag{4.26}$$

and

$$[\Theta_T u,\Theta_T u]_\Sigma=[a,a]. \tag{4.27}$$

The definitions (3.3) and (3.5) of Fh and $\Theta_T(\lambda)$, together with the definition (4.13) of the inner product, give us

$$\begin{aligned}[Fh,\Theta_T u]_\Sigma&=\sum_{k=0}^{\infty}[J_{T^*}Q_{T^*}T^{*n+k}h,\Theta_k a]=\\&=-[J_{T^*}\Theta_{T^*}T^{*n}h,TJ_Ta]+\sum_{k=1}^{\infty}[J_{T^*}Q_{T^*}T^{*n+k}h,J_{T^*}Q_{T^*}T^{*k-1}J_TQ_Ta]=\\&=-(J_TQ_TT^{*n+1}h,a)+\sum_{k=1}^{\infty}(J_TQ_TT^{k-1}(I-TT^*)T^{*n+k}h,a)=\\&=-\lim_{k\to\infty}(T^{*n+k+1}h,T^{*k}J_TQ_Ta)=0,\end{aligned}$$

since $T\in C_{\cdot 0}$. This proves (4.25); to prove (4.26), note that we have, for $m<n$,

$$\begin{aligned}[\Theta_T u,\Theta_T v]_\Sigma&=\sum_{k=0}^{\infty}[\Theta_k a,\Theta_{k+n-m}b]=\\&=-[TJ_Ta,J_{T^*}Q_{T^*}T^{*n-m-1}J_TQ_Tb]+\\&+\sum_{k=1}^{\infty}[J_{T^*}Q_{T^*}T^{*k-1}J_TQ_Ta,J_{T^*}Q_{T^*}T^{*k+n-m-1}J_TQ_Tb]=\\&=-(T^{n-m}J_TQ_Ta,J_TQ_Tb)+\sum_{k=1}^{\infty}(T^{k+n-m-1}(I-TT^*)T^{*k-1}J_TQ_Ta,J_TQ_Tb)=\\&=-\lim_{k\to\infty}(T^{*k}J_TQ_Ta,T^{*k+n-m}J_TQ_Tb)=0.\end{aligned}$$

The remaining identity (4.27) follows immediately from (3.9).

Corollary 4.8. *Suppose* $T \in C_{\cdot 0}$. *Then the indefinite inner products* $[.,.]$ *and* $[.,.]_\Sigma$ *coincide on the dense linear manifold of* $H(T)$ *consisting of finite linear combinations of functions of the form* $\mathrm{U}^n Fh$, *where* $h \in \mathscr{H}$ *and* $n \geqq 0$.

Proof. We can obtain from (4.10) the formula

$$\mathrm{U}^n Fh = FT^n h + \Theta_T \sum_{k=0}^{n-1} \mathrm{U}^k Q_T T^{n-k-1} h, \tag{4.28}$$

showing that Theorem 4.7 applies to functions in the manifold consisting of all linear combinations of functions of the form $\mathrm{U}^n Fh$. The fact that this manifold is dense in $H(T)$ can easily be proved by noting that only the zero function in $H(T)$ can be orthogonal to all functions of the form (4.28).

5. Reproducing kernels

We assume, as in previous sections, that T is an operator with spectrum in the closed unit disc and with trivial residual space. In Section 3, we represented the range of the Fourier representation Φ_* in the form

$$H(T) = F\mathscr{H} + \Theta_T H^2(\mathscr{D}_T),$$

and the operator T was used explicitly in the construction of this space. By contrast, when the characteristic function is bounded, we have $H(T) = H^2(\mathscr{D}_{T^*})$ ([12], [5]), and thus a knowledge of only the space $\mathscr{D}_{T^*}$ suffices to construct $H(T)$. In this section, we show how $H(T)$ can be described in terms of the characteristic function Θ_T, without explicit reference to T, and use this in the following sections to obtain a functional model in terms of Θ_T.

The space $\Theta_T H^2(\mathscr{D}_T)$ already has a description in terms of Θ_T alone, since a knowledge of $\mathscr{D}_T$, the domain space of Θ_T, is all that is required to describe the space $H^2(\mathscr{D}_T)$. However, the description of $F\mathscr{H}$ in terms of Θ_T is not so immediate.

By Theorem 4.5, the space $H(T)$ contains all functions of the form $k_\mu a$, where

$$[k_\mu a](\lambda) = k(\mu, \lambda)a = (1-\lambda\bar{\mu})^{-1}a, \quad a \in \mathscr{D}_{T^*}, \ |\mu| < 1,$$

and $k(\mu, \lambda)$ is a reproducing kernel for the indefinite inner product of $H(T)$. If we consider the orthogonal projection of $k_\mu a$ onto $F\mathscr{H}$, we should obtain a reproducing kernel for the indefinite inner product of $F\mathscr{H}$. In Theorem 5.1 below, we show that the kernel so obtained is

$$K(\mu, \lambda) = (1-\lambda\bar{\mu})^{-1}(I-\Theta_T(\lambda)\Theta_T(\mu)^*). \tag{5.1}$$

We prove the following reproducing property: if

$$[K_\mu a](\lambda) = K(\mu, \lambda)a \tag{5.2}$$

for $a \in \mathscr{D}_{T^*}$ and $|\mu|<1$, then $K_\mu a \in F\mathscr{H}$ and

$$[Fh, K_\mu a] = [(Fh)(\mu), a] \tag{5.3}$$

for all $h \in \mathscr{H}$.

We also show in Theorem 5.1 that the function

$$K(\mu, \lambda) = K(\mu, \lambda)J_{T^*} \tag{5.4}$$

is a reproducing kernel for the Hilbert space inner product on $F\mathscr{H}$: if

$$[K'_\mu a](\lambda) = K'(\mu, \lambda)a \tag{5.5}$$

for $a \in \mathscr{D}_{T^*}$ and $|\mu|<1$, then $K'_\mu a \in F\mathscr{H}$ and

$$(Fh, K'_\mu a) = ([Fh](\mu), a)$$

for all $h \in \mathscr{H}$. Note that the indefinite and Hilbert space inner products coincide on $F\mathscr{H}$; the only reason that separate kernels are needed for the two inner products is that they don't coincide on $\mathscr{D}_{T^*}$.

Theorem 5.1. *The subspace $F\mathscr{H}$ of $H(T)$ is the closed linear span of functions of the form $K_\mu a$ (defined by (5.2)), where $a \in \mathscr{D}_{T^*}$ and $|\mu|<1$. The functions $K(\mu, \lambda)$ and $K'(\mu, \lambda)$, defined by* (5.1) *and* (5.4), *are reproducing kernels for the indefinite and the Hilbert space inner products, respectively, on $F\mathscr{H}$.*

Proof. From the representation of the function $k_\mu a$ given by (4.21), we obtain

$$[F(I-\bar{\mu}T)^{-1}Q_{T^*}a](\lambda) = (1-\lambda\bar{\mu})^{-1}(I-\Theta_T(\lambda)\Theta_T(\mu)^*)a = [K_\mu a](\lambda),$$

showing that the functions $K_\mu a$ are in $F\mathscr{H}$. By Proposition 4.1, the inner products on $F\mathscr{H}$ are given by

$$[Fh, Fh'] = (Fh, Fh') = (h, h'),$$

for all $h, h' \in \mathscr{H}$. For the functions given by (5.2) and (5.5), we therefore have, for all $h \in \mathscr{H}$,

$$[Fh, K_\mu a] = (h, (I-\bar{\mu}T)^{-1}Q_{T^*}a) = [J_{T^*}Q_{T^*}(I-\mu T^*)^{-1}h, a] = [(Fh)(\mu), a]$$

and

$$(Fh, K'_\mu a) = (h, (I-\bar{\mu}T)^{-1}Q_{T^*}J_{T^*}a) = (J_{T^*}Q_{T^*}(I-\mu T^*)^{-1}h, a) = ([Fh](\mu), a).$$

Therefore, $K(\mu, \lambda)$ and $K'(\mu, \lambda)$ are reproducing kernels for the two inner products. Since the indefinite inner product of $\mathcal{D}_{T^*}$ is nondegenerate, (5.3) also shows that only the zero function Fh is orthogonal to every function $K_\mu a$, with $a \in \mathcal{D}_{T^*}$ and $|\mu|<1$, and thus the space of linear combinations of such functions is dense in $F\mathcal{H}$.

The inner products on $\Theta_T H^2(\mathcal{D}_T)$ can also be given in terms of reproducing kernels. Recall that $\Theta_T(\mu)^*$ denotes the adjoint of $\Theta_T(\mu)$ with respect to the indefinite inner products on $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$. We will denote by $\Theta_T(\mu)^{(*)}$ the adjoint of $\Theta_T(\mu)$ with respect to the Hilbert space inner products on $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$.

Theorem 5.2. *The function*

$$L(\mu, \lambda) = (1-\lambda\bar{\mu})^{-1}\Theta_T(\lambda)\Theta_T(\mu)^*$$

is reproducing for the indefinite inner product, and the function

$$L'(\mu, \lambda) = (1-\lambda\bar{\mu})^{-1}\Theta_T(\lambda)\Theta_T(\mu)^{(*)}$$

is reproducing for the Hilbert space inner product on $\Theta_T H^2(\mathcal{D}_T)$.

Proof. If $L_\mu a$ and $L'_\mu a$ are defined for $a \in \mathcal{D}_{T^*}$ and $|\mu|<1$ by $[L_\mu a](\lambda) = L(\mu, \lambda)a$ and $[L'_\mu a](\lambda) = L'(\mu, \lambda)a$, then, clearly, $L_\mu a \in \Theta_T H^2(\mathcal{D}_T)$ and $L'_\mu a \in \Theta_T H^2(\mathcal{D}_T)$.

For every $u \in H^2(\mathcal{D}_T)$ we have, using (4.2) for the indefinite inner product on $\Theta_T H^2(\mathcal{D}_T)$, and the reproducing property (4.18) of $k(\mu, \lambda)$ on $H^2(\mathcal{D}_T)$,

$$[\Theta_T u, L_\mu a] = [u, k_\mu \Theta_T(\mu)^* a] = [u(\mu), \Theta_T(\mu)^* a] = [\Theta_T(\mu)u(\mu), a],$$

proving the reproducing property for the indefinite inner product. Similarly,

$$(\Theta_T u, L'_\mu a) = (\Theta_T(\mu)u(\mu), a),$$

proving the reproducing property for the Hilbert space inner product.

Note that the reproducing kernel $k(\mu, \lambda)$ for the indefinite inner product of $H(T)$ can be obtained as the sum of $K(\mu, \lambda)$ and $L(\mu, \lambda)$. We can obtain a reproducing kernel for the Hilbert space inner product of $H(T)$ by considering

$$k'(\mu, \lambda) = K'(\mu, \lambda) + L'(\mu, \lambda). \tag{5.6}$$

Theorem 5.3. *The function* $k'(\mu, \lambda)$, *defined by* (5.6), *is a reproducing kernel for the Hilbert space inner product of* $H(T)$.

Proof. This follows immediately from Theorems 5.1 and 5.2, and the fact that the spaces $F\mathcal{H}$ and $\Theta_T H^2(\mathcal{D}_T)$ are orthogonal complements in the Hilbert space inner product of $H(T)$.

6. The space $H(T)$

In the preceding sections, the space $H(T)$ was described for an arbitrary operator with trivial residual space. In this section, we obtain a description of a space $H(\Theta)$, for an operator valued analytic function Θ. The function Θ will be assumed to satisfy conditions that are known to be valid for the characteristic function of a $C_{\cdot 0}$ operator. These assumptions will be sufficient to guarantee that Θ is the characteristic function of a completely non-unitary operator T with trivial residual space, and we will then have $H(\Theta)=H(T)$.

Throughout this section, we suppose that Θ is an operator valued analytic function, defined on the open unit disc, and taking values that are operators from a Krein space $\mathcal{D}$ to a Krein space $\mathcal{D}_*$. For $|\lambda|<1$ we can write

$$\Theta(\lambda) = \sum_{n=0}^{\infty} \lambda^n \Theta_n, \tag{6.1}$$

where, for each $n \geqq 0$, Θ_n is a bounded operator from $\mathcal{D}$ to $\mathcal{D}_*$.

We assume that Θ is fundamentally reducible, i.e., that there are fundamental symmetries on $\mathcal{D}$ and $\mathcal{D}_*$ commuting with $\Theta(0)^*\Theta(0)$ and $\Theta(0)\Theta(0)^*$, respectively (see [8]). We also assume that Θ is purely contractive, i.e. if $|\lambda|<1$, then

$$[\Theta(\lambda)a, \Theta(\lambda)a] < [a, a] \quad \text{for} \quad a \in \mathcal{D},\ a \neq 0,$$

and

$$[\Theta(\lambda)^* b, \Theta(\lambda)^* b] < [b, b] \quad \text{for} \quad b \in \mathcal{D}_*,\ b \neq 0.$$

As usual, we are using $[.,.]$ to denote the indefinite inner products on $\mathcal{D}$ and $\mathcal{D}_*$, and $\Theta(\lambda)^*$ denotes the adjoint of $\Theta(\lambda)$ with respect to these inner products.

It follows from the above hypotheses that Θ is the characteristic function of a uniquely determined completely non-unitary operator T and, conversely, the characteristic function of any completely non-unitary operator satisfies these hypotheses (see [8], [1]). Since Θ is analytic in the open unit disc, it also follows from [1] that T has spectrum in the closed unit disc. We will also be assuming that Θ satisfies the additional condition

$$\sum_{n=0}^{\infty} \Theta_n^* \Theta_n = I, \tag{6.2}$$

in the strong operator topology, where the operators Θ_n are given by (6.1). It was shown previously, in (3.9), that Θ satisfies (6.2) if it is the characteristic function of a $C_{\cdot 0}$ operator. It is not known if T is necessarily in $C_{\cdot 0}$ when Θ satisfies (6.2).

In this paper, we will be constructing a different functional model for T than that given by Ball in [1], but we will be appealing to Ball's model to be able to assert that $\Theta = \Theta_T$ for some completely non-unitary operator T acting on a Hilbert space

$\mathcal{H}$. The assumption (6.2) on Θ implies the condition of trivial residual space for T, which we considered earlier. The proof of this, in Theorem 6.1 below, closely resembles the proof, in [9], Theorems 4.2 and 5.5, of the fact that $\mathcal{R}=\{0\}$ for a $C_{\cdot 0}$ operator.

We use the same notation below as we have used previously. In particular, F is the function given by (3.3).

Theorem 6.1. *Suppose Θ is a fundamentally reducible, purely contractive analytic function, satisfying the condition* (6.2), *and let T be the completely non-unitary operator such that $\Theta=\Theta_T$. If $Fh+\Theta_T u=0$ for some $h\in\mathcal{H}$ and $u\in H^2(\mathcal{D})$, then $h=0$ and $u=0$. T has trivial residual space: $\mathcal{R}=\{0\}$.*

Proof. By Theorem 4.2, it suffices to prove the first part: if $Fh+\Theta_T u=0$ for $h\in\mathcal{H}$ and $u\in H^2(\mathcal{D}_T)$, then $h=0$ and $u=0$.

If we assume that $Fh+\Theta_T u=0$, then we obtain, from the nth coefficient in the Taylor series expansion of $Fh+\Theta_T u$,

$$J_{T^*}Q_{T^*}T^{*n}h+\sum_{k=0}^{n}\Theta_k u_{n-k}=0, \tag{6.3}$$

where

$$u(\lambda)=\sum_{k=0}^{\infty}\lambda^k u_k.$$

Define a sequence $\{h_n\}_{n\geq 0}$ in $\mathcal{H}$ by

$$h_n=T^{*n}h+\sum_{k=1}^{n}T^{*k-1}J_T Q_T u_{n-k},$$

for $n\geqq 0$. Then we have

$$h_0=h, \tag{6.4}$$

and, for each $n\geqq 0$,

$$h_n-Th_{n+1}=(I-TT^*)T^{*n}h-TJ_TQ_Tu_n+\sum_{k=1}^{n}(I-TT^*)T^{*k-1}J_TQ_Tu_{n-k}=$$

$$=Q_{T^*}[J_{T^*}Q_{T^*}T^{*n}h-TJ_Tu_n+\sum_{k=1}^{n}J_{T^*}Q_{T^*}T^{*k-1}J_TQ_Tu_{n-k}]=$$

$$=Q_{T^*}[J_{T^*}Q_{T^*}T^{*n}h+\sum_{k=0}^{n}\Theta_k u_{n-k}]=0$$

by (6.3). Thus, for all $n\geqq 0$, we have

$$Th_{n+1}=h_n, \tag{6.5}$$

and by induction we obtain, for $0\leqq n\leqq N$,

$$T^{N-n}h_N=h_n. \tag{6.6}$$

We also have, for all $n \geqq 0$,

$$h_{n+1} = T^{*n+1}h + \sum_{k=1}^{n+1} T^{*k-1} J_T Q_T u_{n+1-k} = \tag{6.7}$$

$$= T^{*n+1}h + \sum_{k=0}^{n} T^{*k} J_T Q_T u_{n-k} = T^* h_n + J_T Q_T u_n.$$

Thus, using (6.7) and (6.5), we get

$$J_T Q_T u_n = h_{n+1} - T^* h_n = (I - T^* T) h_{n+1} = J_T Q_T (Q_T h_{n+1}).$$

Since $J_T Q_T$ is injective on $\mathscr{D}_T$, it follows that

$$u_n = Q_T h_{n+1} \tag{6.8}$$

for all $n \geqq 0$.

Since $u \in H^2(\mathscr{D}_T)$, we can write for the indefinite inner product (using (6.8))

$$[u, u] = \sum_{n=0}^{\infty} [u_n, u_n] = \lim_{N\to\infty} \sum_{n=0}^{N-1} [Q_T h_{n+1}, Q_T h_{n+1}] =$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} [Q_T h_n, Q_T h_n] = \lim_{N\to\infty} \sum_{n=1}^{N} [Q_T T^{N-n} h_N, Q_T T^{N-n} h_N],$$

by (6.6). Thus we obtain the telescoping series

$$[u, u] = \lim_{N\to\infty} \sum_{n=1}^{N} \big(T^{*N-n}(I - T^*T) T^{N-n} h_N, h_N\big) = \tag{6.9}$$

$$= \lim_{N\to\infty} (\|h_N\|^2 - \|T^N h_N\|^2) = \lim_{N\to\infty} \|h_N\|^2 - \|h_0\|^2,$$

by (6.6) again. It follows, from the existence of the limit in (6.9), that the sequence $\{h_n\}_{n\geqq 0}$ must be bounded.

The condition (6.2) on Θ is equivalent to

$$\lim_{n\to\infty} Q_T T^n T^{*n} J_T Q_T = 0,$$

in the strong operator topology (cf. (3.9)). Therefore, for every $k \in \mathscr{H}$, we have

$$\|T^{*n} Q_T k\|^2 = \big(Q_T T^n T^{*n} J_T Q_T (J_T k), k\big) \to 0 \quad \text{as} \quad n \to \infty.$$

Using the boundedness of the sequence $\{h_n\}_{n\geqq 0}$ and property (6.6), we can conclude that

$$(k, Q_T h_n) = (Q_T k, T^{N-n} h_N) = (T^{*N-n} Q_T k, h_N) \to 0 \quad \text{as} \quad N \to \infty.$$

Consequently, $Q_T h_n = 0$ for each $n \geqq 0$; by (6.8) and (6.4), this implies that $u = 0$ and that $Q_T h = 0$.

From (6.4) and (6.6) we can also conclude that, for each $k \in \mathscr{H}$ and $n \geqq 0$,

$$(k, Q_T T^n h) = (k, Q_T T^{n+N} h_N) = (T^{*n+N} Q_T k, h_N) \to 0 \quad \text{as} \quad N \to \infty,$$

and thus

(6.10) $$Q_T T^n h = 0 \quad \text{for all} \quad n \geqq 0.$$

Since $u=0$, we have $Fh=0$, and this implies that

(6.11) $$Q_{T^*} T^{*n} h = 0 \quad \text{for all} \quad n \geqq 0.$$

We complete the proof by showing that the two conditions (6.10) and (6.11) together imply that $h=0$. The subspace $\mathscr{H}_0$ of all vectors $h \in \mathscr{H}$ satisfying (6.10) and (6.11) is invariant for T; this follows from the relations $Q_{T^*} Th = TQ_T h$ and

$$Q_{T^*} T^{*n} Th = Q_{T^*} T^{*n-1}(T^* Th) = Q_{T^*} T^{*n-1} h \quad (n \geqq 1)$$

when $Q_T h=0$. By symmetry, $\mathscr{H}_0$ is invariant for T^* as well. The relations $Q_T h=0$ and $Q_{T^*} h=0$ imply that $\mathscr{H}_0$ reduces T to a unitary operator; since T is assumed to be completely non-unitary, we have $\mathscr{H}_0 = \{0\}$.

The space $F\mathscr{H}$ will be modelled by following the representation given in Theorem 5.1. We define, as before,

$$K(\mu, \lambda) = (1-\lambda\bar{\mu})^{-1}(I - \Theta(\lambda)\Theta(\mu)^*)$$

and

$$K_\mu(\lambda) = K(\mu, \lambda).$$

Consider the space $\mathbf{H}_0$ of all finite linear combinations of functions of the form $K_\mu a$, where $a \in \mathscr{D}_*$ and $|\mu|<1$. We impose on $\mathbf{H}_0$ an inner product $[.,.]$ by means of the formula (5.3):

$$[u, K_\mu a] = [u(\mu), a],$$

for all $u \in \mathbf{H}_0$ and for all $a \in \mathscr{D}_*$ and $|\mu|<1$. Part of the proof of Theorem 1 of [8] shows that this inner product is positive definite. If $\mathbf{H}$ denotes the completion of the space $\mathbf{H}_0$ to a Hilbert space, then standard reproducing kernel arguments can be used to show that $\mathbf{H}$ can be identified with a space of functions analytic in the open unit disc. Since $\mathscr{R} = \{0\}$, Theorem 5.1 shows that $\mathbf{H} = F\mathscr{H}$.

We define the space $H(\Theta)$ as

$$H(\Theta) = \mathbf{H} + \Theta H^2(\mathscr{D}).$$

Since $\mathbf{H}$ can be identified with $F\mathscr{H}$, Theorem 6.1 implies that every function in $H(\Theta)$ has a unique representation in the form $h+\Theta u$, with $h \in \mathbf{H}$ and $u \in H^2(\mathscr{D})$. Suppose $v=h+\Theta u$ and $v'=h'+\Theta u'$ are two functions in $H(\Theta)$; we can define indefinite and Hilbert space inner products on $H(\Theta)$ by

(6.12) $$[v, v'] = (h, h') + [u, u']$$

and

$$(6.13) \qquad (v, v') = (h, h') + (u, u').$$

The first of the inner products on the right sides of (6.12) and (6.13) is the inner product on the Hilbert space $\mathbf{H}$; the second of the inner products is the indefinite inner product (in (6.12)) and the Hilbert space inner product (in (6.13)) on $H^2(\mathcal{D})$. With these inner products, $H(\Theta)$ is a Krein space, with fundamental symmetry J given by

$$J(h+\Theta u) = h+\Theta(Ju) \quad (h\in\mathbf{H}, u\in H^2(\mathcal{D})).$$

A comparison of the constructions of the spaces $H(\Theta)$ and $H(T)$ shows that $H(\Theta)=H(T)$.

Note that we have constructed $H(\Theta)$ in terms of Θ alone; we needed to use the fact that $\Theta=\Theta_T$ for some operator T only to prove some properties of $H(\Theta)$ from the assumptions on Θ. It would be more desirable to be able to construct the space $H(\Theta)$ without any reference to the underlying operator; the stumbling block is finding a direct product of the uniqueness of the representation $h+\Theta u$ for $h\in\mathbf{H}$ and $u\in H^2(\mathcal{D})$.

7. Functional model for an operator

In the first part of this section, we assume that T is an operator with spectrum in the closed unit disc and with trivial residual space. Such an operator is automatically completely non-unitary, since a subspace of $\mathcal{H}$ which reduces T to a unitary operator is in the residual space (see [9], Theorem 3.1). We present here a model for this operator, based on the function space $H(T)$ constructed earlier. We will finish the section by presenting a model in terms of an operator valued analytic function Θ.

Let $\mathbf{K}_+=H(T)$, and let $\mathbf{U}$ denote multiplication by the independent variable, as in Section 4. Then the Fourier representation Φ_* is a unitary map from $\mathcal{K}_+$ onto $\mathbf{K}_+$, preserving both the indefinite and the Hilbert space inner products. The subspace $\mathcal{H}$ of $\mathcal{K}_+$ is identified with the subspace $\mathbf{H}$ of $\mathbf{K}_+$, defined as the orthogonal complement of $\Theta_T H^2(\mathcal{D}_T)$ in $\mathbf{K}_+$:

$$(7.1) \qquad \mathbf{H} = \mathbf{K}_+ \cap [\Theta_T H^2(\mathcal{D}_T)]^\perp.$$

The representation of $H(T)$ in the form $F\mathcal{H}+\Theta_T H^2(\mathcal{D}_T)$, and the forms (4.2) and (4.3) of the inner products on $H(T)$, show that either of the two inner products could be used for the orthogonal complement in (7.1), and that Φ_* maps $\mathcal{H}$ onto $\mathbf{H}$.

If we define $U_+=U|\mathcal{K}_+$, then we have

$$T^* = U_+^*|\mathcal{H}$$

(cf. [12]). It follows immediately that, if we define

$$\mathbf{T}^* = \mathbf{U}^*|\mathbf{H},$$

then we have

$$\Phi_* Th = \mathbf{T}\Phi_* h$$

for all $h \in \mathscr{H}$.

The following theorem summarizes the main properties of the model, which is based on the Sz.-Nagy and Foiaş model (see [12] and [10]) of an operator. We present a model only for the part of the dilation on $\mathscr{K}_+$; the remainder of the space could be modelled very simply by including functions of the form

$$v(\lambda) = \sum_{n=-\infty}^{-1} \lambda^n v_n,$$

with square summable coefficients $v_n \in \mathscr{D}_{T^*}$, but notational convenience would be sacrificed.

We use the function space $H(T)$ in place of the space $H^2(\mathscr{D}_{T^*})$ of the Sz.-Nagy and Foiaş model. The model is simplified by the fact that we are working with operators for which $\mathscr{R} = \{0\}$.

Theorem 7.1. *The Fourier representation Φ_* of $M_+(\mathscr{L}_*)$ is a unitary operator from $\mathscr{K}_+$ onto $\mathbf{K}_+$, preserving both the indefinite and the Hilbert space inner products. If U is the Davis dilation of T, restricted to the subspace $\mathscr{K}_+$, then $\Phi_* U = \mathbf{U}\Phi_*$.*

The subspace $\mathscr{H}$ of $\mathscr{K}_+$ is mapped by Φ onto the subspace $\mathbf{H}$ of $\mathbf{K}_+$, defined by (7.1), and the indefinite and Hilbert space inner products coincide on $\mathbf{H}$. If $\mathbf{T}$ is the operator on $\mathbf{H}$ whose adjoint is defined by

$$\mathbf{T}^* u = \mathbf{U}^* u,$$

for $u \in \mathbf{H}$, then we have, for all $h \in \mathscr{H}$, $\Phi_* Th = \mathbf{T}\Phi_* h$.

When the characteristic function Θ_T is uniformly bounded on the open unit disc, the space $H^2(\mathscr{D}_{T^*})$ can be used as the range of Φ_* [5]. In that case Φ_* is bounded, with bounded inverse, but does not preserve the Hilbert space inner products. Since the shift on $H^2(\mathscr{D}_{T^*})$ is an isometry, in the Hilbert space inner product, the analogue of Theorem 7.1 gives the result that, when Θ_T is bounded, U is similar to a unitary operator on a Hilbert space ([11], and [9], Theorem 7.2).

When $H(T)$ is used as the range of Φ_*, Theorem 7.1 above shows that the Hilbert space inner product is preserved by Φ_*. We lose, however, the property that the shift is an isometry in the Hilbert space inner product: the operator $\mathbf{U}$ on $H(T)$ is a shift in the Kreins space sense, preserving the indefinite inner product, but not necessarily the Hilbert space inner product. Indeed, $\mathbf{U}$ need not be power bounded when $T\in C_{\cdot 0}$, and hence need not be similar to a unitary operator on a Hilbert space.

Example 7.2. Let T be the adjoint of the operator defined in Example 3.1, so that $T\in C_{\cdot 0}$. Suppose $u=Fx\in H(T)$, for the vector $x\in\mathscr{H}$ used in Example 3.1. Then, by (4.28), we have

$$[\mathbf{U}^n u](\lambda) = \lambda^n u(\lambda) = [FT^n x](\lambda)+\Theta_T(\lambda)\sum_{k=0}^{n-1}\lambda^{n-k-1}Q_T T^k x$$

and thus

$$\|\mathbf{U}^n u\|^2 = \|T^n x\|^2+\sum_{k=0}^{n-1}\|Q_T T^k x\|^2. \tag{7.2}$$

Since we are working with the adjoint of the operator in Example 3.1, (3.4) shows that the sequence $\{J_T Q_T T^k x\}_{k\geq 0}$ is not square summable. Since J_T is unitary, the sequence $\{Q_T T^k x\}_{k\geq 0}$ is not square summable, and so, by (7.2), $\mathbf{U}$ is not power bounded.

We finish the section by assuming that $\Theta(\lambda)\colon \mathscr{D}\to\mathscr{D}_*$ is a fundamentally reducible, purely contractive analytic function, which satisfies condition (6.2), We present here a model for an operator having Θ as its characteristic function, based on the function space $H(\Theta)$ constructed in the preceding section. We know, from the previous section, that Θ is the characteristic function of a unique completely non-unitary operator T, acting on a Hilbert space $\mathscr{H}$, with spectrum in the closed unit disc, and with trivial residual space. Thus, we can describe the model directly, using the above results and the fact that $H(T)=H(\Theta)$. Note that the space $\mathbf{H}$, defined by (7.1), is the same as the space $\mathbf{H}$ defined in Section 6.

Theorem 7.3. *Suppose* $\Theta(\lambda)\colon \mathscr{D}\to\mathscr{D}_*$ *is a fundamentally reducible, purely contractive analytic function, which satisfies condition* (6.2), *Define* $\mathbf{K}_+=H(\Theta)$, *and*

$$\mathbf{H} = \mathbf{K}_+\cap[\Theta H^2(\mathscr{D})]^{\perp},$$

where the orthogonal complement is taken in either of the two inner products of $H(\Theta)$. *Then* $\mathbf{H}$ *is a Hilbert space, and the operator* $\mathbf{T}$ *on* $\mathbf{H}$, *defined by*

$$\mathbf{T}^* u = \mathbf{U}^* u,$$

for $u\in\mathbf{H}$, *has characteristic function which coincides with* Θ.

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CENTRE FOR MATHEMATICAL ANALYSIS
THE AUSTRALIAN NATIONAL UNIVERSITY
CANBERRA, A.C.T. 2601

and

THE OHIO STATE UNIVERSITY
MARION, OHIO 43302
U.S.A.

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Hyponormal composition operators on weighted Hardy spaces

NINA ZORBOSKA

Let $\beta=\{\beta_n\}_{n=0}^{\infty}$ be a sequence of positive numbers with $\beta_0=1$ and $\frac{\beta_{n+1}}{\beta_n}\to 1$ as $n\to\infty$. The set $H^2(\beta)$ of formal complex power series $f(z)=\sum_{n=0}^{\infty} a_n z^n$ such that

$$\|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty$$

is a Hilbert space of functions analytic in the unit disc with the inner product

$$(f, g)_\beta = \sum_{n=0}^{\infty} a_n \bar{b}_n \beta_n^2$$

for f as above and $g(z)=\sum_{n=0}^{\infty} b_n z^n$. For details see [9].

If φ is an analytic function mapping the unit disc D into itself, we define the composition operator C_φ on the space $H^2(\beta)$ by $C_\varphi f=f\circ\varphi$. The operators C_φ are not necessarily defined on all of $H^2(\beta)$. They are everywhere defined in some special cases: on the classical Hardy space H^2 (the case when $\beta_n=1$ for all n) — see for example [7], and on a general space $H^2(\beta)$ if the function φ is analytic on some open set containing the closed unit disc having supremum norm strictly smaller than one (see[11]). There are a lot of other known properties of composition operators, on the classical Hardy space H^2 (see for example [1], [6] and [7]), and on more general space $H^2(\beta)$ (see [4], [5], [8], [10] and [11]).

In this article we are interested in the hyponormality of composition operators and their adjoints. The inspiration for this work was Cowen's and Kriete's article [2] in which, among the other results, they get a nice correlation between hyponormality of composition operators on H^2 and the Denjoy—Wolff point of the inducing map. Their proofs use some properties of the spectrum and spectral radius of a composition operator on H^2 which are still not known in the case of general spaces

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$H^2(\beta)$. Nevertheless, taking a different approach, we can get some results on spaces $H^2(\beta)$.

We say that the operator A on a Hilbert space $\mathscr{H}$ is hyponormal if $A^*A-AA^*\geqq 0$, or equivalently if $\|A^*f\|\leqq\|Af\|$ for all f in $\mathscr{H}$.

For a sequence β as above and a point ω in D, let

$$k_\omega^\beta(z)=\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}(\bar{\omega}z)^n.$$

Then the function k_ω^β is a point evaluation for $H^2(\beta)$; i.e., for f in $H^2(\beta)$,

$$(f, k_\omega^\beta)_\beta = f(\omega).$$

Note that $k_0^\beta=1$ (the function identically equal to 1), and that $C_\varphi^* k_\omega^\beta = k_{\varphi(\omega)}^\beta$.

Theorem 1. *If C_φ is hyponormal on the space $H^2(\beta)$, then $\varphi(0)=0$.*

Proof. Let C_φ be hyponormal on $H^2(\beta)$, and k_0^β be point evaluation at 0. Then $\|C_\varphi^* f\|_\beta\leqq\|C_\varphi f\|_\beta$ for all f in $H^2(\beta)$, and if $f=k_0^\beta$ we have

$$\|C_\varphi^* k_0^\beta\|_\beta^2 = \|k_{\varphi(0)}^\beta\|_\beta^2 = \sum_{n=0}^{\infty}\frac{1}{\beta_n^2}|\varphi(0)|^{2n} \leqq \|C_\varphi k_0^\beta\|_\beta^2 = \|k_0^\beta\|_\beta^2 = 1,$$

which implies, since $\beta_0=1$, that $\varphi(0)=0$.

Theorem 1.5 in [2] states that if C_φ^* is hyponormal on H^2, then φ is univalent in D; the proof also applies to a general space $H^2(\beta)$. Also, by Theorem 1.4 in [2], if C_φ^* is hyponormal and not normal on H^2, then the Denjoy—Wolff α of φ (for definition and properties see [1]) is such that $|\alpha|=1$ and $\varphi'(\alpha)<1$. This result is not true in all spaces $H^2(\beta)$, as we can see from the following. Note that the spaces we are going to work with are "the small spaces $H^2(\beta)$" which consist of functions continuous on the closed unit disc. These spaces provide examples of some other interesting composition operators (for example, compact ones with no fixed point in the unit disc (see [8] and [10])).

First we need the following lemma.

Lemma 1. (Lemma 4.3 from [3].) *If A is hyponormal on $\mathscr{H}$, then for all $f\neq 0$ in $\mathscr{H}$, and for all $n>0$,*

$$\|A^n f\|\geqq\frac{\|Af\|^n}{\|f\|^{n-1}}.$$

Lemma 2. *Let the sequence β be such that $\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}<\infty$ and let C_φ^* be hyponormal on $H^2(\beta)$. Then $\varphi(0)=0$.*

Proof. By Lemma 1, for any $n \geqq 0$ we have

$$\|(C_\varphi^*)^n k_0^\beta\|_\beta^2 \geqq \frac{\|C_\varphi^* k_0^\beta\|_\beta^{2n}}{\|k_0^\beta\|_\beta^{2(n-1)}}.$$

But

$$\|C_\varphi^* k_0^\beta\|_\beta^2 = \|k_{\varphi(0)}^\beta\|_\beta^2 = \sum_{k=0}^\infty \frac{1}{\beta_k^2} |\varphi(0)|^{2k} = C_1^2$$

and

$$\|(C_\varphi^*)^n k_0^\beta\|_\beta^2 = \|k_{\varphi^{(n)}(0)}^2\| = \sum_{k=0}^\infty \frac{1}{\beta_k^2} |\varphi^{(n)}(0)|^{2k} \leqq \sum_{k=0}^\infty \frac{1}{\beta_k^2} = C_2^2$$

where $\varphi^{(n)}(0)$ is the n-th iteration of φ at 0.

We have that $\|k_0^\beta\|_\beta = 1$, and so

$$C_2^2 \geqq \|(C_\varphi^*)^n k_0^\beta\|_\beta^2 \geqq \|C_\varphi^* k_0^\beta\|_\beta^{2n} = C_1^{2n}.$$

If $C_1 > 1$, then $C_1^n \to \infty$, which is a contradiction with the previous inequality, and so $C_1 = 1$; i.e., $\varphi(0) = 0$.

Lemma 3. *If C_β^* is hyponormal on $H^2(\beta)$ and $\varphi(0)=0$, then $\varphi(z)=az$ where $|a| \leqq 1$.*

Proof. We use the idea of the proof of Theorem 2.4 in [7]. Let $\varphi(z) = a_1 z + a_2 z^2 + a_3 z^3 + \dots$ and $f_k = \frac{z^k}{\beta_k}$. Then $\{f_k\}_{k=0}^\infty$ is an orthonormal basis for $H^2(\beta)$ and $\varphi(z) = \sum_{n=1}^\infty a_n \beta_n f_n$. Now

$$\|C_\varphi^* f_1\|_\beta^2 = \sum_{k=0}^\infty |(C_\varphi^* f_1, f_k)|^2 = \sum_{k=0}^\infty |(f_1, C_\varphi f_k)|^2 = \sum_{k=0}^\infty \frac{1}{\beta_k^2} |(f_1, \varphi^k)|^2 = \frac{1}{\beta_1^2} |a_1 \beta_1|^2.$$

Also

$$\|C_\varphi f_1\|_\beta^2 = \frac{1}{\beta_1^2} \|\varphi\|_\beta^2 = \frac{1}{\beta_1^2} \sum_{n=1}^\infty |a_n|^2 \beta_n^2.$$

The operator C_φ^* is hyponormal; i.e., for any f in $H^2(\beta)$, $\|C_\varphi f\|_\beta^2 \leqq \|C_\varphi^* f\|_\beta^2$. If $f = f_1$, we get that

$$\frac{1}{\beta_1^2} \sum_{n=1}^\infty |a_n|^2 \beta_n^2 \leqq \frac{1}{\beta_1^2} |a_1 \beta_1|^2$$

which implies that $0 = a_2 = a_3 = \dots$.

Theorem 2. *Let the sequence β be such that $\sum \frac{1}{\beta_n^2} < \infty$, and C_φ^* be hyponormal on $H^2(\beta)$. Then $\varphi(z) = az$, where $|a| \leqq 1$.*

Proof. The proof follows immediately from Lemma 2 and Lemma 3.

SCHWARTZ proved in [7] that a composition operator C_φ is normal on H^2 if and only if $\varphi(z)=az$ where $|a|\leqq 1$. Using the above results we can easily prove that the same statement holds for all spaces $H^2(\beta)$.

Theorem 3. *The operator C_φ is normal on $H^2(\beta)$ if and only if $\varphi(z)=az$ with $|a|\leqq 1$.*

Proof. It is trivial that if $\varphi(z)=az$, $|a|\leqq 1$, then C_φ is normal on $H^2(\beta)$. Conversely, if C_φ is normal, then C_φ is hyponormal and, by Theorem 1, $\varphi(0)=0$. But we also have C_φ^* hyponormal, and by Lemma 3, $\varphi(z)=az$ with $|a|\leqq 1$.

As a consequence of Theorem 2 and Theorem 3, we get an interesting example of family of spaces $H^2(\beta)$, where the only cohyponormal composition operators are the ones which are normal.

Corollary. *If $\sum 1/\beta_n^2<\infty$ and C_φ^* is hyponormal on $H^2(\beta)$, then C_φ^* is normal on $H^2(\beta)$.*

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THE UNIVERSITY OF MANITOBA
MATHEMATICS AND ASTRONOMY
WINNIPEG, MANITOBA R3T 2N2
CANADA

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Essentially normal composition operators on L^2

THOMAS HOOVER and ALAN LAMBERT

1. Preliminaries. Let (X, Σ, m) be a complete, sigma-finite measure space and let T be a Σ-measurable mapping $(T^{-1}\Sigma \subset \Sigma)$ of X into X. The *composition operator* C induced by T on the set of complex valued, measurable functions on X is defined by $Cf = f \circ T$. Throughout this article $L^2 = L^2(X, \Sigma, m)$. For $S \in \Sigma, L^2(S)$ is the L^2 space of functions on S, with the appropriate restrictions of Σ and m. We will regard this space as the subspace of L^2 consisting of those functions with support in S. In general the support of the function f will be denoted S_f. For f in L^∞, M_f will denote the operator of multiplication by f on L^2. We will be concerned with those composition operators C which are bounded linear operators on L^2. A detailed description of the general properties of such operators is given in [3]. In particular, it is shown that C is a bounded operator on L^2 if and only if

(i) $m \circ T^{-1}$ is absolutely continuous with respect to m, and

(ii) $\dfrac{dm \circ T^{-1}}{dm} \in L^\infty$.

Conditions (i) and (ii) are assumed to hold throughout. We set

$$h = \frac{dm \circ T^{-1}}{dm}.$$

We will make use of the following notation. For f in L^2 or measurable and non-negative, $E(f)$ is the conditional expectation $E(f|T^{-1}\Sigma)$. For $f \in L^2$, $E(f)$ is the orthogonal projection of f onto $L^2(X, T^{-1}\Sigma, m)$. Verifications of the following properties are found in [1], [2], and [5].

(iii) $\|C\|^2 = \|h\|_\infty$.

(iv) For each f there is a function F such that $E(f) = F \circ T$. If $E(f) = G \circ T$ as well, then $F = G$ on S_h. In particular the function $h \cdot [E(f)] \circ T^{-1}$ is well defined even if T is not invertible. In fact, $C^*f = h \cdot [E(f)] \circ T^{-1}$, $C^*Cf = hf$, and $CC^*f = h \circ TE(f)$.

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(v) For measurable f and g, $E((f\circ T)\cdot g)=(f\circ T)Eg$. For $f\in L^\infty$ this equation has the operator theoretic form $M_{f\circ T}E=EM_{f\circ T}$.

2. Essential normality. In [5] R. WHITLEY proved that C is normal if and only if T is invertible and bi-measurable, and $h=h\circ T$. Recall that an operator A is *essentially normal* if its image in the Calkin algebra is a normal element. Equivalently A is essentially normal if and only if A^*A-AA^* is compact. R. K. SINGH and T. VELUCHAMY ([4]) have examined the question of essential normality for certain composition operators. Their result in this regard is stated below.

Theorem. *If (X, Σ, m) is completely nonatomic, and if C is essentially normal with dense range, then C is normal.*

In this article we will develop characterizations of essentially normal composition operators. It will be shown that the dense range hypothesis in the above result is unnecessary. We first note that in the atomic case it is possible to have a non-normal, essentially normal composition operator.

2.1. Example. Let $X=\mathbf{N}=\{1, 2, \dots\}$ and let m be the counting measure. Set $T(1)=1$ and $T(n+1)=n$. Then C is a rank one perturbation of the unilateral shift. In particular, it is an essentially normal operator with index -1, and so is not normal.

For convenience, let $D=C^*C-CC^*=M_h-M_{h\circ T}E$. We will examine D with respect to the orthogonal decomposition of L^2 as $EL^2\oplus(I-E)L^2$. We note that EL^2 consists of those L^2 functions which are $T^{-1}\Sigma$ measurable. The range of C is dense in EL^2 ([1]). Also, $(I-E)L^2$ consists of those L^2 functions f for which $\int_{T^{-1}A} f\,dm=0$ for every Σ-set A.

2.2. Lemma. *D is compact if and only if both $M_h(1-E)$ and $M_{h-h\circ T}E$ are compact.*

Proof. D is compact if and only if both DE and $D(I-E)$ are compact. But $D=M_h-M_{h\circ T}E$, so

$$DE=(M_h-M_{h\circ T}E)E=M_{h-h\circ T}E,$$

and

$$D(I-E)=M_h(I-E).$$

2.3. Corollary. *Suppose that D is compact. Then $M_{h\cdot(h-h\circ T)}$ is compact.*

Proof. $M_h(1-E)$ and $M_{h-h\circ T}E$ are compact. But

$$M_h(M_{h-h\circ T}E)^*+(M_h(I-E))M_{h-h\circ T}=M_hEM_{h-h\circ T}+M_h(I-E)M_{h-h\circ T}=$$
$$=M_hM_{h-h\circ T}=M_{h\cdot(h-h\circ T)}.$$

Write $X=X_c\cup\{a_i: i\in J)$ where m is completely nonatomic on X_c and $\{a_i: i\in J)$ consists of the atoms for m. Let $A=T^{-1}X_c$ and $A_i=T^{-1}a_i$, $i\in J$. These sets are pairwise disjoint, so that the corresponding subspaces of L^2 are orthogonal. Note that for any measurable set S, $L^2(T^{-1}S)$ is a reducing subspace for D, because if $S_f\subset T^{-1}S$, then

$$hf-h\circ TEf = hf-h\circ TE(f\chi_{T^{-1}S}) = hf-h\circ T(Ef)\chi_{T^{-1}S} = 0 \quad \text{off} \quad T^{-1}S.$$

We have established the following result.

2.4. Theorem. *C is essentially normal if and only if $D|_{L^2(A)}$ and $D|_{L^2(A_i)}$ $(i\in J)$ are compact, and*

$$\lim_{i\to\infty}\|D|_{L^2(A_i)}\| = 0.$$

This result is strengthened somewhat by Lemma 2.6 below. Its proof depends on the following fact.

2.5. Lemma. *If S is a subset of X_c with $0<m(S)<\infty$, then there is a subset A of S with*

$$\frac{1}{4}m(S) < m(A) < \frac{3}{4}m(S).$$

Proof. Suppose no such set A exists. Then for every measurable subset E of S, either $m(E)<\frac{1}{4}m(S)$ or $m(E)>\frac{3}{4}m(S)$. Let $\mathscr{E}=\left\{E\subset S: m(E)>\frac{3}{4}m(S)\right\}$. If E and F are in $\mathscr{E}$, then

$$m(E\cap F) = m(E)+m(F)-m(E\cup F) > \frac{1}{2}m(S).$$

Thus $E\cap F\in\mathscr{E}$. Let $\alpha=\inf\{m(E): E\in\mathscr{E}\}$, and let $\{E_n\}$ be a decreasing sequence of sets in $\mathscr{E}$ whose measures converge to α. Let $G=\bigcap E_n$. Then $m(G)=\alpha\geqq\frac{3}{4}m(S)$. Now, there is a measurable subset B of G with $0<m(B)<m(G)$. But then neither B nor $G-B$ are in $\mathscr{E}$. It then follows that both B and $G-B$ must have measures less than $\frac{3}{4}m(S)$, which implies that the measure of G is less than $\frac{1}{2}m(S)$. This contradicts the location of G in $\mathscr{E}$.

2.6. Lemma. *If $D|_{L^2(A)}$ is compact then it is* 0.

Proof. Assume $D_0=D|_{L^2(A)}$ is compact. Since D is selfadjoint and reduced by $L^2(A)$, D_0 is selfadjoint. In particular, if D_0 is not 0 then it has a nonzero eigenvalue r. Let $\mathscr{E}_r$ be the corresponding finite dimensional eigenspace, and let φ be any L^∞ function with $S_\varphi\subset X_c$. Then $S_{\varphi\circ T}=T^{-1}S_\varphi\subset A$. Now, $M_{\varphi\circ T}L^2(A)\subset L^2(A)$

and for any f in $L^2(X)$,

$$M_{\varphi\circ T}Df = (\varphi\circ T)(hf - h\circ TEf) = (h)\cdot(\varphi\circ T)\cdot f - (h\circ T)\cdot E((\varphi\circ T)\cdot f) = DM_{\varphi\circ T}f.$$

It follows that $M_{\varphi\circ T}$ leaves $\mathcal{E}_r$ invariant. But $\mathcal{E}_r$ is finite dimensional and so there is a function $f\in\mathcal{E}_r$ other than 0, and a scalar λ such that $(\varphi\circ T)f=\lambda f$ a.e. dm. In particular, $\varphi\circ T=\lambda$ on a set of positive measure. This shows that every $L^\infty(X_c)$ function is constant on a set of positive measure. But by definition X_c is completely nonatomic. Let S be a set of finite, positive measure in X_c. Via Lemma 2.5 we partition S into two measurable sets, each of measure no more than 3/4 that of S. Define the function f_1 to take the values 1/2, 1 respectively on the sets. Repeat this procedure by replacing S by each of the sets of constancy of f_1 and defining f_2 to take the value of f_1 on one part of each of the original two subsets and to be 1/4, 3/4 respectively on the remaining two sets. Continuation of this procedure gives rise to a monotonically decreasing sequence of functions whose pointwise limit is bounded and not constant on any set of positive measure in X_c. Indeed, we have for each x,

$$0 \leqq f_n(x) - f_{n+1}(x) \leqq \frac{1}{2^{n+1}},$$

so that

$$f_n(x) - f(x) \leqq \frac{1}{2^n}.$$

Thus, for any $r>0$ and any positive integer n,

$$\{x\colon f(x) = r\} \subset \left\{x\colon r \leqq f_n(x) \leqq r + \frac{1}{2^n}\right\}.$$

But this latter set contains at most two sets of constancy for f_n, so

$$m\left\{x\colon r \leqq f_n(x) \leqq r + \frac{1}{2^n}\right\} \leqq 2\cdot\left(\frac{3}{4}\right)^n m(S).$$

It then follows that $f\neq r$ a.e. dm. This contradiction completes the proof of the lemma.

Note that the result of Singh and Veluchamy as stated in Section 1 of this paper follows as a special case of Lemma 2.6, for in the completely nonatomic case $A=X$. But then $D=0$, i.e. C is normal. It is interesting to see that one basic property from Whitley's characterization of normality carries over to the general essentially normal setting.

2.7. Corollary. *If C is essentially normal then $h=h\circ T$ a.e. on $T^{-1}X_c$.*

Proof. Assume that C is essentially normal. Then $D|_{L^2(T^{-1}X_c)}=0$. Let Y be a subset of X_c of finite measure. Since $h\in L^\infty$ we have $m(T^{-1}Y)=\int_Y h\,dm<\infty$ and in particular $\chi_{T^{-1}Y}\in L^2(T^{-1}X_c)$. But then we see that

$$0 = D\chi_{T^{-1}Y} = h\cdot\chi_{T^{-1}Y}-(h\circ T)\cdot E(\chi_{T^{-1}Y}) = (h-h\circ T)\chi_{T^{-1}Y}.$$

It then follows that $h=h\circ T$ a.e. on X_c.

We will conclude this paper with an example establishing the existence of an essentially normal composition operator for which $h>0$ a.e. and for which there is an atom a with $T^{-1}a$ infinite. First we investigate the structure of the sets $T^{-1}a_i$, $i\in J$, when C is essentially normal. Let a be an atom for m and let $B=T^{-1}a$. Then $D|_{L^2(B)}$ is compact. Let $f\in L^2(B)$. Since m is sigma-finite and h is essentially bounded, B is a set of finite measure. Noting that Ef is constant on $B=T^{-1}a$, we see that

$$\int_B f\,dm = \int_{T^{-1}a} f\,dm = \int_a h\cdot(Ef)\circ T^{-1}dm = m(a)h(a)(Ef)\circ T^{-1}(a) =$$

$$= m(a)\frac{m(T^{-1}a)}{m(a)}(Ef)\circ T^{-1}(a) = m(B)(Ef)\circ T^{-1}(a).$$

It then follows that $Ef\equiv\dfrac{1}{m(B)}\int_B f\,dm$ on B.

Also, for x in B, $h\circ T(x)=h(a)=\dfrac{m(B)}{m(a)}$. In particular $(M_{h\circ T}E)|_{L^2(B)}$ is the rank one operator $f\to\dfrac{1}{m(a)}\int_B f\,dm$. But then the compactness of $D|_{L^2(B)}$ implies $M_h|_{L^2(B)}$ is compact. This in turn shows that

$$B\cap S_h = \{b_k\colon k\in K\}$$

where each b_k is an atom.

2.8. Example. Let O be the origin in the plane and let $X=\{O\}\cup(\mathbf{N}\times\mathbf{N})$. Define m by $m(O)=1$; $m(i,j)=1/2^{ij}$. Finally, define T on X by

$$T(O) = O;\ T(i,1) = O;\ T(i,j) = (i,j-1) \quad \text{for} \quad j>1.$$

Then

$$T^{-1}(O) = \{O\}\cup(\mathbf{N}x\{1\}),$$

and

$$h(O) = \frac{m(T^{-1}O)}{m(O)} = 1+\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = 2,$$

while

$$h(i,j) = \frac{m(T^{-1}(i,j))}{m(i,j)} = \frac{m(i,j+1)}{m(i,j)} = \frac{2^{-i(j+1)}}{2^{-ij}} = 2^{-i}.$$

For f supported on $T^{-1}O$, $Df=hf-2\int_{T^{-1}O} f\,dm$. Since $\lim_{n\to\infty} h(n,1)=0$, $D|_{L^2(T^{-1}O)}$ is compact. On the other hand,

$$(Df)(i,j+1) = 2^{-i}f(i,j+1)-2^{-i}f(i,j+1) = 0.$$

Thus C is essentially normal.

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(T.H.)
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HAWAII
HONOLULU, HAWAII 96822
U.S.A.

(A.L.)
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NORTH CAROLINA AT CHARLOTTE
CHARLOTTE, NORTH CAROLINA 28223
U.S.A.

Bibliographie

Automata, Languages and Programming. Proceedings of the 16th ICALP Conference held in Stresa, Italy, July 1989. Edited by G. Ansiello, M. Decani-Ciancaglini and S. Ronchi Della Rocca (Lecture Notes in Computer Science, 372), XI+788 pages, Springer-Verlag, Berlin—Heidelberg—York—Tokyo, 1989.

This volume contains the following five invited talks: C. Böhm: Subduing Self-Application; H. Ehrig, P. Pepper, F. Orejas: On Recent Trends in Algebraic Specification; D. Eppstein, Z. Galil: Parallel Algorithmic Techniques for Combinatorial Computation; D. Perrin: Partial Commutations; J. C. Reynolds: Syntactic Control of Interference, Part 2. As the reader may expect the majority of the 45 accepted contributions is more or less tied up with Complexity Theory. They range from adding tensor rank as a new item to the *NP*-completeness heap (J. Hastad), developing a systematic theory based on algebraic automata theory to analyse the inner structure of the complexity class NC^1 (P. McKenzie and D. Therien) to the papers of Allender and Hemachandra offering new oracle constructions and optimal lower bounds. Numerous other fields such as rewriting systems (e.g. Dershowitz, Kaplan and Plaisted's paper on infinite normal forms), factors of biinfinite words (Beauquier, Pin) are represented, too.

This book is a valuable source of information for specialists working in different branches of Computer Science.

J. Virágh (Szeged)

P. Bamberg—S. Sternberg, A course in mathematics for students of physics: 2, XVII+444 pages, Cambridge University Press, Cambridge—New York—Port Chester—Melbourne—Sydney, 1990.

Students with interest in physics need strong and well-organized knowledges in mathematics, as well. Most effective way to organize their mathematical education, is to present mathematics as a powerful resource and means of expression in solving and representing physical problems, result and discoveries.

This natural approach is the starting point and leading idea of Bamberg's and Sternberg's work. Second volume contains chapters based on algebraic topology, exterior differential calculus, theory of functions of complex variable. Connecting fields of physics are: electrical networks, electrostatistics, magnetostatistics, Maxwell-equations, classical thermodynamics.

The subject not only covers a course but it is suitable for individual studying: all the important topics are encountered, furthermore the volumes are self-contained. The reader finds detailed demonstrations, with well-motivated arguments at each step. Numerous figures are nice and clear, important statements and conclusion are conspicuous by logical emphasis, and typographical way, as well.

This new volume is an interesting reading for a mathematician, as well, who wants to strengthen or broaden his or her familiarity with physics.

J. Kozma (Szeged)

N. Bourbaki, Elements of Mathematics, Algebra II, Chapters 4—7, VIII+436 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong, 1990.

This is a new and expanded (English) version of Bourbaki's Algebra Chapters 4—7 (translated from the French by P. M. Cohn and J. Howie). The English translation of the first three chapters of the Algebra was published in 1989 by Springer (see our review in the same *Acta* vol. **54**, p. 410).

Chapter 4 deals with polynomials, rational fractions and power series over commutative rings. New sections on symmetric tensors, polynomial mappings and symmetric functions have been added. The completely rewritten Chapter 5 is devoted to commutative fields and field extensions. After the Galois theory (with an application to finite fields) the transcendental extensions are studied (e.g. p-bases, separability criterions, regular extensions), which are not usual parts of textbooks. In Chapter 6 one can read on ordered groups and fields. The last, Chapter 7 deals with modules over principal ideal domains. New sections on semi-simple endomorphisms and Jordan decomposition have been inserted.

As usual in the volumes of the "Les structures fondamentales de l'analyse" each chapter ends with exercises and most of them also with historical remarks.

As a closing remark we repeat the last two sentences from our previous review on N. Bourbaki's Algebra I and Commutative Algebra: "The works of N. Bourbaki are not easy peaces of reading, but everybody can enjoy them, who likes the strict axiomatic treatment. In my opinion, these masterpieces must have places in every good mathematical library".

Lajos Klukovits (Szeged)

Victor Bryant, Yet another introduction to analysis, VIII+290 pages, Cambridge University Press, Cambridg—New York—Port Chester—Melbourne—Sydney, 1990.

Analysis is notoriously one of the most difficult subjects to present in the classroom. Suppose you have a definite conception on the introduction of analysis and you wish to find books having characteristic features similar to your conception. Although everyone believes that "a new introduction to analysis springs up every other day", the probability to find appropriate books is a surprisingly small positive number. The subject-matter is many-sided. Your task is not only to make clear some notions but at the same time to take preliminary steps towards deeper topics. Who has right to present a new introduction? In my opinion every experienced teacher having an original idea has right to write such an introduction. This in not a hopeless case because the theme is similar to classical music, e.g. Beethoven's Violin concerto, there exist several different but authentic performances. (However, sometimes you can hear really bad ones as well.)

I think that the author of this book has several greater and smaller ideas. (I liked his articles in Math. Gazette very much.) The most characteristic feature of the work is that the new notions are unsophisticated, the proofs are not only clear, but in several cases first you have a water proof, a sketch, then a water-tight proof. (You cannot find even the shadow of "deus ex machina".) Let us have only one characteristic example: After examples one obtains a guess, that any sequence will have either an increasing subsequence or a decreasing subsequence (or possibly both). The author declares the theorem, then he gives a water proof. The keystone of this proof is that the points (n, x_n) (where we assume that $x_n > 0$) represent people on the roots of their hotels on the Costa Bom, and each hotel with a sea-view will have a special symbol. Then we find an exact water-tight proof. The style is fresh and imaginative.

If you are going to enumerate the titles of the paragraphs you find questions only, e.g. in the fourth chapter (Calculus at last): How do we work out gradients? How does that lead to differentiation? How does that help us to find averages and approximations? Finally, the reviewer's question: Why do not you try this interesting "Introduction"? Surely you will have some answers.

L. Pintér (Szeged)

Category Theory and Computer Science, Edited by D. H. Pitt, D. E. Rydeheard, P. Dybjer, A. M. Pitts and A. Poigné (Lecture Notes in Computer Science, 389), Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong, 1989.

This volume is the collection of 21 papers presented at the third conference on Category Theory and Computer Science held in Manchester, UK, September 5—8, 1989. The proceedings of the preceding two conferences in the series were published as volumes 240 and 283 of Springer LNCS. The following lines are from the introduction.

"One of the key ideas is the representation of programming languages as categories. This is particularly appropriate for languages based upon typed lambda calculi where the types become objects in a category and lambda terms (programs) become arrows. Conversions between programs are treated as equality, or alternatively, making the conversions explicit, as 2-cells. Composition is substitution of programs for free variables. Multiple variables are handled by admitting categories with finite products. This treatment enforces a stratification based upon the types of variables and expressions. For example, languages with type variables lead to indexed (or fibered) categories. Constructs in programming languages correspond to structure within categories, and categories with sufficient structure delimit the semantics of a language."

The volume can be recommended to theoretical computer scientists and graduate students with interest in semantics of programming languages or in foundational issues of computer science.

Z. Ésik (Szeged)

John B. Conway, A Course in Functional Analysis (Graduate Texts in Mathematics, 96) XVI + 399 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong, 1990.

The text is divided into eleven chapters, and at the end of the book three Appendices can be found. The first two chapters introduce the basic concepts of Hilbert spaces and Hilbert space operators and develop the main theorems. Here the complete spectral theory of compact normal operators is worked out. Chapter 3 defines Banach spaces and presents the basic theorems, such as the Hahn—Banach theorem and the open mapping and closed graph theorems, Chapter 4 summarizes the essentials of the theory of locally convex spaces. The main objects of the study in Chapter 5 are the weak topology on a Banach space and the weak-star topology on its dual. Chapter 6 is devoted to the general theory of linear operators on a Banach space. Chapter 7 gives a glimpse into the theory of Banach algebras and spectral theory and applies this to the study of operators on a Banach space. In Chapter 8 the notion of a C^*-algebra is explored which is intimately connected with the theory of operators on a Hilbert space. It turns out that any C^*-algebra is isomorphic to a subalgebra of the algebra of bounded operators on a Hilbert space. Chapter 9 develops the spectral theory of bounded normal operators on a Hilbert space as an application of the representation theory of Abelian C^*-algebras. Chapter 10 generalizes the spectral theory for unbounded operators. Chapter 11 studies certain properties of operators on a Hilbert space, that are invariant under compact perturbations, and proves the basic properties of the Fredholm index. The appendices shortly summarize the notions of linear spaces and topology and determine the dual spaces of L^p and $C_0(x)$ spaces. There are, at the end of every sections, several exercises of varying degrees of difficulty with different purpose in mind.

The book is a pearl of the mathematical literature, and it is highly recommended to anybody interested in functional analysis.

L. Gehér (Szeged)

Robert Dautray—Jacques-Louis Lions, Mathematical Analysis and Numerical Methods for Science and Technology (Vol. 4 Integral Equations and Numerical Methods), X+465 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong, 1990.

This is the fourth volume of the planned six volumes. The enumeration of the titles gives some information on the topics: Mixed Problems and the Tricomi Equation; Integral Equations: Part A. Solution Methods Using Analytic Functions and Sectionally Analytic Functions, Part B. Integral Equations Associated with Elliptic Boundary Value Problems in Domains in R^3; Numerical Methods for Stationary Problems; Approximation of Integral Equations by Finite Elements. Error Analysis; Appendix "Singular Integrals".

In general the discussion begins with physical introduction (or hypotheses), this is a clear treatment with references, if necessary. Then comes the equation with the corresponding conditions, and after this the various methods of solutions. The reader has a well-organized book with serious mathematical notions and procedures which are in the closest connection with important applications. The fascinating thing is that the investigation of this book goes "without tears". The methods seem to be natural and easy to understand. This reminds me of one of G. B. Shaw's play (Cashel Byron's Profession (not the best among the Shaw's works)) in which Cashel says that the real artistic work does not show any struggle with the theme (free interpretation). Such a natural lightness (which covers difficult problems) is the main characteristic feature of this work.

For a reader who has not seen the former volumes we cite their titles: Vol. 1: Physical Origins and Classical Methods, Vol. 2: Functional and Variational Methods, Vol. 3: Spectral Theory and Applications.

L. Pintér (Szeged)

B. A. Davey—H. A. Priestley, Introduction to Lattices and Order, VIII+248 pages, Cambridge University Press, Cambridge—New York—Port Chester—Melbourne—Sydney, 1990.

From the preface: "This is the first textbook devoted to ordered sets and lattices and to their contemporary applications. It acknowledges the increasingly major role order theory is playing on the mathematical stage and is aimed at students of mathematics and at professionals in adjacent areas, including logic, discrete mathematics and computer science."

I recommend this book to all mentioned above.

The treatment of Scott's information systems as algebraic semilattices, fixpoint theory with pointing out its role in computer science, Boolean algebras applied to a fragment of propositional calculus, and Priestley's duality theory between distributive lattices and certain topological spaces are some of the interesting parts of the book.

For those intending to apply the theory of lattices and ordered sets the most interesting chapter is perhaps the last one entitled "Formal Concept Analysis". Formal concept analysis was introduced by R. Wille, and the fast development of this recent field is mostly due to R. Wille and other members of his Darmstadt group. The starting point of concept analysis is so natural that it has applications not only in lattice theory but in many other sciences distinct from mathematics as well.

G. Czédli (Szeged)

The Dilworth Theorems (Selected Papers of Robert P. Dilworth), Edited by K. Bogart, R. Freese, J. Kung, XXVI+465 pages, Birkhäuser, Boston—Basel—Berlin, 1990.

This excellent book gives the reader much more than an almost complete collection of Dilworth's contributions to lattice theory, universal algebra and combinatorics. The book is organized into chapters, including Chain Partitions in Ordered Sets, Complementation, Decomposition Theory, Modular and Distributive Lattices, Geometric and Semimodular Lattices, and Multiplicative Lattices.

Besides Dilworth's reprinted papers these chapters contain related articles by leading experts of the field. Further, Dilworth himself has written backgrounds to each chapter. Thus each chapter not only shows how the present stage of a given research field includes and has developed Dilworth's ideas but it contains an up-to-date survay of the field.

The book is recommended to those interested in the theory of lattices and ordered sets. It gives an introduction to many fields of these theories, and it is useful to experts as well.

G. Czédli (Szeged)

Brian F. Doolin—Clyde F. Martin, Introduction to differential geometry for engineers, (Pure and applied mathematics, 136), XII+163 pages, Marcel Dekker, Inc., New York—Basel—Hong Kong, 1990.

This is a carefully written real introductory book for differential geometry. It is written mainly for the engineers and therefore it does not suppose a well prepared mathematical knowledge for the readers.

Its aim, to introduce the reader to this field of mathematics, is reached, in fact, through a very detailed and concrete treatment. Just this fact is why I recommend it not only to the engineers, who will certainly be grateful for this book, but also to the mathematician students who are just studying differential geometry. Although the book is short, all the really basic concepts of the topic are included.

The authors have no doubt about the book's purpose and in spite of the very much details they do not lose their way: only the necessary and important objects are enlightened in details. To collect only the essential concepts of the subject is really a good way for an introductory book. It makes the topic very natural and easily understandable.

In sum, we recommend this book to all who are interested in a basic introduction to the foundation of differential geometry.

Á. Kurusa (Szeged)

B. A. Dubrovin—A. T. Fomenko—S. P. Novikov, Modern geometry — Methods and applications. Part III. Introduction to homology theory, (Graduate texts in mathematics, 124), IX+416 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong—Barcelona, 1989.

All the people, mathematicians, physicists and students, who read the first two volumes of the Modern geometry (Part I.: GTM 93; Part II.: GTM 104) know what a great experience were to read them. Therefore, it is not surprise that there were big expectations for the third volume, that is published now after five years in highly accessible language.

Nevertheless, all the expectations are now satisfied and the mathematician's and physicist's community has now a very valuable reference and text in the homology theory. This volume is written just as clearly as the first two were and also their style are the same. A lot of concrete examples and the descriptiveness characterize this book.

Since the abstract notions can easily cover up the real ideas in such an abstract topic like the homology theory, it is an advantage to use the abstract terminology only in the case it is necessary. In this way, the reader, by my opinion, can understand the ideas behind the abstractions more easily and the abstract notions appear more naturally. The authors chose successfully this heavier way and the book became marvellous.

For a short sum of the topics treated in the book here are the main titles: Homology and cohomology; Computational recipes; Critical points of smooth functions and homology theory; and finally Cobordisms and smooth structures.

In sum, this book must be on the shelf of all the students, mathematicians and physicists who have any interest in the homology theory.

Á. Kurusa (Szeged)

Ciprian Foiaş—Arthur E. Frazho, The Commutant Lifting Approach to Interpolation Problems (Operator Theory: Advances and Applications, 44), XIII+632 pages, Birkhäuser Verlag, Basel—Boston—Berlin, 1990.

In 1967 D. Sarason introduced an ingenious new method for solving classical interpolation problems. Actually, he proved that for every operator A in the commutant of the compression $T=P_M S|M$ of the simple unilateral shift S (to a semi-invariant subspace M) there exists a bounded analytic function φ on the unit disc such that $A=\varphi(T)$ and $\|A\|=\|\varphi\|_\infty$. Then he pointed out the way how the interpolation theorems due to Carathéodory and Nevanlinna—Pick can be derived from this description of the commutant. Shortly afterwards, in 1968 B. Sz.-Nagy and C. Foiaş extended Saranson's result proving that every operator A intertwining the arbitrarily chosen Hilbert space contractions T and T' can be lifted, in a norm-preserving manner, to an operator B intertwining the minimal isometric dilations V_+ and V'_+. This is the so-called Commutant Lifting Theorem which has been proved a powerful tool in handling different problems in mathematics.

The purpose of this monograph is "to present a unified approach, based on the geometric framework of the commutant lifting theorem, to solve many classical and modern interpolation problems arising in mathematics, engineering and geophysics". The subjects treated include, among others, the block versions of the Carathéodory, Nevanlinna—Pick, Hermite—Fejér interpolation problems in both their classical and tangential forms, the Adamjan—Arov—Krein representation of Hankel operators, the characterization of left and right inverses of Toeplitz operators, and a general Schur type fractional representation of the solutions in the commutant lifting theorem.

Explicit formulas and algorithms are provided. Several proofs are given for the fundamental commutant lifting theorem illuminating the different faces of this theorem. Separate chapters are devoted to the applications in H^∞ control theory and in connection with the layered medium model in geophysics.

The book is essentially self-contained, only some knowledge of elementary real, complex and functional analysis is assumed. A chart helps the reader showing the connection between the different chapters (which are also written as self-contained as possible).

This monograph can be warmly recommended to graduate students who want to get acquainted with this exciting, rich field of mathematics. At the same time it will certainly be an indispensable handbook for specialists in operator theory, interpolation theory, control theory and signal processing.

L. Kérchy (Szeged)

Bernard R. Gelbaum—John M. H. Olmsted, Theorems and Counterexamples in Mathematics (Problem Books in Mathematics), XXXIV+305 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong, 1990.

In my younger years the authors' former book: Counterexamples in Analysis was one of my favourites. I have good reason to be thankful for its clear and ingenious way of enlightening ideas in analysis. Even now I have a copy of this work on my bookself and sometimes in my hands.

In the last thirty years the number of mathematical branches increases in great steps. (See e.g. the Subject Index of the *MR*.) One can survey only a small part of the new results. Some of the notions which were not "elementary" thirty years ago have become "elementary" by now. See for example the elementary problems in The American Mathematical Monthly. In the Preface the authors say: "The object of the body of the text is more to enhance what the reader already knows than to review definitions and notations that have become part of every mathematician's working context". In my opinion in this book one finds several interesting examples, results also in branches which are relatively unknown to the reader. Therefore he/she will inquire about these themes, too. For example Dantzig's simplex algorithm was not unknown for me. Moreover I have read L. Lovász's article: A new linear programming algorithm—better or worse than the simplex method? (The Math. Intelligencer vol. 2, no. 3, 1980), but the remarks on Smale's and Karmankar's work in this book were new for me and I would like to know more about them.

Naturally it is impossible to enumerate the examples which were interesting for me, but let us mention some of them. The first one is the Kakeya problem: "A unit line segment can be rotated through 360° within an arbitrarily small polygonal area." The presentation of this astonishing problem with the remarks is interesting in case you have heard about the Besicovitch's solution and about the Perron trees, too. The short history of the Bieberbach conjecture makes the reader eager to know more about this famous problem and the proof of the conjecture given by de Branges in 1985. (Perhaps Korevaar's and Pommerenke's refering articles could have been suggested to the reader.) Another famous conjecture can be found in paragraph "Exotica in differential topology" the Poincaré's conjecture. The results of Smale and Freedman are mentioned.

The book is warmly recommended to the general mathematical public. (Maybe it is because of prejudice on the part of the reviewer but he thinks that the Analysis is the best chapter of the work.)

L. Pintér (Szeged)

Geometry and Robotics, Workshop, Toulouse, France, May 1988, Proceedings, Edited by J.-D. Boissonat—J.-P. Laumond (Lecture Notes in Computer Science, 391), VI+413 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong, 1989.

A lot of relatively distant fields of geometry enter into relation via their connection with robotics. Theory of curves, computational geometry, projective geometry, algebraic topology give rise of several questions in computer science especially in robotics. Furthermore they play a peculiar role in solving problems in recent times.

A workshop was held at Toulouse in 1988, scientific program of which was the base of this volume. It contains 20 contrubutions by French authors. The understanding of the papers do not presume any deeper preliminary knowledges of computer science or geometry, so it can be a useful reading for everyone interested in current topics of robotics.

J. Kozma (Szeged)

D. H. Greene—D. E. Knuth, Mathematics for the analysis of algorithms (Progress in Computer Science and Applied Logic, 1), VIII+132 pages, Birkhäuser, Boston—Basel—Berlin, 1990.

This book contains some fundamental mathematical techniques which are necessary for the analysis of algorithms.

Chapter 1 starts with binomial identities and afterwards inverse relations and the hypergeometric series are treated. Chapter 2 is devoted mainly to linear and nonlinear recurrence relations. Chapter 3 deals with operator method by means of which one can obtain such characteristics as expected values or variances from probability generating functions. The last chapter considers asymptotic analysis which is very useful tool especially for to average case analysis of algorithms (in detail the following methods and theorems are treated: Abelian theorem, Tauberian theorems, Stieltjes integration and asymptotics, Euler's summation formula, Darboux's method, residue calculus, the saddle point method).

For specialists the rich bibliography increases the value of the book, and both teachers and students will evaluate the appendices containing exam problems from which in this third edition further new ones are added).

The book is warmly recommended to all researchers, teachers, students interested in analysis of algorithms.

J. Németh—A. Varga (Szeged)

Grosse Augenblick aus der Geschichte der Mathematik, herausgegeben von Róbert Freud, 263 Seiten, Akadémiai Kiadó, Budapest, 1990.

Das ist die deutsche Übersetzung der ungarischen Originalausgabe von 1981. Mit diesem Buch laden die Leser die Autoren zu einer abenteuerlichen Reise in der Welt der Mathematik ein.

Dieser Band entsteht aus acht unabhängigen Kapitel, die sind die Folgenden:

1. Schon die alten Griechen haben das gewußt (von János Surányi).
2. Sind Gleichnungen lösbar (von Róbert Freud).
3. Wie ist die mathematische Analysis entstanden (von Ákos Császár).
4. „Aus dem Nichts habe ich eine neue, andere Welt erschaften.“ Was ist die Bolyai—Lobatschewskische Geometrie (von György Bizám).
5. Ideale Zahlen und die Fermatsche Vermutung (von Edit Gyarmati).

6. Wie sah Hilbert die Zukunft der Mathematik? (von Ákos Császár.)
7. Ein sonderbarer Lebensweg, Ramanujan (von Pál Turán).
8. Im Reich des Zufalls herrscht nicht mehr der Zufall (von István Vincze).

Jedes Kapitel endet mit Aufgaben, dessen Lösung kann am Ende des Bandes gefunden werden.

Für dieses Buch soll nur eine geringe Vorbildung gehabt werden, die Mathematik, die in Ober- (Mittel)- schulen gelernt wird, ist ganz genügend.

Ich hoffe, daß jeder diese Abenteuer genießen wird, der die Mathematik für einen Teil der allgemeinen menschlichen Kultur hält.

Lajos Klukovits (Szeged)

Niccolo Guicciardini, The Development of Newtonian Calculus in Britain 1700—1800, XII+ 228 pages, Cambridge University Press, Cambridge—New York—Port Chester—Melbourne—Sydney, 1989.

Newton was one of the investors of differentiation and integration. It is very interesting that the development of the calculus in Britain and in other countries in Europe remained separated for over a century. This book is dealing with both the research and teaching of this calculus called the calculus of "fluxions", over the whole period. The book begins with an overture which contains the fundamental elements of Newton's calculus presenting Newton's published work on the calculus of fluxions. The first three chapters are devoted to the early diffusion of the calculus of fluxions from 1700 to 1730 and to the research in pure mathematics done by early Newtonians (Roger Cotes, James Stirling, Brook Taylor, Colin Maclaurin) and furthermore to the controversy on foundations of the calculus originated by Berkeley's Analyst (1734). The next three chapters deal with the middle period of the fluxional school from 1736 to 1785, considering the production of new treatises and improvements in applications of the calculus of fluxions and the attempts made by some British mathematicians to develop new techniques in the calculus. The last three chapters are devoted to the reform of the calculus from 1775 to 1820. This part of the book is based on completely unknown material.

The chapters are followed by six Appendices containing important information (textbooks, chairs of mathematics, military academies, subject index, manuscript sources) and finally the book ends with a rich bibliography containing more than 600 references.

I am sure that this book is very useful for science historians and philosophers studying this period, but it is recommended to any student or teacher of mathematics, too.

J. Németh (Szeged)

Domingo A. Herrero, Approximation of Hilbert space operators, Volume 1, Second edition (Pitman Research Notes in Mathematics Series, 224), 332 pages, Longman Scientific & Technical, England, 1990.

The approximation theory of Hilbert space operators is a rapidly developing field of the operator theory. This book gives a systematic study of approximation problems (in operator norm) related with operator classes which are invariant under similarity. More precisely, the problems considered here are to characterize the closure of such classes and to give exact formulas or at least estimates for the distance of operators from such classes. After giving an "apéritif" in finite dimension and developing the necessary technical means the cases of nilpotent, algebraic and polynomially compact operators are treated. Disregarding from the proofs of some fundamental theorems connected with C^*-algebras the book is self-contained.

The theory elaborated here is completed in the second volume by C. Apostol, L. A. Fialkow, D. A. Herrero and D. Voiculescu. The progress has been made since the publication of the second volume in 1984 is described in this second edition of the first volume in the form of additional Notes and Remarks at the end of the corresponding chapters and in an Appendix. This Appendix contains, among others, a metatheorem which asserts that the closure of a similarity invariant class of operators with "sufficient structure" can be described in terms of the different parts of the spectra of the operators.

This book can serve as an excellent introduction for beginners as well as a good reference for the experts in the operator theory.

L. Kérchy (Szeged)

R. W. Hockney—J. W. Eastwood, Computer Simulation using Particles, XXII + 540 pages, Adam Hilger, Bristol and Philadelphia, 1988.

The combination of computer experiment, and theory proves much more effective in obtaining physically useful results than any one approach or pair of approaches. To obtain results, theory uses mathematical analysis and numerical evaluation, physical experiment uses apparatus and data analysis, and the computer experiment uses computer plus simulation program.

Covering all aspects of particle techniques of simulation — from mathematical models to simulation programs — this book presents case study examples in astrophysics, plasmas, semiconductors and condensed matter physics. The unifying aspects of the diversity of phenomena are similarities of the mathematical models of the physical systems and similarities of the numerical schemes used.

The secret of success in computer experiments is to devise the appropriate model. The best choice of model depends on the relevant physical length and timescales. There is a clear one-to-one correspondence between the physical and computer model particles in the molecular dynamics simulation. At the other extreme, the identity of the atomic building blocks in the vortex fluid simulation model is completely lost. A third type of particle model lies between the two extremes: dilute plasmas, galaxy, and microscopic semiconductor device simulations fall into this category.

Each steps of a computer experiment introduces constraints: Simplifying assumptions in the development of the mathematic description of physical phenomena in one hand and discretization of the continuous differential or integral equations of the mathematical model in order to allow solution on computers in the other hand.

The book is divided into the following chapters: Computer experiments using particle models; A one-dimensional plasma model; The simulation program; Time integration schemes; The particle-mesh force calculation; The solution of the field equations; Callisionless particle models; Particle-particle-particle-mesh (P^3M) algorithms; Plasma simulation; Semiconductor device simulation; Astrophysics; Solids, liquids, and phase changes.

This book was originally written as a textbook for a final-year undergraduate course in scientific computing at Reading University. The material is of wider interest, and the book can be recommended equally to graduate students and computational scientists and engineers.

I. K. Gyémánt (Szeged)

Roger A. Horn—Charles R. Johnson, Matrix Analysis, XIII + 561 pages, Cambridge University Press, Cambridge—London—New York—New Rochelle—Melbourne—Sydney, 1990.

This book is reprinted and corrected edition of its first published edition in 1987. It contains nine chapters and an appendix. Two views of matrix analysis are reflected in the choice and method of topics in this book. One of them is pure algebraic, and the other one prefers those topics in linear algebra that are important for the applications in mathematical analysis, such as differential equations, optimization and approximation theory. The text starts with an usual introductory part defining and discussing the basic concepts and results of linear algebra, including determinants, eigenvalues and eigenvectors, the characteristic polynomial, similarity, unitary equivalence and canonical forms of matrices. Then Hermitian matrices are introduced. Here variational methods for investigating eigenvalues of Hermitian matrices are emphasized. In normed vectorspaces the algebraic, geometric and analytic properties of matrix norms are discussed. The perturbation theory of Hermitian matrices in some detail is treated. Positive definite matrices and the polar and singular value decompositions and their applications to matrix approximation problems are considered. The last chapter discusses componente-wise nonnegative and positive matrices which arise in many applications in probability theory, economics, engineering etc. At the end of the text an Appendix can be found presenting some basic theorems which are used in the book. A lot of exercises and problems are given in the book. The problems are listed at the end of every sections, they are of various difficulties and types.

The text can be easily understood for students, too, and is highly recommended to anyone having some background in linear algebra and mathematical analysis.

L. Gehér (Szeged)

Taqdir Husain, Orthogonal Schauder Bases, (Monographs and Textbooks in Pure and Applied Mathematics, 143), XVII + 283 pages, Marcel Dekker, Inc., New York—Basel—Hong Kong, 1991.

The general theory of Schauder bases in topological vector spaces particularly in Banach spaces is very well-known. The basic importance of this theory is originated in representation of certain functions by Fourier series.

It is well known fact that each separable Hilbert space has a Schauder basis, but it is true that a separable Banach space need not have a Schauder basis, furthermore it can be proved that a Banach space with Schauder basis is reflexive iff its basis is shrinking and boundedly complete. The author of this monograph and some of his colleagues were motivated by questions arisen in the bases theory in topological algebras. From this direction of research a lot of very interesting results have been developed in the theory of Schauder bases.

The main goal of this monograph is to give complete overview on the research done so far on this subject during the last several years. Most of the results presented here are already published but new material also can be found.

The chapter headings are: Rudiments of Topological Vector Spaces; Elements of Topological Algebras; Orthogonal Bases in Topological Algebras; Unconditional Orthogonal Bases; Continuity of Homomorphisms and Functionals; Orthogonal M-Bases; Multipliers of Topological Algebras.

At the end of the book an Appendix containing introductory material of set theory, abstract algebra and topology can be found; furthermore complete bibliography with 85 references enriches the monograph. The style of the book is clear, the theorems and proofs are presented in easily understable manner.

This monograph is highly recommended to functional and mathematical analysts, algebraists, and applied mathematicians and graduate students, too.

J. Németh (Szeged)

Inequalities (fifty years on from Hardy, Littlewood and Pólya), Edited by W. Norrie Everitt (Lecture Notes in Pure and Applied Mathematics, 129), IX+283 pages, Marcel Dekker, Inc., New York—Basel—Hong Kong, 1991.

London Mathematical Society organised an International Conference on Inequalities in July 13—17, 1987, at the University of Birmingham, England. The aim of the Society was not only to encourage the study of inequalities in mathematics but also to express the indebtedness of the subject to the work of G. H. Hardy, J. E. Littlewood and G. Pólya in writing the book Inequalities, which was first published by the Cambridge University Press in 1934. Of the 14 plenary lectures given to the Conference, 13 are presented in this volume and listed below:

Variational Inequalities (Calvin D. Ahlbrandt); The Grunsky Inequalities (J. M. Anderson); Hardy—Littlewood Integral Inequalities (William Desmond Evans and W. Norrie Everitt); Inequalities in Mathematical Physics (Jack Gunson); Inequalities and Growth Lemmas in Function Theory (Walter K. Hayman); Norm Inequalities for Derivatives and Differences (Man Kam Kwong and Anton Zettl); Bounds on Schrödinger Operators and Generalized Sobolev-Type Inequalities with Applications in Mathematics and Physics (Elliott H. Lieb); Inequalities Related to Carleman's Inequality (E. Russell Love); Some Comments on the Past Fifty Years of Isoperimetric Inequalities (Lawrence E. Payne); Operator Inequalities and Applications (Johann Schröder); Rearrangements and Partial Differential Equations (Giorgio G. Talenti); Inequalities in the Theory of Function Spaces: A Tribute to Hardy, Littlewood and Pólya (Hans Triebel); Differential Inequalities (Wolfgang Walter).

J. Németh (Szeged)

I. M. James, Introduction to Uniform Spaces, (London Mathematical Society Lecture Note Series, 144), IV+148 pages, Cambridge University Press, Cambridge—New York—Port Chester—Melbourne—Sydney, 1990.

The book essentially consists of two parts. The first unit includes classical approach with basic results: uniform structure uniform spaces, induced and coinduced uniform structures, uniform topology, completeness and completion.

Chapter 5 is devoted to the notion of topological groups. It leads through theories, discussed in the second unit (Chapter 6—8) which covers the theory of uniform transformation groups, uniform spaces over a base, uniform covering spaces.

As regards such kind of treatment, the author meditates on it, as follows: "Although it has been recognized from the start that topological groups can be regarded as uniform spaces, I do not believe it has been fully appreciated that it is possible to develop a theory of uniform transformation groups." And we have to agree with him.

This arrangement of the subject may be hardly supported by the fact that (in the presented form) the material can properly cover a (one-semester) course on uniform spaces.

Above mentioned intrinsic demand appears in three other aspects, each of them is perfectly realized. Firstly, exercises can be found at the end of the book which help the reader to conceive the subject. Secondly, the author explains and refers some new results (e.g. theory of uniform spaces over a topological base space, the fiberwise uniform spaces, uniform spreads). Thirdly, the author confines himself to present a brief and coherent treatment, which is the main merit of the volume at the same time.

The notion of uniform space is presented without the need of any topological background in Chapters 1—2. It makes possible to observe basic concepts and results of the theory in a self-contained way: taking uniform space out of the standard material of general topology. The only necessary rudiments are concepts in connect with filters, which can be found in the Appendices.

J. Kozma (Szeged)

K. Jänich, Analysis für Physiker und Ingenieure (Springer Lehrbuch), 2. Auflage, XI+419 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong—Barcelona, 1990.

This is a book for science and engineering students. It consists of three main parts: Function theory (complex analysis), ordinary differential equations and special functions of mathematical physics.

The basic ideas and methods are explained slowly in various forms just as in the best lectures. Taking into account the students one of the problems of the authors of similar works is to find the adequate phase of mathematical rigor. Whether this corresponds to your taste you can decide after reading the presentation of Cauchy's integral theorem.

Clear and careful exposition characterizes the whole work. Every chapter (we have fourteen ones) ends with a test containing ten examples. The right answers can be found at the end of the book. Well chosen exercises (with hints) help the student. The number of figures are unusually great and they are of first class.

This work is a great step for students in engineering and physics and makes them interested in further mathematical studies which are necessary to their profession.

L. Pintér (Szeged)

Klaus Jänich, Topologie (dritte Auflage), IX+215 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong—Barcelona, 1990.

The main purpose of this book is to give a glance into the methods of general topology to anyone who can use topology in his special study or research. The text is divided into ten chapters. The first three chapters discuss the basic concepts and theorems concerning topological spaces, topological vectorspaces and quotient topology. The fourth chapter is devoted to metric spaces, the embedding theorem for metric spaces into complete metric spaces is worked out both in cases of general metric spaces and normed linear spaces. Chapter 5 introduces the concepts of homotopy, category and functors. Chapter 6 gives the two countability axioms and investigates their rules in special theorems. In Chapter 7 simplicial complexes, cell complexes and *CW*-complexes are examined, Chapter 8 is devoted to the classical extension theorems for continuous functions and partition of unity on paracompact spaces. In Chapter 9 covering spaces and fundamental group are treated. In the last chapter the Tychonoff theorem and its applications can be found. At the end of the text a short glimpse into set theory is given.

L. Gehér (Szeged)

H. F. Jones, Groups, Representations and Physics, XIV+207 pages, Adam Hilger, Bristol and New York, 1990.

This is an introductory text on groups and their linear representations intented primarily for advanced undergraduates and postgraduates in solid state atomic and elementary particle physics.

The first four chapters deal with the basic concepts of groups and representations, such as, for example, subgroups, conjugacy classes, cosets, characters, Schur's lemmas and properties of irreducible representations. The notions and proofs are illustrated on a number of examples, using finite groups. This first part of the book is completed by a chapter treating some important physical applications of finite groups in solid state and molecular physics.

The second part of the book is devoted to continuous (Lie) groups, concentrating on aspects important in physical applications. The rotation group $SO(3)$ and angular momentum theory, the special unitary groups $SU(N)$ and their use in describing elementary particles and their interactions, the fundamental role of the Poincare group and its representations in relativistic physics are among the subjects dealt with by the author here. Additional topics, for example Dirac's notation in quantum mechanics and the invariant measure for $SO(3)$, are treated in the five appendices.

The book is self-contained and clearly written. Its main text is complemented by a list of problems added at the end of each chapter, with solutions sketched at the end of the book. It should provide the interested student of physics or mathematics with a firm grounding in the basics of group theory and its physical applications.

László Fehér (Szeged)

Wilbur Knorr, Textual Studies in Ancient and Medieval Geometrie, XVII + 852 pages, Birkhäuser, Boston—Basel—Berlin, 1989.

This is an important study in the documentary history of ancient and early medieval technical texts and the first attempt to give a complete survey of the existing evidence from antiquity on three special problems: the cube duplication, the angle trisection and the circle quadrature.

At each problem W. Knorr critically examines the extant manuscripts to determine those that appear the most trustworthy (not necessarily the earliest). Through their collation one seeks to construct a text that is the closest possible approximation to the original form, but, where the evidence is questionable, to identify among the variants those most likely to be candidates for the original reading.

In this book he traces out the transmissions and development of a specific set of ancient mathematical works connected to these three problems. Among the works by ancient Greek commutators of particular interest in this study are the following: Hero, Menelaus, Pappus, Theon and Hypatia (all) of Alexandria, Proclus, Eutocius of Askalan, John Philoponus and Simplicius.

Parts I and III are based on these commentaries and use some Hebrew traditions and translations, too. The complete Part III is devoted to a single work, Dimension of the Circle by Archimedes.

Part II deals with Arabic geometric texts and their ancient sources connected with cube duplication and angle trisection due to Abû Bakr al-Haravi, Ahmed ibn Mûsâ, Thabit ibn Qurra, al-Sijzî, Abu Sahl al-Quhi and Abu Ja^{c}far.

There are several texts in Greek and Arabic in the book, some of them in facsimile (these later are Arabic).

We recommend this volume to those who are interested not only in the history of science (ancient and early medieval geometrie) but can enjoy a careful philological examination of the ancient texts.

Lajos Klukovits (Szeged)

D. König, Theory of Finite and Infinite Graphs, 426 pages, Birkhäuser, Boston—Basel—Berlin, 1990.

In the first chapter the author introduces the basis concepts. He is dealing with the connected graphs, walks, components in details. The second chapter is an overview on the Euler trails and Hamiltonian cycles. Examining the problem for finite, undirected graphs König makes a transition to directed and infinite graphs as well. The next part of the book gives different solutions (Wiener's, Tremiaux', Tarry's) for the Labirinth Problem. Acyclic Graphs are considered in Chapter 4. The inquiring reader can find more details about the centers of trees in the next chapter. Basis concepts of the infinite graphs and the directed graphs have been introduced in Chapters 6—7.

Logic, theory of games and group theoretical applications of the directed graphs are mentioned in Chapter 8. Directed and undirected cycles and stars are considered with their compositions in the subsequent two chapters. Factorizations are examined in the remaining part of the book for different type of graphs (regular finite and infinite graphs).

Commentaries of W. T. Tutte and a "Biographical Sketch" of T. Gallai complete the book. It is a special pride that a lot of professors are mentioned by König from Szeged (L. Fejér, T. Grünwald, A. Haar, L. Kalmár) who discussed the content of this book by D. König.

Gábor Galambos (Szeged)

K. Königsberger, Analysis 1 (Springer-Lehrbuch) XI+360 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong—Barcelona, 1990.

This textbook is a very good introduction to real analysis. The material presented here much more than the subject of a usual "calculus book". Its building and style is very clear. Every chapter contains in necessary measure fundamental facts, definitions, statements, proofs and beautiful applications and finally each chapter ends with rich collection of examples.

After the foundations of numbers (real and complex) the concept of functions, sequences, series are treated. Later the theory of continuous functions and its application for the exponential function is developed. The next chapter (differential calculus) is followed by introduction of trigonometric functions and linear differential equations with constant coefficients. The second part of differential equations is treated after the integration of functions. The last four chapters deal with such very important subjects of analysis as local and global approximation of functions (Taylor polynomials, Bernoulli-polynomials, approximation theorem of Weierstrass) Fourier series (pointwise convergence, Bessel-approximation, L^2 convergence) and the investigation of the gamma-function.

This excellent book is warmly recommended to teachers, who can find in it a lot of ideas, beautiful proofs and examples and to students who will surely find the enjoy of discovery in this book.

J. Németh (Szeged)

Mathematics and Cognition: A Research Synthesis by the International Group for the Psychology of Mathematics Education, Edited by P. Nesher—J. Kilpatrick, (ICMI Study Series), 180 pages, Cambridge University Press, Cambridge—New York—Port Chester—Melbourne—Sydney, 1990.

Are there any significant difference with respect their efficiency between verbal interaction and reading mathematics texts? What can a teacher do in order to eliminate the difficulties or to make a best of advantages and, after all, to make a synthesis of these methods?

All the mathematics teachers and educators have to face the problem of cognition during his or her every-day educational work. Such problems in the process of teaching and learning call the attention to scientific analysis of mathematics and cognition.

This book is not purely a collection of interesting studies, but is a real "Research Synthesis", as the subheading promises it. Indeed, the reader finds a homogeneous presentation of different aspects of the problem indicated in the title.

The introductionary essay (by E. Fischbein) gives a brief survey of the history of researches devoted to psychological aspects of mathematics and education. Self-evident fact is that this is the history of the International Comission for Mathematics Instruction (ICMI) and the International Group for the Psychology of Mathematics Education (PME).

The seven studies are written on the same uniformly high level: Epistemology and Psychology of Mathematics Education (G. Vergnaud), Psychological Aspects of Learning Early Arithmetic

(J. C. Bergeron and N. Herscovics), Language and Mathematics (Colette Laborde), Psychological Aspects of Learning Geometry (R. Hershkowitz), Cognitive Processes Involved in Learning School Algebra (C. Kiernan), Advanced Mathematical Thinking (T. Dreyfus), Future perspectives for Research in the Psychology of Mathematics Education (N. Balacheff).

Each of them besides the author(s) has some contributors. They all belong to a group which had started the work on selected topics in Montreal (1987), and contained it in Veszprém, Hungary (1988). That's why the mindful reader gets familiar with theoretical and practical aspects, the classical and new results of the respective topics at the same time. An example: Euclidean geometry plays an important role in mathematics education in two respects as well. It is the science of the surrounding space on one hand, and a tool which is especially suitable to demonstrate mathematical structures, on the other hand. A lot of problems of mathematical imagination are presented via concrete geometric concepts, and ramifying theories respond to the practical questions of teaching geometry (e.g. visualization, deductive proofs).

This volume should be a pleasure for mathematicians and mathematics teachers interested in these exciting problems of education of high level.

J. Kozma (Szeged)

Neville de Mestre, The Mathematics of Projectiles in Sport (Australian Mathematical Society Lecture Series 6), XI+175 pages, Cambridge University Press, Cambridge—New York—Port Chester—Melbourne—Sydney, 1990.

When I was about 15 years old football and table-tennis were my favourite sports. Especially in table-tennis we had problems without solutions (then). We knew from experience how to shot a low ball if we wish it to bounce on the opponent's side of the table. (The success was not complete in every case.) The trajectories of the shots were sometimes unexpected. Several similar problems on the motion of projectiles take its origin in various games. A representative collection of them is contained in Chapter 8. Some of them: Shot-put and hammer throw; Basketball; Tennis, table-tennis and squash; Badminton; Golf; Cricket; Baseball; Soccer; Discus, frisbeee and flying ring; Long jump, high jump and ski jump; Boomerangs.

The first seven chapters contain the basic principles in mechanics and dynamics and the necessary mathematical techniques. Chapter headings are: Motion under gravity alone; Motion in a linear resisting medium; Motion in a non-linear resisting medium; The basic equations and their numerical solution; Small drag or small gravity; Corrections due to other effects; Spin effects; Projectiles in sport and recreation.

The concept of mathematical modelling of real problems is attractive in every case but especially here because the sports are "near" to the students. This is a book for almost everyone because only basic knowledge of classical dynamics, calculus, differential equations and their numerical solution is assumed. The problems in the text are presented in such a way that arouses the reader's interest. I found that, like most good texts, those topics which appeared difficult to grasp at the beginning of the book had become easy by the time I had reached the end. It is a pity that this book did not exist forty years ago.

L. Pintér (Szeged)

P. J. Nicholls, The Ergodic Theory of Discrete Groups (L.M.S. Lectures Note Series, 143) XI+221 pages, Cambridge University Press, New York—Port Chester—Melbourne—Sydney, 1989.

Denote by B the unit ball in R^n and by S the unit sphere. A point $\xi\in S$ is a limit point for a discrete group Γ of Möbius transforms preserving B if for every point $x\in B$ the orbit $\Gamma(x)=$

$\{\gamma(x):\ \gamma\in\Gamma$ accumulates at $\xi\}$. The subset $\Lambda(\Gamma)$ of S of limit points is the limit set of Γ. This book presents an introduction to the theory of measures on the limit set of discrete groups which has recently been developed by S. J. Patterson, D. Sullivan and others and which has emerged as one of the most powerful tools in the theory of discrete groups. The book assumes a working knowledge of graduate level analysis and topology; the particular results of ergodic theory needed for applications are fully developed from the classical ergodic theorems. The chapter headings are: Preliminaries; The Limit Set; A Measure on the Limit Set; Conformal Densities; Hyperbolically Harmonic Functions; The Sphere at Infinity; Elementary Ergodic Theory; The Geodesic Flow; Geometrically Finite Groups; Fuchsian Groups.

L. I. Szabó (Szeged)

Alfredo M. Ozorio de Almeida, Hamiltonian Systems, Chaos and Quantization (Cambridge Monographs on Mathematical Physics), IX+238 pages, Cambridge University Press, Cambridge—New York—New Rochelle—Melbourne—Sydney, 1988.

The theory of classical dynamical systems has undergone a rapid development in the last few decades. Out of the several branches of this field the author deals only with conservative systems. The preservation of volume in phase space — though it might seem a great simplification in the description of the qualitative behaviour of the solutions — does not make the problem much easier. A lot of beautiful and very deep mathematical results have been achieved concerning only Hamiltonian systems. The first part of the book provides a simple and nontechnical introduction to the fundamental notions of chaotic motion: structural stability, normal forms and KAM theory. Its language is rather the one of theoretical physics, but the important theorems of Hartman and Grobman, Peixoto, Smale, Birkhoff, Moser, Kolmogorov and Arnold are outlined and their proofs are explained in simple terms.

The second part of the book is devoted to the question of chaos in semiclassical quantum mechanics. While classical Hamiltonian dynamics is well developed, its quantum counterpart is still in its "physical period". The results are formulated in less rigorous terms, conjectures based on empirical results from computer calculations are not rare. The deeper structure of this part of the theory is much less understood, compared with the classical case. Nevertheless there are many interesting achievements in this field as well, mainly due to M. V. Berry and the Bristol group. Having been published in the physical literature, the quantum mechanical results are much less known to mathematicians. The systematic elaboration of these problems is in a somewhat chaotic phase, and it is a great merit of the second part of this book, that it provides an order in this many sided topic. According to the reviewer's opinion, this is the more valuable part of the volume, because there are a number of excellent books on classical systems, while — as far as I know — comprehensive monographs on quantum chaos have not been published so far.

M. G. Benedict (Szeged)

Reminiscences about a Great Physicist Paul Adrien Maurice Dirac, Edited by B. N. Kursunoglu and E. P. Wigner, XVIII+297 pages, Cambridge University Press, Cambridge—New York—Port Chester—Melbourne—Sydney, 1990.

No doubt, Paul Dirac was one of the greatest physicists of the century and all times. The first formulation of the structure of quantum mechanics, the quantum theory of radiation, the relativistic theory of the electron, the prediction of antimatter, the theory of magnetic monopoles, the statistics of half integer spin particles known as Fermi—Dirac statistics, are the most celebrated results of him. His book on the principles of quantum mechanics has been the fundamental textbook for generations of physicists. The principal features of the Hilbert space structure of quantum mechanics were laid down first in this book (without mentioning the name of this concept, of course). It was then criticised by von Neumann for its incorrect use of "continuous bases", and it has been proposed that the spectral decomposition theorem had to be used instead. This latter variant, however, has never become popular among physicists. It is well-known, that to the contrary, it was the Dirac formulation, that gave rise to L. Schwartz's theory of distributions. Interestingly enough, a more recent development of functional analysis, the introduction of the notion of the rigged Hilbert space by I. M. Gelfand and others, presents itself essentially the rigorous mathematical variant of the original Dirac method of quantum mechanics. In his equation for the relativistic electron Dirac used a special Clifford algebra and introduced spinors. The theory of the magnetic monopole is a construction of a nontrivial fibre bundle etc. Thus it is difficult to exaggerate the influence of Dirac on XX-th century mathematics.

The book contains personal reminiscences of colleagues, friends and pupils of Dirac, his wife, Margit Wigner, Eugene Wigner, R. Peierls, F. Hoyle, N. Mott, A. Salam, W. Lamb are among the authors. The book is very enjoyable, with several anecdotes about the man with a reputation of silence. It is recommended to everybody who is interested in the personality of an extraordinary great man, and in the history and development of physics and science of our century. Let us close this review with one of Dirac's famous sentences: "It is the essential mathematical beauty of the physical theory, which I feel is the real reason for believing in it."

M. G. Benedict (Szeged)

Reinhold Remmert, Theory of Complex Functions (Graduate Texts in Mathematics, 122), XIX+453 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong, 1990.

This book is a translation of the second edition of Funktionentheorie I, Grundwissen Mathematik 5, Springer-Verlag 1989, but it should be noted that several valuable improvements are made.

The book is consisting of three parts. The main topic of the first one is an introduction to the theory of complex variable (complex numbers, continuous functions, differential calculus, holomorphy and conformality, modes of convergence in function theory, power series, transcendental functions). The title of the second part is: "The Cauchy Theory" and the complex integral calculus, the integral theorem, integral formula, power series development are treated. The Cauchy—Weierstrass—Riemann theory is the main topic of the last part including for example the fundamental theorems about holomorphic functions, the fundamental theorem of algebra, Schwarz' lemma, isolated singularities, meromorphic functions, Laurent series and Fourier series and finally the residue calculus and its applications. At the end of the book short biographies of Abel, Cauchy, Eisenstein, Euler, Riemann and Weierstrass can be found. The book includes many examples and practice exercises.

Very useful parts of the book are the discussions of the historical evolution of the theory, biographical sketches of important contributors and citations (original language together with English translation) from their classical works.

I am sure that any teacher and student will enjoy reading this book because they will find in it not only very interesting historical remarks but many beautiful ideas and examples, too.

J. Németh (Szeged)

Konrad Schmüdgen, Unbounded Operator Algebras and Representation Theory (Operator Theory: Advances and Applications, 37), 380 pages, Birkhäuser Verlag, Basel—Boston—Berlin, 1990.

This monograph provides a thorough treatment of *-algebras of unbounded operators, and that of *-representations of general *-algebras. The main discussion is divided into two quite independent parts, consisting of Chapters 2—7 and 8—12, respectively. Chapter 1 is of introductory nature.

The main topics discussed in Part I are O^*-algebras and related topologies. An O^*-algebra is — roughly speaking — a *-algebra of unbounded operators acting on a dense common domain $\mathscr{D}$ in a Hilbert space, provided that this algebra contains the identity and each element leaves $\mathscr{D}$ invariant. The related topologies are considered on $\mathscr{D}$ or on the algebra itself, or even on space of associated sesquilinear forms. Among others, generalised Calkin algebra and one more special type of *-algebras are discussed in detail. The first part is finished by studying commutants of O^*-algebras.

Part II deals with *-representations of *-algebras by unbounded operators in a Hilbert space. After detailed discussion of general *-representations, some particular cases are considered. Special attention is paid to infinitesimal representation associated with unitary representation of a Lie group. The last chapter is devoted to the decompositions of closed operators and *-representations.

This book is a monograph of a theory having been rapidly developed in the last two decades. Besides the essential contribution to this developement by research papers, the author often improves the original proofs and results in this book. He introduces new concepts, unifies the terminology and the notation, and enlights the general theory from several points of views by giving examples and counter-examples. The theme of this comprehensive treatment is connected also with physics (e.g. quantum field theory).

The book is written in concise, lucid style. "Symbol Index" and "Key Index" help the reader to orient himself in the material. Each chapter ends with "Notes", where historical and bibliographical comments are presented.

The reader is often referred to textbooks, monographs, lecture notes and research papers listed in the rich "Bibliography". He/she has to be familiar with functional analysis (applying some topology) and operator theory. At any rate, one who wants to study (by learning or by doing research work) any branches of the theme indicated in the title, hardly can do without this book.

E. Durszt (Szeged)

TAPSOFT '89. Proceedings of the International Joint Conference on Theory and Practice of Software Development, Barcelona, Spain, March 1988 (LNCS, 351), Edited by J. Diaz and F. Orejas) X+383 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1989.

TAPSOFT '89 consisted of three parts: Advanced Seminar on Foundations of Innovative Software Development; CAAP (Colloquium on Trees in Algebra and Programming); CCIPL (Colloquium on Current Issues in Programming Languages).

The current Volume 1 of the Proceedings includes the four most theoretical invited papers of the Seminar plus the 20 CAAP contributions.

The invited talks were given by C.A.R. Hoare (The Varieties of Programming Language), J. L. Lassez and K. McAlloon (Independence of Negative Constraints), P. Lescanne (Completion Procedures as Transition Rules+Control), M. Wirsing, M. Broy (A Modular Framework for Specification and Information).

The CAAP papers can be grouped according to the following four major topics: Logic Programming, Prolog and its derivatives; Term Rewriting Systems; Graph Grammars; Algebraic Specifications.

This book may be a useful tool both for software experts with a stronger theoretical interest and computer scientists working on related fields.

J. Virágh (Szeged)

Toeplitz Operators and Spectral Function Theory. Essays from the Leningrad Seminar on Operator Theory. Edited by N. K. Nikolskii (Operator Theory: Advances and Applications, 42), 425 pages, Birkhäuser Verlag, Basel—Boston—Berlin, 1989.

As the editor remarks in the Preface: "the volume contains selected papers on the Spectral Function Theory Seminar, Leningrad Branch of Steklov Mathematical Institute. The papers are mostly devoted to the theory of Toeplitz and model operators."

The first article consists of an introductory and the first chapters of N. K. Nikolskii's publication to be published elsewhere. The present part is a survey of general properties of multiplicities and maxi-formulae. A maxi-formula is a formula of type $\mu(\mathfrak{A})=\sup_k \mu(\mathfrak{A}|X_k)$ where $\mathfrak{A}$ is a subalgebra of $L(X)$, $X=\operatorname{span}\{X_k\colon k\geqq 1\}$ and the X_k's are invariant subspaces for $\mathfrak{A}$.

The investigation on multiplicities is continued by B. M. Solomyak and A. L. Volberg in two joint papers. The first one contains the computation of the multiplicity of a Toeplitz operator with symbol analytic in the closed unit disc. The obtained formula is generalised for matrix symbol case in the second paper.

Using the Sz.-Nagy—Foiaş model theory, V. I. Vasyunin computes the multiplicity of a contraction with finite defect indices.

V. V. Peller discusses the following question: Under what conditions belongs the operator $f(T_\varphi)-T_{f\circ\varphi}$ to the Schatten—von Neumann class S_p or, in particular, to the trace class?

D. V. Yakubovich deals also with Toeplitz operators. Applying Riemann surfaces, he construts a similarity model for certain Toeplitz operators.

S. R. Treil' presents some recent results concerning the spectral theory of vector valued functions.

The reader holds in his hands a new outlet of the known workshop, where operator theory and complex analysis have been handled in a fruitful unit. This collection provides a good survey of the discussed area, presents new results and lists a great number of references. One, whose field of interest meets Toeplitz operators and/or multiplicity theory, surely is going to have a look at this work. And then, he/she will certainly read (at least the majority of) the articles.

E. Durszt (Szeged)

V. S. Varadarajan, An introduction to harmonic analysis on semisimple Lie groups (Cambridge studies in advanced mathematics, 16), X+316 pages, Cambridge University Press, Cambridge—New York—Port Chester—Melbourne—Sydney, 1989.

The well-known author, V. S. Varadarajan, of this book gives a very nice introduction to the subject of harmonic analysis on semisimple Lie groups. The book is intended to advanced undergraduates and to beginning graduate students.

Therefore it deals mainly with the simplest nontrivial semisimple Lie group $SL(2,\mathbf{R})$. In this way, the author can keep the requirements minimal contrary to a general treatment of the subject when a deep level algebra, geometry etc. would be needed. Since all the major themes come naturally up in the case of $SL(2, \mathbf{R})$ the reader can understand then easily the general statements and their proofs also.

Well, one could think now that this book is a variant of S. Lang's well-known book but only the approach is the same. A number of topics are included in this book that is not treated in Lang's book such as the Schwartz space, wave packets and so on.

It is worth noting that the book begins with a brilliant introductory chapter. Reading this chapter the reader will have a great mind to learn all of this subject. We note also that appendices on functional analysis and Lie groups are included offering the reader some basic definitions and results of the indicated subject.

In sum, we warmly recommend this book to all who want to learn the basics of harmonic analysis on semisimple Lie groups and especially to students interested in this subject.

Á. Kurusa (Szeged)

E. B. Vinberg, Linear Representations of Groups (Basler Lehrbücher, A Series of Advanced Textbooks in Mathematics, Vol. 2), (translated from Russian by A. Iacob), VII+146 pages, Birkhäuser Verlag, Basel—Boston—Berlin, 1989.

This nice book is an excellent introductory work into the linear representation of groups. The treatment is in good accordance with the intrinsic structure of the topic. It is divided into four chapters and (0+)11 sections. In the preliminary (0th) section with the aid of examples basic concepts of representation theory are introduced: exponential function, matrix representation and linear representation, action of a group.

Chapter I is devoted to the simplest results of the theory: from invariant representations to the complexification.

Chapter II and III deal with the representation of finite and compact groups, while Chapter IV is about representation of Lie groups.

The book satisfies all the requirements of a university textbook. As a consequence of its conciseness the reader can concentrate the logical treatment. Most important cases and examples are especially stressed (e.g. "A very important example"). Proofs take a prominent part of the material. They are presented not only for completeness reasons. Passing them, one loses a lot of niceties which play significant role in the explanation.

Problems and questions which do not need immediate presentation and solution are gathered at the end of each section. However, it is worth dealing with them in order to obtain a wider knowledge of the topic (answers and hints are also given).

Four appendices contain important technical details: presentation of groups by means of generators and relations, tensor products, the convex hull of a compact set, conjugate elements in groups.

The book is recommended to mathematics students of undergraduate as well as graduate courses.

J. Kozma (Szeged)

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