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INSTITUTUM BOLYAIANUM UNIVERSITATIS SZEGEDIENSIS

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On the lattice of complete congruences of a complete lattice: On a result of K. Reuter and R. Wille

G. GRÄTZER, H. LAKSER and B. WOLK

1. Introduction. For a complete lattice L, let $\operatorname{Com} L$ denote the lattice of *complete* congruence relations of L. Obviously, $\operatorname{Com} L$ is a complete lattice; however, unlike $\operatorname{Con} L$, the lattice of congruence relations of a lattice L, it is not distributive in general. In fact, in [4], K. REUTER and R. WILLE raise the question whether every complete lattice K can be represented in the form $\operatorname{Com} L$ for some complete lattice L.

K. REUTER and R. WILLE [4] prove the following

Theorem. *Let K be a complete distributive lattice in which every element is the (infinite) join of (finitely) join-irreducible elements. Then K is isomorphic to the lattice of complete congruences of some complete lattice L.*

They quote [1, pp. 69 and 58]: the condition of the Theorem holds for every distributive dually continuous lattice, and in particular, for every completely distributive complete lattice.

The proof of K. Reuter and R. Wille is based on an earlier paper of R. WILLE [5] on complete congruence relations of concept lattices. In this note we show how the approaches of [2] and [3] apply.

In Sec. 2 and 3, we present two essentially equivalent proofs of the Theorem. The first uses sequences and it is purely computational; it assumes no background in lattice theory. The second is based on ideals of partial lattices and uses some knowledge of lattice theory; this approach may help visualize the proof.

In Sec. 4, we show that the complete lattice L of the Theorem can be chosen to be sectionally complemented. We also compare the constructions of [4], Sec. 2, and 3. Finally, we find the "simplest" complete lattice L such that $\operatorname{Com} L$ is not distributive.

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2. Construction with sequences. Let K be a complete distributive lattice; let J denote the set of join-irreducible elements of K. We assume that every element u of K is a join of join-irreducible elements, that is, $u=\bigvee_K((u]\cap J)$, where $(u]=\{p\in K \mid p\leqq u\}$.

To construct the lattice L, take the lattice $Q=M_3^J$, the J-th power of the lattice M_3. (In forming the direct power, J is regarded as an unordered set.) The elements of the lattice M_3 will be denoted by o, a, b, c, i, where o is the zero, a, b, c are the atoms, and i is the unit. For $s\in Q$ and $p\in J$, $s(p)$ will denote the p-th component of s. For $s\in Q$, let $T(s)=\{p\in J \mid s(p)=i\}$ and $\tau(s)=\bigvee_K T(s)$. We define $\bar{s}\in Q$ as follows:

$$\bar{s}(p)=\begin{cases} i, & \text{if } p\leqq\tau(s) \text{ in } K \text{ and } s(p)>o \text{ in } M_3; \\ s(p), & \text{otherwise.}\end{cases} \tag{1}$$

We call s *closed* iff $s=\bar{s}$. We construct L as the set of all closed $s\in Q$, partially ordered componentwise.

Claim 1. *Let* $S\subseteqq L$. *Then* $u=\bigwedge_Q S$ *is again closed.*

Proof. Take a $p\in J$ such that $u(p)>o$. Since $u(p)=\bigwedge_Q(s(p)\mid s\in S)$ and $u(p)$ is completely meet-irreducible in M_3, it follows that $u(p)=s(p)$ for some $s\in S$. Now $u\leqq s$, hence $\bar{u}\leqq\bar{s}=$(since s is closed)$=s$, hence $\bar{u}(p)\leqq\bar{s}(p)=s(p)=u(p)$, and therefore u is closed.

Thus L is a $\wedge$-sublattice of Q. It follows that L is a complete lattice, in which

$$\bigvee_L S=\overline{\bigvee_Q S} \quad \text{for} \quad S\subseteqq L. \tag{2}$$

For $z\in K$, we define a congruence, θ^z, on Q as follows:

$$u\equiv v \pmod{\theta^z} \quad \text{iff} \quad u(p)=v(p) \quad \text{for all} \quad p\not\leqq z. \tag{3}$$

Obviously, θ^z is the kernel of the projection of $Q=M_3^J$ onto $M_3^{J-(z]}$.

Claim 2. *Let* $u, v\in Q$. *Then*

$$u\equiv v \pmod{\theta^z} \quad \textit{implies that} \quad \bar{u}\equiv\bar{v} \pmod{\theta^z}.$$

Proof. Let $u\equiv v \pmod{\theta^z}$. We want to prove that $\bar{u}(p)=\bar{v}(p)$ for $p\not\leqq z$. Since $u(p)=v(p)$, we can assume that $u(p)\neq\bar{u}(p)$, by symmetry. By (1), in K,

$$p\leqq\tau(u)=\bigvee_K T(u)=\bigvee_K(T(u)-(z])\vee_K\bigvee_K(T(u)\cap(z]).$$

Since p is join-irreducible in K, this implies that $p\leqq\bigvee_K(T(u)-(z])$ or $p\leqq\bigvee_K(T(u)\cap(z])$. The latter would imply that $p\leqq z$, contradicting that $p\not\leqq z$. Hence $p\leqq\bigvee_K(T(u)-(z])$.

By (3), $u(q)=v(q)$ for $q\notin(z]$. Hence $T(u)-(z]=T(v)-(z]$. Since $u(p)\neq\bar{u}(p)$, therefore, by (1), $u(p)>o$ and so $v(p)>o$. Finally,

$$p \leqq \bigvee_K (T(u)-(z]) = \bigvee_K (T(v)-(z]) \leqq \bigvee_K T(v) = \tau(v),$$

hence $\bar{v}(p)=i$ by (1). Therefore, $\bar{u}(p)=\bar{v}(p)$.

By Claim 2, the restriction, θ_L^z, of θ^z to L is a complete congruence relation on L. To complete the proof of the Theorem we have to prove that every complete congruence relation of L is of this form.

Let θ be a complete congruence of L. Set

$$P = \{p\in J \mid \text{there exist } u^p, v^p\in L,\ u^p \equiv v^p \ (\mathrm{mod}\ \theta),\ u^p(p) \neq v^p(p)\}.$$

We claim that $\theta=\theta_L^z$ with $z=\bigvee_K P$.

Obviously, $\theta\leqq\theta_L^z$.

For $x\in M_3$ and $Y\subseteq J$, let x_Y denote the element of Q defined by

$$x_Y(p) = \begin{cases} x, & \text{for } p\in Y, \\ o, & \text{otherwise.} \end{cases}$$

Note that $\bar{x}_Y=x_Y$, since $x_Y(p)$ is either o or x, and so $x_Y\in L$.

For convenience of notation, if $x\in M_3$ and $Y\subseteq K$, then we write x_Y for $x_{Y\cap J}$.

For $Y=\{y\}$, we write x_y for $x_{\{y\}}$. Note that $\{x_Y \mid x\in M_3\}$ is a sublattice of L isomorphic to M_3. For all $Y\subseteq J$, $o_Y=0$, the zero of L.

Since, for all $p\in P$,

$$u^p \equiv v^p \ (\mathrm{mod}\ \theta),$$

it follows, by taking the meet of both sides with i_p, that

$$u^p(p)_p \equiv v^p(p)_p \ (\mathrm{mod}\ \theta),$$

and so

$$i_p \equiv o_p \ (\mathrm{mod}\ \theta).$$

By the completeness of θ,

(4) $$i_P \equiv 0 \ (\mathrm{mod}\ \theta).$$

Now consider $s=i_P\vee_L b_{(z]-P}$. Obviously, $\tau(s)=z$, hence, $s=i_{(z]}$. Thus joining both sides of (4) with $b_{(z]-P}$ yields

$$i_{(z]} \equiv b_{(z]-P} \ (\mathrm{mod}\ \theta).$$

Thus

$$i_{(z]-P} \equiv b_{(z]-P} \ (\mathrm{mod}\ \theta),$$

so

$$i_{(z]-P} \equiv 0 \ (\mathrm{mod}\ \theta).$$

Consequently,

$$i_{(z]} \equiv 0 \pmod{\theta},$$

completing the proof of $\theta_L^z \leqq \theta$, and the proof of the Theorem.

3. Construction with ideals. We are given K and J as in Sec. 2. First, we construct a partial lattice, M, as in [3, pp. 81—84]: the elements of M are 0, for every $p \in J$, the elements p, p_1, and p_2, and for $p, q \in J$, $p > q$, the element $p(q)$; if p is a maximal element of J, we set $p = p_1 = p_2$. For $p > q$, we form the six-element lattice, $M(p, q)$, with elements $0, p_2, q_1, q_2, q$, and $p(q)$; the operations are defined by

$$q_1 \wedge q_2 = 0, \quad q_1 \vee q_2 = q, \quad p_2 \wedge q = 0,$$
$$p_2 \vee q_1 = p(q), \quad p_2 \wedge q_2 = p(q), \quad p_2 \vee q = p(q).$$

In the partial lattice M, all the elements p_1 and p_2 $(p \in J)$ are atoms; any two elements have a meet; two elements have a join iff they belong to an $M(p, q)$ and then their join is the join in $M(p, q)$. Note that $J \subseteq M$.

The partial lattice M is *atomic* (every element is a join of atoms), hence every complete congruence relation is determined by its *kernel*, i.e., by the congruence class containing 0.

Every congruence of M extends uniquely to a congruence of the lattice, $\operatorname{Id} M = Q$, of ideals of M. Since Q is atomic, it follows from [3, p. 147] that an element S of Q is standard iff for any atom u of Q such that $u \nleqq S$, the atoms of M in $S \vee u$ are the atoms of S and u.

For an ideal I of M, we define $\tau(I) = \bigvee_K (I \cap J)$. We call I *closed* iff for all $p \in J$ and $p \leqq \tau(I)$, if p_1 or $p_2 \in I$, then $p \in I$. Using the fact that all $p \in J$ are join-irreducible, it is easy to verify that if $I \cap J$ is finite, then I is closed. Every ideal I has a closure $\bar{I}$, the smallest closed ideal containing I, and the closed ideals of M form a lattice $\operatorname{Cd} M = L$.

For $a \in K$, let I_a be the ideal of M generated by $J \cap (a]$. Obviously, I_a is a closed ideal. We claim that I_a is standard. Indeed, let u be an atom of L, such that $u \nleqq I_a$; then there is a $p \in J$ with $u = (p_1]$ (or $(p_2]$) and $p \nleqq a$. Obviously, $\tau(I_a \vee_Q u) = a$, hence $I_a \vee_Q u$ is closed, implying that the only atom of $I_a \vee_L u$ not in I_a is u.

Let θ_a be the standard congruence relation associated with the standard element I_a of L. We claim that $a \to \theta_a$ is an isomorphism between K and $\operatorname{Com} L$. Since $a \to \theta_a$ is obviously order preserving, it is sufficient to prove that it is one-to-one and onto.

θ_a is a complete congruence on L. Indeed, $I \equiv J \pmod{\theta_a}$ iff the atoms of $I - I_a$ and $J - I_a$ are the same; thus θ_a preserves $\bigcap (= \bigwedge_Q = \bigwedge_L)$ and it also preserves $\bigvee_L$ since the kernel, $(I_a]$, is principal. Conversely, let θ be a complete congruence on L. Since θ is complete, the kernel must be principal, generated by an ideal S of M.

Let $a=\bigvee_K J\cap S$. We claim that $\theta=\theta_a$; equivalently, that $S=I_a$. The ideals S and I_a are equal iff they contain the same atoms. So let $p_i\in S$ ($i=1$ or 2), i.e., $p_i\equiv 0$ (mod θ); then $p\equiv 0$ (mod θ) also holds: if p is maximal in J, it holds by virtue of $p=p_i$; otherwise, take a $q>p$ in J and compute in $M(q,p)$ that $p_i\equiv 0$ (mod θ) implies that $p\equiv 0$ (mod θ). Conversely, let $p_i\in I_a$ ($i=1$ or 2), i.e., $p\leqq a=\bigvee_K(J\cap S)$. Then $p\leqq\tau(S)$, and S is closed, hence $p_i\in S$. This, again, completes the proof of Theorem.

4. Concluding remarks. A lattice L with zero is *sectionally complemented* if every interval $[0, a]$ is complemented. See, e.g., [3, Sec. III. 3 and III. 4] for the significance of this property. Using our first proof we can somewhat strengthen the Theorem.

Addendum to Theorem. *The complete lattice L of the Theorem can be chosen to be sectionally complemented.*

Proof. Let $u, t\in L$, and let $u<t$. We have to construct a $v\in L$ with $u\wedge_L v=0$ and $u\vee_L v=t$. Set

$$A=\{p\in J\mid t(p)=i \text{ and } u(p)=o\}.$$

For $p\in J$, define $u^*(p)$ as a complement of $u(p)$ in $[o, t(p)]$. Now we describe v; for $p\in J$, define

$$v(p)=\begin{cases} u^*(p), & \text{if } u^*(p) \text{ is the unique complement of } u(p) \text{ in } [o, t(p)],\\ u^*(p), & \text{if } p\not\leqq \bigvee_K A,\\ o, & \text{otherwise.}\end{cases}$$

Obviously, $v\in Q$. Furthermore, $T(v)=A$. Hence v is closed, and so $v\in L$. Now, $u(p)\wedge v(p)=o$ holds in M_3 by definition for all $p\in J$, so $u\wedge_L v=0$. Finally, $(u\vee_Q v)(p)=u(p)\vee u^*(p)=t(p)$ except if $u^*(p)$ is not the unique complement of $u(p)$ in $[o, t(p)]$ and $p\leqq\bigvee_K A$; in this case, $(u\vee_Q v)(p)=u(p)\vee o=u(p)$. However, $T(u\vee_Q v)\supseteqq T(v)=A$ and $u(p)\in\{a, b, c\}$ (otherwise, $u^*(p)$ would be the unique complement of $u(p)$ in $[o, t(p)]$), hence by (1), $\overline{u\vee_Q v}(p)=i=t(p)$, proving that $u\vee_L v=t$.

It is reasonable to ask how the constructions of [4], Sec. 2, and 3 compare. Let K be the three-element chain. It can be computed that the construction of [4] yields a lattice isomorphic to L_1 which can be represented as M_3^2 with the elements $\langle a, a\rangle$, $\langle a, b\rangle$, and $\langle a, i\rangle$ removed. Sec 2 yields a lattice L_2 which can be represented as M_3^2 with the elements $\langle a, i\rangle$, $\langle b, i\rangle$, and $\langle c, i\rangle$ removed. Note that L_1 and L_2 both have 22 elements but they are not isomorphic. Finally, Sec. 3 produces the six-element lattice $M(p, q)$.

Finally, in [4, Section 4], K. REUTER and R. WILLE produce examples of complete lattices L such that Com L is not distributive. We think the following example is the simplest.

Let L be N (the set of nonnegative integers with the usual partial ordering) with two additional elements: a, i. Let 0 be the zero, and i the unit of L. Let $a \wedge n = 0$ and $a \vee n = i$ for all $n \in N$, $n \neq 0$. Obviously, L is a complete lattice. We define three complete congruences, α, β, and γ on L:

nontrivial classes

$$\alpha\colon [2n+1, 2n+2], \quad \text{for} \quad n = 0, 1, 2, \ldots$$

$$\beta\colon [2n+1, 2n+2], \quad \text{for} \quad n = 1, 2, \ldots$$

$$\gamma\colon [2n, 2n+1], \quad \text{for} \quad n = 1, 2, \ldots.$$

It is easy to check that α, β, γ generate a sublattice isomorphic to N_5 in Com L.

Observe that L is a "minimal" example. If Com L is nondistributive, then L must contain a chain, C, of the type $\omega+1$ or its dual, otherwise Com L is isomorphic to Con L, and hence distributive. $L-C$ is nonempty; indeed, if $L=C$, then Com L = Com C, and Com C is isomorphic to Con ω, which is distributive. We conclude that $L-C$ must contain at least one element. In our example, it contains exactly one element.

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Endomorphism monoids in small varieties of bands

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Introduction

The study of the relationship between an algebra A and its endomorphism monoid End (A) has gradually, in the course of two decades, crystallized into a general framework, which we find worthwhile to outline here.

As soon as we have a class $\mathscr{K}$ of algebras, the assignment $A \to \mathrm{End}\,(A)$ for $A \in \mathscr{K}$ defines the class $\mathscr{M}$ of monoids M isomorphic to End (A) for some $A \in \mathscr{K}$, i.e. the monoids *representable* in $\mathscr{K}$. The class $\mathscr{K}$ is said to be *monoid universal* if all monoids are representable in $\mathscr{K}$. If every finite monoid M is representable by a finite algebra in $\mathscr{K}$ then we say that $\mathscr{K}$ is *finite monoid universal.*

The *problem of representability* of a given monoid M in a given class $\mathscr{K}$ of algebras is just one aspect of the relationship between $A \in \mathscr{K}$ and End (A). Another, in a way complementary aspect of this relationship is the *problem of determinancy* of $A \in \mathscr{K}$ by End (A): to what extent the knowledge of End (A) (up to isomorphism) determines the structure of A (within the class $\mathscr{K}$)? The class $\mathscr{K}$ is said to be *k-determined*, for a cardinal k, if any set of pairwise non-isomorphic algebras from $\mathscr{K}$ with the same (up to isomorphism) endomorphism monoid has the cardinality strictly less than k.

Since both representability and determinacy are tied to the algebraic structure of the algebras of a given class, it is natural to consider in the first place the varieties of algebras (of a given similarity type); the lattice of subvarieties can serve as a sort of a structural hierarchy in which universality is an increasing property and determinacy a decreasing property.

When we try to elucidate the nature of universality/determinacy in this lattice of subvarieties setting, we are naturally led to the notion of (categorical) universality:

*) The results were presented at the Colloquium on Semigroups and its Applications held in Wien, July 4—8, 1988.

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A variety $\mathscr{K}$ is said to be *universal* if the category of all graphs and compatible mappings can be fully embedded into $\mathscr{K}$. If, moreover, there exists a full embedding from the category of all graphs and compatible mappings into $\mathscr{K}$ such that it maps finite graphs into finite algebras then $\mathscr{K}$ is called *finite-to-finite universal.*

All known monoid universal varieties are also universal but, in general, it does not hold. It is an open problem whether for varieties the monoid universality and the categorical universality are equivalent. The categorical universality of a variety $\mathscr{V}$ excludes any k-determinacy of $\mathscr{V}$, for the reason that simply for any cardinal k, the discrete category of k graphs can be fully embedded into $\mathscr{V}$. More generally, any monoid M has a proper class of pairwise non-isomorphic representing objects in $\mathscr{V}$ (see [7] or [9]).

So much for the general framework of the present study.

Our subject proper — endomorphism monoids of bands (i.e., idempotent semigroups) — does not ideally fit into the above general scheme for the obvious reason that bands admit all constant self maps as endomorphisms, thus the endomorphism monoid of any band has left zeros, thus no variety of bands is monoid universal (and also not universal). However, as it is shown in the previous work [3] of the authors, monoid universality (and even more, universality) is there, only as if buried by a layer of superfluous morphisms. A natural way how to dispose of the undesirable morphisms is to strengthen the structure of the representing objects — the bands in our case. It may come as a surprise that even very small varieties of bands can be made universal by enriching their operational type by two or three nullary operation symbols, i.e. by turning the bands in question into 2 or 3-pointed bands (1-pointed would not do).

Every band variety is determined, within the variety of all bands, by a single equation $u=v$, a useful means to refer to the variety as $[u=v]$ (especially if there is no other commonly accepted name for its members).

Figure 1 visualizes the meet semilattice T_0 which is isomorphic to the bottom of the lattice of band varieties, see [1, 4, 5]. The nodes of T_0 represent the following band varieties:

$a_0=[x=y]$ — trivial bands

$a_1=[xy=x]$ — left zero semigroups

$a_2=[xy=yx]$ — semilattices

$a_3=[yx=x]$ — right zero semigroups

$a_4=[xyz=xzy]$ — left normal bands

$a_5=[xyz=xz]$ — rectangular bands

$a_6 = [yzx = zyx]$ — right normal bands

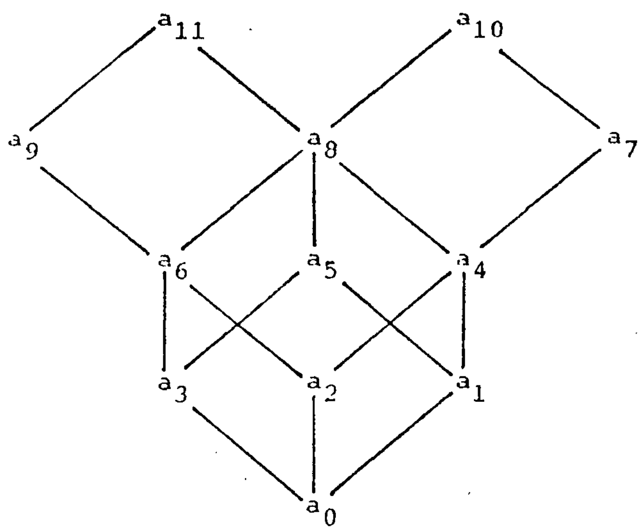
$a_7 = [xyx = xy]$ — semilattices of left zero semigroups

$a_8 = [xyzu = xzyu]$ — normal bands

$a_9 = [xyx = yx]$ — semilattices of right zero semigroups

$a_{10} = [xyz = xyxz]$ — left distributive bands

$a_{11} = [xyz = xzyz]$ — right distributive bands.



The meet semilattice T_0
Figure 1

It is readily seen that no number of nullary operations added to semilattices or rectangular bands makes them monoid universal.

The aim of this paper is to prove

Theorem 1.1. *The variety of rectangular bands and the variety of semilattices with an arbitrary number of nullary operations added is not universal.*

Theorem 1.2. *A variety $\mathcal{V}$ of bands with two nullary operations added is universal if and only if $\mathcal{V}$ contains either the variety of semilattices of left zero semigroups or the variety of semilattices of right zero semigroups.*

Theorem 1.3. *A variety $\mathcal{V}$ of bands with three nullary operations added is universal if and only if the variety of semilattices is a proper subvariety of $\mathcal{V}$.*

It should be said that the very "undesirable" morphisms, removed by the additional nullary operations in order to achieve universality, are very precious for the determinacy of small band varieties: semilattices are 3-determined [10], normal bands are 5-determined [11], semilattices of left (or right) zero semigroups are 3-determined and left (or right) distributive bands are 5-determined [3].

The results of this paper raise the question whether there exist other strengthenings of the structure of bands to obtain a universal category. The authors [3] showed that also the variety of bands with a unary operation * satisfying the identities $xx^*x=x$ and $x^{**}=x$ is universal. It is an open question whether we can restrict ourselves to the *-bands (here the unary operation, moreover, satisfies the identity $x^*y^*=(yx)^*$), or to a subvariety of *-bands.

The semigroup theoretical notions used in this paper can be found in the monographs [2] or [8].

The rest of the paper is devoted to the proof of Theorems 1.1, 1.2, and 1.3. The proof is divided into three parts. The proof of the universality of the 2-pointed variety $[xyx=xy]$ (or $[xyx=yx]$) is contained in Section 2, and the proof of the universality of the 3-pointed variety $[xyz=xzy]$ (or $[yzx=zyx]$) is the aim of Section 3. Common to both parts is the use of unary varieties. Denote by $I(1, 1)$ the variety of algebras with two unary idempotent operations and $I(1, 1, 0)$ its 1-pointed version. It is known

Theorem 1.4 [9]. *$I(1, 1)$ and $I(1, 1, 0)$ are finite-to-finite universal.*

Our universality proofs construct a full embedding of $I(1, 1)$ or $I(1, 1, 0)$ into the variety in question.

The final section is devoted to the proof of non-universality of some pointed varieties of bands. This finishes the proof of Theorems 1.1, 1.2, and 1.3.

2. Universality of 2-pointed semilattices of left zero semigroups

Denote by $(S, *)$ the groupoid given by the following table (see on the next page).

Then the following holds:

Proposition 2.1. *The groupoid $(S, *)$ is a semigroup belonging to the variety $[xyx=xy]$ of semilattices of left zero semigroups. Moreover, $B=\{b_i;\ i\in 2\}$, $C=\{c_i;\ i\in 2\}$, $D=\{d_i;\ i\in 2\}$, $E=\{e_i;\ i\in 4\}$ are all non-singleton $\mathscr{D}$-classes of $(S, *)$.*

Proof by a direct inspection.

Assume that $(X, \varphi_0, \varphi_1)$ is an algebra from $I(1, 1)$ such that $X\cap S=\emptyset$. Denote by X_0, X_1, X_2 three disjoint copies of X, the element $x\in X$ in the copy X_i, $i\in 3$

$(S, *)$	a_0	a_1	t_0	t_1	t_2	b_0	b_1	c_0	c_1	d_0	d_1	e_0	e_1	e_2	e_3
a_0	a_0	b_0	t_0	b_0	t_0	b_0	b_0	c_0	c_1	d_1	d_1	e_0	e_1	e_3	e_3
a_1	b_1	a_1	b_1	t_1	t_1	b_1	b_1	c_1	c_1	d_0	d_1	e_1	e_1	e_2	e_3
t_0	t_0	b_0	t_0	b_0	t_0	b_0	b_0	c_1	c_1	d_1	d_1	e_1	e_1	e_3	e_3
t_1	b_1	t_1	b_1	t_1	t_1	b_1	b_1	c_1	c_1	d_1	d_1	e_1	e_1	e_3	e_3
t_2	t_0	t_1	t_0	t_1	t_2	b_0	b_1	c_1	c_1	d_1	d_1	e_1	e_1	e_3	e_3
b_0	b_0	b_0	b_0	b_0	b_0	b_0	b_0	c_1	c_1	d_1	d_1	e_1	e_1	e_3	e_3
b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	c_1	c_1	d_1	d_1	e_1	e_1	e_3	e_3
c_0	c_0	c_0	c_0	c_0	c_0	c_0	c_0	c_0	c_0	e_0	e_0	e_0	e_0	e_0	e_0
c_1	c_1	c_1	c_1	c_1	c_1	c_1	c_1	c_1	c_1	e_1	e_1	e_1	e_1	e_1	e_1
d_0	d_0	d_0	d_0	d_0	d_0	d_0	d_0	e_2	e_2	d_0	d_0	e_2	e_2	e_2	e_2
d_1	d_1	d_1	d_1	d_1	d_1	d_1	d_1	e_3	e_3	d_1	d_1	e_3	e_3	e_3	e_3
e_0	e_0	e_0	e_0	e_0	e_0	e_0	e_0	e_0	e_0	e_0	e_0	e_0	e_0	e_0	e_0
e_1	e_1	e_1	e_1	e_1	e_1	e_1	e_1	e_1	e_1	e_1	e_1	e_1	e_1	e_1	e_1
e_2	e_2	e_2	e_2	e_2	e_2	e_2	e_2	e_2	e_2	e_2	e_2	e_2	e_2	e_2	e_2
e_3	e_3	e_3	e_3	e_3	e_3	e_3	e_3	e_3	e_3	e_3	e_3	e_3	e_3	e_3	e_3

Figure 2

is denoted by x_i. We shall define a groupoid $\Phi'(X, \varphi_0, \varphi_1)=(Y, \cdot)$ which is a co-extension of S (i.e. there exists a surjective homomorphism $f\colon (Y, \cdot)\to(S, *)$) as follows:

$$Y = (S\setminus\{t_i;\ i\in 3\})\cup(\cup\{X_i;\ i\in 3\})$$

and if $y, z\in Y$ then:

$y\cdot z=u*v$ if the following hold:

$y\in S$ and $y=u$ or $y\in X_i$ and $u=t_i$ for some $i\in 3$,

$z\in S$ and $z=v$ or $z\in X_i$ and $v=t_i$ for some $i\in 3$, $u*v\in Y$;

$y\cdot z=u_k$ if there exist $u, v\in X$ with $y=u_i$, $z=v_j$, and $t_i^* t_j=t_k$ for some $i, j, k\in 3$;

$y\cdot z=u_k$ if $y=u_i\in X_i$, $z\in\{a_0, a_1\}$ and $t_i*z=t_k$ for some $i, k\in 3$;

$y\cdot z=(\varphi_k(u))_k$ if $y=a_i$, $z=u_j\in X_j$ and $a_i*t_j=t_k$ for some $i, j, k\in 3$.

Denote by ψ a mapping from Y to S such that

$$\psi(y) = y \quad \text{for} \quad y \in S, \qquad \psi(y) = t_i \quad \text{for} \quad y \in X_i,\ i \in 3.$$

We have

Proposition 2.2. *$\Phi'(X, \varphi_0, \varphi_1) = (Y, \cdot)$ is a semigroup belonging to the variety $[xyx = xy]$ for every $(X, \varphi_0, \varphi_1) \in I(1, 1)$. Furthermore, $\psi: (Y, \cdot) \to (S, *)$ is a surjective homomorphism and $B, C, D, E, X_i, i \in 3$ are all non-singleton $\mathcal{D}$-classes of $(Y, \cdot)$.*

Proof. That ψ is a homomorphism is straightforward. We show that $(Y, \cdot)$ is a semigroup. Let $x, y, z \in Y$ and we investigate the equality

$$(*) \qquad (x \cdot y) \cdot z = x \cdot (y \cdot z).$$

Since ψ is a homomorphism and $(S, *)$ is a semigroup we obtain that $(*)$ holds for every $x, y, z \in Y$ with $(x \cdot y) \cdot z \in S$ or $x \cdot (y \cdot z) \in S$. If $(x \cdot y) \cdot z \in Y \setminus S$ then $(x \cdot y) \cdot z \in \in \bigcup \{X_i;\ i \in 3\}$, and moreover, $(x \cdot y) \cdot z \in X_i$ if and only if $x \cdot (y \cdot z) \in X_i$. Assume that $(x \cdot y) \cdot z \in X_0$ then $x, y, z \in X_0 \cup X_2 \cup \{a_0\}$. If $x \in X_0$ then $(*)$ holds because x is a left zero with respect to the set $X_0 \cup X_2 \cup \{a_0\}$. If $x = u_2$ for some $u \in X$ then for every $v \in X_0 \cup X_2 \cup \{a_0\}$ we have $u_2 \cdot v \in \{u_0, u_2\}$ and hence we again obtain $(*)$. Finally, assume $x = a_0$. If $y = u_i$ for some $u \in X$, $i \in \{0, 2\}$ then we have $(x \cdot y) \cdot z = (\varphi_0(u)_0) \cdot z = \varphi_0(u)_0$ and $y \cdot z \in \{u_0, u_2\}$, hence $x \cdot (y \cdot z) = \varphi_0(u)_0$ and $(*)$ hold. If $y = a_0$ then $z = u_i$ for some $u \in X$, $i \in \{0, 2\}$ and hence $(x \cdot y) \cdot z = a_0 \cdot z = \varphi_0(u)_0$ and $x \cdot (y \cdot z) = x \cdot \varphi_0(u)_0 = \varphi_0(\varphi_0(u))_0 = \varphi_0(u)_0$ because φ_0 is idempotent. Analogously we prove $(*)$ if $(x \cdot y) \cdot z \in X_1$. Finally, if $(x \cdot y) \cdot z \in X_2$, then $x, y, z \in X_2$ and because X_2 is a left zero subsemigroup of $(Y, \cdot)$ we conclude that $(*)$ holds and hence $(Y, \cdot)$ is a semigroup. The rest is obvious.

Define a functor Φ from $I(1, 1)$ into the 2-pointed variety $[xyx = xy]$ of all 2-pointed semilattices of left zero semigroups. For an algebra $(X, \varphi_0, \varphi_1)$ from $I(1, 1)$ define $\Phi(X, \varphi_0, \varphi_1) = (Y, \cdot, c_0, d_0)$ where $\Phi'(X, \varphi_0, \varphi_1) = (Y, \cdot)$. For a homomorphism $f: (X, \varphi_0, \varphi_1) \to (X', \varphi_0', \varphi_1')$ in $I(1, 1)$ define a mapping Φf:

$$\Phi f(x_i) = f(x)_i \quad \text{for every} \quad x \in X,\ i \in 3, \quad \text{and} \quad \Phi f(s) = s \quad \text{for every} \quad s \in S.$$

If $u \in \{a_0, a_1\}$, $v \in \bigcup \{X_i;\ i \in 3\}$ then $\Phi f(uv) = \Phi f(u) \cdot \Phi f(v)$ because f is a homomorphism, for the remaining case we obtain by a direct inspection that Φf is a homomorphism. Thus we can summarize:

Proposition 2.3. *Φ is an embedding of $I(1, 1)$ into the 2-pointed $[xyx = xy]$.*

We prove that Φ is full. Assume that $\Phi(X, \varphi_0, \varphi_1) = (Y, \cdot, c_0, d_0)$, $\Phi(X', \varphi_0',$ $\varphi_1') = (Y', \cdot, c_0, d_0)$ are algebras from the 2-pointed variety $[xyx = xy]$ and let $f: (Y, \cdot, c_0, d_0) \to (Y', \cdot, c_0, d_0)$ be a homomorphism. Then we have

Lemma 2.4. *For every* $u \in S \cap Y$ *we have* $f(u)=u$.

Proof. Since f preserves the nullary operations we have $f(c_0)=c_0, f(d_0)=d_0$. Hence $f(C) \subseteqq C$, $f(D) \subseteqq D$. Furthermore, an arbitrary $\mathscr{D}$-class containing an arbitrary element of the set $\bigcup \{X_i;\ i\in 3\} \cup B \cup \{a_0, a_1\}$ is greater than the $\mathscr{D}$-classes C and D. Thus $f(\bigcup \{X_i;\ i\in 3\} \cup B \cup \{a_0, a_1\}) \subseteqq \bigcup \{X_i';\ i\in 3\} \cup B \cup \{a_0, a_1\}$. Moreover, a_0 is a unique element of the set $\bigcup \{X_i;\ i\in 3\} \cup B \cup \{a_0, a_1\}$ with $a_0 \cdot c_0 = c_0$ and a_1 is a unique element of the set $\bigcup \{X_i;\ i\in 3\} \cup B \cup \{a_0, a_1\}$ with $a_1 \cdot d_0 = d_0$. Hence $f(a_0)= =a_0, f(a_1)=a_1$. Since the subsemigroup generated by $\{a_0, a_1, c_0, d_0\}$ is $S \cap Y$ we obtain that f is identical on the set $S \cap Y$.

Lemma 2.5. *There exists* $g\colon X \to X'$ *such that for every* $x \in X$, $i \in 3$ *we have* $f(x_i)=g(x)_i$.

Proof. Choose $x \in X$. By Lemma 2.4 we conclude that $f(x_2) \in \bigcup \{X_i';\ i\in 3\} \cup \cup B \cup \{a_0, a_1\}$. If $f(x_2) \in X_0' \cup \{a_0, b_0\}$ then $b_1 = f(b_1) = f(x_2 \cdot a_1 \cdot a_0) = f(x_2) \cdot f(a_1) \cdot \cdot f(a_0) = b_0$ — a contradiction, if $f(x_2) \in X_1' \cup \{a_1, b_1\}$ then $b_0 = f(b_0) = f(x_2 \cdot a_0 \cdot a_1) = =f(x_2) \cdot f(a_0) \cdot f(a_1) = b_1$ — a contradiction. Thus $f(X_2) \subseteqq X_2'$. Set $g\colon X \to X'$ with $f(x_2) = g(x)_2$ for every $x \in X$. Then we have $f(x_0) = f(x_2 \cdot a_0) = f(x_2) \cdot f(a_0) = g(x)_2 \cdot a_0 = =g(x)_0$ and $f(x_1) = f(x_2 \cdot a_1) = f(x_2) \cdot f(a_1) = g(x)_2 \cdot a_1 = g(x)_1$.

Lemma 2.6. *The mapping* g *of Lemma* 2.5 *is a homomorphism from* $(X, \varphi_0, \varphi_1)$ *into* $(X', \varphi_0', \varphi_1')$.

Proof. Consider $x \in X$, then $g(\varphi_0(x))_0 = f(\varphi_0(x)_0) = f(a_0 \cdot x_2) = f(a_0) \cdot f(x_2) = =a_0 \cdot g(x)_2 = \varphi_0'(g(x))_0$ and hence $g \circ \varphi_0 = \varphi_0' \circ g$. By the dual argument we obtain $g \circ \varphi_1 = \varphi_1' \circ g$, whence g is a homomorphism.

Since for a homomorphism g from Lemma 2.5 we have $\Phi g = f$ we have proved that Φ is a full embedding and thus Theorem 1.4 completes the proof of the following

Theorem 2.7. *The variety* $[xyx=xy]$ *with two nullary operations added is finite-to-finite universal.*

Hence we immediately obtain

Corollary 2.8. *The variety* $[xyx=yx]$ *with two nullary operations added is finite-to-finite universal.*

Proof. Obviously, a semigroup $(T, \cdot)$ belongs to the variety $[xyx=xy]$ if and only if the semigroup $(T, \oplus)$ belongs to the variety $[xyx=yx]$ where $t \oplus u = u \cdot t$ for every $t, u \in T$. Hence Corollary 2.8 immediately follows from Theorem 2.7.

3. The universality of the 3-pointed variety $[xyz=xzy]$

For an algebra $A=(X, \varphi, \psi, q)\in I(1,1,0)$ denote by X_i, $i\in 2$ two disjoint copies of the set X, for an element $x\in X$ denote by x_i the corresponding element in the copy X_i, $i\in 2$. Define an algebra ΦA in the 3-pointed variety $[xyz=xzy]$. The underlying set of ΦA is $(X_0\times\{a_1, a_2, a_7, a_{10}\})\cup(X_1\times\{a_3, a_9, a_{11}\})\cup((X_0\cup X_1)\times \times\{a_4, a_5, a_6, a_8\})\cup\{(0, a_0)\}$. For $x, y\in X$, $m, n\in 2$, $i, j\in 12$ if $a_i\wedge a_j=a_k$ in the semilattice T_0 and if (x_m, a_i), (y_n, a_j) are elements of the underlying set of ΦA then

$$(x_m, a_i)(y_n, a_j)=\begin{cases}(x_m, a_k) & \text{if } k>3,\\ (\bar{\psi}(x_m)_1, a_3) & \text{if } k=3,\\ (x_0, a_2) & \text{if } k=2,\\ (\bar{\varphi}(x_m)_0, a_1) & \text{if } k=1,\\ (0, a_0) & \text{if } k=0,\end{cases}$$

moreover $(0, a_0)$ is a zero in ΦA, where $\bar{\varphi}, \bar{\psi}\colon (X_0\cup X_1)\to X$ are the mappings defined $\bar{\varphi}(x_0)=x$, $\bar{\varphi}(x_1)=\varphi(x)$, $\bar{\psi}(x_0)=\psi(x)$, $\bar{\psi}(x_1)=x$ for every $x\in X$. By a direct inspection we obtain that the definition of the binary operation is correct and that ΦA is a strong semilattice of left zero semigroups, thus by [8] it is a left normal band. The three added nullary operations are (q_0, a_5), (q_0, a_7), (q_1, a_9).

For a homomorphism $f\colon A\to B$ where $A=(X, \varphi, \psi, q)$, $B=(Y, \varphi', \psi', r)$ denote by f' the mapping defined as follows: $f'(x_m, a_i)=(f(x)_m, a_i)$ for every $x\in X$, $m\in 2$, $i\in 12\setminus\{0\}$, $f'(0, a_0)=(0, a_0)$. By a direct inspection we obtain that f' maps the underlying set of ΦA into the underlying set of ΦB, furthermore the restriction of f' to ΦA is a homomorphism. Thus if the restriction of f' to ΦA and ΦB is denoted by Φf then we obtain

Proposition 3.1. *Φ is an embedding of $I(1,1,0)$ into the 3-pointed variety $[xyz=xzy]$.*

Proof. By a direct inspection.

We prove that Φ is a full embedding. For the purpose assume that $A, B\in I(1,1,0)$ where $A=(X, \varphi, \psi, q)$, $B=(Y, \varphi', \psi', r)$ and that $f\colon \Phi A\to\Phi B$ is a homomorphism in the 3-pointed variety $[xyz=xzy]$.

Lemma 3.2. *The structural homomorphism of f is the identity.*

Proof: Since T_0 is the structural semilattice of ΦA and ΦB we get that the structural homomorphism g of f is an endomorphism of T_0. Since f preserves the nullary operations we conclude that $g(a_i)=a_i$ for every $i\in\{5, 7, 9\}$. Moreover, g preserves the order and thus $g(a_i)=a_i$ for $i\in\{10, 11\}$. Since g is an endomorphism and $\{a_i;\, i\in\{5, 7, 9, 10, 11\}\}$ generates T_0 we conclude that g is the identity.

Define mappings $f_0, f_1\colon X \to Y$ such that

$$f(x_0, a_{10}) = (f_0(x)_0, a_{10}) \quad \text{for every} \quad x \in X,$$

$$f(x_1, a_{11}) = (f_1(x)_1, a_{11}) \quad \text{for every} \quad x \in X.$$

Lemma 3.3. *For every* $i \in \{1, 2, 4, 5, 6, 7, 8, 10\}$, *we have* $f(x_0, a_i) = (f_0(x)_0, a_i)$ *for every* $x \in X$.

For every $i \in \{3, 4, 5, 6, 8, 9, 11\}$, *we have* $f(x_1, a_i) = (f_1(x)_1, a_i)$ *for every* $x \in X$.

Proof. For every $x \in X$ and $i \in \{1, 2, 4, 5, 6, 7, 8\}$ we have $(x_0, a_i) = (x_0, a_{10}) \cdot \cdot (x_0, a_i)$ and hence

$$f(x_0, a_i) = f(x_0, a_{10}) f(x_0, a_i) = (f_0(x)_0, a_{10}) f(x_0, a_i) = (f_0(x)_0, a_i).$$

Hence we obtain the first assertion. The proof of the second assertion is dual.

Corollary 3.4. $f_0 = f_1$.

Proof. We apply Lemma 3.3 and the fact that

$$(f_0(x)_0, a_2) = f(x_0, a_2) = f\big((x_1, a_{11})(x_0, a_2)\big) = f(x_1, a_{11}) f(x_0, a_2) =$$
$$= (f_1(x)_1, a_{11})(f_0(x)_0, a_2) = (f_1(x)_0, a_2)$$

for every $x \in X$.

Lemma 3.5. f_0 *is a homomorphism of* $I(1, 1, 0)$ *from* A *into* B.

Proof. Obviously $f_0(q) = r$. We have

$$(\varphi(f_0(x))_0, a_1) = (f_0(x)_1, a_5)(f_0(x)_0, a_1) = f\big((x_1, a_5)(x_0, a_1)\big) =$$
$$= f(\varphi(x)_0, a_1) = (f_0(\varphi(x))_0, a_1).$$

Thus f_0 commutes with φ. By duality we obtain that f_0 commutes with ψ. Hence f_0 is a homomorphism.

Since $\Phi f_0 = f$ we conclude that Φ is a full embedding from $I(1, 1, 0)$ into the 3-pointed variety $[xyz = xzy]$. Theorem 1.4 completes the proof of the following:

Theorem 3.6. *The variety* $[xyz = xzy]$ *with three nullary operations added is finite-to-finite universal.*

If we apply the same idea as in the proof of Corollary 2.8 we obtain

Corollary 3.7. *The variety* $[yzx = zyx]$ *with three nullary operations added is finite-to-finite universal.*

4. Non-universality of pointed varieties of bands

First we investigate the variety of rectangular bands, and the variety of semilattices. If B is a rectangular band then B is a product of a left zero semigroup L and a right zero semigroup R. It is well known that $f: B \to B$ is an endomorphism of B if and only if $f = g \times h$ where $g: L \to L$, $h: R \to R$. Hence we obtain:

Proposition 4.1. *For any cardinal α, no α-pointed rectangular band B represents a non-trivial group as* End (B).

We prove an analogous result for semilattices:

Proposition 4.2. *For any cardinal α, no α-pointed semilattice S represents a non-trivial group of a finite order as* End (S).

Proof. Assume the contrary, let S be an α-pointed semilattice such that its endomorphism monoid is isomorphic to a non-trivial group G of finite order. First, for every $g \in \mathrm{End}(S)$ and for every $x \in S$ if $g(x) \neq x$ then x and $g(x)$ are incomparable because there exists a natural number n with $g^n(x) = x$. For every endomorphism $g \in \mathrm{End}(S)$ define $f: S \to S$ such that $f(x) = xg(x)$ for every $x \in S$. Obviously, $f \in \mathrm{End}(S)$ and $f(x)$ and x are comparable for every $x \in S$. Moreover, f is identical if and only if g is identical and this is a contradiction with the fact that G is non-trivial.

Propositions 4.1 and 4.2 complete the proof of Theorem 1.1. Moreover, Theorems 1.1 and 3.6, and Corollary 3.7 complete the proof of Theorem 1.3. Thus it suffices to finish the proof of Theorem 1.2. For this purpose we shall investigate the 2-pointed variety of normal bands.

Proposition 4.3. *Let B be a normal band with a structural semilattice S. If f is an endomorphism of S such that $f(s) \leqq s$ for every $s \in S$ then there exists an endomorphism $g: B \to B$ with a structural morphism f such that for every $\mathcal{D}$-class D of B with $f(D) = D$ and for every $x \in D$ we have $g(x) = x$.*

Proof. By [8], B is a strong semilattice of rectangular bands, i.e. for every $s \in S$ there exists a rectangular band $D(s)$ (it is the $\mathcal{D}$-class corresponding to s) and for every pair $s, t \in S$ with $s \leqq t$ there exists a homomorphism $\mu_{t,s}: D(t) \to D(s)$ such that

a) for every $s \in S$, $\mu_{s,s}$ is the identity;

b) for every triple $s, t, u \in S$ with $s \leqq t \leqq u$ we have

$$\mu_{t,s} \circ \mu_{u,t} = \mu_{u,s};$$

c) $B = \bigcup \{D(s);\, s \in S\}$ and $\{D(s);\, s \in S\}$ are pairwise disjoint;

d) for every $s, t\in S$, $x\in D(s), y\in D(t)$ we have

$$xy = \mu_{s,s\wedge t}(x)\,\mu_{t,s\wedge t}(y)$$

where the former product is in B and the latter one is in $D(s\wedge t)$.

For every $x\in D(s)$, $s\in S$ define $g(x)=\mu_{s,f(s)}(x)$. By a) through d) we easily obtain that g is an endomorphism of B with the required properties.

Lemma 4.4. *Let S be a semilattice with an element $d\in S$. If $f\in \mathrm{End}\,(S_d)$ where $S_d=\{s\in S;\ s\geqq d\}$ is a subsemilattice of S then $g\colon S\to S$ defined by $g(s)=f(s)$ for $s\in S_d$, $g(s)=d\wedge s$ for $s\in S\setminus S_d$ is an endomorphism of S.*

Proof. Clearly g is correctly defined. Let $x, y\in S$. If $x, y\in S_d$ then also $x\wedge y\in S_d$ and since $f\in \mathrm{End}\,(S_d)$ we obtain $g(x)\wedge g(y)=g(x\wedge y)$. If $y\in S\setminus S_d$ then $g(x\wedge y)=x\wedge y\wedge d$. If $x\in S_d$ then $x, f(x)\geqq d$, whence $x\wedge y\wedge d=f(x)\wedge y\wedge d=$ $=g(x)\wedge g(y)$; if $x\in S\setminus S_d$ then obviously $g(x)\wedge g(y)=x\wedge y\wedge d$. If $x\in S\setminus S_d$ the proof is analogous.

Theorem 4.5. *No 2-pointed normal band B represents a nontrivial group as* End (B).

Proof. Assume that B is a normal band with two added nullary operations a_i, $i\in 2$ such that End (B) is a group (i.e. every endomorphism of B is an automorphism). Let S be the structural semilattice of B, assume that elements b_i, $i\in 2$ of S correspond to the $\mathcal{D}$-classes containing a_i, $i\in 2$. If there exists $s\in S$ such that $s\geqq b_i$ for $i\in 2$ and s is not the unity of S then consider the endomorphism h of S such that $h(x)=s\wedge x$ for every $x\in S$. Since $s\geqq b_i$ we have $h(b_i)=b_i$. By Proposition 4.3 there exists a band endomorphism g of B with structural endomorphism h and $g(a_i)=a_i$ for $i\in 2$. Thus g is an endomorphism of B and because neither h nor g is an automorphism, this is a contradiction. Hence we can assume that only the unity 1 in S is greater than b_i, $i\in 2$. Set $c=b_0\wedge b_1$ and let $d\in S$ with $d\leqq c$. Denote $S_d=$ $=\{s\in S;\ s\geqq d\}$ and define $f\colon S_d\to S_d$ as follows:

$f(x)=x$ if $x=1$,

$f(x)=b_i$ if $x\neq 1$ and $x\geqq b_i$ for an $i\in 2$,

$f(x)=c$ if $x\ngeqq b_i$ for any $i\in 2$ and $x\geqq c$,

$f(x)=d$ if $x\ngeqq c$ and $x\geqq d$.

By a direct inspection we obtain that f is an endomorphism of S_d with $f(x)\leqq x$ for every $x\in S_d$ and $f(b_i)=b_i$ for $i\in 2$. If we use Lemma 4.4 we obtain an endomorphism h of S with $h(x)\leqq x$ for every $x\in S$ and $h(b_i)=b_i$ for $i\in 2$. Finally, if we apply Proposition 4.3 we obtain a 2-pointed band endomorphism g of B with structural endomorphism h. Since g is an automorphism we conclude that h is an automorphism of S, thus $S_d = S \subseteqq \{1, b_0, b_1, c, d\}$ where 1 is the unity of S (if it exists). It is routine to verify that B is rigid.

The proof of Theorem 1.2 follows from Theorems 2.7, 4.5, and from Corollary 2.8. In fact, we have proved stronger results than Theorems 1.1, 1.2, and 1.3:

Corollary 4.6. *For a variety $\mathscr{V}$ of bands with k nullary operations the following are equivalent:*

a) $\mathscr{V}$ is finite-to-finite universal;

b) $\mathscr{V}$ is universal;

c) $\mathscr{V}$ is monoid universal;

d) $\mathscr{V}$ is finite monoid universal;

e) there exist a non-trivial group G of finite order and an algebra $A \in \mathscr{V}$ with $\mathrm{End}(A) \cong G$;

f) either $k \geqq 2$ and $\mathscr{V} \supseteqq [xyx = xy]$ or $k \geqq 2$ and $\mathscr{V} \supseteqq [xyx = yx]$ or $k \geqq 3$ and the variety of all semilattices is a proper subvariety of $\mathscr{V}$.

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Loops with and without subloops

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In the theory of local loops, those loops which have one-parameter subloops are of considerable importance. M. A. Akivis [1] gave an interesting example of a quasigroup which has r-parameter subloops in every direction [2]. At the "*Web Geometry Conference*" held at Szeged, 1987, K. H. Hofmann raised the problem of exhibiting a simple example in which *there is no* one-parameter subloop in any direction.

Searching for a satisfactory answer to this problem we shall investigate the question: *Is the existence of one-parameter subloops a general property of local loops?*

The aim of our considerations are as follows:

1. Firstly, we exhibit a class of loops which have subloops in every direction, and examine associativity conditions for this loop-class.
2. Secondly, we give (analytic) examples of elastic loops in our class which are not groups on the one hand, and a further example for elastic loops whose one-parameter subloops are not one-parameter subgroups on the other hand. Hence we separate analytic elastic loops from right alternative analytic loops since the latter ones are necessarily power-associative (see [6]).
3. Thirdly, we exhibit a class of loops which have one-parameter subloops only in one direction.
4. Finally, we give an example of a loop without one-parameter subloops at all.

All the results of the present paper are based on the existence of *canonical coordinate systems* [3] and the following main feature of loops ([4], Theorem 1). If f is a local loop of class C^k $(k \geqq 2)$ and D is a canonical coordinate system then every one-parameter subloop is locally a straight line through the origin in D.

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I. Preliminaries

Definition 1. Let $\mathscr{F}$ be an n-dimensional differentiable manifold. A partial mapping f *of* class C^k

$$f\colon \mathscr{F}\times\mathscr{F}\to\mathscr{F}\colon (\mathbf{x},\mathbf{y})\to\mathbf{z}\quad (\mathbf{x},\mathbf{y},\mathbf{z}\in\mathscr{F})$$

is called *local loop-multiplication of class* C^k and $(\mathscr{F};f)$ is called a *local loop of class* C^k if the following conditions are satisfied.

a) The multiplication is a local quasigroup, that is, there exist open neighbourhoods $\mathscr{V}, \mathscr{U}\ (\mathscr{V}\subset\mathscr{U}\subset\mathscr{F})$ such that $f\colon \mathscr{V}\times\mathscr{V}\to\mathscr{U}$ and $f(\mathbf{x},\mathbf{y})=\mathbf{z}\in\mathscr{U}$ for all $\mathbf{x},\mathbf{y}\in\mathscr{V}$. Furthermore for arbitrary elements $\mathbf{x}\in\mathscr{V}$, $\mathbf{z}\in\mathscr{V}$ (respectively, $\mathbf{y}\in\mathscr{V}$, $\mathbf{z}\in\mathscr{V}$) there exists one and only one $\mathbf{y}\in\mathscr{U}$ (respectively, $\mathbf{x}\in\mathscr{V}$) for which $f(\mathbf{x},\mathbf{y})=\mathbf{z}$.

b) The loop has a unit element, that is, there is an element $\mathbf{e}\in\mathscr{V}$ such that $f(\mathbf{x},\mathbf{e})=f(\mathbf{e},\mathbf{x})=\mathbf{x}$ for all $\mathbf{x}\in\mathscr{V}$.

c) The loop-multiplication is of class C^k.

We shall consider charts $(\mathscr{U}_i, \varphi_i)$ for which $\varphi_i\colon \mathscr{U}\to\mathscr{W}_i\subseteqq\mathbf{R}^n$; $\mathbf{e}\mapsto\mathbf{0}$, where $\mathbf{0}$ is the origin of $\mathbf{R}^n$.

A loop on an m-dimensional manifold $\mathscr{F}$ is called an *m-parameter loop.* Instead of $(\mathscr{F};f)$ we shall frequently write f.

Since the canonical coordinate-system defined in [3] plays an important role in our considerations further on, we recall its definition.

Definition 2. Let us consider a loop f of class C^k $(k\geqq 2)$. We shall say that a coordinate-system φ given by the chart $(\mathscr{U},\varphi)$, $f\colon\mathscr{U}\to\mathbf{R}^n$, $\varphi(\mathbf{e})=\mathbf{0}$, is a *canonical coordinate system (CCS)* with respect to f if in these coordinates we have

$$F(\mathbf{x},\mathbf{x}) = 2\mathbf{x}$$

for all $\mathbf{x}\in\varphi(\mathscr{V})$, where

$$F=\varphi\circ f\circ(\varphi^{-1}\times\varphi^{-1}).$$

Further on, by a loop we mean a local loop of class C^k $(k\geqq 2)$.

Definition 3. Let $(\mathscr{F};f)$ and $(\mathscr{G};g)$ be two loops, and let ξ be a local map $\xi\colon\mathscr{F}\to\mathscr{G}$ of class C^k. If ξ is a local embedding, then $(\mathscr{F};f)$ is called a *local m-parameter subloop of* $(\mathscr{G};g)$.

II. Maximal families of one-parameter subloops

Our purpose is now to describe a class of loops having subloops in every direction.

Definition 4. A local multiplication

$$F\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n;\ \mathscr{V} \times \mathscr{V} \to \mathscr{U}$$

(where $\mathscr{V}$, $\mathscr{U}$ are appropriate neighbourhoods of $\mathbf{0}$) is called an *(α, β)-multiplication* if for all $\mathbf{x}, \mathbf{y} \in \mathscr{V}$

$$F(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y} + \alpha(\mathbf{y})\mathbf{x} + \beta(\mathbf{x})\mathbf{y} \tag{1}$$

where $\alpha, \beta\colon \mathbf{R}^n \to \mathbf{R}\colon \mathbf{0} \to 0$ are real-valued functions of class C^k.

Proposition 1. *An (α, β)-multiplication is a local loop-multiplication on a neighbourhood $\mathscr{V}$ of the origin $\mathbf{0}$ of $\mathbf{R}^n$, the unit element is the origin $\mathbf{0}$.*

Proof. First of all let us show that (1) defines a local loop-multiplication of class C^k.

a) As α and β are defined on $\mathbf{R}^n$, $F(\mathbf{x}, \mathbf{y})$ is well-defined. The multiplication is locally solvable from left and right since

$$D_1 F(\mathbf{0}, \mathbf{0}) = I, \quad D_2 F(\mathbf{0}, \mathbf{0}) = I$$

where D_1 and D_2 denotes the partial derivative with respect to the first, respectively, second variable belonging to $\mathbf{R}^n$, and where I is the identity map in $\mathbf{R}^n$.

b) The origin $\mathbf{0}$ of $\mathbf{R}^n$ is the unit element since we have

$$F(\mathbf{x}, \mathbf{0}) = \mathbf{x} + \mathbf{0} + \alpha(\mathbf{0})\mathbf{x} + \beta(\mathbf{x})\mathbf{0} = \mathbf{x},$$

and similarly

$$F(\mathbf{0}, \mathbf{y}) = \mathbf{y}.$$

c) The loop-multiplication is of class C^k because α and β are of class C^k, as well.

A loop $(\mathbf{R}^n, F)$ with (α, β)-multiplication is called an $(\mathbf{R}^n; \alpha, \beta)$-loop.

Now we are going to show that these loops have subloops in every direction.

Theorem 1. *For every vector subspace $\mathscr{A}$ of $\mathbf{R}^n$ the restriction $F|_{\mathscr{A} \times \mathscr{A}}$ of an (α, β)-multiplication is a loop-multiplication on a neighbourhood of $\mathbf{0} \in \mathscr{A}$.*

Proof. Let $\mathscr{A}$ be a vector subspace of $\mathbf{R}^n$. Then for $\mathbf{x}, \mathbf{y} \in \mathscr{A}$ the element

$$\mathbf{z} = F(\mathbf{x}, \mathbf{y}) = [1 + \alpha(\mathbf{y})]\mathbf{x} + [1 + \beta(\mathbf{x})]\mathbf{y}$$

obviously belongs to $\mathscr{A}$, as well. Similarly if $\mathbf{x}, \mathbf{z}$ (respectively $\mathbf{y}, \mathbf{z}$) are elements of

$\mathscr{A}$, then $\mathbf{y}$ (respectively $\mathbf{x}$) also belongs to $\mathscr{A}$. This implies that the restriction of F to $\mathscr{A}\times\mathscr{A}$ is a local subloop of $(\mathbf{R}^n; F)$.

The subloop $(\mathscr{A}; F)$ is called an $(\mathscr{A}; \alpha, \beta)$-loop.

Let us emphasize two details of the above result.

Corollary. *The loop* $(\mathbf{R}^n; \alpha, \beta)$ *has r-parameter subloops in every r-dimensional subspace of* $\mathbf{R}^n$. *In particular, it has one-parameter subloops in every direction.*

Theorem 2. *Let F be an (α, β)-multiplication defined on a neighbourhood $\mathscr{V}\subset\mathbf{R}^n$. Then the loop $(\mathbf{R}^n; \alpha, \beta)$ $(n\geqq 2)$ is a local group on a sufficiently small neighbourhood $\mathscr{W}$ of $\mathbf{0}$ if and only if*

$$\alpha(F(\mathbf{x}, \mathbf{y})) = \alpha(\mathbf{x})+\alpha(\mathbf{y})+\alpha(\mathbf{x})\alpha(\mathbf{y})$$

(for every $\mathbf{x}, \mathbf{y}\in\mathscr{W}$*).*

Proof. 1) For the necessity we show that $(\mathbf{R}^n; \alpha, \beta)$ is a group if and only if for all $\mathbf{x}, \mathbf{y}\in\mathscr{W}$ the relations

$$\alpha(F(\mathbf{x}, \mathbf{y})) = \alpha(\mathbf{x})+\alpha(\mathbf{y})+\alpha(\mathbf{x})\alpha(\mathbf{y}) \tag{2a}$$

$$\beta(F(\mathbf{x}, \mathbf{y})) = \beta(\mathbf{x})+\beta(\mathbf{y})+\beta(\mathbf{x})\beta(\mathbf{y}) \tag{2b}$$

are satisfied. Indeed, $(\mathbf{R}^n; \alpha, \beta)$ is a group if and only if F is associative, that is,

$$F(F(\mathbf{x}, \mathbf{y}), \mathbf{z}) = F(\mathbf{x}, F(\mathbf{y}, \mathbf{z}))$$

holds for all $\mathbf{x}, \mathbf{y}, \mathbf{z}\in\mathscr{W}$ ($\mathscr{W}$ is a sufficiently small neighbourhood of $\mathbf{0}$). By a straightforward calculation we see that this identity is equivalent to the following one:

$$[\alpha(F(\mathbf{y}, \mathbf{z}))-[\alpha(\mathbf{y})+\alpha(\mathbf{z})+\alpha(\mathbf{y})\alpha(\mathbf{z})]]\,\mathbf{x} = [\beta(F(\mathbf{x}, \mathbf{y}))-[\beta(\mathbf{x})+\beta(\mathbf{y})+\beta(\mathbf{x})\beta(\mathbf{y})]]\,\mathbf{z}$$

for all $\mathbf{x}, \mathbf{y}, \mathbf{z}\in\mathscr{W}$, which is equivalent to (2a) and (2b).

2) In order to show that condition (2a) is sufficient we shall prove that (2a) implies (2b). For this purpose we show that (2a) implies the linearity of $\alpha-\beta$ via the commutativity of the one-parameter subloops.

By (2a) we have

$$\alpha(F(\mathbf{x}, \mathbf{y})) = \alpha(F(\mathbf{y}, \mathbf{x})) \tag{3}$$

for all $\mathbf{x}, \mathbf{y}\in\mathscr{W}$. Whenever for a direction a $D_{\mathbf{a}}\alpha(\mathbf{0})\neq 0$, we get from (3) that

$$F(\mathbf{x}, \mathbf{y}) = F(\mathbf{y}, \mathbf{x})$$

for all $\mathbf{x}, \mathbf{y}$ belonging to the one-parameter subloop of direction $\mathbf{a}$. That is, this one-parameter subloop is commutative. However, if for the direction a $D_{\mathbf{a}}\alpha(\mathbf{0})=0$, we introduce new defining functions

$$\alpha^*(\mathbf{x}) = \alpha(\mathbf{x})+\langle \mathbf{a}, \mathbf{x}\rangle, \qquad \beta^*(\mathbf{x}) = \beta(\mathbf{x})-\langle \mathbf{a}, \mathbf{x}\rangle$$

to obtain the $(\mathbf{R}^n;\ \alpha^*, \beta^*)$-loop on $\mathcal{W}$. Since

$$u\mathbf{a}+v\mathbf{a}\alpha^*(v\mathbf{a})\,u\mathbf{a}+\beta^*(u\mathbf{a})v\mathbf{a} = u\mathbf{a}+v\mathbf{a}\alpha(v\mathbf{a})\,u\mathbf{a}+\beta(u\mathbf{a})v\mathbf{a}$$

for all sufficiently small real number u and v the one-parameter subloops of the direction $\mathbf{a}$ of the loops $(\mathbf{R}^n;\ \alpha^*, \beta^*)$ and $(\mathbf{R}^n;\ \alpha, \beta)$ are the same. Since $D_{\mathbf{a}}\alpha^*(\mathbf{0})\neq 0$ when $D_{\mathbf{a}}\alpha(\mathbf{0})=0$, we obtain that this one-parameter subloop is commutative.

From the commutativity it immediately follows that

$$\alpha(v\mathbf{x})\,u\mathbf{x}+\beta(u\mathbf{x})v\mathbf{x} = \alpha(u\mathbf{x})v\mathbf{x}+\beta(v\mathbf{x})\,u\mathbf{x}$$

holds in every direction $\mathbf{x}$ for all sufficiently small nonzero u and v in $\mathbf{R}$, which yields

$$\frac{\alpha(v\mathbf{x})-\beta(v\mathbf{x})}{v} = \frac{\alpha(u\mathbf{x})-\beta(u\mathbf{x})}{u}.$$

The left hand side does not depend on u, so we get

(4)

$$\alpha(v\mathbf{x})-\beta(v\mathbf{x}) = v\cdot\lim_{u\to 0}\frac{\alpha(u\mathbf{x})-\alpha(\mathbf{0})}{u}-v\cdot\lim_{u\to 0}\frac{\beta(u\mathbf{x})-\beta(\mathbf{0})}{u} = v\cdot D_{\mathbf{x}}\big(\alpha(\mathbf{0})-\beta(\mathbf{0})\big)$$

for each $\mathbf{x}\in\mathcal{W}$. Thus

$$\alpha(\mathbf{z})-\beta(\mathbf{z}) = \langle D(\alpha-\beta)(\mathbf{0}), \mathbf{z}\rangle.$$

That is, $\alpha-\beta$ is a linear function on the appropriate neighbourhood.

Now we are ready to prove (2b) from (2a). If in the left and right hand side of (2a) we substitute $\lambda+\beta$ for α (where λ is a linear function), we obtain firstly

$$\alpha\big(F(\mathbf{x},\mathbf{y})\big) = \lambda\big(F(\mathbf{x},\mathbf{y})\big)+\beta\big(F(\mathbf{x},\mathbf{y})\big) = \lambda(\mathbf{x})+\lambda(\mathbf{y})+\alpha(\mathbf{y})\lambda(\mathbf{x})+\beta(\mathbf{x})\lambda(\mathbf{y})+$$
$$+\beta\big(F(\mathbf{x},\mathbf{y})\big) = [\lambda(\mathbf{x})+\lambda(\mathbf{y})+\lambda(\mathbf{y})\lambda(\mathbf{x})+\beta(\mathbf{y})\lambda(\mathbf{x})+\beta(\mathbf{x})\lambda(\mathbf{y})]+\beta\big(F(\mathbf{x},\mathbf{y})\big),$$

secondly

$$\alpha(\mathbf{x})+\alpha(\mathbf{y})+\alpha(\mathbf{x})\alpha(\mathbf{y}) =$$
$$= [\lambda(\mathbf{x})+\lambda(\mathbf{y})+\lambda(\mathbf{x})\lambda(\mathbf{y})+\lambda(\mathbf{x})\beta(\mathbf{y})+\beta(\mathbf{x})\lambda(\mathbf{y})]+[\beta(\mathbf{x})+\beta(\mathbf{y})+\beta(\mathbf{x})\beta(\mathbf{y})],$$

hence

$$\beta\big(F(\mathbf{x},\mathbf{y})\big) = \beta(\mathbf{x})+\beta(\mathbf{y})+\beta(\mathbf{x})\beta(\mathbf{y})$$

which is just (2b) and the theorem is proved.

Remarks 1. In accordance with the proof above, the $(\mathbf{R}^n;\ \alpha, \beta)$-loop is not a group if $\alpha-\beta$ is not linear (for example if β is identically 0, and α is not linear). Thus relation (1) defines a non-trivial loop, in general. 2. By Theorem 1, r-parameter subloops $(\mathscr{A};\alpha,\beta)$ of $(\mathbf{R}^n;\alpha,\beta)$ exist for all subspace $\mathscr{A}\subseteqq\mathbf{R}^n$. Since Theorem 2 holds for these loops, too, we obtain that the r-parameter $(r\geqq 2)$ subloops of $(\mathbf{R}^n;\alpha,\beta)$ are not r-parameter subgroups if $\alpha-\beta$ is not linear.

We now consider some other algebraic identities which are weaker than associativity. Let us recall some definitions.

Definition 5. A loop has the *left-inverse property (right-inverse property)* if for each $\mathbf{x}\in\mathscr{V}$ there is an element $\bar{\mathbf{x}}_l\in\mathscr{V}$ $(\bar{\mathbf{x}}_r\in\mathscr{V})$ such that for every $\mathbf{y}\in\mathscr{V}$

(5) $$F(\bar{\mathbf{x}}_l, F(\mathbf{x},\mathbf{y}))=\mathbf{y},\quad \text{respectively,}\quad F(F(\mathbf{y},\mathbf{x}),\bar{\mathbf{x}}_r)=\mathbf{y},$$

in particular, for $\mathbf{y}=\mathbf{0}$

(6) $$F(\mathbf{x}_l,\mathbf{x})=\mathbf{0},\quad \text{respectively,}\quad F(\mathbf{x},\bar{\mathbf{x}}_r)=\mathbf{0}.$$

The loop F said to be *left alternative (right alternative)* if for all $\mathbf{x},\mathbf{y}\in\mathscr{V}$

(7) $$F(F(\mathbf{x},\mathbf{x}),\mathbf{y})=F(\mathbf{x},F(\mathbf{x},\mathbf{y})),\quad \text{respectively,}\quad F(\mathbf{y},F(\mathbf{x},\mathbf{x}))=F(F(\mathbf{y},\mathbf{x}),\mathbf{x}).$$

The loop has the *property of elasticity* if for all $\mathbf{x},\mathbf{y}\in\mathscr{V}$

(8) $$F(\mathbf{x},F(\mathbf{y},\mathbf{x}))=F(F(\mathbf{x},\mathbf{y}),\mathbf{x}).$$

Theorem 3. *An* $(\mathbf{R}^n;\alpha,\beta)$*-loop* $(n\geqq 2)$ *is a group if and only if it*
a) *possesses the left-inverse property (right-inverse property),*
b) *possesses the left alternative property (right alternative property).*

Theorem 4. *Whenever* $\alpha=\beta$ *and* α *is linear then the corresponding* $(\mathbf{R}^n;\alpha,\beta)$*-loops* $(n\geqq 2)$ *are elastic. Such an* $(\mathbf{R}^n;\alpha,\beta)$*-loop of dimension* n $(n\geqq 2)$ *is a group if and only if* $\alpha\equiv 0$.

Proof of Theorem 3. a) If we use expression (1) to reformulate (5) and (6) we get

(5′) $$[1+\alpha[\mathbf{x}+\mathbf{y}+\alpha(\mathbf{y})\mathbf{x}+\beta(\mathbf{x})\mathbf{y}]]\cdot\bar{\mathbf{x}}_l+[1+\alpha(\mathbf{y})+\beta(\bar{\mathbf{x}}_l)+\alpha(\mathbf{y})\beta(\bar{\mathbf{x}}_l)]\cdot\mathbf{x}+$$
$$+[\beta(\mathbf{x})+\beta(\bar{\mathbf{x}}_l)+\beta(\mathbf{x})\beta(\bar{\mathbf{x}}_l)]\cdot\mathbf{y}=\mathbf{0},$$

(6′) $$[1+\alpha(\mathbf{x})]\cdot\bar{\mathbf{x}}_l+[1+\beta(\bar{\mathbf{x}}_l)]\cdot\mathbf{x}=\mathbf{0}$$

for all $\mathbf{x}$, $\mathbf{y}$ and $\bar{\mathbf{x}}_l\in\mathscr{V}$. Subtracting (6′) from (5′) we get

$$[\alpha[\mathbf{x}+\mathbf{y}+\alpha(\mathbf{y})\mathbf{x}+\beta(\mathbf{x})\mathbf{y}]-\alpha(\mathbf{x})]\,\bar{\mathbf{x}}_l+\alpha(\mathbf{y})[1+\beta(\bar{\mathbf{x}}_l)]\,\mathbf{x}+$$
$$+[\beta(\mathbf{x})+\beta(\bar{\mathbf{x}}_l)+\beta(\mathbf{x})\beta(\bar{\mathbf{x}}_l)]\,\mathbf{y}=\mathbf{0}$$

for all $\mathbf{x}$, $\mathbf{y}$ and $\bar{\mathbf{x}}_l\in\mathscr{V}$. In view of (6′) we obtain

$$[\alpha[\mathbf{x}+\mathbf{y}+\alpha(\mathbf{y})\mathbf{x}+\beta(\mathbf{x})\mathbf{y}]-\alpha(\mathbf{x})-\alpha(\mathbf{y})-\alpha(\mathbf{x})\alpha(\mathbf{y})]\,\bar{\mathbf{x}}_l+[\beta(\mathbf{x})+\beta(\bar{\mathbf{x}}_l)+\beta(\mathbf{x})\beta(\bar{\mathbf{x}}_l)]\,\mathbf{y}=\mathbf{0}$$

for all $\mathbf{x}, \mathbf{y}$ and $\bar{\mathbf{x}}_l \in \mathscr{V}$. Notice that $\bar{\mathbf{x}}_l = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$ (see (6)). Hence relation (5) holds for the loop $(\mathbf{R}^n;\ \alpha, \beta)$ if and only if the following two identities are satisfied

$$\alpha[\mathbf{x}+\mathbf{y}+\alpha(\mathbf{y})\mathbf{x}+\beta(\mathbf{x})\mathbf{y}] = \alpha(\mathbf{x})+\alpha(\mathbf{y})+\alpha(\mathbf{x})\alpha(\mathbf{y}) \tag{9a}$$

$$0 = \beta(\mathbf{x})+\beta(\bar{\mathbf{x}}_l)+\beta(\mathbf{x})\beta(\bar{\mathbf{x}}) \tag{9b}$$

for all $\mathbf{x}, \mathbf{y} \in \mathscr{V}$. Since condition (2a) is the same as (9a), this local loop is a local group. As a group always possesses left-inverse property, part a) of the theorem is proved in the case of left inverse loops.

Notice that from (9b) and (6′) we can express $\bar{\mathbf{x}}_l$ as follows:

$$\bar{\mathbf{x}}_l = \frac{1}{[1+\alpha(\mathbf{x})]\cdot[1+\beta(\mathbf{x})]}\,\mathbf{x}. \tag{$*$}$$

For right inverse $(\mathbf{R}^n;\ \alpha, \beta)$-loops the proof can be carried out in the same way as above. Furthermore, the right and left inverse of an element are clearly the same by $(*)$ above.

b) Expressing identity (7) in terms of (1) we obtain that (7) is equivalent to the following two identites

$$\alpha[\mathbf{x}+\mathbf{y}+\alpha(\mathbf{y})\mathbf{x}+\beta(\mathbf{x})\mathbf{y}] = \alpha(\mathbf{x})+\alpha(\mathbf{y})+\alpha(\mathbf{x})\alpha(\mathbf{y}), \tag{10a}$$

$$\beta[2\mathbf{x}+\alpha(\mathbf{x})\mathbf{x}+\beta(\mathbf{x})\mathbf{x}] = 2\beta(\mathbf{x})+\beta(\mathbf{x})\beta(\mathbf{x}). \tag{10b}$$

We see that (10a) and (2a) are equivalent. Thus from the left alternative property it follows that an $(\mathbf{R}^n;\ \alpha, \beta)$-loop is a local group. The converse is obvious, furthermore the right alternative case is similar, therefore part b) is proved.

Proof of Theorem 4. From expression (1) we obtain that (8) is equivalent to the following equality;

$$\alpha(\mathbf{x})+\alpha(\mathbf{y})+\alpha(\mathbf{x})\alpha(\mathbf{y})-\alpha[\mathbf{x}+\mathbf{y}+\alpha(\mathbf{x})\mathbf{y}+\beta(\mathbf{y})\mathbf{x}] =$$
$$= \beta(\mathbf{x})+\beta(\mathbf{y})+\beta(\mathbf{x})\beta(\mathbf{y})-\beta[\mathbf{x}+\mathbf{y}+\alpha(\mathbf{y})\mathbf{x}+\beta(\mathbf{x})\mathbf{y}]$$

for all $\mathbf{x}, \mathbf{y} \in \mathscr{V}$. It is clear that if $\alpha \equiv \beta$, then this equality is an identity for all $\mathbf{x}, \mathbf{y} \in \mathscr{V}$. In other words, for $\alpha \equiv \beta$ the loop $(\mathbf{R}^n;\ \alpha, \beta)$ is elastic. Let now $\alpha \equiv \beta$ be linear function. Let us suppose that $(\mathbf{R}^n;\ \alpha, \beta)$ is a group, and $n \geqq 2$. Then (2a) is fulfilled and has the form

$$\alpha(F(\mathbf{x}), \mathbf{y}) = \alpha(\mathbf{x})+\alpha(\mathbf{y})+\alpha(\mathbf{x})\alpha(\mathbf{y})$$

$$\alpha[\mathbf{x}+\mathbf{y}+\alpha(\mathbf{y})\mathbf{z}+\alpha(\mathbf{x})\mathbf{y}] = \alpha(\mathbf{x})+\alpha(\mathbf{y})+\alpha(\mathbf{x})\alpha(\mathbf{y})$$

$$\alpha(\mathbf{x})+\alpha(\mathbf{y})+\alpha(\mathbf{y})\alpha(\mathbf{x})+\alpha(\mathbf{x})\alpha(\mathbf{y}) = \alpha(\mathbf{x})+\alpha(\mathbf{y})+\alpha(\mathbf{x})\alpha(\mathbf{y})$$

$$\alpha(\mathbf{x})\alpha(\mathbf{y}) = 0$$

for all $\mathbf{x}, \mathbf{y} \in \mathscr{V}$. However, this means that $\alpha \equiv \beta \equiv 0$. Thus for the linear function $\alpha \equiv \beta$ which is not identically zero the loop $(\mathbf{R}^n; \alpha, \beta)$ is elastic, but not a group.

Another (less obvious) example is an $(\mathbf{R}^n; \alpha, \beta)$-loop $(n \geqq 1)$ for which $\alpha(\mathbf{x})$ is a quadratic function, e.g. $\alpha(\mathbf{x}) = \|\mathbf{x}\|^2$. It is easy to see that in this case the one-parameter subloops are not one-parameter subgroups. Indeed, for arbitrary $s, t, u \in \mathbf{R}$ we have

$$F\big(F(s\mathbf{x}, t\mathbf{x}), u\mathbf{x}\big) - F\big(s\mathbf{x}, F(t\mathbf{x}, u\mathbf{x})\big) =$$

$$= stu \cdot \|\mathbf{x}\|^2 \cdot \big[(3t+2s+2u)(s-u) + \|\mathbf{x}\|^2 t[t(s^3-u^3) + 2t^2(s^2-u^2) + t^3(s-u)]\big]\, \mathbf{x}.$$

Hence these subloops are not subgroups because the difference vector has a positive norm provided $0 < u < s < t$. Consequently, these loops are not groups.

Remark. L. V. Sabinin and P. O. Mikheev [5] proved that analytical right-alternative loops are power associative (that is, $F(\mathbf{x}^m, \mathbf{x}^n) = \mathbf{x}^{m+n}$ for arbitrary $\mathbf{x} \in \mathscr{F}$ and any integers m and n). However, we can show that the analogous statement for elastic loops is false. As a power-associative loop has one-parameter subgroups in every direction (see Kuz'min [6]), our analytical elastic $(\mathbf{R}^n; \alpha, \alpha)$-loops $(\alpha(\mathbf{x}) = \|\mathbf{x}\|^2)$ can not be power associative.

III. Loops without one-parameter subloops

In final part of this paper we exhibit loops which do not have any non-trivial one-parameter subloops whatsoever. However, let us start with another class of loops which have one-parameter subloops in one unique direction. For this purpose we give a slight modification of the loops given by (1).

Definition 6. Let us define a local multiplication $(\alpha, -\alpha; \varrho, \mathbf{a})$ in $\mathbf{R}^n$ as follows

$$F_2\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n\colon \mathscr{V} \times \mathscr{V} \to \mathscr{U}$$

where $\mathscr{V}, \mathscr{U}$ are appropriate neighbourhoods of 0, furthermore for all $x, y \in \mathscr{V}$ we have

$$F_2(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y} + \alpha(\mathbf{y})\mathbf{x} - \alpha(\mathbf{x})\mathbf{y} + \varrho(\mathbf{x}, \mathbf{y})\mathbf{a} \tag{11}$$

for all $\mathbf{x}, \mathbf{y} \in \mathscr{V}$. Here α is the same as in Definition 4; furthermore $\mathbf{a} \in \mathbf{R}^n$ is a point (sufficiently close to $\mathbf{0}$) different from $\mathbf{0}$, and ϱ is a real function of class C^k:

$$\varrho\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}\colon \mathscr{V} \to \mathscr{U}^* \subset \mathbf{R}$$

such that $\varrho(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$ or $\mathbf{x} = \mathbf{y}$. (That is, $\varrho(\mathbf{x}, \mathbf{0}) = \varrho(\mathbf{0}, \mathbf{y}) = \varrho(\mathbf{z}, \mathbf{z}) = 0$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathscr{V}$, and ϱ does not vanish in any other case).

Such functions ϱ exist, e.g. the function ϱ defined as follows:

(12) $$\varrho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 \|\mathbf{x}-\mathbf{y}\|^2.$$

Proposition 2. *The local* $(\alpha, -\alpha; \varrho, \mathbf{a})$*-multiplication given by* (13) *defines a local loop of class* C^k *which has one-parameter subloop in the direction of* $\mathbf{a}$, *exclusively.*

Proof. We show that F_2 is a loop-multiplication.

a) Since α and ϱ are of class C^k, F_2 is well defined in $\mathscr{V}$. $F_2(\mathbf{x}, \mathbf{y})=\mathbf{z}$ can be solved from left and right because the derivative of F_2 with respect to the first and second variable (belonging to $\mathbf{R}^n$) is the identity map.

b) The origin $\mathbf{0}$ in $\mathbf{R}^n$ is the unit element since

$$F_2(\mathbf{x}, \mathbf{0}) = \mathbf{x}+\mathbf{0}+\alpha(\mathbf{0})\mathbf{x}-\alpha(\mathbf{x})\mathbf{0}+\varrho(\mathbf{x}, \mathbf{0})\mathbf{a} = \mathbf{x}$$

and similarly $F_2(\mathbf{0}, \mathbf{y})=\mathbf{y}$, for all $\mathbf{x}, \mathbf{y}\in\mathscr{V}$.

c) α and ϱ are of class C^k, therefore F_2 is also of class C^k.

Following this, let us notice that the coordinate system in which F_2 is defined, is a CCS. Indeed, according to Definition 2 we have

$$F_2(\mathbf{x}, \mathbf{x}) = \mathbf{x}+\mathbf{x}+\alpha(\mathbf{x})\mathbf{x}-\alpha(\mathbf{x})\mathbf{x}+\varrho(\mathbf{x}, \mathbf{x})\mathbf{a} = 2\mathbf{x}$$

for all $\mathbf{x}\in\mathscr{V}$. So, owing to Theorem 1 in [4], if there exists a one-parameter subloop, then it is locally a straight line. It is obvious that there exists one-parameter subloop in the direction of $\mathbf{a}$.

Let us suppose that there exists one-parameter subloop in the direction of $\mathbf{d}$. Then the elements of this subloop are of form $t\cdot\mathbf{d}$, $t\in(-T_0, T_0)$, at least locally. Let $\mathbf{x}=s_1\mathbf{d}\neq\mathbf{0}$ and $\mathbf{y}=s_2\mathbf{d}\neq\mathbf{0}$ $(s_1\neq s_2)$ be two different elements of it. According to (11) we have

$$F_2(\mathbf{x}, \mathbf{y}) = F_2(s_1\mathbf{d}, s_2\mathbf{d}) = [s_1+s_2+s_1\alpha(s_2\mathbf{d})-s_2\alpha(s_1\mathbf{d})]\mathbf{d}+\varrho(s_1\mathbf{d}, s_2\mathbf{d})\mathbf{a}+r\mathbf{d}$$

for some $r\in\mathbf{R}$. Since for these s_1 and s_2: $\varrho(s_1\mathbf{d}, s_2\mathbf{d})\neq 0$, directions $\mathbf{a}$ and $\mathbf{d}$ must be the same, which completes the proof.

This loop is called an $(\mathbf{R}^n; \alpha, -\alpha; \varrho, \mathbf{a})$-loop.

Our next and last example is that of a loop without nontrivial one-parameter subloops. For this purpose we shall modify the previous construction.

Definition 7. Let us define a local multiplication $(\alpha, -\alpha; \varrho, \mathbf{a}; \sigma, \mathbf{b})$ in $\mathbf{R}^n$ $(n\geqq 2)$ in the following manner

$$F_3\colon \mathbf{R}^n\times\mathbf{R}^n \to \mathbf{R}^n\colon \mathscr{V}\times\mathscr{V}\to\mathscr{U}$$

where $\mathscr{V}$, $\mathscr{U}$ are appropriate neighbourhoods of the origin $\mathbf{0}$, furthermore for all

$\mathbf{x}, \mathbf{y} \in \mathscr{V}$ we have

$$F_3(\mathbf{x}, \mathbf{y}) = \mathbf{x}+\mathbf{y}+\alpha(\mathbf{y})\mathbf{x}-\alpha(\mathbf{x})\mathbf{y}+\varrho(\mathbf{x}, \mathbf{y})\mathbf{a}+\sigma(\mathbf{x}, \mathbf{y})\mathbf{b} \tag{13}$$

where α and σ are the same as in Definition 6, $\mathbf{a}$ and $\mathbf{b}$ are linearly independent (fixed) points (directions) in $\mathbf{R}^n$ (sufficiently close to $\mathbf{0}$). Further we suppose that σ possesses all the properties of ϱ and that, in additional, from $\varrho(\mathbf{x}_0, \mathbf{y}_0)=\sigma(\mathbf{x}_0, \mathbf{y}_0)$ it follows that $\varrho(\mathbf{x}_0, \mathbf{y}_0)=\sigma(\mathbf{x}_0, \mathbf{y}_0)=\mathbf{0}$ (that is that σ and ϱ are different, except where they vanish).

Functions σ with the required properties exist, e.g. the one defined by the following formula:

$$\sigma(\mathbf{x}, \mathbf{y}) = h(\mathbf{x})\|\mathbf{y}\|^2\|\mathbf{x}-\mathbf{y}\|^2, \tag{14}$$

where $h\colon \mathbf{R}^n \to \mathbf{R}^n$ and $h(\mathbf{x})$ vanishes if and only if $\mathbf{x}=\mathbf{0}$, and $h(\mathbf{x}) \neq \|\mathbf{x}\|^2$. Furthermore h is assumed to be of class C^k.

Proposition 2. *Any multiplication given by* (13) *defines a local loop of class C^k which has no one-parameter subloop at all.*

This loop is called an $(\mathbf{R}^n; \alpha, -\alpha; \varrho, \mathbf{a}; \sigma, \mathbf{b})$-loop.

Proof. In the same way as above we can prove that conditions a), b) and c) for loops are fulfilled.

It can be stated, again, that F_3 is defined in a CCS. Thus if there exists a one-parameter subloop of F_3 in the direction of $\mathbf{d}$, then for two different elements $\mathbf{x}=s_1\mathbf{d}$, $\mathbf{y}=s_2\mathbf{d}$ $(0 \neq s_1 \neq s_2 \neq 0)$ we have

$$F_3(\mathbf{x}, \mathbf{y}) = F_3(s_1\mathbf{d}, s_2\mathbf{d}) =$$

$$= [s_1+s_2+s_1\alpha(s_2\mathbf{d})-s_2\alpha(s_1\mathbf{d})]\mathbf{d}+\varrho(s_1\mathbf{d}, s_2\mathbf{d})\mathbf{a}+\sigma(s_1\mathbf{d}, s_2\mathbf{d})\mathbf{b} = r\mathbf{d}.$$

Thus we obtain that

$$\varrho(s_1\mathbf{d}, s_2\mathbf{d})\mathbf{a}+\sigma(s_1\mathbf{d}, s_2\mathbf{d})\mathbf{b} = t\mathbf{d}$$

for some $t=t(s_1, s_2)\in(-T_0, T_0)$. As for $\mathbf{d}$ we have unique expression $\mathbf{d}=\varkappa_1\mathbf{a}+\varkappa_2\mathbf{b}$, we get

$$t\mathbf{d} = (t\varkappa_1)\mathbf{a}+(t\varkappa_2)\mathbf{b} = \varrho(s_1\mathbf{d}, s_2\mathbf{d})\mathbf{a}+\sigma(s_1\mathbf{d}, s_2\mathbf{d})\mathbf{b}$$

for all allowable $s_1 \neq s_2$, hence we obtain that

$$\varrho(s_1\mathbf{d}, s_2\mathbf{d}) = t\varkappa_1 \quad \text{and} \quad \sigma(s_1\mathbf{d}, s_2\mathbf{d}) = t\varkappa_2. \tag{15}$$

Let us first suppose that $\mathbf{d}=\mathbf{a}$. That is, there exists a one-parameter subloop in the direction of $\mathbf{a}$. Then we have

$$\varrho(s_1\mathbf{a}, s_2\mathbf{a})\mathbf{a}+\sigma(s_1\mathbf{a}, s_2\mathbf{a})\mathbf{b} = t\mathbf{a}.$$

Since $\varrho(s_1\mathbf{a}, s_2\mathbf{a})\neq 0$, $\sigma(s_1\mathbf{a}, s_2\mathbf{a})\neq 0$ and $t\neq 0$, it follows that $\mathbf{b}=\mathbf{a}$, which contradicts the assumption for a and b. In the case of $\mathbf{d}=\mathbf{b}$ we get the same result.

Let us now suppose that $\mathbf{a}\neq\mathbf{d}\neq\mathbf{b}$. Then from relations (15) we obtain that $\varkappa_1\neq 0$, $\varkappa_2\neq 0$, consequently for all $0\neq s_1\neq s_2\neq 0$

$$\varrho(s_1\mathbf{d}, s_2\mathbf{d}) = \frac{\varkappa_1}{\varkappa_2}\sigma(s_1\mathbf{d}, s_2\mathbf{d}). \tag{16}$$

But, we shall now give a ϱ and σ such that this equation does not hold identically in s_1 and s_2. With relations (12) and (14) equation (16) becomes

$$\|s_1\mathbf{d}\|^2\,\|s_2\mathbf{d}\|^2\,\|(s_1-s_2)\mathbf{d}\|^2 = \frac{\varkappa_1}{\varkappa_2}h(s_1\mathbf{d})\,\|s_2\mathbf{d}\|^2\,\|(s_1-s_2)\mathbf{d}\|^2$$

and consequently

$$\|s_1\mathbf{d}\|^2 = \frac{\varkappa_1}{\varkappa_2}h(s_1\mathbf{d})$$

for all sufficiently small $s_1\neq 0$. Now the function

$$h(\mathbf{x}) = e^{\|\mathbf{x}\|^2}\cdot\|\mathbf{x}\|^2$$

can not satisfy identically the previous equation, which completes the proof.

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The heat kernel for p-forms on manifolds of bounded geometry

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1. Introduction

In [3] an approximation result for the eigenvalues below the essential spectrum of the Laplace operator was proved for open manifolds. These eigenvalues were approximated by the eigenvalues of some sequence of semicombinatorical Laplace operators. The essential assumptions were completeness and bounded geometry up to a certain order. Using this, the first author proved, following a paper of DONNELLY [6], the existence of a good fundamental solution of the heat operator acting on functions, or what is the same, the existence of a good heat kernel. For p-forms this existence was presumed. It is widely believed that the existence result holds for p-forms and several authors refer to [4] for example. But in [4] a rigorous proof was given only for functions. Further, in [5] there is a nice existence proof for functions and a uniqueness proof for p-forms. The paper [4] does not contain an existence proof for p-forms. As a matter of fact, we have never seen such a proof until now. This is in some sense understandable because it needs some nontrivial facts that have to be established. In this paper we present an existence proof for a good heat kernel on open manifolds of bounded geometry of infinite order, as expressed by Theorem 4.1. The uniqueness then follows from [5].

The paper is organized as follows. In Section 2 we introduce and summarize some facts on manifolds of bounded geometry. Section 3 is devoted to the main technical lemmas concerning the construction of the heat kernel. Finally, in Section 4 we present the main results of this paper.

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2. Manifolds of bounded geometry

Let (M^N, g) be open and complete. Denote the curvature tensor of (M^N, g) by R and the Levi—Civita derivative by ∇. We consider the following conditions:

$$r_{\mathrm{inj}}(M) = \inf_{x \in M} r_{\mathrm{inj}}(x) > 0, \tag{1}$$

i.e. the injectivity radius possesses a positive lower bound.

(B_m) There exist bounds C_i such that $|\nabla^i R| \leqq C_i$, $0 \leqq i \leqq m$.
Assuming condition (1), we consider further the condition

(BC^m) For every $\varepsilon > 0$, $0 < \varepsilon < r_{\mathrm{inj}}(M)$, and multiindex

$$\alpha = \{\alpha_1, \ldots, \alpha_N), \alpha_i \geqq 0, |\alpha| = \alpha_1 + \ldots + \alpha_N \leqq m$$

and every choice of an orthonormal base in all points $x \in M$ there exist constants C_α independent of x such that $|D^\alpha g_{ij}| \leqq C_\alpha$, $y \in B_\varepsilon(x)$, in normal coordinates x^i defined in the open ball $B_\varepsilon(x)$.

Remarks. 1. (B_0) is equivalent to the boundedness of the sectional curvature. 2. (BC^m) is independent of the choice of the orthonormal base in $T_x M$. This follows from the chain rule, the triangle inequality and the compactness of the orthogonal group $O(n)$. 3. The boundedness of the $|D^\alpha(g_{ij})|$, $|\alpha| \leqq m$, implies the boundedness of the $|D^\alpha(g^{ij})|$. For $|\alpha| = 0$ this is seen from

$$(g_{ij})(g^{ij}) = E. \tag{2.1}$$

Assuming the validity for $|\alpha| = m-1$, we obtain the validity for m, applying D^α to (2.1), expressing $D^\alpha(g_{ij})$ by the $D^\beta(g_{kl})$, $|\beta| \leqq m$ and $D^\gamma(g^{rs})$, $|\gamma| \leqq m-1$ and applying the induction assumption.

We summarize some relations between the above conditions.

Proposition 2.1. *Let* (M^N, g) *be open, complete and satisfying* (1).

(a) (BC^m) *implies* (B_{m-2}),

(b) (BC^∞) *and* (B_∞) *are equivalent,*

(c) (B_0) *implies* (BC^0),

(d) (B_i) *implies* (BC^i).

Proof. (a) The curvature tensor can be expressed by derivatives of the g_{ij}, g^{kl} of order $\leqq 2$.

(b) We refer to [4], page 33.

(c) The boundedness of the g_{ij}, assuming (B_0), is just Lemma 1 of [9].

(d) In [10] it was shown that (B_i) implies the boundedness of the Christoffel symbols Γ^j_{ik}. From

$$(2.2) \qquad \Gamma_{j,ik} = g_{rj}\Gamma^r_{ik}, \quad \Gamma_{j,ik}+\Gamma_{i,jk} = (\partial/\partial x^k)g_{ij}$$

and (c) we obtain the assertion.

Examples of open manifolds satisfying (1) and (B_∞) are the Riemannian homogeneous spaces, in particular the symmetric spaces of noncompact type.

The existence problem for metrics satisfying (1) and (B_m) is more subtle. The condition (B_∞) does not imply (1), as cusp manifolds of constant curvature $K=-1$ show. CHEEGER, GROMOV and TAYLOR presented in Theorem 4.7 of [4] explicit lower bounds for the injectivity radius $r_{\mathrm{inj}}(x)$ by relative volume estimates, assuming additionally curvature bounds. As a trivial conclusion, the injectivity radius of an open manifold in general is governed by the curvature and by additional geometrical entities.

Let us list up some classes of open manifolds admitting a natural construction of metrics of bounded geometry.

Proposition 2.2. *The following classes of smooth open manifolds admit a natural construction of complete metrics satisfying* (1) *and* (B_∞).

(a) *Reductive homogeneous spaces G/H, G beeing a Lie group and H a compact subgroup.*

(b) *Coverings of closed manifolds.*

(c) *Open manifolds which are built up by infinitely often gluing together a finite number of bordisms (manifolds with so called almost periodic ends, cf.* [7]*). In particular, any infinite connected sum of a finite number of closed manifolds or manifolds with a finite number of ends, each of them collared, belong to that class.*

(d) *Leaves of a foliation of a compact manifold.*

(e) *Every finite connected sum of open manifolds, each of them admitting a metric of the above type.*

Proof. (a) Every such manifold admits a metric making it to a Riemannian homogeneous space.

(b) Equip the closed manifold with any metric and take its lift.

(c) If $M_1, \ldots, M_r$ are the nondiffeomorphic boundaries, fix a metric g_ϱ at M_ϱ, extend g_ϱ as a product metric to collar neighbohoods and then to the bordisms. For a collared end there is a simpler construction by fixing a product metric at each end and extending the end metrics to the remaining compact part of M.

(d) This item was proved in [8].

(e) The proof is trivial.

Remark. Natural construction here means that the construction of the metric is in a certain sense adapted to the topology of M. Nevertheless, a much more general existence theorem holds true. Namely, we reformulate Theorem 2′ of [8] as

Theorem 2.3. *Every open manifold admits a complete Riemannian metric satisfying* (1) *and* (B_∞).

In the present paper we call an open, complete manifold (M^N, g) satisfying (1) and (B_∞) a manifold of bounded geometry (of infinite order).

In the next Section we need some properties of the Christoffel symbols for such manifolds which we establish now. We recall that (B_∞) implies (BC^∞) according to (2.2). Fixing $x \in M$ and $0 < \varepsilon < r_{\mathrm{inj}}(M)$, we consider geodesic polar coordinates $(r, u) = (r, u^1, \dots, u^{N-1}) = (x^1, \dots, x^N)$ around x. Then according to the tensorial transformation rule, we have

$$|D^\alpha g_{ij}(y)| \leqq C_\alpha \tag{2.3}$$

and

$$|D^\alpha g^{kl}(y)| \leqq C'_\alpha \tag{2.4}$$

for all $y \in U_\varepsilon(x)$, C_α, C'_α independent of x. From the definitions of the Christoffel symbols, (2.3) and (2.4) we immediately obtain

Proposition 2.4. *Let (M^N, g) be of bounded geometry, $x \in M$, $0 < \varepsilon < r_{\mathrm{inj}}(x)$, $(r, u^1, \dots, u^{N-1})$ geodesic polar coordinates. Then there exist constants K_α independent of x such that*

$$|D^\alpha \Gamma^k_{ij}(y)| \leqq K_\alpha \tag{2.5}$$

for all $y \in U_\varepsilon(x)$.

3. The main estimates for the heat kernel construction

Let (M^N, g) be open, complete, oriented and of bounded geometry. We denote by Ω resp. Ω^p_0 the vector space of all smooth p-forms with compact support, by ${}^2\Omega^p$ the vector space of all measurable square integrable p-forms and by $D(\bar{\Delta}) \subset {}^2\Omega^p$ the domain of the closure of the Laplace operator

$$\Delta = d\delta + \delta d \colon \Omega^p_0 \to \Omega^p_0 .$$

Since $\bar{\Delta}$ is nonnegative and selfadjoint, the spectral theorem implies representations

$$\bar{\Delta} = \int_0^\infty \lambda \, dE_\lambda, \quad e^{-t\bar{\Delta}} = \int_0^\infty e^{-t\lambda} \, dE_\lambda .$$

If $e^{-t\bar{\Delta}}$ can be written as an integral operator, the kernel of the latter is called the heat kernel of (M^N, g) for p-forms. One asks then for the properties of the kernel.

An integral kernel always exists according to the Schwartz kernel theorem, but this kernel has on open manifolds no importance since it has no mapping properties between L^q-spaces, $1 \leqq q \leqq \infty$.

Let us now make the definitions precise. A two-point form E^p with values $E^p(t, x, y) \in \wedge^p T_x M \otimes \wedge^p T_y M$ is called a good global heat kernel, if it satisfies the following conditions:

(H1) $E^p(t, x, y)$ is smooth for $t > 0$.

(H2) $(\partial/\partial t + \Delta) E^p(t, x, y) = 0$, where we apply Δ acts on E^p as a section depending on y.

(H3) $\lim\limits_{t \to 0^+} \int\limits_M E^p(t, x, y) \wedge * \omega_0(y) = \omega_0(x)$ for all $x \in M$ and

$$\omega_0 \in \Omega_0^p, \quad \text{i.e.} \quad E^p(t, x, y) \to \delta_{x,y}.$$

(H4) There exist constants $C_1, C_2 > 0$, depending on l, m, n, such that for all $x, y \in M$, $0 < t < \infty$

$$|(\partial/\partial t)^l \nabla^m \nabla^n E^p(t, x, y)| \leqq C_1 t^{-N/2-(m+n)/2-1} \exp\left(-C_2 r^2(x, y)/t\right).$$

(H5) The heat kernels $E^p(t, x, y)$ and $E^{p+1}(t, x, y)$ are related by $\bar{d}_x(E^p(t, x, y) = \bar{\delta}_y E^{p+1}(t, x, y)$.

The main aim of this paper is an existence proof for a good heat kernel, assuming (M^N, g) to be of bounded geometry. The method of proof consists in summing up iterated convolutions of a certain initial expression, where the convergence is guaranteed by some majorization.

Let us start with preparatory lemmas. Assume $0 < \varepsilon < r_{\text{inj}}$, $\Phi \in C_0^\infty(\mathbf{R})$, $\varphi(\alpha) = 1$ for $|\alpha| < \varepsilon/2$ and $\varphi(\alpha) = 0$ for $|\alpha| > 1$. Then we define $\eta \colon M \times M \to \mathbf{R}$ by means of $\eta(x, y) := \eta(r(x, y))$. We define a smooth two-point form ${}^{(1)}E(t, x, y)$ as follows:

$${}^{(1)}E^p(t, x, y) := (4\pi t)^{-N/2} \exp\left(-r^2(x, y)/4t\right) \sum_{i=0}^{k} t^i U_i(x, y) \eta(x, y) =: S^{p_k}(t, x, y) \eta(x, y),$$

where $U_i(x, y)$, $0 \leqq i \leqq k$ are some smooth two-point p-forms, k fixed.

Lemma 3.1. *The two-point forms $U_i(x, y)$, $0 \leqq i \leqq k$, can be choosen such that*

(i) $(\partial/\partial t + \Delta_y) S^{p_k}(t, x, y) = (4\pi t)^{-N/2} t^{k-N/2} \exp\left(-r^2(x, y)/4t\right) \Delta_y U_k(x, y)$.

(ii) *There exists some constant $D_l > 0$ such that for all $0 \leqq i \leqq k$, $0 \leqq l \leqq k$, we have $|\Delta_y^l U_i(x, y)| \leqq D_l$.*

Proof. The two-point forms $U_i(x, y)$, $0 \leqq i \leqq k$, are the classical Hadamard coefficients. Existence, uniqueness, and recursion formulae for the $U_i(x, y)$ are shown in the literature [2, 11, 12]. The calculation of these Hadamard coefficients leads to a system of differential equations of the following form:

$$\big(r(\partial/\partial r)+G+k\big)U_k(x,\cdot\,)=\Delta U_{k-1}(x,\cdot\,),\quad r(x,\cdot\,)<\varepsilon/2,$$
$$U_0(x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}=g_{i_1j_1}(x)\dots g_{i_pj_p}(x),\quad U_{-1}(x,\cdot\,):=0,$$

where G means some matrix function.

An integral recursion formula for the $U_k(x,y)$ is given by

$$U_k(x,y)=-U_0(x,y)\varrho^{-k}\int\limits_0 r^{k-1}U_0(x,z)^{-1}\Delta_y U_{k-1}(x,z)\,dr \tag{3.1}$$

where $\varrho:=r(x,y)$, $r:=r(x,z)$ and z lies on the geodesic connecting x and y.

Assertion (ii) follows from an analoguous integral recursion formula obtained by covariant differentiation of (3.1) and from the assumption of bounded geometry.

Further we set

$${}^{(1)}R^p(t,x,y):=(\partial/\partial t+\Delta_y)\,{}^{(1)}E^p(t,x,y).$$

Now we will estimate $|{}^{(1)}R^p(t,x,y)|$.

Lemma 3.2. *There exist constants $A_1(T)>0$, $A_2(T)>0$ depending on $T>0$ such that for all $0<t\leqq T$, $x,y\in M$*

$$|{}^{(1)}R^p(t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}|\leqq A_1(T)\,t^{k-N/2}\exp\big(-A_2(T)\,r^2(x,y)/t\big). \tag{3.2}$$

Proof. We use the following well-known formula $\Delta(f\Phi)=(\Delta f)\Phi+f\Delta\Phi-$ $-2\nabla_{\operatorname{grad} f}\Phi$, for $f\in C^\infty(M)$, $\Phi\in\Omega^p$. With that we obtain

$$(\partial/\partial t+\Delta_y)S^{p_k}(t,x,y)\eta(x,y)=(4\pi)^{-N/2}t^{k-N/2}\exp\big(-r^2(x,y)/4t\big)\Delta_y U_k(x,y)+$$
$$+S^{p_k}(t,x,y)\Delta_y\eta(x,y)-2\nabla_{\operatorname{grad}\eta}S^{p_k}(t,x,y).$$

The estimation of the first term follows from Lemma 3.1 and our assumption of bounded geometry. In the expressions of the second and third ones there occur factors η' and η'' which are zero for $r(x,y)<\varepsilon/2$. So they decrease exponentially to zero as $t\to+0$. Furthermoore $|\Delta_y\eta(x,y)|$ is uniformly bounded because of bounded geometry. These arguments yield the desired estimations.

Corollary 3.3. *Let (M^n,g) be open, complete and of bounded geometry. Then there exist constants $A_1(T)>0$, $A_2(T)>0$ depending on $T>0$ such that for all $0<t\leqq T$ and $x,y\in M$*

$$|{}^{(1)}R^p(t,x,y)|_{(x,y)}\leqq A_1'(T)\,t^{k-N/2}\exp\big(-A_2(T)\,r^2(x,y)/t\big), \tag{3.3}$$

where $|\cdot|_{(x,y)}$ means the pointwise norm of two-points forms.

Proof. The assertion follows from (3.2) and from the fact that for manifolds of bounded sectional curvature and our choice of coordinate systems the pointwise Riemannian norm and the euclidean norm are equivalent (cf. [9]).

Now we define for two-point forms A^p and B^p their convolution according to

$$A^p * B^p := \int_0^t \int_M A^p(s, x, z) \wedge * B^p(t-s, z, y)\, d_{\mathrm{vol}_z}\, ds,$$

i.e.:

$$A^p * B^p = \sum_{\substack{i_1<\dots<i_p \\ j_1<\dots<j_p}} (A^p * B^p)_{i_1, \dots, i_p, j_1, \dots, j_p}\, dx^{i_1} \wedge \dots \wedge dx^{i_p} \otimes dy^{j_1} \wedge \dots \wedge dy^{j_p}$$

with

$$(A^p * B^p)_{i_1, \dots, i_p, j_1, \dots, j_p} :=$$

$$:= \sum_{k_1<\dots<k_p} \int_0^t \int_M A^p_{i_1, \dots, i_p, k_1, \dots, k_p}(s, x, z) B^{p, k_1, \dots, k_p}_{j_1, \dots, j_p}(t-s, z, y)\, d_{\mathrm{vol}_z}\, ds.$$

We set ${}^{(i)}R^p := {}^{(1)}R^p * \dots * {}^{(1)}R^p$ i-times, and assume $k > N/2$.

Lemma 3.4. *Let be $T>0$ and I a positive integer. Then there exist constants $A_3, A_4>0$ such that for $0<t\leqq T$, $1\leqq i\leqq I$, $x, y\in M$ and all $i_1<\dots<i_p$, $j_1<\dots<j_p$ we have*

$$|{}^{(i)}R^p(t, x, y)_{i_1, \dots, i_p, j_1, \dots, j_p}| \leqq A_3 t^{k-N/2+i-1} \exp\left(-A_4 r^2(x, y)/t\right). \tag{3.4}$$

Proof. We perform mathematical induction. For fixed x by definition of ${}^{(1)}R^p$ the y-support of ${}^{(i)}R^p_{i_1 \dots, i_p, j_1, \dots, j_p}$ is contained in $B_{i\varepsilon}(x)$. We consider $i=2$. Then, denoting $\sum_{(k)} := \sum_{k_1<\dots<k_p}$ we have

$$\begin{aligned} &|{}^{(2)}R^p(t, x, y)_{i_1, \dots, i_p, j_1, \dots, j_p}| = \\ &= \Big|\sum_{(k)} \int_0^t \int_M {}^{(i)}R^p(s, x, z)_{i_1, \dots, i_p, k_1, \dots, k_p}\, {}^{(i)}R^p(t-s, z, y)^{k_1, \dots, k_p}_{j_1, \dots, j_p}\, d_{\mathrm{vol}_x}\, ds\Big| = \\ &= \Big|\int_0^t \int_M \Big[\sum_{(k)} {}^{(i)}R^p(s, x, z)_{i_1, \dots, i_p, k_1, \dots, k_p}\, {}^{(i)}R^p(t-s, z, y)_{i_1, \dots, i_p, j_1, \dots, j_p} \times \\ &\qquad \times g^{k_1 i_1} \dots g^{k_p i_p}\Big]\, d_{\mathrm{vol}_x}\, ds\Big| \leqq A_1^2 D_2\, {}^{(2)}\tilde{R}(t, x, y), \end{aligned} \tag{3.5}$$

with some constant $D_2>0$ and

$$\begin{aligned} {}^{(2)}\tilde{R}(t, x, y) := \int_0^t \int_{B_\varepsilon(x)\cap B_\varepsilon(y)} & s^{k-(N/2)}(t-s)^{k-(N/2)} \exp\left(-A_2 r^2(x, z)/s\right) \times \\ & \times \exp\left(-A_2 r^2(z, y)/(t-s)\right) d_{\mathrm{vol}_z}\, ds. \end{aligned}$$

For the estimation of ${}^{(2)}\tilde{R}(t, x, y)$ we need

Lemma 3.5.

$$r^2(x, z)/s + r^2(z, y)/(t-s) \geqq r^2(x, y)/t + \left(t/s(t-s)\right)[r(x, z) - sr(x, y)/t]^2.$$

For the simple proof which immediately follows from the triangle inequality, we refer to [3].

Lemma 3.5 now implies

$$|{}^{(2)}\tilde{R}(t, x, y)| \leqq \exp(-A_2 r^2(x, y)/t) \int_0^t \int_{B_\varepsilon(x)\cap B_\varepsilon(y)} s^{k-(N/2)}(t-s)^{k-(N/2)} \times$$
$$\times \exp\left[-A_2\left(r(x, z)-r(x, y)(s/t)\right)^2\left(t/s(t-s)\right)\right] d_{\mathrm{vol}_x}\, ds \leqq t^{k-(N/2)+1}[k-(N/2)+1]^{-1} \times$$
$$\times \exp(-A_2 r^2(x, y)/t) \int_0^t \int_{S^{N-1}} \int_0^\varepsilon (t-s)^{k-(N/2)} \times$$
$$\times \exp\left[-A_2\left(r(x, z)-r(x, y)(s/t)\right)^2\left(t/s(t-s)\right)\right] \Theta_x(z)\, dr_x\, du_x\, ds,$$

where (r_x, u_x) are the geodesic polar coordinates of $z \in B_\varepsilon(x)$ and $\Theta_x(z) :=$ $:= (\det g_{ij})^{1/2}(z)$. According to the Rauch comparison theorem and our assumption of bounded geometry there exists a constant $D_3 > 0$ independent of x such that $|\Theta_x(z)| \leqq D_3$ for all $z \in B_\varepsilon(x)$.

We set $\varrho := r(x, y)$, $r := r(x, z)$. Then there remains the integral $I(s) :=$ $:= \int_0^\varepsilon \exp[-A_2 t(r-(s/t)\varrho)^2 s(t-s)]\, dr$ to be estimated. But $I(s)$ decreases at least like $s^{N/2}$ resp. $(t-s)^{N/2}$ for $s \to +0$ resp. $s \to -t$. For $|{}^{(2)}\tilde{R}(t, x, y)|$ this implies the estimate

$$|{}^{(2)}\tilde{R}(t, x, y)| \leqq t^{k-(N/2)+1}[k-(N/2)+1]^{-1} \exp(-A_2 r^2(x, y)/t) D_3 \Big(\int_{S^{N-1}} du\Big) \times$$
$$\times \int_0^t s^{k-(N/2)} I(s)\, ds \leqq D_3 D_4 D_5 t^{k-(N/2)+1}[k-(N/2)+1]^{-1} \exp(-A_2 r^2(x, y)/t),$$

where

$$\int_{S^{N-1}} du \leqq D_4$$

and

$$\int_0^t (t-s)^{k-(N/2)} I(s)\, ds \leqq D_5.$$

Using these estimates and (3.5), we obtain for $0 < t \leqq T$ and $x, y, \in M$

$$(3.6) \qquad |{}^{(2)}R^p(t, x, y)_{i_1, \ldots, i_p, j_1, \ldots, j_p}| \leqq$$
$$\leqq A_1^2 D_2 D_3 D_4 D_5 t^{k-(N/2)+1}[k-(N/2)+1]^{-1} \exp(-A_2 r^2(x, y)/t).$$

Using the estimate of Corollary 3.3, and its iteration, we obtain

$$(3.7) \qquad |{}^{(i)}R(t, x, y)_{i_1, \ldots, i_p, j_1, \ldots, j_p}| \leqq A_1^i D_2^{i-1}\, {}^{(i)}\tilde{R}(t, x, y),$$

where ${}^{(i)}\tilde{R}(t, x, y)$ is the i-fold convolution of $t^{k-(N/2)} \exp(-A_2 r^2(x, y)/t)$.

In the sequel we need also an estimate for ${}^{(i)}\tilde{R}(t, x, y)$, which we establish again by mathematical induction, namely

$$(3.8)\qquad |{}^{(i)}\tilde{R}(t, x, y)| \leqq (D_3 D_4 D_5)^{i-1} t^{k-(N/2)+i-1} \times$$
$$\times \exp\left(-A_2 r^2(x, y)/t\right)[k-(N/2)+1]^{-1} \dots [k-(N/2)+i-1]^{-1}.$$

For $i=2$ this is already proved. The induction step $i \to i+1$ shall be done below. Assuming (3.8) for a moment, we obtain from the induction assumption and the estimation of the first i convolution factors of ${}^{(i+1)}\tilde{R}(t, x, y)$

$$|{}^{(i+1)}\tilde{R}(t, x, y)| \leqq (D_3 D_4 D_5)^{i-1}[k-(N/2)+1]^{-1} \dots [k-(N/2)+i-1]^{-1} \times$$
$$\times \int_0^t \int_{B_{i\varepsilon}(x) \cap B_\varepsilon(y)} (t-s)^{k-(N/2)+i-1} s^{k-(N/2)} \exp\left(-A_2 r^2(x, z)/s\right) \times$$
$$\times \exp\left(-A_2 r^2(z, y)/(t-s)\right) d_{\mathrm{vol}_z}\, ds.$$

Denoting the last integral by J, we get

$$|J| \leqq \int_0^t \int_{S^{N-1}} \int_0^\varepsilon (t-s)^{k-(N/2)+i-1} s^{k-(N/2)} \exp\left(-A_2 r^2(x, z)/s\right) \times$$
$$\times \exp\left(-A_2 r^2(z, y)/(t-s)\right) \Theta_x(z)\, dr\, du\, ds \leqq D_3 D_4 t^{k-(N/2)+i}[k-(N/2)+i]^{-1} \times$$
$$\times \exp\left(-A_2 r^2(x, y)/t\right) \int_0^t \int_0^\varepsilon s^{k-(N/2)} \exp\left(-A_2 t\left(r-(s/t)\varrho\right)^2/s(t-s)\right) dr\, ds \leqq$$
$$\leqq D_3 D_4 D_5 t^{k-(N/2)+1}[k-(N/2)+i]^{-1} \exp\left(-A_2 r^2(x, y)/t\right).$$

This finishes the induction for ${}^{(i+1)}\tilde{R}(t, x, y)$ and shows

$$|{}^{(i+1)}R^p(t, x, y)_{i_1, \dots, i_p, j_1, \dots, j_p}| \leqq A_1^{i+1}(D_2 D_3 D_4 D_5)^i \times$$
$$\times \exp\left(-A_2 r^2(x, y)/t\right) t^{k-(N/2)+i}[k-(N/2)+1]^{-1} \dots [k-(N/2)+i]^{-1}.$$

Furthermore, there exist constants $A_3, A_4 > 0$ such that for $0 < t \leqq T$, $2 \leqq i \leqq I$

$$(3.9)\qquad A_1^i (D_2 D_3 D_4 D_5)^{i-1} \exp\left(-A_2 r^2(x, y)/t\right)[k-(N/2)+1]^{-1} \dots$$
$$\dots [k-(N/2)+i-1]^{-1} \leqq A_3 \exp\left(-A_4 r^2(x, y)/t\right)$$

which finishes the proof of the Lemma.

Lemma 3.6. *For every* $T > 0$ *there exist contstants* $A_5, A_6 > 0$ *such tath*

$$(3.10)\quad |{}^{(2m)}R^p(t, x, y)_{i_1, \dots, i_p, j_1, \dots, j_p}| \leqq A_5 A_6^m t^{k-(N/2)+2m-1}(m!)^{-1} \exp\left(-A_2 r^2(x, y)/t\right)$$

for all $0 < t \leqq T$ *and all positive integers* m.

Proof. The calculations in the proof of Lemma 3.4 give for $i=2m$

$$|^{(2m)}R^p(t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq A_1^{2m}(D_2D_3D_4D_5)^{2m-1}\times$$
$$\times t^{k-(N/2)+2m-1}[k-(N/2)+1]^{-1}\dots[k-(N/2)+2m-1]^{-1}\exp\big(-A_2r^2(x,y)/t\big).$$

Using

$$[k-(N/2)+1]\dots[k-(N/2)+2m-1] > m!(m+1)\dots(2m-1)$$

we obtain

$$A_5A_6^m(m!)^{-1}t^{k-(N/2)+2m-1}\exp\big(-A_2r^2(x,y)/t\big),$$

as an upper bound for the right-hand side of (3.10) (where $A_5, A_6>0$).

Lemma 3.7. *For every $T>0$ there exist constants $A_7, A_8>0$ such that*

(3.11) $$|^{(2m+1)}R^p(t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq A_7A_8^m(m!)^{-1}t^{k-(N/2)+2m}\exp\big(-A_2r^2(x,y)/t\big)$$

for all $0<t\leqq T$ and all positive integers m.

Proof.

$$|^{(2m+1)}R^p(t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq A_1^{2m+1}(D_2D_3D_4D_5)^{2m}(m!)^{-1}t^{k-(N/2)+2m}\times$$
$$\times\exp\big(-A_2r^2(x,y)/t\big) \leqq A_7A_8^m(m!)^{-1}t^{k-(N/2)+2m}\exp\big(-A_2r^2(x,y)/t\big).$$

Let us define $Q^p := \sum\limits_{i=1} (-1)^{i\,(i)}R^p$, i.e.

$$Q^p(t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p} = \sum_{i=1}(-1)^{i\,(i)}R^p(t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}.$$

Lemma 3.8. *For all $T>0$ the series for $Q^p_{i_1,\dots,i_p,j_1,\dots,j_p}$ converges absolutely and uniformly. There exist constants $A_9, A_{10}>0$, depending on T, such that*

(3.12) $$|Q^p(t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq A_9t^{k-(N/2)}\exp\big(-A_{10}r^2(x,y)/t\big)$$

for all $0<t\leqq T$, $x,y\in M$, $i_1<\dots<i_p$, $j_1<\dots<j_p$.

Proof. The convergence follows from the preceding two lemmas since $Q^p_{i_1,\dots,i_p,j_1,\dots,j_p}$ can be estimated from above by an exponential series. Furthermore, we obtain from (3.2), (3.4), (3.10), (3.11):

$$|^{(1)}R^p(t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq A_1t^{k-(N/2)}\exp\big(-A_2r^2(x,y)/t\big),$$
$$\sum_{m=1}|^{(2m)}R^p(t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq A_5\exp\big(-A_2r^2(x,y)/t\big)\times$$
$$\times\Big[\sum_{m=1}(A_6^m/m!)t^{k-(N/2)+2m-1}\Big] \leqq D_5\exp\big(-D_6r^2(x,y)/t\big),$$
$$\sum_{m=1}|^{(2m+1)}R^p(t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq A_7\exp\big(-A_2r^2(x,y)/t\big)\times$$
$$\times\Big[\sum_{m=1}(A_8^m/m!)t^{k-(N/2)+2m}\Big] \leqq D_7\exp\big(-D_8r^2(x,y)/t\big),$$

where D_5, D_6, D_7, D_8 are positive constants. This provides the asserted estimate.

4. The main results

We set $E^p := {}^{(1)}E^p - Q^p * {}^{(1)}E^p$ and show that E^p is the asserted heat kernel.

Main Theorem 4.1. *Let (M^N, g) be open, complete, and of bounded geometry. Then there exists a good global heat kernel $E^p(t, x, y)$ satisfying the conditions* (H1)–(H4), *and*

(H5) $E^p(t, x, y) = E^p(t, y, x)$ *for all* $x, y \in M$ *(symmetry),*

(H6) $E^p(t+s, x, y) = \int\limits_M E^p(t, x, z) \wedge * E^p(s, z, y)$ *(semigroup property).*

Moreover, E^p is uniquely determined.

Proof. (H1) Smoothness is a local property and it is sufficient to establish it for all compact subsets. The kernels ${}^{(1)}E^p$ and ${}^{(1)}R^p$ are smooth by construction. On compact subsets one can differentiate ${}^{(1)}R^p$ under the integral sign, thus establishing the smoothness of ${}^{(1)}R^p$. Also on compact subsets the series for Q^p and its derivatives converge uniformly according to the estimates of Section 3.

(H2) Using, once again, the argument of uniform convergence, we obtain for $0 < t \leqq T$ and $k > N/2+2$ (c.f. [2])

$$(\partial/\partial t + \Delta) E^p = (\partial/\partial t + \Delta)({}^{(1)}E^p - Q^p * {}^{(1)}E^p) = {}^{(1)}R^p - Q^p - Q^p * {}^{(1)}R^p =$$
$$= {}^{(1)}R^p - \sum_{i=1} (-1)^i \, {}^{(1)}R^p - \sum_{i=2} (-1)^i \, {}^{(1)}R^p = 0.$$

(H3) For $\omega_0 \in \Omega_0^p$ there holds

$$\int\limits_M E^p(t, x, y) \wedge * \omega_0(y) = \sum_{i_1 < \ldots < i_p} \Big[\int\limits_M E^p(t, x, y) \wedge * \omega_0(y) \Big]_{i_1, \ldots, i_p, j_1, \ldots, j_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p},$$

where

$$\Big[\int\limits_M E^p(t, x, y) \wedge * \omega_0(y) \Big]_{i_1, \ldots, i_p, j_1, \ldots, j_p} =$$
$$= \sum_{(k)} \int\limits_M E^p(t, x, y)_{i_1, \ldots, i_p, k_1, \ldots, k_p} \omega_0(y)^{k_1, \ldots, k_p} d_{\mathrm{vol}_y} =$$
$$= \sum_{(k)} \int\limits_M {}^{(1)}E^p(t, x, y)_{i_1, \ldots, i_p, k_1, \ldots, k_p} \omega_0(y)^{k_1, \ldots, k_p} d_{\mathrm{vol}_y} -$$
$$- \sum_{(k)} \int\limits_M (Q^p * {}^{(1)}E^p)(t, x, y)_{i_1, \ldots, i_p, k_1, \ldots, k_p} \omega_0(y)^{k_1, \ldots, k_p} d_{\mathrm{vol}_y}.$$

Introducing normal coordinates centered at x, the formula

$$\lim_{t \to 0^+} \int\limits_M \sum_{(k)} {}^{(1)}E^p(t, x, y)_{i_1, \ldots, i_p, k_1, \ldots, k_p} \omega_{0, l_1, \ldots, l_p}(y) \times$$
$$\times g^{k_1 l_1}(y) \ldots g^{k_p l_p}(y) \, d_{\mathrm{vol}_y} = \omega_{0, i_1, \ldots, i_p}(x)$$

follows just like in the Euclidean case. The estimate (3.12) for Q^p gives

$$(4.1)\qquad |t^{-1}Q^p(t, x, y)_{i_1,\dots,i_p, j_1,\dots,j_p}| \leqq A_9 t^{k-(N/2)+1}\exp\left(-A_{10}r^2(x, y)/t\right).$$

The calculation $Q^p *^{(1)}E^p = t(t^{-1}Q^p *^{(1)}E^p)$, use of (4.1) and a version of Lemma 3.5 lead to an estimation from above by

$$A_9 t\int_0^t\int_M s^{k-(N/2)+1}(t-s)^{k-(N/2)}\exp\left(-A_{10}r^2(x, z)/s\right)\exp\left(-A_2 r^2(z, y)/(t-s)\right)d_{\mathrm{vol}_z}\,ds \leqq$$

$$\leqq t\int_0^t\int_M \exp\left(-A_{11}r^2(x, y)/t\right)s^{k-(N/2)+1}(t-s)^{k-(N/2)}\times$$

$$\times\exp\left[-A_{12}\left(r(x, z)-r(x, y)(s/t)\right)^2\left(t/s(t-s)\right)\right]d_{\mathrm{vol}_z}\,ds$$

where A_{11}, A_{12} are positive constants. Therefore we have for $\omega_0\in\Omega_0^p$

$$\left|t\sum_{(k)}\int_M (t^{-1}Q^p *^{(1)}E^p)_{i_1,\dots,i_p, k_1,\dots,k_p}\,\omega_0(y)^{k_1,\dots,k_p}\,d_{\mathrm{vol}_y}\right| \leqq$$

$$\leqq t\Big\{A_9\int_M \exp\left(-A_{11}r^2(x, y)/t\right)\int_0^t\int_M s^{k-(N/2)+1}(t-s)^{k-(N/2)}\times$$

$$\times\exp\left[-A_{12}\left(r(x, z)-r(x, y)(s/t)\right)^2\left(t/s(t-s)\right)\right]\max_{k_1<\dots<k_p}|\omega_0(y)^{k_1,\dots,k_p}|\,d_{\mathrm{vol}_z}\,ds\,dy\Big\}.$$

Since $\operatorname{supp}\omega_0$ is compact, we can cover it by a finite number of ε-balls, $\varepsilon<r_{\mathrm{inj}}$, and apply for the estimation of all three integrals the estimates of Section 3. Thus we prove that the expression $\{\dots\}$ remains bounded as $t\to 0^+$ and $\lim\limits_{t\to 0^+}\int_M (Q^p *^{(1)}E^p)\wedge\wedge *\omega_0(y)=0$. (H3) is proved.

(H6) In order to show the semigroup property

$$E^p(t, x, y) = \int_M E^p(s, x, z)\wedge *E^p(t-s, z, y)$$

we prove that

$$F^p(t, x, y) := \int_M E^p(s, x, z)\wedge *E^p(t-s, z, y)$$

has the properties of a heat kernel. The uniqueness theorem of [5] then ensures $F^p(t, x, y)=E^p(t, x, y)$. In fact, from (H2) for $E^p(t, x, y)$ we obtain

$$(\partial/\partial t+\Delta_y)F^p(t, x, y) = \int_M E^p(s, x, z)\wedge *E^p(t-s, z, y)+$$

$$+\int_M E^p(s, x, z)\wedge *\Delta_y E^p(t-s, z, y) = \int_M E^p(s, x, z)\wedge *(-\Delta_y)E^p(t-s, z, y)+$$

$$+E^p(s, z, x)\wedge *\Delta_y E^p(t-s, z, y) = 0.$$

Property (H3) of $E^p(t, x, y)$ implies

$$\lim_{t\to 0^+} F^p(t, x, y) = \lim_{t\to 0^+} \int_M E^p(s, x, z)\wedge * E^p(t-s, z, y) = \delta_{x,z}\delta_{z,y} = \delta_{x,y}$$

since $t\to 0^+$ implies $s\to 0^+$, $t-s\to 0^+$

(H7) (Symmetry). Using property (H3), we get

$$\lim_{s\to 0^+} \int_M E^p(s, x, z)\wedge * E^p(t-s, y, z) = E^p(t, y, x),$$

$$\lim_{s\to t^-} \int_M E^p(s, x, z)\wedge * E^p(t-s, y, z) = E^p(t, x, y).$$

Then, according to Duhamel's principle for forms,

$$E^p(t, x, y) - E^p(t, y, x) = \int_0^t \partial/\partial s \int_M E^p(s, x, z)\wedge * E^p(t-s, y, z) =$$

$$= \int_0^t \int_M [\Delta_z E^p(s, x, z)\wedge * E^p(t-s, y, z) - E^p(s, x, z)\wedge * \Delta_x E^p(t-s, y, z)] =$$

$$= \int_0^t \int_M \{\bar{d}_z E^p(s, x, z)\wedge * \bar{d}_z E^p(t-s, y, z) + \bar{\delta}_z E^p(s, x, z)\wedge * \bar{\delta}_z E^p(t-s, y, z) -$$

$$-[\bar{d}_z E^p(s, x, z)\wedge * \bar{d}_z E^p(t-s, y, z) + \bar{\delta}_z E^p(s, x, z)\wedge * \bar{\delta}_z E^p(t-s, y, z)]\} = 0.$$

Here we essentially used the completeness of (M^N, g).

(H4). (Estimates for the derivatives.) Assume $T>0$, $0<t\leqq T$ and $k>N/2+$ $+(m+n)/2+1$, and begin with $l=m=n=0$. Then the proof is done according to the estimates for ${}^{(1)}E$ and Q^p. Next we consider $\nabla_y^n E^p(t, x, y)$. There holds

$$|\nabla_y^n E^p(t, x, y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq |\nabla_y^n {}^{(1)}E^p(t, x, y)_{i_1,\dots,i_p,j_1,\dots,j_p}| +$$
$$+ |\nabla_y^n (Q^p * {}^{(1)}E^p)(t, x, y)_{i_1,\dots,i_p,j_1,\dots,j_p}|.$$

We start with the estimation of the first term.

Lemma 4.2. *There exist positive constants A_{13}, $A_{14}(n, T)$ such that for all $0<t\leqq$ $\leqq T$, $x, y\in M$, $i_1<\dots<i_p$, $j_1<\dots<j_p$*

$$|\nabla_y^n {}^{(1)}E^p(t, x, y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq A_{13} t^{-N/2-n/2} \exp(-A_{14} r^2(x, y)/t). \tag{4.2}$$

The proof is by mathematical induction. For $n=0$ it is done. Assume the assertion for $1,\dots,n-1$. Since (M^N, g) has bounded geometry, there exists a uniformly locally finite cover of M^N by geodesic ε-balls, $0<\varepsilon<r_{\mathrm{inj}}$. According to [1] there exists a constant $D_9>0$, independent of x, such that for all $y\in B_\varepsilon(x)$

$$\left|\nabla_y^n \exp(-r^2(x, y)/4t)\right| \leqq D_9 \left|(\partial^n/\partial r^n) \exp(-r^2(x, y)/4t)\right|. \tag{4.3}$$

This follows from Lemma 3 of [1] and the boundedness of the Christoffel symbols together with their derivatives. Using the inequality $e^{-\alpha} \leqq (\alpha e)^{-1}$, we obtain

$$|\partial/\partial r \exp(-r^2/4t)| \leqq [\exp(-r^2/4t)(r^2/4t)]^{1/2} \exp(-r^2/8t) \leqq e^{-1} t^{-1/2} \exp(-r^2/8t).$$

Iteration and application of the product rule gives

$$|(\partial^n/\partial r^n) \exp(-r^2/4t)| \leqq D_{10} t^{-N/2-n/2} \exp(-D_{11} r^2/t) \tag{4.4}$$

with positive constants D_{10}, $D_{11}(n, T)$. According to Lemma 4 of [1] there exists a constant $D_{12}>0$ such that

$$|\nabla_y^n(r(x, y))| \leqq D_{12}. \tag{4.5}$$

Lemma 3.1 (ii), (4.3)—(4.5) and the derivation rules applied to ${}^{(1)}E^p$ provide the asserted estimation.

In order to estimate the second term, we use the uniform convergence of the integrals and can therefore differentiate under the integral sign:

$$|\nabla_y^n(Q^p * {}^{(1)}E^p)(t, x, y)_{i_1, \dots, i_p, j_1, \dots, j_p}| \leqq \int_0^t \int_M \sum_{(k)} |Q^p(s, x, z)_{i_1, \dots, i_p, k_1, \dots, k_p}| \times$$

$$\begin{aligned} &\times |\nabla_y {}^{(1)}E^p(t-s, z, y)_{i_1, \dots, i_p, j_1, \dots, j_p}| \, |g^{k_1 i_1}(z)| \dots |g^{k_p i_p}(z)| \, d_{\mathrm{vol}_z} \, ds \leqq \\ &\leqq D_{13} t^{k-N/2-n/2} \exp\left(-D_{14} r^2(x, y)/t\right), \quad D_{13}, D_{14}(n, T) > 0. \end{aligned} \tag{4.6}$$

Now (4.2) and (4.6) provide the asserted estimate. From the symmetry of $E^p(t, x, y)$ in x, y we obtain the analogous estimate

$$|\nabla_x^m E^p(t, x, y)_{i_1, \dots, i_p, j_1, \dots, j_p}| \leqq A_{15} t^{-N/2-m/2} \exp\left(-A_{16} r^2(x, y)/t\right), \quad A_{15}, A_{16}(m, T) > 0.$$

We now turn to t-derivatives of E^p. Clearly,

$$\begin{aligned} |(\partial^l/\partial t^l) E^p(t, x, y)_{i_1, \dots, i_p, j_1, \dots, j_p}| &\leqq |(\partial^l/\partial t^l) {}^{(1)}E^p(t, x, y)_{i_1, \dots, i_p, j_1, \dots, j_p}| + \\ &+ |(\partial^l/\partial t^l)(Q^p * {}^{(1)}E^p)(t, x, y)_{i_1, \dots, i_p, j_1, \dots, j_p}|. \end{aligned}$$

We start with the first term. Since

$${}^{(1)}E^p = f_1(\eta) \sum_{i=0}^{k} t^i U_i(x, y),$$

and the U_i and η are independent of t, it suffices to estimate $(\partial^l/\partial t^l) f_1$:

$$f_1(t, x, y) := (4\pi t)^{-N/2} \exp\left(-r^2(x, y)/4t\right),$$

$$(4\pi)^{N/2} (\partial/\partial t)\left(t^{-N/2} \exp\left(-r^2(x, y)/4t\right)\right) = \left(-(N/2) t^{-N/2-1} + t^{-N/2}(r^2/4t^2)\right) \exp(-r^2/4t).$$

Use of the inequality $e^{-\alpha} \leqq (\alpha e)^{-1}$, like in the proof of Lemma 4.2, gives

$$t^{-N/2} \exp(-r^2/4t)(r^2/4t^2) \leqq 2 e^{-1} t^{-N/2-1} \exp(-r^2/8t).$$

Iterating this procedure and applying the product rule, we obtain

$$|(\partial^1/\partial t^1)\,{}^{(1)}E^p(t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq$$
$$\leqq D_{15}\,t^{-N/2-l}\exp\bigl(-D_{16}r^2(x,y)/t\bigr),\quad D_{15},D_{16}(l,T)>0.$$

We estimate the second term as follows:

$$|(\partial^l/\partial t^l)(Q^p*{}^{(1)}E^p)(t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq \Bigl|(\partial^l/\partial t^l)\sum_{(k)}\int_0^t\int_{B_\varepsilon(y)}(4\pi)^{-N/2}(t-s)^{-N/2}\times$$
$$\times\exp\bigl(-r^2(z,y)/4(t-s)\bigr)\sum_{i=0}^k t^i U_{i,i_1,\dots,i_p,k_1,\dots,k_p}(z,y)\times$$
$$\times Q^p(s,x,z)_{l_1,\dots,l_p,j_1,\dots,j_p}\,g^{k_1l_1}(z)\dots g^{k_pl_p}(z)\,d_{\mathrm{vol}_z}\,ds\Bigr|\leqq$$
$$\leqq\sum_{(k)}\Bigl|(\partial^{l-1}/\partial t^{l-1})\Bigl\{\Bigl[\lim_{s\to t^-}\int_{B_\varepsilon(y)}-\lim_{s\to 0^+}\int_{B_\varepsilon(y)}\Bigr](4\pi)^{-N/2}(t-s)^{-N/2}\exp\bigl(-r^2(z,y)/4(t-s)\bigr)\times$$
$$\times\sum_{i=0}^k t^iU_{i,i_1,\dots,i_p,k_1,\dots,k_p}\,Q^p(s,x,z)_{l_1,\dots,l_p,j_1,\dots,j_p}\,g^{k_1l_1}(z)\dots g^{k_pl_p}(z)\,d_{\mathrm{vol}_z}\,ds\Bigr\}\Bigr|\leqq$$
$$\leqq D_{17}\,t^{k-N/2-1}\exp\bigl(-D_{18}r^2(x,y)/t\bigr),\quad D_{17},D_{18}(l,T)>0.$$

Gathering the results, we have

$$|\partial^l/\partial t^l\,E^p(t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}|\leqq A_{17}\,t^{-N/2-1}\exp\bigl(-A_{18}r^2(x,y)/t\bigr),$$
$$A_{17},A_{18}(l,T)>0.$$

Iterating the derivatives ∇_x^m, ∇_y^n, $\partial^l/\partial t^l$, using the above estimates and the fact that $\nabla_y^i(x,y)$ is bounded thanks to the bounded geometry, we finally obtain the asserted estimate for $0<t\leqq T$. In order to establish (H4) completely, we have still to consider the behaviour of the derivatives for $t\to\infty$. To do this, we essentially use the semi-group property (H6). Until now we have proved

$$|(\partial^l/\partial t^l)\nabla_x^m E^p(2t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}|\leqq$$
$$\leqq\int_M C_{1,m,l}(T)\,t^{-N/2-m/2-1}\exp\bigl(-C_{2,m,l}(T)\,r^2(x,z)/t\bigr)C_{1,0,0}(T)\,t^{-N/2}\times$$
$$\times\exp\bigl(-C_{2,0,0}(T)\,r^2(z,y)/t\bigr)\,d_{\mathrm{vol}_z}.$$

Without loss of generality we assume $C_{2,m,1}(T)>C_{2,0,0}(T)$. Lemma 3.5 now gives

$$|(\partial^l/\partial t^l)\nabla_x^m E^p(2t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}|\leqq C_{1,m,l}(T)C_{1,0,0}(T)\,t^{-N-m/2-1}\times$$
$$\times\exp\bigl(-C_{2,0,0}(T)\,r^2(x,y)/2t\bigr)\int_M\exp\bigl[\bigl(-C_{2,0,0}(T)/t\bigr)(2/1)\bigl(r(x,z)-r(x,y)/2\bigr)^2\bigr]\,d_{\mathrm{vol}_z}.$$

We denote the latter integral by $I_2(t)$ and state as induction assumption

$$|(\partial^l/\partial t^l)\nabla_x^m E^p(\tilde{k}t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}|\leqq C_{1,m,l}(T)C_{1,0,0}(T)^{\tilde{k}-1}\times$$
$$\times t^{-m/2-1}(t^{-N/2})^{\tilde{k}}\exp\bigl(-C_{2,0,0}(T)\,r^2(x,y)/t\bigr)I_2(t)\dots I_{\tilde{k}}(t),$$

where

$$I_{\tilde{k}}(t) := \int_M \exp\left[-C_{2,0,0}(T)/t\big(\tilde{k}/(\tilde{k}-1)\big)\big(r(x,z)-((\tilde{k}-1)/\tilde{k})\,r(x,y)\big)^2\right] d_{\mathrm{vol}_z}.$$

From the semigroup property we obtain

$$\begin{aligned}
&\left|(\partial^l/\partial t^l)\nabla_x^m E^p\big((\tilde{k}+1)\,t,x,y\big)_{i_1,\dots,i_p,j_1,\dots,j_p}\right| \leqq \\
&\leqq \int_M C_{1,m,l}(T)\,C_{1,0,0}(T)^{\tilde{k}-1}\,t^{-m/2-1}\,(t^{-N/2})^{\tilde{k}}\times \\
&\times \exp\big(-C_{2,0,0}(T)\,r^2(x,z)/\tilde{k}t\big)\,I_2(t)\dots I_{\tilde{k}}(t)\,C_{1,0,0}(T)\times \\
&\times \exp\big(-C_{2,0,0}(T)\,r^2(z,y)/t\big)\,t^{-N/2}\,d_{\mathrm{vol}_z} \leqq C_{1,m,l}(T)\big(C_{1,0,0}(T)\big)^{\tilde{k}}\,t^{-m/2-1}\,(t^{-N/2})^{\tilde{k}+1}\times \\
&\times \exp\big(-C_{2,0,0}(T)\,r^2(x,y)/t\big)\,I_2(t)\dots I_{\tilde{k}}(t)\times \\
&\times \int_M \exp\left[\big(-C_{2,0,0}(T)/t\big)\big((\tilde{k}+1)/\tilde{k}\big)\big(r(x,z)-(\tilde{k}/(\tilde{k}+1))\,r(x,y)\big)^2\right] d_{\mathrm{vol}_z} \leqq \\
&\leqq C_{1,m,l}(T)\big(C_{1,0,0}(T)\big)^{\tilde{k}}(t^{-N/2})^{\tilde{k}+1}\,I_2(t)\dots I_{\tilde{k}}(t)\,I_{\tilde{k}+1}(t)\times \\
&\times t^{-m/2-1}\exp\big(-C_{2,0,0}(T)\,r^2(x,y)/t\big),
\end{aligned}$$

where we denoted the last integral by $I_{\tilde{k}+1}(t)$. There exist constants $D_{19}>0$, $k_0>0$, such that for all $\tilde{k}>k_0$

$$t^{-m/2-1}(t^{-N/2})^{\tilde{k}-1} \leqq D_{19}(\tilde{k}t)^{-N/2-m/2-1}.$$

Moreover,

$$I_{\tilde{k}}(t) \to 0 \quad \text{for} \quad t \to 0, \quad I_{\tilde{k}}(t) \to I(t) \quad \text{for} \quad \tilde{k} \to \infty,$$

where

$$I(t) := \int_M \exp\left[-C_{2,0,0}(T)/t\big(r(x,z)-r(x,y)\big)^2\right] d_{\mathrm{vol}_z}.$$

This implies the existence of a constant $k_1>0$ and $t_0 \in]0, T]$ such that for all $0<t\leqq t_0$ and $\tilde{k}>k_0$, $\tilde{k}>k_1$

$$|I_{\tilde{k}}(t)| \leqq \big(C_{1,0,0}(T)\big)^{-1}.$$

Therefore, there exists a constant $D_{18}>0$, dependent on t_0, k_0, k_1, such that for all $0<t\leqq t_0$, $x, y\in M$, $\tilde{k}>k_0$, $\tilde{k}>k_1$

$$\begin{aligned}
&|(\partial^l/\partial t^l)\nabla_x^m E^p(kt,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq \\
&\leqq D_{18}(\tilde{k}t)^{-N/2-m/2-1}\exp\big(-C_{2,0,0}(T)\,r^2(x,y)/t\big).
\end{aligned}$$

Since $t\in]0, t_0[$ was arbitrary, we have the estimation for arbitrary large $\tilde{k}t$.

In a similar manner we estimate $|\nabla_y^n E^p(t, x, y)_{i_1,\dots,i_p,j_1,\dots,j_p}|$. We have for $0<t\leqq T$

$$|\nabla_y^n E^p(2t, x, y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq \int_M C_{1,0}(T)\, t^{-N/2} \times$$

$$\times \exp\left(-C_{2,0}(T)\, r^2(x, z)/t\right) C_{1,n}(T)\, t^{-N/2-n/2} \exp\left(-C_{2,n}(T)\, r^2(z, y)/t\right) d_{\mathrm{vol}_z} \leqq$$

$$\leqq C_{1,n}(T)\, C_{1,0}(T)\, (t^{-N/2})^2\, t^{-n/2} \exp\left(-C_{2,0}(T)\, r^2(x, y)/t\right) \times$$

$$\times \int_M \exp\left[-C_{2,0}(T)/t(2/1)\left(r(x, z)-(1/2)\, r(x, y)\right)^2\right] d_{\mathrm{vol}_z}.$$

The last integral shall be denoted by $I_2(t)$. Furthermore, we assumed without loss of generality $C_{2,n}(T)>C_{2,0}(T)$. This implies

$$|\nabla_y^n E(t, x, y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq C_{1,0}(T)\, C_{1,0}(T)\, I(t)(t^{-N/2})^2 \times$$

$$\times t^{-n/2} \exp\left(-C_{2,0}(T)\, r^2(x, y)/2t\right).$$

By mathematical induction,

$$|\nabla_y^n E^p(\tilde{k}t, x, y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq C_{1,n}(T)\left(C_{1,0}(T)\right)^{\tilde{k}-1} I_2(t) \dots$$

$$\dots I_{\tilde{k}}(t)(t^{-N/2})^{\tilde{k}}\, t^{-n/2} \exp\left(-C_{2,0}(T)\, r^2(x, y)/\tilde{k}t\right).$$

There exist constants $t_0, k_0, k_1, D_{19}>0$, D_{19} dependent on t_0, k_0, k_1, such that for all $0<t\leqq t_0$, $\tilde{k}>k_0$, $\tilde{k}>k_1$

$$|\nabla_y^n E^p(t, x, y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq D_{19}(\tilde{k}t)^{-N/2-n/2} \exp\left(-C_{2,0}(T)\, r^2(x, y)/\tilde{k}t\right).$$

The time value $t\in]0, t_0[$ was arbitrary and we obtain the estimate for arbitrary large $\tilde{k}t$. Iterating both estimates and using once again the boundedness of $\nabla^l r$, we finally obtain (H4).

(H5). $$\bar{d}_x E^p(t, x, y) = \bar{\delta}_y E^{p+1}(t, x, y).$$

The property (H3), $\lim_{t\to 0^+} E^p(t, x, y)=\delta_{x,y}$, implies

$$\lim_{s\to 0^+} \int_M E^{p+1}(s, x, z) \wedge * \bar{d}_z E^p(t-s, z, y) = \bar{d}_x E^p(t, x, y),$$

$$\lim_{s\to t^-} \int_M \bar{\delta}_z E^{p+1}(s, x, z) \wedge * E^p(t-s, z, y) = \bar{\delta}_y E^{p+1}(t, x, y).$$

Using this, Duhamel's principle and the heat equation, we obtain

$$\bar{d}_x E^p(t,x,y)-\bar{\delta}_y E^{p+1}(t,x,y)=$$

$$=\int_0^t (\partial/\partial s)\int_M E^{p+1}(s,x,z)\wedge *\bar{d}_z E^p(t-s,z,y)=$$

$$=\int_0^t\int_M [\bar{\Delta}_z E^{p+1}(s,x,z)\wedge *\bar{d}_z E^p(t-s,z,y)-E^{p+1}(s,x,z)\wedge *\bar{d}_z\bar{\Delta}_z E^p(t-s,z,y)]=$$

$$=\int_0^t\int_M [\bar{\delta}_z E^{p+1}(s,x,z)\wedge *\bar{\delta}_z\bar{d}_z E^p(t-s,z,y)-$$

$$-\bar{d}_z\bar{\delta}_z E^{p+1}(s,x,z)\wedge *\bar{d}_z E^p(t-s,z,y)]=0.$$

since (M^N, g) is complete.

This finishes the proof of the main Theorem 4.1.

As it is well known, the existence of a good heat kernel has many good consequences in global analysis, for instance in spectral theory, in the theory of semigroups, and for the existence of characteristic numbers. We do not intend to present all this here, but restrict ourselves to a special class of applications. For many purposes one is interested to invert the Laplace operator Δ outside the L_2-harmonic forms. Denote by ${}^2\Omega^p$ the space of square integrable measurable p-forms on M^N, by ${}^2\Omega^p_k$ the set of all smooth p-forms ω such that $\|\omega\|, \|\Delta\omega\|, \ldots, \|\Delta^k\omega\|<\infty$ ($\|\cdot\|=$ $=L_2$-norm) and by ${}^2\Omega^{p,k}$ the completion of ${}^2\Omega^p_k$ with respect to ${}^2\|\cdot\|_k$,

$${}^2\|\omega\|_k := \|\omega\|+\|\Delta\omega\|+\ldots+\|\Delta^k\omega\|.$$

Let H denote the projection onto

$${}^2\mathscr{H}^p=\{\omega\in\Omega^p\cap{}^2\Omega^p \mid \bar{d}\omega=\bar{\delta}\omega=0\}=\ker\bar{\Delta}$$

(since (M^N, g) is complete).

Then one is searching for an operator G satisfying

$$\Delta G\omega=\omega-H\omega$$

and, if possible, for a meaningful integral representation of G. This G is called Green's operator.

Theorem 4.3. *Let (M^N, g) be open, complete, and of bounded geometry. Assume further that $\bar{\Delta}=\bar{\Delta}_p$ has positive eigenvalues below the essential spectrum. Then*

$$G\omega(x)=\int_0\int_M E^p(t,x,y)\wedge *(\omega-H\omega)(y)\,d_{\mathrm{vol}_y}$$

is a Green operator and has the following properties:

(a) $\|G\omega\|\leqq(2\lambda_1)^{-1/2}\|\omega\|$ *for* $\omega\in\Omega^p_0$, *where λ_1 is the first nonzero eigenvalue of $\bar{\Delta}$. Hence G can be extended to a bounded linear operator $G\colon {}^2\Omega^p\to{}^2\Omega^p$.*

(b) $G\omega \in {}^2\Omega^{p,k}$ *for arbitrary large* k.
(c) $\omega = H\omega + \bar{d}\delta G\omega + \delta\bar{d}G\omega$ *is the Hodge decomposition.*

Proof. A complete proof is given in [3] under the assumption of the existence of a good heat kernel. This existence we have just now established.

Using Theorem 4.3, we can establish the approximation theorem for the eigenvalues below the essential spectrum by the eigenvalues of semicombinatorical Laplace operators associated to sequences of uniform triangulations also for $0 \leqq p \leqq N$. For $p=0$ this was completely proved in [3], for $p>0$ under the assumption of the existence of a good heat kernel.

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On a problem posed by I. Z. Ruzsa

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In his paper titled "On the small sieve II. Sifting by composite numbers" (*Journal of Number Theory*, **14** (1982), 260—268) I. Z. RUZSA posed the following problem:

For $a \in \mathbf{N}$ and $b \in \mathbf{Z}$ let $R(a, b)$ denote the residue class b mod a. Consider all systems $a_1, \ldots, a_m$ (m not fixed) of natural numbers $1 \leqq a_1 < \ldots < a_m \leqq n$ for which there exist integers $b_1, \ldots, b_m$ such that

$$(*) \qquad \bigcup_{j=1}^{m} R(a_j, b_j) \supset \{1, \ldots, n\}.$$

Put

$$\mu(n) := \min \sum_{j=1}^{m} \frac{1}{a_j}$$

where the minimum has to be taken over all those systems. What can be said abou the behaviour of $\mu(n)$ for $n \to \infty$?

Since $(*)$ implies

$$2n \sum_{j=1}^{m} \frac{1}{a_j} \geqq \sum_{j=1}^{m} \left(\left[\frac{n}{a_j} \right] + 1 \right) \geqq n$$

the lower estimate $\mu(n) \geqq \frac{1}{2}$ follows at once. Ruzsa mentions that he can improve it to

$$\mu(n) \geqq \log \frac{2^5 3^6}{5^2 23^2} = 0.567\,545\,38 \ldots$$

and he also gives the upper estimate

$$\mu(n) \leqq \log \frac{5}{2} + O\left(\frac{1}{n}\right).$$

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Ruzsa's problem appears to be very delicate but it gives rise to another one which can be solved: Denote by $\mathscr{A}(n)$ the family of all subsets $A\subset\{1, \ldots, n\}$ with the property

$$\sum_{a\in A}\left(\left[\frac{n}{a}\right]+1\right) \geqq n$$

and put

$$\nu(n) := \min_{A\in\mathscr{A}(n)} \sum_{a\in A} \frac{1}{a}.$$

Obviously $\nu(n)\leqq\mu(n)$. I could show

$$\nu(n) = \log\frac{2^5\cdot 3^6}{23^3}+O(n^{-1/3}).$$

Ruzsa simplified my proof. His modifications also led to the better error term $O(1/n)$. With his kind permission I present this simpler version.

Let $Y(n)$ denote the set of all $y=(y_1, \ldots, y_n)\in\mathbf{R}^n$ which fulfil

$$0 \leqq y_j \leqq 1 \quad (1 \leqq j \leqq n) \quad\text{and}\quad \sum_{j=1}^{n} y_j\left(\left[\frac{n}{j}\right]+1\right) \geqq n.$$

Put

$$\nu^*(n) := \min_{y\in Y(n)} \sum_{j=1}^{n} y_j\frac{1}{j}.$$

Obviously $\nu^*(n)\leqq\nu(n)$. It will be shown that

$$\nu^*(n) = \log\frac{2^5\cdot 3^6}{23^3}+O\left(\frac{1}{n}\right) \tag{1}$$

and

$$\nu(n) \leqq \nu^*(n)+O\left(\frac{1}{n}\right). \tag{2}$$

If we put $\beta_j := \frac{j}{n}\left(\left[\frac{n}{j}\right]+1\right)$ and denote by $Z(n)$ the set of all $z=(z_1, \ldots, z_n)\in\mathbf{R}^n$ which fulfil

$$0 \leqq z_j \leqq \frac{1}{j} \quad (1 \leqq j \leqq n) \quad\text{and}\quad \sum_{j=1}^{n} z_j\beta_j \geqq 1$$

then

$$\nu^*(n) = \min_{z\in Z(n)} \sum_{j=1}^{n} z_j.$$

Let $(\xi_1, \ldots, \xi_n) \in Z(n)$ be such that $v^*(n) = \sum_{j=1}^{n} \xi_j$. Then

$$0 \leqq \xi_j \leqq \frac{1}{j} \ (1 \leqq j \leqq n) \quad \text{and} \quad \sum_{j=1}^{n} \xi_j \beta_j = 1.$$

The quantity

$$\gamma = \gamma(n) := \min_{\xi_j > 0} \beta_j$$

turns out to be crucial in the argumentation. One has

$$\beta_j < \gamma \Rightarrow \xi_j = 0, \tag{3}$$

$$\beta_j > \gamma \Rightarrow \xi_j = \frac{1}{j}. \tag{4}$$

Here (3) is evident and (4) can be seen this way: Let m, $1 \leqq m \leqq n$, be such that $\beta_m = \gamma$. Then $\xi_m > 0$. If there existed some k, $1 \leqq k \leqq n$, with $\beta_k > \gamma$ and $\xi_k < \frac{1}{k}$ then $k \neq m$ and

$$\varepsilon := \min\left\{\frac{\xi_m}{\beta_k}, \frac{1}{\beta_m}\left(\frac{1}{k} - \xi_k\right)\right\} > 0.$$

Now put

$$\xi'_m := \xi_m - \varepsilon\beta_k, \quad \xi'_k := \xi_k + \varepsilon\beta_m, \quad \xi'_j := \xi_j \quad \text{for} \quad j \neq m, k.$$

Since $0 \leqq \xi'_j \leqq \frac{1}{j}$ and $\sum_{j=1}^{n} \xi'_j\beta_j = \sum_{j=1}^{n} \xi_j\beta_j = 1$, we have $(\xi'_1, \ldots, \xi'_n) \in Z(n)$. This implies

$$v^*(n) \leqq \sum_{j=1}^{n} \xi'_j = v^*(n) - \varepsilon(\beta_k - \gamma) < v^*(n)$$

which is absurd.

Now put $\delta = \delta(n) := \gamma(n) - 1$. From $\beta_j \leqq 2$ and

$$\beta_j - 1 = \frac{j}{n}\left(1 - \left(\frac{n}{j} - \left[\frac{n}{j}\right]\right)\right) \geqq \frac{j}{n}\left(1 - \frac{j-1}{j}\right) = \frac{1}{n}$$

we see that in particular $\frac{1}{n} \leqq \delta \leqq 1$. If k is an integer, $1 \leqq k \leqq n$, then

$$\beta_j = \frac{j}{n}(k+1) \quad \text{for} \quad \frac{n}{k+1} < j \leqq \frac{n}{k}.$$

We shall show next that

$$\delta(n) \geqq \frac{1}{2500} =: \vartheta \quad \text{for all } n. \tag{5}$$

We put $x := \min\left\{\frac{1}{\delta}, \sqrt{n}\right\}$ and have

$$1 = \sum_{j=1}^{n} \xi_j \beta_j \geqq \sum_{k<x} \sum_{n\gamma/(k+1)<j\leqq n/k} \xi_j \beta_j .$$

Since in the inner sum $\beta_j > \frac{\gamma}{k+1}(k+1) = \gamma$. We infer from (4) that

$$1 \geqq \sum_{k<x} \sum_{n\gamma/(k+1)<j\leqq n/k} \frac{1}{j}\frac{j}{n}(k+1) = \frac{1}{n}\sum_{k<x}(k+1)\left(\left[\frac{n}{k}\right] - \left[\frac{n\gamma}{k+1}\right]\right) \geqq$$

$$\geqq \frac{1}{n}\sum_{k<x}(k+1)\left(\frac{n}{k} - \frac{n\gamma}{k+1} - 1\right) = \sum_{k<x}\frac{1}{k} - \delta\sum_{k<x}1 - \frac{1}{n}\sum_{k<x}(k+1) \geqq \sum_{k<x}\frac{1}{k} - 3.$$

Therefore $\sum_{k<x} \frac{1}{k} \leqq 4$ which implies $x \leqq 50$.

If $x = \sqrt{n}$ then $\sqrt{n} \leqq 50$ and therefore $\delta \geqq \frac{1}{n} \geqq \frac{1}{2500}$. If $x = \frac{1}{\delta}$ then $\frac{1}{\delta} \leqq 50$, and therefore $\delta \geqq \frac{1}{50}$.

Now comes

$$\delta(n) = \frac{5}{18} + O\left(\frac{1}{n}\right). \tag{6}$$

To establish that we start from

$$1 = \sum_{j=1}^{n} \xi_j\beta_j = \sum_{k<1/\delta}\Big(\sum_{n/(k+1)<j\leqq n\gamma/(k+1)} \xi_j\beta_j + \sum_{n\gamma/(k+1)<j\leqq n/k} \xi_j\beta_j\Big) +$$

$$+ \sum_{1/\delta\leqq k\leqq n} \sum_{n/(k+1)<j\leqq n/k} \xi_j\beta_j = S_1 + S_2 + S_3 .$$

We estimate S_3. If $j < \frac{n}{k}$ then $\beta_j < 1 + \frac{1}{k} \leqq 1 + \delta = \gamma$. If $k > \frac{1}{\delta}$ then $\beta_j \leqq 1 + \frac{1}{k} < 1 + \delta = \gamma$. Thus by (3) there is at most one term ξ_i, β_i, $i = \frac{n}{1/\delta}$, in S_3 which may not vanish. Therefore, by (5),

$$S_3 = \xi_i\beta_i \leqq 2\frac{1}{i} = \frac{2}{\delta n} \leqq \frac{2}{\vartheta n}$$

We estimate S_1. If $j < \frac{n\gamma}{k+1}$ then $\beta_j < \frac{\gamma}{k+1}(k+1) = \gamma$. Thus by (3) there is at most one term $\xi_i\beta_i$ in the innermost sum which may not vanish and

$$\xi_i\beta_i \leqq \frac{1}{i}\frac{i}{n}(k+1) = \frac{k+1}{n}.$$

Therefore, by (5),

$$S_1 \leqq \sum_{k<1/\delta} \frac{k+1}{n} \leqq \frac{2}{\delta^2 n} \leqq \frac{2}{\vartheta^2 n}.$$

Finally we evaluate S_2. In S_2 we have $\beta_j > \frac{\gamma}{k+1}(k+1) = \gamma$ which by (4) implies $\xi_j = \frac{1}{j}$. Therefore

$$S_2 = \frac{1}{n}\sum_{k<1/\delta}(k+1)\left(\left[\frac{n}{k}\right]-\left[\frac{n\gamma}{k+1}\right]\right) =$$

$$= \frac{1}{n}\sum_{k<1/\delta}(k+1)\left(\frac{n}{k}-\frac{n\gamma}{k+1}+r\right) \quad \text{where } |r| \leqq 1$$

$$= \sum_{k<1/\delta}\left(\frac{1}{k}-\delta\right)+R \quad \text{where } |R| \leqq \frac{2}{\vartheta^2 n}.$$

Thus we obtain

$$1-\frac{6}{\vartheta^2 n} \leqq \sum_{k<1/\delta}\left(\frac{1}{k}-\delta\right) \leqq 1+\frac{2}{\vartheta^2 n}.$$

If we put

$$f(t) := \sum_{k<1/t}\left(\frac{1}{k}-t\right) \quad \text{for} \quad 0 < t \leqq 1$$

then one easily verifies that the following inequality holds true:

$$|t_1-t_2| \leqq |f(t_1)-f(t_2)| \quad \text{for} \quad 0 < t_1, \quad t_2 \leqq 1.$$

Since $f\left(\frac{5}{18}\right) = 1$ we obtain

$$\left|\delta-\frac{5}{18}\right| \leqq |f(\delta)-1| \leqq \frac{6}{\vartheta^2 n}.$$

Proof of (1).

$$v^*(n) = \sum_{j=1}^{n}\xi_j = \sum_{k<1/\delta}\Big(\sum_{n/(k+1)<j\leqq n\gamma/(k+1)}\xi_j + \sum_{n\gamma/(k+1)<j\leqq n/k}\xi_j\Big) +$$

$$+ \sum_{1/\delta\leqq k\leqq n}\ \sum_{n/(k+1)<j\leqq n/k}\xi_j = T_1+T_2+T_3.$$

Because of $\beta_j > 1$ we have

$$T_1+T_3 \leqq S_1+S_3 \ll \frac{1}{n}.$$

Further for $n \geqq n_0$ by (6) one has

$$T_2 = \sum_{k=1}^{3} \sum_{n\gamma/(k+1) < j \leqq n/k} \frac{1}{j} = \sum_{k=1}^{3} \left(\log \frac{k+1}{\gamma k} + O\left(\frac{1}{n}\right) \right) = \log \frac{4}{\gamma^3} + O\left(\frac{1}{n}\right).$$

But (6) implies

$$\gamma^3 = (1+\delta)^3 = \left(\frac{23}{18}\right)^3 + O\left(\frac{1}{n}\right).$$

Proof of (2). Since $\frac{1}{\delta}$ is no integer we may write $k > \frac{1}{\delta}$ in T_3. Therefore $\xi_j = 0$ in T_3. In T_1 there are at most 3 terms with $\xi_j < \frac{1}{j}$. These are replaced by $\frac{1}{j} \leqq \frac{4}{n}$. If we denote the new system by ξ'_j then we have

$$\xi'_j = 0 \quad \text{or} \quad \frac{1}{j} \quad (1 \leqq j \leqq n) \quad \text{and} \quad \sum_{j=1}^{n} \xi'_j \beta_j \geqq \sum_{j=1}^{n} \xi_j \beta_j \geqq 1.$$

Therefore

$$v(n) \leqq \sum_{j=1}^{n} \xi'_j \leqq \sum_{j=1}^{n} \xi_j + \frac{12}{n} = v^*(n) + \frac{12}{n}.$$

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Well-quasiordering depends on the labels

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1. Introduction

A well-quasiordering (WQO) is a quasiordered set containing no infinite decreasing chain and no infinite antichain. A considerable part of the results on WQO is of the form that a concrete category Q is WQO, where $a \leqq b$ means that there exists a Q-morphism from a to b.

As an example let us mention the category T of finite trees with tree embeddings (see [2]). The recent solution of the Wagner's conjecture by ROBERTSON and SEYMOUR ([5]) is also of the form that certain category is WQO.

Other categories are trivially WQO, for example the category F of finite sets and injective mappings or the category H of finite linearly ordered sets and strictly increasing mappings. Still, we come to non-trivial questions, if we introduce more involved orderings:

Let A be a WQO and let Q be a concrete category with finite objects and injective morphisms. We consider a class $Q(A)$ of objects of Q "labeled by" elements of A at each point. We put $a \leqq b$ if there is a morphism from a to b which increases the labels (not necessarily strictly). Now the question is: Is it true that

(1) $$Q(A) \text{ is WQO whenever } A \text{ is WQO?}$$

(1) was proved for F, H by HIGMAN [1] and for T by NASH-WILLIAMS [4]. Of course, it would be useful if (1) were implied by a simpler condition, say,

(2) $$Q(\gamma) \text{ is WQO for any } \gamma \in \mathrm{Ord}.$$

Although this is not known in general, it was proved recently by one of the authors ([3]) for a considerably broad class of categories (for all subcategories of H). Let us note that, by an easy cardinality argument, (2) is equivalent to

(3) $$Q(\omega_1) \text{ is WQO.}$$

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So, for subcategories of H, (3)$\to$(1). However, it has not been known if (1) were not implied by a still weaker condition, say (even!),

$$Q(2) \text{ is WQO.} \tag{4}$$

It is the purpose of this paper to present a bunch of counterexamples of this kind. To be exact, we show that the set

$$M = \{\gamma | (\exists Q)(((\forall \beta < \gamma) Q(\beta) \text{ is WQO}) \& (Q(\gamma) \text{ is not WQO}))\} \tag{5}$$

is confinal in ω_1. We also prove that

$$M \supseteqq \omega,$$

showing that (4) does not imply $Q(3)$ to be WQO.

2. Preliminaries

2.1. Conventions and notation. The cardinality of a set X is denoted by $|X|$. For the ordinals, we use that definition where γ is identified with

$$\{\beta | \beta < \gamma\}.$$

In particular, this will apply to natural numbers. A quasiordering is a reflexive and transitive relation. In a quasiordering, a sequence (a_i) (finite or infinite) is called *bad* if

$$i < j \to a_i \nleqq a_j$$

and is called *good* if

$$i < j \to a_i \leqq a_j.$$

Each infinite sequence contains an infinite good subsequence or an infinite bad subsequence (Ramsey theorem). A quasiordering is called WQO, if no infinite sequence is bad. This definition is equivalent to that used in the Introduction by the Ramsey theorem. For a category C and objects a, b the symbol $C(a, b)$ designates the corresponding hom-set and the symbol Id_a designates the identity on a. For a concrete category, let the forgetful functor be denoted by U.

2.2. Definition. In this paper, a QO-category is a concrete category with finite objects and injective morphisms. For a QO-category Q and a quasiordering A, put

$$Q(A) = \{z = (u_z, c_z) | c_z \text{ is an object of } Q \text{ and } u_z\colon c_z \to A\}.$$

The *quasiordering* on $Q(A)$ is given by $z \leqq t$ if there exists a Q-morphism $\varphi\colon c_z \to c_t$ such that $u_t \circ U(\varphi) \geqq u_z$ (pointwise). We also say that $z \leqq t$ via the morphism φ. In the sequel, we shall use the symbol M for the set defined by formula (5) of the Introduction.

3. The results

To warm up, we start with a special theorem which demostrates the basic idea of our construction.

3.1. Theorem. $M \supseteqq \omega$.

Proof. Let a category Q_k consist of finite sets a_n $(n \in \omega)$ where each a_n is a disjoint union of k sets $a_n^0, \ldots, a_n^{k-1}$. Moreover, we shall assume

$$\lim_{n \to \omega} |a_n^i| = \omega \quad \text{for each } i \in k.$$

The hom-set $Q_k(a_n, a_m)$ will be

(a) $\emptyset$ if $n > m$,

(b) $\{\mathrm{Id}_{a_n}\}$ if $n = m$,

(c) $\{\varphi\colon a_n \to a_m \,|\, \varphi \text{ injective and } (\forall i \in k)(\varphi(a_n^i) \subseteqq \bigcup_{j \leqq i} a_m^j) \mathbin{\&} (\exists i \in k)(\varphi(a_n^i) \nsubseteqq a_m^i)\}$
if $n < m$.

To see that $Q_k(k)$ is not WQO, let $z_n(u_n, a_n)$ where u_n sends a_n^i to i for each $i \in k$. It is easily seen that $(z_n)_{n \in \omega}$ is a bad sequence. To prove that $Q_k(i)$ is WQO for $i < k$, introduce an auxilliary category $\bar{Q}_k$ with the same objects as Q_k and with the same morphisms from a_n to a_m for $n \geqq m$, while for $n < m$

$$\bar{Q}_k(a_n, a_m) = \{\varphi\colon a_n \to a_m \,|\, \varphi \text{ injective and } (\forall i \in k)(\varphi(a_n^i) \subseteqq \bigcup_{j \leqq i} a_m^j)\}.$$

Let $(z_t) = ((u_t, a_{n(t)}))$ be a bad sequence in $Q_k(i)$, $i < k$. Of course, we have

$$\lim_{t \to \omega} n(t) = \omega,$$

since (z_t) is bad. By HIGMAN's theorem [1], we may assume that (z_t) is good with respect to $\bar{Q}_k(i)$. Let, in $\bar{Q}_k(i)$, $z_s \leqq z_t$ via $\varphi_{s,t}\colon a_{n(s)} \to a_{n(t)}$. Now, since $i < k$, there is a $j \in i$ such that, for each $K \in \omega$, there exist $p, r \in k$, $p > r$, and $t(K) \in \omega$ such that

$$|\{x \in a^p_{n(t(K))} \,|\, u_{t(K)}(x) = j\}| \geqq K,$$
$$|\{x \in a^r_{n(t(K))} \,|\, u_{t(K)}(x) = j\}| \geqq K.$$

Without loss of generality, we may assume p, r fixed and $t(0) = 0$. Put $t = t(|a^r_{n(0)}| + 1)$. Thus, there exist $x \in a_0^p$, $y \in a_0^r$ such that

$$u_0(x) = u_t(y) = j$$

$$y \notin \operatorname{Im} \varphi_{0,t}.$$

We conclude that, in $Q_k(i)$, $z_0 \leqq z_t$ via a mapping φ given by

$$\varphi(z) = \varphi_{0,t}(z) \quad \text{if} \quad z \neq x$$
$$\varphi(x) = y,$$

contradicting our assumption.

3.2. Theorem. *M is confinal in ω_1.*

Proof below in 3.8.

3.3. The Constructions. Let $\omega \leqq \gamma \leqq \omega_1$. The there exists a bijection s_γ: $\omega \to \gamma$. Let the objects of C_γ be the sets

$$a_\gamma(n) = \{s_\gamma(0), \ldots, s_\gamma(n-1)\}.$$

The hom-set $C_\gamma(a_\gamma(k), a_\gamma(n))$ will be

(1) $\emptyset$ for $k > n$,

(2) $\{\mathrm{Id}_{a_\gamma(n)}\}$ for $k = n$,

(3) the set of all injective mappings φ: $a_\gamma(k) \to a_\gamma(n)$ such that, for some $j < k$,

(a) $\varphi(s_\gamma(j)) < s_\gamma(j)$,

(b) for $i < j$, $\varphi(s_\gamma(i)) = s_\gamma(i)$,

(c) for $i \leqq j$, $\beta < s_\gamma(i) \to \varphi(\beta) < s_\gamma(i)$.

3.4. Lemma. *C_γ is a QO-category.*

Proof. What remains to show is that, for $k < m < n$,

$$\varphi \in C_\gamma(a_\gamma(k), a_\gamma(n)), \quad \psi \in C_\gamma(a_\gamma(m), a_\gamma(n))$$
$$\psi \circ \varphi \in C_\gamma(a_\gamma(k), a_\gamma(n)).$$

To this end, let φ, ψ satisfy the statement of (3) with constants $j < k, \bar{j} < m$, respectively. We will show that $\psi \circ \varphi$ satisfies it with the constant $\min(j, \bar{j})$. We distinguish two cases:

Case 1. $\bar{j} \geqq j$: The proof of (a), (b), (c) for $\psi \circ \varphi$ is contained in the following computations. (By (a) for φ, $j \leqq \bar{j}$ and (c) for ψ)

$$\psi \circ \varphi(s_\gamma(j)) < s_\gamma(j).$$

For $i < j$ (by (b) for φ, ψ and $j \leqq \bar{j}$)

$$\psi \circ \varphi(s_\gamma(i)) = \psi(s_\gamma(i)) = s_\gamma(i).$$

For $i \leqq j$, $\beta < s_\gamma(i)$, (by (c) for φ, ψ and $j \leqq \bar{j}$)

$$\psi \circ \varphi(\beta) < s_\gamma(i).$$

Case 2. $\bar{j}<j$: Compute again. (By (b) for φ and by (a) for ψ)

$$\psi\circ\varphi(s_\gamma(\bar{j}))=\psi(s_\gamma(\bar{j}))=s_\gamma(\bar{j}).$$

For $i<\bar{j}$, (by (b) for φ,ψ)

$$\psi\circ\varphi(s_\gamma(i))=\psi(s_\gamma(i))=s_\gamma(i).$$

For $i\leqq\bar{j}$, $\beta<s_\gamma(i)$, (by (c) for φ,ψ)

$$\psi\circ\varphi(\beta)<s_\gamma(i).$$

3.5. Lemma. *$C_\gamma(\gamma)$ is not* WQO.

Proof. Introduce a sequence $z_n=(u_n,c_n)$ in $C_\gamma(\gamma)$ by putting

$$c_n=a_\gamma(n)$$
$$u_n(s_\gamma(i))=s_\gamma(i).$$

By condition (a) in 3.3 (3) we see easily that $(z_n)_{n\in\omega}$ is bad in $c_\gamma(\gamma)$.

3.6. Auxilliary definition. Let us call a pair (β,α), $\alpha,\beta\in\omega_1$ *admissible,* if there exist a $\gamma\in\omega_1$, $\gamma\geqq\alpha$, an increasing sequence $(n(i))_{i\in\omega}$, a number $K\in\omega$ and a bad sequence $z_i=(u_i,a_\gamma(n(i)))$ in $C_\gamma(\omega_1)$ such that for each $i\in\omega$

$$|\{\delta<\alpha|\delta\in a_\gamma(n(i))\,\&\,u_i(\delta)\geqq\beta\}|<K.$$

3.7. Lemma.

(1) *If $C_\gamma(\beta)$ is not* WQO *then (β,γ) is admissible.*

(2) *If (β,α) is admissible and $\bar{\alpha}<\alpha$ then $(\beta,\bar{\alpha})$ is admissible.*

(3) *$(0,\omega)$ is not admissible.*

(4) *If $(\beta,\alpha+\omega)$ is admissible then there exists a $\bar{\beta}<\beta$ such that $(\bar{\beta},\alpha)$ is admissible.*

Proof. (2) and (3) are obvious. Note that in (1) we may use $K=1$. We shall prove (4).

Consider the entities $\gamma, n(i), K, z_i$ constituting the admissibility of $(\beta,\alpha+\omega)$. Let

$$p=\max\{i|\alpha\leqq s_\gamma(i)<\alpha+K\}.$$

Further, let $\alpha_{p+1}=\gamma$,

$$\{s_\gamma(0),\ldots,s_\gamma(p)\}=\{\alpha_0<\alpha_1<\ldots<\alpha_p\}.$$

For $i\in p+1$, $t\in\omega$, define $b_i(t)\in F(\omega_1)$ (recall that F is the category of finite sets and injective mappings) in the following way:

$$c_i(t)=\{s_\gamma(j)|j<n(t),\quad \alpha_i<s_\gamma(j)<\alpha_{i+1}\},$$
$$b_i(t)=(u_i|c_i(t),\ c_i(t)).$$

By Ramsey and Higman's theorems there exists an increasing sequence $(t_x)_{x\in\omega}$

such that $n(t_x)>p$ and for $x<y$, $i\in p+1$

$$(*) \qquad u_{t_x}(\alpha_i) \leqq u_{t_y}(\alpha_i)$$

$$(**) \qquad b_i(t_x) \leqq b_i(t_y) \text{ in } F(\omega_1) \text{ via some mapping } \varphi_i(t_x, t_y).$$

Without loss of generality, we may assume $t_x=x$. Now, by $(*)$ and by the definition of K there exists an $\alpha\leqq\bar{\alpha}<\alpha+K$ such that for all t

$$u_t(\bar{\alpha}) < \beta.$$

We will show that $(u_0(\bar{\alpha}), \bar{\alpha})$ is admissible, concluding the proof of (4) by (2).

In fact, for the new K we may take $n(0)$: If, for some $t>0$, there are more than $n(0)$, of $j<n(t)$ with $u_t(n_\gamma(j))\geqq u_0(\bar{\alpha})$ then there exists at least one such j that neither $j\leqq p$ nor $s_\gamma(j)$ lies in the image of $\varphi_i(0,t)$ for any i. Now define $\varphi\colon a_\gamma(n(0))\to\to a_\gamma(n(t))$ by

$$\varphi(s_\gamma(i)) = s_\gamma(i) \quad \text{whenever} \quad i\in p+1, s(i) \neq \bar{\alpha}$$

$$\varphi(\bar{\alpha}) = s_\gamma(j)$$

$$\varphi(\delta) = \varphi_i(0, t)(\delta) \quad \text{if} \quad \delta\in c_i(0).$$

We see easily that $\varphi\in C_\gamma(a_\gamma(n(0)), a_\gamma(n(t)))$ and

$$z_0 \leqq z_t \quad \text{via } \varphi,$$

contradicting the assumption that (z_i) is bad.

3.8. Proof of Theorem 3.2. Define $\gamma(\beta)$ inductively by

$$\gamma(0) = \omega$$

$$\gamma(\beta+1) = \gamma(\beta)+\omega$$

$$\gamma(\beta) = (\lim_{i\to\omega}\gamma(\beta_i))+\omega \quad \text{for} \quad \beta_i\nearrow\beta.$$

It follows from 3.7. (2), (4) that $(\beta, \gamma(\beta))$ is not admissible for any β. Thus, by 3.7. (1), $c_{\gamma(\beta)}(\beta)$ is WQO. This, together with Lemma 3.5, concludes the proof of Theorem 3.2.

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Embedding results pertaining to strong approximation

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1. The aim of the paper is to make a step toward answering an open problem of our previous paper [2] and to extend another result published in the same paper.

In order to quote the known results we have to recall some notions and notations.

Let $f(x)$ be a continuous and 2π-periodic function and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

be its Fourier series. Let $s_n = s_n(x) = s_n(f; x)$ and $\tau_n = \tau_n(x) = \tau_n(f, x)$ denote the n-th partial sum and the classical de la Vallée Poussin mean of (1), i.e.

$$\tau_n(x) = \frac{1}{n} \sum_{k=n+1}^{2n} s_k(x), \quad n = 1, 2, \ldots.$$

We denote by $\|\cdot\|$ the usual supremum norm.

Let $\omega(\delta)$ be a nondecreasing continuous function on the interval $[0, 2\pi]$ having the properties: $\omega(0)=0$, $\omega(\delta_1+\delta_2) \leqq \omega(\delta_1)+\omega(\delta_2)$ for any $0 \leqq \delta_1 \leqq \delta_2 \leqq \delta_1+\delta_2 \leqq 2\pi$. Such a function is called a modulus of continuity. The modulus of continuity of f will be denoted by $\omega(f; \delta)$.

We define the following classes of continuous functions:

$$H^{\omega} := \{f\colon \omega(f; \delta) = O(\omega(\delta))\},$$

$$S_p(\lambda) := \{f\colon \|\sum_{n=0}^{\infty} \lambda_n |s_n - f|^p\| < \infty\}$$

and

$$V_p(\lambda) := \{f\colon \|\sum_{n=1}^{\infty} \lambda_n |\tau_n - f|^p\| < \infty\},$$

where $\lambda = \{\lambda_n\}$ is a monotonic sequence of positive numbers and $0 < p < \infty$.

V. G. Krotov and the author ([1]) proved the following result.

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Theorem A. *If* $\{\lambda_n\}$ *is a positive monotonic sequence,* ω *is a modulus of continuity and* $0<p<\infty$, *then*

$$\sum_{k=1}^{n}(k\lambda_k)^{-1/p}=O\left(n\omega\left(\frac{1}{n}\right)\right) \tag{2}$$

implies

$$S_p(\lambda)\subset H^{\omega}. \tag{3}$$

Conversely, if there exists a number Q *such that* $0\leqq Q<1$ *and*

$$n^{Q}\lambda_n\uparrow, \tag{4}$$

then (3) *implies* (2).

Since the de la Vallée Poussin means τ_n usually approximate the function f, in the sup norm, better than the partial sums s_n do, so we may expect that under reasonable conditions the following embedding relations will hold

$$S_p(\lambda)\subset V_p(\lambda)\subset H^{\omega}. \tag{5}$$

In [2], A. Meir and me, verified some results pertaining to (5). More precisely the following theorems were proved:

Theorem B. *If* $p\geqq 1$ *and* $\{\lambda_n\}$ *is a monotonic (nondecreasing or nonincreasing) sequence of positive numbers satisfying the condition*

$$\lambda_n/\lambda_{2n}\leqq K^*,\quad n=1,2,\ldots, \tag{6}$$

then

$$S_p(\lambda)\subset V_p(\lambda) \tag{7}$$

holds.

Theorem C. *Let* $\{\lambda_n\}$ *be a monotonic sequence of positive numbers, furthermore let* ω *be a modulus of continuity and* $0<p<\infty$. *Then condition* (2) *implies*

$$V_p(\lambda)\subset H^{\omega}. \tag{8}$$

If $p\geqq 1$ *and there exists a number* Q *such that* $0\leqq Q<1$ *and* (4) *holds, then, conversely,* (8) *implies* (2).

To decide whether $S_p(\lambda)\subset V_p(\lambda)$, i.e. (7), holds when $0<p<1$; it was left as an open problem.

Making many unsuccessful attempts to prove (7) or its converse, at the present time, I have the conjecture that neither $S_p(\lambda)\subset V_p(\lambda)$ nor $V_p(\lambda)\subset S_p(\lambda)$ hold generally, but I have not been able to verify this statement.

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*) $K, K_1, \ldots$ will denote positive constants, not necessarily the same at each occurence.

However it turned out that if one defined a new subclass of $V_p(\lambda)$, which one could call "strong" $V_p(\lambda)$-class, and denoted by $V_p^{(s)}(\lambda)$, i.e.

$$V_p^{(s)}(\lambda) := \left\{f\colon \left\| \sum_{n=1}^{\infty} \lambda_n \left(\frac{1}{n} \sum_{k=n+1}^{2n} |s_k - f|\right)^p \right\| < \infty\right\},$$

then under restriction (6) $S_p(\lambda) \subset V_p^{(s)}(\lambda)$ also holds for $p \geqq 1$, and $S_p(\lambda) \supset V_p^{(s)}(\lambda)$ is already true for $0 < p \leqq 1$ if $\lambda_{2n} \leqq K\lambda_n$. First we prove these statements.

Compare the definitions of $V_p(\lambda)$ and $V_p^{(s)}(\lambda)$, it is obvious that for any positive p and for any λ

$$V_p^{(s)}(\lambda) \subset V_p(\lambda) \tag{9}$$

always holds.

It is clear that (8) and (9) imply

$$V_p^{(s)}(\lambda) \subset H^{\omega}. \tag{10}$$

Secondly we prove that (10) also implies relation (2) for any positive p if (4) holds.

This result is a mild sharpening of the second part of Theorem C for $p \geqq 1$; and by (9) it extends the previous statement for any positive p. The latter result is the more important one.

2. We prove the following results.

Theorem 1. *Let $\lambda = \{\lambda_n\}$ be a monotonic sequence of positive numbers. The following embedding relations hold:*

$$S_p(\lambda) \subset V_p^{(s)}(\lambda) \quad \text{if} \quad p \geqq 1 \quad \text{and} \quad \lambda_n = O(\lambda_{2n}); \tag{11}$$

and

$$S_p(\lambda) \supset V_p^{(s)}(\lambda) \quad \text{if} \quad 0 < p \leqq 1 \quad \text{and} \quad \lambda_{2n} = O(\lambda_n). \tag{12}$$

Theorem 2. *Let $\lambda = \{\lambda_n\}$ be a monotonic sequence of positive numbers, furthermore let ω be a modulus of continuity and $p > 0$. If there exists a number Q such that $0 \leqq Q < 1$ and* (4) *holds, then the embedding relation* (10) *implies relation* (2).

Theorem C and Theorem 2 convey as an immediate consequence the following result.

Corollary. *Under condition* (4) *the embedding relation* (8) *implies relation* (2) *for any positive p.*

3. To prove our theorems we require some lemmas.

Lemma 1 ([1]). *If $a_n \geqq 0$ and the function*

$$f(x) \sim \sum_{n=1}^{\infty} a_n \sin nx$$

belongs to the class H^ω, then

$$\sum_{k=1}^{n} ka_k = O\left(n\omega\left(\frac{1}{n}\right)\right).$$

Lemma 2. *If $0<p<\infty$, $\lambda_n\uparrow$ or $\lambda_n\downarrow$ and there exists a number Q, $0\leqq Q<1$, such that* (4) *holds, then the function*

$$f(x) := \sum_{n=1}^{\infty} \frac{1}{n}(n\lambda_n)^{-1/p}\sin nx$$

belongs to the class $V_p^{(s)}(\lambda)$.

Proof. It is easy to see that

$$\sum_{n=1}^{\infty}\frac{1}{n}(n\lambda_n)^{-1/p} = \sum_{n=1}^{\infty} n^{-(1+(1/p)(1-Q))}(n^Q\lambda_n)^{-1/p}<\infty,$$

so f is a continuous function, and $f(0)=f(\pi)=0$.

To prove that $f\in V_p^{(s)}(\lambda)$ we fix $0<x<\pi$ and choose N such that

$$\frac{1}{N+1}<x\leqq\frac{1}{N}.$$

We make the following estimates:

$$\sum_{n=1}^{\infty}\lambda_n\left\{\frac{1}{n}\sum_{k=n+1}^{2n}|s_k(x)-f(x)|\right\}^p \leqq$$

$$\leqq \sum_{n=1}^{N/2}\lambda_n\left\{\frac{1}{n}\sum_{k=n+1}^{2n}\left(\left|\sum_{m=k+1}^{N+1}\frac{1}{m}(m\lambda_m)^{-1/p}\sin mx\right|+\left|\sum_{m=N+2}^{\infty}\frac{1}{m}(m\lambda_n)^{-1/p}\sin mx\right|\right)\right\}^p+$$

$$+\sum_{n=N/2}^{\infty}\lambda_n\left\{\frac{1}{n}\sum_{k=n+1}^{2n}\left|\sum_{m=k+1}^{\infty}\frac{1}{m}(m\lambda_m)^{-1/p}\sin mx\right|\right\}^p\equiv\sum\nolimits_1+\sum\nolimits_2,$$

where

$$\sum\nolimits_1\leqq K\sum_{n=1}^{N/2}\lambda_n\left\{\frac{1}{n}\sum_{k=n+1}^{2n}\left|\sum_{m=k+1}^{N+1}\frac{1}{m}(m\lambda_m)^{-1/p}\sin mx\right|\right\}^p+$$

$$+K\sum_{n=1}^{N/2}\lambda_n\left\{\frac{1}{n}\sum_{k=n+1}^{2n}\left|\sum_{m=N+2}^{\infty}\frac{1}{m}(m\lambda_m)^{-1/p}\sin mx\right|\right\}^p\equiv\sum\nolimits_{11}+\sum\nolimits_{12}.$$

First we assume that $\lambda_n\downarrow$. By our assumption, we can choose a positive Q such that $1>Q>1-p$ and $n^Q\lambda_n\uparrow$. Then $\dfrac{Q-1}{p}>-1$, so for any $n<k<N$ we have

that

$$\sum_{m=k+1}^{N+1} \frac{1}{m}(m\lambda_m)^{-1/p}\sin mx \leqq x \sum_{m=k+1}^{N+1} (m\lambda_m)^{-1/p} =$$

$$= x \sum_{m=k+1}^{N+1} (m^Q \lambda_m m^{1-Q})^{-1/p} \leqq x(n^Q \lambda_n)^{-1/p} \sum_{m=n+1}^{N+1} m^{(Q-1)/p} \leqq$$

$$\leqq Kx(n^Q \lambda_n)^{-1/p} N^{1+(Q-1)/p},$$

whence we get that

$$\sum\nolimits_{11} \leqq K_1 \sum_{n=1}^{N/2} \lambda_n x^p \lambda_n^{-1} n^{-Q} N^{p+Q-1} \leqq K_2 x^p N^p \leqq K_3.$$

Furthermore

$$\sum\nolimits_{12} \leqq \sum_{n=1}^{N/2} \lambda_n \left| \sum_{m=N+2}^{\infty} \frac{1}{m}(m\lambda_m)^{-1/p}\sin mx \right|^p \leqq$$

$$\leqq \sum_{n=1}^{N/2} \lambda_n \left\{ \sum_{m=N+2}^{\infty} \frac{1}{m}(m^Q \lambda_m m^{1-Q})^{-1/p} \right\}^p \leqq$$

$$\leqq \sum_{n=1}^{N/2} \lambda_n \left\{ (N^Q \lambda_N)^{-1/p} \sum_{m=N}^{\infty} m^{-1-(1-Q)/p} \right\}^p \leqq$$

$$\leqq \sum_{n=1}^{N} \lambda_n (N^Q \lambda_N)^{-1} N^{-(1-Q)} =$$

$$= N^{-1} \lambda_N^{-1} \sum_{n=1}^{N} n^Q \lambda_n n^{-Q} \leqq N^{Q-1} \sum_{n=1}^{N} n^{-Q} \leqq K.$$

To estimate $\sum_2$ we use the following inequality

$$\left| \sum_{m=k+1}^{\infty} \frac{1}{m}(m\lambda_m)^{-1/p}\sin mx \right| \leqq \frac{K}{kx}(k\lambda_k)^{-1/p}$$

for any $0<x<\pi$. Hence

$$\sum\nolimits_2 \leqq K_1 \sum_{n=N/2}^{\infty} \lambda_n n^{-p} x^{-p} n^{-1} \lambda_n^{-1} \leqq K_2 \sum_{n=N/2}^{\infty} n^{-p-1} x^{-p} \leqq K_3 (xN)^{-p} \leqq K_4.$$

Collecting these estimates we get that $f \in V_p^{(s)}(\lambda)$ in the case $\lambda_n \downarrow$.

The proof in the case $\lambda_n \uparrow$ is easier, then we can simply replace condition (4) by $\lambda_n \uparrow$ in some parts of the previous proof. Therefore we omit the details.

The proof is completed.

4. Now we can prove our theorems.

Proof of Theorem 1. For $p\geqq 1$, by Hölder's inequality, the inequality

$$\frac{1}{n}\sum_{k=n+1}^{2n}|s_k-f|\leqq\left\{\frac{1}{n}\sum_{k=n+1}^{2n}|s_k-f|^p\right\}^{1/p}$$

holds, whence

$$\sum_{n=1}^{\infty}\lambda_n\left\{\frac{1}{n}\sum_{k=n+1}^{2n}|s_k-f|\right\}^p\leqq\sum_{n=1}^{\infty}\lambda_n\left\{\frac{1}{n}\sum_{k=n+1}^{2n}|s_k-f|^p\right\}\leqq \tag{13}$$

$$\leqq\sum_{k=2}^{\infty}|s_k-f|^p\sum_{n=k/2}^{k}\lambda_n/n\equiv\sum\nolimits_3$$

follows. By $\lambda_n=O(\lambda_{2n})$ we have

$$\sum\nolimits_3\leqq K\sum_{k=2}^{\infty}\lambda_k|s_k-f|^p. \tag{14}$$

Inequalities (13) and (14) imply (11).

In the case $0<p\leqq 1$ we use the inequality

$$\frac{1}{n}\sum_{k=n+1}^{2n}|s_k-f|\geqq\left\{\frac{1}{n}\sum_{k=n+1}^{2n}|s_k-f|^p\right\}^{1/p}, \tag{15}$$

which can also be proved by Hölder inequality, and the estimate

$$\lambda_k=O\Big(\sum_{n=k/2}^{k-1}\lambda_n/n\Big), \tag{16}$$

it follows from $\lambda_{2n}=O(\lambda_n)$. Then, by (15) and (16),

$$\sum_{n=2}^{\infty}\lambda_n|s_n-f|^p\leqq K\sum_{n=2}^{\infty}\Big(\sum_{k=n/2}^{n-1}\lambda_k/k\Big)|s_n-f|^p\leqq$$

$$\leqq K\sum_{k=1}^{\infty}\lambda_k/k\sum_{n=k+1}^{2k}|s_n-f|^p\leqq K\sum_{k=1}^{\infty}\lambda_k\left(\frac{1}{k}\sum_{n=k+1}^{2k}|s_n-f|\right)^p$$

holds, whence (12) clearly follows.

Proof of Theorem 2. Let us consider the function given in Lemma 2, i.e. let

$$f_0(x):=\sum_{n=1}^{\infty}\frac{1}{n}(n\lambda_n)^{-1/p}\sin nx.$$

Then, by Lemma 2, $f_0 \in V_p^{(s)}(\lambda)$. The assumption $V_p^{(s)}(\lambda) \subset H^\omega$ conveys that $f_0 \in H^\omega$ also holds. Hence, using Lemma 1, relation (2) follows, that is, (10) really implies (2).

The proof is completed.

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On the strong and very strong summability of orthogonal series

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1. Let $\{\varphi_n(x)\}$ be an orthonormal system on the interval (0, 1). We shall consider real orthogonal series

$$(1.1)\qquad \sum_{n=0}^{\infty} c_n \varphi_n(x) \quad \text{with} \quad \sum_{n=0}^{\infty} c_n^2 < \infty.$$

By the Riesz—Fischer theorem series (1.1) converges in L^2 to a square-integrable function $f(x)$. Let us denote the partial sums of (1.1) by $s_n(x)$.

As introductory sample results we recall the following theorems:

Theorem A (A. ZYGMUND [15]). *If series* (1.1) $(C, 1)$*-summable almost everywhere then it is also strongly* $(C, 1)$*-summable almost everywhere, i.e.*

$$\frac{1}{n+1} \sum_{k=0}^{n} |s_k(x)-f(x)|^2 \to 0 \quad a.e.$$

Theorem B (K. TANDORI [13]). *If*

$$\sum_{n=4}^{\infty} c_n^2 \log\log^2 n < \infty$$

then series (1.1) *is very strongly* $(C, 1)$*-summable on* (0, 1) *almost everywhere, i.e.*

$$\frac{1}{n+1} \sum_{k=0}^{n} |s_{\nu_k}(x)-f(x)|^2 \to 0$$

for any increasing sequence $\{\nu_k\}$ *of natural numbers on* (0, 1) *almost everywhere.*

Theorem C (K. TANDORI [12]). *There exist an orthonormal system* $\{\varphi_n(x)\}$ *and coefficients* d_n *with* $\sum_{n=0}^{\infty} d_n^2 < \infty$ *such that the series*

$$\sum_{n=0}^{\infty} d_n \varphi_n(x)$$

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is strongly $(C, 1)$-summable almost everywhere but it is nowhere very strongly $(C, 1)$-summable.

In other words Theorem C states that the strong $(C, 1)$-summability does not imply the very strong $(C, 1)$-summability, generally.

The analogues of Theorems A and B for other summability methods have been proved individually. E.g. for Riesz-means J. MEDER [5] and L. LEINDLER [1], for $(C, \alpha>0)$-means G. SUNOUCHI [11], for Euler-means H. SCHWINN [9] and for generalized Abel-method L. LEINDLER [3] proved similar results.

On the other hand. F. MÓRICZ [6] proved that for an arbitrary regular Toeplitz T-summability method it is not true that if series (1.1) is T-summable then it is strongly T-summable, too.

The Móricz's result gives a reason for writing of a new paper, namely in the present paper we prove the analogues of Theorems A and B for a large class of general summability methods; and shall apply them to verify that the so-called generalized de la Vallée Poussin method also belongs to these summability methods. It will be easy to see that some of the above mentioned summability methods also belong to the class to be treated in Theorem 1. Roughly speaking one of the aims of the present paper is to verify that for a large class of summability methods the summability implies the strong summability for orthogonal series.

We mention that H. SCHWINN [10] also investigated the latter problem, and proved a slightly sharper result, but his proof quite differs from our one.

Theorem C shows that it cannot be expected that a general summability method should imply the very strong summability. But we shall show that if a coefficient-condition, e.g. of the form

$$\sum_{n=0}^{\infty} c_n^2 \varrho_n^2 < \infty \quad (\varrho_n \leqq \varrho_{n+1}),$$

implies the summability — as in Theorem B — then this condition will imply the very strong summability, too.

Let $\alpha:=(\alpha_{nk})$ be a positive regular Toeplitz-matrix satisfying the usual conditions: $\alpha_{nk}\geqq 0$; $\lim_{n\to\infty} \alpha_{nk}=0$ $(k=0, 1, \ldots)$; $\lim_{n\to\infty} \sum_{k=0}^{\infty} \alpha_{nk}=1$. We say that *series* (1.1) *is α-summable to $f(x)$ if*

$$\alpha_n(x) := \sum_{k=0}^{\infty} \alpha_{nk} s_k(x) \to f(x)$$

almost everywhere; and it will be called *strongly α-summable if*

$$\alpha_n |x| := \sum_{k=0}^{\infty} \alpha_{nk} |s_k(x)-f(x)|^2 \to 0$$

almost everywhere; and very strongly α-summable if for any increasing sequence $\{\nu_k\}$ of natural numbers

$$\alpha_n|\nu;\ x| := \sum_{k=0}^{\infty} \alpha_{nk}|s_{\nu_k}(x)-f(x)|^2 \to 0$$

holds almost everywhere.

We say that an α-summability method is an $N(\mu_m)$-*summability* if there exists an increasing sequence $\{\mu_m\}$ of natural numbers such that if series (1.1) is α-summable then $s_{\mu_m}(x)\to f(x)$ always holds almost everywhere, i.e. the convergence of the partial sums $s_{\mu_m}(x)$ is a necessary condition of the α-summability of series (1.1) for any orthonormal system $\{\varphi_n\}$ and for any coefficients c_n with $\sum_{n=0}^{\infty} c_n^2<\infty$. It is known that the $(C, \alpha>0)$-methods and generalized Abel-methods (for the latter see L. Rempulska [7]) are $N(2^m)$-summability methods and the Euler-method is an $N(m^2)$-method (see O. A. Ziza [14] and H. Schwinn [8]).

Now we recall the definition of the generalized ordinary and strong de la Vallée Poussin summability methods (see [2]). Let $\lambda=\{\lambda_n\}$ be a nondecreasing sequence of natural numbers for which $\lambda_0=1$ and $\lambda_{n+1}-\lambda_n\leqq 1$. Series (1.1) is (V, λ)-*summable* if

$$V_n(\lambda;\ x) := \frac{1}{\lambda_n}\sum_{k=n-\lambda_n+1}^{n} s_k(x)\to f(x)$$

almost everywhere, *strongly* (V, λ)-*summable* if

$$V_n|\lambda;\ x| := \frac{1}{\lambda_n}\sum_{k=n-\lambda_n+1}^{n} |s_k(x)-f(x)|^2\to 0$$

almost everywhere; and *very strongly* (V, λ)-*summable* if for any increasing sequence $\{\nu_k\}$ of natural numbers

$$V_n|\lambda, \nu;\ x| := \frac{1}{\lambda_n}\sum_{k=n-\lambda_n+1}^{n} |s_{\nu_k}(x)-f(x)|^2\to 0$$

almost everywhere.

It is obvious that if $\lambda_n=n$ then the (V, λ)-means reduce to the $(C, 1)$-means, and if $\lambda_n=\left[\frac{n}{2}\right]$ $(n\geqq 2)$, where $[\beta]$ denotes the integer part of β, then we get the classical de la Vallée Poussin means.

In [2] we proved that for any λ the (V, λ)-summability is an $N(\nu_m)$-summability with $\nu_0=1$ and $\nu_m := \sum_{k=0}^{m-1}\lambda_{\nu_k}$, $m\geqq 1$; furthermore that if

$$\sum_{m=1}^{\infty}\Big\{\sum_{n=\nu_m+1}^{\nu_{m+1}} c_n^2\Big\}\log^2 m<\infty \tag{1.2}$$

then series (1.1) is (V, λ)-summable; moreover very strongly (V, λ)-summable.

2. Now we can formulate our theorems:

Theorem 1. *If a positive regular Toeplitz-matrix* $\alpha=(\alpha_{nk})$ *generates an* $N(\mu_m)$ *summability and satisfies the following additional conditions: there exist a natural number p and a positive M constant such that*

$$\alpha_{nk} \leqq M \sum_{i=-p}^{p} \alpha_{(\mu_m-i)k} \quad \textit{for} \quad \mu_{m-1} < n < \mu_m \tag{2.1}$$

and

$$\sum_{\nu=1}^{\infty} \sum_{i=\mu_{m-1}+1}^{\mu_m} \alpha_{\mu_\nu i} \leqq M \tag{2.2}$$

hold for all m and k, then the α-summability of series (1.1) *implies its strong α-summability.*

Theorem 2. *Under the assumptions of Theorem* 1, *if the following condition*

$$\sum_{n=1}^{\infty} c_n^2 \gamma_n^2 < \infty, \quad \gamma_n \leqq \gamma_{n+1}, \tag{2.3}$$

implies the α-summability of series (1.1) *for any orthonormal system, then* (2.3) *also implies the very strong α-summability of series* (1.1).

Using these theorems we verify the following theorems:

Theorem 3. *If series* (1.1) *is* (V, λ)*-summable then it is strongly* (V, λ)*-summable, too.*

Theorem D. *Condition* (1.2) *implies that series* (1.1) *is very strongly* (V, λ) *summable.*

We remark that Theorem D was proved in [2] as we stated above, but its proof is totally different from to be given here.

3. We require the following lemma.

Lemma ([4], Lemma 3). *Let* $\varkappa>0$ *and* $\{\beta_n\}$ *be an arbitrary sequence of non-negative numbers. Assuming that the condition*

$$\sum_{n=1}^{\infty} \beta_n \Big\{\sum_{k=n}^{\infty} c_k^2\Big\}^{\varkappa} < \infty \tag{3.1}$$

implies a "certain property $T=T(\{s_n(x)\})$*" of the partial sums* $s_n(x)$ *of* (1.1) *for any orthonormal system, then* (3.1) *implies that the partial sums* $s_{\nu_n}(x)$ *of* (1.1) *also have the same property T for any increasing sequence* $\{\nu_n\}$, *i.e. if* $(3.1)\Rightarrow T(\{s_n(x)\})$ *then* $(3.1)\Rightarrow T(\{s_{\nu_n}(x)\})$ *for any increasing sequence* $\{\nu_n\}$.

4. Now we can prove our theorems. For the sake of brevity, from now on, convergence and summability have the meaning of convergence and summability almost everywhere in (0, 1).

Proof of Theorem 1. Since the α-summability now implies the convergence of the partial sums $s_{\mu_m}(x)$, thus putting $\nu_k:=\mu_m$ for $k=\mu_{m-1}+1,\ \mu_{m-1}+2, \ldots$ $\ldots, \mu_m$, $m=1, 2, \ldots$; $\nu_0=0$ and $\nu_1=1$; we can see by the following obvious inequality

$$\alpha_n |x| \leqq 2 \sum_{k=0}^{\infty} \alpha_{nk}\big(|s_k(x)-s_{\nu_k}(x)|^2+|s_{\nu_k}(x)-f(x)|^2\big)$$

and on account of the regularity of α-summability, that in order to prove Theorem 1 it is enough to verify that

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \alpha_{nk} |s_k(x)-s_{\nu_k}(x)|^2 = 0 \tag{4.1}$$

holds almost everywhere.

By (2.1) we have for any $\mu_{m-1}<n<\mu_m$

$$\sum_{k=0}^{\infty} \alpha_{nk} |s_k(x)-s_{\nu_k}(x)|^2 \leqq M \sum_{k=0}^{\infty} \sum_{i=-p}^{p} \alpha_{(\mu_{m-i})k} |s_k(x)-s_{\nu_k}(x)|^2, \tag{4.2}$$

therefore if we can prove

$$\lim_{m\to\infty} \sum_{k=0}^{\infty} \alpha_{\mu_m k} |s_k(x)-s_{\nu_k}(x)|^2 = 0 \tag{4.3}$$

almost everywhere, then, by (4.2), (4.1) will be proved.

An elementary calculation shows on account of (2.2) that

$$\sum_{m=1}^{\infty}\sum_{k=0}^{\infty} \alpha_{\mu_m k} \int_0^1 |s_k(x)-s_{\nu_k}(x)|^2\,dx \leqq \sum_{m=1}^{\infty}\sum_{i=1}^{\infty}\sum_{k=\mu_{i-1}+1}^{\mu_i} \alpha_{\mu_m k} \sum_{k=\mu_{i-1}+1}^{\mu_i} c_k^2 =$$

$$= \sum_{i=1}^{\infty}\sum_{k=\mu_{i-1}+1}^{\mu_i} c_k^2 \sum_{m=1}^{\infty}\sum_{k=\mu_{i-1}+1}^{\mu_i} \alpha_{\mu_m k} \leqq M \sum_{k=0}^{\infty} c_k^2 < \infty,$$

whence by B. Levi's theorem (4.3) follows.

This has completed the proof.

Proof of Theorem 2. Putting $\gamma_0=0$ then condition (2.3) and

$$\sum_{n=1}^{\infty} (\gamma_n^2-\gamma_{n-1}^2) \sum_{k=n}^{\infty} c_k^2 < \infty$$

are obviously equivalent. Hence we can already see that the statement of Theorem 2 is a consequence of Theorem 1 and our Lemma.

Proof of Theorem 3. In order to prove Theorem 3 it is enough to verify that if

$$\mu_0 = 0 \quad \text{and} \quad \mu_m := \sum_{k=0}^{m-1} \lambda_{\mu_k}, \tag{4.4}$$

then for this sequence $\{\mu_m\}$ conditions (2.1) and (2.2) are fulfilled.

Since $\lambda_{i+1}-\lambda_i \leqq 1$ for any i and $\mu_m-\mu_{m-1}=\lambda_{\mu_{m-1}}$, therefore $\lambda_{\mu_m} \leqq 2\lambda_{\mu_{m-1}}$, whence for any $\mu_{m-1}<n<\mu_m$

$$\frac{1}{\lambda_n} \leqq \frac{1}{\lambda_{\mu_{m-1}}} \leqq \frac{2}{\lambda_{\mu_m}} \tag{4.5}$$

holds, which verifies that (2.1) also holds with $M=2$ and $p=1$. Since then for any n

$$\alpha_{nk} = \begin{cases} 0 & \text{for } \ k < n+1-\lambda_n \ \text{ and } \ k > n, \\ \dfrac{1}{\lambda_n} & \text{for } \ n+1-\lambda_n \leqq k \leqq n; \end{cases} \tag{4.6}$$

and thus by (4.5) for $\mu_{m-1}<n<\mu_m$

$$\alpha_{nk} \leqq 2(\alpha_{\mu_{m-1}k}+\alpha_{\mu_m k})$$

always holds because $\mu_{m-1}+1-\lambda_{\mu_{m-1}} \leqq n+1-\lambda_n$ by $\lambda_{i+1}-\lambda_i \leqq 1$.

Namely if $n+1-\lambda_n \leqq k \leqq \mu_{m-1}$ then

$$\alpha_{nk} = \frac{1}{\lambda_n} \leqq \frac{1}{\lambda_{\mu_{m-1}}} = \alpha_{\mu_{m-1}k}$$

and if $\mu_{m-1}<k\leqq n<\mu_m$ then by (4.5)

$$\alpha_{nk} = \frac{1}{\lambda_n} \leqq \frac{2}{\lambda_{\mu_m}} = 2\alpha_{\mu_m k}$$

hold.

Next we show that (2.2) is also fulfilled for the α-matrix given by (4.6) and for sequence $\{\mu_m\}$ defined under (4.4).

By (4.6) it is clear that if $\nu \leqq m-1$ then

$$\sum_{i=\mu_{m-1}+1}^{\mu_m} \alpha_{\mu_\nu i} = 0 \quad (\mu_\nu < i), \tag{4.7}$$

and if $\nu \geqq m$ then

$$\sum_{i=\mu_{m-1}+1}^{\mu_m} \alpha_{\mu_\nu i} \leqq \frac{1}{\lambda_{\mu_\nu}} [\mu_m-(\mu_\nu-\lambda_{\mu_\nu})]^+ =: A_{m,\nu}, \tag{4.8}$$

where $[\beta]^+$ denotes the positive part of β. On account of the definition $\{\mu_m\}$ and the property $\lambda_{\mu_m} \leqq 2\lambda_{\mu_{m-1}}$ we can verify that for any $\nu \geqq m$

$$A_{m,\nu} \leqq \left(\frac{1}{2}\right)^{\nu-m} \tag{4.9}$$

holds. Namely an easy calculation shows that

$$A_{m,\nu} = \left[1 - \frac{\mu_\nu - \mu_m}{\lambda_{\mu_\nu}}\right]^+ = [1 - \lambda_{\mu_\nu}^{-1} \sum_{k=m}^{\nu-1} \lambda_{\mu_k}]^+ \leqq$$

$$\leqq [1 - (2\lambda_{\mu_{\nu-1}})^{-1} \sum_{k=m}^{\nu-1} \lambda_{\mu_k}]^+ = \left[\frac{1}{2} - (2\lambda_{\mu_{\nu-1}})^{-1} \sum_{k=m}^{\nu-2} \lambda_{\mu_k}\right]^+ \leqq$$

$$\leqq \left[\frac{1}{4} - (4\lambda_{\mu_{\nu-2}})^{-1} \sum_{k=m}^{\nu-3} \lambda_{\mu_k}\right]^+ \leqq \ldots \leqq \left(\frac{1}{2}\right)^{\nu-m}.$$

Collecting the results of (4.7), (4.8) and (4.9) we get (2.2) with $M=2$. So we can apply Theorem 1 which obviously proves Theorem 3.

Proof of Theorem D. Let

$$\beta_0 = \beta_1 = \ldots = \beta_{\mu_1} = 1;$$

$$\beta_n := \frac{\log m}{(\mu_{m+1} - \mu_m)m} \quad \text{for} \quad \mu_m < n \leqq \mu_{m+1}, \quad m = 1, 2, \ldots;$$

and $\varkappa=2$. A standard calculation shows that for these β_n and $\varkappa$ (3.1) holds if and only if (1.2) is fulfilled. Namely if (3.1) holds then by

$$\sum_{n=1}^\infty \beta_n \sum_{k=n}^\infty c_k^2 \geqq \sum_{m=1}^\infty \sum_{n=\mu_m+1}^{\mu_{m+1}} \beta_n \sum_{k=\mu_{m+1}+1}^\infty c_k^2 =$$

$$= \sum_{m=1}^\infty \frac{\log m}{m} \sum_{\nu=m+1}^\infty \sum_{k=\mu_\nu+1}^{\mu_{\nu+1}} c_k^2 = \sum_{\nu=2}^\infty \sum_{k=\mu_\nu+1}^{\mu_{\nu+1}} c_k^2 \sum_{m=1}^{\nu-1} \frac{\log m}{m}$$

(1.2) also holds. Conversely if (1.2) is fulfilled then the following inequalities

$$\sum_{m=1}^\infty \Big(\sum_{k=\mu_m+1}^{\mu_{m+1}} c_k^2\Big) \log^2 m \geqq \frac{1}{4} \sum_{m=3}^\infty \Big(\sum_{k=\mu_m+1}^{\mu_{m+1}} c_k^2\Big) \sum_{\nu=2}^{m-1} \frac{\log \nu}{\nu} =$$

$$= \frac{1}{4} \sum_{\nu=2}^\infty \frac{\log \nu}{\nu} \sum_{m=\nu+1}^\infty \sum_{k=\mu_m+1}^{\mu_{m+1}} c_k^2 \geqq \frac{1}{4} \sum_{\nu=2}^\infty \frac{\log(\nu+1)}{\nu+1} \sum_{k=\mu_{\nu+1}+1}^\infty c_k^2 =$$

$$= \frac{1}{4} \sum_{m=3}^\infty \frac{\log m}{m} \sum_{k=\mu_m+1}^\infty c_k^2 \geqq \frac{1}{4} \sum_{m=3}^\infty \sum_{n=\mu_m+1}^{\mu_{m+1}} \beta_n \sum_{k=n}^\infty c_k^2 = \frac{1}{4} \sum_{n=\mu_3+1}^\infty \beta_n \sum_{k=n}^\infty c_k^2$$

prove (3.1).

On account of this equivalence and our Lemma the statement of Theorem D is proved.

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Note on Fourier series with nonnegative coefficients

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1. Let $f(x)$ be a continuous and 2π-periodic function and let

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{1}$$

be its Fourier series. Denote $s_n = s_n(x)$ the n-th partial sum of (1). If $\omega(\delta)$ is a non-decreasing continuous function on the interval $[0, 2\pi]$ having the properties

$$\omega(0) = 0, \quad \omega(\delta_1 + \delta_2) \leqq \omega(\delta_1) + \omega(\delta_2)$$

for any $0 \leqq \delta_1 \leqq \delta_2 \leqq \delta_1 + \delta_2 \leqq 2\pi$, then it will be called modulus of continuity.

Define the following classes of functions

$$H^\omega = \{f\colon \|f(x+h) - f(x)\| = O(\omega(h))\}, \tag{2}$$

$$(H^\omega)^* = \{f\colon \|f(x+h) + f(x-h) - 2f(x)\| = O(\omega(h))\}, \tag{3}$$

where $\|\cdot\|$ denotes the usual maximum norm. If $\omega(\delta) = \delta^\alpha$ then we write $\operatorname{Lip}\alpha$ instead of H^{δ^α}.

In 1948 G. G. LORENTZ [7] proved a theorem containing a coefficient-condition for $f \in \operatorname{Lip}\alpha$ in the case if the sequence of the Fourier coefficients is monotonic. Namely he proved the following result.

Theorem A ([7]). *Let $\lambda_n \downarrow 0$ and let λ_n be the Fourier sine or cosine coefficients of φ. Then $\varphi \in \operatorname{Lip}\alpha$ $(0 < \alpha < 1)$ if and only if $\lambda_n = O(n^{-1-\alpha})$.*

Later this result was generalized by R. P. BOAS [1] in 1967 as follows:

*) This result was partly obtained while the author visited to the Ohio State University, Columbus, U.S.A. in the academic years 1985—86 and 1986—87.

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Theorem B ([1]). *Let $\lambda_n \geqq 0$ and let λ_n be the sine or cosine coefficients of φ. Then $\varphi \in \operatorname{Lip} \alpha$ $(0<\alpha<1)$ if and only if*

$$\sum_{k=n}^{\infty} \lambda_k = O(n^{-\alpha}), \tag{4}$$

or equivalently

$$\sum_{k=1}^{n} k\lambda_k = O(n^{1-\alpha}). \tag{5}$$

In 1980 L. Leindler in connection with certain investigations in the theory of strong approximation by Fourier series, defined some function classes which are more general than Lip α but narrower than H^{ω}. Namely he gave the following definition.

Let $\omega_\alpha(\delta)$ $(0 \leqq \alpha \leqq 1)$ denote a modulus of continuity having the following properties:

(i) for any $\alpha' > \alpha$ there exists a natural number $\mu = \mu(\alpha')$ such that

$$2^{\mu\alpha'} \omega_\alpha(2^{-n-\mu}) > 2\omega_\alpha(2^{-n}) \quad \text{holds for all} \quad n \geqq 1; \tag{6}$$

(ii) for every natural number ν there exists a natural number $N(\nu)$ such that

$$2^{\nu\alpha} \omega_\alpha(2^{-n-\nu}) \leqq 2\omega_\alpha(2^{-n}) \quad \text{if} \quad n > N(\nu). \tag{7}$$

Using $\omega_\alpha(\delta)$ L. Leindler defined the function class Lip ω_α in the following way

$$\operatorname{Lip} \omega_\alpha = \{f \colon \|f(x+h) - f(x)\| = O(\omega_\alpha(h))\}.$$

Recently the author of the present paper generalized the result of R. P. Boas formulated in Theorem B and some other ones for Lip ω_α instead of Lip α.

For example we proved the following

Theorem C ([8]). *Let $\lambda_n \geqq 0$ and λ_n be the Fourier sine or cosine coefficients of φ. Then*

$$\varphi \in \operatorname{Lip} \omega_\alpha \quad (0 < \alpha < 1)$$

if and only if

$$\sum_{k=n}^{\infty} \lambda_n = O\left(\omega_\alpha\left(\frac{1}{n}\right)\right), \tag{8}$$

or equivalently

$$\sum_{k=1}^{n} k\lambda_k = O\left(n\omega_\alpha\left(\frac{1}{n}\right)\right). \tag{9}$$

The question of further generalizations for arbitrary $\omega(\delta)$ and H^{ω} can naturally be arisen.

The first results in this direction were already given by A. I. Rubinstein ([9]) for cosine series, furthermore V. G. Krotov and L. Leindler ([3], see also in [6]) for the sine case. Their results read as follows

Theorem D ([9]). *Let f be an even function belonging to H^{ω} and let a_n be its Fourier coefficients with $a_n \geqq 0$ $(n=1, 2, \ldots)$. Then*

$$\sum_{k=n}^{\infty} a_k = O\left(\omega\left(\frac{1}{n}\right)\right) \tag{10}$$

and

$$\frac{1}{n}\sum_{k=1}^{n} k a_k = O\left(\frac{1}{n}\int_{1/n}^{\delta_0} \frac{\omega(t)}{t^2}\,dt\right) \tag{11}$$

hold for some fix $\delta_0 > 0$.

If ω satisfies the condition

$$\delta \int_{\delta}^{\delta_0} \frac{\omega(t)}{t^2}\,dt = O(\omega(\delta)) \tag{12}$$

then conditions (10) *and* (11) *are sufficient for*

$$f \in H^{\omega}.$$

It should be noted that (10) implies (11) for any ω, namely

$$\sum_{k=1}^{n} k a_k = \sum_{k=1}^{n} \sum_{i=k}^{n} a_i = O(1) \sum_{k=1}^{n} \omega\left(\frac{1}{k}\right) = O(1) \int_{1/n}^{\delta_0} \frac{\omega(t)}{t^2}\,dt,$$

and thus for the special moduli of continuity ω satisfying relation (12) the condition (10) itself is a sufficient condition.

Theorem E ([3] Lemma 3, see also in [6]). *If $\lambda_n \geqq 0$ and*

$$g(x) = \sum_{n=1}^{\infty} \lambda_n \sin nx$$

belongs to the class H^{ω} then

$$\sum_{k=1}^{n} k\lambda_k = O\left(n\omega\left(\frac{1}{n}\right)\right). \tag{13}$$

The aim of this paper is to show that neither (10) nor (13) is sufficient for the corresponding function to be in H^{ω}; furthermore to give sufficient condition for $f \in H^{\omega}$ in both cases. We also show that (10) is a necessary and sufficient condition for f to belong to the class $(H^{\omega})^*$ (which is broader than H^{ω}, so this result in this sense is a little sharper than that of Rubinstein). Finally it will be proved that (10) and (13), respectively, is not only a necessary but also sufficient condition for $f \in H^{\omega}$ and $g \in H^{\omega}$, if the coefficients a_k and b_k form monotonically decreasing sequences.

2. Now we formulate our results.

Theorem 1. *If $\lambda_n \geqq 0$ and λ_n are the Fourier sine or cosine coefficients of φ, then the conditions*

$$\sum_{k=1}^{n} k\lambda_k = O\left(n\omega\left(\frac{1}{n}\right)\right) \tag{14}$$

and

$$\sum_{k=n}^{\infty} \lambda_k = O\left(\omega\left(\frac{1}{n}\right)\right) \tag{15}$$

imply

$$\varphi \in H^{\omega}. \tag{16}$$

Remark. The well-known Weierstrass function

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos 2^n x}{2^n}$$

shows that (15) itself is not sufficient for $\varphi \in H^{\omega}$ (since $f \notin H^{\omega}$ if $\omega(\delta)=\delta$ and (15) is obviously satisfied).

The example

$$g(x) = \sum_{k=1}^{\infty} \frac{1}{4k^2} \sin 3^{4k^2} x$$

proves that from (14) alone (16) does not follow. This function was constructed by A. I. Rubinstein ([9]) in connection with lacunary series. He proved that $g \notin H^{\omega}$, for $\omega(\delta) = \dfrac{1}{|\log_3 \delta|}$. At the same time it can easily be checked that (14) holds.

Theorem 2. *If $a_k \geqq 0$ and a_k are the Fourier cosine coefficients of f then*

$$f \in (H^{\omega})^* \tag{17a}$$

if and only if

$$\sum_{k=n}^{\infty} a_k = O\left(\omega\left(\frac{1}{n}\right)\right). \tag{17b}$$

Remark. Notice that (17b) implies

$$\sum_{k=1}^{n} k a_k \leqq K \sum_{k=1}^{n} \omega\left(\frac{1}{k}\right) \tag{18}$$

and using the standard estimation we have that (18) implies

$$f \in H^{\omega_*},$$

where

$$\omega_*(t) := t \sum_{k=1}^{[1/t]} \omega\left(\frac{1}{k}\right).$$

In fact, since from (17a) (18) follows we have that

$$\sum_{k=n}^{\infty} a_k = O\left(\omega\left(\frac{1}{n}\right)\right) \tag{19}$$

implies $f \in H^{\omega_*}$ and Theorem 2 gives that the same condition (17a) implies

$$f \in (H^{\omega})^*,$$

too. Thus the following question can be arisen: whether

$$f \in H^{\omega_*} \Leftrightarrow f \in (H^{\omega})^* \tag{20}$$

or not.

We can prove that

$$f \in (H^{\omega})^* \Rightarrow f \in H^{\omega_*} \tag{21}$$

but the converse is false. Really, from Theorem 2 we have that

$$f \in (H^{\omega})^* \Rightarrow \sum_{k=n}^{\infty} a_k = O\left(\omega\left(\frac{1}{n}\right)\right)$$

which assures that $f \in H^{\omega_*}$, so (21) is proved. In order to prove that

$$f \in H^{\omega_*} \not\Rightarrow f \in (H^{\omega})^* \tag{22}$$

we consider the following function

$$f(x) = \sum_{k=1}^{\infty} \frac{\log n}{n^2} \cos nx \tag{23}$$

and let $\omega(t) = t$, that is, $\omega_*(t) = t \log t$. From Theorem 4 of [8] it follows that

$$f \in H^{\delta \log \delta} \quad (= H^{\omega_*})$$

because both

$$\sum_{h=n}^{\infty} \frac{\log k}{k^2} = O\left(\frac{\log n}{n}\right) \tag{24}$$

and

$$\left\| \sum_{k=1}^{n} \frac{\log k}{k} \sin kx \right\| = O(\log n) \tag{25}$$

hold. And at the same time

$$f \notin (H^{\delta})^* = (H^{\omega})^*,$$

because

$$\frac{1}{2} |f(0+2h) + f(0-2h) - 2f(0)| = \sum_{n=1}^{\infty} \frac{\log n}{n^2} (1 - \cos 2nh) =$$

$$= 2 \sum_{n=1}^{\infty} \frac{\log n}{n^2} \sin^2 nh \geqq 2h^2 \sum_{n=1}^{[1/h]} \log n \frac{\sin^2 nh}{n^2 h^2} \geqq 2h |\log h|,$$

which gives that

$$\|f(x+h)+f(x-h)-2f(x)\| \neq O(h),$$

that is, $f\notin(H^\delta)^*$ and so (22) is proved.

Theorem 3. *If* $b_k\downarrow 0$ *and* $g(x)=\sum_{k=1}^{\infty} b_k \sin kx$ *then*

(26a) $$g\in H^\omega$$

if and only if

(26b) $$\sum_{k=1}^{n} kb_k = O\left(n\omega\left(\frac{1}{n}\right)\right).$$

Theorem 4. *If* $a_k\downarrow 0$ *and* $f(x)=\sum_{k=1}^{\infty} a_k \cos kx$ *then*

(27a) $$f\in H^\omega$$

if and only if

(27b) $$\sum_{k=n}^{\infty} a_k = O\left(\omega\left(\frac{1}{n}\right)\right).$$

3. We require the following lemmas.

Lemma 1. *Let* $\{a_n\}$ *be a sequence of nonnegative numbers and* ω *be a modulus of continuity. Then*

(28) $$\sum_{k=n}^{\infty} a_k = O\left(\omega\left(\frac{1}{n}\right)\right)$$

implies

(29) $$\sum_{k=1}^{n} k^2 a_k = O\left(n^2\omega\left(\frac{1}{n}\right)\right).$$

Proof.*) Using (28) we have

(30) $$\sum_{k=1}^{n} k^2 a_k = \sum_{k=1}^{n}(2k-1)\sum_{i=k}^{n} a_i \leqq 2\sum_{k=1}^{n} k\omega\left(\frac{1}{k}\right) = I.$$

Since for any ω the inequality

(31) $$\frac{\omega(x_1)}{x_1} \leqq 2\,\frac{\omega(x_2)}{x_2} \quad (0<x_2\leqq x_1)$$

*) This very elegant proof is due to L. Leindler; the author's original one was much more complicated.

(see for example [11] p. 103) holds I can be estimated as follows

(32) $$I \leqq 2n \cdot 2n\omega\left(\frac{1}{n}\right) = 4n^2\omega\left(\frac{1}{n}\right).$$

Thus (30) and (32) give (29).

Lemma 2. *Let* $a_k \geqq 0$ *and* a_k *be the Fourier cosine coefficients of* f. *Then*

(33) $$\sum_{k=n}^{\infty} a_k = O\left(\omega\left(\frac{1}{n}\right)\right) \quad \textit{and} \quad \left\|\sum_{k=1}^{n} k a_k \sin kx\right\| = O\left(n\omega\left(\frac{1}{n}\right)\right)$$

imply

(34) $$f \in H^{\omega}.$$

This lemma can be proved in the same way as Theorem 4 in [8] for $\omega_1(\delta)$.

4. Proofs.

Proof of Theorem 1. We detail the proof just for cosine series. Set

(35) $$|f(x+2h)-f(x)| = 2\left|\sum_{k=1}^{\infty} \lambda_k \sin k(x+h) \sin kh\right| \leqq$$

$$\leqq 2\Big(\sum_{k=1}^{[1/h]} \lambda_k \sin kh + \sum_{k=[1/h]}^{\infty} \lambda_k\Big) = I + II.$$

Since

(36) $$I \leqq Kh \sum_{k=1}^{[1/h]} k\lambda_k \frac{\sin kh}{kh} \leqq K_1 h \sum_{k=1}^{[1/h]} k\lambda_k,$$

from (14) it follows that

(37) $$I = O(\omega(h)).$$

By using (15) we have that

(38) $$II = O(\omega(h)).$$

So (35), (36), (37) and (38) give that

$$f \in H^{\omega}.$$

Theorem 1 is completed.

Proof of Theorem 2. Suppose that (17b) holds. Then

(39) $$|f(x+2h)+f(x-2h)-2f(x)| = 4\left|\sum_{k=1}^{\infty} a_k \sin^2 kh \cos kx\right| \leqq$$

$$\leqq 4\sum_{k=1}^{\infty} a_k \sin^2 kh = 4h^2 \sum_{k=1}^{[1/h]} k^2 a_k \frac{\sin^2 kh}{k^2 h^2} + \sum_{k=[1/h]}^{\infty} a_k.$$

The first item of the last formula does not exceed $O(\omega(h))$ because of Lemma 1; and from (17b) we get that the second one is also $O(\omega(h))$. So (17b)$\Rightarrow$(17a) is proved.

Turning to prove (17a)$\Rightarrow$(17b) first we note that the proof will be led by the same way as A. I. Rubinstein did in [9]. Let $I_n(x, g)$ be the Jackson polynomial defined by

$$I_n(x, g) = \frac{3}{2n\pi(2n^2+1)} \int_{-\pi}^{\pi} g(t) \left(\frac{\sin n \frac{t-x}{2}}{\sin \frac{t-x}{2}} \right)^4 dt. \tag{40}$$

This polynomial can be written in the following form

$$I_n(x, g) = \frac{a_0}{2} + \sum_{k=1}^{2n-2} \varrho_k^{(n)} (a_k \cos kx + b_k \sin kx), \tag{41}$$

where a_k, b_k are the Fourier coefficients of g and $\varrho_k^{(n)}$ are defined as follows

$$\varrho_k^{(n)} = \frac{1}{2n(2n^2+1)} \left[\frac{(2n-k+1)!}{(2n-k-2)!} - 4 \frac{(n-k+1)!}{(n-k-2)!} \right] \quad \text{for} \quad 1 \leqq k \leqq n-2, \tag{42}$$

$$\varrho_k^{(n)} = \frac{1}{2n(2n^2+1)} \frac{(2n-k+1)!}{(2n-k-2)!} \quad \text{for} \quad n-2 < k \leqq 2n-2.$$

Formula (42) was given by G. P. Safranova ([10]).

Consider the following difference for

$$f(x) = \sum_{k=1}^{\infty} a_k \cos kx$$

$$f(0) - I_n(0; f) = \sum_{k=1}^{2n-2} (1 - \varrho_k^{(n)}) a_k + \sum_{k=2n-1}^{\infty} a_k. \tag{43}$$

It can be proved that the order of approximation by polynomial (40) is $O\left(\omega\left(\frac{1}{n}\right)\right)$ for

$$g \in (H^\omega)^*$$

(see for example [2] pp. 496—497).

Using this fact and that $1 - \varrho_k^{(n)} \geqq 0$ we have from (43)

$$\sum_{k=2n-1}^{\infty} a_k = O\left(\omega\left(\frac{1}{n}\right)\right),$$

which was to be proved.

Theorem 2 is completed.

Proof of Theorem 3. The statement (26a)$\Rightarrow$(26b) was proved by V. G. Krotov and L. Leindler (see Theorem E). Now we suppose (26b). It is obvious that to prove (26a) it is sufficient to show

(44) $$|g(h)-g(0)| \leqq K_1 \cdot \omega(h)$$

and

(45) $$|g(x)-g(x+h)| \leqq K_2 \omega(h), \quad \text{for} \quad 0 < h \leqq x \leqq \pi.$$

First we prove (44).

Set

(46) $$|g(h)-g(0)| \leqq \Big|\sum_{k=1}^{[1/h]} b_k \sin kh\Big| + \Big|\sum_{k=[1/h]}^{\infty} b_k \sin kh\Big| = \mathrm{I}+\mathrm{II}.$$

Using (26b) we can estimate I as follows

(47) $$\mathrm{I} \leqq h \sum_{k=1}^{[1/h]} kb_k \frac{\sin kh}{kh} \leqq Kh \sum_{k=1}^{[1/h]} kb_k = O(\omega(h)).$$

From the well-known inequality

(48) $$\Big|\sum_{k=n}^{m} a_k \sin kx\Big| \leqq \frac{4}{x} a_n \quad (a_n \downarrow,\ x \in (0, \pi))$$

it follows that

(49) $$\mathrm{II} \leqq 4 \frac{1}{h} b_{[1/h]}.$$

But taking into account that $b_k \downarrow$, from (26b) we have

(50) $$b_n = O\left(\frac{\omega\left(\frac{1}{n}\right)}{n}\right).$$

From (49) and (50) we get

(51) $$\mathrm{II} = O(\omega(h)).$$

Using (46), (47) and (51) we obtain (44).

Now we verify (45). Consider

(52) $$|g(x+h)-g(x)| = \Big|\sum_{k=1}^{\infty} b_k (\sin kx - \sin k(x+h))\Big| \leqq$$

$$\leqq \Big|\sum_{k=1}^{[1/h]} b_k \cos k(x+h) \sin kh\Big| + \Big|\sum_{k=[1/h]}^{\infty} b_k \sin kx - \sin k(x+h)\Big| \leqq$$

$$\leqq \Big|\sum_{k=1}^{[1/h]} b_k \sin kh\Big| + \Big|\sum_{k=[1/h]}^{\infty} b_k \sin kx\Big| + \Big|\sum_{k=[1/h]}^{\infty} b_k \sin k(x+h)\Big| = \mathrm{I}+\mathrm{II}'+\mathrm{II}''.$$

By (47) we have

(53) $$\mathrm{I} = O(\omega(h)).$$

Taking into account again (48), (49) and the condition $0<h\leqq x$ we have that the magnitude of either II′ or II″ is $O(\omega(h))$, that is,

$$\mathrm{II}'+\mathrm{II}''=O(\omega(h)). \tag{54}$$

Thus (52), (53) and (54) give (45) which is the desired statement.

Theorem 3 is completed.

Proof of Theorem 4. Using Theorem 2 and the fact that $H^\omega\subset(H^\omega)^*$ the statement (27a)⇒(27b) can immediately be obtained. Concerning the opposite direction, by Lemma 2, it is enough to show that

$$\Big\|\sum_{k=1}^{n} ka_k\sin kx\Big\|\leqq K\cdot n\omega\left(\frac{1}{n}\right). \tag{55}$$

Let $x\in(0,\pi)$ be fixed and let ν denote $\left[\frac{1}{x}\right]$; if $n>\frac{1}{x}$, then split up the left hand side of (55) into two parts as follows

$$\Big\|\sum_{k=1}^{n} ka_k\sin kx\Big\|\leqq\Big\|\sum_{k=1}^{\nu} ka_k\sin kx\Big\|+\Big\|\sum_{k=\nu=1}^{n} ka_k\sin kx\Big\|=\mathrm{I}+\mathrm{II}. \tag{56}$$

Estimating I we get

$$\mathrm{I}\leqq K_1x\sum_{k=1}^{\nu}k^2a_k. \tag{57}$$

Taking into account the monotonity of a_k and (27b) we have

$$ka_k=O\left(\omega\left(\frac{1}{k}\right)\right). \tag{58}$$

From (58) it follows that

$$x\sum_{k=1}^{\nu}k^2a_k\leqq K_2x\sum_{k=1}^{\nu}k\omega\left(\frac{1}{k}\right)\leqq K_3n\omega\left(\frac{1}{n}\right). \tag{59}$$

In the last step we used again inequality (31) and $n>\nu$.

Thus from (57) and (59)

$$\mathrm{I}\doteq O\left(n\omega\left(\frac{1}{n}\right)\right) \tag{60}$$

can be obtained.

Now we estimate the second item in (56). Since

$$\mathrm{II}=\Big\|\sum_{k=\nu}^{n} ka_k\sin kx\Big\|\leqq\Big\|\sum_{k=\nu}^{n}\sum_{i=k}^{n}a_i\sin ix\Big\|+\nu\sum_{i=\nu}^{n}a_i=\mathrm{II}'+\mathrm{II}'' \tag{61}$$

and using again (48) and (58)

$$\mathrm{II}' \leqq K \sum_{k=\nu}^{n} \frac{a_k}{x} \leqq K_1 \nu \sum_{k=\nu}^{n} k\omega\left(\frac{1}{k}\right)\frac{1}{k^2} \leqq K_2 n\omega\left(\frac{1}{n}\right). \tag{62}$$

And for II″ using (58) we immediately obtain that

$$\mathrm{II}'' \leqq K_3 n\omega\left(\frac{1}{n}\right), \tag{63}$$

and (61), (62), (63) give that

$$\mathrm{II} = O\left(n\omega\left(\frac{1}{n}\right)\right). \tag{64}$$

Thus (60) and (64) together give (56) which was to be proved.

Theorem 4 is completed.

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On Fourier series with nonnegative coefficients

J. NÉMETH

1. Let $f(x)$ be a continuous and 2π periodic function and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

be its Fourier series. Denote by $s_n = s_n(x) = s_n(f; x)$ the n-th partial sum of (1).

If $\omega(\delta)$ is a nondecreasing continuous function on the interval $[0, 2\pi]$ having the following properties

$$\omega(0) = 0, \quad \omega(\delta_1 + \delta_2) \leqq \omega(\delta_1) + \omega(\delta_2)$$

for any $0 \leqq \delta_1 \leqq \delta_2 \leqq \delta_1 + \delta_2 \leqq 2\pi$ then it will be called modulus of continuity. As usually $W^r H^\omega$ and $W^r (H^\omega)^*$ denote the following function classes:

$$W^r H^\omega = \{f\colon \|f^{(r)}(x+h) - f^{(r)}(x)\| = O(\omega(h))\}, \tag{2}$$

$$W^r (H^\omega)^* = \{f\colon \|f^{(r)}(x+h) + f^{(r)}(x-h) - 2f^{(r)}(x)\| = O(\omega(h))\}, \tag{3}$$

where $f^{(r)}$ denotes the r-th derivative of f, and $\|\cdot\|$ denotes the usual supremum norm. For $r=0$ and $\omega(\delta) = \delta^\alpha$ $H^\omega = H^{\delta^\alpha}$ is called the Lipschitz class of order α.

L. LEINDLER ([3]) defined the so called generalized Lipschitz-classes as follows. For $0 \leqq \alpha \leqq 1$ let $\omega_\alpha(\delta)$ denote a modulus of continuity having the following properties

(i) for any $\alpha' > \alpha$ there exists a natural number $\mu = \mu(\alpha')$ such that

$$2^{\mu\alpha'} \omega_\alpha(2^{-n-\mu}) > 2\omega_\alpha(2^{-n}) \quad \text{holds for all} \quad n \geqq 1; \tag{4}$$

(ii) for every natural ν there exists a natural number $N(\nu)$ such that

$$2^{\nu\alpha} \omega_\alpha(2^{-n-\nu}) \leqq 2\omega_\alpha(2^{-n}) \quad \text{if} \quad n \geqq N(\nu). \tag{5}$$

Using such modulus of continuity, H^{ω_α} defines the generalized Lipschitz class.

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For any positive β and p L. LEINDLER ([2]) defined the following strong means and function classes:

$$h_n(f,\beta,p)=\left\|\left\{\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_k(x)-f(x)|^p\right\}^{1/p}\right\|, \tag{6}$$

$$H(\beta,p,r,\omega)=\left\{f:\ h_n(f,\beta,p)=O\left(n^{-r}\omega\left(\frac{1}{n}\right)\right)\right\}, \tag{7}$$

and in [3] and [4] he proved the following relations:

$$\left.\begin{array}{ll} H(\beta,p,r,\omega_\alpha)\equiv W^rH^{\omega_\alpha} & \text{for}\quad 0<\alpha<1; \\ W^rH^{\omega_1}\subset H(\beta,p,r,\omega_1)\equiv W^r(H^{\omega_1})^* & \text{for}\quad \alpha=1 \end{array}\right\}\ \text{if}\quad \beta>(r+\alpha)p. \tag{8}$$

In [8] we gave coefficient-conditions assuring that a function should belong to H^{ω_α} (and so in certain cases to $H(\beta,p,\omega_\alpha)$).

For example the following theorem was proved.

Theorem A (Theorem 1 of [8]). *Let* $\lambda_n\geqq 0$ *and* λ_n *be the Fourier sine or cosine coefficients of* $\varphi(x)$. *Then*

$$\varphi\in H^{\omega_\alpha}\quad (0<\alpha<1)$$

if and only if

$$\sum_{k=n}^{\infty}\lambda_n=O\left(\omega_\alpha\left(\frac{1}{n}\right)\right), \tag{9}$$

or equivalently

$$\sum_{k=1}^{n}k\cdot\lambda_k=O\left(n\omega_\alpha\left(\frac{1}{n}\right)\right). \tag{10}$$

It is clear that in order to obtain coefficient-conditions of type (9) for f to belong to $H(\beta,p,r,\omega_\alpha)$ instead of $H(\beta,p,\omega_\alpha)$ it is sufficient to give conditions assuring that f should belong to $W^rH^{\omega_\alpha}$ or equivalently, under the restriction $\lambda_n\geqq 0$, to $H(\beta,p,r,\omega_\alpha)$. In other words it is sufficient to find coefficient conditions for the derivatives of f to be in H^{ω_α}.

In the special cases $\omega(\delta)=\delta^\alpha$ coefficient-conditions for $f\in H^{\delta^\alpha}$ and $f\in W^rH^{\delta^\alpha}$ were given by G. G. LORENTZ ([7]), R. P. BOAS ([1]) and LING-YAU CHAN ([6]).

2. Theorems. Throughout the rest of the paper we shall assume that the Fourier coefficients a_n, b_n are nonnegative and

$$g(x)=\sum_{k=1}^{\infty}b_k\sin kx,\quad f(x)=\sum_{k=1}^{\infty}a_k\cos kx,$$

furthermore f and g are continuous functions on $[0,\pi]$.

Theorem 1. *If* $0<\alpha<1$ *then for any* $r\geqq 1$

$$g\in W^rH^{\omega_\alpha}$$

if and only if

$$\sum_{k=1}^{n} k^{r+1} b_k = O\left(n\omega_\alpha\left(\frac{1}{n}\right)\right),$$

or equivalently

$$\sum_{k=n}^{\infty} k^{r} b_k = O\left(\omega_\alpha\left(\frac{1}{n}\right)\right).$$

Theorem 2. *If* $0<\alpha<1$ *then for any* $r\geqq 1$

$$f\in W^r H^{\omega_\alpha}$$

if and only if

$$\sum_{k=1}^{n} k^{r+1} a_k = O\left(n\omega_\alpha\left(\frac{1}{n}\right)\right),$$

or equivalently

$$\sum_{k=n}^{\infty} k^{r} a_k = O\left(\omega_\alpha\left(\frac{1}{n}\right)\right).$$

Theorem 3. *If* $\alpha=1$ *and* r *is odd, then*

$$g\in W^r (H^{\omega_1})^*$$

if and only if

$$\sum_{k=1}^{n} k^{r+2} b_k = O\left(n^2\omega_1\left(\frac{1}{n}\right)\right).$$

Theorem 4. *If* $\alpha=1$ *and* r *is even* $(r\geqq 2)$, *then*

$$g\in W^r H^{\omega_1}$$

if and only if

$$\sum_{k=1}^{n} k^{r+1} b_k = O\left(n\omega_1\left(\frac{1}{n}\right)\right).$$

Theorem 5. *If* $\alpha=1$ *and* r *is even* $(r\geqq 0)$, *then*

$$f\in W^r (H^{\omega_1})^*$$

if and only if

$$\sum_{k=1}^{n} k^{r+2} a_k = O\left(n^2\omega_1\left(\frac{1}{n}\right)\right).$$

Theorem 6. *If* $\alpha=1$ *and* r *is odd, then*

$$f\in W^r H^{\omega_1}$$

if and only if

$$\sum_{k=1}^{n} k^{r+1} a_k = O\left(n\omega_1\left(\frac{1}{n}\right)\right).$$

Theorem 7. *If* $\alpha=1$ *and* r *is odd, then*

$$g\in W^r H^{\omega_1}$$

if and only if

$$\sum_{k=1}^{n} k^{r+2} b_k = O\left(n^2\omega_1\left(\frac{1}{n}\right)\right) \quad \text{and} \quad \left\|\sum_{k=1}^{n} k^{r+1} b_k \sin kx\right\| = O\left(n\omega_1\left(\frac{1}{n}\right)\right).$$

Theorem 8. *If* $\alpha=1$ *and* r *is even* $(r\geqq 0)$, *then*

$$f\in W^r H^{\omega_1}$$

if and only if

$$\sum_{k=1}^{n} k^{r+2} a_k = O\left(n^2\omega_1\left(\frac{1}{n}\right)\right) \quad \text{and} \quad \left\|\sum_{k=1}^{n} k^{r+1} a_k \sin kx\right\| = O\left(n\omega_1\left(\frac{1}{n}\right)\right).$$

3. Lemmas.

Lemma 1 (Lemma 2 of [8]). *If* $\mu_k\geqq 0$ *and* $\delta>\beta>0$, *then*

$$\sum_{k=1}^{n} k\cdot\mu_k = O\left(n^\delta\omega_{\delta-\beta}\left(\frac{1}{n}\right)\right)$$

if and only if

$$\sum_{k=n}^{\infty} \mu_k = O\left(\omega_{\delta-\beta}\left(\frac{1}{n}\right)\right).$$

Lemma 2. (Lemma 2 of [6]). *For each integer* $j\geqq 0$ *the quantity*

$$G(j,u) = \sin u - u + \frac{u^3}{3!} - \dots + \frac{(-1)^{j+1}}{(2j+1)!}u^{2j+1}$$

is of constant sign for all $u>0$. *Furthermore if* $0<u\leqq 1$, *then*

$$|G(j,u)| \geqq \frac{u^{2j+3}}{(2j+3)!\,2}.$$

Lemma 3 (Lemma 3 of [6]). *For each integer* $j\geqq 0$ *the quantity*

$$F(j,u) = \cos u - 1 + \frac{u^2}{2!} - \dots + (-1)^{j+1}\frac{u^{2j}}{(2j)!}$$

is of constant sign for all $u>0$. *Furthermore, if* $0\leqq u\leqq 1$, *then*

$$|F(j.u)| \geqq \frac{u^{2j+2}}{(2j+2)!\,2}.$$

Lemma 4 (Theorem 2 of [8]).

$$g\in H^{\omega_1}$$

if and only if

$$\sum_{k=1}^{n} kb_k = O\left(n\omega_1\left(\frac{1}{n}\right)\right).$$

Lemma 5 (Theorem 3 of [8]).

$$f\in(H^{\omega_1})^*$$

if and only if

$$\sum_{k=n}^{\infty} a_k = O\left(\omega_1\left(\frac{1}{n}\right)\right).$$

Lemma 6 (Theorem 4 of [8]).

$$f\in H^{\omega_1}$$

if and only if

$$\sum_{k=n}^{\infty} a_k = O\left(\omega_1\left(\frac{1}{n}\right)\right) \quad \textit{and} \quad \left\|\sum_{k=1}^{n} ka_k \sin kx\right\| = O\left(n\omega_1\left(\frac{1}{n}\right)\right).$$

4. Proofs. Since the proofs of all theorems above mentioned can be done in the same way as LING-YAU CHAN did in [6] (by using Theorem A and Lemma 1—Lemma 6 instead of those used in [6]) we here show only the proof of Theorem 1 for $r=1$.

Let us suppose that $0<\alpha<1$ and

$$\sum_{k=1}^{n} k^2 b_k = O\left(n\omega_\alpha\left(\frac{1}{n}\right)\right). \tag{11}$$

By Lemma 1 we get that (11) is equivalent to

$$\sum_{k=n}^{\infty} kb_k = O\left(\omega_\alpha\left(\frac{1}{n}\right)\right). \tag{12}$$

So $\sum_{k=1}^{\infty} kb_k$ is convergent series, that is, the series

$$\sum_{k=1}^{\infty} kb_k \cos kx \tag{13}$$

is convergent uniformly which allows us to differentiate the series

$$\sum_{k=1}^{\infty} b_k \sin kx \tag{14}$$

term by term, which gives that

$$g'(x) = \sum_{k=1}^{\infty} kb_k \cos kx. \tag{15}$$

Using Theorem A and (11) we have that

$$g'\in H^{\omega_\alpha},$$

that is

$$g \in W^1 H^{\omega_\alpha},$$

which proves our Theorem 1 in the case $r=1$ in one direction.

For the other direction we assume that

$$g' \in H^{\omega_\alpha}, \tag{16}$$

that is,

$$g \in W^1 H^{\omega_\alpha}.$$

From (16) it is obtained that

$$|g'(t)-g'(0)| = O\big(\omega_\alpha(t)\big). \tag{17}$$

Integrating both sides over $(0, x]$ we have

$$|g(x)-xg'(0)| = O\big(x\omega_\alpha(x)\big). \tag{18}$$

Using (18) we have that

$$g(x) = O(x). \tag{19}$$

But (19), by using Lemma 4 (for $\omega_1(\delta)=\delta$) and the fact that

$$|g(x)-g(0)| = O(x)$$

implies $g(x) \in H^\delta$, what gives that

$$\sum_{k=1}^{n} kb_k = O(1) \tag{20}$$

that gives that the series

$$\sum_{k=1}^{\infty} kb_k \cos kx \tag{21}$$

is convergent uniformly, so the series

$$g(x) = \sum_{k=1}^{\infty} b_k \sin kx \tag{22}$$

can be differentiated term by term, that is,

$$g'(x) = \sum_{k=1}^{\infty} kb_k \cos kx \quad \text{and} \quad g'(0) = \sum_{k=1}^{\infty} kb_k. \tag{23}$$

Combining (18) and (23) we have

$$\sum_{k=1}^{\infty} b_k(\sin kx - kx) = O\big(x\omega_\alpha(x)\big). \tag{24}$$

Using Lemma 2 (for $u=kx$) we get from (24)

(25) $$\sum_{k=1}^{[1/x]} b_k(\sin kx-kx) = O(x\omega_\alpha(x)).$$

Using again Lemma 2 (for $u=kx$) we have

(26) $$\sum_{k=1}^{[1/x]} k^3 b_k x^3 = O(x\cdot\omega_\alpha(x)).$$

Putting $\left[\frac{1}{x}\right]=n$ we have that

(27) $$\sum_{k=1}^{n} k^3 b_k = O\left(n^2\omega_\alpha\left(\frac{1}{n}\right)\right).$$

Using Lemma 1 from (27) we obtain the desired

$$\sum_{k=1}^{n} k^2 b_k = O\left(n\omega_\alpha\left(\frac{1}{n}\right)\right).$$

The proof of Theorem is completed for $r=1$.

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Pointwise limits of nets of multilinear maps

ÁRPÁD SZÁZ

Introduction. Motivated by the fact that most of the standard integrals are pointwise limits of the nets of their approximating sums which are either linear or bilinear maps (see [7] and [8]), we establish the most important algebraic and topological properties of the pointwise limit of a net of multilinear maps.

More concretely, using our former results on bounded nets [14] and multi-preseminorms [15], we show that the pointwise limit of a net of multilinear maps being equicontinuous at the origin is a selectionally boundedly uniformly continuous multilinear relation whose domain is a closed set whenever the range space is complete.

Having had the necessary definitions, it becomes clear that particular cases of this assertion greatly improve a useful continuity criterion for multilinear maps [3, (18.2) Theorem], a general convergence theorem for net integrals [7, Theorem 3.8] and a part of a generalized Banach—Steinhaus theorem [1, 7. (5)].

1. Prerequisites. Instead of topological vector spaces, it is often more convenient to use preseminormed spaces [9]. A preseminormed space over $\mathbf{K}=\mathbf{R}$ or $\mathbf{C}$ is an ordered pair $X(\mathcal{P})=(X, \mathcal{P})$ consisting of a vector space X over $\mathbf{K}$ and a nonvoid family $\mathcal{P}$ of preseminorms on X. A preseminorm on X is a subadditive real-valued function p on X such that $p(\lambda x)\leqq p(x)$ for all $|\lambda|\leqq 1$ and $x\in X$, and $\lim\limits_{\lambda\to 0} p(\lambda x)=0$ for all $x\in X$. Note that these latter properties imply, in particular, that $p(0)=0$ and $p(\lambda x)\leqq p(\mu x)$ for all $|\lambda|\leqq|\mu|$ and $x\in X$.

If $X(\mathcal{P})$ is a preseminormed space, then because of [4, Theorem 6.3], the family of all surroundings

$$B_p^r = \{(x, y)\colon\ p(x-y) < r\},$$

where $p\in\mathcal{P}$ and $r>0$, is a subbase for a uniformly $\mathcal{U}_{\mathcal{P}}$ on X. However, this fact is only of minor importance for us now since among $\mathcal{U}_{\mathcal{P}}$ and the various structures on X induced by $\mathcal{U}_{\mathcal{P}}$ we shall actually need only the induced net convergence $\lim_{\mathcal{P}}=\lim_{\mathcal{U}_{\mathcal{P}}}$ which can also be naturally derived directly from $\mathcal{P}$.

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If $X(\mathcal{P})$ is a preseminormed space, then $\lim_{\mathcal{P}}$ is a relation between nets (x_α) and points x in X such that, after a customary convention in the notation, we have $x \in \lim_{\alpha} {}_{\mathcal{P}}\, x_\alpha$ if and only if $\lim_{\alpha} p(x_\alpha - x) = 0$ for all $p \in \mathcal{P}$. As usual a net (x_α) in $X(\mathcal{P})$ is called a convergent net (a null net) if $\lim_{\alpha} {}_{\mathcal{P}}\, x_\alpha \neq \emptyset$ $(0 \in \lim_{\alpha} {}_{\mathcal{P}}\, x_\alpha)$. Moreover, two nets (x_α) and (y_α) in $X(\mathcal{P})$ are called coherent [12] if $(x_\alpha - y_\alpha)$ is a null net. Note that several useful properties of the convergence $\lim_{\mathcal{P}}$ can be easily derived from the usual properties of the convergence of nets of real numbers by using the above properties of preseminorms.

On the other hand, a net (x_α) in $X(\mathcal{P})$ is called a bounded net (a Cauchy net) if

$$\lim_{\lambda \to 0} \overline{\lim_{\alpha}}\, p(\lambda x_\alpha) = 0 \quad \big(\lim_{(\alpha, \beta)} p(x_\alpha - x_\beta) = 0\big)$$

for all $p \in \mathcal{P}$. In [14], we have proved that all Cauchy nets in $X(\mathcal{P})$ are bounded. And a net (x_α) in $X(\mathcal{P})$ is a bounded net (a Cauchy net) if and only if for any subnet (y_β) of (x_α) and any null net (λ_β) of scalars $(\lambda_\beta y_\beta)$ is a null net (any two subnets (z_ν) and (w_ν) of (x_α) are coherent).

Another remarkable feature of this new definition of bounded nets is that a nonvoid subset A of $X(\mathcal{P})$ may henceforth be called bounded if the identity function $(x)_{x \in A}$ of A is bounded as a net whenever A is considered to be directed such that $x \leqq y$ for all $x, y \in A$. Note that A is therefore bounded if and only if

$$\lim_{\lambda \to 0} \sup_{x \in A} p(\lambda x) = 0$$

for all $p \in \mathcal{P}$. And thus nets contained in bounded subsets of $X(\mathcal{P})$ are necessarily bounded.

Having the above definition of bounded nets, we may also define a function f from a subset D of $X(\mathcal{P})$ into another preseminormed space $Y(Q)$ to be boundedly uniformly continuous if $(f(x_\alpha))$ and $(f(y_\alpha))$ are coherent nets in $Y(Q)$ whenever (x_α) and (y_α) are bounded coherent nets in D. Note that f may be called uniformly continuous if it maps coherent nets into coherent nets. Thus, if f is uniformly continuous, then f is also boundedly uniformly continuous. On the other hand, if f is boundedly uniformly continuous, then f is necessarily continuous and the restrictions of f to bounded subsets of D are uniformly continuous.

In the sequel, we shall also need a straightforward notion of a product preseminormed space from [10]. If $X_i(\mathcal{P}_i)$ is a preseminormed space for each i in a nonvoid set I, and moreover

$$X = \bigtimes_{i \in I} X_i \quad \text{and} \quad \mathcal{P} = \bigcup_{i \in I} \mathcal{P}_i \circ \pi_i,$$

where π_i is the projection of X onto X_i and $\mathcal{P}_i \circ \pi_i = \{p \circ \pi_i \colon p \in \mathcal{P}_i\}$, then the preseminormed space $X(\mathcal{P})$ is called the Cartesian product of the spaces $X_i(\mathcal{P}_i)$ and the notation

$$X(\mathscr{P}) = \bigtimes_{i\in I} X_i(\mathscr{P}_i)$$

is used. An important consequence of this definition is that a net (x_α) in $X(\mathscr{P})$ is convergent, Cauchy, resp. bounded if and only if each of its coordinate nets $(x_{\alpha i})$ has the corresponding property.

Finally, a real-valued function p on a product vector space $X = \bigtimes_{i=1}^{n} X_i$ is called a multi-preseminorm [15] if it is a preseminorm in each of its variables separately, and moreover

$$p(x_1, \ldots, x_{i-1}, \lambda x_i, x_{i+1}, \ldots, x_n) = p(x_1, \ldots, x_{k-1}, \lambda x_k, x_{k+1}, \ldots, x_n)$$

for all $x=(x_i)\in X$, scalar λ and $i, k=1,2,\ldots,n$. The importance of this notion lies mainly in the fact that a multilinear map f from a product preseminormed space

$$X(\mathscr{P}) = \bigtimes_{i=1}^{n} X_i(\mathscr{P}_i)$$

into an arbitrary preseminormed space $Y(\mathscr{Q})$ is boundedly uniformly continuous if and only if the multi-preseminorm $q\circ f$ is continuous at the origin of $X(\mathscr{P})$ for all $q\in\mathscr{Q}$.

2. Multilinear relations. Since the pointwise limit of a net of multilinear maps is, in general, only a relation which need not be defined on the whole product space, the usual concept of a multilinear map [3, p. 72] has to be subtantially extended.

For this, we need a straightforward notion of a linear relation from [17] which is mainly motivated by the fact that the inverse of a linear function is a linear relation.

Definition 2.1. A relation f from a vector space X over $\mathbf{K}$ into another Y is called *linear* if

$$f(x)+f(y) \subset f(x+y) \quad \text{and} \quad \lambda f(x) \subset f(\lambda x)$$

for all $x, y\in X$ and $\lambda\in\mathbf{K}$.

Remark 1.2. Note that, in other words, this means only that f is a linear subspace of the product space $X\times Y$ such that the set $f(x)=\{y\colon (x, y)\in f\}$ is not empty for all $x\in X$.

After having this self-evident definition now we can easily define a sufficiently general notion of a multilinear relation whose insufficient particular case has already been considered in [18].

Definition 2.3. Let X_i be a vector space over $\mathbf{K}$ for all $i=1, 2, \ldots, n$, and

$$X = \mathop{\times}_{i=1}^{n} X_i.$$

For each $x=(x_i)\in X$ and $i=1, 2, \ldots, n$, denote by φ_{xi} the function defined on X_i by

$$\varphi_{xi}(t) = (x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n).$$

A subset D of X will be called *multilinear* if the set

$$D_{xi} = \overset{-1}{\varphi}_{xi}(D)$$

is a linear subspace of X_i for all $x\in X$ and $i=1, 2, \ldots, n$.

A relation f from a multilinear subset D of X into a vector space Y over $\mathbf{K}$ will be called *multilinear* if the partial relation

$$f_{xi} = f \circ \varphi_{xi}$$

is a linear relation from D_{xi} into Y for all $x\in X$ and $i=1, 2, \ldots, n$.

Remark 2.4. Instead of "$x\in X$" we might only write "$x\in D$" in the above definition. However, this would lead to a further generalization which we do not need here.

Moreover, instead of "multilinear" we may also say "n-linear". Thus, "linear" and "bilinear" can be identified as "1-linear" and "2-linear", respectively.

Concerning multilinear sets and relations, we will only list here a few basic theorems without proofs.

Theorem 2.5. *If X is as in Definition* 2.3, *then*

$$X_0 = \{x\in X\colon\ x_i = 0 \quad \textit{for some} \quad i = 1, 2, \ldots, n\}$$

is the smallest multilinear subset of X.

Theorem 2.6. *If D is a multilinear subset of $X = \mathop{\times}_{i=1}^{n} X_i$, then*

$$D = \bigcup_{x\in X} \bigcup_{i=1}^{n} (\mathbf{K}x_1)\times\ldots\times(\mathbf{K}x_{i-1})\times D_{xi}\times(\mathbf{K}x_{i+1})\times\ldots\times(\mathbf{K}x_n).$$

Remark 2.7. This latter theorem, which is also true under a more general definition of multilinear sets, has been pointed out to me by György Szabó.

Theorem 2.8. *If f is a multilinear relation from a multilinear subset D of X into Y, then $f(0)$ is a linear subspace of Y and $f(x)=f(0)$ for all $x\in X_0$.*

Theorem 2.9. *If f is a multilinear relation from X into Y, then there exists a multilinear function φ from X into Y such that*

$$f(x) = \varphi(x)+f(0)$$

for all $x\in X$.

Remark 2.10. Note that if φ is a multilinear function from a multilinear subset D of X into Y and M is a linear subspace of Y, then the relation f defined on D by $f(x)=\varphi(x)+M$ is also multilinear.

By an immediate application of the above assertions, we can at once state the next simple

Example 2.11. A subset D of $\mathbf{K}^n$ is multilinear if and only if either $D=(\mathbf{K}^n)_0$ or $D=\mathbf{K}^n$.

A relation f from $D=\mathbf{K}^n$ or $(\mathbf{K}^n)_0$ into Y is multilinear if and only if there exist a vector $y\in Y$ and a linear subspace M of Y such that

$$f(x) = \Big(\prod_{i=1}^{n} x_i\Big) y+M \quad \text{for all} \quad x\in D.$$

More difficult examples for multilinear sets and relations can be easily obtained from the following obvious, but important theorem which needs only a few properties of convergent nets in preseminormed spaces.

Theorem 1.12. *If (f_α) is a net of multilinear maps from $X=\mathop{\times}\limits_{i=1}^{n} X_i$ into a preseminormed space $Y(\mathcal{Q})$, then the set*

$$D = \{x\in X\colon\ (f_\alpha(x)) \text{ converges in } Y(\mathcal{Q})\}$$

is a multilinear subset of X and the relation f defined on D by

$$f(x) = \lim_{\alpha}{}_{\mathcal{Q}} f_\alpha(x)$$

is a multilinear relation from D into Y.

Remark 2.13. Note that f is a function if and only if $Y(\mathcal{Q})$ is separated in the sense that for each $y\in Y$ with $y\neq 0$ there exists $q\in\mathcal{Q}$ such that $q(y)\neq 0$.

Therefore, in separated preseminormed spaces we may usually restrict ourselves to multilinear functions. But, unfortunately separated preseminormed spaces are often insufficient.

3. Equicontinuity. Before defining a suitable new notion of equicontinuity, which is necessary to rightly state our main results about the pointwise limit of a net of multilinear maps, we shall briefly deal with a corresponding concept of pointwise boundedness.

Definition 3.1. A net (f_α) of functions from a set X into a preseminormed space $Y(\mathcal{Q})$ will be called *pointwise bounded* if $(f_\alpha(x))$ is a bounded net in $Y(\mathcal{Q})$ for all $x \in X$.

Remark 3.2. A nonvoid set $\{f_\alpha\}_{\alpha \in \Gamma}$ of functions from X into $Y(\mathcal{Q})$ may henceforth be called pointwise bounded if the family $(f_\alpha)_{\alpha \in \Gamma}$ is pointwise bounded as a net whenever Γ is considered to be directed such that $\alpha \leqq \beta$ for all $\alpha, \beta \in \Gamma$.

For a preliminary illustration of the appropriateness of these unusual definitions, we can now easily prove a useful characterization of pointwise boundedness in terms of multi-preseminorms.

Theorem 3.3. *If (f_α) is a net of multilinear maps from $X = \mathop{\times}\limits_{i=1}^{n} X_i$ into $Y(\mathcal{Q})$, then the following assertions are equivalent:*

(i) *(f_α) is pointwise bounded;*

(ii) *$M_q = \overline{\lim_\alpha}\, q \circ f_\alpha$ is a multi-preseminorm on X for all $q \in \mathcal{Q}$.*

Proof. Because of the fact that $q \circ f_\alpha$ is a multi-preseminorm on X for all α and some of the basic properties of upper limit, it is clear that M_q is always multi-subadditive and

$$M_q(\varphi_{xi}(\lambda x_i)) \leqq M_q(x) \quad \text{and} \quad M_q(\varphi_{xi}(\mu x_i)) = M_q(\varphi_{xk}(\mu x_k))$$

for all $|\lambda| \leqq 1$, $\mu \in \mathbf{K}$, $x \in X$ and $i, k = 1, 2, \ldots, n$.
Moreover, since

$$M_q(\varphi_{xi}(\lambda x_i)) = \overline{\lim_\alpha}\, q(\lambda f_\alpha(x))$$

for all $q \in \mathcal{Q}$, $\lambda \in \mathbf{K}$, $x \in X$ and $i = 1, 2, \ldots, n$, it is also clear that

$$\lim_{\lambda \to 0} M_q(\varphi_{xi}(\lambda x_i)) = 0$$

for all $q \in \mathcal{Q}$, $x \in X$ and $i = 1, 2, \ldots, n$ if and only if (i) holds. Thus, it remains only to show that M_q is necessarily real-valued for all $q \in \mathcal{Q}$ if (i) holds. For this, note that if $x \in X$ and $p = M_q \circ \varphi_{x1}$, then

$$M_q(x) = p(x_1) = p(m(m^{-1}x)) \leqq mp(m^{-1}x_1)$$

for all $m \in \mathbf{N}$, whence because of

$$\lim_{m \to \infty} p(m^{-1}x) = 0,$$

it is evident that $M_q(x) < \infty$.

Remark 3.4. Hence, it is clear that a nonvoid set $\{f_\alpha\}$ of multilinear maps from X into $Y(\mathcal{Q})$ is pointwise bounded if and only if $M_q = \sup_\alpha q \circ f_\alpha$ is a multi-preseminorm on X for all $q \in \mathcal{Q}$.

Having in mind a particular case of the last statement of Section 1 and our basic concept of boundedness of a net, it seems now quite reasonable to introduce a suitable new notion of equicontinuity.

Definition 3.5. A net (f_α) of multilinear maps from a product preseminormed space

$$X(\mathscr{P}) = \mathop{\times}_{i=1}^{n} X_i(\mathscr{P}_i)$$

into another preseminormed space $Y(\mathscr{Q})$ will be called *equicontinuous at the origin of* $X(\mathscr{P})$ if the function

$$M_q = \overline{\lim_\alpha}\, q\circ f_\alpha$$

is continuous at the origin of $X(\mathscr{P})$ for all $q\in\mathscr{Q}$.

Remark 3.6. A nonvoid set $\{f_\alpha\}_{\alpha\in\Gamma}$ of multilinear maps from $X(\mathscr{P})$ into $Y(\mathscr{Q})$ may henceforth be called equicontinuous at the origin of $X(\mathscr{P})$ if the family $(f_\alpha)_{\alpha\in\Gamma}$ is equicontinuous at the origin of $X(\mathscr{P})$ as a net whenever Γ is considered to be directed such that $\alpha\leqq\beta$ for all $\alpha,\beta\in\Gamma$.

To let the reader feel the appropriateness of these apparently very strange definitions, we first show that this particular equicontinuity does already imply pointwise boundedness.

Theorem 3.7. *If* (f_α) *is a net of multilinear maps from*

$$X(\mathscr{P}) = \mathop{\times}_{i=1}^{n} X_i(\mathscr{P}_i)$$

into $Y(\mathscr{Q})$ *such that* (f_α) *is equicontinuous at the origin of* $X(\mathscr{P})$, *then* (f_α) *is pointwise bounded.*

Proof. If $x\in X$ and $q\in\mathscr{Q}$, then we clearly have

$$q(\lambda f_\alpha(x)) \leqq q(|\lambda| f_\alpha(x)) = q\big(f_\alpha(\sqrt[n]{|\lambda|}\,x_1, \ldots, \sqrt[n]{|\lambda|}\,x_n)\big)$$

and hence

$$\overline{\lim_\alpha}\, q(\lambda f_\alpha(x)) \leqq M_q(\sqrt[n]{|\lambda|}\,x_1, \ldots, \sqrt[n]{|\lambda|}\,x_n)$$

for all $\lambda\in\mathbf{K}$. Hence, because of the continuity of M_q at 0,

$$\lim_{\lambda\to 0}\overline{\lim_\alpha}\, q(\lambda f_\alpha(x)) = 0$$

follows. And this shows that (f_α) is pointwise bounded.

Remark 3.8. Hence, it is clear that if $\{f_\alpha\}$ is a set of multilinear maps from $X(\mathscr{P})$ into $Y(\mathscr{Q})$ such that $\{f_\alpha\}$ is equicontinuous at the origin of $X(\mathscr{P})$ then $\{f_\alpha\}$ is pointwise bounded.

Next, we prove a useful characterization of equicontinuity which, together with Theorem 3.3, provides subtantial motivation for introducing and studying multi-preseminorms.

Theorem 3.9. *If (f_α) is a net of multilinear maps from*

$$X(\mathscr{P}) = \mathop{\mathsf{X}}_{i=1}^{n} X_i(\mathscr{P}_i)$$

into $Y(\mathscr{Q})$, then the following assertions are equivalent:

(i) *(f_α) is equicontinuous at the origin of $X(\mathscr{P})$;*

(ii) *$M_q = \overline{\lim_\alpha}\, q\circ f_\alpha$ is a boundedly uniformly continuous multi-preseminorm on $X(\mathscr{P})$ for all $q\in\mathscr{Q}$.*

Proof. If (i) holds, then by Theorem 3.7, (f_α) is pointwise bounded. Thus, by Theorem 3.3, $M_q = \overline{\lim_\alpha}\, q\circ f_\alpha$ is a multi-preseminorm on X for all $q\in\mathscr{Q}$. On the other hand, by [15, Theorem 2.7] a multi-preseminorm which is continuous at the origin is necessarily boundedly uniformly continuous. Therefore, (ii) also holds.

The converse implication (ii)$\Rightarrow$(i) is trivial since bounded uniform continuity always implies continuity.

Remark 3.10. Hence, it is clear that a nonvoid set $\{f_\alpha\}$ of multilinear maps from $X(\mathscr{P})$ into $Y(\mathscr{Q})$ is equicontinuous at the origin of $X(\mathscr{P})$ if and only if $M_q = \sup_\alpha q\circ f_\alpha$ is a boundedly uniformly continuous multi-preseminorm on $X(\mathscr{P})$ for all $q\in\mathscr{Q}$.

4. Main results. To easily prove our main results about the topological properties of the pointwise limit of an equicontinuous net of multilinear maps, we also neeed a somewhat deeper characterization of equicontinuity.

Theorem 4.1. *If (f_α) is a net of multilinear maps from*

$$X(\mathscr{P}) = \mathop{\mathsf{X}}_{i=1}^{n} X_i(\mathscr{P}_i)$$

into $Y(\mathscr{Q})$, then the following assertions are equivalent:

(i) *(f_α) is equicontinuous at the origin of $X(\mathscr{P})$;*

(ii) *$\lim_\nu \overline{\lim_\alpha}\, q\big(f_\alpha(x_\nu)-f_\alpha(y_\nu)\big)=0$ for all $q\in\mathscr{Q}$ whenever (x_ν) and (y_ν) are bounded coherent nets in $X(\mathscr{P})$.*

Proof. Assume that (i) is true, and moreover (x_ν) and (y_ν) are bounded coherent nets in $X(\mathscr{P})$ and $q\in\mathscr{Q}$. If $I=\{1, 2, \dots, n\}$ and χ_A is the characteristic function of

$A \subset I$, then according to [3, (18.3) Lemma], we have

$$f_\alpha(x_\nu)-f_\alpha(y_\nu) = \sum_{\emptyset \neq A \subset I} f_\alpha\big(\chi_A(x_\nu-y_\nu)+(\chi_I-\chi_A)y_\nu\big)$$

for all α and ν, where the multiplication is taken in the usual pointwise sense. Hence, because of the subadditivity of q and $\overline{\lim}$, it follows that

$$\overline{\lim_\alpha}\, q\big(f_\alpha(x_\nu)-f_\alpha(y_\nu)\big) \leqq \sum_{\emptyset \neq A \subset I} M_q\big(\chi_A(x_\nu-y_\nu)+(\chi_I-\chi_A)y_\nu\big)$$

for all ν, where again $M_q = \overline{\lim_\alpha}\, q \circ f_\alpha$.

On the other hand, if $\emptyset \neq A \subset I$, then by our former results mentioned in Section 1, it is clear that

$$\big(\chi_A(x_\nu-y_\nu)+(\chi_I-\chi_A)y_\nu\big) \quad \text{and} \quad \big((\chi_I-\chi_A)y_\nu\big)$$

are bounded coherent nets in $X(\mathscr{P})$. Moreover, since $f_\alpha\big((\chi_I-\chi_A)y_\nu\big)=0$ for all α and ν, it is also clear that

$$M_q\big((\chi_I-\chi_A)y_\nu\big) = 0$$

for all ν. Thus, by a particular case of Theorem 3.9, we also have

$$\lim_\nu M_q\big(\chi_A(x_\nu-y_\nu)+(\chi_I-\chi_A)y_\nu\big) = 0$$

for all $\emptyset \neq A \subset I$. Using these latter equalities, from our previous inequality, we can immediately infer that

$$\lim_\nu \overline{\lim_\alpha}\, q\big(f_\alpha(x_\nu)-f_\alpha(y_\nu)\big) = 0,$$

which shows that (ii) is also true.

To prove the converse implication (ii)$\Rightarrow$(i), note that if (x_ν) is a null net in $X(\mathscr{P})$, then by defining $y_\nu=0$ for all ν, we can at once state that (x_ν) and (y_ν) are bounded coherent nets in $X(\mathscr{P})$ such that $f_\alpha(y_\nu)=0$ for all α and ν. Therefore, if (ii) holds, then we also have

$$\lim_\nu \overline{\lim_\alpha}\, q\big(f_\alpha(x_\nu)\big) = 0$$

for all $q \in \mathscr{Q}$. Consequently, the function $M_q = \overline{\lim_\alpha}\, q \circ f_\alpha$ is continuous at the origin of $X(\mathscr{P})$ for all $q \in \mathscr{Q}$, and thus (i) also holds.

R e m a r k 4.2. Hence, it is clear that a nonvoid set $\{f_\alpha\}$ of multilinear maps from $X(\mathscr{P})$ into $Y(\mathscr{Q})$ is equicontinuous at the origin of $X(\mathscr{P})$ if and oly if $\lim_\nu \sup_\alpha q\big(f_\alpha(x_\nu)-$ $-f_\alpha(y_\nu)\big)=0$ for all $q \in \mathscr{Q}$ whenever (x_ν) and (y_ν) are bounded coherent nets in $X(\mathscr{P})$.

To partly express a very strong continuity property of the pointwise limit of an equicontinuous net of multilinear maps, we also need the next straightforward

Definition 4.3. A relation f from a subset D of a preseminormed space $X(\mathcal{P})$ into another preseminormed space $Y(\mathcal{Q})$ will be called *selectionally boundedly uniformly continuous* if each selection function φ for f is boundedly uniformly continuous.

Remark 4.4. Note that a selectionally boundedly uniformly continuous relation is, in particular, lower semicontinuous in the usual topological sense [6, p. 32].

Now, having all the necessary preparations, we can easily state and prove the following important addition to Theorem 2.12 which greatly improve the second assertion of [16].

Theorem 4.5. *If (f_α) is a net of multilinear maps from*

$$X(\mathcal{P}) = \bigtimes_{i=1}^{n} X_i(\mathcal{P}_i)$$

into $Y(\mathcal{Q})$ which is equicontinuous at the origin of $Y(\mathcal{P})$, then the relation f defined by

$$f(x) = \lim_{\alpha}{}_{\mathcal{Q}} f_\alpha(x)$$

is a selectionally boundedly uniformly continuous relation from its domain D into $Y(\mathcal{Q})$.

Proof. Assume that φ is a selection function for f and (x_ν) and (y_ν) are bounded coherent nets in D. If $q \in \mathcal{Q}$, then because of the subadditivity of q and the assumption that $\varphi(x) \in \lim_{\alpha}{}_{\mathcal{Q}} f_\alpha(x)$ for all $x \in D$, we clearly have

$$q\big(\varphi(x_\nu)-\varphi(y_\nu)\big) \leqq q\big(\varphi(x_\nu)-f_\alpha(x_\nu)\big)+q\big(f_\alpha(x_\nu)-f_\alpha(y_\nu)\big)+q\big(f_\alpha(y_\nu)-\varphi(y_\nu)\big)$$

and

$$\lim_{\alpha} q\big(\varphi(x_\nu)-f_\alpha(x_\nu)\big) = 0 \quad \text{and} \quad \lim_{\alpha} q\big(f_\alpha(y_\nu)-\varphi(y_\nu)\big) = 0$$

for all α and ν, respectively. Hence, it follows that

$$q\big(\varphi(x_\nu)-\varphi(y_\nu)\big) \leqq \lim_{\alpha} q\big(f_\alpha(x_\nu)-f_\alpha(y_\nu)\big)$$

for all ν. Hence, by Theorem 4.1, it is clear that

$$\lim_{\nu} q\big(\varphi(x_\nu)-\varphi(y_\nu)\big) = 0.$$

Consequently, φ is a boundedly uniformly continuous function of D into $Y(\mathcal{Q})$, and thus the selectional bounded uniform continuity of f is proved.

Since each preseminormed space can be naturally embedded into a complete one, we may usually assume that $Y(\mathcal{Q})$ is complete. In this particular case, the above theorem can be supplemented by the next important

Theorem 4.6. *If (f_α) is a net of multilinear maps from*

$$X(\mathscr{P}) = \bigtimes_{i=1}^{n} X_i(\mathscr{P}_i)$$

into $Y(\mathscr{Q})$ which is equicontinuous at the origin of $X(\mathscr{P})$, and $Y(\mathscr{Q})$ is, in addition, complete, then the set

$$D = \{x \in X\colon (f_\alpha(x)) \text{ converges in } Y(Q)\}$$

is a closed subset of $X(\mathscr{P})$.

Proof. Assume that $x \in X$ and (x_ν) is a net in D such that

$$x \in \lim_{\nu}{}_{\mathscr{P}}\, x_\nu .$$

If $q \in \mathscr{Q}$, then because of the subadditivity of q and $\overline{\lim}$, we clearly have

$$\overline{\lim_{(\alpha,\beta)}}\, q\big(f_\alpha(x)-f_\beta(x)\big) \leqq \overline{\lim_{(\alpha,\beta)}}\, q\big(f_\alpha(x)-f_\alpha(x_\nu)\big) +$$
$$+ \overline{\lim_{(\alpha,\beta)}}\, q\big(f_\alpha(x_\nu)-f_\beta(x_\nu)\big) + \overline{\lim_{(\alpha,\beta)}}\, q\big(f_\beta(x_\nu)-f_\beta(x)\big)$$

for all ν, where (α, β) runs in the corresponding product directed set. Moreover, since convergent nets are Cauchy nets, we also have

$$\lim_{(\alpha,\beta)} q\big(f_\alpha(x_\nu)-f_\beta(x_\nu)\big) = 0$$

for all ν. On the other hand, because of $q(-y)=q(y)$ and the definition of upper limit, it is also clear that

$$\overline{\lim_{(\alpha,\beta)}}\, q\big(f_\alpha(x)-f(x_\nu)\big) = \overline{\lim_{(\alpha,\beta)}}\, q\big(f_\beta(x_\nu)-f_\beta(x)\big) = \overline{\lim_{\alpha}}\, q\big(f_\alpha(x_\nu)-f_\alpha(x)\big)$$

for all ν. Consequently, we have

$$\overline{\lim_{(\alpha,\beta)}}\, q\big(f_\alpha(x)-f_\beta(x)\big) \leqq 2\,\overline{\lim_{\alpha}}\, q\big(f_\alpha(x_\nu)-f_\alpha(x)\big)$$

for all ν. Hence, by noticing that (x_ν) and (x) are bounded coherent nets in $X(\mathscr{P})$ and thus by Theorem 4.1

$$\lim_{\nu}\, \overline{\lim_{(\alpha,\beta)}}\, q\big(f_\alpha(x_\nu)-f_\alpha(x)\big) = 0,$$

we can infer that

$$\lim_{(\alpha,\beta)} q\big(f_\alpha(x)-f_\beta(x)\big) = 0.$$

This shows that $(f_\alpha(x))$ is a Cauchy net in $Y(\mathscr{Q})$. Hence, by the completeness of $Y(\mathscr{Q})$, it is clear that $x \in D$. And thus, we have proved that D is closed in $X(\mathscr{P})$.

Remark 4.7. Particular cases of Theorems 4.5 and 4.6 can be used to derive some essential extensions of a general convergence theorem for net integrals [7, Theorem 3.8].

However, to realize the usefulness of Theorems 4.5 and 4.6 in integration, the reader is rather advised to derive first a uniform convergence theorem for the classical Reimann—Stieltjes integral.

5. Supplements. By using Theorem 4.1, we can also easily prove a remarkable characterization of equicontinuity of a net (f_α) of multilinear maps from $X(\mathscr{P})$ into $Y(\mathscr{Q})$ in terms of the induced uniformities $\mathscr{U}_\mathscr{P}$ and $\mathscr{U}_\mathscr{Q}$.

Theorem 5.1. *If (f_α) is a net of multilinear maps from*

$$X(\mathscr{P}) = \bigtimes_{i=1}^{n} X_i(\mathscr{P}_i)$$

into $Y(\mathscr{Q})$, then the following assertions are equivalent:

(i) *(f_α) is equicontinuous at the origin of $X(\mathscr{P})$;*

(ii) *$\underline{\lim}_\alpha (f_\alpha^{-1} \circ V \circ f_\alpha)(x)$ is a neighbourhood of x in $X(\mathscr{P})$ for all $V \in \mathscr{U}_\mathscr{Q}$ and $x \in X$.*

Proof. If (ii) does not hold, then because of the definition of the induced uniformities and [9, Remark 3.9], there exist $x \in X$, $q \in \mathscr{Q}$ and $\varepsilon > 0$ such that the ball $B_p^{m^{-1}}(x)$ is not contained in the set

$$\underline{\lim}_\alpha (f_\alpha^{-1} \circ B_q^\varepsilon \circ f_\alpha)(x) = \bigcup_\alpha \bigcap_{\beta \geqq \alpha} f_\alpha^{-1}(B_q^\varepsilon(f_\alpha(x)))$$

for all $p \in \mathscr{P}^*$ and $m \in \mathbf{N}$. Thus, for each $\nu = (p, m) \in \Delta = \mathscr{P}^* \times \mathbf{N}$ there exists $x_\nu \in B_p^{m^{-1}}(x)$ such that

$$\overline{\lim_\alpha}\, q(f_\alpha(x_\nu) - f_\alpha(x)) \geqq \varepsilon.$$

Hence, it is clear that $(x_\nu)_{\nu \in \Delta}$ is a net in $X(\mathscr{P})$ such that $x \in \lim_{\nu} {}_\mathscr{P} x_\nu$, but

$$\overline{\lim_\nu}\, \overline{\lim_\alpha}\, q(f_\alpha(x_\nu) - f_\alpha(x)) \geqq \varepsilon,$$

and thus (i) cannot hold because of Theorem 4.1.

Thus, we have proved that (i) implies (ii). To prove the converse implication, note that even the particular case of (ii) when $x = 0$ does already imply (i).

Remark 5.2. Hence, it is clear that a nonvoid set $\{f_\alpha\}$ of multilinear maps from $X(\mathscr{P}$, into $Y(\mathscr{Q})$ is equicontinuos at the origin of $X(\mathscr{P})$ if and only if $\bigcap_\alpha (f_\alpha^{-1} \circ V \circ f_\alpha)(x)$ is a neighbourhood of x in $X(\mathscr{P})$ for all $V \in \mathscr{U}_\mathscr{Q}$ and $x \in X$.

Remark 5.3. By using the topological refinement

$$\hat{\mathscr{U}}_\mathscr{P} = \{R \subset XxX \colon \forall x \in X \colon \exists U \in \mathscr{U}_\mathscr{P} \colon U(x) \subset R(x)\}$$

of $\mathscr{U}_\mathscr{P}$ [13], the assertions of Theorem 5.1 and Remark 5.2 can be rephrased in the more instructive form that the net (f_α) (set $\{f_\alpha\}$) is equicontinuous at the origin of $X(\mathscr{P})$ if and only if

$$\underline{\lim}_{\alpha} f_\alpha^{-1}\circ V\circ f_\alpha \in \hat{\mathscr{U}}_{\mathscr{P}} \quad \left(\bigcap_{\alpha} f_\alpha^{-1}\circ V\circ f_\alpha \in \hat{\mathscr{U}}_{\mathscr{P}}\right)$$

for all $V\in\mathscr{U}_{\mathscr{Q}}$.

Note that the "only if parts" of the above assertions are much weaker then the corresponding parts of Theorems 3.9 and 4.1 and Remarks 3.10 and 4.2. In principle, $\lim_{\mathscr{P}}$ and $\mathscr{U}_{\mathscr{P}}$ should be equivalent tools in $X(\mathscr{P})$. However, actually we do not even know that which subfamily of $\hat{\mathscr{U}}_{\mathscr{P}}$ could be used to express the bounded uniform continuity of a function f from $X(\mathscr{P})$ ino $Y(\mathscr{Q})$.

Whenever the net (f_α) of multilinear maps from $X(\mathscr{P})$ into $Y(\mathscr{Q})$ is pointwise convergent in the usual sense that the net $(f_\alpha(x))$ converges in $Y(\mathscr{Q})$ for all $x\in X$, then the converse of Theorem 4.5 is also true. In fact, in this particular case, we can even prove a little more.

Theorem 5.4. *If (f_α) is a pointwise convergent net of multilinear maps from*

$$X(\mathscr{P}) = \bigtimes_{i=1}^{n} X_i(\mathscr{P}_i)$$

into $Y(\mathscr{Q})$ and f is the relation defined on X by

$$f(x) = \lim_{\alpha}{}_{\mathscr{Q}} f_\alpha(x),$$

then the following assertions are equivalent:

(i) *(f_α) is equicontinuous at the origin of $X(\mathscr{P})$;*

(ii) *f is selectionally boundedly uniformly continuous;*

(iii) *f is lower semicontinuous at the origin of $X(\mathscr{P})$.*

Proof. Because of Theorem 4.5 and Remark 4.4, we need only show that (iii) also implies (i). For this, assume that (iii) holds, and let $q\in\mathscr{Q}$ and $M_q=\overline{\lim}_{\alpha}\, q\circ f_\alpha$. If $\varepsilon>0$, then by the definition of $\mathscr{U}_{\mathscr{Q}}$, the ball $B_q^\varepsilon(0)$ is a neighbourhood of 0 in $Y(\mathscr{Q})$. Thus, because of $0\in f(0)$ and (iii), the set $U=f^{-1}(B_q^\varepsilon(0))$ is a neighbourhood of 0 in $X(\mathscr{P})$. If $x\in U$, then by the definition of U, there exists $y\in B_q^\varepsilon(0)$ such that $y\in f(x)$. Hence, it is clear that

$$q(f_\alpha(x)) \leqq q(f_\alpha(x)-y)+q(y) \leqq q(f_\alpha(x)-y)+\varepsilon$$

for all α, and

$$\lim_{\alpha} q(f_\alpha(x)-y) = 0.$$

Consequently, we have

$$M_q(x) = \overline{\lim_{\alpha}}\, q(f_\alpha(x)) \leqq \varepsilon.$$

Hence, it is clear that M_q is continuous at the origin of $X(\mathscr{P})$, and thus by Definition 3.5, (i) also holds.

Remark 5.5. Note that to obtain (i) we have only used a particular case of (iii).

As an immediate consequence of Theorem 5.4, we can easily get the essential improvement of [3, (18.2) Theorem] proved directly in [15].

Corollary 5.6. *If f is a multilinear map from*

$$X(\mathscr{P}) = \bigtimes_{i=1}^{n} X_i(\mathscr{P}_i)$$

into $Y(\mathscr{Q})$, then the following assertions are equivalent:

(i) *f is boundedly uniformly continuous;*

(ii) *f is continuous at the origin of $X(\mathscr{P})$;*

(iii) *$q \circ f$ is continuous at the origin of $X(\mathscr{P})$ for all $q \in \mathscr{Q}$.*

Proof. To apply Theorem 5.4, note that f is a selection function for the relation F defined on X by

$$F(x) = \lim_{\alpha} {}_{\mathscr{Q}} f_\alpha(x),$$

where α runs in an arbitrary nonvoid directed set.

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A model for a general linear bounded operator between two Hilbert spaces

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The main result of this paper is a theorem asserting that every bounded linear operator between two Hilbert spaces is unitary equivalent with a certain particular operator, the "model", in a similar sense with that used for contractions in [5]. This is accomplished by proving a model theorem for a contraction between two Hilbert spaces inspired by the techniques used in Ch. I, Sec. 10 from [7] then by proving a model theorem for an invertible linear bounded operator between two Hilbert spaces whose inverse is a contraction and then by the use of the canonical decomposition of every linear bounded operator as a direct sum of a contraction, an operator whose inverse is a contraction and an isometry (see [4], [6]). The model for the contraction is used also to prove a result concerning dilation of the couple (T, T^*).

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1. A model for a contraction between two Hilbert spaces

Let $\mathscr{H}_1, \mathscr{H}_2$ be two separable Hilbert spaces and $T: \mathscr{H}_1 \to \mathscr{H}_2$ a contraction, that is a bounded linear operator with $\|T\| \leqq 1$. Then $T^*: \mathscr{H}_2 \to \mathscr{H}_1$ is also a contraction. Define

$$D = (I_{\mathscr{H}_1} - T^*T)^{1/2}, \quad D_* = (I_{\mathscr{H}_2} - TT^*)^{1/2}, \quad \mathscr{E}_1 = \overline{D\mathscr{H}_1}, \quad \mathscr{E}_2 = \overline{D_*\mathscr{H}_2}$$

where $I_{\mathscr{H}}$ denotes the identity operator in $\mathscr{H}$. The norms in the two Hilbert spaces $\mathscr{H}_1, \mathscr{H}_2$ will be denoted respectively by $\|\cdot\|_1, \|\cdot\|_2$.

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We observe that $((T^*T)^k)_{k=0}^{\infty}$ is a decreasing sequence of selfadjoint contractions, consequently $Q_1=\lim_k (T^*T)^k$ exists in the strong sense and $0\leqq Q_1\leqq I_{\mathscr{H}_1}$. Since $Q_1(I_{\mathscr{H}_1}-T^*T)h=0$ for $h\in\mathscr{H}_1$, Q_1 is the orthogonal projection onto $\ker(I_{\mathscr{H}_1}-T^*T)$. Similarly $Q_2=\mathrm{s}-\lim_k (TT^*)^k$ is the orthogonal projection onto $\ker(I_{\mathscr{H}_2}-TT^*)$. In particular $Q_1\mathscr{H}_1$ and $Q_2\mathscr{H}_2$ are closed subspaces of $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively.

The definitions of Q_1 and Q_2 show that

$$Q_1=T^*Q_2T,\quad Q_2=TQ_1T^*. \tag{1.1}$$

Let $W\colon Q_1\mathscr{H}_1\to Q_2\mathscr{H}_2$ be defined by

$$WQ_1h=Q_2Th,\quad h\in\mathscr{H}_1. \tag{1.2}$$

Then by (1.1) one can easily see that

$$\|WQ_1h\|_2=\|Q_2Th\|_2=\|Q_1h\|_1,$$

such that W is an isometry.

Since, by (1.1), $Q_2(\ker T^*)=\{0\}$, it results that $Q_2T\mathscr{H}_1$ is dense in $Q_2\mathscr{H}_2$, such that, by (1.2), W has dense range in $Q_2\mathscr{H}_2$. It results that W is a unitary operator. A computation shows (see [7] Ch. I, Sec. 10) that for every $h\in\mathscr{H}_1$

$$\sum_{k=0}^{n}\|D(T^*T)^kh\|_1^2+\sum_{k=1}^{n}\|D_*T(T^*T)^kh\|_2^2=$$

$$=\sum_{k=0}^{n}((T^*T)^{2k}-(T^*T)^{2k+1}h,h)+\sum_{k=0}^{n}((T^*T)^{2k+1}-(T^*T)^{2k+2}h,h)=$$

$$=\|h\|_1^2-\|T^*T)^{n+1}h\|_1^2.$$

Taking limits we have

$$\|h\|_1^2=\sum_{k=0}^{\infty}\|D(T^*T)^kh\|_1^2+\sum_{k=0}^{\infty}\|D_*T(T^*T)^kh\|_2^2+\|Q_1h\|_1^2,\quad h\in\mathscr{H}_1. \tag{1.3}$$

By similar computations

$$\|h'\|_2^2=\sum_{k=0}^{\infty}\|D_*(TT^*)^kh'\|_2^2+\sum_{k=0}^{\infty}\|DT^*(TT^*)^kh'\|_1^2+\|Q_2h'\|_2^2,\quad h'\in\mathscr{H}_2. \tag{1.4}$$

For a Hilbert space $\mathscr{E}$, $H^2(\mathscr{E})$ denotes the vectorial Hardy space (see [7], Ch. V Sec. 1 or [5], Sec. 0). For

$$u(z)=\sum_{k=0}^{\infty}z^ka_k,\quad |z|<1$$

the norm is defined by

$$\|u\|^2_{H^2(\mathscr{E})} = \sum_{k=0}^{\infty} \|a_k\|^2_{\mathscr{E}}.$$

We denote by $S_{\mathscr{E}}$ the unilateral shift on $H^2(\mathscr{E})$, ([5] Sec. 0). Let

$$(1.5)\qquad \begin{gathered} V_1\colon \mathscr{H}_1 \to H^2(\mathscr{E}_1)\oplus H^2(\mathscr{E}_2)\oplus Q_1\mathscr{H}_1, \\ V_1 h = [\sum_{k=0}^{\infty} z^k D(T^*T)^k h]\oplus[\sum_{k=0}^{\infty} z^k D_* T(T^*T)^k h]\oplus Q_1 h. \end{gathered}$$

From (1.3) we have $\|V_1 h\|^2 = \|h\|_1^2$, where the square of the norm in the direct sum is the sum of the squares of the norms of the components. Let

$$(1.6)\qquad \begin{gathered} V_2\colon \mathscr{H}_2 \to H^2(\mathscr{E}_1)\oplus H^2(\mathscr{E}_2)\oplus Q_2\mathscr{H}_2 \\ V_2 h' = [\sum_{k=0}^{\infty} z^k DT^*(TT^*)^k h']\oplus[\sum_{k=0}^{\infty} z^k D_*(TT^*)^k h']\oplus Q_2 h', \quad h'\in\mathscr{H}_2. \end{gathered}$$

From (1.4) it follows that $\|V_2 h'\|^2 = \|h'\|_2^2$. From the previous definitions

$$(1.7)\qquad V_2 Th = [\sum_{k=0}^{\infty} z^k DT^*(TT^*)^k Th]\oplus[\sum_{k=0}^{\infty} z^k D_*(TT^*)^k Th]\oplus Q_2 Th =$$

$$= [\sum_{k=0}^{\infty} z^k D(T^*T)^{k+1} h]\oplus[\sum_{k=0}^{\infty} z^k D_* T(T^*T)^k h]\oplus Q_2 Th = [S^*_{\mathscr{E}_1}\oplus I_{H^2(\mathscr{E}_2)}\oplus W]V_1 h$$

for ewery $h\in\mathscr{H}_1$, and

$$(1.8)\qquad V_1 T^* h' = [\sum_{k=0}^{\infty} z^k D(T^*T)^k T^* h']\oplus[\sum_{k=0}^{\infty} z^k D_*(TT^*)^{k+1} h']\oplus Q_1 T^* h' =$$

$$= [I_{H^2(\mathscr{E}_1)}\oplus S^*_{\mathscr{E}_2}\oplus W^*]V_2 h'$$

for every $h'\in\mathscr{H}_2$. Therefore the following model theorem is proved.

Theorem 1.1. *Let $T\colon \mathscr{H}_1\to\mathscr{H}_2$ be a contraction. There exist the Hilbert spaces $\mathscr{E}_1, \mathscr{E}_2$, the closed subspaces $\mathscr{K}_1\subset H^2(\mathscr{E}_1)\oplus H^2(\mathscr{E}_2)$, $\mathscr{K}_2\subset H^2(\mathscr{E}_1)\oplus H^2(\mathscr{E}_2)$ and the unitary operators*

$$V_1\colon \mathscr{H}_1 \to \mathscr{K}_1\oplus Q_1\mathscr{H}_1,\quad V_2\colon \mathscr{H}_2\to\mathscr{K}_2\oplus Q_2\mathscr{H}_2,\quad W\colon Q_1\mathscr{H}_1\to Q_2\mathscr{H}_2$$

such that

$$(1.9)\qquad T = V_2^*(S^*_{\mathscr{E}_1}\oplus I_{H^2(\mathscr{E}_2)}\oplus W)V_1,$$

$$(1.10)\qquad T^* = V_1^*(I_{H^2(\mathscr{E}_1)}\oplus S^*_{\mathscr{E}_2}\oplus W^*)V_2.$$

2. A model for the inverse of a contraction

Let $T: \mathcal{H}_1 \to \mathcal{H}_2$ an invertible contraction. T^* is then invertible, too. We proceed to exhibit a model for T^{-1}.

Lemma 2.1.

$$\|Dh\|_1^2 = \sum_{n=1}^{\infty} \|D_*^n Th\|_2^2 \quad \textit{for every} \quad h \in \mathcal{H}_1. \tag{2.1}$$

Proof. First we observe that $\|D_*\| < 1$. Indeed,

$$\|D_*\|^2 = \sup_{\|h'\|_2=1} \|D_* h'\|_2^2 = \sup_{\|h'\|_2=1} (1 - \|T^* h'\|_1^2) = 1 - \inf_{\|h'\|_2=1} \|T^* h'\|_1^2 < 1.$$

Then $\|D_*^2\| < 1$, so $(I - D_*^2)^{-1} = \sum_{n=0}^{\infty} D_*^{2n}$. But $(I - D_*^2)^{-1} = (TT^*)^{-1}$ and so

$$\sum_{n=1}^{\infty} D_*^{2n} = D_*^2 (TT^*)^{-1}. \tag{2.2}$$

We observe that

$$T^* D_*^2 (TT^*)^{-1} T = T^* (I_{\mathcal{H}_2} - TT^*)(T^*)^{-1} = I_{\mathcal{H}_1} - T^*T = D^2.$$

Then

$$\|Dh\|_1^2 = (D^2 h, h)_1 = \big(T^* D_*^2 (TT^*)^{-1} Th, h\big)_1 = \big(D_*^2 (TT^*)^{-1} Th, Th\big)_2 =$$

$$= \Big(\sum_{n=1}^{\infty} D_*^{2n} Th, Th\Big)_2 = \sum_{n=1}^{\infty} (D_*^{2n} Th, Th)_2 = \sum_{n=1}^{\infty} \|D_*^n Th\|_2^2.$$

The lemma is proved.

From (1.3) and (2.1) it results

$$\|h\|_1^2 = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \|D_*^n T(T^*T)^k h\|_2^2 + \sum_{k=0}^{\infty} \|D_* T(T^*T)^k h\|_2^2 + \|Q_1 h\|_1^2 \tag{2.3}$$

for every $h \in \mathcal{H}_1$. From (1.4) and (2.1) it results

$$\|h'\|_2^2 = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \|D_*^n (TT^*)^{k+1} h'\|_2^2 + \sum_{k=0}^{\infty} \|D_* (TT^*)^k h'\|_2^2 + \|Q_2 h'\|_2^2 \tag{2.4}$$

for every $h' \in \mathcal{H}_2$.

Let $\mathcal{M} = \{u \in H^2(\mathcal{E}_2) \mid u(\lambda) = \sum_{n=1}^{\infty} \lambda^n D_*^n h', \ |\lambda| < 1, \ h' \in \mathcal{H}_2\}$. $\mathcal{M}$ is a closed subspace of $H^2(\mathcal{E}_2)$. Indeed, let $(u_j)_{j \geq 0}$ be a sequence in $\mathcal{M}$, $u_j \to u$, $u \in H^2(\mathcal{E}_2)$, $u_j(\lambda) = \sum_{n=1}^{\infty} \lambda^n D_*^n h_j'$, $|\lambda| < 1$; $u(\lambda) = \sum_{n=1}^{\infty} \lambda^n a_n$, $|\lambda| < 1$, then

$$\|u_j - u_k\|^2 = \sum_{n=1}^{\infty} \|D_*^n (h_j' - h_k')\|_2^2 = \|DT^{-1}(h_j' - h_k')\|_1^2 \to 0 \quad \text{as} \quad j, k \to \infty.$$

We have $\|T^{-1}(h'_j-h'_k)\|_1^2=\|h'_j-h'_k\|_2^2+\|DT^{-1}(h'_j-h'_k)\|_1^2$. But, since $\|T^{-1}h'\|^2\geqq\geqq\|T\|^{-2}\|h'\|_2^2$ it results $(\|T\|^{-2}-1)\|h'_j-h'_k\|_2^2\to 0$ as $j,k\to\infty$, so there exists $h'=\lim_j h'_j$ and then $D_*^n h'_j\to D_*^n h'$ for every $n\geqq 1$ as $j\to\infty$. But $D_*^n h'_j\to a_n$ as $j\to\infty$, so $a_n=D_*^n h'$ and thus u is in $\mathcal{M}$.

Let $\tilde V_1\colon \mathcal{H}_1\to H^2(\mathcal{M})\oplus H^2(\mathcal{E}_2)\oplus Q_1\mathcal{H}_1$ be defined by

$$\tilde V_1 h=\Big[\sum_{k=0}^{\infty} z^k h_k\Big]\oplus\Big[\sum_{k=0}^{\infty} z^k D_*T(T^*T)^k h\Big]\oplus Q_1 h,\quad h\in\mathcal{H}_1 \tag{2.5}$$

where

$$h_k(\lambda)=\sum_{n=1}^{\infty}\lambda^n D_*^n T(T^*T)^k h,\quad \text{for}\quad |\lambda|<1. \tag{2.6}$$

(2.1) implies $\|h_k\|^2_{H^2(\mathcal{E}_2)}=\|D(T^*T)^k h\|_1^2$ and (2.3) implies $\|\tilde V_1 h\|^2=\|h\|_1^2$ for every $h\in\mathcal{H}_1$.

Let $\tilde V_2\colon \mathcal{H}_2\to H^2(\mathcal{M})\oplus H^2(\mathcal{E}_2)\oplus Q_2\mathcal{H}_2$ be defined by

$$\tilde V_2 f=\Big[\sum_{k=0}^{\infty} z^k f_k\Big]\oplus\Big[\sum_{k=0}^{\infty} z^k D_*(TT^*)^k f\Big]\oplus Q_2 f,\quad f\in\mathcal{H}_2 \tag{2.7}$$

where

$$f_k(\lambda)=\sum_{n=1}^{\infty}\lambda^n D_*^n (TT^*)^{k+1} f,\quad \text{for}\quad |\lambda|<1. \tag{2.8}$$

(2.1) implies $\|f_k\|^2_{H^2(\mathcal{E}_2)}=\|DT^*(TT^*)^k f\|_1^2$ and (2.4) implies $\|\tilde V_2 f\|^2=\|f\|_2^2$ for all $f\in\mathcal{H}_2$.

In order to find a model for T^{-1} we compute $\tilde V_1 T^{-1} f$ for $f\in\mathcal{H}_2$.

$$\tilde V_1 T^{-1} f=\Big[\sum_{k=0}^{\infty} z^k g_k\Big]\oplus\Big[\sum_{k=0}^{\infty} z^k D_*(TT^*)^k f\Big]\oplus Q_1 T^{-1} f \tag{2.9}$$

where

$$g_k(\lambda)=\sum_{n=1}^{\infty}\lambda^n D_*^n (TT^*)^k f,\quad \text{for}\quad |\lambda|<1. \tag{2.10}$$

Then

$$g_k(\lambda)-f_k(\lambda)=\sum_{n=1}^{\infty}\lambda^n D_*^{n+2}(TT^*)^k f\quad \text{for}\quad |\lambda|<1. \tag{2.11}$$

Observe that $\mathcal{M}$ is invariant for $S^*_{\mathcal{E}_2}$ and let us denote

$$S_*=S^*_{\mathcal{E}_2}|_{\mathcal{M}}. \tag{2.12}$$

(2.11) becomes $g_k-f_k=S_*^2 g_k$, so

$$f_k=(I-S_*^2)g\ . \tag{2.13}$$

For a Hilbert space $\mathscr{E}$ and $A \in B(\mathscr{E})$ a linear bounded operator, we denote by $A_\times$ the operator of multiplication by A from $H^2(\mathscr{E})$ to $H^2(\mathscr{E})$:

$$(A_\times u)(z) = \sum_{k=0}^{\infty} z^k A u_k, \quad \text{for} \quad u(z) = \sum_{k=0}^{\infty} z^k u_k, \quad |z| < 1.$$

Lemma 2.2. *The operator $(I_{\mathscr{M}} - S_*^2)_\times \colon H^2(\mathscr{M}) \to H^2(\mathscr{M})$ is invertible.*

Proof. We will prove that $I_{\mathscr{M}} - S_*^2 \colon \mathscr{M} \to \mathscr{M}$ is invertible. Let $S_{\mathscr{E}_2} \colon H^2(\mathscr{E}_2) \to H^2(\mathscr{E}_2)$ be the unilateral shift

$$(S_{\mathscr{E}_2} u)(z) = \sum_{k=0}^{\infty} z^{k+1} u_k, \quad \text{for} \quad u(z) = \sum_{k=0}^{\infty} z^k u_k, \quad |z| < 1.$$

We observe first that $(S_*^2)^* = P_{\mathscr{M}} S_{\mathscr{E}_2}^2|_{\mathscr{M}}$.

Let $u \in \ker (I_{\mathscr{M}} - S_*^2)^* = \ker (I_{\mathscr{M}} - P_{\mathscr{M}} S_{\mathscr{E}_2}^2|_{\mathscr{M}})$. Then $u = P_{\mathscr{M}} u = P_{\mathscr{M}} S_{\mathscr{E}_2}^2 u \Leftrightarrow \Leftrightarrow P_{\mathscr{M}}(u - S_{\mathscr{E}_2}^2 u) = 0$ or equivalently $(u - S_{\mathscr{E}_2}^2 u)$ is in $\mathscr{M}^\perp$ and this implies $(u - S_{\mathscr{E}_2}^2 u) \perp u$ from which it results

$$(u, u) = (S_{\mathscr{E}_2}^2 u, u). \tag{2.14}$$

Let $u(\lambda) = \sum_{n=1}^{\infty} \lambda^n D_*^n h'$, $|\lambda| < 1, h' \in \mathscr{H}_2$. (2.14) becomes

$$\sum_{n=1}^{\infty} \|D_*^n h'\|_2^2 = \sum_{n=3}^{\infty} (D_*^{n-2} h', D_*^n h') = \sum_{n=3}^{\infty} (D_*^{n-1} h', D_*^{n-1} h') = \sum_{n=2}^{\infty} \|D_*^n h'\|_2^2.$$

Then $\|D_* h'\|_2 = 0$ since the series are convergent by Lemma 2.1, so $D_* h' = 0$ and this implies $u = 0$, so

$$\ker (I_{\mathscr{M}} - S_*^2)^* = \{0\}. \tag{2.15}$$

Next we prove that $I_{\mathscr{M}} - S_*^2$ is bounded from below. Let $u(\lambda) = \sum_{n=1}^{\infty} \lambda^n D_*^n h'$, $h' \in \mathscr{H}_2$, $|\lambda| < 1$, then

$$\|(I_{\mathscr{M}} - S_*^2) u\|_{H^2(\mathscr{E}_2)}^2 = \sum_{n=1}^{\infty} \|(D_*^n - D_*^{n+2}) h'\|_2^2 = \sum_{n=1}^{\infty} \|D_*^n (TT^*) h'\|_2^2 =$$

$$= \sum_{n=1}^{\infty} \|(TT^*) D_*^n h'\|_2^2 \geqq c^2 \sum_{n=1}^{\infty} \|D_*^n h'\|_2^2 = c^2 \|u\|_{H^2(\mathscr{E}_2)}^2.$$

Here we used the fact that TT^*, being positive and invertible, is bounded from below, i.e.:

$$\|TT^* h'\|_2 \geqq c \|h'\|_2 \quad \text{for every } h' \in \mathscr{H}_2, \text{ with } c > 0.$$

So

$$\|(I_{\mathscr{M}} - S_*^2) u\|_{H^2(\mathscr{E}_2)} \geqq c \|u\|_{H^2(\mathscr{E}_2)}, \quad c > 0. \tag{2.16}$$

(2.15) and (2.16) prove that there exists $(I_{\mathcal{M}}-S_*^2)^{-1}\colon \mathcal{M}\to\mathcal{M}$ and then there exists $(I_{\mathcal{M}}-S_*^2)_\times^{-1}\colon H^2(\mathcal{M})\to H^2(\mathcal{M})$. So the lemma is proved.

Lemma 2.2, (1.2), (2.9) and (2.13) imply

$$\tilde{V}_1T^{-1}f=\Big[\sum_{k=0}^{\infty}z^k(I_{\mathcal{M}}-S_*^2)^{-1}f_k\Big]\oplus\Big[\sum_{k=0}^{\infty}z^kD_*(TT^*)^kf\Big]\oplus W^{-1}Q_2f=$$

$$=[(I_{\mathcal{M}}-S_*^2)_\times^{-1}\oplus I_{H^2(\mathcal{E}_2)}\oplus W^{-1}]\tilde{V}_2f.$$

So we have proved

Theorem 2.3. *Let $T\colon \mathcal{H}_1\to\mathcal{H}_2$ be an invertible contraction. There exist the Hilbert spaces $\mathcal{E}_2$, $\mathcal{M}$, the subspaces (closed, linear) $\mathcal{K}_1$ and $\mathcal{K}_2$ of $H^1(\mathcal{M})\oplus H^2(\mathcal{E}_2)$ and the unitary operators $\tilde{V}_1\colon \mathcal{H}_1\to\mathcal{K}_1\oplus Q_1\mathcal{H}_1$, $\tilde{V}_2\colon \mathcal{H}_2\to\mathcal{K}_2\oplus Q_2\mathcal{H}_2$ such that*

$$T^{-1}=\tilde{V}_1^*[(I_{\mathcal{M}}-S_*^2)_\times^{-1}\oplus I_{H^2(\mathcal{E}_2)}\oplus W^{-1}]\tilde{V}_2 \tag{2.17}$$

where S_* is defined by (2.12) *and W by* (1.2).

3. A model for a general bounded linear operator

To apply the Theorems 1.1 and 2.3 to a general linear bounded operator $T\colon \mathcal{H}_1\to\mathcal{H}_2$, let us denote as in [4], [6]

$$D_T=[(I_{\mathcal{H}_1}-T^*T)^+]^{1/2},\quad X_T=[(I_{\mathcal{H}_1}-T^*T)^-]^{1/2}$$

where, for $A=A^*$, $A^+=\dfrac{|A|+A}{2}$, $A^-=\dfrac{|A|-A}{2}$.

Let $\mathcal{D}_T=\overline{D_T\mathcal{H}_1}$ be the defect space of T, $\mathcal{D}_T^1=\ker(I-T^*T)$, $\mathcal{X}_T=\overline{X_T\mathcal{H}_1}$ the excess space of T, and consider the corresponding spaces $\mathcal{D}_{T^*}, \mathcal{D}_{T^*}^1, \mathcal{X}_{T^*}$ for T^*.

Then $\mathcal{H}_1=\mathcal{D}_T\oplus\mathcal{X}_T\oplus\mathcal{D}_T^1$, $\mathcal{H}_2=\mathcal{D}_{T^*}\oplus\mathcal{X}_{T^*}\oplus\mathcal{D}_{T^*}^1$ and from the relations $TD_T=D_{T^*}T$, $TX_{T^*}=X_{T_*}T$ (see the proof in [4]) it results $T\mathcal{D}_T\subset\mathcal{D}_{T^*}$, $T\mathcal{X}_T\subset\mathcal{X}_{T^*}$ and obviously $T\mathcal{D}_T^1\subset\mathcal{D}_{T^*}^1$. Define the operators $T_1=T|_{\mathcal{D}_T}\colon \mathcal{D}_T\to\mathcal{D}_{T^*}$, $T_2=T|_{\mathcal{X}_T}\colon \mathcal{X}_T\to\mathcal{X}_{T^*}$ and $T_3=T|_{\mathcal{D}_T^1}\colon \mathcal{D}_T^1\to\mathcal{D}_{T^*}^1$. T_1 is a strict contraction and $(|T_1|^n)_{m=1}^\infty$ converges strongly to 0 as $n\to\infty$ (see [4], [6]). T_2 is an invertible operator and T_2^{-1} is a contraction. T_3 is an isometry.

In order to obtain the model for T we apply Theorem 1.1 for T_1 with $\mathcal{H}_1$ replaced by $\mathcal{D}_T$ and $\mathcal{H}_2$ replaced by $\mathcal{D}_{T^*}$ and Theorem 2.3 for T_2^{-1} with $\mathcal{H}_1$ replaced by $\mathcal{X}_{T^*}$ and $\mathcal{H}_2$ replaced by $\mathcal{X}_T$.

4. Some results concerning the dilation of a contraction and its adjoint

Let $\mathcal{H}$ be a separable Hilbert space and $T: \mathcal{H} \to \mathcal{H}$ a contraction. For the sake of simplifying the presentation we suppose that

$$(4.1) \qquad (T^*T)^n \to 0 \quad \text{and} \quad (TT^*)^n \to 0 \quad \text{strongly, as} \quad n \to \infty.$$

The main results remain valid without this assumption. From (4.1), $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{H}$ and by Theorem 1.1 we have the subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$ of $H^2(\mathcal{H}) \oplus H^2(\mathcal{H})$ and the unitary operators $V_1: \mathcal{H} \to \mathcal{K}_1$, $V_2: \mathcal{H} \to \mathcal{K}_2$ such that $V_2 T = (S^* \oplus I) V_1$ and $V_1 T^* = (I \oplus S^*) V_2$ (where we denoted $S^*_{\mathcal{H}}$ by S^* and $I_{H^2(\mathcal{H})}$ by I).

Define $J = V_1 V_2^*$. J is an unitary operator from $\mathcal{K}_2$ to $\mathcal{K}_1$. Using the (easy to prove) fact that $\dim \mathcal{K}_2 = \dim \mathcal{K}_1 = \infty$, the orthogonals being considered in $H^2(\mathcal{H}) \oplus H^2(\mathcal{H})$, we define $\tilde{J}: L^2(\mathcal{H}) \oplus L^2(\mathcal{H}) \to L^2(\mathcal{H}) \oplus L^2(\mathcal{H})$

$$(4.2) \quad \tilde{J} = J \oplus (\text{unitary operator } \mathcal{K}_2^{\perp} \to \mathcal{K}_1^{\perp}) \oplus (\text{identity of } H^2_-(\mathcal{H}) \oplus H^2_-(\mathcal{H}))$$

(for the definition of $L^2(\mathcal{H})$ see [6], Ch. V); $H^2_-(\mathcal{H}) = L^2(\mathcal{H}) \ominus H^2(\mathcal{H})$).

Let Z^* be the backward shift on $L^2(\mathcal{H})$. if

$$u(z) = \sum_{n=-\infty}^{\infty} z^n u_n, \quad |z| = 1,$$

then

$$(Z^* u)(z) = \sum_{n=-\infty}^{\infty} z^n u_{n+1}, \quad |z| = 1.$$

Define

$$(4.3) \qquad U = \tilde{J}^* (I_{L^2(\mathcal{H})} \oplus Z^*), \quad V = (Z^* \oplus I_{L^2(\mathcal{H})}) \tilde{J}.$$

U and V are unitary operators on $L^2(\mathcal{H}) \oplus L^2(\mathcal{H})$. Let us identify $\mathcal{H}$ with $\mathcal{K}_2$ by the mean of V_2. Then we state

Theorem 4.1. *For every polynomial p in two variables,*

$$p(T, T^*) = P_{\mathcal{H}} p(V, U)|_{\mathcal{H}}$$

where by $P_{\mathcal{H}}$ we denote the projection onto $\mathcal{H}$.

The proof relies on direct computation and is omitted. Next we show that in the case of a normal contraction T satisfying the hypothesis (4.1), the operator $\tilde{J}$ of (4.2) can be choosed such that the operators U and V defined in (4.3) commute.

Theorem 4.2. *Let $T: \mathcal{H} \to \mathcal{H}$ be a normal contraction satisfying $(TT^*)^n \to 0$ strongly as $n \to \infty$. Then the operator $\tilde{J}$ in* (4.2) *can be constructed such that U and V defined in* (4.3) *satisfy $UV = VU$.*

Proof. The proof that follows was suggested by the referee, replacing the more complicated original one. T normal implies $D_*=D$ and $\mathscr{E}_1=\mathscr{E}_2=\mathscr{H}$ by hypotheses $(TT^*)^n\to 0$.

Let $T=\hat{W}R$ be the polar decomposition of T. Then $\hat{W}$ can be a unitary operator, $\hat{W}R=R\hat{W}$ and $\hat{W}D=D\hat{W}$. Define the operator $\hat{U}$ on $H^2(\mathscr{H})$ by

$$\hat{U}\Big(\sum_{k=0}^{\infty} z^k h_k\Big) = \sum_{k=0}^{\infty} z^k \hat{W}^2 h_k.$$

$\hat{U}$ is a unitary operator that commutes with S^*, the backward shift on $H^2(\mathscr{H})$. The operator $\tilde{U}$ defined by

$$\tilde{U} = \begin{pmatrix} 0 & I_{H^2(\mathscr{H})} \\ \hat{U} & 0 \end{pmatrix}$$

with respect to $H^2(\mathscr{H})\oplus H^2(\mathscr{H})$ is a unitary operator that satisfies

$$(S^*\oplus S^*)\tilde{U} = \tilde{U}(S^*\oplus S^*). \tag{4.4}$$

Then

$$\tilde{U}\Big(\Big(\sum_{k=0}^{\infty} z^k DT^*(TT^*)^k h\Big)\oplus\Big(\sum_{k=0}^{\infty} z^k D(TT^*)^k h\Big)\Big) =$$

$$= \tilde{U}\Big(\Big(\sum_{k=0}^{\infty} z^k D\hat{W}^* R^{2k+1} h\Big)\oplus\Big(\sum_{k=0}^{\infty} z^k D_* R^{2k} h\Big)\Big) = \Big(\sum_{k=0}^{\infty} z^k D_* R^{2k} h\Big)\oplus\Big(\sum_{k=0}^{\infty} z^k D\hat{W}RR^{2k} h\Big) =$$

$$= \Big(\sum_{k=0}^{\infty} z^k D(T^*T)^k h\Big)\oplus\Big(\sum_{k=0}^{\infty} z^k D_* T(T^*T)^k h\Big) =$$

$$= V_1V_2^*\Big(\Big(\sum_{k=0}^{\infty} z^k DT^*(TT^*)^k h\Big)\oplus\Big(\sum_{k=0}^{\infty} z^k D_*(TT^*)^k h\Big)$$

for every $h\in\mathscr{H}$. This shows that $\tilde{U}\mathscr{K}_2=\mathscr{K}_1$ and $\tilde{U}|_{\mathscr{K}_2}=V_1V_2^*$. Since $\tilde{U}$ is a unitary operator it results $\tilde{U}\mathscr{K}_2^{\perp}=\mathscr{K}_1^{\perp}$ and so we can choose $\tilde{J}$ such that

$$\tilde{J}|_{H^2(\mathscr{H})\oplus H^2(\mathscr{H})} = \tilde{U}.$$

For this $\tilde{J}$ we have, due also to (4.4),

$$UV|_{H^2(\mathscr{H})\oplus H^2(\mathscr{H})} = \tilde{U}^*(S^*\oplus S^*)\tilde{U} = S^*\oplus S^* = VU|_{H^2(\mathscr{H})\oplus H^2(\mathscr{H})}.$$

Since by (4.2), (4.3) the same is true for $H^2_-(\mathscr{H})\oplus H^2_-(\mathscr{H})$ it results $UV=VU$ and the theorem is proved.

We remark at the end that we can drop the assumption (4.1) from Theorems 4.1 and 4.2 without altering the results.

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AF-algebras with unique trace

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An AF C*-algebra is, by definition, the norm closure of an increasing sequence of finite dimensional C*-algebras. In some sense, these are the simplest noncommutative infinite dimensional C*-algebras.

Our interest in AF-algebras with unique trace is related to the problem of constructing subfactors with a given index of the hyperfinite type II_1 von Neumann factor R. For this, one is led to find a sequence of increasing finite dimensional C*-algebras and to take their weak closure in the GNS representation given by a tracial state. If there is only one tracial state, the finite hyperfinite von Neumann algebra one obtains is a factor, hence it is R if it is infinite dimensional.

One way to guarantee the uniqueness of the trace is to fit the situation described in Remark 3: one can apply then either the quoted theorem of Elliott (stated in K-theoretic language) or the Perron—Frobenius theory on matrices with positive entries.

Our approach gives the desired conclusion for a wider class of AF algebras (the matrix given in Remark 2 is not primitive) and establishes some additional properties.

Statement of the result

Let A be a unital AF C*-algebra, inductive limit of the finite dimensional algebras $\mathbf{C}\cdot 1\subset A_1\subset A_2\subset A_3\subset\ldots$ (1 is the unit of A).

We denote by $m_k=(m_1^k, m_2^k, \ldots, m_{c_k}^k)$ the dimension vector of the algebra A_k and by $R_k=(r_{ij}^k)_{i=1,\ldots,c_k;\; j=1,\ldots,c_{k+1}}$ the inclusion matrix for $A_k\subset A_{k+1}$ $(k\geqq 1)$. In particular, ${}^tR_k m_k=m_{k+1}$.

If w is a real vector, $w\geqq 0$ means that its entries are nonnegative.
For $w=(w_1, \ldots, w_n)\in\mathbf{R}^n$, $w\geqq 0$, $w\neq 0$, we define

$$\chi(w):=\Big(\sum_{k=1}^{n} w_k\Big)^{-1}\min\Big\{\sum_{i\in I} w_i \,\Big|\, I\subset\{1, 2, \ldots, n\},\ \mathrm{card}\,(I)\geqq n/2\Big\}.$$

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We consider the multiplicative group $G=\bigcup_{k=1}^{\infty}\mathcal{U}(A_k)$ and its action on A by inner automorphisms.

$$g\in G \overset{\Theta}{\longmapsto} \operatorname{Ad} g\in \operatorname{Int}(A)\subset \operatorname{Aut}(A)$$

$$g(x)\equiv(\operatorname{Ad} g)(x)=gxg^{-1}\quad (g\in G,\, x\in A).$$

We prove the following

Theorem. *With the notations introduced above, let*

$$\varepsilon_k:=\min_{j=i,\ldots,c_{k+1}}\chi\big((m_i^k r_{ij}^k)_{i=1,\ldots,c_k}\big)\quad (k\geqq 1).$$

If

$$(*)\qquad \sum_{k=1}^{\infty}\varepsilon_k=\infty,$$

then:

(a) *there is a unique normalized trace, denoted by τ, on A;*
(b) *τ is faithful if and only if A is simple;*
(c) *the action Θ is mixing with respect to the trace τ, i.e.*

$$(\forall)\, x, y\in A_h,\quad (\exists)\, g_n\in G\quad (n\in\mathrm{N})\quad \textit{such that}\quad \lim_{n\to\infty}\tau\big(g_n(x)y\big)=\tau(x)\tau(y).$$

There are conditions which imply $(*)$ and depend only on the inclusion matrices R_k.

Corollary. *With the R_k's introduced above, let*

$$\delta_k:=\min_{i,j} r_{ij}^k/\max_{i,j} r_{ij}^k\quad (i=1,\ldots,c_k;\ j=1,\ldots,c_{k+1})$$

and

$$\tilde{\varepsilon}_k:=\min_{j=1,\ldots,c_{k+1}}\chi\big((r_{ij}^k)_{i=1,\ldots,c_k}\big).$$

If

$$(1)\qquad \sum_{k=2}^{\infty}\delta_{k-1}\tilde{\varepsilon}_k=\infty,$$

or

$$(2)\qquad \sum_{k=2}^{\infty}\delta_{k-1}\delta_k=\infty,$$

then:

(i) *the algebra A is simple and has a unique normalized trace τ, which is faithful;*
(ii) *the action Θ is mixing with respect to the trace τ.*

Namely, we shall prove that (2)$\Rightarrow$(1)$\Rightarrow$($*$).

Remark 1. Condition $(*)$ depends effectively on the particular sequence of algebras A_n defining A. Indeed, let $m_1=(1,1,1,1)$, and for $k\geqq 1$,

$$R_{2k+1}=\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{pmatrix},\quad R_{2k}=\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{pmatrix};\quad \text{hence}\quad R_{2k-1}R_{2k}=\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}.$$

Then the sum in $(*)$ is zero for the sequence $A_1\subset A_2\subset A_3\subset\ldots$, but it is infinite for the sequence $A_1\subset A_3\subset A_5\subset\ldots$.

Remark 2. Condition $(*)$ does not imply any of the equivalent conditions in (b): let $m_1=(1,1,1)$, and

$$R_k=\begin{pmatrix}1&1&2\\0&1&0\\2&1&1\end{pmatrix}\quad \text{for all}\quad k\geqq 1.$$

Then $\varepsilon_k=1/2$, but the (unique) trace on A has the weights $((1/2)3^{-k+1}, 0, (1/2)3^{-k+1})$ on A_k, hence it is not faithful. One can also see from the Bratteli diagram that A is not simple.

Remark 3. As a special case of the Corollary (part (i)), we can treat the situation dealt with in a theorem of ELLIOTT (Th. 6.1. in [2]), namely when $R_k=R$ for all k, where R is a primitive matrix, i.e. there is a nonzero p such that R^p has positive entries. Indeed, if we consider the sequence

$$A_1\subset A_{p+1}\subset A_{2p+1}\subset A_{3p+1}\subset\ldots$$

(which also defines A), the inclusion matrices will be constantly R^p; hence, the δ_k's will be all equal and nonzero (because R^p has no zero entry), and then clearly (2) holds.

The proof of Elliott follows different ideas.

Notations and steps of the proof

Let

$$A_n=\bigoplus_{l=1}^{c_n}A_n^l,\quad A_n^l\cong \mathrm{Mat}_{m_l^n}(\mathbf{C})$$

be the factor decomposition of the A_n's. For $x\in A_n$, we denote by $[x]_n^l$ its A_n^l-component and by $\alpha_n^l(x)$ the normalized trace of $[x]_n^l\in A_n^l$:

$$\alpha_n^l(x)=\mathrm{tr}([x]_n^l)=(1/m_l^n)\,\mathrm{Tr}([x]_n^l)$$

(we denote by Tr the canonical trace on a full matrix algebra — i.e. the sum of all diagonal entries — and by tr the normalized one).

If $v=(v_1 \dots v_k)\in \mathbf{C}^k$ is a vector, we write $\omega(v)$ for the "oscillation" of v, i.e.

$$\omega(v) := \max_{i,j=1,\dots,k} |v_i - v_j|.$$

Now for any $x \in A_n$, we introduce the vector $\alpha_n(x) := (\alpha_n^1(x), \alpha_n^2(x), \dots, \alpha_n^{c_n}(x))$ and the value $\omega(\alpha_n(x))$. We denote

$$A_\infty := \bigcup_{n=1}^{\infty} A_n.$$

The proof will be divided in a sequence of lemmas.

The first step is to show that for any $x \in A_\infty$, $\lim_{n\to\infty} \omega(\alpha_n(x))=0$, i.e. the entries of $\alpha_n(x)$ tend to become mutually equal. It is here that we use condition $(*)$. This implies that as n goes to infinity, the entries of $\alpha_n(x)$ converge to some complex number $\tau(x)$. This result is derived in Lemma 3, using the results of the previous two lemmas.

In Lemma 4 we check that the map $x \in A_\infty \mapsto \tau(x) \in \mathbf{C}$ defines a tracial state on A_∞ and we show that this is the unique one. So assertion (a) of the Theorem will be proved.

In Lemma 5, using a characterization of simplicity for AF algebras in terms of the inclusion matrices, we prove that the above defined trace is faithful if and only if the algebra A is simple, i.e. assertion (b) of the Theorem.

Assertion (c) of the Theorem (that the action Θ is mixing with respect to the trace τ) is proved in Lemma 6, after some remarks on finite dimensional C^*-algebras.

Finally, in Lemma 7, we show that $(2)\Rightarrow(1)\Rightarrow(*)$ and that if (1) or (2) hold, then the algebra A is simple. Using these facts, the Corollary follows easily from the Theorem.

We emphasize that the whole proof depends on the fact that $\lim_{n\to\infty} \omega(\alpha_n(x))=0$. This is deduced from condition $(*)$ by the estimate given in Lemma 2. One can look for other estimates in order to obtain the same fact from other conditions. Our estimates is insensitive to the equality of all rows of Q, when $\|Q\|_\omega=0$, regardless of ε (see the notations in Lemma 2). We have chosen it because of its relative simplicity.

The proofs

First of all we clarify how the inclusion matrices R_k and the dimension vectors m_k allow the computation of $\alpha_{n+1}(x)$ from $\alpha_n(x)$. Let $Q_n=(q_{ij}^n)_{i=1,\dots,c_{n+1};\ j=1,\dots,c_n}$ be the matrix given by $q_{ij}^n=m_j^n r_{ji}^n/m_i^{n+1}$, i.e.

$$Q_n = \begin{pmatrix} m_1^{n+1} & & 0 \\ & \ddots & \\ 0 & & m_{c_{n+1}}^{n+1} \end{pmatrix}^{-1} {}^tR_n \begin{pmatrix} m_1^n & & 0 \\ & \ddots & \\ 0 & & m_{c_n}^n \end{pmatrix},$$

and $1_m=(1, 1, \dots, 1)\in \mathbf{C}^m$. Note that $Q_n(1_{c_n})=1_{c_{n+1}}$ because $m_l^{n+1}=\sum_{k=1}^{c_n} m_k^n r_{kl}^n$ $(l=1, \dots, c_{n+1})$.

Lemma 1. *For any* $x\in A_n$ *we have*

(a) $\alpha_{n+1}(x)=Q_n\alpha_n(x)$,

(b) $\min_{1\leqq k\leqq c_n} \operatorname{Re}\alpha_n^k(x)\leqq \operatorname{Re}\alpha_{n+1}^l(x)\leqq \max_{1\leqq k\leqq c_n} \operatorname{Re}\alpha_n^k(x)$,

$\min_{1\leqq k\leqq c_n} \operatorname{Im}\alpha_n^k(x) \leqq \operatorname{Im}\alpha_{n+1}^l(x) \leqq \max_{1\leqq k\leqq c_n} \operatorname{Im}\alpha_n^k(x)$ *for all* $l=1, \dots, c_{n+1}$.

Proof. (a) Using the information given by the inclusion matrix, it follows that

$$\alpha_{n+1}^l(x) = \operatorname{Tr}([x]_{n+1}^l)/m_l^{n+1} = \Big(\sum_{k=1}^{c_n} r_{kl}^n \operatorname{Tr}([x]_n^k)\Big)/m_l^{n+1} = \Big(\sum_{k=1}^{c_n} m_k^n r_{kl}^n \alpha_n^k(x)\Big)/m_l^{n+1}.$$

(b) This is a consequence of the relation $Q_n(1_{c_n})=1_{c_{n+1}}$ and of the fact that Q_n has real nonnegative entries (hence $\alpha_{n+1}^l(x)$ is a weighted average of the entries of $\alpha_n(x)$).

Let us study the matrices $Q=(q_{ij})_{i=1,\dots,n;\ j=1,\dots,m}$ with real nonnegative entries which satisfy $Q(1_m)=1_n$. Note that if $v\in \mathbf{R}^m$ and $\omega(v)=0$, then $\omega(Q(v))=0$ ($\omega(w)=0 \Leftrightarrow w$ is proportional to the vector 1_m). Since ω defines a seminorm on any $\mathbf{R}^p$, from the above remark we see that Q induces a linear map $\tilde{Q}$: $\mathbf{R}^m/\omega \to \mathbf{R}^n/\omega$, where $\mathbf{R}^p/\omega$ denotes the quotient space $\mathbf{R}^p/\{v\in \mathbf{R}^p|\omega(v)=0\}$. Hence,

$$\|Q\|_\omega := \sup\{\omega(Q(v))|v\in \mathbf{R}^n,\ \omega(v)\leqq 1\}$$

is finite. Clearly

$$\omega(Q(v)) \leqq \|Q\|_\omega\,\omega(v) \quad \text{and} \quad \|Q_1Q_2\|_\omega \leqq \|Q_1\|_\omega \|Q_2\|_\omega$$

whenever Q_1Q_2 is defined.

Lemma 2. *Let* $Q=(q_{ij})_{i=1,\dots,n;\ j=1,\dots,m}$ *be a matrix with real nonnegative entries which satisfies* $Q(1_m)=1_n$. *Then* $\|Q\|_\omega\leqq 1-\varepsilon$, *where*

$$\varepsilon := \min_{i=1,\dots,n} \chi((q_{ij})_{j=1,\dots,m}).$$

Proof. It is enough to show that if $v=(v_1, \dots, v_m)\in \mathbf{R}^m$, $w=(w_1, \dots, w_m)\in \mathbf{R}^m$ are such that $v\geqq 0$, $w\geqq 0$, $\sum_{k=1}^m v_k=1$, $\sum_{k=1}^m w_k=1$, $\chi(v)\geqq\varepsilon$, $\chi(w)\geqq\varepsilon$, then

$$|\langle\alpha, v\rangle-\langle\alpha, w\rangle| \leqq (1-\varepsilon)\,\omega(\alpha)$$

for any $\alpha=(\alpha_1, \dots, \alpha_m)\in \mathbf{R}^m$, where $\langle\cdot,\cdot\rangle$ stands for the canonical scalar product of $\mathbf{R}^m$. The desired result will then follow by considering $v=(q_{ik})_{k=1,\dots,m}$, $w=(q_{jk})_{k=1,\dots,m}$ for all $1\leqq i, j\leqq n$.

Let $a=\min_k \alpha_k$, $b=\max_k \alpha_k$, $I=[a,b]^m$. Then $\alpha\in I$, $\omega(\alpha)=b-a$. Since the map $f\colon I\to\mathbf{R}$, $f(u):=\langle u, v\rangle-\langle u, w\rangle$ is an affine map, $f(I)=\operatorname{co} f(\operatorname{ext} I)$, where $\operatorname{ext} I$ denotes the set of extreme points of I and co stands for convex hull.

Let $\beta\in\operatorname{ext} I$, $\beta=(\beta_1, \dots, \beta_m)$. Then $\beta_k\in\{a, b\}$ for any $k=1, \dots, m$. Denote

$$K_a=\{k\,|\,1\leqq k\leqq m,\ \beta_k=a\},\quad K_b=\{k\,|\,1\leqq k\leqq m,\ \beta_k=b\}.$$

One of the sets K_a and K_b has at least $n/2$ elements. Suppose $\operatorname{card} K_a\geqq n/2$. Since $\langle\beta, w\rangle\geqq a$, we have

$$f(\beta)=\Big(a\sum_{k\in K_a} v_k+b\sum_{k\in K_b} v_k\Big)-\langle\beta, w\rangle=$$

$$=\Big[b-(b-a)\sum_{k\in K_a} v_k\Big]-\langle\beta, w\rangle\leqq b-(b-a)\chi(v)-a\leqq(1-\varepsilon)(b-a).$$

For v instead of w we also obtain $f(\beta)\geqq-(1-\varepsilon)(b-a)$.

The case $\operatorname{card} K_b\geqq n/2$ can be treated similarly and we obtain the same results. Thus for any $\beta\in\operatorname{ext} I$ we have

$$-(1-\varepsilon)(b-a)\leqq f(\beta)\leqq(1-\varepsilon)(b-a),$$

hence

$$f(I)\subset[-(1-\varepsilon)(b-a),\ (1-\varepsilon)(b-a)],$$

and therefore

$$|f(\alpha)|\leqq(b-a)(1-\varepsilon)=\omega(\alpha)(1-\varepsilon).$$

Recall that $\prod_{n=1}^{\infty}(1-\eta_n)=0$ whenever $0\leqq\eta_n\leqq 1$ and $\sum_{n=1}^{\infty}\eta_n=\infty$. Therefore, by condition $(*)$ we have

$$(**)\qquad \prod_{n=n_0}^{\infty}(1-\varepsilon_n)=0\quad(\forall)\, n_0\geqq 1.$$

Note that due to Lemma 1(a), the ε_n's defined in the Theorem have the same meaning for the matrices Q_n as ε for the matric Q in Lemma 2.

For $v=(v_1, \dots, v_m)\in\mathbf{C}^m$, define

$$\|v\|_\infty:=\max_{k=1,\dots,m}|v_k|.$$

Now we can prove

Lemma 3. *For any* $x \in A_{n_0}$ *we have*

(a) $\lim_{n\to\infty} \omega(\alpha_n(x)) = 0$,

(b) $\lim_{n\to\infty} \|\alpha_n(x) - \tau(x)\cdot 1_{c_n}\|_\infty = 0$ *for some* $\tau(x) \in \mathbf{C}$.

Proof. Let $n \geqq n_0$. Since both ω and $\|\cdot\|_\infty$ are seminorms, we can deal separately with the real and imaginary parts of $\alpha_n(x)$. Denote by $\operatorname{Re}\alpha_n(x)$ and $\operatorname{Im}\alpha_n(x)$ the vectors whose entries are the real and the imaginary parts, respectively, of the entries of $\alpha_n(x)$. By Lemma 1(a), we see that

$$\operatorname{Re}\alpha_{n+1}(x) = Q_n(\operatorname{Re}\alpha_n(x)),\ \operatorname{Im}\alpha_{n+1}(x) = Q_n(\operatorname{Im}\alpha_n(x)).$$

Lemma 2 implies that

$$\omega(\operatorname{Re}\alpha_{n+1}(x)) \leqq \|Q_n\|_\omega\, \omega(\operatorname{Re}\alpha_n(x)) \leqq (1-\varepsilon_n)\,\omega(\operatorname{Re}\alpha_n(n)).$$

Iterating we get

$$\omega(\operatorname{Re}\alpha_{n+1}(x)) \leqq \prod_{k=n_0}^{n} (1-\varepsilon_k)\,\omega(\operatorname{Re}\alpha_{n_0}(x))$$

and then, by $(**)$,

$$\lim_{n\to\infty} \omega(\operatorname{Re}\alpha_n(x)) = 0.$$

Since

$$\omega(\operatorname{Re}\alpha_n(x)) = \max_{1\leqq l\leqq c_n} \operatorname{Re}\alpha_n^l(x) - \min_{1\leqq l\leqq c_n} \operatorname{Re}\alpha_n^l(x),$$

Lemma 1(b) implies that

$$\lim_{n\to\infty} \|\operatorname{Re}\alpha_n(x) - a1_{c_n}\|_\infty = 0 \quad \text{for some} \quad a \in \mathbf{R}.$$

The vectors $\operatorname{Im}\alpha_n(x)$ can be treated similarly.

Lemma 4. (a) *The mapping* $x \in A_\infty \overset{\tau}{\longmapsto} \tau(x) \in \mathbf{C}$ *is a continuous normalized trace on* A_∞ *which can be extended by continuity to the whole* A.

(b) *Any normalized trace on* A_∞ *equals* τ.

Proof. (a) Linearity follows from the fact that

$$\alpha_n(ax+by) = a\alpha_n(x) + b\alpha_n(y) \quad \text{for any} \quad x, y \in A_n \quad \text{and} \quad a, b \in \mathbf{C}.$$

It is easy to see that $\alpha_n(1) = 1_{c_n}$, hence $\tau(1) = 1$. Since $|\operatorname{tr}([x]_n^l)| \leqq \|[x]_n^l\| \leqq \|x\|$, we see that $\|\alpha_n(x)\|_\infty \leqq \|x\|$, and hence $|\tau(x)| \leqq \|x\|$. Similarly, $\alpha_n(x^*x) \geqq 0$, hence $\tau(x^*x) \geqq 0$.

That τ is a trace follows from the relation

$$\alpha_n(xy) = \alpha_n(yx) \quad \text{for any} \quad x, y \in A_n,$$

which is a consequence of the definition of α_n.

(b) Let μ be any normalized trace on A. Since the factors A_n^l have unique normalized traces, the restriction of μ to the algebra A_n is described by a nonnegative vector $t_n=(t_n^1, t_n^2, \ldots, t_n^{c_n})$ with $\sum_{k=1}^{c_n} t_n^k=1$. If $x\in A_n$, then

$$\mu(x) = \sum_{k=1}^{c_n} t_n^k \alpha_n^k(x).$$

Then for all $x\in A_n$.

$$|\mu(x)-\tau(x)| = \Big|\sum_{k=1}^{c_n} t_n^k[\alpha_n^k(x)-\tau(x)]\Big| \leqq \sum_{k=1}^{c_n} t_n^k \|\alpha_n(x)-\tau(x)\,1_{c_n}\|_\infty = \|\alpha_n(x)-\tau(x)\,1_{c_n}\|_\infty.$$

Hence Lemma 3 (a) implies that $\mu(x)=\tau(x)$ for all $x\in A_\infty$.

Lemma 5. *Suppose* $(*)$ *holds and* τ *is the above defined trace. Then* τ *is faithful if and only if the algebra* A *is simple.*

Proof. Denote by e_n^l the minimal central projection of A_n corresponding to A_n^l. It is known that A is simple if and only if for any $n\geqq 1$ and any $1\leqq l\leqq c_n$, there is a $p>n$ such that the inclusion matrix $R_{n,p}=(r_{ij}^{n,p})_{i=1,\ldots,c_n;\ j=1,\ldots,c_p}$ for $A_n\subset A_p$ has only nonzero entries on the l-th row (i.e. A_n^l "enters" in all factor summands of A_p)—just look at the description of the ideals in the Bratteli diagram of A. Since

$$\alpha_p^i(e_n^l) = (r_{li}^{n,p}\, m_l^n)/m_i^p, \quad i=1, \ldots, c_p,$$

we see that the above condition on the inclusion matrix is equivalent to the fact that $\alpha_p(e_n^l)$ has only nonzero entries.

Suppose first that τ is faithful. Choose $n\geqq 1$ and $1\leqq l\leqq c_n$. Since $\tau(e_n^l)\neq 0$, and

$$\lim_{p\to\infty} \|\alpha_p(e_n^l)-\tau(e_n^l)\,1_{c_p}\|_\infty = 0,$$

we infer that for p large enough, all the entries of $\alpha_p(e_n^l)$ are nonzero. Thus by the above remark, A must be simple.

The converse implication is obvious since $J:=\{x\in A \mid \tau(x^*x)=0\}$ is a bilateral ideal and $1\notin J$.

For proving the mixing property of Θ we need two elementary and possible well-known results which we record below.

For a finite dimensional C*-algebra N, with a fixed system of matrix units and $x\in N$, we denote by Diag (x) the set of values which are on the diagonal of x.

Remark 4. Let $x\in \mathrm{Mat}_n(\mathbf{C})\cong\mathcal{B}(\mathbf{C}^n)$, $x=x^*$. Then there is a unitary $u\in\mathrm{Mat}_n(\mathbf{C})$ such that $\mathrm{Diag}\big((\mathrm{Ad}\,u)(x)\big)$ has only one element (namely $\mathrm{tr}\,(x)$). (This statement also holds for $x\neq x^*$ but its proof would be more intricate.)

To see this, notice first that since $x=x^*$, there is an orthogonal basis of C^n with respect to which x has diagonal form, hence the corresponding matrix has real entries.

If we consider $(\operatorname{Ad} u)(x)$ instead of x, where u is the unitary matrix that describes the change of coordinates, we may assume that $x \in \operatorname{Mat}_n(\mathbf{R})$.

We shall obtain the assertion by induction. Let $n=2, x=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{Mat}_2(\mathbf{R})$. We define

$$u_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \in \mathscr{U}(\operatorname{Mat}_2(\mathbf{R})), \quad t \in [0, \pi/2].$$

Since $(\operatorname{Ad} u_0)(x)=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $(\operatorname{Ad} u_{\pi/2})(x)=\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$, and $t \mapsto (\operatorname{Ad} u_t)(x)$ is a continuous function with values in $\operatorname{Mat}_2(\mathbf{R})$, the Darboux property of it implies that there is a $t \in [0, \pi/2]$ such that $(\operatorname{Ad} u_t)(x)$ has equal diagonal entries. Moreover, $(\forall)\lambda \in \mathbf{R}$, $\min\{a, d\} \leqq \lambda \leqq \max\{a, d\} \Rightarrow (\exists)\, t \in [0, \pi/2]$, such that

$$(\operatorname{Ad} u_t)(x) = \begin{pmatrix} \lambda & * \\ * & * \end{pmatrix}. \tag{3}$$

The statement is proved for $n=2$. Assume we have proved it for $n-1$, $n \geqq 3$. Let $x=(a_{ij}) \in \operatorname{Mat}_n(\mathbf{R})$. If x has different diagonal entries, one of them, say a_{11}, differs from $\operatorname{tr}(x)$. We may assume that $a_{11} < \operatorname{tr}(x)$. There must be an $i_0 \neq 1$ such that $a_{i_0 i_0} > \operatorname{tr}(x)$. We may consider $i_0=2$. Due to (3), there is a unitary

$$\tilde{u}_t = \begin{pmatrix} u_t & 0 \\ 0 & I_{n-2} \end{pmatrix} \in \operatorname{Mat}_n(\mathbf{R})$$

such that

$$x' := (\operatorname{Ad} \tilde{u}_t)(x) = \begin{pmatrix} \operatorname{tr}(x) & * \\ * & x'' \end{pmatrix},$$

where $x'' \in \operatorname{Mat}_{n-1}(\mathbf{R})$. By the inductive assumption there is a $u'' \in \operatorname{Mat}_{n-1}(\mathbf{C})$ such that $\operatorname{Diag}((\operatorname{Ad} u'')(x''))$ has only one value, namely $\operatorname{tr}(x'')$. But $\operatorname{tr}(x'')=\operatorname{tr}(x)$, hence if

$$u' = \begin{pmatrix} 1 & 0 \\ 0 & u'' \end{pmatrix}$$

then $\operatorname{Diag}((\operatorname{Ad} u'\tilde{u}_t)(x))$ has only one value.

Remark 5. Let N be a finite dimensional C*-algebra with a fixed system of matrix units, and let μ be a normalized trace on N. If $x, y \in N$ and y has a diagonal form, then

$$|\mu(xy)-\mu(x)\mu(y)| \leqq \|y\|\, \Delta_N(x),$$

where $\Delta_N(x)=\max\{|a-a'| \,|\, a, a' \in \operatorname{Diag}(x)\}$.

This follows by an easy computation. Suppose that $N=\bigoplus_{i=1}^{m} \mathrm{Mat}_{n_i}(\mathrm{C})$, and let $t=(t_1, \dots, t_m)$ be the vector of the weights of the minimal projections of the factor summands of N in the trace μ (so that $\sum_{i=1}^{m} n_i t_i=1$). Let the diagonal entries of x and y be $a_1^1, a_2^1, \dots, a_{n_1}^1, a_1^2, \dots, a_{n_2}^2, \dots, a_1^m, \dots, a_{n_m}^m$ and $b_1^1, b_2^1, \dots, b_{n_1}^1, b_1^2, \dots, b_{n_2}^2, \dots, \dots, b_1^m, \dots, b_{n_m}^m$, respectively (the upper index indicates the factor summand of N).

Then

$$\mu(x) = \sum_{l=1}^{m} t_l \sum_{i=1}^{n_l} a_i^l, \quad \mu(y) = \sum_{k=1}^{m} t_k \sum_{j=1}^{n_k} b_j^k, \quad \mu(xy) = \sum_{k=1}^{m} t_k \sum_{j=1}^{n_k} a_j^k b_j^k$$

(because y has a diagonal form). Since $1= \sum_{l=1}^{m} \sum_{i=1}^{n_l} t_l$,

$$|\mu(xy)-\mu(x)\mu(y)| = \Big|\sum_{l=1}^{m} \sum_{k=1}^{m} \sum_{i=1}^{n_l} \sum_{j=1}^{n_k} t_l t_k (a_j^k b_j^k - a_i^l b_j^k)\Big| \leqq$$
$$\leqq \Big(\sum_{l=1}^{m} \sum_{k=1}^{m} \sum_{i=1}^{n_l} \sum_{j=1}^{n_k} t_l t_k\Big) \max_{k,l,i,j} |a_j^k - a_i^l| \max_{k,j} |b_j^k| = \Delta_N(x)\|y\|.$$

Lemma 6. *Suppose* $(*)$ *holds and* τ *is the trace on* A *given in Lemma* 4. *Then the action* Θ *is mixing with respect to* τ.

Proof. Choose the systems of matrix units in the A_n's such that the matrix units of A_n are sums of matrix units of A_{n+1} for all n. Let $x, y \in A_\infty$, $x=x^*$, $y=y^*$. We may assume $x, y \in A_{n_0}$. Since y is selfadjoint, there is a $u_0 \in \mathcal{U}(A_{n_0})$ such that $(\mathrm{Ad}\, u_0)(y)$ is diagonal in the matrix units system of A_{n_0}; moreover, this will hold in all A_n, $n \geqq n_0$.

From the Remark 4, we infer that for $n \geqq n_0$ there is a $u_n \in \mathcal{U}(A_n)$ such that

$$\mathrm{Diag}\big([(\mathrm{Ad}\, u_n)(x)]_n^l\big) = \big\{\mathrm{tr}\big([x]_n^l\big)\big\} = \{\alpha_n^l(x)\} \quad \text{for all} \quad l = 1, \dots, c_n.$$

Hence $\Delta_{A_n}\big((\mathrm{Ad}\, u_n)(x)\big)=\omega(\alpha_n(x))$. Since $\lim_{n\to\infty} \omega(\alpha_n(x))=0$ and $\tau\big((\mathrm{Ad}\, u)(x)\big)=\tau(x)$, by Remark 5, we see that

$$\big|\tau\big((\mathrm{Ad}\, u_0^* u_n)(x) y\big)-\tau(x)\tau(y)\big| =$$
$$= \big|\tau\big((\mathrm{Ad}\, u_n)(x)(\mathrm{Ad}\, u_0)(y)\big)-\tau\big((\mathrm{Ad}\, u_n)(x)\big)\tau\big((\mathrm{Ad}\, u_0)(y)\big)\big| \leqq$$
$$\leqq \|(\mathrm{Ad}\, u_0)(y)\|\, \Delta_{A_n}\big((\mathrm{Ad}\, u_n)(x)\big) = \|y\|\, \omega\big(\alpha_n(x)\big) \to 0 \quad \text{as} \quad n \to \infty.$$

So we proved the mixing property for $x, y \in (A_\infty)_h$. That it also holds for any $x, y \in A_h$ can be proved using an obvious approximation argument.

Lemma 7. (a) *With the notations of the Corollary,*

$$1/2\, \delta_{k-1} \delta_k \leqq \delta_{k-1} \tilde{\varepsilon}_k \leqq \varepsilon_k \quad (k \geqq 2);$$

hence $(2) \Rightarrow (1) \Rightarrow (*)$.

(b) *If any of* (1) *or* (2) *holds, then the algebra A is simple; hence, by Theorem, part* (b), *the unique normalized trace on A is faithful.*

Proof. (a) Since $m_k={}^tR_{k-1}m_{k-1}$, we see that

$$\big(\max_{i,j} r_{ij}^{k-1}\big)\sum_{l=1}^{c_{k-1}} m_l^{k-1} \geqq \sum_{l=1}^{c_{k-1}} r_{lj}^{k-1} m_l^{k-1} = m_j^k \geqq \big(\min_{i,j} r_{ij}^{k-1}\big)\sum_{l=1}^{c_{k-1}} m_l^{k-1}$$

for any fixed $j=1, \dots, c_k$. Hence

$$\min_j m_k^j/\max_j m_k^j \geqq \min_{i,j} r_{ij}^{k-1}/\max_{i,j} r_{ij}^{k-1} = \delta_{k-1}. \tag{4}$$

The result can now be obtained using the following straightforward inequalities: for any nonnegative nonzero vectors $w=(w_1, \dots, w_n)$, $a=(a_1, \dots, a_n)$ we have

$$(1/2)\min_i w_i/\max_i w_i \leqq \chi(w);$$

$$\big(\min_i a_i/\max_i a_i\big)\chi(w) \leqq \chi\big((a_1w_1, a_2w_2, \dots, a_nw_n)\big).$$

The first one of these inequalities gives $(1/2)\delta_k \leqq \tilde{\varepsilon}_k$, while the second one and (4) give $\delta_{k-1}\tilde{\varepsilon}_k \leqq \varepsilon_k$.

(b) Both (1) and (2) imply that there is an infinity of R_k's with no zero entries. This implies that A is simple, by the same argument as that used in the proof of Lemma 5.

This concludes the proof of the Theorem.

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Hyponormal operators on uniformly convex spaces

MUNEO CHŌ

Dedicated to Professor Jun Tomiyama on his 60th birthdy

1. Introduction. Let X be a complex Banach space. We denote by X^* the dual space of X and by $B(X)$ the space of all bounded linear operators on X.

Let us set

$$\pi = \{(x,f)\in X\times X^*:\ \|f\| = f(x) = \|x\| = 1\}.$$

The *spatial numerical range* $V(T)$ and the *numerical range* $V(B(X), T)$ of $T\in B(X)$ are defined by

$$V(T) = \{f(Tx):\ (x,f)\in\pi\}$$

and

$$V(B(X),T) = \{F(T): F\in B(X)^* \ \text{ and } \ \|F\| = F(I) = 1\},$$

respectively.

Definition 1. If $V(T)\subset\mathbf{R}$, then T is called *hermitian*. An operator $T\in B(X)$ is called *hyponormal* if there are hermitian operators H and K such that $T=H+iK$ and the commutator $C=i(HK-KH)$ is non-negative, that is

$$V(C)\subset\mathbf{R}^+ = \{a\in\mathbf{R}:\ a\geqq 0\}.$$

An operator N is called *normal* if there are hermitian operators H and K such that $N=H+iK$ and $HK=KH$. A normal operator N on a Banach space X has the following properties:

(1) $\mathrm{co}\,\sigma(N)=\overline{V(N)}=V(B(X), N)$.

(2) If $Nx_n\to 0$ for a bounded sequence $\{x_n\}$ in X, then $Hx_n\to 0$ and $Kx_n\to 0$.

Definition 2. Let X be Banach space. X will be said to be *uniformly convex* if to each $\varepsilon>0$ there corresponds a $\delta>0$ such that the conditions $\|x\|=\|y\|=1$ and $\|x-y\|\geqq\varepsilon$ imply $\dfrac{\|x+y\|}{2}\leqq 1-\delta$.

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X will be said to be *uniformly c-convex* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|y\| < \varepsilon$ whenever $\|x\| = 1$ and $\|x + \lambda y\| \leqq 1 + \delta$ for all complex numbers λ with $|\lambda| \leqq 1$.

X will be said to be *strictly c-convex* if $y = 0$ whenever $\|x\| = 1$ and $\|x + \lambda y\| \leqq 1$ for all complex numbers λ with $|\lambda| \leqq 1$.

All uniformly convex spaces, for example $\mathscr{L}^p(S, \Sigma, \mu)$ and $\mathscr{C}_p(\mathscr{H})$ for $1 < p < \infty$, are uniformly c-convex and all uniformly c-convex spaces are strictly c-convex.

$\mathscr{L}^1(S, \Sigma, \mu)$ and the trace class $\mathscr{C}_1(\mathscr{H})$ are the typical examples of uniformly c-convex spaces. See [7] and [9].

For an operator $T \in B(X)$, the spectrum, the approximate point spectrum, the point spectrum, the kernel, and the dual of T are denoted by $\sigma(T)$, $\sigma_\pi(T)$, $\sigma_p(T)$, $\operatorname{Ker}(T)$ and T^*, respectively.

For an operator $T = H + iK$ we denote the operator $H - iK$ by $\bar{T}$.

The following are well-known for $T \in B(X)$:

(1) $\overline{\mathrm{co}}\, V(T) = V(B(X), T)$, where $\overline{\mathrm{co}}\, E$ is the closed convex hull of E.

(2) $\mathrm{co}\, \sigma(T) \subset \overline{V(T)}$, where $\mathrm{co}\, E$ and $\bar{E}$ are the convex hull and the closure of E, respectively.

We now give a concrete example of a hyponormal operator on a uniformly c-convex space. Let $\mathscr{H}$ be a Hilbert space. Then the trace class $C_1(\mathscr{H})$ is a two sided ideal of $B(\mathscr{H})$.

Given $A, B \in B(\mathscr{H})$ we define

$$\delta_{A,B}(T) = AT - TB \quad (T \in \mathscr{C}_1(\mathscr{H})).$$

Then $\delta_{A,B}$ is an operator on a uniformly c-convex space $\mathscr{C}_1(\mathscr{H})$. It is easy to see that if A and B^* are hyponormal then $\delta_{A,B}$ is a hyponormal operator on $\mathscr{C}_1(\mathscr{H})$ (see Theorem 4.3 in [9]).

The following theorem derives from Lemma 20.3 and Corollary 20.10 in [4].

Theorem A. *If H is hermitian and $Hx = 0$ for $x \in X$ ($\|x\| = 1$), then there exists $f \in X^*$ such that $(x, f) \in \pi$ and $H^* f = 0$.*

2. Hyponormal operators on uniformly convex spaces. The following theorem was shown by K. Mattila [9].

Theorem B. *Let X be uniformly c-convex and let $T = H + iK$ be a hyponormal operator on X. If there exists a sequence $\{x_n\}$ of unit vectors in X such that*

$$(T - (a + ib))x_n \to 0,$$

then $(H - a)x_n \to 0$ and $(K - b)x_n \to 0$.

We shall show the following (converse to the theorem above):

Theorem 1. *Let X be uniformly convex and let $T=H+iK$ be a hyponormal operator on X.* (1) *If $a\in\sigma(H)$, then there exist some real number b and sequence $\{x_n\}$ of unit vectors for which $(H-a)x_n\to 0$ and $(K-b)x_n\to 0$, so that in particular, $a+ib\in\sigma(T)$.* (2) *Similarly, if $b'\in\sigma(K)$, then there exist some real number a' and sequence $\{y_n\}$ of unit vectors for which $(H-a')y_n\to 0$ and $(K-b')y_n\to 0$, so that in particular, $a'+ib'\in\sigma(T)$.*

We need the following

Theorem C ([9], Theorem 2.4). *Let X be strictly c-convex and let $C\geqq 0$ be hermitian. If $f(Cx)=0$ for some $(x,f)\in\pi$, then $Cx=0$.*

Proof of Theorem 1. (1) Since H is hermitian, so it follows that $a\in\sigma_\pi(H)$. Consider the extension space X^0 of X and the faithful representation $B(X)\to B(X^0)$: $T\to T^0$ in the sense of DE BARRA [1]. Then a is an eigenvalue of H^0. If x^0 is in $\operatorname{Ker}(H^0-a)$ such that $\|x^0\|=1$, then by Theorem A there exists $f^0\in X^{0*}$ such that $f^0(x^0)=\|f^0\|=1$ and $(H^0-a)^*f^0=0$.

Since T is hyponormal we can let that $C=i(HK-KH)\geqq 0$; then $C^0\geqq 0$ and

$$f^0(C^0x^0) = i\hat{x}\big(K^{0*}(H-a)^{0*}f^0\big)-if^0\big(K^0(H^0-a)x^0\big) = 0,$$

where $\hat{x}$ is the Gel'fand representation of x. Since the space X^0 is uniformly convex ([1], Theorem 4), by Theorem C, it follows that $C^0x^0=0$. Therefore, it is easy to see that $\operatorname{Ker}(H^0-a)$ is invariant for K^0. So there exist a sequence $\{x_n\}$ of unit vectors and a real number b such that $(H-a)x_n\to 0$ and $(K-b)x_n\to 0$.

(2) is the same. So the proof is complete.

Theorem 2. *Let X be uniformly convex and let $T=H+iK$ be a hyponormal operator on X. Then*

$$\operatorname{co}\sigma(T) = \overline{V(T)} = V(B(X),T).$$

Proof. It is well-known that $\operatorname{co}\sigma(T)\subset\overline{V(T)}\subset V\big(B(X),T\big)$. We assume that $\operatorname{Re}\sigma(T)\subset\{a\in\mathbf{R}\colon a\geqq 0\}$. Then, by Theorem 1, it follows that $\sigma(H)\subset\{a\in\mathbf{R}\colon a\geqq 0\}$. So it follows that $V\big(B(X),H\big)\subset\{a\in\mathbf{R}\colon a\geqq 0\}$ and so $\operatorname{Re}V\big(B(X),T\big)\subset\{a\in\mathbf{R}\colon a\geqq 0\}$. Since $\alpha T+\beta$ is hyponormal for every $\alpha,\beta\in\mathbf{C}$, it follows that $\operatorname{co}\sigma(T)=V\big(B(X),T\big)$. So the proof is complete.

Theorem D ([9], Theorem 2.5). *Let X be uniformly c-convex and let $C\geqq 0$ be a hermitian operator on X. If there are sequences $\{x_n\}\subset X$ and $\{f_n\}\subset X^*$ such that $\|x_n\|=\|f_n\|=1$ for each n, $f_n(x_n)\to 1$ and $f_n(Cx_n)\to 0$, then $Cx_n\to 0$.*

Lemma 3. *Let $T=H+iK$ be a hyponormal operator. If $\overline{T}T$ and $T\overline{T}$ are not invertible, then $0\in\partial\sigma(\overline{T}T)$ and $0\in\partial\sigma(T\overline{T})$, respectively, where ∂ denotes 'the boundary of'.*

Proof. We may only prove that $\sigma(\bar{T}T)$ and $\sigma(T\bar{T})$ are included in the half-plane $\{\alpha\in\mathbf{C}\colon \operatorname{Re}\alpha\geqq 0\}$. Since $V(H^2)$ and $V(K^2)$ are included in $\{\alpha\in\mathbf{C}\colon \operatorname{Re}\alpha\geqq 0\}$, it follows that $V(\bar{T}T)=V(H^2+K^2+C)\subset V(H^2)+V(K^2)+V(C)\subset\{\alpha\in\mathbf{C}\colon \operatorname{Re}\alpha\geqq 0\}$, where $C=i(HK-KH)\geqq 0$. Therefore, $\sigma(\bar{T}T)$ is included in $\{\alpha\in\mathbf{C}\colon \operatorname{Re}\alpha\geqq 0\}$. Also, since $\sigma(\bar{T}T)-\{0\}=\sigma(T\bar{T})-\{0\}$, it follows that $\sigma(T\bar{T})\subset\{\alpha\in\mathbf{C}\colon \operatorname{Re}\alpha\geqq 0\}$.

So the proof is complete.

Lemma 4. *Let X be uniformly c-convex and let $T=H+iK$ be a hyponormal operator on X. If $\bar{T}T$ is not invertible, then $T\bar{T}$ is not invertible.*

Proof. By Lemma 3, there exists a sequence $\{x_n\}$ of unit vectors in X such that $\bar{T}Tx_n\to 0$. We let that $C=i(HK-KH)\geqq 0$. Then, for a sequence $\{f_n\}$ in X^* such that $(x_n, f_n)\in\pi$, we get that $f_n(Cx_n)\to 0$. So, by Theorem D, $Cx_n\to 0$. Therefore, $T\bar{T}x_n=(H^2+K^2-C)x_n\to 0$.

So the proof is complete.

Theorem 5. *Let X and X^* be uniformly c-convex and let $T=H+iK$ be a hyponormal operator on X. Then*

$$\sigma(T)=\{z\in\mathbf{C}\colon \bar{z}\in\sigma_\pi(\bar{T})\}.$$

Proof. Since $T-z$ is hyponormal for every $z\in\mathbf{C}$, it is sufficient to show that $0\in\sigma(T)$ if and only if $0\in\sigma_\pi(\bar{T})$. Assume that 0 belongs to $\sigma(T)$. By Lemma 4, we may assume that $T\bar{T}$ is not invertible.

Therefore, by Lemma 3, 0 belongs to $\partial\sigma(T\bar{T})$. It follows that there exists a sequence $\{x_n\}$ of unit vectors in X such that $T\bar{T}x_n\to 0$. Since T is hyponormal, by Theorem B it follows that $\bar{T}^2x_n\to 0$. By the spectral mapping theorem for approximate point spectrum, 0 belongs to $\sigma_\pi(\bar{T})$.

Conversely, assume that 0 belongs to $\sigma_\pi(\bar{T})$. Then it follows that $0\in\sigma(T\bar{T})=\sigma(\bar{T}^*T^*)$. Similarly, 0 belongs to $\sigma_\pi(\bar{T}^*T^*)$. Here, $\bar{T}^*$ is hyponormal on a uniformly c-convex space X^*. Therefore, 0 belongs to $\sigma(T^*)=\sigma(T)$.

So the proof is complete.

Theorem 6. *Let X be strictly c-convex and let $T=H+iK$ be a hyponormal operator on X. Suppose that λ is an extreme point of* $\operatorname{co}\overline{V(T)}$ *such that $\lambda\in V(T)$. Let $f(Tx)=\lambda$ for some $(x,f)\in\pi$. Then $Tx=\lambda x$.*

Proof. Each linear mapping $u(z)=\alpha z+\beta$ $(z\in\mathbf{C})$, where $\alpha,\beta\in\mathbf{C}$, $\alpha\neq 0$, maps $V(T)$ onto $V(u(T))$ and $\overline{V(T)}$ onto $\overline{V(u(T))}$. In addition $u(T)$ is hyponormal. Hence, we can suppose that $\lambda\in\mathbf{R}$ and $\operatorname{Re}z\leqq\lambda$ $(z\in V(T))$. Since $f(Hx)=\lambda=\max\{\alpha\colon \alpha\in\overline{V(H)}\}$, it follows by Theorem C that $Hx=\lambda x$. If $x'\in\operatorname{Ker}(H-\lambda)$ such that $\|x'\|=1$, then there exists $f'\in X^*$ such that $(x',f')\in\pi$ and $(H-\lambda)^*f'=0$.

It follows that

$$f'(Cx') = i\widehat{x}'\big(K^*(H-\lambda)^*f'\big) - if'\big(K(H-\lambda)x'\big) = 0$$

where $C=i(HK-KH)\geqq 0$.

By Theorem C, $Cx'=0$. Hence, it follows that $(H-\lambda)Kx'=0$. Therefore, it is easy to see that $\operatorname{Ker}(H-\lambda)$ is invariant for K. Let K_1 be the restriction of K to $\operatorname{Ker}(H-\lambda I)$. Let $y\in\operatorname{Ker}(H-\lambda)$ with $\|y\|=1$ and $g\in(\operatorname{Ker}(H-\lambda))^*$ such that $\|g\|=g(y)=1$. Then

$$Ty = \lambda y+iKy = \lambda y+iK_1 y\in\operatorname{Ker}(H-\lambda)$$

and

$$g(Ty) = \lambda+ig(K_1 y).$$

Here, $g(Ty)\in V(T)$. Since λ is an extreme point of $\operatorname{co}\overline{V(T)}$ and $\operatorname{Re} z\leqq\lambda$ $\big(z\in V(T)\big)$, it follows that $V(K_1)\subset\mathbf{R}^+$ or $V(-K_1)\subset\mathbf{R}^+$. Let $f_1=f|\operatorname{Ker}(H-\lambda)$. We have then $f_1(K_1x)=f(Kx)=0$ and $\|f_1\|=f_1(x)=1$. Since $\operatorname{Ker}(H-\lambda)$ is strictly c-convex, it follows that $K_1x=Kx=0$, by Theorem C.

So the proof is complete.

3. Doubly commuting n-tuples of hyponormal operators

Definition 3. For commuting operators T_1 and T_2 such that $T_j=H_j+iK_j$ (H_j and K_j hermitian, $j=1, 2$), T_1 and T_2 are called *doubly commuting* if $\overline{T}_1T_2=T_2\overline{T}_1$. If T_1 and T_2 are doubly commuting, then H_j and K_j commute with H_l and K_l for $j\neq l$.

Let $\mathbf{T}=(T_1, \dots, T_n)$ be a commuting n-tuple of operators on X. Let $\sigma(\mathbf{T})$ be the Taylor joint spectrum of $\mathbf{T}$. We refer the reader to TAYLOR [11].

The spatial joint numerical range $V(\mathbf{T})$ and the joint numerical range $V\big(B(X), \mathbf{T}\big)$ of $\mathbf{T}$ are defined by

$$V(\mathbf{T}) = \big\{\big(f(T_1x), \dots, f(T_nx)\big)\in\mathbf{C}^n\colon (x, f)\in\pi\big\}$$

and

$$V\big(B(X), \mathbf{T}\big) = \big\{\big(F(T_1), \dots, F(T_n)\big)\in\mathbf{C}^n\colon F\in B(X)^* \text{ and } \|F\| = F(I) = 1\big\}.$$

The *joint numerical radius* $v(\mathbf{T})$ and the *joint spectral radius* $r(\mathbf{T})$ of $\mathbf{T}=(T_1, \dots, T_n)$ are defined by

$$v(\mathbf{T}) = \sup\{|z|\colon z\in V(\mathbf{T})\}$$

and

$$r(\mathbf{T}) = \sup\{|z|\colon z\in\sigma(\mathbf{T})\}.$$

Theorem E (V. WROBEL [14], Corollary 2.3). *Let* $\mathbf{T}=(T_1, \dots, T_n)$ *be a commuting n-tuple of operators. Then*

$$\operatorname{co}\sigma(\mathbf{T})\subset\overline{V(\mathbf{T})}.$$

Theorem 7. *Let X be uniformly convex, and let $\mathbf{T}=(T_1, \ldots, T_n)$ be a doubly commuting n-tuple of hyponormal operators on X. Then*

$$\operatorname{co}\sigma(\mathbf{T}) = \overline{V(\mathbf{T})} = V(B(X), \mathbf{T}).$$

Proof. By Theorem E, it is clear that $\operatorname{co}\sigma(\mathbf{T})\subset\overline{V(\mathbf{T})}\subset V(B(X), \mathbf{T})$. Assume that $\operatorname{co}\sigma(\mathbf{T})\subsetneqq V(B(X), \mathbf{T})$. Suppose that $\alpha=(\alpha_1, \ldots, \alpha_n)\in V(B(X), \mathbf{T})-\operatorname{co}\sigma(\mathbf{T})$. Then there exists a linear functional Φ on $\mathbf{C}^n$ and a real number r such that

$$\operatorname{Re}\Phi(z) < r < \operatorname{Re}\Phi(\alpha) \quad (z\in\operatorname{co}\sigma(\mathbf{T})).$$

Let $\Phi(z)=t_{11}z_1+\ldots+t_{1n}z_n$ $(z=(z_1, \ldots, z_n)\in\mathbf{C}^n)$, and choose a non-singular $n\times n$ matrix M with $(t_{11}, \ldots, t_{1n})$ as its first row. Then

$$\operatorname{Re} z_1 < r < \operatorname{Re}\beta_1 \quad (z = (z_1, \ldots, z_n)\in\sigma(M\mathbf{T})),$$

where $(\beta_1, \ldots, \beta_n)=M\alpha$. Therefore, $\operatorname{co}\sigma(\Sigma_j t_{1j}T_j)\subsetneqq V(B(X), \Sigma_j t_{1j}T_j)$. Since $\Sigma_j t_{1j}T_j$ is a hyponormal operator on a uniformly convex space, this yields a contradiction to Theorem 2.

So the proof is complete.

Corollary 8. *Let X be uniformly convex and let $\mathbf{T}=(T_1, \ldots, T_n)$ be a doubly commuting n-tuple of hyponormal operators on X. Then $r(\mathbf{T})=v(\mathbf{T})$.*

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Restrictions of positive self-adjoint operators

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A densely defined positive symmetric operator in a Hilbert space has a positive self-adjoint extension within the same space. This theorem is well known for a long time and forms a solid part of our knowledge of the theory of unbounded operators in Hilbert space. Hence the restrictions of positive self-adjoint operators to a dense linear subspace are completely characterized by the properties of symmetry and positiveness. The same problem for an arbitrary linear subspace has so far remained unsolved.

The main aim of this note is to give a necessary and sufficient condition for the existence of a positive self-adjoint operator whose restriction to a linear subspace of a Hilbert space is given. Our theorem contains, as a special case, the above mentioned classical result as well as its generalisation given in 1970 by ANDO and NISHIO [1, Theorem 1; Corollary 1] for closed initial operators. Our method of proof follows the proof used in 1983 by the first named author [2, Theorem] in the bounded operator case. Further properties of our extension presented here generalise the results of [3], [4], [5].

This work is a result of a visit in April 1988 of the second named author at the Eötvös University, Budapest.

Let A be a (linear) operator defined on a linear subspace $\mathscr{D}$ of a (complex) Hilbert space $\mathscr{H}$ with values in the space $\mathscr{H}$. Here $\mathscr{D}$ is not assumed to be closed or dense, nor A is assumed to have a closed graph. Throughout the paper we assume that A is symmetric and positive, that is, A has the following property:

(1) $$0 \leqq (Ax, x) \quad \text{for each} \quad x \text{ in } \mathscr{D}.$$

Of course, (1) is necessary for the existence of a positive self-adjoint extension.

Starting with assumption (1) we define a semi inner product $\langle .\,,.\rangle$ on $\mathscr{D}$ by

$$\langle x, y\rangle := (Ax, y) \quad \text{for } x \text{ and } y \text{ in } \mathscr{D}.$$

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A new Hilbert space appears by the usual construction: let $\mathscr{D}_0=\{x\in D\colon (Ax, x)=0\}$ be the kernel of $\langle\cdot,\ \cdot\rangle$ and let Q be the quotient map of $\mathscr{D}$ with respect to $\mathscr{D}_0$, that is,

$$Qx = x+\mathscr{D}_0 \quad \text{for all } x \text{ in } \mathscr{D},$$

then $Q(\mathscr{D})$ is a pre-Hilbert space with inner product

$$\langle Qx, Qy\rangle := (Ax, y) \quad \text{for } x, y \text{ in } \mathscr{D}. \tag{2}$$

Now $\hat{\mathscr{H}}$ will denote the completion of $Q(\mathscr{D})$.

Assume first for a moment that x belongs to $\mathscr{D}_0$ if and only if $Ax=0$. Then the formula

$$V(Qx) := Ax \quad \text{for } x \text{ in } \mathscr{D} \tag{3}$$

defines a linear map V from $Q(\mathscr{D})$ into $\mathscr{H}$ factoring A through Q. At the same time we observe that V^* extends Q. Indeed, the identity

$$(VQx, y) = (Ax, y) = \langle Qx, Qy\rangle \quad \text{for } x \text{ and } y \text{ in } \mathscr{D} \tag{4}$$

shows that $V^*y=Qy$. If moreover we assure that $\mathscr{D}(V^*)$ is dense in $\mathscr{H}$, in other words that V^{**} exists, then (3) gives us that $V^{**}V^*$ is a self-adjoint positive extension of A. This is because the closure of V is equal to V^{**} and because V^* is a closed operator with adjoint V^{**}.

Theorem 1. *Let A be a positive linear operator defined on a linear subspace $\mathscr{D}$ of a Hilbert space $\mathscr{H}$. The following two statements are equivalent:*

(i) *A has a positive self-adjoint extension $\tilde{A}$ in $\mathscr{H}$;*

(ii) *$\mathscr{D}_*:=[y\in\mathscr{H}\colon \sup\{|(Ax, y)|^2\colon x\in\mathscr{D},\ (Ax, x)\leqq 1\}<\infty]$ is dense in $\mathscr{H}$.*

Proof. Assume first (i). Then the domain $\mathscr{D}(\tilde{A})$ of $\tilde{A}$ is dense in $\mathscr{H}$. Hence the inclusion $\mathscr{D}(\tilde{A})\subset\mathscr{D}_*$ proves (ii); indeed, to prove that an element y from $\mathscr{D}(\tilde{A})$ belongs to $\mathscr{D}_*$ it is enough to see that for each x from $\mathscr{D}$, $Ax=\tilde{A}x$ holds and

$$|(Ax, y)|^2 = |(\tilde{A}x, y)|^2 \leqq (\tilde{A}x, x)(\tilde{A}y, y) = (Ax, x)(\tilde{A}y, y).$$

Assume now that (ii) holds true. The operator V (see (3)) is then well defined. Indeed, if x is a vector from $\mathscr{D}$ such that $(Ax, x)=0$ then one can show that $(Ax, y)=0$ holds true for each y from $\mathscr{D}_*$. Since $\mathscr{D}_*$ is assumed to be dense in $\mathscr{H}$, we obtain $Ax=0$. Moreover the domain $\mathscr{D}(V^*)$ of V^* is just $\mathscr{D}_*$. Hence V^* is densely defined by the assumption (ii). Here we arrive at the situation mentioned before, and $V^{**}V^*$ is a positive self-adjoint extension of A. The proof of Theorem 1 is complete.

Corollary 1. *Let $A\colon \mathscr{D}\to\mathscr{H}$ be a positive linear densely defined operator. Then A has a positive self-adjoint extension in $\mathscr{H}$.*

Proof. Arguing similarly as in the proof of the implication (i)$\Rightarrow$(ii) of Theorem 1, we show that $\mathcal{D}\subset\mathcal{D}_*$. Thus the condition (ii) of Theorem 1 is satisfied. Hence (i) of Theorem 1, which is our present assertion holds true.

Corollary 2. *For the positive linear operator $A\colon \mathcal{D}\to\mathcal{H}$ the following statements are equivalent:*

(i′) *A has a continuous positive extension $\tilde{A}$ on $\mathcal{H}$;*

(ii′) $\mathcal{D}_*=\mathcal{H}$;

(iii′) *there exists a constant $m\geqq 0$ such that*

$$\|Ax\|^2 \leqq m(Ax, x) \quad \textit{for each } x \textit{ from } \mathcal{D}.$$

Proof. Since $\mathcal{H}=\mathcal{D}(\tilde{A})\subset\mathcal{D}_*$ holds true for each continuous positive extension $\tilde{A}$ of A, the implication (i′)$\Rightarrow$(ii′) is immediate. Notice also that $\mathcal{D}(V^*)=\mathcal{D}_*$. So if (ii′) holds true then V^* is an everywhere defined closed operator, that is, V^* is continuous indeed. Hence $V^{**}V^*$ is a continuous positive linear extension of A on $\mathcal{H}$. This proves (ii′)$\Rightarrow$(i′).

If (iii′) holds, the operator V defined by (3) is continuous. Consequently $V^{**}V^*$ is a continuous positive extension of A. Conversely, (i′) implies (iii′) with $m:=\|\tilde{A}\|$.

Corollary 3. *Let $A\colon \mathcal{D}\to\mathcal{H}$ be a positive linear operator with a positive self-adjoint extension $\tilde{A}\colon \tilde{\mathcal{D}}\to\mathcal{H}$. Then $\mathbf{A}:=V^{**}V^*$ has the following properties:*

(iv) $\mathcal{D}(\tilde{A}^{1/2})\subseteqq\mathcal{D}(\mathbf{A}^{1/2})$;

(v) $\|\mathbf{A}^{1/2}x\|^2\leqq\|\tilde{\mathbf{A}}^{1/2}x\|^2$ *for each x in* $\mathcal{D}(\tilde{A}^{1/2})$.

Proof. Starting with positive self-adjoint operator $\tilde{A}$, we can construct the subspace $\tilde{\mathcal{D}}_0$, the quotient map $\tilde{Q}$, the completion $\tilde{\mathcal{H}}$ and the operator $\tilde{V}$ factoring $\tilde{A}$ through $\tilde{Q}$ in the same way as we have obtained $\mathcal{D}_0$, Q, $\hat{\mathcal{H}}$ and V, respectively, from A. Then $\tilde{A}=\tilde{V}^{**}\tilde{V}^*$, because both of these operators are self-adjoint. As in [4], we define an isometry T from $\hat{\mathcal{H}}$ into $\tilde{\mathcal{H}}$(=the completion of $\tilde{Q}(\tilde{\mathcal{D}})$) by the following identity:

$$T(Qx) = \tilde{Q}x \quad \text{for all } x \text{ from } \mathcal{D}.$$

That T is an isometry follows from

$$\langle\tilde{Q}x, \tilde{Q}x\rangle = (\tilde{A}x, x) = (Ax, x) = \langle Qx, Qx\rangle \quad \text{for each } x \text{ in } \mathcal{D}.$$

Since, moreover,

$$(\tilde{V}T)(Qx) = \tilde{V}(TQx) = \tilde{V}\tilde{Q}x = \tilde{A}x = Ax = VQx$$

holds true for each x from $\mathcal{D}$, we conclude that

$$\tilde{V}T|_{Q(\mathcal{D})} = V.$$

Hence, using the fact that T^* is a contraction, we have that

$$\|\mathrm{A}^{1/2}x\|^2 = \|V^*x\|^2 = \|T^*\tilde{V}^*x\|^2 \leqq \|\tilde{V}^*x\|^2 = \|\tilde{A}^{1/2}x\|^2$$

holds for each x in $\mathscr{D}(\mathrm{A}^{1/2}) \cap \mathscr{D}(\tilde{A}^{1/2})$. Now, since $\tilde{A}$ extends A, it follows that

$$\mathscr{D}(\tilde{V}^*) = \tilde{\mathscr{D}}_* \subset \mathscr{D}_* = \mathscr{D}(V^*),$$

and therefore

$$\mathscr{D}(\tilde{A}^{1/2}) = \mathscr{D}\big((\tilde{V}^{**}\tilde{V}^*)^{1/2}\big) = \mathscr{D}(\tilde{V}^*) \subset \mathscr{D}(\tilde{V}^*) = \mathscr{D}\big((V^{**}V^*)^{1/2}\big) = \mathscr{D}(\mathrm{A}^{1/2}).$$

This completes the proof.

Corollary 4. *Let $A\colon \mathscr{D} \to \mathscr{H}$ be a linear operator bounded below by m, that is, such that*

$$m\|x\|^2 \leqq (Ax, x) \quad \textit{holds for all } x \textit{ in } \mathscr{D}.$$

A admits a self-adjoint extension with the same bound if and only if the subspace

$$[y \in \mathscr{H}\colon \sup\{|(Ax - mx, y)|^2\colon x \in \mathscr{D}, (Ax, x) \leqq 1 + m\|x\|^2\} < \infty]$$

is dense in $\mathscr{H}$.

Proof. Since for each self-adjoint extension $\tilde{A}$ of A with a bound m, $\tilde{A} - mI$ is a positive self-adjoint extension of the positive (symmetric) operator $A - mI$, the conclusion of Corollary 4 follows from Theorem 1.

Corollary 5. *Any densely defined semibounded linear operator in Hilbert space has a self-adjoint extension with the same bound.*

Proof. Corollary 5 follows from Corollary 4 via arguments used in the proof of Corollary 1.

An extension of [5, Theorem] is the following

Theorem 2. *Let $A\colon \mathscr{D} \to \mathscr{H}$ be a positive linear operator with a positive self-adjoint extension $\tilde{A}$. Let B and C be continuous linear operators on $\mathscr{H}$ leaving $\mathscr{D}$ invariant and such that*

(vi) $ABx = C^*Ax,\ ACx = B^*Ax$ *for all x in $\mathscr{D}$.*

Then, with $\mathrm{A} = V^{**}V^*$ in Theorem 1, we have

(vii) $\mathrm{A}Bx = C^*\mathrm{A}x,\ \mathrm{A}Cx = B^*\mathrm{A}x$ *for all x in $\mathscr{D}(\mathrm{A})$.*

Proof. We define, as in the proof of [5], continuous linear operators $\hat{B}$ and $\hat{C}$ on $Q(\mathscr{D})$ as follows

(5) $$\hat{B}(Qx) = Q(Bx),\ \hat{C}(Qx) = Q(Cx) \quad \text{for each } x \text{ in } \mathscr{D}.$$

To show that $\hat{B}$ and $\hat{C}$ are well-defined and continuous we find estimates for the norm of $\hat{B}(Qx)$ and $\hat{C}(Qx)$ step by step. First we have for any x in $\mathscr{D}$ that

$$\langle \hat{B}(Qx), \hat{B}(Qx)\rangle = (ABx, Bx) = (C^*Ax, Bx) = (Ax, CBx) = \langle Qx, Q(CBx)\rangle \leqq$$
$$\leqq \langle Qx, Qx\rangle^{1/2} \langle Q(CBx), Q(CBx)\rangle^{1/2}.$$

Repeating this argument we obtain

$$\langle \hat{B}(Qx), \hat{B}(Qx)\rangle \leqq \langle Qx, Qx\rangle^{1/2+\dots+1/2^n} \langle Q(CB)^{2^{n-1}}x, Q(CB)^{2^{n-1}}x\rangle^{1/2^2} =$$
$$= \langle Qx, Qx\rangle^{1-1/2^n} (Ax, (CB)^{2^n}x)^{1/2^n} \leqq$$
$$\leqq \langle Qx, Qx\rangle^{1-1/2^n} \|Ax\|^{1/2^n} \|(CB)^{2^n}\|^{1/2^n} \|x\|^{1/2^n}.$$

Passing with n to infinity we get

(6) $\langle \hat{B}(Qx), \hat{B}(Qx)\rangle \leqq r(CB)\langle Qx, Qx\rangle$ for each x from $\mathscr{D}$,

where $r(CB)(\leqq\|CB\|)$ stands for the spectral radius of CB. (6) tells us that $\hat{B}$ is a well-defined continuous linear operator. $\hat{B}$ has norm not exceeding $r(CB)^{1/2}$. A similar argument applies to show that $\hat{C}$ is also continuous and its norm does not exceed the same value $r(BC)^{1/2}=r(CB)^{1/2}$. Thus both $\hat{B}$ and $\hat{C}$ have unique continuous extensions on $\hat{\mathscr{H}}$ which we also denote by $\hat{B}$ and $\hat{C}$, respectively, as this causes no confusion.

Now we see that $\hat{B}$ and $\hat{C}^*$, hence also $\hat{C}$ and $\hat{B}^*$, coincide since on $Q(\mathscr{D})$ they agree:

$$\langle Qx, \hat{C}^*(Qy)\rangle = \langle \hat{C}(Qx), Qy\rangle = \langle Q(Cx), Qy\rangle = (ACx, y) = (Ax, By) =$$
$$= \langle Qx, B(Qy)\rangle$$

holds true for each x and y in $\mathscr{D}$. On the other hand V interwines $\hat{B}$ and C^* (respectively $\hat{C}$ and B^*). Indeed, if x belongs to $\mathscr{D}$ then

$$V\hat{B}(Qx) = VQ(Bx) = A(Bx) = C^*Ax = C^*V(Qx),$$
$$V\hat{C}(Qx) = VQ(Cx) = A(Cx) = B^*Ax = B^*V(Qx).$$

Hence $C^*V\subset V\hat{B}$ and $B^*V\subset V\hat{C}$. Since C^* is bounded, we get

$$\hat{C}V^* = \hat{B}^*V^* \subset (V\hat{B})^* \subset (C^*V)^* = V^*C.$$

Similar argument shows that $\hat{B}V^*\subset V^*B$. Thus

(viii) $V^*By = \hat{B}V^*y$, $V^*Cy=\hat{C}V^*y$ for every y from $\mathscr{D}_*$.

Returning to the proof of (vii) we see that for each $x\in\mathscr{D}(\mathbf{A})$ and for each $y\in\mathscr{D}_*$ the following identities hold true (using (viii))

$$\langle V^*Bx, V^*y\rangle = \langle \hat{B}V^*x, V^*y\rangle = \langle V^*x, \hat{B}^*V^*y\rangle = \langle V^*x, \hat{C}V^*y\rangle =$$
$$= \langle V^*x, V^*(Cy)\rangle = (C^*V^{**}V^*x, y) = (C^*\mathbf{A}x, y).$$

As a consequence we have that, for each x from $\mathscr{D}(\mathrm{A})$, V^*Bx belongs to $\mathscr{D}(V^{**})$ and at the same time

$$C^*\mathrm{A}x = V^{**}V^*Bx = \mathrm{A}Bx.$$

The other equality of (vii) can be shown similarly. This completes the proof.

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On the local spectral radius of a nonnegative element with respect to an irreducible operator

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1. Introduction

The local spectral radius of a nonnegative element of a partially ordered Banach space with respect to a general positive linear continuous operator has been studied in [2]. The main results there gave, among others, sufficient conditions that the local spectral radius be a singularity of the local resolvent function, characterized the distinguished eigenvalues outside the essential spectrum, and sought positive solutions u of the equation $(\lambda-T)u=x$ for positive λ and positive x.

If T is a reducible positive operator, then we may, in general, clearly find nonnegative elements x of the space E such that the local spectral radius $r_T(x)$ of x with respect to T is strictly smaller than the (global) spectral radius $r(T)$ of T. The situation is more delicate, if the operator T is irreducible. The first main result of this paper, Theorem 7, lists four groups of fairly natural conditions, each of which is sufficient for any nonzero x in the positive cone E_+ to ensure that $r_T(x)=r(T)$, assuming T is irreducible. The preceding Propositions 1 through 5 and Remark 6 formulate some more general conditions ensuring $r_T(x)=r(T)$ even if T is reducible, whereas Example 8 shows that the irreducibility of T alone is not sufficient.

The second main result, Theorem 12, yields three groups of conditions, each of which guarantees that the equation $(r(T)-T)u=x$ has no solution u in all of E, assuming that T is irreducible and $x\in E_+\setminus\{0\}$. The preliminary results contain also here more general conditions. Several examples illustrate the irredundancy of some conditions, or that some other group of conditions is not sufficient.

In the third part of the results we show that if T is irreducible and $r(T)>0$ is a pole of its resolvent, then some conditions ensure that the equation $(r(T)-T)u=$

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$=(1-P)x$, where P denotes the spectral projection corresponding to the set $\{r(T)\}$, has a positive solution u for all x in E_+. It is also shown that some extra conditions are really needed to ensure the existence of a positive solution u. Further, we show that the algebraic eigenspace to the spectral radius of a compact, nonnegative operator need not have a basis of nonnegative elements, and discuss some connections to works of U. G. Rothblum [8], H. D. Victory, jr. [11] and J. Kölsche [5].

2. Preliminaries and notations

Let E be a real Banach space and let T be a linear continuous operator from E into E. By $N(T)$ and $R(T)$ we denote the kernel and the range of T, respectively. As usual ([9], p. 261]), we sometimes identify T with its complex extension $\tilde{T}$. In this spirit, e.g., for x in E we define

$$r_T(x) = \limsup_{n\to\infty} \|T^n x\|^{1/n},$$

$$\Omega_T(x) = \{\lambda\in\mathbf{C} \mid r_T(x) < |\lambda|\}$$

and

$$x_T\colon \Omega_T(x) \to E \quad \text{with} \quad x_T(\lambda) = \sum_{k=0}^{\infty} \lambda^{-k-1} T^k x.$$

We call $r_T(x)$ the local spectral radius of the element x with respect to the operator T, x_T the local resolvent function of the element x with respect to the operator T in its main component $\Omega_T(x)$. Of course $(\lambda-T)x_T(\lambda)=x$ for all $\lambda\in\Omega_T(x)$. We recall some results from [2] which will be used several times in this paper.

Unless explicitely stated otherwise, in the following E will always denote a partially ordered real Banach space with positive cone E_+, and T is a nonnegative operator in E. If $x\neq 0$ is a nonnegative element in E, then

(I) *$r_T(x)$ is a singularity of x_T if E_+ is normal or there is a pole μ of x_T with $|\mu|=r_T(x)$*; see [2, Theorems 6 and 10].

(II) *If E_+ is normal and there exist a $u\in E_+$ and a $\mu\geqq 0$ such that $(\mu-T)u=x$, then $r_T(x)\leqq\mu$*; see [2, Theorem 6].

(III) *If $r_T(x)$ is a pole of x_T, then there exist a $u\in E_+$ and a $\mu>0$ with $(\mu-T)u=$ $=x$ if and only if $r_T(x)<\mu$*; see [2, Theorem 10].

The proof for the last two assertions depends essentially on the following inequality: If $u\geqq 0$ and $\mu\geqq 0$ such that $(\mu-T)u=x$ then

$$0 \leqq \frac{(-1)^n}{n!} x_T^{(n)}(\lambda) \leqq \frac{u}{(\lambda-\mu)^n}$$

for all $n=0, 1, 2, \ldots$ and all $\lambda>\max\{\mu, r_T(x)\}$. This inequality was proved in [2, Proposition 5] with the help of the iterated local resolvent; we give here a very simple proof. From $u \geqq 0$ it follows that $\frac{(-n)^n}{n!} u_T^{(n)}(\lambda) = \sum_{k=0}^{\infty} \binom{k+n}{n} \lambda^{-k-n-1} T^k u \geqq 0$ for all $n=0, 1, 2, \dots$ and all $\lambda > r_T(u)$. From $(\mu - T)u = x$ it follows that $r_T(x) \leqq r_T(u) \leqq \max\{\mu, r_T(x)\}$ and

$$u_T(\lambda) = -\frac{x_T(\lambda) - u}{\lambda - \mu} \quad \text{for} \quad \lambda > \max\{\mu, r_T(x)\}.$$

Differentiating this equality n times and multiplying by $\frac{(-1)^n}{n!}$ we get for $\lambda > \max\{\mu, r_T(x)\}$

$$0 \leqq \frac{(-1)^n}{n!} u_T^{(n)}(\lambda) = -\sum_{j=0}^{n} \frac{(-1)^j}{j!} \frac{x_T^{(j)}(\lambda)}{(\lambda-\mu)^{n-j+1}} + \frac{u}{(\lambda-\mu)^{n+1}} \leqq$$

$$\leqq -\frac{(-1)^n}{n!} \frac{x_T^{(n)}(\lambda)}{\lambda-\mu} + \frac{u}{(\lambda-\mu)^{n+1}}$$

since each summand in the sum is nonnegative, because $(-1)^j x_T^{(j)}(\lambda) \geqq 0$ if $x \geqq 0$ and $\lambda > r_T(x)$. The last inequality is equivalent to the wanted inequality.

3. Results and proofs

Proposition 1. *Let the spectral radius $r(T)$ be a pole of the resolvent $R(\cdot, T)$ of T, and let x be a quasi-interior point in the sense of* [9, p. 241] *of the positive cone E_+. Then $r_T(x) = r(T)$.*

Proof. Let p denote the order of the pole $r = r(T)$, and let $\sum_{k=-p}^{\infty} (\lambda - r)^k Q_k$ be the Laurent expansion of $R(\lambda, T)$ around r. It is well-known that $Q_{-p} \geqq 0$. Assume that $r_T(x) < r(T)$. Then $Q_{-p} x = 0$ and, since x is quasi-interior, we obtain that $Q_{-p} = 0$, a contradiction.

A slightly stronger condition on the spectral radius than in the next proposition was used in [10, Lemma 4] for similar purposes.

Proposition 2. *Let E be a Banach lattice. Let $r(T)$ be a limit point of the set $]-\infty, r(T)[\, \cap \varrho(T)$, and let x be a quasi-interior point of E_+. Then $r_T(x) = r(T)$.*

Proof. Let $\lambda_0 > r_T(x)$ and $z = x_T(\lambda_0)$. Let E_z denote the principal ideal generated by z. It is well-known that E_z with the cone $E_0 = E_+ \cap E_z$ is an (AM)-space with respect to the norm $\|y\|_z = \inf\{\alpha \in \mathbf{R}_+ : |y| \leqq \alpha z\}$.

The restriction T_0 of T to E_z satisfies

$$T_0 z = T x_T(\lambda_0) = \lambda_0 x_T(\lambda_0) - x \leqq \lambda_0 z.$$

Hence the z-norm of T_0 satisfies $\|T_0\|_z \leqq \lambda_0$, and for the corresponding spectral radius we have $r(T_0) \leqq \lambda_0$.

Assume $r_T(x) < r(T)$, and let $r_T(x) < \lambda_0 < r(T)$. Then $z = x_T(\lambda_0) = \sum_{n=0}^{\infty} \lambda_0^{-n-1} T^n x$ is also a quasi-interior point of E_+, hence the ideal E_z above is dense in the topology of E. By assumption, there exists $\mu \in \varrho(T)$ such that $\lambda_0 < \mu < r(T)$. The operator T_0 above is celarly positive with respect to the cone E_0 and $r(T_0) \leqq \lambda_0$, hence the resolvent $(\mu - T_0)^{-1}$, acting in E_z, is also positive with respect to E_0. Since E is a Banach lattice and E_z is dense in E, the closure of E_0 in the topology of E is E_+. Hence the resolvent $(\mu - T)^{-1}$, acting in E, is also positive with respect to E_+. However, this contradicts $\mu < r(T)$ and [9, App. 2.3, p. 263].

The next result is contained in [7, Theorem 9.1], and can be stated in our terminology as follows.

Proposition 3. *If x is an interior point of the normal cone E_+, then $r_T(x) = r(T)$.*

In fact, a bit more is proved in [7]: under the given conditions we have $r(T) = \lim_{n\to\infty} \|T^n x\|^{1/n}$ (which is clearly equal to $r_T(x)$).

The conditions in the next two propositions were used in [6, Theorem 16.2] for other purposes.

Proposition 4. *Let the cone E_+ be normal and generating, and the E_+-positive operator T be bounded from above by the element v in E_+. If x is a quasi-interior point of E_+, then $r_T(x) = r(T)$.*

Proof. Let $\lambda > r_T(x)$. Then $x_T(\lambda)$ is also a quasi-interior point of E_+, further $r_T(x_T(\lambda)) = r_T(x)$. We have $(\lambda - T)x_T(\lambda) = x \geqq 0$, therefore $Tx_T(\lambda) \leqq \lambda x_T(\lambda)$. By assumption and [6, Theorem 16.2], $r(T) \leqq \lambda$ for any $\lambda > r_T(x)$. Hence $r(T) \leqq r_T(x)$, whereas the converse inequality always holds.

Proposition 5. *Let the cone E_+ be normal and generating, $x \in E_+ \setminus \{0\}$, and the E_+-positive operator T be bounded from above by the element x. Then $r_T(x) = r(T)$.*

Proof. Let $u \in E_+$. There is a positive number $\beta = \beta(u)$ such that $Tu \leqq \beta x$. Hence $T^n u \leqq \beta T^{n-1} x$ for every $n = 1, 2, \ldots$. Since E_+ is normal, there is $\gamma \in \mathbf{R}_+$ such that $\|T^n u\| \leqq \gamma\beta \|T^{n-1} x\|$ for all n. Therefore $r_T(u) \leqq r_T(x)$ for every u in E_+. Since E_+ is generating, we have by [2, Lemma 4]

$$r(T) = \max \{r_T(u) \colon u \in E_+)\} \leqq r_T(x) \leqq r(T).$$

Remark 6. Assume that T is a positive continuous linear operator acting in the partially ordered Banach space E, the continuous operator A, acting in E, commutes with T and the real number λ satisfies $|\lambda|>r_T(x)$ for some x in E (A and x need not be nonnegative). Then $r_T(Ax)\leqq r_T(x)$ and $r_T(x_T(\lambda))=r_T(x)$. Therefore the assertions of Propositions 1 through 5 remain valid (i.e. $r_T(x)=r(T)$) if instead of x the element Ax or the element $Ax_T(\lambda)$ satisfies (together with T and E) the respective assumptions. The proofs are slight modifications of those given above, thus they will be omitted.

Note further that if T is irreducible, $\lambda>r(T)$ and $x\in E_+\setminus\{0\}$, then $R(\lambda, T)$ commutes with T and $TR(\lambda, T)x$ is a quasi-interior point in E_+. Hence the following theorem is a simple corollary to Propositions 1 through 4 and the remarks above (note that the condition in Proposition 5 is of different, i.e. of more individual, nature).

Theorem 7. *Assume that the irreducible positive continuous linear operator T acting in the partially ordered Banach space E and E satisfy one of the following conditions:*

(i) *$r(T)$ is a pole of the resolvent $R(\cdot, T)$,*

(ii) *$r(T)$ is a limit point of the set $]-\infty, r(T)[\cap\varrho(T)$, and E is a Banach lattice,*

(iii) *the cone E_+ is normal and solid* (i.e. *has a nonvoid interior*),

(iv) *T is bounded from above by an element v in the normal and generating cone E_+.*

Then for any x in $E_+\setminus\{0\}$ we have $r_T(x)=r(T)$.

The following example will show that the irreducibility of T alone does not guarantee that $r_T(x)=r(T)$ for every x in $E_+\setminus\{0\}$.

Example 8. Let E be the real sequence space l^p $(1\leqq p<\infty)$ or c_0 with the usual cone $E_+=\{x=(x_i)_{i=1}^{\infty}\in E\colon\ x_i\geqq 0$ for $i=1, 2, \ldots\}$. Then x is quasi-interior in E_+ if and only if every $x_i>0$. We shall denote this by $x\gg 0$. Let S be the left shift in E defined by $(Sx)_i=x_{i+1}$ $(i=1, 2, \ldots)$. Let $f\in E'$ act as $fx=x_1$, and let $a=(a_i)\in E$, $a\gg 0$. Define $T\colon E\to E$ by $T=f\otimes a+S$, i.e.

$$(Tx)_i = a_i x_1+x_{i+1} \quad (i = 1, 2, \ldots).$$

T is then a positive irreducible operator. Indeed, for each x in $E_+\setminus\{0\}$ take $k=\min\{i\colon x_i>0\}$. Then $(T^j x)_i=x_{i+j}$ $(1\leqq j<k)$, $(T^k x)_i=a_i x_k+x_{i+k}\geqq a_i x_k>0$ for every $i=1, 2, \ldots$. Thus $T^k x\gg 0$, hence T is irreducible.

It is well known that the Fredholm domain of S is the complement of the unit circle. Since T is a one-dimensional perturbation of S, their essential spectra are identical (cf. [4, Theorem IV. 5.35]). Hence $r(T)\geqq 1$.

Now let $0<q<\dfrac{1}{2}$, and consider the particular case of the operator T when $a=(q^i)_{i=1}^{\infty}$. For any $x\in E$ and $\lambda\in\mathbf{R}$ the equality $Tx=\lambda x$ is equivalent to

$$q^i x_1+x_{i+1} = \lambda x_i \quad (i = 1, 2, \ldots).$$

This holds for $\lambda \neq q$ if and only if $(x_i) \in E$, where

$$x_i = \left(\lambda^{i-1} - \sum_{k=1}^{i-1} q^k \lambda^{i-1-k}\right) x_1 = \left(\lambda^{i-1} \frac{\lambda-2q}{\lambda-q} + \frac{q^i}{\lambda-q}\right) x_1 \quad (i = 2, 3, \ldots).$$

Let λ satisfy $2q \leqq \lambda \leqq 1$, and let $x_1 > 0$. Then $x = (x_i) \in E$, and $x \gg 0$ is an eigenvector corresponding to the eigenvalue $\lambda < 1 \leqq r(T)$. Hence $r_T(x) = \lambda < r(T)$ as stated.

Proposition 9. *If the positive cone E_+, the element x in E_+ and the positive operator T satisfy one of the conditions in Propositions* 1, 3, 4 *or* 5, *further in the cases of Propositions* 4 *or* 5 *we have, in addition,* $r(T) > 0$, *then the equation* $(r(T) - T)u = x$ *has no solution u in E.*

Proof. (P) will denote that we are considering the case when the conditions of Proposition P $(P = 1, 3, 4, 5)$ are satisfied.

(1) Assume that there is a solution u in E, and that $R(\lambda, T) = \sum_{k=-p}^{\infty} (\lambda - r)^k Q_k$ is the Laurent expansion of the resolvent around the pole $r = r(T)$ of exact order $p \geqq 1$. Then $Q_{-p} \geqq 0$, $Q_{-p} \neq 0$, and $Q_{-k} = (T - r)^{k-1} Q_{-1}$ for $k = 1, 2, \ldots, p$. By assumption, $Q_{-p}x = (r - T)Q_{-p}u = -Q_{-p-1}u = 0$. Since x is quasi-interior, we obtain $Q_{-p} = 0$, a contradiction.

(3) Since the cone E_+ is normal and solid, a result of M. Krein and M. Rutman (cf. [9, p. 267]) shows that $r(T)$ is an eigenvalue of the dual T' with corresponding eigenvector $f \neq 0$ in the dual cone E'_+. Should a solution $u \in E$ exist, then we should have (denoting the dual pairing by $\langle \cdot, \cdot \rangle$) $\langle f, x \rangle = \langle (r(T) - T')f, u \rangle = 0$, and this contradicts the fact that $f \in E'_+$, $f \neq 0$, and x is an interior point in E_+.

(4) Since $r(T) > 0$, our assumptions imply that $r(T)$ is an eigenvalue of the dual T' with eigenvector f in the dual cone (cf. [7, Proof of Theorem 5.5]). The rest as in case (3).

(5) Let E_x denote the linear manifold of x-measurable elements y of E (cf. [6, p. 34], [7, p. 80]), i.e. those satisfying $-\alpha x \leqq y \leqq \alpha x$ for some $\alpha \in \mathbf{R}_+$. If we set $\|y\|_x = \inf \{\alpha \in \mathbf{R}_+ : -\alpha x \leqq y \leqq \alpha x\}$ then, since E_+ is a normal cone, E_x is a Banach space with respect to the norm $\| \cdot \|_x$, and $E_+ \cap E_x$ is a closed solid normal cone in E_x. Now E_+ is generating and T is bounded from above by x, therefore $R(T) \subset E_x$ and E_x is invariant under T. Assume that there is a solution u, then we obtain from $r(T)u = Tu + x$ and $r(T) > 0$ that $u \in E_x$. It is fairly straightforward to show (cf. [5, p. 80]) that the spectral radii of the operator T in E and in E_x are identical, so we come to the situation of case (3) in the space E_x, and we reach a contradiction.

Corollary 10. *If one of the conditions in Proposition* 9 *is fulfilled,* $\lambda \in \mathbf{R}$, *and the equation* $(\lambda - T)u = x$ *has a solution* $u \geqq 0$, *then* $\lambda > r(T)$.

Proof. By the preceding results, we have $r_T(x)=r(T)$, and for $\lambda=r_T(x)$ there is no solution u in all of E. On the other hand, [2, Theorems 6 and 10] show that there is no solution $u\in E_+$ under the given conditions if $0\leqq\lambda<r_T(x)$. It is clear that there is no solution u in E_+ for $\lambda\in\mathbf{R}\setminus\mathbf{R}_+$. Hence $\lambda>r(T)$.

Remark 11. If the operator A commutes with T, and the element $x=Az$ satisfies (together with T and E) the conditions of Proposition 9, then the equation $(r(T)-T)u=z$ has no solution u in all of E. Indeed, assuming the contrary, the element Au would satisfy $(r(T)-T)Au=x$, which is impossible. The case $A=-$ identity operator is of interest in the next theorem.

Theorem 12. *Let the positive operator T in E be irreducible, satisfy together with E one of the conditions* (i), (iii) *or* (iv) *of Theorem* 7, *in the last case let* $r(T)>0$, *and let* $z\in E_+\cup(-E_+)$ *and* $z\neq 0$. *Then the equation* $(r(T)-T)u=z$ *has no solution u in all of E.*

Proof. Let $\lambda>r(T)$ and $A=TR(\lambda, T)$. Then $x=Az=TR(\lambda, T)z$ if $z\in E_+\setminus\setminus\{0\}$ and $x=-Az=-TR(\lambda, T)z$ if $z\in(-E_+)\setminus\{0\}$ is a quasi-interior element of the cone E_+, since T is irreducible. Hence x satisfies conditions (1), (3), or (4) in (see the proof!) Proposition 9, and Remark 11 shows that there is no solution u in E to the equation $(r(T)-T)u=z$.

Remark 13. Much stronger conditions on T and E are imposed in [1; Theorem 1.13] to obtain the assertion of Theorem 12.

It is clear that the assertions of Proposition 9 or Theorem 12 are not valid without extra conditions such as (1), (3), (4) or (5) and (i), (iii) or (iv), respectively. This is shown by Example 8, where T is irreducible and there are quasi-interior elements x in E_+ such that $r_T(x)<r(T)$. Then the element $u=x_T(r(T))$ belongs to E_+ by [2; Lemma 4], and satisfies $(r(T)-T)u=x$.

If V is the Volterra operator defined by $(Vx)(t)=\int\limits_0^t x(s)ds$ for $x\in L^2(0, 1)$, then V clearly satisfies condition (4) of Proposition 9 except that we have $r(V)=0$. The elements $u(t)\equiv -1$ and $x(t)\equiv t$ satisfy here $(r(V)-V)u=x$, and x is quasi-interior point in the (usual) cone E_+. Hence the requirement of the positivity of the spectral radius in Proposition 9 is not redundant.

The following example shows that the conditions in Proposition 2 are not sufficient to ensure that $(r(T)-T)u=x$ has no solution u in E for any x in E_+.

Example 14. Let $X=\bigcup\limits_{n=0}^{\infty}[2n, 2n+1]\subset\mathbf{R}$ and let $E=C_0(X)$ with the usual positive cone E_+. Let T be the operator of multiplication by $f(t)=(1+t)^{-1}t$ in E. Then $r(T)=1$, and $[(1-T)u](t)=(1+t)^{-1}u(t)$. If $x(t)\equiv(1+t)^{-1}e^{-t}$ then x is quasi-interior in E_+, and the studied equation has the solution $u(t)=e^{-t}$. The element u is quasi-interior in E_+, and the spectrum of the operator T, i.e. the set $\overline{f(X)}\subset \mathbf{R}$, clearly satisfies the condition in Proposition 2.

The next example will show that the series for the main component of the local resolvent function can converge at $r=r_T(x)$ for an E_+-positive operator T and a quasi-interior point x in E_+. Its sum $u=\sum_{n=0}^{\infty} r^{-n-1}T^n x$ is then a positive solution of the equation $(r-T)u=x$.

Example 15. Let $E=c_0$ with the usual positive cone E_+, let T be the left shift in E, and let $x=(1/n^2)_{n=1}^{\infty}$. Then $\|T^k x\|=(k+1)^{-2}$, hence $r_T(x)=1$. Further, the sum $u=\sum_{n=0}^{\infty} T^n x$ exists in E and its j-th component u_j is $\sum_{n=j}^{\infty} n^{-2}$. The solution u of $(r-T)u=x$ is a quasi-interior point of E_+.

Let $T\geqq 0$ be irreducible, and let $r=r(T)>0$ be a pole of the resolvent $R(\cdot, T)$. Then r is a pole of order one ([9], App. 3.2]). Therefore the residuum of $R(\cdot, T)$ at r is the projection P of E on $N(r-T)$ along $R(r-T)$, hence the equation $(r-T)v=(1-P)x$ has solutions v for all $x\in E$.

Proposition 16. *Let $T\geqq 0$ be irreducible, let $r=r(T)>0$ be a pole of its resolvent and let P be the residuum of $R(\cdot, T)$ at r. If E_+ contains interior points, or else T is finite dimensional, then the equation $(r-T)u=(1-P)x$ has solutions $u\geqq 0$ for all $x\in E$ in the first case, and for all $x\geqq 0$ in the second one.*

Proof. $N(r-T)$ is one-dimensional and generated by a quasi-interior element u_0 of E_+ ([9, App. 3.2]). Let v be a solution of $(r-T)v=(1-P)x$, then $(r-T)\cdot(v+\lambda u_0)=(1-P)x$ for all λ. If E_+ has interior elements, then u_0 is such. In this case x can be an arbitrary element of E, and we can choose λ such that $v+\lambda u_0$ is an interior point of E_+.

Consider now the second case, and let $x\geqq 0$. There exists a μ with $Px=\mu u_0$. Then we have

$$v+\lambda u_0 = \frac{1}{r}[x+Tv+(\lambda r-\mu)\, u_0] \quad \text{for all } \lambda.$$

Now we prove that there exists a λ such that $Tv+(\lambda r-\mu)u_0\geqq 0$. Then $v+\lambda u_0\geqq 0$, since $x\geqq 0$. Let $R_0=\bigcup_{k\in\mathbf{N}} \{z\in R(T)\colon -ku_0\leqq z\leqq ku_0\}$. Then R_0 is a linear subspace which is dense in $R(T)$; this follows from $ru_0=Tu_0\in R(T)$ and the fact that $E_0=\bigcup_{k\in\mathbf{N}} \{y\in E\colon -ku_0\leqq y\leqq ku_0\}$ is T-invariant, and is dense in E, since u_0 is a quasi-interior element of E_+. Since T is finite dimensional (i.e. $\dim R(T)<\infty$), we have $R_0=R(T)$ and we can find a λ such that $Tv+(\lambda r-\mu)u_0\geqq 0$.

The question naturally arises whether the conditions in Proposition 16 are redundant. We now give an example of a compact, irreducible operator T such that $r=r(T)>0$, and the equation $(r-T)u=(1-P)x$ has solutions $u\geqq 0$ for some $x\geqq 0$, $x\neq 0$, and has no solution $u\geqq 0$ for other $x\geqq 0$, $x\neq 0$. A consequence of this example will be discussed at the end of this paper.

Example 17. Let $E=c_0$ or $E=l^p$ $(1\leqq p<\infty)$ with the cone E_+ of nonnegative sequences in E, $a=(a_i)\in E'$ (here we identify E' with the corresponding sequence space), and $b=(b_i)\in c_0$. We consider the operator

$$T = a\otimes e^1+SM_b,$$

where e^k is the sequence with 1 in the kth position and 0 in the others, S is the right shift and M_b is the operator of multiplication by b. We have for $x=(x_i)\in E$

$$(Tx)_i = \begin{cases} \sum_{j=1}^{\infty} a_j x_j & \text{if} \quad i = 1, \\ b_{i-1}x_{i-1} & \text{if} \quad i > 1. \end{cases}$$

It is well known that M_b is compact and that the weighted shift SM_b is compact and quasinilpotent [3, Problem 80 for $E=l^2$]. Therefore T, being a one-dimensional perturbation of SM_b, is compact.

Clearly T is non-negative if and only if $a\geqq 0$ and $b\geqq 0$. T is irreducible if $a\gg 0$ and $b\gg 0$, i.e. $a_i>0$ and $b_i>0$ for all i; this follows from

$$(Tx)_1 = \sum_{j=1}^{\infty} a_j x_j, \quad (T^n x)_n = b_{n-1}\cdot\ldots\cdot b_1 (Tx)_1 \quad \text{for} \quad n \geqq 2.$$

Let $\lambda\neq 0$ be an eigenvalue of T and $v=(v_i)$ be a corresponding eigenvector $\neq 0$; this is equivalent to

$$a_1\lambda^{-1}+a_2 b_1\lambda^{-2}+\ldots+a_i b_{i-1}\cdot\ldots\cdot b_1\lambda^{-i}+\ldots = 1$$

and

$$v_i = b_{i-1}\cdot\ldots\cdot b_1\lambda^{-i+1}v_1 \quad \text{if} \quad i \geqq 1,$$

here and in what follows we put $b_{i-1}\cdot\ldots\cdot b_1=1$ if $i=1$. Since $b\in c_0$, we have $(b_{i-1}\cdot\ldots\cdot b_1\lambda^{-i+1}v_1)\in E$ for all $\lambda\neq 0$ and all v_1, and the power series

$$f(\mu) = \sum_{i=1}^{\infty} a_i b_{i-1}\cdot\ldots\cdot b_1\mu^i$$

converges for all μ. Therefore $\lambda\neq 0$ is an eigenvalue of T if and only if $f(1/\lambda)=1$.

Let us assume that $a\gg 0$ and $b\gg 0$. Then $f(1/\lambda)$ is strictly decreasing for $\lambda>0$, $\lim_{\lambda\to 0} f(1/\lambda)=\infty$ and $\lim_{\lambda\to\infty} f(1/\lambda)=0$. Thus there exists exactly one $r>0$ with $f(1/r)=1$. This r is the spectral radius of T, by the Krein—Rutman Theorem, and is a pole of multiplicity one of $R(\cdot, T)$, since T is irreducible and compact ([9, App.

3.2]). Let P be, as in Proposition 16, the residuum of $R(\cdot, T)$ at r. Then P is a projection on the subspace spanned by $\hat{v}=(b_{i-1}\cdot\ldots\cdot b_1 r^{-i+1})_{i=1}^{\infty}$. If $x=(x_i)$, $u=(u_i)$ and $(r-T)u=(1-P)x$, then for $i\geqq 2$

$$u_i = b_{i-1}\cdot\ldots\cdot b_i r^{-i+1}u_1-(i-1)b_{i-1}\cdot\ldots\cdot b_1 r^{-i}\hat{b}_0+$$
$$+r^{-1}x_i+b_{i-1}r^{-2}x_{i-1}+\ldots+b_{i-1}\cdot\ldots\cdot b_2 r^{-i+1}x_2$$

where $\hat{b}_0$ is uniquely determined by $Px=\hat{b}_0\hat{v}$. If $x_2>0$, but $x_i=0$ for $i\neq 2$, then $x\geqq 0$, $x\neq 0$. Therefore $\hat{b}_0>0$, and

$$u_i = b_{i-1}\cdot\ldots\cdot b_2 r^{-i}[rb_1u_1+rx_2-(i-1)b_1\hat{b}_0] \quad \text{if} \quad i\geqq 2.$$

Clearly, it is not possible to choose u_1 in such a way that u_i is non-negative for all i. Therefore the equation $(r-T)u=(1-P)e^2$ has no solution $u\geqq 0$. Nearly the same argument proves that $(r-T)u=(1-P)x$ has no solution $u\geqq 0$ if x is a "finite sequence", $x\geqq 0$, $x\neq 0$.

On the other hand, if we take x such that $x_1=0$ and

$$x_i = b_{i-1}\cdot\ldots\cdot b_1 r^{-i+1}x_0 \quad \text{if} \quad i>1$$

where $x_0>0$, then $x\geqq 0$, $x\neq 0$, and

$$u_i = b_{i-1}\cdot\ldots\cdot b_1 r^{-i}[ru_1+(i-1)(x_0-\hat{b}_0)] \quad \text{if} \quad i\geqq 1.$$

We show that $x_0>\hat{b}_0$ in this case. There exist solutions u of $(r-T)u=(1-P)x$; for the first coordinate in this equation we get using $f(1/r)=1$ and u_i as above,

$$-\sum_{i=2}^{\infty} a_i b_{i-1}\cdot\ldots\cdot b_1 r^{-i}(i-1)(x_0-\hat{b}_0) = x_1-\hat{b}_0 = -\hat{b}_0,$$

and this implies $\hat{b}_0<x_0$. Therefore, for these special $x\in E$ we have nonnegative solutions u of the equation $(r-T)u=(1-P)x$, if we choose a solution with $u_1\geqq 0$.

This example can also be used to show that the algebraic (or generalized) eigenspace to the spectral radius of a compact, non-negative operator need not have a basis of non-negative elements.

Example 18. Let $E=l^p\times l^p$ $(1\leqq p<\infty)$ and

$$T=\begin{pmatrix} T_1 & S_1 \\ 0 & T_1 \end{pmatrix}$$

where T_1 is the operator of the last example and S_1 is a compact, non-negative, nonzero operator in l^p. E is an order continuous Banach lattice, T is compact and non-negative, and $r=r(T)=r(T_1)>0$ is a pole of order 2 of $R(\cdot, T)$. Let $x=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\in E$, then $(r-T)^2x=0$ is equivalent to

$$(*) \qquad (r-T_1)^2x_1 = [(r-T_1)S_1+S_1(r-T_1)]x_2 \quad \text{and} \quad (r-T_1)^2x_2 = 0.$$

Since T_1 is irreducible and compact, $r=r(T_1)$ is a pole of order 1 of $R(\cdot, T_1)$, therefore $(*)$ is equivalent to $(r-T_1)x_2=0$, $(r-T_1)^2x_1=(r-T_1)S_1x_2$, and the last equation has a solution $\hat{x}_1$. If $x_2\neq 0$, then x_2 generates $N(r-T_1)$, so we have $(r-T_1)\hat{x}_1= =S_1x_2-\hat{\lambda}x_2=(1-P_1)S_1x_2$ for some $\hat{\lambda}$, where P_1 is the residuum of $R(\cdot, T_1)$ at r. Therefore $(r-T)^2x=0$ is equivalent to

$$(r-T_1)x_2 = 0 \quad \text{and} \quad (r-T_1)x_1 = (1-P_1)S_1x_2.$$

For each $x\geqq 0$ in $N((r-T)^2)$ with $(r-T)x\neq 0$ we have to and may choose $x_2\geqq 0$, $x_2\neq 0$, in $N(r-T_1)$, therefore x_2 is a quasi-interior element in l^p_+. Since $S_1\geqq 0$, $S_1\neq 0$, we have $S_1x_2\geqq 0$, $S_1x_2\neq 0$. Now we have to look for a solution $x_1\geqq 0$ of $(r-T_1)x_1=(1-P_1)S_1x_2$. But such a solution does not exist in general, since T_1 is the operator of the last example and we can obtain each non-negative, non-zero element in l^p as S_1x_2 by an appropriate choice of S_1 (as a one dimensional non-negative operator).

As a final remark we recall that U. G. Rothblum [8, Theorem 3.1] has shown that for a non-negative matrix the algebraic eigenspace to its spectral radius has a basis of non-negative elements. Generalizing a result of H. D. Victory, jr. [11, Theorem 1] on integral operators in L^p-spaces, J. Kölsche [5, Satz IV. 2.2] has proved: Given $\varepsilon>0$ arbitrarily, for a non-negative, eventually compact operator T in an order continuous Banach lattice there exists a basis for the algebraic eigenspace of T to $r(T)$ such that every vector in this basis has norm 1 but its negative part has norm smaller than or equal to ε. The last example shows that, in general, ε has to be positive in this assertion.

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Der Bidual von F-Banachverbandsalgebren

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In der Arbeit [5] haben C. B. HUIJSMANS und B. DE PAGTER das Arens-Produkt im Ordnungsbidual von Archimedischen f-Algebren untersucht. Die Frage, ob bei den spezielleren F-Banachverbandsalgebren der Bidual, versehen mit dem Arens-Produkt, stets wieder eine F-Banachverbandsalgebra ist, bleibt dabei offen.

In der vorliegenden Arbeit werden wir diese Frage bejahen und zeigen, daß der Bidual als direkte Summe seines Annullatorbandes und dessen orthogonalen Komplements dargestellt werden kann, wobei letzteres algebraisch- und verbandsisomorph zu einem Vektorverbandsideal in der Banachverbandsalgebra $C_0(\mathcal{M})''$ ist. Unter anderem ergibt sich, daß die Banachverbandsalgebren $C_0(X)$ die einzigen F-Banachverbandsalgebren sind, deren Bidual ein algebraisches Einselement mit Norm 1 besitzt.

Vorbemerkungen

Wir benutzen in dieser Arbeit die auf dem Gebiet der Banachverbände übliche Terminologie und Bezeichnungsweise. Der leichteren Lesbarkeit wegen wollen wir kurz ein paar für uns wichtige Begriffe in Erinnerung rufen.

Eine reelle Banachverbandsalgebra A ist ein reeller Banachverband A, welcher gleichzeitig eine reelle (lineare assoziative) Algebra mit den beiden folgenden Eigenschaften ist: $xy \geqq 0$ und $\|xy\| \leqq \|x\| \|y\|$ für alle positiven Elemente x und y von A.

Eine reelle Banachverbandsalgebra A ist eine F- bzw. FF-Banachverbandsalgebra, falls sie die folgende Eigenschaft F bzw. FF besitzt:

F: $\inf(a, b) = 0$ impliziert $\inf(ca, b) = 0 = \inf(ac, b)$ für $a, b, c \in A$ und $c \geqq 0$;

FF: $\inf(a, b) = 0$ impliziert $a \cdot b = 0$ für $a, b \in A$.

Wie man leicht sieht, ist jede F-Banachverbandsalgebra auch eine FF-Banachverbandsalgebra und somit nach [11, § 2] kommutativ.

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Für einen lokalkompakten Hausdorffraum X bezeichne $C_0(X)$ die reelle Banachverbandsalgebra aller stetigen reellen Funktionen f auf X mit der Eigenschaft, daß für jedes $\varepsilon>0$ die Menge $\{x\in X\colon |f(x)|\geqq\varepsilon\}$ kompakt ist, versehen mit der Supremumsnorm und der kanonischen punktweisen Multiplikation und Ordnung. Falls K ein kompakter Hausdorffraum ist, stimmt die Banachverbandsalgebra $C_0(K)$ natürlich mit der Banachverbandsalgebra $C(K)$ überein. Alle Banachverbandsalgebren $C_0(X)$ sind F-Banachverbandsalgebren.

Die Banachalgebra der beschränkten Endomorphismen eines Banachverbandes E bezeichnen wir mit $\mathscr{L}(E)$ und den Spektralradius eines Elements a einer reellen normierten Algebra mit $r(a)$. Bekanntlich gilt $r(a)=\lim\limits_{n\to\infty}(\|a^n\|)^{1/n}$.

Ist E ein Banachverband und u ein positives Element von E, so bedeutet E_u das von u im verbandstheoretischen Sinn erzeugte Hauptideal. Es läßt sich bekanntlich mit einem Funktionenverband $C(K)$ (K kompakter Hausdorffraum) identifizieren, und diese Identifizierung werden wir die kanonische Identifizierung eines Hauptideals nennen. Dem Element u entspreche dabei stets die Einsfunktion e_K auf K.

Sei nun A eine F-Banachverbandsalgebra, c ein Element des positiven Kegels A_+ und T_c die linksreguläre Darstellung von c auf A, d.h. $T_c x=cx$ für alle $x\in A$. Nach [6,1.1] gilt dann $T_c x\leqq\|T_c\|x$ für alle $x\in A_+$, wobei $\|T_c\|$ die Operatornorm von T_c in der Banachalgebra $\mathscr{L}(A)$ bezeichnet. Für $a\in A$ gehören also die linksregulären Darstellungen T_a zum Zentrum $Z(A)$ des zugrunde liegenden Banachverbandes A, wobei $Z(A)$ aus allen Operatoren $T\in\mathscr{L}(A)$ besteht mit der Eigenschaft: Es gibt eine von T abhängige Konstante $\gamma\in\mathbf{R}_+$ mit

$$|Tx|\leqq\gamma|x| \quad \text{für alle} \quad x\in A.$$

Mit Hilfe einer kleinen zusätzlichen Überlegung erhält man aus dem Beweis von [6,1.1] sofort das folgende Resultat über positive Orthomorphismen: Es sei E ein Banachverband und T ein positiver Orthomorphismus auf E, d.h. T besitzt die Eigenschaft:

$$\inf(x,y)=0 \quad \text{impliziert} \quad \inf(Tx,y)=0.$$

Dann gilt $Tx\leqq r(T)x$ für alle $x\in E_+$.

Aus der Definition einer F-Banachverbandsalgebra folgt unmittelbar, daß die linksregulären Darstellungen positiver Elemente positive Orthomorphismen sind. Mit der vorhergehenden Aussage über Orthomorphismen erhalten wir die im folgenden wichtige Ungleichung:

Satz 0.1. *Es sei A eine F-Banachverbandsalgebra. Dann gilt $ab\leqq r(a)b$ für alle positiven Elemente a und b.*

Aus Satz 0.1 folgt sofort, daß in einer F-Banachverbandsalgebra jedes Vektorverbandsideal auch ein Ringideal ist.

Eine Subalgebra einer Banachverbandsalgebra $C_0(X)$, welche gleichzeitig auch ein Vektorunterverband ist, nennen wir eine Verbandssubalgebra. Eine Anwendung der Stone—Weierstrass-Theorie ergibt den folgenden, für uns wichtigen Approximationssatz:

Satz 0.2. *Es sei X ein lokalkompakter Hausdorffraum und F eine Punkte trennende Verbandssubalgebra der Banachverbandsalgebra $C_0(X)$, welche nirgendwo verschwindet, d.h. zu jedem $t\in X$ existiert eine Funktionen $k\in F$ mit $k(t)\neq 0$. Dann gibt es zu jeder positiven Funktion $g\in C_0(X)$ und jedem $\varepsilon>0$ eine positive Funktion $f\in F$ mit $\|f\|\leqq 1$ und $\|g-f\cdot g\|\leqq\varepsilon$, d.h. die nach oben gerichtete Menge $B:=\{h\in F\colon h\geqq 0$ und $\|h\|\leqq 1\}$ ist ein approximierendes Einselement in $C_0(X)$.*

1. Die Gelfand-Theorie von F-Banachverbandsalgebren

Wichtiges Hilfsmittel bei der Behandlung von F-Banachverbandsalgebren ist in der Arbeit [6] die Algebra der Zentrumsoperatoren, welche eine Algebra vom Typ $C(K)$ ist. In der Arbeit [5] übernimmt diese Rolle die Algebra der Orthomorphismen. Wir werden dagegen im folgenden wesentlichen Gebrauch von der Gelfand-Theorie machen.

Da F-Banachverbandsalgebren spezielle FF-Banachverbandsalgebren sind, gelten für sie bezüglich der Gelfand-Theorie die in der Arbeit [11] gemachten Aussagen. Wir wollen die wichtigsten Punkte kurz angeben (s. [11, Kap. 2]).

Es sei A eine F-Banachverbandsalgebra und $\mathcal{M}$ die Menge der nichttrivialen komplexen Homomorphismen auf A. Wir setzen $\mathcal{M}\neq\emptyset$ voraus. Dann sind die Elemente von $\mathcal{M}$ sogar reelle Vektorverbandshomomorphismen. Die Menge $\mathcal{M}$, als Teilmenge des topologischen Duals A' betrachtet und versehen mit der von der schwachen Topologie $\sigma(A', A)$ induzierten Topologie, ist ein lokalkompakter Hausdorffraum und kann als der Raum der maximalen regulären Ideale von A angesehen werden. Die Gelfand-Transformation $\Phi\colon a\to\hat{a}$ von A nach $C_0(\mathcal{M})$, definiert durch $\hat{a}(\mu):=\mu(a)$ für alle $a\in A$ und $\mu\in\mathcal{M}$, ist ein Algebra- und Vektorverbandshomomorphismus. Ihr Wertevorrat $\hat{A}$ ist daher eine dichte Verbandssubalgebra der Banachverbandsalgebra $C_0(\mathcal{M})$. Für alle $a\in A$ gilt $r(a)=\|\hat{a}\|$ und das Radikal

$$\operatorname{rad}(A) := \{a\in A\colon r(a) = 0\}$$

von A ist sowohl ein abgeschlossenes Algebraideal als auch ein Vektorverbandsideal.

Ist $\operatorname{rad}(A)=A$, so folgt aus 0.1 sofort $a\cdot b=0$ für alle $a, b\in A$, d.h. die Multiplikation ist trivial. Wir werden daher von nun an stets $\operatorname{rad}(A)\neq A$ voraussetzen, was auch $\mathcal{M}\neq\emptyset$ impliziert.

Unter dem Annullator Ann (B) einer kommutativen Algebra B versteht man die Menge $\{x \in B: x \cdot B = \{0\}\}$. Bei der folgenden Charakterisierung des Radikals handelt es sich im Grunde um schon bekannte Ergebnisse (s. [1, 3.1.3] und [8, S.89]).

Satz 1.1. *Es sei A eine F-Banachverbandsalgebra. Dann gilt:*
(i) $\mathrm{rad}(A) = \mathrm{Ann}(A) = \{x \in A: x^2 = 0\}$,
(ii) *das Vektorverbandsideal* $\mathrm{rad}(A)$ *ist sogar ein Band.*

Beweis. *Zu* (i): Sei $a \in \mathrm{rad}(A)$ und $b \in A$. Dann gilt $|a| \in \mathrm{rad}(A)$, $|a \cdot b| \leqq \leqq |a||b| \leqq r(|a|)|b| = 0$ und somit $a \cdot b = 0$, d.h. $\mathrm{rad}(A) \subseteqq \mathrm{Ann}(A)$. Da die Beziehung $\mathrm{Ann}(A) \subseteqq \mathrm{rad}(A)$ offensichtlich ist, erhalten wir $\mathrm{rad}(A) = \mathrm{Ann}(A)$. Für $y \in \mathrm{Ann}(A)$ ist $y^2 = 0$. Ist andererseits $x \in A$ und $x^2 = 0$, so gilt $r(x) = 0$ und somit $x \in \mathrm{rad}(A) \subseteqq \subseteqq \mathrm{Ann}(A)$. Es ist also auch $\mathrm{Ann}(A) = \{x \in A: x^2 = 0\}$.

Zu (ii): Sei $x, a \in A_+$ und $\{a_j: j \in J\}$ ein nach oben gerichtetes Netz positiver Elemente in $\mathrm{Ann}(A)$ mit $a = \sup\{a_j: j \in J\}$. Da die Multiplikation ordnungsstetig ist, folgt $ax = x \cdot a = \sup\{xa_j: j \in J\} = 0$. Es ist also $a \in \mathrm{Ann}(A)$ und $\mathrm{Ann}(A)$ ein Band. Aus $\mathrm{Ann}(A) = \mathrm{rad}(A)$ ergibt sich die Behauptung.

Mit dem vorhergehenden Satz erhalten wir für ordnungsvollständige F-Banachverbandsalgebren einen erwähnenswerten Darstellungssatz, dessen Beweis klar ist.

Satz 1.2. *Es sei A eine ordnungsvollständige F-Banachverbandsalgebra. Dann ist A direkte Summe der beiden Ringideale* $\mathrm{Ann}(A)$ *und* $\mathrm{Ann}(A)^{\perp}$. *Das Band* $\mathrm{Ann}(A)^{\perp}$ *ist für sich betrachtet eine halbeinfache F-Banachverbandsalgebra, welche algebraisch- und verbandsisomorph zu einer dichten Verbandssubalgebra der Banachverbandsalgebra $C_0(\mathscr{M})$ ist, wobei $\mathscr{M}$ den Raum der maximalen regulären Ideale von A bezeichnet.*

2. Der Bidual von F-Banachverbandsalgebren

Zunächst möchten wir das Arens-Produkt in Erinnerung rufen. Es sei A eine F-Banachverbandsalgebra, A' ihr topologischer Dual und A'' ihr Bidual. Damit keine Verwechslung mit der natürlichen Multiplikation $\cdot$ bei Funktionenalgebren auftreten kann, werden wir die Multiplikation in A mit $*$ bezeichnen. Für $f \in A$ und $\mu \in A'$ sei das Element $\mu_f \in A'$ definiert durch $\mu_f(g) = \mu(f * g)$ für alle $g \in A$. Nach 0.1 gilt für alle $f \in A_+$ und $\mu \in A'_+$ die Ungleichung

$$\mu_f \leqq r(f)\mu.$$

Sei nun $G \in A''$. Bei der Definition des Arens-Produkts spielt der folgende, zu G gehörige Operator $T_G \in \mathscr{L}(A')$, definiert durch $(T_G\mu)(f) := G(\mu_f)$ für alle $\mu \in A'$ und $f \in A$, eine wichtige Rolle. Mit diesem Operator T_G ist für $F, G \in A''$ in A'' das

Arens-Produkt $F*G$ wie folgt definiert:

$$F*G(\mu) := F(T_G\mu)$$

für alle $\mu\in A'$. Durch eine Routinerechnung läßt sich zeigen, daß der Bidual A'', versehen mit diesem Arens-Produkt, eine Banachverbandsalgebra ist.

Mit Hilfe der vorhergehenden Ungleichung und der Kommutativität der Multiplikation erhalten wir, daß die Operatoren T_G zum Zentrum von A' gehören.

Satz 2.1. *Sei A eine F-Banachverbandsalgebra, $\mu\in A'_+$ und $G\in A''_+$. Dann gilt $T_G\mu\leqq\|G\|\,\mu$ und $\|T_G\mu\|\leqq G(\mu)$.*

Beweis. Sei $f\in A_+$ und $S:=\{g\in A_+\colon\ \|g\|\leqq 1\}$. Dann gilt

$$\|\mu_f\| = \sup\{\mu(f*g)\colon g\in S\} = \sup\{\mu(g*f)\colon g\in S\} =$$
$$= \sup\{\mu_g(f)\colon\ g\in S\} \leqq \sup\{r(g)\mu(f)\colon\ g\in S\} \leqq \mu(f),$$

da $r(g)\leqq 1$ für $g\in S$. Hieraus erhalten wir

$$(T_G\mu)(f) = G(\mu_f) \leqq \|G\|\,\|\mu_f\| \leqq \|G\|\,\mu(f).$$

Es gilt also $T_G\mu\leqq\|G\|\,\mu$.

Für $g\in S$ gilt $\mu_g\leqq r(g)\mu\leqq\mu$. Es ist somit

$$\|T_G\mu\| = \sup\{(T_G\mu)(g)\colon\ g\in S\} = \sup\{G(\mu_g)\colon\ g\in S\} \leqq G(\mu).$$

Satz 2.2. *Es sei A eine F-Banachverbandsalgebra. Dann ist der Bidual A'', versehen mit dem Arens-Produkt, wieder eine F-Banachverbandsalgebra und somit eine kommutative Algebra.*

Beweis. Es sei $F, G, H\in A''_+$. Es genügt zu zeigen: $\inf(F,G)=0$ impliziert $\inf(H*F,G)=0=\inf(F*H,G)$.

Sei $\mu\in A'_+$. Nach 2.1 gilt $\|T_F\mu\|\leqq F(\mu)$. Hieraus folgt

$$(H*F)(\mu) = H(T_F\mu) \leqq \|H\|\,\|T_F\mu\| \leqq \|H\|\,F(\mu).$$

Es ist also $H*F\leqq\|H\|\,F$.

Nach 2.1 gilt auch $T_H\mu\leqq\|H\|\,\mu$. Hieraus ergibt sich

$$(F*H)(\mu) = F(T_H\mu) \leqq \|H\|\,F(\mu).$$

Es ist also auch $F*H\leqq\|H\|\,F$.

Sei nun $\inf(F,G)=0$. Aus $H*F\leqq\|H\|\,F$ und $F*H\leqq\|H\|\,F$ folgt dann sofort

$$\inf(H*F,G) = 0 = \inf(F*H,G).$$

Nach 2.1 gehören die Operatoren $T_G(G\in A'')$ zur Banachverbandsalgebra $Z(A')$ der Zentrumsoperatoren von A'. In Analogie zu [7] nennen wir die Abbildung $G\to T_G$ von A'' nach $Z(A')$ Arens-Homomorphismus. Mit Hilfe von Satz 2.2 erhalten wir, daß diese Abbildung ein Vektorverbandshomomorphismus ist, was wir später benötigen werden (vgl. [5, 5.2]).

Satz 2.3. *Sei A eine F-Banachverbandsalgebra. Dann ist der Arens-Homomorphismus ein Algebra- und Vektorverbandshomomorphismus.*

Beweis. Wie man leicht nachprüft, ist der Arens-Homomorphismus ein positiver Algebrahomomorphismus.

Sei nun $G \in A''$. Da A'' eine F-Banachverbandsalgebra ist, gilt $G^+ * G^- = 0$. In $Z(A')$ bedeutet dies $T_{G^+} \cdot T_{G^-} = 0$. Da $Z(A')$ bekanntlich vom Typ $C(K)$ ist, erhalten wir $\inf(T_{G^+}, T_{G^-}) = 0$. Es ist also der Arens-Homomorphismus auch ein Vektorverbandshomomorphismus.

Als Beispiel betrachten wir nun den Bidual der F-Banachverbandsalgebra $C_0(X)$. Es wird sich später zeigen, daß diese Biduale bei der Darstellung der Biduale allgemeiner F-Banachverbandsalgebren eine bedeutende Rolle spielen. Wie es scheint, ist bis jetzt der Bidual der Banachalgebren $C_0(X)$ nur im Rahmen der B^*-Algebren-Theorie behandelt worden. Wir wollen nun diesen Bidual mit verbandstheoretischen Mitteln charakterisieren.

Es sei X ein lokalkompakter Hausdorffraum. Da der Banachverband $C_0(X)$ ein AM-Raum ist, ist der Dual $C_0(X)'$ ein AL-Raum und somit der Bidual $C_0(X)''$ ein AM-Raum mit Einheit. Wir können also den Banachverband $C_0(X)''$ mit einem Banachverband $C(\Omega)$ identifizieren, wobei Ω ein gewisser kompakter Hausdorffraum ist und die Einsfunktion E auf Ω dem Normfunktional auf $C_0(X)'$ entspricht (s. [10, Kap. II, 7.4 und 9.1]). Wir nennen diesen Banachverband $C(\Omega)$ die kanonische Identifizierung von $C_0(X)''$. Auf $C(\Omega)$ haben wir nun zwei Produkte, das von $C_0(X)$ herrührende Arens-Produkt $*$ und das natürliche Funktionenprodukt.

Satz 2.4. *Es sei X ein lokalkompakter Hausdorffraum und $C(\Omega)$ die kanonische Identifizierung des Banachverbandes $C_0(X)''$. Dann gilt $F * G = F \cdot G$ für alle $F, G \in C(\Omega)$.*

Beweis. Nach 2.2 ist $C(\Omega)$, versehen mit dem Arens-Produkt $*$, eine (kommutative) F-Banachverbandsalgebra. Auf Grund von [2, 28.7 und 28.8] ist die Einsfunktion E Einselement für das Arens-Produkt $*$. Nach [11, 1.7] stimmt daher die Multiplikation $*$ mit der natürlichen punktweisen Multiplikation überein.

Wir kehren nun wieder zu der Situation einer allgemeinen F-Banachverbandsalgebra zurück. Im folgenden sei A eine F-Banachverbandsalgebra und $\mathcal{M}$ der Raum der maximalen regulären Ideale von A. Nach 1.2 ist der Bidual A'' direkte Summe von $\operatorname{Ann}(A'')$ und $\operatorname{Ann}(A'')^\perp$. Wir werden jetzt diese beiden Summanden genauer untersuchen.

Der Spektralradius r definiert auf A die Halbnorm $x \to r(x)$, welche Spektralhalbnorm genannt und wieder mit r bezeichnet wird. Es sei nun $\tilde{A} := \{\mu \in A' :$ Es gibt eine von μ abhängige Zahl $c \in \mathbf{R}_+$ mit $|\mu(x)| \leqq c r(x)$ für alle $x \in A\}$. Die Menge $\tilde{A}$ besteht also aus allen Linearformen in A', welche auch bezüglich der Spektralhalbnorm r stetig sind. Aus diesem Grunde möchten wir die Menge $\tilde{A}$ den spektralen Dual von A nennen. Offensichtlich ist $\tilde{A}$ ein linearer Teilraum von A'.

Zunächst zeigen wir folgenden Zusammenhang zwischen der Gelfand-Transformation und dem spektralen Dual auf, was im übrigen auch in jeder kommutativen Banachalgebra gilt.

Satz 2.5. *Es sei A eine F-Banachverbandsalgebra. Dann ist der spektrale Dual $\tilde{A}$ gleich dem Wertevorrat der adjungierten Abbildung Φ' der Gelfand-Transformation Φ.*

Beweis. Es gilt offensichtlich $\Phi'(C_0(\mathcal{M})') \subseteqq \tilde{A}$.

Sei nun $\tilde{\mu} \in \tilde{A}$. Da $\tilde{\mu}$ auf $\operatorname{rad}(A)$ verschwindet, ist das folgende Funktional μ_0 auf dem linearen Unterraum $\hat{A} (= \Phi(A))$ von $C_0(\mathcal{M})$ wohl definiert und linear: Für $f \in \hat{A}$ sei $\mu_0(f) := \tilde{\mu}(g)$, wobei $g \in A$ und $f = \Phi(g)$ ist. Da $\|f\| = r(g)$ gilt, ist μ_0 auf $\hat{A}$ stetig. Nach dem Satz von Hahn-Banach gibt es nun ein $\mu \in C_0(\mathcal{M})'$ mit $\mu(k) = \mu_0(k)$ für alle $k \in \hat{A}$. Für dieses μ gilt dann offensichtlich $\tilde{\mu} = \Phi' \mu$. Es ist also $\tilde{\mu} \in \Phi'(C_0(\mathcal{M})')$.

Satz 2.6. *Es sei A eine F-Banachverbandsalgebra. Dann ist der spektrale Dual $\tilde{A}$ ein Ideal in A'. Ferner ist die adjungierte Abbildung Φ' der Gelfand-Transformation Φ ein Vektorverbandsisomorphismus von dem Banachverband $C_0(\mathcal{M})'$ auf das Ideal $\tilde{A}$ in A'.*

Beweis. Da Φ ein Vektorverbandshomomorphismus ist, ist Φ' intervallerhaltend, d.h. es gilt $\Phi'[0, \mu] = [0, \Phi'\mu]$ für alle $\mu \in C_0(\mathcal{M})'_+$. Hieraus folgt, daß $\Phi'(C_0(\mathcal{M})')$ ein Ideal ist. Nach 2.5 ist somit $\tilde{A}$ ein Ideal in A'. Da $\hat{A}$ dicht in $C_0(\mathcal{M})$ ist, ist Φ' injektiv. Ferner ist jeder injektive, intervallerhaltende positive Operator auch ein Vektorverbandshomomorphismus. Es ist also Φ' ein Vektorverbandsisomorphismus von $C_0(\mathcal{M})'$ auf $\tilde{A}$.

Da $C_0(\mathcal{M})'$ ein AL-Raum ist, ist auf $C_0(\mathcal{M})'$ jede positive Linearform ordnungsstetig. Nach dem vorhergehenden Satz hat daher das Ideal $\tilde{A}$ die Eigenschaft, daß für jedes $F \in A''_+$ die Einschränkung $F|_{\tilde{A}}$ von F auf das Ideal $\tilde{A}$ ordnungsstetig ist. Es bezeichne $(A')'_n$ das Band der ordnungsstetigen Linearformen *in* A'' und $(A')'_\sigma$ dessen orthogonales Komplement.

Für $F \in A''_+$ bezeichne $N(F)$ den absoluten Kern von F in A', also $N(F) := \{\mu \in A' : F(|\mu|) = 0\}$. Zwischen $\tilde{A}$ und $(A')'_\sigma$ besteht folgende interessante Beziehung:

Satz 2.7. *Es gilt $(A')'_\sigma \subseteqq \tilde{A}^0$, wobei $\tilde{A}^0$ die Polare von $\tilde{A}$ in A'' ist.*

Beweis. Sei $F \in (A')'_\sigma$ und $F > 0$. Dann gilt $F \perp G$ für alle $G \in (A')'_n$. Bekanntlich folgt hieraus $N(F)^\perp \subseteqq N(G)$ für alle $G \in (A')'_n$ mit $G \geqq 0$. Da die kanonische Einbettung von A in A'' in dem Band $(A')'_n$ enthalten ist, trennt $(A')'_n$ die Punkte in A'. Aus der vorhergehenden Enthaltenseinsbeziehung folgt daher $N(F)^\perp = \{0\}$. Es gilt also $A' = N(F)^{\perp\perp}$.

Sei nun $\mu \in \tilde{A}$ und $\mu > 0$. Da $A' = N(F)^{\perp\perp}$ gilt, existiert in $N(F)$ ein positives, nach oben gerichtetes Netz $\{\mu_\beta \colon \beta \in B\}$ mit $\mu = \sup\{\mu_\beta \colon \beta \in B\}$. Aus der Ordnungsstetigkeit von $F|_{\tilde{A}}$ folgt $F(\mu) = \sup\{F(\mu_\beta) \colon \beta \in B\} = 0$. Hieraus ergibt sich $F \in \tilde{A}^0$. Es gilt also $(A')'_\sigma \subseteqq \tilde{A}^0$.

Da die Gelfand-Transformation ein Vektorverbandshomomorphismus ist, ist die Spektralnorm r eine M-Verbandshalbnorm auf A, d.h. $|x| \leqq |y|$ impliziert $r(x) \leqq r(y)$ für alle $x, y \in A$ und es ist $r(\sup(x, y)) = \sup(r(x), r(y))$ für alle $x, y \in A_+$.

Satz 2.8. *Es sei A eine F-Banachverbandsalgebra. Für den spektralen Dual $\tilde{A} \subseteqq A'$ gilt:*

(i) $\mu_f \in \tilde{A}$ *für alle* $\mu \in A'$ *und* $f \in A$;

(ii) $\mu = \sup\{\mu_f \colon f \in A_+$ und $r(f) \leqq 1\}$ *für alle* $\mu \in \tilde{A}_+$;

(iii) $T_G(A') \subseteqq \tilde{A}$ *für alle* $G \in A''$;

(iv) *für* $G \in A''$ *gilt genau dann* $T_G = 0$, *wenn* $G \in \tilde{A}^0$.

Beweis. *Zu* (i): Für $f, g \in A$ und $\mu \in A'$ gilt $|g| * |f| \leqq r(|g|)|f|$ nach 0.1 und somit

$$|\mu_f(g)| = |\mu(f * g)| = |\mu(g * f)| \leqq |\mu|(|g| * |f|) \leqq r(|g|)|\mu|(|f|) \leqq \|\mu\| \, \|f\| \, r(g).$$

Es ist also $\mu_f \in \tilde{A}$.

Zu (ii): Diese Gleichung beweisen wir mit Hilfe von 2.6 und der zu A gehörigen F-Banachverbandsalgebra $C_0(\mathcal{M})$ wie folgt: Für alle $f \in A$ und $\mu \in C_0(\mathcal{M})'$ gilt $\Phi'(\mu_{\hat{f}}) = (\Phi'\mu)_f$. Hieraus folgt $\Phi'^{-1}(\nu_f) = (\Phi'^{-1}\nu)_{\hat{f}}$ für alle $\nu \in \tilde{A}$ und $f \in A$.

Sei nun $\mu \in \tilde{A}_+$, $\nu \in C_0(\mathcal{M})'_+$ und $S := \{f \in A_+ \colon r(f) \leqq 1\}$. Wir beweisen zunächst $\nu = \sup\{\nu_{\hat{f}} \colon f \in S\}$. Für $f \in S$ gilt $\|\hat{f}\| = r(f) \leqq 1$ und somit $\nu_{\hat{f}} \leqq \nu$. Ferner ist die Menge $\{\nu_{\hat{f}} \colon f \in S\}$ nach oben gerichtet, da r eine M-Verbandshalbnorm ist. Sei $g \in C_0(\mathcal{M})_+$ und $\varepsilon > 0$. Da $\hat{A}$ eine Punkte trennende, nirgendwo verschwindende Verbandssubalgebra von $C_0(\mathcal{M})$ ist, gibt es nach 0.2 ein $f_0 \in S$ mit $\|g - \hat{f}_0 \cdot g\| \leqq \varepsilon$. Hieraus folgt

$$\nu(g) = \sup\{\nu(\hat{f} \cdot g) \colon f \in S\} = \sup\{\nu_{\hat{f}}(g) \colon f \in S\}.$$

Es ist also $\nu = \sup\{\nu_{\hat{f}} \colon f \in S\}$. Insgesamt erhalten wir

$$\Phi'^{-1}(\mu) = \sup\{(\Phi'^{-1}\mu)_{\hat{f}} \colon f \in S\} = \sup\{\Phi'^{-1}(\mu_f) \colon f \in S\}.$$

Das bedeutet aber $\mu = \sup\{\mu_f \colon f \in S\}$.

Zu (iii): Sei $f, g \in A$, $\mu \in A'$ und $G \in A''$. Dann ist

$$|\mu_f(g)| = |\mu(f*g)| \leqq |\mu|(|f|*|g|) \leqq r(|f|)|\mu|(|g|) \leqq r(f)\|\mu\|\,\|g\|.$$

Es gilt also $\|\mu_f\| \leqq \|\mu\| r(f)$. Hieraus folgt

$$|(T_G\mu)(f)| = |G(\mu_f)| \leqq \|G\|\,\|\mu_f\| \leqq \|G\|\,\|\mu\|\,r(f)$$

und somit $T_G\mu \in \tilde{A}$.

Zu (iv): $\Rightarrow$: Sei $G \in A''$ und $T_G = 0$. Nach 2.4 gilt dann auch $T_{|G|} = 0$. Wir nehmen nun an $|G| \notin \tilde{A}^0$. Dann gilbt es ein $\mu \in \tilde{A}_+$ mit $|G|(\mu) > 0$. Nach (ii) gilt $\mu = \sup\{\mu_f : f \in A_+ \text{ und } r(f) \leqq 1\}$, wobei die Menge in der geschweiften Klammer nach oben gerichtet ist. Da $|G|_{|\tilde{A}}$ ordnungsstetig ist, gibt es dann ein $f_0 \in A_+$ mit $r(f_0) \leqq 1$ und $|G|(\mu_{f_0}) > 0$. Dies bedeutet aber $T_{|G|}(\mu) > 0$, was ein Widerspruch ist. Es ist also $|G| \in \tilde{A}^0$ und somit auch $G \in \tilde{A}^0$.

$\Leftarrow$: Sei $f \in A$, $\mu \in A'$ und $F \in \tilde{A}^0$. Dann gilt $\mu_f \in \tilde{A}$ nach (i). Hieraus folgt $(T_F\mu)(f) = F(\mu_f) = 0$. Es ist also $T_F = 0$.

Die angekündigte Charakterisierung der Bänder $\operatorname{Ann}(A'')$ und $\operatorname{Ann}(A'')^\perp$ lautet nun

Satz 2.9. *Es sei A eine F-Banachverbandsalgebra und $C(\Omega)$ die kanonische Identifizierung des Biduals $C_0(\mathcal{M})''$ der zu A gehörigen Banachverbandsalgebra $C_0(\mathcal{M})$. Ferner bezeichne Φ'' die biadjungierte Abbildung der Gelfand-Transformation. Dann gilt:*

(i) $\operatorname{Ann}(A'') = \tilde{A}^0$;

(ii) *Es ist $\Phi''(A'')$ ein Vektorverbandsideal in $C(\Omega)$. Die Einschränkung der Abbildung Φ'' auf* $\operatorname{Ann}(A'')^\perp$ *ist ein Algebra- und Vektorverbandsisomorphismus von* $\operatorname{Ann}(A'')^\perp$ *auf das Vektorverbandsideal $\Phi''(A'')$ in der Funktionenalgebra $C(\Omega)$.*

Beweis. *Zu* (i): Sei $F \in \tilde{A}^0$. Nach 2.8 (iv) ist dann $T_F = 0$. Dies bedeutet $H * F = 0$ für alle $H \in A''$. Es ist somit $F \in \operatorname{Ann}(A'')$. Sei $G \in \operatorname{Ann}(A'')$. Dann muß $T_G = 0$ sein. Nach 2.8 (iv) gilt dann $G \in \tilde{A}^0$.

Zu (ii): Nach [3, 6.1] ist die Abbildung Φ'' von A'' nach $C(\Omega)$ multiplikativ. Da Φ und Φ' Vektorverbandshomomorphismen sind, ist Φ'' ein intervallerhaltender Vektorverbandshomomorphismus (s. [10, Kap. III, Ex. 24]). Es ist also $\Phi''(A'')$ ein Ideal in $C(\Omega)$. Aus $\tilde{A} = \Phi'(C_0(\mathcal{M})')$ folgt $\tilde{A}^0 = \Phi''^{-1}\{0\}$.

Hieraus ergibt sich, daß die Einschränkung von Φ'' auf $\operatorname{Ann}(A'')^\perp (= \tilde{A}^{0\perp})$ injektiv ist. Insgesamt erhalten wir, daß diese Einschränkung ein Algebra- und Vektorverbandsisomorphismus auf das Ideal $\Phi''(A'')$ in $C(\Omega)$ ist.

Nach dem vorhergehenden Satz können wir also $\operatorname{Ann}(A'')^\perp$ mit dem Vektorverbandsideal $\Phi''(A'')$ in $C(\Omega)$ identifizieren. Es sei nun $\Omega_0 := \{t \in \Omega$: es gibt ein $F \in \Phi''(A'')$ mit $F(t) \neq 0\}$. Dann ist der Unterraum Ω_0 von Ω lokalkompakt. Nach Satz 1.2 ist $\operatorname{Ann}(A'')^\perp$ algebraisch- und verbandsisomorph zu einer dichten Verbandssubalgebra der Banachverbandsalgebra $C_0(\mathcal{M}'')$, wobei $\mathcal{M}''$ den Raum der maximalen regulären Ideale von A'' bedeutet. Der Raum Ω_0 läßt sich nun mit $\mathcal{M}''$ identifizieren.

Satz 2.10. *Der lokalkompakte Raum Ω_0 ist homöomorph zum Raum $\mathcal{M}''$ der maximalen regulären Ideale von A''.*

Beweis. Es bezeichne B das Vektorverbandsideal $\Phi''(A'')$ in $C(\Omega)$. Auf Grund der in 2.9 (ii) angegebenen Isomorphie kann die Punktmenge $\mathcal{M}''$ als die Menge aller nichttrivialen, multiplikativen reellen Vektorverbandshomomorphismen auf B betrachtet werden. Da B ein Vektorverbandsideal ist, trennt B die Punkte in Ω_0. Wir betrachten nun die Abbildung $\tau\colon \Omega_0 \to \mathcal{M}''$, definiert durch $\tau(t)(F) = F(t)$ für alle $F \in B$ und $t \in \Omega_0$. Die Abbildung τ ist injektiv, da B die Punkte trennt. Sie ist aber interessanterweise sogar surjektiv, wie man wie folgt einsieht.

Sei $F \in B$, $F > 0$, $\mathcal{O} := \{t \in \Omega\colon F(t) > 0\}$ und $\beta\mathcal{O}$ die Stone—Čech-Kompaktifizierung von $\mathcal{O}$. Da $C(\Omega)$ ordnungsvollständig ist, ist $\beta\mathcal{O}$ homöomorph zu $\bar{\mathcal{O}}$ in Ω. Es bezeichne $\tilde{F}$ die eindeutig bestimmte stetige Fortsetzung von F auf $\beta\mathcal{O}$. Wegen der Homöomorphie zwischen $\beta\mathcal{O}$ und $\bar{\mathcal{O}}$ gilt dan $\tilde{F} \equiv 0$ auf $\beta\mathcal{O} \setminus \mathcal{O}$. Auf $C(\beta\mathcal{O})$ definieren wir die folgende Multiplikation $**$:

$$G_1 ** G_2 := \tilde{F} \cdot G_1 \cdot G_2 \quad \text{für alle} \quad G_1, G_2 \in C(\beta\mathcal{O}).$$

Es sei $C(\Omega)_F$ das von F in $C(\Omega)$ erzeugte Hauptideal. Wir betrachten die folgende Abbildung φ von $C(\Omega)_F$ nach $C(\beta\mathcal{O})$, welche definiert ist durch $\varphi(G)(t) := \dfrac{1}{F(t)} G(t)$ für alle $G \in C(\Omega)_F$ und alle $t \in \mathcal{O}$ und $\varphi(G) \equiv \widetilde{\varphi(G)}$ auf $\beta\mathcal{O} \setminus \mathcal{O}$, wobei $\widetilde{\varphi(G)}$ die kanonische Fortsetzung von $\varphi(G)$ ist. Wie man leicht nachprüft, ist φ ein Algebra- und Vektorverbandsisomorphismus.

Es sei nun $\mu \in \mathcal{M}''$ und $\mu(F) \neq 0$. Es ist also μ ein multiplikativer reller Verband-homomorphismus auf B. Dasselbe gilt für die Einschränkung μ_F auf $C(\Omega)_F$. Mit μ_F ist dann auch die Linearform $\mu \circ \varphi^{-1}$ ein nichttrivialer multiplikativer reeller Vektorverbandshomomorphismus auf der F-Verbandsalgebra $C(\beta\mathcal{O})$, versehen mit der Multiplikation $**$. Hieraus folgt, es gibt ein $t \in \mathcal{O}(\subseteqq \Omega_0)$ mit $\mu_F \circ \varphi^{-1} = F(t)\varepsilon_t$, wobei ε_t das Dirac-Maß im Punkt t bezeichnet. Dies bedeutet aber, daß $\mu_F = \varepsilon_t$ auf $C(\Omega)_F$ gilt.

Der Punkt t bei der Darstellung von μ_F hängt auf den ersten Blick von F ab. Daß er aber unabhängig vom gewählten F ist, kann man zeigen, indem man zu $F_1 \in B$ mit $F_1 > 0$ und $F_1 \neq F$ das Hauptideal $C(\Omega)_{\sup(F, F_1)}$ betrachtet und feststellt, daß der zu F bzw. F_1 gehörige Punkt jeweils mit dem zu $\sup(F, F_1)$ gehörigen Punkt identisch ist. Wir erhalten also, daß es zu μ ein $t \in \Omega_0$ gibt mit $\mu = \varepsilon_t$. Die Abbildung τ ist also surjektiv. Die Homöomorphie zwischen $\mathcal{M}''$ und Ω_0 folgt nun sofort aus [9, 3.2.5].

Im Hinblick auf die Frage, ob A'' halbeinfach ist bzw. ein algebraisches Einselement besitzt, erhalten wir aus dem Vorhergehenden folgende Aussagen.

Satz 2.11. *Der Bidual A'' einer F-Banachverbandsalgebra A ist genau dann halbeinfach, wenn der spektrale Dual $\tilde{A}$ dicht im Dual A' ist. Eine notwendige Bedingung dafür, daß A'' halbeinfach ist, ist $A''=(A')'_n$.*

Beweis. Nach 1.1 (i) und 2.9 (i) gilt $\operatorname{rad}(A'')=\operatorname{Ann}(A'')=\tilde{A}^0$. Es ist also A'' genau dann halbeinfach, wenn $\tilde{A}^0=\{0\}$ ist, wenn also $\tilde{A}$ dicht in A' ist. Die zweite Aussage fogt sofort aus 2.7.

Der Bidual der F-Banachverbandsalgebren $C_0(X)$ besitzt ein algebraisches Einselement mit Norm 1.

Satz 2.12. *Die Banachverbandsalgebren $C_0(X)$ (X lokalkompakter Hausdorffraum) sind die einzigen F-Banachverbandsalgebren, bei denen der Bidual ein algebraisches Einselement mit Norm 1 besitzt.*

Beweis. Es sei A eine F-Banachverbandsalgebra mit der Eigenschaft, daß A'' ein algebraisches Einselement e mit $\|e\|=1$ besitzt. Ferner sei $C(K_e)$ die kanonische Identifizierung des Ideals A''_e. Wie wir in [11, S. 204] gezeigt haben, ist dann die F-Banachverbandsalgebra A'' isometrisch isomorph zur Banachverbandsalgebra $C(K_e)$. Da A in A'' isometrisch eingebettet ist, erhalten wir $\|a^n\|=\|a\|^n$ für alle $a\in A$ und $n\in\mathrm{N}$. Auf Grund des Satzes von Stone-Weierstrass ist dann die Gelfand-Transformation ein isometrischer Algebra- und Vektorverbandsisomorphismus von A auf die Banachverbandsalgebra $C_0(\mathscr{M})$, wobei $\mathscr{M}$ der Raum der maximalen regulären Ideale von A ist.

Für den in Satz 2.3 betrachteten Arens-Homomorphismus ergeben sich nun folgende Aussagen.

Satz 2.13. *Es sei A eine F-Banachverbandsalgebra. Dann ist der Kern des Arens-Homomorphismus mit dem Band* Ann (A'') *identisch. Der Arens-Homomorphismus ist genau dann surjektiv, wenn A topologisch-algebraisch- und verbandsisomorph zur Banachverbandsalgebra $C_0(\mathscr{M})$ ist.*

Beweis. Es bezeichne S den Arens-Homomorphismus. Nach 2.8 (iv) stimmt der Kern von S mit dem Band $\tilde{A}^0$, welches gleich $\operatorname{Ann}(A'')$ ist, überein.

Sei nun S surjektiv. Ferner sei $G\in A''$ mit $S(G)=T_G=I\in Z(A')$, wobei I die Identität auf A' ist. Nach 2.8 (iii) gilt dann $A'=T_G(A')\subseteqq\tilde{A}$. Dies bedeutet $\tilde{A}=A'$. Hieraus folgt, daß A'' topologisch isomorph zu $C_0(\mathscr{M})''$ ist.

Da auf der Funktionenalgebra $C_0(\mathscr{M})''$ die Normbeziehung $\|F^n\|=\|F\|^n$ für alle $F\in C_0(\mathscr{M})''$ und alle $n\in\mathrm{N}$ gilt, sind auf A'' und somit auch auf A die gegebene Norm und die Spektralhalbnorm r äquivalent. Es ist daher in $C_0(\mathscr{M})$ die dichte Verbands-subalgebra $\hat{A}$ abgeschlossen. Es gilt also $\hat{A}=C_0(\mathcal{M})$. Nach dem Satz von der offenen Abbildung ist dann die Gelfand-Transformation auch ein topologischer Isomorphismus.

Falls wir die im Satz angegebene Isomorphie zwischen A und $C_0(\mathcal{M})$ voraussetzen, können wir auf A eine neue äquivalente Norm einführen, so daß A auch isometrisch isomorph zu $C_0(\mathcal{M})$ ist. Wählen wir nun die kanonische Identifizierung $C(\Omega)$ von $C_0(\mathcal{M})''$, so läßt sich $C_0(\mathcal{M})'$ mit $C(\Omega)'_n$ identifizieren. Nach Satz 2.4 gilt daher

$$\int_\Omega F\, d(T_G\mu) = \int_\Omega F*G\, d\mu = \int_\Omega F\cdot G\, d\mu$$

für alle $F, G\in C(\Omega)$ und alle $\mu\in C_0(\mathcal{M})'$. Hieraus folgt die Darstellung

$$T_G\mu = G\cdot\mu \quad \text{für alle} \quad G\in C(\Omega)$$

und $\mu\in C_0(\mathcal{M})'$. Da aber bekanntlich zu jedem $R\in Z(C_0(\mathcal{M})')$ genau ein $H\in C(\Omega)$ existiert mit $R\mu=H\cdot\mu$ für alle $\mu\in C_0(\mathcal{M})'$, ist in der betrachteten Situation der Arens-Homomorphismus offensichtlich bijektiv.

Beispiele. Die Ergebnisse der Arbeit wollen wir nun an den Banachverbänden c_0, l_1 und l_2 veranschaulichen. Mit der koordinatenweisen Multiplikation sind sie auch F-Banachverbandsalgebren. Alle drei haben denselben Raum $\mathcal{M}$ der maximalen regulären Ideale, nämlich $\mathbf{N}$, versehen mit der diskreten Topologie. Sie haben also auch denselben spektralen Dual $C_0(\mathcal{M})'$ und dasselbe $C_0(\mathcal{M})''$. Es ist

$$C_0(\mathcal{M})\cong c_0,\quad C_0(\mathcal{M})'\cong l_1,\quad C_0(\mathcal{M})''\cong l_\infty \quad \text{und} \quad C_0(\mathcal{M})''\cong C(\Omega) \quad \text{mit} \quad \Omega=\beta\mathbf{N}.$$

Zu $A=c_0$: Aus $A\cong C_0(\mathcal{M})$ folgt $\tilde{A}=A'$ und somit $A''\cong C_0(\mathcal{M})''=C(\beta\mathbf{N})$.

Zu $A=l_1$: Hier gilt $A'\cong l_\infty\cong C(\beta\mathbf{N})$ und $A''\cong C(\beta\mathbf{N})'$. Es ist $\tilde{A}\cong l_1\subsetneqq l_\infty\cong A'$. Wir erhalten $\mathrm{Ann}\,(A'')\cong l_1^0\cong\{\mu\in C(\beta\mathbf{N})'$: Träger von $\mu\subseteq\beta\mathbf{N}\setminus\mathbf{N}\}$ und $\mathrm{Ann}\,(A'')^\perp\cong$ $\cong l_1$. Satz 2.9 (ii) besagt nun, daß l_1 als ein Vektorverbandsideal in $C(\beta\mathbf{N})$ betrachtet werden kann, was offensichtlich stimmt.

Zu $A=l_2$: Bei diesem Beispiel gilt $A=A''$. Interessant ist aber, daß hier $\tilde{A}(\cong l_1)$ dicht in $A'(\cong l_2)$ ist, daß also nach 2.10 der Bidual A'' und somit A halbeinfach sein muß.

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Reflexivity and direct sums

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1. Introduction. Let H be a Hilbert space and let $B(H)$ be the set of (bounded linear) operators on H. If $\mathcal{S}\subset B(H)$, then the commutant $\mathcal{S}'$ of $\mathcal{S}$ is the set of all operators that commute with every operator in $\mathcal{S}$. Also Lat $\mathcal{S}$ denotes the set of (closed linear) subspaces of H that are left invariant under every operator in $\mathcal{S}$, and Alg Lat $\mathcal{S}$ denotes the set of all operators T such that Lat $\mathcal{S}\subset$Lat T. A unital weakly closed subalgebra $\mathcal{A}$ of $B(H)$ is reflexive if $\mathcal{A}=\text{Alg Lat }\mathcal{A}$, and an operator T is reflexive if the weakly closed algebra $\mathcal{A}(T)$ generated by 1 and T is reflexive. A commutative subalgebra $\mathcal{A}$ of $B(H)$ is hyporeflexive if $\mathcal{A}=\mathcal{A}'\cap\text{Alg Lat }\mathcal{A}$, and an operator T is hyporeflexive if $\mathcal{A}(T)$ is. Much work has been done on reflexivity (see e.g., [1]—[7], [10], [11], [14]—[27], [30]—[32]). Recently W. WOGEN [31], answering a question of P. ROSENTHAL and D. SARASON, has constructed a class of operators that are not hyporeflexive.

This paper contains two main ideas. The first idea deals with very general types of shifts, and we prove, for a large class of operators T, that Alg Lat $T\subset\{T\}'$. For such operators, the problems of reflexivity and hyporeflexivity coincide. In some cases we show that the elements of Alg Lat T correspond to formal power series in T. As a consequence we show that every operator-weighted shift is hyporeflexive and that every operator-weighted shift with 1—1 weights of rank at least 2 is reflexive. In particular, the direct sum of two weighted shifts with nonzero scalar weights is reflexive.

The second idea concerns reflexive graphs. Suppose $\mathcal{A}$ is a reflexive algebra of operators and $\pi\colon\mathcal{A}\to B(H_\pi)$ is a homomorphism. We deal with the problem of when the graph of π is a reflexive subalgebra of $B(H\oplus H_\pi)$. In particular, we show that if the algebra $\mathcal{A}$ has property D of [17] and if π is continuous in the weak operator topology, then the graph of π is reflexive. We use these results to show that if T is polynomially bounded and S is the unilateral shift operator, then $S\oplus T$ is reflexive.

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We also show that if S is a nonreductive subnormal operator, then there is a nonempty open set Ω of complex numbers such that $S\oplus T$ is reflexive for every operator T whose spectrum is contained in Ω.

2. Shifted vectors. The principal technique of this section shows that $\text{Alg Lat } T\subset\{T\}'$ for many operators T. Much of what will be done is valid in the context of a complex locally convex topological vector space X; indeed, some of the results hold in an arbitrary vector space. If E is a subset of X, then we will write $\text{sp } E$ for the linear span of E and $\overline{\text{sp}}\, E$ for the closed linear span of E. A biorthogonal base for a closed subspace M of X is a finite or infinite sequence $\{e_k\}$ whose closed linear span is M for which there exists a corresponding dual sequence $\{\varphi_k\}$ in X^*, the dual space of all continuous linear functionals on X, such that $\varphi_j(e_k)=\delta_{jk}$, for all j and k, where δ is the Kronecker δ, and such that $M\cap(\bigcap_k \ker\varphi_k)=\{0\}$. Equivalently $\{e_k\}$ is a biorthogonal base for M if and only if $\{e_k\}$ is a spanning set for M, e_j is not in $\overline{\text{sp}}\,\{e_k\colon k>j\}$ for every j, and $\bigcap_j \overline{\text{sp}}\,\{e_k\colon k\geqq j\}=\{0\}$.

If $\{e_k\}$ is a biorthogonal base for a closed subspace M and $\{\varphi_k\}$ is the corresponding dual base, then, for every x in M, the sequence $\{\varphi_k(x)\}$ determines x. Hence we will write $x\sim\sum_k \varphi_k(x)e_k$ to indicate this relationship. Note that for x in $\text{sp}\,\{e_k\colon k\geqq 0\}$, $x=\sum_k \varphi_k(x)e_k$, but in general the series need not converge. Clearly if $x\sim\sum_k a_k e_k$, $y\sim\sum_k b_k e_k$, and if α and β are scalars, then $\alpha x+\beta y\sim\sum_k(\alpha a_k+\beta b_k)e_k$.

Let T be a continuous linear operator on X. We will write $M(x)=M_T(x)$ for the smallest invariant linear manifold of T that contains the vector x, i.e. $M(x)=\text{sp}\,\{T^k x\colon k\geqq 0\}$. A vector x is called a shifted vector for T in case the nonzero vectors in $\{T^k x\colon k\geqq 0\}$ form a biorthogonal base for $M(x)^-$. Let the order of x, $\text{ord}\,(x)$, be the dimension of $\text{sp}\,\{T^k x\colon k\geqq 0\}$. The following lemma is easily established, and the proof is left to the reader.

Lemma 2.1. *The following are equivalent:*

(1) *x is a shifted vector for T;*

(2) $\bigcap_j M(T^j x)^-=\{0\}$.

It may be tempting to think that x is a shifted vector for T if $T^n x\notin M(T^{n+1}x)^-$ for $0\leqq n<\text{ord}\,(x)$. However, if one lets $\{e_n\colon n\geqq 0\}$ be an orthonormal basis for a Hilbert space H and lets T be the operator defined by $Te_0=e_0$, and $Te_n=(n/n+1)e_{n+1}$ for $n=1, 2, 3, \ldots$, and lets $x=e_0+e_1$, the temptation quickly fades away.

We remark that if x is a shifted vector for T and $\{T^k x\}$ has a dual sequence $\{\varphi_k\}$, then for each polynomial p, we have $\varphi_k(p(T)x)=\varphi_{k+1}(Tp(T)x)$. Thus if $y\sim\sim\sum_k a_k T^k x$, then $Ty\sim\sum_k a_k T^{k+1}x$. More generally, if $A\in B(X)$, A is invertible, and $AT=TA$, then, for each shifted vector x for T, Ax is a shifted vector for T of the same order as x, and whenever $y\sim\sum_k a_kT^kx$, we have $Ay\sim\sum_k a_kT^kAx$. To see this, first note that $\bigcap_j M(T^jAx)^-=A[\bigcap_j M(T^jx)^-]=\{0\}$; whence, by Lemma 2.1, Ax is a shifted vector for T. If $\{\psi_k\}$ is the corresponding dual base for $\{T^kAx\}$, then $\{A^*\psi_k\}$ is a dual base for $\{T^kx\}$, i.e., $(A^*\psi_k)(T^nx)=\psi_k(AT^nx)=\delta_{kn}$. Thus $\varphi_k=A^*\psi_k$ on $M(x)$ for $k\geqq 0$. If $y\in M(x)$, and $y\sim\sum_k a_kT^kx$, then, for each $k\geqq 0$, we have $a_k=\varphi_k(y)=(A^*\psi_k)(y)=\psi_k(Ay)$. Hence, $Ay\sim\sum_k a_kT^kAx$.

The following notion will be basic for our needs. A pair of vectors x, y is called a shifted pair for T if $\operatorname{sp}\{T^kx\colon k\geqq 0\}\cap\operatorname{sp}\{T^ky\colon k\geqq 0\}=\{0\}$ and $(\{T^kx\colon k\geqq 0\}\cup\{T^ky\colon k\geqq 0\})\setminus\{0\}$ is a biorthogonal base for its closed span $M(x,y)^-=M_T(x,y)^-$. In this case each w in $M(x,y)^-$ is associated with a formal series, $w\sim\sum_k a_kT^kx+\sum_k b_kT^ky$.

Lemma 2.2. *Suppose x and y are shifted vectors for T and $m=\operatorname{ord}(x)<\operatorname{ord}(y)=\infty$. Then $\{x,y\}$ is a shifted pair for T.*

Proof. Note that $\operatorname{ord}(y)=\infty$ implies that T is 1—1 on $M(y)^-$. If $v\in M(x)^-\cap M(y)^-$, then $T^mv=0$ (since $v\in M(x)$) and thus $v=0$ (since T^m is 1—1 on $M(y)^-$). Hence $M(x)^-\cap M(y)^-=0$. Since $M(x)$ is finite dimensional, it follows that $M(x,y)^-=M(x)^-+M(y)^-$ is a direct sum and that the projections onto $M(x)^-$ and $M(y)^-$ are continuous. Thus $\{x,y\}$ is a shifted pair for T.

Lemma 2.3. *Suppose T is a continuous linear transformation on a locally convex space X and that x and y are shifted vectors of orders m and n respectively. Suppose also that $\{x,y\}$ is a shifted pair for T. Let $S\in\operatorname{Alg\,Lat}T$, and suppose $Sx\sim\sum_k a_kT^kx$ and $Sy\sim\sum_k b_kT^ky$. The following are true.*

(1) *Every nonzero vector in $M(x,y)^-$ is a shifted vector for T.*

(2) *Suppose $z\in M(x,y)^-$, $z\sim\sum_k(c_kT^kx+d_kT^ky)$, $c_i\neq 0$ for some i, and $m=n=\infty$. Then $\{z,y\}$ is a shifted pair for T.*

(3) *If $m\leqq n$, then*

(a) *$a_i=b_i$ for $0\leqq i<m$,*

(b) *$Sz\sim\sum_k b_kT^kz$ for every z in $M(x)^-$,*

(c) *$ST=TS$ on $M(x)^-$.*

Proof. (1). If $z\in M(x,y)^-$, then $\{T^kz\colon k\geqq j\}\subset\overline{\operatorname{sp}}(\{T^kx\colon k\geqq j\}\cup\{T^ky\colon k\geqq j\})$, and since

$$\bigcap_{j=0}^{\infty}\overline{\operatorname{sp}}(\{T^kx\colon k\geqq j\}\cup\{T^ky\colon k\geqq j\})=\{0\},$$

it follows that z is a shifted vector for T.

(2) Let $z\sim\sum_k (c_kT^kx+d_kT^ky)$ and let i be the smallest index for which $c_i\neq 0$. It will be shown that $\{z, y\}$ is a shifted pair for T. Let $M_j=\overline{\mathrm{sp}}\left(\{T^kz\colon k\neq j\}\cup\{T^ky\colon k\geqq 0\}\right)$ and $N_j=\overline{\mathrm{sp}}\left(\{T^kz\colon k\geqq 0\}\cup\{T^ky\colon k\neq j\}\right)$ for every j. If $T^jz\in M_j$ for some j, then there is a smallest p such that

$$T^jz=\alpha_pT^pz+\alpha_{p+1}T^{p+1}z+\ldots+\alpha_{j-1}T^{j-1}z+w,$$

$\alpha_p\neq 0$, and $w\in\overline{\mathrm{sp}}\left(\{T^kz\colon k>j\}\cup\{T^ky\colon k\geqq 0\}\right)$. It follows that $T^pz\in M_p$, and thus we will suppose that $T^jz\in\overline{\mathrm{sp}}\left(\{T^kz\colon k>j\}\cup\{T^ky\colon k\geqq 0\}\right)$ and show that this leads to a contradiction. Let φ_k and v_s be continuous linear functionals on X for $k\geqq 0$ and $s\geqq 0$ such that $\varphi_k(T^px)=\delta_{kp}$, $\varphi_k(T^py)=0$, $v_s(T^px)=0$, and $v_s(T^py)=\delta_{sp}$ for all $p\geqq 0$. Then $\varphi_{i+j}(T^jz)=c_i\neq 0$, but $\varphi_{i+j}(T^kz)=0$ for $k>j$ and $\varphi_{i+j}(T^ky)=0$ for all k, which is impossible in view of our supposition. Thus T^jz is not in M_j for every j.

A similar argument will show that T^jy is not in N_j for every $j<n$. As above, it will suffice to show that T^jy belonging to $\overline{\mathrm{sp}}\left(\{T^kz\colon k\geqq 0\}\cup\{T^ky\colon k>j\}\right)$ leads to a contradiction. We have

$$T^jy=\sum_{k\leqq j}\alpha_kT^kz+w,$$

where $w\in\overline{\mathrm{sp}}\left(\{T^kz\colon k>j\}\cup\{T^ky\colon k>j\}\right)$. Applying $\varphi_i, \varphi_{i+1}, \ldots, \varphi_{i+j}$ successively to T^jy, we see that $\alpha_0=\alpha_1=\ldots=\alpha_j=0$. It follows that $T^jy=w$, and hence $1=v_j(T^jy)=v_j(w)=0$, which is a contradiction. Thus T^jy is not in N_j for every j, and consequently $\{z, y\}$ is a shifted pair for T.

(3a) By (1), $x+y$ is a shifted vector for T. Let φ_k and v_k be functionals as in the proof of part (2). For every polynomial p, it is clear that $\varphi_k\left(p(T)(x+y)\right)=v_k\left(p(T)\cdot(x+y)\right)$ for all $k<m$, and thus $\varphi_k(z)=v_k(z)$ for all $k<m$ and every z in $M(x+y)^-=\overline{\mathrm{sp}}\,\{T^k(x+y)\colon k\geqq 0\}$. Since $M(x+y)^-$ is invariant under S and $S(x+y)=Sx+Sy$, it follows that

$$a_k=\varphi_k(Sx)=\varphi_k(Sx+Sy)=v_k(Sx+Sy)=v_k(Sy)=b_k$$

for $k<m$.

(3b) If $z\in M(x)^-$, then z is a shifted vector for T and $\operatorname{ord}(z)\leqq m$. If $m<\infty$, then $\{z, y\}$ is a shifted pair, and if $m=\infty$, then $\{z, y\}$ is a shifted pair by part (2) above. Applying (3a) to the pair $\{z, y\}$, we obtain (3b).

(3c) If $z\in M(x)^-$, then $Tz\in M(x)^-$ and (3b) implies both $Sz\sim\sum_k b_kT^kz$ and $STz\sim\sum_k b_kT^kTz=\sum_k b_kT^{k+1}z$. Also $Sz\sim\sum_k b_kT^kz$ implies that $TSz\sim\sum_k b_kT^{k+1}z$; thus $STz=TSz$, establishing (3c).

Useful results concerning shifted vectors of finite order can be cast in a purely algebraic setting. A linear transformation T on a vector space V over a field F is locally nilpotent if, for each v in V there is a positive integer $n=n_v$ such that $T^nv=0$.

Note that if T is locally nilpotent and $\{a_n\}$ is a sequence in F, the sum $\sum_{k\geqq 0} a_k T^k$ is finite at each vector, and thus the sum defines a linear transformation that commutes with every transformation commuting with T.

Lemma 2.4. *Suppose S and T are commuting linear transformations on a vector space V over a field F such that*

(1) *T is locally nilpotent,*

(2) *S leaves invariant every T-invariant linear subspace of V.*

Then there is a sequence $\{a_n\}$ in F such that $S=\sum_{k\geqq 0} a_k T^k$. Moreover, S commutes with every linear transformation on V that commutes with T.

Proof. It follows from [16, Cor. 5] that there is a net $\{p_\lambda\}$ of polynomials in $F[x]$ such that $p_\lambda(T)\to S$ in the strict topology (i.e., pointwise in the discrete topology on V). If T is nilpotent, then the set of polynomials in T is strictly closed and S is a polynomial in T. We can therefore assume that T is not nilpotent. For each integer $m\geqq 1$, choose a vector $v(m)$ in V so that $T^m v(m)\neq 0$. It follows, for sufficiently large λ, that $p_\lambda(T)v(m)=Sv(m)$. Hence, for $0\leqq k\leqq m$, the coefficients of x^k in $p_\lambda(x)$ are equal to a constant a_k for sufficiently large λ. It follows that $S=\sum_{k\geqq 0} a_k T^k$. It is clear that if A is a linear transformation on V and $AT=TA$, then

$$AS=\sum_{k\geqq 0} a_k AT^k=\sum_{k\geqq 0} a_k T^k A=SA.$$

The following lemma was proved for matrices by DEDDENS and FILLMORE [11]; it was observed in [17, p. 20] and [14] that the result holds in general. Note that if J is an $n\times n$ Jordan nilpotent matrix and $k\geqq 2$ is an integer, then $\dim(\ker J^k/\ker J^{k-2})$ is 2 if $k\leqq n$ and is 1 if $k=n+1$, and is 0 if $k\geqq n+2$. Hence if T is a nilpotent matrix, $n\geqq 2$, $T^n=0$ and $T^{n-1}\neq 0$, then $\dim(\ker T^n/\ker T^{n-2})$ is the number of $(n-1)\times(n-1)$ Jordan blocks plus twice the number of $n\times n$ Jordan blocks in the Jordan canonical form for T. Note that this number is greater than 2 if and only if there is one block of size n and another block of size n or $n-1$.

Lemma 2.5. *Suppose T is a nilpotent linear transformation on a vector space V over a field F. The following are equivalent:*

(1) *every linear transformation leaving invariant each T-invariant linear subspace commutes with T,*

(2) *every linear transformation leaving invariant each T-invariant linear subspace is a polynomial in T,*

(3) *for every x in V and every positive integer n such that $T^n x\neq 0$, there is a y in V such that*

$$\operatorname{sp}\{T^k x\colon k\geqq 0\}\cap\operatorname{sp}\{T^k y\colon k\geqq 0\}=\{0\}\quad\textit{and}\quad T^{n-1}y\neq 0,$$

(4) *if* $n \geqq 2$ *and* $T^{n-1} \neq 0$, *then* $\dim(\ker T^n/\ker T^{n-2}) > 2$.

Theorem 2.6. *Let X be a locally convex space, and let T be a continuous linear transformation of X. Suppose* $\bigcup_{k \geqq 1} \ker T^k$ *is dense in X and, for every* $k \geqq 0$ *with* $T^{k+1} \neq 0$, $\dim(\ker T^{k+2}/\ker T^k) > 2$. *If* $S \in \operatorname{Alg} \operatorname{Lat} T$, *then* $S \in \{T\}''$, *and there exists a sequence* $\{a_k\}$ *of complex numbers such that* $Sx \sim \sum_k a_k T^k x$ *for every shifted vector x for T.*

Proof. The dimension hypothesis implies, via Lemma 2.5, that $T|\ker T^k$ is reflexive for every $k \geqq 1$. If T is nilpotent, then we are done. Otherwise let S be in Alg Lat T, and let S_k and T_k be the restrictions of S and T respectively to $\ker T^k$. By the reflexivity of T_k, there is a polynomial p_k of degree $k-1$ or less such that $S_k = p_k(T_k)$. If $A \in \{T\}'$ and if $A_k = A|\ker T^k$, then clearly $A_k S_k = S_k A_k$, and since $\bigcup_k \ker T^k$ is dense, it follows that $AS = SA$. Hence $S \in \{T\}''$.

Applying Lemma 2.4 to the restriction of T to $\bigcup_k \ker T^k$, we obtain a sequence $\{a_k\}$ of complex numbers such that $Sx \sim \sum_k a_k T^k x$ for every x in $\bigcup_k \ker T^k$. If y is a shifted vector of infinite order, then $Sy \sim \sum_k b_k T^k y$. There are shifted vectors x of arbitrarily large finite order that may be used in conjunction with Lemma 2.2 and part (3a) of Lemma 2.3 to conclude that $b_k = a_k$ for every k.

Corollary 2.7. *If T is a nonnilpotent operator on a Hilbert space such that T is a direct sum of nilpotent operators, then every vector is a shifted vector for T, and* Alg Lat $T \subset \{T\}''$. *If* $S \in \operatorname{Alg} \operatorname{Lat} T$, *then there exists a sequence* $\{a_k\}$ *of complex numbers such that* $Sx \sim \sum_k a_k T^k x$ *for every vector x.*

Proof. The hypothesis of the corollary implies that of the theorem, and therefore Alg Lat $T \subset \{T\}''$ and there exists a sequence $\{a_k\}$ such that $Sx \sim \sum_k a_k T^k x$ for every shifted vector x. That every vector is shifted follows easily from the fact that every vector is a direct sum of shifted vectors.

In a Hilbert space, the hypothesis $\bigcap_{k \geqq 1} (\operatorname{ran} T^k)^- = \{0\}$ in the following theorem is equivalent to T having a strictly lower triangular matrix with respect to an orthogonal direct sum decomposition of the space into a sequence of subspaces.

Theorem 2.8. *Suppose X is a locally convex space and* $T \in B(X)$ *has the property that* $\bigcap_{k \geqq 1} (\operatorname{ran} T^k)^- = \{0\}$, *and, for each integer* $n \geqq 1$, $\dim(X/(T^n)^{-1}[(\operatorname{ran} T^{n+2})^-]) > 2$. *Then*

(1) *every vector in X is a shifted vector for T,*

(2) Alg Lat $T \subset \{T\}''$,

(3) *for each S in* Alg Lat T, *there is a sequence* $\{a_n\}$ *of complex numbers such that, for every vector x in X, we have* $Sx\sim\sum_k a_kT^kx$.

Proof. Suppose that $S\in\operatorname{Alg Lat} T$ and $A\in\{T\}'$ and $n\geq 2$ is a positive integer. Since S, T and A each leave $(\operatorname{ran} T^n)^-$ invariant, they induce operators S_n, T_n and A_n (respectively) on the space $X/(\operatorname{ran} T^n)^-$. Clearly, $S_n\in\operatorname{Alg Lat} T_n$ and $A_n\in\{T_n\}'$. Moreover, T_n is nilpotent, and it follows from the hypothesis on dimensions above and Lemma 2.5 that there is a polynomial $p_n(z)$ (unique modulo $z^n\mathbf{C}[z]$) such that $S_n=p_n(T_n)$. Thus $S_nA_n=A_nS_n$.

Translating back in X, we obtain $\operatorname{ran}(S-p_n(T))\subset(\operatorname{ran} T^n)^-$ and $\operatorname{ran}(AS-SA)\subset(\operatorname{ran} T^n)^-$. Since we have $\operatorname{ran}(S-p_{n+1}(T))\subset(\operatorname{ran} T^{n+1})^-\subset(\operatorname{ran} T^n)^-$, we have that $p_{n+1}(z)=p_n(z)$ modulo $z^n\mathbf{C}[z]$. Hence there is a sequence $\{a_k\}$ of complex numbers such that we can take $p_n(z)=\sum_{k<n}a_kz^k$ for each n. It follows that $Sx\sim\sum_k a_kT^kx$ for every x in X. Also we have $\operatorname{ran}(AS-SA)\subset\bigcap_{k\geq 1}(\operatorname{ran} T^k)^-=\{0\}$, which implies $AS=SA$. Hence $S\in\{T\}''$.

Corollary 2.9. *If T_1 and T_2 are operators on a Hilbert space that are strictly lower triangular with respect to some infinite direct sum decomposition of the space, and if neither T_1 nor T_2 is nilpotent, then for $T=T_1\oplus T_2$ on the direct sum H of the space with itself,* Alg Lat $T\subset\{T\}''$, *and for every S in* Alg Lat T *there exists a sequence $\{a_k\}$ such that $Sx\sim\sum_k a_kT^kx$ for every x in H.*

Remark. It follows from the preceding theorem that if $\{x, y\}$ is a shifted pair and $\operatorname{ord}(x)=\operatorname{ord}(y)=\infty$, then, for any S in Alg Lat T there is a sequence $\{a_k\}$ of complex numbers such that $Sz\sim\sum_k a_kT^kz$ for every z in $M(x, y)^-$.

Theorem 2.10. *Suppose that X is locally convex, $T\in B(X)$ and T has shifted vectors of arbitrarily large finite orders and at least one shifted vector of infinite order. If $S\in\operatorname{Alg Lat} T$, then there exists a sequence $\{a_k\}$ of complex numbers such that $Sx\sim\sum_k a_kT^kx$ for every shifted vector x for T. Moreover, if $A\in\{T\}'$ and A is invertible, then $SA-AS=0$ on the linear span of the set of shifted vectors for T.*

Proof. If y is a shifted vector of infinite order for T, and if S is a linear transformation that leaves invariant all the invariant subspaces of T, then there is a sequence $\{a_k\}$ such that $Sy\sim\sum_k a_kT^ky$. By Lemma 2.2, if x is a shifted vector of finite order, then $\{x, y\}$ is a shifted pair. Thus by part (3a) of Lemma 2.3, $Sx=\sum_{k<m}a_kT^kx$, where $m=\operatorname{ord}(x)$. If z is another shifted vector of infinite order, then $Sz\sim\sum_k b_kT^kz$, and another application of Lemma 2.2 and part (3a) of Lemma 2.3 yields $b_k=a_k$ for $k<m$. Since there are shifted vectors of arbitrarily large finite order, $b_k=a_k$ for every k.

Let A be a continuous invertible linear transformation in $\{T\}'$. If x is a shifted vector for T, then Ax is also a shifted vector for T and

$$SAx \sim \sum_k a_k T^k Ax.$$

On the other hand, $Sx \sim \sum_k a_k T^k x$ implies that $ASx \sim \sum_k a_k T^k Ax$. It follows that $ASx = SAx$, and hence $AS = SA$ on the span of the shifted vectors.

Remark. If X is a Banach space in the preceding theorem, we can drop the assumption that A is invertible, since there is a scalar λ such that $A-\lambda$ is invertible, and every operator that commutes with $A-\lambda$ also commutes with A.

Theorem 2.11. *Suppose that X is a locally convex space, $T \in B(X)$, and $\{M_k : k \in \mathbf{Z}\}$ is a collection of closed linear subspaces with zero intersection such that $T(M_k) \subset M_{k+1} \subset M_k$ for each k in $\mathbf{Z}$. Let $N = \bigcup_{k \in \mathbf{Z}} M_k$ and assume that $S \in \operatorname{Alg\,Lat} T$ and $ST - TS = 0$ on N. Then*

(1) *each vector in N is a shifted vector for T,*

(2) *there is a sequence $\{a_n\}$ of complex numbers such that, for every vector x in N, we have $Sx \sim \sum_k a_k T^k x$.*

Moreover, if, for each integer $n \geqq 2$, we have $\dim(M_{-n}/[(T^{2n})^{-1}(M_n)] \cap M_{-n}) > 2$, then $AT - TA = 0$ on N for every A in $\operatorname{Alg\,Lat} T$.

Proof. Statement (1) is clear. Note that S leaves N invariant, since $S \in \operatorname{Alg\,Lat} T$. For each positive integer n, we can apply Lemma 2.4 to the operators induced by S and T on M_{-n}/M_n to obtain a polynomial $p_n(z)$ such that $(S - p_n(T))(M_{-n}) \subset M_n$. Moreover, it is clear that there is a single formal power series $f(z) = \sum_k a_k z^k$ such that each $p_n(z)$ is a partial sum of $f(z)$. Since $\bigcap_k M_k = 0$, it follows that $Sx \sim \sum_k a_k T^k x$ for every x in N.

Note that the hypothesis $\dim(M_{-n}/[(T^{2n})^{-1}(M_n)] \cap M_{-n}) > 2$ implies, via Lemma 2.5, that the operator induced by T on M_{-n}/M_n, is reflexive, which implies that the operator induced by A on M_{-n}/M_n is a polynomial in the operator induced by T. In particular, $(AT - TA)(M_{-n}) \subset M_n$ for all $n \geqq 2$. Since $\bigcap_k M_k = 0$, it follows that $AT - TA = 0$ on N.

3. Weighted shifts. The results of the preceding section often yield $Sx \sim \sum_k a_k T^k x$ for every vector x (or for at least a dense set of vectors). This suggests that the operator S is in the weakly closed unital algebra $\mathscr{A}(T)$ generated by T; however, the formal power series $\sum_k a_k T^k$ need not converge, and it is not clear that either the sequence of partial sums (or its Cesaro means) need have a convergent subnet in the weak operator topology. For unilateral weighted shift operators on

Hilbert space with scalar weights, A. L. SHIELDS and L. J. WALLEN [29] proved that the commutant coincides with the generated weakly closed algebra. In the course of the proof they show that the sum of a formal power series in a weighted shift is the strong limit of the sequence of Cesaro means of the sequence of partial sums of the formal power series in the shift. We will show that the Shields—Wallen result holds for shifts of a much more general nature.

Suppose that X is a normed linear space and $\{X_i: i\in I\}$ is a linearly independent family of closed linear subspaces of X whose algebraic sum is M. We say that X is the direct sum of the X_i's if there is a family $\{P_i: i\in I\}$ of idempotents in $B(X)$ such that

(1) $P_i|M$ is the projection onto X_i for each $i\in I$,

(2) M is dense in X,

(3) $\sup\{\|\sum_{i\in F} P_i\|: F\subset I, F \text{ finite}\}<\infty$.

It follows that $\sum_{i\in I} P_i=1$ converges in the strong operator topology, since the net of partial sums is bounded and converges strongly to 1 on the dense subset M. It also follows that the set $\left\{\sum_{i\in F} P_i: F\subset I, F \text{ or } I\setminus F \text{ finite}\right\}$ is a bounded Boolean algebra of idempotents. Standard results on bounded Boolean algebras of projections (see [13]) imply that there is a constant K such that, for every function $\alpha: I\to \mathbf{C}$ with finite support, we have

$$(*)\qquad \Big\|\sum_{i\in I}\alpha(i)P_i\Big\|\leqq K\sup_i|\alpha(i)|.$$

Moreover, if X is a Banach space, the preceding inequality holds for every bounded $\alpha: I\to\mathbf{C}$ and $\sum_{i\in I}\alpha(i)P_i$ converges in the strong operator topology.

Note that a c_0-sum or an l^p-sum $(1\leqq p<\infty)$ of subspaces is a direct sum in the above sense; however, an l^∞-sum is not a direct sum since M fails to be dense.

An operator T is a (forward) unilateral operator-weighted shift on a normed space X if there is a sequence $\{X_n: n\in\mathbf{Z}^+\}$ of subspaces of X such that

(1) X is the direct sum of the X_n's,

(2) $T(X_n)\subset X_{n+1}$ for $n\in\mathbf{Z}^+$.

Here $\mathbf{Z}^+$ denotes the set of non-negative integers. If $\mathbf{Z}^+$ is replaced by the set $\mathbf{Z}$ of integers, then T is called a bilateral operator-weighted shift. The restriction operators $T|X_n$ are the weights of the shift. If all of the X_n's are 1-dimensional, then the weighted shift is called a scalar-weighted shift. If, on the other hand, condition (2) above is replaced by

$$(2)'\qquad T(X_0)=0,\quad T(X_{n+1})\subset X_n \quad\text{for}\quad n\in\mathbf{Z}^+,$$

then T is a backwards unilateral operator-weighted shift. If T is an operator of any of the three types defined above we say that T is an operator-weighted shift.

The following is a generalization of the SHIELDS—WALLEN theorem [29].

Theorem 3.1. *Suppose T is an operator-weighted shift on a normed space X that is a direct sum of subspaces $\{X_n\}$. Suppose $A\in B(X)$ and $\{a_n\}$ is a sequence of scalars such that, for each x in $\cup X_n$, we have $Ax\sim \sum_k a_k T^k x$. Let $\{A_n\}$ be the sequence of Cesaro means of the sequence of partial sums of the series $\sum_k a_k T^k$. Then*

(1) $\sup_n \|A_n\| < \infty$,

(2) $A_n \to A$ *in the strong operator topology.*

Proof. First note that since X is the direct sum of the X_n's, it follows, for each x in $\cup X_n$, that the sum $\sum_k a_k T^k x$ converges in norm to Ax. It follows that $\|A_n x - Ax\| \to 0$ for every x in $\cup X_n$. If (1) is true, then $\{x\in X: \|A_n x - Ax\| \to 0\}$ is a closed linear subspace of X containing $\cup X_n$, which implies (2). Hence it suffices to prove (1).

Let P_n denote the projection of X onto X_n, and, for each finite set F of indices, let $Q_F = \sum_{i\in F} P_i$. Let K be as in $(*)$ above. It follows that, for each index n, we have $\{Q_F A_n Q_F\}$ converges in the strong operator topology to A_n; whence, $\|A_n\| \leqq \limsup_F \|Q_F A_n Q_F\|$. We will show that if m is any index and k and n are positive integers, and if $F=\{j: m\leqq j\leqq m+k\}$, then $\|Q_F A_n Q_F\| \leqq K^2\|A\|$. From this it follows that $\|A_n\| \leqq K^2\|A\|$ for $n=1, 2, 3, \ldots$, which proves (1).

Define continuous functions $f, g\colon [-\pi, \pi] \to B(X)$ by

$$f(t) = \sum_{0\leqq j\leqq k} e^{ijt} P_{m+j} \quad\text{and}\quad g(t) = \sum_{0\leqq j\leqq k} e^{-ijt} P_{m+j}.$$

Let

$$K_n(t) = \frac{1}{n+1}\left[\frac{\sin(n+1)\,t/2}{\sin(t/2)}\right]^2$$

be the n^{th} Fejér kernel. A simple computation shows that

$$Q_F A_n Q_F = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,Ag(t)K_n(t)\,dt,$$

and it follows that

$$\|Q_F A_n Q_F\| \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi} \|f(t)\,Ag(t)K_n(t)\|\,dt \leqq K^2\|A\|\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t)\,dt = K^2\|A\|.$$

This completes the proof.

Corollary 3.2. *Suppose that T is scalar-weighted unilateral shift with nonzero weights. Then $\{T\}' = \mathcal{A}(T)$.*

Proof. Suppose e is a nonzero vector in X_0. Then $\{T^n e\}$ is a basis for X_n for $n=0, 1, 2, \ldots$. Suppose $S\in\{T\}'$. Then $Se = \sum_{0\leqq k} a_k T^k e$ for some sequence $\{a_k\}$ of scalars. Since $S\in\{T\}'$, it follows that

$$ST^n e = T^n Se = T^n \sum_{0\leqq k} a_k T^k e = \sum_{0\leqq k} a_k T^k(T^n e).$$

Hence the hypothesis of Theorem 3.1 is satisfied, which implies that $S\in\mathscr{A}(T)$.

Corollary 3.3. *If T is an operator-weighted shift on a normed linear space, then $\{T\}'\cap\text{Alg Lat } T=\mathscr{A}(T)$.*

Proof. It follows from Theorem 2.11 that if $S\in\{T\}'\cap\text{Alg Lat } T$, then S satisfies the hypothesis of Theorem 3.1.

Theorem 3.4. *Suppose T is an operator-weighted shift on a normed space X relative to the direct sum $X=\overline{\sum_n} X_n$. Let $M=\bigcup_n X_n$, and suppose for each x in M and each positive integer n with $T^n x\neq 0$, there is a y in M such that $\text{sp}\,\{T^k x\colon k\geqq 0\}\cap \text{sp}\,\{T^k y\colon k\geqq 0\}=\{0\}$ and $T^n y\neq 0$. Then T is reflexive.*

Proof. Suppose $S\in\text{Alg Lat } T$. In view of Corollary 3.3, we need show only that $S\in\{T\}'$. We first show that if $x, y\in M$ and $\text{sp}\,\{T^k x\colon k\geqq 0\}\cap\text{sp}\,\{T^k y\colon k\geqq 0\}=0$, then $\{x, y\}$ is a shifted pair for T. Once this is done, it will follow from part 3(c) of Lemma 2.3 and the hypothesis of the theorem that $ST=TS$ on M, and, since $X=\overline{\text{sp}}\, M$, it will follow that $S\in\{T\}'$.

Suppose $x, y\in M$ and $\text{sp}\,\{T^k x\colon k\geqq 0\}\cap\text{sp}\,\{T^k y\colon k\geqq 0\}=0$. Suppose $x\in X_k$ and $y\in X_j$. By symmetry, it will suffice to show that if n is a positive integer and $T^n x\neq 0$, then $T^n x\notin\overline{\text{sp}}\,(\{T^i x\colon i\geqq 0,\ i\neq n\}\cup\{T^i y\colon i\geqq 0\})$. However, $T^n x\in X_{k+n}$, and if P is the projection onto X_{k+n}, then $T^n x\in\overline{\text{sp}}\,(\{T^i x\colon i\geqq 0,\ i\neq n\}\cup\{T^i y\colon i\geqq 0\})$ implies that $T^n x=PT^n x\in\overline{\text{sp}}\,(\{PT^i x\colon i\geqq 0,\ i\neq n\}\cup\{PT^i y\colon i\geqq 0\})$, and the last set is either 0 or $\text{sp}\,\{T^{n+k-j}y\}$. This violates the conditions that $T^n x\neq 0$ and $\text{sp}\,\{T^k x\colon k\geqq 0\}\cap\text{sp}\,\{T^k y\colon k\geqq 0\}=0$. It follows that $\{x, y\}$ is a shifted pair for T, and the proof is complete.

The following corollaries are immediate consequences of the theorem.

Corollary 3.5. *Operator-weighted shifts with injective operator weights of rank at least two are reflexive.*

Corollary 3.6. *Every direct sum of at least two scalar-weighted shifts with nonzero weights is reflexive.*

The following is a corollary of Theorem 2.10.

Corollary 3.7. *If T is a unilateral backwards operator-weighted shift with injective weights, and T has a shifted vector of infinite order, then T is reflexive.*

We will show that the unweighted (i.e., all weights 1) backwards unilateral shift operator on the Hilbert space H^2 has no shifted vectors of infinite order. Our demonstration depends on the characterization of noncyclic vectors of the backwards shift in [12]. A vector f is noncyclic for the backward shift T in case there exists a meromorphic pseudocontinuation $f^\sim$ of f to the complement D_e of the closed unit disk in the Riemann sphere such that $f^\sim = G/H$, where G and H are bounded holomorphic functions on D_e. To say $f^\sim$ is a pseudocontinuation of f means that the radial limits of f and $f^\sim$ agree at almost every point of the unit circle. We need the following lemma.

Lemma 3.8. *If f is a noncyclic vector for the backwards shift T, then $f \in M(Tf)^-$ if and only if $f^\sim(\infty)=0$.*

Proof. The proof is virtually the same as that of Theorem 1 in [12]. Suppose

$$f(z) = \sum_{k \geqq 0} a_k z^k, \quad G(z) = \sum_{k \geqq 0} b_k z^{-k} \quad \text{and} \quad H(z) = \sum_{k \geqq 0} c_k z^{-k}.$$

By multiplying both G and H by an appropriate power of z, we may assume that either $b_0 \neq 0$ or $c_0 \neq 0$. Then $f^\sim(\infty) = b_0/c_0$. (If $c_0=0$, then $f^\sim(\infty)=\infty$.) We have $f(e^{i\vartheta})H(e^{i\vartheta}) = G(e^{i\vartheta})$ a.e., and hence as in [12],

$$a_0 c_0 + a_1 c_1 + \ldots = b_0,$$

$$a_1 c_0 + a_2 c_1 + \ldots = 0,$$

$$a_2 c_0 + a_3 c_1 + \ldots = 0,$$

etc. Thus if $f^\sim(\infty) \neq 0$, then $b_0 \neq 0$, and the preceding equations show that there exists a vector that is not orthogonal to f but is orthogonal to $T^k f$ for every $k \geqq 1$. Thus f does not belong to $M(Tf)^-$.

Conversely suppose $f^\sim(\infty)=0$. Let h be a vector that is orthogonal to $M(Tf)^-$. If c_k is the conjugate of the k^{th} Fourier coefficient of h, then all but the first of the sequence of equations in the preceding paragraph hold. Thus if $H_0(z) = \sum_{k \geqq 0} c_k z^{-k}$, then H_0 has radial limits at almost every point of the unit circle, and $H_0(e^{i\vartheta}) f(e^{i\vartheta}) = G_0(e^{i\vartheta})$ defines a function G_0 in $L^1(d\vartheta)$ with $G_0(e^{i\vartheta}) = \sum_{k \geqq 0} b_k e^{-ik\vartheta}$. It follows that if $G_0(z) = \sum_{k \geqq 0} b_k z^{-k}$, then $g^\sim = G_0/H_0$ is a pseudocontinuation of f to D_e and G_0 and H_0 are quotients of bounded holomorphic functions since they are in H^2 and H^1 of D_e respectively. Since the pseudocontinuation of a function is unique, it follows that $g^\sim(\infty)=0$, and thus $b_0=0$. Hence the first equation of the sequence in the preceding paragraph shows that f is also orthogonal to h, and it follows that $f \in M(Tf)^-$.

Proposition 3.9. *The only shifted vectors of the adjoint of the unweighted unilateral shift operator are polynomials.*

Proof. It will be shown that T has no noncyclic shifted vectors of infinite order. This will imply that it has no infinite order shifted vectors. For if f is cyclic, shifted and of infinite order, then Tf is shifted and of infinite order, but it is noncyclic.

Suppose f is any noncyclic vector of infinite order and $f^{\sim}$ is its meromorphic pseudocontinuation. Then since $Tf=(f-f(0))/z$, and since pseudocontinuations are determined by their radial limits (see [12]), it follows that $(Tf)^{\sim}=(f^{\sim}-f(0))/z$. Hence if $f^{\sim}$ has a pole at ∞ of order m, then $(Tf)^{\sim}$ has a pole at ∞ of order $m-1$, and consequently $(T^{m+1}f)^{\sim}(\infty)=0$. By Lemma 3.8, $T^{m+1}f\in\overline{\mathrm{sp}}\,\{T^k f\colon k>m+1\}$. Since $T^{m+1}f\neq 0$, f is not a shifted vector.

4. Reflexive Graphs. In this section we study conditions that make graphs reflexive. We wish to consider a more general version of reflexivity than that of the preceding sections. A linear subspace $\mathscr{S}$ of $B(H)$ is reflexive if $T\in\mathscr{S}$ whenever $Tx\in[\mathscr{S}x]^-$ for every x in H. We say that a linear functional φ on $\mathscr{S}$ is elementary if there are vectors x, y in H such that $\varphi(S)=(Sx, y)$ for every S in $\mathscr{S}$. We say that $\mathscr{S}$ is weakly elementary if every weak-operator continuous linear functional is elementary on $\mathscr{S}$. (In [17] and [7], respectively the terms "property D" and "property A" used. Our notation agrees with that in [3].)

Theorem 4.1. *Suppose that $\mathscr{S}$ is a reflexive linear subspace of $B(H)$ and $\pi\colon \mathscr{S}\to B(M)$ is a linear mapping such that the set of elementary linear functionals φ on $B(M)$ for which $\varphi\circ\pi$ is elementary on $\mathscr{S}$ separates the points of $B(M)$. Then* $\mathrm{Graph}\,(\pi)=\{S\oplus\pi(S)\colon S\in\mathscr{S}\}$ *is a reflexive linear subspace of $B(H\oplus M)$.*

Proof. Suppose that $A\in B(H\oplus M)$ and $Ae\in[\mathrm{Graph}\,(\pi)e]^-$ for every vector e in $H\oplus M$. Clearly, we can write $A=B\oplus C$. Also, since $\mathscr{S}$ is reflexive, it is clear that $B\in\mathscr{S}$. Thus, by replacing A by $A-(B\oplus\pi(B))$, we can assume that $B=0$. We need to show that $C=0$. Suppose that $C\neq 0$. Then there is an elementary functional φ on $B(M)$ such that $\varphi(C)\neq 0$ and such that $\varphi\circ\pi$ is elementary on $\mathscr{S}$. Thus there are vectors x_1, x_2 in H and y_1, y_2 in M such $\varphi(T)=(Ty_1, y_2)$ for all T in $B(M)$ and such that $(Sx_1, x_2)=\varphi(\pi(S))=(\pi(S)y_1, y_2)$ for all S in $\mathscr{S}$. Letting $e=x_1\oplus y_1$, it follows that there is a sequence $\{S_n\}$ in $\mathscr{S}$ such that $(S_n\oplus\pi(S_n))e\to Ae$. Thus $S_n x_1\to 0$ and $\pi(S_n)y_1\to Cy_1$, Hence $0\neq\varphi(C)=(Cy_1, y_2)=\lim(\pi(S_n)y_1, y_2)= =\lim\varphi(\pi(S_n))=\lim(S_n x_1, x_2)=0$. This contradiction shows that $C=0$.

Corollary 4.2. *If $\mathscr{S}$ is a weakly elementary reflexive linear subspace of $B(H)$ and $\pi\colon\mathscr{S}\to B(M)$ is a weakly continuous linear map, then* Graph (π) *is reflexive and weakly elementary.*

It was shown in [5] and [32] that if S is the unilateral shift operator and T is a contraction operator, then $S\oplus T$ is reflexive. An unsolved problem of P. R. Halmos [18] asks whether every polynomially bounded operator is similar to a contraction, and it was shown by W. MLAK [22] that every polynomially bounded operator is similar to the direct sum of a unitary operator and an operator with a weakly continuous H^∞ functional calculus.

Corollary 4.3. *If S is the unilateral shift operator and T is a polynomially bounded operator, then $S \oplus T$ is reflexive and weakly elementary.*

Proof. The result of the preceding corollary shows that $S \oplus T$ is reflexive and elementary when T has a weakly continuous H^∞ functional calculus. The direct sum of such an operator with a unitary operator must be reflexive and elementary by [17].

Suppose that S is a subnormal operator. A result of D. SARASON [27] says that there is a compactly supported Borel measure μ in the plane and an open subset Ω of the plane such that the weakly closed algebra generated by 1 and S is isomorphic to $L^\infty(\mu) \oplus H^\infty(\Omega)$. Call the set Ω the Sarason hull of S. It is shown in [8] that convergence in the weak operator topology in the $H^\infty(\Omega)$ summand implies uniform convergence on compact subsets of Ω. Thus if T is an operator whose spectrum is contained in Ω, then there is an appropriate $H^\infty(\Omega)$ functional calculus. The Sarason hull of the unilateral shift operator is the open unit disk. It was shown by R. OLIN and J. THOMSON [23] that the weakly closed algebra generated by a subnormal operator is weakly elementary. The proof of the preceding Corollary combined with the aforementioned facts yield the following.

Corollary 4.4. *If S is a subnormal operator and T is an operator whose spectrum is contained in the Sarason hull of S, then $S \oplus T$ is reflexive and weakly elementary.*

Note that the definitions of reflexivity and of being elementary for a linear subspace $\mathscr{S}$ of $B(H)$ makes sense when $\mathscr{S}$ is a subset of $B(H, K)$, the set of operators from the Hilbert space H to the Hilbert space K. In this way, it makes sense to talk of a subspace $\mathscr{S}$ of $B(H)$ having a reflexive (or elementary) restriction to a linear subspace M of H, i.e., $\mathscr{S}|M$ is reflexive.

Suppose that $\mathscr{S}$ is a linear subspace of $B(H)$ and $x \in H$. We define $G_{\mathscr{S}}(x)$ to be the set of all vectors y in H such that

(a) $[\mathscr{S}x]^- \cap [\mathscr{S}y]^- = \{0\}$,

(b) $[\mathscr{S}x]^- + [\mathscr{S}y]^-$ is closed, and

(c) if $\{S_n\}$ is a sequence in $\mathscr{S}$ such that $\|S_n x\| \to 0$ and $\{S_n y\}$ is norm convergent, then $S_n y \to 0$.

Note that (a) and (b) imply that the sum in (b) is direct sum of Banach spaces and that (c) implies that $\{Sx + Sy \colon S \in \mathscr{S}\}^-$ is a graph in this direct sum.

Theorem 4.5. *Suppose that M is a subspace of the Hilbert space H and $\mathscr{S}$ is a linear subspace of $B(H)$ such that*

(1) *$\mathcal{S}|M$ is reflexive,*

(2) *$M+\operatorname{span}\left(\bigcup\{G_{\mathcal{S}}(x)\colon x\in M\}\right)$ is dense in H.*

Then $\mathcal{S}$ is reflexive.

Proof. Suppose $T\in B(H)$ and $Te\in[\mathcal{S}e]^{-}$ for every e in H. It follows from (1) that $T|M\in\mathcal{S}|M$; hence we can assume that $T|M=0$. Suppose $x\in M$ and $y\in G_{\mathcal{S}}(x)$. Then there is a sequence $\{S_n\}$ in $\mathcal{S}$ such that $S_n(x+y)\to T(x+y)$. It follows from parts (a) and (b) of the definition of $G_{\mathcal{S}}(x)$ that $S_n x\to Tx=0$ and $S_n y\to Ty$. It follows from part (c) of the definition of $G_{\mathcal{S}}(x)$ that $Ty=0$. Thus, by (2), $T=0$, since $T=0$ on a dense subset of H.

Corollary 4.6. *Suppose $\mathcal{S}$ is a reflexive subspace of $B(H)$ and $\pi\colon\mathcal{S}\to B(M)$ is a linear map such that the set*

$$\{y\in M\colon \exists x\in H \text{ such that } \{Sx\oplus\pi(S)y\colon S\in\mathcal{S}\}^{-} \text{ is a graph}\}$$

is dense in M. Then Graph (π) *is reflexive and π is continuous with respect to the ultraweak (and norm) topologies on $\mathcal{S}$ and $B(M)$.*

The preceding Corollary can also be used to recapture the result, that $S\oplus T$ is reflexive whenever S is the unilateral shift operator and T is a contraction [5], [32]. The basic idea is to let $\mathcal{S}$ be the unital weakly closed algebra generated by S and to define π by $\pi(\varphi(S))=\varphi(T)$ for each φ in H^∞. In the case $\|T\|<1$, it follows that if x is a unit vector in $\ker S^*$, then $\{\varphi(S)x\oplus\varphi(T)y\colon \varphi\in H^\infty\}^{-}$ is a graph for every vector y. This follows from the fact that if $\{\varphi_n\}$ is a sequence in H^∞ and $\varphi_n(S)x\to\varphi(S)x$, then $\varphi_n\to\varphi$ in H^2 and thus uniformly on compact subsets of the open unit disk, which, by the Riesz functional calculus, implies that $\varphi_n(T)\to\varphi(T)$ in norm.

5. Questions and comments.

1. A Donoghue operator is a weighted shift on l^2 with square summable weights that tend monotonically to zero. It is easy to see that a backwards Donoghue operator has no shifted vectors of infinite order. For if f is a vector of infinite order, then D^*f is a cyclic vector. Does $(D\oplus D)^*$ have a shifted vector of infinite order, where D is a Donoghue operator?

2. Suppose S is the unilateral shift operator on l^2. What is the set of all Hilbert space operators T for which $S\oplus T$ is reflexive. This paper shows that the set contains all polynomially bounded operators and all operator-weighted shifts.

3. If T is a nonnilpotent Hilbert space operator that is a direct sum of nilpotent operators, must T be reflexive?

4. The results in this paper on reflexive graphs have been generalized in [15] and have been extended to prove that certain graphs are hyperreflexive. In particular, it is shown in [15] that if S is the unilateral shift operator on l^2 and T is polynomially

bounded, then $S \oplus T$ is hyperreflexive. In [9] K. DAVIDSON proved that the unilateral shift operator is hyperreflexive. What about the direct sum of two weighted shifts, or the operator-weighted shifts on Hilbert space considered in Theorem 3.4?

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Applications of Combinatorics and Graph Theory to the Biological and Social Sciences, Edited by Fred Roberts (The IMA Volumes in Mathematics and its Applications), IX+345 pages, X+156 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong, 1989.

One of the principal motive powers of the development of mathematics is the hard demand on more and more application which appear from the side of living sciences. For the people who are makers or users of pure and applied mathematics, a very interesting experience is to find the meeting point of mathematics and the living nature, biology and social sciences.

This volume contains fifteen exciting overviews concerning the above topics. The leading idea of the book is formulated in the first paper (by F. Roberts), drawing up seven fundamental concepts, as RNA chains as "words" in a 4 letter alphabet, Interval graphs, Competition graphs or niche overlap graphs, Qualitative stability, Balanced signed graphs, Social welfare functions, and Semiorders.

Diversity of human and biological sciences manifests itself in the interesting and multicoloured topics in the remaining forteen papers. The list of authors (in the order of papers) is: J-P. Barthelemy, M. B. Cozzens, N. V. R. Mahadev, J-C. Falmagne, P. C. Fishburn, B. Ganter, R. Wille, E. C. Johnsen, V. Klee, J. C. Lundgren, J. S. Maybee, J. K. Percus, P. H. Sellers, P. D. Straffin Jr.

The volume is mainly based on the proceeding of a workshop which was organized in course of an IMA program on Applied Combinatorics.

J. Kozma (Szeged)

Applied Mathematical Ecology, Edited by Simon A. Levin, Thomas G. Hallam and Louis J. Gross (Biomathematics, 18), XIV+491 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1989.

This book contains the subject-matter of the Second Autumn Course on Mathematical Ecology held at the International Centre for Theoretical Physics in Trieste, Italy in November and December of 1986. The contents and the structure of the book is introduced by the editors in the Preface as follows: "This book is structured primarily by application area. Part II provides an introduction to mathematical and statistical applications in resource management. Biological concepts are interwoven with economic constraints to attack problems of biological resource exploitation, conservation of our natural resources and agricultural ecology. Part III consists of articles on the fundamental aspects of epidemiology and case studies of the diseases rubella, influenza and AIDS, Part IV addresses some problems of ecotoxicology by modelling the fate and effects of chemicals in equatic systems. Part V is directed to several topics in demography, population biology and plant ecology, with emphasis on structured population models."

The list of authors shows that the Autumn Course was participated by the most outstanding experts in Mathematical Ecology from all over the world. Their book must be found on the bookshelf of every specialist wishing to follow the main directions of the development of the field.

L. Hatvani (Szeged)

E. Arbarello—C. Procesi—E. Strickland, Geometry today, Giornate di Geometria, Roma 1984 (Progress in Mathematics, 60), 329 pages, Birkhäuser, Boston—Basel—Stuttgart, 1985.

The meeting "Giornate di Geometria" was held at the "Dipartimento di Matematica, Istituto G. Castelnuovo" during the period 4—9 June 1984. There were many mathematicians on this conference from almost all of the area of geometry. At the same time some top specialists were also there such as S. Donaldson, W. Fulton, P. Griffiths, V. Kac, D. Kazdhan, D. Mumford for example only. The book contains almost all of the talks given at the meeting, hence the reader finds accounts on geometry ranging from algebraic curves to topology, from non linear equations to algebraic groups and number theory.

We recommend this book to all who are interested in the modern geometry.

Árpád Kurusa (Szeged)

P. L. Barz—Y. Hervier, Enumerative Geometry and Classical Algebraic Geometry (Progress in Mathematics, 24), X+252 pages, Birkhäuser, Boston—Basel—Stuttgart, 1982.

This book is based on the conference held at the University of Nice during the period 23—27 June 1981. The major areas of the activity were enumerative geometry, curves and cycles and multiplicities. We mention that half of the papers are written in french. The papers of the book are from L. Gruson, C. Peskine, R. Piene, F. Catanese, W. Fulton, R. Lazarsfeld, D. Laksov, A. Beauville, A. Hirschowitz, M. S. Narasimhan, P. Le Barz, I. Vainsencher and S. L. Kleiman.

We recommend this book to graduate students and resarchers as well.

Árpád Kurusa (Szeged)

P. Biler—A. Witkowski, Problems in Mathematical Analysis, (Pure and Applied Mathematics) v+227 pages, Marcel Dekker, Inc. New York—Basel, 1990.

This is a truly excellent collection of problems in mathematical analysis, although several problems from other mathematical disciplines are also included. The level of problems varies considerably, but most of them are above the level of standard textbook exercises. Most of them require some trick or strong theoretical background. Some of the problems are very hard and have the flavour of research results. The collection was selected from several sources, many of them were taken from the American Mathematical Monthly.

The book, which contains about 1200 problems, is divided into nine sections: real and complex numbers, sequences, series, functions of one variable, functional equation and functions of several variables, real analysis, analytic functions, Fourier series and functional analysis. Each of them gives a very thorough account of the given field through fascinating problems, although I have found the first chapter more entertaining than the other ones. It starts out with the easy exersice that for irrational a and b the power a^b can be rational. But the trick is nice: $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}=2$. However, the next problem, that for a $c > 8/3$ asks for the existence of a ϑ such that $[\vartheta^{c^n}]$ is prime for every n, is a much more challenging one.

Unfortunately the hints given to the problems are very scarce and very often of little use since they rather give the reference to the source of the problem than lending help in the solution. This makes the use of the book rather cumbersome since no one wants to run to the library every time he gets stuck with a particular problem. Sometimes no hint or reference is given at all, which may be puzzling for many readers in case of problems like the one which asks if it is possible to divide a square into an odd number of triangles of the same area (try it!). I found it a pity that the authors do not give more detailed hints or full solutions which would have made the book even more outstanding.

I would very strongly recommend the book to both students and teachers, but everyone who likes problem solving, which, according to many of us, is the heart of mathematics, will find hours, and hours of fun and enjoyment in the problems.

Vilmos Totik (Szeged)

A. Bohm—M. Gadella, Dirac Kets, Gamow Vectors and Gel'fand Triplets. The Rigged Hilbert Space Formulation of Quantum Mechanics, (Lecture Notes in Physics 348), VIII+119 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong, 1989.

This book presents the Rigged Hilbert space formulation of Dirac's bracket formalism of quantum mechanism, preferred by most physicists for its elegance and practicality in actual calculations. It is an extension of the first author's well-known lecture notes (LNP 78) on the subject.

Dirac's formalism of bras and kets has been considered as mathematical nonsense by von Neumann, whose Hilbert space formulation became the standard, mathematically rigorous model of quantum mechanics. The right mathematics for describing Dirac's formalism appeared by the invention of the theory of distributions in the fifties, and the concept of Gel'fand triplets and the nuclear spectral theorem in the sixties, which make sense of the complete system of eigenvectors of self-adjoint operators with continuous spectrum. The discovery of these beautiful mathematical theories was inspired by Dirac's heuristic ideas.

This book not only gives a clear exposition of the mathematics of the Rigged Hilbert space formulation of Dirac's approach to quantum mechanics, in a languagage accessible to physicists, but also presents interesting physical applications concerning decaying states and resonances, by using the concept of Gamow vectors.

The reviewer recommends this volume to everybody interested in quantum mechanics, especially to graduate students studying physics or functional analysis, and university instructors lecturing on quantum mechanics.

László Fehér (Szeged)

David M. Burton, Elementary Number Theory (second edition), XVII+450 pages, Wm. C. Brown Publisher, Dubuque, Iowa, 1989.

The theory of numbers has occupied a unique position in the world of mathematics. This position is due to several facts, e.g., it has an unquestioned historical importance, it has several easily formulated but hardly solvable problems (this is the reason why it arises the interest of many amateurs), it is one of the best subjects for early mathematical instruction. We share Gauss' opinion, ,'Mathematics is the Queen of science, and number theory the Queen of mathematics".

The elementary number theory is an integral part of almost all undergraduate mathematical curriculum, therefore several textbooks are available on this topic. Nevertheless, Burton's very readable book is unique in some sense among them.

In each chapter we can read a historical introduction or/and they end with a historical outline. Evoking a nice old practice, we can find a pearl of quotation from mathematicians, philosophers or writers at the top of every chapter. There are problems at the ends of the chapters (the total amount is about 600) ranging in difficulty from the purely mechanical to challenging theoretical questions. They form an integral part of the text, to require the reader's active participation.

This second edition is an enlarged version of the first one (Allyn and Bacon, 1980). The substantial changes are an entirely new section on cryptography, the enlargement of the section on Fermat numbers, introduction of a variety of new topics, e.g., Merten's conjecture, absolute semiprimes, amicable number pairs, and primes in arithmetical progression. Some 150 additional problems are also included.

The first nine chapters (Some Preliminary Considerations, Divisibility Theory in the Integers, Primes and Their Distribution, The Theory of Congruences, Fermat's Theorem, Number-Theoretic Functions, Euler's Generalization of Fermat's Theorem, Primitive Roots and Indices, The Quadratic Reciprocity Law) can be used as a basic material of a one semester course. The additional four chapters (Perfect Numbers, The Fermat's Conjecture, Representation of Integers as Sums of Squares, Fibonacci Numbers and Continued Fractions) are independent of each other. They may be taken up at pleasure. Despite of the material is mostly classical, there are several hints to modern results, too (only in the second edition). Among the five appendices there are an outline of the prime number theorem and answers to selected problems.

This well-organized textbook is warmly recommended to any undergraduate number theory course.

Lajos Klukovits (Szeged)

Categorical Methods in Computer Science, Edited by H. Ehrig, H. Herrlich, H.-J. Kreowski and G. Preuss (Lectures Notes in Computer Science, 393), VI+350 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1989.

This volume contains the papers presented at the International Workshop on Categorical Methods in Computer Science with Aspects from Topology held in Berlin in September 1988. The material is organized into three parts. The following quotation is from the Preface. "In part 1 we have collected papers concerning categorical foundations and fundamental concepts from category theory in computer science. Applications of categorical methods to algebraic specification languages and techniques, data types, data bases, programming, and process specifications are presented in part 2. The papers on categorical aspects from topology in part 3 mainly concentrate on special adjoint situations like cartesian closedness, Galois connections, reflections, and coreflections, which are of growing interest in categorical topology and computer science."

The volume can be recommended to those interested in categorical methods in computer science.

Z. Ésik (Szeged)

Collected papers of Paul Turán, Edited by P. Erdős. 3 Volumes, XXXVIII+2665 pages. Akadémiai Kiadó, Budapest, 1990.

Paul Turán was born in 1910 in Budapest, Hungary. He died in 1976. He achieved a remarkably prolific career with publishing two books and 246 papers. The three volumes of his collected works contain the collection of most of his papers (some of them written to very specific audience were omitted). Many of the earlier papers are in German bacause the works were reproduced photocopically, a process that amplifies the immense variety of Turán's work. The exceptions to this are only the papers written in Hungarian that were translated to English. Naturally, his books about the power sum method invented by him in 1938 are not included in the list, but there are many research papers dealing with the method and its applications.

The main areas in which Turán worked are as follows: power sum method and its applications (some 70 papers), analytic number theory (cc. 60 p.), elementary number theory (15 p.), function theory (22 p.), approximation theory and interpolation (34 p.), Fourier series (8 p.), differential equations (11 p.), statistical group theory (18 p.), combinatorics (16 p.), numerical solutions of equations (10 p.), polynomials (8 p.).

Also included are some most interesting writings discussing the lifelong achievements of L. Fejér (his master), P. Erdős (his lifelong friend and coauthor), K. Rényi, A. Rényi, A. Baker (for Fields Medal), S. Knapowski (his student and coauthor), S. Ramanujan and young Hungarian mathematicians that were the victims of fascism. He himself was in a nazi labour camp during second world war, where he initiated extremal graph theory. One can read about this in two affectionate obituaries contained in volume I by P. Erdős and G. Halász.

Turán's works have initiated several new directions and stimulated the research of an enormous number of mathematicians, so it is natural that in many areas there have been new developments since the publications of his results. Therefore it is most appropriate that the collected works contain many mathematical notes written by L. Alpár, P. Erdős, G. Halász, J. Pintz, M. Simonovits, J. Szabados, M. Szalay and P. Vértesi on the progress in the subjects of the different papers. Unfortunately no list of these notes is included although it would have made easier for the reader to keep trac of these developments.

Paul Turán's collected papers are not the type of books that one would read from the beginning to the end, although I found it impossible to quickly paging through the volumes because quite often a title, a formula or a problem caught my eyes that forced me to read further. I am certain that browsing among Turán's papers will be truly enjoyable for every mathematician even if his or her field is completely different. No library can afford to miss these volumes, and for many of us it will be very pleasant to have them on our bookshelf.

Vilmos Totik (Szeged)

R. Courant—F. John, Introduction to Calculus and Analysis, Vol. I, XXIII+661 pages; Vol. II, XXIII+954 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong, 1989.

Although the first volume of this book was originally published in 1969 and the second one in 1974 it remained one of the best textbooks introducing several generations of mathematicians to higher mathematics. This book leads the students to the heart of the mathematical analysis preparing them for an active application of their knowledge. The main goal of this book is to exhibit the interaction between mathematical analysis and its various applications emphasizing the role of intuition furthermore the importance of the union of intuitive imagination and deductive reasoning. Numerous examples and problems are given at the end of the chapters. Some are challenging, some even difficult; most of them supplement the material in the text. The book is adressed to students on various level, to mathematicians and engineers. Volume I contains among others the following chapters: Integral and Differential Calculus; The Techniques of Calculus; Applications in Physics and Geometry; Taylor's Expansion; Infinite Sums and Products; Trigonometric Series; Differential Equations.

The most characteristic chapters of Volume II are: Functions of Several Variables and Their Derivatives; Vectors, Matrices, Linear Transformations; Applications; Multiple Integrals; Relations Between Surface and Volume Integrals; Differential Equations; Functions of Complex Variable.

This excellent book is highly recommended both to instructors and students.

J. Németh (Szeged)

CSL '88, Edited by E. Börger, H. Kleine Büning and M. M. Richter (2nd Workshop on Computer Science Logic, Duisburg, FRG, October 1988), Proceedings. (Lectures Notes in Computer Science, 385), VI+399 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1989.

This volume is a collection of 24 papers presented at the workshop "Computer Science Logic" held in Duisburg, from October 3 to 7, 1988. The papers cover a broad class of topics ranging from logical aspects of computational complexity to the acceptance of ω-regular languages under various fairness constrains. Below we briefly discuss three contributions of particular interest to the reviewer.

In the paper "Characterizing complexity classes by general recursive definitions in higher types" by A. Goerdth, it is proved that recursive definitions of rank $n+1$ correspond to the complexity class $\bigcup(\mathrm{DTIME}(\exp_n(p(x)))$: $p(x)$ *a polynomial*). Consequently, due to a hierarchy theorem of complexity classes, rank n recursive definitions form a proper hierarchy.

Star-free regular languages have attracted a lot of interest in theoretical computer science. By McNaughton's theorem, star-free regular sets are exactly those definable by some first-order sentence in a suitably chosen language. The paper "Interval temporal logic and star-free expressions" by D. Lippert relates star-free languages and a generalisation thereof to interval temporal logic, a kind of logic introduced for the specification of digital circuits.

Automata and tree automata have continued to play an important role in establishing decidability of certain logics. In the paper "On the emptiness problem of tree automata and completeness of modal logics of programs" by H. Wagner, it is proved that the non-emptiness problem of alternating tree automata is P-complete. This result is then used to show that the satisfiability problem of Propositional Dynamic Logic with a repeat construct is EXPTIME-complete.

The volume can be recommended to those interested in recent research in logical aspects of theoretical computer science.

Z. Ésik (Szeged)

Gerald A. Edgar, Measure, Topology, and Fractal Geometry, (Undergraduate Texts in Mathematics), +230 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong, 1990.

Nowdays the fractals are in the center of the scientists' interest. Since Benoit Mandelbrot established the notion and phylosophy of fractals, quite a lot of books were published on this subject.

Now here is a mathematics book about fractals. The authors' main aim was to give a systematic discussion of the topological and measure theoretical background and to present the most important ideas of fractal geometry.

In Chapter 1 the most basic examples of fractal sets are introduced, such as the Cantor set, the Sierpinski Gasket, the Koch curve, to motivate the "whole story". Chapter 2 is a very good introduction to the topology of metric spaces and Chapter 3 contains the basics of topological dimension theory (small and large inductive dimensions). Chapter 4 is devoted to the complete and detailed discussion of the self-similarity and the more general "graph self-similarity". Here can be found the description of iterated function systems which is an efficient way of generating fractal sets, discovered is recent years by Michael Barnsley. In Chapter 5 the Lebesgue measure and the general methods of generating outer measures and measures are discussed. In Chapter 6 the Hausdorff measure and the Hausdorff dimension are introduced and various fractal dimensions are compared to each other. Finally in Chapter 7 some additional topics are discussed.

Each section contains several exercises for practicing the use of the notions and theorems. The book is written in a nice style illustrated by a lot of figures.

This text is recommended to students as a first course on fractal geometry but it may be useful to anybody who is interested in the rigorous mathematical background of fractals.

J. Kincses (Szeged)

Bernard d'Espagnat, Reality and the Physicist. Knowledge, duration and the quantum world, 280 pages, Cambridge University Press, Cambridge—New York—New Rochelle—Melbourne—Sydney, 1989.

Since the very beginning of this science, quantum mechanics has always been a source and a field of philosophical debates. The founders of this discipline were fully aware of the fact, that the results of quantum theory are in sharp contradiction with the concepts of classical mechanics. They revealed, that microscopic objects are strongly influenced by the measuring apparatus, and the term of physical phenomenon makes sense only, if we take into account the whole apparatus that produces a result of an observation. This process is a kind of collapse, in which the original state of the system changes radically and irreversibly. According to the orthodox view, this final step takes place in the mind of the observer, which is in contradiction with realism, with the principle of the existence of independent reality. For a long time physicists gave up the idea of digging more deeply into such questions, regarding them to belong to the field of philosophy. They rather made use of the calculation rules of quantum physics, which proved to be a very succesful theory.

The debate has been showing a revival in recent years, because it turned out, that the consequences of most simple and logical assertions about a physical system can be put into the form of inequalities (the Bell inequalities), the validity of which can be tested by (much less simple) experiments. And the experiments show, that these most plausible inequalities are violated, while the predictions of quantum mechanics are confirmed. The novelty of these experiments lies in the fact, that the collapse manifests itself on a macroscopic scale, when the "parts" of a single quantum system are several meters apart.

The book, whose author is a well-known theoretical physicist and philosopher, is intended to clear up the situation, stating as precisely as possible the different views on this problem. It is out of question, that d'Espagnat enriches the concept of independent reality and its relation to physical observations. He places his own views somewhere between those of the positivists and the materialists. The book concentrates on the philosophical aspects of the issue, and almost totally avoids the mathematical technicalities, as well as the description of physical experiments. This is certainly a merit, because the book can be recommended to everybody, interested in natural philosophy and the fundamental problems of the material world. Nevertheless, I would propose that the reader should get acquainted with the article by the same author in the Scientific American (vol. 241, p. 158, 1979), where some part of the background is explained in simple terms. This paper, as well as a short synopsis of more recent experimental work, might have been added as an appendix to the text. But anyway, there are a plenty of deep and interesting thoughts in this book, and it enforces us to think over: how absurd independent reality can be on the quantum level.

M. G. Benedict (Szeged)

H. Gross, Quadratic Forms in Infinite Dimensional Vector Spaces (Progress in Mathematics, 1), XXII+419 pages, Birkhäuser, Boston—Basel—Stuttgart, 1979.

Well, one can say this book is old (ten years have gone since its publication) but I think no one can say it is absolete. Although many new results have born in the last decade, for example H. A. Keller's non classical Hilbert space (Math. Z. 172, 41—49), most of the more important results until 1979 are collected in this book together with the directions of the present researches. Its clear style and carefully considered built up makes it still the best book in the subject in my opinion.

The contents of the book are gathered around some important notions and theorems. These are the sesquilinear forms, the diagonalization of $\aleph_0$-forms, the Witt decomposition for Hermitean $\aleph_0$-forms, the quadratic forms and the theorems of Witt an Arf. Every sections are written almost just like a paper closed with specific reference list and some of them with appendix. This helps the reader very well.

I think this book should be on the shelf of every mathematician who makes research on this subject.

Árpád Kurusa (Szeged)

Günther Hämmerlin—Karl-Heinz Hoffmann, Numerische Mathematik, XII+448 pages, Springer-Verlag, Berlin—Heidelberg—New York (Grundwissen Mathematik, B. 7), 1989.

This book offers all the material of the customary one-year introductory courses and a lot of extras.

Its main merit is the clear and intelligent mathematical treatment of the problems. There are numerous consize proofs, illuminating examples and fascinating historical remarks throughout the text.

Even the first Chapter on numerical calculations and algorithms contains supplements on backward error analysis branch-and-bound algorithms and complexity issues.

In Chapters 2 and 3 Numerical Linear Algebra, i.e., Systems of Linear Equations and the Eigenvalue Problem are treated.

The main body of the book follows: Chapter 4: Approximation, Chapter 5: Interpolation and Chapter 6: Splines. In this part the standard topics are investigated with deep mathematical insight. Moreover, the questions of the (two and) finite dimensional interpolation and approximation are touched on.

Chapter 7: Integration starts with the elementary interpolation quadrature rules, extrapolation methods and departs to the special issues of optimal quadrature rules and Monte Carlo methods.

Chapter 8: Iteration gives the basic material on iteration methods for systems of linear and nonlinear equations.

The final Chapter 9: Linear Programming traces out the theoretical background the different variants of the Simplex Method and ends with the polynomial algorithms of Karmarkar and Khachyan.

A Guide on General Literature, an Index and 270 not-only-routine exercises complete this valuable book. It can be recommended to anybody who wants to get a general overview of the spirit and the methods of the Numerical Analysis.

J. Virágh (Szeged)

Micha Hofri, Probabilistic Analysis of Algorithms (Text and Monographs in Computer Science), XV+240 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1987.

To analyse of an algorithm there are two different basic methods. One of them has the objective to find the running time of the algorithm operation in the worst-case in the term of some specified function.

In the other one the operations of algorithms are shown to be associated with probabilistic concepts and processes. In this sense there are two subclasses: On one hand there are explicitly introduced operations in the algorithm and they are choosed on the basis of random elements (pseudorandom numbers, simulated coin flipping etc.). On the other hand we have the operations of a deterministic algorithm and we consider the input data over which some probability measure can be stipulated.

Among the algorithms for which the book provides detailed analyses, the reader finds examples of both varieties. Chapter 1 shows that the second type brings up methodological and conceptual problems that the first case need not entail. Since the probabilistic analysis of algorithms, as a discipline, draws on a fair number of mathematics Chapter 2 is dealing with some of them as introduction to asymptotics, generating functions, integral transforms, combinatorial calculus, asymptotics from generating functions and some selected results from probability theory.

The remining part of the book gives applications. Chapter 3 presents algorithms over permutations (locating the largest term in a permutation, representations of permutations, analysis of sorting algorithms). Chapter 4 contains algorithms for communications network, and Chapter 5 is dealing with bin packing problems.

This is a good book which is recommended to all people who are working in the given fields.

G. Galambos (Szeged)

Irregularities of Partitions, Edited by G. Halász and V. T. Sós (Algorithms and Combinatorics, 8), VIII+168 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1989.

The problem of uniform distribution of sequences has now become an important part of number theory, and this is also true for Ramsey theory in relation to combinatorics. This volume is the homogeneous account of a workshop held at Fertőd in Hungary. Participants discussed the recent emergence of close links between Ramsey theory in combinatorics and the theory of uniform distribution in number theory.

The titles and authors of papers are: J. Beck and W. W. L. Chen: Irregularities of Point Distributions Relative to Convex Polygons; J. Beck and J. Spencer: Balancing Matrices with Line Shifts II; M. Cochand and P. Duchet: A Few Remarks on Orientation of Graphs and Ramsey Theory; P. Erdős, A. Sárközy and V. T. Sós: On a Conjecture of Roth and Some Related Problems I; Ph. Flajolet, P. Kirschenhofer and R. F. Tichy; Discrepancy of Sequences in Discrete Spaces; P. Frankl, R. L. Graham and V. Rödl: On the Distribution of Monochromatic Configurations; A. Gyárfás: Covering Complete Graphs by Monochromatic Paths; H. Lefmann: Canonical Partition Behavior of Cantor Spaces; L. Lovász and K. Vesztergombi: Extremal Problems for Discrepancy; J. H. Loxton: Spectral Studies of Automata; M. Mendes France: A Diophantine Problem; J. Nesetril and P. Pudlák: A Note on Boolean Dimension of Posets; Zs. Tuza: Intersection Properties and Extremal Problems for Set Systems; G. Wagner: On an Imbalance Problem in the Theory of Point Distribution.

Z. Blázsik (Szeged)

W. Klingenberg, Lineare Algebra und Geometrie, zweite verbesserte Auflage, XIII+293 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong, 1990.

This book consists of ten chapters. The material of the first five chapters has an algebraic character. In accordance with the didactical motivation, the text starts with an exposition of the classical algebraic structures: groups, rings and fields. Then moduls and vector spaces, basis systems, dimension of vector spaces and dual spaces are defined. Matrices are first formally defined and then the connection between matrices and linear operators is showed. Solution of linear equation systems and the notion of determinants are also given. At the end of the algebraic part eigenvalues and normal forms of linear operators are discussed and as an application linear differential equation systems are investigated. With the sixth chapter starts the geometric part. First normed vector spaces are introduced. Affin and projectiv spaces are considered over general finite dimensional vector spaces. If the finite dimensional vector space is endowed with an euclidean norm, then the affin space over this one supplies the euclidean, the projectiv space supplies the elliptic geometry. If the vector space is endowed with a Lorenz metric, then the affine space over this supplies the hyperbolic geometry. The main theorem of the affin and projectiv spaces with which the general collineations are characterized is completed with Staudt theorem concerning bijections of a projectiv line.

L. Gehér (Szeged)

R. Kress, Linear Integral Equations (Applied Mathematical Sciences, 82), XI+299 pages. Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong, 1989.

Sometimes the classical theory of integral equations serves as an introduction to the abstract theory of compact operators, on the other hand the theory of integral equations is derived as an important application of the operator theory. Mostly the numerical methods are treated separately.

The aim of the author of this book is to attach the same value on the theory, the application and on the numerical methods. This is a considerable task from scientific and pedagogical aspects as well. Integral equations are useful for engineers, too. Therefore it is desirable that the work should be readable for them. So presenting a modern introduction one cannot begin by saying "you must have solid backgrounds in differential and integral calculus, in differential equations, in complex function theory, in functional analysis, in numerical methods" and so on. (Something like this is often presumed implicitly.) The author of this book relies on bases which — I think — are expectable from trained readers. Some useful and necessary topics are briefly presented.

Roughly speaking the work consists of four main fields: the Riesz—Fredholm theory for integral equations of the second kind; the classical applications (Laplace and heat equation, singular integral equations); introduction to the numerical solution of the equations and finally, ill-posed integral equations of the first kind. These are done in 18 chapters.

The proofs are clear, detailed in a suitable manner. In several cases the considerations are more elementary and straightforward than as customary.

As an example let us quote here the outline of the third chapter. This consists of three points: Riesz theory for compact operators (in my opinion remarks which may be found for example here such as "The main importance of the results of the Riesz theory for compact operators lies in the fact that we can conclude existence from uniqueness as in the case of finite dimensional linear equations." are valuable for the readers; Spectral theory for compact operators (the former results in terms of spectral analysis); Volterra integral equations (the result is formulated in the classical way and also in terms of spectral theory). Finally, as at the end of the other chapters as well we find interesting problems without solutions but with hints in some cases.

I am sure that after reading this book everyone will like integral equations a bit better which was indeed the author's aim.

L. Pintér (Szeged

Y. A. Kubyshin—J. M. Mourão—G. Rudolph—I. P. Volobujev, Dimensional Reduction of Gauge Theories, Spontaneous Compactification and Model Building, (Lecture Notes in Physics, 349), X+80 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong, 1989.

At present there is a general agreement among physicists that the experimental facts of particle physics are correctly reproduced by the so called standard model. The most challenging problem of theoretical particle physics is to produce a theory unifying the Weinberg-Salam-Glashow model of the electroweak force with the quantum chromodynamics describing the strong interaction, and, if possible, describing gravity as well. The dimensional reduction approach to this problem presented in this monograph is a modern version of the ideas put forward by T. Kaluza and O. Klein in the twenties.

In the first part of the book the authors discuss the dimensional reduction of pure Yang-Mills theories. In particular, they present a general method for calculating the scalar potential. The second part is devoted to the dimensional reduction of gravity and to spontaneous compactification. In the final part matter fiels and some aspects of model building are considered.

Throughout the book, the authors make extensive use of homogeneous spaces of Lie groups and connections on fibre bundles. They exhibit the global aspects of the dimensional reduction method and give all the important formulae in local terms as well.

It seems that till now nobody succeded in constructing a model describing the fundamental interactions in a unified scheme, which is satisfactory in all respects. The dimensional reduction approach to constructing such a model deserves further investigation. This book is primarily intended for researchers and graduate students working on this program. It is also recommended to physicists and mathematicians interested in unified field theories and in applications of differential geometry.

László Fehér (Szeged)

Serge Lang, Undergraduate Algebra (Undergraduate Text in Mathematics.), IX+256 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1987.

The book is a second part of an algebra program which is addressed to undergraduates. The theme of Chapter 1 is the set of the real numbers. After some basic properties such important definitions are introduced as the greatest common divisor, the unique prime factorization and the equivalence relations and congruences. The next two Chapters are dealing with the groups and rings with general definitions on mappings, the homomorphisms and automorphisms. Among the groups the permutation groups, the cyclic groups and the finite Abelian groups are studied in details. In the Chapter on rings there are mentioned some basic theorems on their homomorphisms. In Chapter 4 the polynomials are considered. The Euclidean Algorithm, the greatest common divisor, the unique factorization and the partial fractions are introduced. The Chapter is closed by examinations on polynomials over the integers, the principial rings and the factorial rings. Vector spaces and modules are considered in Chapter 5. After some basic definitions (vector space, subspace, generators, basis, homomorphism, kernel) some theorems are presented on the dimension of a vector space. Subsections are dealing with the linear maps, the modules, the factor modules and the free Abelian groups.

For familiar readers have been suggested the next two Chapters which are dealing with some linear groups as the general linear group $(GL_n(K))$. Theorems are introduced for the structure of $GL_2(F)$ and $SL_2(F)$. The Chapter 7 considers the elements of Field Theory: embeddings, splitting fields are mentioned. Basic theorems on the Galois Theory are given. In Chapter 8 the finite fields are considered. Chapter 9 introduces some theorems on the real and complex numbers, and the book is closed with the examinations on the sets. In this section such well-known theorems are considered as the Zorn-Lemma and the Schroeder—Bernstein Theorem.

This book is an elementary text in the Algebra and so a lot of examples are introduced together with the development of the abstractions. The author intended to write a self contained book. The aim has been obtained.

G. Galambos (Szeged)

W. Y. Lick, Difference Equations from Differential Equations (Lecture Notes in Engineering), X+282 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1989.

In mathematical physics and in other branches of the practical problems the first task is to construct a correct model. (Of course every model has some imperfection.) In mechanics one generally obtains a differential equation. The investigation of the obtained equation is twofolded. By using qualitative methods one gets general properties of the solutions and on the other hand we try to present the solution or an approximation of the solution in an explicit form. In general this is done by translation of the differential equation into accurate, stable and physically realistic difference equation and the investigation of this task is the aim of the author.

There are several methods to form difference equations from differential equations. A brief and clear survey of these methods can be found in the Preface. The advantages and disadvantages of every single method are enumerated. In the author's opinion the volume integral method seems to be superior to other methods, therefore this is the primary method used in this book. The application of this single procedure makes the work easier to understand and at the same time it gives more possibility for deriving new and improved difference equations.

The book consists of five chapters. In the first four ones the volume integral method is applied for ordinary and partial differential equations. (Parabolic, hyperbolic and elliptic partial differential equations are treated separately.) Chapter 5 contains the applications of the ideas and algorithms treated formerly for special problems. Let us list them: currents in aquatic systems; the transport of fine-grained sediments in aquatic systems; chemical vapor deposition; free-surface flows around submerged or floating bodies.

The text is clearly written and well-organized. The emphasis on the important role of the basic physical problem is a characteristic feature of the investigations.

In my opinion this book is valuable not only for phisicists, engineers and computer scientists but for mathematicians who are interested in the qualitative theory of differential equations, as well.

L. Pintér (Szeged)

D. Lüst—S. Theisen, Lectures on String Theory, (Lecture Notes in Physics 346), VIII+346 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1989.

In the past few years string theory has been one of the most active areas of theoretical, mathematical physics. Although its relevance for explaining the mysteries of Nature still has not been proven, there can not be any doubt at all that it greatly contributed to the interaction of mathematics and physics. For example, string theory played an important role in the development of conformal field theory, which involves the fascinating mathematics of the Kac-Moody and the Virasoro algebras.

This introduction to string theory is an expanded version of the lectures given by the authors at the Max-Planck-Institut für Physik und Astrophysik in Munich in fall and winter 1987/1988. The authors present a standard introduction to the bosonic and fermionic strings in the critical dimensions, and give a detailed description of the covariant lattice construction of four-dimensional heterotic strings. They give a clear introduction to conformal field theory, including the supersymmetric version, and emphasize its role in constructing four-dimensional strings.

This book will prove useful for graduate students and researchers interested in string theory and is warmly recommended.

László Fehér (Szeged)

Mathematical Logic and Applications, Edited by J. Shinoda, T. A. Slaman and T. Tugué (Lecture Notes in Mathematics, 1388), 222 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong, 1989.

These are the proceedings of the '87th Meeting on Mathematical Logic and its Applications held at the Research Institute of Mathematical Sciences of Kyoto University during August 3—6, 1987. The authors are C. T. Chong, Y. Kakuda, H. Katsutani, S. Kobayashi, M. Shimoda, J. Shinoda and T. A. Slaman, T. A. Slaman and W. H. Woodin, T. Yamakami and M. Yasugi.

Vilmos Totik (Szeged)

Meyberg-Vachenauer, Höhere Mathematik 1. Differential — und Integralrechnung Vektor — and Matrizenrechnung, XIV+517 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong, 1990.

The text is divided into eight chapters. The first chapter is of introductory character. Here the real and complex numbers, vectors, lines and planes are introduced. In the second chapter the limit of number sequence, the limit value and continuity of functions of one variable are defined. The third chapter is devoted to developing the differentiation theory of functions and its applications- At the end of this chapter the exponential and logarithm functions are introduced and discussed. Chapter 4 deals with the integration theory of functions and applies the theory for determining the length of a curve, the area of a rotation surface and the volume of a rotation body. Numerical in. tegration is also shortly discussed. Chapter 5 introduces the concept of convergence of number series and function series. The power series and especially the Taylor series are examined in detail. Chapter 6 is a glimpse into linear algebra, where the usual notions and theorems are given. Chapter 7 investigates functions of several variables, defines the differentiation of such functions and finally develops the differentiation theory of functions with vector values. Chapter 8 treats the theory of integration of functions of two and three variables and the theory of line and surface integration.

The book is recommended to students in the first two semesters.

L. Gehér (Szeged)

Angelo B. Mingarelli—S. Gotskalk Halvorsen, Non-Oscillation Domains of Differential Equations with Two Parameters (Lecture Notes in Mathematics 1338), XI+109 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1988.

Nowadays the literature of the qualitative theory of the linear second order ordinary differential equations fills almost a whole library. To find some new and interesting result is not an easy task. In this book the authors present an important problem, new results and open questions. The starting point is the investigations of R. A. Moore on the equation $y''+(-\alpha+\beta\cdot B(x))y=0$ (1), where α, β are real parameters, the function B is continuous, periodic of period one and has mean value equals to zero. Several well-known equations (Hill, Mathieu etc.) are of this form with various B. Equation (1) will be called disconjugate on R if and only if every nontrivial solution has at most one zero in R. (1) will be called non-oscillatory on R if and only if every nontrivial solution has at most a finite number of zeros in R. The pairs $(\alpha, \beta)\in R^2$ for which (1) is disconjugate resp. non-oscilatory constitute the disconjugacy domain resp. nonoscillation domain of (1) and these sets are denoted by D resp. N. Moore proved that $D=N$ and N is a closed, convex unbounded set. The main problem of this book is the investigation of sets D and N of the equation $y''+(-\alpha\cdot A(x)+$ $+\beta\cdot B(x))y=0$, where x is nonnegative, the functions A, B are Lebesgue integrable on every compact subset of the nonnegative reals. In cases treated in this work D is colsed, convex, bounded or unbounded set, $D\subset N$, N is convex, but N is not always closed. Naturally more interesting questions arise on D and N and their connections. Chapter headings are: Introduction, Scalar linear ordinary differential equations; Linear vector ordinary differential equations; Scalar Volterra—Stieltjes integral equations.

From the style of this work it seems to me that the authors do not take the topic as their own hunting-field and they would not mind if somebody else should solve some of their problems.

L. Pintér (Szeged)

J. D. Murray, Mathematical Biology (Biomathematics, 19), XIV+767 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1989.

The most intensive development of sciences can, to my mind, be waited in the biology. One of the good omens of this is that more and more biological models are constructed and investigated by mathematical methods. This is the way that creates a good possibility of the interaction of mathematical and biological researches and the established involvement would be useful not only for the development of biology, but the mathematics itself should benefit from this connection.

Murray's new book takes an inspiring influence on the involvement of these two sciences. It contains a lot of models from several branches of the biology, for example from the population ecology, reaction kinetics, biological oscillators, the developmental biology, the evolution, the epidemology and so on. The most important biological laws of studied phenomena can be found in it; therefore, the reader will attain a great practice in modelling of biology.

To understand and follow this book, no serious preliminary biological knowledges are needed. The reader has to be familiar only with the basic calculus and differential equations. The authors have involved only deterministic models described by ordinary differential equations, delay equations, integro-differential equations, partial differential equations and their discrete analogies. The used mathematical tools, as such as the catalogue of singularities in the plane, Poincaré-Bendixson theorem, Routh—Hurwitz conditions, Juri conditions, Hopf bifurcation theorem, the properties of Laplacian operators in bounded domains are collected in an appendix that is a great help to the reader.

The book contains simpler and more particular models, too. So, on the one hand this book is an excellent handbook for investigators working in the field of biomathematical modelling. The reason is not only that it provides a good survey on deterministic models of the biology, but its style is suitable for giving inspirations for further researches. On the other hand, the simpler models in the book assure possibility to use it as an introduction for beginer scientific workers in this branch and also in the teaching differential equations.

The book is easy-to-read. The clearness is assured by numerous figures, diagrams. At the same time, the style is deeply interesting since the results obtained by theoretical methods are compared with experimental dates.

The book is recommended to specialists in biomathematics, differential equations, to biologists interested in mathematics, and to graduated students in mathematics and biology.

J. Terjéki (Szeged)

New Integrals, Proceedings. Coleraine 1988. Edited by P. S. Bullen, P. Y. Lee, J. L. Mawlin, P. Muldowney and W. F. Pfeffer (Lecture Notes in Mathematics, 1419), 202 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong, 1990.

In recognition of the pioneering work done by Ralph Henstock in the field of post-Lebesgue integration theory, the 1988 Summer Symposium on Real Analysis was held in Coleraine. The papers in this volume cover current research in generalised Riemann, Denjoy and Perron integration. The 15 papers contained in this volume are written by R. Henstock, P. S. Bullen, T. S. Chew, S. F. L. de Foglio, C. Pierson-Gorez, J. Kurzweil and J. Jarnik, S. Leader, P. Y. Lee, P. Maritz, P. Muldowney, P. Mikusinski and K. Ostaszewski, W. F. Pfeffer, V. A. Skvortsov, J. D. Stegeman.

The wide range ensures that everybody interested in integral theory will find at least one paper of his own interest.

J. Németh (Szeged)

Number Theory and Dynamical Systems (London Mathematical Society Lecture Note Series, 134), Edited by M. M. Dodson and J. A. G. Vickers, 172 pages, Cambridge University Press, Cambridge—New York—Port Chester—Melbourne—Sydney, 1989.

Fifty years ago the title of this book would have been a great surprise. Nowadays number theory appears in various branches of practical applications. Therefore the connection of number theory and dynamical systems is not so astonishing, but at the same time it invariably holds that the combination of various branches produces significant results.

In connection with number theory and dynamical systems let us mention only two facts. One is the Kolmogorov—Arnold—Moser theorem concerning the question of stability of the solar system. The other is Furstenberg's proof of Szemerédi's theorem on arbitrarily long arithmetic progressions in infinite integer sequences. But we could cite several other examples, too.

This book consists of contributions from a Conference on Number Theory and Dynamical Systems held at the University of York in 1987. Perhaps a little characterizing are the addresses of the contributions: H. Rüssmann: Non-degeneracy in the perturbation theory of integrable dynamical systems; J. A. G. Vickers: Infinite dimensional inverse function theorems and small divisors; S. J. Patterson: Metric Diophantine approximation of quadratic forms; Caroline Series: Symbolic dynamics and Diophantine equations; S. G. Dani: On badly approximable numbers, Schmidt games and bounded orbits of flows; S. Raghavan and R. Weissauer: Estimates for Fourier coefficients of cusp forms; K. J. Falconer: The integral geometry of fractals; J. Harrison: Geometry of algebraic continued fractals; Michel Mendes Frances: Chaos implies confusion; J. V. Armitage: The Riemann hypothesis and the Hamiltonian of a quantum mechanical system.

Not only researchers in dynamical systems or in number theory can find interesting ideas in this volume but every curious mathematician, too.

L. Pintér (Szeged)

Numerical Methods for Ordinary Differential Equations, Proceedings of the Workshop held in L'Aquila, 1987. Edited by A. Bellen, C. W. Gear and E. Russo (Lecture Notes in Mathematics, 1368), VII+136 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1989.

This slim volume contains the following 8 invited lectures of the Workshop. C. Baiocchi: Stability in Linear Abstract Differential Equations. — A. Bellen: Parallelism Across the Steps for Difference and Differential Equations. — D. Di Lena, D. Trigiante: On the Spectrum of Families of Matrices with Applications to Stability Problems. — C. W. Gear: DAEs; ODEs with Constraints and Invarianst. — P. J. van der Houwen, B. P. Sommeijer, G. Pontrelli: A Comparative Study of Chebyshev Acceleration and Residue Smoothing in the Solution of Nonlinear Elliptic Difference Equations. — O. Nevanlinna: A Note on Picard—Lindelöf Iteration. — S. P. Norsett, H. H. Simonsen: Aspects of Parallel Runge—Kutta Methods. — L. F. Shampine: Tolerance Proportionality in ODE Codes.

In these research and survey papers the connections between the classical background of numerical initial value ODE methods and new reserearch aras such as differential-algebraic equations, effective stepsize control and parallel ODE solver algorithms for small — and large — scale parallel architectures are investigated.

This volume may be of interest to researchers and graduate students in the ODE field.

J. Virágh (Szeged)

G. Nürnberg, Approximation by Spline Functions, XI+243 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong, 1989.

Splines play an important role in applied mathematics since they possess high flexibility to approximate efficiently, even nonsmooth functions which are given explicitly or only implicitly, e.g. by differential equations.

The aim of this book is to deal with basic theoretical and numerical aspects of interpolation and best approximation by polynomial splines in one variable.

In Chapter I basicly the unique solvability of interpolation problems for Chebyshev spaces is invesitgated, furthermore the construction of interpolating polynomials is given, and the best approximation by functions from Chebyshev spaces in the uniform norm, L_1-norm, L_2-norm is detailed.

Chapter II is devoted to the following main topics: Weak Chebyshev Spaces; B-Splines; Interpolation by Splines (for example Lagrange and Hermite Interpolation by Splines); Best Uniform Approximation by Splines (Algorithms with fixed knots and free knots are detailed); Best L_1-Approximation by Weak Chebyshev spaces; Best One-Sided L_1-Approximation by Weak Chebyshev Spaces and Quadrature Formulas; Approximation of Linear Functionals and Splines. From Appendix the section on Splines in Two Variables should be mentioned. The fact that a large number of new results presented in this book cannot be found in earlier books on spline makes it really valuable one.

This excellent book can be very useful for graduate courses on splines or approximation theory. Only basic knowledge of analysis and linear algebra is supposed.

The book is warmly recommended to everybody interested in approximation theory.

J. Németh (Szeged)

Ortogonal Polynomials, Theory and Practice, Edited by P. Nevai with the assistance of M. E. H. Ismail, (NATO ASI Series C., 294) xi+472 pages, Kluwer Academic Publishers, Dodrecht, 1990.

A NATO Advanced Study Conference was organized by P. Nevai during May 22, 1989 and June 3, 1989 in Columbus, Ohio on "Orthogonal Polynomials and Their Applications". The volume under review contains the proceedings of this conference. Most of the leading researchers of the theory of orthogonal polynomials and related subjects that live east or west of the USSR attended the conference, so its proceedings provide up to date insight of current research, the available methods and applications.

Two main parts can be distinguished in the book: there are papers the primary aim of which is to introduce the readers to applications of orthogonal polynomials, while others are dealing mainly with the extension of the theory. A few papers can be considered to belong to both parts. The theoretical papers can further be classified as those dealing with the algebraic aspects of the theory and the relation of orthogonal polynomials to special functions and combinatorics, while others discuss the analytic properties of orthogonal polynomials.

Among the applications and interrelation with other branches of mathematics are: coding theory and algebraic combinatorics (E. Bannai), Padé approximation and Julia sets (D. Bessis), digital signal processing (P. Delsarte and Y. Genin), functional analysis (J. Dombrowski), numerical analysis (W. Gautschi), Schroedinger equation (R. Haydock), birth and death processes (M. Ismail, J. Letessier, D. M. Masson and G. Valent), Hopf algebras and quantum groups (T. Koornwinder), group representation (D. Stanton).

A sketchy list of the topics dealing mainly with questions of the theory is the following: characterization theorems for orthogonal polynomials (W. A. Salam), three term recurrence relations and spectral properties (T. S. Chihara; W. Van Assche), rational function extensions on the unit circle (M. M. Djrbashian), special functions and symbolic computer algebraic systems (G. Gasper), moment problems and orthogonal polynomials with respect to exponential weights (D. Lubinsky), root systems (I. G. Macdonald), extensions of the beta integral (M. Rahman), orthogonal matrix polynomials (L. Rodman), complex methods (E. B. Saff), potential theory and n-th root asymptotics (H. Stahl and V. Totik).

This excellent book should serve as a standard reference for researchers in the field, but it can also be recommended to students because many of the papers are of introductory nature.

Vilmos Totik (Szeged)

Gilles Pisier, The Volume of Convex Bodies and Banach Space Geometry, Cambridge Tracts in Mathematics, +250 pages, Cambridge University Press, Cambridge—London—New York—Port Chester—Melbourne—Sydney, 1989.

During the last decade, considerable progress was achieved in the Local Theory, i.e. the part of Banach Space Theory which uses finite dimensional tools to study infinite dimensional spaces. One of the leading schools of this subject is the Israel Seminar on Geometric Aspects of Functional Analysis (three Springer Lecture Notes volumes mark their works). The author of the present book is an outstanding researcher of this topic. The aim of this book is to present a self-contained discussion of a number of recent results. A very powerful method is introduced which is a combination of the classical theory of convex sets, probability theory and approximation theory. One of the main ideas is to get quantitative versions of theorems on convex bodies. For example the quantitative version of the famous result of Dvoretzky, due to V. D. Milman, is the following:

Let B be the unit ball of an n-dimensional Banach space. Given $\varepsilon > 0$, there exists a subspace F with dimension $[\varphi(\varepsilon) \log n]$ ($\varphi(\varepsilon) > 0$ depending only on ε) and an ellipsoid $D \subset F$ such that

$$D \subset B \cap F \subset (1+\varepsilon) D.$$

The book is divided into two parts. The object of the first part (Chapters 1 to 9) is to give detait led proofs of three fundamental results:

(I) The quotient of subspace Theorem due to Milman: For each $0 < \delta < 1$ there is a constant $C = C(\delta)$ such that every n-dimensional normed space admits a quotient of a subpace $F = E_1/E_2$ (with $E_2 \subset E_1 \subset E)$ with dimension dim $F \geqq \delta n$ which is C-isomorphic to a Euclidean space.

(II) The inverse Santalo inequality due to Bourgain and Milman: There are positive constants α and β (independent of n) such that for all balls $B \subset \mathbf{R}^n$ we have

$$\alpha/n \leqq (\mathrm{vol}\,(B)\,\mathrm{vol}\,(B^0))^{1/n} \leqq \beta/n.$$

(The upper bound goes back to a 1949 article by Santalo.)

(III) The inverse Brunn—Minkowski inequality due to Milman: Two balls B_1, B_2 in $\mathbf{R}^n$ can always be transformed (by a volume preserving linear isomorphism) into balls $\tilde{B}_1$, $\tilde{B}_2$ which satisfy

$$\mathrm{vol}\,(\tilde{B}_1 + \tilde{B}_2)^{1/n} \leqq C[\mathrm{vol}\,(\tilde{B}_1)^{1/n} + \mathrm{vol}\,(\tilde{B}_2)^{1/n}]$$

where C is a numerical constant independent of n. Moreover, the polars $\tilde{B}_1^0$, $\tilde{B}_2^0$ and all their multiples also satisfy a similar inequalty.

The second part (Chapters 10 to 15) is devoted to the discussion of recently introduced classes of Banach spaces of weak cotype 2 and weak type 2 and the intersection of these classes, the weak Hilbert spaces.

The book is recommended to researchers in functional analysis but it may be useful to convex geometers, too.

J. Kincses (Szeged)

Philip Protter, Stochastic Integration and Differential Equation. A New Approach (Application of Mathematics, 21), X+302 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong, 1990.

The novelty of this introductory book is that the author defines a semimartingale as a stochastic process wich is a "good integrator" on an elementary class of processes, rather than as a process of general Walsh series is equivalent to the study that can be written as the sum of a local martingale and a finite variation process.

At first an intuitive Riemann-type definition of the stochastic integral as the limit of sums is given for the adapted processes having left continuous paths with right limits. This is sufficient to prove many theorems including Itô's formula. Then it is shown that the "good integrator" definition of a semimartingale is equivalent to the usual one and a general theory of semimartingales are developed. Finally, the author extends the stochastic integral by continuity to predictable integrands, making the stochastic integral a Lebesgue-type integral. These integrands give rise to a presentation of the theory of semimartingale local times. The book is concluded by an introduction to stochastic differential equations and to the theory of flows (existence and uniqueness of solutions, stability, Markov nature of solutions).

The book allows a rapid introduction to some of the deepest theorems of the subject. It is highly recommended both to instructors and students in probability and statistics.

L. Hatvani (Szeged)

q-**Series and Partitions,** Edited by D. Stanton (The IMA Volumes in mathematics and its applications, 18), X+212 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong, 1989.

This is the Proceedings of the Workshop on q-Series and Partitions held at the Institute for Mathematics and its Applications, Minnesota, USA on March 7—11, 1988. It contains up to date research papers on q-series, unimodality, q-special functions and q-orthogonal polynomials.

What is a q-series? In the theory of partitions it is customary to write q as the argument in the generating functions, but many "ordinary" objects in mathematics have their q-analogues. For example the q-analogue of the binmial coefficient $\binom{n}{k}$ is

$$\frac{(1-q)(1-q^2)\dots(1-q^n)}{(1-q)(1-q^2)\dots(1-q^k)(1-q)(1-q^2)\dots(1-q^{n-k})}$$

(note that for $q\to 1-0$ we get back the original definition of the binomial coefficient). G. Gasper's paper in the proceedings under review discusses many such q-analogues.

Identities in terms of q-series often have interpretation in terms of partitions. Perhaps one of the most famous q-identities are the two Rogers—Ramanujan identities the first of which reads as

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\dots(1-q^n)}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}.$$

What does this have to do with partitions? If we look at the coefficiens of q^n on both sides then this identity has the interpretation that the partitions of n into parts which differ by at least 2 are equinumerous with the partitions of n into parts congruent ± 1 modulo 5 (try to verify this "translation"; there is twist in the proof!). G. Andrews's paper discusses diffeent proofs of the Rogers—Ramanujan identities. The paper by D. Zeilberger attempts to classify identities with regard to computer time required for their verification using computer algebra. Computers and symbolic computations appear in other papers in the proceedings, as well.

The papers by D. M. Bressoud, F. M. Goodman and K. M. O'Hara, D. Zeilberger and I. G. Macdonald are related with the recent combinatorial proof of K. M. O'Hara for the unimodality of the Gaussian polynomials, which asserts that the coefficients in the polynomials

$$\frac{(1-q^{n+1})(1-q^{n+2})\dots(1-q^{n+k})}{(1-q)\dots(1-q^k)}$$

are increasing up to a point and decreasing after that. Earlier proofs used very advanced techniques and even K. M. O'Hara's proof was rather involved. Using her ideas a relatively simple elementary proof can be found in Zeilberger's and Macdonald's papers.

The papers by F. G. Garvan, L. Habsinger and D. St. P. Richards discuss integrals in several variables and their q-analogues that are related to Selberg's integral

$$\int\limits_{[0,1]^n} \prod_{i=1}^{n} x_i^{a-1}(1-x_i)^{b-1}|D(x)|^{2c}\,dx = \prod_{i=1}^{n} \frac{\Gamma(a+(n-i)c)\Gamma(b+(n-i)c)\Gamma(ic+1)}{\Gamma(a+b+(2n-i-1)c)\Gamma(c+1)},$$

where $x=(x_1, \ldots, x_n)$ and $D(x)=\prod\limits_{i<j}(x_j-x_i)$ is the Vandermonde determinant.

D. Stanton writes on an elementary approach to the Macdonald identities which expand products of the form

$$\prod_{\substack{a>0 \\ a\in S}} (1-e^a)$$

as certain sums.

The volume ends with four papers by R. Askey, I. M. Gessel, M. H. Ismail and W. Miller, about orthogonal polynomials, their zeros and recurrence coefficients and their q-analogues (such as q-Hermite polynomials).

The book under review is an excellent source for a flourishing and very exciting area and can be recommended both to researchers and to advanced students in analysis, combinatorics and number theory.

Vilmos Totik (Szeged)

Rewriting Techniques and Applications, Proceedings, Chapel Hill 1989. Edited by Nachum Dershowitz (Lecture Notes in Computer Science 355), VII+579 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1989.

This volume contains the proceedings of the Third International Conference on Rewriting Techniques and Applications (RTA—89). The conference was held April 3—5, 1989, in Chapel Hill, North Caroline, U.S.A.

This book contains 34 papers in the following areas: Term rewriting systems, Conditional rewriting, Graph rewriting and grammars, Algebraic semantics, Equational reasoning, Lambda and combinatory calculi, Symbolic and algebraic computation, Equational programming languages, Completion procedures, Rewrite-based theorem proving, Unification and matching algorithms Term-based architectures.

Also included in this volume are short description of a dozen of the implemented equational reasoning systems demonstrated at the meeting.

This book is recommended to everybody working in the theory of Rewrite Systems.

Sándor Vágvölgyi (Szeged)

F. Schipp—W. R. Wade—P. Simon (with the assistance from **J. Pál**), **Walsh series, an introduction to dynamic harmonic analysis,** X+560 pages, Akadémiai Kiadó, Budapest, 1990.

The Walsh system is the simplest nontrivial model for harmonic analysis but shares many properties with the trigonometric system. It has been used to solve some fundamental problems in analysis, e.g., the basis problem. It has played a role in the development of other areas of mathematics, e.g., the fundamental theorem of martingales was proved first by Paley for the Walsh system.

The Walsh functions can be applied in many situations, among others, in data transmission, image enhancement, pattern recognition, etc. Since the Walsh functions take on only the values $+1$

and -1, they are easy to implement on high speed computers and can be used with very little storage space.

This is the first systematic and detailed exposition of the subject, from the foundations up to the most recent results, including many which were not previously published. The book can serve both as an excellent reference book and as a textbook. The reader is merely assumed to be familiar with the notion of and basic theorems on Lebesgue integration. Except for this material, concepts are developed as needed and the book is nearly self-contained. In particular, it is accessible to beginning graduate students and doctoral candidates in various specialities in mathematics and engineering.

The abundance and variety of the material presented in the book makes an exhausting description in a short review quite impossible. Thus, we can only comment on the general plan of the book, and mention some samples of the most characteristic results contained.

Chapter 1 contains a systematic account of the dyadic group, the definition of the Walsh functions in various enumerations, such as they were introduced by Walsh in 1923, by Paley in 1932, and by Kaczmarz in 1948. Separate sections are devoted to the transformations and rearrangements of the Walsh system, showing, in particular, that the Haar and Walsh systems are Hadamard transforms of each other; to Walsh—Fourier series, the Walsh—Dirichlet kernel, Walsh—Fejér kernel, dyadic derivative, and Cesàro summability.

The first half of Chapter 2 presents results which estimate the growth order of Walsh-Fourier coefficients for various classes of functions, e.g., L^p functions, continuous functions, etc., while the second half identifies conditions sufficient for pointwise convergence and absolute convergence of Walsh—Fourier series.

The Walsh functions provide a vehicle to link harmonic analysis and probability theory. The basic tool in this interrelation is the dyadic martingales which play an important role in the development of new spaces such as the dyadic Hardy spaces and dyadic BMO (=bounded mean oscillation). These results are dealt with in Chapter 3. Dyadic Hardy spaces are characterized in two ways: by means of martingale maximal function and of the atomic decomposition. It then proceeds to give an account of duality relations. Among others, H_0' (the dual of H_0, i.e., the collection of bounded linear functionals of H_0) is isometric and homeomorphic to BMO, and VMO$'$ (=vanishing mean oscillation) is isometric and homeomorphic to H_0, whereas the proofs are heavily relied on the dyadic version of the famous Fefferman inequality. The chapter ends with the study of martingale trees, i.e., martingales indexed by the tree-like collection of dyadic intervals. By introducing these nonlinear martingales and generalizing the Burkholder—Gundy theory of martingale transforms, the reader sees that the inequalities of Khintchin, Paley, and Sjölin as well as a.e. convergence of Walsh-Fourier series are all parts of a general theory of nonlinear martingale transforms.

Chapter 4 is devoted to study of convergence in L^p-norm, $p\geqq 1$, and uniform convergence of Walsh—Fourier series. The treatment of summability of Walsh—Fourier series in homogeneous Banach spaces and of sets of divergence is a certain adaptation of the corresponding technique developed by Kahane and Katznelson. Likewise, the adjustment of an integrable function f on a set of small measure in order to obtain a new function whose Walsh—Fourier series converges uniformly is modelled after Menshov's celebrated one for trigonometric series.

The first part of Chapter 5 touches the problem of approximation by Walsh polynomials. In great lines, it follows the trigonometric analogue. The major part of Chapter 5, however, presents the Haar, Walsh, Faber— Schauder, Franklin, and Ciesielski systems as bases and identifies for each of them the subspace of L^1 in which the given system is a basis. By indexing the Haar and Franklin systems in a natural way to make them nonlinear sequences, the authors find that the corresponding canonical isomorphisms induce explicit isomorphisms from the dyadic Hardy spaces and dyadic BMO to their classical trigonometric counterparts. This approach gives a natural way to get classical results from dyadic ones and vice versa. Then the authors show that the Haar and Franklin systems are equivalent bases in L^p for $1<p<\infty$. On the other hand, it turns out that the trigonometric and Walsh—Paley system are not equivalent bases in L^p for $1<p<\infty$, except for $p=2$. Finally, they also answer the long-standing problem of Banach by constructing a separable Banach space, similar in spirit to the dyadic Hardy space, which fails to have a basis. However, their decisive step in the construction is due to Enflo.

In Chapter 6 the authors collect several sufficient conditions ensuring the a.e. convergence of a Walsh-Fourier series. Using the notion of the so-called logarithm spaces, the sharpest result is due to Sjölin which says: If $f \in L\log^+ L\log^+ \log^+ L$, then the Walsh—Fourier series of f converges a.e. Then they prove the Walsh analogue of the famous Kolmogorov example of divergent Fourier series. On the other hand, the Walsh—Fourier series of an integrable function is Cesàro summable a.e. This is proved by exploiting the intimate connection between summability and pointwise dyadic derivative.

A fundamental problem in the theory of general Walsh series is the problem of uniqueness. To go into details, a set E is called a U-set (set of uniqueness) if every Walsh series converging to 0 outside E vanishes identically. Otherwise, E is called an M-set (set of multiplicity). It follows that every countable set is a U-set, while every set of positive measure is an M-set. Thus, it remains a delicate problem, not yet solved, to distinguish among sets of measure zero not normally made in Lebesgue analysis. The up-to-date approach of Chapter 7 is based on the observation that the study of general Walsh series is equivalent to the study of Walsh—Fourier—Stieltjes series of quasimeasures (i.e., finitely additive, real-valued set functions) defined on the dyadic intervals. This allows certain problems to be recast as measure theoretic questions. In some cases this perception provides simple explanations of known results, while in other cases it gives new insight into the nature of the problem itself. For example, the fact that no Walsh series can diverge to $+\infty$ on a set of positive measure is a reflection of the fact that a quasimeasure is either a.e. differentiable or has upper derivative $+\infty$ and lower derivative $-\infty$ a.e.

Chapter 8 is dedicated to the problem of representing measurable functions by Walsh series. This is connected with the term by term dyadic differentiation and the behavior of Walsh series with monotone coefficients, where a Sidon type inequality proved jointly by Schipp and the reviewer plays a crucial role. Then the representation problem is considered here in the more general framework of normalized convergence systems studied mainly by Talaljan.

Chapter 9 treats the questions of the Walsh—Fourier transform, which is the counterpart of the classical (trigonometric) Fourier transform. The fast Walsh transform seems to be more appropriate to implement on a computer than the fast Fourier transform. The inverse dyadic derivative plays a central role in the treatment. The various applications of the Walsh functions are only outlined, since several books have been written about them. For further reading we suggest the books by H. F. Harmuth, K. G. Beauchamp, C. A. Bass, M. Maqusi, etc.

Each chapter ends with Exercises ranging from fairly routine applications of the text material to those that extend the coverage of the book. For the reader's convenience there are seven Appendices containing a number of auxiliary topics at the end of the book. Historical Notes to each chapter separately, References to about 450 papers or books, Author, Subject, and Notational Index complete the book.

The book is carefully and accurately written. The presentation is concise but always clear and well-readable.

Finally, may the reviewer venture to express his particular desire to take some time a second volume in his hands comprising the latest research done in the field of multiple Walsh series as well as providing a rigorous mathematical treatment of the concrete questions occurring in the vast and diverse field of pratical applications. The reviewer's hope is that all this enormously arge material in the authors' unified presentation would prove to be more accessible to anyone interested in dyadic harmonic analysis. Of course, this desire does not affect the value of this almost perfect work at all.

To sum up, this book fills in a gap in the literature. It provides in a polished form a rich and up-to-date material of a fast-growing field whose significance is becoming basic for practice. It is perhaps not exaggerated to assert that this book is of fundamental importance for everybody who wants to keep pace with modern developments in the Dyadic and Classical Analysis.

Ferenc Móricz (Szeged)

C. L. Siegel, Lectures on the Geometry of Numbers, X+160 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1989.

This is the printed version of Siegel's lectures at New York University during 1945—46. The original notes by B. Friedman were rewritten by K. Chandrasekharan with the assistance of R. Suter.

"Geometry of Numbers" is a subject dealing mainly with lattices and their points in prescribed sets in R^n. Its fundamental theorem is Minkowski's First Theorem: A convex body in R^n, having a centre at the origin and having a volume larger than 2^n, must contain at least one point other than the origin with integer coordinates. The first chapter of the book is devoted to this theorem and its generalization involving the so called successive minima of even gauge functions.

The second chapter starts out with the discussion of vector groups which are nothing else than the subgroups of the additive group of R^n. Discerte vector groups correspond to lattices and so to matrices. Such concepts as basis, ranks, characters duals etc. are treated in detail. As an application of the duality theorem Kronecker's approximation theorem is proved together with one-of its generalization. Further applications are given concerning periods of real and complex functions, parquets formed by parallelepipeds. The rest of the chapter deals with the minimum of products of linear forms and of positive definite quadratic forms on lattice points different from the origin. The exact minimum value in R^2 is determined and it yields a proof of Hurwitz' theorem according to which to every irrational a there are infinitely many pairs (p, q) of integers with

$$\left|a-\frac{p}{q}\right| \leqq \frac{1}{\sqrt{5}q^2}.$$

A lattice is a geometric object but for its analytic description we use matrices. Several matrices correspond to the same lattice and these are connected by unimodular transformations. In other words, several lattices have the same set of points as a geometrical entity and it would be advantageous to single out one lattice from the class of all lattices which are equivalent under a unimodular transformation. The problem of finding such a representative for every class of equivalent lattices is called the problem of reductions, and Chapter III is devoted to the theory of reduction. Some applications, as for instance closest packing in two, three and four dimensions are also covered.

The lectures on the geometry of numbers provide an excellent source of learning for undergraduate and graduate students and the book can serve as a basis for a course in the field. Some of the lectures contain more material than what is appropriate for a single lecture, so the active participation of the students seems to be absolutely necessary if one would like to keep up with the pace suggested by the table of contents. The only criticism I make is that some of the proofs are unnecessarily detailed and some parts of Chapter III may seem to be more specialized and less exciting for an average reader than the material in the first two chapters.

Vilmos Totik (Szeged)

R. Silhol, Real Algebraic Surfaces (Lecture Notes in Mathematics, 1392), X+215 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong, 1989.

If a book is written by a specialist then it is in a great danger of becoming too dry or too special for a general reader. Fortunately this book has avoided these trips inspite of the fact that its subject is far away from the basics of mathematics.

The basic idea in this book is to consider the real algebraic surfaces and to regard them as complex algebraic varieties with an antiholomorphic involution. From this point of view there are two classes of the real algebraic surfaces as the Galois group Gal $(\mathbf{C}|\mathbf{R})$ on $H^*(X(\mathbf{C}, \mathbf{Z}))$ determines or only estimates the dimension of $H^*(X(\mathbf{R}), \mathbf{Z}/2)$. The previous type of the surfaces, such as rational surfaces and Abelian surfaces etc., are under a detailed analysis in this book. The main result is the complete classification of these surfaces.

We have to mention two great advantages of the book finally. First of all the two introductory chapters are extremely useful because they make really possible to read the book for non-specialists and graduate students in algebraic geometry. Also the examples throughout the book are useful to understand better the new notions.

Árpád Kurusa (Szeged)

James K. Stayer, Linear Programming and Its Applications (Undergraduate Texts in Mathematics), XII+265 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo—Hong Kong, 1989.

This book is devoted to serve as an introductory text in linear programming. It is divided into two main parts.

The first part consisting of four chapters deals with methods for solving general linear programming problems. Chapter 1 exhibits the usual geometrical representation which is a good preparation for the later texts. The canonical forms are considered in Chapter 2 and the classical Dantzig's simplex algorithm is given as a solving method. The general problem of linear programming is treated in Chapter 3, and different methods are presented to solve it. Finally, Chapter 4 discusses the theory of duality showing the connection between the problems of maximization and minimization.

The second part presents several applications related to linear programming. Firstly, Chapter 5 deals with the two-person zero-sum matrix games. Such traditional applications of linear programming as transportation and assignment problems are treated in Chapter 6, and as solving algorithms the stepping stone method and the Hungarian method are given, respectively. Finally, Chapter 7 deals with networks. Algorithms are presented to solve the network-flow problem, the shortest-path network problem and the minimal-cost-flow network problem.

The book is well-written. It contains a rich collection of examples and exercises. Every algorithm is illustrated in a step-by-step manner. It can be recommended as an excellent text for an introductory course in linear programming.

B. Imreh (Szeged)

John Stillwell, Mathematics and Its History (Undergraduate Texts in Mathematics), X+371 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1989.

At almost all universities mathematics students are those who never get a course in mathematics. They get several separate courses such as calculus, algebra, geometry, topology and so on, and our usual teaching method seems to prevent these different topics from being combined in to a whole. Therefore there are several important questions which are not discussed in the proper place, e.g., the fundamental theorem of algebra, because that is analysis. Thus, if students are to feel they really know mathematics by the time they graduate, there is a need to unify the subject. This feeling is very important for future teachers of mathematics. A course on the history of mathematics does not take this job.

This book has grown from a course given to senior undergraduates at Monash University. The selection of the material has been a success. It covers almost all topics of primary importance. The emphasis is on history as a method for unifying and motivating mathematics. The twenty chapters are the following: The Theorem of Pythagoras, Greek Geometry, Greek Number Theory, Infinity in Greek Mathematics, Polynomial Equations, Analitic Geometry, Projective Geometry, Calculus, Infinite Series, The Revival of Number Theory, Elliptic Functions, Mechanics, Complex Numbers in Algebra, Complex Numbers and Curves, Complex Numbers and Functions, Differential Geometry, Noneuclidean Geometry, Group Theory, Topology, Sets, Logic, and Computation.

This is not a book on the history of mathematics, therefore it uses modern notations. This is debatable only in the first glance. For those readers, who want to read the original texts, there is a long reference at the end of the volume.

In each chapter we can find biographical notes and well selected exercises.

We warmly recommend this gap-filling book to any undergraduate course of mathematics, especially to teachers of mathematics.

Lajos Kulkovits (Szeged)

Josef Stoer, Numerishe Mathematik I (Springer Lehrbuch), XII+314 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong, 1989.

A brief comparison to the previous English translation (Stoer—Bulirsch, Introduction to Numerical Analysis, Springer-Verlag, 1980) reveals two major changes beside a few technical updates.

Chapter 2 on interpolation has a comprehensive supplement on the formal properties and a simple recurrence relation of B-splines.

The second addition in Chapter 4 shows the pecularities of sparse matrix techniques. Efficient pivoting and storage schemes are demonstrated for the sparse Cholesky factorisation algorithm.

The standards of this book stand comparison with most new textbooks and, in the reviewer's opinion, this latest edition will not be the last one.

J. Virágh (Szeged)

J.-O. Strömberg—A. Torchinsky, Weighted Hardy Spaces, (Lecture Notes in Mathematics 1381), IV+193 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong, 1989.

The development of harmonic analysis in the last few years has been centered around spaces of functions of bounded mean oscillation and the weighted inequalities for classical operators. The main goal of this book is to further develop some results in this topic in the general setting of the weighted Hardy spaces and to discuss some applications. The authors derive mean value inequalities for wavelet transforms and introduce halfspace techniques with, for example, nontangential maximal functions and g-functions. This leads to several equivalent definitions of the weighted Hardy spaces. Fourier multipliers and singular integral operators are applied to the weighted Hardy spaces and complex interpolation is considered.

Rich bibliography helps the reader in going back to the origin of the research of this topic. Apparently the book covers the whole spectrum of papers dealing with these very important spaces.

The book is highly recommended to research workers interested in the modern harmonic analysis.

J. Németh (Szeged)

J. L. Balcazar—J. Diaz—J. Gabarro, Structural Complexity I, (EATCS Monographs on Theoretical Computer Science, 11), IX+191 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1988.

There are many different yet related studies in the complexity of algorithms. The subject of structural complexity takes an abstract view of the complexity of computation by looking for inherent mathematical structures inside the problem classes. Many concepts of structural complexity originate in recursion theory, as is clearly demonstrated in the present book.

The first two chapters are included for the sake of completeness and to broaden the accessibility of the volume. Chapter 1 provides a brief exposition of models of computation, such as finite automata, and several versions of Turing machines. Some elementary properties of languages that represent decision problems and classes of languages are discussed. The main purpose is to explain the basic notions and to present a formalism for the remainder of the book. Enough references are provided for those who want a deeper background on the material covered in this chapter.

Chapter 2 starts with a survey of the rate of growth of functions and is followed by a discussion of the running time and work space of Turing machines. Some basic results are presented, e.g. the linear speed-up theorem and the tape compression theorem.

Then, after a thorough treatment of time and space constructible functions, complexity classes are defined in a general setting. This chapter ends with some simulation results, such as Savitch's theorem.

Central complexity classes form the subject matter of Chapter 3. Polynomial time (many-one) reducibility and logarithmic space reducibility are defined and related concepts (completeness, hardness, etc.) are discussed. Some well-known NP-complete problems are presented and QBF is shown to be PSPACE-complete. A separate section is devoted to padding arguments, which provide a useful tool for establishing inequalities between complexity classes.

Other types of reducibilities, namely polynomial time Turing reducibility and SN-reducibility are studied in Chapter 4, giving rise to relativizations of complexity classes. SN-reducibility is then related to self-reducible sets.

Finite sets can be accepted by deterministic finite automata in constant time with no work space whatsoever. The "intrinsically algorithmic approach" taken in preceding chapters thus fails when dealing with finite sets. The "uniform" approach of Chapter 5 measures the sizes of the algorithms accepting finite sets and associates with an infinite set the growth of the sizes of the algorithms that accept initial segments of the set. The unifying concept of "advice" functions is then used to relate the two approaches. Boolean complexity fits nicely in this framework.

The average case behavior of algorithms has become a topic of increasing interest in recent years. By using pseudo-random number generators, it is possible to design algorithms that solve problems with a reasonable rate of probability. Accordingly, Chapter 6 provides a glimpse of probabilistic algorithms. A basic theory of probabilistic complexity classes is developed.

A number of studies in complexity theory depend on the assumption that $P \neq NP$. Uniform diagonalization provides a powerful technique to prove e.g. that there are incomplete problems in $NP-P$, assuming that $P \neq NP$. Uniform diagonalization and its applications are discussed in

Chapter 7. The last chapter deals with the polynomial time hierarchy, the polynomial analogue of Kleene's hierarchy.

The book is well-written, the presentation of the material is sufficiently clear. The necessary prerequisites are a basic knowledge of automata and formal languages. Some acquaintance with recursion theory might be helpful. Each chapter ends with detailed bibliographical remarks and a number of exercises. The book can very well serve as a text for a graduate course in structural complexity.

Z. Ésik (Szeged)

Aimo Törn—Antanas Žilinskas, Global Optimization (Lecture Notes in Computer Science, 350), X+255 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1989.

Global optimization is a part of nonlinear programming: it is aimed at solving nonlinear optimization problems with many local minima. This problem is in general unsolvable, if the algorithm is based only on the evaluation of the objective function and its derivatives. Although such problems are rather frequent in practice, the traditional approach of the users is to accept the first local minimum found as an estimate of the global minimum.

The book by Törn and Žilinskas was the first to cover the broad field of global optimization. Since its publishing, some other volumes have been available, dealing mainly with different subproblems of global optimization (such as deterministic and stochastic methods).

After the definition and characterization of the global optimization problem, the book discusses the covering, the clustering and the random search methods, the method of generalized descent and the algorithms based on statistical models of the objective function. Testing is a crucial part of the evaluation of global optimization methods, since their reliability has to be measured somehow. The book devotes a section to questions arising in testing and applications. The test results are collected very carefully, thus the reader looking for a suitable method can rely on the tables given by the authors.

It must be mentioned that spelling errors make the text somewhat difficult to read. An extensive bibliography of more than 400 references completes the book.

The volume can be warmly recommended (beyond experts of the field) to everyone who must solve nonlinear optimization problems that can be multiextremal.

T. Csendes (Szeged)

L. Trave—A. Titli—A. Tarras, Large Scala Systems: Decentralization, Structure Constrainst and Fixed Modes (Lecture Notes in Control and Information Sciences, 120), XIV+384 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1989.

The models of the present day technological, environmental and societal processes are of high dimensions and complexity, which makes impossible to use the classical mathematical tools developed for system analysis and control. This book gives an excellent survey on the new techniques for the large scale systems characterized by a huge number of input and output variables on subsystems which are generally geographically distributed.

Chapter 1 presents an overview of the well-known results around the problem of stabilization and pole assignment of linear time — invariant dynamic systems subjected to centralized control. Chapter 2 deals with these problems when a specified restricted information pattern is required, which constraints the feedback control structure. Chapter 3 gives the different existing characterizations of fixed modes, namely characterizations in term of transmission zeros of subsystems, characterization in time-domain and in the frequency-domain, and graph-theoretic characterizations. In Chapter 4 it is shown that systems with unstable non structurally fixed modes can be stabilized by using time varying or non-linear feedback control laws which preserves the feedback structure constraints. Chapter 5 presents the different available methods for the design of an appropriate feedback control structure. Chapter 6 considers the problem of the synthesis of feedback gains under structural constraints. Chapter 7 is devoted to the problem of structural robustness.

The results are illustrated by significative examples which make easier their understanding. Some important algorithms are presented in a collection of program packages.

This book will be very useful both as a text and as a monograph in the control of large scale systems.

L. Hatvani (Szeged)

Ferdinand Verhulst, Nonlinear Differential Equations and Dynamical Systems, IX+277 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong, 1990.

Recently the theory and applications of nonlinear differential equations and dynamical systems have strongly attracted the attention of mathematicians and users of mathematics. The reason is that a lot of phenomena in the sciences and economy can be explained by modelling the processes by nonlinear differential equations and applying the new results of the nonlinear dynamics to these models. It is not easy to get acquainted with these results demanding deep mathematical prerequisites. This introductory text bridges the gap between elementary courses in ordinary differential equations and the modern research literature in the field of nonlinear dynamics.

The first part of the book — after giving the basic definitions — deals with the periodic phenomena. The reader can find here a very plastic proof for the Poincaré—Bendixson theorem on the existence of periodic solutions. The second part is devoted to the stability theory. The third part gives an overview on the methods for systems containing a small parameters (perturbation theory, Poincaré—Lindstedt method, averaging). In the last four chapters, which give the most interesting part of the book, more advanced topics like relaxation oscillations, bifurcation theory, chaos in mappings and differential equations, Hamiltonian systems are introduced.

The book is well-written and well-organized. Only the most important proofs are included; the results are illustrated by interesting and important examples from the real world. The chapters are concluded by exercises (at the end of book the reader gets answers and hints to them). After studying this book and solving the exercises the reader will be able to start working on open research problems.

This excellent textbook can be warmly recommended both to beginners and specialists interested in the modern theory of nonlinear differential equations and its applications.

L. Hatvani (Szeged)

Wolfgang Walter, Aanalysis I, zweite Anflage (Grundwissen Mathematic, 3) VIII+385 pages, Springer-Verlag, Berlin—Heidelberg—New York—Paris—Tokyo—Hong Kong, 1990. **Analysis II,** (Grundwissen Mathematic 4) VII+396 pages, Springer-Verlag, Berlin—Heidelberg—New York—Paris—Tokyo—Hong Kong, 1990.

The first volume consists of three parts. The first part summarizes the basic knowledges about real numbers, mathematical induction and polynomials. The second part introduces the concept of convergence of sequences and series of real numbers, defines the limit and continuity of functions, discusses power series and elementary transcendent functions. At the end of this part the complex numbers and functions are introduced. The third part is devoted to Riemannian integral and differentiation of functions. A lot of applications can also be found here. This part ends with complementary notes.

The second volume is divided into 10 paragraphs. The text starts with the introduction of metric spaces, basic topological concepts and continuity of functions in metric spaces. Then the differentiation theory of functions of several variables, the problem of implicite functions and extremal values of functions are developed. The general Moore—Smith convergence is introduced and the Riemannian integral as Moore—Smith limit is showed. The length and differentialgeometric concepts of curves are discussed, the equations of motions are developed and the classical two bodies problem is solved. A paragraph (the sixth) is devoted to Riemann—Stieltjes integral and line integrals. Two paragraphs deal with the Jordan mass, the Riemannian integral in n dimension and the Gauss, Green and Stokes integral theorems. The last two paragraphs introduce the Lebesgue integral, the Fourier series and develop the Hilbert-space theory of Fourier series.

The books are highly recommended to students in the first four semesters.

L. Gehér (Szeged)

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