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On the representation of distributive algebraic lattices. II

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1. Introduction

Around 1980, H. Bauer found a result which implies that countable distributive semilattices with 0 can be represented as semilattices of compact congruences of a lattice, whence it also follows that every lower bounded distributive algebraic lattice with countably many compact elements is the congruence lattice of a lattice. This proof, however, was not published. In [2], we proved that if D_1 and D_2 are finite distributive semilattices with 0 such that D_1 is a 0-subsemilattice of D_2, then D_1 and D_2 have a simultaneous representation (in a sense precisely defined in [3]) as semilattices of compact congruences of lattices L_1 and L_2, respectively. There we promised to show that this idea can be developed to a proof of the countable representation problem. Here we present this proof. We note that independently and by different methods H. DOBBERTIN [1] found another proof of the theorem.

It is easy to show that any finite subset of a distributive semilattice with 0 is contained in a finite distributive 0-subsemilattice. Hence it follows that for any countable distributive semilattice D with 0, there exist finite distributive semilattices $D_1, D_2, D_3, \ldots$ with 0 and embeddings $\varepsilon_i\colon D_i \to D_{i+1}$, $i=1,2,\ldots$, such that D is the direct limit of the family $(\{D_i\}_{i\in N}, \{\varepsilon_i\}_{i\in N})$. Now let D and D_i, $i=1,2,\ldots$, be as above and fixed once and for all. We prove the following

Theorem. *There exist lattices L_i, $i=1,2,\ldots$, such that*
(α) *$D_i \cong \mathrm{Con}(L_i)$ under an isomorphism to be denoted by φ_i, $i=1,2,\ldots$,*
(β) *L_i has an embedding λ_i to L_{i+1}, $i=1,2,\ldots$,*
(γ) *if we denote by $\mathrm{Con}(\lambda_i)$ the mapping of $\mathrm{Con}(L_i)$ to $\mathrm{Con}(L_{i+1})$ induced by λ_i (that is the one that maps $\Theta \in \mathrm{Con}(L_i)$ to the congruence generated by*

*) This paper was left behind by András Huhn in the form of a first draft of a manuscript. Hans Dobbertin was kind to prepare it for publication.

$\{(a\lambda_i, b\lambda_i) \in L_{i+1}^2 | (a, b) \in \Theta\})$, *then the following diagram is commutative*

$$\begin{tikzcd} D_i \arrow[r, "\varepsilon_i"] \arrow[d, "\varphi_i"'] & D_{i+1} \arrow[d, "\varphi_{i+1}"] \\ \mathrm{Con}\,(L_i) \arrow[r, "\mathrm{Con}\,(\lambda_i)"'] & \mathrm{Con}\,(L_{i+1}) \end{tikzcd}$$

where ε_i denotes the identical embedding of D_i to D_{i+1}. In other words Con (λ_i) *represents* id_i.

Corollary. *Every countable distributive semilattice with* 0 *is isomorphic to the semilattice of all compact congruences of a lattice.*

To prove the Corollary from the Theorem, observe that the Con (L_i)'s form the same directed system (up to commuting isomorphisms) that the D_i's, whence their direct limit is also isomorphic with D. On the other hand, the L_i's also form a directed system and the congruence lattice of their direct limit is the direct limit of their congruence lattices (see PUDLÁK [3]). This proves the corollary.

2. The construction of L_j. Proof of (a)

First we define the following lattices. Let $i \leqq j$ be natural numbers. Let $D(i \to j)$ be the distributive lattice whose join-irreducibles are $(a_i, \ldots, a_j), (a_{i+1}, \ldots, a_j), \ldots, (a_j)$, where $a_i, \ldots, a_j$ are join-irreducibles of $D_i, \ldots, D_j$, respectively, and $a_i\varepsilon_i \geqq a_{i+1}, a_{i+1}\varepsilon_{i+1} \geqq a_{i+2}, \ldots$. Let these join-irreducibles be ordered componentwise, that is, let $(a_k, \ldots, a_j) \leqq (a'_l, \ldots, a'_j)$ iff $k \leqq l$ and $a_l \leqq a'_l, \ldots, a_j \leqq a'_j$. Clearly, the set of join-irreducibles and their ordering determines $D(i \to j)$. Let $B(1 \to j)$ be the Boolean lattice whose set of atoms is $\{[a] \mid a$ join-irreducible in $D(1 \to j)\}$. Of course, instead of $[(a_1, \ldots, a_j)]$ etc. we shall write $[a_1, \ldots, a_j]$. Now there are some natural 0-1-embeddings. Each element of $D(i+1 \to j)$ can be identified with an element of $D(i \to j)$ as follows: $x \in D(i+1 \to j)$ is a join of join-irreducibles. These join-irreducibles are, however, join-irreducibles of $D(i \to j)$, too. Thus x can be identified with their join in $D(i \to j)$. This is a lattice 0-1-embedding and from now on we shall consider $D(i+1 \to j)$ as a sublattice of $D(i \to j)$. Note that $D(j \to j) \cong D_j$ and will be identified with it. Furthermore, $D(1 \to j)$ can be considered as a 0—1-sublattice of $B(1 \to j)$, namely $x \in D(1 \to j)$ can be identified with the join of all $[a]$, $a \leqq x$, a join-irreducible.

Now we define lattices $L(1 \to j)$ as follows. Let $M(1 \to j)$ consist of all triples $(x, y, z) \in (B(1 \to j))^3$ satisfying $x \wedge y = x \wedge z = y \wedge z$. Let $L(1 \to j)$ be the set of all those triples in $M(1 \to j)$ also satisfying $z \in D(1 \to j)$. Let $M(i \to j)$ $(i > 1)$ consist of all those triples $(x, y, z) \in (D(i-1 \to j))^3$ satisfying $x \wedge y = x \wedge z = y \wedge z$, and let

$L(i\to j)$ be the set of all those triples satisfying also $z\in D(i\to j)$. Now we describe the operations of $L(1\to l)$ and $L(i\to j)$, $i=2, \dots, j$. The meet operations are the same as in $(B(1\to j))^3$ and in $(D(i-1\to j))^3$, respectively. We shall denote the joins in $(B(1\to j))^3$, $M(1\to j)$, $L(1\to j)$ by $\vee$, $\vee_M$, $\vee_L$, respectively and the join in $(D(i-1\to j))^3$, $M(i\to j)$, $L(i\to j)$ by $\vee$, $\vee_M$, $\vee_L$, respectively. This will cause no confusion. As $D(1\to j)$ is a sublattice of $B(1\to j)$, with every $z\in B(1\to j)$ we can associate an element $\bar{z}\in D(1\to j)$ which is the smallest element of $B(1\to j)$ such that $z\leqq \bar{z}$. Also, with any $z\in D(i-1\to j)$ $(i>1)$ we can associate a $\bar{z}\in D(i\to j)$, which is the smallest element of $D(i\to j)$ such that $z\leqq\bar{z}$. Now it is proven in SCHMIDT [4] that

$$(x, y, z)\vee_M(x', y', z') = (x\vee x', y\vee y', z\vee z')^{\sim},$$

where

$$(x, y, z)^{\sim} = \big(x\vee(y\wedge z), y\vee(x\wedge z), z\vee(x\wedge y)\big) \quad \text{for} \quad (x, y, z)\in\big(B(1\to j)\big)^3,$$

and

$$(x, y, z)\vee_L(x', y', z') = (x\vee x', y\vee y', z\vee z')^{\sim\wedge},$$

where

$$(x, y, z)^{\wedge} = \big(x\vee(y\wedge\bar{z}), y\vee(x\wedge\bar{z}), \bar{z}\big) \quad \text{for} \quad (x, y, z)\in M(1\to j).$$

The same proof as in [4], pp. 82—86 yields that this description remains valid for $(x, y, z)\in D(i-1\to j)$ as well as for $(x, y, z)\in M(i\to j)$. Now $L(1\to j)$ has an ideal isomorphic to $D(1\to j)$, namely the ideal $[(0, 0, 0), (0, 0, 1)]$, where 0 and 1 denote the bounds of $B(1\to j)$. The ideals $[(0, 0, 0), (1, 0, 0)]$ and $[(0, 0, 0), (0, 1, 0)]$ are isomorphic to $B(1\to j)$. Furthermore, the dual ideals $[(0, 1, 0), (1, 1, 1)]$ and $[(1, 0, 0), (1, 1, 1)]$ are isomorphic to $B(1\to j)$. All these proofs can be carried out by using the description of the operation of $L(1\to j)$. In fact, as an example, we prove that $[(1, 0, 0), (1, 1, 1)]$ is isomorphic to $D(1\to j)$. The elements of this interval are the elements $(1, y, z)$ with $z\in D(1\to j)$ and by $y\wedge 1=z\wedge 1=1\wedge 1$ we have $y=z$, that is, the elements of the interval are $(1, z, z)$, $z\in D(1\to j)$. Their meet is always formed componentwise and, using the previous description of the operation, is obvious, that the componentwise join is already invariant under $^{\sim}$ and $^{\wedge}$. Now we are ready to define L_j. Namely, similarly as the $L(1\to j)$, all the $L(i\to j)$, $i=2, \dots, j$, have ideals isomorphic to $D(i-1\to j)$ and to $D(i\to j)$ (the proof is the same), so we can "glue them together" as shown in Figure 1. More exactly we form the direct product of the $L(i\to j)$'s. It has an ideal isomorphic to $L(i\to j)$ for all $i=1, \dots, j$. We glue the bottom of this direct product to the top of $\prod_{i=2}^{j} M(i\to j)$. The latter has dual ideals isomorphic to $M(i\to j)$ for all $i=2, \dots, j$. Now we identify, for all $i=1, 2, \dots, j-1$, the ideal $[(0, 0, 0), (0, 0, 1)]$ of $L(i\to j)$ $(\subseteq \prod_i L(i\to j))$ with the dual ideal $[(0, 0, 1), (1, 1, 1)]$ of a copy of $M(i+1\to j)$.

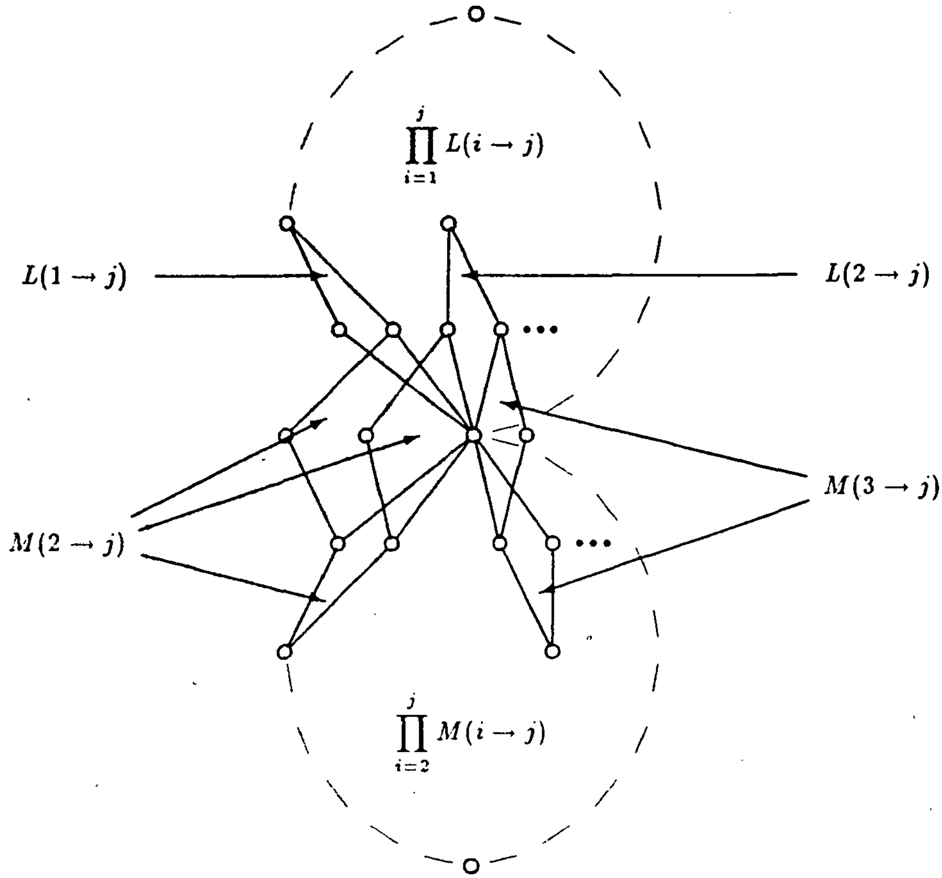

Figure 1

We identify the ideal $[(0,0,0),(0,0,1)]$ of this copy with the dual ideal $[(1,0,0),(1,1,1)]$ of the copy of $M_\varepsilon(i+1 \to j)$ which is a dual ideal in $\prod_{k=2}^{j} M(k \to j)$, and we identify the dual ideal $[(0,0,1),(1,1,1)]$ of this copy with the ideal $[(0,0,0),(0,0,1)]$ of a third copy of $M(i+1 \to j)$. Finally, we identify the dual ideal $[(0,0,1),(1,1,1)]$ of this third copy with the ideal $[(0,0,0),(1,0,0)]$ of $L(i+1 \to j)$ $(\subseteq \prod_{k=1}^{j} L(k \to j))$. The lattice we so obtain is L_j.

Now we have to prove (α). Consider any congruence α of L_j. First of all it splits into a join of congruences of the two direct products and of the joining $M(i \to j)$'s. By perspectivity, the generating pairs of these congruences can be transformed to the upper part $\prod_{i=1}^{j} L(i \to j)$, and there they factorize according to the direct representation, thus α is generated by pairs contained in the $L(i\to j)$'s (considered as ideals of $\prod L(i\to j)$). We shall prove that α is generated by an ideal of the interval $[(0,0,0),(0,0,1)]\cong D_j$ of $L(i\to j)$. As we mentioned, α is a join of principal congruences generated from the $L(i\to j)$'s. We may assume that α itself is such a principal congruence (because the join of ideals of $[(0,0,0),(0,0,1)]\subseteq J(j\to j)$ itself is an ideal).

Let α be generated by the pair $((x,y,z),(x',y',z'))$, where $(x,y,z),(x',y',z')\in \in L(k\to j)$, that is

$$x,y,x',y'\in D(k-1\to j),\quad z,z'\in D(k\to j).$$

Then, forming the meets with $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, we obtain

$$(x,0,0)\,\alpha\,(x',0,0),\quad (0,y,0)\,\alpha\,(0,y',0),\quad (0,0,z)\,\alpha\,(0,0,z').$$

Hence $(x,0,0)\vee_L(0,1,0)=(x,1,0)^{\check{}\,\hat{}}=(x,1,x)^{\hat{}}=(x,1,x)$, thus we have $(x,1,x)\,\alpha\,(x',1,x')$. Forming the meet of both sides with $(0,0,1)$, we get $(0,0,x)\,\alpha\,(0,0,x')$. Similarly $(0,0,y)\,\alpha\,(0,0,y')$. Thus the congruence generated by $((x,y,z),(x',y',z'))$ contains the pairs $((0,0,x),(0,0,x'))$, $((0,y,0),(0,y',0))$, $((0,0,z),(0,0,z'))$. It is also generated by them. We refer to p. 241 of [2] with which our notation coincides. Now $(0,0,x)$, $(0,0,x')$, etc. are contained in the copy $D(k-1\to j)$, which was used for the glueing in Figure 1. Hence α is generated from $L(k-1\to j)$ already (the generators can be transported by perspectivity), that is, by induction, it is generated from $L(1\to j)$, and, finally, with the same computation as above, from $B(1\to j)$. $B(1\to j)$ is Boolean, hence α is generated by an ideal, say, by the pair $((0,0,0),(t,0,0))$, $(0,0,0),(t,0,0)\in L(1\to j)$. Then it is also generated by

$$((0,0,0),(t,0,0))\vee_L((0,1,0),(0,1,0))=((0,1,0),(\bar t,1,\bar t)),$$

that is, by

$$((0,1,0),(\bar t,1,\bar t))\wedge_L((0,0,1),(0,0,1))=((0,0,0),(0,0,\bar t)),$$

which is an ideal of $D(1\to j)$. By induction, it is generated by an ideal of D_j, as claimed.

3. The construction of the embeddings λ_j. Proof of (β)

First of all we define embeddings

$$\beta_{1j}\colon B(1\to j)\to B(1\to j+1)\quad\text{and}\quad \delta_{ij}\colon D(i\to j)\to D(i\to j+1),$$

whenever $i\leqq j$, as follows: The atoms of $B(1\to j)$ are of the form $[a_1,\dots,a_j]$, $a_1\varepsilon_1\geqq \geqq a_2$, $a_2\varepsilon_2\geqq a_3,\dots,a_{j-1}\varepsilon_{j-1}\geqq a_j$ or of the form $[a_2,\dots,a_j]$, $a_2\varepsilon_2\geqq a_3,\dots,a_{j-1}\varepsilon_{j-1}\geqq \geqq a_j$, and so on, or of the form $[a_j]$, where $a_1,\dots,a_j$ are join-irreducibles of $D_1,\dots,D_j$, respectively. (These atoms are unordered.) We associate with $[a_i,\dots,a_j]$

the join of all $[a_i, ..., a_j, a_{j+1}]$ in $B(1\to j+1)$, where $a_j\varepsilon_j \geqq a_{j+1}$, and a_{j+1} is a join-irreducible element in D_{j+1}. With the join of a set of atoms we associate the join of their images. This mapping is then denoted by β_1. β_{1j} clearly preserves 0 and the lattice operations, thus we only have to prove that it is one-to-one. In other words we have to prove that the dual mapping under Stone's duality is onto. This dual mapping associates with the atom $[a_1, ..., a_j, a_{j+1}]$ the atom $[a_1, ..., a_j]$, that is, we have to show that, for every atom $[a_1, ..., a_j]$ of $B(1\to j)$, there is an atom $[a_1, ..., a_j, a_{j+1}]$ of $B(1\to j+1)$ with $a_j\varepsilon_j \geqq a_{j+1}$, and this is evident as $a_j\varepsilon_j \neq 0$. Now we define δ_{ij}. The join-irreducibles of $D(i\to j)$ are of the form $(a_i, ..., a_j)$, $a_i\varepsilon_i \geqq a_{i+1}, ..., a_{j-1}\varepsilon_{j-1} \geqq a_j$, or $(a_{i+1}, ..., a_j)$, $a_{i+1}\varepsilon_{i+1} \geqq a_{i+2}, ..., a_{j-1}\varepsilon_{j-1} \geqq a_j$, and so on, or (a_j), and they are ordered componentwise. For $x\in D(i\to j)$, let $x\delta_{ij}$ be the join of all $(a_k, ..., a_j)$, where $(a_k, ..., a_j)$ is join-irreducible in $D(i\to j)$, $(a_k, ..., a_j)\leqq x$, and $a_j\varepsilon_j \geqq a_j+1$. δ_{ij} is a 0-preserving lattice embedding. The proof is the same as for β_{ij}, but we have to prove Priestley's duality, rather than Stone's duality. We need the following lemmas.

Lemma 1. *Let* $x\in B(1\to j)$. *Then* $\bar{x}\delta_{1j} = \overline{x\delta_{1j}}$.

Lemma 2. *Let* $x\in D(i-1\to j)$, $i-1<j$. *Then* $\bar{x}\delta_{ij} = \overline{x\delta_{i-1,j}}$.

Proof of Lemma 1. Let $(a_1, ..., a_j, a_{j+1})\in D(1\to j+1)$ such that

$$(a_1, ..., a_j, a_{j+1}) \leqq \bar{x}\delta_{1j} \quad \text{and} \quad (a_1, ..., a_j, a_{j+1})$$

is join-irreducible. Then $(a_1, ..., a_j)\in\bar{x}$. Hence there is a join-irreducible element $(b_1, ..., b_j)$ in $D(1\to j)$ such that $(b_1, ..., b_j)\geqq(a_1, ..., a_j)$ and $(b_1, ..., b_j)$ occurs in the join-representation of $\bar{x}$, that is, $[b_1, ..., b_j]$ occurs in the join-representation of x. Then $[b_1, ..., b_j]\leqq x$. Hence $[b_1, ..., b_j, a_j+1]\leqq x\beta_{1j}$, that is, $(a_1, ..., a_j, a_j+1)\leqq$ $\leqq(b_1, ..., b_j, a_j+1)\leqq\overline{x\beta_{1j}}$. Conversely, if $(a_1, ..., a_j, a_j+1)\leqq\overline{x\beta_{1j}}$, then

$$(a_1, ..., a_j, a_j+1) \leqq (b_1, ..., b_j, b_j+1),$$

where $(b_1, ..., b_j, b_j+1)$ occurs in the join-representation of $\overline{x\beta_{1j}}$, that is $[b_1, ..., b_j, b_j+1]$ occurs in the join-representation of $x\beta_{1j}$. Hence $[b_1, ..., b_j, b_j+1]\leqq$ $\leqq x\beta_{1j}$. Then $[b_1, ..., b_j]\leqq x$ (see the definition of β_{1j}), $(b_1, ..., b_j)\leqq\bar{x}$, thus $(a_1, ..., a_j)\leqq\bar{x}$ and $(a_1, ..., a_j, a_{j+1})\leqq\bar{x}\delta_{1j}$.

Proof of Lemma 2. Let $(a_i, ..., a_j, a_{j+1})\leqq\bar{x}\delta_{ij}$, join-irreducible in $D(i\to j+1)$. Then $(a_i, ..., a_j)\leqq\bar{x}$, that is, $(a_i, ..., a_j)\leqq(b_i, ..., b_j)$, where $(b_i, ..., b_j)$ occurs in the join-representation of $\bar{x}$, that is, for a suitable join-irreducible $b_{i-1}\in D_{i-1}$ with $b_{i-1}\varepsilon_{i-1}\geqq b_i$, $(b_{i-1}, b_i, ..., b_j)$ occurs in the join-representation of x. Hence $(b_{i-1}, b_i, ..., b_j, a_{j+1})\leqq x\delta_{i-1,j}$, that is, $(a_i, ..., a_j, a_{j+1})\leqq(b_i, ..., b_j, a_{j+1})\leqq\overline{x\delta_{i-1,j}}$. Conversely, $(a_i, ..., a_j, a_{j+1})\leqq\overline{x\delta_{i-1,j}}$. Then $(a_i, ..., a_j, a_{j+1})\leqq(b_i, ..., b_j, b_{j+1})$, where $(b_i, ..., b_j, b_{j+1})$ occurs in the join-representation of $\overline{x\delta_{i-1,j}}$, that is, for suitable b_{i-1} with $b_{i-1}\varepsilon_{i-1}\geqq b_i$, $(b_{i-1}, b_i, ..., b_{j+1})$ occurs in the join-representation of $x\delta_{i-1,j}$. This means, that $(b_{i-1}, b_i, ..., b_j)\leqq x$. Hence $(b_i, ..., b_j)\leqq \bar{x}$, that is $(a_i, ..., a_j, a_{j+1})\leqq(b_i, ..., b_j, a_{j+1})\leqq \bar{x}\delta_{ij}$.

Now we are ready to prove (β). First we prove that $L(1\to j)$ can be embedded to $L(1\to j+1)$. Consider the elements $(x\beta_{1j}, y\beta_{1j}, z\delta_{1j})\in L(1\to j+1)$ with $x, y\in B(1\to j)$, $z\in D(1\to j)$. These triples form a $\wedge$-subsemilattice of $L(1\to j+1)$. Now consider two such triples (x, y, z), $(x', y', z')\in L(1\to j)$, and let λ_{1j} denote the mapping $(\beta_{1j}, \beta_{1j}, \delta_{1j})$ described above. Then

$$[(x, y, z)\vee_{L(1\to j)}(x', y', z')]\lambda_{1j} = (x\vee x', y\vee y', z\vee z')^{\sim\wedge}\lambda_{1j} =$$
$$= [(x\vee x', y\vee y', z\vee z')^{\sim\wedge}](\beta_{1j}, \beta_{1j}, \delta_{1j}),$$
$$(x, y, z)\lambda_{1j}\vee_{L(1\to j)}(x', y', z')\lambda_{1j} = (x\beta_{1j}, y\beta_{1j}, z\delta_{1j})\vee_{L(1\to j)}(x'\beta_{1j}, y'\beta_{1j}, z'\delta_{1j}) =$$
$$= [(x\vee x', y\vee y', z\vee z')(\beta_{1j}, \beta_{1j}, \delta_{1j})]^{\sim\wedge}.$$

Now it is evident, that the operator $\sim$ and $(\beta_{1j}, \beta_{1j}, \delta_{1j})$ are permutable, and Lemma 1 shows that the same is true for $\wedge$ and $(\beta_{1j}, \beta_{1j}, \delta_{1j})$.

Finally we remark that the embedding λ_{1j} coincides with β_{1j} on $B(1\to j)$ considered as the ideal $[(0, 0, 0), (1, 0, 0)]$ of $L(1\to j)$ and coincides with δ_{1j} on $D(1\to j)$ considered as the ideal $[(0, 0, 0), (0, 0, 1)]$ of $L(1\to j)$.

Now $L(1\to j)$ can also be embedded to $L(i\to j+1)$ $(i\leqq j)$ by the embedding $\lambda_{ij}=(\delta_{i-1,j}, \delta_{i-1,j}, \delta_{i-1,j})$. The proof is the same as above, but we have to use Lemma 2 instead of Lemma 1. Furthermore, λ_{ij} coincides with $\delta_{i-1,j}$ on the copy of $D(i-1\to j)$ used in the glueing of Figure 1 and it coincides with δ_{ij} on the copy of $D(i\to j)$ used in the glueing. Thus we can glue together the λ_{ij}'s to get an embedding λ_j of L_j to L_{j+1}.

4. Proof of (γ)

We need a last lemma.

Lemma 3. *Let $x\in D_{j-1}$. Then $x\delta_{j-1}=x\varepsilon_{j-1}$, where δ_{j-1} stands for $\delta_{j-1,j-1}$ and ε_{j-1} maps D_{j-1} to $D_j\subseteqq D(j-1\to j)$.*

Proof. Let a_j be a join-irreducible element in D_j such that $a_j\leqq\overline{x\delta_{j-1}}$. Then $a_j\leqq b_j$ for some b_j in the join-representation of $\overline{x\delta_{j-1}}$. Thus, for some join-irreducible $b_{j-1}\in D_{j-1}$ with $b_{j-1}\varepsilon_{j-1}\geqq b_j$, (b_{j-1}, b_j) is in the join-representation of $x\delta_{j-1}$. Hence $(b_{j-1}, b_j)\leqq x\delta_{j-1}$, thus $b_{j-1}\leqq x$. Now $x\varepsilon_{j-1}$ is the join of all a'_j with $b'_{j-1}\varepsilon_{j-1}\geqq a'_j$ and b'_{j-1} $(\leqq x)$ join-irreducible. Thus $b_j\leqq x\varepsilon_{j-1}$, whence $a_j\leqq x\varepsilon_{j-1}$.

Conversely, let $a_j \leqq x\varepsilon_{j-1}$. Then $a_j \leqq a_{j-1}\varepsilon_{j-1}$ for some a_{j-1} ($\leqq x$) join-irreducible of D_{j-1}, which can be proved as follows. x is a join of join-irreducibles a_γ, $\gamma \in P$, of D_{j-1}. $a_j \leqq (\bigvee_\gamma a_\gamma)\varepsilon_{j-1} = (\bigvee_\gamma a_\gamma \varepsilon_{j-1})$. As a_j is join-irreducible (hence join-prime), it is less than or equal to one of the components in this join. (Notice, that this is the point of the proof which cannot be generalized to arbitrary directed systems.) Hence $(a_{j-1}, a_j) \leqq x\delta_{j-1}$, that is, $a_j \leqq \overline{x\delta}_{j-1}$.

Now the proof of (γ) is to prove that, for $d \in D_{j-1}$, $d\varepsilon_{j-1}\varphi_j = d\varphi_{j-1}\gamma_{j-1}$, where $\gamma_{j-1} = \mathrm{Con}\,(\lambda_{j-1})$. Now $d\varphi_{j-1}$ is the congruence generated by $[(0,0,0), (0,0,d)]$ of the copy of $L(j-1 \to j-1)$ used in Figure 1 (constructed with $j-1$ instead of j, that is representing L_{j-1}). λ_{j-1} takes this interval to the interval $[(0,0,0), (0,0,d\delta_{j-1})]$ of the copy of $L(j-1 \to j)$ used in the construction of L_j. Thus $d\varphi_{j-1}\gamma_{j-1}$ is generated by this interval. It is also generated (by perspectivity) by the interval $[(0,0,0), (d\delta_{j-1}, 0, 0)]$ of $L(j \to j)$. But then further generating pairs are

$$((0,0,0),(0,0,d\delta_{j-1})) \vee ((0,1,0),(0,1,0)) = ((0,1,0),(\overline{d\delta}_{j-1}, 0, \overline{d\delta}_{j-1}))$$

and

$$((0,1,0),(\overline{d\delta}_{j-1}, 0, \overline{d\delta}_{j-1})) \wedge ((0,0,1),(0,0,1)) = ((0,0,0),(0,0,\overline{d\delta}_{j-1})).$$

Using Lemma 3, we have that $d\varphi_{j-1}\gamma_{j-1}$ is generated by $((0,0,0),(0,0,\overline{d\varepsilon}_{j-1}))$. On the other hand, $d\varepsilon_{j-1}\varphi_{j-1}$ is evidently generated by the pair $((0,0,0),(0,0,\overline{d\varepsilon}_{j-1}))$ of the copy of $L(j \to j)$ used to construct L_j. This completes the proof.

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On the representation of distributive algebraic lattices. III

A. P. HUHN[1])

1. Introduction

Around 1980, unpublished investigations of Heiko Bauer led to the conjecture that every distributive semilattice with 0 of cardinality $\leqq \aleph_1$ is isomorphic to the semilattice of compact congruences of a lattice. In [2], H. DOBBERTIN gave a partial ordering which can be used to prove that on a set of cardinality $\aleph_1$ there is a directed family of finite subsets covering every finite subset such that the Boolean lattice 2^3 is not order-isomorphic to any subset of this family. We shall use this fact to prove the above formulated conjecture.[2]) In more usual terms, this means that every algebraic lattice with 0 having at most $\aleph_1$ compact elements is the congruence lattice of a lattice. We cannot extend the proof for more than $\aleph_1$ compact elements, the reason for that will be discussed in [—][3]). Note that the case of finitely many compact elements was already settled in [3], while the countable case was discussed in [2] and [5].

[1]) This paper was left behind by András Huhn in the form of a first draft of a manuscript. The remarks in footnotes 2, 3, 4 and 6 are due to Hans Dobbertin, who was kind to prepare the paper for publication.

[2]) In [1] the mentioned partial ordering has already been used implicitely in order to prove a theorem (see [1; Thm. 3.4]) which has the following corollary: *every distributive semilattice with 0 of cardinality* $\leqq \aleph_1$ *is the image of a generalized Boolean lattice under a weak-distributive* $\vee$*-homomorphism.* (See [6] for the definition of the notion "weak-distributive".) In the present paper a sharper result is proven, namely "weak-distributive" is replaced by "distributive". The important new idea in András Huhn's proof is the use of "reduced free products".

[3]) Perhaps András Huhn had planned to write another paper to which he made a reference here, but unfortunately no manuscript of it has been found in his inheritance. It is also possible that he wanted to make a here reference to [2].

2. Outline of the proof

Let D be a distributive semilattice with 0. Assume that $|D|=\aleph_1$. First we define a directed family of finite subsets of D. Let $\alpha<\omega_1$ be an ordinal number. For $\alpha=0$, let $h_\alpha=\{0\}$, where 0 is the lower bound of D. For $\alpha=n+1$ ($n\in\mathrm{N}$, N denotes the set of natural numbers), let $h_\alpha=h_n\cup\{a\}$, where $a\in D\setminus h_n$. Now if $\alpha=\omega\beta+n$, $n\in\mathrm{N}$, then we proceed as follows. $\omega\beta$ has a confinal ω-chain $\alpha_0<\alpha_1<\dots$. For $\alpha=\omega\beta$, let $h_\alpha=h_{\alpha_0}\cup\{a\}$ with $a\notin h_\gamma$ for $\gamma<\omega\beta$. For $\alpha=\omega\beta+n+1$, let $h_\alpha=$ $=h_{\alpha_{n+1}}\cup h_{\omega\beta+n}\cup\{a\}$ with $a\notin h_{\omega\beta+n}$. Let H be the set of all h_α, $\alpha<\omega_1$. The inclusion relation orders H, this ordering will be denoted by $\leqq$. h_0 will also be denoted by 0.[4])

Figure 1 shows how $\{h_\gamma\colon \gamma<\omega(\beta+1)\}$ is constructed from $\{h_\gamma\colon \gamma<\omega\beta\}$.

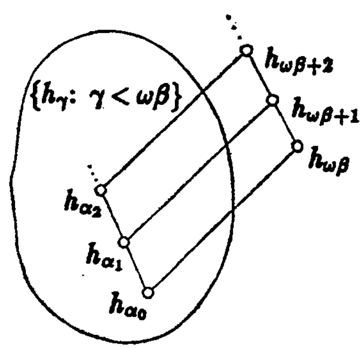

Figure 1

For every $h\in H$, choose a finite distributive 0-subsemilattice D_h of D such that $h\leqq k$, $h, k\in H$, implies $D_h\subseteq D_k$. This can be carried out by induction on α, using the fact that any finite subset of D is included in a finite distributive 0-subsemilattice. Then D is the direct limit of the D_h's. For later purposes we introduce the notation

$$\varepsilon(h,k)\colon \begin{cases} D_h \to D_k \\ d \mapsto d \end{cases}$$

provided that $h\leqq k$ (and therefore $D_h\subseteq D_k$).

[4]) The definition of the family $(h_\alpha)_{\alpha<\aleph_1}$ has to be modified slightly in order to guarantee that D is in fact completely exhausted.

Now, for every $h, i \in H$ with $h \leqq i$, we shall define finite distributive lattices $D(h, i)$ and 0-preserving lattice embeddings

$$\varphi(hg, i)\colon D(h, i) \to D(g, i) \quad \text{for} \quad g \leqq h \leqq i,$$
$$\varphi(h, ij)\colon D(h, i) \to D(h, j) \quad \text{for} \quad h \leqq i \leqq j$$

such that the following diagrams be commutative

$$\begin{tikzcd} D(h, i) \arrow[r, "\varphi(h, ij)"] \arrow[d, "\varphi(hg, i)"'] & D(h, j) \arrow[d, "\varphi(hg, j)"] \\ D(g, i) \arrow[r, "\varphi(g, ij)"'] & D(g, j) \end{tikzcd} \tag{1}$$

$$\begin{tikzcd} D(h, i) \arrow[r, "\varphi(hg, i)"] \arrow[dr] & D(g, i) \arrow[d, "\varphi(gf, i)"] \\ & D(f, i) \end{tikzcd} \tag{2}$$

$$\begin{tikzcd} D(h, i) \arrow[r, "\varphi(h, ij)"] \arrow[dr] & D(h, j) \arrow[d, "\varphi(h, jk)"] \\ & D(h, k) \end{tikzcd} \tag{3}$$

where $g \leqq h \leqq i \leqq j$, $h \leqq i \leqq j \leqq k$, respectively. We denote by $B(0, i)$ the smallest Boolean extension of $D(0, i)$ and by $\chi(0, i)\colon D(0, i) \to B(0, i)$ the canonical embedding (precisely defined later). We also define 0-preserving lattice embeddings $\psi(0, ij)\colon B(0, i) \to B(0, j)$ for $i \leqq j$ such that the following diagrams are commutative

$$\begin{tikzcd} B(0, i) \arrow[r, "\psi(0, ij)"] \arrow[dr] & B(0, j) \arrow[d, "\psi(0, ij)"] \\ & B(0, k) \end{tikzcd} \tag{4}$$

$$\begin{tikzcd} D(0, i) \arrow[r, "\chi(0, i)"] \arrow[d, "\varphi(0, ij)"'] & B(0, i) \arrow[d, "\psi(0, ij)"] \\ D(0, j) \arrow[r, "\chi(0, j)"'] & B(0, j) \end{tikzcd} \tag{5}$$

for $i \leqq j \leqq k$ and $i \leqq j$, respectively. Now let $B(0, -)$ be the direct limit of all $B(0, i)$ and $D(h, -)$ be the direct limit of all $D(h, i)$. Using the above commutativities, we can define embeddings

$$\varphi(hg, -)\colon D(h, -) \to D(g, -),\quad g \leqq h, \quad \text{and} \quad \chi(0, -)\colon D(0, -) \to B(0, -)$$

such that the following diagram is commutative

$$\begin{tikzcd} D(h, -) \arrow[r, "\varphi(hg, -)"] \arrow[dr] & D(g, -) \arrow[d, "\varphi(gf, -)"] \\ & D(f, -) \end{tikzcd} \tag{6}$$

The mappings $\varphi(hg, i)$ have inverses $\varphi'(gh, i)$ (which means that

$$\varphi(hg, i)\varphi'(gh, i) = \mathrm{id}_{D(h, i)},$$

the mappings are carried out in the written order) such that the following diagrams are commutative with $g \leqq h \leqq i \leqq j$ and $f \leqq g \leqq h \leqq i$, respectively.

$$\begin{tikzcd} D(h, i) \arrow[r, "\varphi(h, ij)"] & D(h, j) \\ D(g, i) \arrow[u, "\varphi'(gh, i)"] \arrow[r, "\varphi(g, ij)"'] & D(g, j) \arrow[u, "\varphi(gh, j)"'] \end{tikzcd} \tag{7}$$

$$\begin{tikzcd} D(h, i) & D(g, i) \arrow[l, "\varphi'(gh, i)"'] \\ & D(f, i) \arrow[ul] \arrow[u, "\varphi'(fg, i)"'] \end{tikzcd} \tag{8}$$

The right inverses are monomial 0-preserving weakly distributive $\vee$-homomorphisms (in the sense of SCHMIDT [6]). Also the $\chi(0, i)$'s have 0- and $\vee$-preserving monomial weakly distributive right inverses $\chi'(0, i)$ and we have the following commutativities

$$\begin{tikzcd} D(0, i) \arrow[d, "\varphi(0, ij)"'] & B(0, i) \arrow[l, "\chi'(0, i)"'] \arrow[d, "\psi(0, ij)"] \\ D(0, j) & B(0, j) \arrow[l, "\chi'(0, j)"] \end{tikzcd} \tag{9}$$

for $i \leqq j$. These commutativities allow to carry over the $\chi'(0, i)$ to the direct limit $B(0, -)$ and so we get a 0-preserving monomial weakly distributive $\vee$-homomorphism

$$\chi'(0, -)\colon B(0, -) \to D(0, -)$$

which is a right inverse to $\chi(0, -)$. Similarly, we define $\varphi'(gh, -)$ for $g \leqq h$.

$$\varphi'(gh, -)\colon D(g, -) \to D(h, -)$$

is again a 0-preserving monomial weakly distributive $\vee$-homomorphism and a right inverse to $\varphi(hg, -)$. Again we have that the following diagram is commutative

(10)
$$\begin{tikzcd} D(h, -) & D(g, -) \arrow[l, "\varphi'(hg, -)"'] \\ & D(f, -) \arrow[ul] \arrow[u, "\varphi'(gf, -)"'] \end{tikzcd}$$

for $f \leqq g \leqq h$.

Now we shall consider the congruences θ_h associated with $\chi'(0, -)\varphi'(0h, -)$. These are monomial weakly distributive congruences with the kernel 0 in the sense of SCHMIDT [6]. Thus, if we prove that $B(0, -)/\bigvee_{h \in H} \theta_h$ is isomorphic to D, then we are done by the following theorem of SCHMIDT [6]: if θ_h, $h \in H$, are monomial weakly distributive congruences of the generalized Boolean lattice B, then $B/\bigvee_h \theta_h$ is isomorphic to the semilattice of all compact congruences of a lattice.

3. The main construction

We start to define the $D(h, i)$'s. Motivation: Whenever $h \leqq i$, $D(h, i)$ will be a "reduced free product" of all the D_x, $h \leqq x \leqq i$, in the class of distributive lattices with 0, namely we take free 0-product in the class of distributive lattices and factorize it by a congruence (by the smallest possible) so as to insure that in the factor lattice all the relations $d \leqq d\varepsilon(x, y)$, $h \leqq x \leqq y \leqq i$, $d \in D_x$, hold (here, for brevity d etc. stands for the congruence class of d etc.). This free choice of the $D(h, i)$'s is one of the important ideas of the proof, however, we shall not need in the proof that $D(h, i)$ is really free (relative to the given relations), we only need the description given in the following definition.

Definition. $D(h, i)$ will be the finite distributive lattice with the following $\vee$-irreducibles: j is an irreducible of $D(h, i)$ if j is a mapping of a dual segment P to the poset $[h, i]$ to $\bigcup_{x \in P} D_x$ such that for all $x \in P$, j_x is an irreducible of D_x (0 is not irreducible) and whenever $x \leqq y$, $x, y \in P$, then $j_y \leqq j_x \varepsilon(x, y)$.[5])

[5]) They are ordered componentwise, that is $(j_x | x \in P) \leqq (j'_x | x \in Q)$ if $P \supseteqq Q$ and, for all $x \in Q$, $j_x \leqq j'_x$.

Definition. The irreducibles of $D(h, i)$ are the irreducibles of $D(g, i)$, too, if $g \leqq h$. Therefore we get an embedding $\varphi(gh, i)$, if we map the irreducibles $j \in D(h, i)$ to $j \in D(g, i)$ and extend this map such that the $\vee$ is preserved. (This is a lattice embedding as its dual mapping — by Priestley's duality — maps $(j_x | x \in P)$, P a dual segment of $[g, i]$, to $(j_x | x \in P \cap [h, i])$ and therefore is onto).

Definition. $\varphi(h, ij)$ is defined as follows. The irreducibles of $D(h, i)$ are the choice functions $(j_x | x \in P)$ where P is a dual segment of $[h, i]$. Now $(j_x | x \in P) \varphi(h, ij)$ is the join of all $(j'_x | x \in Q)$ such that Q is a dual segment of $[h, j]$, $Q \cap [h, i] = P$, $j'_x \leqq j_x$ for $x \in P$, and $(j'_x | x \in Q)$ is an irreducible in $D(h, j)$. To arbitrary elements of $D(h, i)$ we extend this mapping in such a way that it preserves joins.

Now the commutativities (1), (2), (3) are evident. To show that $\varphi(h, ij)$ is one-to-one we have to prove that its dual mapping is onto. To do that we first describe how the poset $[h, i]$ is obtained from $[h, j]$. According to Figure 1, j is the greatest element of a finite chain in $\{h_\gamma : \gamma \leqq \omega(\beta+1)\} \setminus \{h_\gamma : \gamma \leqq \omega\beta\}$ for some γ. Omitting this chain we obtain another poset. This remaining poset has a largest element, so we can continue this procedure until the largest element in the remaining poset is i. Now, if we go the other way around, we get $[h, j]$ from $[h, i]$ in such a way, that add finite chains $a_{11}, a_{12}, \ldots, a_{1n_1}$; $a_{21}, a_{22}, \ldots, a_{2n_2}$; $\ldots$; $a_{m1}, a_{m2}, \ldots, a_{mn_m}$ successively to $[h, i]$ as in Figure 2 $(a_{0n_0} = i,\ a_{mn_m} = j)$.

Now we show that the dual map of $\varphi(h, ij)$ is onto. Let $(j_x | x \in P)$ be an irreducible of $[h, i]$, where P is a dual ideal of $[h, i]$. For simplicity, we assume that the adjoined elements are l, m, n, and the chain consisting of the lower covers of these elements is i, h, k.

Now we may choose an irreducible j_l in D_l such that $j_i \varepsilon(i, l) \geqq j_l$. $j_h \varepsilon(h, l) \geqq j_l$, too. Let $x = j_h \varepsilon(h, m)$. Then $x \varepsilon(m, l) \geqq j_l$. x is a join of join-irreducibles: $x = \bigvee_\gamma j_\gamma$ and $j_l \leqq \bigvee_\gamma j_\gamma \varepsilon(m, l)$, thus, for some γ_0, $j_l \leqq j_{\gamma_0} \varepsilon(m, l)$. Define $j_m = j_{\gamma_0}$.

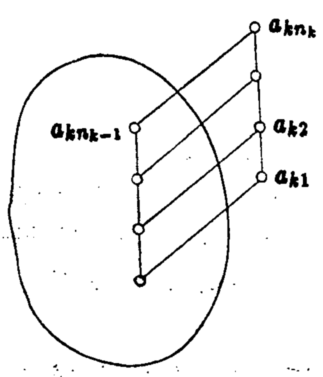

Figure 2

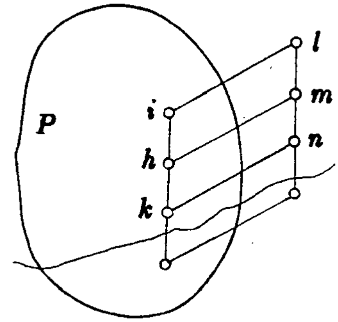

Figure 3

Similarly, we can define j_n, and continuing this procedure we get a vector $(j_x | x \in Q)$ which is mapped to $(j_x | x \in P)$.[6])

To define $B(0, i)$, we agree that the atoms of $B(0, i)$ are $[j_x | x \in P]$ where $(j_x | x \in P)$ is a join-irreducible of $D(0, i)$, the only difference is that in $B(0, i)$ they are, of course, not ordered. The embedding $\chi(0, i)$ is defined as in [5]. Then the commutativity of (4) and (5) is again evident.

Now let $B(0, -)$ be the direct limit of all $B(0, i)$ and $D(h, -)$ be the direct limit of all $D(h, i)$. Then there exist embeddings $\psi(0, i-)$: $B(0, i) \to B(0, -)$ such that the following diagrams are commutative

(11)
$$\begin{tikzcd} D(h, i) \arrow[r, "\varphi(h, ij)"] \arrow[dr] & D(h, j) \arrow[d, "\varphi(h, j-)"] \\ & D(h, -) \end{tikzcd}$$

(12)
$$\begin{tikzcd} B(0, i) \arrow[r, "\psi(0, ij)"] \arrow[dr] & B(0, j) \arrow[d, "\psi(0, j-)"] \\ & B(0, -) \end{tikzcd}$$

with $h \leqq i \leqq j$ and $i \leqq j$, respectively. These commutativities make it possible to define embeddings $\varphi(hg, -)$: $D(h, -) \to D(g, -)$ and $\chi(0, -)$ with $g \leqq h$, such that the following diagrams are commutative:

(13)
$$\begin{tikzcd} D(h, i) \arrow[r, "\varphi(h, i-)"] \arrow[d, "\varphi(hg, i)"'] & D(h, -) \arrow[d, "\varphi(hg, -)"] \\ D(g, i) \arrow[r, "\varphi(g, i-)"'] & D(g, -) \end{tikzcd}$$

(14)
$$\begin{tikzcd} D(0, i) \arrow[r, "\varphi(0, i-)"] \arrow[d, "\chi(0, i)"'] & D(0, -) \arrow[d, "\chi(0, -)"] \\ B(0, i) \arrow[r, "\psi(0, i-)"'] & B(0, -) \end{tikzcd}$$

[6]) By means of some additional observation the case that both i and j lie on a chain added in the inductive construction, can be handled similarly.

Also the following diagrams commute:

$$(15)\qquad \begin{array}{ccc} D(h,i) & \xrightarrow{\varphi(h,i-)} & D(h,-) \\ \uparrow{\scriptstyle \varphi'(gh,i)} & & \uparrow{\scriptstyle \varphi(gh,-)} \\ D(g,i) & \xrightarrow[\varphi(g,i-)]{} & D(g,-) \end{array}$$

$$(16)\qquad \begin{array}{ccc} D(0,i) & \xrightarrow{\varphi(0,i-)} & D(0,-) \\ \uparrow{\scriptstyle \chi'(0,i)} & & \uparrow{\scriptstyle \chi'(0,-)} \\ B(0,i) & \xrightarrow[\psi(0,i-)]{} & B(0,-) \end{array}$$

Hence it follows that the $\varphi'(gh, -)$ and $\chi'(0, -)$ are weakly distributive monomial congruences and so are their composition. (To show the commutativities of (15) and (16) we have to show the commutativities of (7) and (9), but it is the same as Lemmas 1, 2 in [4].)

Now we can finish the proof as follows. The factor lattice by the congruence $\bigvee_h \theta_h$ is the direct limit of the $D(h, -)$'s relative to the morphisms $\varphi(gh, -)$. Let us denote this limit by F. F has subsemilattices isomorphic to the D_h's. Namely, $D(h, h)\cong D_h$, hence $D(h, h)\varphi(h, h-)\cong D_h$.

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Relatively free bands of groups

P. G. TROTTER

The subvarieties of the variety **CS** of all completely simple semigroups, along with their free objects, have been studied by V. V. RASIN [15], P. R. JONES [9] and by M. PETRICH and N. R. REILLY [14]. The lattice of subvarieties of the variety **B** of all bands has been constructed by A. P. BIRJUKOV [1], J. A. GERHARD [6] and C. F. FENNEMORE [5]; the defining laws of these varieties are known.

In this paper we observe that any regular semigroup is a subdirect product of any idempotent separating homomorphic image by any idempotent pure homomorphic image. This enables the construction of free objects of subvarieties of the variety **POBG** of all pseudo orthodox bands of groups in terms of relatively free bands and relatively free completely simple semigroups. It is shown that in any subvariety **V** of the variety **BG** of all bands of groups where $\mathbf{CS} \subseteq \mathbf{V} \nsubseteq \mathbf{POBG}$, the $\mathscr{H}$-classes of elements on 3 or more generators of the free objects are not free in any group variety. It is also shown that the free completely simple semigroup on a finite set is a retract of the free object on a countable set in any variety of completely regular semigroups that contains **CS**.

The first section includes a subdirect product decomposition of a regular semigroup and some preliminary results on varieties; it is shown that $\mathbf{RBG} \cap \mathbf{POBG}$ is a significant lower bound of the set of varieties $\mathbf{V}$, $\mathbf{CS} \subseteq \mathbf{V} \subseteq \mathbf{BG} \setminus \mathbf{POBG}$, where **RBG** is the variety of all regular bands of groups. In the next section models of free objects in subvarieties of **POBG** are described, with an emphasis on those contained in $\mathbf{RBG} \cap \mathbf{POBG}$. The retract and $\mathscr{H}$-class results mentioned above are in the final section.

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1. Definitions and preliminary results

Suppose ϱ is a congruence on a regular semigroup S. Denote by $E(S)$ the set of idempotents of S. Define

$$\textit{trace} \text{ of } \varrho = \operatorname{tr} \varrho = \varrho | E(S)$$

and

$$\textit{kernel} \text{ of } \varrho = \ker \varrho = \{u \in S;\ (u, e) \in \varrho \text{ for some } e \in E(S)\}.$$

By Feigenbaum [4; Theorem 4.1], ϱ is completely determined by its trace and kernel. Note that if τ is also a congruence on S then $\operatorname{tr} \varrho \cap \operatorname{tr} \tau = \operatorname{tr}(\varrho \cap \tau)$. Also, by [8; proof of Lemma II.4.6], $\ker \varrho \cap \ker \tau = \ker(\varrho \cap \tau)$. By [10; Theorem 3.2], there exist least and greatest congruences on S with the same trace as ϱ (denoted respectively $\varrho_{\min}$ and $\varrho_{\max}$), or with the same kernel as ϱ (denoted respectively $\varrho^{\min}$ and $\varrho^{\max}$).

Lemma 1.1. *Let ϱ, τ and λ be congruences on a regular semigroup S such that $\varrho \subseteqq \tau \subseteqq \varrho_{\max}$ and $\varrho \subseteqq \lambda \subseteqq \varrho^{\max}$. Then S/ϱ is isomorphic to the subdirect product $\{(a\tau, a\lambda);\ a \in S\}$ of S/τ by S/λ.*

Proof. Since $\ker \lambda = \ker \varrho \subseteqq \ker \tau$ and $\operatorname{tr} \tau = \operatorname{tr} \varrho \subseteqq \operatorname{tr} \lambda$ then $\ker(\tau \cap \lambda) = \ker \varrho$ and $\operatorname{tr}(\tau \cap \lambda) = \operatorname{tr} \varrho$. So $\varrho = \tau \cap \lambda$ and the result follows (see [12; Proposition II.1.4]).

Throughout the paper $\mathbf{U}$ will denote the variety of all semigroups that have a unary operation, and X will denote a countably infinite set. The free object on X in $\mathbf{U}$ is denoted by $F_X^{\mathbf{U}}$. $F_X^{\mathbf{U}}$ is the smallest subsemigroup of the free semigroup on $X \cup \{(\,,)^{-1}\}$ such that $X \subseteqq F_X^{\mathbf{U}}$ and $(w)^{-1} \in F_X^{\mathbf{U}}$ for all $w \in F_X^{\mathbf{U}}$. We will write $w^{-1} = (w)^{-1}$ and $w^0 = ww^{-1}$.

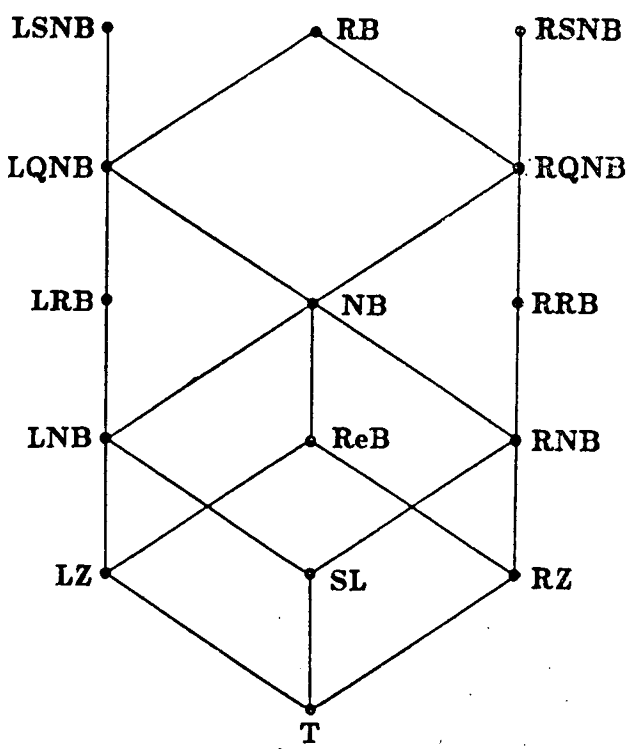
If $\mathbf{V}$ is a subvariety of $\mathbf{U}$ let $F_X^{\mathbf{V}}$ denote the free object in $\mathbf{V}$ on X, and let $\varrho_{\mathbf{V}}$ be the fully invariant congruence on $F_X^{\mathbf{U}}$ such that $F_X^{\mathbf{V}} \cong F_X^{\mathbf{U}}/\varrho_{\mathbf{V}}$. Denote by $L(\mathbf{V})$ the lattice of subvarieties of $\mathbf{V}$ and by $C(\mathbf{V})$ the lattice of fully invariant congruences on $F_X^{\mathbf{V}}$ (both ordered by inclusion). There is a lattice anti-isomorphism between $L(\mathbf{V})$ and $C(\mathbf{V})$ given by $\mathbf{W} \to \varrho_{\mathbf{W}}/\varrho_{\mathbf{V}}$. For $\mathbf{V} \subseteqq \mathbf{W}$ in $L(\mathbf{U})$ let $[\mathbf{V}, \mathbf{W}] = \{\mathbf{Z} \in L(\mathbf{U});\ \mathbf{V} \subseteqq \mathbf{Z} \subseteqq \mathbf{W}\}$. For $Y \subseteqq X$, let $F_Y^{\mathbf{V}}$ denote the subsemigroup of $F_X^{\mathbf{V}}$ generated in $\mathbf{V}$ by Y; $F_Y^{\mathbf{V}}$ is free on Y. We may regard $F_X^{\mathbf{V}}$ as being the set $F_X^{\mathbf{U}}$, subject to the laws of $\mathbf{V}$.

A semigroup is *completely regular* if and only if it is a union of its subgroups. It is well known that the class $\mathbf{CR}$ of all completely regular semigroups is a subvariety of $\mathbf{U}$ defined by the laws $xx^{-1}x = x$, $xx^{-1} = x^{-1}x$ and $(x^{-1})^{-1} = x$. So $\varrho_{\mathbf{CR}}$ is generated by $\{(uu^{-1}u, u), (uu^{-1}, u^{-1}u), ((u^{-1})^{-1}, u);\ u \in F_X^{\mathbf{U}}\}$.

By [10; Theorems 3.6, 4.2 and 4.3], for any $\mathbf{V} \in L(\mathbf{CR})$ then $(\varrho_{\mathbf{V}}/\varrho_{\mathbf{CR}})_{\min}$, $(\varrho_{\mathbf{V}}/\varrho_{\mathbf{CR}})^{\min}$, $(\varrho_{\mathbf{V}}/\varrho_{\mathbf{CR}})_{\max}$ and $(\varrho_{\mathbf{V}}/\varrho_{\mathbf{CR}})^{\max}$ are in $C(\mathbf{CR})$. Let $\mathbf{V}_{\max}$, $\mathbf{V}^{\max}$, $\mathbf{V}_{\min}$ and $\mathbf{V}^{\min}$ denote the varieties in $L(\mathbf{CR})$ that are respectively defined by these congruences.

It is usual when $\mathbf{V} \in L(\mathbf{B})$, the lattice of varieties of bands, to write $\mathbf{VG}$ for $\mathbf{V}_{\max}$. $\mathbf{VG}$ is the variety of all semigroups $S \in \mathbf{CR}$ such that $\mathscr{H}$ is a congruence on S and $S/\mathscr{H} \in \mathbf{V}$.

Let $\mathbf{G}$ denote the variety of all groups, $\mathbf{CS}$ is the variety of all completely simple semigroups, and let $\mathbf{OBG}$ be the variety of all bands of groups that are orthodox. Let $\mathbf{POBG}$ denote the variety (see [7; Proposition 4.1]) of all $S \in \mathbf{BG}$ such that for each $e \in E(S)$, eSe is orthodox; S is called a *pseudo orthodox band of groups.* The following list, from [11], is of the bottom 15 varieties in $L(\mathbf{B})$ along with their defining laws as subvarieties of $\mathbf{B}$: $\mathbf{T}$=trivial variety $(x=y)$; $\mathbf{LZ}$=left zero semigroups $(xy=x)$; $\mathbf{ReB}$=rectangular bands $(xyx=x)$; $\mathbf{SL}$=semilattices $(xy=yx)$; $\mathbf{LNB}$=left normal bands $(xyz=xzy)$; $\mathbf{NB}$=normal bands $(xyzx=xzyx)$; $\mathbf{LRB}$=left regular bands $(xy=xyx)$; $\mathbf{LQNB}$=left quasinormal bands $(xyz=xyxz)$; $\mathbf{RB}$=regular bands $(xyzx=xyxzx)$; $\mathbf{LSNB}$=left seminormal bands $(xyz=xyzxz)$; and the left-right duals $\mathbf{RZ}$, $\mathbf{RNB}$, $\mathbf{RRB}$, $\mathbf{RQNB}$ and $\mathbf{RSNB}$ of $\mathbf{LZ}$, $\mathbf{LNB}$, $\mathbf{LRB}$, $\mathbf{LQNB}$ and $\mathbf{LSNB}$ respectively. If $\mathbf{V} \in L(\mathbf{B})$ is not in the list then $\mathbf{V} \supseteq \mathbf{LSNB} \vee \mathbf{RB}$ or $\mathbf{V} \supseteq \mathbf{RSNB} \vee \mathbf{RB}$.



The following results are to be used later in the text. Define the *content* of $v \in F_X^{\mathbf{U}}$ to be

$$c(v) = \{\text{letters of } X \text{ appearing in } v\},$$

and for $\mathbf{V} \in L(\mathbf{CR})$ define

$$\mathscr{D}_{\mathbf{V}} = \{(u, v);\ u, v \in F_X^{\mathbf{U}} \text{ and } u\varrho_{\mathbf{V}} \mathscr{D} v\varrho_{\mathbf{V}}\}.$$

Theorem 1.2. (i) [2; Theorem 4.2]. *For* $u, v \in F_X^{\mathbf{U}}$, $(u, v) \in \mathscr{D}_{\mathbf{CR}}$ *if and only if* $c(u) = c(v)$.

(ii) $\mathscr{D}_{\mathbf{CR}}$ *is a congruence on* $F_X^{\mathbf{U}}$. *For* $\mathbf{V} \in L(\mathbf{CR})$ *either* $\varrho_{\mathbf{V}} \subseteq \mathscr{D}_{\mathbf{CR}}$ *and* $\mathbf{V} \supseteq \mathbf{SL}$ *or* $\varrho_{\mathbf{V}} \nsubseteq \mathscr{D}_{\mathbf{CR}}$ *and* $\mathbf{V} \subseteq \mathbf{CS}$.

Proof. Since $\mathscr{D}$ is the finest semilattice congruence on any completely regular semigroup then $\mathscr{D}_{\mathbf{CR}}$ is a congruence of $F_X^{\mathbf{U}}$ and $\varrho_{\mathbf{V}} \subseteq \mathscr{D}_{\mathbf{CR}}$ if and only if $\mathbf{V} \supseteq \mathbf{SL}$. If $\mathbf{V} \subseteq \mathbf{CS}$ then $\mathbf{V} \nsupseteq \mathbf{SL}$ and hence $\varrho_{\mathbf{V}} \nsubseteq \mathscr{D}_{\mathbf{CR}}$. Suppose $\varrho_{\mathbf{V}} \nsubseteq \mathscr{D}_{\mathbf{CR}}$. Then by (i) there exists $u, v \in F_X^{\mathbf{U}}$ such that $(u, v) \in \varrho_{\mathbf{V}}$ and $c(u) \neq c(v)$. We may assume that there exists $x \in c(u) \setminus c(v)$. Select finite subsets Y, Z of X and endomorphisms φ, ψ of $F_X^{\mathbf{U}}$ such that $c(x\varphi) = Y = c(z\psi)$ and $c(x\psi) = Z = c(z\varphi)$ for all $z \in X \setminus \{x\}$. Since $\varrho_{\mathbf{V}}$ is fully invariant and $(u, v) \in \varrho_{\mathbf{V}}$ then $(v\varphi, (u^0 v)\varphi), (v\psi, (u^0 v)\psi) \in \varrho_{\mathbf{V}}$ while $c(v\varphi) = Z$, $c(v\psi) = Y$ and $c((u^0 v)\varphi) = Y \cup Z = c((u^0 v)\psi)$. Hence by (i) $F_X^{\mathbf{U}}/\varrho_{\mathbf{V}}$ has just one $\mathscr{D}$-class and is therefore completely simple.

Theorem 1.3. *Suppose* $\mathbf{V} \in L(\mathbf{BG})$. *Then*

(i) $\mathbf{V}_{\max} \in L(\mathbf{OBG})$ *if and only if* $\mathbf{V} \cap \mathbf{B} \nsupseteq \mathbf{ReB}$,

(ii) $\mathbf{V}_{\max} \in L(\mathbf{POBG})$ *if and only if* $\mathbf{V} \cap \mathbf{B} \nsupseteq \mathbf{RB}$, *and*

(iii) $\mathbf{RBG} \cap \mathbf{POBG}$ *is the greatest lower bound in* $L(\mathbf{POBG})$ *of*

$$[\mathbf{CS}, \mathbf{BG}] \setminus L(\mathbf{POBG}).$$

Proof. Note that since $\mathscr{H}$ is the greatest idempotent separating congruence on $F_X^{\mathbf{V}}$, and $\mathscr{H}$ is a band congruence then $\mathbf{V}_{\min} = \mathbf{V} \cap \mathbf{B}$. Also observe that if $\mathbf{Z} \supseteq \mathbf{W}$ in $L(\mathbf{CR})$ then $\mathbf{Z}_{\max} \supseteq \mathbf{W}_{\max}$.

(i) Since $\mathbf{ReB}_{\max} = \mathbf{CS} \nsubseteq \mathbf{OBG}$ then $\mathbf{V}_{\max} \notin L(\mathbf{OBG})$ if $\mathbf{V} \cap \mathbf{B} \supseteq \mathbf{ReB}$. Conversely suppose $\mathbf{V} \cap \mathbf{B} \nsupseteq \mathbf{ReB}$; then $\mathbf{LRB} \supseteq \mathbf{V} \cap \mathbf{B}$ or $\mathbf{RRB} \supseteq \mathbf{V} \cap \mathbf{B}$. By duality, it suffices to assume $\mathbf{V} = \mathbf{V}_{\max} = \mathbf{LRBG}$, and to prove $\mathbf{V} \subseteq \mathbf{OBG}$. In this case $\mathbf{V}$ is defined as a subvariety of $\mathbf{BG}$ by $(xy)^0 = (xyx)^0$. So for any $e, f \in F_X^{\mathbf{U}}$ where $e\varrho_{\mathbf{V}}$ and $f\varrho_{\mathbf{V}}$ are idempotents,

$$ef\,\varrho_{\mathbf{V}}\, ef(ef)^0 f\,\varrho_{\mathbf{V}}\, ef(efe)^0 f\,\varrho_{\mathbf{V}}\, ef(efe)^0 ef\,\varrho_{\mathbf{V}}\, ef(ef)^0 ef\,\varrho_{\mathbf{V}}\, efef.$$

Thus $F_X^{\mathbf{V}}$ is orthodox.

(ii) The free completely simple semigroup with adjoined identity, $(F_X^{\mathbf{CS}})^1$, is not a pseudo-orthodox band of groups but it is a regular band of groups since it satisfies the law $(xyzx)^0=(xyxzx)^0$. Conversely, suppose $\mathbf{V}\cap\mathbf{B}\not\supseteq\mathbf{RB}$; so $\mathbf{V}\cap\mathbf{B}\subseteq\mathbf{LSNB}$ or $\mathbf{V}\cap\mathbf{B}\subseteq\mathbf{RSNB}$. By duality we may assume $\mathbf{V}=\mathbf{V}_{\max}=\mathbf{LSNBG}$. Suppose $e,f,g\in F_X^{\mathbf{U}}$ such that $e\varrho_{\mathbf{V}}, f\varrho_{\mathbf{V}}$ and $g\varrho_{\mathbf{V}}$ are idempotents and $(efe,f),(ege,g)\in\varrho_{\mathbf{V}}$. Since $\mathbf{V}$ is defined in $L(\mathbf{BG})$ by $(xyz)^0=(xyzxz)^0$ then

$$(fg)^0\,\varrho_{\mathbf{V}}\,(fge)^0\,\varrho_{\mathbf{V}}\,(fgefe)^0\,\varrho_{\mathbf{V}}\,(fgf)^0\,\varrho_{\mathbf{V}}\,(fgf)^0f\,\varrho_{\mathbf{V}}\,(fg)^0f,$$

so $fg\,\varrho_{\mathbf{V}}\,fg(fg)^0g\,\varrho_{\mathbf{V}}\,fg(fg)^0fg\,\varrho_{\mathbf{V}}\,fgfg$. Hence $F_X^{\mathbf{V}}\in\mathbf{POBG}$ and the result follows.

(iii) By [7; Theorem 3.1 and Corollary 5.4], $L(\mathbf{BG})$ is modular and $\mathbf{POBG}=\mathbf{CS}\vee\mathbf{B}$. Therefore, since $\mathbf{RBG}\supseteq\mathbf{CS}$,

$$\mathbf{POBG}\cap\mathbf{RBG}=(\mathbf{CS}\vee\mathbf{B})\cap\mathbf{RBG}=\mathbf{CS}\vee(\mathbf{B}\cap\mathbf{RBG})=\mathbf{CS}\vee\mathbf{RB}.$$

By (ii) $\mathbf{CS}\vee\mathbf{RB}$ is a lower bound for $[\mathbf{CS},\mathbf{BG}]\setminus L(\mathbf{POBG})$. Furthermore if $\mathbf{V}\in L(\mathbf{POBG})$ is a lower bound for $[\mathbf{CS},\mathbf{BG}]\setminus L(\mathbf{POBG})$ then $\mathbf{V}\subseteq\mathbf{POBG}\cap\mathbf{RBG}$.

Lemma 1.4. *Suppose* $\mathbf{V}\in L(\mathbf{CR})$ *and* $\mathbf{W}\in[\mathbf{V},\mathbf{V}_{\max}\vee\mathbf{V}^{\max}]$. *Then* $\mathbf{W}=(\mathbf{W}\cap\mathbf{V}_{\max})\vee(\mathbf{W}\cap\mathbf{V}^{\max})$. *Furthermore* $\ker(\varrho_{\mathbf{W}}/\varrho_{\mathbf{CR}})=\ker(\varrho_{\mathbf{W}\cap\mathbf{V}_{\max}}/\varrho_{\mathbf{CR}})$.

Proof. The first statement is by [10; Theorem 5.4]. The second statement is proved in the initial part of the proof of [10; Theorem 5.1].

2. Free pseudo orthodox bands of groups

The lattice $L(\mathbf{CS})$ of completely simple semigroup varieties has been studied by several authors. In particular $F_X^{\mathbf{V}}$ has been characterized for $\mathbf{V}\in L(\mathbf{CS})$ in [9], [14] and [15].

Write $\leqq$ to mean "is embedded in", and omit the embedding details where they are obvious.

Theorem 2.1. (i) *If* $\mathbf{V}\in L(\mathbf{OBG})$ *then*

$$F_X^{\mathbf{V}}\cong\{(u\varrho_{\mathbf{V}\cap\mathbf{B}},u\varrho_{\mathbf{V}\cap\mathbf{G}});\ u\in F_X^{\mathbf{U}}\}\leqq F_X^{\mathbf{V}\cap\mathbf{B}}\times F_X^{\mathbf{V}\cap\mathbf{G}}.$$

(ii) *If* $\mathbf{V}\in[\mathbf{ReB},\mathbf{POBG}]$ *then*

$$F_X^{\mathbf{V}}\cong\{(u\varrho_{\mathbf{V}\cap\mathbf{B}},u\varrho_{\mathbf{V}\cap\mathbf{CS}});\ u\in F_X^{\mathbf{U}}\}\leqq F_X^{\mathbf{V}\cap\mathbf{B}}\times F_X^{\mathbf{V}\cap\mathbf{CS}}.$$

Proof. We have $\mathbf{T}_{\max}=\mathbf{G}$, $\mathbf{T}^{\max}=\mathbf{B}=\mathbf{ReB}^{\max}$ and $\mathbf{ReB}_{\max}=\mathbf{CS}$. By [13; Lemma 1] and [7; Corollary 5.4], $\mathbf{OBG}=\mathbf{B}\vee\mathbf{G}$ and $\mathbf{POBG}=\mathbf{B}\vee\mathbf{CS}$ respectively. By Lemma 1.4 then $\mathbf{V}\supseteq\mathbf{V}\cap\mathbf{G}\supseteq\mathbf{V}^{\min}$ in case (i) and $\mathbf{V}\supseteq\mathbf{V}\cap\mathbf{CS}\supseteq\mathbf{V}^{\min}$ in case (ii). Since $\mathbf{V}_{\min}=\mathbf{V}\cap\mathbf{B}$, the result is by Lemma 1.1.

This result can be refined, given more information on $F_X^{\mathbf{V}\cap\mathbf{B}}$ and $F_X^{\mathbf{V}\cap\mathbf{CS}}$.

The *head* $h(v)$ *of* $v\in F_X^{\mathbf{U}}$ is the first letter of X to appear in v. Dually the *tail*

$t(v)$ is the last letter of X to appear in v. The *initial part* $i(v)$ of v is the word obtained from v by retaining only the first occurrence of each letter from X. Dually define the *final part* $f(v)$ of v. Define $I=\{i(v);\ v\in F_X^{\mathbf{U}}\}$; so $I\subseteqq F_X^{\mathbf{U}}$ consists of finite strings of distinct letters from X. Then

$$(1) \qquad \varrho_{\mathbf{LNB}} = \{(u,v);\ u,v\in F_X^{\mathbf{U}} \text{ where } c(u)=c(v) \text{ and } h(u)=h(v)\}.$$

To see this note that the set is a fully invariant left normal band congruence on $F_X^{\mathbf{U}}$ that is contained in $\varrho_{\mathbf{SL}}\cap\varrho_{\mathbf{LZ}}$. Since the sublattice described in the diagram is convex, the congruence is $\varrho_{\mathbf{LNB}}$.

Likewise

$$(2) \qquad \varrho_{\mathbf{NB}} = \{(u,v)\in\varrho_{\mathbf{LNB}};\ t(u)=t(v)\},$$

$$(3) \qquad \varrho_{\mathbf{LRB}} = \{(u,v);\ u,v\in F_X^{\mathbf{U}} \text{ where } i(u)=i(v)\},$$

$$(4) \qquad \varrho_{\mathbf{LQNB}} = \{(u,v)\in\varrho_{\mathbf{LRB}};\ t(u)=t(v)\},$$

and

$$(5) \qquad \varrho_{\mathbf{RB}} = \{(u,v)\in\varrho_{\mathbf{LRB}};\ f(u)=f(v)\}.$$

Along with the well known results we readily get the following.

Theorem 2.2. *$F_X^{\mathbf{T}}=\{0\}$; $F_X^{\mathbf{LZ}}\cong X$ with multiplication $x\cdot y=x$;*
$F_X^{\mathbf{ReB}}\cong F_X^{\mathbf{LZ}}\times F_X^{\mathbf{RZ}}$; $F_X^{\mathbf{SL}}\cong\{Y\subseteqq X;\ |Y|<\infty\}$ under set union;
$F_X^{\mathbf{LNB}}\cong\{(x,Y);\ x\in Y\subseteqq X,\ |Y|<\infty\}\leqq F_X^{\mathbf{LZ}}\times F_X^{\mathbf{SL}}$;
$F_X^{\mathbf{NB}}\cong\{(x,y,Y);\ x,y\in Y\subseteqq X,\ |Y|<\infty\}\leqq F_X^{\mathbf{ReB}}\times F_X^{\mathbf{SL}}$,
$F_X^{\mathbf{LRB}}\cong I$ with multiplication $a\cdot b=i(ab)$;
$F_X^{\mathbf{LQNB}}\cong\{(a,x);\ a\in I,\ x\in c(a)\}\leqq F_X^{\mathbf{LRB}}\times F_X^{\mathbf{RZ}}$; and
$F_X^{\mathbf{RB}}\cong\{(a,b)\in I\times I;\ c(a)=c(b)\}\leqq F_X^{\mathbf{LRB}}\times F_X^{\mathbf{RRB}}$.

The free objects in other varieties of bands are not so easy to model.

Corollary 2.3. *Suppose $\mathbf{V}\in L(\mathbf{LRBG})$ and $\mathbf{W}=\mathbf{V}\cap\mathbf{G}$. If $\mathbf{V}\in[\mathbf{SL},\mathbf{SLG}]$ then*

$$F_X^{\mathbf{V}}\cong\{(Y,g);\ g\in F_X^{\mathbf{W}},\ c(g)\subseteqq Y\subseteqq X,\ |Y|<\infty\}\leqq F_X^{\mathbf{SL}}\times F_X^{\mathbf{W}}.$$

If $\mathbf{V}\in[\mathbf{LNB},\mathbf{LNBG}]$ then

$$F_X^{\mathbf{V}}\cong\{(x,Y,g);\ g\in F_X^{\mathbf{W}},\ \{x\},\ c(g)\subseteqq Y\subseteqq X,\ |Y|<\infty\}\leqq F_X^{\mathbf{LNB}}\times F_X^{\mathbf{W}}.$$

If $\mathbf{V}\in[\mathbf{LRB},\mathbf{LRBG}]$ then

$$F_X^{\mathbf{V}}\cong\{(a,g)\in I\times F_X^{\mathbf{W}},\ c(g)\subseteqq c(a)\}\leqq F_X^{\mathbf{LRB}}\times F_X^{\mathbf{W}}.$$

Proof. With $F_X^{\mathbf{W}}$ replaced in these descriptions by $F_X^{\mathbf{U}}/\varrho_{\mathbf{W}}$ it can be easily seen by Theorems 2.1 and 2.2 that the respective isomorphisms are given by $u\varrho_{\mathbf{V}}\to(c(u), u\varrho_{\mathbf{W}})$, $u\varrho_{\mathbf{V}}\to(h(u), c(u), u\varrho_{\mathbf{W}})$ and $u\varrho_{\mathbf{V}}\to(i(u), u\varrho_{\mathbf{W}})$.

Select $h\in X$ and let $\{p_{yz};\ y, z\in X\setminus\{h\}\}$ be a set in one to one correspondence with $X\setminus\{h\}\times X\setminus\{h\}$. Put $p_{yz}=e$ if $y=h$ or $z=h$. By [9], [14] or [15], $F_X^{\mathbf{CS}}\cong\mathcal{M}(H, X, X, P)$, a Rees matrix semigroup, where H is the free group with identity e freely generated by $\{exe, p_{yz};\ x, y, z\in X,\ y\neq h\neq z\}$, and P is the matrix with p_{yz} in row y and column z. $\mathcal{M}(H, X, X, P)$ is freely generated in **CS** by $\{(exe, x, x);\ x\in X\}$.

Also by [9], [14] and [15], if $\mathbf{V}\in[\mathbf{ReB}, \mathbf{CS}]$ then there is a unique normal subgroup $N_{\mathbf{V}}$ of H such that $F_X^{\mathbf{V}}\cong\mathcal{M}(H/N_{\mathbf{V}}, X, X, P/N_{\mathbf{V}})$.

Let $\psi\colon F_X^{\mathbf{U}}\to\mathcal{M}(H, X, X, P)$ be the surjective homomorphism given by $x\psi=(exe, x, x)$ for all $x\in X$. Define $\varphi\colon F_X^{\mathbf{U}}\to H$ by $u\psi=(u\varphi, h(u), t(u))$ for all $u\in F_X^{\mathbf{U}}$. Then $x\varphi=exe$, $(xy)\varphi=x\varphi p_{xy}(y\varphi)$ and $u^{-1}\varphi=\left(p_{t(u)h(u)}(u\varphi)p_{t(u)h(u)}\right)^{-1}$ for any $x, y\in X$ and $u\in F_X^{\mathbf{U}}$. It follows that for $\mathbf{V}\in[\mathbf{ReB}, \mathbf{CS}]$ and $u, v\in F_X^{\mathbf{U}}$ then $(u, v)\in\varrho_{\mathbf{V}}$ if and only if $h(u)=h(v)$, $t(u)=t(v)$ and $u\varphi N_{\mathbf{V}}=v\varphi N_{\mathbf{V}}$.

Corollary 2.4. *Let* $\mathbf{V}\in[\mathbf{NB}, \mathbf{RBG}\cap\mathbf{POBG}]$ *and* $\mathbf{W}=\mathbf{V}\cap\mathbf{CS}$. *If* $\mathbf{V}\in[\mathbf{NB}, \mathbf{NBG}]$ *then*

$$F_X^{\mathbf{V}}\cong\{((x, y, Y), (g, x, y));\ g\in H/N_{\mathbf{W}},\ \{x, y\},\ c(g)\subseteqq Y\subseteqq X,\ |Y|<\infty\}\leqq F_X^{\mathbf{NB}}\times F_X^{\mathbf{W}}.$$

If $\mathbf{V}\in[\mathbf{LQNB}, \mathbf{LQNBG}]$ *then*

$$F_X^{\mathbf{V}}\cong\{((a, x), (g, h(a), x));\ g\in H/N_{\mathbf{W}},\ a\in I,\ \{x\},\ c(g)\subseteqq c(a)\}\leqq F_X^{\mathbf{LQNB}}\times F_X^{\mathbf{W}}.$$

If $\mathbf{V}\in[\mathbf{RB}, \mathbf{RBG}\cap\mathbf{POBG}]$ *then*

$$F_X^{\mathbf{V}}\cong\{((a, b), (g, h(a), t(b)));\ g\in H/N_{\mathbf{W}},\ a, b\in I,\ c(g)\subseteqq c(a)=c(b)\}\leqq F_X^{\mathbf{RB}}\times F_X^{\mathbf{W}}.$$

Proof. By Theorems 2.1 and 2.2 it can be readily checked that the respective isomorphisms are given by $u\varrho_{\mathbf{V}}\to((h(u), t(u), c(u)), (u\varphi N_{\mathbf{V}}, h(u), t(u)))$, $u\varrho_{\mathbf{V}}\to((i(u), t(u)), (u\varphi N_{\mathbf{V}}, h(u), t(u)))$ and $u\varrho_{\mathbf{V}}\to((i(u), f(u)), (u\varphi N_{\mathbf{V}}, h(u), t(u)))$.

Note that there are repetitive symbols in the models; $h(a)$ and $t(b)$ are derivable from a and b. The repetitions are included so as to give a simple description of the multiplication.

Since the relatively free objects of **LZG** are known modulo **G** then by the corollaries the relatively free objects of **RBG**$\cap$**POBG** are known modulo **CS** and **G**.

By [12; Theorem IV.4.3], S is a normal band of groups if and only if S is a strong semilattice of completely simple semigroups. We can use Corollary 2.4 to characterize free objects of varieties in [**NB**, **NBG**] in these terms.

Suppose E is a semilattice and $\{S_\alpha;\ \alpha\in E\}$ is a disjoint set of semigroups. Suppose there exists a set of injective homomorphisms $\psi_{\alpha,\beta}\colon S_\alpha\to S_\beta$ for all $\alpha, \beta\in E$ where $\alpha\geqq\beta$, such that $\psi_{\alpha,\alpha}$ is the identity map and $\psi_{\alpha,\beta}\psi_{\beta,\gamma}=\psi_{\alpha,\gamma}$ for all $\alpha, \beta, \gamma\in E$

where $\alpha \geqq \beta \geqq \gamma$. Then $S = \bigcup_{\alpha \in E} S_\alpha$ with multiplication $a \cdot b = a\psi_{\alpha,\alpha\beta} b\psi_{\beta,\alpha\beta}$ for $a \in S_\alpha$ and $b \in S_\beta$ is called a *sturdy semilattice E of semigroups S_α; $\alpha \in E$ with transitive system* $\{\psi_{\alpha,\beta};\ \alpha, \beta \in E\}$ (see [12]).

Corollary 2.5. *If* $\mathbf{V} \in [\mathbf{NB}, \mathbf{NBG}]$ *and* $\mathbf{W} = \mathbf{V} \cap \mathbf{CS}$ *then*

$$F_X^{\mathbf{V}} \cong \{(Y, (g, x, y));\ g \in H/N_{\mathbf{W}},\ \{x, y\},\ c(g) \subseteqq Y \subseteqq X,\ |Y| < \infty\} \leqq F_X^{\mathbf{SL}} \times F_X^{\mathbf{W}}.$$

Hence $F_X^{\mathbf{V}}$ is a sturdy semilattice $F_X^{\mathbf{SL}}$ of semigroups $F_Y^{\mathbf{W}}$; $Y \in F_X^{\mathbf{SL}}$ with transitive system $\{\psi_{Y,Z};\ Y, Z \in F_X^{\mathbf{SL}}\}$ such that $\{x\psi_{\{x\},Y};\ x \in Y\}$ generates $F_Y^{\mathbf{W}}$. Conversely any such sturdy semilattices of semigroups is isomorphic to $F_X^{\mathbf{W}}$.

Proof. The subdirect decomposition is immediate by Corollary 2.4. So $D_Y = \{(Y, g, x, y);\ g \in H/N_{\mathbf{W}},\ \{x, y\},\ c(g) \subseteqq Y\}$ is a $\mathscr{D}$-class of the model and $D_Y \cong F_Y^{\mathbf{W}}$. With $\psi_{Y,Z}\colon D_Y \to D_Z$ given by $(Y, g, x, y) \to (Z, g, x, y)$ for $Z \supseteqq Y$ we see that $F_X^{\mathbf{V}}$ is a sturdy semilattice of the required form. Now suppose S is a sturdy semilattice $F_X^{\mathbf{SL}}$ of $F_Y^{\mathbf{W}}$; $Y \in F_X^{\mathbf{SL}}$ with transitive system $\{\psi'_{Y,Z};\ Y, Z \in F_X^{\mathbf{SL}}\}$ such that $\{x\psi'_{\{x\},Y};\ x \in Y\}$ generates $F_Y^{\mathbf{W}}$ for all Y. Define an automorphism η_Y of $F_Y^{\mathbf{W}}$ by $x\psi_{\{x\},Y}\eta_Y = x\psi'_{\{x\},Y}$ for all $x \in Y$. We have for $Z \supseteqq Y$, $\psi_{\{x\},Y}\eta_Y\psi'_{Y,Z} = \psi'_{\{x\},Y}\psi'_{Y,Z} = \psi'_{\{x\},Z} = \psi_{\{x\},Z}\eta_Z$. By [12; Exercise III.7.12.11] then $S \cong F_X^{\mathbf{V}}$.

3. Free non-pseudo orthodox bands of groups

This section begins with a description of $\mathscr{D}$-classes of relatively free completely regular semigroups that allows easy comparison of some properties of the relatively free objects.

Throughout, Y will denote a finite subset of X and $D_Y = \{u \in F_X^{\mathbf{U}};\ c(u) = Y\}$. D_Y is a unary subsemigroup of $F_X^{\mathbf{U}}$. Let ϱ be a congruence on D_Y such that D_Y/ϱ is completely simple. Select $e_Y = w^0$ for some $w \in D_Y$; so $e_Y\varrho \in E(D_Y/\varrho)$. For $u, v \in F_Y^{\mathbf{U}}$ define

$$e_{Yu,v} = ue_Y(e_Yvue_Y)^{-1}e_Yv. \tag{6}$$

We have $E(D_Y/\varrho) = \{e_{Yu,v}\varrho;\ u, v \in F_Y^{\mathbf{U}}\}$ since $e_{Yu,v}\varrho$ is an idempotent and for $r \in D_Y$, $r^0\ \varrho\ e_{Yr,r}$. For notational convenience write $e_{Y-u} = e_{Ye_Y,u}$ and $e_{Yu-} = e_{Yu,e_Y}$. Define

$$p_{Yu,v} = e_{Y-u}e_{Yv-};\quad u, v \in F_Y^{\mathbf{U}}. \tag{7}$$

By [3; Theorem 3.4], for any $u \in D_Y$ there is a unique $a\varrho$ such that $a\varrho\ \mathscr{H}\ e_Y\varrho$ and $u\ \varrho\ e_{Yu-}ae_{Y-u}$. In fact since $e_{Yu-}\varrho\ \mathscr{L} e_Y\varrho\ \mathscr{R}\ e_{Y-u}\varrho$ then $a\varrho = (e_Yue_Y)\varrho$; so $u\ \varrho\ e_{Yu-}e_Yue_Ye_{Y-u}$. Let H_Y be the unary subsemigroup of $F_X^{\mathbf{U}}$ generated by

$$\{e_Yue_Y,\ p_{Yu,v};\ u, v \in F_Y^{\mathbf{U}}\}. \tag{8}$$

Then by [3; Theorem 3.4], H_Y/ϱ is the $\mathscr{H}$-class of $e_Y\varrho$ in D_Y/ϱ and

$$\{(e_{Yu-}he_{Y-v})\varrho;\ h\in H_Y\}$$

is the $\mathscr{H}$-class of $(e_{Yu-}e_{Y-v})\varrho$, $u, v\in D_Y$.

Suppose $\mathbf{V}\in[\mathbf{CS}, \mathbf{CR}]$. Let $S_{\mathbf{V}Y}$ be the completely simple subsemigroup of $D_Y/\varrho_{\mathbf{V}}$ that is generated by

$$\{(e_{Yx-}e_Yxe_Ye_{Y-x})\varrho_{\mathbf{V}};\ x\in Y\}.$$

Let T be a subsemigroup of a semigroup S and ψ: $S\to T'$ be a homomorphism. Then T is a *retract subsemigroup of S under ψ* if and only if there is an isomorphism φ: $T'\to T$, and $\varphi\psi$ is the identity map.

Theorem 3.1. *Let Y be a finite subset of X and $\mathbf{V}\in[\mathbf{CS}, \mathbf{CR}]$. Then $S_{\mathbf{V}Y}$ is a retract subsemigroup of $F_X^{\mathbf{U}}/\varrho_{\mathbf{V}}$ under $(\varrho_{\mathbf{CS}}/\varrho_{\mathbf{V}})^{\#}$. In particular $S_{\mathbf{V}Y}\cong F_Y^{\mathbf{CS}}$.*

Proof. Let ψ: $F_Y^{\mathbf{U}}/\varrho_{\mathbf{V}}\to F_Y^{\mathbf{U}}/\varrho_{\mathbf{CS}}$ be the surjective homomorphism determined by the action $(x\varrho_{\mathbf{V}})\psi=x\varrho_{\mathbf{CS}}$ for each $x\in Y$. So $\psi\circ\psi^{-1}$ is the restriction of $(\varrho_{\mathbf{CS}}/\varrho_{\mathbf{V}})$ to the subsemigroup $F_Y^{\mathbf{U}}/\varrho_{\mathbf{V}}$ of $F_X^{\mathbf{U}}/\varrho_{\mathbf{V}}$. We have $e_{Yx-}e_Yxe_Ye_{Y-x}\varrho_{\mathbf{V}}xe_Y(e_Yxe_Y)^{-1}e_Yx$, and $(xe_Y(e_Yxe_Y)^{-1}e_Y)\varrho_{\mathbf{CS}}$ is an idempotent that is $\mathscr{R}$-related to $x\varrho_{\mathbf{CS}}$. Hence $((e_{Yx-}e_Yxe_Ye_{Y-x})\varrho_{\mathbf{V}})\psi=x\varrho_{\mathbf{V}}\psi$ and ψ maps $S_{\mathbf{V}Y}$ onto $F_Y^{\mathbf{U}}/\varrho_{\mathbf{CS}}$. But $S_{\mathbf{V}Y}\in\mathbf{CS}$ so there is a surjective homomorphism φ: $F_Y^{\mathbf{U}}/\varrho_{\mathbf{CS}}\to S_{\mathbf{V}Y}$ given by $x\varrho_{\mathbf{CS}}\varphi=(e_{Yx-}e_Yxe_Ye_{Y-x})\varrho_{\mathbf{V}}$. The result follows.

The Theorem can be strengthened in the two variable case.

Theorem 3.2. *If $\mathbf{V}\in L(\mathbf{BG})$ and $\mathbf{W}=\mathbf{V}\cap\mathbf{NBG}$ then $F_{\{x,y\}}^{\mathbf{V}}\cong F_{\{x,y\}}^{\mathbf{W}}$.*

Proof. By [8; Lemma IV.4.6] it can be easily seen that $auva\ \varrho_{\mathbf{B}}\ avua$ for any $a, u, v\in F_{\{x,y\}}^{\mathbf{U}}$. So $(auva)^0\ \varrho_{\mathbf{V}}\ (avua)^0$ and hence $F_{\{x,y\}}^{\mathbf{V}}\in\mathbf{W}$. But $F_{\{x,y\}}^{\mathbf{W}}\in\mathbf{V}$, so the homomorphism $F_{\{x,y\}}^{\mathbf{W}}\to F_{\{x,y\}}^{\mathbf{V}}$ such that $x\to x$, $y\to y$ is an isomorphism.

Now suppose $\mathbf{V}\in L(\mathbf{BG})$ and Y is a finite subset of X. Let $a, b, c, d\in F_Y^{\mathbf{U}}$. If $(a, b), (c, d)\in\varrho_{\mathbf{B}}$ then since $\operatorname{tr}\varrho_{\mathbf{BG}}=\operatorname{tr}\varrho_{\mathbf{B}}$ we have by (6) and (7), $(e_{Ya,c}, e_{Yb,d})\in\varrho_{\mathbf{BG}}$, whence $(p_{Ya,c}, p_{Yb,d})\in\varrho_{\mathbf{BG}}$. So

$$(p_{Ya,c}, p_{Yb,d})\in\varrho_{\mathbf{V}}\quad\text{if}\quad(a, b), (c, d)\in\varrho_{\mathbf{B}}. \tag{9}$$

Also by (7) $p_{Ya,c}\varrho_{\mathbf{V}}(e_Yae_Y)^{-1}e_Yace_Y(e_Yce_Y)^{-1}$ and since $H_Y/\varrho_{\mathbf{V}}$ is a group,

$$e_Yace_Y\ \varrho_{\mathbf{V}}\ e_Yae_Yp_{Ya,c}e_Yce_Y. \tag{10}$$

By (9) and (10) $e_Yae_Y\ \varrho_{\mathbf{V}}\ e_Yaa^0e_Y\ \varrho_{\mathbf{V}}\ e_Yae_Yp_{Ya,a}e_Ya^0e_Y$ so

$$e_Ya^0e_Y\ \varrho_{\mathbf{V}}\ p_{Ya,a}^{-1}. \tag{11}$$

Also $e_Y a e_Y \varrho_V e_Y a^2 a^{-1} e_Y \varrho_V e_Y a e_Y p_{Ya,a} e_Y a e_Y p_{Ya,a} e_Y a^{-1} e_Y$ so

(12)
$$e_Y a^{-1} e_Y \varrho_V (p_{Ya,a} e_Y a e_Y p_{Ya,a})^{-1}.$$

Also note that since $(a^0, (h(a)t(a))^0) \in \varrho_{CS}$ while $e_{Ya-}\varrho_{CS}$ and $e_{Y-b}\varrho_{CS}$ are idempotents then by (6), $(e_{Ya-}, e_{Yh(a)-}), (e_{Y-b}, e_{Y-t(b)}) \in \varrho_{CS}$, so by (7)

(13)
$$p_{Ya,b} \varrho_{CS} p_{Yt(a),h(b)}.$$

Lemma 3.3. *Suppose* $\mathbf{V} \in [\mathbf{CS}, \mathbf{BG}]$. *Then* $\mathbf{V} \in L(\mathbf{POBG})$ *if and only if* $(p_{Ya,b}, p_{Yt(a),h(b)}) \in \varrho_V$ *for some finite subset Y of X such that* $|Y| \geqq 3$ *and for all* $a, b \in F_Y^U$.

Proof. As noted in the proof of Theorem 2.1, $\mathbf{POBG} = \mathbf{ReB}_{\max} \vee \mathbf{ReB}^{\max}$. Then by [10; Theorem 3.4], $\mathbf{POBG} = (\mathbf{ReB}_{\max})^{\max} \cap (\mathbf{ReB}^{\max})_{\max}$. Since $\mathbf{ReB}_{\max} = \mathbf{CS}$ then $\mathbf{POBG} \in [\mathbf{CS}, \mathbf{CS}^{\max}]$ so $\ker \varrho_{\mathbf{POBG}}/\varrho_{CR} = \ker \varrho_{CS}/\varrho_{CR}$. Thus if $\mathbf{CS} \subseteqq \mathbf{V} \subseteqq \mathbf{POBG}$ then $\ker \varrho_{CS}/\varrho_{CR} = \ker \varrho_V/\varrho_{CR}$. Then by (13), since $p_{Ya,b}\varrho_{CR}$ and $p_{Yt(a),h(b)}\varrho_{CR}$ are $\mathscr{H}$-related, $(p_{Yb,a}^{-1} p_{Yt(a),h(b)})\varrho_{CR} \in \ker \varrho_V/\varrho_{CR}$ so $(p_{Ya,b}, p_{Yt(a),h(b)}) \in \varrho_V$ for all $a, b \in F_Y^U$.

Conversely suppose $|Y| \geqq 3$ and $(p_{Ya,b}, p_{Yt(a),h(b)}) \in \varrho_V$ for all $a, b \in F_Y^U$. Then by (8), (10), (11) and (12), H_Y/ϱ_V is the group generated by

$$\{(e_Y x e_Y)\varrho_V, p_{Yx,y}\varrho_V;\ x, y \in Y\}.$$

We begin by showing that $\ker((\varrho_{CS}/\varrho_V)|(F_Y^U/\varrho_V)) = E(F_Y^U/\varrho_V)$. Recall that S_{VY} is a completely simple subsemigroup of D_Y/ϱ_V generated by $\{(e_{Yx-} e_Y x e_Y e_{Y-x})\varrho_V;\ x \in Y\}$. So there is a subgroup K_{VY} of H_Y/ϱ_V such that for each $x, y \in Y$, $\{(e_{Yx-} k e_{Y-y})\varrho_V;\ k \in K_{VY}\}$ is an $\mathscr{H}$-class in S_{VY}. We have $(e_Y x e_Y)\varrho_V \in K_{VY}$. Also, by (7), $(e_{Yy-} p_{Yx,y}^{-1} e_{Y-x})\varrho_V$ is an idempotent; it is $\mathscr{R}$-related to $(e_{Yy-} e_Y y e_Y e_{-yY})\varrho_V$ and $\mathscr{L}$-related to $(e_{Yx-} e_Y x e_Y e_{Y-x})\varrho_V$ so it is in S_{VY}. Hence $p_{Yx,y}\varrho_V \in K_{VY}$. It follows that $H_Y/\varrho_V = K_{VY}$, so the $\mathscr{H}$-classes of S_{VY} are $\mathscr{H}$-classes of D_Y/ϱ_V. Hence, since $D_Y/\varrho_V \in \mathbf{CS}$ and $\ker((\varrho_{CS}/\varrho_V)|S_{VY}) = E(S_{VY})$ by Theorem 3.1 then $\ker((\varrho_{CS}/\varrho_V)|(D_Y/\varrho_V)) = E(D_Y/\varrho_V)$.

Suppose $Z \subseteqq Y$. There is an endomorphism ψ of F_Y^U such that $x\psi = x$ if $x \in Z$ and $x\psi \in Z$ if $x \in Y \setminus Z$. Since ϱ_V is fully invariant then ψ induces an endomorphism φ of F_Y^U/ϱ_V given by $a\varrho_V\varphi = a\psi\varrho_V$. Define $e_Z = e_Y\psi$, so $e_Z\varrho_V = e_Y\varrho_V\varphi$ is an idempotent in D_Z/ϱ_V. Construct $p_{Zu,v}$ by (7) for $u, v \in F_Z^U$. Then

$$p_{Yu,v}\varrho_V\varphi = ((e_Y u e_Y)^{-1} e_Y u v e_Y (e_Y v e_Y)^{-1})\varrho_V\varphi = p_{Zu,v}\varrho_V.$$

Hence $(p_{Zu,v}, p_{Zt(u),h(v)}) \in \varrho_V$ for all $u, v \in F_Z^U$, and as above we get

$$\ker((\varrho_{CS}/\varrho_V)|(D_Z/\varrho_V)) = E(D_Z/\varrho_V).$$

Hence $\ker((\varrho_{CS})/\varrho_V)|(F_Y^U/\varrho_V)) = E(F_Y^U/\varrho_V)$.

Since $(x^0 F_Y^{\mathrm{U}} x^0)/\varrho_{\mathrm{CS}} \in \mathbf{OBG}$ it now follows that $(x^0 F_Y^{\mathrm{U}} x^0)/\varrho_{\mathrm{V}} \in \mathbf{OBG}$. But then $(x^0 y x^0 x^0 z x^0)^0 \, \varrho_{\mathrm{V}} \, (x^0 y x^0)^0 (x^0 z x^0)^0$ for any $x, y, z \in Y$. So $(x^0 y x^0 x^0 z x^0)^0 = (x^0 y x^0)^0 (x^0 z x^0)^0$ is a law in $\mathbf{V}$ and $\mathbf{V} \in L(\mathbf{POBG})$.

The major result of this section can now be proved.

Theorem 3.4. *Suppose* $\mathbf{V} \in [\mathbf{POBG} \cap \mathbf{RBG}, \mathbf{BG}] \setminus L(\mathbf{POBG})$, *and* Y *is a finite subset of* X *such that* $|Y| \geqq 3$. *Then any* $\mathcal{H}$-*class of* $F_X^{\mathrm{U}}/\varrho_{\mathrm{V}}$ *in the* $\mathcal{D}$-*class* D_Y/ϱ_{V} *is not a free group.*

Proof. Suppose $v \in Y$ and $u, w \in F_Y^{\mathrm{U}}$. By (9), (10) and (11) we have

(14) $$e_Y uvwe_Y \; \varrho_{\mathrm{V}} \; e_Y ue_Y p_{Yu,vw} e_Y ve_Y p_{Yv,w} e_Y we_Y,$$

(15a) $$e_Y uvwe_Y \; \varrho_{\mathrm{V}} \; e_Y uv^0 e_Y p_{Yuv,vw} e_Y vwe_Y,$$

(15b) $$e_Y uvwe_Y \; \varrho_{\mathrm{V}} \; e_Y uve_Y p_{Yuv,vw} e_Y v^0 we_Y,$$

(16a) $$e_Y uv^0 e_Y \; \varrho_{\mathrm{V}} \; e_Y ue_Y p_{Yu,v} e_Y v^0 e_Y \; \varrho_{\mathrm{V}} \; e_Y ue_Y p_{Yu,v} p_{Yv,v}^{-1},$$

(16b) $$e_Y v^0 we_Y \; \varrho_{\mathrm{V}} \; p_{Yv,v}^{-1} p_{Yv,w} e_Y we_Y.$$

Then by (15a), (16a) and (14)

(17a) $$\begin{aligned} p_{Yuv,vw} e_Y vwe_Y \; &\varrho_{\mathrm{V}} \; (e_Y uv^0 e_Y)^{-1} e_Y uvwe_Y \\ &\varrho_{\mathrm{V}} \; p_{Yv,v} p_{Yu,v}^{-1} p_{Yu,vw} e_Y ve_Y p_{Yv,w} e_Y we_Y. \end{aligned}$$

Likewise by (15b), (16b) and (14)

(17b) $$e_Y uve_Y p_{Yuv,vw} \; \varrho_{\mathrm{V}} \; e_Y ue_Y p_{Yu,vw} e_Y ve_Y p_{Yv,v}.$$

So by (10) and (17a), and (10) and (17b) respectively

(18a) $$\begin{aligned} e_Y uv^2 we_Y \; &\varrho_{\mathrm{V}} \; e_Y uve_Y (p_{Yuv,vw} e_Y vwe_Y) \\ &\varrho_{\mathrm{V}} \; e_Y ue_Y p_{Yu,v} e_Y ve_Y p_{Yv,v} p_{Yv,v}^{-1} p_{Yu,vw} e_Y ve_Y p_{Yv,w} e_Y we_Y, \end{aligned}$$

(18b) $$\begin{aligned} e_Y uv^2 we_Y \; &\varrho_{\mathrm{V}} \; (e_Y uve_Y p_{Yuv,vw}) e_Y vwe_Y \\ &\varrho_{\mathrm{V}} \; e_Y ue_Y p_{Yu,vw} e_Y ve_Y p_{Yv,v} e_Y ve_Y p_{Yv,w} e_Y we_Y. \end{aligned}$$

Comparing (18a) and (18b) then

$$p_{Yu,v} e_Y ve_Y p_{Yv,v} p_{Yu,v}^{-1} p_{Yu,vw} \; \varrho_{\mathrm{V}} \; p_{Yu,vw} e_Y ve_Y p_{Yv,v}$$

whence

(19) $$(e_Y ve_Y p_{Yv,v})(p_{Yu,v}^{-1} p_{Yu,vw}) \; \varrho_{\mathrm{V}} \; (p_{Yu,v}^{-1} p_{Yu,vw})(e_Y ve_Y p_{Yv,v}).$$

Alternatively we may repeat the above calculation with (14) replaced by $e_Y uvwe_Y \, \varrho_V \, e_Y ue_Y p_{Yu,v} e_Y ve_Y p_{Yuv,w} e_Y we_Y$ to get

(20) $$(p_{Yv,v} e_Y ve_Y)(p_{Yuv,w} p_{Yv,w}^{-1}) \, \varrho_V \, (p_{Yuv,w} p_{Yv,w}^{-1})(p_{Yv,v} e_Y ve_Y).$$

Let

$$\alpha = e_Y ve_Y p_{Yv,v}, \quad \beta = p_{Yu,v}^{-1} p_{Yu,vw}, \quad \gamma = p_{Yv,v} e_Y ve_Y, \quad \delta = p_{Yuv,w} p_{Yv,w}^{-1}.$$

Note that $\alpha\varrho_V$ is not an idempotent. To see this observe that for $v \in Y$ then as in the proof of Theorem 3.1, $(e_{Yv} - e_Y ve_Y e_{Y-v})\varrho_{CS} = v\varrho_{CS}$ which is not an idempotent in F_Y^U/ϱ_{CS}. But $(e_{Yv} - p_{Yv,v}^{-1} e_{Y-v})\varrho_{CS}$ is the idempotent $\mathscr{H}$-related to $v\varrho_{CS}$. Hence $(e_Y ve_Y, p_{Yv,v}^{-1}) \notin \varrho_{CS}$, so $\alpha\varrho_{CS} \neq e_Y \varrho_{CS} = \alpha^0 \varrho_{CS}$. Likewise $\gamma\varrho_{CS} \neq \gamma^0 \varrho_{CS}$.

Let A and B denote the subgroups of the $\mathscr{H}$-class H_Y/ϱ_V of $e_Y \varrho_V$ that are respectively generated by $\{\alpha\varrho_V, \beta\varrho_V\}$ and $\{\gamma\varrho_V, \delta\varrho_V\}$. Assume H_Y/ϱ_V is a free group. By (19) and (20), $\{\alpha\varrho_V, \beta\varrho_V\}$ and $\{\gamma\varrho_V, \delta\varrho_V\}$ are not sets of free generators of free groups, so A and B are free cyclic groups. Say $\lambda\varrho_V$ generates A for some $\lambda \in F_Y^U$ and $\alpha \, \varrho_V \, \lambda^m$, $\beta \, \varrho_V \, \lambda^n$. But $\alpha\varrho_{CS}$, and $\lambda\varrho_{CS}$, are not idempotents while by (13) $\beta\varrho_{CS} = \lambda^n \varrho_{CS}$ is idempotent, so $n=0$. Therefore $(p_{Yu,vw}, p_{Yu,v}) \in \varrho_V$, and likewise $(p_{Yuv,w}, p_{Yv,w}) \in \varrho_V$ for any $u, w \in F_Y^U$ and $v \in Y$. Of course $v = h(vw) = t(uv)$ so by (9) we now have $(p_{Ya,b}, p_{Yt(a),h(b)}) \in \varrho_V$ for all $a, b \in F_Y^U$; thus by Lemma 3.3 $\mathbf{V} \in L(\mathbf{POBG})$. This is a contradiction. Thus H_Y/ϱ_V is not a free group.

Remark. Since the subgroup S_{VY} of F_X^U/ϱ_V is isomorphic to F_Y^{CS} then for $|Y| \geqq 2$ and $\mathscr{H}$-class of S_{VY} is a free group on more than $|Y|$ free generators; that is, it generates the variety $\mathbf{G}$ of all groups. Hence any $\mathscr{H}$-class in D_Y/ϱ_V generates $\mathbf{G}$ and thus lies in no proper subvariety of $\mathbf{G}$.

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Congruences on semigroups of quotients

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Introduction. PETRICH and others [2, 5, 6, 7] have studied semigroups V which are ideal extensions of a semigroup S by the quotient semigroup $T=V/S$. These extensions are classified by their homomorphic image in the translational hull $\Omega(S)$ of S. Most often S is required to be weakly reductive so that S is embedded in $\Omega(S)$. On the other hand, given a right quotient filter Σ on S, the semigroup $Q(S)$ of right quotients of S can be defined and all right S-systems $M\supseteqq S$ for which M/S is torsion can be classified by their homomorphic image in $Q(S)$. Often S is required to be strongly torsion free so that S is embedded in $Q(S)$. When M is strongly torsion free, M is isomorphic to an S-subsystem of $Q(S)$ and so may receive a semigroup structure from $Q(S)$. The author [4] has shown that these two concepts are special instances of a common generalization.

In this paper we study semigroups V containing the strongly torsion free semigroup S with $T=V/S$ torsion, called semigroup extensions of S by torsion T. In this situation T is an (S, S)-system which may not be a semigroup. However, $T^*=T\setminus\{0\}$ has a partial multiplication for pairs $t, t'\in T^*$ with $tt'\notin S$ in V. This partial multiplication is associative. When considering ideal extensions, the (S, S)-system T has a trivial scalar multiplication. In our situation, the (S, S)-system structure on T is not trivial and plays an important role.

In Section 1, the necessary definitions concerning semigroups of quotients are given and the semigroup extensions V of S are characterized in terms of an (S, S)-homomorphism $\theta\colon T^*\to Q$ which preserves any partial multiplication in T^*. This characterization is reminiscent of the characterization of ideal extensions due to CLIFFORD [1]. In Section 2, semigroup congruences υ on V are characterized in terms of the restriction $\sigma=\upsilon|_S$ of υ to S, and the (S, S)-system congruence τ on T inherited from υ when S/σ is strongly torsion free. In Section 3, the semigroup V/υ is shown to be an extension of S/σ by a quotient S-system of T. In Section 4, the special case of extensions determined by partial homomorphisms is considered.

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1. Extensions of semigroups. Let S be a semigroup with zero. (In this paper S will always have a 0 unless otherwise noted.)

Definition. A *right quotient filter* on S is a nonempty collection Σ of right ideals of S satisfying

(i) if $A \in \Sigma$, $A \subseteq B$, a right ideal of S, then $B \in \Sigma$;

(ii) if $A, B \in \Sigma$ then $A \cap B \in \Sigma$;

(iii) if $A \in \Sigma$, $s \in S$ then $s^{-1}A = \{x \in S \mid sx \in A\} \in \Sigma$; and

(iv) if I is a right ideal of S, $A \in \Sigma$, and $a^{-1}I \in \Sigma$ for all $a \in A$, then $I \in \Sigma$.

HINKEL [3] calls such right quotient filters "special".

For $A \in \Sigma$, let $\mathrm{Hom}(A, S) = \{f\colon A \to S \mid f(ax) = f(a)x$ for all $x \in S$, $a \in A\}$. Let $\mathbf{B} = \bigcup_{A \in \Sigma} \mathrm{Hom}(A, S)$, then $\mathbf{B}$ is a semigroup under composition with multiplication of $f\colon A \to S$, $g\colon B \to S$ defined by the composition $f \circ g\colon C \to S$ where

$$C = \{b \in B \mid g(b) \in A\}$$

which is in Σ. Define the relation γ on $\mathbf{B}$ by $f\gamma g$ if and only if there is some $A \in \Sigma$ with $f(a) = g(a)$ for all $a \in A$. γ is a semigroup congruence and $Q = \mathbf{B}/\gamma$ is the *semigroup of right quotients* of S with respect to Σ.

Let M be a right S-system and define a relation δ on M by $m\delta m'$ if and only if for some $A \in \Sigma$, $ma = m'a$ for all $a \in A$. δ is called the *torsion congruence* on M. M is *strongly torsion free* if δ is the identity relation, and M is *torsion* if $\delta = M \times M$. For each $s \in S$, the γ class of the mapping $\lambda_s\colon S \to S$ given by left multiplication by s is denoted by $[s]$ and the mapping $[\ ]\colon S \to Q$ is a semigroup homomorphism. If S is strongly torsion free, $[\ ]$ is a monomorphism and we identify S with its image $[S]$ in Q.

Definition. A *partial (S, S)-algebra* T is a partial groupoid which is, at the same time, an (S, S)-system, for which $(ts)t' = t(st')$ for all $s \in S$ whenever both products are defined.

Let V be a (S, S)-system which is also a semigroup. If V satisfies $(vs)v' = v(sv')$ for all $v, v' \in V$, $s \in S$, we call V an *S-algebra*. A semigroup V containing S as a subsemigroup is clearly an S-algebra. Let $T = V/S$, the Rees quotient (S, S)-system. T has a partial associative multiplication of nonzero elements $t, t' \in T$ if $tt' \notin S$ (as an element of V) inherited from V, and so is a partial (S, S)-algebra. We denote $T \setminus \{0\}$ by T^* and note that $V = T^* \cup S$ as sets.

In general, given a partial (S, S)-algebra T, we wish to define a semigroup multiplication on $V = T^* \cup S$ extending the partial multiplication in T^* and the multiplication in S. If such a multiplication can be defined, we call V a *semigroup extension of S by T.*

Definition. Let Q be a semigroup of right quotients of S with respect to a right quotient filter Σ, and let T be a partial (S, S)-algebra. A mapping $\theta\colon T^* \to Q$ is a *partial homomorphism* if

(i) whenever $t, t' \in T^*$, and tt' is defined, $\theta(tt') = (\theta t)(\theta t')$, and

(ii) if $t \in T^*$, $s \in S$ and $ts \neq 0$ $[st \neq 0]$, then $\theta(ts) = (\theta t)s$ $[\theta(st) = s\theta(t)]$.

When S is strongly torsion free and T is torsion, the desired multiplication on $V = T^* \cup S$ can be defined as shown by the following theorem.

Theorem 1. *Let Σ be a right quotient filter on S and S be strongly torsion free. Let T be a partial (S, S)-algebra. If $\theta\colon T^* \to Q$ is a partial homomorphism satisfying $(\theta a)(\theta b) \in S$ if $a, b \in T^*$, ab undefined and $s(\theta b) \in S$ $[(\theta b)s \in S]$ if $sb = 0$ $[bs = 0]$, then $V = T^* \cup S$ is a semigroup under the multiplication*

$$a*b = \begin{cases} (\theta a)(\theta b) & \text{if } a, b \in T^*, ab \text{ undefined} \\ a(\theta b) & \text{if } a \in S, b \in T^*, ab = 0 \\ (\theta a)b & \text{if } a \in T^*, b \in S, ab = 0 \\ ab & \text{otherwise.} \end{cases}$$

Conversely, every semigroup extension V of S by torsion $T = V/S$ can be constructed in this manner.

Proof. The proof of the direct part of the theorem consists of verifying the associative law. The proof is tedious but not difficult so only the verification that $(a*b)*c = a*(b*c)$ for $a, b, c \in T^*$ is given. If ab, bc, $a(bc)$ and $(ab)c$ are defined, then

$$(a*b)*c = ab*c = (ab)c = a(bc) = a*bc = a*(b*c).$$

If ab and bc are defined while $(ab)c$ is not, then

$$(a*b)*c = ab*c = (\theta ab)(\theta c) = (\theta a\theta b)\theta c = \theta a(\theta b\theta c) = \theta a\theta(bc) = a*bc = \\ = a*(b*c).$$

If bc is defined while ab is not, then

$$(a*b)*c = (\theta a\theta b)*c = (\theta a\theta b)\theta c = \theta a(\theta b\theta c) = \theta a\theta(bc) = a*bc = a*(b*c).$$

Since the case bc undefined, ab defined is similar to the previous case, we consider the case where ab, bc are both undefined. In this case

$$(a*b)*c = (\theta a\theta b)*c = (\theta a\theta b)\theta c = \theta a(\theta b\theta c) = a*(\theta b\theta c) = a*(b*c).$$

Conversely, let $V = T^* \cup S$ be a semigroup extension of S by T where T is torsion. We define $\theta\colon V \to Q$ to be the natural mapping given as follows: for $v \in V$, since T is torsion $v^{-1}S = \{s \in S \mid vs \in S\} \in \Sigma$ so we define θv to be the γ-class of $g\colon v^{-1}S \to S$ given by $g(a) = va$. Clearly θ is a semigroup homomorphism of V into Q whose restriction to S is the identity. By abuse of notation, denote the restriction of θ to T^* by θ. Clearly, θ is a partial homomorphism satisfying $\theta a\,\theta b\in S$ if ab is undefined, and $s(\theta b)\in S$ $[(\theta b)s\in S]$ if $sb=0$ $[bs=0]$.

Moreover, if juxtaposition denotes the multiplication in V, then if $a, b\in T^*$, ab undefined, then $a*b=\theta a\,\theta b=\theta(ab)=ab$; if $as=0$, $a\in T^*$, $s\in S$, then $a*s==(\theta a)s=\theta(as)=as$; if $sa=0$, $a\in T^*$, $s\in S$, then $s*a=s\theta a=\theta(sa)=sa$; and otherwise, ab is defined in T^*, or $ab\in S$ in V, and in both cases $a*b=ab$.

2. Congruences on V. Let Σ be a right quotient filter on S, let S be strongly torsion free, and V be a semigroup containing S with $T=V/S$ a torsion partial (S, S)-algebra. To describe this situation we say that *V is a semigroup extension of S by torsion T.*

Definition. Let σ be a semigroup congruence on S and P be a (S, S)-subsystem of T with the following property:

(1) For each $p\in P^*$ there is $s\in S$ and $A\in\Sigma$ with the property that $pa\sigma sa$ for all $a\in A$.

In this case we say the p is *σ-linked* to s. (Note that T/P inherits a partial multiplication from T.)

Let τ be a 0-restricted multiplication preserving (S, S)-congruence on T/P satisfying

(2) if $x\tau y$, $s\sigma t$ and $xs, yt\in S$ then $xs\sigma yt$.

The relation $(\sigma, P, \tau)=\upsilon$ on $V=T^*\cup S$ is defined as follows:

for $x, y\in T\setminus P$, $x\upsilon y$ if and only if $x\tau y$;

for $x, y\in P^*$, $x\upsilon y$ if and only if there are $s, t\in S$ σ-linked to x and y (respectively) with $s\sigma t$;

for $x\in P^*$, $s\in S$, $x\upsilon s$ if and only if $s\upsilon x$ if and only if there is $t\in S$ σ-linked to x and $t\sigma s$; and

$\upsilon|_S=\sigma$.

A congruence σ on S is *strongly torsion free* if S/σ is a strongly torsion free semigroup with respect to the right quotient filter Σ/σ with base $\{\sigma^\#(A)|A\in\Sigma\}$ where $\sigma^\#$ is the canonical semigroup homomorphism from S to S/σ.

Lemma. *Σ/σ is a right quotient filter on S/σ.*

Proof. Property (i) is clear, while (ii) follows from elementary properties of the function $\sigma^\#$.

(iii) Let $B\in\Sigma/\sigma$ and $t=\sigma^\#(s)\in S/\sigma$. Let $A\in\Sigma$ with $\sigma^\#(A)\subseteq B$. Then $s^{-1}A\in\Sigma$ and $\sigma^\# s\sigma^\#(s^{-1}A)\subseteq B$ and $(\sigma^\# s)^{-1}B\in\Sigma/\sigma$.

(iv) Let $\sigma^{\#}(I)$ be a right ideal of S and $B\in\Sigma/\sigma$. Let $A\in\Sigma$ with $\sigma^{\#}(A)\subseteqq B$. Let $(\sigma^{\#}a)^{-1}\sigma^{\#}(I)\in\Sigma/\sigma$ for all $a\in A$. Then without loss of generality, $a^{-1}I\in\Sigma$ for all $a\in A$ so $I\in\Sigma$ and $\sigma^{\#}(I)\in\Sigma/\sigma$.

Theorem 1. *If σ is a strongly torsion free semigroup congruence on S, then $\upsilon=(\sigma, P, \tau)$ is a semigroup congruence on V whose restriction to S is strongly torsion free. Moreover, every semigroup congruence on V whose restriction to S is strongly torsion free is of this type.*

Proof. To show that υ is an equivalence relation it suffices to verify that for $p\in P$, $s, t\in S$, $p\upsilon s$ and $p\upsilon t$ imply that $s\sigma t$. However, $p\upsilon s$ and $p\upsilon t$ imply the existence of some $x\in S$ σ-linked to p with $x\sigma s$ and $x\sigma t$. Thus $s\sigma t$.

We next verify that υ is a left congruence. The "right" case is dual.

Case 1. Let $t, t'\in T\setminus P$, $c\in V$ and $t\upsilon t'$. Hence $t\tau t'$. If $ct\in T\setminus P$ then $ct'\in T\setminus P$ since τ is 0-restricted. Hence $ct\tau ct'$ or $ct\upsilon ct'$. Next let $ct\in P^*\cup S$, then $ct'\in P^*\cup S$. We consider several subcases.

(a) $ct, ct'\in P^*$: By (1), for some $x, y\in S$, $A\in\Sigma$ $ct\upsilon x$ and $ct'\upsilon y$ so that $cta\sigma xa$, $ct'a\sigma ya$ for all $a\in A$. By (2), $t\tau t'$ implies $cta\sigma ct'a$ which implies $x\sigma y$ since σ is strongly torsion free. Thus $ct\upsilon ct'$ by definition of υ.

(b) $ct\in S$, $ct'\in P^*$: For some $x\in S$ and $A\in\Sigma$, $ct'a\sigma xa$ for all $a\in A$. Hence by (2), $cta\sigma xa$ for all $a\in A$ so $ct\sigma x$ since σ is strongly torsion free. Hence $ct\upsilon ct'$.

(c) The other cases $ct\in P^*$, $ct'\in S$ and $ct, ct'\in S$ are treated similarly.

Case 2. Let $p, p'\in P^*$, $p\upsilon p'$ and $c\in V$. Then $p\upsilon s$, $p'\upsilon s$ for some $s\in S$ by the definition of υ. By (1), for some $A\in\Sigma$, $cpa\sigma csa$ for all $a\in A$. Similarly $cp'a\sigma csa$ for all $a\in A$. Again we consider several cases.

(a) $cp, cp'\in P^*$: Then $cp\upsilon x$ and $cp'\upsilon y$ for some $x, y\in S$. Thus $xa\sigma cpa\sigma cp'a\sigma ya$ for all $a\in A$ and so $x\sigma y$ and $cp\upsilon cp'$.

(b) $cp\in S, cp'\in P^*$: Then $cp'\upsilon x$ for some $x\in S$. Thus for all $a\in A$, $cpa\sigma cp'a\sigma xa$ so $cp\sigma x$ and $cp\upsilon cp'$.

The verification of the remaining cases is either similar to some case considered above or follows immediately from (1) or (2).

Conversely let μ be semigroup congruence on V whose restriction to S is strongly torsion free. Let $P=\{t\in T^*|t\mu s$ for some $s\in S\}\cup\{0\}$, then P is an (S, S)-subsystem of T. Let $\sigma=\mu|_S$ and define τ on T/P by:

$$t\tau t' \text{ if and only if } t\mu t' \quad (t, t'\in T\setminus P);\quad 0\tau 0.$$

Then clearly σ is a strongly torsion free semigroup congruence on S, every element of P^* is σ-linked to an element of S, and τ is a partial multiplication preserving 0-restricted (S, S)-congruence on T/P and conditions (1) and (2) are satisfied. Clearly $\mu\subseteqq\upsilon=(\sigma, P, \tau)$. To see the converse we need to consider two cases.

Case 1. $p\upsilon p'$; $p, p'\in P^*$: Then there are $s, s'\in S$ and $A\in\Sigma$ with $pa\sigma sa$, $p'a\sigma s'a$ for all $a\in A$, and $s\sigma s'$. On the other hand there are $x, x'\in S$ with $p\mu x$, $p'\mu x'$. Thus for all $a\in A$, $xa\sigma pa\sigma sa$ and $x'a\sigma p'a\sigma s'a$ so $x\sigma s\sigma s'\sigma x'$. Thus $p\mu s$, $p'\mu s'$ and $s\mu s'$ so $p\mu p'$.

Case 2. $p\upsilon s$; $p\in P^*$, $s\in S$: There is $x\in S$, $A\in\Sigma$ with $pa\sigma xa$ for all $a\in A$ and $x\sigma s$. However $p\mu x'$ for some $x'\in S$ so $x'a\sigma xa$ for all $a\in A$ or $x\sigma x'$. Thus $p\mu x'\mu x\mu s$ or $p\mu s$.

Corollary 2. *A relation μ on V is a semigroup congruence whose restriction to S is strongly torsion free if and only if μ is of the form (σ, P, τ) for some strongly torsion free semigroup congruence σ on S.*

If P is a nonzero (S, S)-subsystem of T such that $P^*\cup S$ is a strict extension of S (i.e. for all $p\in P^*\cup S$ there is some $s\in S$, $A\in\Sigma$ with $xa=sa$ for all $a\in A$ [4]), then P can be used in (σ, P, τ). In this case condition (1) is automatically satisfied but condition (2) must still hold.

Definition. A semigroup extension V of S by $T=V/S$ is determined by the *partial homomorphism* ω: $T^*\to S$ if (1) ω preserves the partial multiplication and the (S, S)-system multiplication on T, and (2) the multiplication of $a, b\in V$ is given by

$$a*b=\begin{cases}(\omega a)(\omega b) & \text{if } \; a, b\in T^*,\ ab \text{ undefined}\\ (\omega a)b & \text{if } \; a\in T^*,\ b\in S,\ ab=0\\ a(\omega b) & \text{if } \; b\in T^*,\ a\in S,\ ab=0\\ ab & \text{otherwise.}\end{cases}$$

Recall from [4] that if S is strongly torsion free, a semigroup extension V of S by torsion $T=V/S$ is strict if and only if V is determined by a partial homomorphism ω: $T^*\to S$.

When V is determined by a partial homomorphism, we have the following result:

Proposition 3. *Let V be an extension of S determined by a partial homomorphism ω: $T^*\to S$ where $T=V/S$ is torsion, σ be a strongly torsion free semigroup congruence on S, and P be any (S, S)-subsystem of T. Then there exists a multiplication preserving (S, S)-congruence τ on T/P for which $\upsilon=(\sigma, P, \tau)$ is a semigroup congruence on V. Moreover, condition* (2) *on σ and τ is equivalent to*

$$\omega t\sigma\omega t' \text{ if } t\upsilon t' \tag{3}$$

while condition (1) *holds automatically.*

Proof. Let τ be the identity congruence on T/P, then the first statement follows from the remarks preceding the statement of the proposition. If (2) holds, and $t\tau t'$, then for some $A\in\Sigma$ and all $a\in A$, $ta\sigma t'a$ hence $(\omega t)a\sigma(\omega t')a$ but since S is strongly torsion free, $\omega t\sigma\omega t'$. If (3) holds, then $t\tau t'$ and $x\sigma y$ implies $\omega t\sigma\omega t'$ from which $(\omega t)x\sigma(\omega t')y$ and so $tx\sigma t'y$ if tx, $t'y=0$, otherwise $tx\tau t'y$ since $tx\upsilon t'y$ (υ is a congruence) and tx, $t'y$ are both nonzero.

Remark. If T has no nontrivial (S, S)-subsystems then $P=\{0\}$ or $P=T$. Consequently for any semigroup congruence on V, either S is saturated by υ ($P=\{0\}$) or every υ-class intersecting T^* also intersects S; in these cases both conditions (1) and (2) are vacuous.

3. Homomorphic images of V. In this section we describe the homomorphic image of V induced by a congruence $\upsilon=(\sigma, P, \tau)$, where σ is strongly torsion free. Recall that for any semigroup congruence σ on S, $\sigma^\#$ denotes the natural mapping of S onto S/σ.

Theorem 1. *Let V be a semigroup extension of S by torsion $T=V/S$ determined by the partial homomorphism $\theta\colon T^*\to Q$. Let $\upsilon=(\sigma, P, \tau)$ where σ is strongly torsion free. Then υ is a semigroup congruence on V and one of the following two cases occurs:*

(i) $P=T$; *then* $V/\upsilon\cong S/\sigma$; *or*

(ii) $P\neq T$; *then V/υ is an extension of S/σ, by $(V/\upsilon)/(S/\sigma)\cong(T/P)/\tau$ determined by the partial homomorphism $\beta\colon ((T/P)/\tau)^*\to Q(S/\sigma)$ where β is defined by $\beta(\tau^\# t)=\langle\tau^\# t\rangle$, where $\langle\tau^\# t\rangle$ is the equivalence class in $Q(S/\sigma)$ of the mapping*

$$\lambda_{\tau^\# t}\colon \sigma^\#(t^{-1}S)\to S/\sigma \quad \textit{defined by} \quad \lambda_{\tau^\# t}(\sigma^\# a)=\sigma^\#(ta).$$

Proof. That υ is a congruence follows from Theorem 2.1. If $P=T$, the mapping $\varrho(\sigma^\# x)=\upsilon^\# x$ for all $x\in S$ is a semigroup isomorphism from S/σ onto V/υ.

Suppose $P\neq T$. Let $K=T/P$, $V'=V/\upsilon$, and $S'=S/\sigma$. V' is a semigroup extension of $(P^*\cup S)/\upsilon$ by K/υ (by an obvious abuse of notation). From the construction of (σ, P, τ) it is clear that $(P^*\cup S)/\upsilon\cong S/\upsilon\cong S/\sigma$, and $K/\upsilon\cong K/\tau$. Hence we may consider V' as an extension of S' by $K'=K/\tau$. Here S' is strongly torsion free so we may describe this extension by means of a partial homomorphism β defined above. Let $\circ$ be the multiplication in V, $*$ the multiplication in V', and denote the multiplication in T, K and S' by juxtaposition. It remains to show that $*$ satisfies the conditions of Theorem 1.1 in $V'=K'^*\cup S'$.

For any $a', b'\in K'^*$ $(a'=\upsilon^\# a=\tau^\# a)$,

$$a'*b'=(a\circ b)'=\begin{cases}(ab)' & \text{if } ab\in T\setminus P\\ s' & \text{if } ab\in P^*,\ ab\upsilon s\\ [\theta a\theta b]' & \text{if } ab \text{ is undefined}\end{cases}=\begin{cases}a'b' & \text{if } a'b'\neq 0\\ (\beta a')(\beta b') & \text{if } a'b' \text{ is undefined.}\end{cases}$$

If $a'\in S'$, $b'\in K'^*$ then

$$a'*b' = (a\circ b)' = \begin{cases}(ab)' & \text{if } ab\in T\setminus P\\ s' & \text{if } ab\in P^*,\ ab\upsilon s\\ (a\theta b)' & \text{if } ab=0\end{cases} = \begin{cases}a'b' & \text{if } a'b'\neq 0'\\ a'\beta b' & \text{if } a'b'=0'.\end{cases}$$

The case $a'\in K'^*$, $b'\in S'$ is similar to the above case and if $a', b'\in S'$ then $a'*b'= =(a\circ b)'=(ab)'=a'b'$.

Corollary 2. *Under the same hypothesis and notation as in the theorem, if V is also a strict extension of S and $P\neq T$, then V/υ is an extension of S/σ by $(T/P)/\tau$ determined by the partial homomorphism $\varrho\colon ((T/P)/\tau)^*\to S/\sigma$ defined by $\varrho(\tau^\# x)= =\sigma^\# s$ where for some $A\in\Sigma$, $xa=sa$ for all $a\in A$.*

4. Extensions determined by a partial homomorphism. Let V be a semigroup extension of S by torsion T determined by a partial homomorphism $\omega\colon T^*\to S$, σ be a semigroup congruence on S, P be an (S, S)-subsystem of T, τ be a 0-restricted partial multiplication preserving (S, S)-congruence on T/P, and suppose $\omega a\sigma\omega b$ if $a\tau b$ where $a, b\in P$. On V define the relation υ by

$$\begin{aligned}
&a, b\in T\setminus P: && a\upsilon b \text{ iff } a\tau b\\
&a, b\in P^*: && a\upsilon b \text{ iff } \omega a\sigma\omega b\\
&a\in P^*,\ b\in S: && a\upsilon b \text{ iff } b\upsilon a \text{ iff } \omega a\sigma b, \text{ and}\\
&a, b\in S: && a\upsilon b \text{ iff } a\sigma b.
\end{aligned}$$

We write $\upsilon=[\sigma, P, \tau]$.

Theorem 1. *The following statements hold:*

(i) $\upsilon=[\sigma, P, \tau]$ *is a semigroup congruence on* V;

(ii) *if σ is strongly torsion free then* $[\sigma, P, \tau]=(\sigma, P, \tau)$;

(iii) *every semigroup congruence μ on V whose restriction to S is strongly torsion free is of the form* $[\sigma, P, \tau]$;

(iv) *if $P=T$, then* $V/\upsilon\cong S/\sigma$;

(v) *if $P\neq T$ and σ is strongly torsion free then V/υ is an extension of S/σ by $(T/P)/\tau$ determined by the partial homomorphism ω' defined by*

$$\omega'(\tau^\# a) = \sigma^\#(\omega a),\quad a\in T\setminus P, \tag{4}$$

and

(vi) *condition* (3) *is equivalent to the existence of the function* $\omega'\colon ((T/P)/\tau)^*\to S/\sigma$ *satisfying* (4).

Proof. (i) υ is clearly reflexive and symmetric. Let $p\in P^*$ and $s, t\in S$ with $p\upsilon s$, $p\upsilon t$. Then $\omega p\sigma s$, $\omega p\sigma t$ and $s\sigma t$ or $s\upsilon t$. Let $a, b, c\in T^*\cup S$ with $a, b, c\in T^*$, $a\upsilon b$. If $a, b, c\in T^*$ with $ac\in T^*$ then $bc\in T^*$ and $ac\upsilon bc$. If $ac\in P^*$ and $bc\in P^*$ then since $\omega a\sigma\omega b$, $\omega(ac)=\omega a\omega c\sigma\omega b\omega c=\omega(bc)$ or $ac\upsilon bc$. If $ac\in P^*$, $bc\in S$ then $\omega a\sigma\omega b\Rightarrow\omega(ac)\sigma\omega b\omega c\Rightarrow\omega(ac)\sigma bc$. The other cases are either obvious, or follow easily by arguments similar to the above.

(ii) Since in the definition of (σ, P, τ), for any $p\in P^*$, $\omega p\in S$ is σ-linked to p, $[\sigma, P, \tau]\subseteq(\sigma, P, \tau)$. Conversely suppose $p_1, p_2\in P^*$ and $p_1(\sigma, P, \tau)p_2$. Then p_i is σ-linked to $s_i\in S$ $(i=1, 2)$ by $A\in\Sigma$ and $s_1\sigma s_2$. Hence $(\omega p_1)a\sigma s_1 a\sigma s_2 a\sigma(\omega p_2)a$ for all $a\in A$ and since $A\in\Sigma$ and σ is strongly torsion free, $\omega p_1\sigma\omega p_2$ or $p_1[\sigma, P, \tau]p_2$. The cases $p_1\in P^*$, $p_2\in S$ and $p_1\in S$, $p_2\in P^*$ are obtained by similar arguments.

(iii) This follows from ii) and Theorem 2.1.

(iv) This is obvious.

(v) Using the notation in the proof of Theorem 3.1, for $t_1, t_2\in T\setminus P$ we obtain:

$$t_1' = t_2' \Rightarrow t_1\tau t_2 \Rightarrow \omega t_1\sigma\omega t_2 \Rightarrow (\omega t_1)' = (\omega t_2)' \Rightarrow \omega' t_1' = \omega' t_2$$

and so ω' is single-valued. If $t_1't_2'$ is defined, then $t_1t_2\in T\setminus P$ and

$$\omega'(t_1't_2') = \omega'(t_1t_2)' = [\omega(t_1t_2)]' = (\omega t_1\omega t_2)' = (\omega t_1)'(\omega t_2)' = (\omega' t_1')(\omega' t_2')$$

and ω' is a partial homomorphism. If $t_1't_2'$ is undefined and $t_1, t_2\in T\setminus P$ then

$$t_1' * t_2' = (t_1\circ t_2)' = \begin{cases}[\omega(t_1t_2)]' & \text{if } t_1t_2 \text{ is defined}\\ ((\omega t_1)(\omega t_2))' & \text{if } t_1t_2 \text{ is undefined}\end{cases}$$
$$= [(\omega t_1)(\omega t_2)]' = (\omega t_1)'(\omega t_2)' = (\omega' t_1')(\omega' t_2').$$

If $t_1\in T\setminus P$ and $s\in S$, we have

$$t_1' * s' = (t_1\circ s)' = ((\omega t_1)s)' = (\omega t_1)'s' = (\omega' t_1')s'$$

and dually $s' * t_1' = s'(\omega' t_1')$.

(vi) By (v), (3) implies the existence of ω' satisfying (4). Conversely if (4) holds, then

$$t_1\tau t_2 \Rightarrow t_1' = t_2' \Rightarrow \omega' t_1' = \omega' t_2' \Rightarrow (\omega t_1)' = (\omega t_2)' \Rightarrow \omega t_1\sigma\omega t_2$$

and (3) holds.

Condition (4) can be expressed by saying that the following diagram commutes:

$$\begin{tikzcd}
(T/P)^* \arrow[r, "\omega^*"] \arrow[d, "\tau'"'] & S \arrow[d, "\sigma^\#"] \\
((T/P)/\tau)^* \arrow[r, "\omega'"] & S/\sigma
\end{tikzcd}$$

where $\omega^* = \omega|_{T\setminus P}$ and $\tau' = \tau^\#|_{T\setminus P}$.

Comparing Theorem 4.1 with Theorem 3.1, we see that condition (3) in the definition of $\upsilon=[\sigma, P, \tau]$ implies that υ is a semigroup congruence on V, while in Theorem 3.1, we had to suppose that σ is strongly torsion free to prove that (σ, P, τ) is a semigroup congruence on V. In Theorem 4.1, we obtain all semigroup congruences μ on V whose restriction to S is strongly torsion free; if $\mu|_S$ is not strongly torsion free, condition (3) need not hold.

Corollary 2. *Let T be a zero (left zero, right zero) (S, S)-system, then all semigroup congruences $\upsilon=[\sigma, P, \tau]$ on V can be constructed as follows: let σ be a semigroup congruence on S, and on T^* define σ' by*

$$t_1\sigma' t_2 \Leftrightarrow \omega t_1 \sigma \omega t_2. \tag{5}$$

Let P be any (S, S)-subsystem of T, τ be a 0-restricted multiplication preserving equivalence relation (right S-congruence, left S-congruence) on T/P for which $\tau|_{T\setminus P}\subseteqq\sigma'|_{T\setminus P}$. Then (3) holds and $[\sigma, P, \tau]$ is a semigroup congruence on V. Conversely any semigroup congruence $[\sigma, P, \tau]$ on V can be constructed in this fashion. In particular, we obtain all semigroup congruences on V whose restriction to S is strongly torsion free.

Proof. On zero (left zero, right zero) (S, S)-systems all 0-restricted multiplication preserving equivalence relations (right S-congruences, left S-congruences) are (S, S)-congruences. From (5) and $\tau|_{T\setminus P}\subseteqq\sigma'|_{T\setminus P}$ it follows that (3) holds. Hence $[\sigma, P, \tau]$ is a semigroup congruence on V by Theorem 4.1.

Conversely, if $[\sigma, P, \tau]$ is a semigroup congruence on V, then (3) is satisfied and so $\tau|_{T\setminus P}\subseteqq\sigma'|_{T\setminus P}$.

The last statement of the corollary follows from part (iii) of Theorem 4.1.

When T is a zero (S, S)-system, every subset of T containing 0 is an (S, S)-subsystem, while (S, S)-subsystems of the other two types are 0-simple. Thus it is possible to characterize in a simple way a large class of semigroup congruences on V when T is of one of these types of (S, S)-system. Moreover, the extension is determined by a partial homomorphism.

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On maximal clones of co-operations

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In this paper we determine all maximal clones of co-operations on a finite set, presenting a completeness criterion for co-operations in the spirit of Rosenberg's completeness theorem for operations on a finite set (cf. [3]). The result has some consequences for the theory of selective operations [2], too.

Our terminology is based on [1]. Here we present a short summary of the notions we use in this paper. For shortness, the set $\{0, 1, \ldots, l-1\}$ will be denoted by $\mathbf{l}$ for every natural number l. Let A stand for the finite set $\mathbf{n}$ for $n>1$ and let $m>0$ be an integer. An *m-ary co-operation f on A* is a mapping of A into the union of m disjoint copies of A which can be given by and hence identified with a pair of mappings $\langle f_0, f_1\rangle$, where f_0: $A\to\mathbf{m}$ is called the *labelling* and f_1: $A\to A$ is called the *mapping* of f. The *i-th m-ary coprojection* $p^{m,i}$ (a special kind of co-operation) is defined by $p_0^{m,i}(a)=i$ and $p_1^{m,i}(a)=a$ for each $a\in A$ ($i\in\mathbf{m}$). The set of all co-operations and that of all m-ary co-operations on A are denoted by $\mathscr{C}_A$ and $\mathscr{C}_A^m$, respectively. The *variables* of the co-operation $f=\langle f_0, f_1\rangle\in\mathscr{C}_A^m$ are the disjoint copies of A where f maps to, indexed by the elements of $\mathbf{m}$. The i-th copy of A is an *essential variable* of f if its intersection with the range of f is nonempty, i.e. $f_0(x)=i$ for some $x\in A$. The co-operation f is called *essentially k-ary* if $|f_0(A)|=k$. Omitting all non-essential variables of f, we obtain a k-ary co-operation f_e, called the *skeleton* of f. We call a co-operation *essential* if it is injective and essentially at least binary.

Let $f\in\mathscr{C}_A^m$ and $g^{(0)}, g^{(1)}, \ldots, g^{(m-1)}\in\mathscr{C}_A$. The *superposition* $h:=f(g^{(0)}, g^{(1)}, \ldots, g^{(m-1)})$ of f with $g^{(0)}, g^{(1)}, \ldots, g^{(m-1)}$ is the co-operation determined by the equalities $h_0(a)=g_0^{(f_0(a))}(f_1(a))$ and $h_1(a)=g_1^{(f_0(a))}(f_1(a))$ for each $a\in A$. The co-operation f is called the *main component* in this superposition. A set of co-operations on A is called a *clone* if it contains all coprojections and is closed under superposition. The least clone containing a set C of co-operations is called *the clone generated by C* and denoted by $[C]$. C is *complete* if $[C]$ equals $\mathscr{C}_A$. (A co-operation f is called *Sheffer*

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if $\{f\}$ is complete.) The mappings of the set C generate a semigroup $\mathcal{S}(C)$ of selfmaps of A called the *semigroup of* C. We call C *transitive* if $\mathcal{S}(C)$ is transitive. Note that $\mathcal{S}(C)\subseteq\mathcal{S}[C]$.

We remark that the lattice of clones of co-operation on A is finite. This fact can be shown in an easy way using the following remarks:

(1) The relation $\approx$ on $\mathcal{C}_A$ defined for $f, g\in\mathcal{C}_A$ by $f\approx g$ if both of the skeletons of f and g are k-ary and $g_e=f_e(p^{k,0\pi}, p^{k,1\pi}, \dots, p^{k,(k-1)\pi})$ for some permutation π of $\mathbf{k}$ is an equivalence relation with finitely many blocks. (Note that each block of the partition associated with $\approx$ can be represented by an at most $|A|$-ary co-operation and the number of these co-operations is finite.)

(2) Every subclone of $\mathcal{C}_A$ is a union of some blocks of the equivalence $\approx$ defined above. (It is trivial noting that for a clone C from $f\in C$ it follows $g\in C$ for each $g\in\mathcal{C}_A$ with $f\approx g$.)

A *maximal clone* of co-operations on A is a proper subclone C of $\mathcal{C}_A$ such that $C\subset D\subset\mathcal{C}_A$ for no clone D. Similarly to the case of algebras, a pair $\langle A, F\rangle$ with a nonempty set A and $F\subseteq\mathcal{C}_A$ is called a *coalgebra*. We say that $\langle A, F\rangle$ is a *finite coalgebra* if A is finite. $\langle A, F\rangle$ is called *primal* is F is complete. A co-operation $f\in\mathcal{C}_A$ is said to be *constant* if both f_0 and f_1 are constants. The coalgebra $\langle A, F\rangle$ is *functionally complete* if the union of F with the set of constant co-operations on A is complete.

There is a close connection between co-operations and regular selective operations, as follows. Let P and M be nonempty sets, let k be a natural number and let $f_0\colon P\to\mathbf{k}$ and $f_1\colon P\to P$. The k-ary operation f on M^P is called a *regular selective operation* if for every $p\in P$ the p-component of the result of f is the f_1-component of the f_0-th operand. Observe that the mappings f_0 and f_1 can be considered as the labelling and the mapping of a k-ary co-operation on P. Moreover, for any nontrivial M and nonempty P this natural correspondence yields a bijection between the regular selective operations on M^P and the co-operations on P. This bijection is a clone isomorphism. Hence the lattice of clones of regular selective operations on a finite power of a set is isomorphic to the lattice of clones of co-operations on a finite set and our criterion for the maximality of a clone of co-operations provides a description of all maximal proper subclones of the clone of all regular selective operations on a set M^P with P finite (cf. [1], [2]).

Consider a nonempty subset T of A. We say that a co-operation $f\in\mathcal{C}_A$ *preserves* T if T is closed under the mapping f_1. Let π be a partition of A. f *preserves* π if the labelling f_0 is constant on each block of π and f_1 preserves π in the usual sense (i.e. $f_1(a)\equiv_\pi f_1(b)$ holds for every $a, b\in A$ with $a\equiv_\pi b$, where $\equiv_\pi$ is the equivalence associated with π).

We call a co-operation $f\in\mathcal{C}_A$ (x, y)-*gluing* for some distinct $x, y\in A$ if $f(x)=$ $=f(y)$ (i.e. $f_i(x)=f_i(y)$ for $i\in\mathbf{2}$). Note that an arbitrary superposition with an

(x, y)-gluing main component is also (x, y)-gluing. We say that f *glues* in $T \subseteq A$ if f is (x, y)-gluing for some $x, y \in T$. We write $f \| T$ for "f does not glue in T" (i.e., $f|_T$ is injective on T). Let M be a family of subsets of A. M is called *disjoint* if its members are pairwise disjoint and called *uniform* if all its members have the same cardinality. M is *regular* if it is nonempty, disjoint, uniform and distinct from $A^* := \{\{a\}: a \in A\}$. The set of regular families of subsets of A will be denoted by $\mathrm{Rf}(A)$. The family M determines the following relation $\sim_M$ on A: $x \sim_M y$ if $x, y \in S$ for some $S \in M$. If M is disjoint, then $\sim_M$ is an equivalence on the set $\bigcup M := \bigcup_{S \in M} S$. We remark that every member of $\mathrm{Rf}(A)$ can also be considered as a partial equivalence on A.

Let $M \in \mathrm{Rf}(A)$ and $S \in M$ be arbitrary. The co-operation f *preserves S in M* if f_0 is constant on S and f_1 maps S into a member of M, i.e. $f_0(x) = f_0(y)$ and $f_1(x) \sim_M f_1(y)$ for all $x, y \in S$. (Note that the property "f preserves S in M" is not equivalent to the simple property "f preserves S" even in the case of M singleton!) Further, (i) f *weakly preserves S in M* if either f preserves S in M or f glues in S, (ii) *f (weakly) preserves M* if f (weakly) preserves each $S \in M$ in M, and (iii) a subset C of $\mathscr{C}_A$ *(weakly) preserves M* if each $f \in C$ (weakly) preserves M. Denote by C_M the set of co-operations weakly preserving M.

Let $f \in \mathscr{C}_A^m$, $T \subseteq A$ and $|f_0(T)| = k$. We put $\mathrm{ess}_T(f) := k$ and $\mathrm{ess}(f) := \mathrm{ess}_A(f)$. Let $g^{(0)}, g^{(1)}, \ldots, g^{(m-1)} \in \mathscr{C}_A$. The superposition $h = f(g^{(0)}, g^{(1)}, \ldots, g^{(m-1)})$ is called *disjoint* if the ranges of $g_0^{(0)}, g_0^{(1)}, \ldots, g_0^{(m-1)}$ are pairwise disjoint. The following fact is obvious:

Lemma 1. *Let $h = f(g^{(0)}, g^{(1)}, \ldots, g^{(m-1)})$ be a disjoint superposition, let $T \subseteq A$ and for $i \in \mathbf{m}$ put $T_i := T \cap f_0^{-1}(i) = \{x \in T: f_0(x) = i\}$. If $f \| T$ and $g^{(i)} \| f_1(T_i)$ (in particular, if $g^{(i)}$ is non-gluing) for each $i \in m$, then $h \| T$ and* $\mathrm{ess}(h) \geqq \mathrm{ess}(f)$.

A disjoint superposition of form

$$h = f(p^{k,0}, \ldots, p^{k,j-1}, g(p^{k,j}, \ldots, p^{k,j+m'-1}), p^{k,j+m'}, \ldots, p^{k,k-1})$$

will be denoted shortly by $h = f(\ldots, g, \ldots)_j$. Here $h \in \mathscr{C}_A^k$ where $k = m + m' - 1$ for $f \in \mathscr{C}_A^m$ and $g \in \mathscr{C}_A^{m'}$. Obviously we have:

Lemma 2. *Let $T \subseteq A$, $f \in \mathscr{C}_A^m$ and $g \in \mathscr{C}_A^{m'}$. If both f and g preserve T then $f(\ldots, g, \ldots)_j$ preserves T.*

We shall also use the following trivial facts:

Lemma 3. *Let T and T' be proper distinct subsets of A and let $C = \{f \in \mathscr{C}_A: f \text{ preserves } T\}$. Then there is an $f \in C$ not preserving T'.*

Lemma 4. *Let C_1 be a set of selfmaps of A. The semigroup generated by C_1 is transitive if and only if no non-trivial subset of A is preserved by C_1.*

We need some other preparations, as follows:

Lemma 5. *For arbitrary* $M \in \mathrm{Rf}(A)$ *the set* C_M *is a proper subclone of* $\mathscr{C}_A$.

Proof. First observe that $C_M \neq \mathscr{C}_A$ (indeed, there is some $f \in \mathscr{C}_A$ not preserving weakly M). We show that C_M is a clone. Clearly the coprojections preserve M and so it is enough to show the closedness under superposition, i.e. to prove that $h = f(g^{(0)}, g^{(1)}, \ldots, g^{(m-1)}) \in C_M$ for arbitrary m-ary $f \in C_M$ and $g^{(0)}, g^{(1)}, \ldots, g^{(m-1)} \in C_M$.

In order to do so consider a subset $S \in M$. The definition of C_M implies that either f glues in S or f preserves S in M. If $f(x) = f(y)$ for two distinct $x, y \in S$ then $h(x) = g^{(f_0(x))}(f_1(x)) = g^{(f_0(y))}(f_1(y)) = h(y)$, i.e. h glues in S too. Thus assume $f \| S$. Then $f_0(S) = i$ for some $i \in \mathbf{m}$, f_1 is injective on S and $f_1(S) \subseteq S'$ for some $S' \in M$. However, $|S'| = |S|$, whence f_1 maps S bijectively onto S'. If $g^{(i)}$ glues in S', i.e. $g^{(i)}(u) = g^{(i)}(v)$ for two distinct $u, v \in S'$ then (as f maps S onto S') $f_1(x) = u$ and $f_1(y) = v$ for some $x, y \in S$ and so $h(x) = g^{(f_0(x))}(f_1(x)) = g^{(i)}(u) = g^{(i)}(v) = g^{(f_0(y))}(f_1(y)) = h(y)$, i.e. h glues in S too. Thus assume $g^{(i)} \| S'$. Then $g^{(i)}$ preserves S' in M, i.e. $g_0^{(i)}$ is constant on S' and $g_1^{(i)}$ maps S' onto some $S'' \in M$. Since for all $x \in S$, $h_0(x) = g_0^{(i)}(f_1(x))$, we see that h_0 is constant on S and, similarly, $h_1(x) = g_1^{(i)}(f_1(x))$ for all $x \in S$ shows $h_1(S) \subseteq S''$, i.e. h preserves S in M. Therefore, h weakly preserves S in M.

Lemma 6. *Let* $M \in \mathrm{Rf}(A)$ *and suppose that the common cardinality of the members of* M *equals* $k > 1$. *Consider the* m*-ary co-operation* $f \in \mathscr{C}_A \setminus C_M$ *and put* $D := [C_M \cup \{f\}]$. *Let* S *be an arbitrary member of* M *which is not weakly preserved by* f. *Then for every* $\{u, v\} \subseteq S$ *there is a co-operation* $f^* \in D$ *such that* f^* *preserves* S, $f^* \| S$ *and* $f_0^*(u) \neq f_0^*(v)$.

Proof. It will be done in several steps.

Claim 0. For every permutation $\bar{h}$ of S there exists a unary co-operation $h' \in C_M$ preserving the set S, such that h_1' extends $\bar{h}$.

Indeed, put $h_0'(x) = 0$ for all $x \in A$, $h_1'(x) = \bar{h}(x)$ for $x \in S$ and $h_1'(x) = x$ on $A \setminus S$. Then h' obviously preserves M.

Claim 1. There are $\{x, y\} \subseteq S$ and $f' \in D$ such that $f' \| S$ and $f_0'(x) \neq f_0'(y)$.

Indeed, from the choice of S it follows $f \| S$. Furthermore, clearly it suffices to consider the case of f_0 constant on S, i.e. $f_0(S) = j$ for $j \in \mathbf{m}$ and $f_1(x) \not\sim_M f_1(y)$ for some $x, y \in S$. Consider the co-operation h defined as follows:

Suppose $M = \{S_0, S_1, \ldots, S_{q-1}\}$ and $A \setminus \bigcup M = \{w_0, w_1, \ldots, w_{r-1}\}$ where $0 \leqq r \leqq n - qk \leqq n - k$. Let $h \in C_M$ from $\mathscr{C}_A^{q+k}$ defined by

$$(*) \qquad h_0(x) = \begin{cases} i & \text{if } x \in S_i \quad (i \in q) \\ q+j & \text{if } x = w_j \quad (j \in r) \end{cases} \quad \text{and} \quad h_1 = \mathrm{id}_A.$$

Obviously $h_0(f_1(x)) \neq h_0(f_1(y))$. Put $f' = f(\dots, h, \dots)_j$. According to Lemma 1 $f' \| S$ and $f'_0(x) = h_0(f_1(x)) \neq h_0(f_1(y)) = f'_0(y)$.

Claim 2. There are $x, y \in S$ and $f'' \in D$ such that f'' preserves the set S, $f'' \| S$ and $f''_0(x) \neq f''_0(y)$.

Indeed, consider x, y and f' from Claim 1. Suppose $f' \in \mathscr{C}_A^{m'}$, put $J := f'_0(S)$ and, for each $j \in J$, put $R_j := \{u \in A : f'_0(z) = j \text{ and } f'_1(z) = u \text{ for some } z \in S\}$. Further, for each $j \in J$ let $h^{(j)} \in C_M$ be a unary co-operation such that $h_0^j(x) = j$ for all $x \in R_j$, $h^{(j)} \| R_j$ and $h_1^{(j)}(A) \subseteq S$. Such an $h^{(j)}$ exists, because $|R_j| \leqq |S|$. Form the disjoint superposition $f'' = f'(g^{(0)}, g^{(1)}, \dots, g^{(m'-1)})$, where $g^{(j)} = h^{(j)}(p^{m',j})$ for $j \in J$ and $g^{(j)} = p^{m',j}$ otherwise. Lemma 1 implies $f'' \| S$. As $h^{(j)}$ preserves S for each $j \in J$, from the definition of R_j it follows that f'' also preserves S. Furthermore, $f''_0 = f'_0$, hence $f''_0(x) \neq f''_0(y)$ holds too.

To prove the assertion of the lemma consider two arbitrary distinct elements $u, v \in S$. Let x, y and f'' satisfy Claim 2. As $x \neq y$, Claim 0 implies that there exists a unary $h' \in C_M$ with $h'_1(u) = x$ and $h'_1(v) = y$. Put $f^* := h'(f'')$. Since it is a disjoint superposition, f^* preserves S and $f^* \| S$ by virtue of Lemmas 1 and 2. Furthermore, $f_0^*(u) = f''_0(x) \neq f''_0(y) = f_0^*(v)$, as needed.

Lemma 7. *Let D and S be the same as in Lemma 6. For every $i = 1, 2, \dots, k$ there are an i-element subset H of S and $g \in D$ such that g preserves the set S, $g \| S$ and g_0 is injective on H.*

Proof. We proceed by induction on $i = 1, 2, \dots, k$. The assertion is trivial for $i = 1$.

Let $1 \leqq i < k$. Assume the statement is valid for H_i and $g^{(i)} \in \mathscr{C}_A^{m_i}$. Choose an arbitrary element $x \in S \setminus H_i$ and let $H_{i+1} := H_i \cup \{x\}$. If $g_0^{(i)}$ is injective on H_{i+1}, we can put $g^{(i+1)} := g^{(i)}$.

Assume $g_0^{(i)}(y) = g_0^{(i)}(x) = j (\in \mathbf{m}_i)$ for some $y \in H_i$. As $g^{(i)} \| S$, the elements $u = g_1^{(i)}(x)$ and $v = g_1^{(i)}(y)$ are distinct. Hence by Lemma 6 there exists an m^*-ary co-operation $f^* \in D$ such that f^* preserves the set S, $f^* \| S$ and $f^*(u) \neq f^*(v)$. Now put $g^{(i+1)} = g^{(i)}(\dots, f^*, \dots)_j$, where $g^{(i+1)} \in \mathscr{C}_A^{m_{i+1}}$ for $m_{i+1} = m_i + m^* - 1$. Lemma 1 and 2 imply that $g^{(i+1)}$ preserves S and $g^{(i+1)} \| S$. The definition of $g^{(i+1)}$ yields that $g_0^{(i+1)}(x) = f_0^*(u) \neq f_0^*(v) = g_0^{(i+1)}(y)$. As $g^{(i+1)}$ is a disjoint superposition and, for $z_1, z_2 \in H_i$, $g_0^{(i)}(z_1) \neq g_0^{(i)}(z_2)$ implies $g_0^{(i+1)}(z_1) \neq g_0^{(i+1)}(z_2)$, we conclude that $g_0^{(i+1)}$ is injective on H_{i+1} and the lemma is proved.

Corollary 8. *Let the conditions of Lemma 6 be satisfied. Then there exists a co-operation $g \in D$ such that g_0 is injective on S.*

The promised Rosenberg-type criterion for completeness of sets of co-operations is the following.

Theorem. *A set C of co-operations on a finite set A is complete if and only if no regular family of subsets of A is weakly preserved by C.*

Proof. We shall prove the following claim, which is equivalent to the theorem: A set $C \subseteqq \mathscr{C}_A$ is a maximal clone if and only if $C = C_M$ for some $M \in \mathrm{Rf}(A)$.

1. Sufficiency. Let $M \in \mathrm{Rf}(A)$. In accordance with Lemma 5, C_M is a proper subclone of $\mathscr{C}_A$. We verify that C_M is maximal by showing that for arbitrary $f \in \mathscr{C}_A \setminus C_M$ the clone $D := [C_M \cup \{f\}]$ equals $\mathscr{C}_A$. This will be done in two parts.

(i) Suppose that $M \neq A^*$ consists of singletons. Put $\overline{M} := \bigcup M$. Then $h \in \mathscr{C}_A$ weakly preserves M iff it preserves $\overline{M}$. If H is a proper subset of A distinct from $\overline{M}$, then in accordance with Lemma 3 there is a $g \in C_M$ not preserving H. Clearly f does not preserve $\overline{M}$, thus $C_M \cup \{f\}$ preserves no proper subset of A. Then $C_M \cup \{f\}$ is transitive as a consequence of Lemma 4. Further, C_M obviously contains an essentially n-ary co-operation and thus applying Proposition 2 from [1] we obtain that $C_M \cup \{f\}$ is complete, as required.

(ii) Now suppose that the common cardinality of the members of M equals $k > 1$. Then C_M is transitive as C_M contains all the constants in $\mathscr{C}_A$ (as each of them glues in every $S \in M$). We shall construct an essentially n-ary co-operation in D. Let S be an arbitrary member of M being not weakly preserved by f (there is such an S as $f \notin C_M$), and let $\bar{f}$ be a selfmap of A, which maps each member of M bijectively onto S. Consider the unary co-operation $\tilde{f}$ with mapping $\tilde{f}_1$, equal to $\bar{f}$ on $\bigcup M$ and to the identity map otherwise. Clearly $\tilde{f} \in C_M$. Take the co-operation h defined by $(*)$ and the co-operation g from Corollary 8. Form the disjoint superposition

$$g^* := h(\tilde{f}(g(p^{n,0}, p^{n,1}, \ldots, p^{n,k-1})), \ldots$$
$$\ldots, \tilde{f}(g(p^{n,(q-1)k}, p^{n,(q-1)k+1}, \ldots, p^{n,qk-1})), p^{n,qk}, p^{n,qk+1}, \ldots, p^{n,qk+r-1}) \in C_M,$$

where q and r are the same as in $(*)$. From the properties of $h, \tilde{f}$ and g it follows $\mathrm{ess}_{S'}(g^*) = |S'| = k$ for each $S' \in M$. Also we see that $\mathrm{ess}_{A \setminus \cup M}(g^*) = \mathrm{ess}_{A \setminus \cup M}(h) = |A \setminus \bigcup M| = r$. As g^* is a disjoint superposition, its essential arity can be obtained additively: $\mathrm{ess}\,(g^*) = \mathrm{ess}_{(\cup M) \cup (A \setminus \cup M)}(g^*) = \sum_{S' \in M} \mathrm{ess}_{S'}(g^*) + \mathrm{ess}_{A \setminus \cup M}(g^*) = \sum_{S' \in M} |S'| + |A \setminus \bigcup M| = kq + r = n$. This completes the proof of the sufficiency.

Remark. For $M = \{A\}$ the clone C_M is called the Słupecki clone of co-operations on A. It consists of all non-essential co-operations. We see that it is a maximal clone, which occurs in the coalgebraic counterpart of Słupecki's completeness criterion for operations (Proposition 4 in [1]).

2. Necessity. Consider an arbitrary maximal clone C in $\mathscr{C}_A$. We verify that there exists a family $M \in \mathrm{Rf}(A)$ weakly preserved by C. This is enough, since then $C \subseteqq C_M \subset \mathscr{C}_A$ from Lemma 5 and thus C has to equal the clone C_M.

(i) If C is not transitive, then in virtue of Lemma 4 there is a nonempty subset $T \subset A$ preserved by C. However, then $M := \{\{a\} \subset A: a \in T\} \in \mathrm{Rf}(A)$ is preserved by C too.

(ii) Assume in the sequel that C is transitive. Observe that the clone of all gluing co-operations on A is a proper subset of the Słupecki clone on A. Thus C being maximal, it contains a non-gluing co-operation, for else C would be complete according to Proposition 2 in [1].

Consider an (m-ary) non-gluing co-operation $f \in C$ with maximal essential arity for the set of non-gluing co-operations of C. Denote by π the partition of A induced by f_0 and let M_1 be the set of blocks of π with maximal number of elements. The members of M_1 are not singletons, else π would be trivial and hence f essentially n-ary. It follows that $M_1 \in \mathrm{Rf}(A)$.

Claim 0. For arbitrary $T \in M_1$, the restriction of f_1 to T is a bijection from T onto some $T' \in M_1$.

Let $j := f_0(T) (\in \mathrm{m})$ and put $f' := f(\dots, f, \dots)_j \in C$. Obviously, for any $z \in A$, $f_1'(z)$ equals $f_1(f_1(z))$ if $f_0(z) = j$ and $f_1(z)$ otherwise. Lemma 1 implies $f' \| A$ and $\mathrm{ess}(f') \geqq \mathrm{ess}(f)$. It is easy to realize that $\mathrm{ess}(f') > \mathrm{ess}(f)$ iff there are $x, y \in A$ such that $f_0(x) = f_0(y) = j$ and $f_0(f_1(x)) \neq f_0(f_1(y))$, i.e. $f_1(x) \equiv_\pi f_1(y)$ does not hold for some $x, y \in T$. Then it follows from the choice of f that $f_1(x) \equiv_\pi f_1(y)$ for each $x, y \in T$. Further, f is injective, thus f_1 is $1-1$ on T, whence $|f_1(T)| = |T|$. Then $T' := f_1(T) \in M_1$, as needed.

Put the set $M_2 := \{T \in M_1: f_1(\bigcup M_1) \cap T \neq \emptyset\}$ and let $M := \{S \in M_1$: there is $g \in C$ and $S' \in M_2$ such that the restriction of g_1 to S' is a bijection from S' onto $S\}$.

Due to Claim 0, M_2 is nonempty. On the other hand, $M_2 \subseteq M$; thus M is also nonempty and $M \in \mathrm{Rf}(A)$.

We show that M is weakly preserved by C. This property will be obtained as a result of two claims. Let $S \in M$ be arbitrary and let $g \in C$ and $S' \in M_2$ be associated with S in the definition of M. Note that g can be chosen to be unary. Now Claim 0 guarantees that a suitable restriction of f_1 is a bijection onto S' from some $S'' \in M_1$. Let $k := f_0(S'')$.

Claim 1. If $h \in C$ and $h \| S$, then h_0 is constant on S.

Indeed, put $f^* := f(\dots, g(h), \dots)_k \in C$. Then, for arbitrary $z \in A$, $f_1^*(z)$ equals $h_1(g_1(f_1(z)))$ if $f_0(z) = k$ and $f_1(z)$ otherwise. From Lemma 1 it follows $f^* \| A$ and $\mathrm{ess}(f^*) \geqq \mathrm{ess}(f)$. Similarly to the discussion of f' above, $\mathrm{ess}(f^*) > \mathrm{ess}(f)$ iff $h_0(g_1(f_1(x))) = h_0(g_1(f_1(y)))$ does not hold for some $x, y \in S''$. As f_1 and g_1 are $1-1$ when restricted to S'' resp. S', this condition is equivalent to $h_0(u) \neq h_0(v)$ for some $u, v \in S$. However, the choice of f implies that this condition does not hold, as asserted.

Claim 2. If $h\in C$ and $h\|S$ then the restriction of h_1 to S is a bijection from S onto some $S_0\in M$.

Indeed, assume $h\in\mathscr{C}_A^{m_0}$ and let $k_0:=h_0(S)\in\mathbf{m}_0$. Put $h':=h(\dots,f,\dots)_{k_0}\in C$. Obviously $h_0'(z)=f_0(h_1(z))+k_0$ and $h_1'(z)=f_1(h_1(z))$ for $z\in S$. Lemma 1 implies $h'\|S$, thus it follows from Claim 1 that h_0' is constant on S, whence for each $x, y\in S$ we have $f_0(h_1(x))=f_0(h_1(y))$, i.e. $h_1(x)\equiv_\pi h_1(y)$. Note that h_1 is injective on S, since $h\|S$ and h_0 is constant on S. Then, as S is a block of maximal size in π, the restriction of h_1 to S is a bijection from S onto some $S_0\in M_1$. Now consider $S'\in M_2$. The restriction of the mapping of the co-operation $g^*:=g(h)\in C$ to S' is the product of the bijections $g_1|_{S'}$ and $h_1|_S$, hence $g_1^*|_{S'}$ is a bijection from S' to S_0. Thus $S_0\in M$, as required.

This completes the proof of the theorem.

We list some easy consequences of the Theorem (we omit their proofs).

Corollary 9. (Proposition 3 in [1].) *A co-operation on* $\mathbf{n}$ *is Sheffer if and only if it preserves neither non-least partitions nor nonempty proper subsets of* $\mathbf{n}$.

Corollary 10. *A finite coalgebra* $\langle A, F\rangle$ *is*

(i) *primal if and only if no regular family of subsets of A is weakly preserved by F;*

(ii) *functionally complete if and only if no regular family of nonsingleton subsets of A is weakly preserved by F.*

Corollary 11. *No distinct maximal clones of co-operations on a finite set have the same semigroups.*

The last corollary is the coalgebraic counterpart of the well-known fact that maximal clones of operations on a finite set are uniquely determined by the (semigroup of) unary operations they contain.

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Über multiplizitätenfreie Permutationscharaktere

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1. Einleitung. Man erhält — bis auf Ähnlichkeit — alle *transitiven Permutationsdarstellungen* einer endlichen Gruppe G, wenn man G in natürlicher Weise auf den Linksnebenklassen G/H nach Untergruppen H operieren läßt. Der *Charakter* $\pi_{G/H}$ einer solchen Darstellung $(\pi_{G/H}(s)=$Zahl der Fixpunkte von s, $s\in G)$ zerfällt über einem Körper K der Charakteristik 0 in der Form

$$\pi_{G/H} = \sum_{i=1}^{r} e_i\chi_i, \quad e_i \geqq 0,$$

wobei die χ_i gerade die *irreduziblen* Charaktere von G über K sind. Man nennt $\pi_{G/H}$ bzw. die dazugehörige Permutationsdarstellung *multiplizitätenfrei* (über K), wenn $e_i\in\{0, 1\}$ für alle $i=1, \ldots, r$. Ist $\pi_{G/H}$ multiplizitätenfrei über $\mathbf{C}$, so gilt dies für jedes K.

Multiplizitätenfreie Permutationscharaktere sind in mehrfacher Hinsicht von Interesse. Sie haben praktische Bedeutung bei der Bestimmung von (großen) Untergruppen endlicher Gruppen G, bei der Konstruktion primitiver Elemente in Körpererweiterungen und bei der expliziten Erstellung einer irreduziblen Darstellung aus ihrem Charakter (vgl. [5], S. 147 ff., [1]). Ferner sind die häufig untersuchten Permutationsdarstellungen niedrigen Ranges multiplizitätenfrei (vgl. [6]). Auch kennt man im multiplizitätenfreien Fall bemerkenswerte arithmetische Zusammenhänge zwischen den Bahnlängen von H auf G/H und den Charakterdimensionen $\chi_i(1)$, $i=1, \ldots, r$ (vgl. [9], Th. 30.1). Für weitere Motive zum Studium solcher Charaktere siehe [6].

Bis jetzt gibt es keine befriedigende Theorie der multiplizitätenfreien Permutationsdarstellungen bzw. -charaktere. So ist etwa deren Verhältnis zu den primitiven Darstellungen nicht geklärt. D. E. LITTLEWOOD hat einst vermutet, daß alle primitiven Permutationsdarstellungen multiplizitätenfrei sind ([5], S. 147). Dies wurde aufgrund des Gegenbeispiels $G=PSL(2, 11)$, $H=$Diedergruppe der Ordnung 12

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widerlegt, das von J. A. TODD mit Hilfe expliziter Charakterrechnungen gegeben wurde ([8]). Mittlerweile kennt man bei einfachen oder fast einfachen Gruppen weitere Gegenbeispiele (für $G=S_n$ vgl. [6], Sect. 2). In vielen anders gearteten Fällen (z. B. für auflösbare oder Frobeniussche Gruppen) ist die Vermutung jedoch richtig. Im übrigen beschränken sich unsere Kenntnisse multiplizitätenfreier Permutationsdarstellungen m. W. auf einige hinreichende Bedingungen (etwa [9], Th. 29.6) und Spezialfälle (vgl. [6]).

In der vorliegenden Note wird aus einem Grundgedanken, der von J. SAXL bei der Untersuchung der Gruppen $G=S_n$ angewendet worden ist ([6], Beweis von Th. 1), eine relativ leicht überprüfbare *notwendige* Bedingung für die Multiplizitätenfreiheit von $\pi_{G/H}$ — über einem beliebigen Körper K der Charakteristik 0 — entwickelt (Satz 1). Diese Bedingung liefert erhebliche Beschränkungen für die Gruppen $H^*\subseteqq G$, die H enthalten (Satz 2). Ferner gibt sie eine gewisse Erklärung dafür, daß gerade bei einfachen Gruppen nicht-multiplizitätenfreie Permutationsdarstellungen zu erwarten sind. Es wird insbesondere auf ganz einfache Weise gezeigt, daß bis auf endlich viele Ausnahmen alle Gruppen $PSL(2,p)$, p prim, solche Darstellungen besitzen (Satz 3; somit ist das Toddsche Gegenbeispiel $p=11$ keineswegs singulär in dieser Gruppenserie). Schließlich wird Satz 1 verwendet zur teilweisen Bestimmung der Struktur der Gruppe G, wenn $\pi_{G/H}$ multiplizitätenfrei und die Ordnung von H klein ist; G ist auflösbar für $|H|\leqq 4$ (Satz 5).

2. Die Bahnenungleichung und ihre Anwendung. Sei wie oben $\pi_{G/H}=\sum_{i=1}^{r} e_i\chi_i$, sei H^* eine weitere Untergruppe von G und $\pi_{G/H^*}=\sum_{i=1}^{r} e_i^*\chi_i$. Es gelte

(I) *für alle* $i=1,\dots,r$ *ist entweder* $e_i\leqq e_i^*$ *oder* $e_i^*=0$.

Dann folgt für das innere Produkt (bezüglich G) der beiden Permutationscharaktere

$$\langle\pi_{G/H},\pi_{G/H^*}\rangle=\sum_{i=1}^{r} e_ie_i^*\langle\chi_i,\chi_i\rangle\leqq\sum_{i=1}^{r} e_i^{*2}\langle\chi_i,\chi_i\rangle=\langle\pi_{G/H^*},\pi_{G/H^*}\rangle,$$

da $\langle\chi_i,\chi_i\rangle\in\mathbb{N}$ (für beliebiges K). Nach dem Frobeniusschen Reziprozitätsgesetz ist das erste Glied dieser Ungleichung gleich dem inneren Produkt des 1-Charakters mit der Einschränkung von π_{G/H^*} auf H. Dies aber ist gerade die Anzahl der Bahnen von H auf G/H^* (vgl. [3], S. 597) die wir mit $\operatorname{orb}(H,G/H^*)$ bezeichnen wollen. Andererseits ist das letzte Glied der Ungleichung gleich $\operatorname{orb}(H^*,G/H^*)$ (loc. cit.). Wir haben

Satz 1. *Seien H, H^* Untergruppen von G und e_i bzw. e_i^*, $i=1,\dots,r$, die Vielfachheiten der irreduziblen K-Charaktere von G in $\pi_{G/H}$ bzw. π_{G/H^*}. Ist* (I) *erfüllt, so gilt*

(II) $\operatorname{orb}(H,G/H^*)\leqq\operatorname{orb}(H^*,G/H^*)$.

Ist insbesondere $H \subseteqq H^*$, *so ist* (I) *äquivalent zur Aussage* $\operatorname{orb}(H, G/H^*) = \operatorname{orb}(H^*, G/H^*)$.

Nur die zweite Behauptung des Satzes ist noch zu zeigen. Sie folgt aus der obigen Ungleichung für die inneren Produkte, wenn man $e_i \geqq e_i^*$, $i=1, \ldots, r$, und $\operatorname{orb}(H, G/H^*) \geqq \operatorname{orb}(H^*, G/H^*)$ berücksichtigt. Ferner ergibt sich sofort

Korollar 1. *Sei* $\pi_{G/H}$ *multiplizitätenfrei über dem Körper* K *der Charakteristik* 0. *Dann sind* (I) *und* (II) *erfüllt für alle Untergruppen* H^* *von* G.

Korollar 2. *Seien* $G \supseteqq H^* \supseteqq H$ *endliche Gruppen. Folgende Aussagen sind zu* (I) *und* (II) *äquivalent:*

(III) *Für jedes* $t \in G$ *ist* $H^* t \subseteqq H t H^*$.

(IV) *Für jedes* $t \in G$ *stimmen die Indizes* $[H : H \cap H^{*t}]$ *und* $[H^* : H^* \cap H^{*t}]$ *überein* $(H^{*t} = t H^* t^{-1})$.

Beweis. Nach Satz 1 ist (I) genau dann erfüllt, wenn für jedes $t \in G$ die H^*-Bahn $H^* \bar{t}$ ($\bar{t}$=Restklasse von t in G/H^*) gleich der H-Bahn $H\bar{t}$ ist. Dies liefert die Äquivalenz von (I) und (III). Schreibt man die Längen dieser Bahnen als Gruppenindizes, so erkennt man die Äquivalenz von (I) und (IV).

Die nachfolgenden Bedingungen an Gruppen H^*, die den Punktstabilisator H einer multiplizitätenfreien Permutationsdarstellung enthalten, werden wegen ihrer Einfachheit und praktischen Bedeutung als Satz formuliert. Man gewinnt sie ohne Schwierigkeit aus den obigen Korollaren.

Satz 2. *Sei* $\pi_{G/H}$ *multiplizitätenfrei über* K *und* $H^* \subseteqq G$ *eine Gruppe, die* H *enthält.*

1) *Jede weitere solche Gruppe* H^{**}, $H^{**} \supseteqq H$, *ist mit* H^* *vertauschbar (d. h. die Menge* $H^* H^{**} = H^{**} H^*$ *ist eine Gruppe).*

2) *Jedes* $t \in G$ *mit* $H^t \subseteqq H^*$ *normalisiert die Gruppe* H^*. *Insbesondere ist der Normalisator* $N_G(H)$ *in* $N_G(H^*)$ *enthalten.*

3) *Der Index (in* H^**) des Durchschnitts von* H^* *mit einer dazu konjugierten Gruppe teilt* $|H|$.

Anwendungsbeispiele. Sei $\pi_{G/H}$ multiplizitätenfrei über K.

1. Ist $H=1$, so ist $N_G(H)=G$. Nach Satz 2, 2) ist jede Untergruppe H ein Normalteiler in G und mithin G eine abelsche oder hamiltonsche Gruppe (vgl. [3], S. 308). In der Tat ist $\pi_{G/1}$ multiplizitätenfrei über jedem K für abelsches G. Für hamiltonsche Gruppen G und $K=\mathbf{Q}$ gilt: $\pi_{G/1}$ ist genau dann multiplizitätenfrei, wenn $|G|$ nicht durch solche Primzahlen $p \geqq 3$ teilbar ist, für die die Zahl 2 gerade Ordnung in der Primrestgruppe modulo p hat. Diese Tatsache erhält man aus dem Studium der Kreisteilungskörper, über denen die Standard-Quaternionenalgebra

ein Schiefkörper, oder, anders ausgedrückt, -1 nicht Summe zweier Quadrate ist (vgl. [2]).

2. Sei H zyklisch von Primzahlordnung p. Operiert G treu auf G/H, so gibt es keine zyklische p-Gruppe $H^* \neq H$, die H enthält. Sonst wäre nach Satz 2, 3) $H^* \cap H^{*t} \neq 1$ für alle $t \in G$ und damit $H \subseteq \cap H^{*t}$. Da $\cap H^{*t}$ ein Normalteiler von G ist, würde dies auch für H als charakteristische Untergruppe dieser Gruppe gelten.

3. Sei $q = p^k$, p prim, $\mathbf{F}_q$ der Körper mit q Elementen und $G = PSL(2, q)$ die positive Gruppe der projektiven Geraden $\mathbf{F}_q \cup \{\infty\}$. Sei $H^* = ASL(1, q)$ der Stabilisator von ∞ in G. Wegen $\operatorname{orb}(H^*, \mathbf{F}_q \cup \{\infty\}) = 2$ ist π_{G/H^*} multiplizitätenfrei. Sei $H = ASL(1, q')$ $(\subseteq H^*)$ mit $q'|q$. Für $q' \neq q$ ist $\pi_{G/H}$ nicht multiplizitätenfrei. Da alle Bahnen von H auf $\mathbf{F}_q \setminus \mathbf{F}_{q'}$ die Länge $|H|$ haben, hat man nämlich $\operatorname{orb}(H, G/H^*) = 2 + (q - q')/|H| > 2 = \operatorname{orb}(H^*, G/H^*)$, im Widerspruch zu Satz 1. (Alternatives Argument: H^* und $H^{**} = PSL(2, q')$ sind nicht vertauschbar.)

Korollar 3. *Die endliche Gruppe G besitze eine transitive Permutationsdarstellung vom Grad n und vom Rang k (d. i. die Zahl der Bahnen eines Punktstabilisators). Ist $H \subseteq G$ eine Untergruppe und $\pi_{G/H}$ multiplizitätenfrei über K (Charakteristik 0), so gilt $n = \sum_{i=1}^{k} d_i$, wobei $d_i = 0$ oder ein Untergruppenindex von H ist. Insbesondere ist $|H| \geqq n/k$.*

Beweis. Nach der Bahnenungleichung (II) ist k größer oder gleich der Anzahl der Bahnen von H auf der n-elementigen Menge, die der Darstellung vom Rang k zugrundeliegt. Deshalb läßt sich n in der angegebenen Weise schreiben.

Grob gesprochen bedeutet Korollar 3, daß die Existenz von Permutationsdarstellungen niedrigen Ranges verhindert, daß Darstellungen mit einem Punktstabilisator kleiner Ordnung multiplizitätenfrei sein können. Dies wird besonders deutlich im Beweis des folgendes Satzes.

Satz 3. *Sei p eine Primzahl. Ist jede primitive Permutationsdarstellung von $PSL(2, p)$ multiplizitätenfrei über K (Charakteristik 0), so ist*

$$p \in \{2, 3, 5, 7, 19, 23, 31, 47, 59\}.$$

Beweis. Sei $p \geqq 7$. Nach dem Hauptsatz von Dickson ([3], S. 213) enthält $G = PSL(2, p)$ eine maximale Untergruppe H mit

$$H \cong \begin{cases} A_5 & p \equiv \pm 1 \bmod 5 \\ S_4 & p \equiv \pm 1 \bmod 8 \\ A_4 & \text{sonst.} \end{cases}$$

Sei $\pi_{G/H}$ multiplizitätenfrei. Da die gewöhnliche Permutationsdarstellung von G auf

$\mathbf{F}_p \cup \{\infty\}$ den Grad $p+1$ und den Rang 2 hat, ist (nach Korollar 3) $|H| \geqq (p+1)/2$, also $p<120$ im ersten, bzw. $p<48$, $p<24$ im zweiten und dritten Fall. Diese Menge von Ausnahmeprimzahlen wird durch genauere Betrachtung der Untergruppenindizes von H verkleinert zu $\{2, 3, 5, 7, 11, 17, 19, 23, 29, 31, 41, 47, 59\}$. Für $p \geqq 13$ bzw. $p \geqq 11$ besitzt G auch noch Diedergruppen D_{p-1} bzw. D_{p+1} als maximale Untergruppen. Von diesen hat D_{p-1} mehr als zwei Bahnen auf $\mathbf{F}_p \cup \{\infty\}$, sofern $p \equiv 1 \bmod 4$, D_{p+1} dagegen nie. Deshalb kann man die Zahlen 17, 29 und 41 auch noch ausschließen. Die Zahl 11 fällt nach TODD [8] weg.

Bemerkungen. 1. Auch das Toddsche Gegenbeispiel $G=PSL(2, 11)$, $H=D_{12}$, läßt sich ohne Charakterrechnungen behandeln. Denn $PSL(2, 11)$ hat eine 2-fach transitive Darstellung vom Grad 11 ([3], S. 214). Wäre $\pi_{G/H}$ multiplizitätenfrei, müßte nach Korollar 4 die Zahl 11 Summe von höchstens zwei Teilern von 12 sein.

2. Betrachtet man die der hier beschriebenen Methode zugrunde liegenden Tatsachen genauer, so gewinnt Satz 3 sofort folgende Gestalt: Sei χ der eindeutig bestimmte, absolut irreduzible Charakter der Dimension p, der im gewöhnlichen Permutationscharakter (vom Grad $p+1$) von $G=PSL(2,p)$ auftritt. Genau dann ist die Vielfachheit von χ in jedem primitiven Permutationscharakter von G kleiner oder gleich 1, wenn $p=11$ oder eine der Ausnahmeprimzahlen des Satzes 3 ist. Zu den irreduziblen Charakteren von G vergleiche [4], S. 211 ff.

3. Mit Hilfe der Charaktertheorie von $G=PSL(2,p)$ kann man im Satz 3 die Ausnahmeprimzahlen $p \geqq 19$ ausschließen, zumindest für algebraisch abgeschlossenes K. Der Normalisator H eines Singer-Zyklus S von G $\left(|S|=(p+1)/2\right)$ ist nämlich eine Diedergruppe der Ordnung $p+1$ ([3], S. 192) und maximal in G. Ferner gibt es einen zu S gehörigen "Ausnahmecharakter" χ mit folgenden Eigenschaften (vgl. [4], S. 204 ff.): $\chi(1)=p-1$, $\chi(t)=2$ für alle Involutionen t in G und $\chi(s^k)=-(\varepsilon^k+\varepsilon^{-k})$, $k=1, \ldots, (p-1)/2$. Dabei ist $S=\langle s \rangle$ und ε eine primitive $(p+1)/2$-te Einheitswurzel. Daraus ergibt sich (Frobenius-Reziprozität!) $\langle \chi, \pi_{G/H} \rangle = 2$.

Korollar 4. *Sei $\pi_{G/H}$ multiplizitätenfrei, $h=|H|$ und H^* eine p-Untergruppe von G. Ist $p>h$, so gilt*

$$[G:N_G(H^*)] \leqq h(p-1)/(p-h).$$

Beweis. Sei $n=[G:H^*]$, j die Anzahl der H^*-Fixpunkte in G/H^* und l die Anzahl der H^*-Bahnen der Länge $\geqq p$ auf G/H^*. Es ist $j=[N_G(H^*):H^*]$ und $j+lp \leqq n$. Ferner ist nach (II) $j+l=\operatorname{orb}(H^*, G/H^*) \geqq \operatorname{orb}(H, G/H^*) \geqq n/h$, sodaß wegen $l \leqq (n-j)/p$ gilt: $j+(n-j)/p \geqq n/h$. Durch Umformung (beachte $p>h$) erhält man $[G:N_G(H^*)]=n/j \leqq h(p-1)/(p-h)$.

Bevor wir dieses Korollar anwenden, notieren wir die folgenden Hilfssätze, deren einfache Beweise weggelassen werden.

Hilfssatz 1. *Sei P eine p-Sylowgruppe der endlichen Gruppe G und $N=\bigcap_{t\in G} N_G(P)^t$. Dann ist $P\cap N$ ein Normalteiler von G. Falls $p\nmid[G:N]$, ist $N=G$, also P Normalteiler von G.*

Hilfssatz 2. *Sei $\pi_{G/H}$ multiplizitätenfrei über K, N Normalteiler von G, $\bar{G}=G/N$ und $\bar{H}=HN/N$. Dann ist $\pi_{\bar{G}/\bar{H}}$ multiplizitätenfrei.*

Satz 4. *Sei $\pi_{G/H}$ multiplizitätenfrei über K, $h=|H|$ und P eine p-Sylowgruppe von G.*

1) *Ist $p\geqq 4h/3$, so gibt es einen Normalteiler P_0 von G mit $P_0\subseteqq P$ und $[P:P_0]\leqq p$.*

2) *Ist $p\geqq 2h+1$, so ist P abelscher Normalteiler von G.*

Beweis. Im Fall 2) ist nach Korollar 4 $n=[G:N_G(P)]\leqq p-1$, und deshalb nach dem Satz von Sylow $n=1$. Ferner ist jede Untergruppe $H^*\subseteqq P$ normal in P, denn wegen $[G:N_G(H^*)]\leqq p-1$ enthält $N_G(H^*)$ den Sylow-Normalteiler P. Da $p\geqq 3$ ist P abelsch ([3], S. 308).

Zum Beweis von 1) setzen wir $P_0=P\cap N$, N wie im Hilfssatz 1. Die Gruppe $\bar{G}=G/N$ operiert treu und transitiv auf der Menge $G/N_G(P)$, die nach Korollar 4 und dem Sylowschen Satz entweder 1, $p+1$ oder $2p+1$ Elemente hat. Es liege der letzte Fall vor, da nur dann $|\bar{G}|$ durch p^2 teilbar sein kann. Wir fassen $\bar{G}$ als Untergruppe von S_{2p+1} auf. Wäre $\bar{G}=A_{2p+1}$ oder S_{2p+1}, so hätte man wegen Hilfssatz 2 und Korollar 3 den Widerspruch $|\bar{H}|\geqq(2p+1)/2>h$. Ist $\bar{G}$ primitive Untergruppe von S_{2p+1}, $\bar{G}\neq A_{2p+1}$, S_{2p+1}, so teilt p den Index von $\bar{G}$ in S_{2p+1} ([9], Th. 14.1). Ist $\bar{G}$ jedoch imprimitiv, so gilt sogar $p\nmid|\bar{G}|$. Jedenfalls hat man $p^2\nmid|\bar{G}|$ und deshalb $[P:P_0]\leqq p$.

Anwendungsbeispiele. Sei $\pi_{G/H}$ multiplizitätenfrei, $h=|H|$. Sei $G_p=P$ eine p-Sylowgruppe von G.

1. $h=2$. Für $p\geqq 5$ ist G_p abelscher Normalteiler von G. Sei $N=\bigcap_{t\in G} N_G(G_3)^t$ und $\bar{G}=G/N$. Ist $N\neq G$, so ist wegen $[G:N_G(G_3)]\leqq 4$ die Gruppe $\bar{G}$ ähnlich zu einer transitiven Untergruppe von S_4. Nach Hilfssatz 1 teilt 3 die Anzahl von $\bar{G}$. Somit ist $\bar{G}=A_4$ oder S_4. Letzteren Fall kann man wegen Hilfssatz 2 ausschließen, da die Summe der Dimensionen der irreduziblen Charaktere von S_4 gleich 10 (<12) ist. Unser Ergebnis lautet: Der Normalteiler N hat die Struktur $N\cong(A\times P)\rtimes Q$, wo A eine abelsche Gruppe, $2,3\nmid|A|$, P eine 3-Gruppe, Q eine 2-Gruppe und „$\rtimes$" das semidirekte Produkt bezeichnet. Es ist $N=G$ oder $G/N\cong A_4$.

2. $h=3$. Ähnlich wie im Fall $h=2$ bleibt hier eine einzelne Primzahl $>h$ gesondert zu untersuchen, nämlich $p=5$. Sei $N=\bigcap_{t\in G} N_G(G_5)^t$ und $\bar{G}=G/N$, $\bar{G}=1$. Nach Korollar 4 und Hilfssatz 1 ist die Gruppe $\bar{G}$ ähnlich zu einer zweifach transitiven Untergruppe von S_6, d. h., $\bar{G}\cong A_5, S_5, A_6, S_6$. Da $\bar{G}$ aber höchstens sechs

5-Sylowgruppen hat, scheiden S_6 und A_6 aus. Wäre $\bar{G}\cong A_5$ oder S_5, so ließe sich nach Hilfssatz 2 und Korollar 3 die Zahl 5 als Summe höchstens zweier Teiler von 3 darstellen. Somit ist $\bar{G}=1$ und G hat nach dem Satz von Zassenhaus die Struktur $G\cong(A\times G_5)\rtimes Q$, mit einer abelschen Gruppe A, $2,3,5\nmid|A|$, und einer $\{2,3\}$-Gruppe Q.

3. $h=4$. Durch ähnliche, wenn auch weitläufigere Überlegungen (etwa unter Zuhilfenahme der Tabellen in [7]) erhält man in diesem Fall: G hat einen Normalteiler N der Form $N\cong(A\times P\times P')\rtimes Q$, mit abelschem A, $2,3,5,7\nmid|A|$, einer 7-Gruppe P, einer 5-Gruppe P' und einer $\{2,3\}$-Gruppe Q; ferner gilt entweder $G=N$ oder $G/N\cong AGL(1,8)$ oder G/N ist isomorph zu einer Untergruppe von $AGL(1,16)$, $|G/N|=80$.

Diese Strukturanalyse liefert insbesondere

Satz 5. *Sei $\pi_{G/H}$ multiplizitätenfrei über dem Körper K der Charakteristik* 0. *Ist $|H|\leqq 4$, so ist die Gruppe G auflösbar.*

Bemerkungen. 1. Satz 5 ist falsch für $|H|=5,6$, da die Permutationscharaktere $\pi_{A_5/H}$, $H=\langle(12345)\rangle$, bzw. $H=\langle(123),(12)(45)\rangle$, multiplizitätenfrei sind über jedem K.

2. Gelte in der Situation des Anwendungsbeispiels 1 zu Satz 4: $G\supseteqq H^*\supseteqq H$, $H^*\cong A_4$. Dann ist H^* Normalteiler von G, da sonst A_4 eine Untergruppe vom Index 2 hätte (Satz 2, 3). Man schließt jetzt unschwer: $G\cong N\times A_4$ mit $\pi_{N/1}$ multiplizitätenfrei. Die Gruppe N muß sogar abelsch sein (dies ist darauf zurückzuführen, daß der Gruppenring von A_4 den 3. Einheitswurzelkörper enthält, über dem die Standard-Quaternionenalgebra sicher kein Schiefkörper ist; vgl. Bsp. 1 nach Satz 2). In der Tat sind alle Permutationsdarstellungen dieser Art multiplizitätenfrei.

3. Im Beispiel 3 zu Satz 4 lassen sich die Fälle $G/N\cong AGL(1,8)$ bzw. $|G/N|=80$ *nicht* ausschließen. Ist etwa $N=1$, so ist die Summe aller K-irreduziblen Charaktere (K beliebig von Charakteristik 0) in beiden Fällen ein multiplizitätenfreier Permutationscharakter der Form $\pi_{G/H}$, $|H|=4$.

4. Viele Beispiele multiplizitätenfreier Permutationsdarstellungen mit kleinen Punktstabilisatoren liefert die folgende Tatsache: Ist A eine Gruppe, $\pi_{A/1}$ multiplizitätenfrei, und $G=A\rtimes H$, so ist $\pi_{G/H}$ multiplizitätenfrei.

Dank. Die Bemerkung 3 zu Satz 3, die Richtigstellung von Beispiel 3 zu Satz 4 und einige kleinere Verbesserungen gehen auf Hinweise des Referenten zurück, dem ich dafür herzlich danke.

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On additive functions taking values from a compact group

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1. Let G be a metrically compact Abelian topological group, T be the one-dimensional torus. A function $\varphi\colon \mathbf{N}\to G$ will be called additive if $\varphi(mn)=\varphi(m)+\varphi(n)$ holds for every coprime pairs m, n of natural numbers, while if $\varphi(mn)=\varphi(m)+\varphi(n)$ holds for each couple of $m, n\in\mathbf{N}$ then we say that it is completely additive. Let $\mathscr{A}_G$, $\mathscr{A}_G^*$ be the class of additive, and the class of completely additive functions, respectively.

Let $\{x_\nu\}_{\nu=1}^{\infty}$ be an infinite sequence in G. We shall say that it is of property D, if for any convergent subsequence x_{ν_n} the shifted subsequence x_{ν_n+1} has a limit, too. We say that it is of property Δ if $x_{\nu+1}-x_\nu\to 0$ $(\nu\to\infty)$.

Let $\mathscr{A}_G(D)$, $\mathscr{A}_G(\Delta)$ be the set of those $\varphi\in\mathscr{A}_G$ for which the sequence $\{x_n=\varphi(n)\}$ is a property D, Δ, respectively. The classes $\mathscr{A}_G^*(D)$, $\mathscr{A}_G^*(\Delta)$ are defined as follows:

$$\mathscr{A}_G^*(D)=\mathscr{A}_G(D)\cap\mathscr{A}_G^*,\quad \mathscr{A}_G^*(\Delta)=\mathscr{A}_G(\Delta)\cap\mathscr{A}_G^*.$$

It is obvious that $\mathscr{A}_G(\Delta)\subseteqq\mathscr{A}_G(D)$, $\mathscr{A}_G^*(\Delta)\subseteqq\mathscr{A}_G^*(D)$. In [1] we proved that $\mathscr{A}_G^*(\Delta)=\mathscr{A}_G^*(D)$. Recently E. WIRSING [4] proved that $\varphi\in\mathscr{A}_T(D)$ if and only if

$$\varphi(n)\equiv\tau\log n \pmod 1\quad (n\in\mathbf{N}) \tag{1.1}$$

for a $\tau\in\mathbf{R}$. By using Wirsing's theorem we proved in [2] the following assertion.

If $\varphi\in\mathscr{A}_G^*(\Delta)$ $\left(=\mathscr{A}_G^*(D)\right)$ then there exists a continuous homomorphism $\psi\colon \mathbf{R}_x\to G$, $\mathbf{R}_x$ denotes the multiplicative group of the positive reals, such that φ is a restriction of ψ on the set N, i.e. $\varphi(n)=\psi(n)$ for all $n\in\mathbf{N}$. The converse assertion is obvious. If $\psi\colon \mathbf{R}_x\to G$ is a continuous homomorphism, then $\varphi(n):=\psi(n)\in\mathscr{A}_G^*(\Delta)\subseteqq\mathscr{A}_G^*(D)$.

We should like to extend our results for the class $\mathscr{A}_G(D)$. This was done in [3] for $G=T$. Our aim in this paper is to characterize the class $\mathscr{A}_G(\Delta)$ for a general metrically compact Abelian group G.

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Let $\mathbf{N}_1$, $\mathbf{N}_0$ be the set of the odd and the even natural numbers, respectively. For a $\varphi\in\mathcal{A}_G$ let $S(\mathbf{N}_j)$ be the set of limit points of $\{\varphi(n)\mid n\in\mathbf{N}_j\}$ $(j=1,0)$, and let $S(\mathbf{N})$ be the set of limit points of $\{\varphi(n)\mid n\in\mathbf{N}\}$.

Theorem 1. *Let $\varphi\in\mathcal{A}_G(D)$. Then $S(\mathbf{N}_1)$ is a compact subgroup of G, $S(\mathbf{N}_0)=\gamma+S(\mathbf{N}_1)$ with a suitable $\gamma\in G$. There exists a continuous homomorphism $\psi\colon \mathbf{R}_x\to G$ such that $\varphi(n)=\psi(n)$, $n\in\mathbf{N}_1$. The function $u(n):=\varphi(n)-\psi(n)$ is zero for $n\in\mathbf{N}_1$, and $u(2)=u(2^\alpha)$ $(\alpha=1,2,\ldots)$. If $u(2)\in S(\mathbf{N}_1)$, then $2u(2)=0$.*

Conversely, let $\psi\colon\mathbf{R}_x\to G$ be a continuous homomorphism. Let $\beta\in G$ an element for which $\beta\in\psi(G)$ implies that $2\beta=0$. Let $u\in\mathcal{A}_G$ be defined by the relation

$$u(2^\alpha)=\beta\quad(\alpha=1,2,\ldots),\quad u(n)=0\quad\text{for all}\quad n\in\mathbf{N}_1.$$

Then $\varphi=u+\psi\colon \mathbf{N}\to G$ belongs to $\mathcal{A}_G(\Delta)$.

2. To prove our theorem we need some auxiliary results that can be proved by a method that was used by E. Wirsing [4] and in our earlier papers [1], [2].

Lemma 1. *If $\varphi\in\mathcal{A}_G$ and*

$$\varphi(m+2)-\varphi(m)\to 0\quad(m\to\infty,\ m\in\mathbf{N}_1)\tag{2.1}$$

then $\varphi(nm)=\varphi(m)+\varphi(n)$ for each $m,n\in\mathbf{N}_1$.

Proof. We need to prove only that

$$\varphi(p^\alpha)-\varphi(p^{\alpha-1})-\varphi(p)=0\quad(\alpha=1,2,\ldots)\tag{2.2}$$

for each odd prime p. From (2.1) we get that

$$E_m:=\varphi(p^\alpha m)-\varphi(p^\alpha m-2p)\to 0,\quad F_m:=\varphi(p^{\alpha-1}m)-\varphi(p^{\alpha-1}m-2)\to 0,$$

as $m\in\mathbf{N}_1$, $m\to\infty$. Since for $(m(m+2),2p)=1$ the relation

$$E_m=\varphi(p^\alpha)-\varphi(p^{\alpha-1})-\varphi(p)+F_m$$

holds, therefore (2.2) is true.

Without any important modification of the proof of Wirsing's theorem one can get

Lemma 2. *If the conditions of Lemma 1 are satisfied, $G=T$, then $\varphi(n)\equiv\tau\log n \pmod 1$ for all $n\in\mathbf{N}_1$, $\tau\in\mathbf{R}$.*

Upon this result, in the same way as in [2] one can prove easily the next

Lemma 3. *Assume that the conditions of Lemma* 1 *hold. Then there exists a continuous homomorphism* $\psi\colon \mathbf{R}_x \to G$ *such that* $\varphi(n)=\psi(n)$ *for each* $n\in\mathbf{N}_1$.

In the next section we shall prove that $\varphi\in\mathcal{A}_G(\Delta)$ implies (2.1).

3. Let us assume that $\varphi\in\mathcal{A}_G(\Delta)$. Let S denote the set of limit points of $\{\varphi(n)\mid n\in\mathbf{N}\}$, i.e. $g\in S$ if there exists $n_1<n_2<\ldots<n_\nu\in\mathbf{N}$, for which $\varphi(n_\nu)\to g$. Let $\varphi(n_\nu+1)\to g'$. In [1] we proved that g' is determined by g. So the correspondence $F\colon g\to g'$ is a function. Furthermore, it is obvious that $F(S)=S$. Let $p(n)$ and $P(n)$ denote the smallest and the largest prime factor of $n\in\mathbf{N}$.

Let k be an arbitrary integer,

$$\mathsf{R}=\{R_1<R_2<\ldots\} \tag{3.1}$$

be a sequence of natural numbers. We shall say that R belongs to $\mathcal{P}_k$ if for every $d\in\mathbf{N}$, d divides $R_\nu-k$ for every large ν, i.e. if $\nu>\nu_0(\mathsf{R},d)$. Let $\tilde{\mathcal{P}}_k\subseteqq\mathcal{P}_k$ be the set of those $\mathsf{R}\in\mathcal{P}_k$ for which the limit $\lim\limits_{n\to\infty}\varphi(R_n)$ exists. For an arbitrary sequence R let

$$a(\mathsf{R})=\lim_{\nu\to\infty}\varphi(R_\nu)$$

if the limit exists. Furthermore, if R is an infinite subsequence of natural numbers increasing monotonically and k is an integer then $\mathsf{R}+k$ denotes the sequence of the positive elements of $R_\nu+k$ written in increasing order. It is obvious that $\mathsf{R}+k\in\mathcal{P}_k$ if and only if $\mathsf{R}\in\mathcal{P}_0$. Furthermore, if $l<k$, $\mathsf{R}\in\tilde{\mathcal{P}}_l$, then $\mathsf{R}+(k-l)\in\tilde{\mathcal{P}}_k$. If $l>k$, then $\mathsf{R}\in\tilde{\mathcal{P}}_l$ implies only that $\mathsf{R}+(k-l)\in\mathcal{P}_k$. In this case we can assert only that there exists a suitable subsequence of $\mathsf{R}+(k-l)$ that belongs to $\tilde{\mathcal{P}}_k$.

Let

$$K_k:=\{a(\mathsf{R})\mid\mathsf{R}\in\tilde{\mathcal{P}}_k\}. \tag{3.2}$$

It is obvious that

$$F[K_k]=K_{k+1} \tag{3.3}$$

for every integer k, and that

$$\bigcup_{k=-\infty}^{\infty}K_k\subseteqq S. \tag{3.4}$$

Let now $g_1\in K_k$, $g_2\in K_l$, where $k\in\{1,-1\}$. Then there exist $\mathsf{R}\in\tilde{\mathcal{P}}_k$, $\mathsf{S}\in\tilde{\mathcal{P}}_l$ such that $a(\mathsf{R})=g_1$, $a(\mathsf{S})=g_2$. Since $k\in\{1,-1\}$, therefore $p(R_\nu)\to\infty$ $(\nu\to\infty)$. Let now the sequence $Q_\nu=R_{j_\nu}\cdot S_\nu$ be defined as follows: $j_0=0$, $j_\nu>j_{\nu-1}$ such that $p(R_{j_\nu})>P(S_\nu)$. Then $(R_{j_\nu},S_\nu)=1$, and so $\varphi(Q_\nu)=\varphi(R_{j_\nu})+\varphi(S_\nu)\to g_1+g_2$. But

$Q_\nu \equiv kl \pmod d$ for every $d \in \mathbf{N}$ whenever $\nu > \nu_0(d)$, so $\{Q_\nu\} \in \tilde{\mathscr{P}}_{kl}$, i.e. $g_1 + g_2 \in K_{kl}$ So we proved

Lemma 4. *For every integer l*

$$K_1 + K_l \subseteqq K_l, \tag{3.5}$$

$$K_{-1} + K_l \subseteqq K_{-l}. \tag{3.6}$$

(3.5) gives that $K_1 + K_1 \subseteqq K_1$, i.e. that K_1 is a semigroup in G. It is clear that K_1 is closed. The closedness of K_1 implies that K_1 is a compact semigroup in G, and so by [5] (9.16) it must be a group.

Lemma 5. *Let $k \in \mathbf{N}$. Then*

$$K_k = K_1 + \varphi(k), \quad K_{-k} = K_{-1} + \varphi(k). \tag{3.7}$$

Proof. Let $\tau \in K_k$, $\mathsf{R} \in \tilde{\mathscr{P}}_k$, $a(\mathsf{R}) = \tau$. Let $S_\nu := R_{j_\nu} - k$ be a subsequence of $\mathsf{R} - k$ for which $\mathsf{S} \in \tilde{\mathscr{P}}_0$. Then R_{j_ν} can be written as

$$R_{j_\nu} = k[A_\nu + 1], \quad S_\nu = kA_\nu.$$

The sequence $\{A_\nu\} \in \mathscr{P}_0$, therefore $(A_\nu + 1, k) = 1$ for every large ν, so $\varphi(A_\nu + 1) = \varphi(R_{j_\nu}) - \varphi(k)$, consequently

$$\varphi(A_\nu + 1) \to \tau - \varphi(k) \in K_1.$$

So we proved that $K_k - \varphi(k) \subseteqq K_1$.

Let now $\varrho \in K_1$, $\mathsf{R} \in \tilde{\mathscr{P}}_1$ so that $a(\mathsf{R}) = \varrho$. Then the sequence $S_\nu = kR_\nu$ belongs to $\tilde{\mathscr{P}}_k$, $(k, R_\nu) = 1$ if ν is large, $\lim \varphi(S_\nu) = \varphi(k) + \lim \varphi(R_\nu) = \varphi(k) + \varrho \in K_k$. This implies that $K_1 + \varphi(k) \subseteqq K_k$.

The proof of the second relation of (3.7) is the same, and so we omit it.

Lemma 6. *If $g \in K_{-2}$, then*

$$F[g] + F^2[g] = F^2\big[g + F^3[g]\big]. \tag{3.8}$$

Proof. Let us start from the identity $n(n+3) + 2 = (n+1)(n+2)$. If $(n, 3) = 1$, then $(n, n+3) = 1$, furthermore $(n+1, n+2) = 1$ for every $n \in \mathbf{N}$. Let $\{n_\nu\} \in \tilde{\mathscr{P}}_{-2}$ such that $a(\{n_\nu\}) = g \in K_{-2}$. Then $3 \nmid n_\nu$, consequently $\varphi(n_\nu(n_\nu + 3)) = \varphi(n_\nu) + \varphi(n_\nu + 3)$, $\varphi((n_\nu + 1)(n_\nu + 2)) = \varphi(n_\nu + 1) + \varphi(n_\nu + 2)$. Since $\varphi(n_\nu + k) \to F^k[g]$ $(k = 0, 1, 2, 3)$, we get (3.8) immediately.

Since $0 \in K_1$, there exists $\mathsf{R} \in \tilde{\mathscr{P}}_1$, $a(\mathsf{R}) = 0$. Let $R_{j_\nu} - 3$ be a subsequence of $R_\nu - 3$ for which the limit $\lim\limits_\nu \varphi(R_{j_\nu} - 3) = \eta$ exists. Since $\{R_{j_\nu} - 3\}_\nu \in \tilde{\mathscr{P}}_{-2}$, therefore $\eta \in K_{-2}$, and $F^3[\eta] = 0$. Let us apply (3.8) with $g = \eta$. Then we get $F[\eta] = 0$. Since $\eta \in K_{-2}$, therefore $F[\eta] \in K_{-1}$, consequently $0 \in K_{-1}$. Furthermore, $0 = F^3[\eta] = F^2\big[F[\eta]\big] = F^2[0]$. So we proved

Lemma 7. *We have*

$$F^2[0]=0, \tag{3.9}$$

$$0\in K_{-1}. \tag{3.10}$$

Lemma 8. *We have*

$$K_{-1}=K_1. \tag{3.11}$$

Proof. Put $l=1$ in (3.6). We get $K_{-1}+K_1\subseteq K_{-1}$. Since $0\in K_{-1}$, we deduce that $K_1\subseteq K_{-1}$. Let now $l=-1$. Then $K_{-1}+K_{-1}\subseteq K_1$. Since $0\in K_{-1}$ we get that $K_{-1}\subseteq K_1$. Consequently (3.11) is true.

Since $F^2[K_l]=K_{l+2}$ holds for every integer l, we get that $K_{2n+1}=K_1$ for every $n\in\mathrm{N}$. From (3.7) we get that $\varphi(2n+1)\in K_1$. Consequently $S(\mathrm{N}_1)\subseteq K_1$. On the other hand, it is obvious that $K_1\subseteq S(\mathrm{N}_1)$. So we know that

$$S(\mathrm{N}_1)=K_1. \tag{3.12}$$

Since $F[K_m]=K_{m+1}$, we get that $K_1=K_{2n}$ $(n\in\mathrm{N})$, i.e. that $\varphi(2n)-\varphi(2)\in K_1$ for all $n\in\mathrm{N}$, and so $\varphi(2^\alpha)-\varphi(2)\in K_1$ $(\alpha=1,2,\ldots)$. So we get that

$$S=\begin{cases}K_1\cup\{\varphi(2)+K_1\} & \text{if} \quad \varphi(2)\notin K_1,\\ K_1 & \text{if} \quad \varphi(2)\in K_1.\end{cases}$$

Lemma 9. *The function* $F\colon S\to S$ *is continuous.*

For the proof of this quite obvious assertion see [1].

Lemma 10. *If* $g\in K_1$, *then*

$$F[g]=g+F[0]. \tag{3.13}$$

If $h\in K_2$, *then*

$$F^2[h]=h+C, \tag{3.14}$$

where

$$C=\varphi(4)-2\varphi(2)+F[0]. \tag{3.15}$$

Proof. Let $k\in\mathrm{N}_1$, $\mathsf{M}\in\tilde{\mathscr{P}}1$, $a(\mathsf{M})=-\varphi(k)$. Then $(k,M_\nu)=1$, and so $\varphi(kM_\nu)\to 0$, $\varphi(kM_\nu+k)\to F^k[0]=F[0]$. Furthermore, $(k,M_\nu+1)=1$, therefore $\varphi(kM_\nu+k)=$ $=\varphi(k)+\varphi(M_\nu+1)$, $\varphi(M_\nu+1)\to F[-\varphi(k)]$. This implies that

$$F[-\varphi(k)]=-\varphi(k)+F[0]. \tag{3.16}$$

$\{\varphi(k)|k\in\mathrm{N}_1\}$, and so $\{-\varphi(k)|k\in\mathrm{N}_1\}$ is everywhere dense in K_1, F is continuous on K_1, therefore (3.13) is true.

Let now $h=\varphi(2)-\varphi(k)$, k and M as above. Then $\varphi(M_\nu)\to-\varphi(k)=h$. Since $2^2|(2M_\nu+2)$, $2^3\nmid(2M_\nu+2)$, we have

$$\varphi(2M_\nu+2)=\varphi(4)-\varphi(2)+\varphi(M_\nu+1),$$

and so that

$$F^2[h]=\varphi(4)-\varphi(2)+F[-\varphi(k)].$$

Since $-\varphi(k)\in K_1$, from (3.13) we get that $F[-\varphi(k)]=-\varphi(k)+F[0]$, and so that $F^2[h]=h+C$, $h=\varphi(2)-\varphi(k)$ with the C defined in (3.15).

Since $\{-\varphi(k)|k\in\mathbf{N}_1\}$ is everywhere dense in K_1, therefore $\{\varphi(2)-\varphi(k)|k\in\mathbf{N}_1\}$ is everywhere dense in K_2, F^2 being a continuous function, we get (3.14) immediately.

For a sequence x_n let $\Delta x_n:=x_{n+1}-x_n$, $\Delta^2x_n:=x_{n+2}-x_n$.

Lemma 11. *We have*

$$\lim_{m\in\mathbf{N}_1}\Delta\varphi(m)=F[0], \tag{3.16}$$

$$\lim_{m\in\mathbf{N}_0}\Delta^2\varphi(m)=C, \tag{3.17}$$

$$\lim_{m\in\mathbf{N}_1}\Delta^2\varphi(m)=0. \tag{3.18}$$

Furthermore, $C=0$.

Proof. Assume that (3.16) is not true. Then there exists a subsequence $2n_\nu+1$ of positive integers such that $\varphi(2n_\nu+2)-\varphi(2n_\nu+1)\to\delta$, $\delta\neq F[0]$. Then for a suitable subsequence $2n_{j_\nu}+1$ there exists the limit $\lim\varphi(2n_{j_\nu}+1)=\alpha\in K_1$, and $F[\alpha]=\alpha+\delta$. This contradicts (3.13).

The proof of (3.17) is the same and so we omit it.

Since $\Delta^2\varphi(2n-1)=\Delta^2\varphi(4n-2)+\Delta^2\varphi(4n)$, from (3.17) we get that

$$\Delta^2\varphi(2n-1)\to 2C. \tag{3.19}$$

Observe that

$$\Delta\varphi(2n-1)-\Delta\varphi(2n-1)=\Delta^2\varphi(2n)-\Delta^2\varphi(2n-1).$$

From (3.16), (3.17), (3.19) we get that $0=F[0]-F[0]=C-2C$, and so that $C=0$. This proves (3.18).

4. We have almost finished the proof. We know that $\Delta^2\varphi(2n-1)\to 0$. The condition of Lemma 1 is satisfied. Then, by Lemma 3 there exists a continuous homomorphism $\psi\colon\mathbf{R}_x\to G$ such that $\varphi(n)=\psi(n)$ for all $n\in\mathbf{N}_1$. Let $u(n):=\varphi(n)-$

$-\psi(n)$. Then $u\in\mathcal{A}$, $u(n)=0$ for all $n\in N_1$. Since ψ is continuous, therefore $\psi(n+k)-\psi(n)=\psi(1+k/n)\to 0$ as $n\to\infty$ for every fixed k. From (3.16) we get that $u(2n+2)\to F(0)$ as $n\to\infty$, that is $u(2)=u(2^\alpha)=F[0]$ $(\alpha=1, 2, \ldots)$.

If, in addition, $F[0]\in K_1$, then $S=K_1$, and (3.13) can be applied twice. This gives $F^2[g]=F[F[g]]=F[g+F[0]]=g+2F[0]$, that by $F^2[0]=0$ gives that $2F[0]=0$.

By this the first assertion in our Theorem is proved. The converse is obvious.

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Symmetrische Schrotungen im reellen dreidimensionalen projektiven Raum

OTTO RÖSCHEL

Herrn Prof. Dr. F. Hohenberg zum 80. Geburtstag

1. Transformationsgleichungen. Im reellen dreidimensionalen projektiven Raum $\mathbf{P}_3(\mathbf{R})$ weisen wir den Punkten homogene Koordinaten $x_0:x_1:x_2:x_3\neq 0:0:0:0$ zu, die wir zu Vektoren $\mathfrak{x}=(x_0, x_1, x_2, x_3)^t$ zusammenfassen. Dann werden durch

(1) $$\mathfrak{x} = \mathbf{R}\mathfrak{n}^i(t)+\mathbf{R}\mathfrak{f}^i(t) \quad (i = 1, 2)$$

mit $\mathfrak{n}^i(t)=(\dots e^i_j(t)\dots)$, $\mathfrak{f}^i(t)=(\dots f^i_j(t)\dots)$ $(j=0, 1, 2, 3)$ und $e^i_j(t), f^i_j(t)\in C^1(I\subset\mathbf{R})$ zwei *erzeugendenweise aufeinander bezogene C^1-Regelflächenstücke* Φ_1 und Φ_2 des $\mathbf{P}_3(\mathbf{R})$ beschrieben. Den Erzeugenden $e^i(t)$ können *Plücker-Koordinaten*

(2) $$p^i_{jk} := e^i_j f^i_k - e^i_k f^i_j \quad (i = 1, 2;\ j, k = 0, 1, 2, 3)$$

zugewiesen werden, die die *Plückerbedingung*

(3) $$\Omega(e^i(t), e^i(t)) := p^i_{01}p^i_{23}+p^i_{02}p^i_{31}+p^i_{03}p^i_{12} = 0$$

erfüllen. Schneiden der Erzeugenden $e^1(t_0)$ und $e^2(t_0)$ $(t_0\in I)$ ist durch

(4) $$\Omega(e^1(t_0), e^2(t_0)) := \det\big(\mathfrak{n}^1(t_0), \mathfrak{f}^1(t_0), \mathfrak{n}^2(t_0), \mathfrak{f}^2(t_0)\big) = 0$$

gekennzeichnet (vgl. etwa [1]). Wir werden im folgenden stets

(5) $$\Omega(e^1(t), e^2(t)) \neq 0 \quad \forall t\in I$$

verlangen und können dann dem erzeugendenweise aufeinander bezogenen Regelflächenpaar $\{\Phi_1, \Phi_2\}$ einen *symmetrischen projektiven Bewegungsvorgang* zuordnen (H. PRADE beschäftigt sich in [13] ebenfalls mit projektiven Bewegungsvorgängen, die solch einem Regelflächenpaar zugeordnet werden können; er betrachtet die hier untersuchten Zwangläufe jedoch nicht): Je zwei zugeordnete Erzeugende $e^1(t_0)$ und

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$e^2(t_0)$ $(t_0 \in I)$ sind nach (5) windschief und bestimmen daher eine eindeutige *axiale Spiegelung* $S(t_0)$ mit den Fixpunktgeraden $e^i(t_0)$ $(i=1,2)$, die wir *Spiegelungsachsen* nennen wollen. Unterwerfen wir nun das Rastsystem Σ' der stetigen Schar der durch $e^i(t)$ $(i=1,2)$ bestimmten axialen Spiegelungen $S(t)$ $(t \in I)$, so entsteht eine stetige Folge von Bildern $\Sigma(t)$, die untereinander und auch zu Σ' projektiv äquivalent sind; $\Sigma(t)$ stellt somit die Lagen des *Gangsystems* Σ bei einem *projektiven Zwanglauf* Σ/Σ' dar. Diesen Zwanglauf werden wir als *projektive symmetrische Schrotung mit den erzeugendenweise aufeinander bezogenen Grundregelflächen* Φ_1 und Φ_2 nennen.

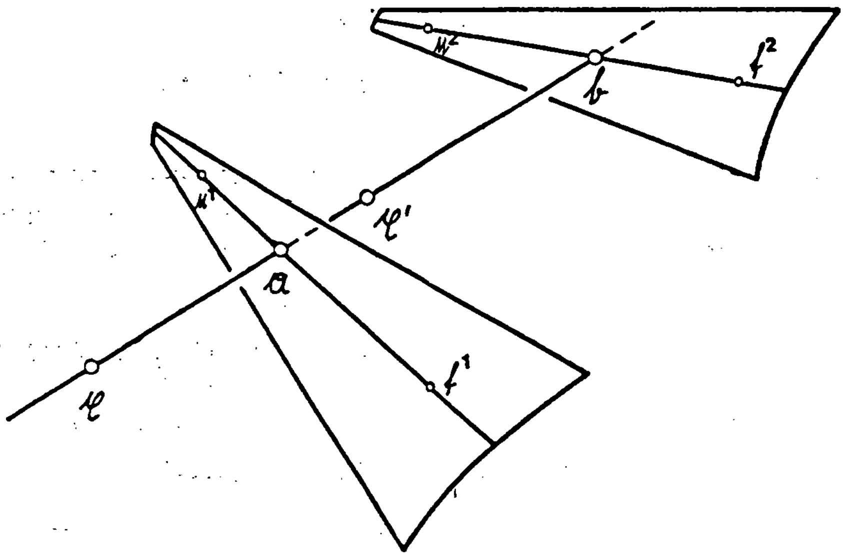

Abbildung 1

Der Punkt $\mathfrak{x}$ wird bei der Spiegelung $S(t)$ auf einem Strahl des durch $e^1(t)$ und $e^2(t)$ bestimmten Netzes in den Punkt $\mathfrak{x}'(t)$ transformiert (vgl. Abb. 1). Dieser Netzstrahl trifft $e^1(t)$ und $e^2(t)$ in den Punkten $\mathfrak{a}$ und $\mathfrak{b}$. Für $\mathfrak{a}$ muß

$$\text{(6)} \qquad \mathfrak{a} = u \cdot \mathfrak{n}^1(t) + v \cdot \mathfrak{f}^1(t) = \lambda\mathfrak{x} + \mu\mathfrak{n}^2(t) + \nu\mathfrak{f}^2(t) \quad (\lambda \neq 0)$$

$(u, v, \lambda, \mu, \nu \in \mathbf{R})$ gelten. Daraus gewinnen wir

$$u = \frac{\lambda \cdot \det(\mathfrak{x}, \mathfrak{f}^1, \mathfrak{n}^2, \mathfrak{f}^2)}{\det(\mathfrak{n}^1, \mathfrak{f}^1, \mathfrak{n}^2, \mathfrak{f}^2)}, \quad v = \frac{\lambda \cdot \det(\mathfrak{n}^1, \mathfrak{x}, \mathfrak{n}^2, \mathfrak{f}^2)}{\det(\mathfrak{n}^1, \mathfrak{f}^1, \mathfrak{n}^2, \mathfrak{f}^2)}$$

(dabei hängen e^i, $\mathfrak{n}^i$, $\mathfrak{f}^i$ von t ab $(i=1,2)$),

$$\text{(7)} \qquad \mu = -\frac{\lambda \cdot \det(\mathfrak{n}^1, \mathfrak{f}^1, \mathfrak{x}, \mathfrak{f}^2)}{\det(\mathfrak{n}^1, \mathfrak{f}^1, \mathfrak{n}^2, \mathfrak{f}^2)}, \quad \nu = -\frac{\lambda \cdot \det(\mathfrak{n}^1, \mathfrak{f}^1, \mathfrak{n}^2, \mathfrak{x})}{\det(\mathfrak{n}^1, \mathfrak{f}^1, \mathfrak{n}^2, \mathfrak{f}^2)}.$$

Damit haben wir zusammen mit (4)

$$(8)\qquad \mathfrak{a} = \frac{\lambda}{\Omega(e^1, e^2)}[\mathfrak{n}^1 \cdot \det(\mathfrak{x}, \mathfrak{k}^1, \mathfrak{n}^2, \mathfrak{k}^2) + \mathfrak{k}^1 \cdot \det(\mathfrak{n}^1, \mathfrak{x}, \mathfrak{n}^2, \mathfrak{k}^2)]$$

und analog

$$(9)\qquad \mathfrak{b} = \frac{\lambda}{\Omega(e^1, e^2)}[\mathfrak{n}^2 \cdot \det(\mathfrak{n}^1, \mathfrak{k}^1, \mathfrak{x}, \mathfrak{k}^2) + \mathfrak{k}^2 \cdot \det(\mathfrak{n}^1, \mathfrak{k}^1, \mathfrak{n}^2, \mathfrak{x})].$$

Auf dem durch $\mathfrak{x}$ laufenden Netzstrahl

$$(10)\qquad \mathfrak{k} = \varrho \cdot \mathfrak{a} + \sigma \cdot \mathfrak{b} \quad (\varrho, \sigma \in \mathbf{R})$$

erhalten wir wegen (6) bis (9) für $\varrho = \sigma = 1$ genau unseren Ausgangspunkt. Der Bildpunkt $\mathfrak{x}'$ ist daher durch

$$(11)\qquad -1 = DV(\mathfrak{a}, \mathfrak{b}, \mathfrak{x}, \mathfrak{x}') = \frac{\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 0 & \varrho \\ 1 & \sigma \end{vmatrix}}{\begin{vmatrix} 1 & \varrho \\ 0 & \sigma \end{vmatrix} \cdot \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{\varrho}{\sigma}$$

gekennzeichnet und wird demnach durch

$$(12)\qquad \mathfrak{x}' = \alpha(\mathfrak{a} - \mathfrak{b}) \quad (\alpha \in \mathbf{R} - \{0\} \text{ beliebig})$$

beschrieben. Nach Unterdrückung des nicht verschwindenden Proportionalitätsfaktors $\alpha(\lambda/\Omega(e^1, e^2))$ erhalten wir als Transformationsgleichung

$$(13)\qquad \begin{aligned} \mathfrak{x}' = {} & \mathfrak{n}^1 \cdot \det(\mathfrak{x}, \mathfrak{k}^1, \mathfrak{n}^2, \mathfrak{k}^2) + \mathfrak{k}^1 \cdot \det(\mathfrak{n}^1, \mathfrak{x}, \mathfrak{n}^2, \mathfrak{k}^2) - \\ & - \mathfrak{n}^2 \cdot \det(\mathfrak{n}^1, \mathfrak{k}^1, \mathfrak{x}, \mathfrak{k}^2) - \mathfrak{k}^2 \cdot \det(\mathfrak{n}^1, \mathfrak{k}^1, \mathfrak{n}^2, \mathfrak{x}). \end{aligned}$$

Unter Verwendung der Plückerkoordinaten (2) gewinnen wir

$$(14)\qquad \begin{aligned} \det(\mathfrak{x}, \mathfrak{k}^1, \mathfrak{n}^2, \mathfrak{k}^2) = {} & (x_0 f_1^1 - x_1 f_0^1) p_{23}^2 + (x_0 f_2^1 - x_2 f_0^1) p_{31}^2 + (x_0 f_3^1 - x_3 f_0^1) p_{12}^2 + \\ & + (x_2 f_3^1 - x_3 f_2^1) p_{01}^2 + (x_3 f_1^1 - x_1 f_3^1) p_{02}^2 + (x_1 f_2^1 - x_2 f_1^1) p_{03}^2. \end{aligned}$$

Analog können die anderen drei Determinanten in (13) berechnet werden. Wenn wir nun zusätzlich

$$(15)\qquad P_{ijkl} = p_{ij}^1 p_{kl}^2 - p_{kl}^1 p_{ij}^2 \quad (i, j, k, l = 0, 1, 2, 3)$$

definieren, können die Transformationsgleichungen (13) in der Form

$$(16)\qquad \mathfrak{x}'(t) =$$

$$= \begin{pmatrix} P_{0123} - P_{0213} + P_{0312} & 2P_{0203} & -2P_{0103} & 2P_{0102} \\ -2P_{1213} & P_{0123} + P_{0213} - P_{0312} & -2P_{0113} & 2P_{0112} \\ -2P_{1223} & 2P_{0223} & -P_{0123} - P_{0213} - P_{0312} & 2P_{0212} \\ -2P_{1323} & 2P_{0323} & -2P_{0313} & -P_{0123} + P_{0213} + P_{0312} \end{pmatrix} \mathfrak{x}$$

angeschrieben werden.

Wir haben damit den

Satz 1. *Zu zwei erzeugendenweise aufeinander bezogenen Grundregelflächen* $\Phi_i = \{e^i(t) | t \in I \subset \mathbf{R}, e^1(t) \cap e^2(t) = \{\ \}\}$ $(i=1, 2)$ *existiert ein eindeutig bestimmter symmetrischer projektiver Zwanglauf* Σ/Σ', *der durch* (13) *bzw.* (16) *beschrieben wird. Die Elemente der diesen Zwanglauf beschreibenden Transformationsmatrix sind bis auf einen Proportionalitätsfaktor quadratische Polynome der Plücker-Koordinaten der Grundregelflächenerzeugenden.*

Formel (16) umfaßt die *Darstellung der symmetrischen Schrotungen in allen dreidimensionalen Cayley—Klein-Räumen:* Man hat dabei zu beachten, daß in diesen Räumen i. a. bereits durch Auszeichnung einer *Grundregelfläche* $\Phi_1 = \{e^1(t) / t \in I \subset \mathbf{R}\}$ ein solcher Zwanglauf eindeutig bestimmt ist, da den Erzeugenden $e^1(t)$ durch die Polarität an der Maßquadrik des Cayley—Klein-Raumes Erzeugende $e^2(t)$ zugeordnet werden, die die zweite Grundregelfläche Φ_2 erfüllen. So wurde Formel (16) für den euklidischen Raum von O. Bottema und B. Roth in [2, S. 319] und für den einfach isotropen Raum von M. Husty in [3] hergeleitet.

2. Momentanbewegung. Wird Formel (16) kürzer durch $\mathfrak{x}'(t) = A(t)\mathfrak{x}$ beschrieben $(\det(A(t)) \neq 0)$, so besitzt die infinitesimale Transformation $T(t_0)$ zum Zeitpunkt $t_0 \in I$ die Darstellung

$$\text{(17)} \qquad \left.\frac{d\mathfrak{x}'}{dt}\right|_{t=t_0} = \dot{A}(t_0) A(t_0) \mathfrak{x}' \quad \text{mit} \quad \dot{A}(t_0) := \left.\frac{dA}{dt}\right|_{t=t_0}.$$

Sie erzeugt eine *einparametrige projektive Transformationsgruppe* $\mathfrak{m}(t_0)$, die wir als *Momentanbewegung des Zwanglaufs* Σ/Σ' (16) zum Zeitpunkt t_0 ansprechen. Um Aussagen über diese Momentanbewegung zu gewinnen, beachten wir, daß offensichtlich

$$\text{(18)} \qquad \mathfrak{m}(t_0) = \lim_{h \to 0} S(t_0 + h) \circ S(t_0) + O(h^2)$$

gilt, und $\mathfrak{m}(t_0)$ damit vom differentialgeometrischen Verhalten erster Ordnung der Erzeugenden $e^i(t_0)$ auf den Grundregelflächen abhängt. Längs den Erzeugenden $e^i(t_0)$ erfüllen die Flächentangenten der Grundregelfläche Φ_i eine *spezielle lineare Geradenkongruenz* $\mathfrak{k}^i(t_0)$. Ist die Flächenerzeugende $e^i(t_0)$ torsal, so besteht $\mathfrak{k}^i(t_0)$ aus den Geraden der Tangentialebene von Φ_i längs $e^i(t_0)$ und des Geradenbündels durch den Gratpunkt (Kuspidalpunkt von $e^i(t_0)$), andernfalls aus den Geraden eines parabolischen Netzes mit der Brennlinie $e^i(t_0)$ ([1, S. 73 und 78 f.]). Bezeichnet man in Analogie zur Terminologie in dreidimensionalen Cayley—Klein-Räumen jene Geraden, die sowohl $\mathfrak{k}^1(t_0)$ als auch $\mathfrak{k}^2(t_0)$ angehören, als *Zentraltangenten* (dieser Begriff der Zentraltangenten ist sehr viel weiter gefaßt als in der euklidischen Regelflächentheorie) der beiden Grundflächen, so gilt mit (18) der

Satz 2. *Die beiden Grundregelflächen Φ_1 und Φ_2 besitzen in zugeordneten Erzeugenden Zentraltangenten, die bei der Momentanbewegung der zugeordneten symmetrischen projektiven Schrotung Fixgeraden sind.*

(Vgl. das euklidische Ergebnis von J. KRAMES [4, S. 397]. Es ist zu bemerken, daß die in Satz 2 angesprochenen Fixgeraden nicht alle Fixgeraden der Momentanbewegung umfassen.)

Es ist unmittelbar einsichtig, daß *Fixpunkte* von $\mathfrak{m}(t_0)$ (wir werden im folgenden die komplexe Erweiterung $\mathbf{P}_3(\mathbf{C})$ des projektiven Raumes vornehmen) auf diesen Zentraltangenten liegen müssen. Zu den Spiegelungen $S(t_0)$ und $S(t_0+h)$ gehören jeweils invariante hyperbolische Geradennetze $\mathfrak{k}(t_0)$ und $\mathfrak{k}(t_0+h)$ mit Brennlinien $e^i(t_0)$ bzw. $e^i(t_0+h)$, deren Schnittgeraden die Fixpunkte von $S(t_0+h)\circ S(t_0)$ enthalten: Auf jeder dieser Schnittgeraden schneiden die Spiegelungsachsen $e^i(t_0)$ und $e^i(t_0+h)$ Punktpaare einer Projektivität aus, deren Doppelpunkte beim Grenzübergang (18) zu den Fixpunkten der Momentanbewegung $\mathfrak{m}(t_0)$ werden, während die Schnittgeraden von $\mathfrak{k}(t_0)$ und $\mathfrak{k}(t_0+h)$ gegen die Zentraltangenten konvergieren.

3. Flächenläufige symmetrische Schrotungen im $\mathbf{P}_3(\mathbf{R})$. Bisher hatten wir zwischen den beiden Grundregelflächen Φ_1 und Φ_2 eine erzeugendenweise Kopplung vorausgesetzt. Läßt man zu, daß die Erzeugenden $e^1\subset\Phi_1$ und $e^2\subset\Phi_2$ voneinander unabhängig sind, so stellt (16) die Transformationsgleichungen eines *flächenläufigen symmetrischen Schrotvorganges* im $\mathbf{P}_3(\mathbf{R})$ dar. Diese flächenläufigen (zweiparametrigen) Bewegungsvorgänge sind geometrisch deshalb interessanter als die in Abschnitt 2 studierten Zwangläufe, weil sie unabhängig von der (willkürlichen) Koppelung der Grundregelflächenerzeugenden sind. Für *algebraische Grundregelflächen* gilt nach komplexer Erweiterung der

Satz 3. *Sind die Grundregelflächen Φ_1 und Φ_2 einer flächenläufigen symmetrischen Schrotung Σ/Σ' nichtzerfallende und verschiedene algebraische Flächen der Ordnung n_1 beziehungsweise n_2, so ist die von einem allgemeinen Punkt des Gangraums Σ bei Σ/Σ' überstrichene Bahnfläche algebraisch von der Ordnung n_1n_2.*

Beweis. Sei P ein allgemeiner Punkt ($P\notin\Phi_1, \Phi_2$), g eine allgemeine *Testgerade,* die die Schnittkurve von Φ_1 und Φ_2 nicht trifft. Wir zeigen, *daß P bei Σ/Σ' im algebraischen Sinn genau n_1n_2-mal nach g gelangt* (vgl. Abb. 2):

P und g spannen eine Ebene ε auf, die Φ_1 und Φ_2 nach zwei algebraischen Kurven k_1 und k_2 schneidet, die bei allgemeiner Lage von g nicht zerfallen und die Ordnungen n_1 und n_2 besitzen. P gelangt genau dann in einen Punkt P^* auf g, wenn $[P, P^*]$ k_1 und k_2 in einem zu P, P^* *harmonischen Punktepaar* trifft. Unterwirft man daher etwa k_1 der *ebenen projektiven Spiegelung an P und g,* so schneiden sich k_2 und die Spiegelkurve k_1^* im algebraischen Sinn in n_1n_2 Punkten X^*, die über

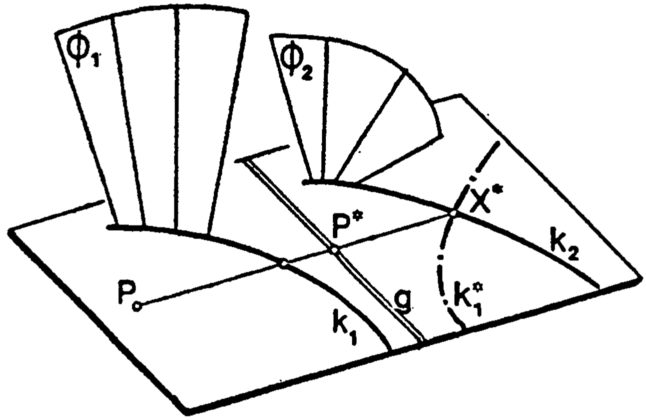

Abbildung 2

$P^* := ([P, X^*], g)$ genau $n_1 n_2$ Lagen von P auf g liefern; die Bahnfläche besitzt somit die Ordnung $n_1 n_2$.

Eine *Reduktion der Bahnflächenordnung* tritt damit genau dann auf, wenn *P der Schnittkurve von Φ_1 und Φ_2 angehört;* die Ordnung wird dann $n_1 n_2 - 1$ und verringert sich jeweils weiter um 1, wenn sich Φ_1 und Φ_2 in P berühren usw.

Die Bahnflächenordnung aller allgemeinen Punkte wird sich nur dann reduzieren, wenn Φ_1 und Φ_2 zusammenfallen. Es gilt der

Satz 4. *Stimmen die beiden Grundregelflächen Φ_1 und Φ_2 einer zweiparametrigen symmetrischen Fchrotung Σ/Σ' mit einer nichtzerfallenden algebraischen Fläche Φ der Ordnung n überein, so ist die von einem allgemeinen Punkt des Gangraumes Σ bei Σ/Σ' überstrichene Bahnfläche algebraisch von der Ordnung $n(n-1)/2$.*

Beweis. Wie im Beweis zu Satz 3 suchen wir die Schnittpunkte der Bahnfläche eines allgemeinen Punktes mit einer allgemeinen Testgeraden g (vgl. Abb. 3):

k und k^* haben nun $n(n-1)$ für uns interessante Schnittpunkte, weil die n Schnittpunkte von k und g nicht in Betracht kommen. Da die übrigen Schnittpunkte von k und k^* bezüglich g und P symmetrisch liegen, kommt P bei Σ/Σ' im algebraischen Sinn genau $n(n-1)/2$ mal auf die Gerade g.

In diesem Fall wird die *Bahnflächenordnung für Punkte der Grundregelfläche* $\Phi = \Phi_1 = \Phi_2$ zu $n(n-1)/2 - 1$.

Flächenläufige symmetrische Schrotungen Σ/Σ' mit *durchwegs ebenen Bahnflächen* werden wir *zweiparametrige symmetrische Darboux-Bewegungen* des $\mathbf{P}_3(\mathbf{R})$ nennen. Diese Definition erfolgt in Anlehnung an die Bezeichnung Darboux-Zwangläufe des euklidischen Raumes (alle Bahnkurven sind eben; vgl. [2, S. 304 f.]). Es gilt der folgende

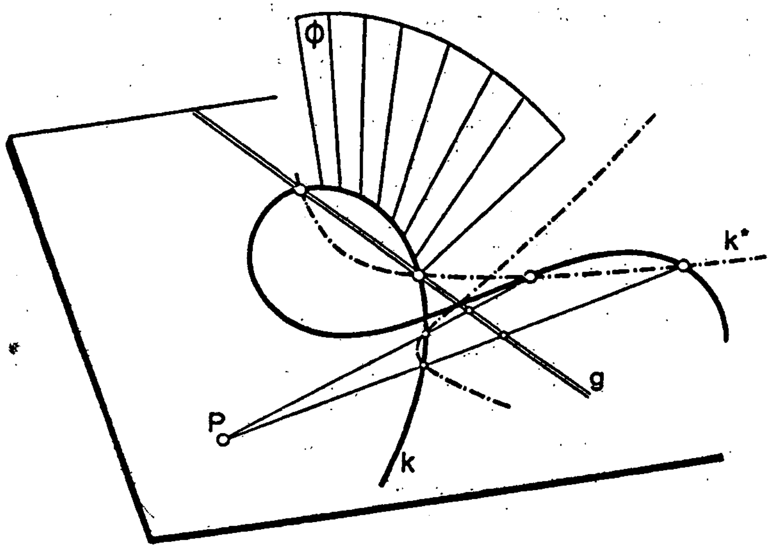

Abbildung 3

Satz 5. *Die zweiparametrigen symmetrischen Darboux-Bewegungen des* $\mathbf{P}_3(\mathbf{R})$ *besitzen als Grundregelflächen* Φ_1 *und* Φ_2 *entweder die Tangentenscharen zweier ebener Kurven in verschiedenen Ebenen des* $\mathbf{P}_3(\mathbf{R})$ *oder es gilt:* Φ_1 *und* Φ_2 *erfüllen einen festen Regulus auf einer nichtzerfallenden Quadrik* Φ. *Im ersten Fall erfüllen die Bahnebenen das von den Trägerebenen von* Φ_1 *und* Φ_2 *aufgespannte Ebenenbüschel, während im zweiten Fall alle allgemeinen Ebenen des* $\mathbf{P}_3(\mathbf{R})$ *als Bahnebenen auftreten.*

Beweis. Die allgemeinen Bahnflächen sind bei flächenläufigen symmetrischen Schrotungen nach Satz 3 und 4 genau dann Ebenen, wenn Φ_1 und Φ_2 entweder selbst Ebenen sind oder ein und denselben Regulus einer Quadrik Φ durchlaufen. Im ersten Fall gehören alle Bahnebenen dem von Φ_1 und Φ_2 aufgespannten Ebenenbüschel an. Der zweite Fall ist nicht trivial: Wird der Regulus Φ_i $(i=1,2)$ in der Normalform

$$p_{01}^i = p_{23}^i = 0, \quad p_{02}^i = 1, \quad p_{03}^i = p_{12}^i = u^i, \quad p_{13}^i = (u^i)^2 \tag{19}$$

mit $u^i \in I \subset \mathbf{R}$ beschrieben, so erhält der zugehörige *flächenläufige symmetrische Schrotvorgang* Σ/Σ' mit (16) die Gestalt

$$\mathfrak{x}' = \begin{pmatrix} u^1+u^2 & -2 & 0 & 0 \\ 2u^1u^2 & -(u^1+u^2) & 0 & 0 \\ 0 & 0 & u^1+u^2 & -2 \\ 0 & 0 & 2u^1u^2 & -(u^1+u^2) \end{pmatrix} \mathfrak{x}. \tag{20}$$

Man bestätigt unschwer, daß jeder nicht auf Φ gelegene Punkt P des Gangraumes Σ als *Bahnfläche eine Ebene* $\Pi(P)$ durchläuft, wobei $\Pi(P)$ genau die *Polarebene von* P *bezüglich der Grundregelfläche* $\Phi=\Phi_1=\Phi_2$ ist. Punkte P von Φ werden auf der zweiten *Erzeugendenschar* von Φ bewegt.

Interessant ist, daß eine Gerade $g\subset\Sigma$, die der Grundquadrik Φ nicht angehört, bei diesen zweiparametrigen symmetrischen Bewegungsvorgängen eine *lineare Geradenkongruenz* $\mathfrak{G}$ durchläuft, deren Leitgeraden von den Bahngeraden der Schnittpunkte von g und Φ gebildet werden. Damit treten je nach Lage von g und Φ *hyperbolische, elliptische oder parabolische lineare Bahngeradenkongruenzen* auf.

4. Flächenläufige symmetrische Schrotungen mit einer reellen Fixebene. Wir wollen versuchen, die beiden Grundregelflächen Φ_1 und Φ_2 so zu bestimmen, daß *die Bahnfläche aller Punkte der Ebene* ω $(x_0=0)$ *diese Ebene* ω *selbst ist*. Bei der axialen Spiegelung an den Erzeugenden e^i der Grundregelflächen Φ_i $(i=1, 2)$ werden nur dann alle Punkte der Ebene ω in dieser Ebene bleiben, wenn *eine der beiden Spiegelungsachsen ganz in* ω *liegt; eine der beiden Grundregelflächen* (o. B. d. A. Φ_2) *muß daher in* ω *enthalten sein*. Besondere Beachtung verdient bei diesen Bewegungsvorgängen die Tatsache, daß offensichtlich eine Änderung der Regelfläche Φ_2 in ω die entstehenden Bahnflächen der Punkte nicht ändert! Es existiert in diesem Fall sogar ein *dreiparametriger symmetrischer Schrotvorgang* im $\mathbf{P}_3(\mathbf{R})$, bei dem jeder Punkt auf einer festen Bahnfläche bleibt. Mit Satz 3 haben wir den

Satz 6. *Ist eine der beiden Grundregelflächen* Φ_1 *einer symmetrischen Schrotung des projektiven Raumes* $\mathbf{P}_3(\mathbf{R})$ *algebraisch von der Ordnung* n, *während die Schar der zweiten Spiegelungsachsen die Geraden einer festen reellen Ebene* ω *erfüllt, so entsteht eine dreiparametrige symmetrische Schrotung, bei der alle Punkte auf algebraischen Bahnflächen der Ordnung* n *gleiten.*

(Wird ω als Fernebene eines im $\mathbf{P}_3(\mathbf{R})$ eingebetteten affinen Raumes $\mathbf{A}_3(\mathbf{R})$ gedeutet, so sind die hier erwähnten symmetrischen Schrotungen dreiparametrige affine Bewegungsvorgänge. In jedem dieser dreiparametrigen Bewegungsvorgänge kann durch Auszeichnung eines nullteiligen Kegelschnitts in ω über die dann in ω vorliegende Polarität ein eindeutiger euklidischer symmetrischer Zwanglauf im Krames'schen Sinne definiert werden. Umgekehrt kann so jede Krames'sche symmetrische Schrotung des euklidischen Raumes in einen zwei- bzw. dreiparametrigen affinen symmetrischen Bewegungsvorgang eingebettet werden.)

Diese symmetrischen Schrotungen werden wir *affine symmetrische Schrotungen* nennen. Bei affinen symmetrischen Schrotungen sind die Bahnflächen der Punkte P im Gegensatz zum allgemeinen Fall *stets Regelflächen*, deren Erzeugenden $\bar{e}(t)$ aus den Grundregelflächenerzeugenden $e^1(t)$ mittels einer perspektiven Raumkollinea-

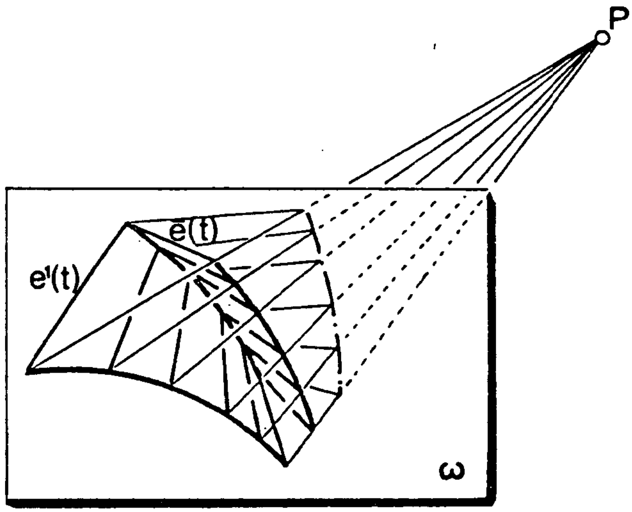

Abbildung 4

tion mit Zentrum P und Fixpunktebene ω hervorgehen; das charakteristische Doppelverhältnis δ hat dabei den Wert $1/2$ (vgl. Abb. 4). Wir haben damit den

Satz 7. *Bei den dreiparametrigen affinen symmetrischen Schrotungen sind die Bahnflächen aller allgemeinen Punkte Regelflächen, die zu der nichtebenen Grundregelfläche Φ_1 projektiv äquivalent sind.*

Die affinen symmetrischen Schrotungen lassen sich wie folgt kennzeichnen:

Satz 8. *Seien Φ_1 und Φ_2 n_1- und n_2-parametrige Geradenscharen* $(n_1 \geqq 1,\ n_2 \geqq 2)$, *$\Sigma/\Sigma'$ der darauf gegründete (n_1+n_2)-parametrige symmetrische projektive Bewegungsvorgang. Wenn dann alle Punkte des Gangraumes Σ beim ganzen Bewegungsvorgang Σ/Σ' auf Bahnflächen gleiten, ist Σ/Σ' notwendig eine affine symmetrische Schrotung.*

Beweis. Wir wählen in Φ_1 eine einparametrige Geradenschar $e^1(t)$ $(t \in I \subset \mathbf{R})$ aus, während die Schar der zweiten Spiegelungsachsen mindestens zweiparametrig ist $(e^2(u, v),\ u, v \in I^2 \subset \mathbf{R})$. Wir setzen voraus, daß $e^1(t)$ nicht in einer Ebene gelegen ist und studieren eine feste Erzeugende $e^1(t_0)$ $(t_0 \in I \subset \mathbf{R})$. Ein nicht auf $e^1(t_0)$ gelegener allgemeiner Punkt P wird bei den axialen Spiegelungen $\{e^1(t_0), e^2(u, v) \mid t_0 = \text{konst.},\ u, v \in I^2 \subset \mathbf{R}\}$ nur dann nicht ein ganzes Gebiet der Ebene $[P, e^1(t_0)] = \varepsilon(t_0)$ überstreichen, wenn die Geradenkongruenz $e^2(u, v)$ in $\varepsilon(t_0)$ eine Leitkurve besitzt. Wird nun $e^1(t)$ geändert, müßte $e^2(u, v)$ in jeder der Ebenen $\varepsilon(t)$ eine Leitkurve besitzen, was aber nur möglich ist, wenn die Kongruenz $e^2(u, v)$ einer Ebene angehört,

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Convergence of Hermite—Fejér interpolation at zeros of generalized Jacobi polynomials

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1. Introduction

The aim of this paper is to find necessary and sufficient conditions for uniform convergence of Hermite—Fejér interpolating processes based at the zeros of generalized Jacobi polynomials. As a by-product of our investigation we also give an answer to a question raised by P. TURÁN [35, Problem XXVII, p. 47] (cf. [36, Sections 2.3.1 and 3.6, pp. 337—338]). If f is a bounded function and w is a nonnegative integrable weight function on the real line, and $x_{1n}(w) > x_{2n}(w) > \ldots > x_{nn}(w)$ are the zeros of the orthonormal polynomials $p_n(w)$ corresponding to w, then the associated Hermite—Fejér interpolating polynomial $H_n(w,f)$ is defined to be the unique polynomial of degree at most $2n-1$ which satisfies

$$H_n(w,f,x_{kn}(w)) = f(x_{kn}(w)) \quad \text{and} \quad H_n'(w,f,x_{kn}(w)) = 0, \quad k = 1, 2, \ldots, n.$$

Ever since the work of L. Fejér, G. Grünwald and G. Szegő there has been a great deal of research performed in conjunction with convergence properties of these polynomials in terms of the weight function w, the point system $\{x_{kn}\}$ and the function f. In particular, when $\{x_{kn}(w^{(a,b)})\}$ are the zeros of the Jacobi polynomials $p_n^{(a,b)}$ which are orthonormal with respect to the Jacobi weight $w^{(a,b)}$ defined by

$$w^{(a,b)}(x) = \begin{cases} (1-x)^a(1+x)^b & \text{for} \quad x \in [-1,1] \\ 0 & \text{for} \quad x \notin (-1,1), \end{cases}$$

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$a>-1$, $b>-1$, one has a complete description of the conditions assuring uniform convergence of the corresponding Hermite—Fejér polynomials $H_n(w^{(a,b)},f)$. Namely, roughly speaking, for negative parameters a and b $\lim H_n(w^{(a,b)},f)=f$ uniformly for all continuous functions f, whereas for nonnegative a and b $\lim H_n(w^{(a,b)},f)=f$ takes place uniformly only under additional conditions on f. An accurate synthesis of the results we are interested in is given by the following five statements.

Proposition 1.1. *Let* $a>-1$, $b>-1$ *and* $0<\varepsilon<1$. *Then*

$$\lim_{n\to\infty} \max_{-\varepsilon\leqq x\leqq\varepsilon} |f(x)-H_n(w^{(a,b)},f,x)| = 0$$

for every function f *continuous in* $[-1,1]$.

Proposition 1.2. *Let* $b>-1$ *and* $-1<\varepsilon<1$. *Then*

$$\sup_{n\geqq 1} \max_{\varepsilon\leqq x\leqq 1} |H_n(w^{(a,b)},f,x)| < \infty$$

for every function f *bounded in* $[-1,1]$ *if and only if* $-1<a\leqq 0$.

Proposition 1.3. *Let* $b>-1$ *and* $-1<\varepsilon<1$. *Then*

$$\lim_{n\to\infty} \max_{\varepsilon\leqq x\leqq 1} |f(x)-H_n(w^{(a,b)},f,x)| = 0$$

for every function f *continuous in* $[-1,1]$ *if and only if* $-1<a<0$.

The above three theorems are condensed from [4, Vol. II, pp. 9—48, 285—317, 361—417, 502—512, 527—562, 767—801], [28, p. 138], [32, Theorem 14.6, pp. 340—344] and [33, Vol. 1, pp. 335—362].

By Markov's theorem on the derivatives of algebraic polynomials (cf. [16, § VI. 6, p. 141]) if $\{Q_n\}$ $(\deg Q_n=n)$ is a uniformly convergent sequence of algebraic polynomials in an interval, then $Q_n^{(r)}$ is $O(n^{2r})$ in the same interval for $r=1,2,\dots$. In view of this observation the following result whose special case of Legendre zeros $(a=b=0)$ was also treated by A. Schönhage [27] and J. Szabados [29] is especially satisfying.

Proposition 1.4 [38, Theorem 2.1, p. 84]. *Let* $-1<\varepsilon<1$ *and let* f *be continuous in* $[-1,1]$. *Let* $a\in[s-1,s)$ *for a fixed positive integer* s, *and let* $b>-1$. *Then*

$$\lim_{n\to\infty} \max_{\varepsilon\leqq x\leqq 1} |f(x)-H_n(w^{(a,b)},f,x)| = 0$$

holds if and only if

$$\lim_{n\to\infty} H_n(w^{(a,b)},f,1) = f(1)$$

and (if $a \geqq 1$*)*

$$\lim_{n\to\infty} n^{-2r}[H_n^{(r)}(w^{(a,b)}, f, x)]|_{x=1} = 0$$

for $r=1, 2, \dots, s-1$.

The following result of J. Szabados is the culmination of research by several authors including L. Fejér [4, Vol. II, pp. 22 and 40], E. Egervári and P. Turán [3], A. Schönhage [27] and G. Freud [9].

Proposition 1.5 [29, Theorems 1 and 3, pp. 470 and 457]. *Let* $b > -1$ *and* $-1 < \varepsilon < 1$. *Let* f *be continuous in* $[-1, 1]$. *Then*

$$\lim_{n\to\infty} H_n(w^{(0,b)}, f, 1) = (1+b)2^{-b-1} \int_{-1}^{1} f(t) w^{(0,b)}(t)\, dt,$$

and

$$\lim_{n\to\infty} \max_{\varepsilon \leqq x \leqq 1} |f(x) - H_n(w^{(0,b)}, f, x)| = 0$$

holds if and only if

$$f(1) = (1+b)2^{-b-1} \int_{-1}^{1} f(t) w^{(0,b)}(t)\, dt.$$

In what follows the function w is a generalized Jacobi weight if it can be represented as

$$w = gw^{(a,b)} \quad \text{where} \quad g(>0) \in C^1 \quad \text{and} \quad g' \in \operatorname{Lip} 1 \quad \text{on} \quad [-1, 1]$$

for some $a > -1$ and $b > -1$. Because of J. Korous' theorem yielding bounds for the corresponding generalized Jacobi polynomials $p_n(w)$ (cf. [32, Theorem 7.1.3, p. 162]) one expects a close relationship between Jacobi and generalized Jacobi polynomials, in particular, between associated approximation procedures. This is indeed the case as shown in the research conducted by V. M. Badkov, A. Máté, V. Totik and us (cf. [1], [2], [11]—[15], [18]—[22] and [24]).

In [24] we dealt with characterizing weighted mean convergence properties of Hermite—Fejér interpolating sequences associated with generalized Jacobi polynomials and we proved the following

Proposition 1.6 [24, Theorem 5, p. 55]. *Let* $0 < p < \infty$, *and let* w *be a generalized Jacobi weight. Let* u *be an unrelated Jacobi weight function. Then*

$$\lim_{n\to\infty} H_n(w, f) = f \quad \text{in} \quad L_p(u) \quad \text{in} \quad [-1, 1]$$

for every function f *continuous in* $[-1, 1]$ *if and only if* $w^{-1} \in L_p(u)$ *in the interval* $[-1, 1]$.

2. Main results

As announced in [23], we can generalize and/or extend the previous six propositions as follows.

Theorem 2.1. *Let w be a generalized Jacobi weight, and let $0<\varepsilon<1$. Then*

$$\lim_{n\to\infty} \max_{-\varepsilon\leqq x\leqq\varepsilon} |f(x)-H_n(w,f,x)| = 0$$

for every function f continuous in $[-1,1]$.

Theorem 2.2. *Let w be a generalized Jacobi weight. Then for every fixed nonnegative integer m there exists a polynomial Π such that R defined by $R(x)=$ $=(1-x)^m\Pi(x)$ satisfies*

$$\liminf_{n\to\infty} n^{-2a}|R(1)-H_n(w,R,1)| \geqq 1.$$

Theorem 2.3. *Let w be a generalized Jacobi weight, and let $-1<\varepsilon<1$. Then*

$$\sup_{n\geqq 1} \max_{\varepsilon\leqq x\leqq 1} |H_n(w,f,x)| < \infty$$

for every function f bounded in $[-1,1]$ if and only if $w(1)\neq 0$.

Theorem 2.4. *Let $w=gw^{(a,b)}$ be a generalized Jacobi weight function, and let $-1<\varepsilon<1$. Then*

$$\lim_{n\to\infty} \max_{\varepsilon\leqq x\leqq 1} |f(x)-H_n(w,f,x)| = 0$$

for every function f continuous in $[-1,1]$ if and only if $w(1)=\infty$.

Theorem 2.5. *Let $w=gw^{(a,b)}$ be a generalized Jacobi weight function, and let $-1<\varepsilon<1$. Let f be continuous in $[-1,1]$. Let $a\in[s-1,s)$ for a fixed positive integer s, and let $b>-1$. Then*

$$\lim_{n\to\infty} \max_{\varepsilon\leqq x\leqq 1} |f(x)-H_n(w,f,x)| = 0$$

holds if and only if

$$\lim_{n\to\infty} H_n(w,f,1) = f(1)$$

and (if $a\geqq 1$)

$$\lim_{n\to\infty} n^{-2r}[H_n^{(r)}(w,f,x)]|_{x=1} = 0$$

for $r=1,2,\ldots,s-1$.

Theorem 2.6. *Let $w=gw^{(0,b)}$ be a generalized Jacobi weight function, and let $-1<\varepsilon<1$. Let f be continuous in $[-1,1]$. Then*

$$\lim_{n\to\infty} H_n(w,f,1) = (2w(1))^{-1} \int_{-1}^{1} f(t)\,d[w(t)(1+t)].$$

Hence,

$$\lim_{n\to\infty}\max_{\varepsilon\le x\le 1}|f(x)-H_n(w,f,x)|=0$$

holds if and only if

$$f(1)=(2w(1))^{-1}\int_{-1}^{1}f(t)\,d[w(t)(1+t)].$$

Needless to say that analogous results can be proved in the interval $[-1,\varepsilon]$ as well, and therefore one can formulate results that are concerned with convergence in the entire interval $[-1, 1]$.

3. Notations

As a rule of thumb, all positive constants whose value is irrelevant and which are independent of the variables in consideration are denoted by "K". Each time "K" is used it may (or may not) take a different value. The symbol "$\sim$" is used to indicate that if A and B are two expressions depending on some variables then $A\sim B\Leftrightarrow|AB^{-1}|\le K$ and $|A^{-1}B|\le K$. We use **N** and **R** to denote the set of positive integers and real numbers, respectively.

Given a weight function w, the leading coefficient of the corresponding orthonormal polynomial $p_n(w)$ is denoted by $\gamma_n(w)$. $K_n(w)$ is the associated reproducing kernel function, that is

$$K_n(w,x,t)=\sum_{k=0}^{n-1}p_n(w,x)p_n(w,t). \tag{3.1}$$

In terms of the Christoffel—Darboux formula (cf. [32, Theorem 3.2.2, p. 43]), $K_n(w)$ can be expressed as

$$K_n(w,x,t)=(\gamma_{n-1}(w)/\gamma_n(w))[p_n(w,x)p_{n-1}(w,t)-p_{n-1}(w,x)p_n(w,t)]/(x-t). \tag{3.2}$$

The Christoffel function $\lambda_n(w)$ is defined by

$$\lambda_n(w,x)=K_n^{-1}(w,x,x). \tag{3.3}$$

The Cotes numbers $\lambda_{kn}(w)$ in the Gauss—Jacobi quadrature formula are given by

$$\lambda_{kn}(w)=\lambda_{kn}(w,x_{kn}(w)). \tag{3.4}$$

The fundamental polynomials of Lagrange interpolation $\ell_{kn}(w)$ associated with the zeros of $p_n(w)$ are defined by

$$\ell_{kn}(w,x)=p_n(w,x)/[p_n'(w,x_{kn}(w))(x-x_{kn}(w))]. \tag{3.5}$$

Another useful expression for $\ell_{kn}(w)$ is the following

$$\ell_{kn}(w, x) = (\gamma_{n-1}(w)/\gamma_n(w))\lambda_{kn}(w)p_{n-1}(w, x_{kn}(w))p_n(w, x)/(x-x_{kn}(w)) \tag{3.6}$$

(cf. [32, formula (3.4.7), p. 48]).

The usual way of expressing the Hermite—Fejér interpolating polynomial $H_n(w,f)$ is in terms of the fundamental polynomials $\ell_{kn}(w)$, and it is given by

$$H_n(w,f,x) = \sum_{k=1}^{n} f(x_{kn}(w))v_{kn}(w, x)\ell_{kn}^2(w, x) \tag{3.7}$$

where $v_{kn}(w)$ is defined by

$$v_{kn}(w, x) = 1-p_n''(w, x_{kn}(w))[p_n'(w, x_{kn}(w))]^{-1}(x-x_{kn}(w)) \tag{3.8}$$

(cf. [32, p. 330—331]). For special orthogonal polynomial systems due to available differential equations $p_n''(w, x_{kn}(w))[p_n'(w, x_{kn}(w))]^{-1}$ can be expressed explicitly in terms of the weight function and the zeros of the orthogonal polynomials, the above expression is convenient when investigating Hermite—Fejér interpolation. However, for general weight functions it is difficult (if not impossible) to handle the derivatives of orthogonal polynomials, and thus this formula is of limited value. On the other hand, G. Freud's formula

$$v_{kn}(w, x) = 1+\lambda_n'(w, x_{kn}(w))\lambda_{kn}(w)^{-1}(x-x_{kn}(w)) \tag{3.9}$$

(cf. [5, p. 113]) involves the Christoffel functions and their derivatives which are much more suitable when the weight function is not one of the classical ones (cf. [5, 8, 24]). If P is a polynomial of degree at most $2n-1$ then in view of the Hermite interpolation formula (cf. [32, pp. 330—331]) we can write

$$P(x) = H_n(w, P, x)+\mathscr{H}_n(w, P', x) \tag{3.10}$$

where

$$\mathscr{H}_n(w,f,x) = \sum_{k=1}^{n} f(x_{kn}(w))(x-x_{kn}(w))\ell_{kn}^2(w, x). \tag{3.11}$$

4. Technicalities

Here, in addition to formulating some useful and known properties of generalized Jacobi polynomials which run parallel to those of Jacobi polynomials, we will also prove a few propositions of technical nature that will subsequently be applied to demonstrate our principal results. In what follows w is a generalized Jacobi weight.

If $x_{kn}(w)=\cos(\theta_{kn}(w))$ where $x_{0n}(w)=1$, $x_{n+1,n}(w)=-1$ and $0\leqq\theta_{kn}(w)\leqq\pi$ then

$$\theta_{k+1,n}(w)-\theta_{kn}(w)\sim 1/n \tag{4.1}$$

uniformly for $0\leqq k\leqq n$ and $n\in\mathbf{N}$ (cf. [18, Theorem 3, p. 367]).

Using Korous' theorem (cf. [32, Theorem 7.1.3, p. 162]), similarly to Jacobi polynomials, the generalized Jacobi polynomials can be estimated in terms of the weight function as follows:

$$|p_n(w,x)|\leqq K\begin{cases}[w(x)(1-x^2)^{1/2}]^{-1/2} & \text{for } x\in[-1+n^{-2},1-n^{-2}]\\ n^{1/2}[w(1-n^{-2})]^{-1/2} & \text{for } x\in[1-n^{-2},1]\\ n^{1/2}[w(-1+n^{-2})]^{-1/2} & \text{for } x\in[-1,-1+n^{-2}],\end{cases} \tag{4.2}$$

uniformly for $n\in\mathbf{N}$ (cf. [32, Theorem 7.32.2, p. 169] or [1, p. 226]),

(4.3)
$$|p_n(w,x)|\sim\begin{cases}n|x-x_{mn}(w)|[w(x)(1-x^2)^{3/2}]^{-1/2} & \text{for } 2x\in[-1+x_{nn}(w),1+x_{1n}(w)]\\ n^{1/2}[w(1-n^{-2})]^{-1/2} & \text{for } 2x\in[1+x_{1n}(w),2]\\ n^{1/2}[w(-1+n^{-2})]^{-1/2} & \text{for } 2x\in[-2,-1+x_{nn}(w)],\end{cases}$$

uniformly for $n\in\mathbf{N}$ where m is the index of the zero $x_{kn}(w)$ which is (one of the) closest to x (cf. [19, Theorem 9.33, p. 171]), and

$$\left|p_{n-1}(w,x_{kn}(w))\right|\sim w(x_{kn}(w))^{-1/2}(1-x_{kn}(w)^2)^{1/4} \tag{4.4}$$

uniformly for $1\leqq k\leqq n$ and $n\in\mathbf{N}$ (cf. [19, Theorem 9.31, p. 170]).

The derivatives of generalized Jacobi polynomials at ±1 satisfy

$$|[p_n^{-1}(w,\pm1)]^{(\ell)}|\leqq Kn^{2\ell}|p_n^{-1}(w,\pm1)| \tag{4.5}$$

uniformly for $n\in\mathbf{N}$ (cf. [20, formula (23), p. 674]). Writing $p_n^{-2}=(p_n^{-1})(p_n^{-1})$, and using (4.5) and the product differentiation rule (Leibnitz's formula) we obtain

$$|[p_n^{-2}(w,\pm1)]^{(\ell)}|\leqq Kn^{2\ell}|p_n^{-2}(w,\pm1)| \tag{4.6}$$

uniformly for $n\in\mathbf{N}$.

For the Christoffel functions and Cotes numbers we have the following estimates

$$\lambda_n(w,x)\sim\begin{cases}n^{-1}w(x)(1-x^2)^{1/2} & \text{for } x\in[-1+n^{-2},1-n^{-2}]\\ n^{-2}w(1-n^{-2}) & \text{for } x\in[1-n^{-2},1]\\ n^{-2}w(-1+n^{-2}) & \text{for } x\in[-1,-1+n^{-2}],\end{cases} \tag{4.7}$$

uniformly for $n\in\mathbf{N}$ (cf. [17, p. 336]) and

$$\lambda_{kn}(w)\sim n^{-1}w(x_{kn}(w))(1-x_{kn}(w)^2)^{1/2} \tag{4.8}$$

uniformly for $1\leqq k\leqq n$ and $n\in\mathbf{N}$ (this follows immediately from estimates (4.1) and (4.7) (cf. 3.4)). The derivatives of the Christoffel functions satisfy

$$(4.9)\qquad |\lambda_n'(w,x)| \leqq K \begin{cases} n^{-1}w(x)(1-x^2)^{-1/2} & \text{for} \quad x\in[-1+n^{-2}, 1-n^{-2}] \\ w(1-n^{-2}) & \text{for} \quad x\in[1-n^{-2}, 1] \\ w(-1+n^{-2}) & \text{for} \quad x\in[-1, -1+n^{-2}], \end{cases}$$

uniformly for $n\in\mathbf{N}$ (cf. [24, formula (23), p. 36]) and

$$(4.10)\qquad \lambda_n'(w, x_{kn}(w)) \leqq Kn^{-1}w(x_{kn}(w))(1-x_{kn}(w)^2)^{-1/2}$$

uniformly for $1\leqq k\leqq n$ and $n\in\mathbf{N}$ (cf. [24, formula (24), p. 36]).

A weight function w is said to belong to Szegő's class ($w\in S$) if it is supported in $[-1, 1]$ and $\log w(\cos\theta)\in L_1$ in $[0, \pi]$. For instance, all generalized Jacobi weights are in Szegő's class. According to the Szegő Theory (cf. [32, Theorem 12.7.1, p. 309]) the leading coefficients $\gamma_n(w)$ of the orthogonal polynomials $p_n(w)$ satisfy

$$(4.11)\qquad 0 < \lim_{n\to\infty} 2^{-n}\gamma_n(w) = \pi^{-1/2}\exp\left\{(2\pi)^{-1}\int_0^\pi \log w(\cos\theta)\,d\theta\right\} < \infty$$

whenever $w\in S$.

The following proposition is a simple but unexpected generalization of (4.2) and (4.4).

Lemma 4.1. *Let $w_1=g_1w^{(a,b)}$ and $w_2=g_2w^{(a,b)}$ be two (not necessarily different) generalized Jacobi weights corresponding to the same parameters $a>-1$ and $b>-1$. Then for every fixed integer ℓ we have*

$$(4.12)\qquad \left|p_{n+\ell}(w_1, x_{kn}(w_2))\right| \leqq Kw(x_{kn}(w_2))^{-1/2}(1-x_{kn}(w_2)^2)^{1/4}$$

uniformly for $1\leqq k\leqq n$ and $n\in\mathbf{N}$.

Proof. By Korous' theorem (cf. [32, Theorem 7.1.3, p. 162]) we have

$$|p_{n+\ell}(w_1, x)| \leqq K[|p_{n+\ell}(w_2, x)| + |p_{n+\ell-1}(w_2, x)|]$$

for $x\in[-1, 1]$. Being orthogonal polynomials, the generalized Jacobi polynomials satisfy the three-term recurrence

$$xp_n(w, x) = a_{n+1}(w)p_{n+1}(w, x) + b_n(w)p_n(w, x) + a_n(w)p_{n-1}(w, x),$$

and since $w>0$ almost everywhere in $[-1, 1]$, we have $\lim a_n(w)=1/2$ and $\lim b_n(w)=0$ (cf. [25], [26], [12, p. 68], [22, Sections 4.5 and 4.13] and (4.11)). Hence by repeated application of the recurrence formula we obtain

$$|p_{n+j}(w_2, x)| \leqq K[|p_{n-1}(w_2, x)| + |p_n(w_2, x)|], \quad x\in[-1, 1],$$

for all fixed j. Now inequality (4.12) follows from (4.4) applied with $w=w_2$.

The next step is to compare Christoffel functions of generalized Jacobi weights.

Lemma 4.2. *Let* $w_1=g_1w^{(a,b)}$ *and* $w_2=g_2w^{(a,b)}$ *be two generalized Jacobi weights corresponding to the same parameters* $a>-1$ *and* $b>-1$. *Then*

$$|g_1(x_{kn}(w_1))\lambda_{kn}(w_1)^{-1}-g_2(x_{kn}(w_1))\lambda_{kn}(w_2)^{-1}|\leqq \leqq Kw_1(x_{kn}(w_1))^{-1}[1-x_{kn}(w_1)^2]^{1/2} \tag{4.13}$$

holds uniformly for $1\leqq k\leqq n$ *and* $n\in\mathbf{N}$.

Proof. Let $w_1=gw_2$. Then the identity

$$g(x)\lambda_n^{-1}(w_1,x)-\lambda_n^{-1}(w_2,x)=\int_{\mathbf{R}}K_n(w_1,x,t)K_n(w_2,x,t)[g(x)-g(t)]w_2(t)\,dt$$

is a straightforward consequence of orthogonality relations. Since we have $g'\in\mathrm{Lip}\,1$, we can write $g(x)-g(t)=g'(x)(x-t)+O([x-t]^2)$. Hence the previous formula becomes

$$\begin{aligned}g(x)\lambda_n^{-1}(w_1,x)-\lambda_n^{-1}(w_2,x)&=\\=g'(x)\int_{\mathbf{R}}K_n(w_1,x,t)K_n(w_2,x,t)(x-t)w_2(t)\,dt&+\\+O(1)\int_{\mathbf{R}}|K_n(w_1,x,t)K_n(w_2,x,t)|(x-t)^2w_2(t)\,dt.&\end{aligned}\tag{4.14}$$

In view of (3.2) the first integral here can explicitly be evaluated in terms of the orthogonal polynomials involved and their leading coefficients. We have

$$\int_{\mathbf{R}}K_n(w_1,x,t)K_n(w_2,x,t)(x-t)w_2(t)\,dt=(\gamma_{n-1}(w_1)/\gamma_n(w_2))p_{n-1}(w_1,x)p_n(w_2,x).\tag{4.15}$$

Using Schwarz's inequality, $w_2\leqq Kw_1$, (3.2) and again orthogonality relations, we can estimate the second integral as follows:

$$\begin{aligned}&\Big[\int_{\mathbf{R}}|K_n(w_1,x,t)K_n(w_2,x,t)|(x-t)^2w_2(t)\,dt\Big]^2\leqq\\&\leqq K\int_{\mathbf{R}}K_n^2(w_1,x,t)(x-t)^2w_1(t)\,dt\int_{\mathbf{R}}K_n^2(w_2,x,t)(x-t)^2w_2(t)\,dt=\\&=K[\gamma_{n-1}(w_1)/\gamma_n(w_1)]^2[p_{n-1}^2(w_1,x)+p_n^2(w_1,x)]\times\\&\times[\gamma_{n-1}(w_2)/\gamma_n(w_2)]^2[p_{n-1}^2(w_2,x)+p_n^2(w_2,x)].\end{aligned}\tag{4.16}$$

Since generalized Jacobi weights w are in Szegő's class, we can use (4.11) to estimate the ratios of the leading coefficients of generalized Jacobi polynomials. Using this observation and inserting (4.15) and (4.16) into (4.14), we obtain

$$|g(x)\lambda_n^{-1}(w_1,x)-\lambda_n^{-1}(w_2,x)|\leqq$$
$$\leqq K[|p_{n-1}(w_1,x)|+|p_n(w_1,x)|][|p_{n-1}(w_2,x)|+|p_n(w_2,x)|].$$

Applying this inequality with $x=x_{kn}(w_1)$ (cf. (3.4)) and using Lemma 1 (cf. (4.12)), Lemma 4.2 follows immediately.

Our next goal is to estimate $v_{kn}(w, x)$ (cf. (3.7)—(3.9)) via improving (4.10) regarding the derivatives of the Christoffel functions. For Jacobi polynomials we have

$$v_{kn}(w^{(a,b)}, x) = 1-[1-x_{kn}^2]^{-1}[a-b+(a+b+2)x_{kn}](x-x_{kn}) \tag{4.17}$$

$(x_{kn}=x_{kn}(w^{(a,b)}))$ (cf. [32, formula (14.5.2), p. 339]). In what follows we will show that the right-hand side of (4.17) is the principal contribution to $v_{kn}(w, x)$ as well.

Lemma 4.3. *Let* $w=gw^{(a,b)}$ *be a generalized Jacobi weight. Then*

$$|\lambda_n'(w, x_{kn}(w))\lambda_{kn}(w)^{-1}+[1-x_{kn}(w)^2]^{-1}[a-b+(a+b+2)x_{kn}(w)]| \leqq K \tag{4.18}$$

uniformly for $1\leqq k\leqq n$ *and* $n\in\mathbf{N}$.

Proof The crux of the matter is the inequality

$$\begin{aligned}|g(x)K_n'(w, x, x)-K_n'(w^{(a,b)}, x, x)| &\leqq \\ \leqq K[|p_{n-2}(w^{(a,b)}, x)|+|p_{n-1}(w^{(a,b)}, x)|+|p_n(w^{(a,b)}, x)|]&\times \\ \times[|p_{n-1}'(w^{(a,b)}, x)|+|p_n'(w^{(a,b)}, x)|] \quad &(x\in[-1, 1]),\end{aligned}$$

$n\in\mathbf{N}$, which is a special case of a general inequality proved in [24, Lemma 1, p. 31]. Setting here $x=x_{kn}(w)$ we can apply Lemma 4.1 to estimate $p_{n+\ell}(w^{(a,b)}, x_{kn}(w))$. Moreover, since

$$p_n'(w^{(a,b)}, x) = \text{const}\, p_{n-1}(w^{(a+1,b+1)}, x)$$

where the constant is of precise order n (cf. [32, formula (4.21.7), p. 63]), we can use (4.1) and (4.2) to estimate $p_{n+\ell}'(w^{(a,b)}, x_{kn}(w))$. We obtain

$$\begin{aligned}|g(x_{kn}(w))K_n'(w, x_{kn}(w), x_{kn}(w))-K_n'(w^{(a,b)}, x_{kn}(w), x_{kn}(w))| &\leqq \\ \leqq Knw^{-1}(x_{kn}(w))(1-x_{kn}(w)^2)^{-1/2}&\end{aligned} \tag{4.19}$$

for $1\leqq k\leqq n$ and $n\in\mathbf{N}$. Now the point is that $K_n'(w^{(a,b)}, x_{kn}(w), x_{kn}(w))$ can be evaluated. By (3.2)

$$K_n'(w) = (\gamma_{n-1}(w)/\gamma_n(w))[p_n''(w)p_{n-1}(w)-p_n(w)p_{n-1}''(w)],$$

and since the Jacobi polynomials satisfy the differential equation

$$(1-x^2)Y'' = -n(n+a+b+1)Y+[a-b+(a+b+2)x]Y'$$

(cf. [32, Theorem 4.2.1, p. 60]) we obtain

$$\begin{aligned}K_n'(w^{(a,b)}, x, x) = (1-x^2)^{-1}\{[a-b+(a+b+2)x]K_n(w^{(a,b)}, x, x)-&\\ -(\gamma_{n-1}(w^{(a,b)})/\gamma_n(w^{(a,b)}))(2n+a+b)p_{n-1}(w^{(a,b)}, x)p_n(w^{(a,b)}, x)\}&\end{aligned}$$

(which, as a matter of fact, immediately yields formula (4.17)). We have $\gamma_{n-1}(w^{(a,b)})/\gamma_n(w^{(a,b)})\to 1/2$, $n\to\infty$ (cf. (4.11)). Therefore, substituting $x=x_{kn}(w)$ here and applying Lemma 4.1, we get

$$\begin{aligned}(4.20)\qquad &\big|K_n'(w^{(a,b)}, x_{kn}(w), x_{kn}(w))-[1-x_{kn}(w)^2]^{-1}\times\\ &\times[a-b+(a+b+2)x_{kn}(w)]K_n(w^{(a,b)}, x_{kn}(w), x_{kn}(w))\big|\leqq\\ &\leqq Knw(x_{kn}(w))^{-1}[1-x_{kn}(w)^2]^{-1/2}\end{aligned}$$

for $1\leqq k\leqq n$ and $n\in\mathbf{N}$. Inequalities (4.19) and (4.20) enable us to conclude

$$\begin{aligned}&\big|g(x_{kn}(w))K_n'(w, x_{kn}(w), x_{kn}(w))-[1-x_{kn}(w)^2]^{-1}\times\\ &\times[a-b+(a+b+2)x_{kn}(w)]K_n(w^{(a,b)}, x_{kn}(w), x_{kn}(w))\big|\leqq\\ &\leqq Knw^{-1}(x_{kn}(w))(1-x_{kn}(w)^2)^{-1/2}\end{aligned}$$

for $1\leqq k\leqq n$ and $n\in\mathbf{N}$. Now we apply Lemma 4.2 with weights $w_1=w$ and $w_2=w^{(a,b)}$. We obtain

$$\begin{aligned}&\big|K_n'(w, x_{kn}(w), x_{kn}(w))-[1-x_{kn}(w)^2]^{-1}\times\\ &\times[a-b+(a+b+2)x_{kn}(w)]K_n(w, x_{kn}(w), x_{kn}(w))\big|\leqq\\ &\leqq Knw^{-1}(x_{kn}(w))(1-x_{kn}(w)^2)^{-1/2}\end{aligned}$$

for $1\leqq k\leqq n$ and $n\in\mathbf{N}$. Since $K_n=\lambda_n^{-1}$ so that $K_n'/K_n=-\lambda_n'/\lambda_n$, and since the right-hand side here is precisely of order $\lambda_{kn}(w)^{-1}$ (cf. (4.8)), the latter inequality is equivalent to (4.18) what we had to prove.

Freud's formula (3.9) and Lemma 4.3 immediately yield

Lemma 4.4. *Let $w=gw^{(a,b)}$ be a generalized Jacobi weight. Then*

$$\begin{aligned}|v_{kn}(w,x)-1-[1-x_{kn}(w)^2]^{-1}[a-b+(a+b+2)x_{kn}(w)](x-x_{kn}(w))|&\leqq\\ \leqq K|x-x_{kn}(w)|&\end{aligned}$$

uniformly for $x\in[-1,1]$, $1\leqq k\leqq n$ and $n\in\mathbf{N}$.

The following three purely technical lemmas deal with Lebesgue function type estimates.

Lemma 4.5. *Let $w=gw^{(a,b)}$ be a generalized Jacobi weight, and let $c\in\mathbf{R}$. Then the asymptotics*

$$(4.21)\qquad \sum_{k=1}^{n}[1-x_{kn}(w)]^{-c}[x-x_{kn}(w)]^2\ell_{kn}^2(w,x)\sim$$

$$\sim p_n^2(w,x)\begin{cases}n^{-1} & \textit{for}\quad a-c+2>0,\\ n^{-1}\log n & \textit{for}\quad a-c+2=0,\\ n^{2(c-a-5/2)} & \textit{for}\quad a-c+2<0,\end{cases}$$

holds uniformly for $n\in\mathbf{N}$ *and* $x\in\mathbf{R}$. *In addition, analogous estimates hold when* $[1-x_{kn}(w)]^{-c}$ *is replaced by* $[1+x_{kn}(w)]^{-c}$ *in the left-hand side of* (4.21).

Proof. By (3.6) we have

$$[1-x_{kn}(w)]^{-c}[x-x_{kn}(w)]^2\ell_{kn}^2(w,x)\,p_n^{-2}(w,x) =$$
$$= (\gamma_{n-1}(w)/\gamma_n(w))^2[1-x_{kn}(w)]^{-c}\lambda_{kn}(w)^2\,p_{n-1}^2(w,x_{kn}(w)).$$

Since $\lim\,[\gamma_{n-1}(w)/\gamma_n(w)]=1/2$ (cf. (4.11)), we can use (4.4) and (4.8) to obtain

$$[1-x_{kn}(w)]^{-c}[x-x_{kn}(w)]^2\ell_{kn}^2(w,x)\,p_n^{-2}(w,x)\sim$$
$$\sim n^{-2}[1-x_{kn}(w)]^{-c+a+3/2}[1+x_{kn}(w)]^{b+3/2}$$

for $n\in\mathbf{N}$, and then (4.21) follows from (4.1) via routine estimates.

Lemma 4.6. *Let* $w=gw^{(a,b)}$ *be a generalized Jacobi weight function, and let* $0<\varepsilon<1$. *Then*

$$\sup_{n\ge 1}\max_{-\varepsilon\le x\le\varepsilon}\sum_{k=1}^{n}[1-x_{kn}(w)]^{-c}|v_{kn}(w,x)|\,\ell_{kn}^2(w,x)<\infty \tag{4.22}$$

for $c\le a+3/2$ *and*

$$\lim_{n\to\infty}\max_{-\varepsilon\le x\le\varepsilon}\sum_{k=1}^{n}[1-x_{kn}(w)]^{-c}|x-x_{kn}(w)|\,\ell_{kn}^2(w,x)=0 \tag{4.23}$$

for $c<a+5/2$.

Proof. First let $c=0$. For $c=0$ formula (4.23) was proved in [24, Lemma 4, (36), p. 40]. The proof of (4.22) with $c=0$ is based on

$$\sup_{n\ge 1}\max_{-\varepsilon\le x\le\varepsilon}\sum_{k=1}^{n}\ell_{kn}^2(w,x)<\infty \tag{4.24}$$

which was verified in [24, Lemma 4, (35), p. 40]. We write

(4.25)

$$\sum_{k=1}^{n}|v_{kn}(w,x)|\,\ell_{kn}^2(w,x)=\sum_{2|x_{kn}|<1+\varepsilon}|v_{kn}(w,x)|\,\ell_{kn}^2(w,x)+\sum_{2|x_{kn}|\ge 1+\varepsilon}|v_{kn}(w,x)|\,\ell_{kn}^2(w,x).$$

By Freud's formula (3.9) and by Lemma 4.3 (cf. (4.18)), $|v_{kn}(w,x)|\le K$ for $2|x_{kn}|<1+\varepsilon$ and $-\varepsilon\le x\le\varepsilon$. Hence (4.24) can be used to estimate the first sum on the right-hand side of (4.25). For $2|x_{kn}|\ge 1+\varepsilon$ and $-\varepsilon\le x\le\varepsilon$ we can apply again (3.9) and (4.18) to obtain $|v_{kn}(w,x)|\le K[1-x_{kn}(w)^2]^{-1}$. Now, in view of (4.4) and (4.8), $\lambda_{kn}(w)p_{n-1}^2(w,x_{kn}(w))[1-x_{kn}(w)^2]^{-1}\sim n^{-1}$. Therefore, the Gauss—Jacobi quadrature formula (cf. [32, Theorem 3.4.1, p. 47]), (3.6) and (4.11) yield

$$\sum_{2|x_{kn}|\ge 1+\varepsilon}|v_{kn}(w,x)|\,\ell_{kn}^2(w,x)\le Kn^{-1}p_n^2(w,x)\sum_{2|x_{kn}|\ge 1+\varepsilon}\lambda_{kn}(w)\le$$
$$\le Kn^{-1}p_n^2(w,x)\sum_{k=1}^{n}\lambda_{kn}(w)=Kn^{-1}p_n^2(w,x)\int_{\mathbf{R}}w$$

for $-\varepsilon \leqq x \leqq \varepsilon$ and $n \in \mathbf{N}$. By (4.3) the generalized Jacobi polynomials are uniformly bounded for $-\varepsilon \leqq x \leqq \varepsilon$. Therefore, the second sum on the right-hand side of (4.25) converges to 0 as $n \to \infty$ uniformly for $-\varepsilon \leqq x \leqq \varepsilon$. Consequently (4.22) and (4.23) hold for $c=0$ which naturally implies their validity for all $c<0$ as well. The extension of (4.22) and (4.23) to all permissable values of c is done via Lemma 4.5 as follows. We write

$$\sum_{k=1}^{n} [1-x_{kn}(w)]^{-c} |v_{kn}(w, x)| \ell_{kn}^2(w, x) = \sum_{2|x_{kn}|<1+\varepsilon} [1-x_{kn}(w)]^{-c} |v_{kn}(w, x)| \ell_{kn}^2(w, x) +$$
$$+ \sum_{2|x_{kn}|\geqq 1+\varepsilon} [1-x_{kn}(w)]^{-c} |v_{kn}(w, x)| \ell_{kn}^2(w, x).$$

To prove (4.22), we can estimate the first sum on the right-hand side here by (4.22) applied with $c=0$, whereas for the second sum Lemma 4.5 can be used in the following way. First, we can assume that $c > a+1$. Second, we do not need to concern ourselves with $p_n^2(w, x)$ since by (4.2) it is uniformly bounded in the interval $[-\varepsilon, \varepsilon]$. Thirdly, we note as before that $|v_{kn}(w, x)| \leqq K[1-x_{kn}(w)^2]^{-1}$ (cf. (3.9) and (4.18)). Thus applying (4.21) with $c+1$ instead of c, inequality (4.22) follows. Formula (4.23) can be proved in a similar way from (4.21) applied with c and then from (4.23) applied with $c=0$.

Lemma 4.7. *Let $w = gw^{(a,b)}$ be a generalized Jacobi weight function, and let $-1<\varepsilon<1$. Then for every nonnegative c we have*

$$\sup_{n\geqq 1} \max_{\varepsilon \leqq x \leqq 1} (1-x)^c \sum_{k=1}^{n} [1-x_{kn}(w)]^{-c} \ell_{kn}^2(w, x) < \infty \tag{4.26}$$

if $c-5/2 < a < c$,

$$\sup_{n\geqq 1} \max_{\varepsilon \leqq x \leqq 1} (1-x)^c \sum_{k=1}^{n} [1-x_{kn}(w)]^{-c-1} |x-x_{kn}(w)| \ell_{kn}^2(w, x) < \infty \tag{4.27}$$

if $c-3/2 \leqq a < c$, and

$$\lim_{n\to\infty} \max_{\varepsilon \leqq x \leqq 1} (1-x)^c \sum_{k=1}^{n} [1-x_{kn}(w)]^{-c} |x-x_{kn}(w)| \ell_{kn}^2(w, x) = 0 \tag{4.28}$$

if $c-5/2 < a < c$.

Proof. Unfortunately, we were unable to find a nontechnical proof, not even one with partially soft features. On the other hand, the computation yielding (4.26)—(4.28) is totally routine, and thus we can (and must) save the reader from the details. Instead, we provide a few hints and instructions as to the nature of the computations. Thus, let $c \geqq 0$ satisfy the appropriate conditions. First, by Lemma 4.6 we can assume $\varepsilon = 1/2$. Second, in view of Lemma 4.5 and inequality (4.2), one needs to consider only those values of k in (4.26)—(4.28) for which $x_{kn}(w)$ is positive. Third, since

$$\ell_{mn}^2(w, x) \leqq \lambda_{mn}(w) \sum_{k=1}^{n} \ell_{kn}^2(w, x)\lambda_{kn}(w)^{-1} = \lambda_{mn}(w)\lambda_n^{-1}(w, x)$$

(cf. (3.3) and [7, formula (1.4.7), p. 25]), we have by (4.1), (4.7) and (4.8)

$$\sup_{n\geqq 1} \max_{\varepsilon\leqq x\leqq 1} (1-x)^c[1-x_{mn}(w)]^{-c}\ell_{mn}^2(w, x) <\infty,$$

$$\sup_{n\geqq 1} \max_{\varepsilon\leqq x\leqq 1} (1-x)^c[1-x_{mn}(w)]^{-c-1}|x-x_{mn}(w)|\ell_{mn}^2(w, x) <\infty$$

and

$$\lim_{n\to\infty} \max_{\varepsilon\leqq x\leqq 1} (1-x)^c[1-x_{mn}(w)]^{-c}|x-x_{mn}(w)|\ell_{mn}^2(w, x) = 0$$

for all nonnegative c. Here (and in what follows) m is the index of one of the zeros $x_{kn}(w)$ which are closest to x. Hence it is sufficient to estimate the sums in (4.26)—(4.28) for which $x\geqq 1/2$, $x_{kn}(w)>0$ and $k\neq m$. For such values of x and $x_{kn}(w)$ we can use (4.1) to verify $(1-x)\leqq K(m/n)^2$, $(1-x_{kn}(w))\sim(k/n)^2$ and $|x-x_{kn}(w)|\sim \sim|m^2-k^2|n^{-2}$. Moreover, in view of expression (3.6) for the fundamental polynomials $\ell_{kn}(w)$, we also need inequalities for $\gamma_{n-1}(w)/\gamma_n(w)$, $\lambda_{kn}(w)$, $|p_{n-1}(w, x_{kn}(w))|$ and $|p_n(w, x)|$. The required estimates are given by formulas (4.11), (4.8), (4.4) and (4.2), respectively (cf. (4.1) as well). Putting all the pieces together, the proof of the lemma is reduced to showing

$$\sup_{n\geqq 1} \max_{1\leqq m\leqq n} m^{-2a+2c-1} \sum_{\substack{k=1\\k\neq m}}^{n} k^{2a-2c+3}|m^2-k^2|^{-2} <\infty \tag{4.29}$$

if $c-5/2<a<c$,

$$\sup_{n\geqq 1} \max_{1\leqq m\leqq n} m^{-2a+2c-1} \sum_{\substack{k=1\\k\neq m}}^{n} k^{2a-2c+1}|m^2-k^2|^{-1} <\infty \tag{4.30}$$

if $c-3/2\leqq a<c$, and

$$\lim_{n\to\infty} \max_{1\leqq m\leqq n} n^{-2}m^{-2a+2c-1} \sum_{\substack{k=1\\k\neq m}}^{n} k^{2a-2c+3}|m^2-k^2|^{-1} = 0 \tag{4.31}$$

if $c-5/2<a<c$. Estimating sums such as the ones in (4.29)—(4.31) is a routine exercise, and it is easily accomplished via splitting up the range of the index k into four subsets given by the inequalities $1\leqq k\leqq[m/2]$, $[m/2]<k<m$, $m<k<2m$ and $2m\leqq k\leqq n$. Or, as an alternative, one can apply [19, Lemma 6.3, p. 109] from which (4.29)—(4.31) follow immediately.

5. Underlying ideas (Part I)

Even though a significant portion of results concerning Hermite—Fejér interpolation is proved via hard analysis, such an approach is not always capable of producing the right result. For instance, if one tries to prove the uniform boundedness of the Hermite—Fejér interpolating polynomials associated with the zeros of Legendre polynomials in $[-1, 1]$ by splitting up the interpolating polynomials and by attempting to prove the uniform boundedness of $\sum |x-x_{kn}|(1-x_{kn}^2)^{-1}\ell_{kn}^2(x)$ and $\sum \ell_{kn}^2(x)$ which comes to one's mind when examining (3.7) and (4.17) with $a=b=0$, then one is destined to fail since the maximums of the latter two expressions are of precise order $\log n$, and thus they are unbounded. In other words, Proposition 1.2 with $a=b=0$ holds for more delicate reasons. These reasons are of the soft variety related to the positivity of the operator sequence $\{H_n(w^{(0,0)})\}$. Since for generalized Jacobi weights of the form $w=gw^{(0,b)}$ both sequences

$$\sum |x-x_{kn}(w)|(1-x_{kn}(w)^2)^{-1}\ell_{kn}^2(w, x)$$

and $\sum \ell_{kn}^2(w, x)$ are also unbounded on $[\varepsilon, 1]$, one is forced again into finding a more sensible and sensitive approach to estimating $\{H_n(w,f)\}$. This is the subject of this section, and we will accomplish it via soft analysis which is based on some quasi-positivity properties of the former sequence.

Theorem 5.1. *Let* $w=gw^{(0,b)}$ *be a generalized Jacobi weight function, and let* $-1<\varepsilon<1$. *Then*

$$\sup_{n\geq 1} \max_{\varepsilon\leq x\leq 1} |H_n(w, f, x)| < \infty$$

for every function f bounded in $[-1, 1]$.

Proof. According to (3.7) we have to prove

$$\sup_{n\geq 1} \max_{\varepsilon\leq x\leq 1} \sum_{k=1}^{n} |v_{kn}(w, x)|\ell_{kn}^2(w, x) < \infty. \tag{5.1}$$

Step 1. Here we will show quasi-positivity of $v_{kn}(w)$ in some sense which helps to reduce the Lebesgue function in (5.1) to an expression which can be subjected to rougher handling without ruining its essential behavior. Our main tool is Lemma 4.4 applied with $a=0$ according to which

$$v_{kn}(w, 1) \geq (1+b)(1-x_{kn}(w))(1+x_{kn}(w))^{-1}-K(1-x_{kn}(w)). \tag{5.2}$$

Hence there exists $d\in[-1, \varepsilon)$ such that $v_{kn}(w, 1)\geq 1$ for $-1<x_{kn}(w)\leq d$. But $v_{kn}(w)$ is a linear function which takes the value 1 at $x_{kn}(w)$. Consequently,

$$v_{kn}(w, x) \geq 1 \quad \text{for} \quad -1 < x_{kn}(w) \leq d \quad \text{and} \quad x\in[\varepsilon, 1]. \tag{5.3}$$

If $d<x_{kn}(w)<1$ then by (5.2)

$$v_{kn}(w,1)\geqq -K(1-x_{kn}(w)),$$

and by (3.9), (4.8) and (4.10) we have $|v'_{kn}(w,x)|\leqq K(1-x_{kn}(w))^{-1}$. Therefore we obtain

$$\begin{aligned} & v_{kn}(w,x)\geqq -\tilde{K}(1-x_{kn}(w))- \\ (5.4)\qquad & -\tilde{K}(1-x)(1-x_{kn}(w))^{-1} \quad \text{for} \quad d<x_{kn}(w)<1 \quad \text{and} \quad x\in[\varepsilon,1] \end{aligned}$$

with an appropriate positive constant $\tilde{K}$. Now by (5.3) and (5.4) we have

$$\begin{aligned} \sum_{k=1}^{n}|v_{kn}(w,x)|\ell_{kn}^2(w,x) &= \sum_{x_{kn}\leqq d}|v_{kn}(w,x)|\ell_{kn}^2(w,x)+\sum_{x_{kn}>d}|v_{kn}(w,x)|\ell_{kn}^2(w,x)\leqq \\ &\leqq \sum_{x_{kn}\leqq d}v_{kn}(w,x)\ell_{kn}^2(w,x)+\sum_{x_{kn}>d}v_{kn}(w,x)\ell_{kn}^2(w,x)+ \\ &+2\tilde{K}\sum_{x_{kn}>d}(1-x_{kn}(w))\ell_{kn}^2(w,x)+2\tilde{K}(1-x)\sum_{x_{kn}>d}(1-x_{kn}(w))^{-1}\ell_{kn}^2(w,x)= \\ &= 1+2\tilde{K}\sum_{{}_{kn}x>d}(1-x_{kn}(w))\ell_{kn}^2(w,x)+2\tilde{K}(1-x)\sum_{x_{kn}>d}(1-x_{kn}(w))^{-1}\ell_{kn}^2(w,x) \end{aligned}$$

since Hermite—Fejér interpolation preserves the constant function. Using the asymptotics for the Cotes numbers (4.8) we obtain from here

$$\begin{aligned} (5.5)\qquad \sum_{k=1}^{n}|v_{kn}(w,x)|\ell_{kn}^2(w,x) &\leqq 1+Kn^{-1}\sum_{k=1}^{n}(1-x_{kn}(w))^{3/2}\lambda_{kn}(w)^{-1}\ell_{kn}^2(w,x)+ \\ &+Kn^{-1}(1-x)\sum_{k=1}^{n}(1-x_{kn}(w))^{-1/2}\lambda_{kn}(w)^{-1}\ell_{kn}^2(w,x) \end{aligned}$$

which is the inequality we were to establish in the first step of the proof.

Step 2. The first sum on the right-hand side of (5.5) can be estimated by applying the same techniques that led to (4.26) in Lemma 4.7. However, we will proceed in a different way which consists of evaluating the sum $\sum(1-x_{kn}(w))\lambda_{kn}(w)^{-1}\ell_{kn}^2(w,x)$ in a closed form. We have

$$\begin{aligned} \sum_{k=1}^{n}(1-x_{kn}(w))\lambda_{kn}(w)^{-1}\ell_{kn}^2(w,x) &= (1-x)\sum_{k=1}^{n}\lambda_{kn}(w)^{-1}\ell_{kn}^2(w,x)+ \\ &+\sum_{k=1}^{n}(x-x_{kn}(w))\lambda_{kn}(w)^{-1}\ell_{kn}^2(w,x). \end{aligned}$$

Here the first sum on the right-hand side equals $\lambda_n^{-1}(w,x)$ (cf. [7, formula (1.4.7), p. 25]), whereas the second one can be obtained from (3.6) and the Lagrange interpolation formula. We get

$$\begin{aligned} &\sum_{k=1}^{n}(1-x_{kn}(w))\lambda_{kn}(w)^{-1}\ell_{kn^2}(w,x)= \\ &=(1-x)\lambda_n^{-1}(w,x)+(\gamma_{n-1}(w)/\gamma_n(w))p_n(w,x)p_{n-1}(w,x). \end{aligned}$$

Thus, applying (4.2), (4.7) and (4.11) we obtain

$$\sup_{n\geq 1}\max_{\varepsilon\leq x\leq 1} n^{-1}\sum_{k=1}^{n}(1-x_{kn}(w))\lambda_{kn}(w)^{-1}\ell_{kn}^{2}(w,x)<\infty$$

from which the inequality

$$\sup_{n\geq 1}\max_{\varepsilon\leq x\leq 1} n^{-1}\sum_{k=1}^{n}(1-x_{kn}(w))^{3/2}\lambda_{kn}(w)^{-1}\ell_{kn}^{2}(w,x)<\infty \tag{5.6}$$

follows as well.

Step 3. The uniform boundedness of the second sum on the right-hand side of (5.5) was established in Lemma 4.7 (cf. (4.26)). This can also be shown via replacing computations by some properties of Christoffel functions as follows. By Cauchy's inequality

$$\Big[\sum_{k=1}^{n}(1-x_{kn}(w))^{-1/2}\lambda_{kn}(w)^{-1}\ell_{kn}^{2}(w,x)\Big]^{2}\leq\sum_{k=1}^{n}\lambda_{kn}(w)^{-1}\ell_{kn}^{2}(w,x)\times$$

$$\times\sum_{k=1}^{n}(1-x_{kn}(w))^{-1}\lambda_{kn}(w)^{-1}\ell_{kn}^{2}(w,x)=\lambda_{n-1}(w,x)\lambda_{n-1}(\tilde{w},x)$$

(here $\tilde{w}$ is defined by $\tilde{w}(x)=(1-x)w(x)$) where we used two identities involving Christoffel functions (cf. [7, formula (1.4.7), p. 25], [6, Lemma 2, formula (15), p. 251] and [19, Lemma 6.1.4, p. 59]). Since both w and $\tilde{w}$ are generalized Jacobi weights, we can use (4.7) to obtain

$$\sup_{n\geq 1}\max_{\varepsilon\leq x\leq 1} n^{-1}(1-x)\sum_{k=1}^{n}(1-x_{kn}(w))^{-1/2}\lambda_{kn}(w)^{-1}\ell_{kn}^{2}(w,x)<\infty. \tag{5.7}$$

Inequality (5.1) follows from (5.5)—(5.7), and so does the theorem.

6. Underlying ideas (Part II)

Here we will be concerned about the connection between uniform convergence of Hermite—Fejér interpolation and its behavior at one single point. In other words, we look behind the scenes that govern the phenomenon described in Theorem 5.1.

For s nonnegative integer define the function u_s by $u_s(x)=(1-x)^s$. Then it turns out that under certain circumstances it is more convenient to approximate $f\in C[-1,1]$ by $u_sH_n(w,fu_s^{-1})$ then by $H_n(w,f)$. Since the former vanishes at $x=1$ for $s>0$, it can only approximate such functions f which also vanish at $x=1$. What is against $H_n(w,f)$ is that if $w(1)=0$ then the sequence of the corresponding Lebesgue functions becomes unbounded at 1 and thus $\lim H_n(w,f)=f$ cannot be expected at the point 1 or uniformly in a neighborhood of 1. What $u_s H_n(w, fu_s^{-1})$ does is that it tempers the quick growth of $p_n(w)$ in such a way that and operator $U_s H_n(w) U_s^{-1}$ (where U_s^{-1} is the multiplication operator defined by the formula $U_s(g)=u_s g$) becomes appropriately balanced with the right choice of s.

The real role of $u_s H_n(w, fu_s^{-1})$ is that it is the principal term in the Hermite—Fejér type interpolating polynomial $H_{n,s}(w,f)$ defined by

$$H_{n,s}(w,f,x_{kn}(w)) = f(x_{kn}(w)), \quad H'_{n,s}(w,f,x_{kn}(w)) = 0,$$

$k=1, 2, \ldots, n$, and

$$H^{(j)}_{n,s}(w,f,1) = 0$$

for $j=0, 1, \ldots, s-1$. The closed formula for $H_{n,s}(w,f)$ is given by

$$H_{n,s}(w,f) = u_s H_n(w, fu_s^{-1}) + u_s \mathcal{H}_n(w, f[u_s^{-1}]') \tag{6.1}$$

(cf. (3.7) and (3.11)) which is easy to verify directly (cf. [38, Section 3.2, p. 88]). It was E. Egerváry and P. Turán [3] who first realized how $H_{n,1}(w,f)$ can be used to investigate uniform convergence of $H_n(w,f)$ for the Legendre weight function $w=w^{(0,0)}$. The process $H_{n,s}(w^{(a,b)},f)$ was fully investigated in [38] where it was shown that it can be used to prove necessary and sufficient conditions for uniform convergence of Hermite—Fejér interpolation at zeros of Jacobi polynomials. The reason for the usefulness of $u_s H_n(w, fu_s^{-1})$ and $H_{n,s}(w,f)$ lies in the representation

(6.2)

$$H_n(w,f,x) = H_{n,s}(w,f,x) + p_n^2(w,x) \sum_{k=0}^{s-1} (1/k!)[H_n(w,f,1)p_n^{-2}(w,1)]^{(k)}(x-1)^k$$

which provides a direct link between $H_n(w,f)$, $u_s H_n(w, fu_s^{-1})$, $p_n(w)$ and $H_n(w,f,1)$. The verification of (6.2) is again easily done by checking out the interpolation conditions. The following is not only a tool necessary for proving one our main results (cf. Theorem 2.5) but the special case $s=1$ is also a de facto solution of P. Turán's Problem XXVII in his collection of "On some open problems of approximation theory" (cf. [35, p. 47]).

Theorem 6.1. *Let $w=gw^{(a,b)}$ be a generalized Jacobi weight function, and let $-1<\varepsilon<1$. Let f be continuous in $[-1,1]$ such that $f(1)=0$. Let $a\geq 0$, $b>-1$, and let s be a fixed positive integer such that $a\in[s-1,s)$. Then*

$$\lim_{n\to\infty} \max_{\varepsilon\leq x\leq 1} |f(x)-u_s(x)H_n(w, fu_s^{-1}, x)| = 0 \tag{6.3}$$

and

$$\lim_{n\to\infty} \max_{\varepsilon\leq x\leq 1} |f(x)-H_{n,s}(w,f,x)| = 0. \tag{6.4}$$

Proof. In view of (3.11), (6.1) and Lemma 4.7 we have

$$\lim_{n\to\infty} \max_{\varepsilon\le x\le 1} |H_{n,s}(w,f,x)-u_s(x)H_n(w,fu_s^{-1},x)| = 0$$

so that it is sufficient to prove (6.3).

Step 1. First we prove (6.3) for the special case when the function f is given by $f(x)=1-x$. Then

$$f(x)-u_s(x)H_n(w,fu_s^{-1},x) = u_1(x)\big[1-u_{s-1}(x)H_n\big(w,(u_{s-1})^{-1},x\big)\big]$$

so that applying (6.1) and (6.2) with $f\equiv 1$ and $s-1$ we obtain

$$f(x)-u_s(x)H_n(w,fu_s^{-1},x) = u_1(x)\big[1-H_{n,s-1}(w,1,x)+u_{s-1}(x)\mathcal{H}_n(w,[u_{s-1}^{-1}]',x)\big]=$$

$$= u_1(x)p_n^2(w,x)\sum_{k=0}^{s-2}(1/k!)[p_n^{-2}(w,1)]^{(k)}(x-1)^k+u_1(x)\big[u_{s-1}(x)\mathcal{H}_n(w,[u_{s-1}^{-1}]',x)\big].$$

Here the first term on the right-hand side can be estimated using (4.2) and (4.6), while the second one by Lemma 4.7 (cf. (3.11) and (4.28) applied with $c=s$). This proves (6.3) for $f=u_1$.

Step 2. Now let f be continuous and $f(1)=0$. The point is that the sequence of operators from $C[-1,1]$ into $C[\varepsilon,1]$ given by $f\to u_sH_n(w,fu_s^{-1})$ is uniformly bounded by (3.7), (3.9), (4.8), (4.10) and Lemma 4.7 (cf. (4.26) and (4.27) applied with $c=s$). Therefore we can finish the proof in the routine fashion as follows. Given $\delta>0$ there exists a polynomial P such that $P(1)=0$ and $|f(x)-P(x)|\le\delta$ for $x\in[-1,1]$ (cf. [34, Theorem 2, p. 259]). Write $P=u_1Q$. With this polynomial P we have

$$(6.5)\quad f(x)-u_s(x)H_n(w,fu_s^{-1},x) = [f(x)-P(x)]-u_s(x)H_n\big(w,(f-P)u_s^{-1},x\big)-$$

$$-u_s(x)H_n\big(w,[Q-Q(x)][u_{s-1}]^{-1},x\big)+Q(x)[u_1(x)-u_s(x)H_n(w,u_1u_s^{-1},x)].$$

By (3.7), (3.9), (4.8), (4.10) and Lemma 4.7 (cf. (4.28) applied with $c=s$)

$$\lim_{n\to\infty}\max_{\varepsilon\le x\le 1}\big|u_s(x)H_n\big(w,[Q-Q(x)][u_{s-1}]^{-1},x\big)\big| = 0$$

since $|Q(x_{kn})-Q(x)|\le K|x_{kn}-x|$, whereas the last term on the right-hand side was taken care of in the first part of the proof. Therefore letting $n\to\infty$ in (6.5) we obtain

$$\limsup_{n\to\infty}\max_{\varepsilon\le x\le 1}|f(x)-u_s(x)H_n(w,fu_s^{-1},x)| \le K\delta$$

from which (6.3) follows.

7. The proofs

On the basis of the results in Sections 4—6 this can be accomplished virtually in a few lines.

Proof of Theorem 2.1. This follows from Lemma 4.6 applied with $c=0$. The details are as follows. By (3.7) and (4.22) the sequence of Hermite—Fejér interpolating polynomials is a bounded sequence of operators from $C[-1, 1]$ to $C[-\varepsilon, \varepsilon]$. By (3.10), (3.11) and (4.23) it converges for polynomials, that is for a dense set of function in $C[-1, 1]$.

Proof of Theorem 2.2. This was de facto proved in [24, Lemma 5, formula (46), p. 43] where it is given with n^{-2a} replaced by $np_n^{-2}(w, 1)$. However, in view of (4.3), they are of the same order.

Proof of Theorem 2.3. First let $w(1)=\infty$. Then by (3.7), (3.9), (4.2), (4.8), (4.10), Lemma 4.5 and Lemma 4.7 (cf. (4.26) and (4.27) applied with $c=0$) the Hermite—Fejér interpolating polynomials are uniformly bounded in $[\varepsilon, 1]$ (here inequality (4.2) and Lemma 4.5 are needed to estimate the expression $\sum [1+x_{kn}(w)]^{-1}[x-x_{kn}(w)]\ell_{kn}^2(w, x)$). If $0<w(1)<\infty$ then this is given in Theorem 5.1. The necessity of the condition $w(1)\neq 0$ follows from Theorem 2.2.

Proof of Theorem 2.4. If $w(1)=\infty$ then by formulas (3.7), (3.10), (3.11) and Lemma 4.7 (cf. (4.28) applied with $c=0$) the Hermite—Fejér interpolating polynomials $H_n(w, P)$ converge uniformly in $[\varepsilon, 1]$ for every fixed polynomial P. Thus Theorem 2.3 yields convergence for every continuous function. The necessity of the condition $w(1)=\infty$ for uniform convergence in $[\varepsilon, 1]$ follows from Theorem 2.2.

Proof of Theorem 2.5. If $\lim H_n(w,f)=f$ uniformly in a left neighborhood of the point 1 then by Markov's theorem (cf. [16, § VI.6, p. 141]) the r-th derivative of $H_n(w,f)$ is $o(n^{2r})$ in the same interval for every $r=1, 2, \ldots$. On the other hand, if we have information concerning the behavior of $H_n(w,f)$ at 1 then we can use Theorem 6.1 (either of (6.3) and (6.4)) and formulas (6.1) and (6.2). First, we can assume without loss of generality that $f(1)=0$ (cf. (3.7), (3.10) and (3.11)). We need to prove

$$\lim_{n\to\infty} \max_{[\varepsilon,1]} p_n^2(w, x) \sum_{k=0}^{s-1} (1/k!)[H_n(w,f, 1)p_n^{-2}(w, 1)]^{(k)}(x-1)^k = 0.$$

This follows immediately by straightforward application of inequalities (4.2), (4.6) and the conditions $H_n^{(r)}(w,f, 1)=O(n^{2r})$, $r=0, 1, \ldots, s$.

Proof of Theorem 2.6. We use an observation by G. Freud in [9, formula (2), p. 176] according to which since $H_n'(w,f)$ vanishes at the zeros of $p_n(w)$ we have $H_n'(w,f)=p_n(w)Q_{n-2}$ where Q_{n-2} is a polynomial of degree at most $n-2$. Thus by orthogonality

$$\int_{-1}^{1} H_n'(w,f,t)[w(t)(1+t)]\,dt=0,$$

and integration by parts yields

$$H_n(w,f,1)=(2w(1))^{-1}\int_{-1}^{1} H_n(w,f,t)\,d[w(t)(1+t)]$$

(cf. [9, formula (4), p. 176]). Now we can use Proposition 1.6 applied with $u=w$ to pass to the limit of the integral which together with Theorem 2.5 proves Theorem 2.6.

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Orthogonal polynomials and their zeros

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Let $d\mu$ be a finite positive Borel measure on the interval $[0, 2\pi)$ such that its support is an infinite set, and let $\{\varphi_n\}_{n=0}^{\infty}$, $\varphi_n(z)=\varphi_n(d\mu, z)=\varkappa_n z^n+\dots$, $\varkappa_n=\varkappa_n(d\mu)>0$, denote the system of orthonormal polynomials associated with $d\mu$, that is,

$$\frac{1}{2\pi}\int_0^{2\pi}\varphi_m(z)\overline{\varphi_n(z)}\,d\mu(\theta)=\delta_{nm},\quad z=e^{i\theta}.$$

The corresponding monic orthogonal polynomials $\varkappa_n^{-1}\varphi_n$ will be denoted by Φ_n. For an nth degree polynomial P the reverse polynomial P^* is defined by $P^*(z)=z^n\overline{P(1/\bar{z})}$. Let $z_{kn}=z_{kn}(d\mu)$ be the zeros of φ_n ordered in such a way that

$$|z_{nn}|\leqq|z_{n-1,n}|\leqq\dots\leqq|z_{1n}|<1 \tag{1}$$

(cf. [7, p. 292]).

P. ALFARO and L. VIGIL (cf. [1, Proposition 1] and [2, Theorem 1]) proved that for every sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ with $|z_n|<1$, $n=1, 2, \dots$, there is a unique measure $d\mu$ (modulo an arbitrary positive constant factor) such that $\varphi_n(d\mu, z_n)=0$ for $n=1, 2, \dots$. This result can be obtained from the recurrence formula

$$\Phi_n(z)=z\Phi_{n-1}(z)+\Phi_n(0)\Phi_{n-1}^*(z) \tag{2}$$

(cf. [7, formula (11.4.7), p. 293]) as follows. By (2) the recurrence coefficients $\Phi_n(0)$

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can be expressed as

$$\Phi_n(0) = -z_{kn}\Phi_{n-1}(z_{kn})/\Phi^*_{n-1}(z_{kn}),$$

and thus one can successively define the monic polynomials ψ_n by $\psi_0=1$ and

$$\psi_n(z) = z\psi_{n-1}(z)-[z_n\psi_{n-1}(z_n)/\psi^*_{n-1}(z_n)]\psi^*_{n-1}(z),$$

$n=1, 2, \ldots$. It is a matter of simple induction to show that $|\psi_n(0)|<1$, $n=1, 2, \ldots$, and then $\{\psi_n\}_{n=0}^{\infty}$ is orthogonal with respect to some $d\mu$ (cf. [4, Theorem 8.1, p. 156]). Since $\psi_n(z_n)=0$, this is the measure $d\mu$ we were looking for.

We point out that P. Alfaro and L. Vigil's result solves the following problem proposed by P. Turán: is there a measure $d\mu$ such that the set $\{z_{kn}(d\mu)\}$ is dense in the unit disk (cf. [9, Problem 67, p. 69]). Namely, the above measure $d\mu$ associated with any sequence $\{z_n\}$ which is dense in the unit disk provides such an example.

In view of this result by P. Alfaro and L. Vigil (and also because of the relation $\Phi_n(0)=\Pi z_{kn}$), one would want to seek for connections between orthogonal polynomials, their zeros and their recurrence coefficients. In spite of the great variety of results of such nature for orthogonal polynomials on the real line, and in spite of the intimate connection between real and complex orthogonal polynomials, there is only a very limited amount of research performed in this direction (cf. J. SZABADOS [6] and R. Askey's comment to paper [34—2] in [8, Vol. 2, p. 542]).

The main purpose of this note is to find a relationship between the quantities r_1, r_2, r_3 and r_4 which are defined as follows:

$$r_1(d\mu) = \limsup_{n\to\infty} |\Phi_n(d\mu, 0)|^{1/n},$$

$$r_2(d\mu) = \inf_k \limsup_{n\to\infty} |z_{kn}(d\mu)|,$$

$$r_3(d\mu) = \{\inf r\colon \sup_n \max_{|z|=r^{-1}} |\Phi^*_n(d\mu, z)| < \infty\}$$

and

$$r_4(d\mu) = \{\inf r\colon D(d\mu, z)^{-1} \text{ is analytic for } |z| < r^{-1}\}$$

where for $|z|<1$ the Szegő function $D(d\mu)$ is given by

$$D(d\mu, z) = \exp\left\{\frac{1}{4\pi}\int_0^{2\pi} \log\mu'(t)\frac{u+z}{u-z}\,dt\right\}, \quad u = e^{it},$$

if $\log\mu'$ is integrable, and $D(d\mu)\equiv 0$ otherwise (cf. [3], [4], [5] and [7]).

Theorem 1. *For every measure $d\mu$ we have $r_1(d\mu)=r_2(d\mu)$. If there is $j\in\{1, 2, 3, 4\}$ such that $r_j(d\mu)<1$ then $r_1(d\mu)=r_3(d\mu)=r_4(d\mu)$.*

Proof.

Step 1. $r_1\le r_2$. Since $\Phi_n(0)=\Pi z_{kn}$ and $|z_{kn}|<1$, we have $|\Phi_n(0)|\le|z_{kn}|^{n-k+1}$ for $k=1, 2, \ldots, n$ (cf. (1)), and thus $r_1\le r_2$ follows.

Step 2. $r_1=1 \Rightarrow r_1=r_2$. This is obvious in view of Step 1 and $r_2 \leqq 1$.

Step 3. $r_1<1 \Rightarrow r_2, r_4 \leqq r_3 \leqq r_1$. By a result of YA. L. GERONIMUS [4, Theorem 8.3, p. 160] the sequence $\{|\Phi_n|\}$ is uniformly bounded on the unit circle. Thus by the maximum principle $\{|z^{-n}\Phi_n(z)|\}$ is uniformly bounded for $|z| \geqq 1$. Repeated application of the recurrence formula (2) leads to

$$\Phi_n^*(z) = 1 + z\sum_0^{n-1} \overline{\Phi_{k+1}(0)}\,\Phi_k(z).$$

Therefore $\lim \Phi_n^* = \Phi^*$ exists uniformly on every disk with radius less than r_1^{-1} which implies $r_3 \leqq r_1$. By formula (8.6) in [4, p. 156]

$$\varkappa_n^2 = \varkappa_0^2 \prod_{l=1}^{n} [1-|\Phi_l(0)|^2]^{-1} \tag{3}$$

so that $r_1<1$ implies the boundedness of the sequence $\{\varkappa_n\}$ which by a theorem of YA. L. GERONIMUS [4, Section 1.2 (15), p. 14] guarantees the integrability of $\log \mu'$. But then by the Szegő theory (cf. [7, Theorem 12.1.1, p. 297]) $\lim \Phi_n^* = D(0)D^{-1}$ holds uniformly on compact subsets of the open unit disk where D denotes the Szegő function. Applying Vitali's theorem we can conclude that $\lim \Phi_n^*(z) = \Phi^*(z)$ exists for every $|z|<r_3^{-1}$ and obtain $\Phi^* = D(0)D^{-1}$, and thus $r_4 \leqq r_3$. In addition, since Φ^* possesses at most a finite number of zeros inside every disk with radius $r<r_3^{-1}$, the number of elements of the sets $\{z\colon |z| \leqq r,\ \Phi_n^*(z)=0\}$ is bounded for every $r<r_3^{-1}$. This follows from Rouche's theorem. In other words, $\{|\{z_{kn}\}_{k=0}^{n} \cap \{z\colon |z| \geqq r\}|\}_{n=0}^{\infty}$ is bounded for every $r>r_3$. Thus $r_2 \leqq r_3$.

Step 4. $r_1 \leqq r_3$. We may assume $r_3<\infty$. Then by Cauchy's formula

$$\overline{\Phi_n(0)} = \frac{1}{2\pi i} \oint_{|z|=r^{-1}} z^{-n-1}\Phi_n^*(z)\,dz = O(r^n)$$

holds for every $r>r_3$. Hence $r_1 \leqq r_3$.

Step 5. $r_1 \leqq r_4$. We may assume $r_4<1$. Then $\log \mu'$ is apriori integrable, and thus we have the Szegő theory at our disposition. Applying formula (5.1.18) in [3, p. 195] and using $\lim \varphi_n^* = D^{-1}$ in $L_2(d\mu)$ (cf. [3, p. 219]) we obtain

$$\overline{\Phi_n(0)} = \frac{1}{2\pi} \int_0^{2\pi} D^{-1}(z)\,\overline{\Phi_n(z)}\,d\mu(\theta), \quad z=e^{i\theta}. \tag{4}$$

Let us denote the Taylor expansion of D^{-1} by $\Sigma c_k z^k$. Then $\limsup |c_k|^{1/k} = r_4$ and by orthogonality

$$\overline{\Phi_n(0)} = \frac{1}{2\pi}\int_0^{2\pi} \Big[\sum_{k=n}^{\infty} c_k z^k\Big] \overline{\Phi_n(z)}\, d\mu(\theta), \quad z = e^{i\theta}.$$

Now using Cauchy's inequality and $\varkappa_0 \leqq \varkappa_n$ (cf. (3)) we obtain $r_1 \leqq r_4$.

Combining the inequalities proved in Steps 1 through 5, we get immediately Theorem 1.

Corollary. *The following assertions are pairwise equivalent:*

(a) $\limsup_{n\to\infty} |z_{1n}(d\mu)| < 1$.

(b) $\limsup_{n\to\infty} |\Phi_n(d\mu, 0)|^{1/n} < 1$.

(c) *$d\mu$ is absolutely continuous and $\mu'(\theta) = g(\theta)$ a.e. where g is a positive analytic function.*

Remark. 1. Note that this corollary characterizes the measures for which all zeros of the corresponding orthogonal polynomials lie in a smaller circle inside the unit circle.

2. There are many other statements equivalent to (a) above. Here are a few of them:

(d) There is $0 < r < 1$ such that $\Phi_n(d\mu, z) = O(r^n)$ for $|z| = r$.

(e) $\limsup_{n\to\infty} \max_{|z|=1} |\Phi_{n+1}(d\mu, z) - z\Phi_n(d\mu, z)|^{1/n} < 1$.

(f) $\limsup_{n\to\infty} \operatorname{ess\,sup}_{|z|=1} |\Phi_n(d\mu, z)z^{-n} - D^{-1}(d\mu, z)| < 1$.

Using the considerations below it is a fairly simple exercise to prove that any of (d)—(f) is equivalent to any of (a)—(c).

Proof of the Corollary. (a)⇒(b) by Theorem 1. That (c) implies (b) follows from the formula

$$\bar{\Phi}_n(0) = \frac{1}{2\pi}\int_0^{2\pi} (D^{-1}(z) - T_{n-1}(\theta))\overline{\Phi_n(z)}\, d\mu(\theta) =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} ((\mu'(\theta))^{-1/2} - T_{n-1}(\theta))\overline{\Phi_n(e^{i\theta})}\, d\mu(\theta) \quad (z = e^{i\theta})$$

(cf. (4)) where T_{n-1} is any trigonometric polynomial of degree at most $n-1$, if we take into account that the Φ_n's are uniformly bounded on $|z|=1$ (see [4, Theorem 4.5]) and that, by the analiticity of $(\mu')^{-1/2}$, we can choose a $0<q<1$ and $\{T_{n-1}\}$ such that

$$\big|(\mu'(\theta))^{-1/2} - T_{n-1}(\theta)\big| \leqq Kq^n \quad (\theta \in [0, 2\pi]).$$

Finally, both (b)⇒(a) and (b)⇒(c) follows if we can show (see Step 3 above) that

(5) $$\Phi^*(z) \neq 0 \quad \text{if} \quad |z| = 1.$$

In fact, (b) implies that $d\mu$ is absolutely continuous (see [4, Theorem 8.5]) and $\mu'(\theta)=D^{-2}(e^{i\theta})$ a.e., hence (b)⇒(c) is an immediate consequence of Step 3; while (b)⇒(a) can be derived from Rouché's theorem, namely there is a neighbourhood U of the unit circumference such that Φ_n^* does *not* have a zero in U (and hence Φ_n does not have a zero in U^{-1}) for large n. (5) follows from

$$\Phi^*(z) = D(0)D^{-1}(z) = (\mu'(\theta))^{-1/2}; \quad z = e^{i\theta},$$

and the analiticity of Φ^* on $|z|=1$ (which was proved above under the assumption (b)), namely $\Phi^*(e^{i\theta_0})=0$ would imply that $\mu'(\theta)\sim(\theta-\theta_0)^{-2}$ in a neighborhood of θ_0 except on a set of measure zero and this contradicts $\mu'\in L^1[\theta_0-\pi, \theta_0+\pi]$. The proof is complete.

Example. Let $1<R\leqq\infty$. Let f be analytic in the open (but not in the closed) disk U_R with radius R centered at 0, and assume $f(0)=1$ and $f(z)\neq 0$ for $|z|\leqq 1$. Let $1<|z_1|\leqq|z_2|\leqq\dots$ be the zeros of f in U_R. Define the measure $d\mu$ by $d\mu(\theta)= =|f(e^{i\theta})|^{-2}\,d\theta$. Then $\lim \Phi_n^*(z)=f(z)$ uniformly for $|z|\leqq r<R$. Hence

(6) $$\lim_{n\to\infty} z_{kn}(d\mu) = (\bar{z}_k)^{-1}$$

holds for every k if f has infinitely many zeros in U_R. If f has finitely many zeros there, say N, then (6) is satisfied for $k=1, 2, \dots, N$. In the former case we have

$$R^{-1} = \limsup_{n\to\infty} |\Phi_n(d\mu, 0)|^{1/n} < \lim_{n\to\infty} |z_{kn}(d\mu)| = |z_k|^{-1},$$

$k=1, 2, \dots$. If, in addition, f is a polynomial of degree, say, m then by the Bernstein—Szegő formula (cf. [3, Theorem 5.4.5, p. 224]) $\Phi_n^*=f$ for $n\geqq m$, and thus $z_{kn}=(\bar{z}_k)^{-1}$ $k=1, 2, \dots, m$, $z_{kn}=0$, $k=m+1, \dots, n$ and $\Phi_n(0)=0$ holds for $n\geqq m$.

We conclude this paper by observing that similarly to P. Alfaro and L. Vigil's result in [1, 2], orthogonal polynomials on the real line are also completely determined by some of their zeros.

Theorem 2. *Let* $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ *be given sequences of real numbers such that*

$$\dots < x_3 < x_2 < x_1 = y_1 < y_2 < y_3 < \dots.$$

Then there exists a unique system of monic polynomials $\{P_n\}_{n=0}^{\infty}$ *orthogonal with respect to a positive measure on the real line such that* $P_n(x_n)=P_n(y_n)=0$ *and* $P_n(t)\neq 0$ *for* $t\notin[x_n, y_n]$, $n=1, 2, \dots$.

Proof. Set $P_0=1$, $A_0=0$ and $b_0=x_1$. Define $\{P_n\}_{n=1}^{\infty}$, $\{A_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ by

$$P_n(x) = (x-b_{n-1})P_{n-1}(x) - A_{n-1}P_{n-2}(x), \tag{7}$$

$$A_n = (x_{n+1}-y_{n+1})\left[\frac{P_{n-1}(x_{n+1})}{P_n(x_{n+1})} - \frac{P_{n-1}(y_{n+1})}{P_n(y_{n+1})}\right]^{-1} \quad \text{and} \quad b_n = x_{n+1} - A_n\frac{P_{n-1}(x_{n+1})}{P_n(x_{n+1})}.$$

(The latter two formulae come from (7) and from the requirement $P_{n+1}(x_{n+1})= =P_{n+1}(y_{n+1})=0$.) Using induction one can show that $P_n(x)\neq 0$ if $x\notin[x_n, y_n]$, $P_n(x_n)=P_n(y_n)=0$ and $A_n>0$ for $n=1, 2, \ldots$. Hence by Favard's theorem (cf. [3, Theorem 2.1.5, p. 60]) $\{P_n\}_{n=1}^{\infty}$ is an orthogonal polynomial system.

If $x_n=-y_n$ for $n=1, 2, \ldots$, then the formula for A_n and b_n above reduces to

$$A_n = x_{n+1}\frac{P_n(x_{n+1})}{P_{n-1}(x_{n+1})} \quad \text{and} \quad b_n = 0.$$

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Noncyclic vectors for the backward Bergman shift

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§ 1. Introduction and notation. The Bergman space $\mathscr{A}^2$ is the Hilbert space of analytic functions f on the unit disk D such that

$$\|f\|^2 = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 |f(re^{i\theta})|^2 r\,dr\,d\theta < \infty.$$

The Bergman shift is the operator S on $\mathscr{A}^2$ defined by $(Sf)(z)=zf(z)$. If we let $e_n=(n+1)^{1/2}z^n$ then $\{e_n\}_{n=0}^\infty$ is an orthonormal basis for $\mathscr{A}^2$ and $Se_n=\left(\dfrac{n+1}{n+2}\right)^{1/2} e_{n+1}$, so S is a weighted shift. The Bergman shift is a subnormal operator so in particular it is hyponormal, so by Theorem 2 in [5], the functions which are contained in finite dimensional S^*-invariant subspaces are the finite linear combinations of the functions of the form $K_{\alpha,n}$ for some $\alpha\in D$ and n a nonnegative integer. In this paper I will give some examples of noncyclic vectors for S^*, which are not contained in finite dimensional S^*-invariant subspaces. I will do this by giving two sufficient conditions for the smallest invariant subspace containing the function $\sum_{k=1}^\infty c_k K_{\alpha_k}$ to be the orthogonal complement of $\{f\colon f(\alpha_k)=0 \text{ for all } k\}$. This is done in § 2.

The theorem in [2] which Theorem 1 in [5] follows from for the special case of the unweighted shift (Theorem 2.1.1) has as one of its consequences that the sum of two noncyclic vectors is noncyclic. In § 3 I will use the second condition given in § 2 to show that this is not true for S^*.

Throughout this paper cyclic will mean cyclic for S^*. If $f\in\mathscr{A}^2$, then $[f]_*$ will be the smallest S^*-invariant subspace containing f. If $\alpha\in D$ and n is a nonnega-

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tive integer then $K_{\alpha,n}$ will be the function in $\mathscr{A}^2$ such that $\langle f, K_{\alpha,n}\rangle = f^{(n)}(\alpha)$ and $K_{\alpha,0}$ will be written K_α when it is convenient.

Since

$$K_{\alpha,n}(z) = \sum_{j=n}^{\infty}(j+1)j\dots(j-n+1)\bar{\alpha}^{j-n}z^j = \frac{(n+1)!\,z^n}{(1-\bar{\alpha}z)^{n+2}},$$

Theorem 1′ in [5] can be stated for the Bergman shift as follows.

Theorem 0. *If f is analytic in a neighborhood of D, then f is either cyclic or a rational function with zero residue at each pole.*

Proof. It suffices to show that the rational functions with zero residue at each pole are the linear combinations of the $K_{\alpha,n}$'s. The residue of $K_{\alpha,n}$ at its only pole $\frac{1}{\bar{\alpha}}$ is

$$\left[(n+1)\left(\frac{-1}{\bar{\alpha}}\right)^{n+2} z^n\right]^{(n+1)}\left(\frac{1}{\bar{\alpha}}\right) = 0,$$

so any lineary combination of the $K_{\alpha,n}$'s has zero residue at all its poles. Conversely, to show that every rational function with zero residue at each pole is a linear combination of the $K_{\alpha,n}$'s it suffices to show that the function $\frac{1}{(1-\bar{\alpha}z)^{n+2}}$ is a linear combination of them, for any $\alpha\in D$ and nonnegative integer n. This is true because

$$\frac{1}{(1-\bar{\alpha}z)^{n+2}} = \sum_{j=0}^{n}\frac{\binom{n}{j}\bar{\alpha}^j z^j}{(1-\bar{\alpha}z)^{j+2}}.$$

§ 2. Some infinite dimensional cyclic invariant subspaces for S^*.

Theorem 1. *If $\{\alpha_k\}_{k=1}^{\infty}$ is a Blaschke sequence of distinct points in D and $\{c_k\}_{k=1}^{\infty}$ is a sequence of nonzero complex numbers such that $f=\sum_{k=1}^{\infty} c_k K_{\alpha_k}\in\mathscr{A}^2$, then $[f]_* = \{g\in\mathscr{A}^2\colon g(\alpha_k)=0$ for all $k\}^{\perp}$.*

Proof. If $g(\alpha_k)=0$ for all k then

$$\langle g, S^{*n}f\rangle = \langle z^n g, f\rangle = \sum_{k=1}^{\infty}\bar{c}_k\alpha_k^n g(\alpha_k) = 0, \quad \text{so} \quad g\in[f]_*^{\perp}.$$

If $h\in H^\infty$ then if $h^*(z)=\overline{h(\bar{z})}$, there is a uniformly bounded sequence of polynomials $\{q_n\}$ with $\|q_n-h^*\|\to 0$. Then $\|q_n(S^*)f-P(\bar{h}f)\| = \left\|P(q_n(\bar{z})f-\bar{h}f)\right\| \leq \leq \|q_n(\bar{z})f-\bar{h}f\|$ which tends to zero by the Lebesgue dominated convergence theorem so $P(\bar{h}f)\in[f]_*$. Hence if $g\perp[f]_*$ then $0=\langle g, P(\bar{h}f)\rangle=\langle hg, f\rangle=\sum_{k=1}^{\infty}\overline{c_k}h(\alpha_k)g(\alpha_k)$ for any h in H^∞. Fix m and let h be an H^∞ function such that $h(\alpha_m)=1$ and $h(\alpha_k)=0$ for $k\neq m$. Then $c_m g(\alpha_m)=0$. Since $c_m\neq 0$, it follows that $g(\alpha_m)=0$.

The next result uses a result of L. BROWN, A. SHIELDS, and K. ZELLER [1] concerning dominating sequences.

Definition. *If $\{\alpha_k\}$ is a sequence of distinct points in D, then $\{\alpha_k\}$ is dominating if for any function h in H^∞, we have $\|h\|_\infty=\sup_k|h(\alpha_k)|$.*

The following is contained in Theorem 3 of [1].

Lemma 1. *If $\{\alpha_k\}_{k=1}^{\infty}$ is a sequence of distinct points in D with all its limit points on ∂D, then the following are equivalent.*

(i) *There exists $\{a_k\}_{k=1}^{\infty}$ such that $0<\sum_{k=1}^{\infty}|a_k|<\infty$ and $\sum_{k=1}^{\infty}a_k\alpha_k^n=0$ for all nonnegative integers n.*
(ii) *$\{\alpha_k\}$ is a dominating sequence.*
(iii) *Almost every boundary point $p=e^{i\theta}$ may be approached nontangentially by points of $\{\alpha_k\}$.*

Theorem 2. *Let $\{\alpha_k\}_{k=1}^{\infty}$ be a sequence of distinct points in D which has all its limit points on ∂D and is not a dominating sequence, and let $\{c_k\}_{k=1}^{\infty}$ be a sequence of nonzero complex numbers such that $\sum_{k=1}^{\infty}\frac{|c_k|}{1-|\alpha_k|^2}<\infty$. If $f=\sum_{k=1}^{\infty}c_kK_{\alpha_k}$, then $[f]_*=\{g\in\mathscr{A}^2\colon g(\alpha_k)=0 \text{ for all } k\}^\perp$.*

Proof. If $g(\alpha_k)=0$ for all k, then for any n, we have

$$\langle g, S^{*n}f\rangle=\sum_{k=1}^{\infty}\overline{c_k}\alpha_k^n g(\alpha_k)=0$$

so $g\in[f]_*^\perp$. If $g\in[f]_*^\perp$ then $\sum_{k=1}^{\infty}\overline{c_k}\alpha_k^n g(\alpha_k)=0$, for any n. For any k, we have

$$|g(\alpha_k)|=|\langle g, K_{\alpha_k}\rangle|\leqq\|g\|\,\|K_{\alpha_k}\|=\frac{\|g\|}{1-|\alpha_k|^2}.$$

So since $\sum_{k=1}^{\infty}\frac{|c_k|}{1-|\alpha_k|^2}<\infty$, the sum $\sum_{k=1}^{\infty}|\overline{c_k}g(\alpha_k)|$ is finite. Thus by Lemma 1, we have $\overline{c_k}g(\alpha_k)=0$ for all k. Since $c_k\neq 0$, it follows that $g(\alpha_k)=0$ for all k.

§ 3. Two noncyclic vectors whose sum is cyclic. In this section I will use Theorem 2 and the results and methods in [3] concerning zero sets for $\mathscr{A}^2$ to give an example of two noncyclic vectors whose sum is cyclic.

Definition. *A set E of points in D is a zero set for $\mathscr{A}^2$ if there exists a function $f\not\equiv 0$ in $\mathscr{A}^2$ with $f(z)=0$ (where $z\in D$) if and only if z is in E.*

The following lemmas are proved in [3].

Lemma 2. *If* $\mu>1$ *and* β *is a positive integer with* $\beta>\mu^2+1$, *then*

$$f(z)=\prod_{j=1}^{\infty}(1+\mu z^{\beta^j})\in\mathcal{A}^2.$$

Lemma 3. *If* $f\in\mathcal{A}^2$, $f(0)\neq 0$ *and* $\{\alpha_1,\alpha_2,\ldots\}$ *are the zeros of* f *indexed so that* $|\alpha_k|\leqq|\alpha_{k+1}|$, *then*

$$\prod_{k=1}^{N}\frac{1}{|\alpha_k|}=O(N^{1/2}).$$

Lemma 4. *Let* $f(z)=\prod_{j=0}^{\infty}(1+\mu z^{\beta^j})$ *where* $\mu>1$ *and* $\beta\geqq 2$ *is an integer. If* $a=\frac{\log\mu}{\log\beta}$ *and* $\{\alpha_1,\alpha_2,\ldots\}$ *are the zeros of* f *indexed so that* $|\alpha_k|\leqq|\alpha_{k+1}|$, *for all* k, *then* $\prod_{k=1}^{N}\frac{1}{|\alpha_k|}>\mathrm{Const}\cdot N^a$.

Lemma 5. *A subset of a zero set for* $\mathcal{A}^2$ *is a zero set for* $\mathcal{A}^2$.

Example 1. Let β be even and $\mu^2+1<\beta<\mu^3$. Then the function $f(z)=\prod_{j=2}^{\infty}(1+\mu z^{\beta^j})$ belongs to $\mathcal{A}^2$. Let E be its zero set and $E_1=\{re^{i\theta}\in E:\ \pi/2\leqq\theta<2\pi\}$. Then E_1 is a zero set by Lemma 5. The set E has β^j equally spaced points on the circle $|z|=\mu^{-\beta^j}$. On the same circle, the set E_1 has $\frac{3}{4}\beta^j$ points. Let $\{z_1,z_2,\ldots\}$ be the points of E and $\{\alpha_1,\alpha_2,\ldots\}$ be the points of E_1 indeed so that $|z_k|\leqq|z_{k+1}|$ and $|\alpha_k|\leqq|\alpha_{k+1}|$ for all k. By Lemma 4, if $a=\frac{\log\mu}{\log\beta}$, then for any N, we have $\prod_{k=1}^{N}\frac{1}{|z_k|}\geqq\mathrm{Const}\cdot N^a$. Thus if $j\geqq 2$ and $N=\beta^2+\ldots+\beta^j$, then

$$\prod_{k=1}^{3N/4}\frac{1}{|\alpha_k|}=\left(\prod_{k=1}^{N}\frac{1}{|z_k|}\right)^{3/4}\geqq\mathrm{Const}\cdot N^{3a/4}=\mathrm{Const}\cdot(3N/4)^{3a/4}.$$

Choose $0<\varphi<\pi/2$ such that $e^{i\varphi}E_1$ is disjoint from E_1 and let $E_2=e^{i\varphi}E_1$. Then E_2 is also a zero set for $\mathcal{A}^2$. If $0<\theta<\pi/2$ then $e^{i\theta}$ is not a nontangential limit point of E_1 and if $\varphi<\theta<\pi/2+\varphi$ then $e^{i\theta}$ is not a nontangential limit point for E_2, so, by Lemma 1, E_1 and E_2 are not dominating.

Let $\{c_k\}$ be a sequence of nonzero complex numbers such that $\sum_{k=1}^{\infty}\frac{|c_k|}{1-|\alpha_k|^2}<\infty$. Let $f_1=\sum_{k=1}^{\infty}c_kK_{\alpha_k}$ and $f_2=\sum_{k=1}^{\infty}c_kK_{e^{i\varphi}\alpha_k}$. Then by Theorem 2,

$$[f_i]_*^{\perp}=\{g\in\mathcal{A}^2:\ g(z)=0\ \text{ for all } z\in E_i\}$$

for $i=1, 2$. If $\varphi<\theta<\pi/2$, then $e^{i\theta}$ is not a nontangential limit point of $E_1\cup E_2$, so, by Lemma 1, $E_1\cup E_2$ is not dominating. Therefore by Theorem 2,

$$[f_1+f_2]_*^\perp = \{g\in\mathscr{A}^2\colon\ g(z)=0 \ \text{ for all } \ z\in E_1\cup E_2\}.$$

If $\{\gamma_1, \gamma_2, \ldots\}$ are the members of $E_1\cup E_2$ indexed so that $|\gamma_k|\leqq|\gamma_{k+1}|$ for all k, then since

$$\prod_{k=1}^{3N/4}\frac{1}{|\alpha_k|} \geqq \mathrm{Const}\cdot(3N/4)^{3a/N},$$

for $N=\beta^2+\ldots+\beta^j$, we have $\prod_{k=1}^{N}\frac{1}{|\gamma_k|}\geqq\mathrm{Const}\cdot N^{3a/2}$, for infinitely many N's. Since $\beta<\mu^3$, we have $a=\dfrac{\log\mu}{\log\beta}>1/3$, so $3a/2>1/2$. Thus by Lemma 3, $E_1\cup E_2$ is not a zero set for $\mathscr{A}^2$, so $[f_1+f_2]_*^\perp=\{0\}$ and thus f_1+f_2 is cyclic.

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The point spectra for generalized Hausdorff operators

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It is the purpose of this paper to show that the point spectra of a large class of generalized Hausdorff matrices is empty. The generalized Hausdorff matrices under consideration were defined independently by ENDL [3] and JAKIMOVSKI [6]. Each matrix $H^{(\alpha)}$ is a lower triangular matrix with nonzero entries

$$h_{nk}^{(\alpha)} = \binom{n+\alpha}{n-k} \Delta^{n-k} \mu_k, \tag{1}$$

where $\{\mu_k\}$ is a real or complex sequence, and Δ is the forward difference operator defined by $\Delta\mu_k = \mu_k - \mu_{k+1}$, $\Delta^{n+1}\mu_k = \Delta(\Delta^n \mu_k)$. Let c denote the space of convergent sequences. The bounded linear operators on c and l^p, $1 \leqq p \leqq \infty$, will be denoted by $B(c)$ and $B(l^p)$, respectively. Although (1) is defined for any real α which is not a negative integer, in this paper α is restricted to be nonnegative.

Let $1<p<\infty$, $H^{(\alpha)} \in B(l^p)$. The author showed in [8] that the point spectrum of $H^{(\alpha)*}$, the adjoint of $H^{(\alpha)}$, contains an open set. Let $C^{(\alpha)}$ denote the generalized Hausdorff matrix generated by $\mu_n = (n+\alpha+1)^{-1}$, q the conjugate index of p. It was also shown in [8] that the spectrum of $I - 2C^{(\alpha)}/q$ is the closed unit disc. For $p=2$, every $H^{(\alpha)} \in B(l^p) \cap B(c)$ is an analytic function of $C^{(\alpha)}$, so the spectral mapping theorem can be used to obtain the spectrum. GHOSH, RHOADES and TRUTT [5] showed that each $H^{(\alpha)} \in B(l^2)$, for integer α, is subnormal. In [8] the author showed that each $C^{(\alpha)}$ is hyponormal.

In order to establish the point spectra results it will first be necessary to extend some results of FUCHS [4]. Define

$$S = S(a_1, a_2, \ldots) = \{\varphi_k(x)\} = (e^{-cx} x^{a_k}\colon\ c > 0;\ k \geqq 1;\ a_1 < a_2 < \ldots\}. \tag{2}$$

The set S is closed in $L^2(0, \infty)$ if, for each $h \in L^2(0, \infty)$ and for each $\varepsilon > 0$, there

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exists a finite linear combination $\Phi(x)$ of the functions φ_k such that

$$\int_0^\infty (h(x)-\Phi(x))^2\,dx < \varepsilon.$$

The set S is said to be complete in $L^2(0, \infty)$ if, for each $h\in L^2(0, \infty)$,

$$\int_0^\infty h(x)\varphi_k(x)\,dx = 0$$

for all $k\geqq 1$ implies $h(x)=0$ a.e. It is well known that the concepts of closed and complete are equivalent.

Theorem 1. *Let $\{s_n\}\subset C$ satisfy $s_n=o(n^{M+\alpha})$, $M>0$, α a nonnegative real number. Define $\{t_n\}$ by*

$$t_n = \sum_{i=0}^{n} \binom{n+\alpha}{n-i}(-1)^i s_i. \tag{3}$$

Then $t_n=0$ for $n=a_1, a_2, \ldots$ implies $s_n=\Gamma(n+\alpha+1)P(n)/n!$, P a polynomial of degree less than M if and only if $S=\{e^{-x/2}x^{a_n}\colon n=0, 1, 2, \ldots\}$ is closed in $L^2(0, \infty)$.

Suppose that $s_n=O(n^{M+\alpha})$, $t_n=0$ for $n=a_1, a_2, \ldots$ implies

$$s_n=\Gamma(n+\alpha+1)P(n)/n!,$$

where the degree of P is less than M.

We may write (3) in the form

$$t_n = \sum_{i=0}^{n}\binom{n+\alpha}{n-k}\frac{(-1)^k\Gamma(i+\alpha+1)P(i)}{i!} =$$

$$= \frac{\Gamma(n+\alpha+1)}{n!}\sum_{i=0}^{n}\binom{n}{i}(-1)^i P(i) = \frac{\Gamma(n+\alpha+1)}{n!}\Delta^n P(0).$$

Since the degree of P is less than n, $t_n=0$ for each $n\geqq[M]+1$, and the set S is closed.

To prove the converse we may assume, without loss of generality, that $\{s_n\}$ is real and that $|s_n|\leqq 1$ for $n<2M+2+s$, $|s_n|\leqq\begin{bmatrix}n+\alpha\\ n-M\end{bmatrix}$ for $n\geqq 2M+2+s$, $s=[\alpha]+1$, replacing s_n by some scalar multiple γs_n, if necessary.

Lemma 1 [1, p. 77]. *Let a_{nk}, b_n be real numbers, with* $\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$. *Then the system of equations*

$$\sum_{k=0}^{\infty} a_{nk} x_k = b_n \quad (n = 0, 1, 2, \ldots)$$

has a solution satisfying $|x_n| \leqq 1$ *if and only if*

$$\Big|\sum_k \lambda_k b_k\Big| \leqq \sum_{n=0}^{\infty} \Big|\sum_k \lambda_k a_{kn}\Big|$$

for every finite set of real multipliers λ_k.

Lemma 2. *Let* $\{a_n\colon n=0, 1, 2, \ldots\}$ *be an increasing sequence of natural numbers,* $\{t_n\}$ *as in* (3). *Then* $t_n=0$ *for* $n=a_1, a_2, \ldots$ *implies* $t_{a_0}=0$ *if and only if*

$$(4)\qquad l\cdot\mathrm{bd}\cdot\left\{\sum_{h=0}^{2M+1+s}\left|\sum_{k=0}^{N}\lambda_k\binom{a_k+\alpha}{a_k-h}\right|+\sum_{h\geqq 2M+2+s}\binom{h+\alpha}{h-M}\left|\sum_{k=0}^{N}\lambda_k\binom{a_k-\alpha}{a_k-h}\right|\right\}=0$$

where $s=[\alpha]+1$, $\lambda_0=1$ *and the* λ_k *for* $k>0$ *run through all sets of real numbers for* $N=1, 2, \ldots$.

Proof of Lemma 2. Consider $t_n=0$ for $n=a_1, a_2, \ldots, t_{a_0}=\gamma>0$ as a system of equations for the unknowns x_n, where $x_n=s_n$ for $n<2M+2+s$, $x_n=\begin{bmatrix}n+\alpha\\ n-M\end{bmatrix}^{-1} s_n$ for $n\geqq 2M+2+s$. From Lemma 1 this system has a solution for $|x_n|\leqq 1$ if and only if the left side of (4) is $\geqq\gamma$. Therefore (4) implies that $\gamma=0$.

Conversely, if $\gamma=0$, then (4) is nonnegative for every choice of the λ_n. But the choice $\lambda_k=0$ for $k>0$ gives the lower bound.

To complete the proof of Theorem 1, we shall show that the condition that S be closed is equivalent to (4). Let the set S in (2) be closed and $a_0\geqq 2M+2+s$. We shall show that (4) is satisfied.

$$(5)\qquad \sum_{h\geqq 2M+2+s}\binom{h+\alpha}{h-M}\left|\sum_{k=0}^{N}\lambda_k\binom{a_k+\alpha}{a_k-h}\right| \leqq \sum_{h\geqq 2M+2s+2}\binom{h+s}{h-M}\sum_{k=0}^{N}|\lambda_k|\binom{a_k+s}{a_k-h} \leqq$$

$$\leqq \left\{\sum_h \binom{h+s}{h-M}^2\binom{h+s}{2M+2s+2}^{-1}\right\}^{1/2}\left\{\sum_h\binom{h+s}{2M+2s+2}\left(\sum_{k=0}^{N}|\lambda_k|\binom{a_k+s}{a_k-h}\right)^2\right\}^{1/2} \leqq$$

$$\leqq A\left\{\sum_h\binom{h+s}{2M+2s+2}\sum_{j,k=0}^{N}|\lambda_j\lambda_k|\binom{a_j+s}{a_j-h}\binom{a_k+s}{a_k-h}\right\}^{1/2},$$

since the first sum is $O(\Sigma h^{-2})$.

(6)
$$\sum_h \binom{h+s}{2s+2M+2}\binom{a_j+s}{a_j-h}\binom{a_k+s}{a_k-h} =$$

$$= \frac{1}{(2s+2M+2)!}\sum_{h=2M+2+s}^{a_k}\frac{(a_j+s)!}{(h-s-2M-2)!(a_j-h)!}\binom{a_k+s}{a_k-h} =$$

$$= \binom{a_j+s}{2s+2M+2}\sum_{i=0}^{a_k-2M-2-s}\binom{a_j-2M-2-s}{i}\binom{a_k+s}{a_k-2M-2-s-i}.$$

For b, c positive noninteger real numbers,

$$(1+t)^b(1+t)^c = \left(\sum_j \binom{b}{j} t^j\right)\left(\sum_j \binom{c}{j} t^j\right) = \sum_n \left(\sum_{j=0}^n \binom{b}{j}\binom{c}{n-j}\right) t^n.$$

Since also $(1+t)^{b+c} = \sum_j \begin{bmatrix} b+c \\ j \end{bmatrix} t^j$,

(7)
$$\sum_{j=0}^n \binom{b}{j}\binom{c}{n-j} = \binom{b+c}{n}.$$

Substituting (7) in (6),

$$\sum_h \binom{h+s}{2s+2M+2}\binom{a_j+s}{a_j-h}\binom{a_k+s}{a_k-h} =$$

$$= \frac{(a_j+a_k-2M-2)!}{(2s+2M+2)!(a_j-2M-2-s)!(a_k-2M-2-s)!} =$$

$$= \frac{1}{(2M+2s+2)!(a_j-2M-2-s)!(a_k-2M-2-s)!}\int_0^\infty e^{-x}x^{a_j+a_k-2M-2}\,dx,$$

and (5) can be written

$$\sum_{h \geqq 2M+2+s}\binom{h+s}{2M+2s+2}\left|\sum_{k=0}^N \lambda_k \binom{a_k+\alpha}{a_k-h}\right| \leqq A\left\{\int_0^\infty e^{-x}x^{2M+2+2s}Q^2(x)\,dx\right\}^{1/2},$$

where A is independent of N and the λ_k's and

$$Q(x) = \sum_{k=0}^N \frac{|\lambda_k| x^{a_k-2M-2-s}}{(a_k-2M-2-s)!} \quad (\lambda_0 = 1).$$

For $h<2M+2+s$,

$$\left|\sum_{k=0}^{N}\lambda_k\binom{a_k+\alpha}{a_k-h}\right| \leqq \sum_{k=0}^{N}|\lambda_k|\binom{a_k+s}{a_k-h} =$$

$$= \frac{1}{(s+h)!}\int_0^\infty \frac{e^{-x}x^{h+s}}{(2M-h+1+s)!}\int_0^x (x-y)^{2M-h+1+s}Q(y)\,dy\,dx =$$

$$= \int_0^\infty Q(y)\,dy \int_y^\infty e^{-x}x^{h+s}(x-y)^{2M-h+1+s}\,dx =$$

$$= \int_0^\infty Q(y)\,dy \int_0^\infty e^{-y-z}(y+z)^{h+s}z^{2M-h+1+s}\,dz <$$

$$< \int_0^\infty e^{-y}Q(y)\,dy \int_0^\infty e^{-z}(y+z)^{h+s+2M-h+1+s}\,dz =$$

$$= \int_0^\infty e^{-y}Q(y)\,dy \int_0^\infty e^{-z}(y+z)^{2M+1+2s}\,dz <$$

$$< 2^{2M+1+2s}\int_0^\infty e^{-y}Q(y)\,dy \int_0^\infty e^{-z}(y^{2M+1+2s}+z^{2M+1+2s})\,dz <$$

$$< B\int_0^\infty e^{-y}Q(y)(1+y^{2M+1+2s})\,dy <$$

$$< B\Big(\int_0^\infty e^{-y}(1+y^{2M+1+2s})^2\,dy\Big)^{1/2}\Big(\int_0^\infty e^{-y}Q^2(y)\,dy\Big)^{1/2} = C\Big(\int_0^\infty e^{-y}Q^2(y)\,dy\Big)^{1/2}.$$

It remains to show that

(8) $$\int_0^\infty e^{-x}Q^2(x)(1+x^{2M+2s+2})\,dx < \varepsilon.$$

Using Lemma 1 and Theorem 4 of [4], the system

(9) $$\{e^{-x/2}(1+x^{2M+2+2s})^{1/2}x^{a_k-2M-2-s}\} \quad (k \geqq 1)$$

is closed since S is closed. Therefore

$$\frac{e^{-x/2}(1+x^{2M+2+2s})^{1/2}x^{a_0-2M-2-s}}{(a_0-2M-2)!}$$

can be approximated arbitrarily close by finite linear combinations of functions from (9). This proves (8).

We shall now show that, if (4) is true for every $a_0 \geqq 2M+2+s$, then S is complete. If (4) is satisfied then, for suitable values of λ_k,

$$\sum_{h \geqq M} \binom{h+\alpha}{h-M} \left| \sum_{k=0}^{N} \lambda_k \binom{a_k+\alpha}{a_k-h} \right| < \varepsilon.$$

It then follows that

$$\sum_{h \geqq M} \binom{h+\alpha}{h-M} \left(\sum_{k=0}^{N} \lambda_k \binom{a_k+\alpha}{a_k-h} \right)^2 < \varepsilon^2. \tag{10}$$

But

$$\sum_{h \geqq M} \binom{h+\alpha}{h-M} \left(\sum_{k=0}^{N} \lambda_k \binom{a_k+\alpha}{a_k-h} \right)^2 = \sum_{j,k=0}^{N} \lambda_j \lambda_k \sum_{h=M}^{a_k} \binom{h+\alpha}{h-M} \binom{a_j+\alpha}{a_j-h} \binom{a_k+\alpha}{a_k-h} =$$

$$= \sum_{j,k=0}^{N} \frac{\lambda_j \lambda_k}{\Gamma(\alpha+M+1)} \sum_{h=M}^{a_k} \frac{\Gamma(a_j+\alpha+1)}{(h-M)!(a_j-h)!} \binom{a_k+\alpha}{a_k-h} =$$

$$= \sum_{j,k=0}^{N} \lambda_j \lambda_k \binom{a_j+\alpha}{M+\alpha} \sum_{i=0}^{a_k-M} \binom{a_j-M}{i} \binom{a_k+\alpha}{a_k-M-i} =$$

$$= \sum_{j,k=0}^{N} \lambda_j \lambda_k \binom{a_j+\alpha}{M+\alpha} \binom{a_j+a_k+\alpha-M}{a_k-M} =$$

$$= \sum_{j,k=0}^{N} \lambda_j \lambda_k \frac{\Gamma(a_j+a_k+\alpha-M+1)}{\Gamma(M+\alpha+1)(a_j-M)!(a_k-M)!} = \frac{1}{\Gamma(M+\alpha+1)} \int_0^\infty e^{-x} R^2(x)\, dx,$$

where

$$R(x) = \sum_{k=0}^{N} \frac{\lambda_k x^{a_k+\alpha/2-M/2}}{(a_k-M)!}.$$

Therefore

$$\frac{1}{\Gamma(M+\alpha+1)} \int_0^\infty e^{-x} R^2(x)\, dx < \varepsilon^2,$$

which implies that

$$e^{-x/2} x^{n-M/2+\alpha/2}, \quad n = 2M+2+s, 2M+3+s, \ldots, \tag{11}$$

can be mean square approximated by linear combinations of the functions $e^{-x/2} x^{a_k-M/2+\alpha/2}$, $k \geqq 1$. From [4, Theorem 5] the set (11) is closed. Thus also is $\{e^{-x/2} x^{a_k-M/2+\alpha/2}\}$. From Lemma 1 of [4] with $p(x) = x^{M/2-\alpha/2}$, S is closed.

Suppose $t_n = 0$ for $n = a_1, a_2, \ldots$, and S is closed. Then one can use condition (4) and mathematical induction to force $t_n = 0$ for all $n \geqq a_0$.

Now suppose that $s_n = o(n^{M+\alpha})$, $\{t_n\}$ satisfies (3) with $t_n = 0$ for $n \geqq 2M+ +s+2$. Note that (3) is the nth term of a diagonal matrix t satisfying $t = \delta^{(\alpha)} s$, where s is the diagonal matrix with entries s_n and $\delta_{nk}^{(\alpha)}=(-1)^k\begin{bmatrix}n+\alpha\\ n-k\end{bmatrix}$. Since $\delta^{(\alpha)}$ is its own inverse, and multiplication is associative, $\delta^{(\alpha)}t=s$; i.e.

$$s_n=\sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}t_k=\sum_{k=0}^{2M+s+1}(-1)^k\binom{n+\alpha}{n-k}t_k=$$

$$=\sum_{k=0}^{\varrho}(-1)^k\binom{n+\alpha}{n-k}t_k=\frac{\Gamma(n+\alpha+1)}{n!}\sum_{k=0}^{\varrho}\frac{(-1)^k n!\,t_k}{(n-k)!\,\Gamma(k+\alpha+1)},$$

where ϱ is the largest integer for which $t_k\neq 0$. Therefore $s_n=\Gamma(n+\alpha+1)P(n)/n!$, where P is a polynomial in n of degree ϱ. Since $s_n=o(n^{M+\alpha})$, $\alpha+\varrho<M+\alpha$, and the degree of P is less than M.

Let $\sigma_p(A)$ denote the point spectrum of an operator A, and write H for $H^{(0)}$.

Theorem 2. (a) *Let* $1<p<\infty$, $H^{(\alpha)}\in B(l^p)\cap B(c)$. *Then* $\sigma_p(H^{(\alpha)})$ *is empty.*

(b) *Let* $H^{(\alpha)}\in B(l)$, $\alpha\geqq 0$. *Then* $B_p(H^{(\alpha)})$ *is empty.*

(c) *Let* $H^{(\alpha)}\in B(c)$. *For* $\alpha>0$, $\sigma_p(H^{(\alpha)})$ *is empty. For* $\alpha=0$, *if* H *is multiplicative, then* $\sigma_p(H)=\{\mu_0\}$.

Proof of (a). Suppose there exists an $x\in l^p$ with $H^{(\alpha)}x=\lambda x$. Then $(H^{(\alpha)}-\lambda I)x=0$. But $H^{(\alpha)}\in B(l^p)\cap B(c)$ implies that $K^{(\alpha)}=H^{(\alpha)}-\lambda I\in B(l^p)\cap B(c)$. Moreover, $K^{(\alpha)}$ is also a generalized Hausdorff matrix. Thus, we are looking for solutions of the system $K^{(\alpha)}x=0$. One may write $K^{(\alpha)}=\delta^{(\alpha)}\mu\delta^{(\alpha)}$, where μ is a diagonal matrix with diagonal entries μ_n and $\delta_{nk}^{(\alpha)}=(-1)^k\binom{n+\alpha}{n-k}$. Since $\delta^{(\alpha)}$ is its own inverse, and each matrix forming $K^{(\alpha)}$ is row finite, the system $K^{(\alpha)}x=0$ is equivalent to $\mu\delta^{(\alpha)}x=0$; i.e.,

$$\mu_n\sum_{i=0}^{n}(-1)^i\binom{n+\alpha}{n-i}x_i=0,\quad n=0,1,2,\ldots\tag{12}$$

Since $H^{(\alpha)}\in B(c)$, so also does $K^{(\alpha)}$, so that μ is a moment sequence. This means that

$$\psi(z)=\int_0^1 t^{z+\alpha}\,d\beta(t)$$

is analytic for $\operatorname{Re}(z)>0$, where β and μ_n satisfy

$$\mu_n=\int_0^1 t^{n+\alpha}\,d\beta(t).$$

From [2], the integer values b_n for which $\psi(b_n)=0$ satisfy the condition $\Sigma_k b_k^{-1}<\infty$. Therefore (12) implies that $t_n=0$ for all values of n except possibly a subset $\{b_n\}$ satisfying $\Sigma_k b_k^{-1}<\infty$. Using Theorem 3 of [4], the set S of integers n for which

$t_n=0$ remains closed. Since $\{x_n\}\subset l^p$, $1<p<\infty$, $x_n=o(n^{1/2+\alpha})$. Applying Theorem 1, $x_n=\Gamma(n+\alpha+1)P(n)/n!$, where $P(x)$ is a polynomial of degree less than $M=1/2$; i.e., P is a constant polynomial. But, unless P is the zero polynomial, $x\notin l^p$, so $H^{(\alpha)}$ has empty point spectrum.

Proof of (b). The author has shown in [7] that $H^{(\alpha)}\in B(l)$ implies $H^{(\alpha)}\in B(c)$. The rest of the proof is the same as that of (a).

Proof of (c). Following the proof of (a), since $\{x_n\}\in c$, $\{x_n\}$ is bounded, hence $x_n=o(n^{1/2+\alpha})$, and again $\sigma_p(H^{(\alpha)})$ is empty, for $\alpha>0$.

For $\alpha=0$, $x_n=o(n^{1/2})$, and the only nonzero sequence satisfying (12) is $e=(1, 1, \ldots)$. With $\alpha=0$, each row sum of H is μ_0. Therefore $\sigma_p(H)=\{\mu_0\}$.

A matrix A is multiplicative if $\lim Ax=t\lim x$ for some scalar t, $x\in c$. In terms of the matrix entries, multiplicativity of A translates into A having all zero column limits. For Hausdorff matrices in $B(c)$ this condition is equivalent to the mass function $\beta(t)$ being continuous from the right at zero, and specifically excludes the compact Hausdorff matrix generated by $\mu_0=1$, $\mu_n=0, n>0$. Theorem 1 does not apply to this matrix since there are too many zeros on the main diagonal, but a direct analysis yields the point spectrum to be $\{0, 1\}$.

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A spectral dilation of some non-Dirichlet algebra

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Let X be a compact Hausdorff space, let $C(X)$ be the algebra of complex-valued continuous functions on X, and let A be a uniform algebra on X. Let $\mathfrak{H}$ be a complex Hilbert space and $L(\mathfrak{H})$ the algebra of all bounded linear operators on $\mathfrak{H}$. I is the identity operator in $\mathfrak{H}$. An algebra homomorphism $f \to T_f$ of A in $L(\mathfrak{H})$, which satisfies

$$T_1 = I \quad \text{and} \quad \|T_f\| \leqq \|f\|$$

is called a representation of A on $\mathfrak{H}$. A representation $\Phi \to U_\Phi$ of $C(X)$ on a Hilbert space $\mathfrak{K}$ is called a spectral dilation of the representation $f \to T_f$ of A on $\mathfrak{H}$ if $\mathfrak{H}$ is a Hilbert subspace of $\mathfrak{K}$ and

$$T_f x = PU_f x \quad \text{for} \quad f \in A \quad \text{and} \quad x \in \mathfrak{H}$$

where P is the orthogonal projection of $\mathfrak{K}$ on $\mathfrak{H}$.

If A is a Dirichlet algebra on X and $f \to T_f$ a representation of A on $\mathfrak{H}$, then there exists a spectral dilation. This was proved by Foiaş and Suciu (cf. [3, Theorem 8.7]). However, it is unknown whether any representation of a non-Dirichlet algebra has a spectral dilation. In this paper we give an example of a uniform algebra which has a spectral dilation for any operator representation and is a subalgebra of a disc algebra, of codimension one.

If $f \to T_f$ is a representation of A on a Hilbert space $\mathfrak{H}$ with the inner product (x, y) $(x, y \in \mathfrak{H})$, then there are measures $\mu_{x,y}$ $(x, y \in \mathfrak{H})$ such that $\|\mu_{x,y}\| \leqq \|x\| \|y\|$ for $x, y \in \mathfrak{H}$ and

$$(T_f x, y) = \int f \, d\mu_{x,y} \quad \text{for} \quad f \in A \quad \text{and} \quad x, y \in \mathfrak{H}$$

(see [3, p. 173]). Let τ be in the maximal ideal space of A and G the Gleason part of τ. We say that the representation $f \to T_f$ of A is G-continuous (G-singular) if

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there exists a system of finite measures $\{\mu_{x,y}\}$ such that $\mu_{x,y}$ is G-absolutely continuous (G-singular) and $(T_f x, y)=\int f\, d\mu_{x,y}$ for all $f\in A$ and all $x, y\in\mathfrak{H}$ (cf. [2, p. 182]). We need the following three lemmas to give a theorem. The first one is a theorem of MLAK [2, Theorem 2.3] and the second one is one result of FOIAŞ and SUCIU (cf. [3, p. 173]).

Lemma 1. *Let $f\to T_f$ be a representation of A on $\mathfrak{H}$. Then $f\to T_f$ is a unique orthogonal sum $T_f=T_f^a\oplus T_f^s$ where the representation $f\to T_f^a$ $(f\to T_f^s)$ of A is G-absolutely continuous (G-singular).*

Lemma 2. *Let $f\to T_f$ be a representation of A on $\mathfrak{H}$. Then there are measures $\mu_{x,y}$ $(x, y\in\mathfrak{H})$ such that $\|\mu_{x,y}\|\leqq\|x\|\,\|y\|$ for $x, y\in\mathfrak{H}$ and*

$$((T_f+T_g^*)x, y)=\int (f+\bar{g})\, d\mu_{x,y}$$

for $f, g\in A$ and $x, y\in\mathfrak{H}$.

A family $\lambda_{x,y}$ $(x, y\in\mathfrak{H})$ of measures on X is called semispectral if it satisfies the following properties:

$$\lambda_{\alpha x+\beta y,z}=\alpha\lambda_{x,z}+\beta\lambda_{y,z}, \tag{1}$$

$$\int\Phi\, d\lambda_{x,y}=\overline{\int\bar{\Phi}\, d\lambda_{y,x}}\quad (\Phi\in C(X)), \tag{2}$$

$$\lambda_{x,x}\geqq 0, \tag{3}$$

$$\|\lambda_{x,y}\|\leqq\gamma\|x\|\|y\| \tag{4}$$

where α and β are complex numbers, and γ is a positive number.

Now we can give an example of a uniform algebra which has a spectral dilation for any operator representation and is not a Dirichlet algebra. Let $\mathbf{T}$ be the unit circle and $\mathscr{A}$ the algebra of those continuous functions on $\mathbf{T}$ which have analytic extensions $\tilde{f}$ to the interior such that $\tilde{f}(0)=f(1)$. Then $\mathscr{A}$ is a uniform algebra on $\mathbf{T}$ and $\mathbf{T}$ is the Shilov boundary of $\mathscr{A}$. The complex homomorphism τ on $\mathscr{A}$ is defined by $\tau(f)=\tilde{f}(0)=f(1)$. Both $d\theta/2\pi$ and the unit point mass δ_1 at 1 represent the same linear functional τ on $\mathscr{A}$. Therefore $\mathscr{A}$ is not a logmodular algebra and hence not a Dirichlet algebra on $\mathbf{T}$ (cf. [1, p. 38]).

Lemma 3. *If μ is an annihilating measure on $\mathbf{T}$ for $\mathscr{A}+\overline{\mathscr{A}}$ then $d\mu=c(d\theta/2\pi-d\delta_1)$ for some constant c.*

Proof. We may assume that μ is a real measure on $\mathbf{T}$. If μ annihilates $\mathscr{A}$ then

$$\int z\, d\mu=\int z^2\, d\mu=\int z^3\, d\mu=\dots$$

because the functions $z-z^2, z^2-z^3, z^3-z^4, \dots$ are all in $\mathscr{A}$. Hence for any positive integer n

$$\int z^n(d\mu-c_1\, d\delta_1)=0,$$

where $c_1=\int z\,d\mu$. By a theorem of F. and M. Riesz (cf. [1, p. 45]), $d\mu-c_1 d\delta_1=$ $=h\,d\theta/2\pi$ for some h in the usual Hardy space H^1. The absolutely continuous part of μ with respect to $d\theta/2\pi$ is a real measure and coincides with $h\,d\theta/2\pi$. Since H^1 has not nonconstant real functions, h is constant. Thus $d\mu=c\,d\theta/2\pi+c_1 d\delta_1$ and $c=-c_1$ because $\int 1\,d\mu=0$.

Theorem. *Let $f\to T_f$ be a representation of $\mathscr{A}$ on a Hilbert space $\mathfrak{H}$. There exists a spectral dilation $\Phi\to U_\Phi$ of $f\to T_f$.*

Proof. By Lemma 1 we may assume that the representation $f\to T_f$ of $\mathscr{A}$ is G-continuous or G-singular, where G is the Gleason part of τ in the maximal ideal space of $\mathscr{A}$. Suppose the representation is G-continuous. By Lemma 2 there are measures $\mu_{x,y}$ $(x, y\in\mathfrak{H})$ such that $\|\mu_{x,y}\|\leqq\|x\|\,\|y\|$ and $((T_f+T_g^*)x, y)=\int (f+\bar{g})\,d\mu_{x,y}$ for $f, g\in\mathscr{A}$ and $x, y\in\mathfrak{H}$. Since the representation of $\mathscr{A}$ is G-continuous, by the definition $\mu_{x,y}$ is absolutely continuous with respect to $d\theta/2\pi+d\delta_1$. Hence

$$d\mu_{x,y} = h_{x,y}\,d\theta/2\pi+c_{x,y}\,d\delta_1$$

where $h_{x,y}$ is in the usual Lebesgue space $L^1(d\theta/2\pi)$ and $c_{x,y}$ is constant.

Put

$$d\lambda_{x,y} = (h_{x,y}+c_{x,y})\,d\theta/2\pi.$$

We shall prove that the family $\lambda_{x,y}$ $(x, y\in\mathfrak{H})$ of measures on $\mathbf{T}$ is semispectral, that is, it satisfies (1)—(4). (4) is clear. $d\mu_{\alpha x+\beta y,z}-(\alpha\,d\mu_{x,z}+\beta\,d\mu_{y,z})$ annihilates $\mathscr{A}+\overline{\mathscr{A}}$. Therefore by Lemma 3 for some constant $a_{x,y,z}$

$$d\mu_{\alpha x+\beta y,z}-(\alpha\,d\mu_{x,z}+\beta\,d\mu_{y,z}) = a_{x,y,z}(d\theta/2\pi-d\delta_1),$$

consequently

$$h_{\alpha x+\beta y,z}-(\alpha h_{x,z}+\beta h_{y,z}) = a_{x,y,z}$$

and

$$c_{\alpha x+\beta y,z}-(\alpha c_{x,z}+\beta c_{y,z}) = -a_{x,y,z}.$$

This implies (1). $d\mu_{x,y}-d\bar{\mu}_{y,x}$ annihilates $\mathscr{A}+\overline{\mathscr{A}}$.

Therefore by Lemma 3 for some constant $b_{x,y}$

$$d\mu_{x,y}-d\bar{\mu}_{y,x} = b_{x,y}(d\theta/2\pi-d\delta_1)$$

consequently

$$h_{x,y}-\bar{h}_{y,x} = b_{x,y} \quad\text{and}\quad c_{x,y}-\bar{c}_{y,x} = -b_{x,y}.$$

This implies (2). By Proposition 7.8 in [3], if $f\in\mathscr{A}$ and $\operatorname{Re} f\geqq 0$ then $\operatorname{Re} T_f\geqq 0$. Hence if $u\in\mathscr{A}+\overline{\mathscr{A}}$ and $u\geqq 0$ then $\int u\,d\mu_{x,x}\geqq 0$. Thus for $u\in\mathscr{A}+\overline{\mathscr{A}}$ with $u\geqq 0$

$$\int u\,d\lambda_{x,x} = \int u(h_{x,x}+c_{x,x})\,d\theta/2\pi = \int uh_{x,x}\,d\theta/2\pi+c_{x,x}\int u\,d\theta/2\pi =$$
$$= \int uh_{x,x}\,d\theta/2\pi+c_{x,x}\int u\,d\delta_1 = \int u\,d\mu_{x,x} \geqq 0.$$

By the Riemann—Lebesgue lemma we know that $z^n \to 0$ in the weak* topology of $L^\infty(d\theta/2\pi)$. Hence the functions $z, z^2, z^3, \ldots$ are all in the weak*-closure of $\mathscr{A}$ because $z^k = (z^k - z^{k-1}) + \ldots + (z^n - z^{n-1}) - z^n$ for $n > k$. Therefore for $u \in C(\mathbf{T})$ with $u \geqq 0 \int u \, d\lambda_{x,x} \geqq 0$ and this implies (3).

Since the family $\lambda_{x,y}$ $(x, y \in \mathfrak{H})$ of measures on $\mathbf{T}$ is semispectral, there is a positive definite map $\Phi \to T'_\Phi$ of $C(\mathbf{T})$ in $L(\mathfrak{H})$ (cf. [3, Theorem 7.1]). By a dilation theorem of Naimark (cf. [3, Theorem 7.5]), we obtain a representation $\Phi \to U_\Phi$ of $C(\mathbf{T})$ on a Hilbert space $\mathfrak{K}$ which is a spectral dilation of $\Phi \to T'_\Phi$. If $f \in \mathscr{A}_0$ then $\int f \, d\theta/2\pi = \int f \, d\delta_1 = 0$ and hence

$$(T'_f x, y) = \int f \, d\lambda_{x,y} = \int f h_{x,y} \, d\theta/2\pi = \int f \, d\mu_{x,y} = (T_f x, y), \quad (x, y \in \mathfrak{H}).$$

Thus $T'_f = T_f$ if $f \in \mathscr{A}$ and the representation $\Phi \to U_\Phi$ is the spectral dilation of $f \to T_f$.

If the representation is G-singular, the family $\mu_{x,y}$ $(x, y \in \mathfrak{H})$ is singular with respect to $d\theta/2\pi + d\delta_1$. Then Lemma 3 implies that it is semispectral immediately, and the proof can be completed as above.

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Generalized Toeplitz kernels and dilations of intertwining operators. II. The continuous case

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I. Matricial Toeplitz kernels and intertwining operators

This paper continues a study about the relation between generalized Toeplitz kernels and the problem of the dilation of the commutant of contractive semigroups, started in [2], where only discrete semigroups were considered. In Section II we shall extend that study to general groups. In Section III the group of the real numbers is considered and the basic results of this paper on dilation theory — theorems (III.11) and (III.13) — are obtained; the last includes a continuous version of the theorem on the dilation of the commutant due to Sz.-Nagy and Foiaş.

In this section we start with preliminary results concerning the relation between intertwining operators, unitary representations of groups, and positive definite matricial functions.

We fix a (topological) group Γ with neutral element e and consider $\mathscr{L}(H)$-valued kernels on Γ, i.e. functions $K\colon \Gamma\times\Gamma\to\mathscr{L}(H)$, where $\mathscr{L}(H)$ is the set of bounded operators on a Hilbert space H. Such a kernel is said to be positive definite, p.d., if

$$\sum_{s,t\in\Gamma}\langle K(s,t)h(s),\, h(t)\rangle_H \geqq 0,$$

for every function $h\colon \Gamma\to H$ whose support $\{t\in\Gamma\colon h(t)\neq 0\}$ is a finite set.

If K is such that $K(st, su)=K(t,u)$ holds for all $s,t,u\in\Gamma$, then K is determined by the function G on Γ given by $G(s)=K(s,e)$; conversely, if a function G on Γ is given, setting $K(s,t)=G(t^{-1}s)$ we get a kernel with the above property; in that case we say that K or — informally speaking — G are Toeplitz kernels. When $H=H_1\oplus H_2$ is the direct sum of two Hilbert spaces, H_1 and H_2, then G is given by a matrix $(G_{jk})_{j,k=1}^2$ where $G_{jk}(s)\in\mathscr{L}(H_j, H_k)$ for all $s\in\Gamma$, and we say that G is a matricial Toeplitz kernel.

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A positive definite matricial Toeplitz kernel can be viewed as a relation between two unitary representations of the given group, in the following sense.

Proposition 1. *For $j=1,2$ let H_j be a Hilbert space and $G_j\colon \Gamma\to\mathcal{L}(H_j)$ a positive definite Toeplitz kernel on the group Γ, such that G_j equals the identity on the neutral element of Γ; let U_j be the minimal unitary dilation of G_j to a Hilbert space F_j. Let $\mathcal{R}(U_1, U_2)$ be the set of intertwining operators between U_1 and U_2, considered as a (closed) subspace of $\mathcal{L}(F_1, F_2)$. Then the relation*

$$g(s) = P^{F_2}_{H_2} WU_1(s)|_{H_1}, \quad i.e. \quad \langle WU_1(s)h_1, h_2\rangle_{F_2} = \langle g(s)h_1, h_2\rangle_{H_2},$$

for all $s\in\Gamma$, $h_1\in H_1$, $h_2\in H_2$, gives a bijection $W\leftrightarrow g$ between the unit ball of $\mathcal{R}(U_1, U_2)$ and the set of functions $f\colon \Gamma\to\mathcal{L}(H_1, H_2)$ such that $G=(G_{jk})^2_{j,k=1}$, $G_{11}=G_1$, $G_{12}=g$, $G_{21}=\tilde{g}$, $G_{22}=G_2$ is a positive definite matricial Toeplitz kernel. If moreover Γ is a topological group and G_1, G_2 are continuous in the weak topology of operators, then all such functions g will be continuous in the strong topology.

Notation. When H is a closed subspace of a Hilbert space F, i^F_H denotes the inclusion of H in F and P^F_H the orthogonal projection of F onto H. If g is a function on Γ, we set $\tilde{g}(s)=g^*(s^{-1})$. If $\{S_t\colon t\in M\}$ is a family of subspaces of F, $\bigvee_{t\in M} S_t$ denotes the minimal closed subspace of F that contains S_t for all $t\in M$.

Proof of Proposition 1. For $j=1,2$, $U_j=\{U_j(s)\colon s\in\Gamma\}\subset\mathcal{L}(F_j)$ is such that $G_j(s)=P^{F_j}_{H_j}U_j(s)|_{H_j}$ holds for all $s\in\Gamma$ and the minimality condition $F_j = \bigvee_{s\in\Gamma} U_j(s)H_j$ is also true; that is the content of Naimark's dilation theorem (see [9]). Let G be as in the above statement; set, for all $s, t\in\Gamma$, $h_1\in H_1$, $h_2\in H_2$,

$$B\big(U_1(s)h_1, U_2(t)h_2\big) := \langle G_{12}(t^{-1}s)h_1, h_2\rangle_{H_2}.$$

Taking in account that the elements $U_j(s)h_j$ span the space F_j, it is easy to verify that G is p.d. if and only if B defines a bounded sesquilinear form on $F_1\times F_2$ of norm $\|B\|\leqq 1$. In that case there exists $W\in\mathcal{L}(F_1, F_2)$ such that $\|W\|=\|B\|\leqq 1$ and $\langle Wf_1, f_2\rangle_{F_2}=B(f_1, f_2)$ hold; moreover, from the equalities

$$\langle WU_1(s)[U_1(u)h_1], U_2(t)h_2\rangle_{F_2} = \langle G_{12}(t^{-1}su)h_1, h_2\rangle_{H_2} =$$
$$= \langle U_2(s)WU_1(u)h_1, U_2(t)h_2\rangle_{F_2}$$

and the minimality condition it follows that $WU_1(s)=U_2(s)W$ is true for all $s\in\Gamma$. Hence, W is a contraction belonging to $R(U_1, U_2)$.

By setting $g(s)=P^{F_2}_{H_2} WU_1(s)|_{H_1}$ for all $s\in\Gamma$, the converse also follows.

We now apply the preceding result to the dilation of the commutant of two semigroups of isometries.

Proposition 2. *Let Γ be a group with neutral element e and Γ_1 a subsemigroup of Γ. Set $\Gamma_1^{-1}=\{s\in\Gamma: s^{-1}\in\Gamma_1\}$ and assume that $\Gamma_1\cap\Gamma_1^{-1}=\{e\}$ and $\Gamma_1\cup\Gamma_1^{-1}=\Gamma$ hold. Let $\{V_1(s): s\in\Gamma_1\}$ and $\{V_2(s): s\in\Gamma_1\}$ be two semigroups of isometries in the Hilbert spaces H_1 and H_2, respectively, and Y a contraction intertwining them, so that*

(2a) $$YV_1(s) = V_2(s)Y, \text{ for every } s\in\Gamma_1, \text{ and } \|Y\| = 1 \text{ hold.}$$

Let a matricial Toeplitz kernel G be associated with the commutator Y by:

(2b) $$\begin{gathered} G_{jj}(s) = V_j(s) \text{ if } s\in\Gamma_1, \quad G_{jj}(s) = V_j(s^{-1}) \text{ if } s\in\Gamma_1^{-1}, \quad j = 1,2; \\ G_{12}(s) = V_2(s)Y \text{ if } s\in\Gamma_1, \quad G_{12}(s) = V_2(s^{-1})Y \text{ if } s\in\Gamma_1^{-1}; \quad G_{21} = \tilde{G}_{12}. \end{gathered}$$

Then G is p.d. if and only if the following conditions hold:

(2c) *for $j=1,2$ there exists a unitary representation U_j of Γ in a Hilbert space F_j that contains H_j and satisfies*

$$U_j(s)|_{H_j} = V_j(s) \text{ for } s\in\Gamma_1, \quad F_j = \bigvee_{s\in\Gamma}[U_j(s)H_j];$$

(2d) *there exists $W\in\mathcal{L}(F_1, F_2)$ that verifies*

$$WU_1(s) = U_2(s)W \text{ for } s\in\Gamma; \quad \|W\| = \|Y\|; \quad P^{F_2}_{H_2}W|_{H_1} = Y.$$

Moreover such a W is unique.

Proof. If $Y=0$, $W=0$ is the only solution of (2d), so we may always assume that $Y\neq 0$ and, by homogeneity, $\|Y\|=1$, as in (2a).

If G is p.d. G_{11} and G_{22} have the same property; let U_1 and U_2 be their minimal unitary dilations, respectively. From $P^{F_j}_{H_j}U_j(s)|_{H_j}=G_{jj}(s)$ and (2b) it follows that $U_j(s)$ is an extension of the isometry $V_j(s)$ for every $s\in\Gamma_1$. Thus (2c) is satisfied. Let W be associated with G as in Proposition 1; then $\|W\|\leqq 1$, W intertwines U_1 and U_2, and $\langle Wh_1, h_2\rangle_{F_2}=\langle G_{12}(e)h_1, h_2\rangle_{H_2}=\langle Yh_1, h_2\rangle_{H_2}$ holds for all $h_1\in H_1$, $h_2\in H_2$; thus $P^{F_2}_{H_2}W|_{H_1}=Y$, so $1=\|Y\|\leqq\|W\|\leqq 1$. Consequently (2d) is also satisfied.

Conversely, assume that (2c) and (2d) hold. From Proposition 1 it follows that it is enough to prove that G_{12} is the same function as $g(s)\overset{\text{def}}{=}P^{F_2}_{H_2}WU_1(s)|_{H_1}$; now, if $s\in\Gamma_1$, then $g(s)=P^{F_2}_{H_2}WV_1(s)=YV_1(s)=G_{12}(s)$; if $s\in\Gamma_1^{-1}$, we have for all $h_1\in H_1$, $h_2\in H_2$,

$$\begin{aligned} \langle g(s)h_1, h_2\rangle_{H_2} &= \langle U_2(s)Wh_1, h_2\rangle_{F_2} = \langle P^{F_2}_{H_2}Wh_1, V_2(s^{-1})h_2\rangle_{H_2} = \\ &= \langle V_2^*(s^{-1})Yh_1, h_2\rangle_{H_2} = \langle G_{12}(s)h_1, h_2\rangle_{H_2}. \end{aligned}$$

The simplest example is perhaps $\Gamma=Z$, the set of integers, $\Gamma_1=Z_1:=\{n\in Z: n\geqq 0\}$. In that case the semigroup $V_j(s)$ is determined by the isometry $V_j(1)$, so that we are concerned with the commutator $YV_1=V_2Y$ of two isometries. Then it is easy to prove ([2], Lemma II.3) that G is p.d. so, if U_1 and U_2 are the minimal unitary extensions of V_1 and V_2, there exists W that verifies $WU_1=U_2W$, $\|W\|=\|Y\|$ and $P^{F_2}_{H_2}W|_{H_1}=Y$. Note that in general the last equality cannot be improved so as to get W to be a strict lifting of Y, i.e., such that $P^{F_2}_{H_2}W=YP^{F_1}_{H_1}$. In fact, the last equation implies $YV_1^*=YP^{F_1}_{H_1}U_1^*i^{F_1}_{H_1}=P^{F_2}_{H_2}WU_1^*i^{F_1}_{H_1}=P^{F_2}_{H_2}U_2^*Wi^{F_1}_{H_1}=V_2^*P^{F_2}_{H_2}Wi^{F_1}_{H_1}$, because U_2 extends V_2; thus $YV_1^*=V_2^*Y$. Now, the last equality is not a consequence of $YV_1=V_2Y$ because if $V_1=V_2=Y=V$ is any non-unitary isometry then $YV_1^*=VV^*\neq \neq I=V_2^*Y$, etc.

Let us now go from the discrete to the continuous case. Set $\Gamma=R=\{\text{real numbers}\}$, $\Gamma_1=R_1=\{s\in R:\ s\geqq 0\}$. In order to apply Proposition 2 we assume that (2a) holds and consider G given by (2b). Working as in [9], page 30, we can prove that G is p.d. whenever the semigroups V_1 and V_2 are weakly continuous. Thus:

Corollary 3. *Let* $\{V_1(s)\}$, $\{V_2(s)\}$, $s\geqq 0$, *be two continuous monoparametric semigroups of isometries in the Hilbert spaces* H_1, H_2, *respectively, and let* $Y\in\mathcal{L}(H_1,H_2)$ *be a contraction intertwining them, i.e., such that*

$$YV_1(s)=V_2(s)Y \quad \textit{for} \quad s\geqq 0, \quad \|Y\|\leqq 1.$$

For $j=1,2$ *let* $\{U_j(s)\}$, $s\in R$, *be a minimal extension of* V_j *to a continuous monoparametric group of unitary operators in a Hilbert space* F_j. *Then there exists a unique operator* $W\in\mathcal{L}(F_1,F_2)$ *such that*

$$WU_1(s)=U_2(s)W, \quad \textit{for every} \quad s\in R; \quad Y=P^{F_2}_{H_2}W|_{H_1}; \quad \|Y\|=\|W\|.$$

II. Generalized Toeplitz kernels and dilations of the commutator of two contractions

When we consider a commutator of two contractions instead of isometries the method of the preceding section does not work. In fact, the associated matricial Toeplitz kernel need not be positive definite. (See [2], II.1b.) Nevertheless a suitable extension of such kind of kernels allows a similar approach to the more general situation.

Let Γ_1 be a sub-semigroup of the group Γ. A generalized Toeplitz kernel (GTK) on (Γ,Γ_1) is by definition a set

$$K=\{(K_{jk}),\ j,k=1,2;\ H_1,H_2\}$$

composed of two Hilbert spaces, H_1 and H_2, and four functions

$$K_{11}\colon \Gamma\to\mathcal{L}(H_1);\quad K_{12}\colon \Gamma_1\to\mathcal{L}(H_1,H_2);\quad K_{21}\colon \Gamma_1^{-1}\to\mathcal{L}(H_2,H_1);\quad K_{22}\colon \Gamma\to\mathcal{L}(H_2).$$

We say that K is positive definite when

$$\sum_{j,k=1,2}\sum_{s,t\in\Gamma}\langle K_{jk}(t^{-1}s)h_j(s),\, h_k(t)\rangle_{H_k} \geqq 0$$

holds for every pair of functions of finite support $h_1\colon \Gamma_1\to H_1$, $h_2\colon \Gamma_1^{-1}\to H_2$.

When $\Gamma_1=\Gamma$ we have a matricial Toeplitz kernel.

Before, in [5], the vectorial case was considered, and in [2] the subject was related to the dilation of a commutant of two contractions. Here we shall consider the general relation between GTK and lifting properties.

We start extending Proposition (I.1).

Proposition 1. *For $j=1,2$ let H_j be a Hilbert space and K_j an $\mathscr{L}(H_j)$-valued positive definite Toeplitz kernel on an abelian group Γ, such that K_j equals the identity on the neutral element e of Γ; call U_j the minimal unitary dilation of K_j to a Hilbert space F_j. Let $\Gamma_1\subset\Gamma$ be a semigroup such that $e\in\Gamma_1$ and every $u\in\Gamma$ can be written as $u=t-s$, $t, s\in\Gamma_1$. Set:*

$$E_+ := \bigvee_{s\in\Gamma_1}[U_1(s)H_1]\subset F_1,\quad E_- := \bigvee_{t\in\Gamma_1}[U_2(-t)H_2]\subset F_2.$$

Then the formula

$$\text{(1a)}\qquad k(s) = P^{E_-}_{H_2}YU_1(s)|_{H_1},\quad s\in\Gamma_1$$

gives a bijection between the operators $Y\in\mathscr{L}(E_+, E_-)$ that satisfy

$$\text{(1b)}\qquad YU_1(s)i^{F_1}_{E_+} = P^{F_2}_{E_-}U_2(s)Y,\ \ for\ \ s\in\Gamma_1,\quad \|Y\| = 1,$$

and the functions k such that $K=\{(K_{jk}), k=1,2;\ H_1, H_2\}$, given by $K_{11}=K_1$, $K_{12}=k$, $K_{21}=\tilde{k}$, $K_{22}=K_2$, is a p.d. GTK on (Γ,Γ_1). Set

$$L(Y) = \{W\in\mathscr{L}(F_1, F_2)\colon\ W\in R(U_1, U_2),\ \|W\| \leqq 1,\ P^{F_2}_{E_-}W|_{E_+}=Y\}.$$

Then

$$\text{(1c)}\qquad g(s) = P^{F_2}_{H_2}WU_1(s)|_{H_1},\quad s\in\Gamma,$$

gives a bijection between $L(Y)$ and the set of functions $g\colon \Gamma\to\mathscr{L}(H_1, H_2)$ such that $G_{11}=K_{11}$, $G_{12}=g$, $G_{21}=\tilde{g}$, $G_{22}=K_{22}$ defines an element $G=(G_{jk})^2_{j,k=1}$ of the class $\mathscr{G}(K)$ of the p.d. matricial Toeplitz kernels that extend K. In particular, $L(Y)$ is non void if and only if $\mathscr{G}(K)$ is non void.

Proof. Assume first that (1b) holds; then K satisfies the following equations for every h_1, h_2 as in the definition of p.d. GTK:

$$\sum_{j,k=1,2}\sum_{s,t\in\Gamma}\langle K_{jk}(s-t)h_j(s),\, h_k(t)\rangle_{H_k} = \sum_{s,t\in\Gamma}\{\langle U_1(s)h_1(s),\, U_1(t)h_1(t)\rangle_{E_+} +$$
$$+2\,\mathrm{Re}\,\langle YU_1(s)h_1(s),\, U_2(t)h_2(t)\rangle_{E_-} + \langle U_2(s)h_2(s),\, U_2(t)h_2(t)\rangle_{E_-}\} =$$
$$= \Big\|\sum_{s\in\Gamma}U_1(s)h_1(s)\Big\|^2_{E_+} + 2\,\mathrm{Re}\Big\langle Y\sum_{s\in\Gamma}U_1(s)h_1(s),\, \sum_{t\in\Gamma}U_2(t)h_2(t)\Big\rangle_{E_-} + \Big\|\sum_{t\in\Gamma}U_2(t)h_2(t)\Big\|^2_{E_-},$$

which is a non-negative real number because $\|Y\|\leqq 1$; thus, K is positive definite.

Conversely, if the last is assumed, set for all h_1, h_2 as above

$$D(h_1, h_2) = \sum \{\langle K_{12}(s-t)h_1(s), h_2(t)\rangle_{H_2}: s\in\Gamma_1, t\in\Gamma_1^{-1}\}.$$

Then D defines a sesquilinear form on $E_+\times E_-$ such that $\|D\|\leqq 1$. So there exists $Y\in\mathcal{L}(E_+, E_-)$ which satisfies $\|Y\|\leqq 1$ and $\langle Ya, b\rangle_{E_-} = D(a, b)$ for all $(a, b)\in E_+\times\times E_-$. The proof of Naimark's dilation theorem shows that we may assume $U_j(s)h_j(t) = h_j(t-s)$ to be always true. Thus

$$\langle YU_1(u)h_1, h_2\rangle_{E_-} = \sum \{\langle K_{12}(s-t)h_1(s-u), h_2(t)\rangle_{H_2}: s\in\Gamma_1, t\in\Gamma_1^{-1}\} =$$
$$= \langle Yh_1, U_2(-u)h_2\rangle_{E_-} = \langle P_{E_-}^{F_2}U_2(u)Yh_1, h_2\rangle_{E_-}.$$

From the definitions of E_+ and E_- it follows that (1b) holds. Our first assertion is proved.

Now let $W\in L(Y)$. From (I.1) we know that G is p.d. For any $s\in\Gamma_1$, $x_1\in H_1$, $x_2\in H_2$ we have that

$$\langle G_{12}(s)x_1, x_2\rangle_{H_2} = \langle P_{H_2}^{F_2}WU_1(s)x_1, x_2\rangle_{H_2} = \langle P_{H_2}^{E_-}(P_{E_-}^{F_2}|_{E_+})U_1(s)x_1, x_2\rangle_{H_2} =$$
$$= \langle P_{H_2}^{E_-}YU_1(s)x_1, x_2\rangle_{H_2} = \langle K_{12}(s)x_1, x_2\rangle_{H_2},$$

so $G\in\mathcal{G}(K)$. If we start by assuming this, we know that (1c) defines a contraction $W\in\mathcal{R}(U_1, U_2)$. For all $x_1\in H_1$, $x_2\in H_2$, $s\in\Gamma_1$, $t\in\Gamma_1^{-1}$ we have:

$$\langle P_{E_-}^{F_2}WU_1(s)x_1, U_2(t)x_2\rangle_{E_-} = \langle WU_1(s)x_1, U_2(t)x_2\rangle_{F_2} = \langle WU_1(s-t)x_1, x_2\rangle_{F_2} =$$
$$= \langle P_{H_2}^{F_2}WU_1(s-t)x_1, x_2\rangle_{H_2} = \langle g(s-t)x_1, x_2\rangle_{H_2} = \langle k(s-t)x_1, x_2\rangle_{H_2} =$$
$$= \langle P_{H_2}^{E_-}YU_1(s-t)x_1, x_2\rangle_{H_2} = \langle P_{H_2}^{E_-}U_2(-t)YU_1(s)x_1, x_2\rangle_{H_2} = \langle YU_1(s)x_1, U_2(t)x_2\rangle_{E_-}.$$

Thus $P_{E_-}^{F_2}W|_{E_+} = Y$.

If K is a p.d. GTK, the (possibly void) set $\mathcal{G}(K)$ is naturally related with the set $\mathcal{U}(K)$ of the minimal unitary dilations of K, i.e., of the unitary representations U of Γ on a Hilbert space F such that:

$$H_1, H_2\subset F;\ F = \bigvee_{j=1,2} \{\bigvee_{s\in\Gamma} [U(s)H_j]\};$$

$$K_{jk}(s-t) = P_{H_k}^F(s-t)|_{H_j}, \quad \text{for} \quad (s, t)\in\Gamma_j\times\Gamma_k, \quad j, k = 1, 2, \quad \text{with} \quad \Gamma_2 = \Gamma_1^{-1}.$$

In fact, if $U\in\mathcal{U}(K)$, $G_{jk}(s) = P_{H_k}^F U(s)|_{H_j}$, for $s\in\Gamma$, defines an element $G = (G_{jk})_{j,k=1}^2$ of $\mathcal{G}(K)$. Conversely, if $G\in\mathcal{G}(K)$, its minimal unitary dilation U satisfies the conditions required to belong to $\mathcal{U}(K)$ and, by its very definition, is related to G by the last equality. Moreover, this correspondence between U and G is a bijection if we identify in $\mathcal{U}(K)$ the representations that are equivalent under unitary isomorphisms that leave invariant all the elements of H_1 and H_2. Thus, if $\mathcal{G}(K)$ non void for every positive definite generalized Toeplitz kernel K on (Γ, Γ_1), it follows that

Naimark's dilation theorem extends to these kernels. In such a case we could say that (Γ, Γ_1) has *Naimark's property*.

When $\Gamma=Z$, $\Gamma_1=Z_1$ (1b) reduces to $YV_+=V_-^*Y$, with V_+, V_- isometries. It is known that $L(Y)$ and $\mathcal{G}(K)$ are both non void and these two facts have been proved independently. Because of (1) each of them can be deduced from the other one. In fact, (Z, Z_1) has Naimark's property [5]. On the other side the lifting of $YV_+=V_-^*Y$ to a commutator of isometries can be obtained as a particular case of the theorem of Sz.-Nagy and Foiaş. More precisely, this theorem is based on a previous result ([9], Proposition II.2.2) which implies that, if V' is a minimal isometric dilation of V_-^* to $E'\supset E_-$, then there exists $Y'\in\mathcal{L}(E_+, E')$ such that $Y'V_+=V'Y'$, $Y=P_{E_-}^{E'}Y'$ and $\|Y'\|\leqq 1$. Now, it is well known that every commutator of isometries can be lifted to a commutator of their minimal unitary extensions (this has also been proved in the previous section); if U_2^* is a minimal unitary dilation of V_-, U_2 has the same property with respect to V_-^* and V'; it follows that there exists $W\in L(Y)$, so that this set is non void. In particular, this gives another proof of the fact that $\mathcal{G}(K)$ is non void (which is certainly less simple than the original one presented in [5]).

Now we can state the relation between GTK and commutators of semigroups of contractions by means of the following extension of proposition (I.2).

Proposition 2. *Let Γ be an Abelian group with neutral element e and Γ_1 a sub-semigroup of Γ. Set $(-\Gamma_1)=\{s\in\Gamma:\ -s\in\Gamma_1\}$ and assume that $\Gamma_1\cap(-\Gamma_1)=\{e\}$ and $\Gamma_1\cup(-\Gamma_1)=\Gamma$. Let $\{T_1(s):\ s\in\Gamma_1\}$ and $\{T_2(s):\ s\in\Gamma_1\}$ be two semigroups of contractions in the Hilbert spaces H_1 and H_2, respectively. Let $X\in\mathcal{L}(H_1, H_2)$ be such that:*

(2a) $$XT_1(s)=T_2(s)X, \quad \text{for} \quad s\in\Gamma_1, \quad \text{and} \quad \|X\|=1.$$

Let the GTK $K=\{(K_{jk}), j, k=1,2;\ H_1, H_2\}$ associated with the commutator (2a) *be defined by*

(2b) $$K_{jj}(s)=T_j(s) \ \text{if} \ s\in\Gamma_1, \quad K_{jj}(s)=T_j^*(-s) \ \text{if} \ s\in(-\Gamma_1), \quad j=1,2;$$
$$K_{12}(s)=T_2(s)X \ \text{for} \ s\in\Gamma_1; \quad K_{21}=\tilde{K}_{12}.$$

Then K is p.d. and $\mathcal{G}(K)$ is non void if and only if the following conditions hold:

(2c) *for $j=1,2$ there exists a unitary representation U_j of Γ in a Hilbert space F_j that contains H_j and satisfies*

$$T_j(s)=P_{H_j}^{F_j}U_j(s)|_{H_j}, \quad \text{for} \quad s\in\Gamma_1,\ F_j=\bigvee_{s\in\Gamma}[U_j(s)H_j];$$

(2d) *there exists $W\in\mathcal{L}(F_1, F_2)$ that satisfies $WU_1(s)=U_2(s)W$ for all*

$$s\in\Gamma;\ \|W\|=\|X\| \ \text{and} \ P_{H_2}^{F_2}W|_{E_1}=XP_{H_1}^{E_1}, \quad \text{where} \quad E_1:=\bigvee_{s\in\Gamma_1}[U_1(s)H_1].$$

Moreover, if these conditions are satisfied, (1c) *gives as in Proposition* 1 *a bijection between* $\mathcal{G}(K)$ *and the set of operators* W *as in* (2d).

Proof. We start assuming K p.d. and $\mathcal{G}(K)$ non void; then K_{11} and K_{22} are also p.d. and (2c) follows from Naimark's dilation theorem. For a given $G\in\mathcal{G}(K)$, (I.1) shows that there exists a contraction $W\in R(U_1, U_2)$ such that $G_{12}(s)=$ $=P_{H_2}^{F_2}WU_1(s)|_{H_1}$ holds for all $s\in\Gamma$. Then, for $x\in H_1$, $s\in\Gamma_1$, we have $P_{H_2}^{F_2}WU_1(s)x=$ $=G_{12}(s)x=K_{12}(s)x=XT_1(s)x=XP_{H_1}^{F_1}U_1(s)x=XP_{H_1}^{E_1}U_1(s)x$ and (2d) follows. Assume conversely that (2c) and (2d) are true. First of all, it is easy to see that U_j is a minimal unitary dilation of K_{jj}; thus the last is positive definite and U_j is essentially unique. Let $G=(G_{jk})_{j,k=1}^2$ be the matricial Toeplitz kernel associated with the commutator (2d); then $G_{jj}=K_{jj}$, $j=1, 2$, and $G_{12}(s)=\tilde{G}_{21}(s)=P_{H_2}^{F_2}WU_1(s)|_{H_1}$, for $s\in\Gamma$. We know that G is p.d.; moreover, for $s\in\Gamma_1$, $U_1(s)H_1\subset E_1$ so we have $K_{12}(s)=T_2(s)X=XT_1(s)=XP_{H_1}^{E_1}U_1(s)|_{H_1}=P_{H_2}^{F_2}WU_1(s)|_{H_1}=G_{12}(s)$. Thus $G\in\mathcal{G}(K)$.

In the next section what has been done up to now will be applied to commutators of continuous monoparametric semigroups of contractions. Here, as an example, we shall recall and complete some results of [2]. The following holds.

Let T_1 and T_2 be contractions in Hilbert spaces H_1 and H_2, respectively, and $X\in\mathcal{L}(H_1, H_2)$ such that $XT_1=T_2X$. Let $V_1\in\mathcal{L}(E_1)$, $V_2\in\mathcal{L}(E_2)$ be the minimal isometric dilations and $U_1\in\mathcal{L}(F_1)$, $U_2\in\mathcal{L}(F_2)$ the minimal unitary dilations of T_1, T_2, respectively. The following two problems are considered:

i) find $Y\in\mathcal{L}(E_1, E_2)$ such that $YV_1=V_2Y$, $P_{H_2}^{E_2}Y=XP_{H_1}^{E_1}$, $\|Y\|=\|X\|$;

ii) find $W\in\mathcal{L}(F_1, F_2)$ such that $WU_1=U_2W$, $P_{H_2}^{F_2}W|_{E_1}=XP_{H_1}^{E_1}$, $\|W\|=\|X\|$.

If $X=0$ both problems have only the trivial solution, so it is also assumed that $\|X\|=1$.

Let $K=\{(K_{jk}), j, k=1, 2; H_1, H_2\}$ be the GTK on (Z, Z_1) given by $K_{jj}(n)=T_j^n$ if $n\geq 0$, $K_{jj}(n)=T_j^{*-n}$ if $n\leq 0$, $j=1, 2$; $K_{12}(n)=XT_1^n$ for $n\geq 0$ and $K_{21}=\tilde{K}_{12}$.

Theorem 3.

a) *Both problems have solutions.*

b) *K is positive definite.*

c) *There is a bijection between the sets of solutions of these problems and with the set* $\mathcal{G}(K)$ *of all the positive definite matricial Toeplitz kernels that extend* K.

d) *This bijection can be obtained as follows: given* $G\in\mathcal{G}(K)$, *let* $F=F_1\vee F_2$ *be the space of the minimal dilation of* G*; then set* $W=P_{F_2}^F|_{F_1}$, *solution of* (i), *and* $Y=P_{E_2}^F|_{E_1}=P_{E_2}^{F_2}W|_{E_1}$, *solution of* (ii).

e) *The solution of these problems is unique if and only if one of the following equalities is satisfied:*

$$\{(I-X^*X)^{1/2}H_1\}^-\oplus\{(U_1-T_1)H_1\}^-=\{(I-X^*X)^{1/2}T_1h+(U_1-T_1)h\colon\ h\in H_1\}^-,$$

$$\{(I-X^*X)^{1/2}H_1\}^-\oplus\{(U_2-T_2)H_2\}^-=\{(I-X^*X)^{1/2}h\oplus(U_2-T_2)Xh\colon\ h\in H_1\}^-.$$

Proof.

a) Follows from (c) and (b) which imply $\mathscr{G}(K)\neq\emptyset$;

b) was proved in [2], Proposition II.1;

c)—d) the assertions concerning Problem (ii) stem from Proposition 2; those concerning Problem (i), from [2], Theorem II.4;

e) follows from (c) and the theorem on the uniqueness of the lifting [1]; also, because $\mathscr{G}(K)$ has only one element if and only if one of these equalities is satisfied ([2], Theorem II.8).

The proof is done.

Remark. The above theorem includes the following result (see [8]) for T_1, T_2, X, U_1, U_2 as before there exists $W\in\mathscr{L}(F_1, F_2)$ such that $WU_1=U_2W$, $\|X\|=\|W\|$ and $X=P^{F_2}_{H_2}W|_{H_1}$ hold.

III. The continuous case

Our task in this section is to show that the results for the discrete case can be extended to the continuous one also. Specifically, we shall show that Proposition II.2 gives positive results when $\Gamma=R$, the set of real numbers, and $\Gamma_1=R_1=\{s\in R\colon s\geqq 0\}$.

Following our general approach we shall first see that (R, R_1) has Naimark's property; in other words, we shall state the dilation theorem for continuous operator-valued GTK, proofs of which were given in [6] and [7] for the scalar case. That result will then be applied to the commutator of two continuous semigroups of contractions.

Our method will be to relate each GTK on (R, R_1) with another on (Z, Z_1) by means of a systematic use of the results concerning semigroups, their dilations and cogenerators, of Sz.-Nagy and Foiaş ([9], Sections III.8 and III.9).

We start with a p.d. GTK on (R, R_1), $K=\{(K_{jk}), j, k=1, 2;\ H_1, H_2\}$, such that the K_{jk} are weakly continuous functions. We keep the notation of the preceding section, in particular that of Propositions II.1 and II.2. Let U_1' and U_2' be the cogenerators of U_1 and U_2, minimal unitary dilations of K_{11} and K_{22}, respectively. It is known that U_1' and U_2' are unitary operators and that the following holds ([9], Theorem III.8.1):

1) $U_j'=\text{strong}\lim\limits_{s\to 0+}\Phi_s[U_j(s)]$, $j=1, 2$, where Φ_s is the holomorphic function in the complex plane minus the point $(1+s)$ given by $\Phi_s(z)=(z-1+s)/(z-1-s)$, for $s\in R_1$.

What follows is based in the next equalities ([9], III.9.6, III.9.10).

2) $F_j \equiv \bigvee_{s\in R} [U_j(s)H_j] = \bigvee_{n\in Z} [U_j'^n H_j], \quad j=1,2;$

$E_+ \equiv \bigvee_{s\in R_1} [U_1(s)H_1] = \bigvee_{n\in Z_1} [U_1'^n H_1],$

$E_- \equiv \bigvee_{-s\in R_1} [U_2(s)H_2] = \bigvee_{-n\in Z_1} [U_2'^n H_2].$

As we said, a GTK on (Z, Z_1), $K'=\{(K'_{jk}), j,k=1,2;\ H_1, H_2\}$ will be associated with K. We start defining K'_{11} and K'_{22} in the natural way:

3) $K'_{jj}(m) := P^{F_j}_{H_j} U_j'^m|_{H_j}, \quad$ for $\quad m\in Z, \quad j=1,2.$

Then, from (1), we get that

3a) $K'_{jj}(m) = \text{strong}\lim_{s\to 0^+} P^{F_j}_{H_j} \Phi_s^{|m|}[U_j(s\cdot \operatorname{sign} m)]|_{H_j}, \quad j=1,2.$

Let D be the sesquilinear form on $E_+\times E_-$ determined by

4) $D(U_1(s)h_1, U_2(t)h_2) = \langle K_{12}(s-t)h_1, h_2\rangle_{H_2}, \quad s\geqq 0, \quad t\leqq 0, \quad h_1\in H_1,$

$h_2\in H_2.$

From the very definition we get the identity

4a) $D(U_1(s-t)h_1, h_2) = D(U_1(s)h_1, U_2(t)h_2) = D(h_1, U_2(t-s)h_2).$

We want to prove the corresponding result for the discrete case, that is,

4b) $D(U_1'^{m-n}h_1, h_2) = D(U_1'^m h_1, U_2'^n h_2) = D(h_1, U_2'^{n-m}h_2)$

for all $m\geqq 0$, $n\leqq 0$, $h_1\in H_1$, $h_2\in H_2$. In order to do that we refer to the identity ([9], III.9.9) and to the one we obtain from it by conjugation. They imply, respectively,

4c) $U_1'^m h_1 = \lim_{s\to 0^+} \sum_{k=0}^{\infty} d_k(s, m)U_1(ks)h_1, \quad m\geqq 0, \quad h_1\in H_1,$

and

4d) $U_2'^n h_2 = \lim_{-s\to 0^+} \sum_{k=0}^{\infty} d_k(-s, -n)U_2(ks)h_2, \quad n\leqq 0, \quad h_2\in H_2,$

where, for $s\in R_1$ and $m\in Z_1$, $\{d_k(s, m)\}_{k=0}^{\infty}$ are the coefficients of the Taylor series of the function Φ_s^m. Since K is positive definite, D is bounded and consequently the following hold:

4e) $D(U_1'^{m-n}h_1, h_2) = \underset{s\to 0^+}{\operatorname{limit}} \sum_{k=0}^{\infty} d_k(s, m-n) D(U_1(ks)h_1, h_2)$,

$$D(U_1'^m h_1, U_2'^n h_2) = \underset{s\to 0^+}{\operatorname{limit}} \sum_{k,j=0}^{\infty} d_k(s, m)\,\overline{d_j(s, -n)}\,D(U_1(ks)h_1, U_2(-js)h_2) =$$

$$= \underset{s\to 0^+}{\operatorname{limit}} \sum_{v=0}^{\infty} \Big[\sum_{j=0}^{v} d_{v-j}(s, m)\, d_j(s, -n)\Big] D(U_1(vs)h_1, h_2).$$

In order to prove the last, recall (4a), set $v=k+j$ and remark that the $d_k(s, m)$ are real numbers. Then the first equality (4b) stems from

$$d_k(s, m+n) = \sum_{j=0}^{k} d_{k-j}(s, m)\, d_j(s, n), \quad s\in R_1, \quad m, n, k\in Z_1,$$

which is a consequence of $\Phi_s^m \Phi_s^n \equiv \Phi_s^{m+n}$. The second equality (4b) can be proved in the same way.

We now complete the definition of K' by setting

5) $\langle K_{12}'(m)h_1, h_2\rangle_{H_2} = D(U_1'^m h_1, h_2), \quad m\in Z_1, \quad h_1\in H_1, \quad h_2\in H_2; \; K_{21}' = \tilde{K}_{12}'$.

From (3a), (4) and (4e) we get the following direct formulas for K' in terms of K.

6) $K_{jj}'(m) = \underset{b\to 0^+}{\text{strong limit}} \sum_{k=0}^{\infty} d_k(s, |m|) K_{jj}[(\operatorname{sign} m)ks], \quad m\in Z, \quad j = 1, 2;$

$$K_{12}'(m) = \underset{s\to 0^+}{\text{strong limit}} \sum_{k=0}^{\infty} d_k(s, m) K_{12}(ks), \quad m\in Z_1.$$

We shall see that K' is p.d. Set $Z_2 = -Z_1$ and let $f_j\colon Z_j \to H_j$ be functions with finite support, $j=1, 2$. From definitions (3) and (5), and the identity (4b) it follows that

$$\sum_{j,k=1,2} \sum_{(m,n)\in Z_j\times Z_k} \langle K_{jk}'(m-n) f_j(m), f_k(n)\rangle_{H_k} = \Big\|\sum_{m\in Z_1} U_1'^m f_1(m)\Big\|_{F_1}^2 +$$

$$+ 2\operatorname{Re} D\Big(\sum_{m\in Z_1} U_1'^m f_1(m), \sum_{n\in Z_2} U_2'^n f_2(n)\Big) + \Big\|\sum_{n\in Z_2} U_2'^n f_2(n)\Big\|_{F_2}^2 \geqq 0,$$

because K p.d. implies $\|D\| \leqq 1$.

Consequently, $\mathcal{U}(K') \approx \mathcal{G}(K') \neq \emptyset$. With each $G'\in\mathcal{G}(K')$ we shall associate a $G\in\mathcal{G}(K)$, getting in that way a bijection and, in particular, proving that $\mathcal{G}(K)$ is non void. In order to do that we refer once more to the relation between matricial Toeplitz kernels and intertwining operators. Given $G' = (G_{jk}')_{j,k=1}^2 \in \mathcal{G}(K')$ let $T_{G'}\in\mathcal{L}(F_1, F_2)$ be the operator determined by

7) $\langle T_{G'} U_1'^m h_1, U_2'^n h_2\rangle_{F_2} = \langle G_{12}'(m-n)h_1, h_2\rangle_{H_2}, \quad m, n\in Z, \quad h_1\in H_1, \quad h_2\in H_2.$

As we know,

7a) $T_{G'}U_1' = U_2'T_{G'}, \quad \|T_{G'}\| \leqq 1,$

and it is clear that

7b) $G_{12}'(m) = P_{H_2}^{F_2}T_{G'}U_1'^m|_{H_1} = P_{H_2}^{F_2}U_2'^mT_{G'}|_{H_1}, \quad \text{for} \quad m\in Z.$

We shall show that $T_{G'}$ also intertwines U_1 and U_2:

7c) $T_{G'}U_1(s) = U_2(s)T_{G'}, \quad \text{for} \quad s\in R.$

It is enough to see it for all $s>0$; in order to do that we refer to a reciprocal ([9], III.9.8) of a formula we have already used; it implies that

8) $$U_j(s)h_j = \operatorname*{limit}_{r\to 1^-} \sum_{k=0}^{\infty} r^k c_k(s)U_j'^k h_j, \quad \text{for} \quad s>0, \quad h_j\in H_j, \quad j=1,2,$$

where $\{c_k(s)\}_{k=0}^{\infty}$ are the Taylor coefficients of the function $e_s(z)=\exp\left(\frac{z+1}{z-1}\right)$, $s>0$. From (8) it follows that $T_{G'}U_1(s)h_1 = \operatorname*{limit}_{r\to 1^-} \sum_{k=0}^{\infty} r^k c_k(s)U_2'^kT_{G'}h_1 = U_2(s)T_{G'}h_1$, so (7c) is proved.

Let $G_{12}\colon R\to\mathscr{L}(H_1, H_2)$ be given by

9) $\langle G_{12}(s)h_1, h_2\rangle_{H_2} = \langle T_{G'}U_1(s)h_1, h_2\rangle_{F_2}, \quad s\in R, \quad h_1\in H_1, \quad h_2\in H_2.$

Setting $G_{11}:=K_{11}, G_{12}, G_{21}:=\tilde{G}_{12}, G_{22}:=K_{22}$ we define a p.d. matricial Toeplitz kernel G. It only remains to see that G extends K. Since

$$D(U_1'^kh_1, h_2) = \langle K_{12}'(k)h_1, h_2\rangle_{H_2} = \langle G_{12}'(k)h_1, h_2\rangle_{H_2} = \langle T_{G'}U_1'^kh_1, h_2\rangle_{F_2},$$

it follows from (8) that, for $s>0$, we have

$$\langle K_{12}(s)h_1, h_2\rangle_{H_2} = D\big(U_1(s)h_1, h_2\big) = \big\langle T_{G'}\big[\operatorname*{limit}_{r\to 1^-} \sum_{k=0}^{\infty} r^k c_k(s)U_1'^kh_1\big], h_2\big\rangle_{F_2} =$$

$$= \langle T_{G'}U_1(s)h_1, h_2\rangle_{F_2} = \langle G_{12}(s)h_1, h_2\rangle_{H_2}.$$

Thus, $G\in\mathscr{G}(K)$. Also, it follows from (9), (8) and (7b) that

10) $$G_{12}(s) = \operatorname*{strong\,limit}_{r\to 1^-} \sum_{k=0}^{\infty} r^k c_k(|s|)G_{12}'[k(\operatorname{sign} s)], \quad \text{for} \quad s\in R.$$

Conversely:

10a) $$G_{12}'(m) = \operatorname*{strong\,limit}_{s\to 0^+} \sum_{k=0}^{\infty} d_k(s, |m|)G_{12}[ks(\operatorname{sign} m)], \quad \text{for} \quad m\in Z.$$

So we have proved the following

Theorem 11. *Let* $K=\{(K_{jk}),\ j,k=1,2;\ H_1,H_2\}$ *be a positive definite generalized Toeplitz kernel on* (R, R_1) *such that the functions* K_{jk} *are continuous in the weak topology of operators. Then there exists a p.d. GTK* K' *on* (Z, Z_1), *such that there is a bijection between* $\mathcal{G}(K)$ *and* $\mathcal{G}(K')$.

The correspondence $K\to K'$ *given by this theorem is reversible; the converse of formula* (6) *is the following:*

$$\text{11a)}\quad K_{jj}(s) = \underset{r\to 1^-}{\text{strong limit}} \sum_{k=0}^{\infty} r^k c_k(|s|)K'_{jj}[(\operatorname{sign} s)k], \quad s\in R, \quad j=1,2;$$

$$K_{12}(s) = \underset{r\to 1^-}{\text{strong limit}} \sum_{k=0}^{\infty} r^k c_k(s)K'_{12}(k), \quad s\in R_1.$$

Theorem (11) allows us to transfer a uniqueness condition from the discrete case ([2], Proposition I.6) to the continuous one.

Corollary 12. *Let* K *be as in theorem* (11). *Then* $\mathcal{G}(K)$ *contains only one element if and only if at least one of the following two conditions is satisfied:*

i) $\{(I-Q^*Q)^{1/2}E\}^- = \{(I-Q^*Q)^{1/2}U_1'E_+\}^-$,

ii) $\{(I-QQ^*)^{1/2}E_-\}^- = \{(I-QQ^*)^{1/2}U_2'E_-\}^-$,

where U_1', U_2' *are the cogenerators of the minimal unitary dilations of* K_{11}, K_{22}, *respectively, and* Q *is the operator from*

$$E_+ = \bigvee_{s\geqq 0} U_1(s)H_1 \quad \textit{to} \quad E_- = \bigvee_{s\leqq 0} U_2(s)H_2,$$

given for $s\geqq 0$, $t\leqq 0$, $h_1\in H_1$, $h_2\in H_2$ *by*

$$\langle QU_1(s)h_1, U_2(t)h_2\rangle_{E_-} = \langle K_{12}(s+t)h_1, h_2\rangle_{H_2}.$$

As an application of what has been done in this section we shall state a theorem on the commutator of two semigroups of contractions which is the continuous version of (II.3).

Theorem 13. *Let* $\{T_1(s)\colon s\geqq 0\}$, $\{T_2(s)\colon s\geqq 0\}$ *be continuous monoparametric semigroups of contraction an Hilbert spaces* H_1, H_2, *respectively, and* $X\in\mathcal{L}(H_1, H_2)$ *such that* $XT_1(s)=T_2(s)X$ *holds for all* $s\geqq 0$. *Let* $\{V_1(s)\colon s\geqq 0\}\subset\mathcal{L}(E_1)$, $\{V_2(s)\colon s\geqq 0\}\subset\mathcal{L}(E_2)$ be *minimal isometric dilations and* $\{U_1(s)\colon s\in R\}\subset\mathcal{L}(F_1)$, $\{U_2(s)\colon s\in R\}\subset\mathcal{L}(F_2)$ *minimal unitary dilations of the semigroups* T_1, T_2, *respectively. The following problems are considered:*

i) *find* $Y\in\mathcal{L}(E_1, E_2)$ *such that* $YV_1(s)=V_2(s)Y$, *for* $s\geqq 0$, $P^{E_2}_{H_2}Y=XP^{E_1}_{H_1}$ *and* $\|Y\|=\|X\|$;

ii) *find* $W\in\mathcal{L}(F_1, F_2)$ *such that* $WU_1(s)=U_2(s)W$, *for* $s\in R$, $P^{F_2}_{H_2}W|_{E_1}=XP^{E_1}_{H_1}$ *and* $\|W\|=\|X\|$.

Discarding the trivial case, it may be assumed by homogeneity that $\|X\|=1$. *Let* $K=\{(K_{jk}), j,k=1,2;\ H_1, H_2\}$ *be the GTK on* (R, R_1) *given by*

$$K_{jj}(s)=T_j(s) \quad \text{if} \quad s\geqq 0, \quad K_{jj}(s)=T_j^*(-s) \quad \text{if} \quad s\leqq 0, \quad j=1,2;$$

$K_{12}(s)=T_2(s)X$ *if* $s\geqq 0$, $K_{21}=\tilde{K}_{12}$. *Then:*

a) *Both problems have solutions.*

b) K *is positive definite.*

c) *There exist bijections between the set of solutions of* (i), *the one of* (ii) *and* $\mathscr{G}(K)$, *and these bijections are determined by*

$$\langle G_{12}(s-t)h_1, h_2\rangle_{H_2} = \begin{cases} \langle WU_1(s)h_1, U_2(t)h_2\rangle_{F_2}, & \text{for} \quad s,t\in R, \quad h_1\in H_1, \quad h_2\in H_2 \\ \langle YV_1(s)h_1, V_2(t)h_2\rangle_{E_2}, & \text{for} \quad s,t\geqq 0, \quad h_1\in H_1, \quad h_2\in H_2. \end{cases}$$

d) *The solution of both problems is unique if and only if at least one of the following equalities is satisfied:*

$$\{(I-X^*X)^{1/2}H_1\}^- \oplus \{(U_1'-T_1')H_1\}^- = \{(I-X^*X)^{1/2}T_1'h+(U_1'-T_1')h\colon h\in H_1\}^-,$$

$$\{(I-X^*X)^{1/2}H_1\}^- \oplus \{(U_2'-T_2')H_2\}^- = \{(I-X^*X)^{1/2}h\oplus(U_2'-T_2')Xh\colon h\in H_1\}^-,$$

where U_1', U_2', T_1', T_2' *are the cogenerators of* U_1, U_2, T_1, T_2, *respectively.*

Proof. *First step:* some properties that we have already used ([9], Sections III.8 and III.9) show that U_j' (V_j') is the minimal unitary (isometric) dilation of T_j', where V_j' denotes the cogenerator of the semigroup V_j; moreover, $XT_1'=T_2'X$ holds; from $WU_1'=U_2'W$ it follows that $WU_1(s)=U_2(s)W$ for all $s\in R$, and from $YV_1'=V_2'Y$, that $YV_1(s)=V_2(s)Y$ for all $s\in R_1$.

Second step: apply Theorem II.3 to T_1', T_2', V_1', V_2', U_1', U_2', calling K' the GTK on (Z, Z_1) that in its statement is called K.

Third step: note that (11a) relates precisely the kernels K and K' we are considering here.

Fourth step: apply Theorem 11 of this section.

Remark. The applications of generalized and matricial Toeplitz kernels to the realization of linear systems and scattering theory are considered in [3] and [4].

Added in proofs. A more conceptual appraach to the concept of section III is given in [10].

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Contractions quasisimilar to an isometry

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1. Introduction. The bounded linear operators T_1 and T_2 on complex, separable Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are *quasisimilar* ($T_1 \sim T_2$) if there are operators $X\colon \mathfrak{H}_1 \to \mathfrak{H}_2$ and $Y\colon \mathfrak{H}_2 \to \mathfrak{H}_1$ with trivial kernel and dense range such that $XT_1 = T_2 X$ and $YT_2 = T_1 Y$. This paper is concerned with the question when a contraction is quasisimilar to an isometry. This problem has been studied before: in [12] for contractions with finite defect indices, [5] for subnormal contractions and [15] for hyponormal contractions. Our main result in this paper (Theorem 2.7) generalizes all these previous ones. We show that a contraction T whose $C_{\cdot 0}$ part has finite multiplicity is quasisimilar to an isometry if and only if its $C_{\cdot 1}$ part is of class C_{11} and its $C_{\cdot 0}$ part is quasisimilar to a unilateral shift. These latter conditions can further be expressed in terms of the inner and outer factors of the characteristic function of T. In § 3, we show that in certain circumstances quasisimilarity to an isometry even implies unitary equivalence and partially verify a conjecture we proposed in [15].

Recall that a contraction T ($\|T\| \leq 1$) is of *class* $C_{\cdot 0}$ (resp. $C_{0\cdot}$) if $T^{*n}x \to 0$ (resp. $T^n x \to 0$) for all x; T is of *class* $C_{\cdot 1}$ (resp. $C_{1\cdot}$) if $T^{*n}x \not\to 0$ (resp. $T^n x \not\to 0$) for all $x \neq 0$. $C_{\alpha\beta} = C_{\alpha\cdot} \cap C_{\cdot\beta}$ for $\alpha, \beta = 0, 1$. Any contraction T can be uniquely triangulated as $T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$, where T_1 and T_2 are of classes $C_{\cdot 1}$ and $C_{\cdot 0}$, respectively (called the $C_{\cdot 1}$ and $C_{\cdot 0}$ *parts* of T). A contraction T can also be decomposed as $U \oplus T'$, where U is a unitary operator and T' is completely nonunitary (c.n.u.); U and T' are called the *unitary part* and *c.n.u. part* of T. T is said to be of *analytic type* if it has no singular unitary direct summand. For such T, the functional calculus, $\varphi(T)$ for $\varphi \in H^\infty$ is well-defined. For the details and other properties of contractions, readers are referred to Sz.-Nagy and Foiaş' book [7].

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Let T_1 and T_2 be operators on $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. We use $T_1 \overset{d}{\prec} T_2$ to denote that there is an operator X: $\mathfrak{H}_1 \to \mathfrak{H}_2$ with dense range and satisfying $XT_1 = T_2X$, and $T_1 \prec T_2$ if the intertwining X is both injective and with dense range (called a *quasiaffinity*). $T_1 \overset{d}{\sim} T_2$ if $T_1 \overset{d}{\prec} T_2$ and $T_2 \overset{d}{\prec} T_1$; $T_1 \sim T_2$ if $T_1 \prec T_2$ and $T_2 \prec T_1$. T_1 is *similar* to T_2 ($T_1 \approx T_2$) if the intertwining operator X is invertible; T_1 is *unitarily equivalent* to T_2 ($T_1 \cong T_2$) if X is unitary. The *multiplicity* μ_T of an operator on $\mathfrak{H}$ is the minimum cardinality of a set $\mathfrak{K} \subseteq \mathfrak{H}$ for which $\mathfrak{H} = \bigvee_{n=0}^{\infty} T^n \mathfrak{K}$. Note that $T_1 \overset{d}{\prec} T_2$ implies $\mu_{T_1} \geqq \mu_{T_2}$. In the following, we use S_n to denote the unilateral shift with multiplicity n acting on H_n^2.

2. Main results. We start with the following proposition.

Proposition 2.1. *Let T be a contraction on $\mathfrak{H}$ and $1 \leqq n < \infty$. Then $T \sim S_n$ if and only if $T \overset{d}{\sim} S_n$. Moreover, in this case, T is of class C_{10}, and there exist quasiaffinities X: $\mathfrak{H} \to H_n^2$ and Y: $H_n^2 \to \mathfrak{H}$ which intertwine T and S_n and such that $XY = \delta(S_n)$ and $YX = \delta(T)$ for some outer function δ in H^∞.*

Proof. Assume that $T \overset{d}{\sim} S_n$. We first show that T is of analytic type. Let $T = U_s \oplus T'$ on $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, where U_s is a singular unitary operator and T' is a contraction of analytic type, and let $Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$: $H_n^2 \to \mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ be an operator intertwining S_n and T and with dense range. Then Y_1 intertwines S_n and U_s and has dense range in $\mathfrak{H}_1$. It can be lifted to an operator $\tilde{Y}_1$ which intertwines the minimal unitary extension U of S_n and U_s (cf. [4, Corollary 5.1]). Since U is absolutely continuous and U_s is singular, $\tilde{Y}_1$ must be the zero operator (cf. [4, Theorem 3]). Hence $Y_1 = 0$ and it follows that $T = T'$ is of analytic type.

Let X: $\mathfrak{H} \to H_n^2$ be an operator intertwining T and S_n and with dense range. Then XY commutes with S_n and has dense range in H_n^2. We may assume that $\|XY\| \leqq 1$. Thus XY is the operator Φ_+ of multiplication by a contractive operator-valued analytic function Φ on H_n^2 which is even outer (cf. [7, Lemma V.3.2]). By [7, Proposition V.6.1], Φ has a scalar multiple $\delta \in H^\infty$: there exists another contractive analytic function Ω such that $\Omega(\lambda)\Phi(\lambda) = \delta(\lambda)I$ and $\Phi(\lambda)\Omega(\lambda) = \delta(\lambda)I$ ($|\lambda| < 1$). Since Φ is an outer function, we may take δ to be outer (cf. [7, Theorem V.6.2]). Let $Z = \Omega_+ X$. Then Z intertwines T and S_n and $ZY = (\Omega_+ X)Y = \Omega_+ \Phi_+ = \delta(S_n)$. Multiplying both sides by Y, we obtain $YZY = Y\delta(S_n) = \delta(T)Y$ (here we need the fact that T is of analytic type). Since Y has dense range, we infer that $YZ = \delta(T)$. Note that δ is outer implies that $\delta(S_n)$ and $\delta(T)$ are quasiaffinities (cf. [7, Proposition III.3.1]). It follows easily that X, Y and Z are all quasiaffinities. This shows that $T \sim S_n$. That T is of class C_{10} can be easily deduced.

Corollary 2.2. *Let T be a contraction of analytic type and $1 \leqq n < \infty$. Then $T \sim S_n$ if and only if $\mu_T = n$ and $T \overset{d}{\prec} S_n$.*

Proof. The assertion follows from Proposition 2.1 and the fact that $\mu_T = n$ implies that $S_n \overset{d}{\prec} T$ (cf. [15, Lemma 2.3]).

When T is subnormal, the preceding corollary was essentially proved by Hastings [5, Proposition 4.1]. For another set of conditions in order that $T \sim S_n$, compare [1, Theorem 2.8].

Corollary 2.3. *Let $T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ on $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ be a contraction of analytic type. If T_1 is not missing and $T_2 \overset{d}{\prec} S_n$, then $\mu_T \geqq n+1$.*

Proof. Since $\mu_T \geqq \mu_{T2} \geqq n$, we may assume that $n < \infty$. Let $X\colon \mathfrak{H}_2 \to H_n^2$ be an operator intertwining T_2 and S_n and with dense range. Let $Y = [0\ X]\colon \mathfrak{H} = \mathfrak{H}_1 \oplus \oplus \mathfrak{H}_2 \to H_n^2$. Then Y intertwines T and S_n and has dense range. If $\mu_T = n$, then $T \sim S_n$ by Corollary 2.2 and so by the proof of Proposition 2.1 Y is injective, which implies that $\mathfrak{H}_1 = \{0\}$, a contradiction. Hence we have $\mu_T \geqq n+1$.

The next theorem characterizes those contractions which are quasisimilar to a unilateral shift with finite multiplicity in terms of their characteristic functions. It generalizes [12, Lemma 1] for contractions with finite defect indices. For any contraction T, let Θ_T denote its characteristic function (consult [7] for its definition and properties).

Theorem 2.4. *Let T be a contraction and $1 \leqq n < \infty$. Then $T \sim S_n$ if and only if T is of class C_{10}, $\mu_T = n$ and there exists a bounded analytic function Ω such that $\Omega\Theta_T = \delta I$ for some outer function δ in H^∞.*

Proof. Assume that $T \sim S_n$. It is easily seen that T is of class C_{10} whence c.n.u. We may consider its functional model, that is, consider T acting on $\mathfrak{H} = H^2_{\mathfrak{D}_*} \ominus \ominus \Theta_T H^2_{\mathfrak{D}}$ by $Tf = P(e^{it} f)$ for $f \in \mathfrak{H}$, where $\mathfrak{D} = \overline{\operatorname{ran}(I - T^*T)^{1/2}}$, $\mathfrak{D}_* = \overline{\operatorname{ran}(I - TT^*)^{1/2}}$ and P denotes the orthogonal projection onto $\mathfrak{H}$ (cf. [7, Proposition VI.2.1]). By Proposition 2.1, there exist quasiaffinities $X\colon \mathfrak{H} \to H_n^2$ and $Y\colon H_n^2 \to \mathfrak{H}$ which intertwine T and S_n and satisfy $XY = \delta(S_n)$ and $YX = \delta(T)$ for some outer function δ in H^∞. Note that $Xf = \Phi f$ for $f \in \mathfrak{H}$ and $Yg = P(\Psi g)$ for $g \in H_n^2$, where Φ and Ψ are bounded analytic functions satisfying $\Phi\Theta_T = 0$ (cf. [7, Theorem VI.3.6]). From $XY = \delta(S_n)$ and $YX = \delta(T)$, we deduce that $\Phi\Psi = \delta$ and $\Psi\Phi - \delta = -\Theta_T \Omega$ for some bounded analytic function Ω. Since Θ_T is an inner function (cf. [7, Proposition VI.3.5]), we have

$$\Omega\Theta_T - \delta = \Theta_T^* \Theta_T(\Omega\Theta_T - \delta) = \Theta_T^*(\Theta_T \Omega - \delta)\Theta_T = \Theta_T^*(-\Psi\Phi)\Theta_T = 0.$$

Therefore $\Omega\Theta_T=\delta I$ as required. The reverse implication follows as in the proof of [12, Lemma 1].

Using Proposition 2.1, Theorem 2.4 and [14, Theorem 2.1], we can obtain the following interesting result.

Theorem 2.5. *Let* $T=\begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ *be a contraction. If* $T_2 \overset{d}{\sim} S_n$ *for some* $1 \leqq n < \infty$, *then* $T \sim T_1 \oplus T_2$.

Proof. If T is c.n.u., then the conclusion follows from the results cited above. For general T, let $T=U\oplus T'$ on $\mathfrak{H}=\mathfrak{K}\oplus\mathfrak{L}$, where U is unitary and T' is c.n.u. Assume that $T=\begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ is acting on $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2$. We first check that $\mathfrak{H}_2\subseteq\mathfrak{L}$. Since $T_2 \overset{d}{\sim} S_n$ implies that T_2 is of class C_{10} by Proposition 2.1, for any $x\in\mathfrak{H}_2$, we have $T^{*m}x=T_2^{*m}x\to 0$ as $m\to\infty$. If $x=x_1\oplus x_2$, where $x_1\in\mathfrak{K}$ and $x_2\in\mathfrak{L}$, then $U^{*m}x_1\to 0$. Since U is unitary, this implies that $x_1=0$ and thus $x=x_2\in\mathfrak{L}$. This proves $\mathfrak{H}_2\subseteq\mathfrak{L}$ which is equivalent to $\mathfrak{K}\subseteq\mathfrak{H}_1$. It is easily seen that

$$T=\begin{bmatrix} U & 0 & 0 \\ 0 & T_1' & * \\ 0 & 0 & T_2 \end{bmatrix} \quad \text{on} \quad \mathfrak{H} = \mathfrak{K}\oplus(\mathfrak{H}_1\ominus\mathfrak{K})\oplus\mathfrak{H}_2.$$

Since $\begin{bmatrix} T_1' & * \\ 0 & T_2 \end{bmatrix}$ is c.n.u., from above we have $\begin{bmatrix} T_1' & * \\ 0 & T_2 \end{bmatrix}\sim T_1'\oplus T_2$ and therefore $T\sim \sim U\oplus T_1'\oplus T_2=T_1\oplus T_2$.

We remark that it is unknown whether the preceding theorem is still valid under $n=\infty$, that is, when $T_2 \overset{d}{\sim} S_\infty$ or $T_2\sim S_\infty$. In a very special case, this is indeed true.

Theorem 2.6. *Let* $T=\begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ *be a contraction. If* T_2 *is similar to an isometry, then* T *is similar to* $T_1\oplus T_2$.

Proof. If T is c.n.u., this follows from [8, Theorem 2.4] and [14, Theorem 2.1]. For the general case, assume that T_2 is similar to the isometry $V=W\oplus S_n$, where W is unitary and S_n is some unilateral shift. It is easily seen that T_2 can be triangulated as $\begin{bmatrix} T_3 & * \\ 0 & T_4 \end{bmatrix}$ with $T_3\approx W$ and $T_4\approx S_n$. Letting $T_5=\begin{bmatrix} T_1 & * \\ 0 & T_3 \end{bmatrix}$, we have $T=\begin{bmatrix} T_5 & * \\ 0 & T_4 \end{bmatrix}$. Since $T_4\approx S_n$, proceeding as in the proof of Theorem 2.5 we obtain $T\approx T_5\oplus T_4$. On the other hand, $T_3\approx W$ implies that $T_5\approx T_1\oplus T_3$ and $T_2\approx T_3\oplus T_4$ (cf. [9, Theorem 2.14]). Thus $T\approx T_1\oplus T_3\oplus T_4\approx T_1\oplus T_2$ as claimed.

Now we are ready for our main result.

Theorem 2.7. *Let* T *be a contraction and* $T=\begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ *be its triangulation of type* $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$. *Assume that* $\mu_{T_2}<\infty$. *Then the following statements are equivalent:*

(1) *T is quasisimilar to an isometry;*

(2) *T_1 is of class C_{11} and T_2 is quasisimilar to a unilateral shift;*

(3) *Θ_e (the outer factor of Θ_T) is outer from both sides, Θ_i (the inner factor of Θ_T) is inner and $*$-outer, and there exists a bounded analytic function Ω such that $\Omega\Theta_i=\delta I$ for some outer function δ in H^∞.*

Moreover, if T is of analytic type and is quasisimilar to the isometry V, then there are quasiaffinities X and Y intertwining T and V such that $XY=\delta(V)$ and $YX=\delta(T)$.

Proof. (1)$\Rightarrow$(2): Assume that $T\sim V=U\oplus S_n$, where U is unitary. Since T_1 and U are of class $C_{\cdot 1}$ and T_2 and S_n are of class $C_{\cdot 0}$, we can easily deduce that $T_2 \overset{d}{\sim} S_n$. This, together with $\mu_{T_2}<\infty$, implies that $T_2\sim S_n$ by Proposition 2.1. On the other hand, $T\prec V$ implies that T is of class $C_{1\cdot}$ whence T_1 is of class C_{11}.

(2)$\Rightarrow$(1): This follows from Theorem 2.5.

(2)$\Leftrightarrow$(3): Since Θ_e and Θ_i correspond to the characteristic functions of T_1 and T_2, respectively, this follows from Theorem 2.4 and [7, Proposition VI.3.5].

The assertion concerning the intertwining quasiaffinities can be deduced easily from [14, Theorem 2.1].

As we remarked in § 1, the preceding theorem generalizes [11, Theorem 3] for contractions with finite defect indices, [5, Corollary to Theorem 4.5] for subnormal contractions and [14, Corollary 3.11] for hyponormal contractions. An example of Hastings [5] shows that (1) may not imply (2) without the assumption $\mu_{T_2}<\infty$. It is interesting to contrast this theorem it with [10, Theorem 2] where "quasisimilarity" is replaced by "similarity" in which case $\mu_{T_2}<\infty$ won't be needed.

3. Some consequences. In this section, we will derive two results for which an operator quasisimilar to an isometry is even unitarily equivalent to it. More precisely, we show that if $V=U\oplus S_n$ is an isometry, where U is unitary and $n<\infty$, and T is a quasinormal operator or $T\in \operatorname{Alg} V$ (the weakly closed algebra generated by T and 1), then $T\overset{d}{\sim}V$ implies $T\cong V$. For the first one, we prove the following more general result.

Proposition 3.1. *If T is a contraction whose c.n.u. part is of class $C_{\cdot 0}$ and $V=U\oplus S_n$ is an isometry with $n<\infty$, then $T\overset{d}{\sim}V$ implies $T\sim V$.*

Proof. Let $T=U'\oplus T'$, where U' is unitary and T' is c.n.u. Since U' and U are of class $C_{\cdot 1}$ and T' and S_n are of class $C_{\cdot 0}$, we deduce from $T\overset{d}{\sim}V$ that $T'\overset{d}{\sim}S_n$. Thus $T'\sim S_n$ by Proposition 2.1 and therefore $T\sim U'\oplus S_n\overset{d}{\sim}V$. [15, Lemma 3.4] yields that $U'\oplus S_n\cong V$. Thus $T\sim V$ as asserted.

Corollary 3.2. *If T is a hyponormal operator and $V=U\oplus S_n$ is an isometry with $n<\infty$, then $T\overset{d}{\sim}V$ implies $T\sim V$.*

Proof. $T \overset{d}{\sim} V$ implies that their spectra are equal [2, Theorem 2], so are their spectral radii: $r(T)=r(V)$. Hence $\|T\|=r(T)=r(V)=1$ showing that T is a contraction. Now the assertion follows from Proposition 3.1 and the fact that c.n.u. hyponormal contractions are of class $C_{\cdot 0}$ [6].

Corollary 3.3. *If T is a quasinormal operator and $V=U\oplus S_n$ is an isometry with $n<\infty$, then $T \overset{d}{\sim} V$ implies $T\cong V$.*

Proof. By Corollary 3.2, we have $T\sim V$. For quasinormal T, this implies $T\cong V$ (cf. [15, Proposition 4.2]).

Now for our final result. In [15], we asked whether for isometry V, $T\in \operatorname{Alg} V$ and $T\sim V$ imply $T\cong V$, and showed that this is indeed the case if $T\approx V$ [15, Proposition 4.6]. We will now verify its validity when $V=U\oplus S_n$ with $n<\infty$. We start with the following. For any operator T, $T^{(n)}$ denotes the direct sum of n copies of T.

Lemma 3.4. *Let T be a contraction. If $T^{(n)} \overset{d}{\sim} S_n$ for some $1\leqq n<\infty$, then $T\sim S_1$.*

Proof. Since $\Theta_{T^{(n)}}=\Theta_T^{(n)}$, Proposition 2.1 and Theorem 2.4 imply that $T^{(n)}$ is of class C_{10} and there exists a bounded analytic function Ω such that $\Omega\Theta_T^{(n)}=\delta I$ for some outer function δ in H^∞. If Φ denotes the (1, 1)-entry of Ω, then $\Phi\Theta_T=\delta I$. Thus, by Theorem 2.4 again, $T\sim S_k$ for some $1\leqq k<\infty$. Since $S_n\sim T^{(n)}\sim S_{kn}$, we conclude that $k=1$ and $T\sim S_1$.

Proposition 3.5. *Let $V=U\oplus S_n$ be an isometry with $n<\infty$. If $T\in \operatorname{Alg} V$ and $T \overset{d}{\sim} V$, then $T\cong V$.*

Proof. Let $U=U_s\oplus U_a$, where U_s and U_a are singular and absolutely continuous unitary operators, respectively. In view of [15, Lemma 4.3], we may assume that V is not unitary. Hence $T\in \operatorname{Alg} V$ implies that $T=W\oplus\varphi(U_a\oplus S_n)$, where $W\in \operatorname{Alg} U_s$ and $\varphi\in H^\infty$ (cf. [13, Lemma 1.3] and [11, Lemma 3.11]). This shows that T is hyponormal and therefore $T \overset{d}{\sim} V$ implies, by Corollary 3.2, that $T\sim V$. If φ is a constant function, then T is normal whence $T\sim V$ implies that V is unitary, a contradiction. Thus φ is nonconstant and therefore $\varphi(S_n)$ is completely nonnormal (cf. [15, Lemmas 4.4 and 4.5]). Hence $T\sim V$ implies that $W\oplus\varphi(U_a)\cong U$ and $\varphi(S_n) \overset{d}{\sim} S_n$ by [5, Proposition 3.5]. We apply Lemma 3.4 to obtain that $\varphi(S_1)\sim S_1$. It follows from [3, Theorem 1] that $\varphi(S_1)\cong S_1$ whence $\varphi(S_n)\cong S_n$ and $T\cong V$ follows.

Added in proof. Takahashi [16] showed that for isometry V, $T\in \operatorname{Alg} V$ and $T\sim V$ imply $T\cong V$ which answered the question asked in [15].

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On the reflexivity of contractions with isometric parts

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For a bounded linear operator T on a Hilbert space, let $\operatorname{Alg} T$ denote the weakly closed algebra generated by T and the identity. Also let $\operatorname{Lat} T$ and $\operatorname{Alg}\operatorname{Lat} T$ denote the lattice of all invariant subspaces of T and the algebra of all operators A such that $\operatorname{Lat} T \subseteqq \operatorname{Lat} A$, respectively. An operator T is said to be *reflexive* if $\operatorname{Alg}\operatorname{Lat} T=\operatorname{Alg} T$. (Note that we always have $\operatorname{Alg} T\subseteqq \operatorname{Alg}\operatorname{Lat} T$.) The first examples of reflexive operators were given by SARASON [7], that is, he proved that normal operators and analytic Toeplitz operators are reflexive. Subsequently DEDDENS [4] proved the reflexivity of isometries, and now various classes of operators are known to be reflexive.

In [9] and [10], WU considered the generalizations of Deddens' result. In [9] the reflexivity was proved for contractions T on $\mathfrak{H}$ such that $T|\mathfrak{M}$ and $T^*|\mathfrak{H}\ominus\mathfrak{M}$ are isometries for some $\mathfrak{M}\in\operatorname{Lat} T$, and in [10] for contractions which have parts similar to the adjoints of unilateral shifts, in particular, for contractions with a unilateral shift summand. The results of [10] were generalized in [2] as conjectured by Wu, that is, it was proved that if T is a contraction and there exists a nonzero operator X such that $XT=SX$ where S is a unilateral shift, then T is reflexive. In this note we prove the reflexivity of a contraction with a unilateral shift part. This result contains the main theorem of [9] as a special case. As an application, we obtain the reflexivity result for a contraction T on a separable Hilbert space such that $u\Theta_T^*$ is an operator-valued H^∞-function for some nonzero scalar H^∞-function u, where Θ_T is the characteristic function of T and $\Theta_T^*(e^{it})=(\Theta_T(e^{it}))^*$ for almost every t, in particular, for a contraction T such that Θ_T is a polynomial. Our proof needs the reflexivity result of [2] stated above. We will extensively use the theory of contractions developed by SZ.-NAGY and FOIAŞ [8].

Theorem 1. *If T is a contraction on a Hilbert space $\mathfrak{H}$ and there exists a nonzero $\mathfrak{M}\in\operatorname{Lat} T$ such that $T|\mathfrak{M}$ is a unilateral shift, then T is reflexive.*

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First let us prove the following lemma.

Lemma 2. *If T is a contraction on $\mathfrak{H}$ and there exists a nonzero $\mathfrak{M}\in\operatorname{Lat} T$ such that $T|\mathfrak{M}$ is a unilateral shift, then there exists a nonzero operator $Y\colon \mathfrak{H}\to L^2$ satisfying the following conditions* (i) *and* (ii); (i) *$YT=WY$ where W is the bilateral shift on L^2 defined by $(Wf)(e^{it})=e^{it}f(e^{it})$ a.e. t, $f\in L^2$,* (ii) *there exists a linear manifold $\mathfrak{L}$ dense in $\mathfrak{H}\ominus\ker Y$ such that $W|\mathfrak{N}_{Yx}$ is a unilateral shift for all $0\neq x\in\mathfrak{L}$, where $\mathfrak{N}_{Yx}=\bigvee\{W^nYx\colon n\geqq 0\}$ (a cyclic subspace for W).*

Proof. By assumption, if $\mathfrak{M}_1$ is a cyclic subspace for T included in $\mathfrak{M}$, then $T|\mathfrak{M}_1$ is unitarily equivalent to the unilateral shift $S=W|H^2$ (cf. [6, Theorem 3.33]), hence there exists an isometry $Z\colon H^2\mapsto\mathfrak{H}$ such that $TZ=ZS$. Let U be the minimal unitary dilation of T acting on $\mathfrak{G}$, thus U is a unitary operator such that $PU|\mathfrak{H}=T$ where P is the orthogonal projection of $\mathfrak{G}$ onto $\mathfrak{H}$, and if $\mathfrak{G}_+=\bigvee_{n\geqq 0}U^n\mathfrak{H}$, then $\mathfrak{G}_+\ominus\mathfrak{H}\in\operatorname{Lat} U$ (cf. [8, Theorem I.4.1 and 4.2]). By the lifting theorem of Sz.-Nagy and Foiaş (cf. [8, Theorem II.2.3] and [5, Corollary 5.1]) there exists an operator $\tilde{Z}\colon L^2\mapsto\mathfrak{G}$ satisfying the conditions (a) $U\tilde{Z}=\tilde{Z}W$, (b) $P\tilde{Z}|H^2=Z$ and (c) $\|\tilde{Z}\|=\|Z\|=1$. Let us show that the operator $Y=\tilde{Z}^*|\mathfrak{H}\colon \mathfrak{H}\mapsto L^2$ is a required one.

Since the condition (a) implies $\tilde{Z}^*U=W\tilde{Z}^*$, to prove $YT=WY$, it suffices to show that $\mathfrak{G}_+\ominus\mathfrak{H}\subseteq\ker\tilde{Z}^*$. Since $\mathfrak{G}_+\ominus\mathfrak{H}\in\operatorname{Lat} U$, $\mathfrak{G}_+\ominus\mathfrak{H}$ is orthogonal to $\bigvee_{n\geqq 0}U^{*n}\mathfrak{H}$. On the other hand, since Z is isometric, it follows from (b) and (c) that $\tilde{Z}|H^2=Z$, and since $\tilde{Z}W^{*n}=U^{*n}\tilde{Z}$ $(n=1,2,\ldots)$ by (a), we see that $\tilde{Z}$ is an isometry and $\operatorname{ran}\tilde{Z}\subseteq\bigvee_{n\geqq 0}U^{*n}\mathfrak{H}$. Therefore it follows that $\mathfrak{G}_+\ominus\mathfrak{H}\subseteq\ker\tilde{Z}^*$. Next to see (ii), let $\mathfrak{M}_0=\{Zp;\ p$ is an analytic polynomial$\}$. Clearly $\mathfrak{M}_0$ is linear and dense in ZH^2. Also since $\tilde{Z}|H^2=Z$, we have $ZH^2\subseteq\mathfrak{H}\ominus\ker Y$. We consider $\mathfrak{L}=\mathfrak{M}_0\oplus\oplus((\mathfrak{H}\ominus\ker Y)\ominus ZH^2)$, which is linear and dense in $\mathfrak{H}\ominus\ker Y$. If $0\neq x=Zp+x_1\in\mathfrak{L}$ where p is a polynomial of degree n and $x_1\in(\mathfrak{H}\ominus\ker Y)\ominus ZH^2$, then $Yx=p+Yx_1$ because $\tilde{Z}|H^2=Z$ and $\tilde{Z}$ is an isometry. Since x_1 is orthogonal to ZH^2, or equivalently Yx_1 is orthogonal to H^2, it follows that $\chi^{-(n+1)}Yx$, where $\chi(e^{it})=e^{it}$, is orthogonal to H^2, so that $Yx=qg$ $(Yx\neq 0)$, where q is a function in L^∞ such that $|q(e^{it})|=1$ a.e. t and g is an outer function in H^2 (cf. [3, Chapter IV, Theorem 6.1 and Corollary 6.4]). This shows $\mathfrak{N}_{Yx}=qH^2$, hence the isometry $W|\mathfrak{N}_{Yx}$ is a unilateral shift. Thus the condition (ii) holds.

Any contraction T can be decomposed uniquely as $T=U\oplus T_1$ where U is a unitary operator and T_1 is a *completely non-unitary (c.n.u.)* contraction, that is, T_1 has no nontrivial unitary direct summand. The operators U and T_1 are called the *unitary part* and the *c.n.u. part* of T, respectively. For a contraction T whose unitary part is absolutely continuous, the H^∞-functional calculus defines a weak*-weak continuous algebra homomorphism, $u\mapsto u(T)$, from H^∞ to $\operatorname{Alg} T$, and T is said to be of class C_0 if $u(T)=0$ for some nonzero $u\in H^\infty$ (cf. [8, Chapter III]).

Proof of Theorem 1. Let $T=U_s\oplus T_1$ on $\mathfrak{H}=\mathfrak{H}_s\oplus\mathfrak{H}_1$ where U_s is a singular unitary operator and T_1 is a contraction whose unitary part is absolutely continuous. It is known that the reflexivity of T is equivalent to that of T_1 (cf. the proof of [9, Theorem 4.1]). Since T has a unilateral shift part, as in the proof of Lemma 2, we have an isometry Z such that $TZ=ZS$ where S is the unilateral shift on H^2. If P_s is the orthogonal projection onto $\mathfrak{H}_s$, then $U_s(P_sZ)=(P_sZ)S$ and it follows from [5, Corollary 5.1 and Theorem 3] that $P_sZ=0$, hence $\operatorname{ran} Z\subseteqq\mathfrak{H}_1$. This shows that T_1 has a unilateral shift part. Thus we may assume that the unitary part of T is absolutely continuous and it suffices to show that for each $A\in\operatorname{Alg Lat} T$, there exists $f\in H^\infty$ such that $A=f(T)$.

Let Y, W and $\mathfrak{L}$ be as in Lemma 2, and let $\tilde{\mathfrak{L}}$ be the set $\{x_1+x_2\colon x_1\in\ker Y$ and $0\neq x_2\in\mathfrak{L}\}$ that is dense in $\mathfrak{H}$. If $x\in\tilde{\mathfrak{L}}$, that is, $x=x_1+x_2$ where $x_1\in\ker Y$ and $0\neq x_2\in\mathfrak{L}$, then since $Yx=Yx_2(\neq 0)$, by Lemma 2 the isometry $W|\mathfrak{N}_{Yx}$ is a unilateral shift and $(W|\mathfrak{N}_{Yx})(Y|\mathfrak{M}_x)=(Y|\mathfrak{M}_x)(T|\mathfrak{M}_x)$ with $Y|\mathfrak{M}_x\neq 0$, where $\mathfrak{M}_x=\bigvee\{T^nx\colon n\geqq 0\}$, so it follows from [2, Theorem 4] that

$$\operatorname{Alg Lat}(T|\mathfrak{M}_x)=\{f(T)|\mathfrak{M}_x\colon f\in H^\infty\}.$$

Here note that the unitary parts of T and $T|\mathfrak{M}_x$ are absolutely continuous. Take $A\in\operatorname{Alg Lat} T$. For each $x\in\tilde{\mathfrak{L}}$, since $\mathfrak{M}_x\in\operatorname{Lat} T\subseteqq\operatorname{Lat} A$ and $A|\mathfrak{M}_x\in\operatorname{Alg Lat}(T|\mathfrak{M}_x)$, by the above fact there is $f_x\in H^\infty$ such that $A|\mathfrak{M}_x=f_x(T)|\mathfrak{M}_x$, in particular, $Ax=f_x(T)x$. Here note that it follows from the identity $WY=YT$ with $Yx\neq 0$ that $T|\mathfrak{M}_x$ is not of class C_0 (cf. [8, Proposition III.4.1]), so that the function f_x is determined uniquely by x. Since $\tilde{\mathfrak{L}}$ is dense in $\mathfrak{H}$, in order to show $A=f(T)$ for some $f\in H^\infty$, it suffices to prove that $f_x=f_y$ for all $x, y\in\tilde{\mathfrak{L}}$. First suppose $x-y\in\ker Y$. Then since $Yx=Yy$ and $\ker Y\in\operatorname{Lat} T\subseteqq\operatorname{Lat} A$, we have

$$(f_x-f_y)(W)Yx=Yf_x(T)x-Yf_y(T)y=YAx-YAy=YA(x-y)=0,$$

and since $Yx\neq 0$, it follows that $f_x=f_y$. Next assume that $x-y\notin\ker Y$. Then since clearly $x-y\in\tilde{\mathfrak{L}}$, there is $f_{x-y}\in H^\infty$ such that

$$f_{x-y}(T)x-f_{x-y}(T)y=f_{x-y}(T)(x-y)=A(x-y)=Ax-Ay=f_x(T)x-f_y(T)y,$$

hence $(f_{x-y}-f_x)(T)x=(f_{x-y}-f_y)(T)y\in\mathfrak{M}_x\cap\mathfrak{M}_y$. Therefore we have

$$f_x(T)(f_{x-y}-f_x)(T)x=A(f_{x-y}-f_x)(T)x=f_y(T)(f_{x-y}-f_x)(T)x,$$

and since $T|\mathfrak{M}_x$ is not of class C_0, $(f_x-f_y)(f_{x-y}-f_x)=0$. Similarly we have $(f_x-f_y)(f_{x-y}-f_y)=0$. This shows $f_x=f_y$ and completes the proof.

Let T be a contraction on a separable Hilbert space. The *characteristic function* Θ_T of T is defined by

$$\Theta_T(\lambda)=[-T+\lambda D_{T^*}(I-\lambda T^*)^{-1}D_T]|\mathfrak{D}_T\quad(|\lambda|<1),$$

where $D_T=(I-T^*T)^{1/2}$, $D_{T^*}=(I-TT^*)^{1/2}$ and $\mathfrak{D}_T=(\operatorname{ran} D_T)^-$. The function Θ_T is an operator-valued H^∞-function whose values are contractions from $\mathfrak{D}_T$ to $\mathfrak{D}_{T^*}:=(\operatorname{ran} D_{T^*})^-$ (cf. [8, Chapter VI]). If T is c.n.u., then it follows from [8, Theorem VII.4.7] that there exists a nonzero $\mathfrak{M}\in\operatorname{Lat} T$ such that $T|\mathfrak{M}$ is a unilateral shift if and only if there exists a nonzero $h\in H^2(\mathfrak{D}_{T^*})$ such that $\Theta_T^* h\in\Delta_T L^2(\mathfrak{D}_T)$, where $H^2(\mathfrak{D}_{T^*})$ (resp. $L^2(\mathfrak{D}_T)$) is the space of $\mathfrak{D}_{T^*}$-valued H^2-functions (resp. $\mathfrak{D}_T$-valued L^2-functions), $\Theta_T^*(e^{it})=(\Theta_T(e^{it}))^*$ a.e. t and $\Delta_T(e^{it})=(I-\Theta_T(e^{it})^*\Theta_T(e^{it}))^{1/2}$ a.e. t.

Now we obtain the reflexivity result for a contraction T such that $u\Theta_T^*$ is an operator-valued H^∞-function for some nonzero scalar function $u\in H^\infty$. If such a contraction T is of class C_{00}, that is, $T^n\to 0$ and $T^{*n}\to 0$ strongly as $n\to\infty$, then since $\Theta_T(e^{it})$ is unitary a.e. t (cf. [8, Proposition VI.3.5]), the condition that $u\Theta_T^*$ is an operator-valued H^∞-function with a nonzero $u\in H^\infty$ means that $u(T)=0$ and so T is of class C_0 (cf. [8, Theorem VI.5.1]). Reflexive contractions of class C_0 were characterized in terms of their Jordan models [1].

Theorem 3. *Let T be a contraction on a separable Hilbert space such that $u\Theta_T^*$ is an operator-valued H^∞-function for some nonzero $u\in H^\infty$. If the c.n.u. part of T is not of class C_{00}, then T is reflexive.*

Proof. By Theorem 1 it suffices to show that T or T^* has a unilateral shift part. Since the characteristic function of a contraction is equal to the one of its c.n.u. part, we may assume that T is a c.n.u. contraction. Since $\Theta_T^*(I-\Theta_T\Theta_T^*)=$ $=\Delta_T^2\Theta_T^*$ and by the assumption for Θ_T the function $u(I-\Theta_T\Theta_T^*)$ is an operator-valued H^∞-function, if $\lim\|T^n x\|\neq 0$ for some x, or equivalently $\Theta_T(e^{it})$ is not coisometric on a set of t's of positive Lebesgue measure (cf. [8, Proposition VI.3.5]), then there is a nonzero $h\in H^2(\mathfrak{D}_{T^*})$ such that $\Theta_T^* h\in\Delta_T L^2(\mathfrak{D}_T)$, and so T has a unilateral shift part by the fact remarked above. Also since $\Theta_{T^*}(e^{it})=(\Theta_T(e^{-it}))^*$ a.e. t for the characteristic function Θ_{T^*} of T^* (cf. [8, p. 239]), the contraction T^* satisfies the same condition as T, that is, $\tilde{u}\Theta_{T^*}^*$ is an operator-valued H^∞-function where $\tilde{u}$ is a function in H^∞ defined by $\tilde{u}(e^{it})=\overline{u(e^{-it})}$ a.e. t. Thus if $\lim\|T^{*n}x\|\neq 0$ for some x, then it follows that T^* has a unilateral shift part.

The following theorem gives a complement of Theorem 3.

Theorem 4. *Let $T=U\oplus T_1$ where U is a unitary operator and T_1 is a contraction of class C_0. Then T is reflexive if and only if the following condition* (i) *or* (ii) *holds:*

(i) *U has a (nontrivial) bilateral shift summand;*

(ii) *T_1 is reflexive.*

Proof. Again we may assume that U is absolutely continuous (cf. the proof of [9, Theorem 4.1]). If U has a bilateral shift summand, then by Theorem 1 T is reflexive. If U has no bilateral shift summand, then by Lemma 5 below we have $\operatorname{Alg} T = \operatorname{Alg} U \oplus \operatorname{Alg} T_1$ and $\operatorname{Lat} T = \operatorname{Lat} U \oplus \operatorname{Lat} T_1$, so $\operatorname{Alg}\operatorname{Lat} T = \operatorname{Alg}\operatorname{Lat} U \oplus \oplus \operatorname{Alg}\operatorname{Lat} T_1$. Therefore it follows from the reflexivity of the unitary operator U (cf. [7]) that T is reflexive if and only if T_1 is. This shows Theorem 4.

The implication (2)⇒(1) in the following lemma was pointed out by P. Y. Wu.

Lemma 5. *Let $T = U \oplus T_1$ on $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$ where U is an absolutely continuous unitary operator and T_1 is a contraction of class C_0. Then the following conditions are equivalent:*

(1) *U has no bilateral shift summand;*

(2) $\operatorname{Lat} T = \operatorname{Lat} U \oplus \operatorname{Lat} T_1$;

(3) $\operatorname{Alg} T = \operatorname{Alg} U \oplus \operatorname{Alg} T_1$.

Proof. (1)⇒(2): Since the inclusion $\operatorname{Lat} U \oplus \operatorname{Lat} T_1 \subseteqq \operatorname{Lat} T$ is obvious, we have to show that any $\mathfrak{M} \in \operatorname{Lat} T$ is decomposed into $\mathfrak{M} = \mathfrak{L} \oplus \mathfrak{N}$ where $\mathfrak{L} \in \operatorname{Lat} U$ and $\mathfrak{N} \in \operatorname{Lat} T_1$. Suppose $\mathfrak{M} \in \operatorname{Lat} T$. Since T_1 is of class C_0, there is a nonzero function $f \in H^\infty$ such that $f(T_1) = 0$. We set $\mathfrak{L} = (f(T)\mathfrak{M})^- \subseteqq \mathfrak{M}$. Then clearly $\mathfrak{L} \in \operatorname{Lat} T$ and $\mathfrak{L} \subseteqq (\operatorname{ran} f(T))^- = (\operatorname{ran} f(U))^- \subseteqq \mathfrak{H}_0$, so $\mathfrak{L}$ is an invariant subspace of U. But since U has no bilateral shift summand, $\mathfrak{L}$ reduces U (cf. [3, Chapter VII, Proposition 5.2]), hence $\mathfrak{L}$ also reduces T. Then the subspace $\mathfrak{N} = \mathfrak{M} \ominus \mathfrak{L}$ is invariant for T and since $f(T)\mathfrak{N} \subseteqq \mathfrak{N}$ and $f(T)\mathfrak{N} \subseteqq f(T)\mathfrak{M} \subseteqq \mathfrak{L}$, we have $f(T)\mathfrak{N} = \{0\}$. But since $f(T) = f(U) \oplus 0$ and obviously $f(U)$ is injective, we conclude $\mathfrak{N} \subseteqq \mathfrak{H}_1$, and therefore $\mathfrak{N} \in \operatorname{Lat} T_1$. This shows (2).

(1)⇒(3): For $n = 1, 2, \dots$, $T^{(n)} = U^{(n)} \oplus T_1^{(n)}$ satisfies the same condition as T, where for an operator A, $A^{(n)}$ denotes the direct sum of n copies of A. Therefore, using the implication (1)⇒(2) proved already, we have $\operatorname{Lat} T^{(n)} = \operatorname{Lat} U^{(n)} \oplus \operatorname{Lat} T_1^{(n)}$. If $A \in \operatorname{Alg} U$ and $B \in \operatorname{Alg} T_1$, then clearly $\operatorname{Lat} U^{(n)} \oplus \operatorname{Lat} T_1^{(n)} \subseteqq \operatorname{Lat}(A \oplus B)^{(n)}$, so that $\operatorname{Lat} T^{(n)} \subseteqq \operatorname{Lat}(A \oplus B)^{(n)}$ for $n = 1, 2, \dots$, hence it follows from Sarason's lemma (cf. [6, Theorem 7.1]) that $A \oplus B \in \operatorname{Alg} T$. This shows $\operatorname{Alg} U \oplus \operatorname{Alg} T_1 \subseteqq \operatorname{Alg} T$. Since the converse inclusion is obvious, we conclude $\operatorname{Alg} T = \operatorname{Alg} U \oplus \operatorname{Alg} T_1$.

(3)⇒(2) is obvious. (2)⇒(1): If U has a bilateral shift summand, then by the proof of Theorem 1 $\operatorname{Alg}\operatorname{Lat} T = \{f(T) : f \in H^\infty\}$. Since the condition (2) implies the inclusion $\operatorname{Alg}\operatorname{Lat} U \oplus \operatorname{Alg}\operatorname{Lat} T_1 \subseteqq \operatorname{Alg}\operatorname{Lat} T$, we have $0 \oplus I \in \operatorname{Alg}\operatorname{Lat} T$, so that there is $f \in H^\infty$ such that $f(U) = 0$ and $f(T_1) = I$, but this is impossible because $f(U) = 0$ implies $f = 0$. This shows (2)⇒(1).

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On normal extensions of unbounded operators. II*)

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This paper continues our study of unbounded subnormal operators. The results contained here may be regarded as reviewing, extending and completing those of [21] (and also of [20] and [22]). The next paper [26] in this series will be devoted to spectral problems as well as to the question of uniqueness of normal extensions.

Subnormal operators in general aspect

1. Let S be a densely defined linear operator in a complex Hilbert space $\mathfrak{H}$. $\mathfrak{D}(S)$, $\mathfrak{N}(S)$ and $\mathfrak{R}(S)$ stands for the domain of S, the null space of S and the range of S, respectively. S is said to be *subnormal* if there is a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ and a normal operator N in $\mathfrak{K}$ such that

$$\mathfrak{D}(S)\subset\mathfrak{D}(N) \quad \text{and} \quad Sf=Nf \quad \text{for each} \quad f\in\mathfrak{D}(S).$$

(A densely defined linear operator N in $\mathfrak{K}$ is said to be *normal* if it is closed and $N^*N=NN^*$. This is the same as to require that $\mathfrak{D}(N)=\mathfrak{D}(N^*)$ and $\|Nf\|=\|N^*f\|$, $f\in\mathfrak{D}(N)$. A normal operator has a spectral representation on the complex plane $\mathbf{C}$.)

The first thing we have to point out is that a subnormal operator must necessarily be closable. Even more we show that $\mathfrak{D}(S)\subset\mathfrak{D}(S^*)$. To see this take $g\in\mathfrak{D}(S)$, then

$$\langle Sf, g\rangle_{\mathfrak{H}} = \langle f, N^*g\rangle_{\mathfrak{K}}, \quad f\in\mathfrak{D}(S)$$

which gives us $g\in\mathfrak{D}(S^*)$ and $S^*g=P_{\mathfrak{H}}N^*g$.

The following characterization of densely defined subnormal operators based on the spectral representation of normal extensions is due to FOIAŞ (cf. [8], p. 248).

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Theorem 1. *A densely defined operator S in $\mathfrak{H}$ is subnormal if and only if there is a (normalized) semispectral measure F in $\mathfrak{H}$ on the complex plane* $\mathbf{C}$ *such that*

$$\langle S^n f, S^m g\rangle = \int_{\mathbf{C}} \lambda^n \bar{\lambda}^m \langle F(d\lambda)f, g\rangle, \quad f, g \in \mathfrak{D}(S), \quad m, n = 0, 1.$$

This theorem seems to be the only known characterization of unbounded densely defined subnormal operators in the general case (without any additional assumption on S). It ought to be noticed that a characterization like this of Foiaş for bounded operators has appeared in [2] (cf. also [6]). However that involves all the powers of the operator S. This requirement is superfluous for bounded operators, while for unbounded ones it leads to unnecessary restriction on behavior of domains of all powers of S.

2. Now we want to discuss the relation between subnormality and quasinormality. Like in the bounded case we have two equivalent possibilities of defining quasinormal operators. Because commutativity of unbounded operators is rather a delicate matter, we wish to discuss this equivalence with more care.

A closed densely defined operator Q in a Hilbert space $\mathfrak{H}$ is said to be *quasinormal* if Q commutes with the spectral measure E of $|Q|:=(Q^*Q)^{1/2}$ i.e. $E(\sigma)Q\subset \subset QE(\sigma)$, σ being a Borel subset of the non-negative part $\mathbf{R}_+$ of the real line $\mathbf{R}$.

Proposition 1. *Q is quasinormal if and only if Q is closed and U commutes with the spectral measure E of $|Q|$, where $Q=U|Q|$ is the polar decomposition of Q.*

Proof. Suppose that U commutes with E. Since E commutes with $|Q|$ (i.e. $E(\sigma)|Q|\subset|Q|E(\sigma)$) we have

$$E(\sigma)Q = E(\sigma)U|Q| = UE(\sigma)|Q|\subset U|Q|E(\sigma) = QE(\sigma).$$

Thus Q is quasinormal.

Suppose now that Q is quasinormal. Since Q commutes with E, U commutes with E on $\overline{\mathfrak{R}(|Q|)}$. Indeed, for each $f\in\mathfrak{D}(|Q|)$ we have

$$\begin{aligned}(UE(\sigma)-E(\sigma)U)|Q|f &= UE(\sigma)|Q|f-E(\sigma)U|Q|f = U|Q|E(\sigma)f-E(\sigma)Qf = \\ &= QE(\sigma)f-E(\sigma)Qf = 0.\end{aligned}$$

Since $E(\{0\})$ is the orthogonal projection onto $\mathfrak{N}(|Q|)$ and $\overline{\mathfrak{R}(|Q|)}^{\perp}=\mathfrak{N}(|Q|)= =\mathfrak{N}(U)$, $(UE(\sigma)-E(\sigma)U)f=UE(\sigma)f=UE(\sigma)E(\{0\})f=UE(\{0\})E(\sigma)f=0$ for each $f\in\overline{\mathfrak{R}(|Q|)}^{\perp}$. Thus U commutes with E. This completes the proof.

The following result as well as its proof is patterned upon that for bounded operators ([6], Prop. 1.7, p. 115) however technically more involved.

Theorem 2. *Every quasinormal operator is subnormal.*

Proof. Let $S=U|S|$ be the polar decomposition of S and let $|S|=\int\limits_0^\infty tE(dt)$ be the spectral representation of $|S|$. Denote by $P_{\mathfrak{N}(|S|)}$ and $P_{\mathfrak{N}(S^*)}$ the orthogonal projections onto $\mathfrak{N}(|S|)$ and $\mathfrak{N}(S^*)$, respectively. Define in $\mathfrak{H}\oplus\mathfrak{H}$ two operators R and $\tilde{U}$ as $R=|S|\oplus|S|$ and

$$\tilde{U}=\begin{bmatrix} U & (I-UU^*)^{1/2} \\ -(I-U^*U)^{1/2} & U^* \end{bmatrix}.$$

It is easy to see [11] that $\tilde{U}$ is a unitary operator which dilates U (the Halmos dilation) and R is a self-adjoint extension of $|S|$. Since U is a partial isometry, $\tilde{U}$ is in fact of the form

$$\tilde{U}=\begin{bmatrix} U & P_{\mathfrak{N}(S^*)} \\ -P_{\mathfrak{N}(|S|)} & U^* \end{bmatrix}.$$

Due to Proposition 1, U and U^* commute with E. Since $I-UU^*=P_{\mathfrak{N}(S^*)}$ and $I-U^*U=P_{\mathfrak{N}(|S|)}$, $P_{\mathfrak{N}(S^*)}$ and $P_{\mathfrak{N}(|S|)}$ commute with E. Consequently $\tilde{U}$ commutes with $E\oplus E$ which is the spectral measure of R. Therefore $\tilde{U}R\subset R\tilde{U}$. This implies that $R\tilde{U}=\tilde{U}\tilde{U}^*R\tilde{U}\subset\tilde{U}(R\tilde{U})^*\tilde{U}\subset\tilde{U}(\tilde{U}R)^*\tilde{U}=\tilde{U}R\tilde{U}^*\tilde{U}=\tilde{U}R$ and $\tilde{U}R=R\tilde{U}$ in consequence. Denote by N the operator $\tilde{U}R$. Since $N^*N=R\tilde{U}^*\tilde{U}R=R^2$ and $NN^*=(R\tilde{U})(R\tilde{U})^*=R\tilde{U}\tilde{U}^*R=R^2$, $N^*N=NN^*$. This means that N is normal.

Let now $f\in\mathfrak{D}(S)=\mathfrak{D}(|S|)$. Then $f\oplus 0\in\mathfrak{D}(R)$. Since $P_{\mathfrak{N}(|S|)}$ commutes with E, $P_{\mathfrak{N}(|S|)}|S|\subset|S|P_{\mathfrak{N}(|S|)}=0$. Thus

$$N(f\oplus 0)=\tilde{U}R(f\oplus 0)=\tilde{U}(|S|f\oplus 0)=U|S|f\oplus(-P_{\mathfrak{N}(|S|)}|S|f)=(U|S|f)\oplus 0=Sf\oplus 0$$

which means that N extends S. This completes the proof.

Corollary 1. *An operator is subnormal if and only if it has a quasinormal extension.*

Proof. We have only to prove that each normal operator N is quasinormal. Indeed, if $N=\int\limits_{\mathbf{C}} zE(dz)$ then $|N|=\int\limits_{\mathbf{C}}|z|E(dz)=\int\limits_0^\infty tF(dt)$, where $F(\sigma)=E(\{z\in\mathbf{C}\colon |z|\in\sigma\})$, σ being a Borel subset of $\mathbf{R}_+$. Since $EN\subset NE$, $FN\subset NF$. This means that N is quasinormal.

Subnormal operators and the complex moment problem

3. The following condition, introduced by HALMOS [11], characterizes [2] bounded subnormal operators in a Hilbert space $\mathfrak{H}$. This is

$$\text{(H)} \qquad \sum_{j,k=0}^{n} \langle S^k f_j, S^j f_k \rangle \geqq 0$$

for all finite sequences $f_0, \dots, f_n \in \mathfrak{H}$. To consider the same condition in unbounded case one needs the linear subspace $\mathfrak{D}^\infty(S)$ of $\mathfrak{H}$

$$\mathfrak{D}^\infty(S) = \bigcap_{n=0}^{\infty} \mathfrak{D}(S^n)$$

(members of $\mathfrak{D}^\infty(S)$ are customarily refered to as *C^∞-vectors*). In this paper we will require that $\mathfrak{D}^\infty(S)$ is big enough (in most cases dense in $\mathfrak{H}$). This requirement makes serious (comparing with Section 1) restriction on subnormal operators because there are symmetric operators (even semi-bounded [4]) with trivial domains of their squares. Moreover the condition (H) considered for $f_0, \dots, f_n \in \mathfrak{D}^\infty(S)$, which is the only possibility to do, is not sufficient for subnormality for S even if $\mathfrak{D}^\infty(S)$ is dense in $\mathfrak{H}$. Let us discuss the following.

Example 1. Take a sequence of real numbers $\{a_{m,n}\}_{m,n=0}^{\infty}$ which is positive definite in the following sense:

$$\sum a_{m+p,n+q} \lambda_{m,n} \bar{\lambda}_{p,q} \geqq 0$$

for each finite sequence $\{\lambda_{m,n}\} \subset \mathbf{C}$, and which is not a two parameter moment sequence (see [1] and [9]). There are two densely defined symmetric operators A and B in some Hilbert space $\mathfrak{H}$ with a common domain $\mathfrak{D} = \mathfrak{D}(A) = \mathfrak{D}(B)$, having a vector $f_0 \in \mathfrak{D}$ such that all the powers $A^m B^n f_0$, $m, n \geqq 0$, span $\mathfrak{D}$, and such that

$$(1) \qquad a_{m,n} = \langle A^m B^n f_0, f_0 \rangle, \quad m, n \geqq 0$$

(cf. again [9]). Moreover A and B commute i.e. $ABf = BAf$ for each $f \in \mathfrak{D}$. Define $T = A + iB$. T satisfies (H) for all finite sequences $f_0, \dots, f_n \in \mathfrak{D} = \mathfrak{D}^\infty(S)$ (even more, $\|Tf\| = \|T^* f\|$, $f \in \mathfrak{D}$, because A and B commute).

Define S as a restriction of T to the linear span of $\{T^n f_0 \colon n \geqq 0\}$.

Neither T nor S is subnormal. If T would be subnormal (then S would be too), then there existed a measure μ on $\mathbf{C}$ (constructed via the spectral measure of a normal extension of T) such that

$$\langle T^n f_0, T^m f_0 \rangle = \int_{\mathbf{C}} z^n \bar{z}^m \, d\mu(z), \quad m, n \geqq 0.$$

Then, due to (1),

$$a_{m,n} = \int_{\mathbf{C}} (\operatorname{Re} z)^m (\operatorname{Im} z)^n \, d\mu(z), \quad m, n \geqq 0.$$

This would mean that $\{a_{m,n}\}_{m,n=0}^{\infty}$ was a two parameter moment sequence, which gives a contradiction.

Thus we have got an example of an operator which has a cyclic vector (an operator S in $\mathfrak{H}$ is said to be *cyclic* with a *cyclic vector* f_0 if $f_0 \in \mathfrak{D}^\infty(S)$ and $\mathfrak{D}(S)$ is a linear span of $\{S^n f_0 \colon n \geqq 0\}$) satisfies (H) on $\mathfrak{D}(S)$ but is not subnormal.

If one would be interested in an example of a non-cyclic operator, one could take a Nelson pair (cf. [17], [5]) to get an operator satisfying (H) on $\mathfrak{D}(S)$ with no normal extension.

As the following proposition shows the condition (H) is satisfied on $\mathfrak{D}(S)$ if and only if S has a formally normal extension (with dense "reducing" domain). Here by a *formally normal* operator in $\mathfrak{H}$ we mean a densely defined operator N in $\mathfrak{H}$ such that $\mathfrak{D}(N) \subset \mathfrak{D}(N^*)$ and $\|Nf\| = \|N^* f\|$ for each $f \in \mathfrak{D}(N)$.

Proposition 2. *Let S be a densely defined operator in $\mathfrak{H}$ such that $S\mathfrak{D}(S) \subset \subset \mathfrak{D}(S)$. Then S satisfies* (H) *for all finite sequences $f_0, \ldots, f_n \in \mathfrak{D}(S)$ if and only if there is a formally normal operator N in some Hilbert space $\mathfrak{K} \supset \mathfrak{H}$ such that*

(i) *$N\mathfrak{D}(N) \subset \mathfrak{D}(N)$ and $N^*\mathfrak{D}(N) \subset \mathfrak{D}(N)$,*

(ii) *$\mathfrak{D}(S) \subset \mathfrak{D}(N)$ and $S \subset N$,*

(iii) *$\mathfrak{D}(N)$ is a linear span of the set*

$$\{N^{*n} f \colon n \geqq 0, \ f \in \mathfrak{D}(S)\}.$$

Proof. The proof of the "if" part of Proposition 2 follows from the equality $N^*Nf = NN^*f$, $f \in \mathfrak{D}(N)$, via direct computation.

To prove the converse, suppose that S satisfies (H) for all finite sequences $f_0, \ldots, f_n \in \mathfrak{D}(S)$. The set $\mathfrak{S} = \mathbf{N} \times \mathbf{N}$ $(\mathbf{N} = \{0, 1, \ldots\})$ equiped with the coordinatewise defined addition and the involution $(m, n)^* = (n, m)$ becomes a *-semigroup. Define the form φ over $(\mathfrak{S}, \mathfrak{D}(S))$ (cf. [23])

$$\varphi((m, n); f, g) = \langle S^m f, S^n g \rangle, \quad f, g \in \mathfrak{D}(S), \quad m, n \in \mathbf{N}.$$

Then like in [24, par. 10], one can show that φ is positive definite i.e.

$$\text{(2)} \qquad \sum_{j,k=1}^{n} \varphi(s_k^* + s_j; f_j, f_k) \geqq 0, \quad f_1, \ldots, f_n \in \mathfrak{D}(S) \quad \text{and} \quad s_1, \ldots, s_n \in \mathfrak{S} \quad (n \geqq 1)$$

($S\mathfrak{D}(S) \subset \mathfrak{D}(S)$ is important here). It follows from Proposition in [23] that there is a family $\{\Phi(s) \colon s \in \mathfrak{S}\}$ of densely defined operators in some Hilbert space $\mathfrak{K}$ with common dense domain $\mathfrak{D}$, a linear operator $V \colon \mathfrak{D}(S) \to \mathfrak{D}$ such that

$\mathfrak{D}\subset\bigcup_{s\in\mathfrak{S}}\mathfrak{D}(\Phi(s)^*)$ and

$$\varphi(s;\, f, g) = \langle\Phi(s)Vf, Vg\rangle, \quad s\in\mathfrak{S}, \quad f, g\in\mathfrak{D}(S),$$

$$\Phi(s)\mathfrak{D}\subset\mathfrak{D} \quad \text{and} \quad \Phi(s)^*\mathfrak{D}\subset\mathfrak{D}, \quad s\in\mathfrak{S},$$

$$\Phi(s)\Phi(t)f = \Phi(s+t)f, \quad s, t\in\mathfrak{S}, \quad f\in\mathfrak{D},$$

$$\Phi(s^*)\subset\Phi(s)^*, \quad s\in\mathfrak{S},$$

$$\mathfrak{D} \text{ is a linear span of } \{\Phi(s)Vf\colon s\in\mathfrak{S}, f\in\mathfrak{D}(S)\}.$$

Set $N=\Phi(1,0)$. Since $(1,0)^*+(1,0)=(1,0)+(1,0)^*$, N is a formally normal operator which satisfies the condition (i) of Proposition 2. Moreover we have

$$\langle S^m f, S^n g\rangle = \langle\Phi(n(1,0)^*+m(1,0))Vf, Vg\rangle = \langle N^m Vf, N^n Vg\rangle,$$
$$m, n\in\mathbf{N} \quad \text{and} \quad f, g\in\mathfrak{D}(S).$$

This implies that V is an isometry from $\mathfrak{D}(S)$ into $\mathfrak{D}$. Identifying $\mathfrak{D}(S)$ with $V\mathfrak{D}(S)$ one can easily check the conditions (ii) and (iii). This completes the proof.

Remark 1. In [21] and in this paper we consider exclusively the operators with invariant domains. If $\mathfrak{D}(S)$ is not invariant for S, we have to replace the condition (H) on $\mathfrak{D}(S)$ by the condition (2).

4. Example 1 shows that the condition (H) itself is *not* sufficient for subnormality even of cyclic operators (however it *is* for weighted shifts — cf. Section 6).

If f_0 is a cyclic vector for S and S satisfies (H) on $\mathfrak{D}^\infty(S)$ then the sequence $\{c_{m,n}\}_{m,n=0}^\infty$ defined by

$$c_{m,n} = \langle S^m f_0, S^n f_0\rangle \quad m, n\in\mathbf{N}$$

is positive definite in the sense that

$$\sum_{\substack{m,n\geq 0\\ p,q\geq 0}} c_{m+q,n+p}\lambda_{m,n}\overline{\lambda_{p,q}} \geq 0$$

for all finite sequences $\{\lambda_{m,n}\}\subset\mathbf{C}$. Unfortunately positive-definiteness of $\{c_{m,n}\}_{m,n=0}^\infty$ does not imply that $\{c_{m,n}\}_{m,n=0}^\infty$ is a complex moment sequence (this is a substance of Example 1). However this gives a hope that subnormality of S (still being cyclic) may be forced by the fact that $\{c_{m,n}\}_{m,n=0}^\infty$ is a complex moment sequence. There is a characterization ([14]) of complex moment sequences in terms of non-negative polynomials which has been originated by M. Riesz. Though this may be interesting rather from the theoretical point of view than applicable to concrete sequences (read: operators — in advance), we will follow this in a context of subnormal operators. It turns out even more: a result of Slinker ([19], Th. 4.2) enables us to prove a M. Riesz-like characterization for *non-cyclic* case.

Theorem 3. *Let S be a densely defined operator in a Hilbert space $\mathfrak{H}$ such that $S\mathfrak{D}(S)\subset\mathfrak{D}(S)$. Then S is subnormal if and only if the following implication holds: if*

$$\{a_{pq}^{ij};\ i, j\in\{1, \dots, m\}\ \text{ and }\ p, q\in\{0, 1, \dots, n\}\}$$

is a sequence of complex numbers such that

$$\text{(i)}\qquad \sum_{i,j=1}^{m}\sum_{p,q=0}^{n} a_{pq}^{ij}\bar{\lambda}^q\lambda^p\bar{z}_i z_j \geqq 0, \quad \text{for all } \lambda, z_1, \dots, z_m\in\mathbf{C},$$

then

$$\text{(ii)}\qquad \sum_{i,j=1}^{m}\sum_{p,q=0}^{n}\sum_{k,l=0}^{r} a_{pq}^{ij}\langle S^{k+p}f_l^j, S^{l+q}f_k^i\rangle \geqq 0,$$

for each finite sequence $\{f_k^i\colon i=1, \dots, m,\ k=0, \dots, r\}\subset\mathfrak{D}(S)$.

Proof. Suppose that S is subnormal and that N is its normal extension in a Hilbert space $\mathfrak{K}\supset\mathfrak{H}$. Notice that

$$(3)\qquad \begin{cases} \mathfrak{D}(S) = \mathfrak{D}^\infty(S)\subset\mathfrak{D}^\infty(N) \quad \text{and} \\ N(\mathfrak{D}^\infty(N))\subset\mathfrak{D}^\infty(N), \quad N^*(\mathfrak{D}^\infty(N))\subset\mathfrak{D}^\infty(N) \quad \text{and} \\ NN^*f = N^*Nf \quad \text{for each} \quad f\in\mathfrak{D}^\infty(N). \end{cases}$$

Define the polynomials p^{ij} $(i, j\in\{1, \dots, m\})$ of two complex variables λ and $\bar{\lambda}$ by

$$(4)\qquad p^{ij}(\lambda, \bar{\lambda}) = \sum_{p,q=0}^{n} a_{pq}^{ij}\bar{\lambda}^q\lambda^p, \quad \lambda\in\mathbf{C}.$$

Then, since $S\subset N$ and (3), we have

$$(5)\qquad \sum_{p,q=0}^{n}\sum_{k,l=0}^{r} a_{pq}^{ij}\langle S^{k+p}f_l^j, S^{l+q}f_k^i\rangle = \sum_{p,q=0}^{n}\sum_{k,l=0}^{r} a_{pq}^{ij}\langle N^pN^{*l}f_l^j, N^qN^{*k}f_k^i\rangle =$$

$$= \sum_{p,q=0}^{n} a_{pq}^{ij}\langle N^ph_j, N^qh_i\rangle = \langle p^{ij}(N, N^*)h_j, h_i\rangle,$$

where

$$(6)\qquad h_i = \sum_{k=0}^{r} N^{*k}f_k^i, \quad i = 1, \dots, m.$$

Thus we have to show that

$$(7)\qquad \sum_{i,j=1}^{m}\langle p^{ij}(N, N^*)h_j, h_i\rangle \geqq 0$$

for all $h_1, \dots, h_m\in\mathfrak{D}^\infty(N)$.

Let E be the spectral measure of N. Since all the complex measures $\langle E(\cdot)h_j, h_i\rangle$, $i, j \in \{1, \dots, m\}$, are absolutely continuous with respect to the non-negative measure $\mu = \sum_{i=1}^{m} \langle E(\cdot)h_i, h_i\rangle$, we find a matrix of summable Borel functions $\{h_{ij}\}_{i,j=1}^{m}$ such that $\langle E(\sigma)h_j, h_i\rangle = \int_\sigma h_{ij}\, d\mu$ for each Borel subset σ of $\mathbf{C}$ and for all i, j.

Let Q be a countable dense subset of $\mathbf{C}$. For $c_1, \dots, c_n \in Q$ we have

$$\int_\sigma \sum_{i,j=1}^{m} h_{ij}(\lambda)\bar{c}_i c_j\, d\mu(\lambda) = \sum_{i,j=1}^{m} \bar{c}_i c_j \langle E(\sigma)h_j, h_i\rangle = \langle E(\sigma)(\sum_{j=1}^{m} c_j h_j), \sum_{j=1}^{m} c_j h_j\rangle \geqq 0$$

for each σ. This implies that

$$\sum_{i,j=1}^{m} h_{ij}(\lambda)\bar{c}_i c_j \geqq 0 \quad \text{a.e.} \quad [\mu]. \tag{8}$$

Since Q is countable, we can find a common Borel subset σ_0 of $\mathbf{C}$ (which does not depend on the choice of the numbers c_k) such that $\mu(\sigma_0) = \mu(\mathbf{C})$ and (8) is fulfilled, first for all $c_k \in Q$ and then, after limit passage, for all complex c_k's.

Thus we have shown that the complex matrix $[h_{ij}(\lambda)]_{i,j=1}^{m}$ is positive definite for each $\lambda \in \sigma_0$. Since, by (i) and (4), the matrix $[p^{ij}(\lambda, \bar{\lambda})]_{i,j=1}^{m}$ is also positive definite, an application of the classical Schur Lemma gives us that

$$[p^{ij}(\lambda, \bar{\lambda}) h_{ij}(\lambda)]_{i,j=1}^{m} \tag{9}$$

is a positive definite matrix for each $\lambda \in \sigma_0$.

Thus

$$\sum_{i,j=1}^{m} \langle p^{ij}(N, N^*)h_j, h_i\rangle = \sum_{i,j=1}^{m} \int_{\mathbf{C}} p^{ij}(\lambda, \bar{\lambda}) h_{ij}(\lambda)\, d\mu(\lambda) =$$

$$= \int_{\sigma_0} \Big(\sum_{i,j=1}^{m} [p^{ij}(\lambda, \bar{\lambda}) h_{ij}(\lambda)]\Big)\, d\mu(\lambda) \geqq 0.$$

(The integrand is non-negative due to (9).) This shows (7).

Now suppose that the implication holds for S. Then S satisfies (H) for all finite sequences $f_0, \dots, f_r \in \mathfrak{D}(S)$ (put $m=1$, $n=0$ and $a_{00}^{11}=1$). Thus, according to Proposition 2, there is a formally normal operator N in some Hilbert space $\mathfrak{K} \supset \mathfrak{H}$, which fulfills the conditions (i), (ii) and (iii) of Proposition 2. Due to a theorem of [19] all we have to prove now is the following implication: if for each $\lambda \in \mathbf{C}$, the polynomial matrix $[p^{ij}(\lambda, \bar{\lambda})]_{i,j=1}^{m}$ is positive definite, then (7) holds for all $h_1, \dots, h_m \in \mathfrak{D}(N)$.

For this let $[p^{ij}]$ be such a matrix of polynomials with coefficients $\{a_{pq}^{ij}\}$ as in (4). Let $h_1, \dots, h_m \in \mathfrak{D}(N)$. Then, by (iii) of Proposition 2, there is a sequence $\{f_k^i\colon i=1, \dots, m,\ k=0, \dots, r\} \subset \mathfrak{D}(S)$ which fulfills the condition (6). Since N is formally normal extension of S, which has property (i) of Proposition 2, we can rewrite all the equalities (5) to obtain

$$\sum_{i,j=1}^{m} \langle p^{ij}(N, N^*)h_j, h_i\rangle = \sum_{i,j=1}^{m} \sum_{p,q=0}^{n} \sum_{k,l=0}^{r} a_{pq}^{ij} \langle S^{k+p}f_l^j, S^{l+q}f_k^i\rangle \geqq 0.$$

This proves (7) and finishes the proof of theorem.

As we have mentioned this characterization of subnormals may be useful in proof. The following application is at hand.

Corollary 2. *Let S be a densely defined operator in H such that $S\mathfrak{D}(S)\subset\subset\mathfrak{D}(S)$. Then*

(a) *If S is subnormal, then among all the subnormal operators T in $\mathfrak{H}$ extending S and such that $T\mathfrak{D}(T)\subset\mathfrak{D}(T)$ there is a maximal one.*

(b) *Suppose that there exists S^{-1} which is densely defined and $S^{-1}\mathfrak{D}(S^{-1})\subset\subset\mathfrak{D}(S^{-1})$. Then if one of the operators S and S^{-1} is subnormal, so is the other.*

Proof. (a) If $\{T_\omega\}$ is a chain (ordered by inclusion) of subnormal operators extending S and such that $T_\omega\mathfrak{D}(T_\omega)\subset\mathfrak{D}(T_\omega)$, then $\bigcup_\omega T_\omega$ is an upper bound, which, due to Theorem 3, has the same properties as T_ω's do. Now an application of the Zorn Lemma gives the conclusion (a).

(b) Let $\{a_{pq}^{ij}\}$ satisfy (i) of Theorem 3. Set $b_{pq}^{ij}=a_{n-p,n-q}^{ij}$ (remind that $0\leqq p, q\leqq n$). Then one can check that $\{b_{pq}^{ij}\}$ satisfies (i) of Theorem 3 too.

Suppose that S is subnormal. Take a finite sequence $\{f_k^i;\ 1\leqq i\leqq m,\ 0\leqq k\leqq r\}\subset\subset\mathfrak{D}(S)$. Then, because in fact $\mathfrak{D}(S)=S\mathfrak{D}(S)$, we have

$$\sum_{i,j=1}^{m} \sum_{p,q=0}^{n} \sum_{k,l=0}^{r} a_{pq}^{ij} \langle (S^{-1})^{k+p}f_l^j, (S^{-1})^{l+q}f_k^i\rangle = \sum_{i,j=1}^{m} \sum_{p,q=0}^{n} \sum_{k,l=0}^{r} b_{pq}^{ij} \langle S^{k+p}g_l^j, S^{l+q}g_k^i\rangle$$

where $g_l^j=S^{-(n+r)}f_{r-l}^j$. Applying Theorem 3 we get the conclusion (b).

A characterization like this in Theorem 3 in a case of cyclic operators appears implicity in Kilpi [14]. What can be easily deduced from [14] is the following.

Proposition 3. *Let S be a densely defined cyclic operator in $\mathfrak{H}$ with a cyclic vector f_0. Then the following conditions are equivalent:*

(i) *S is subnormal;*

(ii) *$\{\langle S^m f_0, S^n f_0\rangle\}_{m,n=0}^{\infty}$ is a complex moment sequence, i.e. there is a non-negative measure μ on $\mathbf{C}$ such that $\langle S^m f_0, S^n f_0\rangle = \int_{\mathbf{C}} z^m \bar{z}^n\, d\mu(z)$, $m, n\in\mathbf{N}$,*

(iii) *If* $\{a_{k,l}\}_{k,l=0}^{m}$ *is a complex matrix such that* $\sum_{k,l=0}^{m} a_{k,l}\lambda^k\bar{\lambda}^l \geqq 0$ *for each* $\lambda \in \mathbf{C}$, *then*

$$\sum_{k,l=0}^{m} a_{k,l}\langle S^k f_0, S^l f_0\rangle \geqq 0.$$

Our characterization in Theorem 3 applied to cyclic operators looks more complicated than that of Kilpi. Because we are unable on this stage, to reduce directly ours to Kilpi's this is why we do not state it explicitely here; though they must necessarily be equivalent.

Subnormal operators and the Stieltjes moment problem. Weighted shifts

5. As we have already known subnormal operator S satisfies the condition (H) for any choice of vectors $f_0, \ldots, f_n \in \mathfrak{D}^\infty(S)$. Taking $g_k = S^k f_k$ and replacing $f_0, \ldots, f_n$ by $g_0, \ldots, g_n$ in (H) we get the condition:

$$\text{(E)} \qquad \sum_{j,k=0}^{n} \langle S^{j+k} f_j, S^{j+k} f_k\rangle \geqq 0$$

for all choices of vectors $f_0, \ldots, f_n$ in $\mathfrak{D}^\infty(S)$, which reminds a condition considered by EMBRY [7] in the bounded case. Going on set $f_j = c_j f$ and $f_j = c_j Sf$ in (E), respectively $(f \in \mathfrak{D}^\infty(S))$ we obtain

$$\sum_{j,k=0}^{n} \|S^{j+k} f\|^2 c_j \bar{c}_k \geqq 0,$$

and

$$\sum_{j,k=0}^{n} \|S^{j+k+1} f\|^2 c_j \bar{c}_k \geqq 0,$$

for all complex numbers $c_1, \ldots, c_n$. This is precisely what is required for the sequence $\{\|S^n f\|^2\}_{n=0}^{\infty}$ to be Stieltjes moment sequence i.e. to be represented as

$$\text{(S)} \qquad \|S^n f\|^2 = \int_0^{+\infty} t^n d\mu(t), \quad n \in \mathbf{N},$$

$\mu = \mu_f$ is a finite non-negative measure.

All what has been said here can be stated as

Proposition 4. *The following implications hold true:*
S is subnormal $\Rightarrow$ *S satisfies* (H) *on* $\mathfrak{D}^\infty(S)$,
S satisfies (H) *on* $\mathfrak{D}^\infty(S) \Rightarrow$ *S satisfies* (E) *on* $\mathfrak{D}^\infty(S)$,
S satisfies (E) *on* $\mathfrak{D}^\infty(S) \Rightarrow$ *S satisfies* (S) *for each f in* $\mathfrak{D}^\infty(S)$.

6. It turns out that the implications in Proposition 4 can be inverted for S being unilateral weighted shift. Recall S is said to be a *unilateral weighted shift* if $Se_n \in (\mathbf{C} \setminus \{0\})\, e_{n+1}$, $n \geqq 0$, where $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis of $\mathfrak{H}$. The domain of S is meant as the linear span of $\{e_n\}_{n=0}^{\infty}$. It is clear that S is a cyclic operator with the cyclic vector e_0.

Theorem 4. *Let S be a unilateral weighted shift. Then the following conditions are equivalent:*

(i) *S is subnormal;*

(ii) *S satisfies* (H) *for all finite sequences $f_0, \ldots, f_n \in \mathfrak{D}(S)$;*

(iii) *S satisfies* (E) *for all finite sequences $f_0, \ldots, f_n \in \mathfrak{D}(S)$;*

(iv) *S satisfies* (S) *for $f = e_0$.*

Since $\mathfrak{D}(S) = \mathfrak{D}^{\infty}(S)$, all the implication but (iv)⇒(i) follow from Proposition 4. To prove the implication (iv)⇒(i) we utilize the following result which may be interesting for itself.

Lemma 1. *Let $f_0 \in \mathfrak{H}$ be a cyclic vector for S. Then the following two conditions are equivalent:*

(a) *$S \subset U \otimes R$, where U is a unitary operator in $\mathfrak{K}_1$, R is a self-adjoint operator in $\mathfrak{K}_2$, $\mathfrak{H} \subset \mathfrak{K}_1 \otimes \mathfrak{K}_2$ and $f_0 = f_1 \otimes f_2$ with some $f_1 \in \mathfrak{K}_1$ and $f_2 \in \mathfrak{D}^{\infty}(R)$;*

(b) *there are two functions $\alpha\colon \mathbf{N} \to \mathbf{C}$ and $\beta\colon \mathbf{Z} \to \mathbf{C}$ such that*

$$\langle S^n f_0, S^m f_0 \rangle = \alpha(n+m)\beta(n-m), \quad n, m \in \mathbf{N}, \tag{10}$$

$$\sum_{m,n=0}^{r} \alpha(m+n) c_m \bar{c}_n \geqq 0, \tag{11}$$

for all finite sequences $c_0, \ldots, c_r \in \mathbf{C}$,

$$\sum_{m,n=0}^{r} \beta(n-m) c_n \bar{c}_m \geqq 0, \tag{12}$$

for all finite sequences $c_0, \ldots, c_r \in \mathbf{C}$.

Proof. Let U, R and f_1, f_2 be as in (a). Because

$$\langle S^n f_0, S^m f_0 \rangle = \langle U^{n-m} f_1, f_1 \rangle \langle R^{n+m} f_2, f_2 \rangle, \quad m, n \in \mathbf{N},$$

a direct computation shows that

$$\alpha(n) = \langle R^n f_2, f_2 \rangle, \quad n \in \mathbf{N},$$

and

$$\beta(m) = \langle U^m f_1, f_1 \rangle, \quad m \in \mathbf{Z},$$

satisfy the condition (11) and (12) respectively.

Suppose that the condition (b) is satisfied. Then $\{\alpha(n)\}_{n=0}^{\infty}$ is a Hamburger moment sequence [18] and $\{\beta(n)\}_{n\in\mathbf{Z}}$ is a trigonometric moment sequence [18]. Consequently there are two positive finite measures μ and ν defined on $\mathbf{R}$ and the unit circle $\mathbf{T}$, respectively, such that

$$(13)\qquad \langle S^n f_0, S^m f_0\rangle = \int_{\mathbf{R}} t^{n+m}\,d\mu(t) \int_{\mathbf{T}} z^{n-m}\,d\nu(z), \quad n, m\in\mathbf{N}.$$

Denote by M_z and M_t the multiplication operators by z and t in $L^2(\mathbf{T}, \nu)$ and $L^2(\mathbf{R}, \mu)$, respectively. Then by (13)

$$\langle S^n f_0, S^m f_0\rangle = \langle M_t^{n+m} 1_\mu, 1_\mu\rangle_{L^2(\mu)} \langle M_z^{n-m} 1_\nu, 1_\nu\rangle_{L^2(\nu)} =$$
$$= \langle (M_z\otimes M_t)^n (1_\nu\otimes 1_\mu), (M_z\otimes M_t)^m (1_\nu\otimes 1_\mu)\rangle_{L^2(\nu\otimes\mu)}, \quad m, n\in\mathbf{N}.$$

This equality allows us to identify $S^n f_0$ with $(M_z\otimes M_t)^n (1_\nu\otimes 1_\mu)$, $n\in\mathbf{N}$, which gives us $S\subset M_z\otimes M_t$. Since M_z is unitary and M_t is self-adjoint, we set $U=M_z$, $R=M_t$, $f_1=1_\nu$ and $f_2=1_\mu$ to get the conclusion. This completes the proof.

Remark 2. If any of the equivalent conditions (a) and (b) of Lemma 1 is satisfied then S is subnormal. Moreover the operator R can be choosen to be positive if (in addition to (11))

$$(11')\qquad \sum_{m,n=0}^{r} \alpha(n+m+1) c_n \bar{c}_m \geqq 0,$$

for all finite sequences $c_0, \ldots, c_r\in\mathbf{C}$, (since then (11) and (11′) imply that $\{\alpha(n)\}_{n\in\mathbf{N}}$ is a Stieltjes moment sequence) and Lemma 1 leads then to an L^2-model of S as the multiplication by z on the complex plane $\mathbf{C}$.

Proof of (iv)⇒(i) of Theorem 4. Let us define $\delta\colon \mathbf{Z}\to\{0, 1\}$ by $\delta(0)=1$ and $\delta(n)=0$ if $n\neq 0$. So we have

(14)
$$\langle S^n e_0, S^m e_0\rangle = \delta(n-m)\|S^n e_0\|^2 = \delta(n-m)\int_0^\infty t^n\,d\mu(t) = \delta(n-m)\int_0^\infty t^{(n+m)/2}\,d\mu(t),$$
$$m, n\in\mathbf{N},$$

where $\mu=\mu_{e_0}$ is the measure given by the integral representation (S). Setting

$$\alpha(n) = \int_0^\infty t^{n/2}\,d\mu(t), \quad n\in\mathbf{N}$$

and

$$\beta(n) = \delta(n), \quad n\in\mathbf{Z},$$

in (14) we get the condition (b) of Lemma 1. An application of Remark 2 completes the proof of our theorem.

7. Now we want to show usefulness of Theorem 4.

Example 2. In [21] we have shown that the *creation* operator is subnormal. This has been ensured by the condition (H) and the presence of analytic vectors for the operator. However, since this operator is a unilateral weighted shift we can use directly the condition (iv) of Theorem 4 instead of checking condition (H) and looking for analytic vectors. To be more precise, recall that the creation operator is defined as

$$A_+ = \frac{1}{\sqrt{2}}\left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right)$$

with $\mathfrak{D}(A_+) = \mathfrak{S}(\mathbf{R})$, the Schwartz space. Since the Hermite functions

$$f_n(x) = e^{x^2/2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}e^{-x^2}, \quad n = 0, 1, \ldots,$$

form an orthogonal basis for $L^2(\mathbf{R})$ and

$$A_+ f_m = \left(-\frac{1}{\sqrt{2}}\right) f_{m+1}, \quad m = 0, 1, \ldots,$$

A_+, when restricted to the linear span $\mathfrak{D}$ of the Hermite function is a weighted shift in $L^2(\mathbf{R})$. Denote this restriction by S. Since

$$\|S^n f_0\|^2 = n!\sqrt{\pi}, \quad n = 0, 1, 2, \ldots,$$

and $\{n!\}_{n=0}^{\infty}$ is a Stieltjes moment sequence, according to Theorem 4, S is subnormal. Since $\bar{A}_+ = (A_+|\mathfrak{D})^-$, A_+ is subnormal.

Theorem 4 allows to produce subnormal operators from simpler ones. As an illustration take a subnormal weighted shift S and define $S_k = S^{*k}S^{k+1}$, k is a positive integer. Then, after some computation — which, in a more general context, will be presented elsewhere [25] — one can show that S_k satisfies the condition (iv) of Theorem 4 and consequently it is subnormal too. In particular, if S is the creation operator then

$$S_1 = \frac{\sqrt{2}}{4}\left\{(1+x^2)x - (3+x^2)\frac{\mathrm{d}}{\mathrm{d}x} - x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^3}{\mathrm{d}x^3}\right\}.$$

8. We pass now to bilateral weighted shifts. In order to prove an analogue of Theorem 4 in this case we need an appropriate version of Lemma 1.

Lemma 2. *Let S be a densely defined operator in $\mathfrak{H}$ such that $\mathfrak{N}(S) = \{0\}$ and $S\mathfrak{D}(S) = \mathfrak{D}(S)$. Suppose there is a vector $f_0 \in \mathfrak{D}(S)$ such that $\mathfrak{D}(S)$ is the linear span of the set $\{S^n f_0\colon n \in \mathbf{Z}\}$. Then the following conditions are equivalent:*

(a) *$S \subset U \otimes R$, where U is a unitary operator in $\mathfrak{K}_1$, R is a self-adjoint operator in $\mathfrak{K}_2$ with $0 \notin \sigma_p(R)$, $\mathfrak{H} \subset \mathfrak{K}_1 \otimes \mathfrak{K}_2$ and $f_0 = f_1 \otimes f_2$ with some $f_1 \in \mathfrak{K}_1$ and $f_2 \in \bigcap_{n \in \mathbf{Z}} \mathfrak{D}(R^n)$;*

(b) *there are two functions* $\alpha, \beta\colon \mathbf{Z}\to\mathbf{C}$ *such that*

$$\langle S^n f_0, S^m f_0\rangle = \alpha(n+m)\beta(n-m), \quad n, m\in\mathbf{Z}, \tag{15}$$

$$\sum_{-r\leqq m,n\leqq r} \alpha(n+m)c_n\bar{c}_m \geqq 0, \tag{16}$$

for all finite sequences $c_{-r}, \ldots, c_r\in\mathbf{C}$, *and* β *satisfies* (12).

The proof of Lemma 2 goes in the same way as that of Lemma 1. However one has to use instead of the Hamburger characterization of moment sequences the following result ([13], [1]). A sequence $\{\alpha(n)\}_{n\in\mathbf{Z}}$ of complex numbers can be represented as

$$\alpha(n) = \int_{\mathbf{R}\setminus\{0\}} t^n\, d\mu(t), \quad n\in\mathbf{Z},$$

with a finite non-negative measure μ if and only if (16) holds.

Remark 3. Each of the equivalent conditions (a) and (b) of Lemma 2 guarantees subnormality of S. If the function $\alpha\colon \mathbf{Z}\to\mathbf{C}$ satisfies the additional condition

$$\sum_{-r\leqq n,m\leqq r} \alpha(n+m+1)c_n\bar{c}_m \geqq 0, \tag{17}$$

for all finite sequences $c_{-r}, \ldots, c_r\in\mathbf{C}$, then the operator R can be choosen to be positive. This happens because, due to the conditions (16) and (17), the sequence $\{\alpha(n)\}_{n\in\mathbf{Z}}$ becomes (cf. [1], [12]) a *two-sided Stieltjes* moment sequence which means that there is a non-negative finite measure μ such that

$$\alpha(n) = \int_{(0,+\infty)} t^n\, d\mu(t), \quad n\in\mathbf{Z}.$$

A densely defined operator S in H is said to be a *bilateral weighted shift* if there is an orthonormal basis $\{e_n\}_{n\in\mathbf{Z}}$ of $\mathfrak{H}$ such that $Se_n\in(\mathbf{C}\setminus\{0\})e_{n+1}$ for each $n\in\mathbf{Z}$. The domain $\mathfrak{D}(S)$ of S is the linear span of $\{e_n\}_{n\in\mathbf{Z}}$.

We have an analogue of Theorem 4 for bilateral weighted shifts.

Theorem 5. *Let S be a bilateral weighted shift. Then the following conditions are equivalent:*

(i) *S is subnormal;*

(ii) *S satisfies* (H) *for all finite sequences* $f_0, \ldots, f_n$ *in* $\mathfrak{D}(S)$;

(iii) *S satisfies* (E) *for all finite sequences* $f_0, \ldots, f_n$ *in* $\mathfrak{D}(S)$;

(iv) *S satisfies* (S) *for each* $f\in\{S^{-2n}e_0\}_{n\geqq 0}$;

(v) $\{\|S^n e_0\|^2\}_{n\in\mathbf{Z}}$ *is a two-sided Stieltjes moment sequence.*

Proof. The only implications which need a proof are (iv)⇒(v) and (v)⇒(i).

(iv)⇒(v): The operator S satisfies all the assumptions of Lemma 2 with $f_0=e_0$. Now we show that the sequence $\{\|S^n e_0\|^2\}_{n\in\mathbf{Z}}$ satisfies the conditions (16) and (17). Let $c_{-r}, \ldots, c_r$ be an arbitrary sequence of complex numbers. Then

$$\sum_{-r\leq n,m\leq r} \|S^{n+m}e_0\|^2 c_n\bar{c}_m = \sum_{n,m=0}^{2r} \|S^{n+m}S^{-2r}e_0\|^2 d_n\bar{d}_m$$

and

$$\sum_{-r\leq n,m\leq r} \|S^{n+m+1}e_0\|^2 c_n\bar{c}_m = \sum_{n,m=0}^{2r} \|S^{n+m+1}S^{-2r}e_0\|^2 d_n\bar{d}_m,$$

where $d_n=c_{n-r}$ for $n\in\{0, 1, \ldots, 2r\}$. Due to (iv), all the sums appearing in the above two equalities are nonnegative. This ensures that $\{\|S^n e_0\|^2\}_{n\in\mathbf{Z}}$ is a two-sided Stieltjes moment sequence.

(v)⇒(i): Like in the proof of Theorem 4 we put

$$\alpha(n) = \int_{(0,+\infty)} t^{n/2}\,d\mu(t), \quad n\in\mathbf{Z}$$

and

$$\beta(n) = \delta(n), \quad n\in\mathbf{Z}.$$

The equality (15) follows from the same argument as its analogue in the proof of Theorem 4. The application of Lemma 2 completes the proof.

Subnormal operators through C^∞-vectors

9. In the papers ([20], [21], [22]) we have studied subnormal operators by means of some of their classes of C^∞-vectors. Here we wish to review and extend these investigations. Recall the definitions.

A vector $f\in\mathfrak{D}^\infty(S)$ is said to be a *bounded vector* of S if there are positive numbers $a=a(f)$ and $c=c(f)$ such that

$$\|S^n f\| \leq ac^n, \quad n = 1, 2, \ldots.$$

A vector $f\in\mathfrak{D}^\infty(S)$ is said to be an *analytic vector* of S if there is a positive number $t=t(f)$ such that

$$\sum_{n=1}^{\infty} \frac{\|S^n f\|}{n!} t^n < +\infty.$$

A vector $f\in\mathfrak{D}^\infty(S)$ is said to be a *quasi-analytic vector* of S if

$$\sum_{n=1}^{\infty} \|S^n f\|^{-1/n} = +\infty.$$

Finally $f \in \mathfrak{D}^\infty(S)$ is said to be a *Stieltjes vector* of S if

$$\sum_{n=1}^{\infty} \|S^n f\|^{-1/2n} = +\infty.$$

Denote by $\mathfrak{B}(S)$, $\mathfrak{A}(S)$, $\mathfrak{Q}(S)$ and $\mathfrak{S}(S)$ the sets of bounded, analytic, quasi-analytic and Stieltjes vectors of S, respectively. It is clear that $\mathfrak{B}(S)$ and $\mathfrak{A}(S)$ are linear subspaces of $\mathfrak{H}$ and $\mathfrak{B}(S) \subset \mathfrak{A}(S) \subset \mathfrak{Q}(S) \subset \mathfrak{S}(S)$. By direct verification we get that $S(\mathfrak{B}(S)) \subset \mathfrak{B}(S)$ and $S(\mathfrak{A}(S)) \subset \mathfrak{A}(S)$. To check that $\mathfrak{Q}(S)$ and $\mathfrak{S}(S)$ share the same property as $\mathfrak{B}(S)$ and $\mathfrak{A}(S)$, use the Carleman inequality [3]:

$$\sum_{n=2}^{r} a_n^{1-(1/n)} \leqq \sum_{n=2}^{r} a_n + 2\sqrt{\sum_{n=2}^{r} a_n} \tag{18}$$

with $a_n = \|S^n f\|^{-\frac{1}{n-1}}$ and $a_n = \|S^n f\|^{-\frac{1}{2(n-1)}}$, respectively.

In [20] we have proved the following theorem.

Theorem I. *Let S be a densely defined linear operator in $\mathfrak{H}$. Suppose that $\mathfrak{D}(S) = \mathfrak{B}(S)$. Then the following conditions are equivalent:*

(i) *S is subnormal;*

(ii) *S satisfies* (H), *for all finite sequences $f_0, \ldots, f_n \in \mathfrak{D}(S)$;*

(iii) *there is an increasing sequence $\{\mathfrak{H}_n\}_{n=1}^{\infty}$ of closed linear subspaces of $\mathfrak{H}$ contained in $\mathfrak{D}(\bar{S})$ such that $\bar{S}\mathfrak{H}_n \subset \mathfrak{H}_n$, each restriction of $\bar{S}$ to $\mathfrak{H}_n$ is a bounded subnormal operator in $\mathfrak{H}$ and $\bigcup_{n=1}^{\infty} \mathfrak{H}_n$ is a core for $\bar{S}$.*

Remark 4. The following comments may be usefull here. Let A be a densely defined closable operator in $\mathfrak{H}$. A linear subspace $\mathfrak{D}$ of $\mathfrak{D}(A)$ is said to be a *core* for A if $\bar{A} = (A|\mathfrak{D})^-$. A closed linear subspace $\mathfrak{G}$ of $\mathfrak{H}$ is said to be *invariant* (resp. *reducing*) for A if $PAP = AP$ (resp. $PA \subset AP$), where P is the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{G}$. If a closed linear subspace $\mathfrak{G}$ of $\mathfrak{H}$ is contained in $\mathfrak{D}(A) \cap \mathfrak{D}(A^*)$ then $\mathfrak{G}$ is reducing for A if and only if $A(\mathfrak{G}) \subset \mathfrak{G}$ and $A^*(\mathfrak{G}) \subset \mathfrak{G}$.

The example of the creation operator indicates that there are closed subnormal operators having no nontrivial bounded vectors. However, if an operator *has* a dense set of bounded vectors, Theorem I provides us with some additional information on its geometrical structure. We show that quasinormal operators we have already considered in Section 2 fall in this class and get, as a by-product, another proof of subnormality of quasinormal operators.

Proposition 5. *Suppose that S is a quasinormal operator in $\mathfrak{H}$. Then $\mathfrak{B}(S)$ is a core for S, there is an increasing sequence $\{\mathfrak{H}_n\}_{n=1}^{\infty}$ of closed linear subspaces of $\mathfrak{H}$ contained in $\mathfrak{D}(S)$ such that each $\mathfrak{H}_n$ reduces S, each restriction of S to $\mathfrak{H}_n$ is a bounded quasinormal operator in $\mathfrak{H}_n$ and $\bigcup_{n=1}^{\infty} \mathfrak{H}_n$ is a core for S.*

Proof. First of all we show that $\mathfrak{B}(S)$ is a core for S. Let $S=U|S|$ be the polar decomposition of S and let E be the spectral measure of $|S|$. Set $\mathfrak{H}_n=E([0,n])\mathfrak{H}$ and $\mathfrak{D}=\bigcup_{n=1}^{\infty}\mathfrak{H}_n$. Take $f\in\mathfrak{H}_m$. Then

$$|S|f=|S|E([0,m])f=E([0,m])|S|f$$

and by Proposition 1,

$$Uf=UE([0,m])f=E([0,m])Uf.$$

This means that each $\mathfrak{H}_m$ and consequently $\mathfrak{D}$ is invariant for $|S|$, U and S. Thus for $f\in\mathfrak{H}_m$ we have

$$\|S^nf\|^2=\|U^n|S|^nf\|^2\leqq\||S|^nf\|^2=$$

$$=\||S|^nE([0,m])f\|^2=\int_0^m t^{2n}\langle E(dt)f,f\rangle\leqq m^{2n}\|f\|^2,$$

so $f\in\mathfrak{B}(S)$. In other words $\mathfrak{D}\subset\mathfrak{B}(S)$. It is easy to see that the equality $|S|=(|S||_{\mathfrak{D}})^-$ implies $S=(S|\mathfrak{D})^-$. So $\mathfrak{D}$ and $\mathfrak{B}(S)$ are cores for S.

Define a bounded operator $S_n=UR_n$, where $R_n=\int_0^n tE(dt)$. Then $\mathfrak{R}(R_n^2)\subset\mathfrak{R}(R_n)\subset\mathfrak{R}(|S|E([0,n]))\subset\mathfrak{R}(|S|)$, so $\mathfrak{R}(R_n^2)\subset\mathfrak{R}(|S|)$. Since U^*U is the orthogonal projection onto $\overline{\mathfrak{R}(|S|)}$, we have $U^*UR_n^2=R_n^2$. By Proposition 1, U commutes with R_n. Therefore $S_n^*S_n=U^*UR_n^2=R_n^2$, which implies $|S_n|=R_n$. Since U commutes with R_n, S_n commutes with $R_n=|S_n|$. This means that S_n is a quasinormal operator. Denote by T_n the operator $S_n|_{\mathfrak{H}_n}$. Then

$$T_n^*T_n=E([0,n])S_n^*S_n|_{\mathfrak{H}_n}=E([0,n])R_n^2|_{\mathfrak{H}_n}=(R_n|_{\mathfrak{H}_n})^2.$$

Thus $|T_n|=|S_n||_{\mathfrak{H}_n}=R_n|_{\mathfrak{H}_n}$. Since S_n commutes with R_n, T_n commutes with $|T_n|$. This means that for each $n\geqq 1$, $S|_{\mathfrak{H}_n}=T_n$ is a bounded quasinormal operator. Since $E([0,n])S\subset SE([0,n])$, $\mathfrak{H}_n$ reduces S. This completes the proof.

Corollary 3. *S is a subnormal operator if and only if there is a subnormal extension $\tilde{S}$ of S in $\tilde{\mathfrak{H}}\supset\mathfrak{H}$ such that $\mathfrak{B}(\tilde{S})$ is a core for $\tilde{S}$.*

Proof. This is an easy consequence of Proposition 5 and Theorem 2.

Corollary 4. *An operator S in $\mathfrak{H}$ is normal if and only if S is formally normal and quasinormal. In particular S is self-adjoint if and only if S is symmetric and quasinormal.*

Proof. Since S is quasinormal, there is a sequence $\{\mathfrak{H}_n\}_{n=1}^{\infty}$ of closed linear subspace of $\mathfrak{H}$ with properties described by Proposition 5. Let $\mathfrak{D}=\bigcup_{n=1}^{\infty}\mathfrak{H}_n$. Then $\mathfrak{D}\subset\mathfrak{B}(S)$ and $\mathfrak{D}$ is a core for S. Since $\mathfrak{H}_n$ is a reducing subspace for S and $\mathfrak{H}_n\subset$

$\subset\mathfrak{D}(S)\cap\mathfrak{D}(S^*)=\mathfrak{D}(S)$, Remark 4 implies that $S\mathfrak{H}_n\subset\mathfrak{H}_n$ and $S^*\mathfrak{H}_n\subset\mathfrak{H}_n$, for each n. Thus $S\mathfrak{D}\subset\mathfrak{D}$ and $S^*\mathfrak{D}\subset\mathfrak{D}$. Since S is formally normal, the nontrivial conclusion of Corollary 4 follows from Theorem 1 of [20].

10. Another result we wish to discuss is one which bears a resemblance to a result of Embry for bounded operators [7].

Theorem 6. *Let S be a densely defined operator in $\mathfrak{H}$ such that $S\mathfrak{D}(S)\subset\mathfrak{D}(S)$. Suppose that $\mathfrak{D}(S)$ is a linear span of the set $\mathfrak{Q}(S)$. Then S is subnormal if and only if S satisfies* (E) *for all finite sequences $f_0, \ldots, f_n\in\mathfrak{D}(S)$.*

This is a stronger version of Theorem 8 of [21] where instead of (E) the condition (H) appears.

In order to prove Theorem 6 we need some lemmas. The first of them gives the full characterization of determinate moment sequences in terms of their representing measures. The proof of it can be done in the same way as that for Hamburger moment sequences (cf. [9], Theorem 8).

Lemma 3. *A Stieltjes moment sequence $\{a_n\}_{n=0}^{\infty}$ with the representing measure μ is determinate if and only if the set of all polynomials of one real variable is dense in $L^2(\mathbf{R}_+, (1+x^2)\mu)$.*

Lemma 4. *Let N be a densely defined operator in $\mathfrak{K}$ such that*

$$\mathfrak{D}=\mathfrak{D}(N)=\mathfrak{D}(N^*N), \quad N(\mathfrak{D})\subset\mathfrak{D} \quad \text{and} \quad N^*N(\mathfrak{D})\subset\mathfrak{D}, \tag{19}$$

$$N(N^*N)f=(N^*N)Nf, \quad f\in\mathfrak{D}, \tag{20}$$

$$\mathfrak{S}((N^*N)) \quad \text{is a total set in } \mathfrak{K}. \tag{21}$$

Then N is closable and $\bar{N}$ is quasinormal.

Proof. Denote by A the symmetric operator N^*N defined on $\mathfrak{D}$. Then $\langle Nf, Ng\rangle=\langle Af, g\rangle$, $f, g\in\mathfrak{D}$. This implies that N is closable. Denote by $\mathfrak{D}_0$ the linear span of $\mathfrak{S}(A)$. Then $A(\mathfrak{D}_0)\subset\mathfrak{D}_0$, $A=N^*N\subset\bar{N}^*\bar{N}$ and, by (21), $\mathfrak{S}(A)$ is a total set in $\mathfrak{K}$. It follows from [16] that $(\bar{A}|_{\mathfrak{D}_0})^-=\bar{A}=\bar{N}^*\bar{N}$. The last equality can be written as $\bar{A}=|\bar{N}|^2$. Since $\mathfrak{D}_0\subset\mathfrak{D}(|N|^2)$ and $\mathfrak{D}_0$ is a core for $|\bar{N}|^2$, $\mathfrak{D}_0$ is a core for $|\bar{N}|$. Now an application of the polar decomposition for $\bar{N}$ gives us that

$$\mathfrak{D}_0 \text{ is a core for } \bar{N}. \tag{22}$$

Let E be the spectral measure of $\bar{A}$, i.e. $\bar{A}=\int\limits_0^\infty tE(dt)$. Then for $f\in\mathfrak{D}$ we have

$$\langle A^nf, f\rangle=\int\limits_0^\infty t^n\langle E(dt)f, f\rangle, \quad n\in\mathbf{N},$$

and

$$\langle A^n Nf, Nf\rangle = \int_0^\infty t^n \langle E(dt)Nf, Nf\rangle, \quad n\in\mathbf{N}.$$

Since $A=N^*N$, (19) and (20) imply

$$\langle A^n Nf, Nf\rangle = \langle A^{n+1}f, f\rangle, \quad n\in\mathbf{N}.$$

Combining these three equalities we obtain

$$(23)\qquad \langle A^{n+1}f, f\rangle = \int_0^\infty t^n\langle E(dt)Nf, Nf\rangle = \int_0^\infty t^n t\langle E(dt)f, f\rangle, \quad f\in\mathfrak{D}, \quad n\geqq 0.$$

Let $f\in\mathfrak{S}(A)$. Due to the Carleman criterion (cf. [18]) the sequence $\{\langle A^n f, f\rangle\}_{n=0}^\infty$ is a determinate Stieltjes moment sequence. Using now the Carleman inequality (18) and again the Carleman criterion we infer that $\{\langle A^{n+1}f, f\rangle\}_{n=0}^\infty$ is a determinate Stieltjes moment sequence. Consequently, due to (23), we have

$$(24)\qquad \langle E(dt)Nf, Nf\rangle = t\langle E(dt)f, f\rangle.$$

Let σ be a Borel subset of $\mathbf{R}_+$. Since $\{\langle A^n f, f\rangle\}_{n=0}^\infty$ is a determinate Stieltjes moment sequence ($f\in\mathfrak{S}(A)$!), Lemma 3 gives us a sequence $\{p_n\}_{n=1}^\infty$ of polynomials, which converges to the indicator function 1_σ of the set σ in $L^2(\mathbf{R}_+, (1+x^2)\mu)$, where $\mu=\langle E(\cdot)f, f\rangle$. One can show then that $\{p_n\}_{n=1}^\infty$ converges to 1_σ in $L^2(\mathbf{R}_+, \mu)$ as well as in $L^2(\mathbf{R}_+, x\mu)$. Since

$$\|E(\sigma)f - p_n(A)f\|^2 = \int_0^\infty |1_\sigma - p_n|^2\, d\mu$$

and, by (24),

$$\|E(\sigma)Nf - p_n(A)Nf\|^2 = \int_0^\infty |1_\sigma(x) - p_n(x)|^2 \langle E(dx)Nf, Nf\rangle =$$

$$= \int_0^\infty |1_\sigma(x) - p_n(x)|^2 x\, d\mu(x),$$

we have $E(\sigma)f = \lim_{n\to\infty} p_n(A)f$ and $E(\sigma)Nf = \lim_{n\to\infty} p_n(A)Nf = \lim_{n\to\infty} Np_n(A)f$. Thus $E(\sigma)f\in\mathfrak{D}(\bar N)$ and $\bar N E(\sigma)f = E(\sigma)Nf$ for each $f\in\mathfrak{S}(A)$. This implies that $E(\sigma)(N|_{\mathfrak{D}_0})\subset \subset \bar N E(\sigma)$ and $E(\sigma)(N|_{\mathfrak{D}_0})^-\subset \bar N E(\sigma)$, in consequence. Due to (22) we obtain

$$(25)\qquad E(\sigma)\bar N\subset \bar N E(\sigma), \quad \text{for each Borel subset } \sigma \text{ of } \mathbf{R}_+.$$

Since $|\bar N| = \bar A^{1/2} = \int_0^{+\infty} t^{1/2} E(dt)$, the spectral measure F of $|\bar N|$ is given by the following formula:

$$(26)\qquad F(\sigma) = E(\varphi^{-1}(\sigma)), \quad \text{for each Borel subset } \sigma \text{ of } \mathbf{R}_+,$$

where $\varphi\colon \mathbf{R}_+ \to \mathbf{R}_+$ is a homeomorphism defined by $\varphi(x)=x^{1/2}$, $x\in\mathbf{R}_+$. The conditions (25) and (26) show that $\bar{N}$ commutes with the spectral measure F of $|\bar{N}|$. This completes the proof of Lemma 4.

The next lemma shows that the condition (E) holds on $\mathfrak{D}^\infty(S)$ if and only if S has a "formally quasinormal" extension with "reducing" domain.

Lemma 5. *Let S be a densely defined operator in $\mathfrak{H}$ such that $S(\mathfrak{D}(S))\subset\mathfrak{D}(S)$. Then S satisfies* (E) *for all finite sequences $f_0, \dots, f_n\in\mathfrak{D}(S)$ if and only if there is a densely defined operator N in some Hilbert space $\mathfrak{K}\supset\mathfrak{H}$ such that*

(i) *N satisfies the conditions* (19) *and* (20),

(ii) *$\mathfrak{D}(S)\subset\mathfrak{D}(N)$ and $S\subset N$,*

(iii) *$\mathfrak{D}(N)$ is a linear span of the set $\{(N^*N)^n f\colon n\geqq 0, f\in\mathfrak{D}(S)\}$.*

Proof. Suppose that N satisfies (i) and (ii). Then (19) and (20) imply (via an induction procedure)

$$\langle (N^*N)^n f, g\rangle = \langle N^n f, N^n g\rangle,\quad f, g\in\mathfrak{D}(N),\quad n\geqq 0,$$

and this can be used to prove the inequality (E) for all finite sequences $f_0, \dots, f_n\in\mathfrak{D}(S)$.

To prove the converse, suppose that S satisfies (E) for all finite sequences $f_0, \dots, f_n\in\mathfrak{D}(S)$. Define the form φ over $(\mathbf{N}, \mathfrak{D}(S))$ (cf. [23]) by

$$\varphi(n, f, g) = \langle S^n f, S^n g\rangle,\quad n\in\mathbf{N},\quad f, g\in\mathfrak{D}(S).$$

$\mathbf{N}$ is a *-semigroup with the identity map as an involution. Since S satisfies (E) for all finite sequences $f_0, \dots, f_n\in\mathfrak{D}(S)$, the form φ is positive definite. Notice also that 1 is a hermitian generator of the *-semigroup $\mathbf{N}$ and $\varphi(0, f, g)=\langle f, g\rangle$ for all $f, g\in\mathfrak{D}(S)$. Thus, by Proposition of [23], there is (under suitable unitary identification — see the proof of Prop. 2) a densely defined symmetric operator A in some Hilbert space $\mathfrak{K}\supset\mathfrak{H}$ such that $A(\mathfrak{D}(A))\subset\mathfrak{D}(A)$, $\mathfrak{D}(A)$ is the linear span of the set $\{A^n f\colon n\geqq 0, f\in\mathfrak{D}(S)\}$, $\mathfrak{D}(S)\subset\mathfrak{D}(A)$ and

$$\langle S^n f, S^n g\rangle = \varphi(n; f, g) = \langle A^n f, g\rangle,\quad n\geqq 0,\quad f, g\in\mathfrak{D}(S). \tag{27}$$

Define an operator N with $\mathfrak{D}(N)=\mathfrak{D}(A)$ by

$$N\Big(\sum_{k=0}^{n} A^k f_k\Big) = \sum_{k=0}^{n} A^k S f_k,\quad f_0, \dots, f_n\in\mathfrak{D}(S),\quad n\geqq 0.$$

It follows from (27) that for all $f_0, \dots, f_n\in\mathfrak{D}(S)$

$$\Big\|\sum_{k=0}^{n} A^k S f_k\Big\|^2 = \sum_{k,l=0}^{n} \langle A^{k+l} S f_k, S f_l\rangle = \sum_{k,l=0}^{n} \langle S^{k+l} S f_k, S^{k+l} S f_l\rangle =$$

$$= \sum_{k,l=0}^{n} \langle S^{k+l+1} f_k, S^{k+l+1} f_l\rangle = \sum_{k,l=0}^{n} \langle A^{k+l+1} f_k, f_l\rangle = \Big\langle A\Big(\sum_{k=0}^{n} A^k f_k\Big), \sum_{l=0}^{n} A^l f_l\Big\rangle.$$

This implies the correctness of the definition of N and shows that $\|Nf\|^2=\langle Af,f\rangle$, $f\in\mathfrak{D}(N)$. Consequently

$$\langle Nf, Ng\rangle = \langle Af, g\rangle, \quad f, g\in\mathfrak{D}(N).$$

This implies that $A=N^*N$. The equality (20) follows from the following ones

$$NA\Big(\sum_{k=0}^{n} A^k f_k\Big) = N\Big(\sum_{k=0}^{n} A^{k+1} f_k\Big) = \sum_{k=0}^{n} A^{k+1} S f_k =$$

$$= A\Big(\sum_{k=0}^{n} A^k S f_k\Big) = AN\Big(\sum_{k=0}^{n} A^k f_k\Big), \quad f_0, \dots, f_n\in\mathfrak{D}(S).$$

The inclusion $S\subset N$ is obvious. This completes the proof of Lemma 5.

Now we are able to pass to the proof of Theorem 6.

Proof of Theorem 6. "Only if" part of Theorem 6 follows from Proposition 4 as well as from Theorem 3.

Conversely, suppose that S satisfies (E) for all finite sequences $f_0, \dots, f_n\in\mathfrak{D}(S)$. Due to Lemma 5, there is a densely defined operator N in some Hilbert space $\mathfrak{K}\supset\mathfrak{H}$, which satisfies the conditions (i), (ii) and (iii) of Lemma 5. Then

$$\mathfrak{Q}(S)\subset\mathfrak{S}(A), \quad A = N^*N. \tag{28}$$

To prove this suppose that $f\in\mathfrak{Q}(S)$. Then Proposition 4 implies that the sequence $\{a_n\}_{n=0}^{\infty}$, where $a_n=\|S^n f\|^2$ for $n\in\mathbf{N}$, is a Stieltjes moment sequence. Thus $a_n^2\leqq a_k a_l$ for $k, l\in\mathbf{N}$ such that $2n=k+l$. Due to Section 1 of [21] we obtain

$$\sum_{n=1}^{\infty} \|S^{2n} f\|^{-1/2n} = +\infty.$$

It follows from (i) and (ii) of Lemma 5 that $\|A^n f\|^2=\langle A^{2n}f,f\rangle=\|S^{2n}f\|^2$, so $\sum_{n=1}^{\infty} \|A^n f\|^{-1/2n}=\sum_{n=1}^{\infty} \|S^{2n} f\|^{-1/2n}=+\infty$. Thus $f\in\mathfrak{S}(A)$.

Now we are in position to use Lemma 4. Indeed, since $\mathfrak{D}(A)$ is a linear span of $\{A^n f\colon n\geqq 0, f\in\mathfrak{D}(S)\}$ and $\mathfrak{D}(S)$ is a linear span of $\mathfrak{Q}(S)$, an application of (28) and $A(\mathfrak{S}(A))\subset\mathfrak{S}(A)$ gives us that $\mathfrak{D}(A)$ is a linear span of $\mathfrak{S}(A)$. It follows from Lemma 4 that $\bar{N}$ is quasinormal and, by Corollary 1, S is subnormal. This completes the proof.

Corollary 5. *Let S be a closed densely defined operator in $\mathfrak{H}$. Suppose that the linear span $\mathfrak{D}$ of $\mathfrak{Q}(S)$ is a core for S and that S satisfies* (E) *for all finite sequences $f_0, \dots, f_n\in\mathfrak{D}$. Then S is a subnormal operator.*

11. Now we show that if an operator S has a dense set of analytic vectors then, similarly as in the case of weighted shifts, the condition (S), when satisfied for all $f\in\mathfrak{D}(S)$, is sufficient for S to be subnormal. This result is an extension of Lambert theorem (cf. [15], Th. 3.1) to the case of unbounded operators. Similarly as in [22] we ask whether this theorem is true for operators having dense set of quasi-analytic vectors.

Theorem 7. *Let S be a densely defined operator in $\mathfrak{H}$ such that $\mathfrak{D}(S)=\mathfrak{A}(S)$. Then S is subnormal if and only if S satisfies* (S) *for each $f\in\mathfrak{D}(S)$.*

Proof. We have only to prove sufficiency. Suppose that S satisfies (S) for each $f\in\mathfrak{D}(S)$. Then for each $f\in\mathfrak{D}(S)$ there is a unique non-negative measure μ_f such that

$$\|S^n f\|^2 = \int_0^\infty t^n\, d\mu_f(t), \quad n=0,1,2,\ldots. \tag{29}$$

Using the polarization formula we define complex measures

$$\mu(\sigma; f, g) = \frac{1}{4}\{\mu_{f+g}(\sigma)-\mu_{f-g}(\sigma)+i\mu_{f+ig}(\sigma)-i\mu_{f-ig}(\sigma)\}$$

for each Borel subset σ of $\mathbf{R}_+$. Since the measure μ_f is uniquely determined we have

$$\mu_{af} = |a|^2\mu_f, \quad a\in\mathbf{C}, \quad f\in\mathfrak{D}(S).$$

This implies that $\mu_f=\mu(\cdot\,; f, f)$, $f\in\mathfrak{D}(S)$ and that the form $\mu(\sigma;\,\cdot, -)$ is hermitian symmetric. It is easy to see that

$$\langle S^n f, S^n g\rangle = \int_0^\infty t^n \mu(dt; f, g), \quad f, g\in\mathfrak{D}(S), \quad n\in\mathbf{N}. \tag{30}$$

Now we prove that $\mu(\gamma;\,\cdot, -)$ is linear with respect to the first variable. To show it is additive we write

$$\langle S^n(f+g), S^n h\rangle = \langle S^n f, S^n h\rangle + \langle S^n g, S^n h\rangle, \quad f, g, h\in\mathfrak{D}(S), \quad n\in\mathbf{N}.$$

Using the polarization formula for the form $(f,g)\to\langle S^n f, S^n g\rangle$ and the integral representation (29) we get

$$\int_0^\infty t^n\, d\nu_1(t) - \int_0^\infty t^n\, d\nu_2(t) + i\Big(\int_0^\infty t^n\, d\nu_3(t) - \int_0^\infty t^n\, d\nu_4(t)\Big) = 0, \quad n\in\mathbf{N},$$

where

$$\nu_1 = \mu_{f+g+h}+\mu_{f-h}+\mu_{g-h}, \qquad \nu_2 = \mu_{f+g-h}+\mu_{f+h}+\mu_{g+h},$$

$$\nu_3 = \mu_{f+g+ih}+\mu_{f-ih}+\mu_{g-ih}, \quad \nu_4 = \mu_{f+g-ih}+\mu_{f+ih}+\mu_{g+ih}.$$

Since the measures ν_k, $k=1, 2, 3, 4$, are non-negative we obtain

$$\int_0^\infty t^n \, d\nu_1(t) = \int_0^\infty t^n \, d\nu_2(t), \quad n\in\mathbf{N},$$

and

$$\int_0^\infty t^n \, d\nu_3(t) = \int_0^\infty t^n \, d\nu_4(t), \quad n\in\mathbf{N}. \tag{31}$$

Each of these Stieltjes moment sequences is determinate. To see this consider the first of them

$$a_n = \int_0^\infty t^n \, d\nu_1(t) = \|S^n(f+g+h)\|^2 + \|S^n(f-h)\|^2 + \|S^n(g-h)\|^2, \quad n\in\mathbf{N}.$$

Since the vectors $f+g+h, f-h, g-h$ are analytic vectors of S, one can prove that there is a positive real number $t>0$ such that

$$\sum_{n=0}^{\infty} \frac{a_n^{1/2}}{n!} t^n < +\infty.$$

This implies that $\sum_{n=1}^{\infty} a_n^{-1/2n} = +\infty$. Due to the Carleman criterion (cf. [18]), $\{a_n\}$ is a determinate Stieltjes moment sequence. The same is true for the other sequence given by (31).

Thus $\nu_1=\nu_2$ and $\nu_3=\nu_4$. This in conclusion implies the required additivity $\mu(\sigma;\, f+g, h)=\mu(\sigma;\, f, h)+\mu(\sigma, g, h)$. By the same trick we can prove that $\mu(\sigma;\, af, g)=a\mu(\sigma;\, f, g)$, first for $a>0$ then for $a<0$ and finally for $a=i$ which exhausts all possibilities.

Thus for each Borel subset σ of $\mathbf{R}_+$, $\mu(\sigma;\, \cdot, -)$ is a hermitian bilinear form and $\mu(\cdot\,; f, f)$ is a non-negative finite measure on $\mathbf{R}_+$ for each $f\in\mathfrak{D}(S)$. Using the generalized Naimark dilation theorem [10] we find a Hilbert space $\mathfrak{K}$, a linear operator $V\colon \mathfrak{D}(S)\to\mathfrak{K}$ and a spectral (normalized) measure E on $\mathbf{R}_+$ in $\mathfrak{K}$ such that

$$\mu(\sigma;\, f, g) = \langle E(\sigma)Vf, Vg\rangle, \quad f, g\in\mathfrak{D}(S), \tag{32}$$

for every Borel subset σ of $\mathbf{R}_+$. According to Theorem 6, the proof of Theorem 7 will be finished if we show that S satisfies (E) for all finite sequences $f_0, \ldots, f_n\in\mathfrak{D}(S)$. Let $f_0, \ldots, f_n\in\mathfrak{D}(S)$. Due to (32)

$$V(\mathfrak{D}(S))\subset\mathfrak{D}\Big(\int_0^\infty t^n E(dt)\Big), \quad n\in\mathbf{N}.$$

Using (30) and (32) we obtain

$$\sum_{j,k=0}^{n}\langle S^{j+k}f_j, S^{j+k}f_k\rangle = \sum_{j,k=0}^{n}\int_0^\infty t^{j+k}\mu(dt; f_j, f_k) = \sum_{j,k=0}^{n}\left\langle\int_0^\infty t^{j+k}E(dt)Vf_j, Vf_k\right\rangle =$$

$$= \sum_{j,k=0}^{n}\left\langle\int_0^\infty t^{j}E(dt)Vf_j, \int_0^\infty t^{k}E(dt)Vf_k\right\rangle = \left\|\sum_{k=0}^{n}\int_0^\infty t^{k}E(dt)Vf_k\right\|^2 \geqq 0.$$

This completes the proof.

The proof of Theorem 7 is similar to that of Theorem 6 in [21]. For reader's convenience we have repeated the most essential parts of it.

Corollary 6. *Let S be a closed densily defined operator in H such that $\mathfrak{A}(S)$ is a core for S. If S satisfies* (S) *for each $f\in\mathfrak{A}(S)$, then S is a subnormal operator.*

In the case when the operator S is invertible, Theorem 7 implies the following

Corollary 7. *Let S be a densely defined operator with the densely defined inverse S^{-1}. Suppose $S\mathfrak{D}(S)\subset\mathfrak{D}(S)$ and $S^{-1}\mathfrak{D}(S^{-1})\subset\mathfrak{D}(S^{-1})$ and S satisfies* (S) *for each $f\in\mathfrak{D}(S)$. Then S is subnormal provided $\mathfrak{D}(S^{-1})=\mathfrak{A}(S^{-1})$.*

Proof. Due to Corollary 2 (b), it is sufficient to show that S^{-1} is subnormal and, due to Theorem 7, it is sufficient to show that S^{-1} satisfies (S) for each $f\in\mathfrak{D}(S^{-1})=\mathfrak{D}(S)$. Take $f\in\mathfrak{D}(S)$ and $c_0, \ldots, c_n\in\mathbf{C}$. Define $g=S^{-2n}f$, $h=S^{-1}f$ and $d_j=c_{n-j}$, $j=0, \ldots, n$. Then

$$\sum_{j,k=0}^{n}\|(S^{-1})^{j+k}f\|^2 c_j\bar{c}_k = \sum_{j,k=0}^{n}\|S^{j+k}g\|^2 d_j\bar{d}_k \geqq 0$$

and

$$\sum_{j,k=0}^{n}\|(S^{-1})^{j+k+1}f\|^2 c_j\bar{c}_k = \sum_{j,k=0}^{n}\|(S^{-1})^{j+k}h\|^2 c_j\bar{c}_k \geqq 0$$

which means that S^{-1} satisfies (S) for each $f\in\mathfrak{D}(S)$.

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Some uniform weak-star ergodic theorems

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0. Introduction. Let $\mathfrak{B}$ be a Banach space and let G be a bounded semigroup of adjoint operators in $\mathfrak{B}^*$. We have proved the following result in [3]:

Suppose $\mathfrak{B}$ is weakly complete and G is commutative and separable. If for every $t \in \mathfrak{B}^*$, the w^*-closed convex hull of the orbit $Gt = \{gt : g \in G\}$ contains exactly one G-invariant element t^G, then the mapping $t \to t^G : \mathfrak{B}^* \to \mathfrak{B}^*$ is a w^*-continuous linear projection P such that $gP = Pg = P$ $(g \in G)$.

(The term "separable" here means that G contains a countable subset G_0 which is dense in G if G is considered in the topology of pointwise w^*-convergence on $\mathfrak{B}^*$.)

According to [4], the above result also holds if instead of the commutativity of G, we only assume its amenability.

In the present paper we are going to prove analogues of the above result for the uniformly closed convex hull of the orbit Gt. The particular case where $\mathfrak{B}^*$ is a W^*-algebra M and G is a group of *-automorphisms of M may be of some interest.

1. Results. Let $\mathfrak{B}$ be a Banach space with dual $\mathfrak{B}^*$ and let G be a bounded semigroup of w^*-continuous linear operators in $\mathfrak{B}^*$. In other words, $\sup \{\|g\| : g \in G\} < \infty$ and for every $g \in G$, there is a unique bounded linear operator g_* acting in $\mathfrak{B}$, such that $(g_*)^* = g$. Let us consider the following two properties of the pair $\mathfrak{B}$, G:

(N) For every $t \in \mathfrak{B}^*$, the norm-closed convex hull of the orbit $Gt = \{gt : g \in G\}$ contains at least one G-invariant element.

(N_1) For every $t \in \mathfrak{B}^*$, the norm-closed convex hull of the orbit Gt contains exactly one G-invariant element, say t^G.

Theorem 1. *Suppose $\mathfrak{B}$ is weakly complete and G is amenable and countable. Then condition (N) implies condition (N_1). If condition (N_1) (or (N)) is satisfied, then the mapping $t \to t^G$ $(t \in \mathfrak{B}^*)$ is a w^*-continuous linear projection P such that $gP = Pg = P$ $(g \in G)$.*

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Theorem 2. *Suppose $\mathfrak{B}$ is weakly complete and G is commutative and separable. Then condition* (N) *implies condition* (N_1). *If condition* (N) (*or* (N_1)) *is satisfied, then the mapping* $t \to t^G$ ($t \in \mathfrak{B}^*$) *is a* w^*-*continuous linear projection* P *such that* $gP = Pg = P$ ($g \in G$).

Proposition. *Let* $\mathfrak{B}^* = M$, *a von Neumann algebra and let G be a countable amenable group of *-automorphisms of M or let* $\mathfrak{B}^* = M$, *a von Neumann algebra in a separable Hilbert space and let G be a commutative group of *-automorphisms of M. Assume that condition* (N) *is satisfied. Then condition* (N_1) *is also satisfied and M is G-finite.* (For this notion, cf. [2].)

2. Proofs.

Proof of Theorem 1. For every $\varphi \in \mathfrak{B}$ and $t \in \mathfrak{B}^*$, let us define the element $f_{\varphi,t} \in l^\infty(G)$ by the equality $f_{\varphi,t}(g) = \varphi(g(t))$ ($g \in G$). Assume that (N) holds and consider a given $t \in \mathfrak{B}^*$. Then there is a sequence $\{v_n\}_{n=1}^\infty$ of elements of the convex hull conv G of G, for which $v_n(t)$ converges in norm to a G-invariant element of $\mathfrak{B}^*$, say t'. Let a positive number ε be given. Using the notation $\|G\| = \sup\{\|g\| : g \in G\}$, we can find a positive integer n_0 such that $\|v_n(t) - t'\| < \varepsilon/\|G\|$ if $n \geqq n_0$. Then $\|gv_n(t) - t'\| = \|g(v_n(t) - t')\| \leqq \|g\|\,\|v_n(t) - t'\| < \|G\|(\varepsilon/\|G\|) = \varepsilon$ uniformly in $g \in G$ for $n \geqq n_0$. Consequently, for a given $\varphi \in \mathfrak{B}$ we have $|f_{\varphi,t}(gv_n) - \varphi(t')| = |\varphi(gv_n(t)) - \varphi(t')| = |\varphi(gv_n(t) - t')| \leqq \|\varphi\|\,\|gv_n(t) - t'\| < \|\varphi\|\,\varepsilon$ for all $g \in G$ if $n \geqq n_0$. Since $\varepsilon > 0$ was arbitrary, we have proved that the constant function on G which is equal to $\varphi(t')$ can be uniformly approximated by convex combinations of the right translates of the element $f_{\varphi,t}$ of $l^\infty(G)$.

Let m now be a right invariant mean on $l^\infty(G)$. The result above implies that $m(f_{\varphi,t}) = \varphi(t')$. In particular, if t'' is another element of the norm-closed convex hull of Gt, then $m(f_{\varphi,t}) = \varphi(t'')$. Consequently, $\varphi(t') = \varphi(t'')$ for every $\varphi \in \mathfrak{B}$, and thus $t' = t''$. Therefore, since $t \in \mathfrak{B}^*$ was arbitrary, we have proved that (N) implies (N_1) (even without assuming the weak completeness of $\mathfrak{B}$ or the countability of G).

Now let $\{G_n\}_{n=1}^\infty$ be a right-hand summing sequence for G, i.e., let $(1/\text{card } G_n)\,\text{card}([G_n \cup G_n g] \setminus [G_n \cap G_n g]) \to 0$ as $n \to \infty$. (For the existence of such a sequence, see [1].) We are going to prove that for $\varphi \in \mathfrak{B}$ and $t \in \mathfrak{B}^*$,

$$(*) \qquad (1/\text{card } G_n) \sum_{h \in G_n} f_{\varphi,t}(h) \to \varphi(t^G)$$

as $n \to \infty$. To prove this, we fix $\varphi \in \mathfrak{B}$ and $t \in \mathfrak{B}^*$ and assume that for some filter F finer than the filter base $\{\{n : n \geqq k\} : k \in \mathbf{N}\}$, $\lim_F (1/\text{card } G_n) \sum_{h \in G_n} f_{\varphi,t}(h)$ exists. Let F_1 be an ultrafilter finer than F. Then for every $f \in l^\infty(G)$ and $g \in G$, the limit $\lim_{F_1} (1/\text{card } G_n) \sum_{h \in G_n} f(hg)$ exists (since F_1 is an ultrafilter) and is independent of

$g\in G$ (because of the summing sequence property, since F_1 is finer than $\{\{n: n\geqq k\}: k\in\mathbf{N}\}$). Consequently, $m(f)=\lim_{F_1}(1/\mathrm{card}\, G_n)\sum_{h\in G_n} f(h)$ is a right invariant mean on $l^\infty(G)$. By the beginning of our proof,

$$m(f_{\varphi,t}) = \lim_{F_1}(1/\mathrm{card}\, G_n)\sum_{h\in G_n} f_{\varphi,t}(h) = \varphi(t^G).$$

This means that $\lim_F(1/\mathrm{card}\, G_n)\sum_{h\in G_n} f_{\varphi,t}(h)=\varphi(t^G)$ for every filter F which is finer than the filter base $\{\{n: n\geqq k\}: k\in\mathbf{N}\}$ and for which $\lim_F(1/\mathrm{card}\, G_n)\sum_{h\in G_n} f_{\varphi,t}(h)$ exists. This means that $\lim_{n\to\infty}(1/\mathrm{card}\, G_n)\sum_{h\in G_n} f_{\varphi,t}(h)=\varphi(t^G)$. Since $\varphi\in\mathfrak{B}$ and $t\in\mathfrak{B}^*$ were arbitrary fixed elements, we have proved $(*)$.

Let us write $w_n=(1/\mathrm{card}\, G_n)\sum_{h\in G_n} h$. Then $w_n\in\mathrm{conv}\, G$ and by $(*)$, $w_n t\to t^G$ in the w^*-topology for every $t\in\mathfrak{B}^*$. From this point we proceed in the same way as in the first paragraph of Proof of Theorem 2 in [3]. For the sake of completeness, we repeat that reasoning here.

Let $\varphi\in\mathfrak{B}$ be given. Then for every $t\in\mathfrak{B}^*$ we have $(w_n^*\varphi-w_m^*\varphi, t)=$ $=(\varphi, (w_n-w_m)t)\to 0$ as $n, m\to\infty$. Therefore, the sequence $\{w_n^*\varphi\}_{n=1}^\infty$ is a weak Cauchy sequence in $\mathfrak{B}$. Since $\mathfrak{B}$ is weakly complete, there is an element $P_*\varphi$ of $\mathfrak{B}$ such that $(w_n^*\varphi, t)\to(P_*\varphi, t)$ for every $t\in\mathfrak{B}^*$ as $n\to\infty$. It is obvious that P_* is a bounded linear operator in $\mathfrak{B}$. Furthermore, letting $n\to\infty$, we obtain that $(\varphi, w_n t)=$ $=(w_n^*\varphi, t)\to(P_*\varphi, t)=(\varphi, (P_*)^*t)$ for $\varphi\in\mathfrak{B}$, $t\in\mathfrak{B}^*$. Consequently, for every $t\in\mathfrak{B}^*$ we have $w_n t\to(P_*)^*t$ $(n\to\infty)$ in the w^*-topology of $\mathfrak{B}^*$ and thus $t^G=(P_*)^*t$ $(t\in\mathfrak{B}^*)$. Since $(P_*)^*$ is obviously w^*-continuous, this completes the proof of Theorem 1.

Remark. The first part of the proof of Theorem 1 shows that if G is a bounded amenable semigroup of linear operators in a Banach space $\mathfrak{C}$ and for every $t\in\mathfrak{C}$, the norm-closed convex hull of the orbit Gt contains at least one G-invariant element, then it contains exactly one G-invariant element. This can be seen in the same way as in the first part of the proof of Theorem 1 if we replace $\mathfrak{B}^*$ by $\mathfrak{C}$ and $\mathfrak{B}$ by $\mathfrak{C}^*$ there.

Proof of Theorem 2. Assume (N). We shall prove that for every $t\in\mathfrak{B}^*$, the w^*-closed convex hull of the orbit Gt contains exactly one G-invariant element. Then Theorem 2 of [3] will imply the statement of Theorem 2 of this paper.

First we prove that for every $t\in\mathfrak{B}^*$, the norm-closed convex hull of Gt contains exactly one G-invariant element. (This follows from the above Remark, but in the commutative case the proof is simpler and we prefer to give an independent proof.) In fact, let t' and t'' be two G-invariant elements in the norm-closed convex hull of Gt and let ε be a positive number. There exist v and w in $\mathrm{conv}\, G$, such that $\|vt-t'\|<\varepsilon$ and $\|wt-t''\|<\varepsilon$. We have $\|t'-t''\|\leqq\|t'-vwt\|+\|vwt-t''\|=$ $=\|w(t'-vt)\|+\|v(wt-t'')\|\leqq\|t'-vt\|+\|wt-t''\|<2\varepsilon$, since $vw=wv$ and $\|v\|\leqq 1$,

$\|w\| \leqq 1$. Since $\varepsilon > 0$ was arbitrary, this proves that $t' = t''$ and thus the norm-closed convex hull of Gt contains exactly one G-invariant element, say t^G.

Now we prove that for every $t \in \mathfrak{B}^*$, the only G-invariant element in the w^*-closed convex hull of Gt is t^G. In fact, let $t \in \mathfrak{B}^*$ and let t_0 be a G-invariant element in the w^*-closure of $[\mathrm{conv}\, G]t$. Given $\varepsilon > 0$, there is $w \in \mathrm{conv}\, G$ such that $\|wt - t^G\| < \varepsilon$. Furthermore, there exists a net v_n in $\mathrm{conv}\, G$, such that $v_n t \to t'$ in the w^*-topology. Then $wv_n t \to wt' = t'$ in the w^*-topology. On the other hand, $\|wv_n t - t^G\| = \|v_n(wt - t^G)\| < \varepsilon$. Consequently, $\|t' - t^G\| \leqq \sup_n \|wv_n t - t^G\| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this proves that $t' = t^G$ is the only G-invariant element in the w^*-closure of $[\mathrm{conv}\, G]t$. This completes the proof of Theorem 2.

Proof of Proposition. The Proposition is a special case of Theorems 1 and 2. We only have to note that if M is a von Neumann algebra in a separable Hilbert space and G is a group of *-automorphisms of M, then G is separable, as was pointed out in [3].

Problem. If $\mathfrak{B}$ is weakly complete and separable, does condition (N_1) imply that the mapping $t \to t^G$ is w^*-continuous on $\mathfrak{B}^*$?

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Reflexive lattices of operator ranges with more than one generator

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Introduction. A linear submanifold (subspace, not necessarily closed) in a Hilbert space $\mathfrak{H}$ is an operator range (paraclosed subspace) if it is the range of some bounded operator on $\mathfrak{H}$—some member of $B(\mathfrak{H})$ (the algebra of bounded linear operators on $\mathfrak{H}$). We refer the interested readers to the article [2] of Fillmore and Williams for detailed discussions of operator ranges. Since the publication of the pioneering work of Foiaş [3] on operator ranges invariant under algebras of operators, much progress has been made by many authors in this direction. However, there are few concrete examples of reflexive lattices of operator ranges have been explicitly described. A reflexive lattice of operator ranges is the lattice of all operator ranges invariant under an algebra of operators. The extreme case of singly generated lattices are described in [6] in terms of the generators. In [1] a description of the operator range lattice invariant under a reflexive algebra with commutative invariant subspace lattice is given. Here we describe the reflexive lattice of operator ranges in terms of the generators. All lattices here will be lattices of operator ranges.

Main results. For fixed positive operators $P_1, P_2, \ldots, P_n$, the reflexive lattice generated by (the ranges of) $P_1, P_2, \ldots, P_n$ will be denoted by $\mathrm{RL}(P_1, P_2, \ldots, P_n)$. This is the lattice invariant under the algebra of operators leaving the ranges $P_1\mathfrak{H}, P_2\mathfrak{H}, \ldots, P_n\mathfrak{H}$ invariant. We wish to represent this lattice as ranges of functions of the generators as in [5] for the case of single generator. For $a>0$, the set of all continuous concave nonnegative nondecreasing functions on $[0, a]$ will be denoted by $\mathbf{K}[0, a]$.

Theorem. *Let $P_1, P_2, \ldots, P_n$ be commuting positive operators on $\mathfrak{H}$ such that there is an orthogonal decomposition $\sum_{j=1}^{n} \oplus \mathfrak{H}_j$ of $\mathfrak{H}$, reducing for every P_i, such that*

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the restriction of P_j to the orthogonal complement $\sum_{j\neq i}\oplus\mathfrak{H}_j$ of $\mathfrak{H}_i$ is invertible, $i=1,2,\dots,n$. Then the following conditions on an operator range $\mathfrak{R}$ are equivalent:

(i) $\mathfrak{R}\in \mathrm{RL}(P_1, P_2, \dots, P_n)$,

(ii) $\mathfrak{R} = (\prod_j \varphi_j(P_j))\mathfrak{H}$, *for some* $\varphi_j\in \mathrm{K}[0, \|P_j\|]$, $j=1,2,\dots,n$.

Proof. Let $\mathscr{A}$ be the algebra of operators on $\mathfrak{H}$ that leave the ranges of $P_1, P_2, \dots, P_n$ invariant, and let $\mathfrak{R}$ be an $\mathscr{A}$-invariant operator range. Since the von Neumann algebra generated by $P_1, \dots, P_n$ and the projections onto $\mathfrak{H}_1, \mathfrak{H}_2, \dots, \mathfrak{H}_n$ is commutative and is contained in $\mathscr{A}$, $\mathfrak{R}$ is the range of some operator in the commutant of this commutative von Neumann algebra by a result of FOIAŞ [3, Lemma 8, p. 890]. It follows that $\mathfrak{R}=(\mathfrak{R}\cap\mathfrak{H}_1)+(\mathfrak{R}\cap\mathfrak{H}_2)+\dots+(\mathfrak{R}\cap\mathfrak{H}_n)$ and each $\mathfrak{R}\cap\mathfrak{H}_i$ is an operator range in the reflexive lattice generated by the range of $P_i|\mathfrak{H}_i$. By [6] Theorem 8, $\mathfrak{R}\cap\mathfrak{H}_i$ is the range of some operator of the form $\varphi_i(P_i|\mathfrak{H}_i)$, where φ_i is in $\mathrm{K}[0, \|P_i|\mathfrak{H}_i\|]$. Extend φ_i to all of $[0, \|P_i\|]$ by defining $\varphi_i(t)=\varphi_i(\|P_i|\mathfrak{H}_i\|)$ for all $t\in(\|P_i|\mathfrak{H}_i\|, \|P_i\|]$. Then φ_i is still a concave function and $\varphi_i(P_i)$ is defined. We claim that $\mathfrak{R}=(\varphi_1(P_1)\dots\varphi_n(P_n))\mathfrak{H}$.

To see the inclusion $\mathfrak{R}\subseteq(\varphi_1(P_1)\dots\varphi_n(P_n))\mathfrak{H}$, let $x\in\mathfrak{R}$. Then $x=x_1+x_2+\dots+x_n$, where $x_i\in\mathfrak{R}\cap\mathfrak{H}_i$, $i=1,2,\dots,n$. We note that if one of $\mathfrak{R}\cap\mathfrak{H}_i\neq\{0\}$, then $\mathfrak{R}\cap\mathfrak{H}_j\neq\{0\}$ for all j. Indeed, let $x_i\in\mathfrak{R}\cap\mathfrak{H}_i$, $x_i\neq 0$. For each j, let $x_j\in P_j\mathfrak{H}_j$, $x_j\neq 0$ (assuming $P_j\mathfrak{H}_j\neq\{0\}$, otherwise we can omit P_j from the discussion at the beginning). Define the operator $Ax=(x, x_i)x_j$ for $x\in\mathfrak{H}$. Then $A\mathfrak{H}\subseteq P_j\mathfrak{H}_j$. Thus $AP_k\mathfrak{H}\subseteq P_k\mathfrak{H}$, $k=1,2,\dots,n$ (since $P_k\mathfrak{H}_j=\mathfrak{H}_j$ for $j\neq k$). Therefore $A\in\mathscr{A}$. Thus, $A\mathfrak{R}\subseteq\mathfrak{R}$ and hence $P_j\mathfrak{H}_j\subseteq\mathfrak{R}$ for all $j=1,2,\dots,n$. In particular, $\mathfrak{R}\cap\mathfrak{H}_j\neq\{0\}$. For a fixed $i=1,2,\dots,n$, it is easy to see that $\varphi_i(P_i)|\mathfrak{H}_j$ is invertible for all $j\neq i$; and $(\prod_{j\neq i}(\varphi_j(P_j)|\mathfrak{H}_i)^{-1})x_i\in\mathfrak{R}\cap\mathfrak{H}_i$. Thus, there is a $y_i\in\mathfrak{H}_i$ such that $\varphi_i(P_i)y_i=(\prod_{j\neq i}(\varphi_j(P_j)|\mathfrak{H}_i)^{-1})x_i$. Let $y=y_1+\dots+y_n$. Then $(\prod_{j=1}^{n}\varphi_j(P_j))y=x$. Therefore $\mathfrak{R}\subseteq(\varphi_1(P_1)\dots\varphi_n(P_n))\mathfrak{H}$.

To see the opposite inclusion, let $y=(\varphi_1(P_1)\dots\varphi_n(P_n))x$, for some $x\in\mathfrak{H}$. Write $x=x_1+x_2+\dots+x_n$ where $x_i\in\mathfrak{H}_i$. Let $z_i=(\prod_{j\neq i}\varphi_j(P_j))x_i\in\mathfrak{H}_i$.

Then obviously, $y=\sum_{i=1}^{n}\varphi_i(P_i)z_i$ is an element of $\varphi_1(P_1)\mathfrak{H}_1+\dots+\varphi_n(P_n)\mathfrak{H}_n=(\mathfrak{R}\cap\mathfrak{H}_1)+\dots+(\mathfrak{R}\cap\mathfrak{H}_n)=\mathfrak{R}$. So $\mathfrak{R}\supseteq(\varphi_1(P_1)\dots\varphi_n(P_n))\mathfrak{H}$. Thus equality holds. This proves the implication (i)$\Rightarrow$(ii). For the converse we note that $(\prod_{j=1}^{n}\varphi_j(P_j))\mathfrak{H}=\bigcap_{j=1}^{n}(\varphi_j(P_j))\mathfrak{H}$, and each $\varphi_j(P_j)\mathfrak{H}$ is $\mathscr{A}$-invariant. The proof is thus complete.

In the special case of $\mathfrak{H}=L^2[0,1]$, we have more definite conclusions when the generators are some special multiplication operators. To simplify the statement, we introduce some notations. For a function f on $[0, 1]$, $Z(f)$ denotes the set of zeros of f. For a sequence of functions $f_1, f_2, \ldots, f_n$, $Z(f_1, f_2, \ldots, f_n)=Z(f_1)\cup \cup Z(f_2)\cup\ldots\cup Z(f_n)$. If G is an open set relative to $[0, 1]$, G is a disjoint union of open intervals together with perhaps one or both of $[0, \alpha)$ and $(\beta, 1]$ for some $\alpha, \beta\in(0, 1)$. A nonnegative continuous function is concave on G if the restriction to each component of G is concave (chords below graph). The set of all such functions will be denoted by $\mathbf{C}(G)$. For each $\varphi\in L^\infty[0, 1]$, the multiplication operator on $L^2[0, 1]$ induced by φ will be denoted by M_φ. The symbol x denotes the identity function on $[0,1]$, and 1 the constant function sending every $t\in[0, 1]$ to 1.

Corollary 1. $\mathrm{RL}\,(M_x, M_{1-x})=\{M_\varphi\mathfrak{H}\colon \varphi\in\mathbf{C}([0, 1]),\ Z(\varphi)\subseteqq\{0, 1\}\}$.

Proof. Let $\mathfrak{R}\in\mathrm{RL}\,(M_x, M_{1-x})$. Then by the proof of the above theorem $\mathfrak{R}=(M_\varphi M_{\psi^\sim})\mathfrak{H}$, where $\psi^\sim(t)=\psi(1-t)$, and φ, ψ are nonnegative, nondecreasing concave functions on $[0, 1]$. Since the functions φ and ψ are nonzero (assuming $\mathfrak{R}\neq\{0\}$) the restrictions $\varphi|[1/2, 1]$ and $\psi|[0, 1/2]$ are bounded from below, we may replace them by a constant functions, viz: the functions taking the constant values $\varphi(1/2)$ and $\psi(1/2)$ on $[1/2, 1]$ and $[0, 1/2]$ respectively. Then it is obvious that $M_\varphi M_{\psi^\sim}=M_{\varphi\psi^\sim}$ and $\varphi\psi^\sim$ is concave near the points 0 and 1. By replacing the restriction of $\varphi\psi^\sim$ to an interval $[\alpha, \beta]$, $\alpha, \beta\in(0, 1)$ by a suitable linear function, we may assume that $\varphi\psi^\sim$ is a concave function on all of $[0, 1]$. Thus, the inclusion $\subseteqq$ of the sets in the corollary holds. The opposite inclusion follows from a result of [4] (see [5, Theorem B]).

With a suitable modification, the above proof can be adapted to a proof of the following

Corollary 2. *Let* $f_1, f_2, \ldots, f_n$ *be nonnegative continuous functions on* $[0, 1]$ *such that* $Z(f_1), \ldots, Z(f_n)$ *are pairwise disjoint. Then* $\mathrm{RL}\,(M_{f_1}, \ldots, M_{f_n})= =\{M_\varphi\mathfrak{H}\colon \varphi\in\mathbf{C}([0; 1]/Z(f_1, \ldots, f_n)),\ Z(\varphi)\subseteqq Z(f_1, \ldots, f_n)\}$.

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Generalized projections for hyponormal and subnormal operators

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0. Sufficient conditions for the existence of certain invariant subspaces of a pure hyponormal operator, T, are obtained. In case T is also subnormal these subspaces are even reducing. In particular, a pure subnormal operator T is shown to be reducible in case $\sigma(T)$ is bisected by the imaginary axis and if, in addition, that part of $\sigma(T)$, which has a projection onto the real axis lying in the absolutely continuous support of $\operatorname{Re}(T)$, is sufficiently sparse near the imaginary axis.

1. Let T be a pure hyponormal operator on the separable Hilbert space $\mathscr{H}$. Thus, $T^*T \geqq TT^*$ and there is no nontrivial reducing subspace of T on which T is normal. In particular, $\sigma_p(T)$ is empty. Let C be a rectifiable, positively oriented, simple closed curve separating the spectrum $\sigma(T)$; thus, $\sigma(T)$ intersects both $\operatorname{int} C$ and $\operatorname{ext} C$, the interior and exterior, respectively, of C. It may be noted that, in general, the set $C \cap \sigma(T)$ may have positive (arc length on C) measure. There will be proved the following

Theorem 1. *Let T be purely hyponormal on $\mathscr{H}$ and satisfy*

$$\int_C \|(T-t)^{-1}x\|\,|dt| < \infty, \quad x \in \mathscr{X}, \tag{1.1}$$

where $\mathscr{X}$ is a set dense in $\mathscr{H}$. Then there exists a linearly independent pair of invariant subspaces $\mathscr{M}_i$ and $\mathscr{M}_e$ of T for which $\mathscr{H} = \mathscr{M}_i \vee \mathscr{M}_e$ and

$$\sigma(T|M_i) = (\sigma(T) \cap \operatorname{int} C)^- \quad \text{and} \quad \sigma(T|M_e) = (\sigma(T) \cap \operatorname{ext} C)^-. \tag{1.2}$$

Further, in case T is also subnormal, $\mathscr{M}_i$ and $\mathscr{M}_e$ are reducing subspaces of T on $\mathscr{H} = \mathscr{M}_i \oplus \mathscr{M}_e$.

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Proof of Theorem 1. Define the "projection" P_C by

$$P_C x = -(2\pi i)^{-1} \int_C (T-t)^{-1} x\, dt, \quad x \in \mathscr{X}, \tag{1.3}$$

so that, by (1.1), $(P_C x, y) = -(2\pi i)^{-1} \int_C ((T-t)^{-1}x, y)\, dt$ is defined as a Lebesgue integral for any x in $\mathscr{X}$ and y in $\mathscr{H}$. Clearly, it may be assumed that $\mathscr{X}$ is a linear manifold. If $\mathscr{M}_i$ and $\mathscr{M}_e$ are the respective closures of the linear manifolds $P_C\mathscr{X}$ and $(I-P_C)\mathscr{X}$, then, in particular, $\mathscr{M}_i$ and $\mathscr{M}_e$ are hyperinvariant subspaces of T. Relation (1.2) now follows from a proof analogous to that of [5], pp. 13—14, and will be omitted. (The set L and the curve C_R of [5] correspond to the present $\mathscr{X}$ and C.) A crucial part of the argument in [5] is that the set $\{x\colon \sigma_T(x) \subset \sigma\}$ is a subspace whenever σ is any nonempty compact subset of the plane and $\sigma_T(x)$ is the local spectrum of any vector x in $\mathscr{H}$. This result is due to STAMPFLI [7] (p. 288, see also p. 295) in case T^* has no point spectrum and to RADJABALIPOUR [6] in the general case.

Also, $\mathscr{M} \equiv \mathscr{M}_i \cap \mathscr{M}_e = \{0\}$. For if $\mathscr{M} \neq \{0\}$, then $\sigma(T|\mathscr{M}) \subset C$ and hence $\sigma(T|\mathscr{M})$ has (area) measure zero. Consequently (cf. [3]), $\mathscr{M} \neq \{0\}$ is a reducing space of T on which T is normal, in contradiction to the hypothesis that T is purely hyponormal.

Before completing the proof of the remainder of Theorem 1 when T is subnormal, there will be proved the following

Lemma. *If T is a pure hyponormal operator satisfying* (1.1) *then*

$$\bar{t} \notin \sigma_p(T^*) \quad \text{for} \quad t \in C - Z, \tag{1.4}$$

where Z is a subset of C of (arc length) measure zero. In case T is also subnormal on $\mathscr{H}$ with the minimal normal extension $N = \int z\, dE_z$ on $\mathscr{K} \supset \mathscr{H}$, then

$$E(C) = 0. \tag{1.5}$$

Proof of Lemma. As noted above, since T is purely hyponormal, $\sigma_p(T)$ is empty. Further, by (1.1), for x fixed in $\mathscr{X}$ and for almost all t on C, $y_t = (T-t)^{-1}x$ is defined. Thus, for each x in $\mathscr{X}$, there exists a set $Z(x)$ on C of arc length measure zero and with the property that $x \in R(T-t)$ for $t \in C - Z(x)$. If $\{x_1, x_2, \ldots\}$ is a countable subset of $\mathscr{X}$ which is dense in $\mathscr{H}$ then $Z = \bigcup_{k=1}^{\infty} Z(x_k)$ is also a zero set. Thus, $\mathscr{R}(T-t)$ is dense in $\mathscr{H}$ for all t in $C-Z$ and, in particular, relation (1.4) follows.

Next, relation (1.5) will be established when T is also subnormal. Let x be any vector in $\mathscr{X}$. For t in $C-Z(x)$ one has $y_t = (T-t)^{-1}x$, hence $x = (T-t)y_t = (N-t)y_t$, and so $(T-t)^{-1}x = \int_{\sigma(N)} (z-t)^{-1} dE_z x$. (Note that $E(\{t\})x = 0$.) Consequently, for any u in $\mathcal{K}$, an application of the Schwarz inequality and (1.1) yields

$$\int_C \Big(\int_{\sigma(N)} |z-t|^{-1} |d(E_z x, u)| \Big) |dt| \leqq \int_C \Big(\int_{\sigma(N)} |z-t|^{-2} \, d\|E_z x\|^2 \Big)^{1/2} \Big(\int_{\sigma(N)} d\|E_z u\|^2 \Big)^{1/2} |dt| =$$

$$= \Big(\int_C \|(T-t)^{-1} x\| \, |dt| \Big) \|u\| < \infty.$$

(Note that $\int_C = \int_{C-Z(x)}$.) Consequently, in view of Fubini's theorem,

$$\int_{\sigma(N)} \Big(\int_C |t-z|^{-1} |dt| \Big) |d(E_z x, u)| < \infty. \tag{1.6}$$

However, $\int_C |t-z|^{-1} |dt| = \infty$ for all z on C. (In fact, otherwise, there would exist some z^* on C for which $\int_C |t-z^*|^{-1} |dt| < \infty$. However, z^* is not an atom of the measure $|dt|$ on C and so $1 \leqq \int_{t^*}^{z^*} |t-z^*|^{-1} |dt| \to 0$ as $t^* \to z^*$, a contradiction.) Hence, by (1.6), $(E(C)x, u) = 0$ for u arbitrary in $\mathcal{K}$ and x arbitrary in $\mathcal{X}$. Thus, for x in $\mathcal{X}$, $E(C)x = 0$ and hence also $0 = N^{*k} E(C) x = E(C) N^{*k} x$ $(k = 0, 1, 2, \ldots)$. Since $\mathcal{X}$ is dense in $\mathcal{H}$ and N is the minimal normal extension of T, the linear span of $\{N^{*k} \mathcal{X}\}$ $(k = 0, 1, 2, \ldots)$ is dense in $\mathcal{K}$ and (1.5) follows. This completes the proof of the Lemma.

The assertion of Theorem 1 when T is purely subnormal now follows from the above Lemma and Corollary 1 of [4], p. 106. In fact, only the hypothesis (5.1) of Corollary 1, corresponding to (1.1) of the present paper, is need to ensure the validity of the assertion of Corollary 1. Indeed, the remaining hypotheses there, namely, that $\{z \in C: \bar{z} \in \sigma_p(T^*)\}$ has measure zero and that $E(C) = 0$, are consequences of (1.1), in view of the Lemma. For completeness, however, an alternate proof of the assertion of Theorem 1 when T is purely subnormal will be given below.

By the Lemma, $E(C) = 0$, and so for x in $\mathcal{X}$ and u in $\mathcal{K}$, one has, by Fubini's theorem,

$$(P_C x, u) = \int_{\sigma(N)-C} \Big[-(2\pi i)^{-1} \int_C (z-t)^{-1} \, dt \Big] \, d(E_z x, u) = \int_{\sigma(N)-C} \Phi(z) \, d(E_z x, u),$$

where $\Phi(z)$ is the characteristic function of $\operatorname{int} C$. Thus, $(P_C x, u) = (E(\operatorname{int} C)x, u)$ for all u in $\mathcal{K}$ and so $P_C x = E(\operatorname{int} C) x$ for all x in $\mathcal{X}$. Let P denote the orthogonal projection $P: \mathcal{K} \to \mathcal{H}$. Since the (orthogonal projection) $E(\operatorname{int} C)$ is bounded on $\mathcal{K}$ and $E(\operatorname{int} C) x = P_C x \in \mathcal{H}$ for x in $\mathcal{X}$, then clearly $E(\operatorname{int} C) P = PE(\operatorname{int} C) P (= PE(\operatorname{int} C))$. Thus $E' = E(\operatorname{int} C) | \mathcal{H}$ is an orthogonal projection and $E' | \mathcal{X} = P_C$. Since $TP_C x = P_C T x$ for x in $\mathcal{X}$ and $\mathcal{X}$ is dense in $\mathcal{H}$, then T commutes with E'. Further, it is clear that $E' \mathcal{H} = \mathcal{M}_i$, and so the spaces $\mathcal{M}_i$ and $\mathcal{M}_e$ defined earlier reduce T and $\mathcal{H} = \mathcal{M}_i \oplus \mathcal{M}_e$. This completes the proof of Theorem 1.

2. For use below, note that if T is purely hyponormal then $\operatorname{Re}(T)$ is absolutely continuous; see [2], p. 46.

Theorem 2. *Let T be purely subnormal on $\mathscr{H}$ and suppose that $\sigma(T)$ intersects both the right and left open half planes $R=\{z: \operatorname{Re}(z)>0\}$ and $L=\{z: \operatorname{Re}(z)<0\}$. In addition, let*

$$\int_\alpha t^{-2}F(t)\,dt < 2\pi, \tag{2.1}$$

where α is the absolutely continuous support of $\operatorname{Re}(T)$ and $F(t)$ is the linear measure of the vertical cross section $\sigma(T)\cap\{z: \operatorname{Re}(z)=t\}$ of $\sigma(T)$. Then there exist subspaces $\mathscr{M}_R$ and $\mathscr{M}_L$ of $\mathscr{H}$ reducing T, satisfying $\mathscr{H}=\mathscr{M}_R\oplus\mathscr{M}_L$ and

$$\sigma(T|\mathscr{M}_R) = (\sigma(T)\cap\mathscr{R})^- \quad \text{and} \quad \sigma(T|M_L) = (\sigma(T)\cap L)^-.$$

Theorem 2 follows from Theorem (*) and its proof in [5] and from Theorem 1 above. In fact, let C denote the positively oriented boundary of the semicircular disk $\{z: \operatorname{Re}(z)>0, |z|<r\}$, where $r>0$ is chosen so large that $\sigma(T)\subset\{z: |z|<r\}$. It was shown in [5] that $\mathscr{X}$ of Theorem 1 above can now be chosen so as to contain the range of $E^A(\beta)$ where $\{E^A\}$ is the spectral family of $A=\operatorname{Re}(T)$ and β is any Borel set of the real line whose closure does not contain 0. This completes the proof of Theorem 2.

For other sufficient conditions ensuring the reducibility of a subnormal operator see the references in Conway [1], pp. 299—300.

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Approximation Theory, Tampa, Proceedings of the seminar held in Tampa, Florida, 1985—1986. Edited by E. B. Saff (Lecture Notes in Mathematics, 1287), VI+228 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

The Institute for Constructive Mathematics at the University of South Florida had its beginnings in 1985. The papers contained in these Proceedings of the Tampa Approximation Seminar prove the quality and variety of research activities conducted of the Institute and show the individual interest of the visitors to the Institute during the academic year 1985—1986.

Contents: P. R. Graves-Morris and J. M. Wilkins, A fast algorithm to solve Kalman's partial realisation problem for single input, multi-output systems; A. Knopfmacher and D. Lubinsky, Analogues of Freud's conjecture for Erdős type weights and related polynomial approximation problems; A. L. Levin and E. B. Saff, Some examples in approximation on the unit disk by reciprocals of polynomials; D. Lubinsky and E. B. Saff, Strong asymptotics for L_p extremal polynomials $(1<p\leqq\infty)$ associated with weights on $[-1, 1]$; L. S. Luo and J. Nuttall, Asymptotic behavior of the Christoffel function related to a certain unbounded set; H. N. Mhaskar, Some discrepancy theorems; J. Palagallo-Price and T. E. Price, Properties of projections obtained by averaging certain polynomial interpolants; L. Reichel, Boundary collocation in Fejér points for computing eigenvalues and eigenfunctions of the Laplacian; B. Shekhtman, On the geometry of real polynomials; H. Stahl, A note on a theorem by H. N. Mhaskar and E. B. Saff: "Where does the sup norm of a weighted polynomial live? (a generalization of incomplete polynomials)"; H. Stahl, Existence and uniqueness of rational interpolants with free and prescribed poles; J. Waldvogel, Zero-free disks in families of analytic functions.

J. Németh (Szeged)

N. H. Bingham—C. M. Goldie—J. L. Teugels, Regular Variation (Encyclopedia of Mathematics and its Applications, Vol. 27), XIX+491 pages, Cambridge University Press, Cambridge—London—New York—New Rochelle—Melbourne—Sydney, 1987.

The publication of this book is a major mathematical event.

The theory of regularly varying functions was initiated by Jovan Karamata in 1930. A positive measurable function f defined on a half-line (a, ∞) with $a>0$ is called regularly varying (at ∞) of index $\varrho\in\mathbf{R}$ if $(*)$ $f(\lambda x)/f(x)\to\lambda^{\varrho}$, as $x\to\infty$, for each $\lambda>0$. (Measurability can usually be replaced by the Baire property for most of the basic results.) If R_ϱ denotes the class of all such functions then the functions in R_0 are called slowly varying, and for $f\in R_\varrho$ we have $f(x)=x^{\varrho}l(x)$ with some $l\in R_0$. The notion of regular variation at zero rather than ∞, and then at any other point, is straightforward, every result at ∞ has a corresponding counterpart.

Karamata himself used his basic results on regular variation in Tauberian theorems and the theory was further developed by his Yugoslav School. As the authors write in their preface "The great potential of regular variation for probability theory and its applications was realised by William Feller, whose book [*An Introduction to Probability Theory and its Applications*, Vol. II, Wiley, New York, 1968 and 1971] did much to stimulate interest in the subject. Another major stimulus — again from a probabilistic viewpoint — was provided by Laurens de Haan in his 1970 thesis [*On Regular Variation and its Applications to the Weak Convergence of Sample Extremes*, Math. Centre Tract 32, Amsterdam], while Eugene Seneta gave a treatment of the basic theory of the subject in his monograph of 1976 [*Functions of Regular Variation*, Lecture Notes in Mathematics 506, Springer, Berlin]."

The first chapter is the essential Karamata theory (pp. 1—60). This is based on results like the uniform convergence theorem (stating that if $l\in R_0$ then the convergence in $(*)$ above holds uniformly on each compact λ-set in $(0, \infty)$), the representation theorem (stating that if $l\in R_0$ then $l(x)=c(x)\exp\left\{\int\limits_a^x b(t)\,dt/t\right\}$, $x\geqq a$, for some $a>0$, where $c(x)$ is measurable and $c(x)\to c\in(0, \infty)$, $b(x)\to 0$ as $x\to\infty$), the characterisation theorem (stating that if for a positive measurable f relation $(*)$ above holds for a λ-set of positive measure and with an unspecified limiting function $g(\lambda)$ on the right side, then it holds for all $\lambda>0$ and necessarily $g(\lambda)=\lambda^\varrho$, $\lambda>0$, for some $\varrho\in\mathbf{R}$) and the absolutely basic Karamata theorem, with many variants and refinements, stating very roughly that $f\in R_\varrho$ if and only if certain integral functions of f behave near ∞ as if $f(x)$ were constant times x^ϱ. There are many variants, versions or extensions of everything, monotone equivalents, asymptotic inverses and conjugates and various related notions and properties are discussed extremely intelligently together with special cases such as smooth variation and monotonicity with first applications as Karamata's Tauberian theorem for Laplace—Stieltjes transforms. Regularly varying sequences receive a separate discussion.

Chapter 2 (Further Karamata theory, pp. 61—126) is devoted to the investigation of the classes *ER* of extended regularly varying functions f and *OR* of O-regularly varying functions f (of positive measurable or Baire functions) for which $\lambda^d\leqq f_*(\lambda)\leqq f^*(\lambda)\leqq\lambda^c$, $1\leqq\lambda<\infty$, for some c and d, and for which $0<f_*(\lambda)\leqq f^*(\lambda)<\infty$, $1\leqq\lambda<\infty$, respectively, where, as $x\to\infty$, $f_*(\lambda)=\liminf f(\lambda x)/f(x)$ and $f^*(\lambda)=\limsup f(\lambda x)/f(x)$, and to related classes. These are functions of bounded or positive increase or decrease, the classes $R_{-\infty}$ and R_∞, quasi-monotone and near-monotone functions, various subclasses of R_0, functions with Pólya peaks, Beurling slow variation, self-neglecting and self-controlled functions, to mention a few for those who know what these are or have the right sense of imagination.

Taking logarithms, relation $(*)$ above, with a general limiting function, can be written as $\varphi(\lambda x)-\varphi(x)\to h(\lambda)$. Chapter 3 (de Haan theory, pp. 127—192) provides the extended modern theory when the left side here is replaced by the ratio $(\varphi(\lambda x)-\varphi(x))/\psi(x)$, where ψ is some auxiliary function, with the corresponding O-, o-, E- and other versions or extensions.

Chapter 4 (Abelian and Tauberian theorems, pp. 193—258) and Chapter 5 (Mercerian theorems, pp. 259—283) together constitute a virtually complete and beautifully constructed account of that part of classical analysis which is defined by these names, obtained by full-force application of the results in the first three chapters, with many far-reaching extensions and complements. All integral transforms of convolution type and all matrix transforms receive detailed attention where some form or other of regular variation plays some rôle in the result. These five chapters form a completely integrated and unrivalled unit which will be difficult to surpass before the twenty-second century.

And now come two little pearls. The first is Chapter 6 (pp. 284—297) on applications to analytic number theory (partitions, the prime number theorem and the order of sums of multiplicative functions), while the second is Chapter 7 (pp. 298—325) with applications to complex analysis concerned mainly with the growth of entire functions.

The last Chapter 8 (Applications to probability theory, pp. 326—422) is a masterpiece in itself. It offers a fantastically rich field of applications of regular variation and here we restrict this review to listing section headings: tail-behaviour and transforms, infinite divisibility, stability and domains of attraction, further central limit theory, self-similarity, renewal theory, regenerative phenomena, relative stability, fluctuation theory, queues, occupation times, branching processes, extremes, records, maxima and sums.

Six short appendices (pp. 423—444) with indications of further fields of applications and technical necessities complete the main body of the text.

However, the remaining forty-seven pages are very important to the excellence of the book. There is a list of references of 645 different items, each one supplied with a list of all the page numbers where it is cited. Then there is an index of named theorems. This is followed by a seven-page comprehensive index of notation and a sixteen-page very detailed general index concludes this encyclopedia of regular variation. These, together with the extremely clever structuring of the material into chapters, sections and subsections, the page headings and the nine-page table of contents make the book very easily usable. This is just one sign of the authors' sense of scholarship. Throughout, all Serbian, Croatian, French, German, Hungarian, Russian or Scandinavian accent marks are proper and are at their own place. All second- or third-named authors have a separate entry in the bibliography with a reference to the first-named author. There are *no* misprints in this book. (The three trivial typos this reviewer found were probably left intentionally by the three authors: to satisfy reviewers who believe that perfect works are impossible.)

This is a perfect work of art in every sense of the word. The language is perfect, the taste is perfect, the typography is perfect and, above all, the mathematics is perfect. The amount of knowledge brought together and of the work that went into this book is truly fascinating. There should be dozens of mathematicians sweating on their problems at this late hour of the night, or early hour of the morning, all over the world who would only need to look up page x of it and exclaim 'heureka'. Many-many results are new: brand new or completely polished versions of older results, when, needless to say, the authors always give the original sources just as when they follow somebody else in the proof even if they greatly simplified and polished that proof. And they are never tired to do so, even when they give five different proofs for the uniform convergence theorem in Chapter 1. In a sense everything is new here: every word of the subject is redigested and the whole comprehensive theory and its many applications are unified and integrated. The writing style is very modest, the mathematical and general intellect shines through, each page ticks, it is sheer delight to read the book.

It is a classic right away. A book for all seasons.

Sándor Csörgő (Szeged)

H. G. Dales—W. H. Woodin, An Introduction to Independence for Analysts (London Mathematical Society Lecture Note Series, 115), XIII+241 pages, Cambridge University Press, Cambridge—New York—Melbourne, 1987.

Let X be an infinite compact space and let $C(X)$ be the Banach algebra of all continuous functions. A famous question, first discussed by Kaplansky in 1948, asks if every algebra norm on $C(X)$ is necessarily equivalent to the given uniform norm. In 1976, using the continuum hypothesis (CH) H. G. Dales and J. R. Esterle, independently of each other, showed that there are algebra norms on $C(X)$ which are not equivalent to the uniform norm. More surprisingly, also in 1976 R. M. Solovay and W. H. Woodin proved that the existence of such norms is independent of the basic axioms of set theory (ZFC).

As the authors write in the preface: "The purpose of this book is to explain what it means for a proposition to be independent of set theory, and to describe how independence results can be proved by the technique of forcing." A full proof of the independence of (CH) from (ZFC) is given and the first proof of the theorem of Solovay and Woodin, accessible not only to logicians but intelligible also to analysts, is provided here. The authors include a discussion of Martin's Axiom, "which can be used to establish independence results without the necessity of knowing any of the technicalities of forcing".

This book offers analysts a good possibility to get acquainted with the powerful technique of forcing and with its application in the resolution of a deep problem in analysis. It can be recommended also to students of set theory as an introductory work.

László Kérchy (Szeged)

Dependence in Probability and Statistics. A Survey of Recent Results (Oberwolfach, 1985). Edited by E. Eberlein and M. S. Taqqu (Progress in Probability and Statistics, Vol. 11), XI+473 pages, Birkhäuser, Boston—Basel—Stuttgart, 1986.

This is a fine collection of a large number of excellent survey papers and a smaller number of equally excellent research papers on various kinds of dependent random variables, concentrating mainly on limit theorems.

Section 2 is on various mixing conditions with papers by R. Bradley, M. Peligrad, W. Philipp, M. Denker, C. M. Goldie and G. J. Morrow and by N. H. Bingham, while Section 3 contains the papers by P. Gaenssler and E. Haeusler and by E. Eberlein on martingale types of dependence. Section 4 carries the articles by A. R. Dabrowski, E. Waymire and by R. H. Burton and E. Waymire on positive and Gibbs dependence, the papers by F. Avram and M. S. Taqqu and by R. A. Davis and S. Resnick on moving averages in independent variables belonging to the domain of attraction of a non-normal stable law constitute Section 5.

Advances in dependent extreme value theory are sketched in the papers of G. O'Brien, J. Hüsler, and W. Vervaat in Section 6. Finally, Section 1 is on the recent hot topic of long-range dependence with papers by T. C. Sun and H. C. Ho, L. Giraitis and D. Surgailis, M. S. Taqqu and J. Levy, M. Maejima, N. Kôno, H. Dehling, the last paper being here a bibliographical guide by M. S. Taqqu to some 286 items. The preface of the two editors provides an intelligent guide to the collection itself, which will probably be indispensable for anyone with dependencies.

Sándor Csörgő (Szeged)

Luc Devroye, A Course in Density Estimation (Progress in Probability and Statistics, Vol. 14), XIX+183 pages, Birkhäuser, Boston—Basel—Stuttgart, 1987.

This seems like a most enjoyable book on density estimation using the L_1 criterion. The larger part of it appears as a lighter edition of the author's research monograph with L. Győrfi, *Nonparametric Density Estimation: The L_1 View,* Wiley, New York, 1985. It is based on the notes of a course the author has taught at Stanford University in 1986. Indeed, it is a first-class textbook for a graduate course with many examples, figures and exercises. The author should indeed be commended for having made the results of a very fresh and sophisticated research available for a wide public in just a little more than no time at all. In comparison to the earlier research monograph, however, some new material is also found in the present book. (Indeed, the opposite would have been very much uncharacteristic for the author.) These are chapters on robustness, minimum distance estimation, estimation of monotone densities, and on relative stability. The book can be enthusiastically recommended to every statistician: students, instructors, research workers, and the layman for that matter.

Sándor Csörgő (Szeged)

Differential Geometry, Calculus of Variations, and their Applications, Edited by G. M. Rassias and Th. M. Rassias (Lecture Notes in Pure and Applied Mathematics, Vol. 100), XIII+521 pages, Marcel Dekker, Inc., New York—Basel, 1985.

This book contains a series of papers dedicated to the memory of Leonhard Euler (1707—1783) on 200th anniversary of his death. His discoveries and significant contributions were devoted to every area of the mathematical sciences that existed in his day: calculus of variations; differential geometry of surfaces; the geometric origins of topology and combinatorics; particle, rigid body and celestial mechanics etc. The pure and applied aspects of mathematics and mechanics were not separated yet in that time. Lagrange, Laplace and Gauss were influenced directly by Euler's work, thus his activity belongs to the foundaments of the modern science. The papers in this volume are written by the authorities of the fields: dynamical systems, differential topology and geometry, calculus of variations, differential equations, control theory, and history and philosophy of sciences.

Péter T. Nagy (Szeged)

H. Edelsbrunner, Algorithms in Combinatorial Geometry, ETACS Monographs on Theoretica Computer Science, XV+423 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

Computational geometry is a rapidly expanding part of mathematics today and several books have been published on this topic. This is not "just another book" but certainly one of the best ones. The theory emerged as the unification of computational technics and results of combinatorial geometry, and this book follows this line. The author's aim was "to demonstrate that computational and combinatorial investigations in geometry are doomed to profit from each other". According to this intention the book is divided into three parts.

The first part is devoted to the combinatorial geometry. It contains the fundamental geometric structures (arrangements of hyperplanes, configurations of points, convex polytopes, Voronoi diagrams), the main combinatorial tools and basic results of the complexity of families of cells (the Euler relation, the Dehn—Sommerville equations, an asymptotic version of the upper bound theorem).

The second part contains the computational methods, the organization of data structures of arrangements and the most important geometric algorithms (construction of convex hulls, linear programming, point location search).

The third part presents applications of the first two parts proving that the combination of these two fields results a really fruitful method.

Each chapter contains a problem section including exercises as well as research problems and the chapters end with a complete and updated bibliographical notes.

The book can be useful to specialists as a reference book but it is also recommended to everybody interested in the present advances in computational geometry.

J. Kincses (Szeged)

F. Forgó, Nonconvex Programming, 188 pages, Akadémiai Kiadó, Budapest, 1988.

Nonconvex programming deals with the class of mathematical programming problems in which a local maximum-point is not necessarily a global maximum-point. This book is devoted to provide a survey of the basic research directions of the nonconvex programming except its two major fields, the integer programming and the global optimization which are treated in a number of excellent monographs.

The book consists of ten chapters. The first three ones comprise such topics and techniques as optimality conditions, nonlinear duality, convex and concave envelopes of functions, direct and implicit enumerations, branch and bound method, different cuts which will be used in subsequent chapters.

Chapter 4 deals with the problem of maximizing a quasi-convex function over a polytope. Cutting-plane methods based on different kinds of cuts such as convexity, polaroid and shallow cuts are given for solving the problem in question. For the case of convex objective function the method of Falk and Hoffmann is presented. This part ends with the treatment of the Tuy—Zwart method.

Chapter 5 studies the problem of maximizing a linear objective function with convex inequality constraints, and an indirect cutting-plane algorithm is discussed.

The general case is studied in Chapter 6, where a continuous function is to be maximized over a compact subset of the n-dimensional Euclidean space. For solving it a branch and bound algorithm developed by Horst is presented, then some bounding techniques are discussed. Finally, the special case of separable objective function is investigated.

Chapter 7 is devoted to the nonconvex quadratic programming problems. Such methods are presented which more or less utilize the quadratic nature of the objective function.

A special nonconvex problem, the fixed charge problem, and some methods of solution are investigated in Chapter 8.

Chapter 9 deals with techniques for converting constrained problems to unconstrained ones and gives an explicit formula for the optimal solution of a nonconvex programming problem in terms of a multiple integral.

Finally, Chapter 10 contains a partition algorithm to decompose the nonconvex programming problem.

The book is well-written. The material is well-organized, the proofs are clear, a subject index helps the reader. It may be recommended to mathematicians, operation researchers, and computer scientists.

B. Imreh (Szeged)

S. Gallot—D. Hulin—J. Lafontaine, Riemannian Geometry (Universitext), XII+248 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

It is a great fun to read this book. The authors had found the ideal rate of abstractions and examples. When a new definition or theorem occurs the reader will meet a detailed recurrent study of the most important examples of Riemannian geometry like spheres, tori, projective spaces, etc. At the same time, throughout the book there are several exercises (the solutions of most of them are given at the end of the book) to help to understand the text.

The book is divided into five chapters. The first one is a quick introduction to differential manifolds. The next two chapters contain the basics of Riemannian geometry until Myer's and

Milnor's theorems. Chapter IV deals with analysis on manifolds and Chapter V is about Riemannian submanifolds.

Summing up, this is a modern, well built and useful book, and we warmly recommend it to all who need a good introduction to Riemannian geometry.

Árpád Kurusa (Szeged)

M. Göckeler—T. Schücker, Differential geometry, gauge theories, and gravity (Cambridge monographs on Mathematical physics), XII+230 pages, Cambridge University Press, New York—New Rochelle—Melbourne—Sydney, 1987.

This book is an introduction to those concepts of differential geometry which are fundamental for applications in elementary particle theory and general relativity. On the mathematical side, the only prerequisites are linear algebra and real analysis. The physical part of the book is essentially self-connected, but it is useful if the reader is already motivated by some knowledge of Yang—Mills theory, general relativity, and the Dirac equation.

The first three chapters contain an elementary account of differential forms in $\mathbf{R}^n$. This machinery is used then to reinterpret and rewrite basic quantities and equations of Yang—Mills theory and general relativity in geometric, coordinate-free terms. Next, the reader is acquainted with the notion and some applications of the Lie derivative. This is followed by chapters providing the rudiments of manifolds and Lie groups.

In Chapter 9 the authors present an introduction to fiber bundles and connections on them. This is a topic of growing importance in applications. The following chapter illustrates the theory on the examples of the Dirac monopole, the 't Hooft—Polyakov monopole, Yang—Mills and gravitational instantons.

Chapter 11 treates the algebraic (Clifford algebra, spinor representations) and analytic (Dirac operator, spin structures) aspects of the Dirac equation. The concept of Kähler fermions is also touched upon here. The next chapter is devoted to a subject of more advanced character, to the algebraic approach to anomalies. The final sections contain some background material on anomalous graphs.

This book is intended for graduate students in theoretical physics in the first place. The reviewer warmly recommends it also to everybody else searching for a well written, elementary introduction to modern differential geometry with emphasis on applications in particle theory and relativity.

László Fehér (Szeged)

Lj. T. Grujic—A. A. Martynyuk—M. Ribbens-Pavella, Large Scale Systems Stability under Structural and Singular Perturbations (Lecture Notes in Control and Information Sciences, 92), XVI+366 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

Almost one hundred years ago, 1892, A. M. Lyapunov founded the mathematical stability theory in his famous doctoral dissertation. Previously stability concepts had been used only for mechanical systems. He has not only formulated the abstract definitions of stability concepts for arbitrary differential systems but established methods of investigation of these properties. One of them, the so-called direct method is suitable for finding conditions of stability properties via the system state differential equation without use of its solutions. This method has been proved to be extremely useful not only in mechanics but in many fields of the applications of differential equations such as control theory, reaction kinetics, population dynamics, biology and so on.

These lecture notes, which are a revised and completed version of their original Russian edition, are devoted to a recently developed branch of stability theory, to large scale systems stability. It is also based upon Lyapunov's direct method.

The first two chapters give an up-to-date survey on the state of Lyapunov's direct method. It was an excellent idea to start some sections with citations of the original definitions of stability concepts and fundamental theorems from Lyapunov's work. The reader can follow the arch of the one hundred years' development realizing that Lyapunov's original theorems are important and actual even today. Chapter I entitled Outline of the Lyapunov Stability Theory in General gives new versions of the definitions of stability concepts and theorems involving the earlier generalizations. The absolute stability is also treated.

Chapter II (Comparison Systems) contains the theory and application of the comparison method with scalar, vector and matrix functions. (The theory of comparison matrix functions initiated by A. A. Martynyuk was available earlier only in papers.)

The second part of the book is devoted to large-scale systems. The main idea here is to decompose the whole system into interconnected subsystems and then to find an aggregation form of the system yielding conditions under which the desired property of the original system can be deduced from the same properties of its interconnected subsystems and from qualitative properties of their interactions.

Chapter V (Large-Scale Power Systems Stability), which is essentially revised and completed in comparison with the original Russian edition, gives a good example for the process of mathematical modelling from the introduction of the physical problem, through the mathematical formulation and treatment until the interpretation of the results.

This book — which should be found on the book shelf of every mathematician, engineer, and any other user of mathematics interested in stability theory — is worthy of celebrating the oncoming hundredth anniversary of the publication of Lyapunov's fundamental work.

L. Hatvani (Szeged)

Jack Carl Kiefer, Introduction to Statistical Inference, Edited by G. Lorden (Springer Texts in Statistics), VIII+334 pages, 60 illustrations, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1987.

This book is unique and is best in its kind. It gives a systematic development of decision-theoretic statistics, and it does this as a first course in mathematical statistics. It is based upon lecture notes of the late Professor Kiefer, one of the great masters of the subject to be compared only to Neyman and Wald, on whose work he builds here. So this is a posthumus book and it would have been a very great loss to the whole international statistical community if these notes had remained only in the privileged possession of those individuals who were fortunate enough to be around Cornell where Kiefer has developed them. It is a gift to all of us. The editor and the publisher should be thanked for making it available.

The first three short chapters (Introduction, Specification of a statistical problem, Classifications of statistical problems; pp. 1—30) introduce the basic decision-theoretic notions such as decision or procedure, loss function, operating characteristic, risk function and admissibility. Chapter 4 (Some criteria for choosing a procedure; pp. 31—80) explains the Bayes, the minimax and the unbiasedness criteria, and gives the essentials on randomized procedures and the methods of maximum likelihood and moments. Following the important Chapter 5 (Linear unbiased estimation; pp. 81—136) concentrating on the general linear model, least squares, orthogonalization and the Gauss—Markov Theorem, the whole Chapter 6 (pp. 137—157) is devoted to sufficiency.

The criteria of completeness, unbiasedness, sufficiency, invariance and asymptotic efficiency are discussed at length in Chapter 7 (pp. 158—245) in the context of point estimation, where more on minimax procedures and maximum likelihood are naturally found. Chapter 8 (pp. 246—286) is on hypothesis testing with less than twenty pages on "common normal theory tests", and the main body of the book concludes with Chapter 9 (pp. 287—311) on confidence intervals. Three short appendices, a list of fifteen references and an index complete the volume.

What is so special in such a book? *It is the lucidity of the mind and, as a result, the simplicity* of the language. Every sentence has a clear meaning (and this in itself would be sufficient to make the book unique) and Kiefer always means something. *Every* single-minded direction gets its share from him, sometimes in rather sharp terms, thus those who decide to cite Kiefer against something for their own benefit should be careful enough to leaf one or two before they do so. This is the work of a thinker. With the possible exception of Charles Stein alone, every living statistician will find something interesting or new in this book. And, at the same time, this is a textbook of introductory statistics for (good) students with minimal mathematical background but with a necessary maturity, seriousness and interest. This is achieved by a very large amount of examples and homework problems with fascinating notes and suggestions from the author. Instructors with the necessary characteristics just listed for students will want to have a copy of the book, independently of the nature of the statistics course they teach.

Sándor Csörgő (Szeged)

A. Kertész, Lectures on artinian Rings, Edited by R. Wiegandt, 427 pages, Akadémiai Kiadó, Budapest, 1987.

The text is a substantially extended and completed translation of the original German edition "Vorlesungen über artinsche Ringe" of the late A. Kertész. The present edition realizes the ideas and intentions of A. Kertész left behind in his notes.

Rather than being a comprehensive account of the theory of artinian rings, this book provides a well-written elementary text on ring theory centered on the basic theorems on artinian rings. Moreover, its scope is considerably wider than the title suggests and the main topic is developed within the framework of those modern generalizations which resulted from the systematic use of the artinian approach.

The book consists of fifteen chapters from which only nine have been treated in the German edition. The first four chapters are developments of the general theory of rings and modules, and requires practically no previous knowledge of that topic. This introduction to rings, modules, prime and Jacobson radical is carried out with care in an almost leisure manner.

The following part of the book deals with artinian rings and with generalizations without assuming the existence of unit element. For artinian rings it presents the classical theorems on semi-simple, primary and simple rings as well as on projective and injective modules. There we also find the general theory of rings of linear transformations, Jacobson's Density theory, the Wedderburn—Artin structure theorem, and Maschke's theorem. Steinfeld's theory on quasi-ideals is developed and used in giving ideal-theoretical characterization of semi-simple rings. The Litoff—Anh theorem on local matrix rings and Vámos' theorem on characterizing artinian modules by finitely embedded modules are also included, which have not been treated in the German edition. A full account of the additive structure for artinian rings is given including the fundamental theory of Fuchs and Szele.

The last six chapters of the book were written by A. Betsch, A. Widinger, and R. Wiegandt. During the last decades many new branches of ring theory have been developed and several important results have been proved involving artinian rings and modules. Thus it has become highly desirable to supplement the text with important topics such as Goldie's theory on rings of quotients, quasi-Frobenius rings, and Connell's theorem on artinian group rings. This part includes also a general decomposition theorem on strictly artinian rings, and investigations of linearly compact rings. In the study of rings with minimum condition on principal right ideals, the splitting theorem due to Ayoub and Huynh is also treated.

For better understanding, each chapter ends by a set of exercises with hints for solutions.

Writing this book the contributors and the editor made an excellent job, and the extended English version enlivens the reputation of the original German edition.

N. V. Loi (Budapest)

Serge Lang, Calculus of Several Variables, Third Edition (Undergraduate Texts in Mathematics), XII+503+A91+I4 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1987.

Sometimes one pays less attention to the functions of several variables than to the functions of one variable. Once a famous mathematician told that: "In several variables everything goes just as in one variable." It was true for him, but the teachers know that in general this is not true for the students. We have several problems teaching this theme.

This book was previously published in 1973 and 1979, and therefore it is widely known. In a self-contained presentation it covers all essential topics in the calculus of several variables. Having read this book the reader will be familiar e.g. with the mathematics of mechanics.

Perhaps the best way to characterize the method of the book is to sketch the discussion of two, slightly embarrassing problems. The paragraph on inverse mappings contains three examples after the definition, then the inverse mapping theorem comes: Let $F\colon U\to R^n$ be a C^1-map. Let P be a point of U. If the Jacobian determinant $\Delta_F(P)$ is not equal to 0, then F is locally C^1-invertible at P. The proof of this theorem is beyond the scope of this book. Then we have three examples again. The next paragraph contains ten proposed exercises, the answers can be found at the end of the book. In the paragraph on implicit functions, after a short introduction the implicit function is stated in the form: Let U be open in R^2 and let $f\colon U\to R$ be a C^1-function. Let (a, b) be a point of U, and let $f(a, b)=c$. Assume that $D_2 f(a, b)\neq 0$. Then there exists an implicit function $y=\varphi(x)$ which is C^1 in some interval containing a, and such that $\varphi(a)=b$. Before the proof we can find four examples. The next paragraph consists of various interesting exercises.

In connection with mathematical analysis in the former century it has been said that while Berlin found Göttingen lacking in rigour, Göttingen found Berlin lacking in ideas. These standpoints are problematical today as well. In my opinion "this book is between Berlin and Göttingen", it has a proper level in rigour and in ideas.

L. Pintér (Szeged)

Ricardo Mañé, Ergodic Theory and Differentiable Dynamics (Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 8), XII+317 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

Modelling systems, especially in mechanics, one often comes to a measure space and a measurable map on it such that the measure is invariant with respect to the map. The theme of ergodic theory is the dynamic behavior of such measure-preserving maps. The first theorem of the theory was proved by Poincaré. His celebrated recurrence theorem says that if the evolution of a system is described by a vector field whose divergence vanishes identically, then the system returns infinitely often to configurations arbitrarily close to the initial one, except for a set of initial configurations with zero Lebesgue measure, i.e. except for a set which can be neglected from the probabilistic point of view.

Around the turn of the century the work of Boltzmann and Gibbs on statistical mechanics raised the following mathematical problem: Given a measure-preserving map T of a space $(X, \mathcal{A}, \mu)$ and an integrable function $f\colon X \to R$, find conditions under which the limit

$$\lim_{n\to\infty} \frac{f(x)+f(Tx)+\ldots+f(T^{n-1}(x))}{n}$$

exists and is constant almost everywhere. Birkhoff proved that for any T and f the limit exists almost everywhere, and a necessary and sufficient condition for its value to be constant almost everywhere is that there exists no set $A\in\mathcal{A}$ such that $0<\mu(A)<1$ and $T^{-1}(A)=A$. Maps which satisfy this condition are called *ergodic*.

It can be very difficult to decide whether or not a map occurring in statistical mechanics is ergodic. For example, Gibbs initiated the study of billiards as models for a perfect gas. In a billiard spheres move with constant velocity within a bounded region colliding with one another and with the boundary in a perfectly elastic way. In the thirties Birkhoff gave an abstract formulation of the problem, but only in the sixties, starting with Sinai's work, were any billiard proved to be ergodic. The first example of a convex ergodic billiard was given by Bunimovich in 1974, but no examples of ergodic billiards with convex C^∞ boundary are known.

The book is an excellent survey on the ergodic theory of differentiable dynamical systems.

Chapter 0 summarizes the basic definitions and theorems of measure theory. This is a quick review, but the reader can find results with proofs on derivatives with respect to sequences of partitions, which cannot be found in standard references. Chapter 1 entitled Measure-Preserving Maps starts with a brilliant introduction outlining the basic problems of the ergodic theory, then presents the main kinds of dynamical systems around which ergodic theory has developed. Chapter II (Ergodicity) contains the classical concepts and results including Birkhoff's Theorem, Kolmogorov—Arnold—Moser Theorem, Gaussian and Markov Shifts. Chapter III (Expanding Maps and Anosov Diffeomorphisms) and Chapter IV (Entropy) are devoted to contemporary ergodic theory. A good part of the information is contained in the great number of exercises which give the reader the opportunity of working actively and individually in the field.

The book can be highly recommended either as an introduction or as a monograph for mathematicians and physicists.

L. Hatvani (Szeged)

Bernard Maskit, Kleinian Groups (Grundlehren der mathematischen Wissenschaften, 287), XIII+326 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1988.

The fractional linear transformation group $PSL(2, \mathbf{C})$ with complex coefficients on the extended complex plane $\mathbf{C}\cup\{\infty\}$ has fundamental importance in theoretical and applied mathematics. This group is isomorphic to the orientation preserving conformal transformation group of the euclidean plane, to the isometry group of the hyperbolic space and to the rotation group of the pseudo-euclidean space-time and contains as subgroup the isometry groups of the euclidean and non-euclidean planes. The discrete subgroups of $PSL(2, \mathbf{C})$ were investigated already by Felix Klein in the relation with the space-form problem of the hyperbolic geometry. This theory had a large development in the last century and has many applications in complex analysis for the investigation of automorphic functions in topology, differential equations and number theory.

The new aspects of this theory are connected with the geometry and topology of 3-manifolds. The fundamental results of W. P. Thurston and his active school show that one could analyse discrete subgroups of $PSL(2, C)$ using 3-dimensional hyperbolic geometry.

The present book is an introduction to the theory of Kleinian groups which are subgroups of $PSL(2, C)$ acting freely and discontinuously at some point $Z \in C \cup (\infty)$. The methods of hyperbolic geometry are used consequently in the treatment. The book is designed for using as a textbook for a one year advanced graduate course in Kleinian groups. The first three chapters give an introduction to the basic notions and results concerning fractional linear transformations, discontinuous groups acting on the plane and the theory of covering spaces. Chapters IV—VII contain the explanation of the general theory and can be used as foundation of Thurston's work, too. Chapter VIII is a collection of examples of Kleinian groups with diverse properties. The last two chapters give a study of special groups and discusse their structure theory.

The chapters are followed by a set of exercises which are quite uneven in terms of difficulty and also by notes giving a brief historical outline of the theory.

The reader is assumed to be familiar in group theory, topology, analytical and differential geometry of hyperbolic spaces. The book is highly recommended to everyone interested in the related fields of mathematics.

Péter T. Nagy (Szeged)

Non-Linear Equations in Classical and Quantum Field Theory, Proceedings, Meudon and Paris VI, France 1983/84. Edited by N. Sanchez (Lecture Notes in Physics, 226), VIII+400 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1985.

Field Theory, Quantum Gravity and Strings, Proceedings, Meudon and Paris VI, France 1984/85. Edited by H. J. de Vega and N. Sanchez (Lecture Notes in Physics, 246), VI+381 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1986.

Field Theory, Quantum Gravity and Strings II, Proceedings, Meudon and Paris VI, France 1985/86. Edited by H. J. de Vega and N. Sanchez (Lecture Notes in Physics, 280), VI+245 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

These three volumes contain the lectures delivered at the series of seminars on current developments in mathematical physics held alternately at DAPHE-Observatoire de Meudon and LPTHE-Universite Pierre and Marie Curie (Paris). The series of seminars started in October 1983 and these volumes account for the lectures (60 altogether) read up to October 1986. The lectures delivered by outstanding experts together provide the reader with a comprehensive review of recent advances and trends in mathematical physics. The following list of key-words can give only a taste of the variety of topics covered in this collection.

The central themes of the first volume are integrable non-linear theories and methods to solve them. Among the key-words are: Lax pairs, Bäcklund transformations, Yang—Baxter and Kac—Moody algebras. The models reviewed include self-dual Yang—Mills fields, Bogomolny—Prasad—Sommerfield monopoles and sigma models.

A number of lectures in the second volume of this set are devoted to the superstring attempt of unification of interactions and to the related topic of conformally invariant two dimensional models. Other reviews treate Kaluza—Klein theories, quantum cosmology and stochastic quantization. Exact solvability is amongst the key-words of most frequent occurrence here too.

In the third volume the reader finds lectures on string theory, quantum gravity, integrable systems, soliton dynamics, twistor theory, dynamical symmetries and critical phenomena.

This collection of stimulating and comprehensive reviews encourages further the interaction between different fields of theoretical physics and mathematics. It should have a place on the shelves of every theoretical physics and mathematics library.

László Fehér (Szeged)

Nonlinear Semigroups, Partial Differential Equations and Attractors, Proceedings, Washington, D.C., 1985. Edited by T. L. Gill and W. W. Zachary (Lecture Notes in Mathematics, 1248), IX+185 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

Reading the classic textbooks and monographs in partial differential equations nowadays one can realize in surprise that everything was linear at that time. During the last three decades it has been pointed out that nonlinear structures are of interest, e.g. that the chaotic behaviour of some nonlinearities offers new explanations for some misterious phenomena. The methods of nonlinear theory have been developing so fast, and so many books have appeared on them that now one must think: everything is nonlinear.

These lecture notes are the proceedings of the symposium on the topics involved in the title held at Howard University in Washington, D. C. on August 5—8, 1985. In the reviewer's opinion, all the articles are of such interest and importance that each of them has to be cited: Joel D. Avrin, Convergence Properties of Strongly-Damped Semilinear Wave Equations; S. A. Belbas, Numerical Solution of Certain Nonlinear Parabolic PDE; Melvyn S. Berger, The Explicit Solution of Nonlinear ODE's and PDE's; Whei-Ching C. Chan and Shui-Nee Chow, Uniform Boundedness and Generalized Inverses in Liapunov—Schmidt Method for Subharmonics; Hans Engler, Existence of Radially Symmetric Solutions of Strongly Damped Wave Equations; H. Engler, F. Neubrander, and J. Sandefur, Strongly Damped Semilinear Second Order Equations; Lawrence C. Evans, Nonlinear Semigroup Theory and Viscosity Solutions of Hamilton—Jacobi PDE; Jerome A. Goldstein, Evolution Equations with Nonlinear Boundary Conditions; Jack K. Habe, Asymptotically Smooth Semigroups and Applications; John Mallet—Paret and George R. Sell, The Principle of Spatial Averaging and Inertial Manifolds for Reaction Diffusion Equations; Robert H. Martin, Jr., Applications of Semigroup Theory to Reaction-Diffusion Systems; Jeffrey Rauch and Michael C. Reed, Ultra Singularities in Nonlinear Waves; M. C. Reed and J. J. Blum, A Reaction-Hyperbolic System in Physiology; Eric Shechter, Compact Perturbations of Linear M-Dissipative Operators Which Lack Gihman's Property; Thomas I. Seidman, Two Compactness Lemmas; Andrew Vogt, The Riccati Equation: When Nonlinearity Reduces to Linearity.

L. Hatvani (Szeged)

Numerical Analysis, Proceedings of the Fourth IIMAS Workshop held at Guanajnato, Mexico, July 23—27, 1984. Edited by J. P. Hennart (Lecture Notes in Mathematics, 1230), X+234 pages, Springer-Verlag, Berlin—Heidelberg, 1986.

This volume contains 18 selected items (mainly of the invited lecturers) from the 29 papers delivered at the Fourth Workshop on Numerical Analysis hosted by the National University of Mexico.

The program of the workshop was centered on the following main areas: optimization problems, the solution of systems of both linear and nonlinear equations, and the numerical aspects of differential equations. Most of the papers deal with special problems/methods of these fields. Moreover, many practical hints and experimental results are provided, too.

The authors' motivations vary from practical problems, e.g. the planning of semiconductor devices and the stability of capillary waves, to 'pure' (numerical) mathematics such as the derivation of new Runge—Kutta formulae and convergence results on the secant methods in Hilbert space.

Let us quote some titles just to give a taste of the book:

Goldfarb: Efficient primal algorithm for strictly convex quadratic programs; Falk and Richter: Remarks on a continuous finite element scheme for hyperbolic equations; Elman and Streit: Polynomial iteration for nonsymmetric indefinite linear systems.

Although a part of the contributions is available in a more polished form in journal this book may be a valuable guide for the specialists working in these subfields to the directions of current interest.

J. Virágh (Szeged)

Tadao Oda, Convex Bodies and Algebraic Geometry; An Introduction to the Theory of Toric Varieties (Ergebnisse der Mathematik and ihrer Grenzgebiete, 3. Folge, Band 15), VII+212 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1988.

The beginners learning algebraic geometry usually have difficulties with the lot of new and abstract notions familiarity of which is necessary to understand the theory. The purpose of this book is to give an introduction to algebraic geometry, especially to the theory of toric varieties, using the language of the visuable convex geometry. The author writes in the introduction: "For this reason, we chose to construct toric varieties as complex analytic spaces, so that they can be understood more easily without much prior knowledge of algebraic geometry. Not only can some of the important complex analytic properties of these spaces be translated into easily visualized elementary geometry of convex figures, but many interesting examples of complex analytic spaces can be easily constructed by means of this theory." Chapter 1 is devoted to the basic notions and facts about toric varieties. Chapter 2 contains results on the cohomology of compact toric varieties and the imbedding theory into projective spaces. Chapter 3 contains a study of the automorphism group using holomorphic differential forms. Chapter 4 deals with applications of the theory to the investigation of singularities. In Appendix the basic results of convex geometry are collected without proofs.

Péter T. Nagy (Szeged)

Nicolae H. Pavel, Nonlinear Evolution Operators and Semigroups. Application to Partial Differential Equations (Lecture Notes in Mathematics, 1260), VI+285 pages, Springer-Verlag,) Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

In the last two decades a very successful new branch has appeared in the theory of differential equations. It has been pointed out that semigroups and evolution equations techniques can be widely used to solve problems related to partial differential equations and functional differential equations. This allows these equations to be treated as suitable ordinary differential equations in infinite dimensional Banach spaces.

The book presents some of the fundamental results and recent research on nonlinear evolution operators and semigroups and their applications. Most of the results involved were available earlier only in papers.

The first chapter is devoted to the construction and main properties of nonlinear evolution operator associated with nonautonomous differential inclusions. Chapter 2 is concerned with nonlinear semigroups generated by dissipative operators (Crandall—Liggett Theory). The most interesting chapter, the third one, shows how to apply the abstract results of the theory to unify the treatments of several types of partial differential equations arising in physics and biology (equation of long water waves of small amplitude, Porous Medium Equation, the heat equation, Schrödinger Equation, Semilinear Schrödinger Equation, and so on).

L. Hatvani (Szeged)

Probability Theory and Mathematical Statistics, Proceedings of the 5th Pannonian Symposium on Mathematical Statistics, Visegrád, Hungary, 20—24 May, 1985. Edited by W. Grossmann, J. Mogyoródi, I. Vincze and W. Wertz, XIII+457 pages, Akadémiai Kiadó, Budapest and D. Reidel Publishing Company, Dordrecht, 1988.

The proceedings of the 3rd and 4th Symposia have been reviewed in these *Acta*, 47 (1984), page 513 and 51 (1987), page 283.

Just as the financial support shrinks as the symposium moves over to Hungarian territory from the Austrian, the number of participants decreases. Accordingly, the proceedings reduce from two volumes to one. Fortunately, however, the level of the quality achieved by the proceedings of the 4th Symposium has been maintained.

Part A (pages1—234) containes the papers on various probability problems by G. Baróti, N. L. Bassily, E. Csáki and A. Földes, G. Elek and K. Grill, I. Fazekas, S. Fridli and F. Schipp, J. Galambos and I. Kátai, B. Gyires, I. Gyöngy, M. Janžura, I. Kalmár, A. Kováts, L. Lakatos, E. G. Martins and D. D. Pestena, T. F. Móri, T. Nemetz and J. Ureczky, P. M. Peruničić, D. Plachky, T. Pogány, G. J. Székely, I. Vincze, and by A. Zempléni. Part B then consists of the papers on diverse statistical topics and applications written by J. Anděl, G. Apoyan and Yu. Kotojants, J. Hurt, P. Kosik and K. Sarkadi, A. Pázmán, Z. Prášková, L. Rüschendorf, A. K. Md. E. Saleh and P. K. Sen, L. Szeidl, G. Terdik, R. Thrum, J. Tóth, S. Veres (2 papers), J. A. Víšek, L. Vostrikova, P. Volf, and by W. Wefelmeyer.

Those who liked the "Pannonian" flavour in the preceding Proceedings will want to savour it in this volume as well.

Sándor Csörgő (Szeged)

Probability Theory and Mathematical Statistics, Proceedings of the Fifth Japan—USSR Symposium, Kyoto, Japan, July 8—14, 1986. Edited by S. Watanabe and Yu. V. Prohorov (Lecture, Notes in Mathematics, 1299), VIII+589 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1988.

The volume contains 61 papers, 20 of which are by authors from the USSR, 2 by visitors in Japan from Czechoslovakia and France, and the remaining 39 articles are written by Japanese authors, 2 with co-authors from France and the USA. There are 2 papers describing the work of G. Maruyama who deceased three days before the symposium, 5 papers which could be classified as belonging to statistical theory, and the great majority of the rest is in probability theory with a few contributions representing related fields such as probabilistic number theory, ergodic theory or information theory. A number of the papers are expository in nature, most of them are proper research articles on a wide variety of different topics.

Sándor Csörgő (Szeged)

Maurice Roseau, Vibrations in Mechanical Systems. Analytical Methods and Applications, XIV+515 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

The vibrations and prediction of their effects are of great importance in construction of machines and devices. The change in time of the mechanical variables is governed by ordinary or partial differential equations. Based on the analysis of differential equations various theories have been developed such as linearized or non-linearized, and very often of an asymptotic nature. These theories deal with the conditions of stability and resonance and the coupling of modes in non-linear systems. In this book such methods are developed which deal with free and induced vibrations in discrete or continuous mechanical structures.

In each of the twelve chapters the reader can find well selected illustrations to the theories and methods. Numerous important examples, known and original, are discussed in a complex and thoroughful way. To show the variety of the subject covered in this book it is enough to cite some items of the contents: Forced Vibrations, Vibrations in Lattices, Gyroscopic Coupling, Stability of Linear Systems, The Stability of Operation of Non-Conservative Mechanical Systems, Flexible Vibrations of Beams, Longitudinal and Torsional Vibrations of Bars, Vibrations of Elastic Solids and of Plane Elastic Plates, Vibrations in Periodic Media, Model Analysis, Synchronisation Theory, Stability of a Column Under Compression, The Method of Amplitude Variation, Rotating Machinery, Non-Linear Waves and Solitons.

This volume is a translation of the French original published in 1984. Several chapters have been taught to graduate students at the Pierre and Marie Curie University in Paris.

The book is useful for mathematicians dealing with applications of differential equations and is recommended to students and researchers interested in mechanics and mechanical engineering.

I. K. Gyémánt (Szeged)

Kennan T. Smith, Power Series from a Computational Point of View (Universitext), VIII+132 pages, Springer-Verlag, New York—Berlin—Heidelberg, 1987.

The author summarizes his aims as follows:

"The purpose of this book is to explain the use of power series in performing calculations, such as approximating definite integrals or solutions to differential equations. This focus may seem narrow but, in fact, such computations require the understanding and use of many of the important theorems of elementary analytic function theory, ... These computations provide an effective motivation for learning the theorems, and a sound basis for understanding them."

In the refree's opinion, the title could be paraphrased such as "Power Series from the Point of View of Complex Analysis" because most of the material is hard-core mathematics. The chapter headings are the following: Taylor polynomials, Sequences and Series, Power Series and Complex Differentiability, Local Analytic Functions, Analytic continuations. So the minimal prerequisite would be an introductory analysis course. In this case, however, some elementary parts of the second chapter could be omitted. For the convenience of the reader the theorems, definitions and formulae are numbered and cross-referenced throughout the text. However, references such as "According to the next section, this is Taylor's formula for $\log(1+x)$ centered at a, but this is not needed" (pp. 25), or "Referring to the picture in Section 4, use Definition 1.7 and Theorems 2.4 and 2.8 to show that ..." (pp. 84) are rather awkward. The book ends with a useful index of notions.

After each chapter a set of selected problems can be found. They belong mainly to the two categories "prove the following theorem" or "compute the definite integral/Taylor polynomial of the following function". There are only a few scattered indications of computer practice one of them, e.g. "Write the FORTRAN program to compute..." (pp. 26).

Finally, I miss the links with Numerical Analysis from the book. (The rare exceptions are the trapezoid and the Simpson's rule mentioned in some problems.)

J. Virágh (Szeged)

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