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A JÓZSEF ATTILA TUDOMÁNYEGYETEM KÖZLEMÉNYEI

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JÓZSEF ATTILA TUDOMÁNYEGYETEM BOLYAI INTÉZETE

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Endliche Loops und ihre Unterloopverbände

HUBERTA LAUSCH

Die vorliegende Arbeit setzt die bereits in [8] begonnene Untersuchung endlicher Loops und ihrer Unterloopverbände fort. Auch hier zeigt sich, daß ohne zusätzliche Forderungen an die Struktur der Loops, wie z. B. Potenz- oder Diassoziativität, zentrale Nilpotenz etc., kaum Aussagen darüber möglich sind, welche Auswirkungen die Struktur des Unterloopverbandes auf die Struktur der Loop hat und umgekehrt.

Der erste Abschnitt behandelt endliche Loops mit modularen Unterloopverbänden und gibt eine vollständige Beschreibung von endlichen Loops mit booleschen Unterloopverbänden. Wesentlichstes Ergebnis des zweiten Abschnitts ist, daß endliche Loops mit schwach booleschen Unterloopverbänden von (höchstens) zwei Elementen erzeugt werden können (Satz 2.1). Abschnitt 3 wendet sich den endlichen zentral nilpotenten Loops zu. Beispielsweise erfüllt der Unterloopverband einer endlichen zentral nilpotenten Loop stets die Jordan—Dedekind-Kettenbedingung (Satz 3.1).

Der abschließende Abschnitt 4 ist den endlichen kommutativen Moufang-Loops gewidmet. In Satz 4.1 wird das Resümee aus den bisherigen Ergebnissen über endliche kommutative Moufang-Loops und ihre Unterloopverbände gezogen. Sodann ergibt sich für endliche kommutative Moufang-Loops mit modularen Unterloopverbänden eine interessante Analogie zu den endlichen nilpotenten Gruppen: Eine endliche kommutative Moufang-Loop besitzt genau dann einen modularen Unterloopverband, wenn alle ihre Unterloops quasinormal sind (Korollar 4.4). Ferner erhält man für endliche kommutative Moufang-Loops mit modularen Unterloopverbänden die schöne Strukturausage, daß die Faktorloop $G/Z(G)$ einer derartigen Loop G nach ihrem assoziativen Zentrum $Z(G)$ eine elementar-abelsche 3-Gruppe ist (Satz 4.7). Endliche kommutative Moufang-Loops mit modularen Unterloopverbänden sind also insbesondere nilpotent der Klasse zwei.

Die Bezeichnungen sind im wesentlichen wie in [3] bzw. [5] gewählt; mit $L(G)$ bezeichnen wir den Unterloopverband einer Loop G.

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1. Loops mit modularen und booleschen Unterloopverbänden

Für endliche Loops mit modularen Unterloopverbänden kann man keine wirklich schönen Strukturaussagen erwarten, wie bereits die Diskussion spezieller modularer Verbände (projektiver Geometrien) in [8] gezeigt hat. Ganz im Gegensatz zu den Gruppen brauchen endliche Loops mit modularen Unterloopverbänden nicht direkte Produkte von Unterloops zu sein; dies belegen die Beispiele 3.2 und 3.3 in [8]. Doch spielt (ähnlich wie für Gruppen) die Vertauschbarkeit zweier Unterloops einer Loop G eine gewisse Rolle. Zwei Unterloops U, V der Loop G heißen *vertauschbar,* wenn $U \cup V = UV = VU$ gilt, und man nennt eine Unterloop U von G *quasinormal* in G, wenn U mit allen Unterloops von G vertauschbar ist. Auch für Loops hat man die folgende, [11, Theorem 5, p. 5] entsprechende Aussage:

1.1. Satz. *Sind zwei Unterloops U, V der Loop G vertauschbar, so gilt für alle Unterloops $W \supseteq U$ die modulare Identität $(U \cup V) \cap W = U \cup (V \cap W)$.*

Daher besitzen Loops, in denen alle Unterloops quasinormal sind, modulare Unterloopverbände. Insbesondere trifft dies für hamiltonsche Loops zu, in denen alle Unterloops normal, also erst recht quasinormal sind.

Für potenzassoziative p-Loops, wo p eine Primzahl ist, gilt:

1.2. Satz. *Sei G eine potenzassoziative p-Loop. Dann bilden die Elemente der Ordnung p zusammen mit* 1 *eine charakteristische Unterloop. Ist G sogar diassoziativ, so bilden die Elemente der Ordnung p zusammen mit* 1 *eine charakteristische kommutative Unterloop.*

Beweis. Seien, $a, b \in G$ zwei Elemente der Ordnung p. Da $L(\langle a, b\rangle)$ wegen der Modularität endliche Länge hat, ist $\langle a, b\rangle$ endlich. Gäbe es ein Element $g \in \langle a, b\rangle$ der Ordnung p^i, $1 < i \in \mathrm{N}$, so wäre $\langle g\rangle \cap \langle a\rangle = \langle g\rangle \cap \langle b\rangle = \langle 1\rangle$. Wegen der Modularität von $L(\langle a, b\rangle)$ müßte $\langle g\rangle = \langle a, b\rangle$ gelten und $\langle a, b\rangle$ wäre zyklische Gruppe. Also hat jedes von 1 verschiedene Element von $\langle a, b\rangle$ die Ordnung p, was die erste Behauptung zeigt. Für diassoziatives G ist $\langle a, b\rangle$ nach [11, Proposition 1.7, p. 14] eine Gruppe der Ordnung p^2, also kommutativ.

Als Folgerung aus dem obigen Satz ergibt sich, daß alle Elemente einer Ordnung $\leqq p^n$, $n \in \mathrm{N}$, einer potenzassoziativen p-Loop G eine charakteristische Unterloop von G bilden.

Nun geben wir eine vollständige Beschreibung endlicher Loops, deren Unterloopverbände boolesch, d. h. distributiv und komplementiert sind. Dazu rekapitulieren wir einige wichtige Ergebnisse aus [8]: Endliche Loops (und alle ihre Unterloops) mit distributiven Unterloopverbänden sind monogen. Als Konsequenz davon sind potenzassoziative Loops mit distributiven Unterloopverbänden zyklische Gruppen. Hilfreich ist ferner die Beobachtung, daß die Frattiniunterloop einer Loop mit einem komplementierten Unterloopverband trivial ist, vgl. [11, Proposition 1.14; p. 26]. Falls der Unterloopverband der Loop G die triviale boolesche Algebra ist, sind keine genaueren Aussagen über die Struktur von G möglich, da es für jede ungerade natürliche Zahl $n \geqq 5$ eine Loop der Ordnung n ohne echte Unterloops gibt, s. [5, p. 93 f.]. Zur Vereinfachung der Schreibweise führen wir die folgende Bezeichnung ein: Für Elemente $a_j \in G$, $j \in J$, $J \subseteqq \{1, \dots, n\}$ bezeichne $\prod_{j \in J} a_j$ das Produkt aller a_j, $j \in J$, in beliebiger Reihenfolge und mit beliebiger Klammerung des Produktes. Damit gilt:

1.3. Satz. *Der Unterloopverband einer Loop G ist genau dann eine boolesche Algebra der Länge n, wenn n Unterloops $A_i = \langle a_i \rangle$, $1 \neq a_i \in A_i$, $i = 1, \dots, n$, ohne echte Unterloops in G existieren, die paarweise trivialen Durschschnitt haben und für welche gilt:*

(a) $G = \langle A_i | i = 1, \dots, n \rangle = \langle \prod_{i=1,\dots,n} a_i \rangle$.

(b) *Für jede Unterloop* $\langle 1 \rangle \neq U < G$ *gibt es eine geeignete Teilmenge* $J \subset \{1, \dots, n\}$ *mit* $U = \bigcup_{j \in J} A_j = \langle \prod_{j \in J} a_j \rangle$.

(c) *Sind* $U = \langle \prod_{j \in J} a_j \rangle$ *und* $V = \langle \prod_{k \in K} a_k \rangle$ *mit* $J, K \subset \{1, \dots, n\}$ *zwei Unterloops von G, so hat man* $U \cup V = \langle \prod_{i \in J \cup K} a_i \rangle$, $U \cap V = \langle \prod_{r \in J \cap K} a_r \rangle$, *falls* $J \cap K \neq \emptyset$ *und* $U \cap V = \langle 1 \rangle$ *für* $J \cap K = \emptyset$.

Zum Beweis von Satz 1.3 hat man nur zu beachten, daß der Verband aller Teilmengen von $\{1, \dots, n\}$ eine boolesche Algebra der Länge n ist.

Für potenzassoziative Loops ergibt sich unmittelbar aus [8, Satz 1.2] und [12, Corollary 2]:

1.4. Korollar. *Die endlichen potenzassoziativen Loops mit booleschen Unterloopverbänden sind genau die zyklischen Höldergruppen.*

2. Loops mit schwach booleschen Unterloopverbänden

Der Unterloopverband $L(G)$ einer Loop G heißt *schwach boolesch*, wenn das Intervall $[G/A]$ für jedes Atom A von $L(G)$ boolesch ist. Endliche Loops mit schwach booleschen Unterloopverbänden haben die folgende bemerkenswerte Eigenschaft:

2.1. Satz. *Sei G eine endliche Loop mit schwach booleschem Unterloopverband $L(G)$. Dann ist G von höchstens zwei Elementen erzeugbar.*

Beweis. Falls es in $L(G)$ genau ein Atom gibt, so ist $L(G)$ offensichtlich distributiv und G kann nach [8, Satz 1.2] sogar von nur einem Element erzeugt werden.

Seien nun $A_1, \ldots, A_n$ die Atome von $L(G)$. Falls das Intervall $[G/A_i]$ für alle $i=1, \ldots, n$ die boolesche Algebra der Länge 1 ist, so stellt $L(G)$ eine projektive Gerade dar und wird daher nach [8, Satz 4.1] von höchstens zwei Elementen erzeugt. Ist $[G/A_i]$ für mindestens ein $i \in \{1, \ldots, n\}$ eine boolesche Algebra der Länge $l > 1$, so wird G bereits von zwei maximalen Unterloops aus $[G/A_i]$ erzeugt. Sei nun A_i so gewählt, daß $[G/A_i]$ eine boolesche Algebra maximaler Länge in $L(G)$ ist. Das Atom A_i wird offenbar von einem Element erzeugt, sei $A_i = \langle a_i \rangle$, $1 \neq a_i \in A_i$. Nun seien B_j, $j=1, \ldots, m$, die Atome von $[G/A_i]$. Dann ist $L(B_j)$, $j=1, \ldots, m$, entweder eine Kette der Länge zwei und B_j wird von einem Element $b_j \in B_j$, $b_j \notin A_i$, erzeugt, oder B_j ist das Erzeugnis von A_i und einer weiteren Unterloop C von G. Dabei ist C notwendigerweise ein Atom von $L(G)$, denn sonst wäre $[G/A_i]$ keine boolesche Algebra maximaler Länge. Also gilt in jedem Fall $B_j = \langle b_j, a_i \rangle$, $1 \neq b_j \notin A_i$, $j=1, \ldots, m$. Nun ist G aber die Vereinigung aller Atome B_j, $j=1, \ldots, m$, von $[G/A_i]$, also gilt $G = \langle b_j, a_i | j=1, \ldots, m \rangle$. Wir betrachten die von dem Produkt $b_1 \ldots b_m$ (mit beliebiger Klammerung) und dem Element a_i erzeugte Unterloop $H = \langle b_1 \ldots b_m, a_i \rangle$ von G. Trivialerweise hat man $A_i < H$. Wäre H eine echte Unterloop von G, so wäre entweder $H = A_i$ oder H wäre Vereinigung von $r < m$ Atomen B_j. In beiden Fällen wäre H also in einer maximalen Unterloop M von G enthalten; o. B. d. A. dürfen wir $H < \langle b_1, \ldots, b_{m-1}, a_i \rangle$ annehmen. Dann folgt $b_1 \ldots b_m \in \langle b_1, \ldots, b_{m-1}, a_i \rangle$, was nach Voraussetzung nicht möglich ist. Daher gilt $H = G$ und die Behauptung ist gezeigt.

Für diassoziative Loops hat Satz 2.1 eine interessante Konsequenz, die eine wesentliche Verallgemeinerung von [10, Korollar 3 in Abschnitt 2] darstellt und die insbesondere für Moufang-Loops gilt:

2.2. Korollar. *Der Unterloopverband einer endlichen diassoziativen Loop G ist genau dann schwach boolesch, wenn G eine Gruppe von einem der folgenden Typen ist:*

(1) *zyklische Höldergruppe,*

(2) $\mathbf{Z}(p^2)$ *für eine Primzahl p,*

(3) $\mathbf{Z}(p) \times \mathbf{Z}(p)$ *für eine Primzahl p,*

(4) *direkt unzerlegbare Höldergruppe mit primzyklischer Kommutatorgruppe.*

Abschließend geben wir noch ein Beispiel für eine nicht diassoziative Loop mit einem schwach booleschen Unterloopverband.

2.3. Beispiel. Die Loop $G = \{1, \ldots, 16\}$ sei gegeben durch die Permutationen (zur Schreibweise s. [8, Abschnitt 3])

$$R_2 = (1\ 2)(3\ 4)(5\ 6)(7\ 13\ 10\ 16\ 15\ 12)(8\ 11)(9\ 14),$$

$$R_3 = (1\ 3\ 2\ 4)(5\ 10\ 14\ 16\ 12\ 11\ 9\ 13\ 7\ 6\ 8\ 15),$$

$$R_4 = (1\ 4\ 2\ 3)(5\ 15\ 10\ 12\ 8\ 14\ 7\ 16\ 6\ 9)(11\ 13),$$
$$R_5 = (1\ 5\ 2\ 6)(3\ 14\ 10\ 15\ 7\ 11)(4\ 13\ 9\ 12\ 16\ 8),$$
$$R_6 = (1\ 6\ 2\ 5)(3\ 15\ 9\ 11\ 10\ 4\ 16\ 7\ 12\ 13\ 14\ 8),$$
$$R_7 = (1\ 7\ 8\ 9\ 10)(2\ 16\ 14\ 12)(3\ 6\ 13\ 5)(4\ 15\ 11),$$
$$R_8 = (1\ 8\ 10\ 7\ 9)(2\ 15\ 13\ 3\ 12\ 6\ 4\ 11\ 16\ 5\ 14),$$
$$R_9 = (1\ 9\ 7\ 10\ 8)(2\ 12\ 3\ 11\ 15\ 14\ 13\ 6\ 16)(4\ 5),$$
$$R_{10} = (1\ 10\ 9\ 8\ 7)(2\ 11)(3\ 16\ 13\ 15\ 4\ 14\ 6)(5\ 12),$$
$$R_{11} = (1\ 11\ 12\ 9\ 3\ 10\ 6\ 14\ 15\ 16)(4\ 8\ 5\ 7\ 2\ 13),$$
$$R_{12} = (1\ 12\ 15)(2\ 16\ 13)(3\ 5\ 8\ 16\ 4\ 9\ 6\ 7)(11\ 14),$$
$$R_{13} = (1\ 13\ 16\ 10\ 5\ 11\ 7\ 14)(2\ 9\ 15\ 3\ 8)(4\ 6\ 12),$$
$$R_{14} = (1\ 14\ 4\ 12\ 10\ 3\ 7\ 5\ 9\ 16\ 11\ 6\ 15\ 2\ 8\ 13),$$
$$R_{15} = (1\ 15\ 8\ 6\ 10\ 11\ 5\ 16\ 9\ 2\ 14\ 3\ 13\ 12)(4\ 7),$$
$$R_{16} = (1\ 16\ 3\ 9\ 4\ 10\ 2\ 7\ 15\ 6\ 11)(5\ 13\ 8\ 12\ 14).$$

Die Loop G besitzt die Unterloops $A_1=\{1, 2\}$, $A_2=\{1, 7, 8, 9, 10\}$, $B_1=\{1, 2, 3, 4\}$, $B_2=\{1, 2, 5, 6\}$. Dabei sind A_1 und A_2 die Atome von $L(G)$; das Intervall $[G/A_1]$ ist eine boolesche Algebra der Länge 2, das Intervall $[G/A_2]$ eine boolesche Algebra der Länge 1. Ferner gilt $G=\langle A_1, A_2\rangle=\langle B_1, B_2\rangle=\langle B_1, A_2\rangle=\langle B_2, A_2\rangle$. Außerdem kann G von nur einem Element erzeugt werden, man hat nämlich $G=\langle 11\rangle=\langle 12\rangle=$ $=\langle 13\rangle=\langle 14\rangle=\langle 15\rangle=\langle 16\rangle$.

3. Zentral nilpotente endliche Loops

Im folgenden Abschnitt 4 werden wir uns mit endlichen kommutativen Moufang-Loops beschäftigen. Da endlich erzeugte kommutative Moufang-Loops zentral nilpotent sind (s. [5, Theorem 10.1, p. 157]), lohnt es sich, zunächst endliche zentral nilpotente Loops zu untersuchen.

Bekanntlich erfüllt der Untergruppenverband $L(G)$ einer endlichen Gruppe G genau dann die Jordan—Dedekind-Kettenbedingung, wenn G überauflösbar ist. Daher erfüllen insbesondere die Untergruppenverbände endlicher nilpotenter Gruppen die Jordan—Dedekind-Kettenbedingung. Für endliche zentral nilpotente Loops hat man die entsprechenden Sätze wie für endliche nilpotente Gruppen, da die Ordnung jeder Unterloop die Ordnung jeder sie enthaltenden Unterloop teilt. Ferner ist eine Unterloop einer zentral nilpotenten Loop G genau dann maximal in G, wenn sie normal in G ist und Primzahlindex in G hat, s. [4, Theorem 7B, Lemma 7F + Corollary]. Weiter existiert stets eine Hauptreihe $G=A_0\supset A_1\supset\ldots\supset A_r=\langle 1\rangle$, wobei jede Faktorloop A_{i-1}/A_i keine echten Unterloops besitzt und A_i Primzahlindex in A_{i-1} hat. Daher folgt mit genau den gleichen Argumenten wie für Gruppen (vgl. [3, p. 177] und [11, Theorem 9, p. 9]):

3.1. Satz. *Der Unterloopverband einer endlichen zentral nilpotenten Loop erfüllt die Jordan—Dedekind-Kettenbedingung.*

Ferner gestatten endliche zentral nilpotente Loops noch die folgenden Aussagen:

3.2. Satz. *Eine endliche zentral nilpotente Loop G hat genau dann einen komplementierten Unterloopverband $L(G)$, wenn G ein direktes Produkt elementar-abelscher p-Gruppen ist.*

Beweis. Da G endlich ist, ist die Frattiniunterloop $\Phi(G)$ von G verschieden von G und wegen der Komplementiertheit von $L(G)$ gilt $\Phi(G)=\langle 1\rangle$. Daher ist G nach [5, Theorem 2.2, p. 98] eine abelsche Gruppe und mit [11, Proposition 1.15, p. 16] folgt nun die Behauptung.

Aus Satz 3.2 und [8, Satz 1.2] ergibt sich unmittelbar:

3.3. Korollar. *Ist der Unterloopverband einer endlichen zentral nilpotenten Loop G eine boolesche Algebra, so ist G eine zyklische Höldergruppe.*

Als weitere Folgerung aus Satz 3.2 erhält man mit Hilfe von [12, Corollary of Theorem 1]:

3.4. Korollar. *Der Unterloopverband $L(G)$ einer endlichen zentral nilpotenten Loop G ist genau dann orthomodular, wenn G direktes Produkt von P-Gruppen mit paarweise teilerfremden Ordnungen ist.*

4. Endliche kommutative Moufang-Loops

In den vorhergehenden Abschnitten haben wir gesehen, daß man für endliche kommutative Moufang-Loops und ihre Unterloopverbände einige schöne Aussagen machen kann. Dies rechtfertigt eine gesonderte Behandlung endlicher kommutativer Moufang-Loops. Zunächst seien hier noch einmal die wichtigsten Ergebnisse zusammengestellt.

4.1. Satz. *Für eine endliche kommutative Moufang-Loop G mit dem Unterloopverband $L(G)$ gelten:*

(1) *$L(G)$ erfüllt die Jordan—Dedekind-Kettenbedingung.*
(2) *$L(G)$ distributiv $\Leftrightarrow$ G zyklische Gruppe.*
(3) *$L(G)$ komplementiert $\Leftrightarrow$ G direktes Produkt elementar-abelscher p-Gruppen.*
(4) *$L(G)$ boolesch $\Leftrightarrow$ G zyklische Höldergruppe.*

(5) *$L(G)$ schwach boolesch $\Leftrightarrow$ G ist abelsche Gruppe von genau einem der folgenden Typen: zyklische Höldergruppe, $\mathbf{Z}(p^2)$, $\mathbf{Z}(p)\times\mathbf{Z}(p)$.*

Im folgenden soll die Struktur von endlichen kommutativen Moufang-Loops mit modularen Unterloopverbänden hergeleitet werden. Es zeigt sich, daß für endliche kommutative Moufang-Loops — ähnlich wie für endliche nilpotente Gruppen, vgl. [11, Theorem 8, p. 8] — die Umkehrung von Satz 1.1 gilt. Dazu benötigen wir zunächst das folgende Lemma.

4.2. Lemma. *Die Unterloops U und V einer endlichen kommutativen Moufang-Loop G sind genau dann vertauschbar, wenn $|(U\cup V)\colon U|=|V\colon (U\cap V)|$ gilt.*

Beweis. Sei zunächst $|(U\cup V)\colon U|=|V\colon (U\cap V)|$. Dann gilt $|U\cup V|=|U|\cdot|(U\cup V)\colon U|=|U|\cdot|V\colon (U\cap V)|=|UV|=|VU|$, woraus wegen $UV=VU\subseteqq\langle U, V\rangle$ sofort $U\cup V=UV=VU$ folgt. Seien nun U und V vertauschbar. Dann hat man $|U\cup V|=|UV|=|U|\cdot|V\colon (U\cap V)|$ und andererseits $|U\cup V|=|(U\cup V)\colon U|\cdot|U|$. Damit folgt die behauptete Identität.

Jetzt können wir den gewünschten Satz zeigen:

4.3. Satz. *Sei G eine endliche kommutative Moufang-Loop und seien U und V ein modulares Paar von Unterloops von G. Dann sind U und V vertauschbar.*

Beweis. Als endliche kommutative Moufang-Loop ist G direktes Produkt einer endlichen abelschen Gruppe A von zu 3 teilerfremder Ordnung und einer kommutativen 3-Moufang-Loop B. Daher läßt sich G als direktes Produkt der p-Sylowgruppen $S_1, \ldots, S_n$ von A und der 3-Moufang-Loop B schreiben, also $G=S_1\times\ldots\times S_n\times B$, s. [7, Proposition 1.3]. Somit ist auch jede Unterloop U von G in der Form $U=U_1\times\ldots\times U_n\times U_{n+1}$ mit $U_i=U\cap S_i$ $(i=1, \ldots, n)$, $U_{n+1}=U\cap B$ darstellbar. Seien also $U=U_1\times\ldots\times U_{n+1}$ und $V=V_1\times\ldots\times V_{n+1}$ ein modulares Paar von Unterloops von G. Wir bezeichnen die Ordnungen von $U, V, U\cup V, U\cap V, U_i, V_i, U_i\cup V_i$ bzw. $U_i\cap V_i$ mit $u, v, m, d, u_i, v_i, m_i$ bzw. d_i. Wegen $U\cup V=(U_1\cup V_1)\times\ldots\times(U_{n+1}\cup V_{n+1})$ folgt $m=\prod_{i=1}^{n+1} m_i$ und $d=\prod_{i=1}^{n+1} d_i$, und md ist durch uv teilbar. Andererseits wird das Intervall $[(U\cup V)/U]$ von $L(G)$ wegen der Gültigkeit der modularen Identität isomorph auf das Intervall $[V/(U\cap V)]$ abgebildet. Da G zentral nilpotent ist, sind auch $U, V, U\cup V$ und $U\cap V$ zentral nilpotent und es existiert je eine Hauptreihe zwischen $U\cup V$ und U sowie zwischen V und $U\cap V$, s. [4, Abschnitt 7]; dabei ist die Länge einer Hauptreihe zwischen $U\cup V$ und U nicht größer als die Länge einer Hauptreihe zwischen V und $U\cap V$. Daher ist die Anzahl der Primfaktoren in $|(U\cup V)\colon U|=m/u$ nicht größer als die Anzahl der Primfaktoren in $|V\colon (U\cap V)|=v/d$. Wegen $uv|md$ muß $md=uv$, also $|(U\cup V)\colon U|=|V\colon (U\cap V)|$ gelten. Nach Lemma 4.2 hat man somit $U\cup V=\langle U, V\rangle=UV=VU$.

Aus den Sätzen 4.3 und 1.1 folgt unmittelbar:

4.4. Korollar. *Eine endliche kommutative Moufang-Loop besitzt genau dann einen modularen Unterloopverband, wenn alle ihre Unterloops quasinormal sind.*

Eine wichtige Klasse von Loops, in denen alle Unterloops quasinormal sind, sind die hamiltonschen Loops. Wegen der Diassoziativität von Moufang-Loops ergibt sich mit [5, Theorem 7.2, p. 87] sofort, daß endliche hamiltonsche kommutative Moufang-Loops abelsche Gruppen sind. Von besonderem Interesse ist daher die Struktur nicht hamiltonscher kommutativer Moufang-Loops mit modularen Unterloopverbänden, die im folgenden untersucht wird.

4.5. Lemma. *Sei $G=A\times B$ eine endliche kommutative Moufang-Loop mit einer abelschen Gruppe A von zu 3 teilerfremder Ordnung und einer kommutativen 3-Moufang-Loop B. Dann ist der Unterloopverband $L(G)$ von G genau dann modular, wenn $L(B)$ modular ist.*

Beweis. Wegen [5, p. 101] ist G in der angegebenen Form $G=A\times B$ darstellbar und mit $L(G)$ ist auch $L(B)$ modular. Da $L(A)$ als Untergruppenverband einer abelschen Gruppe von vornherein modular ist, ist umgekehrt für die Modularität von $L(G)$ die Modularität von $L(B)$ hinreichend (vgl. den Beweis von Theorem 4, p. 5, in [11]).

Somit können wir uns darauf beschränken, endliche kommutative 3-Moufang-Loops mit modularen Unterloopverbänden zu untersuchen. Ein wichtiges Hilfsmittel dazu liefert das folgende Lemma:

4.6. Lemma. *Eine kommutative 3-Moufang-Loop B vom Exponenten 3 hat genau dann einen modularen Unterloopverband, wenn B eine elementar-abelsche 3-Gruppe ist.*

Beweis. Angenommen, B ist keine Gruppe. Dann gibt es unter den von drei Elementen erzeugten Unterloops mindestens eine nicht assoziative Unterloop $U=\langle x, y, z\rangle$ von B. Wegen [7, Lemma 1.6] hat U mindestens die Ordnung 81, während das Komplexprodukt $\langle x, y\rangle\langle z\rangle$ von der Ordnung 27 ist. Daher kann $L(B)$ nach Korollar 4.4 nicht modular sein.

Das vorstehende Lemma zeigt insbesondere, daß die von allen Elementen der Ordnung 3 erzeugte Unterloop S einer endlichen kommutativen Moufang-Loop G mit modularem Unterloopverband eine elementar-abelsche 3-Gruppe ist, da wegen der Gültigkeit von $(xy)^3=x^3y^3$ für alle $x, y\in G$ alle Elemente $\neq 1$ von S die Ordnung 3 haben. Eine weitaus wichtigere Konsequenz aus Lemma 4.6 ist jedoch die folgende Aussage:

4.7. Satz. *Es sei G eine endliche kommutative Moufang-Loop und $Z(G)$ ihr assoziatives Zentrum. Ist der Unterloopverband $L(G)$ von G modular, so ist die Faktorloop $G/Z(G)$ eine elementar-abelsche 3-Gruppe. Insbesondere ist G also nilpotent der Klasse* 2.

Beweis. Nach [9, Theorem 1.8, p. 9] ist $G/Z(G)$ eine kommutative Moufang-Loop vom Exponenten 3. Da mit $L(G)$ auch $L(G/Z(G))$ modular ist, muß $G/Z(G)$ wegen Lemma 4.6 eine abelsche Gruppe vom Exponenten 3 sein.

Übrigens hat Satz 4.7 eine interessante Parallele für endliche Gruppen mit modularen Untergruppenverbänden, sogenannte M-Gruppen: Endliche M-Gruppen sind stets metabelsch, s. [11, Theorem 15, p. 18].

Das kleinste Beispiel für eine nicht assoziative kommutative 3-Moufang-Loop mit modularem Unterloopverband ist die von den Elementen x, y, z mit den definierenden Relationen $x^3=y^3=z^9=1$, $(x, y, z)=z^3$ erzeugte Loop B der Ordnung 81, s. [7, Proposition 6.1]: Alle echten Unterloops von B sind Gruppen, also ist das Komplexprodukt zweier Unterloops, die zusammen eine echte Unterloop von B erzeugen, gleich dem Erzeugnis dieser Unterloops; das Komplexprodukt zweier Unterloops, deren Erzeugnis B ist, hat die Ordnung 81 und ist daher gleich B. Somit sind alle Unterloops von B quasinormal.

Literaturverzeichnis

[1] A. A. ALBERT, Quasigroups. I, *Trans. Amer. Math. Soc.*, 54 (1943), 507—519; II, *Trans. Amer. Math. Soc.*, 55 (1944), 401—419.

[2] L. BÉNÉTEAU, L'irréductibilité centrale dans les boucles de Moufang commutatives et dans systèmes triples de Hall, *Discrete Math.*, 45 (1983), 31—44.

[3] G. BIRKHOFF, *Lattice Theory*, Colloq. Publ., Vol. XXV., Amer. Math. Soc. (Providence, Rhode Island, 3rd ed., 1967).

[4] R. H. BRUCK, Contributions to the theory of loops, *Trans. Amer. Math. Soc.*, 60 (1946), 245—354.

[5] R. H. BRUCK, *A Survey of Binary Systems*, Springer-Verlag (Berlin—Heidelberg, 1958).

[6] M. HALL, *The Theory of Groups*, Chelsea (New York, 1976).

[7] T. KEPKA, P. NĚMEC, Commutative Moufang Loops and Distributive Groupoids of Small Orders, *Czechoslovak Math. J.*, 31 (1981), 633—669.

[8] H. LAUSCH, Loops mit distributiven und geometrischen Unterloopverbänden, *J. Geom.*, 26 (1986), 62—76.

[9] YU. I. MANIN, *Cubic Forms*, North Holland, American Elsevier (Amsterdam—London—New York, 1974).

[10] P. PLAUMANN, K. STRAMBACH, G. ZACHER, Gruppen mit geometrischen Abschnitten im Untergruppenverband, *Monatsh. Math.*, 99 (1985), 117—145.

[11] M. Suzuki, *Structure of a Group and the Structure of its Lattice of Subgroups*, Springer-Verlag (Berlin—Göttingen—Heidelberg, 1956).

[12] G. Whitson, Finite groups whose subgroup, composition subgroup, or normal subgroup lattice is an ortholattice, *Algebra Universalis*, 8 (1978), 123—127.

MATHEMATISCHES INSTITUT
UNIVERSITÄT WÜRZBURG
AM HUBLAND
8700 WÜRZBURG, WEST GERMANY

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On composition of idempotent functions

A. KISIELEWICZ

The general problem of composition of functions was raised by W. SIERPIŃSKI [13]. Since then the problem has been extensively investigated in many-valued logic, synthesis of automata, and recently, in universal algebra (cf. [11], [1], [3], [4]). There are some results and problems showing that idempotent clones play here a special role (cf. [2], [3], [8], [10,] [12], [14], see also [7]). In this paper some further special properties of idempotent clones are established, and examples are provided to show that our theorems do not hold in the general (nonidempotent) case.

The results are stated in Section 3. Before we introduce some definitions (Section 1) and give background information (Section 2). Proofs are given in Section 4.

1. Definitions. A *clone* is a composition closed set of functions (on a fixed universe A) containing all projections (cf. [12]). For two clones $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{B}\supseteqq\mathbf{A}$ we say that $\mathbf{A}$ is a *subclone* of $\mathbf{B}$, while $\mathbf{B}$ is an *extension* of $\mathbf{A}$. If $\mathbf{A}\neq\mathbf{B}$ and $\mathbf{A}$ is not a *trivial clone* (i.e. consisting of projections only), then $\mathbf{A}$ is said to be a *proper subclone* of $\mathbf{B}$. If m is the least integer such that there is an essentially m-ary function in $\mathbf{B}-\mathbf{A}$, then B is called an *m-ary extension* of $\mathbf{A}$.

For any set $\mathbf{F}$ of functions, $P_n(\mathbf{F})$ denotes the set of essentially n-ary functions in $\mathbf{F}$, and $p_n(\mathbf{F})$ is the cardinality of $P_n(\mathbf{F})$. Moreover, we denote $S(\mathbf{F})=\{n\colon p_n(\mathbf{F})>0\}$.

A function $f\colon A^n\to A$ is *idempotent* if it satisfies $f(x,\ldots,x)=x$ identically. If, in addition, it satisfies

$$f(f(x_1^1,x_2^1,\ldots,x_n^1),\,f(x_1^2,x_2^2,\ldots,x_n^2),\ldots,f(x_1^n,x_2^n,\ldots,x_n^n))=f(x_1^1,x_2^2,\ldots,x_n^n),$$

then it is called *diagonal*. If every function in a clone is idempotent (diagonal), then the clone itself is called *idempotent (diagonal)*.

Other, undefined concepts are standard and can be found in corresponding papers given in our references. Throughout the paper we make use of the fact that clones can be identified with sets of polynomials of universal algebras (cf. [11]).

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2. Examples. The algebraic property of idempotency is of special interest in studying clones, because for any clone **A**, the idempotent functions in **A**, as it is easy to check, form a composition closed set (the full idempotent subclone of **A**). Consequently, studying minimal clones leads to studying certain idempotent clones (see [2], [12]). Also, p_n-sequences (and free spectra) of idempotent clones are completely different in nature from those of nonidempotent clones (see [4], [7], [8]). Diagonal clones are rather exceptional among idempotent clones and are fully described. Properties of diagonal clones mentioned below are derived from [9] and [6].

(2.1) *Diagonal clones.* A clone **D** generated by a single essentially r-ary diagonal function $d(x_1, \dots, x_r)$ is called an *r-dimensional diagonal clone (algebra).* $S(\mathbf{D}) = \{2, 3, \dots, r\}$ and $p_n(\mathbf{D})$ is finite for all n. A diagonal clone **D** is *finitely generated* iff it is as above. Otherwise, $S(\mathbf{D}) = \{2, 3, \dots\}$ and $p_n(\mathbf{D}) \geqq \aleph_0$ for all $n \geqq 2$. For any diagonal clone **D**, $P_n(\mathbf{D})$ with $n \geqq 2$, if not empty, is a generating set for **D**. Finally, if a clone **A** is generated by diagonal functions only, and has no nondiagonal binary functions, then it is a diagonal clone; it is finitely generated if and only if $p_2(\mathbf{A})$ is finite. Also, the structure of m-ary extensions of r-dimensional diagonal clones with $m > r+1$ is described (see [14]).

(2.2) *Boolean reducts.* For the full idempotent subclone **I** of the clone (of polynomials) of any Boolean group $\langle G, + \rangle$ we have $S(\mathbf{I}) = \{3, 5, 7, \dots\}$ and $p_n(\mathbf{I}) = 1$ for odd $n \geqq 3$ (see [8], p. 234). In this paper such clones are called simply *Boolean reducts*. The structure of m-ary extensions of Boolean reducts with $m \geqq 5$ is described in [14].

(2.3) *Counter-examples.* Let C be the union of two infinite disjoint sets A and B and two further elements a and b. For any $n \geqq 1$ we define two functions on C: $f_n(x_1, \dots, x_n) = a$ if $x_1, \dots, x_n \in A$ and are pairwise distinct, and $f_n = b$ otherwise. Similarly, $g_n(x_1, \dots, x_n) = a$ if $x_1, \dots, x_n \in B$ and are pairwise distinct, and $g_n = b$ otherwise. It is easy to check that any set of functions f_i, g_i containing the constant b is a clone. Thus, for any set of positive integers S, there exist a clone **B** and a subclone **A** of **B** such that $S(\mathbf{B} - \mathbf{A}) = S$. For these clones $P_n(\mathbf{B}) \cup \{b\}$ is always a subclone. Also, examples of clones without constants and having the same properties can be given using constructions applied in [3].

3. Results. Our main result concerns the difference $\mathbf{B} - \mathbf{A}$ of an idempotent clone **B** and its subclone **A**. In the general case, by example (2.3), the set $S(\mathbf{B} - \mathbf{A})$ can be arbitrary. If **B** is assumed to be idempotent, the situation is very different:

Theorem 1. *Let* **B** *be an idempotent clone and* **A** *its proper subclone. Then one of the following conditions holds:*

(i) $S(\mathbf{B}-\mathbf{A})=\{m, m+1, \ldots\}$ *for some* $m \geqq 2$,
(ii) $S(\mathbf{B}-\mathbf{A})=\{2, 3, \ldots, r\}$ *for some* $r \geqq 2$,
(iii) $S(\mathbf{B}-\mathbf{A})=\{2, 3, \ldots, r\} \cup \{m, m+1, \ldots\}$ *for some* $r \geqq 2$ *and* $m > r+1$,
(iv) $S(\mathbf{B}-\mathbf{A})=\{3, 5, 7, \ldots\} \cup \{m, m+1, \ldots\}$ *for some even* $m > 5$.

Moreover, conditions (ii)—(iv) *determine the structure of the clone* **B**. *Namely,*
1. *if* (ii) *holds, then* **B** *is an r-dimensional diagonal clone,*
2. *if* (iii) *holds, then* **B** *is an m-ary extension of an r-dimensional diagonal clone,*
3. *if* (iv) *holds, then* **B** *is an m-ary (or* $(m-1)$*-ary) extension of a Boolean reduct.*

Corollary. *The difference* $\mathbf{B}-\mathbf{A}$ *of an idempotent clone* **B** *and its proper subclone* **A** *is always infinite, unless* **B** *is a finitely generated diagonal clone.*

Theorem 1 is actually a classification of the differences $\mathbf{B}-\mathbf{A}$, analogous to that of [14]. It is of some interest that from such a theorem one can derive a result concerning composition of functions:

Theorem 2. *If* **B** *is an idempotent clone which can be generated by (at most) k-ary functions, then for any* $n \geqq k$, *the set* $P_n(\mathbf{B})$ *of essentially n-ary functions in* **B**, *if not empty, is a generating set for* **B**.

In addition to the examples in (2.3), many others can be constructed showing that our theorem fails to hold for nonidempotent clones.

4. Proofs. At first, we give the proof of Theorem 1 which is based on several lemmas. We use techniques and constructions worked out in [5] and [8]. Throughout, **B** is assumed to be an idempotent clone, and **A** its subclone. The numbers in question are always integers. For every $k \geqq 2$ we consider the following property of the clone **B**:

(+) for every function $f(x_1, \ldots, x_n) \in P_n(\mathbf{B})$ with $n \geqq k$ there exists a function $\bar{f}(x_1, \ldots, x_{n+1}) \in P_{n+1}(\mathbf{B})$ in **B** such that $\bar{f}(x_1, \ldots, x_n, x_i)=f(x_1, \ldots, x_n)$ for some i.

Lemma 1. *If* **B** *satisfies* (+) *for some k, then for every* $m \geqq k$, $m \in S(\mathbf{B}-\mathbf{A})$ *implies* $\{m, m+1, \ldots\} \subseteq S(\mathbf{B}-\mathbf{A})$.

Proof. $m \in S(\mathbf{B}-\mathbf{A})$ means that there is an essentially m-ary function $f(x_1, \ldots, x_m)$ in $\mathbf{B}-\mathbf{A}$. Then, $\bar{f}(x_1, \ldots, x_{m+1}) \notin \mathbf{A}$, since (by substitution $x_{m+1}=x_i$) it generates $f(x_1, \ldots, x_m)$. It follows that $m+1 \in S(\mathbf{B}-\mathbf{A})$. Now the result follows easily by induction.

Lemma 2. *Let* $g(x, y)=x \cdot y$ *be a binary function in* **B**, *not diagonal. If* $f=f(x_1, \ldots, x_n) \in P_n(\mathbf{B})$ $(n \geqq 2)$, *then for some* i, $\bar{f}=f(x_1, \ldots, x_i \cdot x_{n+1}, \ldots, x_n) \in P_{n+1}(\mathbf{B})$.

Proof. At first, note that $\bar{f}$ obviously depends on each of the variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, since substituting $x_{n+1}=x_i$ in $\bar{f}$ we get f depending on these variables.

Now, suppose that $f(x\cdot y, x_2, \ldots, x_n)$ does not depend on y. Then by substituting $y=x$, $f(x\cdot y, x_2, \ldots, x_n)=f(x, x_2, \ldots, x_n)$. Consequently, $f((x\cdot y)\cdot z, x_2, \ldots, \ldots, x_n)=f(x, x_2, \ldots, x_n)$. Similarly, if $f(x\cdot y, x_2, \ldots, x_n)$ does not depend on x, then $f((x\cdot y)\cdot z, x_2, \ldots, x_n)=f(z, x_2, \ldots, x_n)$. By analogous arguments for all indices i, and in view of the idempotency of f, we infer that, if $f(x_1, \ldots, x_i\cdot x_{n+1}, \ldots, x_n)$ is not essentially $(n+1)$-ary for any i, then $(x\cdot y)\cdot z=f((x\cdot y)\cdot z, \ldots, (x\cdot y)\cdot z)$ does not depend on y. Consequently, $(x\cdot y)\cdot z=x\cdot z$. Similarly, $x\cdot(y\cdot z)=x\cdot z$. This means that $x\cdot y$ is diagonal, a contradiction.

Lemma 3. *If there is a binary nondiagonal function in* **B**, *then condition* (i) *of Theorem* 1 *holds.*

Proof. By Lemma 2, **B** satisfies condition (+) for $k=2$. Now, if m is the least integer such that $m\in S(\mathbf{B}-\mathbf{A})$, then in view of Lemma 1, $S(\mathbf{B}-\mathbf{A})=\{m, m+1, \ldots\}$, as required.

Lemma 4. *If* **B** *is a diagonal clone, then* $S(\mathbf{B}-\mathbf{A})=\{2, \ldots, r\}$ *for some* $r\geqq 2$ *whenever* **B** *is finitely generated, and* $S(\mathbf{B}-\mathbf{A})=\{2, 3, \ldots\}$ *otherwise.*

Proof. If **B** is finitely generated, then the result is by (2.1). Suppose that **B** is not finitely generated and $m\notin S(\mathbf{B}-\mathbf{A})$. Then $P_m(\mathbf{B})=P_m(\mathbf{A})$. However, as $P_m(\mathbf{B})$ generates the clone **B** (cf. (2.1)), $P_m(\mathbf{A})$ also does, and so $\mathbf{A}=\mathbf{B}$, a contradiction.

Lemma 5. *If* **B** *is an m-ary extension of a diagonal clone* **D** *for some* $m\geqq 2$, *then* **D** *is contained in the clone generated by* $P_m(\mathbf{B})$.

Proof. If there is a diagonal function in $P_m(\mathbf{B})$, then by (2.1) $P_m(\mathbf{D})$ generates **D**, and since $P_m(\mathbf{B})\supset P_m(\mathbf{B})$, the result follows. In the opposite case, **D** is an r-dimensional diagonal clone for some $r<m$ $(r\geqq 2)$. In this case we apply the method of diagonal decomposition of the clone **B** with respect to **D** (see [8], p. 244).

Let $f(x_1, \ldots, x_m)$ be a nondiagonal essentially m-ary function in **B**, which exists by assumption. Then

$$f(x_1, \ldots, x_m) = \langle f^1(x_1, \ldots, x_m), \ldots, f^r(x_1, \ldots, x_m)\rangle$$

and each f^i is either essentially m-ary or equal to $h_i(x_k)$. Indeed, if e.g. $f^1(x_1, \ldots, x_m)$ depended on exactly k variables with $1<k<m$, say $f^1(x_1, \ldots, x_m)=g(x_1, \ldots, x_k)$, then we would have $\langle g(x_1, \ldots, x_k), h_2(x_1), \ldots, h_r(x_1)\rangle\in P_k(\mathbf{B})$. This function is nondiagonal (by properties of diagonal decomposition) contradicting the assumptions in our lemma. Moreover, at least one f^i must be essentially m-ary, since otherwise f would be a diagonal function.

So, suppose that e.g. $f^1(x_1, \ldots, x_m)$ is essentially m-ary.

$$f(x_1, \ldots, x_{m-1}, x_{m-1}) = h_1(x_k) \quad \text{for some} \quad k \ (1 \leqq k \leqq m-1)$$

(otherwise we can infer a contradiction, as above). Put

$$g(x_1, \ldots, x_m) = \langle f^1(x_1, \ldots, x_m),\ h_2(x_{i_2}), \ldots, h_r(x_{i_r})\rangle$$

where $i_2, \ldots, i_r$ are pairwise distinct and $k \notin \{i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, m\}$. Then $g \in P_m(\mathbf{B})$ and $g(x_1, \ldots, x_{m-1}, x_{m-1}) = \langle h_1(x_k), h_2(x_{i_2}), \ldots, h_r(x_{i_r})\rangle$ is an essentially r-ary diagonal function, and by (2.1), it generates the clone $\mathbf{D}$. This completes the proof.

Lemma 6. *Suppose that $P_2(\mathbf{B})$ is finite, nonempty, and consists of diagonal functions only. If $\mathbf{B}$ is not a diagonal clone, then either $S(\mathbf{B}-\mathbf{A}) = \{m, m+1, \ldots\}$ for some $m \geqq 2$, or $S(\mathbf{B}-\mathbf{A}) = \{2, \ldots, r\} \cup \{m, m+1, \ldots\}$ for some $r \geqq 2$ and $m > r+1$ In the latter case $\mathbf{B}$ is an m-ary extension of an r-dimensional diagonal clone.*

Proof. Denote by $\mathbf{D}$ the clone generated by $P_2(\mathbf{B})$. By (2.1) it is an r-dimensional diagonal clone for some $r \geqq 2$, and consits of all diagonal functions in $\mathbf{B}$. In other words, since by assumption $\mathbf{B} \neq \mathbf{D}$, $\mathbf{B}$ is an m-ary extension of $\mathbf{D}$ for some $m \geqq 2$, just as the second part of the lemma states. Moreover, $\mathbf{B}$ satisfies $(+)$ for $k=m$. Indeed, it is enough to set $\bar{f} = fL$, where L is a mapping defined in [8], Section 5.4.

Now, if s is the least integer such that $s \in S(\mathbf{B}-\mathbf{A})$ and $s \geqq m$, then in view of Lemma 1, $S(\mathbf{B}-\mathbf{A}) = \{s, s+1, \ldots\}$, as required. It remaines to consider the case when $s < m$, which means that there exists a diagonal essentially s-ary function in $\mathbf{B}-\mathbf{A}$ $(s \geqq 2)$. By means of (2.1), it follows that the full diagonal subclone of $\mathbf{A}$ is contained properly in $\mathbf{D}$, and consequently, $\{2, \ldots, r\} \subseteq S(\mathbf{B}-\mathbf{A})$. On the other hand, by Lemma 5, $m \in S(\mathbf{B}-\mathbf{A})$, and since $\mathbf{B}$ satisfies $(+)$ for $k=m$, $\{m, m+1, \ldots\} \subseteq$ $\subseteq S(\mathbf{B}-\mathbf{A})$.

Now, if $r \geqq m-1$, then $S(\mathbf{B}-\mathbf{A}) = \{2, 3, \ldots\}$, while if $r < m-1$, then as $\mathbf{B}$ is an m-ary extension of $\mathbf{D}$, $S(\mathbf{B}-\mathbf{A}) = \{2, \ldots, r\} \cup \{m, m+1, \ldots\}$. This completes the proof.

Lemma 7. *If $P_2(\mathbf{B})$ is infinite and consists of diagonal functions only, then condition* (i) *in Theorem* 1 *holds.*

Proof. In view of Lemma 1 it is enough to show that $\mathbf{B}$ satisfies condition $(+)$ for $k=2$. Applying again the method of diagonal decomposition [8], we construct a suitable mapping.

Let $f(x_1, \ldots, x_m) \in P_m(\mathbf{B})$, $m \geqq 2$. By virtue of (2.1) (for every m) there exists an essentially $(m+1)$-ary diagonal function in $\mathbf{B}$. This function generates an $(m+1)$-dimensional diagonal clone, a subclone of $\mathbf{B}$. We decompose $\mathbf{B}$ just with respect to this diagonal clone. Thus, we have

$$f(x_1, \ldots, x_m) = \langle f^1, f^2, \ldots, f^{m+1}\rangle.$$

Since each f^i depends on at least one variable, there are a variable x_k and indicies i_1, i_2 such that both f^{i_1} and f^{i_2} depend on x_k. Replacing in f^{i_1} the variable x_k by

x_{m+1}, we obtain an essentially $(m+1)$-ary function $\bar{f}$ which yields f, with the substitution $x_{m+1}=x_k$ as required.

Lemma 8. *If* $p_2(\mathbf{B})=0$ *and* $\mathbf{B}$ *is not a q-ary extension of a Boolean reduct for any* $q\geqq 4$, *then* $S(\mathbf{B}-\mathbf{A})=\{m, m+1, \ldots\}$ *for some* $m\geqq 3$.

Proof. It is enough to show that $\mathbf{B}$ satisfies condition $(+)$ for $k=3$. For the proof one should consider three cases corresponding to Sections 3, 4 and 6 of [5] (note that in Section 7, actually, 4-ary extensions of Boolean reducts are considered), and observe that the constructions given there satisfy the requirements of our condition $(+)$ for $k=3$.

Lemma 9. *If* $\mathbf{B}$ *is an m-ary extension of a Boolean reduct* $\mathbf{I}$ *with* $m\geqq 4$, *then either* $S(\mathbf{B}-\mathbf{A})=\{q, q+1, \ldots\}$ *for some* $q\geqq m$ *or* $S(\mathbf{B}-\mathbf{A})=\{3, 5, 7, \ldots\}\cup\cup\{m, m+1, \ldots\}$.

Proof. For $f\in P_n(\mathbf{B})$ with $n\geqq m-1$ put $\bar{f}=fL_1$, where L_1 is the mapping defined in [8], Section 5.2. It follows (by properties of L_1) that $\mathbf{B}$ satisfies condition $(+)$ for $k=m-1$. (In [7] it is assumed that $m\geqq 5$, but the construction works also for $m=4$, since the conditions (i), (ii) in [8], p. 242, hold for $m=4$ as well (cf. [5], p. 111)).

Now, if q is the least integer such that $q\in S(\mathbf{B}-\mathbf{A})$ and $q\geqq m$, then by Lemma 1, $S(\mathbf{B}-\mathbf{A})=\{q, q+1, \ldots\}$.

In turn, $q<m$ means that one of the essentially n-ary functions of $\mathbf{I}$ with $3\leqq n<m$ is in $\mathbf{B}-\mathbf{A}$, and as each Boolean reduct function generates $\mathbf{I}$, we have $\{3, 5, 7, \ldots\}\subseteq\subseteq S(\mathbf{B}-\mathbf{A})$. Since $\mathbf{B}$ satisfies $(+)$ for $k=m-1$ (applying this condition to Boolean reduct functions), we get $S(\mathbf{B}-\mathbf{A})=\{3, 5, 7, \ldots\}\cup\{m, m+1, \ldots\}$ regardless as to whether m is even or odd. The proof is complete.

Lemma 10. *If* $p_2(\mathbf{B})=p_3(\mathbf{B})=0$, *then* $S(\mathbf{B}-\mathbf{A})=\{m, m+1, \ldots\}$ *for some* $m\geqq 4$.

Proof. Let s be the least integer $(\geqq 4)$ with the property $p_s(\mathbf{B})>0$. Then $\mathbf{B}$ satisfies $(+)$ for $k=s$. To see this, it is enough to put $\bar{f}=fL_1$, where L_1 is as in the previous proof. The result follows by Lemma 1.

Now, Theorem 1 is a consequence of Lemmas 3, 4, 6, 7, 8, 9, and 10. The Corollary is an immediate consequence of Theorem 1. We prove Theorem 2.

To this end suppose that $P_n(\mathbf{B})$ is nonempty (i.e. $n\in S(B)$), and denote by $\mathbf{A}$ the subclone of $\mathbf{B}$ generated by $P_n(\mathbf{B})$. Then we have $n\notin S(\mathbf{B}-\mathbf{A})$. Now, if $\mathbf{A}=\mathbf{B}$, then the result is true. Hence, suppose that $\mathbf{A}$ is a proper subclone of $\mathbf{B}$ and apply Theorem 1. Observe that in cases (ii)—(iv) of Theorem 1, the second part of the theorem combined with Urbanik's result [14] yields that we always have $S(\mathbf{B}-\mathbf{A})=\emptyset$.

This contradicts the fact that $n \in S(\mathbf{B})$, while $n \notin S(\mathbf{B}-\mathbf{A})$. It follows that under our assumptions case (i) in Theorem 1 holds for some $m > n$. In particular, for every $i \leqq n$, $i \notin S(\mathbf{B}-\mathbf{A})$, i.e. $P_i(\mathbf{B}) = P_i(\mathbf{A})$. Since by assumption $k \leqq n$, and $\mathbf{B}$ is generated by k-ary functions, it follows that $\mathbf{A}=\mathbf{B}$, completing the proof of Theorem 2.

References

[1] J. Berman, Free spectra of 3-element algebras, in: *Universal Algebra and Lattice Theory*, (Proc. Conf. Puebla, 1982), Springer-Verlag, Lecture Notes 1004 (1983); pp. 10—53.

[2] B. Csákány, All minimal clones on the three-element set, *Acta Cybernet.* **6** (1983), 227—238.

[3] G. Grätzer, Composition of functions, in: *Proc. Conf. on Universal Algebra at Queens University* (Kingston, Ontario, 1969); pp. 1—106.

[4] G. Grätzer, Universal algebra and lattice theory: a story and three research problems, in: *Universal Algebra and its Links with Logic, Algebra, Combinatorics and Computer Science* (P. Burmeister et al. eds.) Heldermann (Berlin, 1984); pp. 1—13.

[5] G. Grätzer and J. Płonka, On the number of polynomials of an idempotent algebra. II, *Pacific J. Math.*, **14** (1973), 99—113.

[6] A. Kisielewicz, A remark on diagonal algebras, *Algebra Universalis*, **12** (1981), 200—204.

[7] A. Kisielewicz, Characterization of p_n-sequences for nonidempotent algebras, *J. Algebra*, **108** (1987), 102—115.

[8] A. Kisielewicz, The p_n-sequences of idempotent algebras are strictly increasing. *Algebra Universalis*, **13** (1981), 233—250.

[9] J. Płonka, Diagonal algebras, *Fund. Math.*, **58** (1966), 309—321.

[10] R. Quackenbusch, On the composition of idempotent functions, *Algebra Universalis*, **1** (1971), 7—12.

[11] I. Rosenberg, Completeness properties of multiple-valued logic algebras, in: *Computer Science and Multiple-Valued Logic*, (2nd. ed.), North-Holland (Amsterdam, 1984); pp. 144—186.

[12] I. Rosenberg, Minimal clones I: The five types, in: *Lectures in Universal Algebra* (Proc. Conf. (Szeged, 1983), Colloq. Math. Soc. J. Bolyai, vol. 43, North-Holland (Amsterdam, 1986); pp. 405—427.

[13] W. Sierpiński, Sur les functions de plusieurs variables, *Fund. Math.*, **33** (1945), 169—173.

[14] K. Urbanik, On algebraic operations in idempotent algebras, *Colloq. Math.*, **13** (1965), 129—157.

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF WROCŁAW
WYBRZEŻE WYSPIAŃSKIEGO 27
50-370 WROCŁAW, POLAND

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Modifications of congruence permutability

IVAN CHAJDA

One of the most important congruence properties is congruence permutability. Recall that an algebra A is *congruence permutable* if $\Theta \cdot \Phi = \Phi \cdot \Theta$ for each two congruences $\Theta, \Phi \in \mathrm{Con}\, A$. This importance follows especially from two basic properties: If Θ, Φ are congruences on A, then

(a) $\Theta \cdot \Phi$ is a congruence on A if and only if $\Theta \cdot \Phi = \Phi \cdot \Theta$;

(b) if $\Theta \cdot \Phi = \Phi \cdot \Theta$, then $\Theta \vee \Phi = \Theta \cdot \Phi$ in $\mathrm{Con}\, A$.

This implies among other things that congruence permutability makes it easy to investigate other congruence conditions, e.g. congruence regularity, distributivity, subdirect reducibility of subalgebras, etc. However, there are broad classes of algebras which are not congruence permutable, but are very useful in applications. The aim of this paper is to show that such varieties can satisfy some weak modifications.

1. Ideal permutable algebras

A variety $\mathcal{V}$ is *permutable* if each $A \in \mathcal{V}$ is congruence permutable. Permutable varieties were characterized by A. I. MAL'CEV [6]. Let A be an algebra with a nullary operation 0 (briefly an algebra with 0). A is *permutable at* 0 (see [1], [5]) if

$$[0]_{\Theta \cdot \Phi} = [0]_{\Phi \cdot \Psi}$$

for each $\Theta, \Phi \in \mathrm{Con}\, A$. Varieties of such algebras were characterized in [1], [2], [5]. In 1963, G. GRÄTZER [3] introduced an intermediate property. An algebra A has *weakly associative congruences* if for each subalgebra B of A and every $\Theta, \Phi \in \mathrm{Con}\, A$,

$$[[B]_\Theta]_\Phi = [[B]_\Phi]_\Theta.$$

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Lemma 1. *Let A be an algebra with a ternary polynomial $p(x, y, z)$ and unary polynomials $f_\alpha(x)$ ($\alpha \in I$) such that for each a, b of A there exists $\beta \in I$ with*

$$p(a, a, b) = b, \quad p(a, b, b) = f_\beta(a).$$

Then A has weakly associative congruences.

Proof. Let B be a subalgebra of A and $\Theta, \Phi \in \mathrm{Con}\, A$. Suppose $b \in [[B]_\Phi]_\Theta$. Then there exist $a \in B$ and $c \in A$ with $b\ \Theta\ c\ \Phi\ a$. Hence $b = p(a, a, b)\ \Phi\ p(a, c, b)$ $\Theta\ p(a, b, b) = f_\beta(a)$. However, $a \in B$ implies $f_\beta(a) \in B$ and thus $b \in [[B]_\Theta]_\Phi$. The converse inclusion can be proved analogously.

Example 1. Let $A = \{0, a, b\}$ be a three-element algebra with 0 and a binary operation "$-$" given by the table

$$\begin{array}{c|ccc} - & 0 & a & b \\ \hline 0 & 0 & 0 & 0 \\ a & a & 0 & a \\ b & b & b & 0 \end{array} .$$

A has unary polynomials $f_1(x) = x$ and $f_0(x) = x - x = 0$. Put $p(x, y, z) = z - (y - x)$. Then

$$p(x, x, z) = z - (x - x) = z - 0 = z,$$

$$p(x, z, z) = z - (z - x) = \begin{cases} f_1(x) & \text{for} \quad z = x \\ f_0(x) & \text{for} \quad z \neq x. \end{cases}$$

By Lemma 1, A has weakly associative congruences. However, A is not congruence permutable since

$$\langle a, b \rangle \in \Theta(0, a) \cdot \Theta(0, b) \quad \text{and} \quad \langle a, b \rangle \notin \Theta(0, b) \cdot \Theta(0, a).$$

Denote by $F_{\mathcal{V}}(x_1, \ldots, x_n)$ the free algebra of the variety $\mathcal{V}$, generated by the free generators $x_1, \ldots, x_n$. We show that for varieties of algebras, the concept of weakly associative congruences gives nothing new.

Theorem 1. *Let $\mathcal{V}$ be a variety of algebras. The following conditions are equivalent:*

(1) *$\mathcal{V}$ is permutable;*

(2) *every $A \in \mathcal{V}$ has weakly associative congruences.*

Proof. Evidently, (1)$\Rightarrow$(2). We prove (2)$\Rightarrow$(1). Let $\mathcal{V}$ satisfy (2). Clearly $F_{\mathcal{V}}(x)$ is a subalgebra of $F_{\mathcal{V}}(x, y, z)$ and $z \in [[F_{\mathcal{V}}(x)]_\Phi]_\Theta$ for $\Phi = \Theta(x, y)$, $\Theta = \Theta(y, z)$. By (2), this implies $z \in [[F_{\mathcal{V}}(x)]_\Theta]_\Phi$, i.e. there exist elements $b \in F_{\mathcal{V}}(x)$, $c \in F_{\mathcal{V}}(x, y, z)$

such that $\langle z, c\rangle \in \Theta(x, y)$, $\langle c, b\rangle \in \Theta(y, z)$. Hence $c=p(x, y, z)$, $b=f(x)$ for some ternary or unary polynomials p or f, respectively. The foregoing relations yield

$$z = p(x, x, z), \quad p(x, z, z) = f(x).$$

By putting $z=x$ we obtain $x=p(x, x, x)=f(x)$, hence $z=p(x, x, z)$, $p(x, z, z)=x$, which implies permutability, see [6].

On the other hand, this concept can be modified so that subalgebras of A are required to be of a special kind.

Definition 1 (see [5]). Let $\mathcal{K}$ be a class of algebras of the same type with 0. An $(n+m)$-ary polynomial $p(\vec{x}, \vec{y})$ is an *ideal polynomial in* $\vec{y}$ if $p(\vec{x}, \vec{0})=0$ is an identity in $\mathcal{K}$. A subset $I\neq\emptyset$ of $A\in\mathcal{K}$ is an *ideal* if for each ideal polynomial $p(\vec{x}, \vec{y})$ in $\vec{y}$,

$$\vec{a}\in A^n, \quad \vec{\imath}\in I^m \quad \text{imply} \quad p(\vec{a}, \vec{\imath})\in I.$$

Evidently, the intersection of any system of ideals is an ideal, thus we introduce the ideal $I(x)$ generated by a single element $x\in A$ as the intersection of all ideals containing x. From Lemma 1.2 in [5], we obtain immediately:

Lemma 2. *Let A be an algebra with* 0. *For each $c\in A$,*

$$I(c) = \{p(\vec{a}, c);\ \vec{a}\in A^n,\ p(\vec{x}, y) \text{ is an } (n+1)\text{-ary ideal polynomial in } y\}.$$

Definition 2. An algebra A with 0 has *ideal permutable congruences* if $[[I]_\Theta]_\Phi = [[I]_\Phi]_\Theta$ for each Θ, $\Phi\in\operatorname{Con} A$ and every ideal I of A. A variety $\mathcal{V}$ with 0 has *ideal permutable congruences* if each $A\in\mathcal{V}$ has this property.

Remark 1. Clearly, every permutable variety with ideals has also ideal permutable congruences. On the other hand, putting $I=\{0\}$ we obtain immediately that permutability at 0 is a special kind of ideal permutability of congruences.

Lemma 3. *For a variety $\mathcal{V}$ with* 0, *the following conditions are equivalent:*
(1) *$F_\mathcal{V}(x, y, z)$ has ideal permutable congruences;*
(2) *there exists a ternary polynomial $p(x, y, z)$ such that*

$$p(x, x, z) = z \quad \textit{and} \quad p(x, z, z)\in I(x).$$

Proof. Let $I(x)$ be the ideal of $F_\mathcal{V}(x, y, z)$ generated by x.

(1)⇒(2): Put $\Theta=\Theta(y, z)$, $\Phi=\Theta(x, y)$. Then $z\in[[I(x)]_\Phi]_\Theta$. By (1), $z\in[[I(x)]_\Theta]_\Phi$, whence (2) is evident.

(2)⇒(1): Let I be an ideal of $F_\mathcal{V}(x, y, z)$, let Θ, $\Phi\in\operatorname{Con} F_\mathcal{V}(x, y, z)$ and $c\in[[I]_\Phi]_\Theta$. Then there exist elements $a\in F_\mathcal{V}(x, y, z)$ and $i\in I$ with $\langle c, a\rangle\in\Theta$, $\langle a, i\rangle\in\Phi$. Put $d=p(i, a, c)$. By (2), we have

$$\langle c, d\rangle = \langle p(i, i, c), p(i, a, c)\rangle\in\Phi, \quad \langle d, p(i, c, c)\rangle = \langle p(i, a, c), p(i, c, c)\rangle\in\Theta,$$

and $p(i, c, c)\in I(i)\subseteq I$. Thus $c\in[[I]_\Theta]_\Phi$.

Theorem 2. *For a variety $\mathcal{V}$ with 0, the following conditions are equivalent:*

(1) *$\mathcal{V}$ has ideal permutable congruences;*

(2) *there exist a ternary polynomial $p(x, y, z)$ and a binary ideal polynomial $f(z, x)$ in x such that*

$$p(x, x, z) = z \quad \text{and} \quad p(x, z, z) = f(z, x).$$

Proof. (1)⇒(2): If $\mathcal{V}$ has ideal permutable congruences, then $F_{\mathcal{V}}(x, y, z)$ also has this property. By Lemma 3, there exists a ternary polynomial $p(x, y, z)$ with $p(x, x, z)=z$ and $p(x, z, z)\in I(x)$. By Lemma 2, there exists an $(n+1)$-ary ideal polynomial $g(x_1, \ldots, x_n, x)$ in x such that $p(x, z, z)=g(x_1, \ldots, x_n, x)$. Since $F_{\mathcal{V}}(x, y, z)$ has three generators x, y, z, we can take $n=3$ and $x_1=x$, $x_2=y$, $x_3=z$, thus $p(x, z, z)=g(x, y, z, x)$. Moreover, $p(x, z, z)$ does not depend on y, thus g also has this property. Hence $p(x, z, z)=f(z, x)$ for some ideal polynomial $f(z, x)$ in x.

(2)⇒(1): Since for $i\in I$ we have $f(z, i)\in I(i)$, this implication can be proved in a routine way.

2. Permutability in lattice varieties

By a *lattice variety* we mean a variety $\mathcal{V}$ of type $\tau=\{\vee, \wedge, \ldots\}$ such that the reduct of $A\in\mathcal{V}$ onto $\{\vee, \wedge\}$ is a lattice.

Remark 2. It is known that every relatively complemented distributive lattice is congruence permutable. Denote by $r(x, y, z)$ the relative complement of y in the interval $[x\wedge y\wedge z, x\vee y\vee z]$. Denote by $\mathcal{R}$ the class of all relatively complemented distributive lattices. Clearly $\mathcal{R}$ is a lattice variety of type $\varrho=\langle\vee, \wedge, r\rangle$. Since $r(x, x, z)=z$ and $r(x, z, z)=x$, $\mathcal{R}$ is permutable.

Remark 3. Let L be a pseudocomplemented lattice with 0. Put $f(x, y)=x\wedge y^*$ (y^* is the pseudocomplement of y). Denote by $\mathcal{S}$ the class of all pseudocomplemented lattices. Then $\mathcal{S}$ is a lattice variety of type $\sigma=\langle\vee, \wedge, {}^*, 0\rangle$, where * is the unary operation of pseudocomplementation. Clearly,

$$f(x, x) = x\wedge x^* = 0, \quad f(x, 0) = x\wedge 0^* = x,$$

thus f satisfies the identites of [1] ensuring permutability at 0 (however, $\mathcal{S}$ is not permutable).

Theorem 3. *A lattice variety $\mathcal{V}$ with 0 has ideal permutable congruences if and only if there exists a ternary polynomial $p(x, y, z)$ such that $p(x, x, z)=z$ and $p(x, z, z)\leqq x$.*

Proof. Let L be a lattice with 0. Clearly, $p(x, y)=x\wedge y$ is an ideal polynomial in y and

$$I(c) = \{p(a, c);\ a\in L \text{ and } p(x, y) = x\wedge y\} = \{b\in L;\ b \leqq c\}.$$

By Theorem 2, $\mathscr{V}$ has ideal permutable congruences if and only if $F_{\mathscr{V}}(x, y, z)$ has this property. Hence, the assertion follows directly from Lemma 3.

Let L be a lattice and a, b be elements of L. An element $a*b$ is a *relative pseudocomplement of a with respect to b* (see [4]) if for any $x\in L$ we have

$$(*) \qquad a\wedge x \leqq b \quad \text{if and only if} \quad x \leqq a*b.$$

Denote by $\mathscr{P}$ the class of all relative pseudocomplemented lattices with 0, i.e. $\mathscr{P}$ is a class of type $\pi=\langle\vee, \wedge, *, 0\rangle$ where $*$ is a binary operation of relative pseudocomplement.

Theorem 4. *Each lattice variety $\mathscr{V}\subseteq\mathscr{P}$ (of type π) has ideal permutable congruences. $\mathscr{P}$ is not permutable.*

Proof. Put $p(x, y, z)=(y*x)\wedge z$. Then by $(*)$,

$$p(x, x, z) = x*x\wedge z = 1\wedge z = z, \quad p(x, z, z) = z*x\wedge z \leqq x.$$

By Theorem 3, $\mathscr{V}$ has ideal permutable congruences. By [4], a distributive lattice is congruence permutable if and only if it is relative complemented. Since there exist relative pseudocomplemented distributive lattices that are not relative complemented, the second statement is evident.

3. Weak permutability

The concept of congruence permutability can be localized.

Definition 3. Let $c\in A$. A is *weakly permutable in c* if

$$\langle x, c\rangle\in\Theta \quad \text{and} \quad \langle c, y\rangle\in\Phi \quad \text{imply} \quad \langle x, y\rangle\in\Phi\cdot\Theta$$

for every $x, y\in A$ and each $\Theta, \Phi\in\mathrm{Con}\, A$. A variety $\mathscr{V}$ with a nullary operation c is *weakly permutable in c* if each $A\in\mathscr{V}$ has this property.

Theorem 5. *Let $\mathscr{V}$ be a variety with a nullary operation c. The following conditions are equivalent:*

(1) *$\mathscr{V}$ is weakly permutable in c;*

(2) *there exists a binary polynomial $f(x, y)$ such that*

$$f(x, c) = x = f(c, x).$$

Proof. (1)⇒(2): Since $\langle x, c\rangle \in \Theta(x, c)$ and $\langle c, y\rangle \in \Theta(y, c)$ in $F_{\mathscr{V}}(x, y)$, (1) implies the existence of $d \in F_{\mathscr{V}}(x, y)$ with $\langle x, d\rangle \in \Theta(y, c)$ and $\langle d, y\rangle \in \Theta(x, c)$. Hence $d = f(x, y)$ for a binary polynomial f and (2) is evident.

(2)⇒(1) can be proven in the routine way.

Example 2. The variety of ($\vee$-semi-) lattices with 0 and the variety of additive groupoids with 0 are weakly permutable in 0, as we may put $f(x, y) = x \vee y$ resp. $f(x, y) = x + y$.

Remark 4. An algebra A is congruence permutable if and only if it is weakly permutable in each element c of A. If $\mathscr{V}$ is a permutable variety and $m(x, y, z)$ is its Mal'cev polynomial, put $f_y(x, z) = m(x, y, z)$ for each $y \in A \in \mathscr{V}$. Then

$$f_y(x, y) = m(x, y, y) = x, \quad f_y(y, x) = m(y, y, x) = x$$

are the polynomials desired by Theorem 5.

Definition 4. Let $p(x, y)$ be a binary polynomial in a variety $\mathscr{V}$. $\mathscr{V}$ is *weakly permutable in* $p(x, y)$ if

$$\langle x, p(x, y)\rangle \in \Theta \quad \text{and} \quad \langle p(x, y), y\rangle \in \Phi \quad \text{imply} \quad \langle x, y\rangle \in \Phi \cdot \Theta$$

for each $A \in \mathscr{V}$, for every $x, y \in A$ and every $\Theta, \Phi \in \operatorname{Con} A$.

Theorem 6. *Let $p(x, y)$ be a binary polynomial in $\mathscr{V}$. The following conditions are equivalent:*

(1) *$\mathscr{V}$ is weakly permutable in $p(x, y)$;*

(2) *there exists a binary polynomial $q(x, y)$ such that*

$$q(x, p(x, y)) = x, \quad q(p(x, y), y) = y.$$

The proof is analogous to that of Theorem 4.

Example 3. Every variety of lattices is weakly permutable in $x \vee y$ (or in $x \wedge y$).

4. Relational properties

H. Werner [7] proved that permutability of $\mathscr{V}$ is equivalent to some properties of compatible relations. We proceed to show how this can be modified for our modifications of permutability. A binary relation R on an algebra A is *compatible* if it has the substitution property with respect to all operations of A, i.e. R is a subalgebra of $A \times A$.

Definition 5. Let R be a compatible binary relation on an algebra A. R is *ideal-transitive* if $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in R$ imply $\langle a, c'\rangle \in R$ for some $c' \in I(c)$. R is *ideal-symmetric* if $\langle a, b\rangle \in R$ implies $\langle b, a'\rangle \in R$ for some $a' \in I(a)$.

Theorem 7. *For a variety $\mathscr{V}$ with 0, the following conditions are equivalent:*

(1) *$\mathscr{V}$ has ideal permutable congruences;*

(2) *every reflexive compatible binary relation R on $A \in \mathscr{V}$ is ideal-transitive;*

(3) *every reflexive compatible binary relation R on $A \in \mathscr{V}$ is ideal-symmetric.*

Proof. (1)⇒(2): If $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in R$, then also $\langle p(b, b, a), p(c, b, b)\rangle \in R$ for the ternary polynomial $p(x, y, z)$ derived in Theorem 2, thus $\langle a, c'\rangle \in R$ for $c' = p(c, b, b) \in I(c)$.

(1)⇒(3): Analogously, $\langle a, b\rangle \in R$ implies $\langle p(a, a, b), p(a, b, b)\rangle \in R$ thus $\langle b, a'\rangle \in R$ for $a' = p(a, b, b) \in I(a)$.

(2)⇒(1): Let R be a reflexive compatible binary relation on $F_{\mathscr{V}}(x, y, z)$ generated by the pairs $\langle z, y\rangle$ and $\langle y, x\rangle$. Then $\langle z, y\rangle \in R$ and $\langle y, x\rangle \in R$ and, by (2), there exists $d \in I(x)$ such that $\langle z, d\rangle \in R$. Hence, there exists a 5-ary polynomial $q(x_1, x_2, x_3, x_4, x_5)$ with $z = q(y, z, x, y, z)$ and $d = q(x, y, x, y, z)$. Put $p(x, y, z) = q(x, z, x, y, z)$. Then

$$p(x, x, z) = q(x, z, x, x, z) = z, \quad p(x, z, z) = q(x, z, x, z, z) \in I(x).$$

(3)⇒(1): Let R be a reflexive compatible binary relation on $F_{\mathscr{V}}(x, y)$ generated by the pair $\langle x, y\rangle$. Then $\langle x, y\rangle \in R$ and, by (3), $\langle y, c\rangle \in R$ for some $c \in I(x)$. Hence, there exists a ternary polynomial $p(x, y, z)$ with $y = p(x, x, y)$, $c = p(x, y, y)$.

Theorem 8. *Let $\mathscr{V}$ be a variety with a nullary operation c. The following conditions are equivalent:*

(1) *$\mathscr{V}$ is weakly permutable in c;*

(2) *every compatible relation R on $A \in \mathscr{V}$ satisfies: if $\langle a, c\rangle \in R$ and $\langle c, b\rangle \in R$, then $\langle a, b\rangle \in R$.*

Proof. (1)⇒(2) is evident. We prove (2)⇒(1). Let R be a compatible binary relation on $F_{\mathscr{V}}(x, y)$ generated by two pairs $\langle x, c\rangle$ and $\langle c, y\rangle$. Then $\langle x, c\rangle \in R$, $\langle c, y\rangle \in R$ imply $\langle x, y\rangle \in R$, i.e. there exists a binary polynomial f such that

$$\langle x, y\rangle = f(\langle x, c\rangle, \langle c, y\rangle).$$

By writing this componentwise, we obtain (2) of Theorem 5.

The proof of the following theorem is analogous.

Theorem 9. *Let $p(x, y)$ be a binary polynomial of the variety $\mathscr{V}$. The following conditions are equivalent:*

(1) *$\mathscr{V}$ is weakly permutable in $p(x, y)$;*

(2) *every compatible binary relation R on $A \in \mathscr{V}$ satisfies: if $\langle a, p(a, b)\rangle \in R$ and $\langle p(a, b), b\rangle \in R$, then $\langle a, b\rangle \in R$.*

References

[1] I. Chajda, A localization of some congruence conditions in varieties with nullary operations, *Ann. Univ. Sci. Budapest, Sect. Math.*, to appear.
[2] J. Duda, Arithmeticity at 0, *Czechoslovak Math. J.*, 37 (1987), 197—206.
[3] G. Grätzer, On the Jordan—Hölder theorem for universal algebras, *Publ. Math. Inst., Hung. Acad. Sci, Ser. A*, 3 (1963), 397—406.
[4] G. Grätzer, *General Lattice Theory*, Akademie-Verlag (Berlin, 1978).
[5] H.-P. Gumm, A. Ursini, Ideals in universal algebras, *Algebra Universalis* 19 (1984), 45—54.
[6] A. I. Mal'cev, On the general theory of algebraic systems, *Mat. Sb.*, 35 (1954), 3—20. (Russian).
[7] H. Werner, A Mal'cev condition for admissible relations, *Algebra Universalis*, 3 (1973), 263.

LIDOVÝCH MILICÍ 22
750 00 PŘEROV, CZECHOSLOVAKIA

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Two finiteness conditions for finitely generated and periodic semigroups

GIUSEPPE PIRILLO

1. Introduction. In this paper we present two finiteness conditions for a finitely generated and periodic semigroup. The first condition requires that the function which counts the number of elements of the first i generations grows less rapidly than $i(i+3)/2$. The second one requires that the semigroup be repetitive and that there should exist a positive integer p such that each element of the semigroup has order smaller than p.

2. Notations and preliminaries. Let A be an *alphabet*, A^+ (resp. A^*) the *free semigroup* (resp. *free monoid*) on A. For any *word* $w \in A^+$, $|w|$ will be the *length* of w. A word v is a *factor* of a word w if there exist two words $u, u' \in A^*$ such that $w = uvu'$.

Let S be a *semigroup*, G a finite set of *generators* of S and $\bar{G}$ be a copy of G. Let φ: $\bar{G}^+ \to S$ be the (epi-)morphism defined by $\varphi(\bar{g}) = g$, for each $\bar{g} \in \bar{G}$. Suppose that in $\bar{G}$ a total order $<$ is given and consider the lexicographic order induced by $<$ on $\bar{G}^i$, for each positive integer i (i.e., given two words $w, w' \in \bar{G}^i$ we say that w *precedes* w' *in the lexicographic order* if there exists a positive integer j, $1 \leqq j \leqq i$, such that

$$w = ua_jv, \quad w' = ub_jv'$$

where u, v, u' are words of $\bar{G}^*$, a_j and b_j are letters of G such that $a_j < b_j$).

Definition 1. We say that a word $w \in G^+$ is the canonical word of an element $s \in S$ if:

1) $\varphi(w) = s$,

2) for any other word $w' \in \bar{G}^+$ such that $\varphi(w') = s$ we have either

a) $|w| < |w'|$, or

b) $|w| = |w'|$ and w precedes w' in the lexicographic order.

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Fact 1. *A factor of a canonical word is a canonical word.*

Namely, if v is a factor of w and w is the canonical word of an element $s \in S$, then there exists another element $s' \in S$ such that v is the canonical word of s'.

Now consider the following subsets of S:

$$G^i = \varphi(\bar{G}^i), \quad P_i = \bigcup_{j=1}^{i} G^j, \quad R_i = P_i - P_{i-1}$$

(where $P_0 = \emptyset$) and the functions p, r from the set of positive integers into the set of positive integers defined by

$$p(i) = \operatorname{card} P_i, \quad r(i) = p(i) - p(i-1),$$

for each positive integer i.

Definition 2. We say that a finitely generated semigroup has linear growth if there exists a positive integer k such that $p(i) \leqq ki$, for each positive integer i.

For future reference we state below without proof a theorem due to JUSTIN [4].

Theorem 1. *For a finitely generated semigroup, the following conditions are equivalent.*

a) *There exists a finite subset F of $\bar{G}^+$ such that the canonical word of each element of the semigroup belongs to F or has a factorization $w = uv^n u'$ where $u, v, u' \in F$ and n is a positive integer.*

b) *There exists a positive integer m such that $r(i) \leqq m$, for each positive integer i.*

c) *The semigroup has linear growth.*

d) *There exists a positive integer i such that $p(i) < i(i+3)/2$.*

e) *There exists a positive integer d such that $r(d) \leqq d$.*

3. Two conditions of finiteness for finitely generated semigroup. The Burnside problem for semigroups has been recently studied by several authors (see, for example, DE LUCA [2], DE LUCA and RESTIVO [3], RESTIVO and REUTENAUER [6]).

We present here two conditions which are natural in the study of repetitive semigroups (see definition below) and are necessary and sufficient conditions for the finiteness of finitely generated and periodic semigroups.

Our first result is the following proposition.

Proposition 1. *Let S be a finitely generated semigroup. The following conditions are equivalent:*

a) *S is finite.*

b) *S is periodic and has linear growth.*

Proof. (a)$\to$(b) is trivial. Using Theorem 1, the proof of (b)$\to$(a) is just a remark. In fact, the finiteness of F (see condition a) of Theorem 1 and the periodicity of S gives a suitable positive integer q such that

$$S = \varphi(F) \cup \varphi(\{Fv^n F\colon v \in F,\ n \leqq q\}),$$

that is S is finite.

So, we have proved, without much effort, that if S is a periodic semigroup such that $p(i) < i(i+3)/2$ for a suitable non-negative integer i, then S is finite.

Now, let us introduce following definition.

Definition 3. Given a (finite) alphabet A and a semigroup S, a morphism $\alpha\colon A^+ \to S$ is called repetitive if for each integer k there exists a positive integer $l_\alpha(k)$ such that each word $w \in A^+$ of length at least $l_\alpha(k)$ can be factorized as follows:

$$w = w_0 w_1 \dots w_k w_{k+1}$$

where $w_0, w_{k+1} \in A^*$, $w_1, w_2, \dots, w_k \in A^+$, and $\alpha(w_1) = \alpha(w_2) = \dots = \alpha(w_k)$.

Definition 4. A semigroup S is called repetitive if, for each finite alphabet A, each morphism $\alpha\colon A^+ \to S$ is repetitive.

We can prove the following proposition.

Proposition 2. *Let S be a finitely generated semigroup. The following conditions are equivalent:*

a) *S is finite.*

b) *S is periodic, repetitive and there exists a positive integer p such that each element of S has order at most p.*

Proof. The only non-trivial part of (a)$\to$(b) is "S finite"$\to$"S repetitive". This has been proved by Justin [5] (see also [7]).

(b)$\to$(a). Let G be a finite set of generators of S. Let $\bar{G}$ and $\varphi\colon \bar{G} \to S$ be as in the preceding paragraph. By way of contradiction, let S be infinite. We have that the subset of $\bar{G}^+$ of the canonical words of the elements of S is infinite and so there exists a canonical word w of length greater that $l_\varphi(p+1)$.

By the repetitivity of φ we have $w = w_0 w_1 \dots w_p w_{p+1} w_{p+2}$ where $w_0, w_{p+2} \in \bar{G}^*$, $w_1, \dots, w_p, w_{p+1} \in \bar{G}^+$ and $\varphi(w_1) = \dots = \varphi(w_p) = \varphi(w_{p+1})$. Now considering the property of p one easily sees that the word $w_1 w_2 \dots w_p w_{p+1}$ is too long to be a canonical word of an element of S. This is in contradiction with Fact 1.

Remark. In the proof of Proposition 2 we can make use only of the repetitivity of the epimorphism φ.

Proposition 2 provides us with one of the few criteria to establish if an infinite semigroup is repetitive.

Let us show that the semigroup $S=A^+\cup\{0\}/\sim$ (where A is a finite alphabet with at least three elements, 0 is a zero and $\sim$ is the congruence generated by the relation R on A^+ defined by $ww\ R\ 0$ for each $w\in A^+$) is non-repetitive.

In fact, the semigroup S is infinite (this is a consequence of the Thue construction of infinite square-free words over each alphabet with at least three elements, see [1]), evidently periodic and its elements have at most order 2. So, by Proposition 2, S cannot be repetitive.

On the contrary, the semigroup $S'=A'/\approx$ (where A is a finite alphabet, $\approx$ is the congruence generated by the relation R' defined by $ww\ R'\ w$ for each $w\in A^+$) is finite (see again [1]) and therefore repetitive (see [5]).

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References

[1] J. Berstel, Chapter 2 in: M. Lothaire, *Combinatorics on word,* Addison—Wesley (Reading, Mass. 1983).

[2] A. de Luca, On the Burnside problem for semigroups and languages, Atti Sem. Mat. Fis. Univ. Milano, **53** (1983).

[3] A. de Luca and A. Restivo, A finiteness condition for finitely generated semigroups, *Semigroup Forum,* **28** (1984), 123—134.

[4] J. Justin, Groupes et semigroupes à croissance lineaire, *C. R. Acad. Sci. Paris,* **273** (1971), 212—214.

[5] J. Justin, Generalization du Théorème de van der Waerden sur les semigroupes répétitifs, *J. Combin. Theory Ser. A,* **12** (1972), 357—367.

[6] A. Restivo and C. Reutenauer, On the Burnside problem, *J. Algebra,* **89** (1984), 102—104.

[7] M. P. Schützenberger, *Quelques problèmes combinatoires de la théorie des automates* (rédigé par J.-F. Perrot) Cours professé à l'Institut de Programmation, Fac. Sciences de Paris (1966—67).

UNIVERSITA DEGLI STUDI
INSTITUTO MATEMATICO „ULISSE DINI"
VIALE MORGAGNI 87/A
50134 FIREZNE, ITALY

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Неприводимые правые правоальтернативные представления

Ц. ДАШДОРЖ

В работах [2], [1] А. М. Слинько и И. П. Шестаков определили понятия правого представления и правого модуля для алгебр произвольного многообразия. В настоящей работе изучается неприводимые правые правоальтернативные представления (бесконечномерных) правоальтернативных алгебр. В частном случае альтернативных алгебр и представлений утверждения теорем 1 и 2 дают результат работы И. П. Шестакова [1]. Напомним, что для конечномерных правоальтернативных алгебр исчерпывающее описание неприводимых правых правоальтернативных представлений получено И. П. Шестаковым [1].

Пусть Φ — ассоциативно-коммутативное кольцо с единицей. Алгебра называется правоальтернативной, если она удовлетворяет тождествам

$$(xy)y = xy^2, \quad ((xy)z)y = x((yz)y).$$

Пусть A некоторая правоальтернативная алгебра над Φ, Y — некоторый Φ-модуль, $\mathrm{End}_\Phi(Y)$ алгебра эндоморфизмов Φ-модуля Y.

Определение 1. Если для Φ-линейного отображения $\varrho\colon A \to \mathrm{End}_\Phi(Y)$ выполняются равенства

$$(a^2)^\varrho = (a^\varrho)^2, \quad (ab \cdot a)^\varrho = a^\varrho b^\varrho a^\varrho \tag{1}$$

для любых элементов a, b из A, то модуль Y называется правым правоальтернативным модулем над A, ϱ — правым правоальтернативным представлением алгебры A.

Будем использовать обозначение va вместо va^ϱ, $v \in Y$, $a \in A$, и

$$(a, b) = a^\varrho b^\varrho - (ab)^\varrho, \quad (a, b)^* = b^\varrho a^\varrho - (ab)^\varrho, \quad a, b \in A;$$

$$(A, A) = \{(a, b) \mid a, b \in A\}, \quad (A, A)^* = \{(a, b)^* \mid a, b \in A\}.$$

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Определение 2. Если $\mathrm{Ann}\, Y=\{a\in A \mid Ya=0\}=0$, то A-модуль Y называется точным.

Определение 3. Модуль Y называется неприводимым, если 0 является единственным собственным подмодулем A-модуля Y.

Пусть $\mathbf{C}$ — алгебра Кэли—Диксона с канонической инволюцией $^-: \mathbf{C}\to\mathbf{C}$, а $\mathrm{reg}\,\mathbf{C}$, $\overline{\mathrm{reg}\,\mathbf{C}}$ модули, получающиеся введением на векторном пространстве $Y=\mathbf{C}$ соответственно следующих действий алгебры $\mathbf{C}$: если $v\in Y$, $a\in\mathbf{C}$, то $v\cdot a=va$ для модуля $\mathrm{reg}\,\mathbf{C}$, $v\cdot a=v\bar{a}$, для модуля $\overline{\mathrm{reg}\,\mathbf{C}}$, где через va обозначено произведение элементов v и a в алгебре $\mathbf{C}$.

Основным результатом работы является следующая:

Теорема 1. *Пусть Y — точный неприводимый правый правоальтернативный модуль над правоальтернативной алгеброй A. Тогда либо*

(1) *A примитивное ассоциативное кольцо и Y правый или левый ассоциативный A-модуль, либо*

(2) *A является алгеброй Кэли—Диксона над своим центром и $Y_A\in\{\mathrm{reg}\,\mathbf{C}, \overline{\mathrm{reg}\,\mathbf{C}}\}$.*

Рассмотрим некоторые определения и утверждения из теорий йордановых алгебр, необходимые для доказательства основной теоремы.

Определение 4. Если для Φ-модуля J с единицей 1 определена композиция $v_x(y)$, квадратичная по x и линейная по y, удовлетворяющая аксиомам

$$v_1 = id_J, \quad v_x V_{x,y} = V_{y,x} v_x = v_{v_x(y),x}, \quad v_{v_x(y)} = v_x v_y v_x,$$

где

$$v_{x,z} = v_{x+z} - v_x - v_z, \quad V_{x,y}(z) = v_{x,z}(y) = \{xyz\},$$

то тройка $(J, v, 1)$ называется квадратичной йордановой алгеброй с 1. В этом случае единица определяет операцию $x^2=v_x(1)$ возведения в квадрат. Если для Φ-модуля J определены композиции $v_x(y)$, x^2 и модуль $J^{\#}=1\cdot\Phi+J$ является квадратичной йордановой алгеброй с 1, то J называется квадратичной йордановой алгеброй.

Известно, что если A правоальтернативная алгебра, то операторы $v_x(y)=xy\cdot x$, $x^2=xx$ определяют на A структуру квадратичной йордановой алгебры обозначим её через A^+. В дальнейшем под йордановой алгеброй мы будем понимать только квадратичную йорданову алгебру.

Определение 5. Если для некоторого Φ-линейного отображения $\varrho: J\to\mathrm{End}_\Phi(Y)$ выполняются равенства

$$(a^2)^\varrho = (a^\varrho)^2, \quad \{aba\}^\varrho = a^\varrho b^\varrho a^\varrho \tag{2}$$

где $\{aba\}=\dot{v}_a(b)$ для любых элементов a, b из J, то модуль Y называется специальным йордановым модулем над J, а отображение ϱ — специальным представлением алгебры J. Заметим, что если Y правый правоальтернативный модуль над A, то Y является специальным йордановым модулем над A^+.

Точность и неприводимость для специального йорданового модуля определяется аналогично соответствующим определением для правоальтернативного модуля. Элемент $a\in J$ называется абсолютным делителем нуля йордановой алгебры J, если $v_a(J^\#)=0$, где $J^\#=1\cdot\Phi+J$, 1 — формальная единица. Йорданова алгебра невырождена, если она не содержит ненулевых абсолютных делителей нуля. Известно, что если I, K — идеалы йордановой алгебры J, то Φ-модуль $v_I(K)$, порожденный множеством

$$\{v_a(k)\,|\,a\in I,\ k\in K\}$$

также является идеалом в J. Йорданова алгебра J первична, если для любых двух идеалов K, L алгебры J из $v_K(L)=0$ следует либо $K=0$, либо $L=0$.

Лемма 1. *Если Y специальный йорданов модуль над J, то множество* $\mathrm{Ann}\,Y=\{a\in J\,|\,Ya=0\}$ *является идеалом в J, а $J/\mathrm{Ann}\,Y$ специальной йордановой алгеброй.*

Доказательство. Пусть ϱ — специальное представление алгебры J. Ясно, что отображение ϱ: $a\to a^\varrho$, где $a\in J$, $a^\varrho\in\mathrm{End}_\Phi(Y)$, является гомоморфизмом йордановой алгебры J в йорданову алгебру $\mathrm{End}_\Phi(Y)^+$. Таким образом, ядро этого гомоморфизма

$$\mathrm{Ker}\,\varrho=\{a\in J\,|\,Ya=0\}=\mathrm{Ann}\,Y$$

является идеалом в J. В силу изоморфизма $\mathrm{Im}\,\varrho\cong J/\mathrm{Ann}\,Y$ фактор-алгебра $J/\mathrm{Ann}\,Y$ специальна, так как $\mathrm{Im}\,\varrho=\{a^\varrho|a\in J\}$ йорданова подалгебра, порожденная операторами a^ϱ в $\mathrm{End}_\Phi(Y)^+$. Лемма доказана.

Таким образом, если Y точный специальный J-модуль, то в силу леммы 1 алгебра J-специальна. Обозначим через $R_Y(J)$ ассоциативную подалгебру алгебры $\mathrm{End}_\Phi(Y)$, порождённую множеством $\{a^\varrho|a\in J\}$.

Лемма 2. *Пусть Y точный неприводимый специальный модуль над йордановой алгеброй J. Тогда алгебра J первична и невырождена.*

Доказательство. Докажем невырожденность алгебры J. Только что мы заметили, что алгебра J специальна. В силу теоремы А. М. Слинько—В. Г. Скосырского [2; стр. 355], имеем $M(J)\subseteq\mathrm{Loc}\,(J)$, где $M(J)$ радикал Маккриммона, а $\mathrm{Loc}\,(J)$ локально-нильпотентный радикал. Далее, по теореме В. Г. Скосырского [2] имеем $\mathrm{Loc}\,(J)\subseteq\mathrm{Loc}\,(R_Y(J))$. Ясно, что представление ϱ

алгебры J индуцирует неприводимое представление $\bar{\varrho}$ алгебры $R_Y(J)$ в модуле Y и $\mathrm{Jac}\,(R_Y(J))\subseteq\mathrm{Ker}\,\bar{\varrho}$, где $\mathrm{Jac}\,(R_Y(J))$ радикал Джекобсона алгебры $R_Y(J)$. Следовательно, если $a\in M(J)$, то $a\in\mathrm{Jac}\,(R_Y(J))$ и $a^{\varrho}=a^{\bar{\varrho}}=0$ т. е. $a\in\mathrm{Ker}\,\varrho=0$.

Докажем невырожденность алгебры J. Пусть $\{KLK\}=\{\{klk\}|k\in K,\ l\in L\}=0$, где K, L ненулевые идеалы в J. Если $K\cap L=M\neq 0$, то в силу невырожденности M получаем $\{MMM\}\neq 0$. Но $\{MMM\}\subseteq\{KLK\}=0$. Таким образом $K\cap L=0$, ясно что $K\circ L\subseteq K\cap L=0$, здесь

$$K\circ L=\{k\circ l\,|\,k\circ l=(k+l)^2-k^2-l^2,\ k\in K,\ l\in L\}$$

т. е. $K\circ L=0$. Из точности модуля Y следует, что $YK\neq 0$. Используя линеаризацию равенство $(a^{\varrho})^2=(a^{\varrho})^2$ для любого $a\in J$, мы имеем

$$vk\cdot a=-va\cdot k+v(a\circ k)\in YK.$$

Отсюда следует, что YK подмодуль J модуля Y. Ввиду неприводимости Y имеем $YK=Y$. Тогда найдутся элементы $k\in K$, $l\in L$, такие, что $Yk\cdot l\neq\{0\}$. Заметим, что

$$vk\cdot l=-vl\cdot k+v(k\circ l)=-vl\cdot k$$

для любого $v\in V$. Значит, $Yk\cdot l=Yl\cdot k$. Далее, в силу линеаризованного тождества (1) для любых элементов $a\in J$ и $v\in Y$ имеем

$$(vk\cdot l)a=-(va\cdot l)k+v\{kla\}=-(va\cdot l)k$$

так как $\{kla\}\in K\cap L=0$. Отсюда следует, что $Yk\cdot l=Yl\cdot k$ есть J-подмодуль модуля Y. Ввиду неприводимости модуля Y имеем $Y=Yk\cdot l$. Однако по тождеству (1)

$$Y=Yk\cdot l=(Yk\cdot l)k\cdot l=Y\{klk\}\cdot l=\{0\},$$

что противоречит неприводимости модуля Y. Лемма доказана.

Следствие 1. *Если Y — неприводимый правый правоальтернативный модуль над A, то* $\mathrm{Ann}\,Y=\{a\in A|Ya=0\}$ *является идеалом в A.*

Доказательство. Если $b\in\mathrm{Ann}\,Y$, то для любого элемента $a\in A$ имеем $Y(ab+ba)=Ya\cdot b+Yb\cdot a=0$. Значит, $\mathrm{Ann}\,Y$ является идеалом в йордановой алгебре A^+. Таким образом, Y точный неприводимый йорданов модуль над алгеброй $\bar{A}=A^+/\mathrm{Ann}\,Y$. По лемме 2 алгебра $\bar{A}$ первична и невырождена. Значит, $\mathrm{Ann}\,Y$ является идеалом в A согласно лемме Тэди [5].

Рассмотрим свободную ассоциативную Φ-алгебру от множества порождающих $X=\{x_1,x_2,\ldots\}$ и свободную специальную йорданову алгебру от X; $SJ[X]\subseteq\mathrm{Ass}\,[X]^+$. В работе [4] построены вполне характеристический идеал T

алгебры $SJ[X]$ и функция натурального аргумента $f(k)$, $k \geqq 3$, $f(3)=0$, такие что для любых элементов $t \in T^{\langle f(k)\rangle}$, $a_1, \ldots, a_k \in SJ[X]$, справедливо включение

$$\{a_1 \ldots a_i t a_{i+1} \ldots a_k\} = a_1 \ldots t \ldots a_k + a_k \ldots t \ldots a_1 \in SJ[X], \quad 1 \leqq i \leqq k.$$

Обозначим $T_m = T^{\langle f(m)\rangle}$. Пусть J специальная йорданова алгебра и R её ассоциативная обертывающая алгебра. Ясно, что алгебра J является гомоморфным образом некоторой свободной специальной алгебры $SJ[X]$. Пусть $T_m(J)$ гомоморфный образ идеала $T_m(SJ[X])$. Если I идеал в J, то $v_I(J) = \{IJI\} = \check{I}$ тоже является идеалом в J. Множество

$$\operatorname{Ann}(R, T^\infty) = \{a \in R \mid a\check{T}_m(J) = \check{T}_m(J)a = 0 \ \text{ для некоторого } \ m \geqq 1\}$$

является идеалом в алгебре R [7].

Доказательство теоремы 1. Точный неприводимый правоальтернативный модуль Y является точным неприводимым специальным модулем над йордановой алгеброй A^+. По лемме 2 йорданова алгебра A^+ первична и невырождена, следовательно, алгебра A альтернативна [7]. Значит, согласно теореме Слейтера [2] A либо ассоциативна, либо является кольцом Кэли—Диксона.

(а) Предположим, что $T(A^+) \neq 0$. В силу точности A^+-модуля Y алгебра $R_Y(A^+)$ является ассоциативной обертывающей алгеброй для A^+. В [7] доказано, что для любых элементов $a, b \in A$ выполняется включение

$$(a, b) R_Y(A^+)(a, b)^* \subseteqq \tilde{B}(R_Y(A^+)),$$

где $\tilde{B}(R_Y(A^+))$ полный прообраз радикала Бэра $B(\overline{R_Y(A^+)})$ при гомоморфизме

$$R_Y(A^+) \to R_Y(A^+)/\operatorname{Ann}(R_Y(A^+), T^\infty) = \overline{R_Y(A^+)}.$$

Докажем, что $Y\tilde{B}(R_Y(A^+)) = 0$. Пусть $v \in Y$, $W \in \tilde{B}(R_Y(A^+))$ и $vW \neq 0$. Из неприводимости ассоциативного $R_Y(A^+)$-модуля Y следует, что $vWW' = v$ для некоторого оператора $W' \in R_Y(A^+)$. Тогда для любого натурального n имеем $v(WW')^n = v \neq 0$. Найдется $m \geqq 1$ такое что $(WW')^m \in \operatorname{Ann}(R_Y(A^+), T^\infty)$, то есть для некоторого натурльного числа к имеем $(WW')^m \check{T}_k = 0$, $v(WW')^m \check{T}_k = v\check{T}_k = 0$. Множество $Y_1 = \{v \in Y \mid v\check{T}_k = 0\}$ является ненулевым подмодулем йорданова A^+-модуля Y. Значит, из неприводимости A^+-модуля Y следует, что $Y = Y_1$ т. е. $Y\check{T}_k = 0$. Поскольку Y — точный модуль, то $\check{T}_k = 0$. Это противоречит тому, что $\check{T}_k \neq 0$. Таким образом, получаем $Y(a, b)R_Y(A^+)(a, b)^* = 0$. Если $Y(a, b) \neq 0$, то $Y(a, b)R_Y(A^+) = Y$, что влечет $(a, b)^* = 0$. Следовательно, для любых элементов $a, b \in A$, либо $(a, b) = 0$, либо $(a, b)^* = 0$.

Лемма 3. *Пусть $Y = Y_1 \cup Y_2$, где Y_1 и Y_2 подгруппы аддитивной группы $(Y, +)$. Тогда либо $Y = Y_1$ либо $Y = Y_2$.*

Доказательство. Допустим что $Y \neq Y_1$ и $Y \neq Y_2$. Пусть $v_1 \in Y_1$, $v_1 \notin Y_2$, $v_2 \in Y_2$, $v_2 \notin Y_1$. Тогда $0 \neq v_1 + v_2 \notin Y_1$ и $v_1 + v_2 \notin Y_2$. Противоречие. Лемма доказана.

Зафиксируем элемент $a \in A$. Множества

$$A_1 = \{b \in A \mid (a, b) = 0\}, \quad A_2 = \{b \in A \mid (a, b)^* = 0\}$$

являются аддитивными подгруппами в $(A, +)$. По лемме 3 либо $A_1 = A$, либо $A_2 = A$ т. е. для любого элемента $a \in A$ либо $(a, A) = 0$ либо $(a, A)^* = 0$. Пусть $A_1 = \{b \in A \mid (b, A) = 0\}$ и $A_2 = \{b \in A \mid (b, A)^* = 0\}$. Тогда снова по лемме 3 получаем, что либо $(A, A) = 0$, либо $(A, A)^* = 0$ т. е. либо $A_1 = A$ либо $A_2 = A$.

(б) Пусть $T(A^+) = 0$. Напомним, что центром альтернативного кольца A называется множество

$$z(A) = \{z \in A \mid [z, A] = (z, A, A) = 0\}$$

где $(z, x, y) = (zx)y - z(xy)$ ассоциатор элементов $z, x, y \in A$. Через $Z^*(A)$ обозначим множество обратимых элементов центра $Z(A)$. В ассоциативном случае по теореме Маркова—Роуэна [6] и в альтернативном случае по теореме Слейтера [2] получаем, что $A_1 = Z^*(A)^{-1}A$ является простой конечномерной алгеброй над полем частных $K = Z^*(A)^{-1}z(A)$ или алгеброй Кэли—Диксона над полем K. В первом случае известно, что $\dim_K A_1 \leqq 4$, так как в алгебре A_1 тоже выполняется тождество $T(A^+) = 0$.

Лемма 4. *Пусть A — альтернативная алгебра, Y — неприводимый правый правоальтернативный модуль над A, $z \in Z(A)$, $a \in A$. Тогда $Y(z, a) = 0$.*

Доказательство. В силу линеаризованных тождеств (1) для любых элементов $v \in Y$, $b \in A$ имеем

$$(v, z, a)b = (vz \cdot a)b - (v \cdot za)b =$$

$$= -(vb \cdot a)z + v(za \cdot b + ba \cdot z) + vb \cdot za - v(za \cdot b + b \cdot za) =$$

$$= -(vb \cdot a)z + vb \cdot az = -(vb, a, z) = (vb, z, a).$$

Отсюда следует, что подмножество (Y, z, a) является подмодулем A-модуля Y.

Допустим, что $Y(z, a) = (Y, z, a) \neq 0$. Тогда ввиду неприводимости модуля Y имеем $Y(z, a) = Y$. Заметим, что $v(z, a) = v(a, z)^* = -v(z, a)^*$. Поэтому $v(z, a)(a, z)^* = 0$, так как $(a, b)(a, b)^* = 0$, (см. [2]). Но тогда $Y = Y(z, a)(z, a) = 0$. Полученное противоречие доказывает лемму.

Определим на Y структуру правого правоальтернативного A_1-модуля. Проверим сначала, что если $0 \neq v \in Y$, $0 \neq z \in Z(A)$, то $vz \neq 0$. Действительно, если $vz = 0$ то по лемме 4 имеем $0 = vz \cdot A = v \cdot zA = v \cdot Az = vA \cdot z = Yz$, откуда $z \in \operatorname{Ann} Y$, $z = 0$.

Для $v_0z=v_1\neq 0$ положим $v_1z^{-1}=v_0$. Далее положим $v\cdot z^{-1}a=vz^{-1}\cdot a$. Докажем корректность такого определения. Пусть $z_1^{-1}a=z^{-1}a_1$. Тогда $za=$ $=z_1a_1$. Сначала докажем равенство

$$vz^{-1}\cdot z_1^{-1}=vz_1^{-1}\cdot z^{-1}=v(zz_1)^{-1} \tag{3}$$

для любых $v\in Y$, $z, z_1\in Z^*(A)$. Пусть $w=vz^{-1}\cdot z_1^{-1}$, $w_1=vz_1^{-1}\cdot z^{-1}$. Отсюда следуют, что $wz_1\cdot z=v$, $wz\cdot z_1=v$, т. е.

$$w\cdot z_1z=wz_1\cdot z=v=w_1z\cdot z_1=w_1\cdot z_1z,\quad (w-w_1)\cdot z_1z=0,\quad w=w_1.$$

В силу леммы 4 и равенства (3) имеем

$$vz^{-1}z_1^{-1}za=vz_1^{-1}\cdot a=v\cdot z_1^{-1}a,\quad vz^{-1}z_1^{-1}z_1a=vz^{-1}\cdot a_1=v\cdot z^{-1}a_1$$

т. е.

$$v\cdot z_1^{-1}a=v\cdot z^{-1}a_1. \tag{4}$$

Используя (3) и (4), легко получаем, что

$$v(z^{-1}a\cdot z^{-1}a)=v(z^{-2}a^2)=vz^{-2}\cdot a^2=(vz^{-1}\cdot z^{-1})a^2=$$
$$=(vz^{-1}\cdot a^2)z^{-1}=(vz^{-1}\cdot a)a\cdot z^{-1}=(vz^{-1}\cdot a)z^{-1}\cdot a=(v\cdot z^{-1}a)\cdot z^{-1}a,$$
$$v\big((z^{-1}a\cdot z_1^{-1}b)z^{-1}a\big)=\big((v\cdot z^{-1}a)z_1^{-1}b\big)\cdot z^{-1}a.$$

Точность A_1-модуля Y и ясна.

Таким образом, в первом случае (т. е. когда A_1 — ассоциативна) либо Y является правым правоальтернативным модулем над $A_1=K$, либо $\dim_K A_1=4$. Чтобы закончить доказательство можно было сослаться на [3], но ради полноты изложения продолжим рассуждение. Если $A_1=K$, то $(A_1, A_1)=0$ (см. [2; стр. 277]).

Пусть $\dim_K A_1=4$. В A_1 можно выбрать базу 1, v_1, v_2, v_3. Рассмотрим симметрический многочлен

$$d=(x_1x_2-x_3)x_4(x_2x_1-x_4).$$

Полагая в d, $x_1=a^\varrho$, $x_2=b^\varrho$, $x_3=(ab)^\varrho$, $x_4=c^\varrho$, где $a, b, c\in A_1$, получим, что d является симметрическим многочленом от трех букв v_1^ϱ, v_2^ϱ, v_3^ϱ. Значит, по теореме Кона [2] d является йордановым многочленом от v_1^ϱ, v_2^ϱ, v_3^ϱ. Поэтому из $1d=0$ легко следует, что $d=0$ на A_1^ϱ.

В работе [1] доказана, что универсальная алгебра правых правоальтернативных представлений $\mathfrak{R}(A_1)$ изоморфна прямой сумме алгебры A_1 и антиизоморфной ей алгебры A_1^0. Докажем, что $(a, b)\mathfrak{R}(A_1)(a, b)^*=0$, где $a, b\in A_1$. Пусть $(a, b)=x\oplus y$, $x\in A_1$, $y\in A_1^0$, $(a, b)^*=x_1\oplus y_1$, $x_1\in A_1$, $y_1\in A_1^0$. Учитывая равенство $d=0$ и изоморфную вложимость алгебры A_1 в $\mathfrak{R}(A_1)$ посредством отображения $c\to c\oplus c^0$ имеем $(x\oplus y)(c\oplus c^0)(x_1\oplus y_1)=xcx_1\oplus yc^0y_1=0$, $xcx_1=0$ и $yc^0y_1=0$ для любого элемента $c\in A_1$. Если $x\neq 0$ и $x_1\neq 0$, то $xA_1x_1=0$, что противоречит первичности алгебры A_1. Отсюда следует, что либо $x=0$, либо $x_1=0$. Аналогично, либо $y=0$, либо $y_1=0$. Поэтому для любого элемента $c\oplus d^0\in\mathfrak{R}(A_1)$ имеем

$$(a, b)(c\oplus d^0)(a, b)^* = xcx_1\oplus yd^0y_1 = 0\oplus 0 = 0,$$

т. е. $(a, b)\mathfrak{R}(A_1)(a, b)^*=0$.

В алгебре $R_Y(A_1)$ тоже выполняется равенство $(a, b)R_Y(A_1)(a, b)^*=0$, так как $R_Y(A_1)$ является гомоморфным образом алгебры $\mathfrak{R}(A_1)$. Из равенства $Y(a, b)R_Y(A_1)(a, b)^*=0$, рассуждая как в случае (а) получим, что либо $(A_1, A_1)=$ $=0$ либо $(A_1, A_1)^*=0$.

Если алгебра A_1 альтернативна, то $A_1=\mathbf{C}$ и из неприводимости A_1-модуля Y имеем (см. [2; стр. 275]) $Y_\mathbf{C}\in\{\operatorname{reg}\mathbf{C}, \overline{\operatorname{reg}}\,\mathbf{C}\}$. Из неприводимости A-модуля $Y=\mathbf{C}$ следует, что $A=\mathbf{C}$. Теорема доказана.

Теорема 2. *Пусть Y неприводимый правый правоальтернативный модуль над правоальтернативной алгеброй A. Тогда либо $(A, A)=0$, либо $(A, A^*)=0$, либо $Y_{\bar A}\in\{\operatorname{reg}\mathbf{C}, \overline{\operatorname{reg}}\,\mathbf{C}\}$, где $\bar A=A/\operatorname{Ann} Y$.*

Доказательство. Ясно, что всякий неприводимый A-модуль Y является точным неприводимым $\bar A$-модулем, где $\bar A=A/\operatorname{Ann} Y$. Из $v\bar a=v(a+\operatorname{Ann} Y)=$ $=va$ следует, что $(\bar A, \bar A)=0$ или $(\bar A, \bar A)^*=0$ тогда и только тогда, когда, $(A, A)=0$ или $(A, A)^*=0$. Теорема доказана.

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Литература

[1] А. М. Слинько, И. П. Шестаков, Правые представления алгебр, *Алгебра и логика*, **13** (1974), 544—587.

[2] К. А. Жевлаков и др., *Кольца, близкие к ассоциативным*, Наука (Москва, 1978).

[3] И. П. Шестаков, Неприводимые представления альтернативных алгебр, Мат. заметки, **26** (1979), 673—684.

[4] K. McCrimmon, E. I. Zelmanov, *The structure of prime quadratic Jordan algebras*, Preprint, University of Munich, 1984.

[5] A. Thedy, Radicals of right alternative and Jordan rings, *Comm. Algebra*, **12** (1984), 857—887.

[6] Л. А. Бокуть, *Ассоциативные кольца*. II (Новосибирск, 1980).

[7] Ц. Дашдорж, Невырожденные правоальтернативные кольца, Деп. в ВИНИТИ за №7129-В.

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On α_1^λ-products of automata

Z. ÉSIK

1. Introduction

In [3] we introduced α_1^λ-products and gave an algebraic characterization of (homomorphically) complete classes of automata for the α_1^λ-product:

Theorem 1.1. *A class $\mathscr{K}$ of automata is complete for the α_1^λ-product if and only if for every simple group G there exists an $\mathbf{A}\in\mathbf{P}_{1\alpha_1}^\lambda(\mathscr{K})$ such that G is a divisor of the characteristic semigroup of $\mathbf{A}$, written $G|S(\mathbf{A})$.*

Further, we proved the following result.

Theorem 1.2. *Let $\mathscr{K}$ be a class of automata.*

(i) *If $\mathscr{K}$ contains a nonmonotone automaton, i.e. an automaton in $\mathscr{K}$ has a nontrivial cycle, then $\mathbf{A}\in\mathbf{HSP}_{\alpha_1}^\lambda(\mathscr{K})$ if and only if for every simple group G with $G|S(\mathbf{A})$ there exists an automaton $\mathbf{B}\in\mathbf{P}_{1\alpha_1}^\lambda(\mathscr{K})$ with $G|S(\mathbf{B})$.*

(ii) *If $\mathscr{K}$ consists of monotone automata one of which is not discrete, then $\mathbf{HSP}_{\alpha_1}^\lambda(\mathscr{K})$ is the class of all monotone automata.*

(iii) *If $\mathscr{K}$ consists of discrete automata one of which is not trivial then $\mathbf{HSP}_{\alpha_1}^\lambda(\mathscr{K})$ is the class of all discrete automata.*

(iv) *Otherwise, i.e. if $\mathscr{K}$ consists of trivial automata, then $\mathbf{HSP}_{\alpha_1}^\lambda(\mathscr{K})$ is the class of all trivial automata.*

The aim of this paper is to give a graph theoretic characterization of complete classes for the α_1^λ-product and to give a description of the classes of the form $\mathbf{HSP}_{\alpha_1}^\lambda(\mathscr{K})$ on the basis of graph theoretic terms. We believe this solution to be the final one as regards α_1^λ-products. The proofs are based on the fact that the symmetric group of degree $n-1$ ($n>1$) can be "realized" in a biconnected graph on n vertices. For recent results on α_0-products and α_1-products see [2] and [1].

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2. Notions and notations

An automaton is a system $\mathbf{A}=(A, X, \delta)$ with finite nonvoid sets A and X, the state set and input set, respectively, and transition $\delta\colon A\times X\to A$. The transition extends to a mapping $\delta\colon A\times X^*\to A$ in the usual way, where X^* is the free semigroup with unit element λ generated by X. The characteristic semigroup of $\mathbf{A}$, denoted $S(\mathbf{A})$, is the transformation semigroup on A consisting of all the mappings $\delta_u\colon A\to A$, $\delta_u(a)=\delta(a, u)$ $(a\in A,\ u\in X^*)$.

Given a system of automata $\mathbf{A}_t=(A_t, X_t, \delta_t)$ and a family of feedback functions

$$\varphi_t : A_1\times\ldots\times A_n\times X \to X_t\cup\{\lambda\},$$

$t=1, \ldots, n$, the g^λ-product of the $\mathbf{A}_t$'s with respect to X and φ is defined to be the automaton $\mathbf{A}$ with state set $A_1\times\ldots\times A_n$, input set X, and transition

$$\delta\big((a_1, \ldots, a_n), x\big) = \big(\delta_1(a_1, u_1), \ldots, \delta_n(a_n, u_n)\big)$$

where $(a_1, \ldots, a_n)\in A_1\times\ldots\times A_n$, $x\in X$ and

$$u_t = \varphi_t(a_1, \ldots, a_n, x),$$

$t=1, \ldots, n$. If none of the feedback functions $\varphi\,(a_1, \ldots, a_n, x)$ depends on the state variables a_s with $s>t$, we have an α_1^λ-product.

Given a (nonvoid) class $\mathcal{K}$ of automata, we set:

$\mathbf{P}^\lambda_{\alpha_1}(\mathcal{K})$: all α_1^λ-products of automata from $\mathcal{K}$,

$\mathbf{P}^\lambda_{1\alpha_1}(\mathcal{K})$: all α_1^λ-products with a single factor of automata from $\mathcal{K}$ (i.e. $n=1$ above),

$\mathbf{S}(\mathcal{K})$: all subautomata of automata from $\mathcal{K}$,

$\mathbf{H}(\mathcal{K})$: all homomorphic images of automata from $\mathcal{K}$.

Recall that a class $\mathcal{K}$ is called (homomorphically) complete for the α_1^λ-product if and only if $\mathbf{HSP}^\lambda_{\alpha_1}(\mathcal{K})$ is the class of all automata.

By a semigroup (group) we shall mean a finite semigroup (group). We write $S_1|S_2$ for two semigroups S_1 and S_2 if S_1 is a homomorphic image of a subsemigroup of S_2. If S_1 is a group, this just means that S_1 is a homomorphic image of a subgroup of S_2. The following statement is known e.g. from [4]:

Proposition 2.1. *If $G|G_1\times\ldots\times G_n$ for a simple group G and a direct product of groups $G_1, \ldots, G_n$ $(n>0)$, then $G|G_i$ for some i.*

3. Some useful facts

To investigate α_1^λ-products of automata we introduce the (directed) graph $D(\mathbf{A})$ of an automaton $\mathbf{A}=(A, X, \delta)$ as follows. We put $D(\mathbf{A})=(V, E)$ where the vertex set V is just the state set A and

$$E = \{(a, b)\in A\times A \mid a \neq b,\ \exists x\in X\ \delta(a, x) = b\}.$$

We see that E does not contain loop edges, henceforth, by a (directed) graph we shall always mean a graph without loop edges.

Take a graph $D=(V, E)$. We say that D is connected if for every pair a, b of different vertices there is a (directed) path from a to b. A maximal connected subgraph of D is a connected graph $D'=(V', E')$ with $V'\subseteq V$, $E'\subseteq E$ and such that whenever $D''=(V'', E'')$ is a connected graph satisfying $V'\subseteq V''\subseteq V$ and $E'\subseteq E''\subseteq E$, we have $V'=V''$, $E'=E''$.

A cycle is a graph $D=(V, E)$ with $V=\{a_1, \ldots, a_n\}$, $n>1$, and $E=\{(a_1, a_2), \ldots, (a_{n-1}, a_n), (a_n, a_1)\}$. Thus, cycles are connected graphs. Connected graphs other than cycles and having at least two vertices will be referred to biconnected graphs.

Take a graph D with vertex set $V=\{a_1, \ldots, a_n\}$ and place a pebble p_i onto a_i for every $i=1, \ldots, n$. Suppose we are allowed to move the pebbles according to the following three rules:

R1: Each step, an arbitrary number of pebbles can be moved. (Thus, some pebbles may stay where they are.)

R2: Each step, a pebble on a vertex a can be moved to a vertex b only if (a, b) is an edge.

R3: Once two or more pebbles hit the same vertex, they cannot be separated, i.e. have to be moved jointly.

Suppose that after a (possibly zero) number of steps p_i is on vertex a_{j_i}, $i=1, \ldots, n$. To this sequence of transformations we assign the mapping $V\to V$ given by $a_i\to a_{j_i}$, $i=1, \ldots, n$. Denote by $S(D)$ the set of all mappings obtained in this way. Clearly, $S(D)$ is a transformation semigroup on V. We let $G(D)$ denote the group of all permutations in $S(D)$. The following observation easily comes from the definitions:

Fact 3.1. *Let* $\mathbf{A}$ *be an automaton and* $D=D(\mathbf{A})$. *Then, for every* $\mathbf{B}\in\mathbf{P}_{1\alpha_1}^\lambda(\{\mathbf{A}\})$, $S(\mathbf{B})$ *is a subsemigroup of* $S(D)$. *Further, there exists an automaton* $\mathbf{C}\in\mathbf{P}_{1\alpha_1}^\lambda(\{\mathbf{A}\})$ *with* $S(\mathbf{C})=S(D)$.

Our game can be further generalized. Take a graph $D=(V, E)$ and fix a nonvoid subset V' of V, say $V'=\{a_1, \ldots, a_n\}$. Put pebble p_i onto a_i, $i=1, \ldots, n$, and move the pebbles in the graph according to R1, R2 and R3. Suppose that after a (possibly zero) number of steps the pebbles get back to the vertices in V', i.e. for every i, p_i is located on a vertex a_{j_i} in V'. We obtain a mapping $V' \to V'$ that assigns a_{j_i} to a_i. The collection of all these mappings is a transformation semigroup on V', denoted $S(D, V')$. Put $G(D, V')$ for the group of all permutations in $S(D, V')$. The following statement is obvious.

Fact 3.2. *$S(D, V')|S(D)$ and $G(D, V')|S(D)$.*

The next assertion is a reformulation of a well-known fact.

Fact 3.3. *If G is a subgroup of $S(D)$ then there is a nonvoid subset V' of the vertex set of D such that G is isomorphic to a subgroup of $G(D, V')$.*

Directly from Fact 3.3 and the observation that it is impossible to move a pebble back in a maximal connected subgraph if it has been moved out, we obtain:

Fact 3.4. *If G is a subgroup of $S(D)$ then G has maximal connected subgraphs $D_1, \ldots, D_n$ $(n>0)$ such that for some nonvoid subsets V_i of the vertex sets of the graphs D_i it holds that G is isomorphic to a subgroup of the direct product $G(D_1, V_1)\times \ldots \times G(D_n, V_n)$.*

Fact 3.5. *Let G be a simple group. Then $G|S(D)$ if and only if $G|G(D', V')$ for a maximal connected subgraph D' of D and a nonvoid subset V' of the vertex set of D'.*

Proof. Suppose that $G|S(D)$. There is a subgroup H of $S(D)$ which can be mapped homomorphically onto G. By Fact 3.4, H is isomorphic to a subgroup of a direct product $G(D_1, V_1)\times \ldots \times G(D_n, V_n)$ where the graphs D_i are maximal connected subgraphs of D and for every i, V_i is a nonvoid subset of the vertex set of D_i. Thus, $G|G(D_1, V_1)\times \ldots \times G(D_n, V_n)$. From Proposition 2.1, $G|G(D_i, V_i)$ for some i.

Conversely, $G|G(D', V')$ and $G(D', V')|S(D)$ yield $G|S(D)$.

Suppose we are given a graph $D=(V, E)$ with $V=\{a_0, \ldots, a_n\}$, $n \geqq 1$, i.e. D has at least two vertices. Set $V_i=V-\{a_i\}$, $i=0, \ldots, n$. Fix a pair of different integers $i, j \in \{0, \ldots, n\}$ and define the mapping $\psi_{i,j}\colon V_j \to V_i$ by

$$\psi_{i,j}(a_k) = \begin{cases} a_j & \text{if } i = k, \\ a_k & \text{otherwise.} \end{cases}$$

Let us say that $\psi_{i,j}$ has a realization in D if starting with pebble p_k located on a_k, $k=0, \ldots, n$, $k \neq j$, the placement that p_k is located on $\psi_{i,j}(a_k)$, $k=0, \ldots, n$, $k \neq j$, can be achieved by a sequence of moves according to R1, R2, R3. Obviously, if $\psi_{i,j}$ can be realized for every pair of different integers $i, j \in \{0, \ldots, n\}$, then for every $i \in \{0, \ldots, n\}$, $G(D, V_i)$ is the group of all permutations on V_i: to interchange two pebbles on a_{i_1} and a_{i_2} ($a_{i_1}, a_{i_2} \in V_i$, $a_{i_1} \neq a_{i_2}$), take a realization of $\psi_{i_1, i}$ followed by a realization of ψ_{i_2, i_1} and a realization of ψ_{i, i_2}.

Conversely, suppose that D is connected and for every $i \in \{0, ..., n\}$, $G(D, V_i)$ is the group of all permutations on V_i. It then follows that $\psi_{i,j}$ can be realized for every choise of i and j ($i, j \in \{0, ..., n\}$, $i \neq j$). Take a path $a_i = b_0, b_1, ..., b_t = a_j$ from a_i to a_j. If the length of this path is 1, i.e. $t=1$, just move the pebble on a_i to a_j, the others stand still. If $t>1$, since the permutation $(b_0 b_{t-1} ... b_1)$ is in $G(D, V_j)$, we can move the pebbles on $b_0, ..., b_{t-1}$ onto the vertices $b_{t-1}, b_0, ..., b_{t-2}$, respectively, so that the rest of the pebbles get back to their initial positions. To achive the final situation just move the pebbles on $b_0, ..., b_{t-1}$ one vertex forward along the path $b_0, ..., b_t$.

4. The main results

In this section we give a graph theoretic characterization of complete classes for the α_1^λ-product. Further, we give a complete description of the classes of the form $\mathbf{HSP}^\lambda_{\alpha_1}(\mathscr{K})$.

We start with two lemmas. In these lemmas the following designations will be used. Given a path $a_0, ..., a_n$, $n \geqq 1$, so that a_n is free and for each $i=0, ..., n-1$ there is a pebble on a_i, by moving the pebbles along the path $a_0, ..., a_n$ we shall mean the transformation that, in a single step, we move each pebble on a_i to a_{i+1}, $i=0, ..., n-1$. This definition extends to the case $n=0$: the placement of the pebbles remains unchanged. Given a cycle $a_0, ..., a_{n-1}$ $(n \geqq 2)$ with at most one pebble on a_i, $i=0, ..., n-1$, by rotating the pebbles around the cycle we shall mean the transformation obtained by moving the pebble on a_i to $a_{i+1 \bmod n}$ for every i, provided that there was a pebble on a_i.

Lemma 4.1. *Let $D=(V, E)$ be a graph with $D=\{a_0, ..., a_{n+m}\}$, $n, m \geqq 1$, $E=\{(a_0, a_1), ..., (a_{n+m-1}, a_{n+m}), (a_{n+m}, a_0), (a_n, a_0)\}$. Then for every pair i, j of different integers in $\{0, ..., n+m\}$, $\psi_{i,j}$ can be realized in D.*

Proof. Fix an integer $i \in \{0, ..., n+m\}$. We shall show that $G(D, V_i)$ is the group of all permutations on V_i. Since $a_0, ..., a_{n+m}$ is a cycle in D, we may restrict ourselves to $i=n+1$. To see that $G(D, V_{n+1})$ is the group of all permutations on V_{n+1} if suffices to prove that the cyclic permutation $(a_0 ... a_n a_{n+2}, ..., a_{n+m})$ and the transposition $(a_{n-1} a_n)$ are in $G(D, V_{n+1})$.

Place pebble p_i onto a_i, $i=0, ..., n, n+2, ..., n+m$. Move p_n from a_n to a_{n+1}, then rotate the pebbles around the cycle $a_0, ..., a_{n+m}$. We see that $(a_0 ... a_n a_{n+2} ... a_{n+m}) \in G(D, V_{n+1})$. For the transposition $(a_{n-1} a_n)$, apply the following procedure:

Step 1. Move p_n from a_n to a_{n+1}.

Step 2. Check if p_n is located on a_{n+m}, if so, go to Step 3. Move the pebbles along the path $a_{n+m}, a_0, \ldots, a_n$. (It is garanteed that a_n is free when this transformation applies.) Next, rotate the pebbles n times around the cycle $a_0, \ldots, a_n$, and after that, move the pebbles along the path $a_n, \ldots, a_{n+m}$ and go back to Step 2.

Step 3. Before this step applies, the placement of the pebbles is this: for every $i\in\{0, \ldots, n-1\}$, p_i is located on a_i; a_n is free; for every $i\in\{n+2, \ldots, n+m\}$, p_i is on a_{i-1}; p_n is on a_{n+m}. Move p_{n-1} from a_{n-1} to a_n and then rotate the pebbles around the cycle $a_0, \ldots, a_n$ until a_0 gets free, we see that a_0 is free, p_{n-1} is located on a_1, and for every $i\in\{0, \ldots, n-2\}$, p_i is on a_{2+i}. Now move p_n from a_{n+m} to a_0, rotate the pebbles $n-1$ times around the cycle $a_0, \ldots, a_n$, and move the pebbles along the path $a_{n+1}, \ldots, a_{n+m}$.

Lemma 4.2. *Let $G=(V,E)$ be a graph with $V=\{a_0, \ldots, a_{n+m+l}\}$, $n\geqq 0$, $m, l\geqq 1$, and $E=\{(a_0, a_1), \ldots, (a_{n+m-1}, a_{n+m}), (a_{n+m}, a_0), (a_n, a_{n+m+1}), \ldots, \ldots, (a_{n+m+l-1}, a_{n+m+l}), (a_{n+m+l}, a_0)\}$. Then, for every pair of different integers $i, k\in\{0, \ldots, n+m+l\}$, $\psi_{i,k}$ can be realized in D.*

Proof. Place p_t onto a_t, $t=0, \ldots, n+m+l$, $t\neq k$. First we show that we may restrict the consideration to the case that $k=n$. Either $k\in\{0, \ldots, n+m\}$ or $k\in \in\{0, \ldots, n, n+m+1, \ldots, n+m+l\}$. If $k\in\{0, \ldots, n+m\}$ rotate the pebbles around the cycle $a_0, \ldots, a_{n+m}$ until a_n gets free, then move p_i to a_n so that the rest of the pebbles get back to the position they were after the rotations. Finally, rotate the pebbles around the cycle $a_0, \ldots, a_{n+m}$ so that p_i gets onto a_k. The pebbles p_t other than p_i get back to a_t, respectively. Similar procedure applies when $k\in\{0, \ldots, n+m+1, \ldots, \ldots, n+m+l\}$.

Let $k=n$. Because the assumptions $i\in\{0, \ldots, n+m\}$ and $i\in\{0, \ldots, n, n+m+1, \ldots, \ldots, n+m+l\}$ are symmetrical, we may suppose $i\in\{0, \ldots, n+m\}$. We shall realize $\psi_{i,n}$ in five steps.

Step 1. Rotate the pebbles once around the cycle $a_0, \ldots, a_n, a_{n+m+1}, \ldots, a_{n+m+l}$. Observe that a_{n+m+1} becomes free and p_{n+m+l} gets onto a_0.

Step 2. Rotate the pebbles around the cycle $a_0, \ldots, a_{n+m}$ until p_i hits a_n. Then move p_i from a_n to a_{n+m+1}, so that a_n becomes free.

Step 3. When this step applies, one of the vertices $a_0, \ldots, a_{n+m}$ is free, and exactly one of $p_{n+m+1}, \ldots, p_{n+m+l}$, say p_t, is in the cycle $a_0, \ldots, a_{n+m}$ (p_{n+m+l} for the first time). Check if p_i is on a_{n+m+l}, if so, go to Step 4. Otherwise rotate the pebbles around the cycle $a_0, \ldots, a_{n+m}$ until p_t gets onto a_n, and rotate the pebbles once around the cycle $a_0, \ldots, a_n, a_{n+m+1}, \ldots, a_{n+m+l}$. Go to Step 3.

Step 4. Observe that the placement of the pebbles is this. The cycle $a_0, \ldots, a_{n+m}$ contains p_{n+m+1} and the pebbles p_j with $j\in\{0, \ldots, n+m\}$, $j\neq i$, $j\neq n$. Thus, one of

$a_0, \ldots, a_{n+m}$ is free. The relative order of the pebbles p_j ($j \in \{0, \ldots, n+m\}$, $j \neq i$, $j \neq n$) is their original order. Further, p_i is on a_{n+m+l}, p_{n+m+2} is on $a_{n+m+1}, \ldots, \ldots, p_{n+m+l}$ is on $a_{n+m+l-1}$. It is now clear that the pebbles in the cycle $a_0, \ldots, a_{n+m}$ can be arranged in such a way that a_0 gets free and after moving the pebbles along the path $a_{n+m+1}, \ldots, a_{n+m+l}, a_0$ (so that p_i gets onto a_0), the relative order of the pebbles p_j, $j \in \{0, \ldots, n+m\}$, $j \neq n$, in the cycle $a_0, \ldots, a_{n+m}$ will be just as desired.

Step 5. We have p_{n+m+1} free. The pebbles $p_{n+m+2}, \ldots, p_{n+m+l}$ are back on $a_{n+m+2}, \ldots, a_{n+m+l}$, respectively. Further, the cycle $a_0, \ldots, a_{n+m}$ contains the pebbles p_j $j \in \{0, \ldots, n+m\}$, $j \neq n$, and the pebble p_{n+m+1}. The relative order of the pebbles p_j ($j \in \{0, \ldots, n+m\}$, $j \neq n$) is just as desired. Rotate the pebbles around the cycle $a_0, \ldots, a_{n+m}$ until p_{n+m+1} gets onto a_n then move p_{n+m+1} from a_n to a_{n+m+1}. The pebbles $p_{n+m+1}, \ldots, p_{n+m+l}$ are now back on $a_{n+m+1}, \ldots, a_{n+m+l}$, respectively. Further, it is clear that the pebbles in the cycle $a_0, \ldots, a_{n+m}$ can be arranged so that p_i is on a_n, and for $j \in \{0, \ldots, n+m\}$, $j \neq i$, $j \neq n$, p_j is on a_j.

Theorem 4.3. *$S_n | S(D)$ for every biconnected graph D on $n+1$ vertices.*

Proof. Let $D=(V, E)$ with $V=\{a_0, \ldots, a_n\}$. We are going to show that $\psi_{i,j}$ can be realized in D for every possible pair of different integers i, j. Consequently, $G(D, V_i)$ is the group of all permutations on V_i for every i ($0 \leqq i \leqq n$). Hence the result follows by Fact 3.2.

Put pebble p_t onto a_t for every $t \in \{0, \ldots, n\}$, $t \neq j$. Take a path

$$a_i = b_0, b_1, \ldots, b_k = a_j$$

from a_i to a_j. If $k=1$, $\psi_{i,j}$ can be realized obviously. We proceed by induction on k. Assume $k>1$. There are an $m \in \{0, \ldots, k-1\}$ and a path

$$a_j = b_k, b_{k+1}, \ldots, b_{k+l} = b_m$$

with $\{b_0, \ldots, b_k\} \cap \{b_{k+1}, \ldots, b_{k+l-1}\} = \emptyset$. We distinguish two cases.

Case $m \neq 0$. Let us rotate the pebbles l times around the cycle $b_m, \ldots, b_k, b_{k+1}, \ldots, b_{k+l-1}$. We see that b_m is free now. By induction hypothesis, p_1 can be moved from a_i to b_m in such a way that meanwhile all the other pebbles get back to the vertex they were before. Finally, rotate the pebbles $k-m$ times around the cycle $b_m, \ldots, b_k, b_{k+1}, \ldots, b_{k+l-1}$. Obviously, we obtained a realization of $\psi_{i,j}$.

Case $m=0$. We have a cycle

$$b_0, b_1, \ldots, b_k, b_{k+1}, \ldots, b_{k+l-1}.$$

Two subcases arise according to whether this cycle contains all the vertices of D or not.

Subcase $V=\{b_0, \ldots, b_{k+l-1}\}$. Since D is biconnected, there is at least one edge in E other than the edges $(b_0, b_1), \ldots, (b_{k+l-2}, b_{k+l-1}), (b_{k+l-1}, b_0)$. The result follows by Lemma 4.1.

Subcase $V\neq\{b_0, \ldots, b_{k+l-1}\}$. Take a vertex $c\in V-\{b_0, \ldots, b_{k+l-1}\}$ closest to the cycle $b_0, \ldots, b_{k+l-1}$. We then have paths $b_t=c_0, c_1, \ldots, c_u=c$ and $c=d_0, \ldots, d_v=b_s$ for $t, s\in\{0, \ldots, k+l-1\}$ such that the sets $\{b_0, \ldots, b_{k+l-1}\}$, $\{c_1, \ldots, c_u\}$ and $\{d_1, \ldots, d_{v-1}\}$ are pairwise disjoint. The result follows by Lemma 4.2.

Theorem 4.4. *Let $D=(V, E)$ be a cycle with n vertices. Then for every group G, $G|S(D)$ if and only if $G|Z_m$ for some $m\leqq n$.*

Proof. It suffices to show that a group is isomorphic to a subgroup of $S(D)$ if and only if it is isomorphic to a subgroup of Z_m with $m\leqq n$.

Suppose that H is isomorphic to a subgroup of $S(D)$. From Fact 3.3, there is a subset V' of the vertex set of D such that H is isomorphic to a subgroup of $G(D, V')$. Let m be the cardinality of V'. We prove that $G(D, V')$ is a cyclic group of order m.

Set $V=\{a_1, \ldots, a_n\}$ and $V'=\{a_{i_1}, \ldots, a_{i_m}\}$ so that $a_1, \ldots, a_n$ is a cycle and $i_1<\ldots<i_m$. Place pebble p_j onto a_{i_j}, $j=1, \ldots, m$. Rotate the pebbles once around the cycle $a_1, \ldots, a_n$. If each of the pebbles p_j is on the vertex $a_{i_{j+1}}$, or on a_{i_1} if $j=m$, we see that the cyclic permutation $(a_{i_1}\ldots a_{i_m})$ is in $G(D, V')$. Otherwise, rotate those pebbles around the cycle $a_1, \ldots, a_n$ for which it does not hold. In a finite number of steps we obtain a realization of the cyclic permutation $(a_{i_1}\ldots a_{i_m})$. Thus, $(a_{i_1}\ldots a_{i_m})\in$ $\in G(D, V')$. On the other hand, since by our rules and the structure of D the pebbles can never pass each other, every permutation in $G(D, V')$ is a power of the cyclic permutation $(a_{i_1}\ldots a_{i_m})$.

Conversely, it is clear from the above proof that if H is isomorphic to a subgroup of a cyclic group Z_m with $m\leqq n$ then H is isomorphic to a subgroup of $G(D, V')$ for every subset V' of V with m elements. Thus, Fact 4.2 yields $G|S(D)$.

Let $\mathscr{K}$ be a class of automata. Set $D(\mathscr{K})=\{D \mid \exists\, \mathbf{A}\in\mathscr{K}\;\; D$ is a subgraph of $D(\mathbf{A})\}$, where the notion of a subgraph of a graph is used in the usual sense. With the concept of $D(\mathscr{K})$ and that of a biconnected graph we are able to characterize complete classes for the α_1^λ-product:

Theorem 4.5. *A class $\mathscr{K}$ is complete for the α_1^λ-product if and only if for every positive integer n, $D(\mathscr{K})$ contains a biconnected graph on at least n vertices.*

Proof. If $D(\mathscr{K})$ does not contain biconnected graphs then, by Theorem 4.4, Fact 3.5 and Fact 3.1, every simple group dividing $S(\mathbf{A})$ for some $\mathbf{A}\in\mathbf{P}_{1\alpha_1}^\lambda(\mathscr{K})$ is commutative. If n is the highest integer such that $D(\mathscr{K})$ contains a biconnected graph on n vertices then, again by Theorem 4.4, Fact 3.5 and Fact 3.1, every simple group dividing $S(\mathbf{A})$ for an $\mathbf{A}\in\mathbf{P}_{1\alpha_1}^\lambda(\mathscr{K})$ is either commutative or a divisor of S_n. In either case, $\mathscr{K}$ cannot be complete for the α_1^λ-product by Theorem 1.1.

For the converse, suppose that for every positive integer n there exists a biconnected graph in $D(\mathcal{K})$ having at least n vertices. Take a simple group G. There is a positive integer n with $G|S_n$. By Theorem 4.3, Fact 3.2 and Fact 3.1, it is easy to see that $S_n|S(\mathbf{A})$ for some $\mathbf{A}\in\mathbf{P}^\lambda_{1\alpha_1}(\mathcal{K})$. Thus, $\mathcal{K}$ is complete for the α_1^λ-product by Theorem 1.1.

In exactly the same way we obtain the following result:

Theorem 4.6. *Let $\mathcal{K}$ be a class of automata. If $\mathcal{K}$ is not complete for the α_1^λ-product then three cases arise.*

(i) *There is a highest integer n such that $D(\mathcal{K})$ contains a biconnected graph on n vertices. Then $\mathbf{A}\in\mathbf{HSP}^\lambda_{\alpha_1}(\mathcal{K})$ if and only if for every simple group G with $G|S(\mathbf{A})$, either $G|S_{n-1}$ or $G|G(D)$ for a biconnected graph $D\in D(\mathcal{K})$ on n vertices or G is a prime group of order p and $D(\mathcal{K})$ contains a cycle of length at least p.*

(ii) *$D(\mathcal{K})$ does not contain biconnected graphs but there is at least one cycle in $D(\mathcal{K})$. Then $\mathbf{A}\in\mathbf{HSP}^\lambda_{\alpha_1}(\mathcal{K})$ if and only if for every simple group with $G|S(\mathbf{A})$, G is a prime group of order p such that $D(\mathcal{K})$ contains a cycle of length at least p.*

(iii) *Otherwise, i.e. if there is no cycle in $D(\mathcal{K})$, then $\mathbf{HSP}^\lambda_{\alpha_1}(\mathcal{K})$ is the class of all monotone automata or the class of all discrete automata or the class of all trivial automata, just as in Theorem* 1.2.

Corollary 4.7. *There are a countable number of classes of automata of the form $\mathbf{HSP}^\lambda_{\alpha_1}(\mathcal{K})$.*

References

[1] Z. Ésik, Complete classes of automata for the α_1-product, *Found. Control Engineering*, **11** (1986), 95—107.

[2] Z. Ésik and P. Dömösi, Complete classes of automata for the α_0-product, *Theoret. Comput. Sci.*, **47** (1986), 1—14.

[3] Z. Ésik and J. Virágh, On products of automata with identity, *Acta Cybernet.*, **7** (1986), 299—311.

[4] A. Ginzburg, *Algebraic theory of automata*, Academic Press (New York—London, 1968).

A. JÓZSEF UNIVERSITY
BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

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On general connections satisfying $\nabla I = \omega \otimes I$

NAOTO ABE, HIROAKI NEMOTO and SEIICHI YAMAGUCHI

0. Introduction

The notion of general connections was initiated by T. OTSUKI in 1958 [10]. He obtained various result [11—21]. A. MOÓR studied Riemannian manifolds with general connections, he called them Otsuki spaces [2—7]. T. OTSUKI [21, 22] and H. NAGAYAMA [8] applied general connections to the theory of relativity. Recently N. ABE [1] defined general connections on arbitrary vector bundles and H. NEMOTO [9] studied the geometry of submanifolds in a Riemannian manifold with a general connection.

One of the appealing facts in the theory of general connection is the fact that the covariant derivative of the identity endomorphism does not necessarily vanish. In this paper, we will study the case where the identity endomorphism is recurrent.

1. Preliminaries

In this section we review the theory of general connections along [1, 9, 11]. Throughout this paper, we assume that all objects are smooth and all vector bundles are real. Let M be a manifold, TM the tangent bundle and $C(M)$ the ring of real-valued functions on M. Let V and W be vector bundles over M. The fibre of V at $p \in M$ will be denoted by V_p and the dual bundle of V is denoted by V^*. The space of cross-sections of V will be denoted by $\Gamma(V)$. By Hom (V, W) we will denote the vector bundle of which fibre Hom (V, W) at p is the vector space Hom (V_p, W_p) of linear maps from V_p to W_p. In particular, Hom (V, V) will be denoted by End (V). Let HOM (V, W) be the space of vector bundle homomorphisms from V to W. Especially HOM (V, V) will be denoted by END (V). Let I_V be the identity endomorphism of V. Note that HOM (V, W) can be naturally identified with the space

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$\Gamma(\mathrm{Hom}\,(V, W))$. We will generally use the same symbol to denote a vector bundle homomorphism and the induced linear map on the space of cross-sections.

For $s \in \Gamma(V)$, we will denote the 1-jet of s by $j^1(s)$ and the 1-jet at p by $j^1_p(s)$. Let $J^1(V)$ be the 1-jet bundle of V. Now we define two vector bundle homomorphisms. The vector bundle homomorphism ι: $TM^* \otimes V \to J^1(V)$ is defined to be

$$\iota((df)_p \otimes s(p)) := j^1_p((f - f(p))s) \quad \text{for} \quad f \in C(M), \quad s \in \Gamma(V).$$

The vector bundle homomorphism λ: $J^1(V) \to V$ is defined to be

$$\lambda(j^1_p(s)) := s(p) \quad \text{for} \quad s \in \Gamma(V).$$

Definition 1.1. A vector bundle homomorphism $\gamma \in \mathrm{HOM}\,(V, J^1(V))$ is called a general connection on V. The endomorphism $P^\gamma := \lambda \circ \gamma \in \mathrm{END}\,(V)$ is called the principal endomorphism of γ. The linear operator ∇^γ: $\Gamma(V) \to \Gamma(TM^* \otimes V)$, defined by

$$\nabla^\gamma s := \iota^{-1}(j^1(P^\gamma s) - \gamma(s)) \quad \text{for} \quad s \in \Gamma(V),$$

is called the covariant derivative of γ.

It is easily shown that the covariant derivative ∇^γ of a general connection γ with the principal endomorphism P^γ satisfies

$$\nabla^\gamma(fs) = (df) \otimes P^\gamma s + f \nabla^\gamma s \quad \text{for} \quad f \in C(M), \quad s \in \Gamma(V). \tag{1.1}$$

For $P \in \mathrm{END}\,(V)$, we will denote the set of linear operators on $\Gamma(V)$ into $\Gamma(TM^* \otimes V)$ satisfying (1.1) by $O(V; P)$. Then the following theorem is known [1]:

Theorem A. *If $\nabla \in O(V; P)$ for $P \in \mathrm{END}\,(V)$, then there exists a unique general connection γ such that $P^\gamma = P$ and $\nabla^\gamma = \nabla$.*

Thus we may say that a pair (∇, P) of a linear operator ∇ and an endomorphism P satisfying (1.1) is a general connection on V. Given $v \in TM$ and $p \in M$, we define the linear map ∇_v: $\Gamma(V) \to V_p$ by $\nabla_v s := i_v(\nabla s)$ for $s \in \Gamma(V)$, where i_v is the inner product operator. Similarly, given $X \in \Gamma(TM)$, we define the linear operator ∇_X: $\Gamma(V) \to \Gamma(V)$ by $(\nabla_X s)(p) := \nabla_{X(p)} s$. We call ∇_X the covariant derivative along X. Then we have

$$\nabla_{fX} s = f \nabla_X s \quad \text{and} \quad \nabla_X(fs) = (Xf)Ps + f\nabla_X s \quad \text{for} \quad f \in C(M). \tag{1.1$'$}$$

When $P = I_V$, our general connection (∇, I_V) is nothing but a usual connection on V, that is, the linear operator ∇_X: $\Gamma(V) \to \Gamma(V)$ satisfies $\nabla_{fX} s = f \nabla_X s$ and $\nabla_X(fs) = (Xf)s + f\nabla_X s$.

Definition 1.2. A general connection (∇, P) on V is said to be regular if P is a regular endomorphism.

In the theory of general connection, we can define the product of $\nabla\in O(V; P)$ and $Q\in \mathrm{END}\,(V)$ as follows:

$$({}^{Q}\nabla)_X s := Q(\nabla_X s) \quad \text{and} \quad (\nabla^{Q})_X s := \nabla_X(Qs).$$

Then we have ${}^{Q}\nabla\in O(V; QP)$ and $\nabla^{Q}\in O(V; PQ)$. Hence, if a general connection (∇, P) is regular and Q is the inverse endomorphism of P, then the general connections ${}^{Q}\nabla$ and ∇^{Q} are usual connections. Furthermore we can naturally extend a general connection (∇, P) to general connections on the dual bundle and the tensor bundles. We will use the same symbol (∇, P) for the extensions. For instance, we present here the following formulas:

$$(\nabla_X \eta)(s) = X(\eta(Ps)) - \eta(\nabla_X s),$$

$$(\nabla_X \varphi)s = \nabla_X(\varphi Ps) - P\varphi(\nabla_X s),$$

$$(\nabla_X g)(s, s') = X\big(g(Ps, Ps')\big) - g(\nabla_X s, Ps') - g(Ps, \nabla_X s')$$

for $\eta\in\Gamma(V^*)$, $\varphi\in\Gamma(\mathrm{End}\,(V))$, $g\in\Gamma\big((V\otimes V)^*\big)$ and $s, s'\in\Gamma(V)$. In contrast to the case of usual connections, we must note that ∇I_V does not vanish in general.

Definition 1.3. Let $g\in\Gamma\big((V\otimes V)^*\big)$ be a fibre metric on V. A general connection (∇, P) on V is said to be metric if $\nabla g = 0$, that is,

$$(\nabla_X g)(s, s') = X\big(g(Ps, Ps')\big) - g(\nabla_X s, Ps') - g(Ps, \nabla_X s') = 0$$

for $s, s'\in\Gamma(V)$ and $X\in\Gamma(TM)$.

Definition 1.4. The element $R(\nabla)\in \mathrm{HOM}\,\big(\Lambda^2(TM), \mathrm{End}\,(V)\big)$ defined by

$$R(\nabla)_{X,Y} s := \nabla_X\big(\nabla_Y(Ps)\big) - \nabla_Y\big(\nabla_X(Ps)\big) - P\big(\nabla_{[X,Y]}(Ps)\big) - (\nabla_X I_V)\nabla_Y s + (\nabla_Y I_V)\nabla_X s$$

for $s\in\Gamma(V)$ and $X, Y\in\Gamma(TM)$, is called the curvature tensor field of the general connection (∇, P).

Remark. When the vector bundle is the tangent bundle TM, the curvature tensor field defined above coincides with the one defined by T. Otsuki [11].

In the case of $V = TM$, we can define a torsion tensor field of a general connection (∇, P) as follows:

Definition 1.5. Let $V = TM$. The element $\Psi\in \mathrm{HOM}\,(TM\otimes TM, TM)$ defined by

$$\Psi(X, Y) := \nabla_X Y - \nabla_Y X - P[X, Y]$$

for $X, Y\in\Gamma(TM)$, is called the torsion tensor field of the general connection (∇, P). If $\Psi = 0$, then the general connection on TM is said to be torsion free.

2. General connections of recurrent type

In a theory of general connection we noted that the covariant derivative of the identity endomorphism ∇I_V does not vanish in general. The case of $\nabla I_V=0$ was studied in [9]. The purpose of this paper is to study the case of $(\nabla_X I_V)=\omega(X)I_V$, where ω is some 1-form on M.

Definition 2.1. Let (∇, P) be a general connection on a vector bundle V over M. If the general connection (∇, P) satisfies

$$(\nabla_X I_V)s = \omega(X)s \tag{2.1}$$

for some 1-form ω on M, then we call the general connection (∇, P) to be of recurrent type.

Example. For $\varrho\in C(M)$, we put $P:=\varrho I_V$. Let D be a usual connection on V. If we define a general connection (∇, P) by ${}^PD$, then it is easily seen that the general connection (∇, P) is of recurrent type whose recurrent 1-form ω is given by $\omega= =(1/2)d(\varrho^2)$. For the curvature tensor fields $R(\nabla)$ and $R(D)$, we can get the following formula:

$$R(\nabla) = \varrho^3 R(D),$$

which will be generalized in the folloving section. If ϱ does not vanish everywhere on M, the general connection (∇, P) is regular. Let g be a fibre metric on V and ϱ does not vanish everywhere on M. We define the fibre metric G which is conformal to g by $G:=\varrho^2 g$. Then we obtain that

$$\begin{aligned}(\nabla_X g)(s, s') &:= X(g(Ps, Ps'))-g(\nabla_X s, Ps')-g(Ps, \nabla_X s') = \\ &= X(g(\varrho s, \varrho s'))-g(\varrho D_X s, \varrho s')-g(\varrho s, \varrho D_X s') = \\ &= X(G(s, s'))-G(D_X s, s')-G(s, D_X s') = (D_X G)(s, s').\end{aligned}$$

Hence we know that the general connection (∇, P) is a metric general connection with respect to g if and only if the usual connection D is a metric connection with respect to G. Especially when $V=TM$, it is clear that the general connection (∇, P) is torsion free if and only if the usual connection D is torsion free. This type of general connections was treated by T. Otsuki [21] and H. Nagayama [8] when $V=TM$.

3. Regular general connections of recurrent type

In this section we study the case that the general connection (∇, P) is recurrent type and regular.

At first, we prepare several formulas for a regular general connection. Let Q be the inverse endomorphism of P, that is,

$$PQ = QP = I_V.$$

Thus the products ${}^Q\nabla$ and ∇^Q are usual connections on V and are denoted by D and D' respectively. The following equations were proved in [9].

$$(3.1)\qquad (\nabla_X I_V)s = P(D_X P)s = (D'_X P)(Ps),$$

$$(3.2)\qquad R(\nabla)_{X,Y}s = P^2R(D)_{X,Y}(Ps)+P(D_XP)(D_YP)s-P(D_YP)(D_XP)s =$$
$$= PR(D')_{X,Y}(P^2s)+(D'_XP)(D'_YP)(Ps)-(D'_YP)(D'_XP)(Ps).$$

Remark. When $V=TM$, these formulas are first proved by T. Otsuki in [11, 18].

Lemma 3.1. *Let (∇, P) be a regular general connection of recurrent type on V. Then we have the following equations:*

$$(3.3)\qquad (D_XP)s = (D'_XP)s = \omega(X)Qs,$$

$$(3.4)\qquad (D_XQ)s = (D'_XQ)s = -\omega(X)Q^3s,$$

where ω is the recurrent 1-form and Q is the inverse endomorphism of P.

Proof. From (2.1) and (3.1), we obtain

$$P(D_XP)s = (D'_XP)(Ps) = (\nabla_X I_V)s = \omega(X)s,$$

from which we get (3.3). Since D is a usual connection, we have

$$D_Xs = D_X(PQs) = (D_XP)Qs+PD_X(Qs) =$$
$$= (D_XP)Qs+P\{(D_XQ)s+QD_Xs\} = (D_XP)Qs+P(D_XQ)s+D_Xs.$$

Hence we find by (3.3) that

$$(D_XQ)s = -Q(D_XP)Qs = -\omega(X)Q^3s.$$

Similarly we get $(3.4)_2$.

As a regular general connection (∇, P) can be naturally related to usual connections $D:={}^Q\nabla$ and $D':=\nabla^Q$, we give a relation among the curvature tensor fields of $R(\nabla)$, $R(D)$ and $R(D')$.

Theorem 3.2. *Let* (∇, P) *be a regular general connection of recurrent type on* V. *Then we have the following equations:*

(3.5) $R(\nabla)_{X,Y}s = P^3R(D)_{X,Y}s+2d\omega(X,Y)Ps = P^3R(D')_{X,Y}s+4d\omega(X,Y)Ps,$

where

$$2d\omega(X,Y) = X(\omega(Y))-Y(\omega(X))-\omega([X,Y]).$$

Proof. At first, substituting (3.3) into (3.2), we have

$$R(\nabla)_{X,Y}s = P^2R(D)_{X,Y}(Ps).$$

Using (3.3) and (3.4), we calculate $D_XD_Y(Ps)$ and $D_{[X,Y]}(Ps)$ as follows:

$$D_XD_Y(Ps) = D_X\{(D_YP)s+PD_Xs\} = D_X\{\omega(Y)Qs+PD_Ys\} =$$
$$= X(\omega(Y))Qs+\omega(Y)(D_XQ)s+\omega(Y)QD_Xs+(D_XP)D_Ys+PD_XD_Ys =$$
$$= X(\omega(Y)Qs-\omega(X)\omega(Y)Q^3s+\omega(Y)QD_Xs+\omega(X)QD_Ys+PD_XD_Ys,$$
$$D_{[X,Y]}(Ps) = (D_{[X,Y]}P)s+PD_{[X,Y]}s = \omega([X,Y])Qs+PD_{[X,Y]}s.$$

Hence we obtain

$$R(D)_{X,Y}(Ps) = PR(D)_{X,Y}s+\{X(\omega(Y))-Y(\omega(X))-\omega([X,Y])\}Qs,$$

from which we get $(3.5)_1$. By similar calculations we get $(3.5)_2$.

4. Regular metric general connections of recurrent type

In this section we will deal with a regular metric general connection (∇, P) of recurrent type.

Let g be a fibre metric on V and P be regular. Now we define the new metric G by

(4.1) $$G(s,s') := g(Ps, Ps').$$

It is known that when $V=TM$ and g is a Riemannian metric, G is also a Riemannian metric. Furthermore if the regular metric general connection is torsion free, the product ${}^{Q}\nabla$ is the Levi—Civita connection with respect to G [9].

Lemma 4.1. *Let* (∇, P) *be a regular metric general connection of recurrent type on* V. *Then we obtain*

(4.2) $$(D_Xg)(s,s') = -\omega(X)g(Ts,s'),$$

where we put

$$T := Q^2+Q^{*2}$$

and Q^* *is defined by* $g(Q^*s,s') := g(s,Qs')$.

Proof. As D is a usual connection, we get

$$X(g(Ps, Ps')) = D_X(g(Ps, Ps')) =$$
$$= (D_X g)(Ps, Ps')+g((D_X P)s, Ps')+g(Ps, (D_X P)s')+g(PD_X s, Ps')+g(Ps, PD_X s')=$$
$$= (D_X g)(Ps, Ps')+g((D_X P)s, Ps')+g(Ps, (D_X P)s')+g(\nabla_X s, Ps')+g(Ps, \nabla_X s'),$$

where we used ${}^P D_X s=\nabla_X s$. Therefore, substituting (3.3) and $X(g(Ps, Ps'))= =g(\nabla_X s, Ps')+g(Ps, \nabla_X s')$ into the above equation, we obtain

$$(D_X g)(Ps, Ps') = -g((D_X P)s, Ps')-g(Ps, (D_X P)s') =$$
$$= -\omega(X)\{g(Qs, Ps')+g(Ps, Qs')\}.$$

Changing s to Qs and s' to Qs', we find (4.2).

Theorem 4.2. *Let (∇, P) be a regular metric general connection of recurrent type on V. If $G(s, s')=g(s, s')$, that is $g(Ps, Ps')=g(s, s')$, then the recurrent 1-form ω vanishes identically.*

Proof. At first, we note that $g(Ps, s')=g(s, Qs')$ because of $g(Ps, Ps')=g(s, s')$. Moreover, by virtue of Lemma 4.1 and $Dg=DG=0$, we obtain

$$\omega(X)\{g(Q^2s, s')+g(s, Q^2s')\} = 0,$$

for any $X\in\Gamma(TM)$ and $s, s'\in\Gamma(V)$. Suppose that there is a point $p\in M$ such that $\omega\neq 0$ at p, then $\omega\neq 0$ on some open neighborhood U of p. Thus, on U, we have

$$g(Q^2s, s')+g(s, Q^2s') = 0,$$

from which we have

$$P^4 = -I_V.$$

Then from (3.3), we can easily get the following equation:

$$-D_X s = D_X(-s) = D_X(P^4 s) = 4\omega(X)P^2 s-D_X s,$$

which yields that

$$4\omega(X)P^2 s = 0.$$

Since P is regular, this implies that $\omega=0$ on U. This is a contradiction. Therefore, there are no points $p\in M$ such that $\omega\neq 0$ at p.

5. Regular metric general connections of recurrent type on TM

In Section 4, we mentioned that if the general connection (∇, P) on TM is torsion free, regular and metric with respect to g, then D is the Levi—Civita connection with respect to G. On the other hand, there is the Levi—Civita connection $\bar{D}$ with respect to the original metric g. From now on, we study the relation between D and $\bar{D}$.

From the definition of $\bar{D}$, we have

$$(5.1) \qquad X(g(Y,Z)) = g(\bar{D}_X Y, Z)+g(Y, \bar{D}_X Z), \quad \bar{D}_X Y-\bar{D}_Y X = [X,Y].$$

On the other hand, by Lemma 4.1, we also obtain

$$(D_X g)(Y, Z) = X(g(Y,Z))-g(D_X Y, Z)-g(Y, D_X Z) = -\omega(X)g(TY, Z).$$

Substituting (5.1) into above equation, we have

$$(5.2) \qquad g(D_X Y-\bar{D}_X Y, Z)+g(Y, D_X Z-\bar{D}_X Z) = \omega(X)g(TY, Z).$$

Since both D and $\bar{D}$ are torsion free, we get

$$g(D_X Y-\bar{D}_X Y, Z)+g(Y, D_X Z-\bar{D}_X Z)+g(D_Y Z-\bar{D}_Y Z, X)+$$
$$+g(Z, D_Y X-\bar{D}_Y X)-g(D_Z X-\bar{D}_Z X, Y)-g(X, D_Z Y-\bar{D}_Z Y) = 2g(D_X Y-\bar{D}_X Y, Z).$$

From (5.2), the left hand side of the above equation equals

$$\omega(X)g(TY, Z)+\omega(Y)g(TZ, X)-\omega(Z)g(TX, Y).$$

Therefore, we have

$$(5.3) \qquad 2(D_X Y-\bar{D}_X Y) = \omega(X)TY+\omega(Y)TX-g(TX, Y)W,$$

where W is the vector field defined by $g(W, X):=\omega(X)$ and we used $g(TX, Y)= =g(X, TY)$. For brevity, we set

$$(5.4) \qquad S(X,Y):=(1/2)\{\omega(X)TY+\omega(Y)TX-g(TX,Y)W\}.$$

Then (5.3) is rewritten as

$$(5.5) \qquad D_X Y = \bar{D}_X Y+S(X,Y).$$

Now, we consider the relation between the curvature tensor fields $R(D)$ and $R(\bar{D})$. Using (5.5) twice, we have

$$D_X D_Y Z = \bar{D}_X \bar{D}_Y Z+(\bar{D}_X S)(Y,Z)+S(\bar{D}_X Y, Z)+$$
$$+S(Y, \bar{D}_X Z)+S(X, \bar{D}_Y Z)+S(X, S(Y,Z)).$$
$$D_{[X,Y]} Z = \bar{D}_{[X,Y]} Z+S([X,Y], Z).$$

Hence it follows from above equations and $\bar{D}_X Y-\bar{D}_Y X=[X, Y]$ that

(5.6)

$$R(D)_{X,Y} Z = R(\bar{D})_{X,Y} Z+(\bar{D}_X S)(Y,Z)-(\bar{D}_Y S)(X,Z)+S(X, S(Y,Z))-S(Y, S(X,Z)).$$

To express the right hand side of (5.6) more precisely, we prepare several formulas. At first, we put

$$(5.7) \qquad U:=Q^2 \quad \text{and} \quad U^*:=Q^{*2}.$$

Then we have

$$T = U + U^*.$$

From $(3.4)_1$, we easily get

$$(D_X U)Y = -2\omega(X)U^2 Y. \tag{5.8}$$

Let us calculate $(D_X U^*)Y$.

$$g((D_X U^*)Y, Z) = g(D_X(U^*Y), Z) - g(U^*D_X Y, Z) =$$
$$= X(g(U^*Y, Z)) - (D_X g)(U^*Y, Z) - g(U^*Y, D_X Z) - g(U^*D_X Y, Z) =$$
$$= (D_X g)(Y, UZ) + g(D_X Y, UZ) + g(Y, (D_X U)Z) + g(Y, UD_X Z) +$$
$$+ \omega(X)g(TU^*Y, Z) - g(Y, UD_X Z) - g(D_X Y, UZ) =$$
$$= -\omega(X)g(TY, UZ) - 2\omega(X)g(Y, U^2 Z) + \omega(X)g(TU^*Y, Z).$$

Therefore we find that

$$(D_X U^*)Y = -\omega(X)[U^*TY - TU^*Y + 2U^{*2}Y]. \tag{5.9}$$

Using (5.8) and (5.9), we compute $(D_X T)Y$.

$$(D_X T)Y = (D_X U)Y + (D_X U^*)Y = -\omega(X)[2U^2Y + 2U^{*2}Y + U^*TY - TU^*Y]. \tag{5.10}$$

Next, we compute $(\bar{D}_X T)Y$ by the aids of (5.4), (5.5) and (5.10).

$$(\bar{D}_X T)Y = \bar{D}_X(TY) - T\bar{D}_X Y = (D_X T)Y - S(X, TY) + TS(X, Y) = \tag{5.11}$$
$$= -\omega(X)[2U^2Y + 2U^{*2}Y + U^*TY - TU^*Y] -$$
$$-(1/2)[\omega(TY)TX - \omega(Y)T^2X - g(TX, TY)W + g(TX, Y)TW].$$

By virtue of these equations, we can get the following:

$$(\bar{D}_X S)(Y, Z) - (\bar{D}_Y S)(X, Z) = \tag{5.12}$$
$$= (1/2)\{[(\bar{D}_X \omega)(Y) - (\bar{D}_Y \omega)(X)]TZ + [(\bar{D}_X \omega)(Z)TY - (\bar{D}_Y \omega)(Z)TX] -$$
$$-[g(TY, Z)\bar{D}_X W - g(TX, Z)\bar{D}_Y W] - (1/2)\omega(TZ)[\omega(Y)TX - \omega(X)TY] +$$
$$+(1/2)\omega(Z)[\omega(Y)T^2X - \omega(X)T^2Y] - (1/2)[\omega(Y)g(TX, Z) - \omega(X)g(TY, Z)]TW -$$
$$-\omega(Z)[2\omega(X)U^2Y - 2\omega(Y)U^2X + 2\omega(X)U^{*2}Y - 2\omega(Y)U^{*2}X +$$
$$+\omega(X)U^*TY - \omega(Y)U^*TX - \omega(X)TU^*Y + \omega(Y)TU^*X] -$$
$$-(1/2)\omega(Z)[\omega(TY)TX - \omega(TX)TY - \omega(Y)T^2X + \omega(X)T^2Y] +$$
$$+[2\omega(X)g(U^2Y, Z) - 2\omega(Y)g(U^2X, Z) +$$
$$+2\omega(X)g(U^{*2}Y, Z) - 2\omega(Y)g(U^{*2}X, Z) + \omega(Y)g(TU^*X, Z) -$$
$$-\omega(X)g(TU^*Y, Z) - \omega(Y)g(U^*TX, Z) + \omega(X)g(U^*TY, Z)]W +$$
$$+(1/2)[\omega(TY)g(TX, Z) - \omega(TX)g(TY, Z)]W\}.$$

The following equation follows from (5.4) and (5.12).

$$(\bar{D}_X S)(Y,Z)-(\bar{D}_Y S)(X,Z)+S(X,S(Y,Z))-S(Y,S(X,Z)) =$$
$$= d\omega(X,Y)TZ+(1/2)\{[(\bar{D}_X\omega)(Z)TY-(\bar{D}_Y\omega)(Z)TX]-$$
$$-[g(TY,Z)\bar{D}_X W-g(TX,Z)\bar{D}_Y W]\}-(1/4)\{|\omega|^2[g(TY,Z)TX-g(TX,Z)TY]+$$
$$+\omega(X)\omega(Z)[4U^2Y+4U^{*2}Y+2U^*TY-2TU^*Y+T^2Y]-$$
$$-\omega(Y)\omega(Z)[4U^2X+4U^{*2}X+2U^*TX-2TU^*X+T^2X]-$$
$$-\omega(X)g(4U^2Y+4U^{*2}Y+2U^*TY-2TU^*Y+T^2Y,Z)W+$$
$$+\omega(Y)g(4U^2X+4U^{*2}X+2U^*TX-2TU^*X+T^2X,Z)W.$$

Therefore, we obtain the following theorem:

Theorem 5.1. *Let* (∇, P) *be a torsion free regular metric general connection of recurrent type on* TM, D *the product* ${}^{Q}\nabla$ *and* $\bar{D}$ *the Levi—Civita connection with respect to* G. *Then the curvature tensor fields* $R(D)$ *and* $R(\bar{D})$ *satisfy the following equation.*

(5.13) $$R(D)_{X,Y}Z = R(\bar{D})_{X,Y}Z+d\omega(X,Y)TZ+$$
$$+(1/2)\{[(\bar{D}_X\omega)(Z)TY-(\bar{D}_Y\omega)(Z)TX]-[g(TY,Z)\bar{D}_X W-g(TX,Z)\bar{D}_Y W]\}-$$
$$-(1/4)\{|\omega|^2[g(TY,Z)TX-g(TX,Z)TY]+$$
$$+\omega(X)\omega(Z)AY-\omega(Y)\omega(Z)AX-\omega(X)g(AY,Z)W+\omega(Y)g(AX,Z)W]\},$$

where we put

$$A = 4U^2+4U^{*2}+2U^*T-2TU^*+T^2.$$

6. Regular metric general connections of recurrent type whose principal endomorphism is symmetric

In this section, we study the case that the principal endomorphism P is symmetric with respect to g, that is,

(6.1) $$g(PX,Y) = g(X,PY).$$

As a consequence of this, we easily get the following:

(6.2) $$Q^* = Q \quad \text{and} \quad U = U^*,$$

(6.3) $$T = 2U,$$

(6.4) $$A = 12U^2.$$

Then the equation (5.13) is rewritten as

$$(6.5)\qquad R(D)_{X,Y}Z = R(\bar{D})_{X,Y}Z+2d\omega(X,Y)UZ+$$
$$+[(\bar{D}_X\omega)(Z)UY-(\bar{D}_Y\omega)(Z)UX]-[g(UY,Z)\bar{D}_XW-g(UX,Z)\bar{D}_YW]-$$
$$-|\omega|^2[g(UY,Z)UX-g(UX,Z)UY]+3[\omega(Y)\omega(Z)U^2X-\omega(X)\omega(Z)U^2Y+$$
$$+\omega(X)g(U^2Y,Z)W-\omega(Y)g(U^2X,Z)W].$$

Proposition 6.1. *Let* (∇, P) *be a torsion free regular metric general connection of recurrent type on TM. If the general connection* (∇, P) *satisfies* $g(PX, Y)=g(X, PY)$, *then the* 1*-form* ω *is closed.*

Proof. Let $\{e_i\}$ be a local orthonormal frame field with respect to g and $\{f^i\}$ the dual frame of $\{e_i\}$. Then, from (6.5), we have

$$(6.6)\qquad f^i(R(D)_{e_i,Y}Z)=f^i(R(\bar{D})_{e_i,Y}Z)+2d\omega(e_i,Y)f^i(UZ)+$$
$$+[(\bar{D}_{e_i}\omega)(Z)f^i(UY)-(\bar{D}_Y\omega)(Z)f^i(Ue_i)]-$$
$$-[g(UY,Z)f^i(\bar{D}_{e_i}W)-g(Ue_i,Z)f^i(\bar{D}_YW)]-$$
$$-|\omega|^2[g(UY,Z)f^i(Ue_i)-g(Ue_i,Z)f^i(UY)]+$$
$$+3[\omega(Y)\omega(Z)f^i(U^2e_i)-\omega(e_i)\omega(Z)f^i(U^2Y)+$$
$$+\omega(e_i)g(U^2Y,Z)f^i(W)-\omega(Y)g(U^2e_i,Z)f^i(W)],$$

where we used the summation convention. Thus we get

$$(6.7)\qquad K(D)(Y,Z) = K(\bar{D})(Y,Z)+2d\omega(UZ,Y)+(\bar{D}_{UY}\omega)(Z)-(\bar{D}_Y\omega)(Z)\operatorname{Tr}U-$$
$$-g(UY,Z)f^i(\bar{D}_{e_i}W)+g(U\bar{D}_YW,Z)-|\omega|^2(g(UY,Z)\operatorname{Tr}U-g(U^2Y,Z))+$$
$$+3(\omega(Y)\omega(Z)\operatorname{Tr}U^2-\omega(U^2Y)\omega(Z)+\omega(W)g(U^2Y,Z)-\omega(Y)g(U^2W,Z))$$

where $K(D)(Y,Z)$, $K(\bar{D})(Y,Z)$ denote the Ricci curvature tensor fields with respect to G and g respectively. Changing Y and Z in (6.7) and subtracting this from (6.7), we obtain

$$(6.8)\qquad 2d\omega(Y,Z)\operatorname{Tr}U = 0,$$

since $K(D)(Y,Z)$ and $K(\bar{D})(Y,Z)$ are symmetric. As $\operatorname{Tr}U=\operatorname{Tr}Q^2=|Q|^2\neq 0$, we have

$$d\omega = 0.$$

This proves our proposition.

In this case, (6.5) reduces to

$$(6.9)\qquad K(D)_{X,Y}Z = K(D)_{X,Y}Z+[(\bar{D}_X\omega)(Z)UY-(\bar{D}_Y\omega)(Z)UX]-$$
$$-[g(UY,Z)\bar{D}_XW-g(UX,Z)\bar{D}_YW]-|\omega|^2[g(UY,Z)UX-g(UX,Z)UY]+$$
$$+3[\omega(Y)\omega(Z)U^2X-\omega(X)\omega(Z)U^2Y+\omega(X)g(U^2Y,Z)W-\omega(Y)g(U^2X,Z)W].$$

Remark. Excepting Proposition 6.1, our results are true in the case that the metrics are pseudo-Riemannian.

References

[1] N. ABE, General connections on vector bundles, *Kōdai Math. J.*, **8** (1985), 322—329.
[2] A. MOÓR, Otsukische Übertragung mit rekurrentem Maßtensor, *Acta Sci. Math.*, **40** (1978), 129—142.
[3] A. MOÓR, Über die Veränderung der Länge der Vektoren in Weyl-Otsukischen Räumen, *Acta Sci. Math.*, **41** (1979), 1973—185.
[4] A. MOÓR, Über nicht-holonome Otsukische Räume, *Acta Math. Acad. Sci. Hungar.*, 33 (1979), 369—378.
[5] A. MOÓR, Über die aus einem gemischten Tensor gebildeten Otsukischen Räume, *Acta Math. Sci. Hungar.*, 39 (1982), 225—232.
[6] A. MOÓR, Otsukische Räume mit einem zweifach rekurrenten metrischen Grundtensor, *Period. Math. Hungar.*, **13** (1982), 129—135.
[7] A. MOÓR, Über die Begründung von Finsler—Otsukischen Räumen und ihre Dualität, *Tensor New Ser.*, 37 (1982), 121—129.
[8] H. NAGAYAMA, A theory of general relativity by general connections. I, *TRU Math.*, **20** (1984), 173—187.
[9] H. NEMOTO, On differential geometry of general connections, *TRU Math.*, **21** (1985), 67—94.
[10] T. OTSUKI, Tangent bundles of order 2 and general connections, *Math. J. Okayama Univ.*, **8** (1958), 143—180.
[11] T. OTSUKI, On general connections. I, *Math. J Okayama Univ.*, 9 (1960), 99—164.
[12] T. OTSUKI, On general connections. II, *Math. J. Okayama Univ.*, **10** (1961), 113—124.
[13] T. OTSUKI, On normal general connections, *Kōdai Math. Sem. Rep.*, **13** (1961). 152—166.
[14] T. OTSUKI, On metric general connections, *Proc. Japan Acad.*, 37 (1962), 183—188.
[15] T. OTSUKI, A note on metric general connections, *Proc. Japan Acad.*, **38** (1962), 509—513.
[16] T. OTSUKI, General connections $A\Gamma A$ and parallelism of Levi-Cività, *Kōdai Math. Sem. Rep.*, **14** (1962), 40—52.
[17] T. OTSUKI, On basic curves in spaces with normal general connections. I, *Kōdai Math. Sem. Rep.*, **14** (1962), 110—118.
[18] T. OTSUKI, On curvatures of spaces with normal general connections. I, *Kōdai Math. Sem. Rep.*, **15** (1963), 52—61.
[19] T. OTSUKI, On curvatures of spaces with normal general connections. II, *Kōdai Math. Sem. Rep.*, **15** (1963), 184—194.
[20] T. OTSUKI, A note on general connections, *Proc. Japan Acad.*, **41** (1965), 6—10.
[21] T. OTSUKI, A construction of spaces with general connections which have points swallowing geodesics, *Math. J. Okayama Univ.*, **24** (1982), 157—162.
[22] T. OTSUKI and C. S. HOUH, Ricci's formula for normal general connections and its applications, *Kōdai Math. Sem. Rep.*, **15** (1963), 184—194.

(N. A.)
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
SCIENCE UNIVERSITY OF TOKYO
TOKYO, 162, JAPAN

(H. N.)
MATHEMATICAL LABORATORY
COLLEGE OF AGRICULTURE AND VET. MED.
NIHON UNIVERSITY
KAMEINO FUJISAWA, KANAGAWA, 252, JAPAN

(S. Y.)
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
SCIENCE UNIVERSITY OF TOKYO
TOKYO, 162, JAPAN

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On the boundedness of solutions of nonautonomous differential equations

WU JIANHONG* and L. HATVANI**

Dedicated to L. Pintér on his 60th birthday

1. Introduction

In the study of existence of periodic solutions and almost periodic solutions as well as behavior of limiting sets of solutions of ordinary differential equations, the uniform boundedness and uniform ultimate boundedness of solutions are frequently needed [1—4, 9]. These properties of solutions can be regarded as either the instability of infinity or a special case of some kind of stability of a set. Therefore, there exists a close relation between Lyapunov's direct method and the boundedness of solutions. A typical result showing this relation is Theorem 10.4 in [3]. In this theorem the uniform ultimate boundedness is guaranteed by the existence of an appropriate Lyapunov function having a negative definite derivative along the solutions. However, in practice it is very difficult to construct such a Lyapunov function. For example, for mechanical systems the total mechanical energy, which is a typical Lyapunov function, never has a negative definite derivative along the motions with respect to the generalized coordinates.

The purpose of this paper is to study the boundedness and ultimate boundedness of solutions of nonautonomous differential equations by Lyapunov's direct method when the derivative of the Lyapunov function along the solutions is only semidefinite. The results generalize V. M. MATROSOV's theorem [5] on the asymptotic stability to the boundedness of solutions. An application is given to the boundedness of the motions of a holonomic scleronomic mechanical system of n degrees of freedom being under the action of potential, dissipative and gyroscopic forces.

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2. Notations and definitions

Consider the system

$$\dot{x} = X(t, x), \tag{2.1}$$

where $(t, x) \in \mathbf{R}^+ \times \mathbf{R}^n$, $\mathbf{R}^+ = [0, \infty)$ and X: $\mathbf{R}^+ \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous. Throughout this paper, for simplicity, we assume that for any $(t_0, x_0) \in \mathbf{R}^+ \times \mathbf{R}^n$, there exists a unique solution $x(t; t_0, x_0)$ of (2.1) through (t_0, x_0) defined for all $t \geqq t_0$.

Definition 2.1 [3]. A solution $x(t; t_0, x_0)$ of (2.1) is *bounded*, if $\sup_{t \geqq t_0} |x(t; t_0, x_0)| < \infty$.

The solutions of (2.1) are *uniformly bounded* (U.B.) if for every $\alpha > 0$ there exists a $\beta(\alpha) > 0$ such that $[t_0 \geqq 0,\ |x_0| < \alpha,\ t \geqq t_0]$ imply $|x(t; t_0, x_0)| < \beta(\alpha)$.

The solutions of (2.1) are *equiultimately bounded* (E.U.B.) for some bound B if for every $\alpha > 0$ and $t_0 \geqq 0$ there exists a $T(t_0, \alpha) > 0$ such that $[|x_0| < \alpha,\ t \geqq t_0 + T(t_0, \alpha)]$ imply $|x(t; t_0, x_0)| < B$.

The solutions of (2.1) are *uniformly ultimately bounded* (U.U.B.) for some bound B if for every $\alpha > 0$ there exists a $T(\alpha) > 0$ such that $[t_0 \geqq 0,\ |x_0| < \alpha,\ t \geqq t_0 + T(\alpha)]$ imply $|x(t; t_0, x_0)| < B$.

By a *pseudo wedge* W we mean a continuous and strictly increasing function W: $\mathbf{R}^+ \to \mathbf{R}^+$ with $W(r) > 0$ if $r > 0$. A pseudo wedge W is called *unbounded* if $\lim_{r \to \infty} W(r) = +\infty$.

Denote by $[a]_+$ and $[a]_-$ the positive and negative part of the real number a, respectively, that is, $[a]_+ = \max\{a, 0\}$, $[a]_- = \max\{-a, 0\}$.

Definition 2.2 [5]. A measurable function λ: $\mathbf{R}^+ \to \mathbf{R}^+$ is said to be *integrally positive* if $\int_J \lambda(t)\,dt = \infty$ holds on every set $J = \bigcup_{m=1}^{\infty} [a_m, b_m]$ such that $a_m < b_m \leqq a_{m+1}$ and $b_m - a_m \geqq \delta > 0$ $(m = 1, 2, \ldots)$ for a constant $\delta > 0$.

Definition 2.3 [7]. A measurable function λ: $\mathbf{R}^+ \to \mathbf{R}^+$ is said to be *weakly integrally positive* if for every $\delta > 0$, $\Delta > 0$ and for every set $J = \bigcup_{m=1}^{\infty} [a_m, b_m]$ with $a_m + \delta \leqq b_m \leqq a_{m+1} < b_m + \Delta$ $(m = 1, 2, \ldots)$ the relation $\int_J \lambda(t)\,dt = \infty$ holds.

Lemma 2.1. *If a measurable function λ: $\mathbf{R}^+ \to \mathbf{R}^+$ is integrally positive, then for every $\alpha > 0$ and $\delta > 0$ there exists a positive integer $K(\alpha, \delta)$ such that for every set $J = \bigcup_{m=1}^{K} [a_m, b_m]$ with $a_m < a_m + \delta \leqq b_m \leqq a_{m+1}$ for $1 \leqq m \leqq K-1$, we have $\int_J \lambda(t)\,dt \geqq \alpha$.*

Proof. It is easy to see that λ is integrally positive if and only if for every $\delta>0$ the inequality

$$\liminf_{t\to\infty}\int_t^{t+\delta}\lambda(s)\,ds>0 \tag{2.2}$$

holds. Consequently, for any given $\delta>0$ there are $T=T(\delta)>0$ and $\mu(\delta)>0$ such that $t\geqq T(\delta)$ implies

$$\int_t^{t+\delta}\lambda(s)\,ds\geqq\mu(\delta).$$

Let $\alpha>0$ and $\delta>0$ be given, and define $K(\alpha,\delta)=[T(\delta)/\delta]+1+[\alpha/\mu(\delta)]+1$, where $[a]$ denotes the integer part of $a\in\mathbf{R}$, that is, $[a]=\max\{z\colon z$ is an integer with $z\leqq a\}$. Then the number $K(\alpha,\delta)$ has the property mentioned in the assertion.

The following assertion can be easily proved by making use of (2.2).

Lemma 2.2. *If a measurable function* $\lambda\colon\mathbf{R}^+\to\mathbf{R}^+$ *is integrally positive, then*

$$\lim_{T\to\infty}\int_{t_0}^{t_0+T}\lambda=\infty \tag{2.3}$$

uniformly with respect to $t_0\in\mathbf{R}^+$.

Remark 2.1. The property of weak integral positivity and property (2.3) are independent of one another. E.g. $\lambda(t)=1/(1+t)$ is weakly integrally positive, but it does not satisfy (2.3) and so it is not integrally positive. On the other hand, weak integral positivity and (2.3) together do not imply integral positivity. E.g., the function

$$\lambda(t)=\begin{cases}1/(1+t) & n\leqq t\leqq n+1/2\\ 1 & n+1/2<t<n+1\end{cases}$$

is weakly integrally positive and satisfy (2.3) but it is not integrally positive.

With a continuous function $V\colon\mathbf{R}^+\times\mathbf{R}^n\to\mathbf{R}$ we associate the function

$$\dot V_{(2.1)}(t,x)=\limsup_{h\to0+}(1/h)\{V\big(t+h,\,x+hX(t,x)\big)-V(t,x)\},$$

which called the derivative of V with respect to (2.1).

It can be proved (see [3], p. 3) that if V is locally Lipschitz, then for an arbitrary solution $x(t)$ of (2.1) we have

$$V\big(t_2,x(t_2)\big)-V\big(t_1,x(t_1)\big)=\int_{t_1}^{t_2}\dot V\big(t,x(t)\big)\,dt,\quad (t_1,t_2\in\mathbf{R}^+).$$

3. The theorems and their proofs

Theorem 3.1. *Suppose that there exist nonnegative constants B and D, nonnegative locally Lipschitz functions $V(t, x)$, $P(t, x)$ and continuous $K(t, x)$ defined for $t \geqq 0$, $|x| \geqq B$ satisfying the following conditions:*

(i) $W_1(|x|) \leqq V(t, x) \leqq W_2(|x|)$, *where W_1 and W_2 are unbounded pseudo wedges;*

(ii) *the derivative of V with respect to* (2.1) *satisfies the inequality*

$$\dot{V}_{(2.1)}(t, x) \leqq -K(t, x) \quad \text{for} \quad t \geqq 0, \quad |x| \geqq B; \tag{3.1}$$

(iii) *for each $M > B$ there are $k = k(M) > 0$ and $H = H(M) \geqq 0$ such that $[t \geqq 0, B \leqq |x| \leqq M, P(t, x) \geqq H]$ imply $K(t, x) \geqq k$;*

(iv) *for each $M > B$ there exists an $L(M) > 0$ such that $[t \geqq 0, B \leqq |x| \leqq M, H(M) \leqq P(t, x) \leqq 2H(M)]$ imply $\dot{P}_{(2.1)}(t, x) \leqq L(M)$;*

(v) *for each $M > B$ there is a $T(M) > 0$ such that for any solution $x(t)$ of* (2.1) *with $B \leqq |x(t)| \leqq M$ and $P(t, x(t)) \leqq 2H(M)$ for $t_0 \leqq t \leqq t_0 + T(M)$ there exists $s \in [t_0, t_0 + T(M)]$ with $|x(s)| < D$.*

Then the solutions of (2.1) *are U.B. and U.U.B.*

Proof. For any $\alpha > 0$, define $\beta(\alpha) = W_1^{-1}(W_2(\max\{B, \alpha\}))$. It is easy to prove that $[t_0 \geqq 0, |x_0| \leqq \alpha]$ imply $|x(t; t_0, x_0)| \leqq \beta(\alpha)$ for $t \geqq t_0$. Therefore, the solutions of (2.1) are U.B. Throughout the remainder of this proof we use the notations $x(t) = x(t; t_0, x_0)$, $V(t) = V(t, x(t))$ and $\dot{V}(t) = \dot{V}_{(2.1)}(t, x(t))$.

To prove the uniform ultimate boundedness, we consider the following two cases:

(a) there exists a $t_2 \geqq t_0$ with $|x(t_2)| \leqq B$;

(b) $|x(t)| \geqq B$ for all $t \geqq t_0$.

In case (a) $|x(t)| \leqq \beta(B)$ for $t \geqq t_2$.

In case (b) we have $\dot{V}(t) \leqq -K(t, x(t))$ for all $t \geqq t_0$. By (iii) there exist $k = k(\beta(\alpha)) > 0$ and $H = H(\beta(\alpha)) > 0$ such that $P(t, x(t)) \geqq H$ implies $K(t, x(t)) \geqq k$. Let $\bar{t} \geqq t_0$ be fixed, and choose a constant $S = S(\alpha) > W_2(\beta(\alpha))/k$. Then by (3.1) the nonnegativeness of V implies the existence of a $t_3 \in [\bar{t}, \bar{t} + S(\alpha)]$ such that $P(t_3, x(t_3)) < H$. By (v), there exists $T = T(\beta(\alpha)) > 0$ such that if $P(t, x(t)) < 2H$ for $t \in [t_3, t_3 + T]$, then there is an $s \in [t_3, t_3 + T]$ with $|x(s)| < D$, which implies $|x(t)| < \beta(D)$ for $t \geqq t_3 + T$, especially, for $t \geqq \bar{t} + S + T$.

Therefore, only two cases may occur:

(b_1) $P(t, x(t)) < 2H$ for all $t \in [t_3, t_3 + T]$.

In this case, $|x(t)| < \beta(D)$ for $t \geqq \bar{t} + T + S$.

(b_2) there exists $t_4 \in [t_3, t_3 + T]$ with $P(t_4, x(t_4)) \geqq 2H$.

In this case, there are t_5, t_6 such that $t_3 < t_5 < t_6 \leqq t_4$, $P(t_5, x(t_5)) = H$, $P(t_6, x(t_6)) =$

$=2H$ and $H<P(t,x(t))<2H$ for $t_5<t<t_6$. By (iv), we get $t_6-t_5\geqq H/L(\beta(\alpha))$. On the other hand, by $\dot V(t)\leqq -K(t,x(t))\leqq -k$ for $t\in[t_5,t_6]$ we obtain

$$V(t_6)\leqq V(t_5)-kH/L(\beta(\alpha)). \tag{3.2}$$

Since in case (b) $\dot V(t)\leqq -K(t,x(t))\leqq 0$ for all $t\geqq t_0$, we get $V(\bar t+S+T)\leqq \leqq V(\bar t)-kH/L(\beta(\alpha))$. Let $\bar t=t_0+m(S+T)$, where m is a nonnegative integer. Then from the argument above we get either

(c_m) $\quad |x(t)|\leqq \max\{\beta(B),\beta(D)\}$ for $t\geqq t_0+(m+1)(S+T)$,

or

(d_m) $\quad V(t_0+(m+1)(S+T))\leqq V(t_0+m(S+T))-kH/L(\beta(\alpha))$.

Choose a positive integer $N=N(\alpha)$ such that

$$N(\alpha)kH/L(\beta(\alpha))>W_2(\beta(\alpha)). \tag{3.3}$$

Then by the nonnegativeness of V, (d_m) holds for at most $m=0,1,\ldots,N-1$, and thus $|x(t)|<\max\{\beta(B),\beta(D)\}$ for $t\geqq t_0+N(S+T)$. This completes the proof.

Remark 3.1. Using the same argument as one above, the comparison method and Lemma 2.1, we can prove the following assertion:

If conditions (i), (iii)—(v) of Theorem 3.1 are satisfied and if for each $M>B$ there exists a weakly integrally positive function λ_M: $\mathbf{R}^+\to\mathbf{R}^+$ such that

$$\dot V_{(2.1)}(t,x)\leqq -\lambda_M(t)K(t,x)+F(t,V(t,x)) \quad \text{for} \quad t\geqq 0$$

and $B\leqq|x|\leqq M$, where F: $\mathbf{R}^+\times\mathbf{R}^+\to\mathbf{R}^+$ is continuous, the solutions of $\dot z=F(t,z)$ are uniformly bounded, and $\int_0^\infty \sup_{0\leqq z\leqq r} F(t,z)\,dt<\infty$ for $r\geqq 0$, then the solutions of (2.1) are U.B. and E.U.B. If, in addition, λ_M is integrally positive, then the solutions of (2.1) are U.B. and U.U.B.

Remark 3.2. If conditions (i), (iii) and (v) of Theorem 3.1 are satisfied and if

(a) $\dot V_{(2.1)}(t,x)\leqq -\lambda(t)K(t,x)+F(t,V(t,x))$ for $t\geqq 0$ and $|x|\geqq B$, where λ: $\mathbf{R}^+\to\mathbf{R}^+$ is measurable and satisfies condition (2.3), and F is of the same kind as in Remark 3.1;

(b) for any $M>0$ there exists a $\mu=\mu(M)>0$ such that $[B\leqq|x|\leqq M,\ H(M)\leqq \leqq P(t,x)\leqq 2H(M)]$ imply

$$\dot V_{(2.1)}(t,x)\leqq -\mu\dot P_{(2.1)}(t,x)+F(t,V(t,x)),$$

then the solutions of (2.1) are U.B. and U.U.B.

To prove this remark it is sufficient to replace (3.2) and (3.3) in the proof of Theorem 3.1 by

$$V(t_6) \leqq V(t_5) - \mu(\beta(\alpha))H(\beta(\alpha)) + \int_{t_5}^{t_6} \max\{F(t, z) : 0 \leqq z \leqq W_2(\beta(\alpha))\}\, dt$$

and

$$N\mu(\beta(\alpha))H(\beta(\alpha)) > W_2(\beta(\alpha)) + \int_0^\infty \max\{F(t, z) : 0 \leqq z \leqq W_2(\beta(\alpha))\}\, dt,$$

respectively.

Remark 3.3. Condition (iv) in Theorem 3.1 can be weakened as follows: for any $M > B$ there exists a continuous function L_M: $\mathbf{R}^+ \to \mathbf{R}^+$ such that $\int_0^t L_M$ is uniformly continuous on $[0, \infty)$ and either

$$[\dot P_{(2.1)}(t, x)]_+ \leqq L_M(t) \quad \text{for} \quad t \geqq 0, \quad B \leqq |x| \leqq M \quad \text{and} \quad H(M) \leqq P(t, x) \leqq 2H(M),$$

or

$$[\dot P_{(2.1)}(t, x)]_- \leqq L_M(t) \quad \text{for} \quad t \geqq 0, \quad B \leqq |x| \leqq M \quad \text{and} \quad H(M) \leqq P(t, x) \leqq 2H(M).$$

Remark 3.4. Condition (i) in Theorem 3.1 can be replaced by $0 \leqq V(t, x) \leqq \leqq W_2(|x|)$ if the solutions of (2.1) are U.B.

Example 3.1. Consider a Liénard equation with forcing term

$$\ddot x + f(x)\dot x + g(t, x) = e(t), \tag{3.4}$$

where $f(x)$, $g(t, x)$, $\partial g(t, x)/\partial t$ and $e(t)$ are continuous for $(t, x) \in \mathbf{R}^+ \times \mathbf{R}$ and $\int_0^\infty |e(s)|\, ds < \infty$. Besides, we assume that there exist unbounded pseudo wedges W_1, W_2, a continuous W_3: $\mathbf{R}^+ \to \mathbf{R}^+$ with $W_3(r) > 0$ for $r > 0$ and an integrally positive function λ: $\mathbf{R}^+ \to \mathbf{R}^+$ such that

$$W_1(|x|) \leqq \int_0^x g(t, x)\, dx \leqq W_2(|x|),$$

$$g(t, x)F(x) - \int_0^x (\partial g(t, r)/\partial t)\, dr \geqq \lambda(t) W_3(|x|),$$

where $F(x) = \int_0^x f(s)\, ds$. Obviously, (3.4) is equivalent to

$$\dot x = y - F(x), \quad \dot y = -g(t, x) + e(t). \tag{3.5}$$

Let $V(t,x,y)=\left[y^2+2\int\limits_0^x g(t,r)\,dr\right]^{1/2}+\int\limits_t^\infty |e(s)|\,ds$, then

$$[y^2+2W_1(|x|)]^{1/2}\leqq V(t,x,y)\leqq [y^2+2W_2(|x|)]^{1/2}+\int\limits_0^\infty |e(s)|\,ds$$

$$\dot V_{(3.5)}(t,x,y)\leqq -\lambda(t)W_3(|x|)[y^2+2W_2(|x|)]^{-1/2}.$$

Let $K(t,x,y)=W_3(|x|)[y^2+2W_2(|x|)]^{-1/2}$, $P(t,x,y)=|x|$, $B=1$ and $H=1$. Then for each $M>1$ and for $t\geqq 0$, $1\leqq|x|+|y|\leqq M$ and $|x|\geqq 1$, we have $K(t,x,y)\geqq \geqq\min\{W_3(r)\colon 1\leqq r\leqq M\}\,(M^2+2W_2(M))^{-1/2}$. Therefore, conditions (i)—(iv) of Theorem 3.1 hold (see also Remark 3.1). Now we check condition (v).

Let $E=\max\{|F(x)|+1\colon |x|\leqq 2\}$, $D=E+2$, and for $M>1$ define $T(M)= =2M+1$. Suppose that $(x(t),y(t))$ is a solution of (3.5) with $1\leqq|x(t)|+|y(t)|\leqq M$ and $|x(t)|\leqq 2$ for $t\in[t_0,t_0+T(M)]$. If $|x(t)|+|y(t)|\geqq E+2$ for all $t\in[t_0,t_0+T(M)]$, then $|y(t)|\geqq E$, e.g. $y(t)\geqq E$, and consequently $\dot x(t)=y(t)-F(x(t))\geqq E-\max\limits_{|x|\leqq 2}F(x)\geqq \geqq 1$. Hence we obtain the inequality $2M\geqq|x(t_0+T(M))-x(t_0)|\geqq T(M)=2M+1$, which is a contradiction. Therefore, there is an $s\in[t_0,t_0+T(M)]$ with $|x(s)|+ +|y(s)|<D=E+2$, i.e. condition (v) in Theorem 3.1 holds.

Consequently, under our conditions the solutions of (3.5) are U.B. and U.U.B.

Notice that if $P(t,x)=|x|$, then condition (iv) in Theorem 3.1 can be dropped. (Indeed, if condition (i)—(iii), (v) are satisfied for $P(t,x)=|x|$, then all the conditions of the theorem are satisfied for the new auxiliary function $\tilde P(t,x)=V(t,x)$. If, in addition, H in (iii) is constant, then (v) obviously holds. This special case initiates the following generalization of T. YOSHIZAWA's theorem ([3], Theorem 10.4):

Theorem 3.2. *Suppose that there exist a constant $B\geqq 0$, a locally Lipschitz function $V(t,x)$ and a continuous function $K(t,x)$ defined for $t\geqq 0$ and $|x|\geqq B$ satisfying the following conditions:*

(i) $W_1(|x|)\leqq V(t,x)\leqq W_2(|x|)$, *where W_1 and W_2 are unbounded pseudo wedges;*

(ii) $\dot V_{(2.1)}(t,x)\leqq -\lambda(t)K(t,x)$ *for $t\geqq 0$ and $|x|\geqq B$, where $\lambda\colon \mathbf{R}^+\to\mathbf{R}^+$ is measurable with $\lim\limits_{t\to\infty}\int\limits_{t_0}^t \lambda(s)\,ds=\infty$ for any $t_0\geqq 0$;*

(iii) *for each $M>B$ there exists $k(M)>0$ such that $B\leqq|x|\leqq M$ implies $K(t,x)\geqq k(M)$.*

Then the solutions of (2.1) *are U.B. and E.U.B. If, in addition, λ satisfies condition* (2.3), *then the solutions of* (2.1) *are U.B. and U.U.B.*

Proof. For any $\alpha>0$, define $\beta(\alpha)=W_1^{-1}(W_2(\max\{B,\alpha\}))$. Let $x(t;t_0,x_0)$ be a solution of (2.1) with $|x_0|<\alpha$. Then $|x(t;t_0,x_0)|<\beta(\alpha)$ for all $t\geqq t_0$, i.e. the solutions are U.B.

For a given $t_0 \geqq 0$ choose $T(t_0, \alpha) > 0$ such that

$$\int_{t_0}^{t_0+T(t_0,\alpha)} \lambda(s)\,ds > W_2(\beta(\alpha))/k(\beta(\alpha)).$$

It is easy to prove that $|x(t; t_0, x_0)| < \beta(B)$ for all $t \geqq t_0 + T(t_0, \alpha)$.

The second conclusion can be proved similarly.

The following theorem is a generalization of V. M. MATROSOV's stability theorem [5] to the boundedness of solutions.

Theorem 3.3. *Suppose that there exist a constant $B \geqq 0$ and nonnegative locally Lipschitz functions $V(t, x)$, $W(t, x)$, $P(t, x)$, a continuous function $F(t, u)$ defined for $t \geqq 0$, $|x| \geqq B$, $u \geqq 0$ and such that*

(i) $W_1(|x|) \leqq V(t, x) \leqq W_2(|x|)$, *where W_1 and W_2 are unbounded pseudo wedges;*

(ii) *for every $M > B$ there is a measurable function λ_M: $\mathbf{R}^+ \to \mathbf{R}^+$ such that*

$$\dot{V}_{(2.1)}(t, x) \leqq -\lambda_M(t) P(t, x) + F(t, V(t, x)) \quad \textit{for} \quad t \geqq 0 \quad \textit{and} \quad B \leqq |x| \leqq M,$$

where

(a) *λ_M is weakly integrally positive;*

(b) *the solutions of the equation $\dot{z} = F(t, z)$ are U.B., and $\int_0^\infty [\sup_{0 \leqq z \leqq r} F(t, z)]\,dt < \infty$ for every $r > 0$;*

(iii) *for every $M > B$ there exists a continuous function L_M: $\mathbf{R}^+ \to \mathbf{R}^+$ such that $\int_0^t L_M$ is uniformly continuous on $\mathbf{R}^+$ and either $[\dot{P}_{(2.1)}(t, x)]_+ \leqq L_M(t)$ or $[\dot{P}_{(2.1)}(t, x)]_- \leqq L_M(t)$ for $t \geqq 0$, $B \leqq |x| \leqq M$;*

(iv) *for every $M > B$ there exists a constant $A(M) > 0$ such that $|W(t, x)| \leqq A(M)$ for $t \geqq 0$ and $B \leqq |x| \leqq M$;*

(v) *there exists a constant $D \geqq B$ and for any $M > B$ there exists a continuous function W_3: $\mathbf{R}^+ \to \mathbf{R}^+$ with $W_3(r) > 0$ for $r \geqq D$ such that*

$$\max\{P(t, x), |\dot{W}_{(2.1)}(t, x)|\} \geqq W_3(|x|) \quad \textit{for} \quad t \geqq 0 \quad \textit{and} \quad D \leqq |x| \leqq M.$$

Then the solutions of (2.1) are U.B. and E.U.B. If, in addition, $\lambda_M(t)$ is integrally positive, then the solutions of (2.1) are U.B. and U.U.B.

Proof. First we show that under the assumptions of the theorem condition (v) in Theorem 3.1 is satisfied.

For any $M > D$, choose $H(M) > 0$ such that $2H < \alpha(M) = \min_{D \leqq r \leqq M} W_3(r)$ and define $T(M) = [2A(M) + 1]/\alpha$. Let $x(t)$ be a solution of (2.1) with $B \leqq |x(t)| \leqq M$ and $P(t, x(t)) \leqq 2H(M)$ for $t \in [t_0, t_0 + T(M)]$. If $|x(t)| \geqq D$ for all $t \in [t_0, t_0 + T(M)]$ then according to condition (v) we get $|\dot{W}_{(2.1)}(t, x(t))| \geqq \alpha$, hence $2A(M) \geqq$

$|W(t_0+T(M), x(t_0+T(M)))-W(t_0, x(t_0))|\geqq\alpha T(M)=2A(M)+1$, which is a contradiction. Therefore, condition (v) of Theorem 3.1 holds.

An application of Theorem 3.1, Remark 3.1 and Remark 3.3 completes the proof.

Remark 3.5. Condition (v) of Theorem 3.3 can be weakened by asking there is a constant $D\geqq B$ such that for every $M>D$ there are $B_2(M)>0$ and a continuous function μ_M: $\mathbf{R}^+\to\mathbf{R}^+$ with property (2.3) and such that $[t\geqq 0,\ D\leqq|x|\leqq M,\ P(t,x)\leqq B_2]$ imply $|\dot W_{(2.1)}(t,x)|\geqq\mu_M(t)$.

An application of this theorem to a holonomic scleronomic mechanical system will be given in Section 4.

As we have seen so far, the key step in the application of Theorem 3.1 is to check condition (v). Now we establish a sufficient condition for this property by Lyapunov's direct method.

Lemma 3.1. *Suppose that there exist* $H_0>0$, $D>B$ *and a locally Lipschitz function* $Q(t,x)$ *defined on the set* $\{(t,x)\colon t\geqq 0,\ |x|\geqq D,\ P(t,x)\leqq 2H_0\}$ *such that*

(i) *for each* $M>D$ *there are continuous functions* $\gamma, g\colon \mathbf{R}^+\to\mathbf{R}$ *and a number* $H\in(0,H_0]$ *such that* γ *has property* (2.3), *the function* $\int_0^t [g(s)]_+\,ds$ *is bounded on* $\mathbf{R}^+$, *and* $[t\geqq 0,\ D\leqq|x|\leqq M,\ P(t,x)\leqq 2H]$ *imply* $\dot Q_{(2.1)}(t,x)\leqq-\gamma(t)+g(t)$;

(ii) *for each* $M>D$ *there exists* $L(M)>0$ *with* $|Q(t,x)|\leqq L(M)$ *for* $t\geqq 0$ *and* $D\leqq|x|\leqq M$.

Then condition (v) *of Theorem* 3.1 *holds with these numbers* H *and* D.

Proof. Let $M>D$ be given and let a solution $x(t)$ of (2.1) satisfy $B\leqq|x(t)|\leqq M$ and $P(t,x(t))\leqq 2H(M)$ for $t\in[t_0, t_0+T(M)]$, where $T(M)>0$ is a constant such that

$$\int_{t_0}^{t_0+T(M)} \gamma(s)\,ds > 2L(M)+\int_0^\infty [g(s)]_+\,ds \quad \text{for all } t_0\geqq 0.$$

If $|x(t)|\geqq D$ for $t\in[t_0, t_0+T(M)]$, then we get

$$-L(M)\leqq Q(t_0+T(M), x(t_0+T(M)))\leqq L(M)-\int_{t_0}^{t_0+T(M)}\gamma(t)\,dt+\int_0^\infty [g(s)]_+\,ds$$

which yields a contradiction to the choice of $T(M)$. Consequently, there is $s\in\in[t_0, t_0+T(M)]$ with $|x(s)|<D$, and the proof is complete.

Example 3.2. Consider the equation

$$\ddot x+a(t)\dot x+f(x)=e(t) \tag{3.6}$$

and suppose that the continuous functions a, e: $\mathbf{R}^+\to\mathbf{R}$, f: $\mathbf{R}\to\mathbf{R}$ satisfy the following conditions:

(i) $a(t)\geqq 0$ for $t\in\mathbf{R}^+$, a is weakly integrally positive, and there exist constant $\tilde{a}>0$, $T>0$ such that $[t_0\geqq 0,\ t\geqq T]$ imply $(1/t)\int_{t_0}^{t_0+t} a(s)\,ds\leqq\tilde{a}$;

(ii) $e\in L^1[0,\infty)$;

(iii) there is an $r_0>0$ such that $xf(x)>0$, $|f(x)|>0$ provided $|x|>r_0$, and $F(x)=\int_0^x f(s)\,ds\to\infty$, as $|x|\to\infty$.

Then the solutions of equation (3.6) and their derivatives are U.B. and E.U.B. If, in addition, the function $a(t)$ is integrally positive, then the solutions and their derivatives are U.B. and U.U.B.

Equation (3.6) is equivalent to the system

$$\dot{x}=y,\quad \dot{y}=-f(x)-a(t)y+e(t). \tag{3.7}$$

Define $V(t,x,y)=[y^2+2F(x)]^{1/2}+\int_t^\infty |e(s)|\,ds$. Then

$$\dot{V}_{(3.7)}(t,x,y)\leqq -a(t)y^2[y^2+2F(x)]^{-1/2}.$$

Choose $K(t,x,y)=y^2[y^2+2F(x)]^{-1/2}$, $P(t,x,y)=y^2$. Then

$$[\dot{P}_{(3.7)}(t,x,y)]_+=[-f(x)y-a(t)y^2+e(t)y]_+\leqq |f(x)|\,|y|+|e(t)|\,|y|.$$

Let $B>0$ be fixed arbitrarily. For $M>B$ let $K_M=\max\{|f(x)|:\ 0\leqq|x|\leqq M\}$ and suppose $B\leqq|x|+|y|\leqq M$. Then $[\dot{P}_{(3.7)}(t,x,y)]_+\leqq[K_M+|e(t)|]M$ and $\int_0^t (K_M+|e(s)|)M\,ds$ is uniformly continuous in $\mathbf{R}^+$. Consequently, conditions (i)—(iv) of Theorem 3.1 (see also Remark 3.3) are met with arbitrary $H>0$, and the solutions are U.B.

Now define $D=r_0+1$, $H_0=1/2$, and

$$Q(t,x,y)=\begin{cases} y & \text{if } x\geqq r_0,\\ -y & \text{if } x\leqq -r_0,\end{cases}$$

whose derivative is

$$\dot{Q}_{(3.7)}(t,x,y)=\begin{cases} -f(x)-a(t)y+e(t) & \text{if } x\geqq r_0,\\ f(x)+a(t)y-e(t) & \text{if } x\leqq -r_0.\end{cases}$$

For a given $M>D$ introduce the notation $m(M)=\min\{|f(x)|:\ r_0\leqq|x|\leqq M\}$. By the conditions, $m(M)>0$, and $[t\geqq 0,\ D\leqq|x|+|y|\leqq M,\ y^2\leqq 2H]$ imply the inequality

$$\dot{Q}_{(3.7)}(t,x,y)\leqq -m(M)+a(t)[2H]^{1/2}+e(t).$$

Let $H=\min\left\{\frac{1}{2}[m(M)/(\tilde{a}+1)]^2, \frac{1}{2}\right\}$, $\gamma(t)=m(M)-(2H)^{1/2}a(t)$ and $g(t)=|e(t)|$. Then $\dot{Q}_{(3.7)}(t,x,y)\leqq -\gamma(t)+g(t)$ and for sufficiently large $T>0$,

$$\int_{t_0}^{t_0+T} \gamma(t)\,dt = m(M)T-(2H)^{1/2}\int_{t_0}^{t_0+T} a(t)\,dt \geqq m(M)\tilde{a}/(\tilde{a}+1)T\to\infty$$

as $T\to\infty$ uniformly with respect to $t_0\geqq 0$, and so all the conditions of Lemma 3.1 are satisfied.

This completes the proof.

Consider now the system

$$\dot{x} = X(t,x,y),\quad \dot{y}=Y(t,x,y) \tag{3.8}$$

where $x\in\mathbf{R}^m$, $y\in\mathbf{R}^k$; X: $\mathbf{R}^+\times\mathbf{R}^{m+k}\to\mathbf{R}^m$ and Y: $\mathbf{R}^+\times\mathbf{R}^{m+k}\to\mathbf{R}^k$ are continuous. The following theorem shows that the function Q in Lemma 3.1 can be constructed from the reduced subsystem

$$\dot{y} = Y(t,0,y). \tag{3.9}$$

Theorem 3.4. *Suppose that*

(i) *There exist constants B, $H\geqq 0$ and a locally Lipschitz function $V(t,x,y)$ defined for $t\geqq 0$ and $|x|+|y|\geqq B$ such that*

(a) $W_1(|x|+|y|)\leqq V(t,x,y)\leqq W_2(|x|+|y|)$, *where W_1 and W_2 are unbounded pseudo wedges;*

(b) $\dot{V}_{(3.8)}(t,x,y)\leqq -\lambda(t)K(x,y)$ *for $t\geqq 0$ and $|x|+|y|\geqq B$, where $\lambda(t)$ is weakly integrally positive, $K(x,y)\geqq 0$ for $|x|+|y|\geqq B$, and for any $M>B$ there exists $k(M)>0$ such that $K(x,y)\geqq k(M)$ for $H\leqq|x|$, $B\leqq|x|+|y|\leqq M$;*

(ii) *there exist a constant $B_1>0$, a continuous N: $\mathbf{R}^+\to\mathbf{R}^+$ with $N(s)>0$ for $s\geqq B_1$ and a locally Lipschitz function $Q(t,y)$ defined for $t\geqq 0$ and $|y|\geqq B_1$ such that*

(c) $0\leqq Q(t,y)\leqq W_3(|y|)$, *where W_3 is a pseudo wedge;*

(d) $\dot{Q}_{(3.9)}(t,y)\leqq -W_4(|y|)$ *for $|y|\geqq B_1$, where W_4 is a pseudo wedge;*

(e) $|Q(t,y)-Q(t,\tilde{y})|\leqq N(\max\{|y|,|\tilde{y}|\})|y-\tilde{y}|$;

(iii) *for any $M>0$ there exists $L(M)>0$ such that $|X(t,x,y)|\leqq L(M)$ if $|x|+|y|\leqq M$;*

(iv) *there exist continuous P_1, P_2: $\mathbf{R}^+\to\mathbf{R}^+$ with $P_1(s)>0$ for $s\geqq B_1$ such that $|Y(t,x,y)-Y(t,0,y)|\leqq P_1(|y|)P_2(|x|)$;*

(v) $\lim_{r\to\infty} W_4(r)/(P_1(r)N(r))=\infty$.

Then the solutions of (3.8) *are U.B. and E.U.B. If, in addition, λ is integrally positive, then the solutions of* (3.8) *are U.B. and U.U.B.*

Proof. Obviously, (i)—(iv) of Theorem 3.1 hold with $P(t,x,y)=|x|$. Choose $D>0$ such that $D-2H\geqq B_1$, $W_4(r)/N(r)P_1(r)\geqq\max\{P_2(s)\colon |s|\leqq 2H\}+1$

for $s \geqq D-2H$. Then if $D \leqq |x|+|y| \leqq M$, $|x| \leqq 2H$, then $|y| \geqq D-2H \geqq B_1$, and thus

$$\dot{Q}_{(3.8)}(t, y) \leqq \dot{Q}_{(3.9)}(t, y)+N(|y|)|Y(t, x, y)-Y(t, 0, y)| \leqq -W_4(|y|)+$$

$$+N(|y|)P_1(|y|)P_2(|x|) \leqq -N(|y|)P_1(|y|)\left[\frac{W_4(|y|)}{N(|y|)P_1(|y|)}-P_2(|x|)\right]-$$

$$-N(|y|)P_1(|y|) \leqq -\inf\{N(r)P_1(r)\colon B_1 \leqq r \leqq M\}.$$

Therefore, condition (v) of Theorem 3.1 holds by Lemma 3.1, and so the proof is complete.

Example 3.3. Consider now the system

$$\dot{x} = f_1(t, x)+by, \quad \dot{y} = f_2(t, x)+dy+e(t), \tag{3.10}$$

where $f_1, f_2 \in C(\mathbf{R}^+ \times \mathbf{R}, \mathbf{R})$ with $f_1(t, 0)=0$, $f_2(t, 0)=0$, $e(t)$ is a bounded continuous function on $\mathbf{R}^+$ with $e \in L^1[0, \infty)$, b, d are constants with $db \neq 0$. Besides, we assume

(i) $\sup\{|f_1(t, x)|+|f_2(t, x)|\colon t \geqq 0, |x| \leqq M\} < \infty$ for any $M>0$;

(ii) $[df_1(t, x)-bf_2(t, x)]/x \geqq \alpha(x)>0$ for $t \geqq 0$ and $x \neq 0$, where α is continuous and $\lim\limits_{|x|\to\infty} \int_0^x \alpha(r)r\,dr = \infty$;

(iii) $[f_1(t, x)+dx][bf_2(t, x)-df_1(t, x)]-\int_0^x [(d\partial f_1(t, r)/\partial t)-(b\partial f_2(t, r)/\partial t)]\,dr \geqq$ $\geqq \lambda(t)\beta(x)$, where $\lambda(t)$ is integrally positive, β is continuous with $\beta(x)>0$ if $x \neq 0$.

Under these conditions the solutions of (3.10) are U.B. and U.U.B.

Indeed, let

$$V(t, x, y) = \left[(dx-by)^2+2\int_0^x [df_1(t, r)-bf_2(t, r)]\,dr\right]^{1/2}+b\int_t^\infty |e(s)|\,ds.$$

Then

$$\dot{V}_{(3.10)}(t, x, y) \leqq$$

$$\leqq \frac{-[bf_2(t, x)-df_1(t, x)][f_1(t, x)+dx]+\int_0^x \left[d\frac{\partial}{\partial t}f_1(t, r)-b\frac{\partial}{\partial t}f_2(t, r)\right]dr}{\left[(dx-by)^2+2\int_0^x [df_1(t, r)-bf_2(t, r)]\,dr\right]^{1/2}} \leqq$$

$$\leqq -\lambda(t)K(x, y),$$

where

$$K(x, y) = \beta(x)\left[(dx-by)^2+2\sup_{t \geqq 0}\int_0^x [df_1(t, r)-bf_2(t, r)]\,dr\right]^{-1/2}.$$

It is easy to prove that for any $M>0$ there exists $k=k(M)>0$ such that $[|x|+|y|\leqq M,\ |x|\geqq H]$ imply $K(x, y)>k(M)$. Therefore, (i) of Theorem 3.4 holds.

On the other hand, for the subsystem

$$\dot{y} = dy+e(t) \tag{3.11}$$

and for $Q(t, y)=y^2/2$, $N(r)=r$, we have

$$\dot{Q}_{(3.11)}(t, y) \leqq d|y|\left[|y|+(1/d)\sup_{t\geqq 0}|e(t)|\right] \leqq (1/2)\,dy^2 \quad \text{for} \quad |y| \geqq -(2/d)\sup_{t\geqq 0}|e(t)|.$$

Therefore, after making the choice $P_1(r)=1$, $P_2(r)=\sup\{|f_2(t, x)|\colon t\geqq 0,\ |x|\leqq r\}$ all the conditions of Theorem 3.4 are met, and our assertion is true.

Theorem 3.5. *For system* (3.8), *suppose that*

(i) *there exist continuous functions* $P_1, P_2\colon \mathbf{R}^+\to\mathbf{R}^+$ *with* $P_1(s)>0$ *for* $s>0$ *such that* $|Y(t, x, y)-Y(t, 0, y)|\leqq P_1(|y|)P_2(|x|)$;

(ii) *there exist a constant* $B_1>0$ *and a locally Lipschitz function* $V_1(t, x, y)$ *defined for* $t\geqq 0$, $|x|\geqq B_1$ *and* $y\in\mathbf{R}^k$ *such that*

$$W_1(|x|) \leqq V_1(t, x, y) \leqq W_2(|x|),$$

$$\dot{V}_{1(3.8)}(t, x, y) \leqq -W_3(|x|) \quad \textit{for} \quad t \geqq 0, \quad |x| \geqq B_1 \quad \textit{and} \quad y\in\mathbf{R}^k,$$

where W_1 *and* W_2 *are unbounded pseudo wedges and* $W_3\colon \mathbf{R}^+\to\mathbf{R}^+$ *is continuous with* $W_3(r)>0$ *for* $r\geqq B_1$;

(iii) *there exist a constant* $B_2>0$, *a locally Lipschitz function* $V_2(t, y)$ *defined for* $t\geqq 0$ *and* $|y|\geqq B_2$, *and a positive continuous function* $N\colon \mathbf{R}^+\to\mathbf{R}^+$ *with* $N(r)>0$ *for* $r\geqq B_2$ *and such that*

$$W_4(|y|) \leqq V_2(t, y) \leqq W_5(|y|),$$

$$\dot{V}_{2(3.9)}(t, y) \leqq -W_6(|y|) \quad \textit{for} \quad |y| \geqq B_2,$$

$$|V_2(t, y)-V_2(t, \tilde{y})| \leqq N\big(\max\{|y|, |\tilde{y}|\}\big)|y-\tilde{y}|,$$

where W_4, W_5 *are unbounded pseudo wedges,* W_6 *is nonnegative and continuous with* $\lim_{r\to\infty} W_6(r)/(N(r)P_1(r))=\infty$.

Then the solutions of (3.8) *are U.B. and U.U.B.*

Proof. First, we shall prove the uniform boundedness. For any $\alpha>\max\{B_1, B_2\}$, there exist $\beta(\alpha)$, $\beta_1(\alpha)$ and $\beta_2(\alpha)>0$ such that $W_1(\beta(\alpha))>W_2(\alpha)$, $\beta_2(\alpha)>\beta_1(\alpha)>\alpha$, $W_6(s)/N(s)P_1(s)-\max_{r\leqq\beta(\alpha)} P_2(r)\geqq 1$ for $s\geqq\beta_1(\alpha)$, and $W_4(\beta_2(\alpha))>W_5(\beta_1(\alpha))$. Then for any solution $(x(t), y(t))$ with $|x(t_0)|<\alpha$, and $|y(t_0)|<\alpha$, we have $x(t)<\beta(\alpha)$ and $|y(t)|<\beta_2(\alpha)$ for $t\geqq t_0$.

If this is not true, then only two cases may occur:

Case 1. There exist $t_2>t_1>t_0$ with $|y(t_1)|=\beta_1(\alpha)$, $|y(t_2)|=\beta_2(\alpha)$, $\beta_1(\alpha)<|y(t)|<\beta_2(\alpha)$ for $t\in(t_1, t_2)$ and $|x(t)|<\beta(\alpha)$ for $t\in[t_0, t_2)$.

Case 2. There exist $t_4>t_3>t_0$ such that $|x(t_3)|=\alpha$, $|x(t_4)|=\beta(\alpha)$, $\alpha<|x(t)|<\beta(\alpha)$ for $t\in(t_3, t_4)$ and $|y(t)|\leqq\beta_2(\alpha)$ for $t\in[t_3, t_4)$.

In Case 1, for $t\in[t_1, t_2]$, we have

$$\dot V_{2(3.8)}(t, y(t)) \leqq -W_6(|y(t)|)+N(|y(t)|)P_1(|y(t)|)P_2(|x(t)|) \leqq$$
$$\leqq -N(|y(t)|)P_1(|y(t)|)[W_6(|y(t)|)/(N(|y(t)|))P_1(|y(t)|)-P_2(|x(t)|)] \leqq$$
$$\leqq -N(|y(t)|)P_1(|y(t)|) \leqq 0.$$

Therefore, $W_4(\beta_2(\alpha))\leqq V_2(t_2, y(t_2))\leqq V_2(t_1, y(t_1))\leqq W_5(\beta_1(\alpha))$. This contradicts $W_4(\beta_2(\alpha))>W_5(\beta_1(\alpha))$.

In Case 2, for $t\in[t_3, t_4]$, we have $\dot V_1(t, x(t), y(t))\leqq 0$, thus

$$W_1(\beta(\alpha)) \leqq V_1(t_4, x(t_4), y(t_4)) \leqq V_1(t_3, x(t_3), y(t_3)) \leqq W_2(\alpha),$$

which contradicts $W_1(\beta(\alpha))>W_2(\alpha)$.

Therefore, $|x(t; t_0, x_0, y_0)|<\beta(\alpha)$ and $|y(t; t_0, x_0, y_0)|<\beta_2(\alpha)$ for $t\geqq t_0$ if $|x_0|<\alpha$ and $|y_0|<\alpha$. This completes the proof of uniform boundedness.

Let $v_1(\alpha)=\min\{W_3(r)\colon B_1+1\leqq r\leqq\beta(\alpha)\}$ and $T_1(\alpha)=W_2(\alpha)/v_1(\alpha)$. If $|x(t)|\geqq B_1+1$ holds for $t\in[t_0, \bar t]$ $(\bar t>t_0+T_1(\alpha))$ then

$$W_1(B_1+1) \leqq V_1(\bar t, x(\bar t), y(\bar t)) \leqq V_1(t_0, x(t_0), y(t_0))-v_1(\alpha)(\bar t-t_0) <$$
$$< W_2(\alpha)-v_1(\alpha)W_2(\alpha)/v_1(\alpha) = 0,$$

which yields a contradiction. Therefore, there exists $t_5\in[t_0, t_0+T_1(\alpha)]$ with $|x(t_5)|\leqq B_1+1$. Following the same argument as in the proof of uniform boundedness, we get $|x(t)|<\beta(B_1+1)$ for $t\geqq t_5$, especially for $t\geqq t_0+T_1(\alpha)$.

Choose $B_3>B_2$ with $W_6(s)/N(s)P_1(s)-\max\{P_2(r)\colon |r|<\beta(B_1+1)\}\geqq 1$ for $s\geqq B_3$. If $|y(t)|\geqq B_3$ for $t\geqq t_0+T_1(\alpha)$, then there exists $v_2(\alpha)>0$ such that $P_1(|y(t)|)N(|y(t)|)\geqq v_2(\alpha)$, and so

$$\dot V_{2(3.8)}(t, y(t)) \leqq -P_1(|y(t)|)N(|y(t)|)[W_6(|y(t)|)/N(|y(t)|)P_1(|y(t)|)-P_2(|x(t)|)] \leqq$$
$$\leqq -N(|y(t)|)P_1(|y(t)|) \leqq -v_2(\alpha).$$

Therefore, if $|y(t)|\geqq B_3$ for $t\in[t_0+T_1(\alpha), t_0+T_1(\alpha)+\bar t]$, then

$$V_2(t_0+T_1(\alpha)+\bar t, y(t_0+T_1(\alpha)+\bar t)) \leqq$$
$$\leqq V_2(t_0+T_1(\alpha), y(t_0+T_1(\alpha)))-v_2(\alpha)\bar t \leqq W_5(\beta(\alpha))-v_2(\alpha)\bar t.$$

If $\bar t\geqq T_2(\alpha)$, where $T_2(\alpha)=(W_5(\beta_2(\alpha))-W_4(B_3))/v_2(\alpha)$, then

$$W_4(B_3) \leqq V_2(t_0+T_1(\alpha)+\bar t, y(t_0+T_1(\alpha)+\bar t)) < W_5(\beta_2(\alpha))-v_2(\alpha)T_2(\alpha) \leqq W_4(B_3),$$

which yields a contradiction. Therefore, there exists $t_6\in[t_0+T_1(\alpha),\ t_0+T_1(\alpha)+T_2(\alpha)]$ with $|y(t_6)|<B_3$, and thus $|x(t_6)|<B_4$ and $|y(t_6)|<B_4$, where $B_4=\max\{B_3,\ \beta(B_1+1)\}$. This implies $|x(t)|<\beta(B_4)$ and $|y(t)|<\beta_2(B_4)$ for $t\geqq t_0+T_1(\alpha)+T_2(\alpha)$. This completes the proof.

Sometimes in practice it is very difficult to find a Lyapunov function satisfying the condition $V_1(t,x,y)\leqq W_2(|x|)$ (see Example 3.4). Now we give a modification of Theorem 3.5 asking the much milder property $V_1(t,x,y)\leqq W_2(|x|+|y|)$.

Theorem 3.6. *Suppose that*

(i) *conditions* (i), (iii) *of Theorem* 3.5 *hold;*

(ii) *there exist a constant* $B_1>0$ *and a continuous function* $V_1(t,x,y)$ *defined for* $t\geqq 0$, $(x,y)\in\mathbf{R}^{m+k}$ *and such that*

$$W_1(|x|)\leqq V_1(t,x,y)\leqq W_2(|x|+|y|),$$

$$\dot V_{1(3.8)}(t,x,y)\leqq -W_3(x,y),$$

where W_1 *and* W_2 *are unbounded pseudo wedges, and* $W_3\colon \mathbf{R}^{m+k}\to\mathbf{R}^+$ *is continuous and* $|x|\geqq B_1$ *implies* $W_3(x,y)>0$;

(iii) *for any* $M>0$ *there exists* $L(M)>0$ *such that* $[t\geqq 0,\ |x|+|y|\leqq M]$ *imply* $|X(t,x,y)|\leqq L(M)$;

Then the solutions of (3.8) *are U.B. and U.U.B.*

Proof. Obviously, by (ii) for any $\alpha>0$, if $|x_0|+|y_0|<\alpha$, then $|x(t;t_0,x_0,y_0)<$ $<W_1^{-1}(W_2(\alpha))=\beta(\alpha)$ provided that $(x(t;t_0,x_0,y_0),\ y(t;t_0,x_0,y_0))$ exists. Following the same argument as in the proof of Theorem 3.5, there exists $\beta_2(\alpha)>0$ such that $|y(t;t_0,x_0,y_0)|<\beta_2(\alpha)$ provided that $|x_0|+|y_0|<\alpha$ and $(x(t;t_0,x_0,y_0),y(t;t_0,x_0,y_0))$ exists. Then the solutions of (3.8) are U.B. Throughout the remainder of the proof denote $x(t)=x(t;t_0,x_0,y_0)$, $y(t)=y(t;t_0,x_0,y_0)$.

Let $T_1(\alpha)=W_2(\beta(\alpha)+\beta_2(\alpha))/\min\{W_3(x,y)\colon B_1+1\leqq|x|\leqq\beta(\alpha),\ |y|\leqq\beta_2(\alpha)\}$. Then by (ii), for any $\bar t\geqq t_0$ there is a $t_1\in[\bar t,\bar t+T_1(\alpha)]$ with $|x(t_1)|<B_1+1$.

Suppose that for all $t\in[t_1,\bar t+T_1(\alpha)+t^*]$ we have $|x(t)|<B_1+2$ and $|y(t)|\geqq B_3$, where $B_3=B_2$ is a fixed constant such that

$$W_6(r)/N(r)P_1(r)-\max\{P_2(s)\colon 0\leqq s\leqq B_1+2\}\geqq 1\quad\text{for}\quad r\geqq B_3.$$

Then from

$$\dot V_{2(3.8)}(t,y(t))\leqq -N(|y(t)|)P_1(|y(t)|)\left[\frac{W_6(|y(t)|)}{N(|y(t)|)P_1(|y(t)|)}-P_2(|x(t)|)\right]\leqq$$

$$\leqq -\min\{N(r)P_1(r)\colon B_3\leqq r\leqq\beta_2(\alpha)\}=-m$$

we get

$$0\leqq V_2(\bar t+T_1(\alpha)+t^*,\ y(\bar t+T_1(\alpha)+t^*))\leqq$$

$$\leqq V_2(t_1,y(t_1))-m[t^*+T_1(\alpha)+\bar t-t_1]\leqq W_5(\beta_2(\alpha))-m[t^*+T_1(\alpha)+\bar t-t_1].$$

Therefore, $t^* < T_2(\alpha) = [W_5(\beta_2(\alpha))+1]/m$. This shows only two cases may occur:

Case 1. $|x(t)| < B_1+2$ for all $t \in [t_1, \bar{t}+T_1(\alpha)+T_2(\alpha)]$ and there exists $t_2 \in [t_1, \bar{t}+T_1(\alpha)+T_2(\alpha)]$ with $|y(t_2)| < B_3$. In this case, $|x(t)| < \beta(B_1+B_3+2)$ and $|y(t)| < \beta_2(B_1+B_3+2)$ for $t \geqq \bar{t}+T_1(\alpha)+T_2(\alpha)$.

Case 2. There exists $t_3 \in [t_1, \bar{t}+T_1(\alpha)+T_2(\alpha)]$ such that $|x(t_3)| \geqq B_1+2$. In this case, there exist $t_4, t_5 \in [t_1, t_3]$ with $|x(t_4)| = B_1+1$ and $|x(t_5)| = B_1+2$ and $B_1+1 < |x(t)| < B_1+2$ for $t \in (t_4, t_5)$. By condition (iii) $t_5 - t_4 \geqq 1/L(\beta(\alpha)+\beta_2(\alpha))$, and (ii) implies $V_1(\bar{t}+T_1(\alpha)+T_2(\alpha)) \leqq V_1(t_5) \leqq V_1(t_4) - (t_5-t_4)m(\alpha) \leqq V_1(\bar{t}) - v(\alpha)$, where $V_1(t) = V_1(t, x(t), y(t))$, $v(\alpha) = [L(\beta(\alpha)+\beta_2(\alpha))]^{-1} m(\alpha)$, and $m(\alpha) = \min\{W_3(x,y): B_1+1 \leqq |x| \leqq \beta(\alpha),\ |y| \leqq \beta_2(\alpha)\}$. Making the choice $\bar{t} = t_m = t_0 + m[T_1(\alpha)+T_2(\alpha)]$ $(m = 0, 1, 2, \dots)$ we get that either $|x(t)| < \beta(B_1+B_3+2)$ and $|y(t)| < \beta_2(B_1+B_3+2)$ for $t \geqq t_{m+1}$, or

$$V_1(t_{m+1}) \leqq V_1(t_m) - v(\alpha). \tag{3.12}$$

On the other hand, $0 \leqq V_1(t) \leqq W_2(\beta(\alpha)+\beta_2(\alpha))$ for $t \geqq t_0$, and so (3.12) can not be true for $m = 0, 1, \dots, N$, where $N = N(\alpha)$ is a positive integer such that $N(\alpha)v(\alpha) > W_2(\beta(\alpha)+\beta_2(\alpha))$. Therefore, $|x(t)| < \beta(B_1+B_3+2)$ and $|y(t)| < \beta_2(B_1+B_3+2)$ for $t \geqq t_0 + [N(\alpha)+1][T_1(\alpha)+T_2(\alpha)]$. This completes the proof.

Example 3.4. Consider the Liénard equation with forcing term

$$\ddot{x} + f(x)\dot{x} + g(x) = p(t), \tag{3.13}$$

where $f(x)$ and $g(x)$ are continuous for $x \in \mathbf{R}$ and $p(t)$ is continuous for $t \geqq 0$. Besides, we assume that

(i) $f(x) > 1$;
(ii) $x\{g(x) - x[f(x)-1]\} \geqq 0$;
(iii) $\int_0^\infty |p(s)|\, ds < \infty$.

Then the solutions of (3.13) are U.B. and U.U.B.

Proof. System (3.13) is equivalent to

$$\dot{x} = -x+y, \quad \dot{y} = -\{g(x) - x[f(x)-1]\} - [f(x)-1]y + p(t). \tag{3.14}$$

Let $V(t,x,y) = \left[y^2 + 2\int_0^x \{g(r) - r[f(r)-1]\}\, dr\right]^{1/2} + \int_t^\infty |p(s)|\, ds$.

Then

$$\dot{V}_{(3.14)}(t,x,y) \leqq \frac{-[f(x)-1]y^2 - x\{g(x) - x[f(x)-1]\}}{\left[y^2 + 2\int_0^x \{g(r) - r[f(r)-1]\}\, dr\right]^{1/2}} = -W(x,y).$$

Then $|y|>0$ implies $W(x, y)>0$. On the other hand, for the subsystem $\dot{x}=-x$ the auxiliary function $V_2(t, x)=x^2$, $N(r)=2r$ and $W_6(r)=2r^2$ satisfy condition (iii) of Theorem 3.5 and so the solutions of (3.13) are U.B. and U.U.B. by Theorem 3.6.

4. An application to a holonomic scleronomic mechanical system

Consider a holonomic scleronomic mechanical system of n degrees of freedom being under the action of potential, disspative and gyroscopic forces. The motions such a system can be described by the Langrangian equation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}}-\frac{\partial T}{\partial q}=-\frac{\partial \pi}{\partial q}-B\dot{q}+G\dot{q}, \tag{4.1}$$

where $q, \dot{q}\in \mathbf{R}^n$ are the vectors of the generalized coordinates and velocities, respectively, $\pi=\pi(t, q)$ is the potential energy, $T=T(q, \dot{q})=(1/2)\dot{q}^T A(q)\dot{q}$ is the kinetic energy where $A(q)$ is a symmetric $n\times n$ matrix function (v^T denotes the transposed of $v\in \mathbf{R}^n$); $B=B(t, q)$ is the symmetric positive semi-definite $n\times n$ matrix function of dissipation, and $G=G(t, q)$ is the antisymmetric $n\times n$ matrix of the gyroscopic coefficients.

By the Hamiltonian variables $q, p=A(q)\dot{q}$ system (4.1) can be rewritten into the form

$$\dot{q}=\frac{\partial H}{\partial p}, \quad \dot{p}=-\frac{\partial H}{\partial q}+(G-B)\frac{\partial H}{\partial p}, \tag{4.2}$$

where $H=H(t, p, q)$ is the total mechanical energy:

$$H=H(t, q, p)=T+\pi=(1/2)p^T A^{-1}(q)p+\pi(t, q).$$

Choose the auxiliary functions $V=H(t, p, q)$, $W=p^T q$. Their derivatives with respect to (4.2) read as follows:

$$\dot{H}=\left(\frac{\partial H}{\partial p}\right)^T (G-B)\frac{\partial H}{\partial p}+\frac{\partial \pi}{\partial t}=-p^T A^{-1}(q)B(t, q)A^{-1}(q)p+\frac{\partial \pi(t, q)}{\partial t}\leqq$$

$$\leqq -\beta(t, q)\Lambda^{-1}(q)p^T A^{-1}(q)p+\left[\frac{\partial \pi(t, q)}{\partial t}\right]_+$$

where $\beta(t, q)$ denotes the smallest eigenvalue of the matrix $B(t, q)$; $\Lambda(q)$ denotes the largest eigenvalue of $A(q)$. It is known from the mechanics that the kinetic energy is a positive definite quadratic form of the velocities, consequently $\Lambda(q)>0$ for all $q\in \mathbf{R}^n$.

Let

$$A^{-1}(q) = (a_{ij}^{-1}(q))_{nxn},$$

$$d_{ij} = \left(\frac{\partial a_{ij}^{-1}(q)}{\partial q_1}, \ldots, \frac{\partial a_{ij}^{-1}(q)}{\partial q_n}\right) A^{-1}(q)p, \quad D = (d_{ij})_{nxn},$$

$$e_k = \sum_{i,j=1}^{n} \frac{\partial a_{ij}^{-1}(q)}{\partial q_k} p_i p_j, \quad e = (e_1, \ldots, e_n)^T.$$

Then for $P:=p^T A^{-1}(q)p$, its derivative with respect to (4.2) is

$$\dot{P} = \left[-\frac{\partial \pi}{\partial q} - \frac{1}{2}\frac{\partial}{\partial q} p^T A^{-1}(q)p + (G-B)A^{-1}(q)p\right]^T A^{-1}(q)p +$$

$$+p^T A^{-1}(q)\left[-\frac{\partial \pi}{\partial q} - \frac{1}{2}\frac{\partial}{\partial q} p^T A^{-1}(q)p + (G-B)A^{-1}(q)p\right] + p^T Dp =$$

$$= -2\left[\frac{\partial \pi(t, q)}{\partial q}\right]^T A^{-1}(q)p + p^T A^{-1}(q)[(G-B)^T + (G-B)]A^{-1}(q)p -$$

$$-p^T A^{-1}(q)\frac{\partial}{\partial q}[p^T A^{-1}(q)p] + p^T Dp = -2\left[\frac{\partial \pi(t, q)}{\partial q}\right]^T A^{-1}(q)p -$$

$$-2p^T A^{-1}(q)BA^{-1}(q)p - p^T A^{-1}(q)e + p^T Dp;$$

$$[\dot{P}]_+ \leqq \left|\frac{\partial}{\partial q}\pi(t, q)\right| F_2(q, p) + F_3(q, p),$$

where

$$F_2(q, p) = 2|A^{-1}(q)p|, \quad F_3(q, p) = |p|\,|A^{-1}(q)|\,|e| + |D|p^2.$$

Similarly,

$$\dot{W} = \dot{p}^T q + p^T \dot{q} = -\left[\frac{\partial \pi(t, q)}{\partial q}\right]^T q + \frac{1}{2}e^T q + p^T A^{-1}(q)(G-B)^T q + p^T A^{-1}(q)p,$$

$$|\dot{W}| \geqq \left|q^T \frac{\partial \pi(t, q)}{\partial q}\right| - |G(t, q) - B(t, q)| F_5(q, p) - F_4(q, p),$$

where

$$F_4(q, p) = \frac{1}{2}|e|\,|q| + |A^{-1}(q)|p^2, \quad F_5(q, p) = |A^{-1}(q)|\,|q|\,|p|.$$

It is easy to prove that $F_i(q, p)$ are continuous for $p, q \in \mathbf{R}^n$, and for every $M>0$, $\lim_{p\to 0} \sup_{|q|\leqq M} F_i(q, p) = 0$ for $i=2, \ldots, 5$. Therefore, from Theorem 3.3 and Remark 3.5, we get the following

Corollary 4.1. *Suppose that there are $B \geqq 0$ and unbounded pseudo wedges W_1, W_2 such that*

(i) $W_1(|q|) \leqq \pi(t, q) \leqq W_2(|q|)$ *for* $t > 0$ *and* $q \in \mathbf{R}^n$;

(ii) *for every* $M > 0$ *the function* $\beta_M(t) = \min\{\beta(t, q)\colon 0 \leqq |q| \leqq M\}$ *is weakly integrally positive;*

(iii) *there is a continuous function* $r\colon \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ *such that* $r(t, u)$ *is increasing with respect to* u *for every* $t \in \mathbf{R}^+$ *and* $[\partial\pi(t, q)/\partial t]_+ \leqq r(t, \pi(t, q))$ *for* $t \in \mathbf{R}^+$ *and* $q \in \mathbf{R}^n$;

(iv) *for every* $u_0 > 0$ *there is a* $u_1 > u_0$ *with* $\int_0^\infty r(s, u_1)\, ds < u_1 - u_0$;

(v) *for every* $M > 0$ *the function* $|\partial\pi(t, q)/\partial q|$ *is bounded for* $t \geqq 0$ *and* $|q| \leqq M$;

(vi) *for every* $M > B$ *there are* $\mu_M > 0$ *and* $K_M > 0$ *such that* $|q^T \partial\pi(t, q)/\partial q| \geqq \mu_M |G(t, q) - B(t, q)| \leqq K_M$ *for* $t \geqq 0$ *and* $B \leqq |q| \leqq M$.

Then the motions are U.B. and E.U.B.

If, in addition, $\beta_M(t)$ *is integrally positive, then the motions are U.B. and U.U.B.*

References

[1] T. Yoshizawa, Liapunov's function and boundedness of solutions, *Funkcial. Ekvac.*, **2** (1959). 95—112.

[2] T. Yoshizawa, Stability and boundedness of systems, *Arch. Rational Mech. Anal.*, **6** (1960), 409—421.

[3] T. Yoshizawa, *Stability Theory by Liapunov's Second Method*, The Mathematical Society of Japan (Tokyo, 1966).

[4] T. A. Burton, *Voltera Integral and Differential Equations*, Academic Press (New York, 1983).

[5] V. M. Matrosov, On the stability of motion, *Prikl. Mat. Mekh.*, **26** (1962), 885—895.

[6] L. Hatvani, Attractivity theorems for nonautonomous systems of differential equations, *Acta Sci. Math.*, **40** (1978), 271—283.

[7] L. Hatvani, A generalization of the Barbashin—Krasovskiĭ theorems to the partial stability in nonautonomous systems, in: *Qualitative Theory of Differential Equations*, Coll. Math. Soc. J. Bolyai, 30 (Szeged, 1979), pp. 381—409.

[8] Li Senlin and Wu Jianhong, On the stability of differential equations, *Ann. Differential Equations*, **1** (1985), 171—187.

[9] J. Kato, Boundedness theorems on Liénard type differential equations with damping, *J. Northeast Normal Univ.* (China), to appear.

(W. J.)
DEPARTMENT OF MATHEMATICS
HUNAN UNIVERSITY
CHANGSHA, HUNAN 1801
PEOPLE'S REPUBLIC OF CHINA

(L. H.)
BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
H—6720, SZEGED, HUNGARY

On the convergence of the differentiated trigonometric projection operators

P. O. RUNCK, J. SZABADOS and P. VÉRTESI*)

Let $C_{2\pi}$ be the set of 2π-periodic continuous functions and $\mathcal{T}_n$ the set of trigonometric polynomials of order at most n. We will consider projection operators $P_n \in C_{2\pi} \to \mathcal{T}_n$, i.e. linear operators $P_n(f, t)$ with the properties

(i) $P_n(f, t) \in \mathcal{T}_n$ if $f \in C_{2\pi}$

(ii) $P_n(f, t) \equiv f(t)$ if $f \in \mathcal{T}_n$.

Let r be a nonnegative integer, and consider the r times differentiated operator $P_n^{(r)}(f, t)$. One may ask: under what conditions will this operator uniformly converge to $f^{(r)}(t)$? To state a result in the positive direction, we need some definitions. Let

$$\|P_n^{(r)}\| := \sup_{0 \not\equiv f \in C_{2\pi}} \frac{\|P_n^{(r)}(f, t)\|}{\|f\|} \tag{1}$$

be the norm of the r times differentiated operator ($\|\cdot\|$ denotes supremum norm over the real line), and let $E_n(g)$ be the best (uniform) trigonometric approximation of order n of $g \in C_{2\pi}$.

Theorem 1. *If $f^{(r)}(t)$ is continuous and $P_n \in C_{2\pi} \to \mathcal{T}_n$, then*

$$\|f^{(r)}(t) - P_n^{(r)}(f, t)\| = O\big(E_n(f^{(r)}) + E_n(f)\|P_n^{(r)}\|\big).$$

Here the O-sign refers to $n \to \infty$, while r is fixed. Hence a sufficient condition of the uniform convergence is

$$\lim_{n \to \infty} E_n(f)\|P_n^{(r)}\| = 0.$$

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Proof of Theorem 1. Let $T_n(t)$ be the best approximating polynomial of $f(t)$. Then according to a result of CZIPSZER and FREUD [2] on simultaneous approximation

$$\|f^{(k)}(t)-T_n^{(k)}(t)\| \leqq c_0 E_n(f^{(k)}) \quad (k=0, 1, \ldots, r).^{1)}$$

Using this result, as well as property (ii) of the projection operator P_n we get

$$\|f^{(r)}(t)-P_n^{(r)}(f, t)\| \leqq \|f^{(r)}(t)-T_n^{(r)}(t)\| + \|\{T_n(t)-P_n(T_n, t)\}^{(r)}\| +$$
$$+\|P_n^{(r)}(T_n-f, t)\| \leqq c_0 E_n(f^{(r)}) + c_0 E_n(f)\|P_n^{(r)}\|.$$

Now we turn to the divergence phenomena of the operator $P_n^{(r)}(f, t)$. Let $\omega(t)$ be an arbitrary modulus of continuity, and define

(2) $$C_r(\omega) = \left\{f(t) \,|\, f^{(r)}(t) \in C_{2\pi},\ \sup_{t>0} \frac{\omega(f^{(r)}, t)}{\omega(t)} < \infty\right\}.$$

Theorem 2. *Given $r \geqq 0$ and a modulus of continuity $\omega(t)$ such that*

(3) $$\lim_{t\to 0+} \frac{t}{\omega(t)} = 0,$$

further a sequence of projection operators $P_n \in C_{2\pi} \to \mathcal{T}_n$, there exists an $f_r(t) \in C_r(\omega)$ such that

(4) $$\limsup_{n\to\infty} \frac{\|f_r^{(r)}(t)-P_n^{(r)}(f, t)\|}{\omega\left(\frac{1}{n}\right)\log n} > 0.$$

For the proof of Theorem 2 we need the following

Lemma. *Given r and n, there exists a function $g_{nr}(t) \in C_{2\pi}$ such that*

(5) $$\|g_{nr}^{(j)}(t)\| \leqq c_1 n^j \quad (j=0, 1, \ldots, r+1)$$

and

(6) $$\frac{1}{\pi}\int_{-\pi}^{\pi} g_{nr}^{(r)}(t) D_n(t)\,dt \geqq c_2 n^r \log n,$$

where

(7) $$D_n(t) = \frac{\sin\dfrac{2n+1}{2}t}{2\sin\dfrac{t}{2}}$$

is the Dirichlet kernel.

---

[1] In what follows $c_0, c_1, \ldots$ will denote constants depending on r but independent of n.

Proof. We distinguish two cases.

Case 1. r is odd. Then let

(8) $$g_{nr}(t) = (-1)^{(r+1)/2}(\operatorname{sgn} t)\cos\frac{2n+1}{2}t \quad \text{if} \quad \frac{2\pi}{2n+1} \leqq |t| \leqq \frac{2n\pi}{2n+1}.$$

To extend the definition of $g_{nr}(t)$ for $|t|<2\pi/(2n+1)$, let $h_{nr}(t)$ be that uniquely determined algebraic polynomial of degree at most $2r+3$ which satisfies the conditions

(9) $$h_{nr}^{(j)}\left(-\frac{2\pi}{2n+1}\right) = g_{nr}^{(j)}\left(-\frac{2\pi}{2n+1}\right), \quad h_{nr}^{(j)}\left(\frac{2\pi}{2n+1}\right) = g_{nr}^{(j)}\left(\frac{2\pi}{2n+1}\right)$$

$$(j = 0, 1, \dots, r+1).$$

Then let

(10) $$g_{nr}(t) = h_{nr}(t) \quad \text{if} \quad |t| < \frac{2\pi}{2n+1}.$$

Assume

(11) $$h_{nr}(t) = \sum_{k=0}^{2r+3} a_{kn}\left(\frac{2n+1}{2\pi}t\right)^k,$$

then by (9) and (8)

$$h_{nr}^{(j)}\left(-\frac{2\pi}{2n+1}\right) = \left(-\frac{2n+1}{2\pi}\right)^j \sum_{k=j}^{2r+3} a_{kn}k(k-1)\dots(k-j+1)(-1)^k =$$

$$= g_{nr}^{(j)}\left(-\frac{2\pi}{2n+1}\right) = O(n^j)$$

$$(j = 0, 1, \dots, r+1),$$

i.e.

(12) $$\sum_{k=j}^{2r+3} (-1)^k k(k-1)\dots(k-j+1)a_{kn} = O(1) \quad (j = 0, 1, \dots, r+1).$$

Similarly, from the second group of conditions in (9),

(13) $$\sum_{k=j}^{2r+3} k(k-1)\dots(k-j+1)a_{kn} = O(1) \quad (j = 0, 1, \dots, r+1).$$

(12) and (13) together can be considered as a system of linear equations for the unknowns a_{kn}. Since $h_{nr}(t)$ is uniquely determined, this system is uniquely solvable and

$$|a_{kn}| \leqq c_8 \quad (k = 0, 1, \dots, 2r+3).$$

Thus by (10) and (11) we get for $j=0, 1, \dots, r+1$

$$|g_{nr}^{(j)}(t)| = |h_{nr}^{(j)}(t)| \leqq$$

$$\leqq \left(\frac{2n+1}{2\pi}\right)^j \sum_{k=j}^{2r+3} k(k-1)\dots(k-j+1)|a_{kn}| \leqq c_1 n^j \quad \text{if} \quad |t| \leqq \frac{2\pi}{2n+1}.$$

Now $g_{nr}(t)$ is defined on $|t|\leqq 2\pi/(2n+1)$, and extending the definition by translations of length 2π, the only missing interval is $\left(\frac{2n\pi}{2n+1}, \frac{2(n+1)\pi}{2n+1}\right)$ (and its translates). In this interval the construction is similar: let $H_{nr}(t)$ be that uniquely determined algebraic polynomial of degree at most $2r+3$ for which

$$H_{nr}^{(j)}\left(\frac{2n\pi}{2n+1}\right) = g_{nr}^{(j)}\left(\frac{2n\pi}{2n+1}\right), \quad H_{nr}^{(j)}\left(\frac{2(n+1)\pi}{2n+1}\right) = g_{nr}^{(j)}\left(\frac{2(n+1)\pi}{2n+1}\right)$$

$$(j = 0, 1, \dots, r+1),$$

and let

$$g_{nr}(t) = H_{nr}(t) \quad \text{if} \quad \frac{2n\pi}{2n+1} < t < \frac{2(n+1)\pi}{2n+1}.$$

Thus the definition of $g_{nr}(t)$ is complete. Property (5) on the interval $\left[\frac{2n\pi}{2n+1}, \frac{2(n+1)\pi}{2n+1}\right]$ can be easily established.

The only thing remained to prove is (6). Since by (7) $\|D_n(t)\| = n+1/2$, we get from (8) and (5)

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g_{nr}^{(r)}(t)D_n(t)\,dt \geqq \frac{2}{\pi}\left(\frac{2n+1}{2}\right)^r \int_{\frac{2\pi}{2n+1}}^{\frac{2n\pi}{2n+1}} \frac{\sin^2\frac{2n+1}{2}t}{2\sin\frac{t}{2}}\,dt -$$

$$-\frac{6}{2n+1}\|g_{nr}^{(r)}(t)D_n(t)\| \geqq \frac{1}{\pi}\left(\frac{2n+1}{2}\right)^r \sum_{k=1}^{n-1} \int_{\frac{(4k+1)\pi}{2(2n+1)}}^{\frac{(4k+3)\pi}{2(2n+1)}} \frac{dt}{t} - 3c_1 n^r \geqq$$

$$\geqq \frac{1}{\pi}\left(\frac{2n+1}{2}\right)^r 2\sum_{k=1}^{n-1} \frac{1}{4k+3} - 3c_1 n^r \geqq c_2 n^r \log n.$$

Case 2. r is even. Now the definition of $g_{nr}(t)$ starts with

$$g_{nr}(t) = (-1)^{r/2}(\operatorname{sgn} t)\sin\frac{2n+1}{2}t \quad \text{if} \quad \frac{2\pi}{2n+1} \leqq |t| \leqq \frac{2n\pi}{2n+1}$$

instead of (8). The rest of the proof is very similar to Case 1, and we omit the details.

Proof of Theorem 2. Since $P_n(f, t)$ is a projection operator, according to the Berman—Faber—Marcinkiewicz relation we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n(f(\cdot+u), x-u)\,du = S_n(f, x)$$

where

$$S_n(f, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)D_n(t)\,dt$$

is the n^{th} partial sum of the Fourier series of $f(x)$ (see e.g. Lorentz [3], p. 97). Applying this for $f(x)=g_{nr}(x)$, differentiating r times and setting $x=0$ we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n^{(r)}(g_{nr}(\cdot+u), -u)\,du = \frac{1}{\pi}\int_{-\pi}^{\pi} g_{nr}^{(r)}(t)D_n(t)\,dt.$$

Let u_n be a point where $|P_n^{(r)}(g_{nr}(\cdot+u), -u)|$ attains its maximum, then by (6) we get

(14) $$\|P_n^{(r)}(g_{nr}(\cdot+u_n), t)\| \geqq |P_n^{(r)}(g_{nr}(\cdot+u_n), -u_n)| \geqq c_2 n^r \log n.$$

Now define a sequence of integers $n_1<n_2<\dots$ with the following properties: let

(15) $$\omega\left(\frac{1}{n_1}\right) \leqq \frac{c_2}{8c_1}, \quad n_1 > e^{8/c_2}$$

and assume that $n_1, n_2, \dots, n_{j-1}$ has been already defined.

If there exists a k, $1\leqq k\leqq j-1$, such that for infinitely many n's we have

$$\|g_{n_k r}^{(r)}(t)-P_n^{(r)}(g_{n_k r}(\cdot+u_{n_k}), t)\| \geqq c_1\omega(1/n)\log n$$

then this $g_{n_k r}(t)$ will satisfy the requirements of the theorem. If this is not the case, then for sufficiently large n's

$$\|g_{n_k r}^{(r)}(t)-P_n^{(r)}(g_{n_k r}(\cdot+u_{n_k}), t)\| < c_1\omega(1/n)\log n \quad (k=1, \dots, j-1).$$

Now choose n_j in this case such that

(16) $$\|g_{n_k r}^{(r)}(t)-P_{n_j}^{(r)}(g_{n_k r}(\cdot+u_{n_k}), t)\| < c_1\omega(1/n_j)\log n_j \quad (k=1, \dots, j-1)$$

and

(17) $$\frac{2n_{j-1}}{n_j}\omega(1/n_{j-1}) \leqq \omega(1/n_j) \leqq \min\left(\frac{1}{2}\omega(1/n_{j-1}), \frac{c_2}{4c_1\|P_{n_{j-1}}^{(r)}\|}\right)$$

hold. (The left hand side inequality is possible because of (3).)

We may assume that we can construct an infinite sequence of indices this way. Define

$$f_r(t) = \sum_{k=1}^{\infty} \frac{g_{n_k r}(t+u_{n_k})}{n_k^r}\,\omega(1/n_k).$$

Here the right hand side series, even after differentiating r times, uniformly converges by (5) and (17). Moreover, if $0<\delta\leqq h$ then

$$|f_r^{(r)}(t+\delta)-f_r^{(r)}(t)| \leqq \sum_{k=1}^{\infty} \frac{|g_{n_k r}^{(r)}(t+\delta+u_{n_k})-g_{n_k r}^{(r)}(t+u_{n_k})|}{n_k^r}\,\omega(1/n_k).$$

Let $0<h<1/n_1$ and j be that index for which

$$1/n_{j+1} \leqq h < 1/n_j.$$

Then by (5) and (17)

$$|f_r^{(r)}(t+\delta)-f_r^{(r)}(t)| \leqq \sum_{k=1}^{j} \frac{\delta\|g_{n_k r}^{(r)}(t)\|}{n_k^r}\,\omega(1/n_k)+\sum_{k=j+1}^{\infty} \frac{2\|g_{n_k r}^{(r)}(t)\|}{n_k^r}\,\omega(1/n_k) \leqq$$

$$\leqq c_1\delta\sum_{k=1}^{j} n_k\omega(1/n_k)+2c_1\sum_{k=j+1}^{\infty}\omega(1/n_{k+1})\leqq 2c_1 h n_j\omega(1/n_j)+4c_1\omega(1/n_{j+1})\leqq 8c_1\omega(h),$$

i.e. $f_r(t)\in C_r(\omega)$ (cf. (2)).

Finally, to show (4) we obtain by (14), (16), (5), (17) and (15)

$$\|f_r^{(r)}(t)-P_{n_j}^{(r)}(f_r, t)\| = \left\|\sum_{k=1}^{\infty} \frac{g_{n_k r}^{(r)}(t+u_{n_k})-P_{n_j}^{(r)}(g_{n_k r}(\cdot+u_{n_k}), t)}{n_k^r}\,\omega(1/n_k)\right\| \geqq$$

$$\geqq \frac{\left\|P_{n_j}^{(r)}(g_{n_j r}(\cdot+u_{n_j}), t)\right\|}{n_j^r}\,\omega(1/n_j)-\sum_{k=1}^{j-1}\frac{\left\|g_{n_k r}^{(r)}(t+u_{n_k})-P_{n_j}^{(r)}(g_{n_k r}(\cdot+u_{n_k}), t)\right\|}{n_k^r}\,\omega(1/n_k)-$$

$$-\sum_{k=j}^{\infty}\frac{\|g_{n_k r}^{(r)}\|}{n_k^r}\,\omega(1/n_k)-\sum_{k=j+1}^{\infty}\frac{\left\|P_{n_j}^{(r)}(g_{n_k r}(\cdot+u_{n_k}), t)\right\|}{n_k^r}\,\omega(1/n_k)\geqq$$

$$\geqq c_2\omega(1/n_j)\log n_j - c_1\omega(1/n_j)\log n_j\sum_{k=1}^{\infty}\omega(1/n_k)-\sum_{k=j}^{\infty}\omega(1/n_k)-$$

$$-c_1\|P_{n_j}^{(r)}\|\sum_{k=j+1}^{\infty}\omega(1/n_k)\geqq c_2\omega(1/n_j)\log n_j-2c_1\omega(1/n_1)\omega(1/n_j)\log n_j-$$

$$-2\omega(1/n_j)-2c_1\|P_{n_j}^{(r)}\|\,\omega(1/n_{j+1})\geqq c_2\omega(1/n_j)\log n_j-\frac{c_2}{4}\,\omega(1/n_j)\log n_j-$$

$$-\frac{c_2}{4}\,\omega(1/n_j)\log n_j-\frac{c_2}{4}\,\omega(1/n_j)\log n_j=\frac{c_2}{4}\,\omega(1/n_j)\log n_j\quad (j=1,2,\ldots).$$

$\omega(t)=o(t)$ is excluded in Theorem 2, by condition (3). With a slight modification of the proof we can easily get the following statement in this case.

Theorem 3. *Given* $r \geqq 0$, *a sequence of projection operators* $P_n \in C_{2\pi} \to \mathcal{T}_n$, *and a sequence* $\varepsilon_1 \geqq \varepsilon_2 \geqq \dots$, $\lim_{n\to\infty} \varepsilon_n = 0$, *there exists an* $f_r(t) \in C_{2\pi}$ *such that* $f_r^{(r)}(t) \in \in \mathrm{Lip}\, 1$ *and*

$$\limsup_{n\to\infty} \frac{\|f_r^{(r)}(t) - P_n^{(r)}(f, t)\|}{\varepsilon_n \log n/n} > 0. \tag{18}$$

We do not give the details of the proof of this theorem. We only mention that now

$$f_r(t) = \sum_{k=1}^{\infty} \frac{\varepsilon_{n_k}}{n_k^{r+1}} g_{n_k r}(t + u_{n_k})$$

will be the function satisfying (18), where $n_1 < n_2 < \dots$ is a properly chosen sequence of indices.

An obvious consequence of Theorem 1 is that if $f(t) \in C_r(\omega)$ then

$$\|f^{(r)}(t) - P_n^{(r)}(f, t)\| = O\big(n^{-r}\omega(1/n)\|P_n^{(r)}\|\big). \tag{19}$$

Since here $\|P_n^{(r)}\| \geqq c_3 n^r \log n$ for any projection operator P_n (cf. Berman [1]), the best estimate one can obtain from (19) is

$$\|f^{(r)}(t) - P_n^{(r)}(f, t)\| = O\big(\omega(1/n)\log n\big) \quad (f(t) \in C_r(\omega)).$$

This shows that the results of Theorems 2 and 3 are sharp.

In particular, our theorems can be applied to the differentiated partial sums of the Fourier series and to the differentiated interpolating polynomials based on arbitrary systems of nodes.

References

[1] D. L. Berman, On a class of linear operators, *Dokl. Akad. Nauk SSSR*, **85** (1952), 13—16 (in Russian).

[2] I. Czipszer—G. Freud, Sur l'approximation d'une fonction périodique etc., *Acta Math.*, **99** (1958), 33—51.

[3] G. G. Lorentz, *Approximation of Functions*, Holt, Rinehart and Winston (New York, 1966).

(P. O. R.)
UNIVERSITÄT LINZ
A—4045, ÖSTERREICH

(J. SZ.—P. V.)
MATHEMATICAL INSTITUTE
HUNGARIAN ACADEMY OF SCIENCES
REÁLTANODA U. 13—15
1053 BUDAPEST, HUNGARY

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Notes on approximation by Riesz-means

L. LEINDLER

To my dear colleague L. Pintér on his 60th birthday

1. Let $f=f(x)$ be a continuous 2π-periodic function i.e. $f\in C_{2\pi}$, and let

(1) $$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

be its Fourier series. Denote $s_n=s_n(x)=s_n(f;x)$ and $\sigma_n^\alpha=\sigma_n^\alpha(x)=\sigma_n^\alpha(f;x)$ the n-th partial sum and the n-th (C,α)-mean of (1), respectively, i.e.

$$\sigma_n^\alpha(x)=\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha-1}s_\nu(x),\quad A_n^\alpha=\binom{n+\alpha}{n};$$

furthermore $\tilde{f}$ denotes the conjugate function of f, and $f^{(r)}$ is the r-th derivative of f.

Let $E_n(f)$ denote the best approximation of f by trigonometric polynomials of order at most n in the space $C_{2\pi}$, and let $\|\cdot\|$ denote the usual supremum norm.

We define two important strong means:

$$h_n(f,\beta,p;\ x):=\left\{\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_k(x)-f(x)|^p\right\}^{1/p}\quad(\beta,p>0),$$

$$\sigma_n^\gamma|f,p;\ x|:=\left\{\frac{1}{A_n^\gamma}\sum_{k=0}^{n}A_{n-k}^{\gamma-1}|s_k(x)-f(x)|^p\right\}^{1/p}\quad(\gamma,p>0).$$

The first result on strong approximation by Fourier series has been connected with the following classical theorem of S. N. BERNSTEIN [3]:

If $f\in\mathrm{Lip}\,\alpha$ *then*

(2) $$\|\sigma_n^1-f\|=O(n^{-\alpha})\quad\textit{for}\quad 0<\alpha<1$$

and

(3) $$\|\sigma_n^1-f\|=O(n^{-1}\log n)\quad\textit{for}\quad\alpha=1.$$

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Namely G. ALEXITS and D. KRÁLIK [2] sharpened this theorem by proving that the order of approximation given in estimates (2) and (3) can be achieved for the strong means $h_n(f, 1, 1; x)$, too; i.e.

$f \in \mathrm{Lip}\,\alpha$ implies that

$$\|h_n(f, 1, 1;\ x)\| = \begin{cases} O(n^{-\alpha}) & \text{if } 0 < \alpha < 1, \\ O(n^{-1} \log n) & \text{if } \alpha = 1. \end{cases}$$

Improving further the result of Alexits and Králik we ([4], [5]) proved, among others, the following theorems:

Theorem A. *If $f^{(r)} \in \mathrm{Lip}\,\alpha$, $0<\alpha\leqq 1$, $p>0$ and $\beta>(r+\alpha)p$ then*

$$h_n(f, \beta, p) := \|h_n(f, \beta, p;\ x)\| = O(n^{-r-\alpha}).$$

Theorem B. *If $f^{(r)} \in \mathrm{Lip}\,\alpha$, $0<\alpha\leqq 1$, $p>0$ and $(r+\alpha)p<1$ then for arbitrary $\gamma>0$*

$$\sigma_n^\gamma|f, p| := \|\sigma_n^\gamma|f, p;\ x|\| = O(n^{-r-\alpha}).$$

It is clear that these estimations are best possible, namely, by the well-known result of Jackson $f^{(r)} \in \mathrm{Lip}\,\alpha$ implies that $E_n(f) = O(n^{-r-\alpha})$.

The following theorems show that the conditions $\beta>(r+\alpha)p$ and $(r+\alpha)p<1$ are very essential with respect to the order of approximation. If they are not fulfilled then the strong means do not approximate in the order of best approximation.

Theorem C. *If $f^{(r)} \in \mathrm{Lip}\,\alpha$, $0<\alpha\leqq 1$, $p>0$ and $\beta=(r+\alpha)p$ then we have only*

$$h_n(f, \beta, p) = O\big(n^{-r-\alpha}(\log n)^{1/p}\big).$$

Furthermore there exists a function f_1 such that $f_1^{(r)} \in \mathrm{Lip}\,\alpha$, $0<\alpha\leqq 1$, but

$$h_n(f_1, \beta, p;\ 0) \geqq cn^{-r-\alpha}(\log n)^{1/p} \quad (c>0),$$

holds if n is large enough.

Theorem D. *If $f^{(r)} \in \mathrm{Lip}\,\alpha$, $0<\alpha\leqq 1$, $p>0$, $\gamma>0$ and $(r+\alpha)p=1$ then we have only*

$$\sigma_n^\gamma|f, p| = O\big(n^{-r-\alpha}(\log n)^{1/p}\big).$$

Moreover, there exists a function f_2 such that $f_2^{(r)} \in \mathrm{Lip}\,\alpha$, $0<\alpha\leqq 1$, and

$$\sigma_n^\gamma|f_2, p;\ 0| \geqq dn^{-r-\alpha}(\log n)^{1/p} \quad (d>0)$$

holds for sufficiently large n.

Analogous estimations for the conjugate functions have been proved, but now we do not treat them.

Analysing these results we can see that the strong means $\sigma_n^\gamma|f, p; x|$ behave like $\sigma_n^1|f, p; x|=h_n(f, 1, p; x)$, i.e. the strong means $h_n(f, \beta, p; x)$ are more sensible of parameter β regarding the order of approximation.

This phenomenon raises the following problems: If we consider the following regular ordinary Riesz-means

$$R_n(f, \beta, x) := \frac{\beta_n}{(n+1)^\beta} \sum_{k=0}^{n} (k+1)^{\beta-1} s_k(x) \quad \left(\beta_n^{-1} := (n+1)^{-\beta} \sum_{k=0}^{n} (k+1)^{\beta-1}\right)$$

and take the difference

$$\|R_n(f, \beta;\ x)-f(x)\|$$

i.e. if we consider the ordinary approximation instead of strong one for the Riesz-means, then at which value of the parameter β will a jump in the order of approximation appear, also at the parameter $\beta=r+\alpha$ $(p=1)$ as in the strong case? If $r=0$, then will the jump be at $\beta=\alpha$ independently of the value of α, regardless whether $\alpha<1$ or $\alpha=1$? The answer is affermative if $r=0$, and this shows that the analogue of Bernstein's theorem holds for the Riesz-means, but the jump of the order of approximation can appear at any value $\beta\leqq 1$ if the Lipschitz class has the same parameter. But if $r\neq 0$ then a curious phenomenon appears, namely if r is odd then the case $\alpha=1$ will be exceptional. The reason of this exception has its roots in the following classical result of M. Zamansky [10]: $f^{(r)}\in \mathrm{Lip}\, 1$ *if and only if*

$$\|f-R_n(f, r+1)\| = O(n^{-r-1}) \quad \textit{for an odd} \quad r, \quad \textit{and}$$
$$\|\tilde f-R_n(\tilde f, r+1)\| = O(n^{-r-1}) \quad \textit{for an even} \quad r.$$

We mention that the case $r=0$ of this theorem was proved by G. Alexits [1].

Now we formulate the statements mentioned above precisely, and refer to our paper [6] where the statements of Theorem E appear implicitly.

Theorem E. *Let* $f^{(r)}\in \mathrm{Lip}\,\alpha$, $0<\alpha\leqq 1$. *Then*

(i) *if r is even*

$$\|R_n(f, \beta;\ x)-f(x)\| = \begin{cases} O(n^{-r-\alpha}), & \text{if} \quad r+\alpha<\beta, \\ O(n^{-r-\alpha}\log n), & \text{if} \quad r+\alpha=\beta; \end{cases}$$

(ii) *if r is odd*

$$\|R_n(f, \beta;\ x)-f(x)\| = \begin{cases} O(n^{-r-\alpha}) & \text{if} \quad r+\alpha<\beta \\ O(n^{-r-1}) & \text{if} \quad r+1=\beta \quad (\alpha=1) \\ O(n^{-r-\alpha}\log n) & \text{if} \quad r+\alpha=\beta \quad \text{and} \quad \alpha<1, \end{cases}$$

hold true.

Furthermore, if whether r is even or $\alpha<1$, *then there exists a function* f_0 *such that* $f_0^{(r)}\in \mathrm{Lip}\,\alpha$, $0<\alpha\leqq 1$ *and*

(4) $$|R_n(f_0, r+\alpha;\ 0)-f_0(0)| \geqq cn^{-r-\alpha}\log n$$

holds with a positive $c=c(r, \alpha)$ *if n is large enough.*

We mention that analogous results for the conjugate functions also hold, and that the special case $\alpha=1$ of (4) is not proved in [6], but it is true, and our theorem to be proved includes this special case, too.

The results of Theorem A and C (B and D also) were generalized by V. TOTIK [9] as follows:

Theorem F. *If $f\in W^r H^\omega$ then for any $\beta>0$ and $p>0$*

$$h_n(f,\beta,p)=O(H^{\beta,p}_{r,\omega,n}) \tag{5}$$

holds, where

$$H^{\beta,p}_{r,\omega,n}:=\left\{\frac{1}{(n+1)^\beta}\sum_{k=1}^{n}k^{\beta-1}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p\right\}^{1/p}.$$

Furthermore there exists a function f_r such that $f_r\in W^r H^\omega$, but

$$h_n(f_r,\beta,p;\,0)\geqq cH^{\beta,p}_{r,\omega,n}\quad (c>0).$$

The aim of our note is to show that Theorem E can be generalized for the class $W^r H^\omega$, i.e. to prove that the ordinary Riesz-means do not approximate better than the strong Riesz-means on the whole class $W^r H^\omega$ if r is even or if r is odd but $\sum_{k=1}^{n}\omega(1/k)=O(n\omega(1/n))$.

Our theorem reads:

Theorem. *If $f\in W^r H^\omega$ then for any $\beta>0$*

$$\|R_n(f,\beta;\,x)-f(x)\|=O(H^{\beta,1}_{r,\omega,n}) \tag{6}$$

holds.

Furthermore, if whether r is even or r is odd but $\sum_{k=1}^{n}\omega(1/k)=O(n\omega(1/n))$ is fulfilled, then there exists a function f_0 such that $f_0\in W^r H^\omega$ and

$$|R_n(f_0,\beta;\,0)-f_0(0)|\geqq cH^{\beta,1}_{r,\omega,n} \tag{7}$$

hold with a positive $c=c(\beta,r)$.

It is easy to verify that if r is even, $\beta=r+1$ and $\omega(\delta)=\delta$ $(\alpha=1)$ then (7) reduces to (4) as we stated above.

2. To prove our theorem we require the following lemmas.

We may assume, without restriction of generality, that the modulus of continuity ω is always concave. (See [8, p. 45].)

Lemma 1. *If ω is a modulus of continuity, then the function*

$$f^*(x):=\sum_{n=1}^{\infty}\left(\omega(1/n)-\omega(1/(n+1))\right)\cos nx$$

belongs to H^ω.

See Lemma 2.18 of [7] or V. Totik [9].

Lemma 2. *If the modulus of continuity ω satisfies the condition*

$$\sum_{k=1}^{n} \omega(1/k) = O(n\omega(1/n)), \tag{8}$$

then

$$g^*(x) := \sum_{k=1}^{\infty} (\omega(1/k) - \omega(1/(k+1))) \sin kx$$

belongs to H^ω.

Proof. Since

$$E_n(g^*) \leqq \|g^* - s_n(g^*)\| \leqq \omega(1/(n+1))$$

and

$$\omega\left(g^*, \frac{1}{n}\right) \leqq K\frac{1}{n}\sum_{k=0}^{n} E_k(g^*) \leqq K\frac{1}{n}\sum_{k=1}^{n+1} \omega(1/k),$$

so, by (8), $g^* \in H^\omega$.

Now we can start the proof of Theorem.

3. Proof of Theorem. The estimation (6) follows from (5) obviously. To prove the lower estimation (7) we define f_0 as follows:

$$f_0(x) := \sum_{n=1}^{\infty} n^{-r}(\omega(1/n) - \omega(1/(n+1))) \cos nx.$$

Since, by Lemmas 1 and 2, the functions f^* and g^* belong to H^ω and

$$f_0^{(r)}(x) = \begin{cases} \pm f^*(x) & \text{if } r \text{ is even,} \\ \pm g^*(x) & \text{if } r \text{ is odd,} \end{cases}$$

so $f_0 \in W^r H^\omega$.

A standard calculation gives that

$$\begin{aligned} R_n(f_0, \beta;\ 0) - f_0(0) &= \frac{\beta_n}{(n+1)^\beta} \sum_{k=0}^{n} (k+1)^{\beta-1} \sum_{\nu=k+1}^{\infty} \nu^{-r}(\omega(1/\nu) - \omega(1/(\nu+1))) \geqq \\ &\geqq \frac{d(\beta)}{n^\beta} \sum_{k=1}^{n} k^{\beta-1} \sum_{\nu=k}^{n} \nu^{-r}(\omega(1/\nu) - \omega(1/(\nu+1))) \geqq \\ &\geqq \frac{d(\beta)}{n^\beta} \sum_{\nu=1}^{n} \nu^{-r}(\omega(1/\nu) - \omega(1/(\nu+1))) \sum_{k=1}^{\nu} k^{\beta-1} \geqq \\ &\geqq \frac{d_1(\beta)}{n^\beta} \sum_{\nu=1}^{n} \nu^{\beta-r}(\omega(1/\nu) - \omega(1/(\nu+1))) \geqq \\ &\geqq d(\beta, r) n^{-\beta} \sum_{\nu=1}^{n} \omega(1/\nu)\nu^{\beta-r-1} \geqq c(\beta, r) H_{r,\omega,n}^{\beta,1}, \end{aligned}$$

what proves (7).

Finally, we mention that a comparison of the statements of Theorem F and those of Theorem shows that if r is odd and

$$\sum_{k=1}^{n} \omega(1/k) \neq O(n\omega(1/n))$$

then the ordinary Riesz-means can approximate better than the strong ones, e.g. if $\omega(\delta)=\delta$.

Theorems C and E, in the special case $\alpha=1$, and $\beta=r+1$, also show this phenomenon clearly.

References

[1] G. Alexits, Sur l'ordre de grandear de l'approximation d'une fonction par les moyennes de sa série de Fourier, *Mat. Fiz. Lapok* **48** (1941), 410—422.

[2] G. Alexits und D. Králik, Über den Annäherungsgrad der Approximation im starken Sinne von stetigen Funktionen, *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, **8** (1963), 317—327.

[3] S. Bernstein, Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné, *Mémoires Acad. Roy. Belgique*, (2) **4** (1912), 1—104.

[4] L. Leindler, Über die Approximation im starken Sinne, *Acta Math. Acad. Sci. Hungar.*, **16** (1965), 255—262.

[5] L. Leindler, Bemerkungen zur Approximation im starken Sinne, *ibid.* **18** (1967), 273—277.

[6] L. Leindler, On strong summability of Fourier series, *Abstract Spaces and Approximation*, (Proceedings of Conference in Oberwolfach, 1968), 242—247.

[7] L. Leindler, *Strong approximation by Fourier series*, Akadémiai Kiadó (Budapest, 1985).

[8] G. G. Lorentz, *Approximation of functions*, Holt, Rinehart and Winston (New York—Chicago —Toronto, 1966).

[9] V. Totik, On the strong approximation of Fourier series, *Acta Math. Acad. Sci. Hungar.*, **35** (1980), 151—172.

[10] M. Zamansky, Classes de saturation des procedes de sommation des series de Fourier et applications aux series trigonometriques, *Ann. Sci. Ecole Norm. Sup.*, **67** (1950), 161—198.

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Fourier—Stieltjes transforms of vector-valued measures on compact groups

V. S. K. ASSIAMOUA and A. OLUBUMMO

1. Introduction. In recent years, various studies have shown the growing importance of vector-valued measures as can be seen for instance from [1], [3], [4] and many others as well as the numerous references contained in them. To give just one specific example: the Fourier transforms of the distributions studied by BONNET [2] in generalizing the Bochner theorem to noncommutative Lie groups turn out to be vector-valued measures.

In the present paper, we study the Fourier—Stieltjes transforms of vector-valued measures defined on an infinite compact group. Let G be an infinite compact group with Σ as its dual object. We consider measures m on G with values in a Banach space E. Following ASSIAMOUA [1], we define the Fourier—Stieltjes transforms of such measures and obtain analogues of the results in § 28 of HEWITT and ROSS [6]. Among other results, we prove the celebrated Lebesgue theorem and the Parseval—Plancherel—Riesz—Fischer theorem.

2. Preliminaries

2.1. Definition. Let S be a locally compact Hausdorff space and $\mathscr{K}(S)$ the real (resp. complex) vector space of all continuous real (resp. complex) valued functions on S with compact supports. A *vector measure* on S with values in a real (resp. complex) normed linear space E is any linear mapping $m\colon \mathscr{K}(S)\to E$ with the following property: for every compact set $K\subset S$, there exists a positive constant a_K such that if $f\in\mathscr{K}(S)$ and supp $f\subset K$, then ([3], 2, no. 1)

$$\|m(f)\|_E \leqq a_k \sup\{|f(t)|\colon t\in K\}.$$

We note that if S is compact, then $\mathscr{K}(S)$ is equal to the vector space $\mathscr{C}(S, \mathbf{R})$ (resp. $\mathscr{C}(S, \mathbf{C})$) of all continuous functions on S into $\mathbf{R}$ (resp. $\mathbf{C}$) and a vector measure

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on S is a linear mapping $m: \mathcal{K}(S) \to E$ which is continuous in the uniform norm topology since in this case, there exists a constant $a=a_S$ such that

$$\|m(f)\|_E \leqq a\|f\|, \quad f \in \mathcal{K}(S),$$

where $\|f\| = \sup\{|f(t)|: t \in S\}$ is the uniform norm on $\mathscr{C}(S, \mathbf{R})$. If $m: \mathcal{K}(S) \to E$ is a vector measure, we shall write

$$m(f) = \int_S f(t)\, dm(t) \quad \text{or} \quad \int f\, dm.$$

2.2. Definition. An E-valued vector measure is said to be *dominated* if there exists a positive (real-valued) measure μ such that

$$\left\|\int f\, dm\right\|_E \leqq \int |f|\, d\mu, \quad f \in \mathcal{K}(S).$$

If m is dominated, then there exists a smallest positive measure $|m|$ called the *variation* or the *modulus* of m that dominates it.

A positive measure is said to be *bounded* if it is continuous in the uniform norm topology of $\mathcal{K}(S)$ and a dominated vector measure is said to be bounded if it is dominated by a bounded positive measure.

Thus every dominated vector measure on a compact space is bounded. (For these properties of vector measure and the general theory of vector integration, the reader is referred to [3] or [4].) We note also that if E is a Banach space and $S=G$ is a group, then the space $M^1(G, E)$ of all bounded E-valued measures on G is a Banach space with the norm

$$\|m\| = \int \chi_G\, d|m|,$$

where χ_G is the characteristic function of G.

3. The Fourier—Stieltjes transform. We shall now define the Fourier—Stieltjes transform of a vector-valued measure on a compact group G and obtain some of the properties of such transforms.

3.1. Definition. Let G be a compact infinite group and Σ its dual object. For each $\sigma \in \Sigma$, we choose once and for all, an element $U^{(\sigma)}$ in σ, denote its representation space by H_σ, fix a conjugation D_σ on H_σ and put $\overline{U}^{(\sigma)} = D_\sigma U^{(\sigma)} D_\sigma$, ([6], 27.28. C).

As in [1], we define the *Fourier—Stieltjes transform* of a vector-valued measure $m: G \to E$ by

$$\hat{m}(\sigma)(\xi, \eta) = \int_G \langle \overline{U}_t^{(\sigma)} \xi, \eta \rangle\, dm(t), \quad (\xi, \eta) \in H_\sigma \times H_\sigma.$$

Let E be a Banach space. Then the mapping $(\xi, \eta) \to \hat{m}(\sigma)(\xi, \eta)$ from $H_\sigma \times H_\sigma$ into the space $\mathscr{S}(H_\sigma, \times H_\sigma, E)$ of the E-valued continuous sesquilinear mappings on

$H_\sigma \times H_\sigma$, equipped with the norm

$$\|\Phi(\sigma)\| = \sup\{\|\Phi(\sigma)(\xi,\eta)\|_E : \|\xi\|_{H_\sigma} \leqq 1, \|\eta\|_{H_\sigma} \leqq 1\}$$

is continuous ([1], 4.1).

Following HEWITT and ROSS [6], 28.24, we shall write

$$\mathscr{S}(\Sigma, E) = \prod_{\sigma\in\Sigma} \mathscr{S}(H_\sigma \times H_\sigma, E).$$

It is easy to see that, with addition and scalar multiplication defined coordinatewise, $\mathscr{S}(\Sigma, E)$ is a vector space. For $\Phi\in\mathscr{S}(\Sigma, E)$, we put

$$\|\Phi\|_\infty = \sup\{\|\Phi(\sigma)\| : \sigma\in\Sigma\}$$

and denote by $\mathscr{S}_\infty(\Sigma, E)$ the space $\{\Phi\in\mathscr{S}(\Sigma, E) : \|\Phi\|_\infty < \infty\}$. Also we denote by $\mathscr{S}_{00}(\Sigma, E)$ the space

$$\{\Phi\in\mathscr{S}_\infty(\Sigma, E) : \{\sigma\in\Sigma : \Phi(\sigma)\neq 0\} \text{ is finite}\}$$

and by $\mathscr{S}_0(\Sigma, E)$ the space

$$\{\Phi\in\mathscr{S}_\infty(\Sigma, E) : \text{for every } \varepsilon > 0,\ \{\sigma\in\Sigma : \|\Phi(\sigma)\| > \varepsilon\} \text{ is finite}\}.$$

The next theorem is an analogue of HEWITT and ROSS [6], 28.25.

3.2. Theorem.

(i) *The mapping* $\Phi\to\|\Phi\|_\infty$ *is a norm on* $\mathscr{S}_\infty(\Sigma, E)$ *and* $\mathscr{S}_\infty(\Sigma, E)$ *is a Banach space with respect to this norm.*

(ii) $\mathscr{S}_{00}(\Sigma, E)$ *is dense in* $\mathscr{S}_0(\Sigma, E)$.

Proof. (i) It is clear that $\Phi\to\|\Phi\|_\infty$ is a norm. Let $\{\Phi_n\}$ be a Cauchy sequence in $\mathscr{S}_\infty(\Sigma, E)$. Then for every $\sigma\in\Sigma$, $\{\Phi_n(\sigma)\}$ is a Cauchy sequence in $\mathscr{S}(H_\sigma\times H_\sigma, E)$. Since $\mathscr{S}(H_\sigma\times H_\sigma, E)$ is a Banach space, $\{\Phi_n(\sigma)\}$ converges to an element $\Phi(\sigma)$ in $\mathscr{S}(H_\sigma\times H_\sigma, E)$. An argument similar to [6], 28.25 shows that $\Phi=(\Phi(\sigma))$ belongs to $\mathscr{S}_\infty(\Sigma, E)$ and that $\{\Phi_n\}$ tends to Φ.

(ii) Let Φ be an element of $\mathscr{S}_0(\Sigma, E)$. For $n=1, 2, \ldots,$ define the element Φ_n of $\mathscr{S}_{00}(\Sigma, E)$ by

$$\Phi_n(\sigma) = \begin{cases} \Phi(\sigma) & \text{if } \|\Phi(\sigma)\| \geqq 1/n, \\ 0 & \text{if } \|\Phi(\sigma)\| < 1/n. \end{cases}$$

Then plainly $\{\Phi_n\}$ converges to Φ in $\mathscr{S}_0(\Sigma, E)$.

3.3 Lemma. *Every* $\Phi(\sigma)\in\mathscr{S}(H_\sigma\times H_\sigma, E)$ *is determined by the* d_σ^2 *elements* $a_{ij}^\sigma = \Phi(\sigma)(\xi_j, \xi_i)$ *of* E *where* d_σ *is the finite dimension of* H_σ *and* $(\xi_1, \xi_2, \ldots, \xi_{d_\sigma})$ *is an orthonormal basis of* H_σ. *More precisely, we have* $\Phi(\sigma) = \sum_{i,j=1}^{d_\sigma} d_\sigma a_{ij}^\sigma \hat{u}_{ij}^\sigma(\sigma)$ *where* $u_{ij}^\sigma(t) = \langle U_t^{(\sigma)}\xi_j, \xi_i\rangle$.

(Note that for a complex function u, $\hat{u}$ is the Fourier transform that is the Fourier—Stieltjes transform of the measure $u\lambda$, λ being the normalized Haar measure on G.)

Proof. We have

$$\Phi(\sigma)(\xi,\eta)=\sum_{i,j=1}^{d_\sigma}\alpha_j\bar{\beta}_i a_{ij}^\sigma$$

on putting

$$\xi=\sum_{j=1}^{d_\sigma}\alpha_j\xi_j \quad\text{and}\quad \eta=\sum_{i=1}^{d_\sigma}\beta_i\xi_i.$$

Now for a coordinate function u_{ij}^σ: $t\to\langle U_t^{(\sigma)}, \xi_j, \xi_i\rangle$, we have (by [6], 27.19)

$$\hat{u}_{ij}^\sigma(\sigma)(\xi,\eta)=\int_G\langle\overline{U}_t^{(\sigma)}\xi,\eta\rangle u_{ij}^\sigma(t)\,d\lambda(t)=\sum_{k,l}\int_G\alpha_l\bar{\beta}_k\bar{u}_{kl}^\sigma(t)u_{ij}^\sigma(t)\,d\lambda(t)=1/d_\sigma\alpha_j\bar{\beta}_i.$$

Thus

$$\Phi(\sigma)(\xi,\eta)=\sum\alpha_j\bar{\beta}_i a_{ij}^\sigma=\sum d_\sigma\hat{u}_{ij}^\sigma(\sigma)(\xi,\eta)a_{ij}^\sigma.$$

Hence

$$\Phi(\sigma)=\sum_{i,j=1}^{d_\sigma}d_\sigma a_{ij}^{(\sigma)}\hat{u}_{ij}^\sigma(\sigma).$$

3.4. Definition. We shall write $\mathscr{S}_2(\Sigma,E)$ for the vector space

$$\Big\{\Phi\in\mathscr{S}(\Sigma,E)\colon \sum_{\sigma\in\Sigma}d_\sigma\sum_{i,j}\|\Phi(\sigma)(\xi_i,\xi_i)\|_E^2<\infty\Big\}.$$

3.5. Lemma. *Suppose that E is a Hilbert space. Then the mapping*

$$(\Phi,\Psi)\to\langle\Phi,\Psi\rangle=\sum_{\sigma\in\Sigma}d_\sigma\sum_{i,j=1}^{d_\sigma}\langle\Phi(\sigma)(\xi_j,\xi_i),\ \Psi(\sigma)(\xi_j,\xi_i)\rangle$$

is an inner product on $\mathscr{S}_2(\Sigma,E)$.

Proof.

$$\sum\sum d_\sigma|\langle\Phi(\sigma)(\xi_j,\xi_i),\ \Psi(\sigma)(\xi_j,\xi_i)\rangle|\leqq\sum\sum d_\sigma^{1/2}\|\Phi(\sigma)(\xi_j,\xi_i)\|_E d_\sigma^{1/2}\|\Psi(\sigma)(\xi_j,\xi_i)\|_E\leqq$$
$$\leqq\sum\sum(d_\sigma\|\Phi(\sigma)(\xi_j,\xi_i)\|_2^2)^{1/2}\sum\sum(d_\sigma\|\Psi(\sigma)(\xi_j,\xi_i)\|^2)^{1/2}<\infty.$$

This shows that the mapping is well defined and the proof can be easily completed.

4. Properties of Fourier—Stieltjes transforms. Throughout this section, we adopt the following notation: if X is a subset of $M^1(G,E)$, we shall denote by $\hat{X}$ the set $\{\hat{u}\colon u\in X\}$. In the next two theorems we obtain analogues of Theorems 28.36 and 28.39 (i, ii) of [6], respectively.

4.1. Theorem. *The mapping $m\to\hat{m}$ from $M^1(G,E)$ into $\mathscr{S}_\infty(\Sigma,E)$ is linear, injective and continuous.*

Proof. That $m \to \hat{m}$ is linear is clear. We know that it is one-to-one by [1], Lemma 4.1.5. Now,

$$\|\hat{m}(\sigma)\| = \sup\{\|\hat{m}(\sigma)(\xi,\eta)\|_E \colon \|\xi\|_{H_\sigma} \leqq 1 \text{ and } \|\eta\|_{H_\sigma} \leqq 1\} =$$
$$= \sup\Big\{\Big\|\int \langle \overline{U}_t^{(\sigma)}\xi, \eta\rangle\, dm(t)\Big\|_E \colon \|\xi\|_{H_\sigma} \leqq 1,\ \|\eta\|_{H_\sigma} \leqq 1\Big\} \leqq \int \chi_G \, d|m|,$$

since $\overline{U}_t^{(\sigma)}$ is unitary. Thus $\|\hat{m}(\sigma)\| \leqq \|m\|$, $\sigma \in \Sigma$ and $\|\hat{m}\|_\infty \leqq \|m\|$. Hence $\hat{m} \in \mathcal{S}_\infty(\Sigma, E)$ and the mapping is continuous.

4.2. Definition. Let $\mathscr{C}(G, E)$ denote complex Banach space of all continuous E-valued functions on G with pointwise operations and norm given by $\|f\| = \sup\{\|f(t)\|_E \colon t \in G\}$. For $\sigma \in \Sigma$ and a fixed orthonormal basis $(\xi_1, \xi_2, \dots, \xi_{d_\sigma})$ in H_σ, $\mathscr{I}^\sigma(G)$ will denote the subspace of $\mathscr{C}(G, \mathbf{C})$ generated by the coordinate functions u_{ij}^σ. We set $\mathscr{I}^\sigma(G, E) = \{x\varphi \colon x \in E \text{ and } \varphi \in \mathscr{I}^\sigma(G)\}$ and define $\mathscr{I}(G, E)$ to be subspace of $\mathscr{C}(G, E)$ generated by the union $\bigcup_{\sigma \in \Sigma} \mathscr{I}^\sigma(G, E)$.

4.3. Theorem.

(i) *For each $\sigma \in \Sigma$, we have* $\widehat{\mathscr{I}^\sigma(G, E)} = \mathcal{S}(H_\sigma \times H_\sigma, E)$.

(ii) $\widehat{\mathscr{I}(G, E)} = \mathcal{S}_{00}(\Sigma, E)$.

Proof. (i) The result readily follows from Lemma 3.3 since

$$\Phi(\sigma) \in \mathcal{S}(H_\sigma \times H_\sigma, E) \Leftrightarrow$$
$$\Leftrightarrow a_{ij}^\sigma\text{'s in } E \text{ and } u_{ij}^\sigma\text{'s in } \mathscr{I}(G, \mathbf{C}) \text{ such that } \Phi(\sigma) = \sum d_\sigma a_{ij}^\sigma \hat{u}_{ij}^\sigma(\sigma) \Leftrightarrow$$
$$\Leftrightarrow \Phi(\sigma) \in \mathscr{I}^\sigma\widehat{(G, E)}.$$

(ii) Suppose that $f \in \mathscr{I}(G, E)$. Then f may be written $f = \sum_{i=1}^n \alpha_i f_{\sigma_i}$, $\alpha_i \in \mathbf{C}$, $\sigma_i \in \Sigma$ and $f_{\sigma_i} = \sum_{j=1}^n x_j u_j^\sigma$, $x_j \in E$, $u_{ji}^\sigma \in \mathscr{I}^{\sigma_i}(G, \mathbf{C})$. Thus

$$\hat{f}(\sigma)(\xi_1, \xi_m) = \sum_i \alpha_i \sum_j x_j \hat{u}_j^{\sigma_i}(\sigma)(\xi_1, \xi_m) \neq 0 \text{ only if } \sigma = \sigma_i,\ i = 1, 2, \dots, n.$$

Hence $\hat{f} \in \mathcal{S}_{00}(\Sigma, E)$.

Conversely, if $\Phi \in \mathcal{S}_{00}(\Sigma, E)$, then the set $P = \{\sigma \in \Sigma \colon \Phi(\sigma) \neq 0\}$ is finite. Moreover, each $\Phi(\sigma) = \sum_{i,j=1}^{d_\sigma} d_\sigma a_{ij}^\sigma \hat{u}_{ij}^\sigma(\sigma)$. Putting $f = \sum d_\sigma \sum_{i,j=1}^{d_\sigma} a_{ij}^\sigma u_{ij}^\sigma$, we get $\hat{f} = \Phi$ and so $\widehat{\mathscr{I}(G, E)} = \mathcal{S}_{00}(\Sigma, E)$.

4.4. Lemma. *The space $\mathscr{I}(G, E)$ is dense in $\mathscr{C}(G, E)$.*

Proof. We identify $\mathscr{I}(G, E)$ with $\mathscr{I}(G, \mathbf{C}) \otimes_\varepsilon E$, the injective tensor product of $\mathscr{I}(G, \mathbf{C})$ and E, i.e. the tensor product carrying the norm

$$\Big\|\sum_{1 \leqq i \leqq n} x_i \varphi_i\Big\|_\varepsilon = \Big\|\sum_{1 \leqq i \leqq n} \varphi_i \otimes x_i\Big\|_\varepsilon = \sup\Big\{\Big|\sum_{1 \leqq i \leqq n} u(x_i) v(\varphi_i)\Big| \colon \|u\| \leqq 1,\ \|v\| \leqq 1\Big\},$$

$u \in E'$, $v \in \mathscr{I}(G, \mathbf{C})'$ where E' and $\mathscr{I}(G, \mathbf{C})'$ are the topological duals of E and $\mathscr{I}(G, \mathbf{C})$, respectively ([7], 44.2 (3)). Since $\mathscr{I}(G, \mathbf{C})$ is dense in $\mathscr{C}(G, E)$, ([6], 27.39), it follows that $\mathscr{I}(G, E)$ is dense in $\mathscr{C}(G, E)$, because $\mathscr{C}(G, E)$ is norm isomorphic to $\mathscr{C}(G, \mathbf{C})\overline{\otimes}_\varepsilon E$, the completion of $\mathscr{C}(G, \mathbf{C})\otimes_\varepsilon E$, ([7], 44.7 (2)).

4.5. Theorem. *The space $\hat{L}_1(G, E)$ of the Fourier transforms of Haar-integrable functions f: $G \to E$ is dense in $\mathscr{S}_0(\Sigma, E)$.*

Proof. The space $\mathscr{I}(G, E)$ is dense in $L_1(G, E)$ because $\mathscr{I}(G, E)$ is dense in $\mathscr{C}(G, E)$ and $\mathscr{C}(G, E)$ is dense in $L_1(G, E)$ ([4], 7.16). Since $\mathscr{I}(\widehat{G, E}) = \mathscr{S}_{00}(\Sigma, E)$ is dense in $\mathscr{S}_0(\Sigma, E)$, $\hat{L}_1(G, E)$ which contains $\mathscr{I}(\widehat{G, E})$, is dense in $\mathscr{S}_0(\Sigma, E)$.

4.6. Corollary. *If $f \in L_1(G, E)$, then the set $\{\sigma \in \Sigma:\ \hat{f}(\sigma) \neq 0\}$ is countable.*

4.7. Lemma. *Let $L_2(G, E)$ denote the Banach space of the Haar-square integrable functions on G into E. If $f \in L_2(G, E)$, then*

$$f = \sum_\sigma \sum_{i,j} d_\sigma \hat{f}(\sigma)(\xi_j, \xi_i) u_{ij}^\sigma .$$

Proof. If $f = xh$, $x \in E$ and $h \in L_2(G, \mathbf{C})$, then

$$f = \sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j=1}^{d_\sigma} \left(\int xh(t)\bar{u}_{ij}^\sigma(t)\, d\lambda(t)\right) u_{ij}^\sigma$$

(use [6], 27.40 for h). Hence $f = \sum_\sigma d_\sigma \sum_{i,j=1} \left(\int f(t)\bar{u}_{ij}^\sigma(t)\, d\lambda(t)\right) u_{ij}^\sigma$. Since $L_2(G, \mathbf{C})\otimes E$ is dense in $L_2(G, E)$ it is clear that the last equality holds for $f \in L_2(G, E)$. Now,

$$\int f(t)\bar{u}_{ij}^\sigma(t)\, d\lambda(t) = \int \langle \bar{U}_t^{(\sigma)}\xi_j, \xi_i\rangle f(t)\, d\lambda(t) = \hat{f}(\sigma)(\xi_j, \xi_i).$$

Hence $f = \sum_\sigma d_\sigma \sum_{i,j} \hat{f}(\sigma)(\xi_j, \xi_i) u_{ij}^\sigma$.

Finally, we obtain the analogue of [6], 28.43.

4.8. Theorem. *Assume that E is a Hilbert space. Then the mapping $f \to \hat{f}$ is an isometry from $L_2(G, E)$ onto $\mathscr{S}_2(\Sigma, E)$ and so $\mathscr{S}_2(\Sigma, E)$ is a Hilbert space.*

Proof. If E is a Hilbert space, than $L_2(G, E)$ is a Hilbert space so that $f \in L_2(G, E)$ if and only if

$$\|f\|_2^2 = \left\langle \sum_\sigma \sum_{i,j} d_\sigma a_{ij}^\sigma u_{ij}^\sigma,\quad \sum_\sigma \sum_{i,j} d_\sigma a_{ij}^\sigma u_{ij}^\sigma \right\rangle,$$

where $a_{ij}^\sigma = \hat{f}(\sigma)(\xi_j, \xi_i)$, $1 \leq i,\ j \leq d_\sigma$. Hence

$$\|f\|_2^2 = \sum_\sigma \sum_{i,j} d_\sigma^2 \|a_{ij}^\sigma\|_E^2 \|u_{ij}^\sigma\|_2^2 = \sum_\sigma \sum_{i,j} d_\sigma \|\hat{f}(\sigma)(\xi_j, \xi_i)\|_2^2,$$

since $d_\sigma \|u^\sigma_{ij}\|_2^2 = 1$ ([6], 27.40). Thus $\hat{f} \in \mathcal{S}_2(\Sigma, E)$ and

$$\|\hat{f}\|_2^2 = \sum_\sigma \sum_{i,j} d_\sigma \|\hat{f}(\sigma)(\xi_j, \xi_i)\|_E^2 = \|f\|_2^2.$$

Conversely, let $\Phi \in \mathcal{S}_2(\Sigma, E)$. Then $\sum_\sigma \sum_{i,j} d_\sigma \|\Phi(\sigma)(\xi_j, \xi_i)\|_E^2 < \infty$ and hence the set $\{\Phi(\sigma)(\xi_j, \xi_i) \neq 0\}$ is countable, say $\{a_k\}_{k \in \mathbf{N}}$. Put $f_n = \sum_{k=1}^n d_{\sigma_n} a_k u_k$, where u_k replaces u^σ_{ij} whenever $a^\sigma_{ij} = a_k$ is different from zero. Then the functions f_n form a Cauchy sequence in $L_2(G, E)$ whose limit f satisfies $\hat{f} = \Phi$ and the proof is complete.

References

[1] V. S. K. Assiamoua, $L_1(G, A)$-multipliers, *Acta Sci. Math.*, **53** (1989), 309—318.

[2] P. Bonnet, Transformation de Fourier des distributions de type positif sur un groupe de Lie unimodulaire, *J. Funct. Anal.*, **55** (1984), 220—246.

[3] N. Bourbaki, *Integration*, Chap. 6, Hermann (Paris, 1966).

[4] N. Dinculeanu, *Integration in locally compact spaces*, Noordhoff Internat. Publishing (Leyden, 1974).

[5] N. Dunford, J. T. Schwartz, *Linear Operators*, Vol. I, Interscience (New York, 1957).

[6] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, Vol. II, Springer-Verlag (New York—Heidelberg—Berlin, 1970).

[7] G. Köthe, *Topological Vector Spaces*, Vol. II, Springer-Verlag, (New York—Heidelberg—Berlin, 1979).

(V. S. K. A.)
ECOLE DES SCIENCES (U. B.)
B. P. LOME, TOGO

(A. O.)
ONDO STATE UNIVERSITY
ADO-EKITI, NIGERIA

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$L_1(G, A)$-multipliers

V. S. K. ASSIAMOUA

1. Introduction

Throughout this paper, G will be a locally compact group and A a complex Banach algebra.

In [9] MING-KAM CHAN characterizes the $L_1(G, A)$-multipliers for algebras having a weak (hence a strong) bounded approximate identity.

By the present work, we prove that the characterizations remain true even in case A doesn't possess such an approximate identity, using the fact that any Banach algebra is contained in a Banach algebra with identity. Next, we enter upon the situation (not considered in [9]) where G is compact, non abelian. Doing this, we are induced to extend the notion of Fourier—Stieltjes transform of a vector measure.

2. Terminology

2.1. Vector measures. Let S be a locally compact (Hausdorff) space and E, a real or complex normed space. Denote by $\mathscr{K}(S, E)$ the vector space over the same field as E of all continuous functions on S into E, having compact supports, and write $\mathscr{K}(S)$ for $\mathscr{K}(S, \mathbf{R})$.

Let F be a real Banach space. By definition, an F-valued vector measure on S is a linear mapping m: $\mathscr{K}(S)\to F$ such that, for every compact set $K\subset S$, there exists a non negative constant α_K and $\|m(f)\|\leqq\alpha_K \sup_{t\in S}|f(t)|$ for every function f with support in K. $m(f)$ is also written

$$\int_S f\,dm, \quad \int_S f(t)\,dm(t) \quad \text{or} \quad \int f(t)\,dm(t).$$

(See [2], chap VI, § 2, no. 1.)

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Suppose now F is complex and consider the underlying real Banach space F_0. Then any **R**-linear mapping: $\mathcal{K}(S)\to F_0$ extends uniquely to a **C**-linear mapping: $\mathcal{K}(S, \mathbf{C})\to F$. As such, we shall always identify every measure m: $\mathcal{K}(S)\to F_0$ with the corresponding linear mapping (still denoted m) from $\mathcal{K}(S, \mathbf{C})$ into F, and shall call again vector measure, any linear mapping m: $\mathcal{K}(S, \mathbf{C})\to F$ whose restriction to $\mathcal{K}(S)$ is a measure (into F_0).

2.2. Bounded measures. A vector measure is said to be dominated if there exists a positive measure μ such that $\left\|\int f(t)\,dm(t)\right\| \leqq \int |f(t)|\,d\mu(t)$, $f\in\mathcal{K}(S)$.

If m is dominated, then there exists a smallest positive measure $|m|$ called the modulus or the variation of m, that dominates it. A positive measure is said to be bounded if it is continuous in the uniform norm topology of $\mathcal{K}(S)$.

A vector measure is said to be bounded if it is dominated by a bounded positive measure. It is clear that m is bounded if and only if $|m|$ is bounded ([3], § 3).

2.3. Integration. Assume μ is a positive measure on S and E is a Banach space. For a function f: $S\to E$, put

$$N_p(f) = \left(\int^* (\|f(t)\|_E)^p\,d\mu(t)\right)^{1/p}, \quad 1\leqq p \lneqq \infty$$

where $\int^*$ designates the upper integral ([2], chap. IV, § 1, no. 3).

$$N_\infty(f) = \inf\{\alpha\colon \|f(t)\|_E \leqq \alpha, \ \mu\text{-almost everywhere}\}.$$

The vector space (over the same field as E) of all μ-measureable functions f: $S\to E$ such that $N_p(f)<\infty$ is denoted by $\mathcal{L}_p(S, \mu, E)$ or $\mathcal{L}_p(S, E)$... and the corresponding quotient space $\mathcal{L}_p(S, \mu, E)/\mathcal{N}$ with respect to the closed subspace of the negligible functions, by $L_p(S, \mu, E)$ or $L_p(S, E)$. The seminorm N_p induces a norm $\|\cdot\|_{pE}$ on $L_p(S, E)$ which becomes a Banach space. In the sequel, we shall write f for the class $[f]$ as it is usually done.

With the positive measure μ on S is uniquely associated a continuous linear mapping n: $\mathcal{K}(S, E)\to E$ given by the equation

$$n(x\psi) = x\mu(\psi), \quad x\in E, \quad \psi\in\mathcal{K}(S)$$

([3], 2.11). Since $\mathcal{K}(S, E)$ is dense in $\mathcal{L}_1(S, E)$, n has an extension (still called n) to $\mathcal{L}_1(S, E)$. The integral of $f\in\mathcal{L}_1(S, E)$ with respect to μ is the value $n(f)$ denoted:

$$\int_S f(t)\,d\mu(t).$$

(It belongs to E.)

Now, let m be a dominated measure with values in a Banach space F. Then the space $\mathcal{L}_p(S, m, E)$ is by definition the space $\mathcal{L}_p(S, |m|, E)$. We associate with

m in a unique manner a continuous linear mapping (still called m) from $\mathscr{L}_1(S, m, E)$ into a Banach space D, provided there exists a continuous bilinear mapping: $E\times F\to D$. The corresponding integral $\int f(t)\,dm(t)$ belonging to D, is the value $m(f)$ ([3], 8.61. or [2], chap. VI, § 2, no. 7). If $f\in\mathscr{L}_1(S, \mathbf{R})$, put $D=F$ and $E=\mathbf{R}$, and the required bilinear mapping is the multiplication by real numbers: $\mathbf{R}\times F\to F$.

2.4. Convolution. The space $M_1(A)=M_1(G, A)$ of all bounded A-valued measures on G is a Banach algebra with the norm $\|m\|=\int \chi_G\, d|m|$ where χ_G is the characteristic function of G, and the convolution

$$m*n(f) = \int \Big(\int f(st)\,dm(s)\Big)\,dn(t), \quad f\in\mathscr{K}(G) \quad \text{and} \quad m, n\in M_1(A),$$

$$\text{shortly written} \quad \iint f(st)\,dm(s)\,dn(t).$$

Let λ be the left Haar measure on G. Identifying $f\in L_1(G, \lambda, A)$ with the bounded measure $f\lambda$ defined by

$$f\lambda(g) = \int f(t)g(t)\,d\lambda(t), \quad g\in\mathscr{K}(G)$$

then the functions

$$t\to f*g(t) = \int f(s)g(s^{-1}t)\,ds, \quad f, g\in L_1(G, A), \quad ds = d\lambda(s),$$

$$t\to m*f(t) = \int f(s^{-1}t)\,dm(s), \quad m\in M_1(A), \quad f\in L_1(G, A),$$

and

$$t\to f*m(t) = \int f(ts^{-1})\Delta(s^{-1})\,dm(s), \quad f\in L_1(G, A), \quad m\in M_1(A),$$

(where Δ is the modular function of G), belong to $L_1(G, A)$. Consequently, $L_1(G, A)$ appears as a two-sided ideal of $M_1(A)$.

2.4.1. Lemma. *Let τ_s, $s\in G$, be the right translation: $\tau_s f(t)=f(ts^{-1})$. Then*

(i) $\tau_s(f*g) = f*\tau_s g = (\Delta(s)\tau_{s^{-1}}f)*g, \quad f, g\in L_1(G, A),$

(ii) $\tau_s(m*g) = m*\tau_s g, \quad m\in M_1(A)$ *and* $g\in L_1(G, A)$.

The proof is straightforward.

2.4.2. Lemma. *If m is bounded and $m*g=0$ or $g*m=0$ for every $g\in\mathscr{K}(G)$ or for every $g\in\mathscr{K}(G, A)$, then $m=0$.*

See [3], 24.35 for the proof.

From now on, we shall write dt for $d\lambda(t)$.

3. $L_1(G, A)$-multipliers for general locally compact groups

3.1. Definition. A left $L_1(G, A)$-multiplier is a continuous linear operator $T\colon L_1(G, A)\to L_1(G, A)$ such that

$$\tau_s T = T\tau_s,\quad s\in G \quad\text{and}\quad T(xf) = xT(f),\quad x\in A,\quad f\in L_1(G, A). \tag{3.1.1}$$

A right $L_1(G, A)$-multiplier is respectively defined. Now, any result true for left multipliers has its analogue for right multipliers. Therefore we are going to study the left multipliers only, and omit the word "left" and sometimes the symbol "$L_1(G, A)$". By [6], it is not necessary to include continuity in the definition of multipliers. We do it only to avoid superfluous discussions.

We are going to use in the proof of the next theorem a number of facts that we want to point out now.

3.2. Extension of $L_1(G, A)$-multipliers to $L_1(G, \overset{*}{A})$. In the sequel, $\overset{*}{A}$ will be the Banach algebra obtained by joining an identity e to A. We recall that A is a two sided maximal ideal in $\overset{*}{A}$.

3.2.1. Lemma. *The space* $L_1(G, \overset{*}{A})$ *is norm isomorphic to the direct sum* $L_1(G, A)\oplus eL_1(G)$.

Proof. It is known that $\overset{*}{A}\cong A\oplus e\mathbf{C}$. Hence every $F\in L_1(G, \overset{*}{A})$ may be represented as $f=f+e\varphi$, $f\in L_1(G, A)$ and $\varphi\in L_1(G)$. To see it, put $f=P\circ F$ and $e\varphi=(I-P)\circ F$ where P is the projection operator: $\overset{*}{A}\to A$. Indeed, f and φ are integrable ([4] p. 480 and [3], 8.3).

3.2.2. Lemma. *Let T be an $L_1(G, A)$-multiplier. Then T is extendable to an $L_1(G, \overset{*}{A})$-multiplier $\overset{*}{T}$ and there exists a linear operator*

$$\tau\colon L_1(G) \to L_1(G) \quad \textit{such that} \quad T(x\varphi) = x\tau(\varphi),\quad x\in A \quad \textit{and} \quad \varphi\in L_1(G).$$

Proof. The condition $T(xf)=xT(f)$ in (3.1.1) shows that T is an A-module homomorphism on $L_1(G, A)$; hence according to [2], § 1, no. 1, T has an extension $\overset{*}{T}$ which is $\mathbf{C}$- and A-linear on $L_1(G, \overset{*}{A})$. Thus, for $X=x+e\xi$ in $\overset{*}{A}$ and $F=$ $=f+e\varphi$ in $L_1(G, \overset{*}{A})$ we have:

$$\overset{*}{T}(XF) = \overset{*}{T}(xf+\xi f+x\varphi+e\xi\varphi) = x\overset{*}{T}(f)+\xi\overset{*}{T}(f)+x\overset{*}{T}(e\varphi)+\xi\overset{*}{T}(e\varphi).$$

Put $\overset{*}{\tau}=$the restriction of $\overset{*}{T}$ to $eL_1(G)$. Since $\overset{*}{T}|_{L_1(G, A)}=T$, it is clear that

$$\overset{*}{T}(XF) = xT(f)+\xi T(f)+x\overset{*}{\tau}(e\varphi)+\xi\overset{*}{\tau}(e\varphi) = (x+e\xi)T(f)+(x+e\xi)\overset{*}{\tau}(e\varphi) = X\overset{*}{T}(F).$$

The definition of $\overset{*}{\tau}$ proves that $\overset{*}{\tau}(e\varphi)=e\tau(\varphi)$ for some linear mapping $\tau\colon L_1(G)\to\to L_1(G)$. Hence $\overset{*}{T}=T+e\tau$.

For $x\in A$ and $\varphi\in L_1(G)$, $x\varphi\in L_1(G, A)$; then

$$T(x\varphi) = \overset{*}{T}(xe\varphi) = x\overset{*}{\tau}(e\varphi) = x\tau(\varphi).$$

It is easy to deduce that τ is continuous, $\|\tau\|\leqq\|T\|$ and $\tau_S\tau=\tau\tau_S$. Now,

$$\|\overset{*}{T}(F)\|_{1\overset{*}{A}} \leqq \|T(f)\|_{1A}+\|\tau(\varphi)\|_1 \leqq \|T\|\|f\|_{1A}+\|\tau\|\|\varphi\|_1 \leqq$$

$$\leqq \|T\|(\|f\|_{1A}+\|\varphi\|_1) = \|T\|\|F\|_{1\overset{*}{A}}.$$

Hence $\overset{*}{T}$ is continuous. Finally

$$\tau_s\overset{*}{T}(F) = \tau_sT(f)+e\tau_s\tau(\varphi) = T(\tau_sf)+e\tau(\tau_s\varphi) = \overset{*}{T}\tau_s(F).$$

Therefore, $\overset{*}{T}$ is an $L_1(G, \overset{*}{A})$-multiplier.

Now, here is the first main theorem:

3.3. Theorem. *Let T be a continuous linear operator from $L_1(G, A)$ into $L_1(G, A)$. Then the following statements are equivalent:*

(3.1.1) $\quad T\tau_s = \tau_sT, \quad s\in G \quad$ *and* $\quad T(xf) = xT(f), \quad x\in A, \quad f\in L_1(G, A),$

(3.3.1) $\quad T(f*\mu) = T(f)*\mu, \quad f\in L_1(G, A), \quad \mu\in M_1(G, \mathbf{C}),$

(3.3.2) $\quad T(f*g) = T(f)*g, \quad f$ *and* $g\in L_1(G, A),$

(3.3.3) $\quad T(f) = m*f \quad$ *for some* $\quad m\in M_1(G, \overset{*}{A}).$

Proof. (a) Assume (3.1.1) and denote by A' the topological dual of A. Using [3] Corollary 14.21, we claim that, if $t\to\langle f(t), g(t)\rangle$ is negligible for every $g\in L_\infty(G, A')$ then f, $f\in L_1(G, A)$ is negligible because the assertion is true for functions of the form $x'\varphi$, $x'\in A'$ and $\varphi\in\mathcal{K}(G)$, which belong to $L_\infty(G, A')$.

We know that $L_\infty(G, A')\subset L_1(G, A)'$. Let $\mu\in M_1(G, \mathbf{C})$ and T' be the ajoint of T. We have, for $g\in L_\infty(G, A')$,

$$\int\langle T(f*\mu)(t), g(t)\rangle\, dt = \int\Big\langle\int f(ts^{-1})\Delta(s^{-1})\, d\mu(s), T'(g)(t)\Big\rangle\, dt =$$

$$= \iint\langle f(ts^{-1}), T'(g)(t)\rangle\Delta(s^{-1})\, d\mu(s)\, dt.$$

Applying Fubini's theorem, we have:

$$\iint\langle f(ts^{-1}), T'(g)(t)\rangle\Delta(s^{-1})\, d\mu(s)\, dt = \iint\langle f(ts^{-1}), T'(g)(t)\rangle\Delta(s^{-1})\, dt\, d\mu(s) =$$

$$= \iint\langle T(f)(ts^{-1}), g(t)\rangle\Delta(s^{-1})\, dt\, d\mu(s)$$

according to (3.1.1). We apply once again the Fubini's theorem.

$$\iint \langle T(f)(ts^{-1}), g(t)\rangle \Delta(s^{-1})\, dt\, d\mu(s) = \int \Big\langle \int T(f)(ts^{-1})\Delta(s^{-1})\, d\mu(s), g(t)\Big\rangle\, dt =$$
$$= \int \big\langle (T(f)*\mu)(t), g(t)\big\rangle\, dt.$$

Hence $T(f*\mu)=T(f)*\mu$. Therefore (3.1.1)$\Rightarrow$(3.3.1).

(b) Assume (3.3.1). We know that the projective tensor product $L_1(G)\otimes_\pi A$ is dense in $L_1(G, A)$. Then it suffices to prove (3.3.2) for $g=x\varphi$, $x\in A$ and $\varphi\in L_1(G)$.

Now, putting $\mu=\varphi\lambda$ in (3.3.1), $T(f*x\varphi)=T\big(x(f*\varphi)\big)=xT(f)*\varphi=T(f)*x\varphi$. Thus (3.3.1)$\Rightarrow$(3.3.2).

(c) Suppose (3.3.2). Then, for $f\in L_1(G, A)$ and $g\in \mathcal{K}(G, A)$ we have

$$T\tau_s(f)*g = T\big(\tau_s(f)\big)*g = T\big(\Delta(s^{-1})\tau_{s^{-1}}(f*g)\big) = \Delta(s^{-1})T\big(f*\tau_{s^{-1}}(g)\big) =$$
$$= \Delta(s^{-1})\big(T(f)*\tau_{s^{-1}}(g)\big) = \Delta(s^{-1})\tau_{s^{-1}}\big(T(f)*g\big) = \tau_s T(f)*g.$$

Consequently $T\tau_s(f)=\tau_s T(f)$, $f\in L_1(G, A)$ (Lemma 2.4.2) and hence $T\tau_s=\tau_s T$, $s\in G$. Moreover, for $x\in A$, the equalities

$$T(xf)*g = T(xf*g) = T(f*xg) = T(f)*xg = xT(f)*g$$

hold.

Therefore (3.1.1) obtains and hence (3.3.2)$\Rightarrow$(3.1.1). We deduce that (3.1.1), (3.3.1) and (3.3.2) are equivalent. To show that (3.3.3) is equivalent to them, let us suppose (3.1.1). Since $\overset{*}{A}$ has an identity and $L_1(G)$, an approximate identity, $L_1(G, \overset{*}{A})$ possesses an approximate identity. By Lemma 3.2.2., T is extendable to an $L_1(G, \overset{*}{A})$-multiplier $\overset{*}{T}$. Applying [9], Theorem 4 (ii) and results on page 181 § 2 to $\overset{*}{T}$, we conclude that there exists a vector measure m, $m\in M_1(G, \overset{*}{A})$ such that

$$\overset{*}{T}(F) = m*F, \quad F\in L_1(G, \overset{*}{A}),$$

identifying $\overset{*}{A}$ with its canonical image in its second conjugate space $\overset{*}{A}''$ (see also [9], page 186, Remark (2)).

Finally, for $F\in L_1(G, A)$, $T(f)=\overset{*}{T}(f)=m*f$, which is (3.3.3).

Conversely, by Lemma 2.4.1. (ii) and the fact that $m*xf=x(m*f)$, $x\in A$, the implication (3.3.3)$\Rightarrow$(3.1.1) is clear. This ends the proof of the theorem.

3.4. Remark. (i) In the proof of Lemma 3.2.2., we saw that the extension $\overset{*}{T}$ of T has the form $\overset{*}{T}=T+e\tau$ where τ is related to T by the equation

$$T(x\varphi) = x\tau(\varphi), \quad x\in A \quad \text{and} \quad \varphi\in L_1(G).$$

Hence, if A is right faithful i.e. the right annihilator of A is $\{0\}$, then $\overset{*}{T}$ is the unique extention of T which is a multiplier; and uniqueness holds in (3.3.3). This occurs for instance when A has a right approximate identity.

(ii) We shall not repeat the theorem concerning the Fourier transform when G is abelian as found in [9]. We shall rather treat the corresponding statement for compact groups which is a new fact.

4. $L_1(G, A)$-multipliers for compact groups

4.1. Fourier—Stieltjes transform. Let $m \in M_1(A)$. The Fourier—Stieltjes transform $\hat{m}$ of m is well known if G is abelian or if G is compact and $A = \mathbf{C}$ or $\mathbf{R}$. In fact, let G be abelian and denote its character group by $\hat{G}$. Then $\hat{m}$ is defined by the equation:

$$\hat{m}(\Gamma) = \int \bar{\Gamma}(t)\, dm(t), \quad \Gamma \in \hat{G}, \tag{4.1.1}$$

where $\bar{\Gamma}$ is the complex conjugate of Γ [9]. If G is compact and $A = \mathbf{C}$, the equality defining $\hat{m}$ becomes:

$$\langle \hat{m}(\sigma)\xi, \eta \rangle = \int \langle \bar{U}_t^{(\sigma)}\xi, \eta \rangle\, dm(t), \quad \sigma \in \Sigma, \quad (\xi, \eta) \in H_\sigma \times H_\sigma, \tag{4.1.2}$$

where Σ is the dual object of G, $U^{(\sigma)}$, a representative of the equivalent class $\sigma \in \Sigma$ and H_σ, the corresponding representation Hilbert space [5].

Now, suppose G is compact, non abelian and $A \neq \mathbf{C}$ and $\mathbf{R}$. The formula (4.1.2) is no longer meaningful because the mapping:

$$\eta \to \int \langle \bar{U}_t^{(\sigma)}\xi, \eta \rangle\, dm(t)$$

is a function from H_σ into A and as such, it is impossible to express it as a scalar product $\eta \to \langle \hat{m}(\sigma)\xi, \eta \rangle$ in general. The next lemma clarifies the situation.

4.1.3. Lemma. *The mapping* $H_\sigma \times H_\sigma \to A$:

$$(\xi, \eta) \to \int \langle \bar{U}_t^{(\sigma)}\xi, \eta \rangle\, dm(t), \quad m \in M_1(A)$$

is sesquilinear and continuous.

Proof. It is easily checked that the mapping is sesquilinear. Let us show that it is continuous.

Since $\bar{U}_t^{(\sigma)}$ is unitary for every $t \in G$, the inequality

$$\left\| \int \langle \bar{U}_t^{(\sigma)}\xi, \eta \rangle\, dm(t) \right\|_A \leqq \|\xi\|_{H_\sigma} \|\eta\|_{H_\sigma} \|m\|$$

holds, and the lemma obtains.

4.1.4. Definition. We define the Fourier—Stieltjes transform $\hat{m}$ of m, $m \in M_1(A)$ by the equation:

$$\hat{m}(\sigma)(\xi, \eta) \doteq \int \langle \overline{U}_t^{(\sigma)} \xi, \eta \rangle \, dm(t), \quad \sigma \in \Sigma \quad \text{and} \quad (\xi, \eta) \in H_\sigma \times H_\sigma,$$

where $\overline{U}^{(\sigma)}$ is fixed once and for all for each σ [5].

Let E and F be topological vector spaces. Denote by $\mathscr{L}(E, F)$ the space of the continuous linear mappings from E into F, by $\mathscr{B}(E \times E, F)$ the space of the continuous bilinear mappings from $E \times E$ into F and by $\mathscr{S}(E \times E, F)$ the space of the continuous sesquilinear mappings from $E \times E$ into F. We know that $\mathscr{B}(E \times E, F)$ is norm isomorphic to $\mathscr{L}(E, \mathscr{L}(E, F))$ if E and F are Banach spaces [7]. Similarly $\mathscr{S}(E \times E, F)$ is norm isomorphic to $\mathscr{L}(E, \mathscr{L}(E, F))$. Thus, if G is abelian, $\mathscr{S}(H_\sigma \times H_\sigma, A) \cong A$ for, $H_\sigma \cong \mathbf{C}$ in this case, and, if $A = \mathbf{C}$, $\mathscr{S}(H_\sigma \times H_\sigma, A) \cong \mathscr{L}(H_\sigma, H_\sigma)$ for compact groups because $\mathscr{L}(H_\sigma, \mathbf{C}) \cong H_\sigma$. Hence 4.1.4 generalizes (4.1.1) and (4.1.2).

4.1.5. Injectivity of the Fourier—Stieltjes transform

Lemma. *The map $m \to \hat{m}$ from $M_1(A)$ into $\prod_{\sigma \in \Sigma} \mathscr{S}(H_\sigma \times H_\sigma, A)$ is one-to-one.*

Proof. Suppose $\hat{n} = \hat{m}$. Then for any $\sigma \in \Sigma$ and any $(\xi, \eta) \in H_\sigma \times H_\sigma$

$$\int_G \langle \overline{U}_t^{(\sigma)} \xi, \eta \rangle \, dn(t) = \int_G \langle \overline{U}_t^{(\sigma)} \xi, \eta \rangle \, dm(t).$$

In particular $\int_G \langle \overline{U}_t^{(\sigma)} \xi, \eta \rangle \, d(n-m)(t) = 0$ for any σ, and ξ and η in an orthonormal basis of H_σ, $\sigma \in \Sigma$. According to [5] Theorem 27.39 and Remark (a) 27.8, $n-m$ is identically 0 on $\mathscr{K}(G)$. Thus $n = m$. Therefore the map is one-to-one.

4.1.6. Fourier—Stieltjes transform of a convolution

Lemma. *Assume G is compact and consider the set*

$$\hat{M}_1(A, \sigma) = \{\hat{m}(\sigma) \colon m \in M_1(A), \quad \sigma \in \Sigma\}.$$

Define B_σ by

$$B_\sigma(\Phi(\sigma), \hat{m}(\sigma))(\xi, \eta) = \int_G \Phi(\sigma)(\overline{U}_t^{(\sigma)} \xi, \eta) \, dm(t), \quad \Phi \in \prod_{\sigma \in \Sigma} \mathscr{S}(H_\sigma \times H_\sigma, A),$$

$m \in M_1(A)$ and $(\xi, \eta) \in H_\sigma \times H_\sigma$. Then

(i) *B_σ is a bilinear mapping from $\mathscr{S}(H_\sigma \times H_\sigma, A) \times \hat{M}_1(A, \sigma)$ into $\mathscr{S}(H_\sigma \times H_\sigma, A)$.*

(ii) *$\widehat{n * m}(\sigma) = B_\sigma(\hat{n}(\sigma), \hat{m}(\sigma))$, $(n, m) \in M_1(A) \times M_1(A)$.*

Proof. (i) For $\sigma\in\Sigma$, $(\xi,\eta)\in H_\sigma\times H_\sigma$ and $\|\Phi(\sigma)\|=\sup\{\|\Phi(\sigma)(\alpha,\beta)\|_A: \|\alpha\|\leqq\leqq 1;\ \|\beta\|\leqq 1\}$, we have

$$\overset{*}{\int_G} \|\Phi(\sigma)(\overline{U}_t^{(\sigma)}\xi,\eta)\|_A\, dm(t) \leqq \|\Phi(\sigma)\|\,\|\xi\|\,\|\eta\| \int_G \chi_G\, d|m| = \|\Phi(\sigma)\|\,\|\xi\|\,\|\eta\|\,\|m\| < \infty.$$

Hence B_σ is well defined. Thus the continuous (hence $|m|$-measurable) function $t\to\Phi(\sigma)(\overline{U}_t^{(\sigma)}\xi,\eta)$ from G into A is $|m|$- or equivalently m-integrable. Thus $B_\sigma(\Phi(\sigma))$, $\hat{m}(\sigma)\in\mathcal{S}(H_\sigma\times H_\sigma, A)$ since

$$\big\|B_\sigma(\Phi(\sigma),\hat{m}(\sigma))(\xi,\eta)\big\|_A \leqq \int_G \|\Phi(\sigma)(\overline{U}_t^{(\sigma)}\xi,\eta)\|_A\, d|m|(t).$$

It is obvious that B_σ is bilinear.

(ii) Plainly

$$\widehat{m*n}(\sigma)(\xi,\eta) = \int \langle \overline{U}_t^{(\sigma)}\xi,\eta\rangle\, d(m*n)(t) = \iint \langle \overline{U}_{st}^{(\sigma)}\xi,\eta\rangle\, dm(s)\, dn(t) =$$
$$= \iint \langle \overline{U}_s^{(\sigma)}\overline{U}_t^{(\sigma)}\xi,\eta\rangle\, dm(s)\, dn(t) = \int \hat{m}(\sigma)(\overline{U}_t^{(\sigma)}\xi,\eta)\, dn(t) = B_\sigma(\hat{m}(\sigma),\hat{n}(\sigma))(\xi,\eta).$$

Notation. We shall use the notation $\hat{m}\times\hat{n}(\sigma)$ instead of $B_\sigma(\hat{m}(\sigma),\hat{n}(\sigma))$. The second main theorem follows:

4.2. Theorem. *Suppose G is compact. Let T be a continuous linear operator: $L_1(G, A)\to L_1(G, A)$. Then*

(4.2.1) *T is a multiplier*

if and only if

(4.2.2) *there exists a* $\Phi\in\prod_{\sigma\in\Sigma}\mathcal{S}(H_\sigma\times H_\sigma, \overset{*}{A})$ *such that*

$$\widehat{T(f)} = \Phi\times\hat{f},\quad f\in L_1(G, A).$$

Proof. Suppose (4.2.1) and write down $T(f)=m*f$, for some $m\in M_1(G,\overset{*}{A})$ (Theorem 2.2). Then $\widehat{T(f)}=\widehat{m*f}=\hat{m}\times\hat{f}$. We obtain (4.2.2) if we put $\Phi=\hat{m}$.

Conversely, suppose (4.2.2). Then

$$\widehat{T(f*g)}(\sigma)(\xi,\eta) = \int \Phi(\sigma)(\overline{U}_t^{(\sigma)}\xi,\eta)\int f(s)g(s^{-1}t)\, ds\, dt =$$
$$= \iint \Phi(\sigma)(\overline{U}_t^{(\sigma)}\xi,\eta) f(s) g(s^{-1}t)\, ds\, dt$$

and, for $\sigma\in\Sigma$, $\xi\in H_\sigma$ and $\eta\in H_\sigma$ we have

$$\widehat{(T(f)*g)}(\sigma)(\xi,\eta)=\int\widehat{T(f)}(\sigma)(\overline{U}_t^{(\sigma)}\xi,\eta)g(t)\,dt=\iint\Phi(\sigma)(\overline{U}_{su}^{(\sigma)}\xi,\eta)f(s)g(u)\,du=$$

$$=\iint\Phi(\sigma)(\overline{U}_t^{(\sigma)}\xi,\eta)f(s)g(s^{-1}t)\,dt=\widehat{T(f*g)}(\sigma)(\xi,\eta).$$

Therefore $T(f*g)=T(f)*g$, the mapping $m\to\hat{m}$ being one-to-one. We conclude that T is a multiplier since it is supposed to be continuous.

References

[1] O. Akinyele, A multiplier problem, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.*, **57** (1974), 487—490.

[2] N. Bourbaki, *Integration*, Chap. 1—6, Hermann (Paris 1965).

[3] N. Bourbaki, *Algebre*, Chap. 8, Hermann (Paris 1973).

[4] N. Dinculeanu, *Integration on locally compact spaces*, Noordhoff International Publishing (Leyden, 1974).

[5] N. Dunford and J. T. Schwartz, *Linear operators*, Vol. I, Interscience (New York, 1957).

[6] E. Hewitt and K. Ross, *Abstract harmonic analysis*, Vol. II, Springer-Verlag (New York—Heidelberg—Berlin, 1970).

[7] B. E. Johnson, Continuity of centralisers on Banach algebras, *J. London Math. Soc.*, **41** (1966), 639—640.

[8] G. Kothe, *Topological vector spaces*. II, Springer-Verlag (New York—Heidelberg—Berlin, 1979).

[9] R. Larsen, *An introduction to the theory of multipliers*, Springer-Verlag (Berlin—Heidelberg—New York, 1971).

[10] Ming-Kam Chan, Characterizations of the right multipliers for $L_1(G, A)$, *Proc. Edinburgh Math. Soc.*, **22** (1979).

[11] B. J. Tomiuk, Multipliers on Banach algebras, *Studia Math.*, **54** (1975/76), 267—283.

ECOLE DES SCIENCES (U. B.)
B. P. 1515 LOME, TOGO

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Characterization of locally bounded functions with a finite number of negative squares

Z. SASVÁRI

1. Introduction

Throughout the paper G denotes a locally compact commutative group.

Let f be a complex-valued function on G. The function f is called Hermitian if $f(-x)=\overline{f(x)}$ holds for every $x\in G$. If k is a nonnegative integer the Hermitian function f is said to have k negative squares if the Hermitian matrix

$$(1) \qquad (f(x_i-x_j))_{i,j=1}^{n}$$

has at most k negative eigenvalues for any choice of n and $x_1, \ldots, x_n\in G$, and for some choice of $x_1, \ldots, x_n$ the matrix (1) has exactly k negative eigenvalues. This definition reduces to that of a positive definite function in the case $k=0$. We denote by $P_k(G)$ ($P_k^c(G)$) the set of all (continuous) functions on G which have k negative squares.

For a function $f\in P_k^c(G)$, where G is second countable, an integral representation was given in [10]. The bounded functions in $P_k^c(G)$ are exactly the Fourier transforms of such measures on the character group of G which assigne negative measure to k points and which are nonnegative outside of these points [9, 10]. A survey and bibliography about functions with k negative squares can be found in [1, 10, 12].

It is the aim of this note to characterize those functions $f\in P_k(G)$ which are locally bounded, i.e., bounded on every compact set $K\subset G$. As was shown in [11], every measurable function f with k negative squares on an arbitrary locally compact group is locally bounded. Moreover, f has the decomposition $f=f_c+p$, where f_c is a continuous function with k negative squares and p is a positive definite function vanishing almost everywhere on G [10].

If f is not measurable and $k>0$, then it may be unbounded on every open set. To see this let l be a nonmeasurable real-valued function on $\mathbf{R}$ satisfying the

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equation $l(x+y)=l(x)+l(y)$ $(x, y\in \mathbf{R})$. Then the function $f=il$ has one negative square and f is unbounded on every open set $V\subset \mathbf{R}$.

The main result of the present paper is the following

Theorem 1. *Every locally bounded function $f\in P_k(G)$ has the decomposition*

$$f=\gamma_1 f_1+\ldots+\gamma_n f_n+p \tag{2}$$

where

(i) *γ_j is a bounded (continuous or discontinuous) character of G $(j=1, \ldots, n)$;*
(ii) *f_j is a continuous function with k_j negative squares and $k_1+\ldots+k_n=k$;*
(iii) *p is a positive definite function.*

Recall that a complex-valued Hermitian function defined on G is said to be conditionally positive definite if

$$\sum_{i,j=1}^{n} f(x_i-x_j)c_i\bar{c}_j \geqq 0$$

holds for every choice of $x_1, \ldots, x_n\in G$ and for every choice of complex numbers $c_1, \ldots, c_n$ such that $c_1+\ldots+c_n=0$. It is easy to see that a conditionally positive definite function has at most one negative square. For a bibliography about conditionally positive definite functions we refer to [2, 4].

The above theorem has the following

Corollary 1. *Let f be a conditionally positive definite function on G which is bounded on a set of positive Haar measure. Then f has the decomposition*

$$f=f_c+p$$

where f_c is a continuous conditionally positive definite function and p is positive definite.

We remark that a conditionally positive definite function f is bounded if and only if $f=p+m$, where $m\in \mathbf{R}$ and p is a positive definite function [2]. The function $f=il$ introduced above is a conditionally positive definite function which is unbounded on every set $V\subset \mathbf{R}$ of positive Haar measure.

2. Notation and preliminaries

(2.1) Let k be a nonnegative integer. Throughout the paper the symbol Π_k denotes a π_k-space with rank of negativity k. We shall assume familiarity with basic information about π_k-spaces as found in [3, 5].

Let

$$\Pi_k=\Pi_+\oplus\Pi_- \tag{3}$$

be a fixed decomposition of Π_k where Π_+ is a positive subspace and Π_- is a negative k-dimensional subspace. Representing each vector $v \in \Pi_k$ in the form $v = v_+ + v_-$ ($v_+ \in \Pi_+$, $v_- \in \Pi_-$) we introduce a new scalar product $[\,,\,]$ in Π_k by

(4) $$[v, w] = (v_+, w_+) - (v_-, w_-), \quad v, w \in \Pi_k.$$

This scalar product is positive definite and Π_k can be regarded as a Hilbert space with scalar product $[\,,\,]$ and with the norm

(5) $$\|v\| = \sqrt{[v, v]}.$$

The scalar product (v, w) is continuous with respect to the norm (5) in both variables v and w.

Let $\{e_1, \ldots, e_k\}$ be a basis of Π_- such that $[e_i, e_j] = -(e_i, e_j) = \delta_{ij}$. Then we have for any $v \in \Pi_k$

(6) $$\|v\|^2 = [v, v] = (v, v) + 2 \sum_{i=1}^{k} |(e_i, v)|^2.$$

Recall that a linear operator U in Π_k is called unitary if it maps Π_k onto Π_k and preserves the scalar product $(\,,\,)$ of Π_k, i.e.,

$$(Uv, Uw) = (v, w) \quad \text{for all} \quad v, w \in \Pi_k.$$

By a unitary representation of G in Π_k there is meant a mapping $x \to U_x$ of G satisfying the following conditions:

(i) $U_0 = I$ where I is the identity operator in Π_k;
(ii) $U_{x+y} = U_x U_y$ for any $x, y \in G$;
(iii) U_x is a unitary operator in Π_k for all $x \in G$.

We shall need the following correspondence between cyclic unitary representations of G in π_k-spaces and functions of the class $P_k(G)$ [10, Satz 9.2].

Theorem 2. *For an arbitrary function $f \in P_k(G)$ there exists a π_k-space $\Pi_k(f)$ with the following properties:*

(i) *the elements of $\Pi_k(f)$ are complex-valued functions on G, $f \in \Pi_k(f)$, and $\Pi_k(f)$ is invariant under translations;*
(ii) *the linear manifold $T(f)$ spanned by all translations of f is dense in $\Pi_k(f)$;*
(iii) *$x \to U_x$ is a cyclic unitary representation of G in $\Pi_k(f)$, where U_x is defined by*

$$(U_x g)(y) = g(y-x), \quad g \in \Pi_k(f), \quad x, y \in G;$$

(iv) *$g(x) = (g, U_x f)$, $g \in \Pi_k(f)$, $x \in G$;*
(v) *if f is locally bounded then every function $g \in \Pi_k(f)$ is locally bounded.*

We now prove a further assertion.

(vi) *If f is locally bounded then the function $x \to \|U_x\|$ is locally bounded.*[1]

[1] The operator norm is induced by the vector norm (5).

Proof of (vi). It follows from the proof of Satz 9.2 in [10] that

$$\Pi_k(f) = P \oplus N$$

where P is a positive subspace and N is a negative k-dimensional subspace such that every function $h \in N$ is a finite linear combination of translations of f, i.e., $h \in T(f)$. Let $\{e_1, \ldots, e_k\}$ be an orthonormal basis of N. By (6) we have

$$\|U_x g\|^2 = (g, g) + 2 \sum_{i=1}^{k} |(U_x g, e_i)|^2, \quad g \in \Pi_k(f).$$

From $e_i \in T(f)$ and (iv) it follows easily that the function $h_i(-x) = (U_{-x} g, e_i) = (g, U_x e_i)$ is a finite linear combination of translations of g $(i=1, \ldots, n)$. By (v), g is locally bounded, so the function $x \to \|U_x g\|^2$ is locally bounded for every $g \in \Pi_n(f)$. The local boundedness of $x \to \|U_x\|$ follows now from the Banach—Steinhaus Theorem.

(2.2) Let $x \to U_x$ be a representation of G by invertible bounded linear operators on a Hilbert space $\mathfrak{H}$. We say that the representation $x \to U_x$ is locally bounded if the function $x \to \|U_x\|$ is locally bounded. Denote by $\mathfrak{H}_c$ the subspace of continuously translating elements of $\mathfrak{H}$, i.e., the set of all $h \in \mathfrak{H}$ for which $x \to U_x h$ is continuous from G into $\mathfrak{H}$ in its weak topology. Let $\mathscr{V}$ denote the set of all neighbourhoods V of the zero of G, $U_V = \{U_x \colon x \in V\}$, and $\mathscr{C}(U_V h)$ the closed convex hull of the "partial orbit" $U_V h = \{U_x h \colon x \in V\}$. The subspace $\mathfrak{H}_0$ of elements averaging to $0 \in \mathfrak{H}$ is the set of all $h \in \mathfrak{H}$ for which

$$0 \in \bigcap_{V \in \mathscr{V}} \mathscr{C}(U_V h).$$

K. deLeeuw and I. Glicksberg [6, Th. 2.7] proved the following

Theorem 3. *Let $x \to U_x$ be a locally bounded representation of G in a Hilbert space $\mathfrak{H}$. Then $\mathfrak{H}_c$ and $\mathfrak{H}_0$ are closed (U_x)-invariant subspaces and $\mathfrak{H}$ is the orthogonal direct sum of $\mathfrak{H}_c$ and $\mathfrak{H}_0$.*

Let now $f \in P_k(G)$ be a locally bounded function and consider the unitary representation $x \to U_x$ of G in $\Pi_k(f)$. By (vi) this representation is locally bounded with respect to the positive definite scalar product (4). (Note that local boundedness of $x \to U_x$ does not depend on the special decomposition (3).) It follows from the definition of $\mathfrak{H}_c$ and from (iv) that every $h \in \mathfrak{H}_c$ is a continuous function. When $h \in \mathfrak{H}_0$ then h has the following property: for $\varepsilon > 0$ and any $V \in \mathscr{V}$ for which $\sup_{x \in V} \|U_x\| < \infty$ there exist $x_1, \ldots, x_n \in V$ and positive numbers $p_1, \ldots, p_n$ summing to 1 such that

$$\left| \sum_{i=1}^{n} p_i h(x - x_i) \right| < \varepsilon \quad \text{for all} \quad x \in V. \tag{7}$$

Indeed, by the definition of $\mathfrak{H}_0$ there are $x_1, \dots, x_n \in V$ and positive numbers $p_1, \dots, p_n$ summing to 1 such that

$$\left\|\sum_{i=1}^{n} p_i U_{x_i} h\right\| < \varepsilon / \sup_{x\in V} \|U_x f\|.$$

By (iv) we have

$$\left|\sum_{i=1}^{n} p_i h(x-x_i)\right| = \left|\left(\sum_{i=1}^{n} p_i U_{x_i} h\right)(x)\right| = \left|\left(\left(\sum_{i=1}^{n} p_i U_{x_i} h\right), U_x f\right)\right| \leq$$

$$\leq \left\|\sum_{i=1}^{n} p_i U_{x_i} h\right\| \|U_x f\| < \varepsilon \quad \text{for} \quad x \in V.$$

(2.3) Let G^d be the discrete version of G. The character group of G^d is denoted by Γ^d. We introduce the notation Γ_u^d for the set of unbounded characters of G^d; i.e., the set of complex-valued unbounded functions γ on G^d for which $\gamma(0)=1$ and $\gamma(x+y)=\gamma(x)\gamma(y)$ hold. Let

$$\Gamma'^d = \Gamma^d \cup \Gamma_u^d.$$

In the proof of Theorem 1 we shall need the following result which is the discrete version of Folgerung 11.7 in [10] (see also Theorem 3.1 in [8]).

Theorem 4. *For every* $f \in P_k(G)$ *there exist positive integers* k_i, *functions* $f_i \in P_{k_i}(G)$ *and* $\gamma_i \in \Gamma'^d$ $(i=1, \dots, n)$ *with the following properties:*

(a) $f = f_1 + \dots + f_n$;

(b) $k = k_1 + \dots + k_n$;

(c) $f_i \in \Pi_k(f)$ $(i=1, \dots, n)$;

(d) *the only common nonpositive eigenvector of the translation operators* U_x *in* $\Pi_{k_i}(f_i)$ *are* γ_i *and* $\overline{\gamma_i^{-1}}$.[2)]

When f is locally bounded then by (c) and (v) in Theorem 2 the functions f_i are locally bounded as well.

3. Proof of Theorem 1 and Corollary 1

(3.1) Let $f \in P_k(G)$ be a locally bounded function and consider the locally bounded unitary representation $x \to U_x$ of G in $\Pi_k(f)$. By Theorem 4 we can restrict ourselves to the case where the only common nonpositive eigenvectors of the operators U_x are $\gamma, \overline{\gamma^{-1}} \in \Gamma'^d$. Since $\dfrac{|\gamma|}{\gamma}$ is a bounded character of G, the (locally bounded)

[2)] Note that $\gamma_i = \overline{\gamma_i^{-1}}$ if and only if $\gamma_i \in \Gamma^d$.

function $f' = \frac{|\gamma|}{\gamma} f$ has k negative squares. Moreover, the only common nonpositive eigenvectors of the translation operators U_x in $\Pi_k(f')$ are $|\gamma|$ and $|\gamma|^{-1}$ (see for this (11.5)(a) and (3.2)(c) in [10]). Thus, the proof of Theorem 1 will be complete if we verify the following.

Proposition 1. *Let $f \in P_k(G)$ be a locally bounded function. If the only common nonpositive eigenvectors of the operators U_x in $\Pi_k(f)$ are γ and γ^{-1}, and if they are positive then*

$$f = f_c + p,$$

where $f_c \in P_k^c(G)$ and $p \in P_0(G)$.

Proof. We consider the π_k-space $\Pi_k(f)$ as a Hilbert space with the scalar product $[\,,\,]$ in (4). By Theorem 3 $\Pi_k(f)$ is the $[\,,\,]$-orthogonal direct sum of the closed (U_x)-invariant subspaces X_c and X_0. Considering X_0 as subspace of the π_k-space $\Pi_k(f)$ there are three possibilities:

(i) X_0 is a π_l-space ($l \geqq 1$);

(ii) X_0 is degenerate;

(iii) X_0 is a Hilbert space.

In the first case the commuting unitary operators U_x have a common nonpositive eigenvector in X_0 [7] which by our assumption must be γ or γ^{-1}. In the second case the isotropic part N of X_0 is (U_x)-invariant and finite dimensional. Hence the commuting operators U_x have a common eigenvector in N which must be again γ or γ^{-1}. Thus, in both cases we have $\gamma \in X_0$ or $\gamma^{-1} \in X_0$. Suppose for example $\gamma \in X_0$ and let V be an open symmetric neighbourhood of zero such that γ is bounded on V:

$$\gamma(x) < K \quad (x \in V).$$

As $\gamma(-x)\gamma(x) = 1$, we get

$$1/K < \gamma(x) < K \quad (x \in V).$$

Consequently, for any $x, x_i \in V$ ($i = 1, \dots, n$) and arbitrary positive numbers $p_1, \dots, p_n$ summing to 1 we have:

$$\sum_{i=1}^{n} \gamma(x - x_i) p_i = \gamma(x) \sum_{i=1}^{n} \gamma(-x_i) p_i > \gamma(x)/K > 1/K^2,$$

in contradiction to (7). Hence (i) and (ii) are not possible and so X_0 is a Hilbert space.

Let X_c' denote the $(\,,\,)$-orthogonal complement of X_0. Then X_c' is a closed (U_x)-invariant π_k-space and

(8) $$\Pi_k(f) = X_c' \oplus X_0.$$

(the symbol $\oplus$ denotes $(\, ,)$-orthogonal direct sum). On the other hand, X_c' is a Hilbert space with respect to the scalar product $[\, ,]$, and the restriction $x \to U_x'$ of $x \to U_x$ to X_c' is a locally bounded representation of G in X_c'. If $h' \in X_c'$ averages to zero with respect to the representation $x \to U_x'$ then it averages to zero with respect to $x \to U_x$ as well. Since X_0 consists of all $h \in \Pi_k(f)$ averaging to zero we necessarily have $h'=0$. Applying Theorem 3 to the representation $x \to U_x'$ in X_c' we see that every $h \in X_c'$ is continuously translating. Hence the function $x \to [g, U_x h]$ is continuous, from which the continuity of $x \to (g, U_x h)$ follows $(g, h \in X_c')$.

Let now $f=f_c+p$ $(f_c \in X_c',\ p \in X_0)$ be the decomposition of f corresponding to (8). We have

$$f(x) = (f, U_x f) = (f_c+p, U_x f_c+U_x p) = (f_c, U_x f_c)+(p, U_x p).$$

Moreover,

$$f_c(x) = (f_c, U_x f) = (f_c, U_x f_c)+(f_c, U_x p) = (f_c, U_x f_c) \tag{9}$$

and analogously

$$p(x) = (p, U_x p).$$

It follows from (9) that f_c is continuous. The function f is a cyclic vector for $x \to U_x$ and so f_c is cyclic for $x \to U_x'$. Thus, f_c has k negative squares [10, Satz 11.1]. Since X_0 is a Hilbert space (with respect to $(\, ,)$) the function p is positive definite, completing the proof of Proposition 1.

(3.2) Let now f be a conditionally positive definite function which is bounded on a set $A \subset G$ of positive Haar measure. By a well known property of the Haar measure, $A-A$ contains an open set $V \neq \emptyset$. It follows from the inequality

$$\sqrt{|f(x-y)|} \leqq \sqrt{|f(x)|}+\sqrt{|f(y)|}, \quad x, y \in G, \tag{10}$$

that f is bounded on V. Moreover, (10) implies that f is bounded on $y+V$ for every $y \in G$. Since compact sets can be covered by finitely many sets V_i of the form $V_i=y_i+V$, f is locally bounded.

Let us consider the (locally bounded) unitary representation $x \to U_x$ in $\Pi_1(f)$ (we neglect the trivial case where f is positive definite). By [10, (11.5)] the only common nonpositive eigenvector of the operators U_x is $\gamma=1$. Therefore, we can apply Proposition 1 to obtain the decomposition

$$f = f_c+p,$$

where $f_c \in P_1^c(G)$ and $p \in P_0(G)$. All what remains to prove is that f_c is conditionally positive definite. Since $T(f)$ is dense in $\Pi_1(f)$, there is a sequence w_n of finitely supported complex measures on G such that

$$f_c = \lim_{n\to\infty} f * w_n .$$

(the symbol $*$ denotes convolution). By (9) and (iv) in Theorem 2 we have

$$f_c(x) = (f_c, U_x f_c) = \lim_{n\to\infty} (f*w_n, U_x(f*w_n)) =$$
$$= \lim_{n\to\infty} (f*w_n*\tilde{w}_n, U_x f) = \lim_{n\to\infty} f*w_n*\tilde{w}_n(x),$$

where $\tilde{w}_n$ is defined by $\tilde{w}_n(\{-x\})=\overline{w_n(\{x\})}$. It follows immediately from the definition of conditional positive definiteness that the functions $f*w_n*\tilde{w}_n$ and so f_c are conditionally positive definite. The proof of Corollary 1 is complete.

Remark 1. As we have seen, boundedness on a set of positive Haar measure of a conditionally positive definite function implies local boundedness. It would be interesting to know whether a similar assertion holds for functions with a finite number of negative squares.

Remark 2. Corollary 1 probably holds even for noncommutative groups while the problem of characterization of locally bounded functions $f\in P_k(G)$ seems to be very difficult if G is not commutative.

Remark 3. Let G be an arbitrary commutative topological group. We say that a complex-valued function g on G is locally bounded if there exists an open set $V\subset G$ such that g is bounded on $y+V$ for every $y\in G$. Let now $f\in P_k(G)$ be a locally bounded function and consider the representation $x\to U_x$ in $\Pi_k(f)$. It follows by the same arguments as in the proof of property (vi) that the function $x\to\|U_x\|$ is locally bounded. Since Theorem 3 holds for an arbitrary commutative topological group G [6, Th. 2.7] we can repeat the proof of Theorem 1 to get the decomposition (2) of f.

References

[1] T. Ja. Azizov, I. S. Iohvidov, Linear operators in spaces with an indefinite metric and their applications, *Itogi Nauki i Techn.*, Ser. Mat. Anal., 17 (1979), 113—205 (Russian).

[2] Ch. Berg, G. Forst, *Potential Theory on Locally Compact Abelian Groups*, Springer-Verlag (Berlin—Heidelberg—New York, 1977).

[3] J. Bognár, *Indefinite Inner Product Spaces*, Springer-Verlag (Berlin—Heidelberg—New York, 1974).

[4] H. Heyer, *Probability Measures on Locally Compact Groups*, Springer-Verlag (Berlin—Heidelberg—New York, 1977).

[5] I. S. Iohvidov, M. G. Krein and H. Langer, *Introduction to the Spectral Theory of Operators in Spaces with an Indefinite Metric*, Akademie-Verlag (Berlin, 1982).

[6] K. deLeeuw, I. Glicksberg, The decomposition of certain group representations, *J. Anal. Math.*, 15 (1965), 135—192.

[7] M. A. Naimark, On commuting unitary operators in spaces with an indefinite metric, *Acta Sci. Math.*, 24 (1963), 177—189.

[8] Z. SASVÁRI, Indefinite functions on commutative groups, *Monatshefte Math.*, **100** (1985), 223—238.
[9] Z. SASVÁRI, On bounded functions with a finite number of negative squares, *Monatshefte Math.*, **99** (1985), 223—234.
[10] Z. SASVÁRI, Definisierbare Funktionen auf Gruppen, *Dissertationes Math.*, **281** (1989), 1—83.
[11] Z. SASVÁRI, On measurable functions with a finite number of negative squares, *Acta Sci. Math.*, **50** (1986), 359—363.
[12] J. STEWART, Positive definite functions and generalizations, an historical survey, *Rocky Mountain J. Math.*, **6** (1976), 409—434.

Z. SASVÁRI
TECHNISCHE UNIVERSITÄT
SEKTION MATHEMATIK
8027 DRESDEN, MOMMSENSTR. 13
GDR

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Injection of shifts into contractions

L. KÉRCHY

The structure of unilateral shifts is well understood. Hence any relation between a contraction and a unilateral shift can be very useful. Here we only quote a recent result of H. BERCOVICI and K. TAKAHASHI (cf. [1]) claiming that a contraction T is reflexive whenever the set $\mathcal{I}(T, S)=\{A\colon AT=SA\}$ of intertwining operators contains a nonzero element, where S denotes the simple unilateral shift. In 1974 B. SZ.-NAGY and C. FOIAŞ proved the following (cf. [7, Corollary 2]):

Theorem 0. *If T is a contraction of class C_{10} with finite defect indices d_T and d_{T*}, then*

$$S^{(k)} \overset{\text{c.i.}}{\prec} T \prec S^{(k)}, \quad \textit{where} \quad k = d_{T*}-d_T.$$

Here $S^{(k)}$ stands for the unilateral shift of multiplicity k, i.e. for the orthogonal sum of k copies of the simple unilateral shift $S=S^{(1)}$. $T\prec S^{(k)}$ denotes that T is a quasiaffine transform of $S^{(k)}$, i.e. $\mathcal{I}(T, S^{(k)})$ contains a quasiaffinity (an operator with trivial kernel and dense range). The meaning of the notation $S^{(k)} \overset{\text{c.i.}}{\prec} T$ is that $S^{(k)}$ can be completely injected into T, i.e. $\mathcal{I}(S^{(k)}, T)$ contains a subsystem Φ consisting of injections such that $\vee\{\operatorname{ran} A\colon A\in\Phi\}=\operatorname{dom} T$. In connection with other notions concerning contractions readers are referred to the monograph [9].

We remark that, as it was illustrated by an example in [7], the relation $S^{(k)} \overset{\text{c.i.}}{\prec} T$ in Theorem 0 can not be generally replaced by $S^{(k)}\prec T$.

Definition. Let T be a completely non-unitary (c.n.u.) contraction. If the space of T is separable then the number

$$\mu_{*,T} = \operatorname*{ess\,sup}_{\zeta\in\partial\mathbf{D}} \operatorname{rank} \Delta_{*,T}(\zeta)\in[0,\infty]$$

will be called the *$*$-multiplicity* of T. In the general case $\mu_{*,T}$ is defined as the least upper bound of the $*$-multiplicities of the restrictions of T to its separable reducing subspaces.

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Here $\Delta_{*,T}(\zeta)=[I-\Theta_T(\zeta)\Theta_T(\zeta)^*]^{1/2}$ is the defect function of the adjoint of the characteristic function Θ_T of T, and the essential upper bound is taken with respect to the normalized Lebesgue measure m on the boundary $\partial\,\mathbf{D}$ of the open unit disc $\mathbf{D}$.

The $*$-multiplicity $\mu_{*,T}$ of T coincides with the usual multiplicity of the unitary operator $R_{*,T}$ of multiplication by the identical function $\chi(\zeta)=\zeta$ on the Hilbert space $(\Delta_{*,T}L^2(\mathfrak{D}_{T^*}))^-$. (Cf. [3].) Furthermore, we can observe that if T is of class $C_{\cdot 0}$ with $d_T<\infty$, then rank $\Delta_{*,T}(\zeta)=d_{T^*}-d_T$ a.e., whence $\mu_{*,T}=d_{T^*}-d_T$. Now, it is natural to ask how the statement of Theorem 0 alters if $\mu_{*,T}<\infty$ is assumed instead of $d_{T^*}<\infty$.

First we note that by a result of Takahashi (cf. [10, Proposition 2]) $S^{(k)}\overset{\text{c.i.}}{\prec}T$ is already a consequence of the relation $T\prec S^{(k)}$. However $T\prec S^{(k)}$ does not hold in general. This is shown by the following.

Example. Let us consider a contraction T of class C_{10} such that rank $\Delta_{*,T}=\chi_\alpha$ a.e., where χ_α denotes the characteristic function of a Borel set $\alpha\subset\partial\mathbf{D}$ of measure $0<m(\alpha)<1$. (The existence of such a contraction was proved in [4].) Now the $*$-multiplicity of T is 1.

Let us assume that T is the quasi-affine transform of $S^{(k)}$, for some $1\leqq k\leqq\infty$, and let $C\in\mathscr{I}(T,S^{(k)})$ be a quasi-affinity. Let $U^{(k)}$ denote the minimal unitary extension of $S^{(k)}$. The operator C can be considered as an element of $\mathscr{I}(T,U^k)$. In view of [5, Proposition 4] there exists an operator $D\in\mathscr{I}(R_{*,T},U^{(k)})$ such that $C=DX$, where $X\in\mathscr{I}(T,R_{*,T})$ is a canonical intertwining operator. Since $R^*_{*,T}|(\operatorname{ran}X)^\perp$ is always of class C_{10} (cf. [,5 Proposition 4]) and since $R_{*,T}$ is now reductive, it follows that X has dense range. We infer that $(\operatorname{ran}D)^-=(\operatorname{ran}C)^-=$ $=\operatorname{dom}S^{(k)}$, so D can be considered as a quasi-surjective operator from $\mathscr{I}(R_{*,T},S^{(k)})$, whence $D^*\in\mathscr{I}(S^{*(k)},R^*_{*,T})$ is an injection. This yields that $\{0\}=\ker R^*_{*,T}\supset$ $\supset D^*\ker S^{*(k)}\neq\{0\}$, what is a contradiction.

Therefore $T\prec S^{(k)}$ is not true, for any $1\leqq k\leqq\infty$.

In [10] K. Takahashi characterized, in terms of the characteristic function, contractions which are quasi-affine transforms of unilateral shifts of finite multiplicity. While in [11] P. Y. Wu gave a characterization for contractions which are quasi-similar to unilateral shifts of finite multiplicity.

Though, as we have seen, $T\prec S^{(k)}$ $(k=\mu_{*,T})$ loses validity in Theorem 0 if $d_{T^*}=\infty$, we shall prove that the relation $S^{(k)}\overset{\text{c.i.}}{\prec}T$ $(k=\mu_{*,T})$ does remain true in a very general setting. This is expressed in the following theorem, the main result of our paper.

Theorem. *If T is a c.n.u. contraction with $*$-multiplicity* $1 \leqq \mu_{*,T} < \infty$, *then*

$$S^{(k)} \overset{\text{c. i.}}{\prec} T, \quad \textit{where} \quad k = \mu_{*,T}.$$

We remark that injection of shifts into strict contractions was investigated in [8] and [12]. A contraction T is called strict if $\|T\|<1$, in which case $\mu_{*,T}=0$.

In proving our theorem we can assume that T acts on a separable Hilbert space $\mathfrak{H}$. In fact, in the opposite case $\mathfrak{H}$ can be decomposed into the orthogonal sum of separable subspaces reducing for T, and then the characteristic function of T will be the orthogonal sum of the characteristic functions of the restrictions of T. Hence in the sequel every Hilbert space will be supposed to be separable.

Since T is c.n.u. it can be given as a model operator (cf. [9, Chapter VI]). So let $\{\Theta, \mathfrak{E}, \mathfrak{E}_*\}$ be a purely contractive analytic function, its defect function is $\Delta=[I-\Theta^*\Theta]^{1/2}$. Let U_+ denote the operator of multiplication by the identical function $\chi(\zeta)=\zeta$ on the Hilbert space $\mathfrak{K}_+=H^2(\mathfrak{E}_*)\oplus(\Delta L^2(\mathfrak{E}))^-$. The c.n.u. contraction T is defined on the Hilbert space $\mathfrak{H}=\mathfrak{K}_+\ominus\{\Theta w\oplus\Delta w\colon w\in H^2(\mathfrak{E})\}$ as $T= =PU_+|\mathfrak{H}$, where P denotes the orthogonal projection onto $\mathfrak{H}$ in $\mathfrak{K}_+$. The $*$-multiplicity of T is $\mu_{*,T}=\operatorname{ess\,sup}_\zeta \operatorname{rank} \Delta_*(\zeta)$, where $\Delta_*=[I-\Theta\Theta^*]^{1/2}$.

The proof of the Theorem is based on the following.

Lemma. *Let h be a function in $L^2(\mathfrak{E}_*)$ such that $\|h(\zeta)\|_{\mathfrak{E}_*}=1$ a.e. Then for any non-zero function $f\in H^2(\mathfrak{E}_*)$ and for any number $0<c<1$, there exists an analytic function $u\in H^2(\mathfrak{E}_*)$ such that*

$$\|u(\zeta)\|_{\mathfrak{E}_*} \leqq 1 \quad \textit{a.e.,} \tag{1}$$

$$|\langle u(\zeta), h(\zeta)\rangle_{\mathfrak{E}_*}| \geqq c \quad \textit{a.e., and} \tag{2}$$

$$\langle u, f\rangle_{H^2(\mathfrak{E}_*)} \neq 0. \tag{3}$$

Proof. First we show that a function $u\in H^2(\mathfrak{E}_*)$ can be found with the properties (1) and (2). The proof of this is essentially the same as the proof of the Lemma in [6]. For the sake of easy reference we give the details.

Let $\{x_j\}_{j=1}^\infty$ be a dense sequence on the unit sphere of $\mathfrak{E}_*$, and for every j let us consider the function $h_j(\zeta)=\langle x_j, h(\zeta)\rangle_{\mathfrak{E}_*}$ $(\zeta\in\partial\mathbf{D})$, $h_j\in L^2$. Then we have

$$1 = \|h(\zeta)\|_{\mathfrak{E}_*} = \sup_j |h_j(\zeta)|, \quad \text{for a.e.} \quad \zeta\in\partial\mathbf{D}. \tag{4}$$

Let $0<\nu<1$ be arbitrary, and define $\{\alpha_j\}_{j=1}^\infty$ as $\alpha_1=\alpha_1^{(0)}$, $\alpha_j=\alpha_j^{(0)}\setminus\left(\bigcup_{i=1}^{j-1}\alpha_i\right)$ $(j\geqq 2)$, where $\alpha_j^{(0)}=\{\zeta\in\partial\mathbf{D}\colon |h_j(\zeta)|>\nu\}$. The sequence $\{\alpha_j\}_{j=1}^\infty$ consists of pairwise disjoint Borel sets, and by (4) we have

$$m(\partial\mathbf{D}\setminus(\bigcup_j \alpha_j)) = 0. \tag{5}$$

Let $\{\mu_j\}_{j=1}^{\infty}$ be a sequence of positive numbers such that $0<\mu=\sum_{j=1}^{\infty}\mu_j<1$. For every j, let us consider an outer function $\hat{u}_j\in H^\infty$ with absolute value $|\hat{u}_j|=(1-\mu)\chi_{\alpha_j}+\mu_j\chi_{\partial\mathbf{D}\setminus\alpha_j}$ a.e. on $\partial\mathbf{D}$, and let us define $u_j=\hat{u}_jx_j\in H^2(\mathfrak{E}_*)$.

For every j and for a.e. $\zeta\in\alpha_j$ we can write $\sum_{i=1}^{\infty}\|u_i(\zeta)\|_{\mathfrak{E}_*}=1-\mu+\sum_{\substack{i=1\\ i\neq j}}^{\infty}\mu_i\leq 1-\mu+\mu=1$. Hence in view of (5) $\sum_{j=1}^{\infty}\|u_j(\zeta)\|_{\mathfrak{E}_*}\leq 1$ and so $\sum_{j=1}^{\infty}u_j(\zeta)$ strongly converges in $\mathfrak{E}_*$ a.e. on $\partial\mathbf{D}$. The limit function $u(\zeta)=\sum_{j=1}^{\infty}u_j(\zeta)$ satisfies (1), therefore, $u\in L^2(\mathfrak{E}_*)$. Furthermore, Lebesgue's dominated theorem ensures that $\lim_{n\to\infty}\left\|\sum_{j=1}^{n}u_j-u\right\|_{L^2(\mathfrak{E}_*)}=0$, whence $u\in H^2(\mathfrak{E}_*)$.

For every j and for a.e. $\zeta\in\alpha_j$ we have

$$\text{(6)}\qquad |\langle u(\zeta), h(\zeta)\rangle_{\mathfrak{E}_*}| = \Big|\sum_{i=1}^{\infty}\langle u_i(\zeta), h(\zeta)\rangle_{\mathfrak{E}_*}\Big| = \Big|\sum_{i=1}^{\infty}\hat{u}_i(\zeta)h_i(\zeta)\Big| \geq$$

$$\geq |\hat{u}_j(\zeta)||h_j(\zeta)| - \sum_{\substack{i=1\\ i\neq j}}^{\infty}|\hat{u}_i(\zeta)||h_i(\zeta)| \geq (1-\mu)\nu - \sum_{i=1}^{\infty}\mu_i = (1-\mu)\nu-\mu.$$

If μ and ν are chosen sufficiently close to 0 and 1, respectively, then $(1-\mu)\nu-\mu\geq c$, and so (2) is implied by (5) and (6).

Now, let us take real numbers c_1 and c_2 satisfying $c<c_1<c_2<1$. By the previous part of the proof we can find a function $u_1\in H^2(\mathfrak{E}_*)$ such that (1) and (2) hold with c_1/c_2 in place of c. Then for the function $u_2=c_2u_1\in H^2(\mathfrak{E}_*)$ we have

$$\text{(7)}\qquad \|u_2(\zeta)\|_{\mathfrak{E}_*}\leq c_2 \quad \text{a.e., and}$$

$$\text{(8)}\qquad |\langle u_2(\zeta), h(\zeta)\rangle_{\mathfrak{E}_*}|\geq c_1 \quad \text{a.e..}$$

Let δ denote the positive number $\delta=\min\{c_1-c, 1-c_2\}$, and for any integer $n\geq 0$ and for any vector $a\in\mathfrak{E}_*$, $\|a\|\leq\delta$ let us define the function $u_{n,a}\in H^2(\mathfrak{E}_*)$ as $u_{n,a}=u_2+\chi^n a$. By (7) and (8) it easily follows that $u_{n,a}$ has the properties (1) and (2). Let us assume that (3) is not true, for any choice of n and a. Then taking $a=0$ we obtain $\langle u_2, f\rangle_{H^2(\mathfrak{E}_*)}=0$, whence $\langle\chi^n a, f\rangle=\langle u_{n,a}, f\rangle=0$ for every $n\geq 0$ and $a\in\mathfrak{E}_*$, $\|a\|\leq\delta$. But the set $\{\chi^n a\colon n\geq 0, a\in\mathfrak{E}_*, \|a\|\leq\delta\}$ is total in $H^2(\mathfrak{E}_*)$ and $f\in H^2(\mathfrak{E}_*)$, so f must be zero, which is a contradiction.

Therefore, the function $u=u_{n,a}\in H^2(\mathfrak{E}_*)$ possesses the properties (1)—(3) for an appropriate choice of $n\geq 0$ and $a\in\mathfrak{E}_*$, $\|a\|\leq\delta$.

Now we turn to the

Proof of the Theorem. Let k denote the $*$-multiplicity of T: $1\leq k=\mu_{*,T}<\infty$.

1) First we show that there exists an injection A in $\mathscr{I}(S^{(k)}, T)$.

The operator $X_+ \colon \mathfrak{K}_+ \to (\Delta_*(L^2(\mathfrak{E}_*))^-$, $X_+(u \oplus v) = (-\Delta_* u + \Theta v)$ intertwines U_+ with the operator R_* of multiplication by χ on the space $(\Delta_* L^2(\mathfrak{E}_*))^-$, $X_+ \in \mathscr{I}(U_+, R_*)$. In view of the commuting relation $\Delta_* \Theta = \Theta \Delta$ it is immediate that $X_+(\mathfrak{K}_+ \ominus \mathfrak{H}) = \{0\}$, and so the operator $X = X_+ | \mathfrak{H}$ belongs to $\mathscr{I}(T, R_*)$ and the relation

(9)
$$X_+ = XP$$

holds. (A detailed study of the operator X can be found in [5].)

Since $\Delta_*(\zeta)$ is a positive operator of finite rank a.e. and $\operatorname{ess\,sup}_{\zeta} \operatorname{rank} \Delta_*(\zeta) = k$, we conclude that $\Delta_*(\zeta)$ is of the form

(10)
$$\Delta_*(\zeta) = \sum_{j=1}^{k} \delta_j(\zeta) h_j(\zeta) \otimes h_j(\zeta),$$

where

$$h_j \in L^2(\mathfrak{E}_*) \quad \text{for every} \quad 1 \leqq j \leqq k,$$

$$\{h_j(\zeta)\}_{j=1}^k \text{ is an orthonormal system in } \mathfrak{E}_* \text{ a. e. on } \partial \mathbf{D},$$

(11)
$$0 \leqq \delta_j \in L^\infty \quad \text{for every} \quad 1 \leqq j \leqq k,$$

$$1 \geqq \delta_1(\zeta) \geqq \delta_2(\zeta) \geqq \ldots \geqq \delta_k(\zeta) \quad \text{a.e.} \quad \text{on} \quad \partial \mathbf{D}, \quad \text{and}$$

$$m(\alpha_k) > 0, \quad \text{where} \quad \alpha_k = \{\zeta \in \partial \mathbf{D} \colon \delta_k(\zeta) \neq 0\}.$$

(Indeed, the function $\delta_1(\zeta) = \|\Delta_*(\zeta)\|_{\mathfrak{E}_*}$ is measurable, and an easy application of [2, Lemma II.1.1] guarantees the existence of a function $h_1 \in L^2(\mathfrak{E}_*)$ such that $\|h_1(\zeta)\|_{\mathfrak{E}_*} = 1$ a.e. and $h_1(\zeta) \in \ker(\Delta_*(\zeta) - \delta_1(\zeta) I)$, whenever $\ker(\Delta_*(\zeta) - \delta_1(\zeta) I) \neq \{0\}$. The functions $\delta_2 \in L^\infty$ and $h_2 \in L^2(\mathfrak{E}_*)$ can be obtained from $\Delta_* - \delta_1 h_1$ in place of Δ_* in an analogous way; and so on.)

Let $0 < c < 1$ be arbitrary. In virtue of our Lemma, for every $1 \leqq j \leqq k$, we can find a function $u_j \in H^2(\mathfrak{E}_*)$ such that

(12)
$$\|u_j(\zeta)\|_{\mathfrak{E}_*} \leqq 1 \quad \text{a.e., and}$$

(13)
$$|\langle u_j(\zeta), h_j(\zeta) \rangle_{\mathfrak{E}_*}| \geqq c \quad \text{a.e..}$$

Let $\{e_j\}_{j=1}^k$ be an orthonormal basis on a Hilbert space $\mathfrak{G}$. The operator of multiplication by χ on the space $H^2(\mathfrak{G})$ is a unilateral shift of multiplicity k, which will be denoted by $S^{(k)}$. Since on account of (12), for any sequence $\{\xi_j\}_{j=1}^k \subset H^2$, we have

$$\Big\| \sum_{j=1}^{k} \xi_j u_j \Big\|_{H^2(\mathfrak{E}_*)} \leqq \sum_{j=1}^{k} \|\xi_j u_j\|_{H^2(\mathfrak{E}_*)} = \sum_{j=1}^{k} \Big(\int_{\partial \mathbf{D}} |\xi_j|^2 \|u_j\|_{\mathfrak{E}_*}^2 \, dm \Big)^{1/2} \leqq$$

$$\leqq \sum_{j=1}^{k} \|\xi_j\|_{H^2} \leqq k^{1/2} \Big\| \sum_{j=1}^{k} \xi_j e_j \Big\|_{H^2(\mathfrak{G})},$$

it follows that by the definition

$$W\Big(\sum_{j=1}^{k} \xi_j e_j\Big) = \sum_{j=1}^{k} \xi_j u_j, \quad \{\xi_j\}_{j=1}^{k} \subset H^2,$$

we obtain a bounded, linear operator, belonging to $\mathscr{I}(S^{(k)}, U_+)$. Now, in virtue of [9, Theorem I.4.1] the operator

$$A = PW \tag{14}$$

belongs to $\mathscr{I}(S^{(k)}, T)$.

We are going to prove that A is *injective* if c is sufficiently close to 1. First of all we observe that by (9) and (14)

$$XA = X_+ W \tag{15}$$

holds, hence the injectivity of A is a consequence of the injectivity of X_+W.

Let us assume that $X_+ W\big(\sum_{j=1}^{k} \xi_j e_j\big) = 0$, for a sequence $\{\xi_j\}_{j=1}^{k} \subset H^2$. On account of (10) this means that for a.e. $\zeta \in \partial\mathbf{D}$ we have

$$0 = \Big(X_+ W \sum_{j=1}^{k} \xi_j e_j\Big)(\zeta) = -\Delta_*(\zeta) \sum_{j=1}^{k} \xi_j(\zeta) u_j(\zeta) =$$

$$= -\Big[\sum_{i=1}^{k} \delta_i(\zeta) h_i(\zeta) \otimes h_i(\zeta)\Big] \sum_{j=1}^{k} \xi_j(\zeta) u_j(\zeta) = -\sum_{i=1}^{k} \delta_i(\zeta) \Big(\sum_{j=1}^{k} \xi_j(\zeta) \langle u_j(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*}\Big) h_i(\zeta).$$

Making use of (11) we obtain that

$$\sum_{j=1}^{k} \xi_j(\zeta) \langle u_j(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*} = 0, \quad 1 \leqq i \leqq k, \tag{16}$$

for a.e. $\zeta \in \alpha_k$.

Let us introduce the operators $B(\zeta)$, $C(\zeta)$, $D(\zeta)$ ($\zeta \in \partial\mathbf{D}$) acting on $\mathfrak{G}$ such that their matrices $[b_{ij}(\zeta)]_{i,j=1}^{k}$, $[c_{ij}(\zeta)]_{i,j=1}^{k}$, $[d_{ij}(\zeta)]_{i,j=1}^{k}$, respectively, in the basis $\{e_j\}_{j=1}^{k}$ are of the following form:

$$b_{ij}(\zeta) = \langle u_j(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*}, \quad 1 \leqq i, j \leqq k,$$

$$c_{ij}(\zeta) = \begin{cases} b_{ij}(\zeta) & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases}$$

$$d_{ij}(\zeta) = \begin{cases} 0 & \text{if } i = j \\ -b_{ij}(\zeta) & \text{otherwise.} \end{cases}$$

By (13) we see that $|c_{jj}(\zeta)| = |b_{jj}(\zeta)| = |\langle u_j(\zeta), h_j(\zeta)\rangle_{\mathfrak{E}_*}| \geqq c$ a.e. ($1 \leqq j \leqq k$), hence $C(\zeta)$ is invertible and

$$\|C(\zeta)^{-1}\| \leqq c^{-1} \quad \text{a.e..} \tag{17}$$

On the other hand, if $i \neq j$ then by (12) and (13)

$$|d_{ij}(\zeta)| = |b_{ij}(\zeta)| = |\langle u_j(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*}| = |\langle u_j(\zeta) - \langle u_j(\zeta), h_j(\zeta)\rangle_{\mathfrak{E}_*} h_j(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*}| \leqq$$
$$\leqq \|u_j(\zeta) - \langle u_j(\zeta), h_j(\zeta)\rangle_{\mathfrak{E}_*} h_j(\zeta)\|_{\mathfrak{E}_*} = [\|u_j(\zeta)\|^2_{\mathfrak{E}_*} - |\langle u_j(\zeta), h_j(\zeta)\rangle_{\mathfrak{E}_*}|^2]^{1/2} \leqq (1-c^2)^{1/2},$$

and so

$$\|D(\zeta)\| \leqq \sum_{i=1}^{k} \Big(\sum_{j=1}^{k} |d_{ij}(\zeta)|^2\Big)^{1/2} \leqq (1-c^2)^{1/2} k^{3/2} \quad \text{a.e..} \tag{18}$$

Consequently, if c satisfies

$$1 > c > k^{3/2}(k^3+1)^{-1/2}, \tag{19}$$

then $k^{3/2}(1-c^2)^{1/2} < c$, and by the inequalities (17), (18) we infer $\|D(\zeta)\| < \|C(\zeta)^{-1}\|^{-1}$. Then the operator $B(\zeta) = C(\zeta) - D(\zeta) = C(\zeta)[I - C(\zeta)^{-1} D(\zeta)]$ will be invertible and

$$\|B(\zeta)^{-1}\| \leqq \|C(\zeta)^{-1}\| (1 - \|C(\zeta)^{-1}\| \|D(\zeta)\|)^{-1} \leqq \tag{20}$$
$$\leqq c^{-1}\big(1 - k^{3/2}(1-c^2)^{1/2} c^{-1}\big)^{-1} = \big(c - k^{3/2}(1-c^2)^{1/2}\big)^{-1} \quad \text{a.e..}$$

Since the matrix of $B(\zeta)$ coincides with the matrix of the system of equations (16), it follows that $\xi_j(\zeta) = 0$ for every $1 \leqq j \leqq k$ and for a.e. $\zeta \in \alpha_k$. But α_k is of positive measure and the functions ξ_j are from the Hardy class H^2, so we conclude that $\xi_j = 0$, for every $1 \leqq j \leqq k$.

Therefore, taking into consideration (15) we obtain that under the assumption (19) the operator $A \in \mathcal{I}(S^{(k)}, T)$ defined before is injective.

2) To prove that $S^{(k)}$ can be completely injected into T it is enough to show that for any non-zero vector h in $\mathfrak{H}$ the injection $A \in \mathcal{I}(S^{(k)}, T)$ can be chosen in such a way that h is not orthogonal to the range of A.

Let us be given first $0 \neq f \in H^2(\mathfrak{E}_*)$ and $g \in (\Delta L^2(\mathfrak{E}))^-$ such that $f \oplus g \in \mathfrak{H}$. Our Lemma ensures the existence of a function $u_1 \in H^2(\mathfrak{E}_*)$ for which beyond (12) and (13) even the relation $\langle u_1, f\rangle_{H^2(\mathfrak{E}_*)} \neq 0$ holds. In this case $\langle Ae_1, f \oplus g\rangle_{\mathfrak{H}} = \langle P(u_1 \oplus 0), f \oplus g\rangle_{\mathfrak{K}_+} = \langle u_1 \oplus 0, P(f \oplus g)\rangle_{\mathfrak{K}_+} = \langle u_1 \oplus 0, f \oplus g\rangle_{\mathfrak{K}_+} = \langle u_1, f\rangle_{H^2(\mathfrak{E}_*)} \neq 0$, i.e. $f \oplus g$ is not orthogonal onto ran A.

Let us assume now that $0 \neq g \in \mathfrak{H} \cap (\Delta L^2(\mathfrak{E}))^-$. Let $\lambda > 1$ be a real number such that the set $\alpha = \{\zeta \in \partial\mathbf{D}: \lambda^{-1} < \|g(\zeta)\|_{\mathfrak{E}} < \lambda\}$ is of positive measure. Let $\varrho > 0$ be arbitrary and let us consider the functions $\{u_j\}_{j=1}^k \subset H^2(\mathfrak{E}_*)$ occuring in the first part of the proof. Since for any $\xi_1 \in H^2$ we have

$$\|\xi_1(u_1 \oplus \varrho\chi_\alpha g)\|_{\mathfrak{K}_+} = \Big(\int_{\partial\mathbf{D}} |\zeta_1|^2 (\|u_1\|^2_{\mathfrak{E}_*} + \varrho^2 \chi_\alpha \|g\|^2_{\mathfrak{E}})\, dm\Big)^{1/2} \leqq (1 + \varrho^2\lambda^2)^{1/2} \|\xi_1\|_{H^2},$$

it follows that the definition

$$W_\varrho\Big(\sum_{j=1}^{k} \xi_j e_j\Big) = \xi_1(u_1 \oplus \varrho\chi_\alpha g) + \sum_{j=2}^{k} \xi_j u_j \quad (\{\xi_j\}_{j=1}^k \subset H^2)$$

gives a bounded linear operator ($\|W_\varrho\| \leqq k^{1/2}(1+\varrho^2\lambda^2)^{1/2}$) belonging to $\mathcal{I}(S^{(k)}, U_+)$. We define $A_\varrho \in \mathcal{I}(S^{(k)}, T)$ by $A_\varrho = PW_\varrho$. Since $XA_\varrho = X_+W_\varrho$, the injectivity of A_ϱ is again implied by the injectivity of X_+W_ϱ.

For any $\{\xi_j\}_{j=1}^k \subset H^2$ we have

$$X_+W_\varrho\Big(\sum_{j=1}^k \xi_j e_j\Big)(\zeta) = X_+\Big(\sum_{j=1}^k \xi_j u_j \oplus \xi_1 \varrho \chi_\alpha g\Big)(\zeta) =$$

$$= -\Delta_*(\zeta) \sum_{j=1}^k \xi_j(\zeta) u_j(\zeta) + \Theta(\zeta)\xi_1(\zeta)\varrho\chi_\alpha(\zeta) g(\zeta) =$$

$$= \sum_{i=1}^k \Big[-\delta_i(\zeta) \sum_{j=1}^k \xi_j(\zeta)\langle u_j(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*} + \xi_1(\zeta)\varrho\langle \Theta(\zeta)\chi_\alpha(\zeta)g(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*}\Big] h_i(\zeta) \quad \text{a.e.}.$$

Hence $X_+W_\varrho\big(\sum_{j=1}^k \xi_j e_j\big) = 0$ yields that

(21)

$$\delta_i(\zeta) \sum_{j=1}^k \xi_j(\zeta)\langle u_j(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*} - \xi_1(\zeta)\varrho\langle \Theta(\zeta)\chi_\alpha(\zeta)g(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*} = 0, \quad 1 \leqq i \leqq k,$$

holds for a.e. $\zeta \in \alpha_k$.

Let $E_\varrho(\zeta)$ ($\zeta \in \alpha_k$) stand for the operator acting on $\mathfrak{G}$ with matrix in the basis $\{e_j\}_{j=1}^k$ of the form

$$e_{ij}^{(\varrho)}(\zeta) = \begin{cases} \varrho\delta_i(\zeta)^{-1}\langle \Theta(\zeta)\chi_\alpha(\zeta)g(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*} & \text{if } j = 1 \\ 0 \quad \text{otherwise.} \end{cases}$$

By (11) we infer that

$$|e_{i1}^{(\varrho)}(\zeta)| = \varrho|\delta_i(\zeta)|^{-1}|\langle \Theta(\zeta)\chi_\alpha(\zeta)g(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*}| \leqq$$
$$\leqq \varrho|\delta_i(\zeta)|^{-1}\chi_\alpha(\zeta)\|g(\zeta)\|_{\mathfrak{E}} \leqq \varrho\lambda|\delta_k(\zeta)|^{-1}$$

is true for every $1 \leqq i \leqq k$ and a.e. $\zeta \in \alpha_k$, whence

$$\|E_\varrho(\zeta)\| = \Big(\sum_{i=1}^k |e_{i1}^{(\varrho)}(\zeta)|^2\Big)^{1/2} \leqq k^{1/2}\varrho\lambda|\delta_k(\zeta)|^{-1} \quad \text{a.e. on} \quad \alpha_k. \tag{22}$$

Let us consider a Borel set $\beta \subset \alpha_k$ of positive measure and a positive number $\lambda' > 0$ such that $|\delta_k(\zeta)|^{-1} \leqq \lambda'$ for a.e. $\zeta \in \beta$. Let us assume that the functions $\{u_j\}_{j=1}^k$ correspond to a number c satisfying (19). Now, if $\varrho > 0$ fulfils the inequality

$$\varrho k^{1/2}\lambda\lambda' < c - k^{3/2}(1-c^2)^{1/2}, \tag{23}$$

then by (20) and (22) we obtain that $\|E_\varrho(\zeta)\| < \|B(\zeta)^{-1}\|^{-1}$ and so $B_1(\zeta) = B(\zeta) - E(\zeta)$ is invertible a.e. on β. In view of (21) we infer that $\xi_j(\zeta) = 0$ $(1 \leqq j \leqq k)$ a.e. on β, and since $m(\beta) > 0$ that $\xi_j = 0$ $(1 \leqq j \leqq k)$.

Therefore, the operator $A_\varrho \in \mathcal{I}(S^{(k)}; T)$ defined before will be an injection whenever c and $\varrho > 0$ satisfy the inequalities (19) and (23), respectively. At the same time we have

$$\langle A_\varrho e_1, 0 \oplus g\rangle_{\mathfrak{H}} = \langle P(u_1 \oplus \varrho\chi_\alpha g), 0\oplus g\rangle_{\mathfrak{K}_+} = \langle u_1 \oplus \varrho\chi_\alpha g, P(0\oplus g)\rangle_{\mathfrak{K}_+} =$$

$$= \langle u_1 \oplus \varrho\chi_\alpha g, 0\oplus g\rangle_{\mathfrak{K}_+} = \langle \varrho\chi_\alpha g, g\rangle_{L^2(\mathfrak{E})} = \varrho \Big(\int_\alpha \|g\|_{\mathfrak{E}}^2 \, dm\Big)^{1/2} \geqq \varrho\lambda^{-1} m(\alpha)^{1/2} > 0,$$

i.e. g is not orthogonal to ran A_ϱ.

According to [7, Theorem 5], if T is a contraction of class $C_{\cdot 0}$ with finite defect indices d_T, d_{T^*} and if $S^{(k)} \overset{i}{\prec} T$, then $k \leqq d_{T^*} - d_T = \mu_{*,T}$. Hence, under the assumptions of Theorem 0, $\mu_{*,T}$ is the maximum of the multiplicities of those unilateral shifts which can be completely injected into T. The following example shows that this statement fails if $d_{T^*} = \infty$.

Example. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint Borel subsets of $\partial\mathbf{D}$ of positive measure. For every n, let T_n be a contraction of class C_{10} such that rank $\Delta_{*,T_n} = \chi_{\alpha_n}$ a.e. (cf. [4]). Then the orthogonal sum $T = \bigoplus_{n=1}^{\infty} T_n$ is also of class C_{10} with rank $\Delta_{*,T} = \text{rank} \bigoplus_{n=1}^{\infty} \Delta_{*,T_n} = \chi_{\bigcup_{n=1}^{\infty} \alpha_n}$ a.e., whence $\mu_{*,T} = 1$. By our Theorem $S \overset{\text{c.i.}}{\prec} T_n$ for every n, which results in that $S^{(\infty)} \overset{\text{c.i.}}{\prec} T$.

References

[1 H. Bercovici and K. Takahashi, On the reflexivity of contractions on Hilbert space, *J. London Math. Soc.*, (2) **31** (1985).

[2] J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien*, Gauthier—Villars (Paris, 1969).

[3] L. Kérchy, On the commutant of C_{11}-contractions, *Acta Sci. Math.*, **43** (1981), 15—26.

[4] L. Kérchy, On the spectra of contractions belonging to special classes, *J. Funct. Analysis*, **67** (1986), 153—166.

[5] L. Kérchy, On the residual parts of completely non-unitary contractions, *Acta Math. Acad. Sci. Hungar.*, **50** (1987), 127—145.

[6] L. Kérchy, Invariant subspaces of $C_{1\cdot}$-contractions with non-reductive unitary extensions, *Bull. London Math. Soc.*, **19** (1987), 161—166.

[7] B. Sz.-Nagy and C. Foias, Jordan model for contractions of class $C_{\cdot 0}$, *Acta Sci. Math.*, **36** (1974), 305—322

[8] B. Sz.-Nagy and C. Foiaş, Injection of shifts into strict contractions, *Linear operators and approximation. II*, Proc. M. R. I. Oberwolfach, Birkhäuser (Basel, 1974), 29—37.

[9] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*, North Holland — Akadémiai Kiadó (Amsterdam—Budapest, 1970).

[10] K. Takahashi, On quasi-affine transforms of unilateral shifts, *Proc. Amer. Math. Soc.*, **100** (1987), 683—687.
[11] P. Y. Wu, Contractions quasisimilar to an isometry, *Acta Sci. Math.*, **53** (1989), 139—145.
[12] C. Apostol, H. Bercovici, C. Foias and C. Pearcy, Quasiaffine transforms of operators, *Michigan Math. J*, 29 (1982), 243—255.

BOLYAI INSTITUTE
UNIVERSITY SZEGED
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

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On commutativity and spectral radius property of real generalized *-algebras

ZOLTÁN MAGYAR

Introduction. Let A denote a Banach algebra over the real field throughout this paper. Of course, a complex algebra is a real algebra as well, although the spectra will change (cf. [1], p. 70). Assume we have a linear operation $a \to a^*$ on A with the properties

(i) $a^{**}=a$,
(ii) $(ab)^*=b^*a^*$.

Then A is called a *-algebra. If we replace (ii) by

(ii') $(ab)^*=a^*b^*$

then we call A an auto-*-algebra. We say A is a generalized *-algebra if A is either a *-algebra or an auto-*-algebra (cf. [4], [6]). In such an algebra let

$$A_H = \{a \in A;\ a = a^*\},\quad A_J = \{a \in A;\ a = -a^*\},\quad A_N = \{a \in A;\ aa^* = a^*a\}.$$

We call the elements of A_H, A_J and A_N self-adjoint, skew-adjoint and normal, respectively.

In [6] A is called Hermitian if each self-adjoint element has purely real spectrum and A is called skew Hermitian if the spectra of the skew-adjoint elements do not contain any non-zero real number (the spectrum is defined as follows: a complex number z belongs to $\mathrm{Sp}(x)$ if and only if $z \cdot 1 - x$ is not invertible in A_1, where A_1 is the complexification, and unitization if necessary, of A, see [1], p. 70). None of these properties implies the other one as simple examples show. This is a marked difference from the complex case.

We shall retain the above definition of skew Hermitianness but *we shall call A Hermitian if both properties are satisfied.*

Our main results then:

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Theorem 1. *If A is an Hermitian Banach auto-*-algebra then $A/\mathrm{rad}\ A$ is commutative.*

Theorem 2. *If A is an Hermitian Banach generalized *-algebra then $r(a^*a) \geqq r(a)^2$ for any $a \in A$, where r denotes the spectral radius.*

Remarks. If the *-operation is the identical mapping then Theorem 1 reduces to a theorem of I. KAPLANSKY (see Thm. 8 in [2]) and, indeed, that result is the starting point of our proof. We should emphasize that the significance of Kaplansky's theorem for Hermitian auto-*-algebras was first pointed out by T. W. PALMER in [4], though [4] contains a wrong proof assuming the unitary elements form a group, which is not true in an auto-*-algebra. On the other hand, the authors of [6] simply overlooked that the proof of their key Gelfand—Naimark type theorem (Theorem 2.3 in [6]) does not work for auto-*-algebras. Now our Theorem 1 implies that all results of [4] and [6] are true.

Finally we shall include a version of Theorem 2 which answers a question in [6] (see Proposition 3 below).

To prove our theorems we shall need the following simple lemmas.

Lemma 1. *If A is skew Hermitian then every skew-adjoint element has purely imaginary spectrum.*

Proof. Suppose to the contrary that $a \in A_J$, $z \in \mathrm{Sp}\,(a)$ and z is not imaginary. Then z can not be real, since A is skew Hermitian, and hence z^2 is not real. Thus z and z^3 are linearly independent over $\mathbf{R}$, and hence there are $s, t \in \mathbf{R}$ such that $sz + tz^3 = 1$. Then $\mathrm{Sp}\,(sa+ta^3) \ni 1$, while $sa+ta^3$ is skew-adjoint; this is a contradiction.

Lemma 2. *If A is an auto-*-algebra then*

$$\mathrm{rad}\ A_H = A_H \cap \mathrm{rad}\ A$$

(see [1] for the concept of the Jacobson-radical).

Proof. The containment "$\supset$" follows at once from the "quasi-inverse-characterization" of the radical (see [1], p. 125).

Prove "$\subset$". Consider an element $a \in \mathrm{rad}\ A_H$, and an irreducible representation p of A over the real vector space X. Then we have to show $p(a)=0$ (if this is true for all p then $a \in \mathrm{rad}\ A$). If p is irreducible for A_H too, then we are done. If p is not irreducible then for any non-trivial A_H-invariant subspace M set

$$M' := \text{the linear span of } p(A_J)M.$$

Then the relations $A_H A_J \subset A_J$, $A_J A_J \subset A_H$ imply $p(A_H)M' \subset M'$ and $p(A_J)M' \subset M$. Hence $M+M'$ and $M \cap M'$ are invariant for $A_H + A_J = A$, and therefore $X = M \oplus$

$\oplus M'$. Now if M were not irreducible then one could find a non-trivial A_H-invariant subspace L in M, on the other hand, $X=L\oplus L'$ and clearly $L'\subset M'$, which is a contradiction. The same is true for M' since it is another invariant subspace. So we see $p|_{A_H}$ is a direct sum of two irreducible representations and hence $p(a)=0$.

Lemma 3. *Let A be an auto-*-algebra. Then*

$$\mathrm{Sp}_{A_H}(h) = \mathrm{Sp}_A(h) \quad \textit{for any} \quad h\in A_H.$$

Proof. If a self-adjoint element has a quasi-inverse (or inverse) in A, then this quasi-inverse (or inverse) is self-adjoint, too. Thus we get our statement using the well-known characterization of the spectrum (see [1], p. 70).

Lemma. 4. *Factorization by the radical does not effect the spectra except possibly for the number 0 in them.*

Proof. Use the "quasi-inverse-characterization" of the radical (see [1], p. 125) and the fact if x has a left- and a right-quasi-inverse then x is quasi-invertible.

Proof of Theorem 1. First observe that the "*" preserves the radical (use the characterizations of the radical from [1]). Hence $A/\mathrm{rad}\, A$ is a Banach auto-*-algebra and it is Hermitian by Lemma 4. Thus we can assume A is semi-simple. In this case A_H is semi-simple, too, by Lemma 2. If $\|a\|':=\|a^*\|$ then $\|\cdot\|'$ is another Banach algebra norm, hence by Johnson's theorem the two norms are equivalent (see [1], p. 130 for the proof of Johnson's theorem). Thus A_H is closed. Using Lemma 4 we see A_H is a semi-simple Banach algebra in which every element has purely real spectrum. This implies, by Theorem 8 of [2], that

(1) $$A_H \text{ is commutative.}$$

Let $h\to\hat{h}$ be the Gelfand transform on A_H. It is injective, because A_H is semi-simple. Next we will show

(2) $$\text{if } j\in A_J \text{ and } j^2=0 \text{ then } j=0.$$

Consider a fixed $j\in A_J$ for which $j^2=0$. Let $k\in A_J$ be arbitrary and $r\in \mathbf{R}$. Since A is skew Hermitian, thus $\mathrm{Sp}\,(rj+k)$ is imaginary, and hence, using Lemma 3, we have $0\geqq\widehat{(rj+k)^2}=r\widehat{(jk+kj)}+\widehat{k^2}$, for $j^2=0$. This is true for any r, therefore $\widehat{jk+kj}=0$, $jk+kj=0$. Thus $(jk)^2=j(-jk)k=0$, which implies $\widehat{(jk)}=0$, $jk=0$. Since $jk+kj=0$, we have $jk=kj=0$ for any $k\in A_J$. Now let $a\in A$ be arbitrary, and $h=(1/2)(a+a^*)$, $k=(1/2)(a-a^*)$. Then $aj=(h+k)j=hj\in A_J$, and therefore $jaj=0$, $(aj)^2=0$. We get from this $\mathrm{Sp}\,(aj)=\{0\}$ for each $a\in A$, and hence $j=0$ for A is semi-simple.

Next we want to show that

(3) $$khk=k^2h \quad \text{for any} \quad h\in A_H,\ k\in A_J.$$

Let $g=hk-kh$. Since $k^2\in A_H$, thus $k^2h=hk^2$, and hence $gk=-kg$. Therefore $(kg)^2=k\cdot(-kg)\cdot g$ and hence $\widehat{(kg)}^2=-\widehat{k^2}\widehat{g^2}$. Since $k, g\in A_J$ and $kg\in A_H$, thus k^2, g^2 have non-positive real spectra, while $(kg)^2$ has non-negative spectrum. Thus we can infer $\widehat{kg}=0$, $kg=0$, which is exactly (3).

Now we will prove that

$$(4) \qquad kh = hk \quad \text{for any} \quad h\in A_H, \quad k\in A_J.$$

Let $g=hk-kh$. Then $g^2=hkhk-hk^2h+khkh-kh^2k=0$ (use (3) for h, k in the 1st and 3rd term, and for h^2, k in the 4th term). Thus, by (2), we get $g=0$.

Finally, we will show that

$$(5) \qquad jk = kj \quad \text{for any} \quad j, k\in A_J.$$

Since $jk, kj\in A_H$, thus, by (4), $jkj=j^2k$ and $kjk=k^2j$; therefore $0=kjkj-kj^2k+$ $+jkjk-jk^2j$, in other words, $m^2=0$ where $m=kj-jk\in A_H$. Thus $\hat{m}=0$ and (5) is proved.

The theorem is proved by uniting (2), (4) and (5).

Remark. Since the complex radical of a complex algebra is the same as the real radical (cf. [1]), therefore Theorem 1 is valid for complex algebras, too. Of course, one should check that a complex Hermitian algebra is Hermitian in our sense as a real algebra. This follows from the fact if S is the complex spectrum of an element then the "real spectrum" is the set $S\cup\bar{S}$.

Proof of Theorem 2. By Lemma 4 we may again assume A is semi-simple. But then, by Theorem 1, A is a *-algebra anyway. So let A be an Hermitian Banach *-algebra. Let $p(x):=r(x^*x)^{1/2}$ for all $x\in A$. Now A satisfies the conditions of Lemma 3.1 from [6], therefore we can infer

$$(6) \qquad p \text{ is an algebra-seminorm on } A.$$

The proof of Lemma 41.2 in [1] (see p. 225) yields in the real case that

$$(7) \qquad \text{if} \quad 1\in \mathrm{Sp}\,(a) \quad \text{then} \quad p(a)\geqq 1.$$

We assert that

$$(8) \qquad 3p(a)\geqq r(a) \quad \text{for all} \quad a\in A.$$

If $r(a)=0$ then this is clear. If $r(a)>0$ then let $b=r(a)^{-1}a$. We can choose a $z\in \mathrm{Sp}\,(b)$ such that $|z|=1$. Let $c=(z+\bar{z})b-b^2$. Then $1=(z+\bar{z})z-z^2\in \mathrm{Sp}\,(c)$, and hence, by (7) and (6), we have

$$1 \leqq p(c) \leqq |z+\bar{z}|p(b)+p(b)^2 \leqq (2+p(b))\cdot p(b), \quad \text{thus} \quad p(b)\geqq 1/3$$

and (8) is proved.

Applying (8) to a^n we get $r(a)^n = r(a^n) \leqq 3p(a^n)$. Now use the submultiplicativity of p and tend with n to infinity. The theorem is proved.

Remark. Differently from the complex case (cf. [5]), $r(a^*a) \geqq r(a)^2$ does not imply A is Hermitian; e.g., if $A = \mathbf{C}$ (considered as a real algebra) and the * is the identical mapping then $r(a^*a) = r(a)^2$ for all a but A is not Hermitian.

Proposition 3. *Let A be a skew Hermitian Banach generalized *-algebra. Then $r(a^*a) = r(a)^2$ for any normal element a.*

Proof. Let $a \in A_N$ be fixed. Let B be the second commutant of the set $\{a, a^*\}$. Then B is a Banach algebra, closed under the involution and $\mathrm{Sp}_B(b) = \mathrm{Sp}_A(b)$ for any $b \in B$. Further, B is commutative for a is normal. Let f be a multiplicative linear functional on B. Let $f(a) = u$, $f(a^*) = v$. Since A is skew Hermitian, thus, by Lemma 1, $a - a^*$ and $a^2 - (a^*)^2$ both have imaginary spectrum, and hence $u - v$ and $u^2 - v^2$ are imaginary numbers. Thus if $u \neq v$ then $u + v$ is real and $v = \bar{u}$. In any case $|v| = |u|$, and hence $|f(a^*a)| = |f(a)|^2$. This is true for any multiplicative linear functional f on B, therefore $r(a^*a) = r(a)^2$.

References

[1] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer (Berlin—Heidelberg—New York, 1973).

[2] I. Kaplansky, Normed algebras, *Duke Math. J.*, **16** (1949), 399—418.

[3] S. Shirali and J. W. M. Ford, Symmetry in complex involutory Banach algebras, II. *Duke Math. J.*, **37** (1970), 275—280.

[4] T. W. Palmer, Real C^*-algebras, *Pacific J. Math.*, **35** (1970), 195—204.

[5] V. Pták, Banach algebras with involution, *Manuscripta Math.*, **6** (1972), 245—290.

[6] S. H. Kulkarni and B. V. Limaye, Gelfand-Naimark theorems for real Banach *-algebras, *Math. Jap.*, **25** (1980), 545—558.

MATHEMATICAL INSTITUTE OF THE HUNGARIAN ACADEMY OF SCIENCES
REÁLTANODA U. 13—15
1053 BUDAPEST, HUNGARY

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A characterization of (real or complex) Hermitian algebras and equivalent C^*-algebras

ZOLTÁN MAGYAR

0. Introduction

We use the symbol $\mathbf{F}$ to denote a field that is either the real field $\mathbf{R}$ or the complex field $\mathbf{C}$. We call an algebra A over $\mathbf{F}$ a *-algebra if there is a conjugate linear mapping "$*$" from A into A satisfying

(i) $(ab)^* = b^*a^*$ for all $a, b \in A$,

(ii) $(a^*)^* = a$ for all $a \in A$.

We call A an auto-*-algebra if we replace the axiom (i) by the axiom

(i′) $(ab)^* = a^*b^*$ for all $a, b \in A$.

We call A a generalized *-algebra if A is a *-algebra or an auto-*-algebra. An element $a \in A$ is called self-adjoint, if $a = a^*$, skew-adjoint, if $a = -a^*$; and normal, if $aa^* = a^*a$. Denote by A_H, A_J and A_N the sets of all self-adjoint, skew-adjoint and normal elements, respectively.

We will treat Banach generalized *-algebras, that are generalized *-algebras with complete algebra norm. We define the spectrum of an element with respect to an algebra containing it as in [1] (see pp. 19—20 and 70). Then it is known that

$$\max\{|z|;\ z \in \mathrm{Sp}(A, a)\} = \inf_n \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^n\|^{1/n}$$

if $\|\,.\,\|$ is a complete algebra norm on A. We write in this case

$$r(a) := \inf_n \|a^n\|^{1/n}.$$

Let A be a Banach generalized *-algebra. A is called Hermitian if $\mathrm{Sp}(A, a) \subset \mathbf{R}$ for all $a \in A_H$, and skew-Hermitian if $\mathrm{Sp}(A, a) \subset i \cdot \mathbf{R}$ for all $a \in A_J$. Every Hermitian

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algebra over $\mathbf{C}$ is automatically skew-Hermitian, of course. But this assertion is not true for real algebras. We will prove that a real Banach generalized *-algebra A is Hermitian *and* skew-Hermitian if and only if its complexification $A_{\mathbf{C}}$ (see [1] pp. 68—69) is Hermitian (see Theorem 3 below).

We remark that there is an equivalent, but formally weaker, definition of the skew-Hermitian property demanding only $1 \notin \mathrm{Sp}(A, a)$ for all $a \in A_J$. It is not very hard to see that if $\mathrm{Sp}(A, a) \not\subset i \cdot \mathbf{R}$ for some $a \in A_J$ then there are $s, t \in \mathbf{R}$ such that $\mathrm{Sp}(A, sa+ta^3) \ni 1$, and $sa+ta^3 \in A_J$.

A is called a C^*-algebra, if it is isometrically *-isomorphic to a norm-closed *-subalgebra of the Banach *-algebra $B(\mathfrak{H})$ of all bounded $\mathbf{F}$-linear operators on some Hilbert space $\mathfrak{H}$ over $\mathbf{F}$. A is called an equivalent C^*-algebra, if it is homeomorphically *-isomorphic to some C^*-algebra. We will give a characterization of equivalent C^*-algebras in Theorem 1 below, which is a generalization of a result of Pták (see [4]).

We will prove the following characterization of Hermitian and skew-Hermitian algebras: A is Hermitian and skew Hermitian if and only if there is such a *-homomorphism π of A into some $B(\mathfrak{H})$ which preserves the spectral radius (see Theorem 2). In contrast to a lot of characterizations of complex Hermitian algebras, this is valid for real algebras, too.

Our results are based on the following lemma:

Lemma 0.1. *Let A be a Hermitian and skew-Hermitian Banach generalized *-algebra over $\mathbf{F}$. Then there is a Hilbert space $\mathfrak{H}$ over $\mathbf{F}$ and a *-homomorphism π: $A \mapsto B(\mathfrak{H})$ such that $\|\pi(a)\| = r(a^*a)^{1/2}$ for all $a \in A$. Moreover, $r(a) \leqq \|\pi(a)\|$ for all $a \in A$, and* $\mathrm{rad}(A) = \pi^{-1}(\{0\})$. *If A has a unit then π can be chosen so that $\pi(1) = 1$.*

Proof. First we suppose that A is a *-algebra. Let

$$A_p = \{a \in A_H;\ \mathrm{Sp}(A, a) \subset \mathbf{R}_+\}.$$

Then it is known that A_p is a cone and $a^*a \in A_p$ for all $a \in A$ (see [5]). This is also true for the unitization $A+\mathbf{F}$ of A, since $A+\mathbf{F}$ is Hermitian and skew-Hermitian as well. Thus it is not hard to see that we can find for any fixed $a \in A$ a self-adjoint positive functional such that $f(1)=1$ and $f(a^*a)=r(a^*a)$ so that the customary GNS-construction gives us a Hilbert space $\mathfrak{H}$ and a *-homomorphism π of A satisfying $\|\pi(a)\| = r(a^*a)^{1/2}$ for all $a \in A$. (For more detailed description see [2], Lemma 3.1 and [1] § 37. See also [4] for another proof in case $\mathbf{F}=\mathbf{C}$.)

Since $\mathrm{rad}(A) = \{a \in A;\ r(qa)=0$ for every $q \in A\}$ (see [1] p. 126), it is clear that $\mathrm{rad}(A) \subset N$, where $N := \pi^{-1}(\{0\})$. On the other hand, the author has proved in [3], that $r(a) \leqq r(a^*a)^{1/2}$ in a Hermitian and skew-Hermitian Banach *-algebra. Thus N is an ideal consisting of elements of spectrum $\{0\}$ whence $N \subset \mathrm{rad}(A)$. Moreover, we see that $r(a) \leqq \|\pi(a)\|$ for all $a \in A$.

Now we suppose A is an auto-*-algebra. Being a conjugate linear automorphism the "*" maps rad (A) onto itself. Let $B=A/\mathrm{rad}(A)$ and p be the canonical mapping $A\mapsto B$. Then it is known that

$$\mathrm{Sp}(A, a)\setminus\{0\} = \mathrm{Sp}(B, p(a))\setminus\{0\} \quad \text{for all} \quad a\in A. \tag{1}$$

(It is not hard to deduce this fact from Proposition 24.16. (i), p. 125 in [1].)

Therefore B is a Hermitian and skew-Hermitian Banach auto-*-algebra. Moreover, B is semisimple (see [1] p. 126). Thus, by a result of the author (see [3]), B is commutative, and hence B is a *-algebra. Therefore we have a representation π_1 of B satisfying the statements of our lemma, and so by (1) $\pi:=\pi_1\circ p$ is a representation we asked.

1. A characterization of equivalent C*-algebras

Lemma 1.1. *Let A be a Banach-algebra over $\mathbf{F}$, and let g be an entire function on $\mathbf{C}$, satisfying $g'(0)\neq 0$. Further in case $\mathbf{F}=\mathbf{R}$ we assume that the Taylor-series of g at zero has only real coefficients. Then there is a function f: $\mathbf{R}_+\mapsto\mathbf{R}_+$ so that $\|x\|^2\leqq f(c)\cdot\|x^2\|$ whenever x is such that $\|g(tx)\|\leqq c$ for all $t\in\mathbf{R}_+$. ($g(a)$ may be in the unitization $A+\mathbf{F}$ of A, if A does not have a unit. We fix a norm on $A+\mathbf{F}$ in that case.)*

Proof. Let $g(z)=\sum_{n=0}^{\infty}\alpha_n z^n$. If $h(z)=\sum_{n=2}^{\infty}|\alpha_n|\cdot z^n$ then h is an entire function, too. Suppose that $\|g(tx)\|\leqq c$ for all $t\in\mathbf{R}_+$ for some $x\in A$ and $c\in\mathbf{R}_+$. We can assume that $\|x\|=1$ because both sides of the inequality $\|x\|^2\leqq f(c)\cdot\|x^2\|$ are multiplied by $|\lambda|^2$ when we replace x by λx, and the case $x=0$ is trivial. Then let $p=\|x^2\|^{1/3}$, thus we see that $p\leqq 1$ and $\|x^n\|\leqq(p^3)^{[n/2]}\leqq p^n$ for all $n\geqq 2$. Hence we have for all $t\in\mathbf{R}_+$

$$t = \|tx\| = |\alpha_1|^{-1}\cdot\Big\|g(tx)-\alpha_0\cdot 1-\sum_{n=2}^{\infty}\alpha_n t^n x^n\Big\| \leqq |\alpha_1|^{-1}\cdot\big(c+|\alpha_0|\cdot\|1\|+h(tp)\big).$$

Hence $p\neq 0$, and replace $t=p^{-1}$, we see that $p^{-1}\leqq\varphi(c)$, where $\varphi(c)=|\alpha_1|^{-1}\big(c+|\alpha_0|\cdot\|1\|+h(1)\big)$. Thus $\|x^2\|=p^3\geqq\varphi(c)^{-3}$, and so $f(c)=\varphi(c)^3$ satisfies our condition.

Lemma 1.2. *Let A and g be as in Lemma 1.1, and let $\langle x\rangle$ denote the real algebra generated by an element $x\in A$. Then the function f of Lemma 1.1 also satisfies $\|x\|\leqq f(c)\cdot r(x)$ whenever x is such that $\|g(a)\|\leqq c$ for all $a\in\langle x\rangle$.*

Proof. Assume that $\|g(a)\|\leqq c$ for all $a\in\langle x\rangle$ for some $x\in A$ and $c\in\mathbf{R}_+$. Then by Lemma 1.1 we have

$$\|a\|^2 \leqq f(c)\cdot\|a^2\| \quad \text{for all} \quad a\in\langle x\rangle.$$

Writing $a=x^{2^n}$, we can infer by induction that

$$\|x\|^{2^n} \leqq f(c)^{2^n-1} \cdot \|x^{2^n}\|$$

and hence, tending with n to infinity we get $\|x\| \leqq f(c) \cdot r(x)$.

Theorem 1. *Let A be a Banach generalized *-algebra over* **F**. *Then A is an equivalent C^*-algebra if and only if there is a constant C such that*

(i) $\|\sin(h)\| \leqq C$ *for all* $h \in A_H$ *and,*
(ii) $\|\sinh(k)\| \leqq C$ *for all* $k \in A_J$.

Remark. Of course, in case $\mathbf{F}=\mathbf{C}$ (i) is equivalent to (ii).

Proof. First we assume that A is an equivalent C^*-algebra. Then there is a norm p on A so that (A, p) is a C^*-algebra and a constant C such that $\|a\| \leqq C \cdot p(a)$ for all $a \in A$. It is known that a C^*-algebra is Hermitian, skew-Hermitian and its norm equals the spectral radius on normal elements (this is well known for $\mathbf{F}=\mathbf{C}$, and for $\mathbf{F}=\mathbf{R}$ we can canonically embed the subalgebra of $B(\mathfrak{H})$ into $B(\mathfrak{H}_\mathbf{C})$ where $\mathfrak{H}_\mathbf{C}$ is the complexification of the real Hilbert space $\mathfrak{H}$, and thus we can infer the statement). Therefore if $h \in A_H$ then $\mathrm{Sp}(A, h) \subset \mathbf{R}$, and so $\mathrm{Sp}(A, \sin(h)) \subset [-1, 1]$ (see [1], § 7), further $\sin(h) \in A_H$ for the * is norm-preserving in a C^*-algebra, and hence $p(\sin(h))=r(\sin(h)) \leqq 1$, $\|\sin(h)\| \leqq C \cdot p(\sin(h)) \leqq C$. Similarly, if $k \in A_J$ then $\mathrm{Sp}(A, k) \subset i \cdot \mathbf{R}$, $\mathrm{Sp}(A, \sinh(k)) \subset i \cdot [-1, 1]$, $\sinh(k) \in A_J$, and hence $\|\sinh(k)\| \leqq C$.

Now we assume that A satisfies (i) and (ii) with a suitable constant C. First we show that

(1) A is Hermitian and skew-Hermitian.

Observe that if $z \in \mathbf{C} \setminus \mathbf{R}$, then the set $\{\sin(tz);\ t \in \mathbf{R}\}$ is not bounded. This fact implies that $\{r(\sin(th));\ t \in \mathbf{R}\}$ is not bounded if $\mathrm{Sp}(A, h) \not\subset \mathbf{R}$, and similarly $\{r(\sinh(tk));\ t \in \mathbf{R}\}$ is not bounded if $\mathrm{Sp}(A, k) \not\subset i \cdot \mathbf{R}$ for $\sinh(z) = -i \cdot \sin(iz)$. Since $r(a) \leqq \|a\|$, thus (i) and (ii) clearly imply (1).

Now we want to show that

(2) there is a constant M such that $\|a\| \leqq M \cdot r(a)$ for all $a \in A_H \cup A_J$.

We have by Lemma 1.2 and (i) a constant m_1 such that

(3) $$\|a\| \leqq m_1 \cdot r(a) \quad \text{for all} \quad a \in A_H$$

and we have by Lemma 1.1 and (ii) a constant m_2 such that

(4) $$\|a\|^2 \leqq m_2 \|a^2\| \quad \text{for all} \quad a \in A_J.$$

But $a^2 \in A_H$ for $a \in A_J$, thus $\|a^2\| \leqq m_1 \cdot r(a^2) = m_1 \cdot r(a)^2$, and hence (2) is true with $M = \max(m_1, \sqrt{m_1 \cdot m_2})$.

We can apply Lemma 0.1 to A because (1) holds; let π be the corresponding representation. Since $\|\pi(a)\|=r(a^*a)^{1/2}$, we have

(5) $$\|\pi(a)\| = r(a) \quad \text{for all} \quad a\in A_H\cup A_J,$$

and so by (2) we get $\|a\|\leqq M\cdot\|\pi(a)\|$ for all $a\in A_H\cup A_J$. Thus if a is an arbitrary element in A and $h=\dfrac{a+a^*}{2}$, $k=\dfrac{a-a^*}{2}$, then $\|a\|\leqq\|h\|+\|k\|\leqq M(\|\pi(h)\|+\|\pi(k)\|)$ and $\|\pi(h)\|\leqq\|\pi(a)\|$, $\|\pi(k)\|\leqq\|\pi(a)\|$ for the * is norm-preserving on $B(\mathfrak{H})$. Thus we get

(6) $$\|a\| \leqq 2M\cdot\|\pi(a)\| \quad \text{for all} \quad a\in A.$$

We have $\|\pi(a)\|^2=r(a^*a)\leqq\|a^*a\|\leqq\|a^*\|\cdot\|a\|$, and hence by (6) we infer $\|a\|\leqq \leqq 4M^2\cdot\|a^*\|$. Thus $\|a^*\|\leqq 4M^2\cdot\|a\|$ for $a^{**}=a$, and hence

(7) $$\|\pi(a)\|^2 \leqq 4M^2\cdot\|a\|^2 \quad \text{for all} \quad a\in A.$$

It follows from (6) and (7) that π is homeomorphic and $\pi(A)$ is complete. Therefore A is an equivalent C^*-algebra.

2. A characterization of Hermitian algebras

Lemma 2.1. *Let A and B be Banach generalized *-algebras over* $\mathbf{F}$. *Assume that $p\colon A\mapsto B$ is a *-homomorphism satisfying $r(h)\leqq r(p(h))$ for all $h\in A_H$. Then A is Hermitian (resp. skew-Hermitian) whenever B is.*

Remark. The condition $r(h)\leqq r(p(h))$ is equivalent to $r(h)=r(p(h))$ for $\mathrm{Sp}(B, p(h))\subset\mathrm{Sp}(A, h)\cup\{0\}$.

Proof. Suppose that A is not Hermitian (resp. skew-Hermitian) but B is. Then there is an element $h_1\in A_H$ (resp. $k_1\in A_J$) such that $\mathrm{Sp}(A, h_1)\not\subset\mathbf{R}$ (resp. $\mathrm{Sp}(A, k_1)\not\subset i\cdot\mathbf{R}$). If $z\in\mathbf{C}\setminus(\mathbf{R}\cup i\cdot\mathbf{R})$ then $z^2\notin\mathbf{R}$ and hence $\{tz+sz^3;\ t, s\in\mathbf{R}\}=\mathbf{C}$. This implies that there is an element $h\in\{th_1+sh_1^3;\ t, s\in\mathbf{R}\}\subset A_H$ (resp. $k\in\{tk_1+sk_1^3;\ t, s\in\mathbf{R}\}\subset A_J$) such that $i\in\mathrm{Sp}(A, h)$ (resp. $1\in\mathrm{Sp}(A, k)$). Let $c=h^2$ (resp. $c=-k^2$). Then

(1) $$-1\in\mathrm{Sp}(A, c) \quad \text{and} \quad c\in A_H.$$

Further, $p(c)=p(h)^2$ (resp. $p(c)=-p(k)^2$), p is a *-homomorphism, and B is Hermitian (resp. skew-Hermitian); thus we get

(2) $$\mathrm{Sp}(B, p(c))\subset\mathbf{R}_+.$$

Since A is a Banach-algebra, $\mathrm{Sp}(A, c)$ is bounded and hence there is a real number λ such that

(3) $$\lambda>1 \quad \text{and} \quad -\lambda^{-1}\cdot c \text{ has a quasi-inverse } d \text{ in } A.$$

Moreover, $d \in A_H$, because $-\lambda^{-1} \cdot c \in A_H$. Since p is homomorphic, thus $p(d)$ is the quasi-inverse of $-\lambda^{-1} \cdot p(c)$. It is known that if b is the quasi-inverse of a in an arbitrary algebra then $\{t(t-1)^{-1};\ t \in \mathrm{Sp}\,(a)\} = \mathrm{Sp}\,(b)$. (Sketch of the proof: b is the quasi-inverse of a if and only if $1-b$ is the inverse of $1-a$, where $1-a, 1-b \in \in A+\mathbf{F}$ if A does not have a unit in which case $\mathrm{Sp}\,(A, x) = \mathrm{Sp}\,(A+\mathbf{F}, x)$ for all $x \in A$; and hence it is easy to deduce the statement.) Thus we get from (1), (2) and (3) that

(4) there is a negative number (namely $(1-\lambda)^{-1}$) in $\mathrm{Sp}\,(A, d)$

and

(5) $$\mathrm{Sp}\,(B, p(d)) \subset [0, 1).$$

Consider the polynomials $P_n(X) = X(1-X)^n$. Then $P_n(d) \in A_H$, and since $\mathrm{Sp}\,(P_n(a)) = P_n(\mathrm{Sp}\,(a))$ in an arbitrary algebra, thus $r(P_n(d)) > 1$ for sufficient large n by (4), while $r(P_n(p(d))) < 1$ for all n by (5). Thus we have got a contradiction to the assumption of our lemma.

Lemma 2.2. *Let A and B be Banach algebras over* $\mathbf{F}$ *and* $p\colon A \mapsto B$ *be a homomorphism. Then the following conditions are equivalent:*

(i) $r(a) = r(p(a))$ *for all* $a \in A$,

(ii) $\partial \mathrm{Sp}\,(A, a) \subset \partial \mathrm{Sp}\,(B, p(a)) \cup \{0\}$ *for all* $a \in A$.

Proof. First we assume (ii). Let $a \in A$ be fixed and let S be the closed disc about zero in $\mathbf{C}$ with radius $r(p(a))$. Then $\partial\,\mathrm{Sp}\,(A, a) \subset S$, and $\mathrm{Sp}\,(A, a)$ is a bounded set in $\mathbf{C}$, thus $\mathrm{Sp}\,(A, a) \subset S$, $r(a) \leqq r(p(a))$. Therefore (i) holds, for $r(a) \geqq r(p(a))$ is true for any homomorphism p.

Now we assume (i). Fix an element $a \in A$ and a complex number $z \in \partial\,\mathrm{Sp}\,(A, a) \setminus \{0\}$. Suppose that $z \notin \partial\,\mathrm{Sp}\,(B, p(a))$. Since $\mathrm{Sp}\,(B, p(a)) \subset \mathrm{Sp}\,(A, a) \cup \cup \{0\}$, we get $z \notin \mathrm{Sp}\,(B, p(a))$. Choose a sequence of complex numbers $z_n \to z$ such that $z_n \notin \mathrm{Sp}\,(A, a)$. We may assume $z_n \neq 0$ for all n. If $\mathbf{F} = \mathbf{R}$ then let

$$u_n = |z_n|^{-2} \cdot (2 \cdot \mathrm{Re}\,(z_n) a - a^2) \quad \text{and} \quad u = |z|^{-2} \cdot (2 \cdot \mathrm{Re}\,(z) a - a^2),$$

while in case $\mathbf{F} = \mathbf{C}$ let

$$u_n = z_n^{-1} \cdot a \quad \text{and} \quad u = z^{-1} \cdot a.$$

Then we have by [1] (see p. 70):

(1) $$u_n \to u \text{ in } A \text{ and } p(u_n) \to p(u) \text{ in } B,$$

(2) $$u_n \text{ has a quasi-inverse in } A,$$

(3) $$u \text{ does not have a quasi-inverse in } A,$$

(4) $$p(u) \text{ has a quasi-inverse in } B.$$

Further on, u_n and u are polynomials of a, and hence there is a maximal commutative subalgebra A' of A containing u and u_n for all n, and similarly a maximal commutative subalgebra B' of B containing $p(A')$. By (3) there is a character φ on A' such that $\varphi(u)=1$. Thus $\varphi(u_n)\to 1$, and hence, denoting the quasi-inverse of u_n by v_n, $|\varphi(v_n)|\to\infty$. Therefore $r(v_n)\to\infty$ and thus (i) yields

$$r(p(v_n))\to\infty. \tag{5}$$

On the other hand, $1\notin \mathrm{Sp}(B', p(u))$, and hence there is an $\varepsilon>0$ such that $|\psi(p(u))-1|>\varepsilon$ for all characters ψ of B'. Thus if $\|p(u_n)-p(u)\|\leqq\varepsilon/2$ then $|\psi(p(u_n))-1|\geqq\varepsilon/2$ for all ψ, and $|\psi(p(u_n))|\leqq\|p(u_n)\|\leqq$constant, because $p(u_n)\to p(u)$. Hence $r_n:=\max\{|\lambda(\lambda-1)^{-1}|;\ \lambda\in\mathrm{Sp}(B,p(u_n))\}\not\to\infty$, while $r_n=r(p(v_n))$ for $p(v_n)$ is the quasi-inverse of $p(u_n)$. This contradiction to (5) proves our lemma.

Theorem 2. *Let A be a Banach generalized *-algebra over* $\mathbf{F}$. *Then the following conditions are equivalent:*

(i) *A is Hermitian and skew-Hermitian,*

(ii) *there is a Hilbert space $\mathfrak{H}$ and a *-homomorphism π: $A\mapsto B(\mathfrak{H})$ satisfying $\|\pi(a)\|=r(a^*a)^{1/2}$ for all $a\in A$,*

(iii) *there is a π as in* (ii) *and satisfying $r(\pi(a))=r(a)$ for all $a\in A$,*

(iv) *there is a π as in* (ii) *and satisfying*

$$\partial\,\mathrm{Sp}(A,a)\subset\partial\,\mathrm{Sp}(B(\mathfrak{H}),\pi(a))\cup\{0\}\quad \textit{for all}\quad a\in A.$$

Proof. First we prove (i)$\Rightarrow$(iii). Consider the homomorphism π obtained from Lemma 0.1. Then for any $a\in A$ $r(a)^n=r(a^n)\leqq\|\pi(a^n)\|$ for all n, and hence $r(a)\leqq$ $\leqq r(\pi(a))$, thus $r(a)=r(\pi(a))$.

Now we prove (ii)$\Rightarrow$(i). If $h\in A_H$ then $r(h)^2=r(h^2)=r(h^*h)=\|\pi(h)\|^2=r(\pi(h))^2$ and hence by Lemma 2.1 we get (i), because $B(\mathfrak{H})$ is Hermitian and skew-Hermitian.

Since (iii)$\Rightarrow$(ii) is trivial and (iii)$\Leftrightarrow$(iv) was proved in Lemma 2.2, the proof of Theorem 2 is complete.

3. Relation between real and complex Hermitian algebras

Lemma 3.1. *Let A and B be Banach-algebras with unit over* $\mathbf{F}$, *and p: $A\mapsto B$ be a homomorphism satisfying $p(1)=1$ and $r(p(a))=r(a)$ for all $a\in A$. Assume that* $\mathrm{Sp}(B,p(x))\subset\mathbf{R}\setminus\{0\}$ *for some $x\in A$. Then x is invertible in A.*

Proof. Since A is a Banach algebra with unit, there is a real number $\lambda>0$ so that $a=(\lambda+x^2)^{-1}$ exists in A. Then $p(a)=(\lambda+p(x)^2)^{-1}$, and hence $\mathrm{Sp}(B,p(a))\subset(0,1/\lambda)$, $r(p(a))<\lambda^{-1}$. Thus $r(a)<\lambda^{-1}$, and therefore $\lambda^{-1}\notin\mathrm{Sp}(A,a)$, $\lambda\notin\mathrm{Sp}(A,\lambda+x^2)$, and we see that x^2 is invertible in A. Hence x is invertible in A.

Lemma 3.2. *Let A be a generalized *-algebra over* $\mathbf{R}$, *and $A_{\mathbf{C}}$ be its complexification. Then $(A/\mathrm{rad}(A))_{\mathbf{C}}$ is *-isomorphic to $A_{\mathbf{C}}/\mathrm{rad}(A_{\mathbf{C}})$.*

Proof. We want to prove that

$$\text{(1)} \qquad \mathrm{rad}(A_{\mathbf{C}}) = \{(a, b) \in A_{\mathbf{C}};\ a, b \in \mathrm{rad}(A)\}.$$

(We use the symbols of [1], see p. 68.) Let $N = \{a \in A;\ (a, 0) \in \mathrm{rad}(A_{\mathbf{C}})\}$. Clearly N is an ideal of A. If $a \in N$, then $(a, 0)$ is quasi-invertible in $A_{\mathbf{C}}$, hence a is quasi-invertible in A. Thus we obtain

$$\text{(2)} \qquad N \subset \mathrm{rad}(A).$$

Now we fix an element $b \in \mathrm{rad}(A)$ and an irreducible representation p of $A_{\mathbf{C}}$ over the complex linear space X. Suppose that L is a real subspace of X, invariant for the operators $p((a, 0))$ for all $a \in A$. Then $L + i \cdot L$ and $L \cup i \cdot L$ are complex subspaces, invariant for $p(A_{\mathbf{C}})$, and hence, being p an irreducible representation, if L is non-trivial then $X = L \oplus i \cdot L$ as a real linear space. Hence if L_1 is another such subspace then $\{0\} \subsetneqq L_1 \subsetneqq L$ is not possible, that is $a \to p((a, 0))|_L$ is an irreducible representation of A on L. Thus $p((b, 0))|_L = 0$ for $b \in \mathrm{rad}(A)$, and hence $p((b, 0)) = 0$ because X is the complex hull of L. If such L does not exist then $a \to p((a, 0))$ gives an irreducible representation and $p((b, 0)) = 0$, too. Having this for any irreducible representation p of $A_{\mathbf{C}}$ we see that $b \in N$, $\mathrm{rad}(A) \subset N$, and hence by (2) we get

$$\text{(3)} \qquad N = \mathrm{rad}(A).$$

Now consider the mapping $(a, b)' := (a, -b)$ on $A_{\mathbf{C}}$. This is conjugate linear automorphism, hence it preserves the quasi-invertibility, and therefore maps $\mathrm{rad}(A_{\mathbf{C}})$ onto itself. We can infer from this:

$$\mathrm{rad}(A_{\mathbf{C}}) = \{(a, b);\ (a, 0), (0, b) \in \mathrm{rad}(A_{\mathbf{C}})\}.$$

But $-i \cdot (0, b) = (b, 0)$ and hence $\mathrm{rad}(A_{\mathbf{C}}) = \{(a, b);\ a, b \in N\}$, that is, by (3), we can see that (1) holds.

It is easy to deduce from (1) that $(A/\mathrm{rad}(A))_{\mathbf{C}}$ is *-isomorphic to $A_{\mathbf{C}}/\mathrm{rad}(A_{\mathbf{C}})$.

Theorem 3. *Let A be a Banach generalized *-algebra over* $\mathbf{R}$. *Then A is Hermitian and skew-Hermitian if an only if its complexification $A_{\mathbf{C}}$ is a complex Hermitian algebra.*

Proof. Since the spectrum in a real algebra is defined through its complexification, one of the directions is trivial. To prove the other direction let A be Hermitian and skew-Hermitian as well. We may assume A has a unit, because otherwise $A + \mathbf{R}$ is Hermitian and skew-Hermitian while $(A + \mathbf{R})_{\mathbf{C}}$ is *-isomorphic to $A_{\mathbf{C}} + \mathbf{C}$. Then we may also assume A is semi-simple by Lemma 3.2.

Thus by Lemma 0.1 we have a *-homomorphism $\pi\colon A \mapsto B(\mathfrak{H})$, which is now *injective*. Moreover, it is easy to show (see e.g. the proof of Theorem 2) that

(1) π satisfies the conditions of Lemma 3.1.

We want to prove that $A_{\mathbf{C}}$ is Hermitian. Since $1\in A$, it is enough to show that $1+x^2$ is invertible in $A_{\mathbf{C}}$ whenever $x\in(A_{\mathbf{C}})_H$. Fix an $x=(a,b)\in(A_{\mathbf{C}})_H$, then $a\in A_H$ and $b\in A_I$. Let $c=1+a^2-b^2$, $d=ab+ba$, then $1+x^2=(c,d)$. Since the complexification of $B(\mathfrak{H})$ is clearly *-isomorphic to $B(\mathfrak{H}_{\mathbf{C}})$, which is Hermitian, thus $(\pi(c),\pi(d))$ is invertible in $B(\mathfrak{H})_{\mathbf{C}}$, so we have $u, v\in B(\mathfrak{H})$ satisfying

(2) $$u\cdot\pi(c)-v\cdot\pi(d)=1,\quad u\cdot\pi(d)+v\cdot\pi(c)=0$$

and

(3) $$\pi(c)\cdot u-\pi(d)\cdot v=1,\quad \pi(d)\cdot u+\pi(c)\cdot v=0.$$

It is known that the set $A_p=\{h\in A_H;\ \mathrm{Sp}\,(A,h)\subset\mathbf{R}_+\}$ is a cone (see [5]), and hence $a^2-b^2\in A_p$ because $a^2, -b^2\in A_p$. Thus we can infer

(4) c has an inverse h in A_H.

We see from (2) that $v=-u\cdot\pi(dh)$ and so $u\cdot\pi(c+dhd)=1$. Similarly, we can see from (3) that $\pi(c+dhd)\cdot u=1$. Observe that $m=c+dhd\in A_H$ because $d\in A_I$ and $c, h\in A_H$, and hence $\mathrm{Sp}\,(B(\mathfrak{H}),\pi(m))\subset\mathbf{R}$. Applying Lemma 3.1 we get a $k=m^{-1}$ in A, moreover, $\pi(k)=u$. Hence $v=\pi(j)$, where $j=-kdh$. Now by the injectivity of π we can infer that $(k,j)=(1+x^2)^{-1}$ in $A_{\mathbf{C}}$. The proof is complete.

References

[1] F. F. Bonsall and J. Duncan, *Complete normed algebras*, Springer-Verlag (Berlin—Heidelberg—New York, 1973).

[2] S. H. Kulkarni and B. V. Limaye, Gelfand-Naimark theorems for real Banach *-algebras, *Math. Jap.*, **25** (1980), 545—558.

[3] Z. Magyar, On commutativity and spectral radius property of real generalized *-algebras, *Acta Sci. Math.*, **53** (1989), 339—343.

[4] V. Pták, Banach algebras with involution, *Manuscripta Math.*, **6** (1972), 245—290.

[5] S. Shirali and J. W. M. Ford, Symmetry in complex involutory Banach algebras. II, *Duke Math. J.*, **37** (1970), 275—280.

EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF APPLIED ANALYSIS
MÚZEUM KÖRÚT 6—8
1088 BUDAPEST, HUNGARY

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Models for infinite sequences of noncommuting operators

GELU POPESCU

In [1], J. W. BUNCE developed a model theory for n-tuples of not necessarily commuting operators, extending the work of A. E. FRAZHO [4] for pairs of operators. He proved, for a finite number of operators on Hilbert space, versions of the Rota model theorem, the de Branges—Rovnyak model theorem, and the coisometric extension theorem.

The aim of this paper is to extend these results for infinite sequences of noncommuting operators, to generalize some results due to B. SZ.-NAGY [8] and G. C. ROTA [7] [5, Problem 121] and to give some necessary and sufficient conditions for simultaneous similarity. We shall prove all these results without using the theorems above mentioned (for a finite number of operators).

1. Let $\mathscr{H}$ be a Hilbert space and $B(\mathscr{H})$ be the algebra of all bounded operators in $\mathscr{H}$. We recall that a coisometry $V\in B(\mathscr{H})$ is called pure if $V^n\to 0$ (strongly) as $n\to\infty$.

In [1, Proposition 2] J. W. Bunce proved that for any finite family $\{A_i;\ 1\leqq i\leqq n\}$ ($n\in\mathrm{N}$) of operators such that $r(A_i)<1$ for each i, ($r(T)$ denoting the spectral radius of an operator $T\in B(\mathscr{H})$), and $\sum_{i=1}^{n} A_i^*A_i\leqq I_{\mathscr{H}}$ ($I_{\mathscr{H}}$ is the identity on $\mathscr{H}$), there are a Hilbert space $\mathscr{K}\supset\mathscr{H}$ and pure coisometries $\{S_i;\ 1\leqq i\leqq n\}$ acting on $\mathscr{K}$ such that $S_i(\mathscr{H})\subset\mathscr{H}$, $S_i|_{\mathscr{H}}=A_i$ for each i and $S_iS_j^*=0$ for $i\neq j$.

We begin with a theorem which generalizes Proposition 1 of [1] and the above mentioned result, replacing the condition that $r(A_i)<1$ by the condition $A_i^n\to 0$ (strongly) as $n\to\infty$. Let us consider Λ and J to be sets of indices such that card $\Lambda\leqq\aleph_0$.

Proposition 1.1. *Let* $\mathscr{A}_\alpha=\{A_{\alpha,i}\colon i\in\Lambda\}\subset B(\mathscr{H})$ *for each* $\alpha\in J$. *Then the following two conditions are equivalent:*

a) $\sum_{i\in\Lambda} A_{\alpha,i}^*A_{\alpha,i}\leqq I_{\mathscr{H}}$ *for each* $\alpha\in J$.

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b) *There exist a Hilbert space* $\mathcal{K}\supset\mathcal{H}$ *and families of coisometries* $\mathcal{V}_\alpha=\{V_{\alpha,i};\ i\in\Lambda\}\subset B(\mathcal{K})$ $(\alpha\in J)$ *such that*

$$\sum_{i\in\Lambda} V_{\alpha,i}^*V_{\alpha,i}\leqq I_{\mathcal{K}},\quad V_{\alpha,i}(\mathcal{H})\subset\mathcal{H}\quad \text{and}\quad V_{\alpha,i}|_{\mathcal{H}}=A_{\alpha,i}\quad \text{for each}\quad \alpha\in J,\quad i\in\Lambda.$$

One can even require that $V_{\alpha,i}$ *be a pure coisometry for every* $\alpha\in J$ *and* $i\in\Lambda$ *for which* $A_{\alpha,i}^n\to 0$ *(strongly) as* $n\to\infty$.

Proof. It is easy to see that b) implies a).

Conversely, assume that for each $\alpha\in J$, $\sum_{i\in\Lambda} A_{\alpha,i}^*A_{\alpha,i}\leqq I_{\mathcal{H}}$. Consider the Hilbert space $\mathbf{H}=\bigoplus_{\alpha\in J,\, i\in\Lambda}\mathcal{H}_{\alpha,i}$, where $\mathcal{H}_{\alpha,i}$ is a copy of the Hilbert space $\mathcal{H}$, and the operator $T\in B(\mathbf{H})$ defined by

$$T\Big(\bigoplus_{\alpha\in J,\, i\in\Lambda} h_{\alpha,i}\Big)=\bigoplus_{\alpha\in J,\, i\in\Lambda} A_{\alpha,i}h_{\alpha,1}.$$

Note that T is a contraction. Indeed,

$$\Big\|T\Big(\bigoplus_{\alpha\in J,\, i\in\Lambda} h_{\alpha,i}\Big)\Big\|^2=\sum_{\alpha\in J}\Big(\sum_{i\in\Lambda} A_{\alpha,i}^*A_{\alpha,i}h_{\alpha,1},h_{\alpha,1}\Big)\leqq\sum_{\alpha\in J}\|h_{\alpha,1}\|^2\leqq\Big\|\bigoplus_{\alpha\in J,\, i\in\Lambda} h_{\alpha,i}\Big\|^2,$$

for every $\bigoplus_{\alpha\in J,\, i\in\Lambda} h_{\alpha,i}\in\mathbf{H}$.

Let us determine a Hilbert space $\mathbf{K}\supset\mathbf{H}$ and a coisometry $V\in B(\mathbf{K})$ such that $V(\mathbf{H})\subset\mathbf{H}$ and $V|_{\mathbf{H}}=T$. Let $\mathbf{K}=\mathbf{H}\oplus\mathcal{M}$, where $\mathcal{M}$ is a Hilbert space which we shall determine. With respect to this decomposition of $\mathbf{K}$ the matrix of V is

$$V=\begin{pmatrix}T & X\\ 0 & Y\end{pmatrix}$$

where $X\colon\mathcal{M}\to\mathbf{H}$ and $Y\colon\mathcal{M}\to\mathcal{M}$ satisfy the relations:

$$TT^*+XX^*=I_{\mathbf{H}},\quad XY^*=0,\quad YY^*=I_{\mathcal{M}}. \tag{1.1}$$

Since Y is a coisometry, the Wold decomposition of the Hilbert space $\mathcal{M}$ with respect to Y^* is $\mathcal{M}=\mathcal{M}_0\oplus\mathcal{M}_1$ and $Y=S^*\oplus U$, where S is the unilateral shift acting on $\mathcal{M}_0=\bigoplus_{n=0}^{\infty}Y^{*n}(\mathcal{L})$, $\mathcal{L}=\mathcal{M}\ominus Y^*(\mathcal{M})$ is the wandering subspace of S, and U is a unitary operator acting on $\mathcal{M}_1=\mathcal{M}\ominus\mathcal{M}_0=\bigcap_{n=0}^{\infty}Y^{*n}(\mathcal{M})$.

The relation $XY^*=0$ implies $X|_{\mathcal{M}_1}=0$. Therefore, with respect to the decomposition $\mathbf{K}=\mathbf{H}\oplus\mathcal{M}_0\oplus\mathcal{M}_1$, the matrix of V is

$$V=\begin{pmatrix}T & X & 0\\ 0 & S^* & 0\\ 0 & 0 & U\end{pmatrix}$$

where X stands for $X|_{\mathcal{M}_0}$. Therefore $X\colon \mathcal{M}_0 \to \mathbf{H}$ and the relations (1.1) become

(1.2) $$TT^*+XX^* = I_{\mathbf{H}}, \quad XS = 0.$$

Obviously, we can consider

$$\mathcal{M}_0 = \ell^2(\mathcal{L}) = \{(x_1, x_2, \ldots);\ x_i \in \mathcal{L},\ \sum_{i=1}^{\infty} \|x_i\|^2 < \infty\}$$

and $S\colon \mathcal{M}_0 \to \mathcal{M}_0$, $S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$. Since $XS=0$, it follows that $X(0, y_1, y_2, \ldots) = 0$ for every $y_i \in \mathcal{L}$ satisfying $\sum_{i=1}^{\infty} \|y_i\|^2 < \infty$. We embed $\mathcal{L}$ in $\ell^2(\mathcal{L})$ by identifying the element $\ell \in \mathcal{L}$ with the element $(\ell, 0, 0, \ldots) \in \ell^2(\mathcal{L})$. In the sequel X stands for $X|_{\mathcal{L}}$. Thus the relations (1.2) become

(1.3) $$TT^*+XX^* = I_{\mathbf{H}},$$

where $X\colon \mathcal{L} \to \mathbf{H}$.

The relation (1.3) holds for $\mathcal{L} = \mathbf{H}$ and $X = (I_{\mathbf{H}} - TT^*)^{1/2}$. With respect to the decomposition $\mathbf{H} = \bigoplus_{\alpha\in J, i\in\Lambda} \mathcal{H}_{\alpha,i}$ we have $X = \bigoplus_{\alpha\in J, i\in\Lambda} X_{\alpha,i}$, where $X_{\alpha,i}\colon \mathbf{H} \to \mathcal{H}$. Taking into account (1.3), the following relations hold:

(1.4) $$\begin{cases} A_{\alpha,i}A_{\alpha,i}^* + X_{\alpha,i}X_{\alpha,i}^* = I_{\mathcal{H}}, & \text{for } \alpha\in J,\ i\in\Lambda, \\ A_{\alpha,i}A_{\alpha,j}^* + X_{\alpha,i}X_{\alpha,j}^* = 0, & \text{for } \alpha\in J,\ i\neq j,\ i,j\in\Lambda. \end{cases}$$

Let $\{1, 2, \ldots\} = \bigcup_{i\in\Lambda} N_i$ such that $N_i \cap N_j = \emptyset$ $(i\neq j)$ and card $N_i = \aleph_0$ for each $i\in\Lambda$. Setting $N_i = \{n_1^{(i)}, n_2^{(i)}, \ldots\}$, where $n_1^{(i)} < n_2^{(i)} < \ldots$ for each $i\in\Lambda$, we define $Z_i \in B(\ell^2(\mathbf{H}))$ by

$$Z_i(h_1, h_2, \ldots) = (h_{n_1^{(i)}}, h_{n_2^{(i)}}, \ldots), \quad h_j\in\mathbf{H}, \quad \left(\sum_{j\in\Lambda} \|h_j\|^2 < \infty\right).$$

Note that $Z_iZ_i^* = I_{\ell^2(\mathbf{H})}$ $(i\in\Lambda)$, and $Z_iZ_j^* = 0$ $(i\neq j)$.

Consider the Hilbert space $\mathcal{K} = \mathcal{H} \oplus \ell^2(\mathbf{H})$ and define $V_{\alpha,i} \in B(\mathcal{K})$ $(\alpha\in J, i\in\Lambda)$ by the matrix

$$V_{\alpha,i} = \begin{pmatrix} A_{\alpha,i} & W_{\alpha,i} \\ 0 & Z_iS^* \end{pmatrix}$$

where $W_{\alpha,i}$ $(\alpha\in J,\ i\in\Lambda)$ stands for the operator $X_{\alpha,i}\oplus 0\oplus 0\oplus\ldots$. By (1.4) a simple computation shows that for every $\alpha\in J$, $i\in\Lambda$ we have

$$V_{\alpha,i}V_{\alpha,i}^* = I_{\mathcal{K}}, \quad V_{\alpha,i}V_{\alpha,j}^* = 0 \ (i\neq j), \quad V_{\alpha,i}(\mathcal{H}) \subset \mathcal{H} \ \text{ and } \ V_{\alpha,i}|_{\mathcal{H}} = A_{\alpha,i}.$$

Let us prove that, if $A_{\alpha,i}^n \to 0$ (strongly) as $n\to\infty$ for some $\alpha\in J$, $i\in\Lambda$, then $V_{\alpha,i}^n \to 0$ (strongly) as $n\to\infty$ for the same $\alpha\in J$, $i\in\Lambda$. Note that, with respect to the decomposition $\mathcal{K} = \mathcal{H}\oplus\ell^2(\mathbf{H})$, we have

$$V_{\alpha,i}^n = \begin{pmatrix} A_{\alpha,i}^n & \sum_{j=0}^{n-1} A_{\alpha,i}^j W_{\alpha,i}(Z_iS^*)^{n-j-1} \\ 0 & (Z_iS^*)^n \end{pmatrix}.$$

Let $y=(\underbrace{0, \ldots, 0}_{m-1 \text{ times}}, y_m, 0, 0, \ldots)\in\ell^2(\mathrm{H})$, where $m\geqq 1$, and let $n>m$. Since $W_{\alpha,i}(0, h_2, h_3, \ldots)=0$ for every $(0, h_2, h_3, \ldots)\in\ell^2(\mathrm{H})$, it follows that, if there exists $1\leqq p\leqq m$ such that $(Z_iS^*)^p y=(y_m, 0, 0, \ldots)$, then

$$\sum_{j=0}^{n} A^j_{\alpha,i} W_{\alpha,i}(Z_iS^*)^{n-j} y = A^{n-p}_{\alpha,i} W_{\alpha,i}(Z_iS^*)^p y,$$

otherwise we have

$$\sum_{j=0}^{n} A^j_{\alpha,i} W_{\alpha,i}(Z_iS^*)^{n-j} y = A^n_{\alpha,i} W_{\alpha,i} y.$$

In both the cases, since $A^n_{\alpha,i}\to 0$ (strongly) as $n\to\infty$, it follows that

$$\sum_{j=0}^{n} A^j_{\alpha,i} W_{\alpha,i}(Z_iS^*)^{n-j} y \to 0 \quad \text{as} \quad n\to\infty.$$

Since $(Z_iS^*)^n\to 0$ (strongly) too, it follows that $V^n_{\alpha,i}(h\oplus y)\to 0$ as $n\to\infty$, for every $h\in\mathcal{H}$ and all y of the form above mentioned. But, the span of all the vectors y of considered types is the Hilbert space $\ell^2(\mathrm{H})$ and $\|V^n_{\alpha,i}\|\leqq 1$ for each $n\in\{1, 2, \ldots\}$. Thus, we have that $V^n_{\alpha,i}\to 0$ (strongly) as $n\to\infty$ for the same $\alpha\in J$, $i\in\Lambda$ for which $A^n_{\alpha,i}\to 0$ (strongly) as $n\to\infty$. This completes the proof.

We remark that, if card Λ = card $J=1$, we find again the coisometric extension theorem and the de Branges—Rovnyak model theorem (see [9], [5]). The result of E. Durszt and B. Sz.-Nagy [2] is contained also in Proposition 1.1.

2. We say that a family $\mathcal{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ is simultaneously similar to a family $\mathcal{B}=\{B_i\}_{i\in\Lambda}\subset B(\mathcal{K})$ if there exists an invertible operator $R\in B(\mathcal{H}, \mathcal{K})$ so that $A_i=R^{-1}B_iR$ for every $i\in\Lambda$.

In what follows we shall obtain a generalization of a result due to B. Sz.-Nagy [8], that is, an operator $A\in B(\mathcal{H})$ is similar to an isometry if and only if there exist $a\geqq b>0$ such that

$$b\|h\|^2 \leqq \|A^n h\|^2 \leqq a\|h\|^2$$

for every $h\in\mathcal{H}$, $n\in\mathbf{N}$.

We shall also obtain a generalization of Rota's model theorem, for infinite sequences of operators, and we shall give some necessary and sufficient conditions for a family $\{A_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ to be simultaneously similar to a family $\{T_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ of contractions with $\sum_{i\in\Lambda} T_i^*T_i\leqq I_{\mathcal{H}}$.

Let us denote by $F(k, \Lambda)$ the set of all functions from the set $\{1, 2, \ldots, k\}$ to the set Λ. For $\mathcal{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ and $f\in F(k, \Lambda)$, let $A_f=A_{f(1)}A_{f(2)}\ldots A_{f(k)}$.

The following two lemmas are simple extensions of Lemmas 4 and 5 from [1]. We omit the proofs.

Lemma 2.1. *Let $\mathcal{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ such that the series $\sum_{i\in\Lambda} A_i^* A_i$ is convergent in the strong operator topology (if* card $\Lambda=\aleph_0$*).*

a) *If $1\leqq m<n$, then* $\sum_{f\in F(n,\Lambda)} A_f^* A_f=\sum_{q\in F(n-m,\Lambda)} A_q^*\Big(\sum_{g\in F(m,\Lambda)} A_g^* A_g\Big)A_q$.

b) *For any $m,n\geqq 1$* $\Big\|\sum_{f\in F(mn,\Lambda)} A_f^* A_f\Big\|\leqq\Big\|\sum_{g\in F(n,\Lambda)} A_g^* A_g\Big\|^m$.

Lemma 2.2. *Let $\mathcal{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ with $\sum_{i\in\Lambda} A_i^* A_i$ strongly convergent. Then*

$$\lim_{m\to\infty}\Big\|\sum_{f\in F(m,\Lambda)} A_f^* A_f\Big\|^{1/m}=\inf_m\Big\{\Big\|\sum_{f\in F(m,\Lambda)} A_f^* A_f\Big\|^{1/m}\Big\}.$$

Define

$$r(\mathcal{A})=\inf_m\Big\{\Big\|\sum_{f\in F(m,\Lambda)} A_f^* A_f\Big\|^{1/2m}\Big\}.$$

For Λ with card $\Lambda=1$ we find again the well-known formula for the spectral radius of an operator. The case when card $\Lambda<\aleph_0$ is considered in [1].

Proposition 2.3. *Let $\mathcal{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathcal{H})$. The following statements are equivalent:*

a) *There exists a family of contractions $\mathcal{T}=\{T_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ with $\sum_{i\in\Lambda} T_i^* T_i\leqq I_{\mathcal{H}}$ such that $\mathcal{A}$ is simultaneously similar to $\mathcal{T}$.*

b) *There exists a positive invertible operator $P\in B(\mathcal{H})$ such that $\sum_{i\in\Lambda} A_i^* P A_i\leqq P$.*

Proof. Assume a) and let $R\in B(\mathcal{H})$ be an invertible operator such that $A_i=R^{-1}T_iR$ for each $i\in\Lambda$. Since

$$\sum_{i\in\Lambda} T_i^* T_i=(R^{-1})^*\Big(\sum_{i\in\Lambda} A_i^* R^* R A_i\Big)R^{-1}\leqq I_{\mathcal{H}}$$

it follows that $\sum_{i\in\Lambda} A_i^* P A_i\leqq P$, where $P=R^*R$. Conversely, assume b) and consider $T_i=RA_iR^{-1}$, where $R=P^{1/2}$. Thus, $\sum_{i\in\Lambda} T_i^* T_i\leqq I_{\mathcal{H}}$ and the proof is complete.

A necessary condition for simultaneous similarity is the following

Proposition 2.4. *If a family $\mathcal{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ is simultaneously similar to a family $\mathcal{T}=\{T_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ of contractions with $\sum_{i\in\Lambda} T_i^* T_i\leqq I_{\mathcal{H}}$, then there exists $M>0$ such that*

$$\sum_{f\in F(n,\Lambda)}\|A_f h\|^2\leqq M\|h\|^2$$

for every $h\in\mathcal{H}$ and $n\in\mathbf{N}$. In particular it follows that $r(\mathcal{A})\leqq 1$.

Proof. According to Proposition 2.3 there is a positive invertible operator $R\in B(\mathscr{H})$ such that $T_i=RA_iR^{-1}$ for each $i\in\Lambda$. By Lemma 2.1 we have

$$\Big\|\sum_{f\in F(k,\Lambda)} T_f^*T_f\Big\| \leqq \Big\|\sum_{i\in\Lambda} T_i^*T_i\Big\|^k \leqq 1 \quad \text{for any} \quad k\in\mathbf{N}.$$

Since $A_f=R^{-1}T_fR$ $(f\in F(k,\Lambda))$, it follows that

$$\Big\|\sum_{f\in F(k,\Lambda)} A_f^*A_f\Big\| \leqq \|R\|^2\Big\|\sum_{f\in F(k,\Lambda)} T_f^*(R^{-1})^2T_f\Big\| \leqq \|R\|^2\|R^{-1}\|^2$$

for any $k\in\mathbf{N}$. By Lemma 2.2 it is simple to deduce that $r(\mathscr{A})\leqq 1$.

The following proposition is a generalization of the result due to B. Sz.-Nagy [8] (for single operators) and the proof is on the same line.

Proposition 2.5. *Let* $\mathscr{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathscr{H})$. *The following two conditions are equivalent:*

a) *There exists* $\mathscr{V}=\{V_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ *with* $\sum_{i\in\Lambda} V_i^*V_i=I_{\mathscr{H}}$ *such that* $\mathscr{A}$ *is simultaneously similar to* $\mathscr{V}$.

b) *There exist* $a\geqq b>0$ *such that*

$$b\|h\|^2 \leqq \sum_{f\in F(n,\Lambda)} \|A_fh\|^2 \leqq a\|h\|^2$$

for any $h\in\mathscr{H}$ *and* $n\in\mathbf{N}$.

Proof. Assume a) and let $R\in B(\mathscr{H})$ be a positive invertible operator such that $A_i=R^{-1}V_iR$ for each $i\in\Lambda$. Since $\sum_{i\in\Lambda} V_i^*V_i=I_{\mathscr{H}}$, we have, using Lemma 2.1,

$$\sum_{f\in F(n,\Lambda)} V_f^*V_f = I_{\mathscr{H}} \quad \text{for any} \quad n\in\mathbf{N}. \tag{2.1}$$

As in the proof of Proposition 2.4, we obtain that

$$\sum_{f\in F(n,\Lambda)} \|A_fh\|^2 \leqq \|R\|^2\|R^{-1}\|^2\|h\|^2$$

for any $h\in\mathscr{H}$, $n\in\mathbf{N}$. On the other hand using (2.1), we have

$$\sum_{f\in F(n,\Lambda)} \|A_fh\|^2 \geqq \frac{1}{\|R\|^2}\sum_{f\in F(n,\Lambda)} \|V_fRh\|^2 = \frac{1}{\|R\|^2}\|Rh\|^2 \geqq \frac{\|h\|^2}{\|R\|^2\|R^{-1}\|^2}$$

for any $h\in\mathscr{H}$, $n\in\mathbf{N}$.

Conversely, assume condition b) is true. If card $\Lambda=\aleph_0$, we can take $\Lambda=\mathbf{N}=\{1,2,\dots\}$. $\sum_{f\in F(n,\Lambda)} \|A_fh\|^2$ is convergent for $h\in\mathscr{H}$ and $n\in\mathbf{N}$, whence $\sum_{f\in F(n,\Lambda)} (A_fh_1, A_fh_2)$ is convergent for every $h_1, h_2\in\mathscr{H}$ and $n\in\mathbf{N}$. Since for any $h_1, h_2\in\mathscr{H}$ we have that

$$\Big|\sum_{f\in F(n,\Lambda)} (A_fh_1, A_fh_2)\Big| \leqq (a/2)(\|h_1\|^2+\|h_2\|^2)$$

for every $n\in\mathbf{N}$, we can define for any $h_1, h_2\in\mathscr{H}$

$$\langle h_1, h_2\rangle = \operatorname*{LIM}_{n\to\infty} \sum_{f\in F(n,A)} (A_f h_1, A_f h_2),$$

where LIM means a Banach limit.

Taking into account the properties of the Banach limit we see that $\langle\cdot,\cdot\rangle$ is a hermitian bilinear form and

$$\text{(2.2)}\qquad b\|h\|^2 \leqq \langle h, h\rangle = \operatorname*{LIM}_{n\to\infty} \sum_{f\in F(n,A)} \|A_f h\|^2 \leqq a\|h\|^2 \quad \text{for each} \quad h\in\mathscr{H}.$$

By a well-known theorem on the bounded hermitian bilinear form, there exists a selfadjoint operator $P\in B(\mathscr{H})$ such that

$$\langle h_1, h_2\rangle = (Ph_1, h_2) \quad \text{for all} \quad h_1, h_2\in\mathscr{H}.$$

From (2.2) it follows that $bI_{\mathscr{H}}\leqq P\leqq aI_{\mathscr{H}}$, therefore P is a positive invertible operator.

Now we shall show that $P=\sum_{i=1}^{\infty} A_i^* P A_i$. Since the series $\sum_{i\in A}\|A_i h\|^2$ is convergent, for every $\varepsilon>0$, there exists $k_0\in\mathbf{N}$ such that $\sum_{i=k+1}^{\infty}\|A_i h\|^2\leqq\varepsilon/a$ for any $k\geqq k_0$. Thus, for every $k\geqq k_0$ and $n\in\mathbf{N}$ we have:

$$0 \leqq \sum_{i=1}^{\infty} \sum_{f\in F(n,A)} \|A_f A_i h\|^2 - \sum_{i=1}^{k} \sum_{f\in F(n,A)} \|A_f A_i h\|^2 =$$

$$= \sum_{i=k+1}^{\infty} \sum_{f\in F(n,A)} \|A_f A_i h\|^2 \leqq a \sum_{i=k+1}^{\infty} \|A_i h\|^2 \leqq \varepsilon,$$

whence it follows

$$0 \leqq \operatorname*{LIM}_{n\to\infty} \Big(\sum_{i=1}^{\infty} \sum_{f\in F(n,A)} \|A_f A_i h\|^2\Big) - \operatorname*{LIM}_{n\to\infty} \Big(\sum_{i=1}^{k} \sum_{f\in F(n,A)} \|A_f A_i h\|^2\Big) \leqq \varepsilon$$

for any $k\geqq k_0$. Since

$$(Ph, h) = \operatorname*{LIM}_{n\to\infty} \sum_{f\in F(n+1,A)} \|A_f h\|^2 = \operatorname*{LIM}_{n\to\infty} \Big(\sum_{i=1}^{\infty} \sum_{f\in F(n,A)} \|A_f A_i h\|^2\Big)$$

for any $h\in\mathscr{H}$, we have that

$$0 \leqq (Ph, h) - \sum_{i=1}^{k} \Big(\operatorname*{LIM}_{n\to\infty} \sum_{f\in F(n,A)} \|A_f A_i h\|^2\Big) \leqq \varepsilon \quad \text{for any} \quad k\geqq k_0.$$

In other words

$$(Ph, h) = \lim_{k\to\infty} \sum_{i=1}^{k} \Big(\operatorname*{LIM}_{n\to\infty} \sum_{f\in F(n,A)} \|A_f A_i h\|^2\Big) =$$

$$= \sum_{i=1}^{\infty} \langle A_i h, A_i h\rangle = \Big(\Big(\sum_{i=1}^{\infty} A_i^* P A_i\Big)h, h\Big) \quad \text{for every} \quad h\in\mathscr{H}.$$

Therefore, $\sum_{i=1}^{\infty} A_i^* P A_i = P$ and setting $R = P^{1/2}$, $V_i = R A_i R^{-1}$ for each $i \in \Lambda$, we deduce that $\sum_{i \in \Lambda} V_i^* V_i = I_{\mathscr{H}}$. The case when Λ is a finite set is even simpler to deduce. The proof is complete.

We now give a necessary and sufficient condition for simultaneous similarity.

Proposition 2.6. *Let* $\mathscr{A} = \{A_i\}_{i \in \Lambda} \subset B(\mathscr{H})$. *Then the following conditions are equivalent.*

a) *There exists* $\mathscr{T} = \{T_i\}_{i \in \Lambda} \subset B(\mathscr{H})$ *with* $\sum_{i \in \Lambda} T_i^* T_i \leqq I_{\mathscr{H}}$ *such that* $\mathscr{A}$ *is simultaneously similar to* $\mathscr{T}$.

b) *There exist* $D \in B(\mathscr{H})$ *and* $a \geqq b > 0$ *such that*

$$b\|h\|^2 \leqq \sum_{f \in F(n,\Lambda)} \|A_f h\|^2 + \sum_{f \in F(n-1,\Lambda)} \|DA_f h\|^2 + \ldots + \sum_{f \in F(1,\Lambda)} \|DA_f h\|^2 + \|Dh\|^2 \leqq a\|h\|^2$$

for every $h \in \mathscr{H}$, $n \in \mathbf{N}$.

Proof. Assume condition a) is true. Then, according to Proposition 2.3, there exists a positive invertible operator $P \in B(\mathscr{H})$ such that $\sum_{i \in \Lambda} A_i^* P A_i \leqq P$ (we can assume that $\|P\| \leqq 1$). Let

$$D = \Big(P - \sum_{i \in \Lambda} A_i^* P A_i\Big)^{1/2}$$

and for each $h \in \mathscr{H}$ and $n \in \mathbf{N}$ let

$$S_{n,h} = \sum_{f \in F(n,\Lambda)} \|A_f h\|^2 + \sum_{f \in F(n-1,\Lambda)} \|DA_f h\|^2 + \ldots + \sum_{f \in F(1,\Lambda)} \|DA_f h\|^2 + \|Dh\|^2.$$

An easy computation shows that

$$S_{n,h} = \sum_{f \in F(n,\Lambda)} (A_f^* A_f h, h) - \sum_{f \in F(n,\Lambda)} (A_f^* P A_f h, h) + (Ph, h).$$

By Proposition 2.4 there exists $M > 0$ such that

$$S_{n,h} \leqq \sum_{f \in F(n,\Lambda)} (A_f^* A_f h, h) + (Ph, h) \leqq (M+1)\|h\|^2$$

for any $h \in \mathscr{H}$ and $n \in \mathbf{N}$.

On the other hand, since $P \leqq I_{\mathscr{H}}$, we have

$$\sum_{f \in F(n,\Lambda)} (A_f^* A_f h, h) \geqq \sum_{f \in F(n,\Lambda)} (A_f^* P A_f h, h),$$

therefore $S_{n,h} \geqq (Ph, h) \geqq \|R^{-1}\|^{-1} \|h\|^2$ (where $R = P^{1/2}$) for any $h \in \mathscr{H}$ and $n \in \mathbf{N}$.

Now we shall prove that b) implies a). Let us consider the Hilbert space

$$\mathbf{K} = \mathscr{H} \oplus \mathscr{D} \oplus \mathscr{D} \oplus \ldots \quad \text{where} \quad \mathscr{D} = \overline{D\mathscr{H}},$$

and embed $\mathscr{H}$ in $\mathbf{K}$ by identifying the element $h\in\mathscr{H}$ with the element $(h, 0, 0, \ldots)\in\mathbf{K}$. Let $\lambda_i\in\mathbf{C}$ $(i\in\Lambda)$ with $\sum_{i\in\Lambda}|\lambda_i|^2=1$ and define the operators $B_i\in B(\mathbf{K})$ $(i\in\Lambda)$ by

$$B_i(h_0, h_1, \ldots) = (A_ih_0, \lambda_iDh_0, \lambda_ih_1, \ldots).$$

For each $f\in F(n, \Lambda)$ $(n\in\mathbf{N})$ we have:

$$B_f(h_0, h_1, \ldots) = (A_{f(1)}\ldots A_{f(n)}h_0, \lambda_{f(1)}DA_{f(2)}\ldots A_{f(n)}h_0,$$
$$\lambda_{f(1)}\lambda_{f(2)}DA_{f(3)}\ldots A_{f(n)}h_0, \ldots, \lambda_{f(1)}\ldots\lambda_{f(n-1)}DA_{f(n)}h_0,$$
$$\lambda_{f(1)}\ldots\lambda_{f(n)}Dh_0, \lambda_fh_1, \ldots)$$

where λ_f stands for $\lambda_{f(1)}\ldots\lambda_{f(n)}$.

Since $\sum_{f\in F(n,\Lambda)}|\lambda_f|^2=1$, for any $n\in\mathbf{N}$, it is easy to show (by induction) that, for each $k\in\{1, 2, \ldots, n-1\}$,

$$\sum_{f\in F(n,\Lambda)}\|\lambda_{f(1)}\ldots\lambda_{f(k)}DA_{f(k+1)}\ldots A_{f(n)}h\|^2 = \sum_{g\in F(n-k,\Lambda)}\|DA_gh\|^2$$

for any $h\in\mathscr{H}$ and $n\in\mathbf{N}$. Therefore

$$\sum_{f\in F(n,\Lambda)}\|B_f(h_0, h_1, \ldots)\|^2 = \sum_{f\in F(n,\Lambda)}\|A_fh_0\|^2+$$
$$+\sum_{f\in F(n-1,\Lambda)}\|DA_fh_0\|^2+\ldots+\sum_{f\in F(1,\Lambda)}\|DA_fh_0\|^2+\|h_1\|^2+\ldots$$

for any $(h_0, h_1, \ldots)\in\mathbf{K}$ and $n\in\mathbf{N}$. Thus, by the assumption b), there exist $a\geqq h>0$ such that

$$b\|k\|^2\leqq\sum_{f\in F(n,\Lambda)}\|B_fk\|^2\leqq a\|k\|^2 \quad\text{for any}\quad k\in\mathbf{K},\quad n\in\mathbf{N}.$$

According to Proposition 2.5, there exists $\mathscr{V}=\{V_i\}_{i\in\Lambda}\subset B(\mathbf{K})$ with $\sum_{i\in\Lambda}V_i^*V_i=I_{\mathbf{K}}$ such that the family $\mathscr{B}=\{B_i\}_{i\in\Lambda}$ is simultaneously similar to $\mathscr{V}$. Let us notice that $B_i^*|_{\mathscr{H}}=A_i^*$, $B_i^*(\mathscr{H})\subset\mathscr{H}$. Let $Q\in B(\mathbf{K})$ an invertible operator such that $B_i^*=QV_i^*Q^{-1}$ $(i\in\Lambda)$ and consider the invertible operator $Q_0\colon\mathscr{H}\to\mathscr{H}_0$, $Q_0=Q^{-1}|_{\mathscr{H}}$, where $\mathscr{H}_0$ stands for $Q^{-1}\mathscr{H}$. Since $B_i^*(\mathscr{H})\subset\mathscr{H}$ we have that $V_i^*(\mathscr{H}_0)\subset\mathscr{H}_0$ $(i\in\Lambda)$ and

$$A_i^* = B_i^*|_{\mathscr{H}} = Q_0^{-1}(V_i^*|_{\mathscr{H}_0})Q_0.$$

Using the polar decomposition of Q_0, that is $Q_0=U|Q_0|$ where $|Q_0|=(Q_0^*Q_0)^{1/2}$ and $U=Q_0|Q_0|^{-1}$ we obtain $A_i^*=RT_i^*R^{-1}$, where $R=|Q_0|^{-1}$ and $T_i^*=U^*(V_i^*|_{\mathscr{H}_0})U$. Now $\sum_{i\in\Lambda}V_i^*V_i=I_K$ implies that $\sum_{i\in\Lambda}T_i^*T_i\leqq I_{\mathscr{H}}$. The proof is complete.

Remark 2.7. For Λ with card $\Lambda=1$ the above proposition was proved in [3].

Corollary 2.8. *Let* $D_{\mathcal{A}}=\left|I_{\mathscr{H}}-\sum_{i\in\Lambda}A_i^*A_i\right|^{1/2}$. *If there exists* $a>0$ *such that*

$$\sum_{f\in F(n,\Lambda)}\|A_fh\|^2+\sum_{f\in F(n-1,\Lambda)}\|D_{\mathcal{A}}A_fh\|^2+\ldots+\|D_{\mathcal{A}}h\|^2\leqq a\|h\|^2 \tag{2.3}$$

for any $h\in\mathscr{H}$ *and* $n\in\mathbf{N}$, *then the condition* b) *of Proposition* 2.6 *is fulfilled.*

Proof. The upper estimation is trivial. Since for every $h\in\mathscr{H}$, $k\in\mathbf{N}$

$$\sum_{f\in F(k,\Lambda)}(A_f^*D_{\mathcal{A}}A_fh,h)\geqq\sum_{f\in F(k,\Lambda)}(A_f^*A_fh,h)-\sum_{f\in F(k+1,\Lambda)}(A_f^*A_fh,h),$$

we have the lower estimation with $b=1$.

The following corollary is a generalization of Rota's model theorem [7].

Corollary 2.9. *Let* $\mathcal{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ *and suppose that there exists* $a>0$ *such that*

$$\sum_{n=1}^{\infty}\Big(\sum_{f\in F(n,\Lambda)}\|A_fh\|^2\Big)\leqq a\|h\|^2 \tag{2.4}$$

for any $h\in\mathscr{H}$. *Then there exist a Hilbert space* $\mathscr{K}\supset\mathscr{H}$, *a family* $\mathscr{S}=\{S_i\}_{i\in\Lambda}\subset B(\mathscr{K})$ *of pure coisometries satistying* $S_i(\mathscr{H})\subset\mathscr{H}$ $(i\in\Lambda)$ *with orthogonal initial spaces and an invertible operator* $R\in B(\mathscr{H})$ *such that*

$$A_i=R^{-1}(S_i|_{\mathscr{H}})R\quad\text{for each}\quad i\in\Lambda.$$

Proof. According to (2.4) and Proposition 2.6 (with $D=I_{\mathscr{H}}$), there exists $\mathscr{T}=\{T_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ with $\sum_{i\in\Lambda}T_i^*T_i\leqq I_{\mathscr{H}}$ and an invertible operator $R\in B(\mathscr{H})$ such that

$$A_i=R^{-1}T_iR\quad(i\in\Lambda). \tag{2.5}$$

On the other hand, for each $i\in\Lambda$,

$$\sum_{n=1}^{\infty}\|A_i^nh\|^2\leqq\sum_{n=1}^{\infty}\Big(\sum_{f\in F(n,\Lambda)}\|A_fh\|^2\Big)\leqq a\|h\|^2\quad\text{for any}\quad h\in\mathscr{H}.$$

Hence $A_i^n\to0$ (strongly) as $n\to\infty$ and by (2.5) $T_i^n\to0$ (strongly) for any $i\in\Lambda$. Applying Proposition 1.1, there exist a Hilbert space $\mathscr{K}\supset\mathscr{H}$, a family $\mathscr{S}=\{S_i\}_{i\in\Lambda}\subset B(\mathscr{K})$ of pure coisometries with $\sum_{i\in\Lambda}S_i^*S_i\leqq I_{\mathscr{K}}$, $S_i(\mathscr{H})\subset\mathscr{H}$ and $S_i|_{\mathscr{H}}=T_i$ for each $i\in\Lambda$. Thus, $A_i=R^{-1}(S_i|_{\mathscr{H}})R$ $(i\in\Lambda)$ and the proof is complete.

Remark 2.10. For Λ with card $\Lambda=1$ we find again the Rota model theorem, namely, if there exists $a>0$ such that $\sum_{n=1}^{\infty}\|A^nh\|^2\leqq a\|h\|^2$ for any $h\in\mathscr{H}$ (equivalent to $r(A)<1$), then A is similar to a part of a backward shift.

We now give some conditions equivalent to condition (2.4) in Corollary 2.9. The proof of the following proposition is almost identical to that of [1, Proposition 6]. We omit the proof.

Proposition 2.11. *Let* $\mathcal{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathcal{H})$. *The following conditions are equivalent.*

a) *There is a positive operator* $P\in B(\mathcal{H})$ *such that* $(\sum_{i\in\Lambda} A_i^*PA_i)+I_{\mathcal{H}}=P$.

b) *The series* $\sum_{i\in\Lambda} A_i^*A_i$ *and* $\sum_{n=1}^{\infty}(\sum_{f\in F(n,\Lambda)} A_f^*A_f)$ *are strongly convergent.*

c) *The series* $\sum_{i\in\Lambda} A_i^*A_i$ *is strongly convergent and* $r(\mathcal{A})<1$.

d) *There is* $a>0$ *such that* $\sum_{n=1}^{\infty}(\sum_{f\in F(n,\Lambda)}\|A_fh\|^2)\leqq a\|h\|^2$ *for any* $h\in\mathcal{H}$.

Remark 2.12. If $\sum_{i\in\Lambda} A_i^*A_i\leqq rI_{\mathcal{H}}$, where $r<1$, then the family $\mathcal{A}=\{A_i\}_{i\in\Lambda}$ satisfies the condition d) of Proposition 2.11.

3. In this section we generalize the result from [5], namely, an operator $A\in B(\mathcal{H})$ is similar to a contraction if and only if there exists $k\in\mathbf{N}$ such that A^k is similar to a contraction. We shall also obtain a formula for $r(\mathcal{A})$, where $\mathcal{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ with $\sum_{i\in\Lambda} A_i^*A_i$ strongly convergent, which generalizes the well-known formula for the spectral radius of an operator $A\in B(\mathcal{H})$, that is,

$$r(A)=\inf_S\|S^{-1}AS\|,$$

where the infimum is taken for all invertible operators $S\in B(\mathcal{H})$ (see [5, Problem 122]).

Proposition 3.1. *Let* $\mathcal{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathcal{H})$. *The following statements are equivalent:*

a) *There is a family* $\mathcal{C}=\{C_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ *with* $\sum_{i\in\Lambda} C_i^*C_i\leqq I_{\mathcal{H}}$ *such that* $\mathcal{A}$ *is simultaneously similar to* $\mathcal{C}$.

b) *There are* $k\in\mathbf{N}$ *and a family* $\mathcal{T}_k=\{T_{(f)}\}_{f\in F(k,\Lambda)}\subset B(\mathcal{H})$ *with* $\sum_{f\in F(k,\Lambda)} T_{(f)}^*T_{(f)}\leqq I_{\mathcal{H}}$ *such that the family* $\mathcal{A}_k=\{A_f\}_{f\in F(k,\Lambda)}$ *is simultaneously similar to* $\mathcal{T}_k$.

Moreover, a) *implies* b) *for all* $k\in\mathbf{N}$.

Proof. Assume condition a) is true. Let $R\in B(\mathcal{H})$ an invertible operator such that $A_i=R^{-1}C_iR$ $(i\in\Lambda)$. Hence $A_f=R^{-1}C_fR$ for $f\in F(k,\Lambda)$, $k\in\mathbf{N}$. Setting $T_{(f)}=C_f$ for $f\in F(k,\Lambda)$, $k\in\mathbf{N}$, we have that for each $k\in\mathbf{N}$ the family $\mathcal{A}_k$ is simultaneously similar to $\mathcal{T}_k$.

Conversely, assume b) is true. By Proposition 2.3 there is a positive invertible operator $P\in B(\mathcal{H})$ such that

$$\sum_{f\in F(k,\Lambda)} A_f^*PA_f\leqq P. \tag{3.1}$$

Let us consider the positive invertible operator $Q\in B(\mathscr{H})$ given by the relation

$$Q = P+\sum_{n=1}^{k-1}\Big(\sum_{f\in F(n,\Lambda)} A_f^* P A_f\Big).$$

Taking into account (3.1) we have

$$\sum_{i\in\Lambda} A_i^* Q A_i = \sum_{n=1}^{k}\Big(\sum_{f\in F(n,\Lambda)} A_f^* P A_f\Big) \leqq P+\sum_{n=1}^{k-1}\Big(\sum_{f\in F(n,\Lambda)} A_f^* P A_f\Big) = Q.$$

It then follows from Proposition 2.3 that a) is true, so the proof is complete.

Corollary 3.2. *Let $\mathscr{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ be such that there exist $k\in\mathbf{N}$, $0<r\leqq 1$ and*

$$\sum_{f\in F(k,\Lambda)} \|A_f h\|^2 \leqq r\|h\|^2 \quad \textit{for any} \quad h\in\mathscr{H}. \tag{3.2}$$

Then there exists $\mathscr{T}=\{T_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ with $\sum_{i\in\Lambda} T_i^* T_i\leqq I_{\mathscr{H}}$ and such that $\mathscr{A}$ is simultaneously similar to $\mathscr{T}$.

If $0<r<1$, one can even require that $\|T_i\|<1$ for any $i\in\Lambda$.

Proof. Note that the condition (3.2) is equivalent to the condition

$$\sum_{f\in F(k,\Lambda)} A_f^* A_f \leqq rI_{\mathscr{H}}. \tag{3.3}$$

By Proposition 3.1 there exists a family $\mathscr{T}=\{T_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ with $\sum_{i\in\Lambda} T_i^* T_i\leqq I_{\mathscr{H}}$ and such that $\mathscr{A}$ is simultaneously similar to $\mathscr{T}$.

If $0<r<1$, then there is $\varepsilon>1$ such that $\Big\|\varepsilon^{2k}\sum_{f\in F(k,\Lambda)} A_f^* A_f\Big\|\leqq 1$. Considering $B_i=\varepsilon A_i$ $(i\in\Lambda)$ we have $\sum_{f\in F(k,\Lambda)} B_f^* B_f\leqq I_{\mathscr{H}}$ and by Proposition 3.1 there exists a family $\mathscr{C}=\{C_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ with $\sum_{i\in\Lambda} C_i^* C_i\leqq I_{\mathscr{H}}$ such that the family $\mathscr{B}=\{B_i\}_{i\in\Lambda}$ is simultaneously similar to $\mathscr{C}$. Hence, $\mathscr{A}$ is simultaneously similar to the family $\mathscr{T}=\{T_i\}_{i\in\Lambda}$, where $T_i=(1/\varepsilon)C_i$ $(i\in\Lambda)$.

Remark 3.3. If $\mathscr{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ and $r(\mathscr{A})<1$, then the condition (3.2) of Corollary 3.2 is fulfilled.

Corollary 3.4. *If $\mathscr{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ and $r(\mathscr{A})=0$ then for every $\varepsilon>0$, there is a family $\mathscr{T}=\{T_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ with $\sum_{i\in\Lambda} T_i^* T_i\leqq\varepsilon^2 I_{\mathscr{H}}$ $(i\in\Lambda)$ such that $\mathscr{A}$ is simultaneously similar to $\mathscr{T}$.*

Proof. For any $\varepsilon>0$ we have $r(\varepsilon^{-1}\mathscr{A})=\varepsilon^{-1}r(\mathscr{A})=0$, where $\varepsilon^{-1}\mathscr{A}=\{\varepsilon^{-1}A_i\}_{i\in\Lambda}$. Hence, $r(\varepsilon^{-1}\mathscr{A})<1$ and by Remark 3.3 and Corollary 3.2 there is a family $\mathscr{C}=\{C_i\}_{i\in\Lambda}$ with $\sum_{i\in\Lambda} C_i^* C_i\leqq I_{\mathscr{H}}$ such that $\varepsilon^{-1}\mathscr{A}$ is simultaneously similar to $\mathscr{C}$. Setting $\mathscr{T}=\{T_i\}_{i\in\Lambda}$ where $T_i=\varepsilon C_i$ $(i\in\Lambda)$ the proof is complete.

We now use these results for proving the following

Proposition 3.5. *If* $\mathcal{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ *and* $\sum_{i\in\Lambda} A_i^*A_i$ *is strongly convergent, then*

$$r(\mathcal{A}) = \inf_S \{\|\sum_{i\in\Lambda} (S^{-1}A_iS)^*(S^{-1}A_iS)\|^{1/2}\},$$

where the infimum is taken for all invertible operators $S\in B(\mathcal{H})$.

Proof. First we show that for each invertible operator $S\in B(\mathcal{H})$, $r(\mathcal{A})=r(S^{-1}\mathcal{A}S)$, where $S^{-1}\mathcal{A}S$ stands for the family $\{S^{-1}A_iS\}_{i\in\Lambda}$. By the definition of $r(\mathcal{A})$ we have

$$r(S^{-1}\mathcal{A}S) \leqq \inf_k \{\|S\|^{1/k}\|S^{-1}\|^{1/k}\|\sum_{f\in F(k,\Lambda)} A_f^*A_f\|^{1/2k}\} \leqq r(\mathcal{A}).$$

Hence, $r(\mathcal{A})=r(S(S^{-1}\mathcal{A}S)S^{-1})\leqq r(S^{-1}\mathcal{A}S)$. Therefore,

$$r(\mathcal{A}) = r(S^{-1}\mathcal{A}S). \tag{3.4}$$

Using Lemma 2.1 and (3.4) we obtain

$$r(\mathcal{A}) \leqq \inf_S \{\|\sum_{i\in\Lambda} (S^{-1}A_iS)^*(S^{-1}A_iS)\|^{1/2}\}. \tag{3.5}$$

According to Corollary 3.4, if $r(\mathcal{A})=0$ and $0<\varepsilon<1$, then there is a family $\mathcal{T}=\{T_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ with $\sum_{i\in\Lambda} T_i^*T_i\leqq\varepsilon^2 I_{\mathcal{H}}$, $(i\in\Lambda)$ such that $A_i=R^{-1}T_iR$ $(i\in\Lambda)$ for an invertible operator $R\in B(\mathcal{H})$. Therefore,

$$\|\sum_{i\in\Lambda} (RA_iR^{-1})^*(RA_iR^{-1})\|^{1/2} \leqq \varepsilon$$

whence

$$\inf_S \{\|\sum_{i\in\Lambda} (S^{-1}A_iS)^*(S^{-1}A_iS)\|^{1/2}\} = 0.$$

If $r(\mathcal{A})\neq 0$, let us consider the family $\mathcal{B}=\{B_i\}_{i\in\Lambda}$ where $B_i=(\varepsilon/r(\mathcal{A}))A_i$, $0<\varepsilon<1$, $i\in\Lambda$. Since $r(\mathcal{B})<1$, by Remark 3.3 and Corollary 3.2 there exist a family $\mathcal{C}=\{C_i\}_{i\in\Lambda}$ with $\sum_{i\in\Lambda} C_i^*C_i\leqq I_{\mathcal{H}}$ and a positive invertible operator $P\in B(\mathcal{H})$ such that $B_i=P^{-1}C_iP$ $(i\in\Lambda)$. An easy computation shows that

$$\sum_{i\in\Lambda} (PA_iP^{-1})^*(PA_iP^{-1}) = (r(\mathcal{A})/\varepsilon)^2 \sum_{i\in\Lambda} C_i^*C_i,$$

whence

$$r(\mathcal{A}) \geqq \varepsilon\|\sum_{i\in\Lambda} (PA_iP^{-1})^*(PA_iP^{-1})\|^{1/2} \geqq \varepsilon \inf_S \{\|\sum_{i\in\Lambda} (SA_iS^{-1})^*(SA_iS^{-1})\|^{1/2}\}$$

for every $0<\varepsilon<1$. Setting $\varepsilon\to 1$ it follows that

$$r(\mathcal{A}) \geqq \inf_S \{\|\sum_{i\in\Lambda} (SA_iS^{-1})^*(SA_iS^{-1})\|^{1/2}\}. \tag{3.6}$$

The proof is complete.

References

[1] J. W. BUNCE, Models for n-tuples of noncommuting operators, *J. Funct. Anal.*, **57** (1984), 21—30.

[2] E. DURSZT and B. SZ.-NAGY, Remark to a paper of A. E. Frazho: „Models for noncommuting operators", *J. Funct. Anal.*, **52** (1983), 146—147.

[3] GH. ECKSTEIN and A. RÁCZ, On operators similar to contractions, *S. L. O. A. A.* (1978), Timişoara.

[4] A. E. FRAZHO, Models for noncommuting operators, *J. Funct. Anal.*, **48** (1982), 1—11.

[5] P. R. HALMOS, *A Hilbert space problem book*, Van Nostrand (New York, 1967).

[6] P. R. HALMOS, Ten problems in Hilbert space, *Bull. Amer. Math. Soc.*, **76** (1970), 887—933.

[7] G. C. ROTA, On models for linear operators, *Comm. Pure Appl. Math.*, **13** (1960), 469—472.

[8] B. SZ.-NAGY, On uniformly bounded linear transformations in Hilbert space, *Acta Sci. Math.*, **11** (1947), 152—157.

[9] B. SZ.-NAGY and C. FOIAŞ, *Harmonic analysis of operators on Hilbert space*, North-Holland (Amsterdam, 1970).

DEPARTMENT OF MATHEMATICS
THE NATIONAL INSTITUTE FOR
SCIENTIFIC AND TECHNICAL CREATION
BDUL PACII 220, 79622 BUCHAREST, ROMANIA

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On reducing subspaces of composition operators

JAMES GUYKER

If φ is an analytic function mapping the unit disk into itself, RYFF [10] has shown that φ induces a bounded linear operator C_φ on Hardy space H^2 defined by $C_\varphi f = f\circ\varphi$. Many of the basic properties of C_φ depend on the fixed points of φ in the closure of the disk (see [4], [9] for references). If φ is not a rotation about a fixed point, then by the Denjoy—Wolff theorem ([6], [13]) φ has a unique fixed point α such that $|\varphi'(\alpha)|\leqq 1$. In this paper, the reducing subspaces of classes of C_φ are characterized when $|\alpha|<1$ and either φ is univalent or some positive integral power of C_φ is compact. The complementary case when $\alpha=0$ and φ is inner follows from results of NORDGREN [8] and BROWN [2].

Notion. We will assume henceforth that φ is neither a constant nor a Möbius transformation of the disk onto itself, that α is the Denjoy—Wolff fixed point of φ, and that $|\alpha|<1$. Then $|\varphi'(\alpha)|<1$, and there is a natural basis of H^2 with respect to which C_φ is lower triangular with diagonal $[1, \varphi'(\alpha), \varphi'(\alpha)^2, \dots]$. Indeed, let

$$b_n(\alpha, z) = \frac{(1-|\alpha|^2)^{1/2}}{1-z\bar\alpha}\left[\frac{z-\alpha}{1-z\bar\alpha}\right]^n \quad (n=0, 1, \dots),$$

then for $i\leqq j$,

$$\langle C_\varphi b_j, b_i\rangle = \left\langle \frac{1-|\alpha|^2}{1-\varphi(z)\bar\alpha}\left[\frac{\varphi(z)-\varphi(\alpha)}{z-\alpha}\right]^j\left[\frac{1-z\bar\alpha}{1-\varphi(z)\bar\alpha}\right]^j\left[\frac{z-\alpha}{1-z\bar\alpha}\right]^{j-i}, \frac{1}{1-z\bar\alpha}\right\rangle,$$

which is $\varphi'(\alpha)^i$ whenever $i=j$, and is 0 when $i<j$.

Moreover, if f is in H^2, then $f=\sum\langle f, b_n\rangle b_n$ where

$$\langle f, b_n\rangle = \sum_{k=0}^{n}\binom{n}{k}\frac{f^{(k)}(\alpha)}{k!}(-\bar\alpha)^{n-k}(1-|\alpha|^2)^{k+(1/2)}.$$

This follows directly by writing f in terms of its Taylor series, expanding

$$\left[\frac{z-\alpha}{1-z\bar\alpha}\right]^n = \left[-\alpha+(1-|\alpha|^2)\frac{z}{1-z\bar\alpha}\right]^n$$

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by the binomial theorem, and by observing that $\langle (z-\alpha)^k, z^n/(1-z\bar{\alpha})^{n+1}\rangle$ is 0 whenever $n\neq k$, and is 1 when $n=k$, since the adjoint of multiplication by $z/(1-z\bar{\alpha})$ on H^2 maps $h(z)$ into $(h(z)-h(\alpha))/(z-\alpha)$.

We recall that a subspace $\mathcal{M}$ reduces an operator T on a Hilbert space $\mathcal{H}$ if both $\mathcal{M}$ and $\mathcal{H}\ominus\mathcal{M}$ are invariant under T, or equivalently, if the orthogonal projection onto $\mathcal{M}$ commutes with T. If the only subspaces that reduce T are $\{0\}$ and $\mathcal{H}$ itself, then T is said to be irreducible (otherwise T is reducible).

Theorem 1. *If φ is univalent and $\alpha\neq 0$, then C_φ is irreducible.*

When $\alpha=0$, the constants reduce C_φ. In fact, by [4, Theorem 4.1], the kernel of $1-C_\varphi$ contains only the constant functions, so it follows in this case that $\mathcal{M}$ reduces C_φ if and only if $\mathcal{M}=\mathcal{M}_0\oplus\mathcal{M}_1$, where $\mathcal{M}_0$ is either $\{0\}$ or the space of constants, and $\mathcal{M}_1$ reduces the restriction of C_φ to zH^2. A complete description (Theorem 2) of the subspaces $\mathcal{M}_1$ may be obtained under a compactness condition, with univalence weakened to $\varphi'(\alpha)\neq 0$. The study of compact composition operators was initiated by Schwartz in [11], and continued by several authors ([3], [5], [12]). In particular, Caughran and Schwartz [3, Theorem 2] have shown that when some positive integral power of C_φ is compact, the Denjoy—Wolff point always lies inside the disk. Note that $C_\varphi^N=C_{\varphi_N}$ where φ_N is defined inductively by $\varphi_1=\varphi$ and $\varphi_{n+1}=\varphi\circ\varphi_n$.

Theorem 2. *Suppose that C_φ^N is compact for some positive integer N, and that $\varphi'(\alpha)\neq 0$. Then C_φ is reducible if and only if $\alpha=0$. Moreover, if $\alpha=0$, then the restriction of C_φ to zH^2 is reducible if and only if there exists an H^∞ function Ψ which is bounded by one, and a nonnegative integer $p\neq 1$, such that $\varphi(z)=z\Psi(z^p)$; in this case, a subspace $\mathcal{M}$ reduces C_φ restricted to zH^2 if and only if $\mathcal{M}=\bigvee\{b_{ip+j}\colon i\geqq 0, j\in\Gamma\}$ where Γ is an arbitrary subset of $\{1,\dots,p\}$ ($\{1,2,\dots\}$ if $p=0$).*

The reducing subspaces of more general composition operators are formed from cyclic subsets of basis vectors as follows. Let $j\geqq 1$, $p\geqq 0$, and $r\geqq 1$ be integers such that if $p>0$, then $j\leqq p$ and p is relatively prime to r. Let $j_0=j$ and $j_{n+1}=$ $=rj_n-i_np$ $(n=0,1,\dots)$ where i_n is the unique integer which is 0 if $p=0$, and satisfies $i_np<rj_n\leqq(i_n+1)p$ if $p>0$. The set $\{j_n\colon n\geqq 0\}$ will be called the (r,p)-cycle generated by j. Let $p>0$. Then since $1\leqq j_n\leqq p$ for all n, the terms of the sequence j_n repeat. It follows easily that if $j_{m+1}=j_{n+1}$ $(m>n)$, then $j_m=j_n$; and hence $j_{m-n}=j$. Therefore, j is the first term to reappear. Moreover, the set $\{1,\dots,p\}$ ($\{1,2,\dots\}$ if $p=0$) may be written as a disjoint union of (r,p)-cycles. With no additional conditions on φ, we have

Theorem 3. *If $\alpha\neq 0$, then no nontrivial closed span of basis vectors b_n $(n\geqq 0)$ reduces C_φ. If $\alpha=0$, and is of order r, then a nontrivial closed span $\mathcal{M}$ of vectors b_n*

$(n \geqq 1)$ reduces the restriction of C_φ to zH^2 if and only if there exists a nonnegative integer $p \neq 1$, which is relatively prime to r whenever $p \neq 0$, such that $\varphi(z) = z^r \Psi(z^p)$ for some H^∞ function Ψ which is bounded by one, and $\mathcal{M} = \bigvee \{b_{ip+j}: i \geqq 0, j \in \Gamma\}$ where Γ is a union of (r, p)-cycles.

In view of the above results, a natural question is: when either φ is univalent or C_φ^N is compact, are all the reducing subspaces of C_φ closed spans of basis vectors? The related step in the proof of Theorem 2 follows by expressing the span of the first n basis vectors $(n = 0, 1, \ldots)$ in terms of the kernel of some element of the von Neumann algebra generated by 1, C_φ and C_φ^*. A similar argument may be used in the following example.

Example 1. Let $\varphi = \lambda\theta$ where λ is a constant $(0 < |\lambda| < 1)$ and θ is an inner function such that $\theta(0) = 0$. By ([1, Theorem 20], [8, Theorem 1], or [10, Theorem 3]), C_θ is an isometry, so that $C_\varphi^* C_\varphi$ is a diagonal operator with diagonal $(1, |\lambda|, |\lambda|^2, \ldots)$. Therefore, $\bigvee_0^n b_i = \ker \prod_0^n (C_\varphi^* C_\varphi - |\lambda|^i)$ $(n = 0, 1, \ldots)$, and it follows that the reducing subspaces are closed spans of b_n's and are thus described by Theorem 3.

Further evidence is provided by the following result which implies that reducing subspaces are (at least) closed spans of *finite* linear combinations of basis vectors.

Theorem 4. *Suppose that $\|\varphi\|_\infty < 1$ and $\varphi'(\alpha) = 0$. If X commutes with C_φ and (λ_{ij}) is the matrix of X with respect to $\{b_n\}$, then $\lambda_{0j} = 0$ $(j = 1, 2, \ldots)$ and there exists an integer M such that $\lambda_{ij} = 0$ $(i = 1, 2, \ldots)$ for every $j \geqq Mi$.*

Theorem 4 suggests an alternative approach to answering the above question in the affirmative, as illustrated by

Example 2. Let $\alpha = 0$ be of order $r > 1$, and suppose that φ is a polynomial of degree r^M such that $\|\varphi_N\|_\infty < 1$ for some positive integers M and N. Then the reducing subspaces of C_φ are given by Theorem 3: Let P be the projection onto a reducing subspace. Since P commutes with $C_{\varphi_N}^*$, it follows from Theorem 4 that Pb_n is a polynomial for every n; thus, it suffices to show that the degree of Pb_n does not exceed n for every n. Suppose that $n < \deg Pb_n$ for some n, and let i be the least such integer. Setting $j = \deg Pb_i$, we have that

$$\langle PC_{\varphi_N}^{*M} C_{\varphi_N} b_i, b_j \rangle = \langle C_{\varphi_N}^{*M} C_{\varphi_N} Pb_i, b_j \rangle,$$

and hence by straightforward calculations, $\mu^i = \mu^j$ where

$$\mu = [\langle b_r, \varphi \rangle^{1/(r-1)} \langle \varphi, b_{r^M} \rangle^{1/(r^M - 1)}]^{r^{MN} - 1}$$

(so that $0 < |\mu| < 1$). Therefore $i = j$, a contradiction.

The verification of Theorem 3 depends upon a reformulation of the usual multinomial theorem, which subsequently determines how often powers of φ have nonzero coefficients.

Lemma. *Let*

$$f(z) = a_{10}+\sum_{i=1}^{\infty}\sum_{j=0}^{p-1} a_{1,ip+j}z^{ip+j}$$

be a formal power series where a_{10} is nonzero, and for $j=0, 1, \ldots,$ define

$$\hat{a}_{1j} = a_{1,p+j} \quad and \quad \hat{a}_{mj} = \sum_{k=0}^{j} a_{1,p+k}\hat{a}_{m-1,j-k} \quad (m>1).$$

Then for every positive integer n,

$$f(z)^n = a_{10}^n+\sum_{i=1}^{\infty}\sum_{j=0}^{p-1} a_{n,ip+j}z^{ip+j}$$

where

$$a_{n,ip+j} = \sum_{m=1}^{i}\binom{n}{m}a_{10}^{n-m}\hat{a}_{m,(i-m)p+j}.$$

In particular, for fixed i and j, either $a_{n,ip+j}=0$ for all $n\geqq 1$, or $a_{n,ip+j}=0$ for at most $i-1$ values of $n\geqq 1$. If $a_{1p}\neq 0$, then $a_{n,ip}=0$ for at most $i-1$ values of $n\geqq 1$.

The following estimate in H^∞ is essential to the proof of Theorem 4, and may be of independent interest.

Proposition. *If $\varphi^{(i)}(\alpha)=0$ for every $i=1, \ldots, r-1$, then*

$$\|C_\varphi^n b_m\|_\infty \leqq \left\|\frac{\varphi-\alpha}{1-\varphi\bar{\alpha}}\right\|_\infty^{m(r^n-1)/(r-1)}$$

for all nonnegative integers m and n.

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Proof of Theorem 1. Let $\mathcal{M}$ reduce C_φ. By [4, Theorem 4.1], the kernel of $1-C_\varphi$ consists of just the constant functions. Thus we may assume that constants belong to $\mathcal{M}$, and hence so does $C_\varphi^{*n}1=[1-z\overline{\varphi_n(0)}]^{-1}$ $(n=1, 2, \ldots)$.

Let f be orthogonal to $\mathcal{M}$. Then f vanishes on the set $\Omega=\{\varphi_n(0)\colon n\geqq 1\}$. If $\varphi_m(0)=\varphi_{n+m}(0)$ for some positive integers m and n, then $\varphi_m(0)$ is a fixed point of φ_n. But α is also a fixed point of φ_n, and φ_n is not a rotation about a fixed point since otherwise φ would be inner and hence by [7; Theorem 3.17] φ would be a Möbius transformation. Thus it follows that $\varphi_m(0)=\alpha=\varphi_m(\alpha)$, and since φ_m is univalent, we have that $\alpha=0$, a contradiction. Hence, the set Ω consists of distinct points which must cluster in the closure of the disk. However, by Schwarz's lemma,

$$\left|\frac{\varphi_n(z)-\alpha}{1-\varphi_n(z)\bar{\alpha}}\right| \leqq \left|\frac{z-\alpha}{1-z\bar{\alpha}}\right|$$

for every z in the disk. Setting $z=0$, we conclude that Ω must cluster inside the disk. Therefore, f is identically zero.

Proof of the lemma. The formula is obvious for $n=1$, so by induction, we assume it is valid for some n. Multiplying $f(z)^{n+1}=f(z)\cdot f(z)^n$, we have that

$$a_{n+1,ip+j} = a_{10}a_{n,ip+j}+\Big[\sum_{k=0}^{(i-2)p+j} a_{1,p+k}a_{n,(i-1)p+j-k}\Big]+a_{1,ip+j}a_{10}^n =$$

$$= \left[\sum_{m=1}^{i}\binom{n}{m}a_{10}^{n-m+1}\hat{a}_{m,(i-m)p+j}\right]+\left[\sum_{m=2}^{i}\binom{n}{m-1}a_{10}^{n-m+1}\hat{a}_{m,(i-m)p+j}\right]+a_{10}^n a_{1,ip+j} =$$

$$= \sum_{m=1}^{i}\left[\binom{n}{m}+\binom{n}{m-1}\right]a_{10}^{n-m+1}\hat{a}_{m,(i-m)p+j} = \sum_{m=1}^{i}\binom{n+1}{m}a_{10}^{(n+1)-m}\hat{a}_{m,(i-m)p+j}.$$

Thus, the form of $f(z)^n$ follows for every n.

Fix i and j, and let $n\geqq i$. Then $(na_{10}^n)^{-1}a_{n,ip+j}$ is a polynomial in n of degree at most $i-1$. Suppose that $a_{k,ip+j}=0$ for some k such that $1\leqq k\leqq i-1$. It follows that the sum of the first k terms of $(na_{10}^n)^{-1}a_{n,ip+j}$ is equal to

$$\sum_{m=1}^{k}\frac{1}{m!}\left[\frac{(n-1)!}{(n-m)!}-\frac{(k-1)!}{(k-m)!}\right]a_{10}^{-m}\hat{a}_{m,(i-m)p+j},$$

which is divisible by $n-k$. Since each of the last $i-k$ terms contains a factor of $n-k$, we have that $n-k$ divides $(na_{10}^n)^{-1}a_{n,ip+j}$. Therefore, either all of the coefficients of $(na_{10}^n)^{-1}a_{n,ip+j}$ $(n\geqq i)$ are zero, or $a_{n,ip+j}=0$ for at most $i-1$ values of $n\geqq 1$.

Finally, suppose that $a_{1p}\neq 0$. Then the leading coefficient of $(na_{10}^n)^{-1}a_{n,ip}$ $(n\geqq i)$ is $(i!\,a_{10}^i)^{-1}\hat{a}_{i0}=(i!\,a_{10}^i)^{-1}a_{1p}^i\neq 0$. Hence, $a_{n,ip}=0$ for at most $i-1$ values of $n\geqq 1$.

Proof of Theorem 3. Let $\alpha\neq 0$, and suppose that $\mathcal{M}$ is a nontrivial closed span of b_n's $(n\geqq 0)$ which reduces C_φ. Since $\mathcal{M}^\perp$ is of the same form, we may assume that b_0 is in $\mathcal{M}$. Let n be the greatest integer such that b_m belongs to $\mathcal{M}$ for every $m=0, \ldots, n$. Since $\mathcal{M}$ is invariant under C_φ, we have that $f=C_\varphi\sum_0^n(-\bar{\alpha})^m\times$

$\times(1-|\alpha|^2)^{-1/2}b_m$ is in $\mathcal{M}$ and $\langle f, b_{n+1}\rangle=0$. However, by induction,

$$f'(z)=-(n+1)(-\bar{\alpha})^{n+1}\varphi'(z)[1-\varphi(z)\bar{\alpha}]^{-n-2}[\varphi(z)-\alpha]^n$$

and

$$f^{(n+1)}(\alpha)=-(n+1)!(-\bar{\alpha})^{n+1}(1-|\alpha|^2)^{-n-2}\varphi'(\alpha)^{n+1}.$$

Therefore,

$$\langle f, b_{n+1}\rangle=(-\bar{\alpha})^{n+1}(1-|\alpha|^2)^{-1/2}[1-\varphi'(\alpha)^{n+1}]\neq 0,$$

a contradiction. Thus, $\mathcal{M}$ must be trivial.

Next, let $\alpha=0$ be a zero of φ of order r, and let $\mathcal{M}$ be a nontrivial closed span of vectors $b_n=z^n$ $(n\geqq 1)$ which reduces the restriction of C_φ to zH^2. Let us write $\varphi(z)=z^r\Psi(z)$ for some H^∞ function Ψ. If Ψ is a constant function, then the proposed forms of φ and $\mathcal{M}$ clearly follow with $p=0$. Henceforth, using the notation of the lemma, we assume that $\Psi(z)=\sum\langle\Psi, b_n\rangle b_n=a_{10}+a_{1q}z^q+a_{1,q+1}z^{q+1}+\dots$ where a_{10} and a_{1q} are nonzero. Observe that if $\mathcal{M}$ contains z^n, and $a_{n,rq}\neq 0$, then $\mathcal{M}$ also contains z^{n+q}. Indeed, $\varphi(z)^n=z^{rn}\Psi(z)^n$ is in $\mathcal{M}$, and $\langle\varphi(z)^n, z^{r(n+q)}\rangle\neq 0$; therefore, by the given form of $\mathcal{M}$, $z^{r(n+q)}$ belongs to $\mathcal{M}$. Since z^{rm} is orthogonal to $\mathcal{M}$ whenever z^m is, we have that $\mathcal{M}$ contains z^{n+q}.

Let z^n be in $\mathcal{M}$. By the lemma, there exists an integer K such that $a_{k,rq}\neq 0$ for every $k\geqq r^Kn$. Now, z^{r^Kn} is in $\mathcal{M}$. And, if z^{r^Kn+mq} is in $\mathcal{M}$ for some $m\geqq 0$, then $a_{r^Kn+mq,rq}\neq 0$, and, by the above argument, $\mathcal{M}$ contains $z^{r^Kn+(m+1)q}$. Thus, by induction, z^{r^Kn+mq} is in $\mathcal{M}$ for every $m=0, 1, \dots$, and hence, in particular, $z^{r^K(n+q)}$ is in $\mathcal{M}$. Consequently, $\mathcal{M}$ contains z^{n+q} whenever it contains z^n.

For integers i and j, let $i\wedge j$ denote the greatest common divisor of i and j. Let $q(1)=q$, and for $t=2, 3, \dots$, define $q(t)=[r\wedge q(t-1)]^{-1}q(t-1)$. Since $\{q(t)\}$ is a monotonically decreasing sequence of positive integers, there exists a least integer T such that $q(T+1)=q(T)$, i.e., $r\wedge q(T)=1$. Note that $q(T)=r^{-T}\varrho q$ where $\varrho=\prod_1^T[r\wedge q(t)]^{-1}r$. If z^n belongs to $\mathcal{M}$, it follows that $z^{n+q(T)}=z^{r^{-T}(r^Tn+\varrho q)}$ belongs to $\mathcal{M}$. Similarly, the orthogonal complement of $\mathcal{M}$ in zH^2 is invariant under multiplication by $z^{q(T)}$. Therefore, there exists a subset Γ_1 of $\{1, \dots, q(T)\}$ such that $\mathcal{M}$ is the closed span of vectors of the form $z^{iq(T)+j}$ $(i\geqq 0;\ j\in\Gamma_1)$. Furthermore, if j is in Γ_1, then $iq(T)<rj\leqq(i+1)q(T)$ for some integer i, and $rj-iq(T)$ is in Γ_1. Hence, Γ_1 is a union of $[r, q(T)]$-cycles.

If $\Psi=\Psi(z^{q(T)})$, let $p=q(T)$ and $\Gamma=\Gamma_1$; otherwise, let $p(1)=q(T)$. Suppose that for some integer $s\geqq 1$, a positive integer $p(s)$, relatively prime to r, is defined such that $\mathcal{M}=\bigvee\{z^{ip(s)+j}\colon i\geqq 0,\ j\in\Gamma_s\}$ for some union Γ_s of $[r, p(s)]$-cycles, and $\Psi\neq\Psi(z^{p(s)})$. Let $I=\min\{i\colon a_{1,ip(s)+j}\neq 0$ for some j such that $0<j<p(s)\}$, and let $J=\min\{j>0\colon a_{1,Ip(s)+j}\neq 0\}$. By the lemma, $a_{n,Ip(s)+J}=na_{10}^{n-1}a_{1,Ip(s)+J}\neq 0$ for every $n=1, 2, \dots$. As above, $z^{rn+Ip(s)+J}$, and hence z^{rn+J}, belong to $\mathcal{M}$ whenever z^n belongs to $\mathcal{M}$.

Let z^n be in $\mathcal{M}$. Then $n=ip(s)+j$ where j is in Γ_s and $i\geqq 0$. Since Γ_s is a union of cycles, there exists an element j' in Γ_s and an integer i' such that $j=rj'-i'p(s)$. Thus, $n+J=(i-i')p(s)+(rj'+J)$, so that z^{n+J} is in $\mathcal{M}$. Similarly, $zH^2\ominus\mathcal{M}$ is invariant under multiplication by z^J.

Let $p(s+1)=p(s)\wedge J$. Then $p(s+1)$ is relatively prime to r, and there exist integers u and v such that $p(s+1)=up(s)+vJ$. It follows that $\mathcal{M}$ and the orthogonal complement of $\mathcal{M}$ in zH^2 are invariant under multiplication by $z^{p(s+1)}$, and hence, as above, there exists a union Γ_{s+1} of $[r, p(s+1)]$-cycles such that $\mathcal{M}=\bigvee\{z^{ip(s+1)+j}\colon i\geqq 0,\ j\in\Gamma_{s+1}\}$. Therefore, $p(s+1)$ $(s=1, 2, \ldots)$ may be defined recursively provided $\Psi\neq\Psi(z^{p(s)})$ for every s. But this is impossible since $\{p(s)\}$ is a strictly decreasing sequence of positive integers. Consequently, there exists an integer S such that $\Psi=\Psi(z^{p(S)})$, and the forms of φ and $\mathcal{M}$ follow by setting $p=p(S)$ and $\Gamma=\Gamma_S$. Note that $p\neq 1$ since $\mathcal{M}$ is nontrivial.

Conversely, suppose that $\varphi(z)=z^r\Psi(z^p)$ where $r\wedge p=1$ if $p\neq 0$, and that $\mathcal{M}=\bigvee\{z^{ip+j}\colon i\geqq 0,\ j\in\Gamma\}$ for some union Γ of (r,p)-cycles. If $p=0$, then clearly $\mathcal{M}$ is invariant under C_φ; so we assume that $p>1$. If z^n belongs to $\mathcal{M}$, then so does z^{rq+mp} for every $m\geqq 0$. Indeed $n=ip+j$ where j is in Γ and $i\geqq 0$ and there exists an integer i' such that $i'p<rj\leqq(i'+1)p$. Hence, $rj=i'p+j'$, where j' is in Γ by the definition of (r,p)-cycle. Therefore, $z^{rn+mp}=z^{(ri+i'+m)p}z^{j'}$ is in $\mathcal{M}$, and thus, so is $\varphi(z)^n=z^{rn}\Psi(z^p)^n$. It follows that $\mathcal{M}$ is invariant under C_φ.

Finally, $\mathcal{M}^\perp$ is invariant under C_φ since it is the closed span of vectors of the form z^{ip+j} $(i\geqq 0;\ j\in\Gamma')$, where Γ' is the complement in $\{1, \ldots, p\}$ ($\{1, 2, \ldots\}$, if $p=0$) of Γ and is hence the union of (r,p)-cycles.

Proof of Theorem 2. Since $C_\varphi^{*^N}=C_{\varphi_N}^*$ is compact with nonzero eigenvalues $\overline{\varphi'(\alpha)}^{mN}$ $(m=0, 1, \ldots)$, it follows from [4, Theorem 4.1] that $\overline{\varphi'(\alpha)}^m$ is an eigenvalue of C_φ^* of multiplicity one for every m. Thus, by the matrix of C_φ^* with respect to $\{b_n\}$, we have that $\bigvee_0^n b_m=\ker\prod_0^n[C_\varphi^*-\overline{\varphi'(\alpha)}^m]$ for every $n=0, 1, \ldots$. Therefore, by induction, either b_n belongs to $\mathcal{M}$ or is orthogonal to $\mathcal{M}$ for each n, and hence the form of $\mathcal{M}$ is given by Theorem 3.

Proof of the proposition. Using induction on n with m fixed, the case $n=0$ is obvious, so we assume that the inequality holds for some n. Since $\varphi(\alpha)=\alpha$ and $\varphi^{(i)}(\alpha)=0$ $(i=1, \ldots, r-1)$, we have that $\varphi_n^{(i)}(\alpha)=0$ for every $i=1, \ldots, r^n-1$. Hence, $C_\varphi^{n+1}b_m=C_\varphi f$ where $f=b_m(\alpha, \varphi_n)=[(z-\alpha)/(1-z\bar\alpha)]^{mr^n}g$ for some H^∞ function g. Therefore, $\|C_\varphi^{n+1}b_m\|_\infty=\|(\varphi-\alpha)/(1-\varphi\bar\alpha)\|_\infty^{mr^n}\|g(\varphi)\|_\infty$ where $\|g(\varphi)\|_\infty\leqq$ $\leqq\|g\|_\infty=\|f\|_\infty\leqq\|(\varphi-\alpha)/(1-\varphi\bar\alpha)\|_\infty^{m(r^n-1)/(r-1)}$ by the induction hypothesis. The case $n+1$ now follows by combining the above inequalities.

Proof of Theorem 4. Since $\|\varphi\|_\infty<1$, C_φ is compact by [11, Theorem 5.2].

Therefore, by [4, Theorem 4.1], the kernel of $1-C_\varphi^*$ is one-dimensional, and since it is invariant under X^*, we have that $\lambda_{0j}=0$ $(j\geqq 1)$.

Suppose that $\varphi^{(m)}(\alpha)=0$ $(m=1, \ldots, r-1)$ and $\varphi^{(r)}(\alpha)\neq 0$. By direct computations, there exist constants $\mu_{i'}=\mu_{i'}(n)$ such that for every $i\geqq 0$,

$$b_i = \bar{\mu}^{-i(r^n-1)/(r-1)} C_\varphi^{*^n} b_{ir^n} + \sum_{i'<i} \mu_{i'} b_{i'}$$

where $\mu=(1-|\alpha|^2)^{r-1}\varphi^{(r)}(\alpha)(r!)^{-1}$. Moreover, since $\|\varphi\|_\infty<1$, it follows that $\|(\varphi-\alpha)/(1-\varphi\bar{\alpha})\|_\infty<1$, and hence there exists an integer $M\geqq 1$ such that $\|(\varphi-\alpha)/(1-\varphi\bar{\alpha})\|_\infty^M<|\mu|$. Thus,

$$\lambda_{ij} = \langle Xb_j, b_i\rangle_2 = \mu^{-i(r^n-1)/(r-1)} \langle XC_\varphi^n b_j, b_{ir^n}\rangle_2 + \sum_{i'<i} \bar{\mu}_{i'} \lambda_{i'j}$$

and consequently by the proposition, for $j\geqq Mi$ we have that

$$\Big|\lambda_{ij} - \sum_{i'<i} \bar{\mu}_{i'} \lambda_{i'j}\Big| \leqq \|X\|_2 \left(\left\| \frac{\varphi-\alpha}{1-\varphi\bar{\alpha}} \right\|_\infty^M |\mu|^{-1} \right)^{i(r^n-1)/(r-1)}.$$

Therefore, the theorem follows by induction on $i\geqq 1$, and the separate case $i=0$, since the right hand side converges to zero as n tends to infinity.

References

[1] L. de Branges and J. Rovnyak, *Square Summable Power Series*, Holt, Rinehart, and Winston (New York, 1966).

[2] A. Brown, On a class of operators, *Proc. Amer. Math. Soc.*, 4 (1953), 723—728.

[3] J. G. Caughran and H. J. Schwartz, Spectra of compact composition operators, *Proc. Amer. Math. Soc.*, 51 (1975), 127—130.

[4] C. C. Cowen, Composition operators on H^2, *J. Operator Theory*, 9 (1983), 77—106.

[5] J. A. Deddens, Analytic Toeplitz and composition operators, *Canad. J. Math.*, 24 (1972), 859—865.

[6] A. Denjoy, Sur l'itération des fonctions analytiques, *C. R. Acad. Sci. Paris Sér. A.*, 182 (1926), 255—257.

[7] P. L. Duren, *Theory of H^p Spaces*, Academic Press (New York, 1970).

[8] E. A. Nordgren, Composition operators, *Canad. J. Math.*, 20 (1968), 442—449.

[9] E. A. Nordgren, Composition operators on Hilbert spaces, in: *Hilbert Space Operators*, Lecture Notes in Math., 693, Springer-Verlag (Berlin, 1978), pp. 37—63.

[10] J. V. Ryff, Subordinate H^p functions, *Duke Math. J.*, 33 (1966), 347—354.

[11] H. J. Schwartz, *Composition Operators on H^p*, Thesis, (University of Toledo, 1969).

[12] J. H. Shapiro and P. D. Taylor, Compact, nuclear, and Hilbert—Schmidt composition operators on H^2, *Indiana Univ. Math. J.*, 23 (1973), 471—496.

[13] J. Wolff, Sur l'itération des fonctions, *C. R. Acad. Sci. Paris Sér. A.*, 182 (1926), 42—43, 200—201.

DEPARTMENT OF MATHEMATICS
SUNY COLLEGE AT BUFFALO
BUFFALO, NEW YORK 14222
U. S. A.

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The numerical ranges and the smooth points of the unit sphere

FRANK PIETSCHMANN and ADOLF RHODIUS

I. Let S_p be the unit sphere of a complex Banach space (E, p). The set of all smooth points on S_p will be denoted by F_p. The element $x \in S_p$ is a smooth point if and only if the Gâteaux derivative p' at x exists. We denote by $V_{D_p}(T)$ the spatial numerical range of T. If the unit sphere is smooth, then the relation

$$V_{D_p}(T) = \{p'(x, Tx) - ip'(x, iTx) \colon x \in S_p\}$$

holds. We assume that the set F_p is dense in the unit sphere S_p, e.g. this holds for separable or reflexive Banach spaces. We prove that for continuous operators T the closure of the set

$$\{p'(x, Tx) - ip'(x, iTx) \colon x \in F_p\}$$

is the closure of a Lumer numerical range of T.

II. Let D_p be the mapping of S_p into the power set of the dual E' of E defined by

$$D_p(x) = \{f \in E' \colon f(x) = 1, \ |f(y)| \leqq p(y), \ (y \in E)\}.$$

We consider the continuous operator $G \colon E \to E$ with the domain $D(G) \subseteqq S_p$. For a mapping Q_p of $D(G)$ into the power set of E' with

$$\emptyset \neq Q_p(x) \subseteqq D_p(x) \quad (x \in D(G))$$

the set

$$V_{Q_p}(G) = \{f(Gx) \colon f \in Q_p(x), \ x \in D(G)\}$$

is called the numerical range of G corresponding to Q_p. (See [7].) If card $Q_p(x) = 1$ $(x \in D(G))$ holds, then $V_{Q_p}(G)$ is a Lumer numerical range. $V_{D_p}(G)$ is called the spatical numerical range of G.

Theorem 1. *Let T be a continuous operator of S_p into E. If $V_{\tilde{Q}_p}(T|A)$ is a numerical range of the restriction of T to the subset A of S_p with* cl $A = S_p$, *then there exists an extension Q_p of $\tilde{Q}_p$ to the unit sphere S_p such that*

$$\operatorname{cl} V_{\tilde{Q}_p}(T|A) = \operatorname{cl} V_{Q_p}(T).$$

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Proof. Let $x \in S_p \setminus A$. Then there are sequences (x_n) in A and (f_n) with $f_n \in \tilde{Q}_p(x_n)$ and $p(x_n - x) \to 0$. Since the unit ball of E' is weak*-compact, we can choose subnets $(f_\beta)_{\beta \in B}$ of (f_n) and $(x_\beta)_{\beta \in B}$ of (x_n) and an $f_x \in E'$ such that

$$(f_\beta)_{\beta \in B} \text{ is weak *-convergent to } f_x \text{ and } p(x_\beta - x) \to 0.$$

The inequalities

$$|f_n(y)| \leqq p(y) \quad (y \in E,\ n \in N)$$

imply

$$|f_x(y)| \leqq p(y) \quad (y \in E).$$

But since

$$f_\beta(x_\beta) = f_\beta(x_\beta - x) + f_\beta(x); \quad |f_\beta(x_\beta - x)| \leqq p(x_\beta - x)$$

we deduce $f_\beta(x_\beta) \to f_x(x)$ and $f_x(x) = 1$. So we have $f_x \in D_p(x)$. Now we extend the mapping $\tilde{Q}_p$ by the definition

$$Q_p(z) = \begin{cases} \tilde{Q}_p(z) & \text{for } z \in A, \\ \{f_z\} & \text{for } z \in S_p \setminus A. \end{cases}$$

It is clear that the relation $\operatorname{cl} V_{\tilde{Q}_p}(T|A) \subseteqq \operatorname{cl} V_{Q_p}(T)$ holds. It remains to show that the scalar $f_x(Tx)$ is a cluster point of $V_{\tilde{Q}_p}(T|A)$ $(x \in S_p \setminus A)$. By the construction there are nets $(x_\beta)_{\beta \in B}$ of A and $(f_\beta)_{\beta \in B}$ with $f_\beta \in \tilde{Q}_p(x_\beta)$ such that

$$f_\beta(y) \to f_x(y) \ (y \in E) \quad \text{and} \quad p(x_\beta - x) \to 0.$$

The inequality $|f_\beta(Tx_\beta - Tx)| \leqq p(Tx_\beta - Tx)$ and the continuity of T imply $f_\beta(Tx_\beta - Tx) \to 0$. Hence from the relation

$$f_\beta(Tx_\beta) = f_\beta(Tx) + f_\beta(Tx_\beta - Tx)$$

follows $f_\beta(Tx_\beta) \to f_x(Tx)$.

Remark 1. The proof of Theorem 1 shows that there exists an extension Q_p of $\tilde{Q}_p$ satisfying the condition $\operatorname{card} Q_p(x) = 1$ $(x \in S_p \setminus A)$.

Theorem 2. *Let T be a continuous operator of S_p into E. If F_p is dense in S_p, then the set*

$$\operatorname{cl} \{p'(x, Tx) - ip'(x, iTx) \colon x \in F_p\}$$

is the closure of a Lumer numerical range of T corresponding to a mapping Q_p defined on the whole S_p.

Proof. We applicate Theorem 1 putting $A = F_p$. There exists exactly one mapping $\tilde{Q}_p$ of F_p into the power set of E' with $\emptyset \neq \tilde{Q}_p(x) \subseteqq D_p(x)$ $(x \in F_p)$. By [6] holds

$$V_{\tilde{Q}_p}(T|F_p) = \{p'(x, Tx) - ip'(x, iTx) \colon x \in F_p\}.$$

Hence by Theorem 1 the conclusion follows.

Corollary 1. *Let T be a linear continuous operator of S_p into E. If F_p is dense in S_p, then for the numerical radius $v_p(T)$ the following relation holds:*

$$v_p(T) = \sup_{x \in F_p} |p'(x, Tx) - ip'(x, iTx)|.$$

Remark 2. The condition $\operatorname{cl} F_p = S_p$ is fulfilled for separable Banach spaces (see [5]), and for reflexive Banach spaces (see [3]).

Remark 3. Let E be a separable Banach space and let T be a linear continuous operator of S_p into E. While the set

$$\operatorname{cl} \{p'(x, Tx) - ip'(x, iTx) \colon x \in F_p\}$$

is the closure of a Lumer numerical range of T defined on the whole S_p, in general it is not the closure of the spatial numerical range of T. We consider the following example.

Let c_0 be the Banach space of all complex null sequences $x=(x_i)$ equipped with the norm $p(x) = \max |x_i|$. Then $x \in S_p$ is a smooth point on S_p if and only if the relation $|x_i|=1$ holds for exactly one coordinate x_i of x; let be $|x_{i(x)}|=1$. Using the functional f_x defined by

$$f_x(y) = y_{i(x)} \bar{x}_{i(x)} \quad (y = (y_i)),$$

it follows $\tilde{Q}_p(x) = \{f_x\}$ $(x \in F_p)$. For the operator T with

$$Tx = (x_1, 1/2x_2, 1/3x_3, \ldots, 1/nx_n, \ldots) \quad (x \in c_0)$$

one obtains $V_{\tilde{Q}_p}(T|F_p) = \{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$. Therefore the set $\{1/n \colon n \in \mathbf{N}\} \cup \{0\}$ is the closure of a Lumer numerical range of T defined on the whole S_p. The closure of the spatial numerical range of T is the interval $\{\lambda \in \mathbf{R} \colon 0 \leq \lambda \leq 1\}$.

References

[1] F. F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and of elements of normed algebras*, London Math. Soc. Lect. Note series, 2 (Campbridge, 1971).

[2] F. F. Bonsall and J. Duncan, *Numerical Ranges. II*, London Math. Soc. Lect. Note series, 10 (Cambridge, 1973).

[3] J. Lindenstrauß, On nonseparable reflexive Banach spaces, *Bull. Amer. Math. Soc.*, **72** (1966), 967—970.

[4] G. Lumer, Semi-inner-product spaces, *Trans. Amer. Math. Soc.*, **100** (1961), 29—43.

[5] S. Mazur, Über konvexe Mengen in linearen normierten Räumen, *Studia Math.*, **4** (1933), 70—84.

[6] A. Rhodius, Der numerische Wertebereich für nicht notwendig lineare Abbildungen in lokalkonvexen Räumen, *Math. Nachrichten*, **72** (1976), 169—180.

[7] A. Rhodius, Über zu Halbnormen gehörende numerische Wertebereiche linearer Operatoren, *Math. Nachrichten*, **86** (1978), 181—185.

SEKTION MATHEMATIK
PH „K. F. W. WANDER" DRESDEN, PSF 365
DRESDEN 806, DDR

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On the joint Weyl spectrum. II

MUNEO CHŌ

Dedicated to Professor Satoshi Kotō in token of gratitude

1. Introduction

In [3], we studied the joint Weyl spectrum for a commuting pair. In this paper we shall show that the Weyl theorem holds for a commuting pair of normal operators.

Let H be a complex Hilbert space with the scalar product $(\,,\,)$ and the norm $\|\cdot\|$. Let $B(H)$ be the algebra of all bounded linear operators on H and $C(H)$ the algebra of all compact operators in $B(H)$.

Definition 1. Let $T=(T_1, T_2)\subset B(H)$ be a commuting pair. Taylor joint spectrum $\sigma(T)$ of T is defined by $\sigma(T)=\{z=(z_1, z_2)\in \mathbf{C}^2: \alpha(T-z)$ is not invertible on $H\oplus H\}$, where

$$\alpha(T-z) = \begin{pmatrix} T_1-z_1 & T_2-z_2 \\ -(T_2-z_2)^* & (T_1-z_1)^* \end{pmatrix}.$$

Definition 2. Let $T=(T_1, T_2)\subset B(H)$ be a commuting pair. The joint Weyl spectrum $\omega(T)$ of T is defined by

$$\omega(T) = \bigcap\{\sigma(T+C): C = (C_1, C_2)\subset C(H) \text{ and } T+C = (T_1+C_1, T_2+C_2)$$

is a commuting pair$\}$.

$z=(z_1, z_2)$ in $\mathbf{C}^2$ is said to be joint eigenvalue of $T=(T_1, T_2)$ if there exists a non-zero vector x such that

$$T_i x = z_i x \quad (i = 1, 2).$$

$\sigma_p(T)$ is the set of joint eigenvalues of T.

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$z=(z_1, z_2)$ in $\mathbf{C}^2$ is said to be joint residual eigenvalue of $T=(T_1, T_2)$ if there exists a non-zero vector x such that

$$T_i^* x = \bar{z}_i x \quad (i = 1, 2).$$

$\sigma_r(T)$ is the set of joint residual eigenvalues of T.

For a commuting pair $T=(T_1, T_2)$, $\sigma_{pf}(T)$ is the set of joint eigenvalues of finite multiplicity, $\sigma_{rf}(T)$ is the set of joint residual eigenvalues of finite multiplicity, $\sigma_{pfi}(T)$ is the set of isolated points in $\sigma(T)$ which are joint eigenvalues of finite multiplicity and $\sigma_{rfi}(T)$ is the set of isolated points in $\sigma(T)$ which are joint residual eigenvalues of finite multiplicity.

For any operator S on H, we denote by $N(S)$ the null space of S.

2. Theorem

Theorem A (Vasilescu [6]). *For a commutating pair* $T=(T_1, T_2)$, $\alpha(T)$ *is invertible if and only if*

$$\beta(T) = \begin{pmatrix} T_1 & -T_2^* \\ T_2 & T_1^* \end{pmatrix}$$

is invertible on $H \oplus H$.

Theorem B (Chō and Takaguchi [3]). *For a commuting pair* $T=(T_1, T_2)$,

$$\sigma(T) - \omega(T) \subset \sigma_p(T) \cup \sigma_r(T).$$

Lemma 1. *Let* $T=(T_1, T_2)$ *be a commuting pair. Then*

$$\sigma(T) - \sigma_{pf}(T) \cup \sigma_{rf}(T) \subset \omega(T).$$

Proof. Let $z=(z_1, z_2)$ be a joint eigenvalue of infinite multiplicity. Let $C= =(C_1, C_2)$ be in $C(H)$ such that $T+C=(T_1+C_1, T_2+C_2)$ is a commuting pair. For a infinite orthonormal sequence $\{x_n\}$ in $\{x\colon T_i x = z_i x \ (i=1, 2)\}$, we may assume that there exist vectors y_1 and y_2 such that $\lim C_i x_n = y_i$ $(i=1, 2)$. If

$$\beta(T+C-z) = \begin{pmatrix} T_1+C_1-z_1 & -(T_2+C_2-z_2)^* \\ T_2+C_2-z_2 & (T_1+C_1-z_1)^* \end{pmatrix}$$

is invertible, then

$$\lim (x_n \oplus 0) = \beta(T+C-z)^{-1}(y_1 \oplus y_2).$$

It is a contradiction to the choice of $\{x_n\}$. So it follows, by Theorem A, that $z \in \omega(T)$.

Let $z=(z_1, z_2)$ be a joint residual eigenvalue of infinite multiplicity. Then for an infinite orthonormal sequence $\{x_n\}$ in

$$\{x\colon T_i^* x = \bar{z}_i x \ (i = 1, 2)\},$$

we may assume that there exist vectors y_1 and y_2 such that

$$\lim C_i^* x_n = y_i \quad (i = 1, 2).$$

If $\beta(T+C-z)$ is invertible, then

$$\lim (0 \oplus x_n) = \beta(T+C-z)^{-1}(-y_2 \oplus y_1).$$

It is a contradiction. So it follows that $z \in \omega(T)$.

So the proof is complete by Theorem B.

Next following Baxley we consider the following condition $\mathscr{C}_1$: If $\{z_n\}$ is an infinite sequence of distinct points in $\sigma_{pf}(T) \cup \sigma_{rf}(T)$ and $\{x_n\}$ is any sequence of corresponding normalized joint eigenvectors, then the sequence $\{x_n\}$ does not converge.

Lemma 2. *If a commuting pair $T=(T_1, T_2)$ satisfies $\mathscr{C}_1$, then*

$$\sigma(T) - (\sigma_{pfi}(T) \cup \sigma_{rfi}(T)) \subset \omega(T).$$

Proof. We have the identity

$$\sigma(T) - (\sigma_{pfi}(T) \cup \sigma_{rfi}(T)) =$$
$$(\sigma(T) - (\sigma_{pf}(T) \cup \sigma_{rf}(T))) \cup ((\sigma_{pf}(T) \cup \sigma_{rf}(T)) - (\sigma_{pfi}(T) \cup \sigma_{rfi}(T))).$$

So, by the above lemma, it will be sufficient to prove that $z=(z_1, z_2)$ is in $(\sigma_{pf}(T) \cup \sigma_{rf}(T)) - (\sigma_{pfi}(T) \cup \sigma_{rfi}(T))$ and not in the closure of $(\sigma(T) - (\sigma_{pf}(T) \cup \sigma_{rf}(T)))$, then z is in $\sigma(T+C)$ for every $C=(C_1, C_2)$ such that $T+C=(T_1+C_1, T_2+C_2)$ is a commuting pair.

Assume that z is in $(\sigma_{pf}(T) \cup \sigma_{rf}(T)) - (\sigma_{pfi}(T) \cup \sigma_{rfi}(T))$. Then there exist $z_n=(z_1^n, z_2^n)$ $(n=1, 2, \ldots)$ in $\sigma_{pf}(T)$ or in $\sigma_{rf}(T)$ such that $z_n \neq z_m$ $(n \neq m)$ and $\lim z_n = z$. Suppose that the z_n's are in $\sigma_{pf}(T)$, then we can consider a sequence $\{x_n\}$ of unit vectors such that $T_i x_n = z_i^n x_n$ $(i=1, 2)$ for every n. Of course, we can suppose, without loss of generality, that there exist vectors y_1 and y_2 such that $\lim C_i x_n = y_i$ $(i=1, 2)$. If, for $T+C-z=(T_1+C_1-z_1, T_2+C_2-z_2)$, $\beta(T+C-z)$ is invertible, then

$$\lim (x_n \oplus 0) = \beta(T+C-z)^{-1}(y_1 \oplus y_2).$$

It is a contradiction to the condition $\mathscr{C}_1$. So it follows that $z \in \omega(T)$.

When $\{z_n\}$ is in $\sigma_{rf}(T)$, then we can prove that z belongs to $\omega(T)$ in a similar way (see the proof of the lemma above).

So the proof is complete.

Next we shall show that the Weyl theorem holds for a commuting pair of normal operators. We need the following theorem. An easy computation shows that the theorem holds:

Theorem C. *Let $T=(T_1, T_2)$ be a commuting pair of normal operators. Then $\alpha(T)$ is invertible if and only if $T_1^* T_1 + T_2^* T_2$ is invertible.*

Theorem. *Let* $T=(T_1, T_2)$ *be a commuting pair of normal operators. Then the Weyl theorem holds, that is,*

$$\sigma(T)-\sigma_{pfi}(T)=\omega(T).$$

Proof. Since $T=(T_1, T_2)$ is a commuting pair of normal operators, T satisfies the condition $\mathscr{C}_1$. So, by Lemma 2, it suffices to prove that

$$\sigma(T)-\sigma_{pfi}(T)\supset\omega(T).$$

Let us consider a point in $\sigma_{pfi}(T)$. We may assume without loss of generality that this is (0, 0). We define $N=N(T_1^*T_1+T_2^*T_2)$, then $\dim(N)<\infty$. Let P denote the orthogonal projection of H onto N. Then P is a compact operator and $T_iP=PT_i=0$ $(i=1, 2)$. Hence

$$T+Q=(T_1+(1/\sqrt{2})P,\ T_2+(1/\sqrt{2})P)$$

is a commuting pair of normal operators. Since (0, 0) is an isolated point of $\sigma(T)$, so using Theorem C for T_1-z_1 and T_2-z_2 in place of T_1 and T_2, respectively, by continuity arguments we obtain that 0 is an isolated point in the spectrum of $T_1^*T_1+T_2^*T_2$. It follows that

$$(T_1+(1/\sqrt{2})P)^*(T_1+(1/\sqrt{2})P)+(T_2+(1/\sqrt{2})P)^*(T_2+(1/\sqrt{2})P)=T_1^*T_1+T_2^*T_2+P$$

is invertible. So, by Theorem C, $(0, 0)\notin\sigma(T+Q)$ and thus $(0, 0)\notin\omega(T)$.

So the proof is complete.

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References

[1] J. Baxley, Some general conditions implying Weyl's theorem, *Rev. Roum. Math. Pures Appl.*, **16** (1971), 1163—1166.

[2] J. Baxley, On the Weyl spectrum of a Hilbert space operators, *Proc. Amer. Math. Soc.*, **34** (1972), 447—452.

[3] M. Chō and M. Takaguchi, On the joint Weyl spectrum, *Sci. Rep. Hirosaki Univ.*, **27** (1980), 47—49.

[4] M. Chō and M. Takaguchi, Boundary of Taylor's joint spectrum for two commuting operators, *Rev. Roum. Math. Pures Appl.*, **27** (1982), 863—866.

[5] V. Istratescu, *Introduction to linear operator theory*, Dekker (New York, 1981).

[6] F.-H. Vasilescu, On pairs of commuting operators, *Studia Math.* **62** (1978), 203—207.

DEPARTMENT OF MATHEMATICS
JOETSU UNIVERSITY OF EDUCATION
JOETSU, 943, JAPAN

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The accuracy of the normal approximation for U-statistics with a random summation index converging to a random variable

M. AERTS and H. CALLAERT

Introduction

The exact order of the normal approximation has been obtained in [2] for U-statistics with a random summation index L_n where $L_n/n \to \tau$ with τ a constant. In this paper it is shown that the same order bounds can be obtained in the situation that the random index L_n satisfies $L_n/n \to \tau$ where now τ is a positive random variable. Moreover, a sharpening of the moment condition on the kernel is included. The results are valid for U-statistics with kernel of general degree r but in order to avoid a cumbersome notation, the proofs of the main theorems are given for the case that $r=2$. Tools for passing from $r=2$ to an arbitrary degree r are given in the preliminary lemmas which are formulated and proved for general r. For further information we refer to the Ph. D. thesis of one of the authors [1].

The results obtained in this paper are an extension of earlier results for random sums of i.i.d. random variables, proved in [6] and [3]. The proofs of these results use some methods which heavily rely on the i.i.d. structure. However, if one makes use of the structure of a U-statistic together with some technical fine-tuning, it is possible to obtain order bounds which are as sharp as in the i.i.d. case without imposing any additional condition. We also note that an asymptotic normality result contained in Theorem 1 below could in principle be obtained from Theorem 1 of [4]. However, this derivation would require some extra assumptions on the kernel function and no information on the rate of convergence could be gained.

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Preliminary lemmas

In order to create some flexibility in the renormalization of the statistics under consideration we formulate some general lemmas, special cases of which will be needed in the proof of our main theorem. The proof of Lemma 1 is elementary and is left to the reader. Throughout the paper we use the convention $[x]=\min\{k\in\mathbf{N}, x\leq k\}$.

Lemma 1. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and X_n and Y_n two sequences of random variables defined on Ω. Let C be a positive constant and d_n a sequence of nonnegative real numbers. If for some $k\geqq 0$ and some $\alpha>1$, $S_n^{k,\alpha}$ denotes the set on which $Y_n>k\alpha/(\alpha-1)$, then*

$$S_n^{k,\alpha}\cap\left\{\left|\frac{X_n-k}{Y_n-k}-1\right|>\alpha C d_n\right\}\subset S_n^{k,\alpha}\cap\left\{\left|\frac{X_n}{Y_n}-1\right|>C d_n\right\}.$$

Lemma 2. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and X_n and Y_n two sequences of positive random variables defined on Ω. If there exist positive constants c_1 and c_2 and a sequence of positive numbers ε_n with $\varepsilon_n\to 0$ for $n\to\infty$, such that*

$$P\left(\left|\frac{X_n}{[Y_n]}-1\right|>c_1\varepsilon_n\right)=O(\sqrt{\varepsilon_n}),\quad n\to\infty \tag{1}$$

and

$$P(Y_n<c_2\varepsilon_n^{-1/\delta})=O(\sqrt{\varepsilon_n}),\quad n\to\infty,\quad 0<\delta\leqq 1, \tag{2}$$

then, for every integer $k\geqq 0$, there exists a constant M_k such that

$$P\left(\left|\frac{\sqrt{[Y_n]}}{\sqrt{X_n}}\,\frac{X_n(X_n-1)\dots(X_n-k)}{[Y_n]([Y_n]-1)\dots([Y_n]-k)}-1\right|>M_k\sqrt{\varepsilon_n}\right)=O(\sqrt{\varepsilon_n}),\quad n\to\infty. \tag{3}$$

Proof. The proof is by induction. For $k=0$, (3) follows by taking $M_0=\sqrt{c_1}$. Assume that (3) holds true when $k=r-1$ for some $r\in\mathbf{N}_0$. Putting

$$Z_n=\frac{\sqrt{[Y_n]}}{\sqrt{X_n}}\,\frac{X_n(X_n-1)\dots(X_n-r+1)}{[Y_n]([Y_n]-1)\dots([Y_n]-r+1)},$$

the induction hypothesis yields that $P(|Z_n-1|>M_{r-1}\sqrt{\varepsilon_n})=O(\sqrt{\varepsilon_n})$, $n\to\infty$, for some constant M_{r-1}. Now choose M_r such that $M_r\geqq\max(3M_{r-1}, 6c_1)$ and then take n so large that

$$\varepsilon_n<\min\{1, (c_2/2r)^\delta, 9/(M_r^2)\} \tag{4}$$

is satisfied. Since (4) implies that $[c_2\varepsilon_n^{-1/\delta}]>2r$, one has, using the Bonferroni inequality and (2)

$$P\left(\left|Z_n \frac{X_n-r}{[Y_n]-r}-1\right| > M_r\sqrt{\varepsilon_n}\right) \leqq P(Y_n < c_2\varepsilon_n^{-1/\delta})+$$

$$+P\left(\left|(Z_n-1)\left(\frac{X_n-r}{[Y_n]-r}-1\right)+\left(\frac{X_n-r}{[Y_n]-r}-1\right)+(Z_n-1)\right| > M_r\sqrt{\varepsilon_n},\ [Y_n]>2r\right) \leqq$$

$$\leqq O(\sqrt{\varepsilon_n})+P\left(|Z_n-1| > \frac{M_r}{3}\sqrt{\varepsilon_n}\right)+P\left(\left|\frac{X_n-r}{[Y_n]-r}-1\right| > \frac{M_r}{3}\sqrt{\varepsilon_n},\ [Y_n]>2r\right)+$$

$$+P\left(\left|(Z_n-1)\left(\frac{X_n-r}{[Y_n]-r}-1\right)\right| > \frac{M_r}{3}\sqrt{\varepsilon_n},\ [Y_n]>2r\right).$$

It is easy to see, using the choice of M_r, the induction hypothesis, (1), (4), and Lemma 1 with $C=M_r/6$, $\alpha=2$, $d_n=\sqrt{\varepsilon_n}$ and $k=r$, that the second and third terms here are $O(\sqrt{\varepsilon_n})$. But by (4) the fourth term is not greater than

$$P\left(\left|(Z_n-1)\left(\frac{X_n-r}{[Y_n]-r}-1\right)\right| > \frac{M_r}{3}\sqrt{\varepsilon_n},\ [Y_n]>2r,\ |Z_n-1|\leqq 1\right)+P(|Z_n-1|>1) \leqq$$

$$\leqq P\left(\left|\frac{X_n-r}{[Y_n]-r}-1\right| > \frac{M_r}{3}\sqrt{\varepsilon_n},\ [Y_n]>2r\right)+P\left(|Z_n-1| > \frac{M_r}{3}\sqrt{\varepsilon_n}\right),$$

and the lemma follows.

The next lemma, which states the rate of convergence to normality for non-stochastically indexed U-statistics, plays a crucial role in the proof of the main theorem. It determines, together with the asymptotic behaviour of the random index L_n, the final approximation order for random U-statistics.

Lemma 3. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and $X_1, X_2, \ldots$ i.i.d. random variables defined on Ω. Let $U_n=\binom{n}{r}^{-1}\sum^{(n,r)} h(X_{i_1}, \ldots, X_{i_r})$ be a U-statistic with $Eh(X_1, \ldots, X_r)=\theta$ and put $g(X_1)=E\big(h(X_1, \ldots, X_r)-\theta|X_1\big)$. Assume that $\sigma^2=\operatorname{Var} g(X_1)$ is strictly positive, and that for some δ, $0<\delta\leqq 1$, one has that $E|g(X_1)|^{2+\delta}<\infty$ and $E|h(X_1, \ldots, X_r)|^{(4+\delta)/3}<\infty$. Then, one has:*

$$\sup_x \left|P\left\{\frac{\sqrt{n}(U_n-\theta)}{r\sigma}\leqq x\right\}-\Phi(x)\right| = O(n^{-\delta/2}),\quad n\to\infty.$$

Proof. The proof is essentially based on an improvement of a Berry—Esseen bound for more general non-parametric statistics (see [5]). For details of the proof we refer to [1], where it is also shown that the result of Lemma 3 is valid for statistics with structure $\sum_{i=1}^{p} g(X_i)+Y_k$ as used in the proof of our main theorem.

Main result

Theorem 1. *Let $(\Omega, \mathscr{A}, P)$ be a probability space and $X_1, X_2, \ldots$ i.i.d. random variables defined on Ω. Let $U_n = \binom{n}{2}^{-1} \sum_{1 \le i < j \le n} h(X_i, X_j)$ be a U-statistic with $Eh(X_1, X_2) = \theta$ and put $g(X_1) = E\big(h(X_1, X_2) - \theta | X_1\big)$. Assume that $\sigma^2 = \operatorname{Var} g(X_1)$ is strictly positive, and that for some δ, $0 < \delta \le 1$, one has that $E|g(X_1)|^{2+\delta} < \infty$ and $E|h(X_1, X_2)|^{(4+\delta)/3} < \infty$. Further, let ε_n be a sequence of positive numbers tending to zero and such that, for n large, $n^{-\delta} \le \varepsilon_n$. Let $L_n\colon \Omega \to \{2, 3, 4, \ldots\}$ and $\tau\colon \Omega \to (0, \infty)$ be random variables satisfying, for some constants $c_1, c_2 > 0$:*

$$P\left(\left|\frac{L_n}{[n\tau]} - 1\right| > c_1 \varepsilon_n\right) = O(\sqrt{\varepsilon_n}), \quad n \to \infty \tag{5}$$

$$P\left(\tau < \frac{c_2}{n} \varepsilon_n^{-1/\delta}\right) = O(\sqrt{\varepsilon_n}), \quad n \to \infty \tag{6}$$

(7) $\qquad \tau$ *is independent of* X_k, $k = 1, 2, \ldots$

then, one has:

(i) $$\sup_x \left| P\left(\frac{\sqrt{n\tau}}{2\sigma} \binom{n\tau}{2}^{-1} \sum_{1 \le i < j \le L_n} (h(X_i, X_j) - \theta) \le x\right) - \Phi(x)\right| = O(\sqrt{\varepsilon_n}), \quad n \to \infty$$

(ii) $$\sup_x \left| P\left(\frac{\sqrt{L_n}}{2\sigma} (U_{L_n} - \theta) \le x\right) - \Phi(x)\right| = O(\sqrt{\varepsilon_n}), \quad n \to \infty$$

and, if $\sigma_n^2 = \operatorname{Var} U_n$ exists:

(iii) $$\sup_x \left| P(\sigma_{L_n}^{-1}(U_{L_n} - \theta) \le x) - \Phi(x)\right| = O(\sqrt{\varepsilon_n}), \quad n \to \infty.$$

Proof. W.l.g. we assume that $\theta = 0$. The following notation will be used:

$$\mathbf{N}_1 = \{2, 3, 4, \ldots\}$$

$$I_n^{**} = I_n^{**}(\omega) =$$

$$= \{j \in \mathbf{N}_1 \,|\, [n\tau(\omega)](1 - c_1\varepsilon_n) \le j \le L_n(\omega) \quad \text{or} \quad L_n(\omega) < j < [n\tau(\omega)](1 - c_1\varepsilon_n)\},$$

$$I_n^* = I_n^*(\omega) = \{j \in \mathbf{N}_1 \,|\, j < [n\tau(\omega)](1 - c_1\varepsilon_n)\},$$

$$I_k = \{j \in \mathbf{N}_1 \,|\, j < k(1 - c_1\varepsilon_n)\},$$

$$J_n^* = J_n^*(\omega) = \{j \in \mathbf{N}_1 \,|\, [n\tau(\omega)](1 - c_1\varepsilon_n) \le j \le [n\tau(\omega)](1 + c_1\varepsilon_n)\},$$

$$J_k = \{j \in \mathbf{N}_1 \,|\, k(1 - c_1\varepsilon_n) \le j \le k(1 + c_1\varepsilon_n)\},$$

$$\delta_n = \delta_n(\omega) = \begin{cases} 1 & \text{if} \quad [n\tau(\omega)](1 - c_1\varepsilon_n) \le L_n(\omega), \\ -1 & \text{otherwise.} \end{cases}$$

Proof of (*i*). We first prove (*i*) with $n\tau$ replaced by $[n\tau]$. Choose n large enough so that $\varepsilon_n < c_2^\delta$ and, using (6), remark that

$$P([n\tau] < [c_2\varepsilon_n^{-1/\delta}]) \leqq P\left(\tau < \frac{c_2}{n}\varepsilon_n^{-1/\delta}\right) = O(\sqrt{\varepsilon_n}). \tag{8}$$

Hence

$$\sup_x \left| P\left(\frac{\sqrt{[n\tau]}}{2\sigma}\binom{[n\tau]}{2}^{-1}\sum_{1\leqq i<j\leqq L_n} h(X_i, X_j) \leqq x\right) - \Phi(x)\right| \leqq$$

$$\leqq \sup_x \left| P\left(\frac{\sqrt{[n\tau]}}{2\sigma}\binom{[n\tau]}{2}^{-1}\sum_{1\leqq i<j\leqq L_n} h(X_i, X_j) \leqq x,\ [n\tau] \geqq [c_2\varepsilon_n^{-1/\delta}]\right) - \right.$$

$$\left. - \Phi(x)P([n\tau] \geqq [c_2\varepsilon_n^{-1/\delta}])\right| + O(\sqrt{\varepsilon_n}).$$

Putting $\psi(X_i, X_j) = h(X_i, X_j) - g(X_i) - g(X_j)$, the following decomposition holds on the set where $[n\tau] \geqq [c_2\varepsilon_n^{-1/\delta}]$:

$$\frac{\sqrt{[n\tau]}}{2\sigma}\binom{[n\tau]}{2}^{-1}\sum_{1\leqq i<j\leqq L_n} h(X_i, X_j) =$$

$$= \frac{1}{\sigma\sqrt{[n\tau]}}\sum_{i=1}^{L_n} g(X_i) + \frac{1}{\sigma\sqrt{[n\tau]}([n\tau]-1)}\sum_{j\in I_n^*}\sum_{i=1}^{j-1}\psi(X_i, X_j) +$$

$$+\left(\frac{L_n - 1}{[n\tau]-1} - 1\right)\frac{1}{\sigma\sqrt{[n\tau]}}\sum_{i=1}^{L_n} g(X_i) + \frac{\delta_n}{\sigma\sqrt{[n\tau]}([n\tau]-1)}\sum_{j\in I_n^{**}}\sum_{i=1}^{j-1}\psi(X_i, X_j) =$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.$$

Using a Slutsky argument and the Bonferroni inequality, it suffices to prove that

(i.A)

$$\sup_x \left|P(\mathrm{I}+\mathrm{II} \leqq x,\ [n\tau] \geqq [c_2\varepsilon_n^{-1/\delta}]) - \Phi(x)P([n\tau] \geqq [c_2\varepsilon_n^{-1/\delta}])\right| = O(\sqrt{\varepsilon_n}),\quad n\to\infty$$

$$P\left(|\mathrm{III}| > \frac{\sqrt{\varepsilon_n}}{2},\ [n\tau] \geqq [c_2\varepsilon_n^{-1/\delta}]\right) = O(\sqrt{\varepsilon_n}),\quad n\to\infty \tag{i.B}$$

$$P\left(|\mathrm{IV}| > \frac{\sqrt{\varepsilon_n}}{2},\ [n\tau] \geqq [c_2\varepsilon_n^{-1/\delta}]\right) = O(\sqrt{\varepsilon_n}),\quad n\to\infty. \tag{i.C}$$

Proof of (i.A)

$$\sup_x |P(\mathrm{I}+\mathrm{II} \leqq x,\ [n\tau] \geqq [c_2\varepsilon_n^{-1/\delta}]) - \Phi(x)P([n\tau] \geqq [c_2\varepsilon_n^{-1/\delta}])| \leqq \tag{9}$$

$$\leqq \sum_{k=[c_2\varepsilon_n^{-1/\delta}]}^{\infty} P([n\tau] = k)\sup_x \left|P\left(\sum_{i=1}^{L_n} g(X_i) + Y_k \leqq b_k(x) \mid [n\tau] = k\right) - \Phi(x)\right|$$

with $b_k(x)=x\sigma\sqrt{k}$ and $Y_k=\frac{1}{k-1}\sum_{j\in I_k}\sum_{i=1}^{j-1}\psi(X_i, X_j)$. On the summands in the r.h.s. of (9) we use the following inequality:

$$\sup_x\left|P\left(\sum_{i=1}^{L_n}g(X_i)+Y_k\le b_k(x)\,|\,[n\tau]=k\right)-\Phi(x)\right|\le \tag{10}$$

$$\le\sup_x\left|P\left(\sum_{i=1}^{k}g(X_i)+Y_k\le b_k(x)\right)-\Phi(x)\right|+$$

$$+\sup_x\left|P\left(\sum_{i=1}^{L_n}g(X_i)+Y_k\le b_k(x),\ L_n\in J_k\,|\,[n\tau]=k\right)-P\left(\sum_{i=1}^{k}g(X_i)+Y_k\le b_k(x)\right)\right|+$$
$$+P(L_n\notin J_k\,|\,[n\tau]=k).$$

Putting

$$r_k(x)=P\left(\sum_{i=1}^{L_n}g(X_i)+Y_k\le b_k(x),\ L_n\in J_k\,|\,[n\tau]=k\right),$$

$$s_k(x)=P\left(\sum_{i=1}^{k}g(X_i)+Y_k\le b_k(x)\right),$$

$$A_k(x)=\left\{\omega\,\Big|\max_{m\in J_k}\sum_{i=1}^{m}g(X_i)+Y_k\le b_k(x)\right\},$$

$$B_k(x)=\left\{\omega\,\Big|\min_{m\in J_k}\sum_{i=1}^{m}g(X_i)+Y_k\le b_k(x)\right\},$$

one has that $P(A_k(x))\le s_k(x)\le P(B_k(x))$ and $P(A_k(x),\ L_n\in J_k|[n\tau]=k)\le r_k(x)\le$ $\le P(B_k(x))$, where we have used (7) to obtain the last inequality. Since $P(A_k(x))=$ $=P(A_k(x),\ L_n\in J_k|[n\tau]=k)+P(A_k(x),\ L_n\notin J_k|[n\tau]=k)$ it follows that $|r_k(x)-s_k(x)|\le$ $\le P(B_k(x))-P(A_k(x))+P(A_k(x),\ L_n\notin J_k|[n\tau]=k)$ and hence that

$$\sup_x|r_k(x)-s_k(x)|\le\sup_x\left(P(B_k(x))-P(A_k(x))\right)+P(L_n\notin J_k\,|\,[n\tau]=k). \tag{11}$$

An application of Lemma 3 yields that there exists a constant C such that

$$\sup_x\left|P\left(\sum_{i=1}^{k}g(X_i)+Y_k\le b_k(x)\right)-\Phi(x)\right|\le Ck^{-\delta/2}. \tag{12}$$

Applying (11) and (12) on the r.h.s. of (10) and using the obtained inequality in (9) leads to:

$$\sup_x|P(\mathrm{I}+\mathrm{II}\le x,\ [n\tau]\ge[c_2\varepsilon_n^{-1/\delta}])-\Phi(x)P([n\tau]\ge[c_2\varepsilon_n^{-1/\delta}])|\le$$

$$\le\sum_{k=[c_2\varepsilon_n^{-1/\delta}]}^{\infty}P([n\tau]=k)\sup_x\left(P(B_k(x))-P(A_k(x))\right)+$$

$$+2\sum_{k=[c_2\varepsilon_n^{-1/\delta}]}^{\infty}P(L_n\notin J_k\,|\,[n\tau]=k)P([n\tau]=k)+C\sum_{k=[c_2\varepsilon_n^{-1/\delta}]}^{\infty}k^{-\delta/2}P([n\tau]=k).$$

Now, remark that

$$(13)\qquad \sum_{k=[c_2\varepsilon_n^{-1/\delta}]}^{\infty} k^{-\delta/2}P([n\tau]=k) \leqq c_2^{-\delta/2}\sqrt{\varepsilon_n}\,P([n\tau] \geqq [c_2\varepsilon_n^{-1/\delta}]) = O(\sqrt{\varepsilon_n})$$

and that, using (5),

$$(14)\qquad \sum_{k=[c_2\varepsilon_n^{-1/\delta}]}^{\infty} P(L_n \notin J_k \mid [n\tau]=k)P([n\tau]=k) \leqq P(L_n \notin J_n^*) = O(\sqrt{\varepsilon_n}).$$

Hence, it suffices to show that

$$(15)\qquad \sum_{k=[c_2\varepsilon_n^{-1/\delta}]}^{\infty} P([n\tau]=k)\sup_x \big(P(B_k(x))-P(A_k(x))\big) = O(\sqrt{\varepsilon_n}).$$

Putting $p=\min J_k$, $q=\max J_k$, $r=\max I_k$ and remarking that $r=p-1$, it follows from Lemma 2 in [2] that

$$P(B_k(x))-P(A_k(x)) \leqq c\Big\{P\Big(\sum_{i=1}^{p} g(X_i) \leqq b_k(x)-Y_k,\ \sum_{i=1}^{q} g(X_i) \geqq b_k(x)-Y_k\Big)+$$

$$+P\Big(\sum_{i=1}^{p} g(X_i) \geqq b_k(x)-Y_k,\ \sum_{i=1}^{q} g(X_i) \leqq b_k(x)-Y_k\Big)\Big\}$$

for some constant c. We now use Lemma 3 from [2] with X replaced by

$\sum_{i=1}^{p} g(X_i)+Y_k$; Y by $\sum_{i=p+1}^{q} g(X_i)$; b by $\sigma\sqrt{k}$; d by $Ck^{-\delta/2}$ and t by $b_k(x)$.

We then obtain that for constants K and L:

$$\sup_x \big(P(B_k(x))-P(A_k(x))\big) \leqq Kk^{-\delta/2}+Lk^{-1/2}E\Big|\sum_{i=p+1}^{q} g(X_i)\Big| \leqq Kk^{-\delta/2}+\sigma L\sqrt{\frac{q-p}{k}}$$

where the last inequality follows from $E\left|\frac{1}{\sqrt{q-p}}\sum_{i=p+1}^{q} g(X_i)\right| \leqq \sigma$ by the moment inequality and the independence of the X_i's together with $Eg(X_1)=0$. Inserting this result into the l.h.s. of (15), after remarking that $\sqrt{\frac{q-p}{k}} \leqq \sqrt{2c_1\varepsilon_n}$, and using (13), one arrives at the desired order bound $O(\sqrt{\varepsilon_n})$, completing the proof of (i.A).

Proof of (i.B). From (8) and (14) it follows that

$$(16)\qquad P\Big(|\mathrm{III}| > \frac{\sqrt{\varepsilon_n}}{2},\ [n\tau] \geqq [c_2\varepsilon_n^{-1/\delta}]\Big) \leqq O(\sqrt{\varepsilon_n})+$$

$$+\sum_{k=[c_2\varepsilon_n^{-1/\delta}]}^{\infty} P\left(\left|\left(\frac{L_n-1}{k-1}-1\right)\frac{1}{\sigma\sqrt{k}}\sum_{i=1}^{L_n} g(X_i)\right| > \frac{\sqrt{\varepsilon_n}}{2},\ L_n \in J_k \,\Big|\, [n\tau]=k\right)P([n\tau]=k).$$

Using (7) and the fact that $\max_{m\in J_k}|m-k|\leq kc_1\varepsilon_n$, one obtains:

$$P\left(\left|\frac{L_n-1}{k-1}-1\right|\left|\sum_{i=1}^{L_n} g(X_i)\right|>\frac{\sigma\sqrt{k\varepsilon_n}}{2},\ L_n\in J_k|[n\tau]=k\right)\leq$$

$$\leq P\left(\max_{m\in J_k}\left|\frac{m-k}{k-1}\right|\max_{m\in J_k}\left|\sum_{i=1}^{m} g(X_i)\right|>\frac{\sigma\sqrt{k\varepsilon_n}}{2}\right)\leq P\left(\max_{m\in J_k}\left|\sum_{i=1}^{m} g(X_i)\right|>\frac{\sigma(k-1)}{2c_1\sqrt{k\varepsilon_n}}\right).$$

Since $\sum_{i=1}^{m} g(X_i)$, $m=1, 2, \ldots,$ forms a martingale, the Kolmogorov inequality yields that

$$P\left(\max_{p\leq m\leq q}\left|\sum_{i=1}^{m} g(X_i)\right|>\frac{\sigma(k-1)}{2c_1\sqrt{k\varepsilon_n}}\right)\leq\frac{4c_1^2k\varepsilon_n}{\sigma^2(k-1)^2}E\left(\sum_{i=1}^{q} g(X_i)\right)^2=$$

$$=4c_1^2qk\varepsilon_n/(k-1)^2=O(\sqrt{\varepsilon_n})$$

showing that the r.h.s. of (16) is of the order $O(\sqrt{\varepsilon_n})$.

Proof of (i.C). Using the same reasoning as in the proof of (i.B) and remembering that $\delta_n=1$ if $[n\tau](1-c_1\varepsilon_n)\leq L_n$, one has:

$$P\left(|\mathrm{IV}|>\frac{\sqrt{\varepsilon_n}}{2},\ [n\tau]\geq[c_2\varepsilon_n^{-1/\delta}]\right)\leq$$

$$\leq\sum_{k=[c_2\varepsilon_n^{-1/\delta}]}^{\infty}P\left(\max_{m\in J_k}\left|\sum_{j=p}^{m}\sum_{i=1}^{j-1}\psi(X_i, X_j)\right|>\frac{\sigma\sqrt{k\varepsilon_n}(k-1)}{2}\right)P([n\tau]=k)+O(\sqrt{\varepsilon_n}).$$

Further, it is well-known that $V_m=\sum_{j=p}^{m}\sum_{i=1}^{j-1}\psi(X_i, X_j)$, $m=p,\ p+1, \ldots, q$, and also $W_k=\sum_{i=1}^{k}\psi(X_i, X_j)$, $k=1, 2, \ldots, j-1$ are martingales. An application of the Kolmogorov inequality and a theorem in [8] lead to $\left(\text{denote } \frac{4+\delta}{3} \text{ by } s\right)$

$$P\left(\max_{p\leq m\leq q}\left|\sum_{j=p}^{m}\sum_{i=1}^{j-1}\psi(X_i, X_j)\right|>\frac{\sigma\sqrt{k\varepsilon_n}(k-1)}{2}\right)\leq 2^s\sigma^{-s}(k-1)^{-s}(k\varepsilon_n)^{-s/2}E|V_q|^s\leq$$

$$\leq K(k-1)^{-s}(k\varepsilon_n)^{-s/2}\sum_{j=p}^{q}E|W_{j-1}|^s\leq K'(k-1)^{-s}(k\varepsilon_n)^{-s/2}(q-p+1)q$$

where K and K' are constants. A short computation, using $q-p\leq 2kc_1\varepsilon_n$ and $q\leq k(1+c_1\varepsilon_n)$, yields the desired order bound $O(\sqrt{\varepsilon_n})$. To complete the proof of (i), we have to show that $[n\tau]$ can be replaced by $n\tau$. An application of Lemma 1 of [7] yields that it is sufficient to prove that for some constant C

$$P\left(\left|\frac{\sqrt{n\tau(n\tau-1)}}{\sqrt{[n\tau]([n\tau]-1)}}-1\right|>C\sqrt{\varepsilon_n}\right)=O(\sqrt{\varepsilon_n}).\tag{17}$$

That (17) holds follows from Lemma 2 with $X_n=Y_n=n\tau$ and $k=1$, checking by (6) that (1) and (2) are satisfied.

Proof of (ii). As above, it can be shown that $[n\tau]$ may also be replaced by L_n. We take $X_n=L_n$, $Y_n=n\tau$ and $k=1$ in Lemma 2. Since (1) and (2) then coincide with (5) and (6), the proof of (ii) is complete.

Proof of (iii). We first show that

$$P\left(\left|\frac{L_n\sigma_{L_n}^2}{4\sigma^2}-1\right|>C^2\varepsilon_n\right)=O(\sqrt{\varepsilon_n})\quad\text{with}\quad C^2=\frac{2E\psi^2(X_1,X_2)}{c_2\sigma^2}.$$

Using that $n\sigma_n^2=4\sigma^2+\frac{2}{n-1}E\psi^2(X_1,X_2)$, this follows from condition (6) after easy manipulation. Since

$$P\left(\left|\frac{\sigma_{L_n}\sqrt{L_n}}{2\sigma}-1\right|>C\sqrt{\varepsilon_n}\right)\leqq P\left(\left|\frac{L_n\sigma_{L_n}^2}{4\sigma^2}-1\right|>C^2\varepsilon_n\right)=O(\sqrt{\varepsilon_n})$$

a lemma of [7] makes it possible to go from (ii) to (iii). This finishes the proof of the theorem.

We close with a result concerning the case when the indices are independent of the basic sequence. The details of the proof are of course simpler than in the general case (for instance, there is no need for the decomposition of U_n) and therefore are not given here.

Theorem 2. *Let the assumptions of Theorem 1 be fulfilled with* (5) *deleted and* (7) *replaced by:* L_n, τ and X_k, $k=1,2,\ldots$ *are independent for each* $n=1,2,\ldots$. *Then*

(a) *if* $P\left(\left|\frac{L_u}{[n\tau]}-1\right|>c_1\sqrt{\varepsilon_n}\right)=O(\sqrt{\varepsilon_n})$, *the results* (i), (ii) *and* (iii) *of Theorem* 1 *hold;*

(b) *if* $P\left(\frac{L_n}{[n\tau]}<1-\alpha\right)=O(\sqrt{\varepsilon_n})$ *for some constant* $\alpha<1$, *the results* (ii) *and* (iii) *of Theorem* 1 *hold.*

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References

[1] M. Aerts, *Asymptotic distribution theory for random U- and GM-statistics, with application in sequential estimation*, Ph. D. Dissertation, Limburgs Universitair Centrum (1988).

[2] M. Aerts and H. Callaert, The exact approximation order in the central limit theorem for random *U*-statistics, *Sequential Analysis*, **5** (1986), 19—35.

[3] H. Callaert and P. Janssen, A note on the convergence rate of random sums, *Rev. Roumaine Math. Pures Appl.*, **28** (1983), 147—151.

[4] S. Csörgő, On weak convergence of the empirical process with random sample size, *Acta Sci. Math.*, **36** (1974), 17—25, 375—376.

[5] K. O. Friedrich, A Berry—Esseen bound for functions of independent random variables, *Ann. Statistics*, **17** (1989), 170—183.

[6] D. Landers and L. Rogge, A counterexample in the approximation theory of random summation, *Ann. Probability*, **5** (1977), 1018—1023.

[7] R. Michel and J. Pfanzagl, The accuracy of the normal approximation for minimum contrast estimates, *Z. Wahrscheinlichkeitstheorie Verw. Gebiete*, **18** (1971), 73—84.

[8] B. von Bahr and C. G. Esseen, Inequalities for the rth absolute moment of a sum of random variables, $1 \leqq r \leqq 2$, *Ann. Math. Statist.*, **36** (1965), 299—303.

LIMBURGS UNIVERSITAIR CENTRUM
DEPARTMENT OF MATHEMATICS
UNIVERSITAIRE CAMPUS
B-3610 DIEPENBEEK, BELGIUM

Bibliographie

G. M. Adelson-Velsky—V. L. Arlazarov—M. V. Donskoy, Algorithms for Games, X+197 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1988.

The original Russian edition was published in 1978. Though the progress in the development of computer chess programs has been rapid, present translation is still interesting since it deals with the basic ideas of the research problems rather than with specific programs and programming techniques.

The book consists of four chapters. Chapter 1 is devoted to a description of a two-person game with complete information. It contains the definition of the game tree and the position score furthermore brand- and- bound method of searching for the best move in a position. Some simple theorems on the decomposition of a game tree are proved as well as theorems on valuation of positions for finding the best moves.
Chapter 2 is devoted to heuristic methods for choosing a move in a contemplated position. A probabilistic method is used for justifying a heuristic algorithm. Shannon's model with the concept of evaluation function and depth of the search are introduced.
Chapter 3 (entitled The Method of Analogy) is devoted to the definition and study of moves which are independent of positions (analogous positions) having thus analogous consequences in different positions. In Chapter 4 (Algorithms for Games and Probability Theory) constructions of probabilistic models for two-person games and calculation of model scores on a probabilistic basic are investigated.

The authors of the book had written the program of KAISSA which won the First International Championship for Chess Programs in 1974.

In the Appendix the reader finds a list of chess programs which took part in 1974—1977 competitions, and an interesting historical survey of game programming of computer age up to 1978.

Zoltán Blázsik (Szeged)

Brian A. Barsky, Computer Graphics and Geometric Modeling Using Beta-splines (Computer Science Workbench) IX+156 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1988.

Specialist of β-splines B. A. Barsky gives the following conception of this book in the Introduction: "The underlying concept of this work is the synthesis of two useful concepts: the application of tension to a shape; and the study of a parametrically defined shape as fundamental geometric measures." The whole method is based on considering the continuity of these two differential geometric notions, which are of basic importance in the investigation of their geometric shape.

Even the reader who is unfamiliar with the theory of curves and surfaces can easily catch ideas of the considerations concerning continuity of the unit tangent and curvature vectors of a curve given by a piece-wise representation. The visualization problems of these concepts by computer graphic methods are introduced in a clear way, as well.

As to the technical applications, the most important feature of the β-spline curves and surfaces is that the base points render local control possibility. The shape of the curve and the surface can be modified locally, furthermore the so called shape parameters determine the "tightness" and "looseness" of the curves and surfaces as they fit close the control polygon or surface.

Independent chapters are devoted to the cases of uniform, respectively continuously varying shape parameters. In the both cases a method is explained in detail for determination of the β-spline curve (surface). For this purpose the author shows REDUCE computer algebra system developed at the Department of Computer Science at the University of Utah. This is a perspicuous program for the evaluation of the unknown coefficient functions of β-spline curves and surfaces. Boundary conditions (respectively, end conditions) are analyzed in original chapters, including the problem of classification.

One of the greatest merits of the book is the excellent collection of figures and pictures. They help the reader to understand the concepts above more exhaustively, and demonstrate the effectiveness of β-spline representation. Especially remarkable are the figures analyzing the relations between control polygons (surfaces) and shape parameters, by the side of which the reader can see the synthetic image appearing on the monitor. Wide possibilities of β-splines are illustrated by nice colour pictures in the 19th chapter.

The book is recommended to readers interested in ability of β-spline technique. However, it should be a pleasure first of all for those mathematicians and computer scientists who want to deal with computer graphic and design problems. In the latter case, the summary in the 20th chapter with an outlook on further research directions; the enclosed Reduce programs in the Appendix, and the Bibliography on Curves and Surfaces including about 400 references are very useful.

József Kozma (Szeged)

M. Berger—B. Gostiaux, Differential Geometry: Manifolds, Curves and Surfaces (Graduate Texts in Mathematics, 115), XII+474 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1988.

First of all we must sound out that this is an extremely good book. The observant readers must certainly find great pleasure in reading this book, because of its clear style and very nice setting up. Although one has to read this book to know its taste we try to say some words about it.

This book can be regarded as an enlarged and revised version of M. Berger's book "Géométrie Différentielle" (1972). In order to know something about the building up of this book it is worth to quote Berger's words about his aims in lectures he read in Paris in 1969—71 served as the basis of this beautiful book:

"First, to avoid making the statement and proof of Stokes' formula the climax of the course and running out of time before any of its applications could be discussed. Second, to illustrate each new notion with nontrivial examples, as soon as possible after its introduction. And finally, to familiarize geometry-oriented students with analysis and analysis-oriented students with geometry, at least in what concerns manifolds.".

While the first nine chapters are based on the above mentioned book absolutely, the last two chapters are an "attempt to remedy the notorious absence in the original book of any treatment of surfaces in three-space, an omission all the more unforgivable in that surfaces are some of the most common geometrical objects, not only in mathematics but in many branches of physics".

Although we have not detailed the book's contents we suppose these words above can take a fancy to reading of this book, hence we call again attention of everybody interested in differential geometry on graduate level or reading lectures about it to this well illustrated nice book.

Árpád Kurusa (Szeged)

K. H. Borgwardt, The Simplex Method. A Probabilistic Analysis, XI+268 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

Since the Simplex-Method was discovered by George B. Dantzig it has been in the central of interests of researches. One of the most interesting problem in connection with the Method is the discrepancy of its worst-case behaviour and the good practical behaviour of it.

This book — giving a comprehensive probabilistic analysis of the so-called Two-Phase Simplex Method — attempts to resolve this discrepancy.

After an extensive introduction in which the author overviews the known results of the probabilistic analysis of the Symplex Method including some theorems from the field of stochastic geometry, the papers of Smale and Haimowich and of course their own earlier results as well.

Because of the analysis is based on the shadow vertex algorithm, the first chapter reviews this algorithm. The next two sections deal with giving an upper bound for the average number of pivot step of the algorithm. The research culminates in the Theorem 6 of the Chapter 3 in which the author postulates that the average number of pivot steps ($E_{m,n}$) for the complete Simplex Method is polynomial, namely if we have m inequalities with n variables then

$$E_{m,n} = O(m^{1/(n-1)}n^4).$$

Chapter 4 studies the asymptotic average behaviour of the Simplex Method. (The author uses the term "asymptotic" in the sense that $m\to\infty$, and n is fixed.)

Upper bounds have been given in integral form, for certain classes of distributions including the uniform and the Gaussian distributions as well.

In the Chapter 5 the author introduces a modified version of the Two-Phase Simplex Method solving the so-called rotation invariant model with n additional nonnegativity constraints. It has been proved that the expected number of pivot steps of this algorithm is not greater than

$$m^{1/(n-1)}(n+1)^4\frac{2}{5}\pi\left(1+\frac{e\pi}{2}\right).$$

An Appendix including definitions and proofs for Gamafunction and Betafunction closes the book. The book is well-organized readable (in mathematical sence), but *I* have to mention that the lack of some definitions causes that the book is not absolutely "self-contained".

G. Galambos (Szeged)

CAAP '88, 13th Colloquium on Trees in Algebra and Programming, Proceedings, Nancy 1988. Edited by M. Dauchet and M. Nivat (Lecture Notes in Computer Science, 299), VIII+304 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1988.

This volume contains the proceedings of the 13th Colloquium on Trees in Algebra and Programming, held on March 21—24, 1988, in Nancy.

CAAP '88, following the tradition, is devoted to trees, which are a basic structure for Computer Science, and which are explicitly or implicitly studied in a lot of papers in this volume. But

CAAP '88 covers also a wider range of topics in Theoretical Computer Science: Algorithms and complexity on trees and other structures; Abstract data types and term rewriting; Logic, parallelism and concurrency.

We warmly recommend this book to everybody, working in Theoretical Computer Science.

Sándor Vágvölgyi (Szeged)

G. S. Campbell, An Introduction to Environmental Biophysics, XV+159 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1986.

Environmental biophysics is a specialized branch at the borderline of physics and biophysics and mainly concentrates on energy exchange processes taking place in our environment exposed to solar radiation and variations in air humidity. The principal emphasis of the book is to present the differential equation formalism for mass and heat transfer and it gives an introduction into the mathematical physics of rate equations. Special attention has been paid to a quantitative analysis of energy balance using the continuity equation. Then it goes on to apply the general principles to selected examples and in the second part of the book the energy and mass transfer models are applied to exchange processes between organisms and their microenvironment. Throughout the book the basis principles are illustrated by several examples which are rahter useful in gaining an understanding the subject. The illustrations are superb and rahter useful additions to the text. This book is addressed to the physics and biophysics undergraduate student of conventional course background. The author does not review mathematical physics, but it is a useful supplement for those who have met the concepts in other courses. It can be used as a textbook of environmental biophysics or a supplementary reference source of classical mechanics for first year undergraduate courses. At the end of each chapter further problems are presented which can be very useful additions to conventional physics courses.

L. I. Horváth (Szeged),

Classic Papers in Combinatorics, Edited by Ira Gessel and Gian-Carlo Rota, X+489 pages, Birkhäuser, Boston—Basel—Stuttgart, 1987.

Excellent papers from different fields of the combinatorics are presented in this collection. Without giving a complete enumeration on the contents of the book we give some significant results. From the Ramsey theory we meet the basic paper of Ramsey from 1930, the classical paper of Erdős and Szekeres (1935), the Erdős—Rado theorem on the partition calculus. The new results are represented by the Graham's, Leeb's, Rotschild's papers on the categorical underpinning of Ramsey theorem.

Withney's paper (1932) presents the first paper on the theory of matriods. Tutte's paper included in this book roots in the matroid theory. Classical papers are presented from the graph coloring (Brooks, Lovász). The matching theory represent 8 papers among them the opening papers of Hall (1935), Halmos (1958) and Dilworth. Here we can find the lot-cited papers of Ford and Fulkerson, Tutte's paper on factors of graphs and Edmonds's efficient matching algorithms.

One of the editors (Rota) has used Pólya's paper on picturewriting to establish the theory of Möbius functions. His work was extended by Crapo.

On the field of the extremal set theory the first paper is due to Katona. Clements's, Kruskal's, Kleitman's and Erdős's results are cited in this part.

Again an Erdős's paper on the probabilistic method is in the collection. This paper is very important since it helps to prove a lot of existence theorems in the graph theory. Lovász's contribution to the Ulam reconstruction problem is an ingenious use of the inclusion-exclusion priciple.

It was a great pleasure of the refree that the Hungarian matematicians who played significant role in this field of matematics are present in this collection with a weight.

G. Galambos (Szeged)

Underwood Dudley, A Budget of Trisections, XV+169 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1987.

From time to time every mathematical institute receives letters in which the authors "solve,, some famous problems. They prove the Fermat Conjecture, the Goldbach Conjecture, they duplicate the cube with compass and straightedge and so on. Numerous amateurs try the trisection of the angle. (This is impossible with straightedge and compass as was proved by P. L. Wantzel in 1837.) Archimedes trisected the angle using a compass and a straightedge with two scratches on it. This is a non-euclidean construction and you will find some more examples of this kind in the first chapter. The second and third chapters (Characteristics of Trisectors, Three Trisectors) enlighten the personalities of these amateurs. The fourth chapter contains the collection of trisections.

This book is a curious, extraordinary work. I have never seen anything similar to this.

Everyone can read it with minimal mathematical background. The author writes in the Introduction: "What follows, then, is something which has never been done before: it is an effort to do something which may be as impossible as trisecting the angle: namely to put an end to trisections and trisectors".

L. Pintér (Szeged)

Dynamics of Infinite Dimensional Systems, Edited by Shui-Nee Chow and Jack K. Hale (NATO ASI Series, Series F: Computer and Systems Sciences, 37), IX+514 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

This is the Proceedings of the NATO Advanced Study Institute on Dynamics of Infinite Dimensional Systems, held in Lisbon, Portugal, May 19—24, 1986.

In recent years it has become a general method in the researches into partial differential equations (PDE's) and functional differential equations (FDE's) to consider these equations as dynamical systems on functional spaces. The purpose of this workshop was to bring together research workers from the various areas coming with several different backgrounds and interests. The papers investigate asymptotic behaviour of solutions (e.g. stability properties oscillation, bifurcation) for such equations as semilinear and nonlinear parabolic and elliptic PDE's integrodifferential equations dissipative systems, FDE's with finite and infinite delay, infectious disease model, wave equation and reaction diffusion system.

L. Hatvani (Szeged)

Foundations of Logic and Functional Programming, Proceedings, Trento 1986. Edited by M. Boscarol, L. Carlucci Aiello and G. Levi (Lecture Notes in Computer Science 306), IV+218 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1988.

This volume contains ten papers presented at the workshop on "Foundations of Logic and Functional Programming" held in Trento, Italy, December 15—19, 1986.

The titles of invited contributions are: 1. C. Talcott: Rum. An intensional theory of function and control abstractions. 2. L. Cardelli: Typechecking dependent types and subtypes. 3. C. Böhm: Reducing recursion to iteration by means of pairs and N-tuples. 4. J.-L. Lassez, M. J. Maher and K. Marriott: Unification revisited. 5. C. Zaniolo and D. Saccà: Rule rewriting methods for efficient implementations of Horn logic.

The titles of submitted contributions are: 1. P. Miglioli, U. Moscato and M. Ornaghi: PAP: A logic programming system based on a constructive logic. 2. E. Giovannetti and C. Moiso: A completeness result for E-unification algorithms based on conditional narrowing. 3. N. Guarino: Representing domain structure of many-sorted Prolog knowledge bases. 4. A D'Angelo: Horn: An inference engine prototype to implement intelligent systems. 5. E. G. Omodeo: Hints for the design of a set calculus oriented to Automated Deduction.

This book is recommended to everybody working in the theory of Logic and Functional Programming.

Sándor Vágvölgyi (Szeged)

M. Goresky—R. MacPherson, Stratified Morse Theory (Ergebnisse der Mathematik und ihrer Grenzgebiete), XIV+272 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1988.

This book consists of three parts and a nice introduction. This introduction makes absolutely clear basis for the tree distinct subjects of three parts. The parts contain: a systematic exploration of the natural extension of Morse theory to include singular spaces; a large collection of theorems on the topology of complex analytic varieties; the calculation of the homology of the complement of a collection of flat subspaces of Euclidean space.

The only common thing in these parts is the application of the Morse theory, but we think the appearance of these subjects in one book was a very good and natural idea.

To end our review we establish that this book is very nice in its form, contents and also its getting-up. We are sure that it will become a fundamental book of its subject.

Árpád Kurusa (Szeged)

Martin Grötschel—László Lovász—Alexander Schrijver, Geometric Algorithms and Combinatorial Optimization, XII+362 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1988.

In spite of the fact that many of the most frequently used combinatorial algorithms were based on the discrete structure of the problems in the last several years geometric methods have played more significant role in combinatorial optimization.

In the focus of this book states the investigation of two geometrical algorithms: the ellipsoid method and the basis reduction. The first one has been developed by L. G. Khachiyan for linear programs and the authors examined it deeply in their earlier papers as well. The roots of the second method go back to Hermite and Minkowski, and it has been used for the polynomial time solvability of integer linear programming in fixed dimension by Tardos and H. W. Lenstra.

The first two sections of the book contain preliminaries. A list of the main problems (The Weak Optimization Problem, the Weak Violation Problem, the Weak Validity Problem, the Weak Separation Problem, the Weak Membership Problem) are introduced in Chapter 2. The next section contains the description of the ellipsoid method. Applications and specializations of the method are collected in Chapter 4. The algorithms concerning the different characteristics of a convex set are approximations (because of the nature of the method).

The next two sections contain the basis reduction algorithm for lattices and its applications. Different combinations of the ellipsoid method with basis reduction are given for the programming in fixed dimension. The last four chapters contain further applications: Chapter 7 gives some basic examples, in Chapter 8 there is given a deep survey of the basis reductions. Specific fields of the application of the ellipsoid method are discussed in the finishing sections.

The book has a clear style. It may be a useful piece of reading not only for experts but for students as well.

G. Galambos

John L. Kelley—T. P. Srinivasan, Measure and Integral, Volume 1 (Graduate Texts in Mathematics, 116), X+150 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1988.

The measure and integral have been two basic notions of analysis and probability theory since their beginnings. Nowadays they play an important role also in other branches of pure and applied mathematics. This book is a systematic exposition of the theory of measure and integration emphasizing the part of the theory most commonly used in functional analysis.

The book consists of two kinds of text. The body of the text, requiring only a first course in analysis as a background, is a study of abstract measures and integrals. It establishes Borel measures and integration for R. The chapters are followed by supplements, which are more informal and present such parts of the theory as Borel measures and integration for R^n, integration for locally compact Haussdorff spaces, invariant measures for groups, Stieltjes integration, Haar measure, the Bochner integral.

The method of presentation differs from the standard one, namely, at first integrals are constructed, then measures are derived from them. The integral is extended to some R^* valued functions, and measures with R^* values; signed measures and indefinite integrals are also treated.

The well-written book can be highly recommended to mathematicians especially those dealing with functional analysis.

L. Harvani (Szeged)

Neal Koblitz, A Course in Number Theory and Cryptography (Graduate Texts in Mathematics), IV+208 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1987.

Since the fifties number theory has changed in an extremly rapid manner. Only a few decades ago this theory had no practical use. Gauss called it the "Queen of Mathematics". The results are interesting and sometimes surprising, the methods can be delightful, and nowadays there appear more and more new applications. A course is interesting and the publication of a book is justified if it has got some original distinguishing features. In my opinion the reader will enjoy this book. One of its characteristic features is the algorithmic approach, emphasizing estimates of the efficiency of the techniques. Cryptography is in the centre of the discussions. The inclusion of some very recent applications of the theory of elliptic curves seems to be originally new.

The first two chapters — Some topics in elementary number theory and Finite fields and quadratic residues—give a general background. Some of the proofs are omitted (one finds them in introductory textbooks). A characteristic (unusual) topic is the estimation of the number of bit operations needed to perform different tasks by computer.

The following four chapters—Cryptography; Public key; Primality and factoring; Elliptic curves — are similar to a fascinating novel. Especially the chapter on public key supplies astonishing novelties for the readers who are inexperienced in this theme. Let us cite the last sentences of this chapter from which consequences may be drawn on the discussion and further we can see that the book holds the benefits of a lecture: "At the present time there is no known polynomial time algorithm for solving the iterated knapsack problem, i.e., the public key cryptosystem described in the last paragraph. However, there are some promising approaches to generalizing Shamir's algorithm. It is not unlikely that intensive research on this problem would before long produce an efficient algorithm for breaking the iterated knapsack cryptosystem. In any case, most experts, traumatized by Shamir's unexpected breakthrough, do not have much confidence in the security of any public key cryptosystem of this type."

Several various exercises increase the interest of the work, answers and in more difficult cases solutions are given.

Although we can read on the cover: "No background in algebra or number theory is assumed", however, in my opinion the reader needs some experience in the theory and in this case she/he can find great enjoyment in this text and very much of it indeed.

L. Pintér (Szeged)

Max Koecher, Klassische elementare Analysis, 211 pages, Birkhäuser Verlag, Basel—Boston, 1987.

The text is divided into six parts. Chapter 1 is a preparatory part the main idea of which is the investigation of the connection between the classical golden section problem, the Fibonacci numbers and continued fractions. An algebraic application of golden section is also given. Chapter 2 introduces the notions of convergence of sequences and series of real numbers. In Chapter 3 the Riemann integral is defined, the integration methods are acquainted and at the end of the chapter the logarithm function as an integral and its inverse, the exponential function are introduced. Chapter 4 is devoted to algebraic and number theoretic applications. Chapter 5 deals with convergence of function sequences and series, the power series of elementary functions are deduced, the partial fraction decomposition of cotangent function and by using the power series representation of the arctangent function a series of $\frac{\pi}{4}$ are considered. Chapter 6 discusses famous classical problems of elementary analysis. Here Bernoulli polynomials, Euler series, Euler and Poisson summations and the Gamma-Function are investigated.

The book is a pearl of the mathematical literature. It is highly recommended to students for learning analysis in the first two semesters.

L. Gehér (Szeged)

Paul Koosis, The logarithmic integral. I (Cambridge Studies in Advanced Mathematics 12), XI+606 pages, Cambridge University Press, Cambridge—New York—New Rochelle— Melbourne —Sydney, 1988.

The frequent appearance of $\int_{-\infty}^{\infty} M(t)/(1+t^2)\,dt$ (or its transformed form) in more or less different branches of mathematical analysis as well as in their applications naturally raises the question: What is the role of this integral in the analysis? One, who is interested in this theme, should consult with the present book. Moreover, having even a look at its "Contents", the prospective reader will surely find something of his particular interest.

The first two Chapters are devoted to Jensen's formula, the celebrated Szegő's theorem and the familiar Poisson integral. Chapter III, called for more frequently in the subsequent ones, deals with "Entire functions of exponential type", i.e.: entire functions satisfying $|f(z)| \leq Ce^{A|z|}$. The rest of the Chapters are entitled as follows: "IV. Quasianalycity", "V. The moment problem on the real line", "VI. Weighted approximation on the real line", "VII. How small can the Fourier transform of a rapidly decreasing non-zero function be?", "Persistence of the form $dx/(1+x^2)$". An "Addendum" improves the content of Chapter VII by discussing some recent results.

The author pays attention to show how things grow up from simple ideas. The reader, familiar with an introduction to the theory of real and complex functions, and a bit of functional analysis, will find only a few cases, when he needs to look for supplementary material. Exact references help the readers to find way in such situations. "Bibliography for volume I" lists approximately 80 items, including a number of books.

The argumentations are detailed to such an extent that one can follow them easily. However, a large area for the reader's activity is provided by giving "Problems" accompanied with hints (if necessary). By solving these problems (mostly of own interest) one can deeply understand, how to use the methods of the discussed theme, and thus possibly feels to be stimulated to do research work in analysis.

Reading this book, everybody will certainly be caught by the author's enthusiasm: "It is a beautiful material. May the reader learn to love it as I do." Thus, it mustn't escape the reader's attention that this book is completed by "Contents of volume II."

Endre Durszt (Szeged)

Hary Krishna, Computational Complexity of Bilinear Forms (Lecture Notes in Control and Information Sciences, 94), XVI + 166 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1987.

The book contains two parts: the first one has four chapters and is a deep study of the relationship between the computation of biliniear forms and the linear error-correcting codes. The two chapters of the second part describe an application of the class of linear codes showed in Part I.

In details, Chapter 2 discusses the multiplicative complexity of certain noncommutative algorithms that are usable to compute a system of k bilinear forms and establishes a connection between linear (n, k, d) codes and algorithms. Using the property of duality it is shown that the multiplicative complexity of the bilinear forms is the same as the multiplicative complexity of an aperiodic convolution algorithm with length $(k+d+1)$.

In Chapter 3 efficient algorithms are developed for aperiodic convolutions. In Chapter 4 bilinear algorithms — basing on two approaches developed in the previous Chapter — are presented for aperiodic convolution of sequences defined over GF(2) and GF(3).

Chapter 5 shows the decoding procedure for the class of codes obtained from the aperiodic convolution algorithms, moreover it is established that the length- and the errorcorrecting capability of these codes can be varied easily. As a consequence it has been proved that the encoder/decoder can be designed to incorporate a large number of these codes into the same configuration.

The next two chapters deal with the basic automatic repeat request schemes, with their protocols and their generalization.

G. Galambos (Szeged)

Fred Kröger, Temporal Logic of Programs (EATCS Monographs on Theoretical Computer Science, 8), VIII+148 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

Temporal logic is a branch of modal logic. Its basic idea is that the truth of an assertion may depend on a discrete time scale. As a logic of this kind, it can be used to describe properties of programs in a natural way where the execution sequence of a program plays the role of the time scale.

Besides the Introduction, the book consists of seven chapters. Chapter I provides a detailed discussion of syntax and semantics of propositional temporal logic. In addition to the usual modal operators, the nexttime operator and the atnext operator are taken as primitive. If A and B are formulas, then A *atnext* B expresses that A will hold at the next time poirt than B holds. Other temporal operators are introduced as derived ones. Soundness and completeness of an axiomatization of propositional temporal logic is established in Chapter II. Some induction principles are also included. Chapter III is devoted to first order temporal logic. No completeness theorem is stated.

A basic (parallel) programming language is the subject of Chapter IV. Program properties are formalized and classified as safety (or invariance), liveness (or eventuality) and precendence properties. The rest of the book is devoted to program verification using temporal logic. Invariance and procedence properties are discussed in Chapter V and eventuality properties in Chapter VI. Hoare's calculus is embedded in temporal logic in Chapter VII.

It is shown in each case how program verification rules can be derived within the system, these are however the only theorems incorporated. Several examples are discussed.

The volume can be recommended to graduate students with interest in program verification.

Z. Ésik (Szeged—Munich)

Yurii T. Lyubich, Introduction to the Theory of Banach Representation of Groups, VI+223 pages, Birkhäuser Verlag, Basel—Boston—Berlin, 1988.

This is a translation of the original Russian edition. The book consists of five chapters. The first three chapters are devoted to give the mathematical background needed in the last two chapters. Chapter one deals with the basic properties of bounded linear operators in Banach spaces and with commutative Banach algebras. Chapter 2 introduces the notions of topological groups and topological semigroups, a brief reference to invariant measures and means is also given. Chapter 3 gives a glimpse into the elements of general representation theory. Chapter 4 presents the representation theory of compact groups and semigroups in the space of bounded operators of a Banach space. In the final chapter the representation theory of locally compact Abelian groups can be found. In the text a rich collection of exercises and examples is given, serving as the illustration of the ideas.

L. Gehér (Szeged)

Erkki Mäkinen, On context-free derivations (Acta Universitatics Tamperensis, ser. A, vol. 198), 94 pages, Tampere, 1985.

Given a context-free grammar, its Szilard language contains one word for each terminating derivation. Szilard languages also arise with restricted types of derivations such as leftmost derivations, depth-first derivations and breadth-first derivations. The book provides a good survey of results on Szilard languages: basic properties and relation to the Chomsky hierarchy, decision problems, recognition of Szilard languages, etc. Unrestricted Szilard languages are related to m-counter automata and left Szilard languages to simple pushdown automata. The importance of depth-first derivations lies in the fact that depth-first Szilard languages are context-free yet they are more general than leftmost derivations. The last chapter is devoted to the relation of Szilard languages to grammatical similarity.

The book can be recommended to graduate students and computer scientists with interest in formal languages and compiler construction.

Z. Ésik (Szeged—Munich)

Mathematical Foundations of Programming Language Semantics, Edited by M. Main, A. Melton, M. Mislove and D. Schmidt (Lecture Notes in Computer Science, 298), VIII+637 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1988.

This volume is the proceedings of the Third Workshop on Mathematical Foundations of Programming Language Semantics held at Tulane University, New Orleans, Louisiana, in April, 1987. The 32 contributions (4 invited and 28 selected) are organized into six chapters. The subject matter covers a wide range from category theory and λ-calculus to domain theory and implementation issues.

The invited addresses are the following: J. W. Gray: A Categorical Treatment of Polymorphic Operations; The main thesis is that 2-categories provide the right framework for studying polymorphic operations, i.e., operations that behave the same everywhere. J. D. Lawson: The Verstile Continuous Order; A survey of basic properties of continuously ordered sets including two natural topologies. S. D. Brookes: Semantically Based Axiomatics, A discussion on the basic ideas of Hoare's calculus. N. D. Jones et al.: MIX: A Self-Applicable Partial Evaluator for Experiments in Compiler Generation (Extended Abstract). The volume does not contain the text of the invited talks given by G. Plotkin and D. Scott.

The volume can be recommended to researchers and graduate students with interest in semantic issues.

Z. Ésik (Szeged—Munich)

Particle Physics, A Los Alamos Primer, Edited by N. G. Cooper and G. B. West, XI+199 pages, Cambridge University Press, Cambridge—New York—New Rochelle—Melbourne—Sidney, 1988.

Particle physics is one of the most challenging fields for the human thought, and likewise for the budget of those few countries and organizations that can afford to finance the enormous costs of experimental particle physics. This book, which is a collection of articles written by a group of particle physicists at Los Alamos, is divided into two main parts. The first one is a theoretical frame; work. The authors explain here what are meant by the fundamental physical particles as quarks-leptons, gauge bosons, and how the related mathematical ideas: gauge fields, spontaneous symmetry breaking, quantum chromodynamics, etc., have emerged in the last 20 years. The subject is treated on a variety of technical levels and will certainly be enjoyed by anyone who is interested in the modern developments of natural sciences. Physicists working in other fields than particle physics will like this book too because everything is explained on the level of ordinary four dimensional electrodynamics and quantum mechanics. I think also the professional particle physicist may obtain much help reading this book because it demonstrates how to expose this most difficult subject in a simple manner.

The second-part of the book acquaints us with the grandiose experiments have been done so far, or planned in the future by particle physicists. We can learn about "underground science", as the enormous detectors, that are to be detecting proton decay, or neutrino oscillations — predicted by some theories — are located in deep mines. Not less interesting are the details of huge accelerators with diameters of tens of kilometers etc.

Each article is illustrated with several figures that help very much to understand the physical ideas. In the end of the book the authors express their personal viewpoint in a lively discussion about their profession, and also about social psychology of the particle physics community.

Physical theories of the first half of our century yielded us, among others, the theoretical basis of atomic power plants, but at present one can only hope that some time in the future high energy physics will also provide us with practical comfort. And though physicists are convinced that this will come true one day, the situation is more idealistic at present. We can only state that the gigantic and very expensive experiments serve merely to prove that deep mathematical ideas, such as Lie groups, supersymmetry, gauge invariance etc. have their origins in reality. Nevertheless these theories have strong predictive power and will allow mankind to control reality in an ever increasing manner.

M. G. Benedict (Szeged)

S. J. Patterson, An Introduction to the Theory of the Riemann Zeta-Function (Cambridge Studies in Advanced Mathematics, 14), 156 pages, Cambridge University Press, Cambridge—New York—New Rochelle—Melbourne—Sidney, 1988.

One of the most famous problems of mathematics is the so called Riemann Hypothesis. This states that all the zeros of the zeta-function lie on the "critical line" $\left\{z\colon \mathrm{Re}z=\frac{1}{2}\right\}$. (This is one of the several forms of the conjecture.) This was formulated in 1859 by B. Riemann, and it occurs in the eighth problem of the famous 23 unsolved problems presented by D. Hilbert before the International Congress of Mathematicians in 1900.

The following little story told by G. Pólya in a speech characterizes the importance of the problem. Somebody allegedly asked Hilbert, "If you would revive, like Barbarossa, after five hundred years, what would you do?" "I would ask" said Hilbert, "Has somebody proved the Riemann Hypothesis?"

The problem had resisted for over 100 years the efforts of mathematicians.

Several examples prove that the most fruitful and exciting task is to build a bridge over mathematical branches which are seemingly far off. The zeta-function is a meromorphic function, it can be investigated by the techniques of complex analysis and at the same time it yields important and characteristic results concerning the integers. Through the history of the zeta-function a long series of the world's greatest mathematicians (the enumeration is almost impossible) obtained determinant results. Two widely known classical summaries were written by E. Landau and E. C. Titchmarsh.

This book grew out of a lecture course about the Riemann Hypothesis and Weil's point of view concerning it. In determining the direction of the investigations the Riemann Hypothesis plays a central role. Chapter headings are: Historical introduction; The Poisson summation formula and the functional equation; The Hadamard product formula and explicit formulae of prime number theory; The zeros of the zeta-function and the prime number theorem; The Riemann Hypothesis and the Lindelöf Hypothesis; The approximate functional equation; Appendices.

An interested reader having a good background in analysis and number theory should be able to read the main part of the book. For the reviewer one of the most attractive features of this work is the concise but clear style of the treatment. The appendices make the reading of the texts easier. Various exercises in an unusually large number constitute an essential part of the book. (Some hints would be useful for the reader concerning the more difficult examples.) The thorough examination of this book offers the reader a good possibility to study special problems and to do some research. Last but not least the work consists of only 156 pages.

L. Pintér (Szeged)

Efim M. Polishchuk, Continual Means and Boundary Value Problems in Function Spaces, 159 pages, Birkhäuser Verlag, Basel—Boston—Berlin, 1988.

The main purpose of this book is to develop the theory of integration of infiite dimensional spaces and to give applications to boundary value problems for function domains. The text is divided into four parts. In the first part the definitions of uniform and normal functional domains are introduced. The notion of the main value of a functional over a domain is given and explicit formulae for its calculation are deduced. The procedure of functional averaging is shown to result in a Dirac measure, which is a generalized function. At the end of this part several definitions of the functional Laplace operator are presented. The second part is devoted to study the weak Dirichlet problem for normal domains with boundary values from the Gatoux class, furthermore the Poisson equation and the solution of an exterior Dirichlet problem in a function space are considered. In the third part a completely different approach to the functional boundary value problems is proposed. Also boundary value problems for uniform domains are investigated. The final part deals with the extension of some of the previous results to boundary value problems with a general elliptic functional operator using the theory of diffusion processes and the compact extension of a function domain.

The material is as selfcontained as it is possible. The book is recommended to research workers who are familiar with measure theory and functional analysis.

L. Gehér (Szeged)

Recent Developments in Mathematical Physics, Edited by H. Mitter and L. Pittnes, XI+323 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

The demand for mathematical rigor appears in theoretical physics mostly when "rude" physics itself shows that "something is wrong". This is the point where it is worth to try more exact methods, and it turns out very often that behind the new mathematics there is something new in physics as well. Mathematical rigor has the advantage that the physical model, its assumptions and restrictions, can be formulated in a most compact way. The 34 articles, contained in this book are written in this spirit. They are the texts of the lectures given at a meeting in Schladming, Austria, in 1987, in honour of Professor W. Thirring. Both classical and quantum mechanical problems are considered as well as problems in statistical physics and quantum field theory. The book will be interesting for mathematicians and also for physicists who like the mathematical style in theoretical physics. It will be useful for anyone who wants to see at least a part of fields of present day mathematical physics.

M. G. Benedict (Szeged)

Recent Topics in Theoretical Physics (Proceedings in Physics, 24). Edited by H. Takayama, IX+129 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1988.

The volume consists of 10 lectures on recent developments in theoretical physics, held at the Yukawa Memorial Symposium, in 1986, in Nishinomiya. The topics of the lectures can be divided into two classes. The first one is high energy physics and cosmology. The titles contain: superstrings, lattice quantum chromodynamics, the quark structure of the nucleus, solar neutrinos, the very early universe, and gravitational collapse. The rest of the volume is devoted to some most recent developments of solid state theory, such as the quantum Hall effect, diffusion of heavy particles in metals, spin glasses, and pattern formation. The lectures are written on a high level, mostly by well known Japanese specialists. Nevertheless, the style is introductory and the text is aimed at any physicist independently from his special field of research. Concepts unfamiliar to the nonexpert are explained in simple terms. From the book one can learn what is in the centre of interest of theoretical physics now. It can be recommended to mathematicians and experimental physicists as well.

M. G. Benedict (Szeged)

Rewriting Techniques and Applications, Edited by J. P. Jouannaud, 216 pages, Academic Press, London—Orlando—San Diego—New York—Austin—Montreal—Sydney—Tokyo—Toronto, 1987.

This volume contains a selection of papers presented at the first international conference on Rewriting Techniques and Applications held in May 1985, in Dijon, France. The material is reprinted from the Journal of Symbolic Computation, Volume 3, Numbers 182, 1987. The 8 selected papers are: B. Buchberger: History and Basic Features of the Critical-pair/completition procedure; R. V. Book: Thue Systems as Rewriting Systems; N. Dershowitz: Termination of Rewriting; M. Rusinowitch: Path of Subterms Ordering and Recursive Decomposition Ordering Revisited; J. Hsiang: Rewrite Method for Theorem Proving in First Order Theory with Equality; K. A. Yelick: Unification in Combinations of Collapse-free Regular Theories; E. Tiden and S. Arnborg: Unification Problems with One-sided Distributivity; D. Benenav, D. Kapur and P. Narendran: Complexity of Matching Problems.

The first 3 papers are invited and provide good surveys on 3 different topics of symbolic computation. The following is a quotation from the Editorial by Jean-Pierre Jounnaud.

"The paper by Bruno Buchberger relates the history of the most important discovery in term rewriting theory: the notion of a critical pair, and its natural consequence, the completion algorithm. The reader will find his bibliography very helpful.

The paper by Ronald Book synthesies at least ten years of research on Thue Systems, with a particular emphasis on the role of Church—Rosser properties in deciding important questions related to Thue Systems.

The paper by Nachum Dershowitz is a beautiful presentation of the current state of knowledge of termination. Moreover, he gives a new coding of Turing machines by rewrite rules, which leaves open the uniform termination problem of the one rule case only."

The book can be recommended both to researchers and graduate students interested in the field.

Z. Ésik (Szeged—Munich)

Paulo Ribenboim, The Book of Prime Number Records, XXII+476 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1988.

Having read this book the opinion of A. Rényi came to my mind (he was a Hungarian mathematician (1921—1970), mentioned in this book in the Index of Names, too). He wrote somewhere that the really beautiful, interesting, very significant and genuine things will never become ordinary or boring. This is true for the field of mathematics as well. One will never be tired of Euclid's proof stating that there exists infinitely many prime numbers. This proof is astonishing and it is a delightful experience every time just as when climbing up a peak the scenery opens up in front of us.

To tell the truth I didn't quite understand why Rényi mentioned this very example. After reading this book it became obvious for me.

Consider the above mentioned theorem. You find it in Chapter 1 having the title: How many prime numbers are there? Giving Euclid's classical proof you find some records. Denote by $p\#$ the products of all $q \leqq p$, where q and p are primes. The largest known prime of the form $p\#+1$ is $13649\#+1$ and it was discovered by H. Dubner in 1987. (This number has 5862 digits.) Then several other proofs are given for the infinity of prime numbers: Kummer's proof; Pólya's proof (this uses the idea: it is enough to find an infinite sequence of natural numbers $1<a_1<a_2<\ldots$ that are pairwise relative prime); Euler's proof investigating the product of $1/(1-1/p_i)$ which leads to important developments; Thue's proof (this applies the fundamental theorem of unique factorization of natural numbers as product of prime numbers); Perrot's proof requiring the convergence of $\sum (1/n^2)$; Auric's proof; Métrod's proof; Washington's proof done via commutative algebra (this comes from 1980); and the last is Fürstenberg's proof that appeared in 1955 and is based on topological ideas. I think that there are only a few mathematicians, who don't find something new for themselves in this first chapter concerning a well-known theorem.

The further questions (at the same time chapter headings) are: How to recognize whether a natural number is a prime? Are there functions defining prime numbers? How are the prime numbers distributed? Which special kinds of primes have been considered? Heuristic and probabilistic results about prime numbers.

Let us mention only a few of the records: the largest known prime of the form $k\times 2^n+1$ with $n \geqq 2$ is $7\times 2^{54486}+1$ having 16402 digits (J. Young (1987)); the largest known prime of the form n^2+1 is $17^2\times 2^{18502}+1$ (Keller (1984)).

Let us consider another record concerning the famous Waring's problem: for every $k \geqq 2$ there exists a number $r \geqq 1$ such that every natural number is the sum of at most r kth powers. If such a number r exists denote by $g(k)$ the smallest possible one. While these phenomena tend to become more regular for sufficiently large numbers another characteristic number is introduced: denote by $C(k)$ the minimal value of r such that every sufficiently large integer is the sum of r kth powers, obviously $C(k) \leqq g(k)$. Waring's problem (the existence of $g(k)<\infty$ for arbitrary k) was first solved by Hilbert in 1909. Here you have the records on $g(3)$ and $C(3)$. (In the book the reader finds much more in detail.) J. A. Euler (L. Euler's son) $g(3) \geqq 9$. E. Maillet (1985) $g(3) \leqq 21$, $C(3) \geqq 4$, $g(3)$ exists; A. Fleck (1906) $g(3) \leqq 13$; A. Wieferich (1906) $g(3) \leqq 9$, $g(3)$ best possible; E. Landau (1909) $C(3) \leqq 8$; Yu. V. Linnik $C(3) \leqq 7$. Present status: $4 \leqq C(3) \leqq 7$; recent computations of Bohman and Fröberg as well as of Romani (1982) point to the likelihood that $C(3)=4$.

We could enumerate several interesting problems from this work but we have no space. (One of my favourites is the discussion of Dirichlet's famous result on arithmetic progressions, and a related question established by Sierpinski in 1959: Let $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n$ be any digits ($0 \leqq a_i, b_j \leqq 9$), satisfying $b_n = 1, 3, 7$ or 9. Then there exist infinitely many prime numbers p which are written in base 10, with the a_i as initial digits and the b_j as final digits: $p=a_1, a_2, ..., a_m, b_1, b_2, ...$ $..., b_n$.)

I liked this book. (For the reviewer this is the book of prime numbers with records.) It is well written in a conversational style, and with evident enthusiasm. The Bibliography which is compiled carefully is extremly useful. Reading on prime numbers is similar to playing tennis: it is marvellous in your youth and in your old age, too.

L. Pintér (Szeged)

J. L. C. Sanz—E. B. Hinkle—A. K. Jain, Radon and Projection Transform—Based Computer Vision (Algorithms, A Pipeline Architecture, and Industrial Applications), VIII + 123 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1988,

This book provides a description of the applicability of Radon and projection transforms to computer vision and processing. Particularly it deals with novel machine vision architecture ideas that make real-time projection-based algorithms a reality.

The authors concern themselves with several image analysis algorithms for computing (for example the projections of gray-level images along linear patterns, i.e. the Radon transform). They provide fast methods to transform images into projection space representations and to backtrace projection space information to the image domain which are suitable for implementation in a pipeline architecture.

We recommend this book to the beginners and also to the specialists, since it includes a survey of the architecture trends and some novel algorithms in computer vision.

Árpád Kurusa (Szeged)

Jaroslav Smital, On Functions and Functional Equations, VII + 155 pages, Adam Hilger, Bristol and Philadelphia, 1988.

The text consists of five chapters. The introductory one summarizes the elementary ideas concerning functions. The second chapter studies functional equations of several variables, and solves the Cauchy functional equation starting different initial assumptions. The third chapter dealing with iterations is the most important part of the text playing a central role in the book. Chapter 4 gives the application of the iteration method for the study of population growth model. The final chapter investigates linear functional equations, the Abel and the Schröder equations.

Only elementary mathematical knowledge of the reader is supposed.

L. Gehér (Szeged)

Song Jian—Yu Jingyuan, Population System Control, XI + 286 pages, China Academic Publishers, Beijing and Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1988.

During the last century the world's population has increased enormously. Today more than five billion people live on Earth and the population has been increasing further. Will the resources of energy and food be enough for the mankind? Philosophers have always shown great concern about this problem throughout history. The earlier works on population studies, however, used figurative and literary language and methods and were not of a scientific nature. The modern natural sciences are required to have theories that can be quantifiably tested and verified. Recently population studies has been included among these sciences thanks to the use of statistical and qualitative research methodologies. One of the most significant steps in this direction was that researchers have begun to regard the population evolution of a community as a dynamic process. The book gives an excellent account on the latest results of Chinese systems analysts achieved by the investigation of this mathematical model.

Chapter 1 (Introduction) gives a survey on the history and basic ideas of population studies and formulates tasks of population cybernetics. In Chapter 2 the continuous, discrete and stochastic population equations are derived. In the continuous model the state function is the agedistribution density function $p(a, t)$. (Roughly speaking, if $\Delta a>0$ is small, then the total number of people of age between a and $a+\Delta a$ at time t is $p(a, t)\cdot\Delta a$.) The state function in the discrete model is a vector: $x(t)=(x_0(t), x_1(t), \ldots, x_m(t))$, where $x_i(t)$ is the total number of persons in year t whose full age is within the age interval $[i, i+1]$. The model is a first order partial differential equation and difference equation, respectively. In Chapter 3 demographic indeces (average lifetime and life expectancy, net population reproduction rate, average female fertility rate) are expressed by the state functions $p(a, t)$ and $x(t)$. Chapter 4 contains the dynamic analysis of population systems based upon the evolution equations. The most interesting (in reviewer's opinion) Chapter 5 is concerned with stability problems for population systems. The authors prove that the necessary and sufficient condition of stability in Liapunov's sense for a population system is that the total fertility rate should not exceed a critical value. Chapters 6 and 7 are devoted to population projections and policies, and description of the population structure in an ideal society. The concluding Chapter 8 presents an optimization theory of birth control policy and its applications.

This book will be very useful for mathematicians as well as social scientists dealing with population dynamics and population policy.

L. Hatvani (Szeged)

STACS 88, Edited by R. Cori and M. Wirsing (Lecture Notes in Computer Science, 294). IX+404 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1988,

The fifth Symphosium on Theoretical Aspects of Computer Science was held in Berdeaux in February 1988. The volume contains the text of the invited talk "Geometry of Numbers and Integer Programming" by C. P. Schnorr as well as 34 selected contributions that cover a wide range of theoretical computer science: Algorithms, Complexity, Formal Languages, Rewriting Systems and Abstract Data Types, Graph Grammars, Distributed Algorithms, Geometrical Algorithms, Trace Languages, Semantics of Parallelism. In addition to the technical contributions, eight software systems presented at the symphosium are reviewed.

The wide range and high quality ensure that every computer scientist will find at least one paper of his own interest.

Z. Ésik (Szeged—Munich)

Topics in Operator Theory, Constantin Apostol Memorial Issue, Edited by Gohberg, 274 pages, Birkhäuser Verlag, Basel—Boston—Berlin, 1988.

The text starts with a short glimpse of the life and mathematical results of Constantin Apostol. List of his publications is also given. Eleven papers can be found in the book. Their subjects are: operator theory and operator algebras. The first paper contains a result of Constantin Apostol

(On the Spectral Equivalence of operators). The other papers are written by different, in general, well known authors, all of them are dedicated in memory of Constantin Apostol.

The book is highly recommended to research workers interested in the modern functional analysis.

L. Gehér (Szeged)

Stephen Wiggins, Global Bifurcations and Chaos. Analytical methods (Applied Mathematical Sciences, 73), XIV+494 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1988.

The first chostic phenomena arised in deterministic nonlinear dynamical systems fifteen years ago. As it can be followed also in our Review Section, since that time an unusually great number of texts and monographs have been published devoted to the theoretical and applied problems of these phenomena. This book is concerned with the following three fundamental questions exciting both mathematicians and applied scientists: What is meant by the term "chaos"? What mechanism does chaos result? How can one predict when chaos will occur in a specific dynamical system?

It is pointed out in the book that the global bifurcation can often be the mechanism for producing deterministic chaos (the final answers are far from known). The global bifurcation means a qualitative change in the orbit structure of an extended region of phase space.

The first chapter contains the background for ordinary differential equations and dynamical systems (including such notions as conjugacies, invariant manifolds, structural stability, genericity, bifurcations, Poincaré maps) which are derived for a dynamical system to exhibit chaotic behaviour. The reader can find here a clear and exact, easily readable description of the Smale horseshoe which is the prototypical map possessing a chaotic invariant set, and which is absolutely essential for understanding what is meant by the term "chaos". The chapter includes also a good introduction to symbolic dynamics. Chapter 3 is concerned with homoclinic and heteroclinic motions, which typically result global bifurcation and chaotic behaviour in deterministic systems. A homoclinic orbit connects an unstable equilibrium to itself, a heteroclinic one connects two unstable equilibria. In the fourth chapter a variety of perturbation techniques are developed which allow the scientists to detect homoclinic and heteroclinic orbits. These are such generalizations and improvements of the Melnikov—Arnold method which are applicable to arbitrary finite dimensional systems and allow for slowly varying parameters and quasiperiodic excitation.

The book is written in an excellent style. It is selfcontained, requiring only the knowledge of calculus. During the exposition of the complicated notions the author first gives some examples of specific physical systems so that the reader may develop some intuition. After this he gives the exact mathematical definition.

This excellent book can be highly recommended to every mathematician, user of mathematics or student interested in qualitative theory of dynamical systems and its applications.

L. Hatvani (Szeged)

Eberhard Zeidler, Nonlinear Functional Analysis and its Applications IV: Applications to Mathematical Physics, XXIII+975 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1988.

This book is the fourth of a five-volume survey on the main principles and methods of nonlinear functional analysis and its applications. The main goal of the book is to give an exact clear exposition of the field which is self-contained and accessible to the nonspecialists, which combines the classical and modern ideas, and which builds a bridge between the language and thoughts of physicists and mathematicians. The specific nature and importance of the problems of mathematical physics are well expressed by M. Atiyah's opinion, which is cited in the book: "The more I have learned about physics, the more convinced I am that physics provided, in a sense, the deepest applications of mathematics. The mathematical problems that have been solved, or techniques that have arisen out of physics in the past, have been the lifeblood of mathematics. The really deep questions are still in the physical sciences. For the health of mathematics at its research level, I think it is very important to maintain that link as much as possible". However, reading physical literature mathematicians often complain that the presentation is not rigorous and exact enough, and vice versa, physicists find the mathematical methods too abstract and useless for their purposes covering the physical thoughts. The present book helps to solve this difficulty. Mathematicians will feel it comfortable because it uses precise mathematical language, at the same time the reader can learn a lot about the physical interpretation. On the other hand, the physicists can recognize the familiar physical ideas and can get acquainted with their justification.

Similarly to the previous volumes, the chapters are grouped into blocks according to applications:

Applications in Mechanics: Ch. 58. Basic Equations of Point Mechanics; Ch. 59. Dualism Between Wave and Particle, Preview of Quantum Theory, and Elementary Particles.

Applications in Elasticity Theory: Ch. 60. Elastoplastic Wire; Ch. 61. Basic Equations of Nonlinear Elasticity Theory; Ch. 62. Monotone Potential Operators and a Class of Models with Nonlinear Hooke's Law, Duality and Plasticity, and Polyconvexity; Ch. 63. Variational Inequalities and Signorini Problem for Nonlinear Material; Ch. 64. Bifurcation for Variational Inequalities; Ch. 65. Pseudomonotone Operators, Bifurcations, and the von Kármán Plate Equations; Ch. 66. Convex Analysis, Maximal Montone Operators and Elasto-Viscoplastic Material with Linear Hardening and Hysteresis.

Applications in Thermodynamics: Ch. 67. Phenomenological Thermodynamics of Quasi-Equilibrium and Equilibrium States; Ch. 68. Statistical Physics; Ch. 69. Continuation with respect to a Parameter and a Radiation Problem of Carleman.

Applications in Hydrodynamics: Ch. 70. Basic Equations of Hydrodynamics; Ch. 71. Bifurcation and Permanent Gravitational Waves; Ch. 72. Viscous Fluids and the Navier-Stokes Equations.

Manifods and their Applications: Ch. 73. Banach Manifolds; Ch. 74. Classical Surface Theory, the Theorema Egregium of Gauss, and Differential Geometry on Manifolds; Ch. 75. Special Theory of Relativity; Ch. 76. General Theory of Relativity; Ch. 77. Simplicial Methods, Fixed Point Theory, and Mathematical Economics; Ch. 78. Homotopy Methods and One Dimensional Manifolds; Ch. 79. Dynamical Stability and Bifurcation in B—S-spaces.

The chapters are followed by interesting problems supplying the body of the text and encouraging the reader's individual thinking.

Apparently, the book covers the whole spectrum of the significant applications of the nonlinear functional analysis. It will be very useful and inevitably importan tfor mathematicians, physicists and students interested in applications of mathematical methods in physics.

L. Hatvani (Szeged)

Livres reçus par la rédaction

W. Ballmann—M. Gromov—V. Schroeder, Manifolds of nonpositive curvature (Progress in Mathematics, Vol. 61), IV+266 pages, Birkhäuser Verlag, Boston—Basel—Stuttgart, 1985. — Sfr. 92,—.

A. Bultheel, Laurent series and their Padé approximations (Operator Theory, Vol. 27), XI+274 pages, Birkhäuser Verlag, Boston—Basel—Stuttgart, 1987. — Sfr. 92,—.

T. tom Dieck—I. Hambleton, Surgery theory and geometry of representations (DMV Seminar, Vol. 11), VII+115 pages, Birkhäuser Verlag, Boston—Basel—Stuttgart, 1988 — Sfr. 34,—

Differential Geometry. Proceedings, Special Year, Maryland 1981—82. Edited by R. Brooks, A. Gray, B. L. Reinhart (Progress in Mathematics, Vol. 32), VII+259 pages, Birkhäuser Verlag, Boston—Basel—Stuttgart, 1983. — Sfr. 55,—

Discrete groups in geometry and analysis. Papers in honor of G. D. Mostow on his sixtieth birthday Edited by R. Howe (Progress in Mathematics, Vol. 67), XI+212 pages, Birkhäuser Verlag, Boston—Basel—Stuttgart, 1987. — Sfr. 57,—.

I. M. Gelfand, Collected papers, Vol. 2. Edited by S. G. Gindikin, V. W. Guillemin, A. A. Kirillov, B. Kostant, S. Sternberg, X+1039 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1988. — DM 248,—.

S. G. Gindikin, Tales of physicists and mathematicians, XI+157 pages, Birkhäuser Verlag, Boston—Basel—Stuttgart, 1988. — Sfr. 48,—.

M. Jantzen, Confluent string rewriting (EATCS Monographs on Theoretical Computer Science, Vol. 14), IX+126 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1988. — DM 54,—.

M. Koecher, Klassische elementare Analysis, 211 Seiten, Birkhäuser Verlag, Boston—Basel—Stuttgart, 1987. — Sfr. 42,—.

P. Koosis, The logarithmic integral (Cambridge Studies in Advanced Mathematics, Vol. 12), XVI+606 pages, Cambridge University Press, Cambridge—London—New York—New Rochelle—Melbourne—Sydney, 1988. — £ 89.50.

H. Loeffel, Blaise Pascal (Vita Mathematica, Vol. 2), 175 Seiten, Birkhäuser Verlag, Boston—Basel—Stuttgart, 1987. — Sfr. 40,—.

Y. I. Lyubich, Introduction to the theory of Banach representation of groups (Operator Theory, Vol. 30), IX+223 pages, Birkhäuser Verlag, Boston—Basel—Stuttgart, 1988. — Sfr. 118,—.

E. Maor, To infinity and beyond. A cultural history of the infinite, XVI+275 pages, Birkhäuser Verlag, Boston—Basel—Stuttgart, 1987. — Sfr. 103,—.

A. B. Mingarelli—S. G. Halvorsen, Non-oscillation domains of differential equations with two parameters (Lecture Notes in Mathematics, Vol. 1338), XI+109 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1988. — DM 23,—.

P. Molino, Riemann foliations (Progress in Mathematics, Vol. 73), XII+344 pages, Birkhäuser Verlag, Boston—Basel—Stuttgart, 1988. — Sfr. 68,—.

G. L. Naber, Spacetime and singularities. An introduction (London Mathematical Society Student Texts, Vol. 11), IX+178 pages, Cambridge University Press, Cambridge—London—New York—New Rochelle—Melbourne—Sydney, 1988. — £ 39.50.

Number Theory Related to Fermat's Last Theorem. Edited by N. Koblitz (Progress in Mathematics, Vol. 26), X+366 pages, Birkhäuser Verlag, Boston—Basel—Stuttgart, 1982. — Sfr. 94,—.

E. M. Polischchuk, Continual means and boundary value problems in function spaces (Operator Theory, Vol. 31), 160 pages, Birkhäuser Verlag, Boston—Basel—Stuttgart, 1988. — Sfr. 62,—.

R. Remmert—P. Ullrich, Elementare Zahlentheorie, 275 Seiten, Birkhäuser Verlag, Boston—Basel—Stuttgart, 1987. — Sfr. 44,—.

B. A. Rosenfeld, A history of non-euclidean geometry. Evolution of the concept of a geometric space (Studies in the History of Mathematics and Physical Sciences Vol. 12), XI+471 pages, Springer-Verlag, New York—Berlin—Heidelberg—Tokyo, 1988. — DM 184,—.

Topics in operator Theory. Constantin Apostol memorial issue. Edited by I. Gohberg (Operator Theory, Vol. 32), VII+274 pages, Birkhäuser Verlag, Boston—Basel—Stuttgart, 1988. — Sfr. 88,—.

S. Wiggins, Global bifurcations and chaos. Analitical methods (Applied Mathematical Sciences, Vol. 73), XIV+494 pages, Springer-Verlag, New York—Berlin—Heidelberg—Tokyo, 1988. — DM 98,—.

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