ACTA UNIVERSITATIS SZEGEDIENSIS

ACTA SCIENTIARUM MATHEMATICARUM

TOMUS 44
FASC. 1—2

SZEGED, 1982

INSTITUTUM BOLYAIANUM UNIVERSITATIS SZEGEDIENSIS

A JÓZSEF ATTILA TUDOMÁNYEGYETEM KÖZLEMÉNYEI

ACTA SCIENTIARUM MATHEMATICARUM

44. KÖTET
FASC. 1—2

SZEGED, 1982

JÓZSEF ATTILA TUDOMÁNYEGYETEM BOLYAI INTÉZETE

ACTA UNIVERSITATIS SZEGEDIENSIS

ACTA SCIENTIARUM MATHEMATICARUM

TOMUS 44

SZEGED, 1982

INSTITUTUM BOLYAIANUM UNIVERSITATIS SZEGEDIENSIS

A JÓZSEF ATTILA TUDOMÁNYEGYETEM KÖZLEMÉNYEI

ACTA SCIENTIARUM MATHEMATICARUM

44. KÖTET

SZEGED, 1982

JÓZSEF ATTILA TUDOMÁNYEGYETEM BOLYAI INTÉZETE

Acta Sci. Math., 44 (1982), 3—4

Bands of power joined semigroups

STOJAN BOGDANOVIĆ

A *band* is a semigroup in which every element is an idempotent. A semigroup S is called *power joined* if for each pair of elements $a, b \in S$ there exist positive integers m, n with $a^m = b^n$. We say that a semigroup S is a *band of power joined semigroups* if there exists a congruence ϱ such that S/ϱ is a band and each class modϱ is a power joined semigroup. In this case ϱ is called a *band congruence*. One defines analogously semilattices, rectangular bands and left zero bands of power joined semigroups. Bands of power joined semigroups are studied by T. NORDAHL [1] in the *medial* case ($xaby = = xaby$). In the present paper we consider the general case.

For non-defined notions we refer to [2] and [3].

Theorem 1. *A semigroup S is a band of power joined semigroups if and only if*

(A) $$(\forall a, b \in S)(\forall m, n \in N)(\exists r, s \in N)((ab)^r = (a^m b^n)^s).$$

Proof. Let S be a band Y of power joined semigroups $S_\alpha, \alpha \in Y$. For $a \in S_\alpha$, $\alpha \in Y$ and $b \in S_\beta, \beta \in Y$ we have $a^m b^n \in S_{\alpha\beta}$ for every $m, n \in N$, and thus

$$(ab)^r = (a^m b^n)^s \quad \text{for some} \quad r, s \in N.$$

Conversely, let S satisfy condition (A). We define a relation ϱ on a semigroup S as follows:

(1) $$a \varrho b \Leftrightarrow (\exists m, n \in N)(a^m = b^n).$$

It is clear that ϱ is an equivalence on S. Let $a\varrho b$, then

$$(ab)^t = (a^m b^n)^p = (a^{m+m})^p = a^{2pm}.$$

Hence, each ϱ-class is a power joined subsemigroup of S. We shall show that ϱ is a congruence on S. Suppose $a\varrho b$ and $c \in S$. Then $a^m = b^n$ for some $m, n \in N$, and by (A) we have

(2) $$(ac)^k = (a^m c^t)^r \quad \text{for some} \quad k, r \in N,$$

(3) $$(bc)^{k_1} = (b^n c^t)^{r_1} \quad \text{for some} \quad k_1, r_1 \in N.$$

It follows from (2) and (3) that

$$(ac)^{kr_1} = (a^m c^t)^{rr_1} = [(a^m c^t)^{r_1}]^r = [(b^n c^t)^{r_1}]^r = [(bc)^{k_1}]^r = (bc)^{k_1 r}.$$

Received April 3, 1981.

Hence, $ac\varrho bc$. Similarly, we obtain $ca\varrho cb$. Consequently, ϱ is a congruence and since $a\varrho a^2$ for every $a\in S$, we have that S is a band of power joined semigroups.

Theorem 2. *A semigroup S is a semilattice of power joined semigroups if and only if*

(B) $$(\forall a, b\in S)(\forall m, n\in N)(\exists r, s\in N)\big((ba)^r = (a^m b^n)^s\big).$$

Proof. Let S be a semilattice Y of power joined semigroups $S_\alpha, \alpha\in Y$. For $a\in S_\alpha, \alpha\in Y$ and $b\in S_\beta, \beta\in Y$ we have $a^m b^n, ba\in S_{\alpha\beta}$ for every $m, n\in N$. Hence,

$$(ba)^r = (a^m b^n)^s \quad \text{for some} \quad r, s\in N.$$

Conversely, let S satisfy condition (B). Then

(4) $$(ba)^{r_1} = (ab)^{s_1} \quad \text{for some} \quad r_1, s_1\in N.$$

From (B) and (4) we have

(5) $$(ab)^{s_1 r} = (ba)^{rr_1} = (a^m b^n)^{sr_1}$$

for every $m, n\in N$ and for some $r, s\in N$. It follows from (5) and Theorem 1 that the relation ϱ on S (from (1)) is a band congruence and every equivalence class of S mod ϱ is a power joined semigroup. It follows from (4) that $ab\varrho ba$, so ϱ is a semilattice congruence.

Theorem 3. *A semigroup S is a rectangular band of power joined semigroups if and only if*

(C) $$(\forall a, b, c\in S)(\exists r, s\in N)\big((abc)^r = (ac)^s\big).$$

Proof. Let S satisfy condition (C). Then

$$(a^m b^n)^r = \big(a(a^{m-1}b^{n-1})b\big)^r = (ab)^s$$

for every $m, n\in N$ and for some $r, s\in N$. Hence, the condition (A) holds and from this ϱ (from (1)) is a band congruence on S (Theorem 1) and every equivalence class of S mod ϱ is a power joined semigroup. It follows from (C) that ϱ is a rectangular band congruence.

The converse follows immediately.

Corollary. *A semigroup S is a left zero band of power joined semigroups if and only if*

(D) $$(\forall a, b\in S)(\exists r, s\in N)\big((ab)^r = a^s\big).$$

References

[1] T. Nordahl, Bands of power joined semigroups, *Semigroup Forum*, **12** (1976), 299—311.
[2] J. M. Howie, *An Introduction to Semigroup Theory*, Academic Press (1976).
[3] M. Petrich, *Introduction to Semigroups*, Merill Publishing Company (1973).

INSTITUTE OF MATHEMATICS
UNIVERSITY OF NOVI SAD
21001 NOVI SAD, P. O. B. 224
YUGOSLAVIA

Acta Sci. Math., 44 (1982), 5—11

Compact approximants

RICHARD BOULDIN

§ 1. **Introduction.** Interest in approximating a given (bounded linear) operator T on a fixed Hilbert space $\mathfrak{H}$ goes back to [3] and [4], among other references. Each of the preceding sources constructed a compact operator C such that $\|T-C\|$ equaled the distance from T to the compact operators; such an operator C is said to be a compact approximant. Although much attention has been focused on the Calkin algebra and discovering compact approximants with various algebraic properties, only [5] seems to have studied the structure of the set of compact approximants. The main results of [5] show that the set of compact approximants has no extreme points except in the case that a multiple of T is a compact perturbation of a maximal partial isometry and the existence of a finite rank compact approximant is characterized.

This paper attempts to clarify where the investigation of compact approximants stands and to extend it in several directions. The next section compares the methods of [3] and [4] and shows that the resulting compact approximants are essentially the same. The new derivation of the Gohberg—Krein compact approximant will play a key role in several subsequent proofs. Section § 3 gives a simplified criterion for when T has a finite rank compact approximant. A similar criterion is given for T to have a compact approximant which belongs to the Schatten p-class. Section § 4 gives a condition which is necessary and sufficient for T to have a compact approximant with maximal norm.

Throughout this work $U|T|$ will be the polar factorization of T where U is a maximal partial isometry and $|T|$ is $(T^*T)^{1/2}$. For T compact let $s_1(T), s_2(T), \dots$ be the eigenvalues of $|T|$ in nonincreasing order repeated according to multiplicity. If for some $p \geqq 1$ one has

$$\sum_j s_j(T)^p < \infty$$

Received January 30, 1981.

then one says that T belongs to the Schatten p-class C_p which is normed with

$$\|T\|_p = (\sum_j s_j(T)^p)^{1/p}.$$

The quantities $\|T\|_e$ and $r_e(T)$ are defined to be the norm and spectral radius, respectively, of the coset of T in the Calkin algebra.

§ 2. Constructing compact approximants. Since the existence of a compact approximant is proved in [3] as a by-product of the extension of s-numbers from compact operators to bounded operators and the latter is only outlined, a brief development of the Gohberg—Krein compact approximant is offered. Through the use of the characterization of the essential spectrum for a self-adjoint operator, a much quicker derivation is achieved. For any normal operator the essential spectrum coincides with the Weyl spectrum which is all the points in the spectrum except isolated eigenvalues with finite multiplicity. See [p. 376, 6], [2], [1]. First, a fundamental lemma is required.

Lemma 2.1. $\|T\|_e = \||T|\|_e = r_e(|T|)$.

Proof. Let π denote the canonical map of the operators on $\mathfrak{H}$ into the Calkin algebra $\mathscr{C}$. Since $\mathscr{C}$ is a C^*-algebra and π is a *-homomorphism, one knows that $\|\pi(T)\| = \||\pi(T)|\|$ and

$$|\pi(T)| = (\pi(T)^* \pi(T))^{1/2} = (\pi(T^*T))^{1/2} = \pi((T^*T)^{1/2}) = \pi(|T|).$$

Thus, $\|T\|_e = \|\pi(T)\| = \|\pi(|T|)\| = \||T|\|_e$. Since $\pi(|T|)$ is normal in $\mathscr{C}$, its norm equals its spectral radius and the lemma is proved.

It is now clear that the spectrum of $|T|$ in the open interval $(\|T\|_e, \infty)$ consists entirely of isolated eigenvalues with finite multiplicity; let $\{\lambda_1, \lambda_2, \ldots\}$ be a nonincreasing enumeration of that possibly finite set with each eigenvalue repeated according to its multiplicity. Let $E(\cdot)$ be the spectral measure for $|T|$ and denote $E([0, \|T\|_e])\mathfrak{H}$ and $E((\|T\|_e, \infty))\mathfrak{H}$ by $\mathfrak{H}_0$ and $\mathfrak{H}_1$, respectively. Let $\{e_1, e_2, \ldots\}$ be an orthonormal sequence of eigenvectors of $|T|$ such that e_j corresponds to λ_j for $j=1, 2, \ldots$ and note that the spectral representation of $|T|$ restricted to $\mathfrak{H}_1$, denoted $|T|\,|\mathfrak{H}_1$, is

$$\sum_j \langle \cdot, e_j \rangle \lambda_j e_j.$$

If $U|T|$ is the polar factorization of T then the Gohberg—Krein compact approximant of T, denoted by K henceforth, is

$$K = \sum_j \langle \cdot, e_j \rangle (\lambda_j - \|T\|_e) U e_j.$$

Since $\{\lambda_1, \lambda_2, \ldots\}$ cannot have an accumulation point in $(\|T\|_e, \infty)$, either the above sum is finite or $\{\lambda_1, \lambda_2, \ldots\}$ converges to $\|T\|_e$. In either case, it is apparent that

K is the limit of finite rank operators and consequently K is compact. The following calculation shows that K is a compact approximant for T.

$$\|T-K\| = \|U|T|-U(0|\mathfrak{H}_0\oplus(|T|-\|T\|_e)|\mathfrak{H}_1)\| \leq \||T|-0|\mathfrak{H}_0\oplus(|T|-\|T\|_e)|\mathfrak{H}_1\| =$$
$$= \max\{\||T||\mathfrak{H}_0\|, \|T\|_e I|\mathfrak{H}_1\|\} = \|T\|_e.$$

In sharp contrast to the above contruction Holmes and Kripke obtain a compact approximant for T without using the polar factorization of T. They note that if there is an orthogonal projection P with finite codimension such that TP does not assume its norm — i.e. $\|TPx\|=\|TP\|\|x\|$ implies $x=0$ — then $T(I-P)$ is a finite rank compact approximant. In the case that T does not have a finite rank compact approximant, the compact approximant constructed by Holmes and Kripke, denoted by L henceforth, is

$$L = \sum_j \langle\cdot, f_j\rangle(\|Tf_j\|-\|T\|_e)Tf_j/\|Tf_j\|$$

where $\{f_1, f_2, \ldots\}$ is an orthonormal sequence such that $\|Tf_1\|=\|T\|$ and $\|Tf_{j+1}\|=\|TP_j\|$ where P_j is the orthogonal projection onto the orthogonal complement of $\{f_1, \ldots, f_j\}$ for $j=1, 2, \ldots$.

Since $\|Tx\|=\|U|T|x\|=\||T|x\|$, one has $\||T|f_1\|=\||T|\|$ and $\||T|f_{j+1}\|=\||T|P_j\|$ for $j=1, 2, \ldots$. This implies that

$$|T|f_1 = \||T|\|f_1 \quad \text{and} \quad |T|f_{j+1} = \||T|P_j\|f_{j+1} \quad \text{for} \quad j = 1, 2, \ldots.$$

Clearly one can choose $f_j=e_j$ for $j=1, 2, \ldots$ with $\{e_1, e_2, \ldots\}$ given as in the construction of the Gohberg—Krein compact approximant. The formula for L becomes

$$L = \sum_j \langle\cdot, e_j\rangle(\lambda_j-\|T\|_e)Te_j/\|Te_j\| \quad \text{or} \quad L = \sum_j \langle\cdot, e_j\rangle(\lambda_j-\|T\|_e)Ue_j$$

where $\lambda_j=\||T|P_j\|$ for $j=0, 1, \ldots$ and $P_0=I$. Here it is used that

$$Te_j/\|Te_j\| = U|T|e_j/\|U|T|e_j\| = U\lambda_j e_j/\|U\lambda_j e_j\| = Ue_j.$$

It is straightforward to see that the formulas for K and L can be restated in forms which are independent of the choices of bases for the eigenspaces of $|T|$. Thus the following theorem has been proved.

Theorem 2.2. *For any operator T which does not have a finite rank compact approximant the Holmes—Kripke compact approximant L coincides with the Gohberg—Krein compact approximant K.*

A slight refinement of the Holmes—Kripke construction produces a unique compact approximant even in the case that T has a finite rank compact approximant. If n is the infimum of the codimension of orthogonal projections P such that TP does not assume its norm then the Holmes—Kripke construction produces a unique rank n compact approximant which coincides with the Gohberg—Krein compact approximant.

§ 3. Compact approximants in C_p. In [5] it is shown that T has a finite rank compact approximant if and only if there is no infinite dimensional closed subspace $\mathscr{E}\subset\mathfrak{H}$ with $\|Tx\|>\|T\|_e\|x\|$ for all nonzero $x\in\mathscr{E}$. Following [5] the set of compact approximants of T is denoted $\mathfrak{K}_T$.

Theorem 3.1. *The following conditions are equivalent.*

(i) $\mathfrak{K}_T$ *contains a finite rank operator.*

(ii) $|T|$ *has only finitely many eigenvalues in* $(\|T\|_e, \infty)$.

(iii) *The Gohberg—Krein compact approximant K has finite rank.*

Proof. The alternative derivation of the Gohberg—Krein compact approximant makes it clear that (ii) implies (iii) which implies (i). Thus, it suffices to show that (i) implies (ii).

Let A be a finite rank operator in $\mathfrak{K}_T$ and, for the sake of a contradiction, assume $|T|$ has infinitely many eigenvalues $\{\lambda_1, \lambda_2, \ldots\}$ in the open interval $(|T|_e, \infty)$. Let $\mathscr{E}$ be the closed span of the eigenspaces of $|T|$ corresponding to $\{\lambda_1, \lambda_2, \ldots\}$. It is easy to see that

$$\||T|x\| > \|T\|_e\|x\| \quad \text{for every} \quad x\in\mathscr{E}, \quad x\neq 0$$

and so

$$\|Tx\| = \|U|T|x\| = \||T|x\| > \|T\|_e\|x\| \quad \text{for such } x.$$

The argument is finished as in [5]. Since the restriction of A to $\mathscr{E}$ must have nontrivial kernel, there is some nonzero $y\in\mathscr{E}\cap\ker A$ and $\|(T-A)y\|=\|Ty\|>\|T\|_e\|y\|$ which contradicts that $A\in\mathfrak{K}_T$.

For a given operator T it is much easier to construct T^*T and check the number of eigenvalues in $(\|T\|_e^2, \infty)$ than it is to examine all possible subspaces $\mathscr{E}$. It is not difficult to see that if T has infinitely many eigenvalues in $\{z\colon |z|>\|T\|_e\}$ then there is an infinite dimensional subspace $\mathscr{E}$. But the converse of the preceding statement is false. Thus, it appears that the criterion for a finite rank compact approximant cannot be simplified any further.

The results in the preceding theorem can be refined to provide a condition which is necessary and sufficient for $\mathfrak{K}_T$ to contain an operator from the Schatten p-class C_p.

Theorem 3.2. *The following conditions are equivalent.*

(i) $\mathfrak{K}_T$ *contains an operator in* C_p.

(ii) *If* $\{\lambda_1, \lambda_2, \ldots\}$ *is a nonincreasing enumeration of the eigenvalues of* $|T|$ *in* $(\|T\|_e, \infty)$, *repeated according to multiplicity, then*

$$\sum_j (\lambda_j - \|T\|_e)^p < \infty.$$

(iii) *The Gohberg—Krein compact approximant K for T belongs to* C_p.

Proof. The alternative derivation of the Gohberg—Krein compact approximant given in section § 2 makes it reasonably clear that (ii) implies (iii). That (iii) implies (i) is trivial and so it suffices to show that (i) implies (ii).

Of course, the spectrum of $|T|$ in $(\|T\|_e, \infty)$ belongs to the complement of the essential spectrum of $|T|$ and, thus, it consists of isolated eigenvalues $\{\lambda_1, \lambda_2, \ldots\}$ each with finite multiplicity. Furthermore, the only possible accumulation point of $\{\lambda_1, \lambda_2, \ldots\}$ is $\|T\|_e$. Let $\{e_1, e_2, \ldots\}$ be an orthonormal sequence such that e_j is an eigenvector for $|T|$ corresponding to λ_j for $j=1, 2, \ldots$. Each λ_j is repeated according to its multiplicity.

Let A be a C_p-operator in $\mathfrak{K}_T$ and let $U|T|$ be the polar factorization of T. Note that U^*A is a C_p-operator and

$$\||T| - U^*A\| = \|U^*U|T| - U^*A\| \leqq \|T - A\| = \|T\|_e.$$

Furthermore, one has

$$\|T\|_e^2 \geqq \||T| - U^*A\|^2 \geqq \|(|T| - \operatorname{re} U^*A)^2 + (\operatorname{im} U^*A)^2\| \geqq \||T| - \operatorname{re} U^*A\|^2.$$

Thus $\operatorname{re}(U^*A)$ belongs to $\mathfrak{K}_{|T|}$ and it is routine to see that it is a C_p-operator.

Let C denote $\operatorname{re}(U^*A)$ henceforth, let $\alpha_j = \langle Ce_j, e_j\rangle$ for $j=1, 2, \ldots$ and $Ce_j = \alpha_j e_j + x_j$ with $x_j \perp e_j$. Note that

$$\|T\|_e^2 \geqq \||T|e_j - Ce_j\|^2 = \|\lambda_j e_j - Ce_j\|^2 = \|\lambda_j e_j - \alpha_j e_j - x_j\|^2 = (\lambda_j - \alpha_j)^2 + \|x_j\|^2.$$

Thus, $\|T\|_e \geqq |\lambda_j - \alpha_j|$ or $\lambda_j - \|T\|_e \leqq \alpha_j \leqq \lambda_j + \|T\|_e$. This makes it apparent that $\alpha_j \geqq 0$ and $\sum_j (\lambda_j - \|T\|_e)^p \leqq \sum_j \alpha_j^p$. According to [item 5, p. 94, 3]

$$\|C\|_p^p \geqq \sum_j \langle |C| e_j, e_j\rangle^p$$

and since $|C| \geqq C$ it is apparent that

$$\langle |C| e_j, e_j\rangle \geqq \langle Ce_j, e_j\rangle = \alpha_j \quad \text{for} \quad j = 1, 2, \ldots.$$

Thus, it is proved that $\sum_j (\lambda_j - \|T\|_e)^p < \infty$ as desired.

§ 4. A compact approximant with maximal norm. Recall that an operator T is said to "assume its norm" provided there is a nonzero vector f such that $\|Tf\| = \|T\|\|f\|$. Such f is said to be a maximal vector for T. It is easy to see that T assumes its norm if and only if $\|T\|^2$ is an eigenvalue of T^*T. Note that $\|T\|^2\|f\|^2 = \|Tf\|^2 = \langle T^*Tf, f\rangle$ and $\|(\|T\|^2 - T^*T)^{1/2} f\|^2 = 0$ are equivalent. This makes it clear, for example, that any compact operator assumes its norm.

The condition that T assume its norm played a key role in [4] and now it plays a key role in determining when $\mathfrak{K}_T$ contains an operator with maximal norm — i.e. $A \in \mathfrak{K}_T$ such that $B \in \mathfrak{K}_T$ implies $\|B\| \leqq \|A\|$.

Theorem 4.1. *There is a compact approximant A of T, i.e. $A \in \mathfrak{K}_T$, with maximal norm if and only if T assumes its norm.*

Proof. First it is shown that if T does not assume its norm then $\mathfrak{K}_T$ does not contain an operator with maximal norm. For any $B \in \mathfrak{K}_T$ and f a maximal unit vector for B one has

$$\|T\|_e \geqq \|(B-T)f\| \geqq \|Bf\| - \|Tf\| = \|B\| - \|Tf\|$$

or

$$\|T\|_e + \|T\| > \|T\|_e + \|Tf\| \geqq \|B\|.$$

Thus, it would suffice to show that $\|T\|_e + \|T\|$ is the supremum of the norms of operators in $\mathfrak{K}_T$.

Since T does not assume its norm, $|T|$ does not assume its norm. Since $\|T\|$ is not an eigenvalue for $|T|$, it must be an accumulation point for the spectrum. Consequently $\||T|\|_e$ equals $\||T|\|$ and equivalently $\|T\|_e$ equals $\|T\|$. Let $E(\cdot)$ be the spectral measure for $|T|$ and choose a unit vector f_n from $E([\|T\| - 1/n, \|T\|])\mathfrak{H}$. Define C_n by

$$C_n = \langle \cdot, f_n \rangle (2\|T\| - 1/n) f_n.$$

Note that C_n is rank one and $\|C_n\|$ converges to $2\|T\| = \|T\| + \|T\|_e$.

It now suffices for this half of the proof to show that C_n is a compact approximant for $|T|$. Denote $E([0, \|T\| - 1/n))\mathfrak{H}$ and $E([\|T\| - 1/n, \|T\|])\mathfrak{H}$ by $\mathfrak{H}_0$ and $\mathfrak{H}_1$, respectively. Since $\mathfrak{H}_0$ reduces $|T| - C_n$ to $|T|\,|\mathfrak{H}_0$ it suffices to sow that

$$\|(|T| - C_n)|\mathfrak{H}_1\| \leqq \|T\|_e = \||T|\|_e$$

where $A|\mathfrak{H}_1$ denotes the restriction of A to $\mathfrak{H}_1$. Since the above restriction is self-adjoint it clearly suffices to show that

$$\langle (|T| - C_n) g, g \rangle \in [-\|T\|, \|T\|]$$

for every unit vector g in $\mathfrak{H}_1$. Since the numerical range of C_n is $[0, 2\|T\| - 1/n]$, one has

$$-\|T\| = \|T\| - 1/n - (2\|T\| - 1/n) \leqq \|T\| - 1/n - \langle C_n g, g \rangle \leqq$$
$$\leqq \langle (|T| - C_n) g, g \rangle \leqq \|T\| - \langle C_n g, g \rangle \leqq \|T\|.$$

This shows that $\|T\|_e + \|T\| = 2\|T\|$ is the supremum of the norms of the operators UC_n which belong to $\mathfrak{K}_T$, where $U|T|$ is the polar factorization of T. Thus, half of the theorem is proved.

Now it is assumed that T has a maximal vector and it is to be shown that $\mathfrak{K}_T$ contains an operator with norm $\|T\|_e + \|T\|$. Since T assumes its norm, $\|T\|^2$ is an eigenvalue for T^*T and this implies $\|T\|$ is an eigenvalue for $|T|$. First, consider the case that $\|T\|$ has finite multiplicity for $|T|$ and let P be the orthogonal projection onto the corresponding eigenspace. For brevity sake let β denote

$\|T\|_e + \|T\|$. In order to show that $\beta P \in \mathfrak{K}_{|T|}$ it is noted that the restriction of $|T| - \beta P$ to $(I-P)\mathfrak{H}$ is just $|T|\,|(I-P)\mathfrak{H}$. Thus, it suffices to show that

$$\|(|T| - \beta P)|P\mathfrak{H}\| \leqq \|T\|_e.$$

Since $(|T| - \beta P)|P\mathfrak{H}$ is just $-\|T\|_e P|P\mathfrak{H}$, the above inequality is clear and $\beta P \in \mathfrak{K}_{|T|}$. It follows that βUP belongs to $\mathfrak{K}_T$ where $U|T|$ is the polar factorization of T.

It only remains to deal with the case that $\|T\|$ is an infinite dimensional eigenvalue of $|T|$. In this case it is clear that $\|T\|_e = \||T|\|_e = \|T\|$. Let P be the orthogonal projection onto some nontrivial finite dimensional subspace of the eigenspace for $|T|$ correspoding to $\|T\|$. Since $(I-P)\mathfrak{H}$ reduces $|T| - 2\|T\|P$ to $|T|\,|(I-P)\mathfrak{H}$ and $P\mathfrak{H}$ reduces it to $-\|T\|P|P\mathfrak{H}$, it is apparent that $2\|T\|P$ belongs to $\mathfrak{K}_{|T|}$. Thus $2\|T\|UP$ belongs to $\mathfrak{K}_T$ and the proof of the theorem is complete.

References

[1] S. K. Berberian, The Weyl spectrum of an operator, *Indiana Univ. Math. J.*, **29** (1970), 529—544.

[2] P. A. Fillmore, J. G. Stampfli and J. P. Williams, On the essential numerical range, the essential spectrum, and a problem of Halmos, *Acta Sci. Math.*, **33** (1972), 179—192.

[3] I. C. Gohberg and M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Transl. Math. Monographs 18, Amer. Math. Soc. (Providence, 1969).

[4] R. B. Holmes and B. R. Kripke, Best approximation by compact operators, *Indiana Univ. Math. J.*, **21** (1971), 255—263.

[5] C. L. Olsen, Extreme points and finite rank operators in the set of compact approximants, *Indiana Univ. Math. J.*, **24** (1974), 409—416.

[6] F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Ungar (New York, 1955).

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF GEORGIA
ATHENS, GEORGIA 30602
USA.

Acta Sci. Math., 44 (1982), 13—16

Tolerance Hamiltonian varieties of algebras

IVAN CHAJDA

The concept of Hamiltonian algebras was first introduced for groups. *A group* $\mathfrak{G}$ *is Hamiltonian* if every subgroup of $\mathfrak{G}$ is normal, i.e., a class of some congruence on $\mathfrak{G}$. EVANS [6] introduced the Hamiltonian property for loops and KLUKOVITS [7] generalized this concept for universal algebras and varieties: *an algebra* $\mathfrak{A}$ *is Hamiltonian* if every subalgebra of $\mathfrak{A}$ is a class of some congruence on $\mathfrak{A}$; *a variety* $\mathscr{V}$ *is Hamiltonian* if each $\mathfrak{A}\in\mathscr{V}$ has this property.

In [7], Hamiltonian varieties are characterized by a nice $\forall\exists$-condition. Such conditions are also used for characterizations of varieties fulfilling given tolerance identities [3]. It is natural to ask whether the Hamiltonian property can be extended also for tolerances (see e.g. [5]) and which $\forall\exists$-condition characterizes such varieties.

By a *tolerance* on an algebra $\mathfrak{A}=(A, F)$ is meant a reflexive and symmetric binary relation T on A having the *Substitution Property* with respect to F (i.e. T is a subalgebra of the direct product $\mathfrak{A}\times\mathfrak{A}$). Thus each congruence is a tolerance but not vice versa.

Definition 1. Let T be a tolerance on an algebra $\mathfrak{A}=(A, F)$. Call $\emptyset\neq B\subseteqq A$ a *block of* T provided

(i) $B\times B\subseteqq T$,

(ii) B is a maximal subset of A with respect to (i), i.e. if $B\subseteqq C$ and $C\times C\subseteqq T$, then $B=C$.

Clearly, if a tolerance T on $\mathfrak{A}$ is a congruence on $\mathfrak{A}$, every block of T is a congruence class of T and vice versa. Thus blocks of tolerances are generalizations of congruence classes.

The paper [2] contains a characterization of the property that every block of each tolerance on $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}$. The objective of this paper is to describe the converse situation, namely:

Received November 20, 1979.

Definition 2. *An algebra* $\mathfrak{A}$ *is tolerance Hamiltonian* if every subalgebra of $\mathfrak{A}$ is a block of some tolerance on $\mathfrak{A}$. *A variety* $\mathcal{V}$ *is tolerance Hamiltonian* if each $\mathfrak{A}\in\mathcal{V}$ has this property.

Although [1], [2] contain necessary and sufficient conditions under which a subset of an algebra is a block of some tolerance on it, these conditions cannot be used in the way as Mal'cev's Lemma in [7]. The proof of our Theorem 1 is based on a characterization given by Lemma 3 below.

For the sake of brevity, write $\vec{x}_j$ instead of $x_1, \dots, x_n$ and $\vec{y}_i$ instead of $y_1, \dots, y_m$ if the integers n, m are given.

Theorem 1. *Let* $\mathcal{V}$ *be a variety of algebras. The following conditions are equivalent:*

(1) $\mathcal{V}$ *is tolerance Hamiltonian.*

(2) *For every* $(m+n+k)$*-ary polynomial* p *and for every* $(m+n+1)$*-ary polynomial* t *there exists an* $(m+n+1)$*-ary polynomial* q *over* $\mathcal{V}$ *such that*

$$p\big(t(\vec{y}_i, \vec{x}_j, z) \dots, t(\vec{y}_i, \vec{x}_j, z), x_1, \dots, x_n, v_1, \dots, v_k\big) = t(\vec{y}_i, \vec{x}_j, z)$$

implies

$$p\big(y_1, \dots, y_m, t(\vec{y}_i, \vec{x}_j, z), \dots, t(\vec{y}_i, \vec{x}_j, z), v_1, \dots, v_k\big) = q(\vec{y}_i, \vec{x}_j, z).$$

Let us begin the proof of Theorem 1 with some lemmas. If T is a binary relation on a set $\mathfrak{A}$, we denote $[z]_T=\{a\in A; \langle a, z\rangle\in T\}$.

Lemma 1. *Let* $\mathfrak{A}=(A, F)$ *be an algebra and* $z\in B\subseteqq A$. *The following conditions are equivalent:*

(a) $B=[z]_T$ *for some tolerance* T *on* $\mathfrak{A}$.

(b) *For every* $(m+n)$*-ary algebraic function* φ *over* $\mathfrak{A}$,

$$\varphi(z, \dots, z, b_1, \dots, b_n) = z \quad \textit{for some} \quad b_i\in B$$

implies

$$\varphi(a_1, \dots, a_m, z, \dots, z)\in B \quad \textit{for each} \quad a_j\in B.$$

Proof. $(a)\Rightarrow(b)$: Routine.

$(b)\Rightarrow(a)$: Let $R=\{\langle x, x\rangle; x\in A\}\cup\{\langle x, z\rangle; x\in B\}\cup\{\langle z, x\rangle; x\in B\}$. Let T be the set of all $\langle a, b\rangle$ such that $a=\varphi(a_1, \dots, a_k)$, $b=\varphi(b_1, \dots, b_k)$ for some $\langle a_i, b_i\rangle\in R$ and for some algebraic function φ over $\mathfrak{A}$. Clearly T is a tolerance on $\mathfrak{A}$. It only remains to prove $B=[z]_T$. Evidenty, $B\subseteqq[z]_T$. Suppose $c\in[z]_T$. Then $\langle c, z\rangle\in T$, i.e. $c=\psi(a_1, \dots, a_k)$, $z=\psi(b_1, \dots, b_k)$ for some $\langle a_i, b_i\rangle\in R$ and some k–ary algebraic function ψ. We can suppose, that $k=m+n+k'$ $(m\geqq 0, n\geqq 0, k'\geqq 0)$, moreover, $b_i=z$ for $i=1, \dots, m$ and $a_i=z$ for $i=m+1, \dots, m+n$ and $a_i=b_i$ for $i=m+n+1, \dots, k$. Put

$$\varphi(\xi_1, \dots, \xi_{m+n}) = \psi(\xi_1, \dots, \xi_{m+n}, a_{m+n+1}, \dots, a_k).$$

Since $z=\varphi(z, ..., z, b_1, ..., b_n)$, by (b) we obtain $c=\varphi(a_1, ..., a_m, z, ..., z)\in B$ proving the reverse inclusion $[z]_T\subseteq B$.

Lemma 2. *Let* $\mathfrak{A}=(A, F)$ *and let* T *be a tolerance on* $\mathfrak{A}$. *For* $\emptyset\neq B\subseteq A$ *the following conditions are equivalent:*

(a) *B is a block of T.*

(b) $B=\bigcap\{[z]_T;\ z\in B\}$.

Proof. Routine.

Lemma 3. *Let* $\mathfrak{A}=(A, F)$ *and* $\emptyset\neq B\subseteq A$. *The following conditions are equivalent:*

(a) *B is a block of some tolerance on $\mathfrak{A}$.*

(b) *For every $(m+n)$-ary algebraic function φ over $\mathfrak{A}$ and for each $z\in B$,*

$$\varphi(z, \dots, z, b_1, \dots, b_n) = z \quad \textit{for some} \quad b_i\in B$$

implies

$$\varphi(a_1, \dots, a_m, z, \dots, z)\in B \quad \textit{for each} \quad a_j\in B.$$

This follows directly from Lemmas 1 and 2.

Proof of Theorem 1. (1)⇒(2): Let p and t be $(m+n+k)$-ary and $(m+n+1)$-ary polynomials over $\mathscr{V}$, respectively, such that

$$(*)\qquad p\big(t(\vec{y}_i, \vec{x}_j, z), \dots, t(\vec{y}_i, \vec{x}_j, z), x_1, \dots, x_n, v_1, \dots, v_k\big) = t(\vec{y}_i, \vec{x}_j, z).$$

Let $\mathfrak{A}=(A, F)=\mathfrak{F}_{m+n+k+1}$ be the $\mathscr{V}$-free algebra with the set of free generators $\{x_1, ..., x_n, y_1, ..., y_m, v_1, ..., v_k, z\}$ and $\mathfrak{B}=(B, F)=\mathfrak{F}_{m+n+1}$ the $\mathscr{V}$-free algebra with generators $\{x_1, ..., x_n, y_1, ..., y_m, z\}$. Hence $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$. Since $\mathscr{V}$ is tolerance Hamiltonian, B is a block of some tolerance on $\mathfrak{A}$. By Lemma 3, $(*)$ yields $p(y_1, ..., y_m, t(\vec{y}_i, \vec{x}_j, z), ..., t(\vec{y}_i, \vec{x}_j, z), v_1, ..., v_k)\in B$. Since $\mathfrak{B}=\mathfrak{F}_{m+n+1}$ there exists an $(m+n+1)$-ary polynomial q over $\mathscr{V}$ such that (2) of Theorem 1 is valid.

(2)⇒(1): Let $\mathscr{V}$ be a variety fulfilling (2), $\mathfrak{A}=(A, F)\in\mathscr{V}$, $\mathfrak{B}=(B, F)$ a subalgebra of $\mathfrak{A}$ and $z\in B$. Let φ be an arbitrary $(m+n)$-ary algebraic function over $\mathfrak{A}$ and p its generating polynomial, i.e. $\varphi(\xi_1, ..., \xi_{m+n})=p(\xi_1, ..., \xi_{m+n}, c_1, ..., c_k)$ for some $c_1, ..., c_k\in A$. If $\varphi(z, ..., z, b_1, ..., b_n)=z$ for some $b_i\in B$, then, by (2),

$$\varphi(a_1, \dots, a_m, z, \dots, z) = q(a_1, \dots, a_m, b_1, \dots, b_n, z)\in B$$

for each $a_1, ..., a_m\in B$. By Lemma 3, $\mathfrak{A}$ and also $\mathscr{V}$ are tolerance Hamiltonian.

Theorem 2. *The tolerance Hamiltonian property is local, i.e. an algebra* $\mathfrak{A}$ *is tolerance Hamiltonian if and only if every finitely generated subalgebra of* $\mathfrak{A}$ *is a block of some tolerance on* $\mathfrak{A}$.

Proof. It is a direct consequence of Lemma 3: if $\mathfrak{B}=(B, F)$ is a subalgebra of $\mathfrak{A}$ which is not a block of any tolerance on $\mathfrak{A}$ and every finitely generated subalgebra is, then there exist $z\in B$ and an $(m+n)$-ary algebraic function φ over $\mathfrak{A}$ such that $\varphi(z, \ldots, z, b_1, \ldots, b_n)=z$ and $\varphi(a_1, \ldots, a_m, z, \ldots, z)\notin B$ for some $a_1, \ldots, a_m, b_1, \ldots, b_n\in B$. Hence the subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ generated by $\{a_1, \ldots, a_m, b_1, \ldots, b_n, z\}$ is not a block of any tolerance on A which contradicts the assumptions. The converse implication is trivial.

Theorem 3. *The variety of all semilattices is tolerance Hamiltonian (but not Hamiltonian).*

Proof. If p, t are semilattice polynomials fulfilling the assumptions of the condition (2) of Theorem 1, then clearly p does not depend on $v_1, \ldots, v_k$ and the statement of (2) is evident. Thus Theorem 3 is a direct consequence of Theorem 1. By the theorem of KLUKOVITS [7], this variety is evidently not Hamiltonian.

Remark. As it was proved by ZELINKA [8], on every at least three element semilattice there exists a tolerance which is not a congruence.

Theorem 4. *No non-trivial variety of lattices is tolerance Hamiltonian.*

Proof. Let p and t be $(2+0+1)$-ary (i.e. ternary) lattice polynomials given as follows:

$$p(x, y, z) = x\vee(y\wedge z), \quad t(x, y, z) = z.$$

Then we have $p(t(y_1, y_2, z), t(y_1, y_2, z), v_1)=p(z, z, v_1)=z=t(y_1, y_2, z)$, thus the assumptions of (2) of Theorem 1 are valid, but $p(y_1, y_2, v_1)$ is essentially dependent on v_1. Hence, no polynomial q of the required type exists.

References

[1] I. CHAJDA, Partitions, coverings and blocks of compatible relations, *Glasnik Matem. Zagreb,* **14** (1979), 21—26.

[2] I. CHAJDA, Characterizations of relational blocks, *Algebra Universalis,* **10** (1980), 65—69.

[3] I. CHAJDA, Distributivity and modularity of lattices of tolerances, *Algebra Universalis,* **12** (1981), 247—255.

[4] I. CHAJDA, J. DUDA, Blocks of binary relations, *Ann. Univ. Sci. Budapest,* Sectio Math., **22—23** (1979—1980), 3—9.

[5] I. CHAJDA—B. ZELINKA, Lattices of tolerances, *Časop. pěst. matem.,* **102** (1977), 10—24.

[6] T. EVANS, Properties of algebras almost equivalent to identities, *J. London Math. Soc.,* **35** (1962), 53—59.

[7] L. KLUKOVITS, Hamiltonian varieties of universal algebras, *Acta Sci. Math.,* **37** (1975), 11—15.

[8] B. ZELINKA, Tolerance relations on semilattices, *CMUC* **16, 2** (1975), 333—338.

TŘÍDA LIDOVÝCH MILICÍ 22
750 00 PŘEROV
CZECHOSLOVAKIA

Acta Sci. Math., 44 (1982), 17—21

The range of the transform of certain parts of a measure

KERRITH B. CHAPMAN and LOUIS PIGNO

In this note we point out a very elementary condition which provides a uniform treatment for the results in [2, 4, 5] concerning the range of the transform of certain parts of a measure. We assume familiarity with the basic facts of [8].

Let G be a nondiscrete LCA group with character group Γ and let $M(G)$ denote the customary convolution algebra of bounded Borel measures on G. Denote by S the structure semi-group of $M(G)$ and let $\hat{S}$ denote the semi-characters of S; recall that $\hat{S}$ is the maximal ideal space of $M(G)$, see [8]. For $\mu \in M(G)$ let $\hat{\mu}$ denote the Gelfand transform defined on $\hat{S}$ by

$$\hat{\mu}(\chi) = \int_S \chi d\mu$$

where we have identified μ and the image of μ in $M(S)$; we will also let $\hat{}$ denote the usual Fourier—Stieltjes transformation. By $M_0(G)$ we mean the set of $\mu \in M(G)$ such that $\hat{\mu}$ vanishes at infinity, i.e. $\hat{\mu}$ is zero on $\bar{\Gamma} \setminus \Gamma$.

The main result of this paper is the theorem stated below; its proof is quite simple. After stating and proving our theorem, we present two examples which serve to indicate its scope. Example 1 is obtained by adapting the work of B. HOST and F. PARREAU [3]. In order to present Example 2, we prove a proposition by modifying an argument of I. GLICKSBERG and I. WIK [2]. Professor Glicksberg has kindly pointed out (private communication) that the proposition is also a consequence of the main result of [1].

Theorem. Let $h \in \bar{\Gamma} \setminus \Gamma$ *and* $E \setminus \Gamma$. *Then for every* $\mu \in M(G)$,

$$(1) \qquad (h\mu)\hat{}(\Gamma) \subset \hat{\mu}(\Gamma \setminus E)^-$$

if and only if

$$(2) \qquad h \in (\Gamma \setminus \gamma E)^- \quad \textit{for every} \quad \gamma \in \Gamma.$$

Received March 25, 1981.

Proof. Let E satisfy (2) with respect to some $h\in\bar{\Gamma}\setminus\Gamma$. Fix $\gamma_0\in\Gamma$; since $h\in(\Gamma\setminus\gamma_0^{-1}E)^-$ there is a net $\langle\gamma_j\rangle\subset\Gamma$ such that $\gamma_j\to h$ and $\gamma_j\notin\gamma_0^{-1}E$ for all j. Observe that

$$(h\mu)^\wedge(\gamma_0)=\hat{\mu}(\gamma_0 h)=\lim_j\hat{\mu}(\gamma_0\gamma_j)$$

because $\hat{\mu}$ is continuous on $\hat{S}$. Thus (2) implies (1).

Now let $h\in\bar{\Gamma}\setminus\Gamma$ and suppose for every $\mu\in M(G)$, $(h\mu)^\wedge(\Gamma)\subset\hat{\mu}(\Gamma\setminus E)^-$; we want to see that E satisfies (2) with respect to h. With this in mind fix $\gamma_0\in\Gamma$ and let V be any open set of $\hat{S}$ containing $\{h\}$. It suffices to confirm that $V\cap(\Gamma\setminus\gamma_0 E)$ is not empty.

Let $W=\bar{\gamma}_0 V$. Then W is an open set containing $\{\bar{\gamma}_0 h\}$; by the definition of the Gelfand topology on $\hat{S}$ there exist measures $\mu_1,\dots,\mu_n\in M(G)$ and $\varepsilon>0$ such that

$$\bigcap_{i=1}^{n}\{\chi\colon|\hat{\mu}_i(\chi)-\hat{\mu}_i(\bar{\gamma}_0 h)|<\varepsilon\}\subseteqq W.$$

For $\mu\in M(G)$ put $\tilde{\mu}$ equal to the measure such that $(\tilde{\mu})^\wedge=\bar{\hat{\mu}}$ on Γ and let δ_0 be the identity measure in $M(G)$. Define auxiliary measures by:

$$\nu_i=\mu_i-\hat{\mu}_i(\bar{\gamma}_0 h)\delta_0\quad\text{and}\quad\sigma_i=\nu_i*\tilde{\nu}_i;\quad i=1,2,\dots,n.$$

Put $\sigma=\sum_{i=1}^{n}\sigma_i$; now, on the one hand, $\hat{\sigma}(h\bar{\gamma}_0)=0$, while, on the other,

$$(h\sigma)^\wedge(\bar{\gamma}_0)=\hat{\sigma}(h\bar{\gamma}_0)\in\hat{\sigma}(\Gamma\setminus E)^-$$

by hypothesis.

We gather from all this that there is a net $(\gamma_\alpha)\subset\Gamma\setminus E$ such that $\hat{\sigma}(\gamma_\alpha)\to 0$. Now given $\varepsilon>0$ choose α' such that for all $\alpha\geqq\alpha'$

$$|\hat{\sigma}(\gamma_\alpha)|<\varepsilon^2;$$

consequently for all $\alpha\geqq\alpha'$

$$\sum_{i=1}^{n}|\hat{\mu}_i(\gamma_\alpha)-\hat{\mu}_i(\bar{\gamma}_0 h)|^2<\varepsilon^2.$$

Thus $|\hat{\mu}_i(\gamma_\alpha)-\hat{\mu}_i(\bar{\gamma}_0 h)|<\varepsilon$ for $\alpha\geqq\alpha'$, and so $\gamma_\alpha\in W$ for all $\alpha\geqq\alpha'$.

We have now proved that if $\alpha\geqq\alpha'$, $\gamma_0\gamma_\alpha\in V\cap(\Gamma\setminus\gamma_0 E)$; thus $h\in(\Gamma\setminus\gamma_0 E)^-$ and this means that (1) implies (2).

Let G be an infinite compact abelian group; a subset $R\subset\Gamma$ is called a *Rajchman set* if whenever $\mu\in M(G)$ and $\operatorname{supp}\hat{\mu}\subset R$ then $\mu\in M_0(G)$; here $\hat{}$ is the Fourier—Stieltjes transformation. Examples of Rajchman sets can be found in [7]; all the sets considered in [4, 5] are Rajchman sets.

Example I. If R is a Rajchman set then R satisfies (2) with respect to every idempotent $h\in\bar{\Gamma}\setminus\Gamma$; we point out that this fact is more or less implicit in [3]. To be explicit we need to reproduce some details from [3].

To confirm that R satisfies (2) with respect to every $h=h^2\in\bar{\Gamma}\setminus\Gamma$ we fix an $h_0^2=h_0\in\bar{\Gamma}\setminus\Gamma$ and suppose by way of contradiction that there is a $\gamma_0\in\Gamma$ such that $h_0\notin(\Gamma\setminus\gamma_0 R)^-$. Thus, there is an open set V_0 with $h_0\in V_0$ such that $V_0\cap(\Gamma\setminus\gamma_0 R)$ is empty and $1\notin V_0$. For the remainder of the proof, $^-$ is complex conjugation.

By the definition of the Gelfand topology on $\hat{S}$ there exist measures $\mu_1, \ldots, \mu_n\in$ $\in M(G)$ and $\varepsilon>0$ so that $\bigcap_{i=1}^{n}\{\chi\colon |\hat{\mu}_i(\chi)-\hat{\mu}_i(h_0)|<\varepsilon\}$ is open and contained in V_0. Put $A_i=\{z\in\mathbf{C}\colon |z-\hat{\mu}_i(h_0)|<\varepsilon\}$ and consider the open set $\bigcap_{i=1}^{n}\{(h_0\mu_i)\hat{\ }\}^{-1}(A)$; since $h_0=h_0^2$ it follows that $h_0\in\bigcap_{i=1}^{n}\{(h_0\mu_i)\hat{\ }\}^{-1}(A_i)\}$ and therefore

$$W_1=\left\{\bigcap_1^n \hat{\mu}_i^{-1}(A_i)\right\}\cap\left\{\bigcap_1^n (h_0\mu_i)\hat{\ }^{-1}(A_i)\right\}$$

is an open set about h_0. Put $W_1^*=\{\chi\colon\chi\in\overline{W}_1\}$ and define $V_1=W_1\cap W_1^*$; since $h_0=\bar{h}_0$ we see that $V_1\subset V_0$ and V_1 is an open set about h_0. Choose $\beta_1\in\Gamma$ such that $\beta_1, \beta_1^{-1}\in V_1$. Next define $B_1=\{\beta_1, \beta_1^{-1}, 1\}$; let

$$\begin{aligned}W_2 &= \{\chi\colon \chi\in\{(\beta\mu_i)\hat{\ }\}^{-1}(A_i) \text{ for all } i \text{ and all } \beta\in B_1\}\cap\\ &\cap\{\chi\colon \chi\in\{(\beta h_0\mu_i)\hat{\ }\}^{-1}(A_i) \text{ for all } i \text{ and all } \beta\in B_1\}\end{aligned}$$

and $V_2=W_2\cap W_2^*$; evidently $V_2\subset V_1$ and $h_0\in V_2$. Since V_2 is open and B_1 is finite we select $\beta_2\in\Gamma$ such that $\beta_2\in V_2\setminus B_1$.

Put $B_2=\{\beta=\prod_{i=1}^{2}\beta_i^{\delta_i}\colon \delta_i\in\{-1,0,1\}\}\cup\{1\}$; let

$$\begin{aligned}W_3 &= \{\chi\colon \chi\in\{(\beta\mu_i)\hat{\ }\}^{-1}(A_i) \text{ for all } i \text{ and all } \beta\in B_2\}\cap\\ &\cap\{\chi\colon \chi\in\{(\beta h_0\mu_i)\hat{\ }\}^{-1}(A_i) \text{ for all } i \text{ and all } \beta\in \beta_2\}\end{aligned}$$

and $V_3=W_3\cap W_3^*$; evidently $V_3\subset V_2$ and $h_0\in V_3$. Since V_3 is open and B_2 is finite we select $\beta_3\in\Gamma$ such that $\beta_3\in V_3\setminus B_2$. Continuing in this manner we inductively construct a sequence of distinct characters $\langle\beta_j\rangle_1^\infty$ such that $\beta=\prod_{i=1}^{j}\beta_i^{\delta_i}$, $\delta_i\in\{-1,0,1\}$ and $\delta_i\neq 0$ for some i, then $\beta\in V_0$; since $\beta\in V_0\cap\Gamma$, this means that $\beta\gamma_0^{-1}\in R$ for all β of the form $\beta=\prod_{i=1}^{j}\beta_i^{\delta_i}$, $\delta_i\in\{-1,0,1\}$. As shown in [3] (see Theorem 2.8 of [6, p. 21]) there is a dissociate sequence $\langle\omega_p\rangle_1^\infty$ with the property that if ω is of the form $\omega=\prod_{i=1}^{k}\omega_i^{\delta_i}$, $\delta_i\in\{-1,0,1\}$, then ω is also of the form $\omega=\prod_{j=1}^{n}\beta_j^{m_j}$, $m_j\in\{-1,0,1\}$.

Since $\langle\omega_p\rangle_1^\infty$ is dissociate we may now construct a Riesz product $\lambda\in M(G)$ such that $\operatorname{supp}\hat{\lambda}\subset R$ and $\lambda\notin M_0(G)$; this contradicts the fact that R is a Rajchman set and so our discussion is complete.

The above example is not the only one we know: Let $\mathbf{R}$ denote the additive group of real numbers and let $\varphi : \Gamma \to \mathbf{R}$ be a nontrivial homomorphism. A measure $\mu \in M(G)$ is said to *vanish at infinity in the direction of* φ if whenever $\varphi(\gamma_j) \to +\infty$ then $\hat{\mu}(\gamma_j) \to 0$; denote the set of all measures vanishing at infinity in the direction of φ by $M_\varphi(G)$. A subset $R \subset \Gamma$ is said to be *φ-Rajchman* if for $\mu \in M(G)$ and $\operatorname{supp} \hat{\mu} \subset R \Rightarrow \mu \in M_\varphi(G)$. Then it can be shown that if E is φ-Rajchman, E satisfies (2) with respect to various h's. Notice that in general there are φ-Rajchman sets which are not Rajchman sets; let $\Gamma = \{m + n\sqrt{2} : m, n \in \mathbf{Z}\}$ and let φ be the identity homomorphism of Γ into $\mathbf{R}$. Then the set $\{x \in \Gamma : x \geqq 0\}$ is φ-Rajchman but not Rajchman.

Although φ-Rajchman sets and Rajchman sets are the same for the additive group of integers $\mathbf{Z}$, there do exist non-Rajchman subsets of $\mathbf{Z}$ which determine the range of the transform of certain parts of a measure. For the circle group $\mathbf{T}$ put $\mu = \mu_d + \mu_c$ where $\mu \in M(\mathbf{T})$, μ_d is discrete and μ_c continuous.

Let $\beta(\mathbf{Z})$ denote the Bohr compactification of $\mathbf{Z}$ and for $E \subset \mathbf{Z}$ let $\bar{E}$ be the closure of E in $\beta(\mathbf{Z})$. Our result is then:

Proposition. *If $E \subset \mathbf{Z}$ and $\mathbf{Z} \setminus \bar{E}$ is dense in $\beta(\mathbf{Z})$ then for $\mu \in M(\mathbf{T})$*

$$\hat{\mu}_d(\mathbf{Z}) \subset \hat{\mu}(\mathbf{Z} \setminus E)^-.$$

Proof. For $\mu \in M(\mathbf{T})$ write $\mu = \mu_d + \mu_c$; fix $0 < \varepsilon < 1$ and $m_0 \in \mathbf{Z} \setminus \bar{E}$. We see from [2] that there is an infinite sequence $\langle m_i \rangle_1^\infty$ of distinct integers satisfying

$$|\hat{\mu}_c(m_0 + m_n - m_j)| < \frac{\varepsilon}{2} \quad \text{for} \quad j < n. \tag{2.1}$$

Put $H = \langle m_i \rangle_1^\infty$ and consider $\bar{H}$ where the closure is of course taken in $\beta(\mathbf{Z})$. Since $\operatorname{card} H = \infty$, there is an $x \notin \mathbf{Z}$ and a net $m_\alpha \in H$, $\alpha \in A$ such that $m_\alpha \to x \in \beta(\mathbf{Z})$.

Inasmuch as $m_0 \in \mathbf{Z} \setminus \bar{E}$ it follows that there is an $\alpha_0 \in A$ such that for all α and β greater than α_0

$$m_0 + m_\alpha - m_\beta \notin E \tag{2.2}$$

and

$$|\hat{\mu}_d(m_0) - \hat{\mu}_d(m_0 + m_\alpha - m_\beta)| < \frac{\varepsilon}{2}. \tag{2.3}$$

Notice that (2.3) is valid since $\hat{\mu}_d$ is a continuous function on $\beta(\mathbf{Z})$. As a consequence of (2.2) and (2.3) there is a $k \geqq 1$ and an $r > k$ such that $m_0 + m_r - m_k \notin E$ and $|\hat{\mu}_d(m_0) - \hat{\mu}_d(m_0 + m_r - m_k)| < \frac{\varepsilon}{2}$. Since

$$|\hat{\mu}_d(m_0) - \hat{\mu}(m_0 + m_r - m_k)| \leqq |\hat{\mu}_d(m_0) - \hat{\mu}_d(m_0 + m_r - m_k)| + |\hat{\mu}_c(m_0 + m_r - m_k)|,$$

and $r>k$, we gather from (2.1) that

$$|\hat{\mu}_d(m_0)-\hat{\mu}(m_0+m_r-m_k)| \leqq \varepsilon.$$

Thus $\hat{\mu}_d(\mathbf{Z}\setminus\bar{E})\subset\hat{\mu}(\mathbf{Z}\setminus E)^-$ and since $\mathbf{Z}\setminus\bar{E}$ is dense in $\beta(\mathbf{Z})$ we obtain $\hat{\mu}_d(\mathbf{Z})\subset\subset\hat{\mu}(\mathbf{Z}\setminus E)^-$. The proof is complete.

Example II. Let $\mathbf{N}$ be the natural numbers and for each $n\in\mathbf{N}$ put $E_n=\{m : m=\sum_{i=1}^{n}\delta_i 5^i,\ \delta_i\in\{-1,0,1\}\}$; set $E=\bigcup_1^\infty E_n$. Let $\mathbf{D}=\{e^{2\pi ik/5^j} : k\in\mathbf{Z}, j\in\mathbf{N}\}$ and consider E as a subset of $\hat{\mathbf{D}}$ where $\mathbf{D}$ is given the discrete topology. Now the integer accumulation points of E in $\hat{\mathbf{D}}$ belong to E so it follows that E is a closed subset of $\mathbf{Z}$ in the relative tolopogy of $\beta(\mathbf{Z})$. Notice that E has natural density zero so by Wiener's Theorem it follows that if supp $\hat{\varrho}\subset E$ then ϱ is continuous and this in turn implies that $\mathbf{Z}\setminus E=\mathbf{Z}\setminus\bar{E}$ is dense in $\beta(\mathbf{Z})$. Clearly E is not a Rajchman set since it contains the spectrum of an infinite Riesz product.

Remark. An easy application of Theorem 1 and Corollary 2 of [5, p. 2] establishes the following assertion: *Let* $E\subset\Gamma$ *satisfy* (2) *with respect to some* $h=h^2\in\in\bar{\Gamma}\setminus\Gamma$ *and let* S *be an infinite Sidon subset of* Γ; *then* $E\cup S$ *satisfies* (2) *with respect to* h.

The authors wish to thank Professor Colin Graham of Northwestern University for helpful correspondence.

References

[1] I. Glicksberg, Weakly almost periodic functions and generators of invariant filters, *Duke Math. J.*, **42** (1975), 409—422.

[2] I. Glicksberg and I. Wik, The range of Fourier—Stieltjes transforms of parts of measures. *Conference on Harmonic Analysis Lecture Notes in Mathematics*, **266**, 73—77.

[3] B. Host et F. Parreau, Sur les mesures dont les transformées de Fourier—Stieltjes ne tendent pas vers 0 à l'infini, *Colloq. Math.*, **41** (1979), 297—301.

[4] K. Izuchi, Sidon sets and Fourier—Stieltjes transforms of some prime L-ideals, *Math. Scand.*, **40** (1977), 46—50.

[5] K. Izuchi, Cluster points of the characters in the maximal ideal space of the measure algebra, *Rep. Fac. Engrg., Kanagawa Univ.*, **16** (1978), 1—3.

[6] J. M. López and K. A. Ross, *Sidon Sets*, Marcel Dekker, Inc. (New York, 1975).

[7] L. Pigno, Fourier—Stieltjes transforms which vanish at infinity off certain sets, *Glasgow Math. J.*, **19** (1978), 49—56.

[8] J. L. Taylor, *Measure Algebras*, CBMS Regional Conference, No. **16**, 1973.

KANSAS STATE UNIVERSITY
MANHATTAN, KANSAS, U.S.A.

Acta Sci. Math., 44 (1982), 23—34

On random censorship from the right

SÁNDOR CSÖRGŐ and LAJOS HORVÁTH

1. Introduction. BURKE *et al.* [4] introduced the following censorship model. Let X be a real random variable with distribution function $F(t)=\text{pr}\,\{X<t\}$. For a fixed integer $k\geqq 1$ let $A^1, \dots, A^k$ be pairwise disjoint random events, and define the sub-distribution function $F^i(t)=\text{pr}\,\{X<t, A^i\}, i=1, \dots, k$. We are interested in the joint behaviour of the pairs (X, A^i) as expressed by

$$S^i(t) = \exp\left(-\Lambda^i(t)\right), \quad i = 1, \dots, k,$$

where Λ^i is the i-th hazard function $\left(\int\limits_{-\infty}^{t} = \int\limits_{(-\infty,t)}\right)$

$$\Lambda^i(t) = \int\limits_{-\infty}^{t} (1-F(s))^{-1}\, dF^i(s).$$

So let $\{X_n, A_n^1, \dots, A_n^k\}$ be a sequence of independent replicas of $\{X, A^1, \dots, A^k\}$, $n=1, 2, \dots$. *We assume throughout that the functions* $F, F^1, \dots, F^k$ *are continuous.* Define the product-limit estimates

$$\tilde{S}_n^i(t) = 1-\tilde{F}_n^i(t) = \begin{cases} \prod\limits_{\{1\leq j\leq n:\, X_j<t\}} \left(\dfrac{n-R_j^i}{n-R_j^i+1}\right)^{\delta_j^i}, & \text{if} \quad t < \max(X_1, \dots, X_n) \\ 0, & \text{otherwise,} \end{cases}$$

$i=1, \dots, k$, where δ_j^i is the indicator of A_j^i, and R_j^i is the rank of $(X_j, 1-\delta_j^i)$ in the lexicographic ordering of the sequence $(X_1, 1-\delta_1^i), \dots, (X_n, 1-\delta_n^i)$. Finally, introduce the i-th product-limit process

$$Z_n^i(t) = n^{1/2}(S^i(t)-\tilde{S}_n^i(t)),$$

and, for $x=(x_1, \dots, x_k)$, the corresponding vector process

$$Z_n(x) = \left(Z_n^1(x_1), \dots, Z_n^k(x_k)\right).$$

Received July 1, 1980.

However general is this model, the most important special cases are a) and b) below. By working in this generality we merely would like to emphasize the fact that the asymptotic theory of random censorship on the right requires only the above structure. When dealing with random censorship on the left the basic ingredients S^i and $\tilde{S}_n^i$ should be accordingly modified. This is done in [9].

a) Let $X_1^0, X_2^0, \dots$ be a sequence of independent random variables with common continuous distribution function F^0. These are censored on the right by $Y_1, Y_2, \dots$ a sequence of independent random variables, independent of the X^0 sequence, with common continuous distribution function H. One can only observe the sequence $(X_n=\min(X_n^0, Y_n), \delta_n)$, where $\delta_n=\delta_n^1$ is the indicator of $A_n=A_n^1=\{X_n=X_n^0\}$. In this case $k=2, 1-F=(1-F^0)(1-H), F^1(t)=\int_{-\infty}^{t}(1-H)\,dF^0$, thus $S^1(t)=S(t)=1-F^0(t)$, and $\tilde{S}_n^1=\tilde{S}_n$ reduces to the usual product-limit estimate. This is the KAPLAN—MEIER [15] model as defined by EFRON [12]. It was investigated by BRESLOW and CROWLEY [3], MEIER [19], HALL and WELLNER [14], BURKE *et al.* [4] and others. The useful special case when $1-H=(1-F^0)^\beta, \beta>0$, was considered by KOZIOL and GREEN [18], and their model was investigated by CSÖRGŐ and HORVÁTH [7] and KOZIOL [17].

b) For $k>1$ consider k independent sequences $Y_1^i, Y_2^i, \dots$ $(i=1, \dots, k)$ of independent random variables with common continuous distribution function H^i, and let $X_n=\min(Y_n^1, \dots, Y_n^k)$. One observes the sequences $(X_n, \delta_n^i), i=1, \dots, k$, where δ_n^i is the indicator of the event $A_n^i=\{X_n=Y_n^i\}$. This is the competing risks model (giving back the above Kaplan—Meier model for $k=2$) considered by many authors, notably, from the present viewpoint, by YANG [22] and BURKE *et al.* [4]. Here, as BERMAN [2] proved, the above S^i reduces to $S^i(t)=1-H^i(t)$.

On the basis of the Efron-transformed variant of the Breslow—Crowley weak convergence theorem, GILLESPIE and FISHER [13] constructed asymptotic confidence bands for the survival curve $1-F^0$ in the Kaplan—Meier model. However, their Monte Carlo study has shown that sample size $n=200$ is not large enough to apply the asymptotic bands with high precision. Their results were a strong motivation for us to work out a strong approximation theory in [4] for the above general Z_n and related processes. A variant of one of the main approximation theorems is formulated in the next section. This result enabled us to build the approximation rates into the construction of the Gillespie—Fisher type bands, i.e., we could construct "exact" confidence bands ([4]) for the general survival functions S^i under the "i-th risk A^i". We also indicated that these constructions should give reasonable bands for much less sample sizes than the asymptotic ones of Gillespie and Fisher.

HALL and WELLNER [14] utilized Doob's transformation of the Brownian motion into the Brownian bridge, and hence proposed the corresponding transformation of the product-limit process in the Kaplan—Meier model. The resulting process converges weakly to a transformed Brownian bridge. Although Doob's transformation belongs to the statistical folklore nowadays, its use by Hall and Wellner in the present context is a remarkable step in the asymptotic theory of censored empirical processes. The resulting asymptotic confidence bands constructed by HALL and WELLNER [14] enjoy many advantages over those of GILLESPIE and FISHER [13] as they explain it in detail. For example, they reduce in the uncensored case to the classical Kolmogorov bands. Following HALL and WELLNER [14], KOZIOL [17] considered Kolmogorov, Kuiper and Cramér—von Mises statistics corresponding to the transformed product-limit process in the Kaplan—Meier model for testing goodness of fit (cf. Section 3 here).

Following AALEN [1], NAIR [20] proposed another clever transformation of the product-limit process in the Kaplan—Meier model. It is a modification of Efron's transformation, where the limit process is a scale-changed Wiener process. The rescaling depends on censoring, but the Kolmogorov—Smirnov, Kuiper and Cramér—von Mises functionals of this process are distribution-free.

The aim of the present note is to develop strong approximation theorems corresponding to the transformations of Hall and Wellner and of Aalen and Nair in the general setting of the first paragraph of this section. This is done in Sections 3 and 4, respectively, after some preliminaries from Burke *et al.* Convergence rates are deduced from these theorems for the above mentioned statistics in Sections 3 and 4. Using the approximation rates, we show in Section 5 a possibility for making exact the asymptotic bands of Hall and Wellner. This is done again in the general setting. Comulative hazard processes are investigated in a similar manner by CSÖRGŐ and HORVÁTH [8].

2. Preliminaries. Let $T_F = \inf\{t : F(t) = 1\}$ and define

$$d^i(t) = \begin{cases} \int\limits_{-\infty}^{t} (1-F(s))^{-2}\, dF^i(s), & t < T_F \\ \int\limits_{-\infty}^{T_F} (1-F(s))^{-2}\, dF^i(s), & t \geqq T_F, \end{cases} \tag{2.1}$$

$i=1, \ldots, k$. The last integral $\int\limits_{-\infty}^{T_F}$ here is either finite or infinite. Consider

$$\tilde{Z}_n^i(t) = \exp(\Lambda^i(t)) Z_n^i(t),$$

$i=1, \ldots, k$, and for $x=(x_1, \ldots, x_k)$ the corresponding vector process

$$\tilde{Z}_n(x) = (\tilde{Z}_n^1(x_1), \ldots, \tilde{Z}_n^k(x_k)).$$

If a^i denotes the inverse of d^i, then the vector process with components $\tilde{Z}_n^i(a^i(x_i))$,

$x \in (0, \infty)^k$, is the one which was called by BURKE *et al.* [4] as the Efron transform of Z_n. All of our approximations will take place on the infinite cube $(-\infty, T_n]^k$ where T_n is a sequence of numbers satisfying first the condition:

$$(2.2) \qquad T_n < T_F \quad \text{and} \quad 1-F(T_n) \geqq (2\varepsilon n^{-1} \log n)^{1/2}$$

where, throughout this note, ε is some arbitrarily fixed positive number. Let

$$(2.3) \qquad b_n = \left(1-F(T_n)\right)^{-1},$$

and introduce the following (rather messy) rate-sequence

$$(2.4) \qquad r(n) = v(n) + \frac{1}{2} n^{-1/2} \{v(n) + 3(\varepsilon/2)^{1/2} b_n^2 (\log n)^{1/2}\}^2$$

where

$$v(n) = \left[b_n^2 \{(2k+1)5A_1 + (2+(5(2k+1)/A_3))\varepsilon + ((2/3)\varepsilon+\varepsilon^2)^{1/2}\} + b_n^3 4\varepsilon\right] n^{-1/2} \log n +$$
$$+ b_n^2 (12\varepsilon)^{1/2} n^{-1/3} (\log n)^{1/2} + b_n^2 (2k+1) \{A_1 + (\varepsilon/A_3)\} \varepsilon^{1/2} n^{-1/3} (\log n)^{3/2} + b_n 2n^{-1/2}.$$

For $x = (x_1, \ldots, x_k)$ let $\|x\| = \max(|x_1|, \ldots, |x_k|)$ denote the maximum norm. In [4] we constructed a special probability space $(\Omega, \mathcal{A}, P)$ carrying k sequences $\{W_n^i\}$ of Wiener processes such that for the vector process

$$W_n^d(x) = \left(W_n^1(d^1(x_1)), \ldots, W_n^k(d^k(x_k))\right)$$

we have

Theorem A (BURKE, CSÖRGŐ, HORVÁTH [4]). *If T_n satisfies condition (2.2), then*

$$P\{\sup_{x \in (-\infty, T_n]^k} \|\tilde{Z}_n(x) - W_n^d(x)\| > r(n) b_n\} \leqq kQn^{-\varepsilon},$$

where $Q = 10A_2(2k+1) + 100 + 16D$.

The constants A_1, A_2 and A_3 in $r(n)$ (A_2 also in Q) are the C, K, λ constants of Theorem 3 of KOMLÓS, MAJOR and TUSNÁDY [16] (quoted as Theorem 2. A in [4]), respectively. According to TUSNÁDY [21] (cf. also M. CSÖRGŐ and P. RÉVÉSZ [5]) A_1, A_2 and A_3 can respectively be taken as 100, 10 and 1/50. Constant D (in Q) is the absolute constant of Lemma 2 of DVORETZKY, KIEFER and WOLFOWITZ [11]. The smallest available value for D known to us at present is $2\{1+32/(6\pi)^{1/2} + 8/3^{1/2} + 2^{1/2} 4 \exp(71/18)\} \leqq 611$ due to DEVROYE and WISE [10]. But in practice one can probably use without harm the well-known conjecture (which was empirically verified in a number of situations) that D is 2.

We should also point out here that originally (Theorem 5.6 of [4]) we had the factor $r_E(n) = \max_{1 \leqq i \leqq k} \exp\left(\Lambda^i(T_n)\right)$ of $kr(n)$ instead of b_n. But it is not hard to see that $r_E(n) \leqq b_n$ and the above form of Theorem A is more fortunate since the whole rate-sequence $r(n)b_n$ depends on the censoring only through b_n of (2.3).

3. Approximation theorems for the Hall—Wellner transformation. Goodness of fit. Introduce (with d^i of (2.1))

$$K^i(t) = d^i(t)/(1+d^i(t)), \quad -\infty < t < \infty, \quad i = 1, \ldots, k.$$

$K^i(t)$ is a sub-distribution function in general for each i. It is a distribution function (as HALL and WELLNER [14] point out) in the Kaplan—Meiner model ($k=2$). In the competing risks model K^i is a distribution function for those i for which $T_F = T_{H^i} \leqq \min(T_{H^1}, \ldots, T_{H^k})$, where T_{H^i} is defined analogously to T_F. The empirical counterpart of $d^i(t)$ was considered by BURKE *et al.* [4] as

$$d_n^i(t) = \int_{-\infty}^{t} (1-F_n(s))^{-2}\, dF_n^i(s), \quad i=1, \ldots, k,$$

where F_n is the (left continuous) empirical distribution function of $X_1, \ldots, X_n$ and F_n^i is the empirical sub-distribution function defined as

$$F_n^i(t) = n^{-1} \# \{m:\ 1 \leqq m \leqq n, X_m < t \text{ and } A_m^i \text{ occurs}\}, \quad i = 1, \ldots, k.$$

Independently of us but earlier, HALL and WELLNER [14] have also considered d_n^i (in the Kaplan—Meier model) but pointed out that it fails to satisfy their reduction property. Instead they proposed the following modification of it:

$$c_n^i(t) = \int_{-\infty}^{t} (1-F_n(s))^{-1}(1-F_n^+(s))^{-1}\, dF_n^i(s) = n \sum_{\{j:X_j<t\}} (n-j)^{-1}(n-j+1)^{-1}\delta_j^i$$

where F_n^+ is the right-continuous version of F_n. Although we could have worked with d_n^i, we adopted this modification for the sake of accordance.

Lemma 6.2 of [4] estimates the distance of d_n^i and d^i. Using Lemma 4.1 of that paper, it is trivial that if T_n satisfies condition (2.2), then

$$\mathrm{pr}\Big\{\sup_{-\infty<t\leqq T_n} |d_n^i(t)-c_n^i(t)| > 8b_n^3 n^{-1}\Big\} \leqq 2Dn^{-\varepsilon}.$$

Let

$$K_n^i(t) = c_n^i(t)/(1+c_n^i(t)), \quad i = 1, \ldots, k.$$

Evidently

$$|K_n^i(t)-K^i(t)| \leqq |c_n^i(t)-d^i(t)|,$$

and hence, putting together Lemma 6.2 of [4] and the last probability inequality, we obtain

Lemma 3.1. *If T_n satisfies (2.2), then for each $i=1, \ldots, k$*

$$\mathrm{pr}\Big\{\sup_{-\infty<t\leqq T_n} |K_n^i(t)-K^i(t)| > r_1(n)\Big\} \leqq 8Dn^{-\varepsilon},$$

where $r_1(n) = 12(\varepsilon/2)^{1/2} n^{-1/2} b_n^4 (\log n)^{1/2} + 8b_n^3 n^{-1}$.

Consider now

$$\hat{Z}_n^i(t) = (1-K_n^i(t))\exp(\Lambda^i(t))Z_n^i(t), \quad i = 1, \ldots, k,$$

and let $B_n^i(t)=(1-t)W_n^i(t/(1-t))$ be the sequence of Brownian bridges supplied by $\{W_n^i\}$ of Theorem A. For $x=(x_1, \ldots, x_k)$ let

$$\hat{Z}_n(x) = (\hat{Z}_n^1(x_1), \ldots, \hat{Z}_n^k(x_k))$$

and

$$B_n^K(x) = (B_n^1(K^1(x_1)), \ldots, B_n^k(K^k(x_k))).$$

Theorem 3.2. *If T_n satisfies (2.2), then*

$$P\{\sup_{x\in(-\infty, T_n]^k} \|\hat{Z}_n(x)-B_n^K(x)\| > q_1(n)\} \leqq kR_1 n^{-\varepsilon},$$

where $q_1(n)=r(n)b_n+r_2(n)$ *with* $r_2(n)=2\varepsilon^{1/2}r_1(n)\,b_n(\log n)^{1/2}$, *and* $R_1=Q+8D+2= =10A_2(2k+1)+102+24D$.

Proof. It is enough to show that

$$P\{\sup|\hat{Z}_n^i(t)-B_n^i(K^i(t))| > q_1(n)\} \leqq R_1 n^{-\varepsilon}$$

for each $i=1, \ldots, k$, where unspecified sup means $\sup\limits_{-\infty<t\leqq T_n}$. The last probability is not greater than

$$P\{\sup(1-K_n^i(t))\,|\tilde{Z}_n^i(t)-W_n^i(d(t))| > r(n)\,b_n\}+$$
$$+P\{\sup|K_n^i(t)-K^i(t)|\,|W_n^i(d^i(t))| > r_2(n)\} \leqq$$
$$\leqq (Q+8D)n^{-\varepsilon}+P\{\sup|W_n^i(d^i(t))| > 2\varepsilon^{1/2}b_n(\log n)^{1/2}\} \leqq$$
$$\leqq (Q+8D)n^{-\varepsilon}+2P\{|W_n^i(b_n^2)/b_n| > 2\varepsilon^{1/2}(\log n)^{1/2}\}$$

by Theorem A, Lemma 3.1, and the fact that $b_n^2 \geqq d^i(T_n)$. The last probability is less than or equal to $n^{-\varepsilon}$, and hence the theorem.

The components $\hat{Z}_n^i$ of our vector (-vector) process are in fact weighted processes, the weight being $\exp(\Lambda^i(t))$. It is then natural to replace this weight with an empirical counterpart of it and investigate the convergence of the resulting "twice estimated" product-limit process. In principle there are two empirical candidates for doing this. One is the exponential empirical hazard function $\exp(\Lambda_n^i(t))$ (cf.[4]) and the other is the product limit estimate itself. The latter being more natural here, consider

$$\hat{\hat{Z}}_n^i(t) = (1-K_n^i(t))Z_n^i(t)/\tilde{S}_n^i(t) = (1-K_n^i(t))(\exp(-\Lambda^i(t))-\tilde{S}_n^i(t))/\tilde{S}_n^i(t)$$

$i=1, \ldots, k$, and the corresponding vector process

$$\hat{\hat{Z}}_n(x) = (\hat{\hat{Z}}_n^1(x_1), \ldots, \hat{\hat{Z}}_n^k(x_k))$$

for $x=(x_1, \ldots, x_k)$.

For T_n we introduce a slightly stronger regularity condition instead of (2.2):

(3.1) $$T_n < T_F \quad \text{and} \quad 1-F(T_n) \geqq 2n^{-1/2} r_3(n),$$

where

$$r_3(n) = r(n) + 3(\varepsilon/2)^{1/2} b_n^2 (\log n)^{1/2}.$$

By definition (2.4) of $r(n)$ it can be shown that a rough sufficient condition for condition (3.1) to be satisfied is that ($T_n \nearrow T_F$ so slowly that)

(3.2) $$b_n = \big(1-F(T_n)\big)^{-1} \leqq M_\varepsilon (n/\log n)^{1/6}$$

with some constant M_ε depending only on ε, which can be computed from $r_3(n)$.

Just as Lemma 5.1 of [4] was deduced form an approximation theorem, the first statement of the next lemma easily results from Theorem 5.5 of [4] which is the original Breslow—Crowley-type variant of the Efron-type theorem cited here. When deducing it, one also should apply the already mentioned fact that $\exp\big(-\Lambda^i(T_n)\big) \geqq 1-F(T_n)$. The second statement of the lemma follows from the first just as Lemma 4.1 of [4] followed from the Dvoretzky—Kiefer—Wolfowitz bound.

Lemma 3.3. *If T_n satisfies (3.1), then*

$$\mathrm{pr}\Big\{\sup_{-\infty < t \leqq T_n} |Z_n^i(t)| > r_3(n)\Big\} \leqq (Q+6)\, n^{-\varepsilon}$$

and

$$\mathrm{pr}\left\{\sup_{-\infty < t \leqq T_n} \frac{1}{\tilde{S}_n^i(t)} > \frac{2}{\exp\big(-\Lambda^i(T_n)\big)}\right\} \leqq (Q+6) n^{-\varepsilon}.$$

Theorem 3.4. *If T_n satisfies (2.2), then*

$$P\Big\{\sup_{x \in (-\infty, T_n]^k} \|\hat{\hat{Z}}_n(x) - B_n^K(x)\| > q_2(n)\Big\} \leqq kR_2 n^{-\varepsilon},$$

where $q_2(n) = q_1(n) + 2n^{-1/2} b_n^2 \big(r_3(n)\big)^2$ *and* $R_2 = R_1 + 2Q + 12 = 3Q + 14 + 8D = 30A_2(2k+1) + 314 + 56D$.

Proof.

$$\big|\hat{\hat{Z}}_n^i(t) - B_n^i(K^i(t))\big| \leqq \big|\hat{Z}_n^i(t) - B_n^i(K^i(t))\big| + |\hat{\hat{Z}}_n^i(t) - \hat{Z}_n^i(t)| \leqq$$
$$\leqq \big|\hat{Z}_n^i(t) - B_n^i(K^i(t))\big| + n^{-1/2} |Z_n^i(t)|^2 / \{\tilde{S}_n^i(t) \exp\big(-\Lambda^i(t)\big)\},$$

and the theorem follows from Theorem 3.2 and Lemma 3.3.

As to the order of our rate sequences $q_1(n)$ and $q_2(n)$, we note that since

$$r(n) = O\big(\max\{b_n^2 n^{-1/3} (\log n)^{3/2},\ b_n^4 n^{-1/2} \log n\}\big),$$

we have

$$q_1(n) = O\big(\max\{b_n^3 n^{-1/3} (\log n)^{3/2},\ b_n^5 n^{-1/2} \log n\}\big),$$
$$q_2(n) = O\big(\max\{b_n^3 n^{-1/3} (\log n)^{3/2},\ b_n^6 n^{-1/2} \log n\}\big).$$

Now we formulate the corresponding consequences for approximation on the fixed cube $(-\infty, T]^k$ with $T<T_F$. These consequences follow from Theorems 3.2 and 3.4 in the same way as Corollary 5.7 of [4] did. Note that $q_1(n)$, $q_2(n)$ and $r_3(n)$ are understood from now on with b_n replaced in them by the constant

$$b = \big(1-F(T)\big)^{-1}.$$

Corollary 3.5. *If* $n/\log n \geqq 2\varepsilon b^2$, *then*

$$P\Big\{\sup_{x\in(-\infty,T]^k} \|\hat{Z}_n(x)-B_n^K(x)\| > q_1(n)\Big\} \leqq kR_1 n^{-\varepsilon},$$

and if $n^{1/2}/r_3(n) \geqq 2b$, *then*

$$P\Big\{\sup_{x\in(-\infty,T]^k} \|\hat{\hat{Z}}_n(x)-B_n^K(x)\| > q_2(n)\Big\} \leqq kR_2 n^{-\varepsilon}.$$

The rough sufficient condition for the second statement here is (3.2) with b in place of b_n.

The joint weak convergence of the components of $\hat{Z}_n$ and $\hat{\hat{Z}}_n$ follows from this corollary together with rate-of-convergence results. Namely, for many functionals ψ (cf. Corollary of Komlós *et al.* [16] and Corollary 1 of Csörgő [6]) on the space of functions defined on $(-\infty, T]^k$ we obtain

$$\sup_{-\infty<y<\infty} \big|\mathrm{pr}\{\psi(\overset{*}{Z}_n(\cdot)) < y\} - \mathrm{pr}\{\psi(B^K(\cdot)) < y\}\big| = O\big(n^{-1/3}(\log n)^{3/2}\big), \tag{3.2}$$

where $* = \hat{}\,,\ \hat{\hat{}}$ and B^K is a copy of B_n^K since, if T $(<T_F)$ is fixed, then

$$q_1(n) = O\big(n^{-1/3}(\log n)^{3/2}\big) = q_2(n).$$

For example, (3.2) holds for the Kolmogorov, Smirnov and Kuiper statistics considered by Koziol [17].

4. An approximation theorem for the Aalen—Nair transformation. Goodness of fit. Let T be a number such that the inequalities

$$T < T_F, \quad F^i(T) > 0, \quad i = 1, \dots, k, \tag{4.1}$$

hold, and consider the processes

$$\check{Z}_n^i(t) = \tilde{Z}_n^i(t)/(d_n^i(T))^{1/2}, \quad i = 1, \dots, k,$$

proposed by Aalen [1] and Nair [20] in the Kaplan—Meier case $(k=2)$ where $\tilde{Z}_n^i$ and d_n^i are of Sections 2 and 3 respectively. Also, with W_n^i of Theorem A, introduce

$$W_n^{(i)}(t) = W_n^i(d^i(t))/(d^i(T))^{1/2}, \quad i = 1, \dots, k,$$

and for $x=(x_1, \dots, x_k)$ set

$$\check{Z}_n(x) = \big(\check{Z}_n^1(x_1), \dots, \check{Z}_k^n(x_k)\big), \quad \check{W}_n(x) = \big(W_n^{(1)}(x_1), \dots, W_n^{(k)}(x_k)\big).$$

We note that, $i=1, \dots, k$,

$$\{W_n^{(i)}(t):\ -\infty < t \leqq T\} =_{\mathcal{D}} \{W(d^i(t)/d^i(T)):\ -\infty < t \leqq T\} \tag{4.2}$$

and this equality in distribution is in fact the main advantage of the Aalen—Nair transformation. Introduce the notation (in addition to those of the preceding sections)

$$a = \max_{1 \leqq i \leqq k} 1/F^i(T).$$

Theorem 4.1. *If* $n/\log n \geqq \max(2\varepsilon b^2, 8\varepsilon a^2)$, *then*

$$P\{\sup_{x \in (-\infty, T]^k} \|\check{Z}_n(x) - \check{W}_n(x)\| > q_3(n)\} \leqq kR_3 n^{-\varepsilon},$$

where $q_3(n) = b(2a)^{1/2} r(n) + 12(2)^{1/2} \varepsilon b^5 a^{3/2} n^{-1/2} \log n$ *and* $R_3 = Q + 9D + 1 = 10A_2(2k+1) + 25D + 101$.

Proof.

$$P\{\sup_{-\infty < t \leqq T} |\check{Z}_n^i(t) - W_n^{(i)}(t)| > q_3(n)\} \leqq$$

$$\leqq P\{\sup_{-\infty < t \leqq T} (d_n^i(T))^{-1/2} |\tilde{Z}_n^i(t) - W_n^i(d^i(t))| > b(2a)^{1/2} r(n)\} +$$

$$+ P\{\sup_{-\infty < t \leqq T} |W_n^i(d^i(t))| > 2\varepsilon^{1/2} b (\log n)^{1/2}\} + P\{(d^i(T)\, d_n^i(T))^{-1/2} > (2a^2)^{1/2}\} +$$

$$+ P\{|(d^i(T))^{1/2} - (d_n^i(T))^{1/2}| > 12(\varepsilon/2)^{1/2} b^4 (a/2)^{1/2} n^{-1/2} (\log n)^{1/2}\}.$$

Since $d_n^i(T) \geqq F_n^i(T)$ and by an obvious analogue of Lemma 4.1 of [4]

$$P\left\{\frac{1}{F_n^i(T)} \geqq \frac{2}{F^i(T)}\right\} \leqq Dn^{-\varepsilon}, \tag{4.3}$$

provided that $n/\log n \geqq 8\varepsilon a^2$, we obtain, using also Theorem A, that the first term of the above sum is not greater than $(Q+D)n^{-\varepsilon}$. We saw in the proof of Theorem 3.2 that the second term is not greater than $n^{-\varepsilon}$. By (4.3) the third term is majorized by $Dn^{-\varepsilon}$. Using again (4.3), the fourth probability is majorized by

$$Dn^{-\varepsilon} + P\{|d^i(T) - d_n^i(T)| > 12(\varepsilon/2)^{1/2} b^4 n^{-1/2} (\log n)^{1/2}\} \leqq 7Dn^{-\varepsilon},$$

where we used Lemma 6.2 of [4] in the last step. This proves the theorem.

By (4.2) the limit distributions of the Kolmogorov, Smirnov and Kuiper statistics based on the processes $\check{Z}_n^i$ coincide with the distributions of the corresponding functionals of $\{W(s) : 0 \leqq s \leqq 1\}$. These distributions are well known, one of them is tabulated in [7]. If $\psi(\check{Z}_n^i)$ denotes any of these three statistics and $\psi(W)$ denotes its distribution-free limiting random variable, then we have (3.2) for their distribution functions by Theorem 4.1.

Since the Aalen—Nair modified Efron transformation leads to asymptotically distribution-free statistics, this transformation is more advantageous than those of

Hall and Wellner when testing goodness of fit. However, the latter seems much better when constructing confidence bands. This is why we do not spell out the exact probability inequalities in the next section corresponding to the confidence bands arising from the transformation of Aalen and Nair.

The two-sample processes, or, more generally, their vector-process generalization (for the general competing risks model) can be similarly approximated as the one-sample processes in Theorems 3.2, 3.4 and 4.1.

5. Conficence bands. If G is a continuous distribution function and G_n is the n-stage empirical distribution function of a sample corresponding to G, then it follows from Theorem 3 of KOMLÓS *et al.* (1975) that for any $\lambda, \varepsilon>0$ we have

$$-A_2 n^{-\varepsilon}+M\big(\lambda-(A_1+(\varepsilon/A_3))n^{-1/2}\log n\big) \leqq$$

$$\leqq \operatorname{pr}\{G_n(t)-\lambda/n^{1/2} \leqq G(t) \leqq G_n(t)+\lambda/n^{1/2},\ -\infty<t<\infty\} \leqq$$

$$\leqq M\big(\lambda+(A_1+(\varepsilon/A_3))n^{-1/2}\log n\big)+A_2 n^{-\varepsilon},$$

where

$$M(y) = \operatorname{pr}\{\sup_{0\leqq s\leqq 1} |B(s)| < y\}.$$

As we have already noted, A_1, A_2 and A_3 can be taken by TUSNÁDY [21] as 100, 10 and 1/50, respectively. (It would be interesting to search for smaller A_1, A_2 and larger A_3 by Monte Carlo through the above inequalities.) By Remark 1 of [6] and the fact that $\sup_{-\infty<y<\infty} |B(G(y))| \leqq \sup_{0\leqq s\leqq 1} |B(s)|$, the lower half of the above inequality remains valid for discontinuous G as well.

For $0<a<1$ set

$$M_a(y) = \operatorname{pr}\{\sup_{0\leqq s\leqq a} |B(s)| < y\}.$$

The analogues of the above inequalities for the general right censorship model are the following consequences of Corollary 3.5.

Corollary 5.1. *Let $T<T_F$. If $n/\log n \geqq 2\varepsilon b^2$, then for any $\lambda>0$ and $i=1, \dots, k$ we have*

$$-R_1 n^{-\varepsilon}+M_{K^i(T)}\big(\lambda-q_1(n)\big) \leqq$$

$$\leqq \operatorname{pr}\left\{\frac{\tilde{S}_n^i(t)}{1+\dfrac{\lambda}{n^{1/2}(1-K_n^i(t))}} \leqq S^i(t) \leqq \frac{\tilde{S}_n^i(t)}{1-\dfrac{\lambda}{n^{1/2}(1-K_n(t))}},\ -\infty<t\leqq T\right\} \leqq$$

$$\leqq M_{K^i(T)}\big(\lambda+q_1(n)\big)+R_1 n^{-\varepsilon}.$$

If $n^{1/2}/r_3(n) \geqq 2b$, *then for any* $\lambda>0$ *and* $i=1, \ldots, k$ *we have*

$$-R_2 n^{-\varepsilon}+M_{K^i(T)}(\lambda-q_2(n)) \leqq \mathrm{pr}\Big\{\tilde{S}_n^i(t)-\lambda \frac{\tilde{S}_n^i(t)}{n^{1/2}(1-K_n^i(t))} \leqq$$

$$\leqq S^i(t) \leqq \tilde{S}_n^i(t)+\lambda \frac{\tilde{S}_n^i(t)}{n^{1/2}(1-K_n^i(t))}, -\infty<t \leqq T\Big\} \leqq M_{K^i(T)}(\lambda+q_2(n))+R_2 n^{-\varepsilon}.$$

Since $K^i(T) \leqq b^2/(1+b^2), i=1, \ldots, k$, we have $M(\lambda-q_j(n))<M_{b^2/(1+b^2)}(\lambda-q_j(n)) \leqq$ $\leqq M_{K^i(T)}(\lambda-q_j(n))$, $i=1, \ldots, k$; $j=1,2$, thus $M_{K^i(T)}$ can be replaced by either M (as noted by HALL and WELLNER [14]), or $M_{b^2/(1+b^2)}$ in the lower bounds. Since the choice of ε is ours, the only unknown quantity in the lower bounds $R_j n^{-\varepsilon}+$ $+M_{b^2/(1+b^2)}(\lambda-q_j(n))$, $j=1,2$, is b, and this can be estimated by $(1-F_n(T))^{-1}$.

If $k=2$ and we are in the Kaplan—Meier model, then the symmetric bands of the second statement of the above corollary are those of [14] (without rates). Even if we compute with the conjecture $D=2$ but with $A_1=100, A_2=10$ and $A_3=1/50$, a practical application of the lower halves of the above inequalities would demand rather astronomic sample sizes. Nevertheless, the above inequalities constitute the only information presently available for the precision of the bands in question, and if one can dream about future values of the A's as $A_1, A_2 \approx 1/10, A_3 \approx 10$, then this information is not disappointing at all.

References

[1] O. AALEN, Nonparametric inference in connection with multiple decrement models, *Scand. J. Statist.*, **3** (1976), 15—27.

[2] S. M. BERMAN, A note on extreme values, competing risks and semi-Markov processes, *Ann. Math. Statist.*, **34** (1963), 1104—1106.

[3] N. BRESLOW and J. CROWLEY, A large sample study of the life table and product limit estimates under random censorship, *Ann. Statist.*, **2** (1974), 437—453.

[4] M. D. BURKE, S. CSÖRGŐ and L. HORVÁTH, Strong approximation of some biometric estimates under random censorship. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, **56** (1981), 87—112.

[5] M. CSÖRGŐ and P. RÉVÉSZ, *Strong approximation in probability and statistics*, Academic Press (New York, 1981).

[6] S. CSÖRGŐ, Limit behaviour of the empirical characteristic function, *Ann. Probability*, **9** (1981), 130—144.

[7] S. CSÖRGŐ and L. HORVÁTH, On the Koziol-Green model for random censorship, *Biometrika*, **68** (1981), 391—401.

[8] S. CSÖRGŐ and L. HORVÁTH, On cumulative hazard processes under random censorship, *Scand. J. Statist.*, **9** (1982), 13—21.

[9] S. CSÖRGŐ and L. HORVÁTH, Random censorship from the left, *Studia Sci. Math. Hung.*, to appear.

[10] L. P. DEVROYE and G. L. WISE, On the recovery of discrete probability densities from imperfect measurements, *J. Franklin Inst.*, **307** (1979), 1—20.

[11] A. DVORETZKY, J. KIEFER and J. WOLFOWITZ, Asymptotic minimax character of the sample distribution function and of the multinomial estimator, *Ann. Math. Statist.*, **27** (1956), 642—669.

[12] B. EFRON, The two-sample problem with censored data, *Proc. Fifth Berkeley Symp. Math. Statist. Probability*, **4** (1967), 831—853.

[13] M. J. GILLESPIE and L. FISHER, Confidence bands for the Kaplan—Meier survival curve estimate, *Ann. Statist.*, **7** (1979), 920—924.

[14] W. J. HALL and J. A. WELLNER, Confidence bands for a survival curve from censored data, *Biometrika*, **67** (1980), 133—143.

[15] E. L. KAPLAN and P. MEIER, Nonparametric estimation from incomplete observations, *J. Amer. Statist. Assoc.*, **53** (1958), 457—481.

[16] J. KOMLÓS, P. MAJOR and G. TUSNÁDY, An approximation of partial sums of independent r. v.'s and the sample d. f. I., *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, **32** (1975), 111—132.

[17] J. A. KOZIOL, Goodness-of-fit tests for randomly censored dara, *Biometrika*, **67** (1980), 693—696.

[18] J. A. KOZIOL and S. B. GREEN, A Cramér—von Mises statistic for randomly censored data, *Biometrika*, **63** (1976), 465—474.

[19] P. MEIER, Estimation of a distribution function from incomplete observations, *Perspectives in Probability and Statistics*, Academic Press (New York, 1975), pp. 67—87.

[20] V. N. NAIR, Plots and tests for goodness of fit with randomly censored data, *Biometrika*, **68** (1981), 99—103.

[21] G. TUSNÁDY, *Investigations of statistical hypotheses* (Hungarian), Candidate's dissertation. Hungarian Academy of Sciences (1978).

[22] G. YANG, Life expectancy under random censorship, *Stochastic Processes App.*, **6** (1977), 33—39.

BOLYAI INSTITUTE
SZEGED UNIVERSITY
ARADI VÉRTANÚK TERE 1
H—6720 SZEGED, HUNGARY

Acta Sci. Math., 44 (1982), 35—42

Factor lattices by tolerances

GÁBOR CZÉDLI

1. Introduction

Given a lattice L, a binary, reflexive, symmetric *and compatible* relation $\varrho \subseteqq L \times L$ is said to be a tolerance relation (or shortly tolerance) of L. Tolerances of lattices were firstly investigated by CHAJDA and ZELINKA [2]. Recently the importance of this concept has grown: a finite lattice is monotone functionally complete iff it has the trivial tolerances only (cf. KINDERMANN [4]). Moreover, KINDERMANN [4] has shown that the algebraic functions on a finite lattice are just the monotone functions preserving its tolerances.

Our aims in the present paper are to introduce the concept of L/ϱ (i.e., factor lattice by a tolerance ϱ), to give a more handlable description of L/ϱ, and to give a structure-like theorem for lattices with the following consequence: every finite lattice is isomorphic to D/ϱ for a suitable finite distributive lattice D. A characterization for tolerances of lattices will be presented in Theorem 2.

Given a reflexive and symmetric relation ϱ over a non-empty set A, a subset H of A is called a *block* of ϱ if $H^2 \subseteqq \varrho$ but $G^2 \subseteqq \varrho$ for no $H \subset G \subseteqq A$. I.e., H is a block of ϱ if it is maximal with respect to the property: for any $a, b \in H$ $a \varrho b$. Let the set of all blocks be denoted by $\mathscr{C}_\varrho$. On the other hand, certain subsets of $P^+(A)$, the set of non-empty subsets of A, can be called *quasi-partitions* on A (cf. CHAJDA, NIEDERLE, and ZELINKA [1]). The connection of these two concepts (see [1] again) is the following. If ϱ is a reflexive and symmetric relation then $\mathscr{C}_\varrho$ is a quasi-partition. For a quasi-partition $\mathscr{C}$ the relation $\varrho_{\mathscr{C}} = \{(a, b) : \{a, b\} \subseteqq H$ for some $H \in \mathscr{C}\}$ is reflexive and symmetric. The map $\varrho \mapsto \mathscr{C}_\varrho$, from the set of reflexive and symmetric relations on A into the set of quasi-partitions on A, is bijective and its inverse map is $\mathscr{C} \mapsto \varrho_{\mathscr{C}}$. Moreover, a reflexive and symmetric relation ϱ is an equivalence iff $\mathscr{C}_\varrho$ is a partition. Therefore the following notion of factor latti-

Received January 30, 1981.

ces by tolerances seems to be a natural generalization of that of factor lattices by congruences.

For definition, let ϱ be a tolerance of a lattice L. For blocks G and H of and $\circ\in\{\wedge,\vee\}$ we define $G\circ H$ to be the unique block of ϱ for which $\{g\circ h: g\in G, h\in H\}\subseteq G\circ H$. (The correctness of this definition will be shown!). Now L/ϱ, the factor lattice by ϱ, is the set of all blocks of ϱ equipped with the above defined $\wedge$ and $\vee$ operations. I.e., the notation L/ϱ is used instead of $\mathscr{C}_\varrho$ and $L/\varrho=(L/\varrho;\wedge,\vee)$. It is worth mentioning that L/ϱ is the factor lattice in the usual sence whenever the tolerance ϱ happens to be a congruence relation.

2. L/ϱ is an algebra

In this section the correctness of the definition of L/ϱ will be shown. Suppose $G,H\in L/\varrho$. If $g_i\in G$, $h_i\in H$ $(i=1,2)$ then the compatibility of ϱ yields $(g_1\circ h_1, g_2\circ h_2)\in\varrho$. I.e., $\{g\circ h: g\in G, h\in H\}^2\subseteq\varrho$. Now Zorn Lemma applies and $\{g\circ h: g\in G, h\in H\}\subseteq E$ for some $E\in L/\varrho$.

To show the uniqueness of E some preliminaries are needed. In what follows in this section let ϱ be a fixed tolerance of a lattice L.

Lemma 1 (Chajda and Zelinka [2]). *For $a,b\in L$, $(a,b)\in\varrho$ if and only if $[a\wedge b, a\vee b]^2\subseteq\varrho$.*

Lemma 2. *The blocks of ϱ are convex sublattices of L.*

Proof. Let C be a block of ϱ, and suppose $a,b\in C$. For an arbitrary $x\in C$ $a\varrho x$ and $b\varrho x$, whence $a\vee b\varrho x\vee x=x$. I.e., $(C\cup\{a\vee b\})^2\subseteq\varrho$ and the maximality of C yields $a\vee b\in C$. Therefore C is a sublattice. If $a,b\in C$, $u\in L$, and $a\leqq u\leqq b$, then, for any $x\in C$, $a\wedge x\in C$ and $b\vee x\in C$. Thus $a\wedge x\varrho b\vee x$, and Lemma 1 yields $x\varrho c$. Finally, $u\in C$ follows from the maximality of C again. Q. e. d.

For a subset X of L let $[X)$ and $(X]$ denote the dual ideal and ideal generated by X, respectively. We write $[a)$ instead of $[\{a\})$, and dually.

Lemma 3 (Grätzer [3]). *For any convex sublattice C of L the equality $C=[C)\cap(C]$ holds. Moreover, if C is the intersection of a dual ideal D and an ideal I, then $D=[C)$ and $I=(C]$.*

Definition 1. For ideals I_1 and I_2 let $I_1\wedge I_2=I_1\cap I_2$, $I_1\vee I_2=\{x: x\leqq c\vee d$ for some $c\in I_1, d\in I_2\}=(I_1\cup I_2]$, and let $I_1\leqq I_2$ mean $I_1\subseteq I_2$. On the other hand for dual ideals D_1 and D_2 let $D_1\wedge D_2=\{x: x\geqq c\wedge d$ for some $c\in D_1, d\in D_2\}=[D_1\cup D_2)$, $D_1\vee D_2=D_1\cap D_2$, and let $D_1\leqq D_2$ mean $D_1\supseteq D_2$.

The motivation of this definition will be given in the remark to Lemma 4.

Proposition 1. *If* $(C]=(D]$ *for* $C, D\in L/\varrho$ *then* $C=D$.

Proof. First we show that $U=([C)\wedge[D))\cap(C]\in L/\varrho$. Suppose $x_1, x_2\in U$. Then $x_i\geqq c_i'\wedge d_i'$ for $c_i'\in[C)$ and $d_i'\in[D)$, $i=1,2$. Let $c\in C$ and $d\in D$, and set $c_i=c_i'\wedge c$, $d_i=d_i'\wedge d$ $(i=1, 2)$. Then, by Lemma 3, we have $x_i\geqq c\wedge d_i$, $c_i\in C$, and $d_i\in D$ for $i=1,2$. Set $a=x_1\vee c_1\vee x_2\vee c_2$ and $b=x_1\vee d_1\vee x_2\vee d_2$. By $(C]=(D]$ and Lemma 3 we obtain $a\in C$, $b\in D$, and $a\vee b\in C\cap D$. Since $c_1\wedge c_2\in C$ and $d_1\wedge d_2\in D$, $(c_1\wedge c_2, a\vee b)\in\varrho$ and $(d_1\wedge d_2, a\vee b)\in\varrho$ follow. The compatibility of ϱ yields $(c_1\wedge c_2\wedge d_1\wedge d_2, a\vee b)\in\varrho$. But $x_1, x_2\in[c_1\wedge d_1\wedge c_2\wedge d_2, a\vee b]$, whence Lemma 1 implies $(x_1, x_2)\in\varrho$. We have shown that $U^2\subseteqq\varrho$. $U\supseteqq[C)\cap(C]=C$ and the maximality of C yields $U=C\in L/\varrho$. By making use of $(C]=(D]$ we obtain $U\supseteqq[D)\cap(D]=D$ similarly. Therefore $U=D$ as well. Q. e. d.

Proposition 2. *Suppose* $C, D, E\in L/\varrho$ *and* $\{c\vee d: c\in C,\ d\in D\}\subseteqq E$. *Then* $[C)\vee[D)=[E)$.

Proof. Let $\{c\vee d: c\in C, d\in D\}$ be denoted by U. Since $[U)=[C)\cap[D)=$ $=[C)\vee[D)$, $[C)\vee[D)\subseteqq[E)$ follows easily. To show the required equality let $[C)\cap[D)=$ $=[C)\vee[D)\subset E$ be assumed. Then $[E)\setminus([C)\cap[D))\neq\emptyset$, and one can easily see that $E\setminus([C)\cap[D))\neq\emptyset$ as well. Therefore an element a can be chosen so that $a\in E$ and, e.g., $a\notin[C)$. Choosing elements $c\in C$ and $d\in D$ we can assume that $a\leqq c\vee d$. (Otherwise a could be replaced by $(c\vee d)\wedge a$, because $c\vee d, (c\vee d)\wedge a\in E$ and $(c\vee d)\wedge a\notin[C)$.) Evidently we have $a\wedge c\notin C$. For an arbitrary $x\in C$ we can proceed as follows. From $(x\vee c)\vee d\in U\subseteqq E$ and $a\in E$ we obtain $(x\vee c\vee d, a)\in\varrho$. From $x, c\in C$ and Lemma 2 $(x\vee c, x\wedge c)\in\varrho$ follows. By meeting we obtain $(x\vee c, a\wedge x\wedge c)\in\varrho$. From Lemma 1 $(x, a\wedge c)\in\varrho$ can be concluded. Consequently $(C\cup\{a\vee c\})^2\subseteqq\varrho$, a contradiction. Q. e. d.

Now Propositions 1 and 2 and their dual statements imply the correctness of the definition of L/ϱ.

3. L/ϱ is a lattice

Before proving what is stated in the title of this section, a more handlable description of L/ϱ is necessary.

Lemma 4. *Suppose* $E=C\vee D$ *and* $F=C\wedge D$ *for* $C, D, E, F\in L/\varrho$. *Then we have* $[C)\vee[D)=[E)$ *and* $(C]\vee(D]\leqq(E]$. *The dual statement,* $[C)\wedge[D)\geqq[F)$ *and* $(C]\wedge(D]=(F]$, *also holds.*

Remark. If for $X\in\{C, D, E, F\}\subseteqq L/\varrho$ X is an interval $[x_1, x_2]$, and $E=$ $=C\vee D, F=C\wedge D$, then Lemma 4 yields $c_1\vee d_1=e_1, c_2\vee d_2\leqq e_2, c_1\wedge d_1\geqq f_1$, and

$c_2 \wedge d_2 = f_2$. (This is always the case when L is a finite lattice.) This remark can supply a motivation of Definition 1.

Proof. Since $\{c \vee d : c \in C, d \in D\} \subseteq E$, we have $(C] \vee (D] = (\{c \vee d : c \in C, d \in D\}] \subseteq \subseteq (E]$, implying $(C] \vee (D] \leqq (E]$. The rest follows from Proposition 2 and the Duality Principle.

This lemma enables us to strengthen Proposition 1:

Corollary 1. *For $C, D \in L/\varrho$ we have $[C) \leqq [D)$ if and only if $(C] \leqq (D]$.*

Really, Proposition 1 follows from this corollary and Lemma 3.

Proof. Suppose $[C) \leqq [D)$, then $[C) \vee [D) = [D)$. Proposition 2 and the dual of Proposition 1 imply $C \vee D = D$. By making use of Lemma 4 we obtain $(C] \leqq \leqq (C] \vee (D] \leqq (D]$. The Duality Principle yields the converse implication. Q. e. d.

Theorem 1. *For any tolerance ϱ of an arbitrary lattice L, L/ϱ is a lattice again.*

Proof. By the Duality Principle it is enough to show that the $\vee$ operation is commutative and associative, and one of the absorption laws holds. Since the join for dual ideals in Definition 1 is commutative and associative, the commutativity and associativity are straightforward consequences of Proposition 2 and the dual of Proposition 1. To show $C \vee (C \wedge D) = C$, for $C, D \in L/\varrho$, by the dual of Proposition 1 it is enough to check $[C \vee (C \wedge D)) = [C)$. But, by Lemma 4, $[C) \geqq [C) \wedge [D) \geqq \geqq [C \wedge D)$, and so $[C \vee (C \wedge D)) = [C) \wedge [C \wedge D) = [C)$. Q. e. d.

The following theorem deals with the connection between tolerances and corresponding quasi-partitions on lattices. For a tolerance ϱ on a lattice L, $\mathscr{C}_\varrho = L/\varrho$ and $P^+(L)$ were defined in the Introduction.

Theorem 2. *Given a lattice L, for any $\mathscr{C} \subseteq P^+(L)$ the following two conditions are equivalent.*

(a) *$\mathscr{C} = \mathscr{C}_\varrho (= L/\varrho)$ for some tolerance ϱ on L.*

(b) *$\mathscr{C}$ has the following six properties:*

(C1) *The elements of $\mathscr{C}$ are convex sublattices of L;*

(C2) $\bigcup_{C \in \mathscr{C}} C = L$;

(C3) *For any $C, D \in \mathscr{C}$, $[C) = [D)$ is equivalent to $(C] = (D]$;*

(C4) *For any $C, D \in \mathscr{C}$ there exist $E, F \in \mathscr{C}$ such that $[C) \vee [D) = [E)$, $(C] \vee (D] \leqq (E]$, and $[C) \wedge [D) \geqq [F)$, $(C] \wedge (D] = (F]$;*

(C5) *Let $x \in L$, $d \in C \in \mathscr{C}$ be arbitrary. If for any $e \in C \cap (d]$ there exists C_e such that $\{e, x\} \subseteq C_e \in \mathscr{C}$ then $x \in (C]$, and, dually, if for any $f \in C \cap [d)$ there exists C_f such that $\{f, x\} \subseteq C_f \in \mathscr{C}$ then $x \in [C)$;*

(C6) *If U is a convex sublattice of L and for any $a, b \in U$ there exists $D \in \mathscr{C}$ containing both a and b, then $U \subseteq C$ for some $C \in \mathscr{C}$.*

Moreover, if L is a finite lattice then (C5) *and* (C6) *follow already from* (C1), (C2), (C3), *and* (C4).

Proof. (a) *implies* (b). (C1), (C3) and (C4) is involved in Lemma 2, Corrollary 1, and Lemma 4, respectively. Zorn Lemma yields (C2) and (C6). Suppose $x\in L$, $d\in C\in\mathscr{C}_\varrho=L/\varrho$, and for any $e\in C\cap(d]$ there exists $C_e\in L/\varrho$ such that $\{e,x\}\subseteq C_e$. Considering the set $X=\{x\}\cup(C\cap(d])$ we have $X^2\subseteq\varrho$. Extending X to an element of L/ϱ, say E, we obtain $[C)=[C\cap(d])\subseteq[X)\subseteq[E)$, i.e. $[C)\geqq[E)$. Corollary 1 yields $(C]\geqq(E]$. Hence $x\in X\subseteq E\subseteq(E]\subseteq(C]$. The proof of (C5) is completed by the Duality Principle.

(b) *implies* (a). Suppose $\mathscr{C}$ satisfies the requirements of (b) and let ϱ denote $\varrho_{\mathscr{C}}=\{(a,b)\in L^2\colon \{a,b\}\subseteq C$ for some $C\in\mathscr{C}\}$. The relation ϱ is evidently symmetric; and it is reflexive by (C2). If $C,D,E\in\mathscr{C}$, U denotes the set $\{c\vee d\colon c\in C, d\in D\}$, $[C)\vee[D)=[E)$, and $(C]\vee(D]\leqq(E]$ then $U\subseteq E$. Indeed, $U\subseteq[C)\cap[D)=[E)$, $U\subseteq(C]\vee(D]\subseteq(E]$, and, by Lemma 3, $E=[E)\cap(E]$. Now (C4) and the Duality Principle yield the compatibility of ϱ. Therefore ϱ is a tolerance on L, and $\mathscr{C}_\varrho=\mathscr{C}$ has to be shown. Suppose $C\in\mathscr{C}$. Then $C^2\subseteq\varrho$. If $(x,c)\in\varrho$ for any $c\in C$ then $x\in[C)\cap(C]=C$ by (C5) and Lemma 3. Thus $C\in\mathscr{C}_\varrho$ and $\mathscr{C}\subseteq\mathscr{C}_\varrho$. Conversely, if $U\in\mathscr{C}_\varrho$ then $U\subseteq C$ for some $C\in\mathscr{C}$ by (C6). But then both U and C belong to $\mathscr{C}_\varrho$, whence $U=C$. $\mathscr{C}=\mathscr{C}_\varrho$ has been shown.

Finally, suppose L is a finite lattice, $\mathscr{C}\subseteq P^+(L)$ and $\mathscr{C}$ satisfies (C1), (C2), (C3), and (C4). Since any convex sublattice of L is an interval, (C6) evidently holds. Suppose $x\in L$, $d\in C=[a,b]\in\mathscr{C}$ and for any $e\in C\cap(d]$ there exists C_e such that $\{e,x\}\subseteq C_e\in\mathscr{C}$. Then $\{a,x\}\subseteq C_a=[u,v]$. Since $u\leqq a$, we obtain $[C)\vee[C_a)=[C)$. Now (C4) together with (C3) yield $(C]\vee(C_a]=(C]$, i.e., $b\vee v=b$. Hence $x\leqq v\leqq b$, which implies $x\in(C]$. (C5) is satisfied by the Duality Principle. Q. e. d.

Note that usually it is convenient to give $\mathscr{C}_\varrho$ instead of ϱ. For example, let D be a five-element chain, say $D=\{0<1<2<3<4\}$, let $L=D^2\setminus\{(0,4)\}$, a sublattice of D^2, and let $\mathscr{C}_\varrho=\{[(0,0),(2,1)],\ [(3,0),(4,1)],\ [(3,2),(4,4)],\ [(0,2),(2,3)],\ [(1,2),(2,4)]\}$. Then Theorem 2 makes it easy to check that ϱ is a tolerance and L/ϱ is isomorphic to N_5, the five-element non-modular lattice.

Proposition 1 yields that for any tolerance ϱ on a finite lattice L, L/ϱ cannot have more element than L. That is why the following example can be of some interest. Define ϱ over Q, the set of rational numbers, by $\varrho=\{(x,y)\colon |x-y|\leqq 1\}$. Armed with the usual ordering Q turns into a lattice and ϱ is a tolerance on it. By making use of the results of this section it is easy to check that the factor lattice Q/ϱ is isomorphic to R, the set of real numbers with the usual ordering. (Indeed, the map $Q/\varrho\to R$, $C\mapsto\inf C$ is an isomorphism.)

4. Lattices as tolerance-factors of distributive lattices

The first example in the previous section indicates that forming factor lattices by tolerances preserves neither distributivity nor modularity. It is a naturally arising question which lattice identities are preserved. No non-trivial ones, as it will appear from the forthcoming theorem. Let **T**, **I**, **H**, **S**, **P**, and $\mathbf{P}_f$ denote the operators of taking factor lattices by tolerances, isomorphic lattices, homomorphic images, sublattices, direct products, and direct products of finite families, respectively. Note, that $\mathbf{H}\,V \subseteq \mathbf{IT}\,V$ for any class V of lattices. Moreover, as it can be deduced from Theorem 2, $\mathbf{IT}\,V = \mathbf{ITT}\,V$ for any class V of lattices. (To keep the size of the paper limited, the proof, which is similar to that of Homomorphism Theorem, will be omitted.) Let **2** denote the two-element lattice.

Theorem 3. $\mathbf{ISTSP}\{\mathbf{2}\}$ *is the class of all lattices, while* $\mathbf{ITSP}_f\{\mathbf{2}\}$ *is the class of all finite lattices.*

Proof. Only one argument is needed to prove this theorem consisting of two statements, just we have to show that our embeddings are surjective for the case of finite lattices. We have to show that an arbitrary (finite, respectively) lattice L belongs to $\mathbf{ISTSP}\{\mathbf{2}\}$ (to $\mathbf{ITSP}_f\{\mathbf{2}\}$, resp.). First of all we can assume that L is complete, since the map $L \to I(L)$, $x \mapsto (x]$ is an (surjective for finite L) embedding of L into its ideal lattice, i.e., into a complete lattice.

Claim 1. There are complete distributive lattices D_0 and D_1 in $\mathbf{P}\{\mathbf{2}\}$ and injective 0-and 1-preserving maps $\varphi_0\colon L \to D_0$, $\varphi_1\colon L \to D_1$ such that φ_0 preserves arbitrary joins and φ_1 preserves arbitrary meets. If L is finite then $D_0, D_1 \in \mathbf{P}_f\{\mathbf{2}\}$.

Proof. Let D_1 be $P(L \setminus \{0\})$, the Boolean lattice of all subsets of $L \setminus \{0\}$, and define $\varphi_1\colon L \to D_1$ as $x \mapsto (x] \setminus \{0\}$. The completeness of L yields $(\bigwedge (x_\gamma\colon \gamma \in \Gamma)] = \bigcap ((x_\gamma]\colon \gamma \in \Gamma)$, whence the required properties of φ_1 are trivial. Moreover, D_1 is isomorphic to $\mathbf{2}^{|L|-1}$. Q. e. d.

Now let D be $D_0 + D_1$, the ordinal sum of D_0 and D_1. I.e., D is the disjoint union of D_0 and D_1 equipped with the following ordering: $x \leqq y$ iff $x \in D_0$ and $y \in D_1$, or $x, y \in D_i$ and $x \leqq y$ for some $i \in \{0, 1\}$. Note that D is complete and it can be embedded into the direct square of $\mathbf{2}^{|L|-1}$, thus it is in $\mathbf{ISP}\{\mathbf{2}\}$ (in $\mathbf{ISP}_f\{\mathbf{2}\}$ for finite L). With the help of functions in Claim 1 define $\mathscr{C} \subseteq P^+(D)$ by

$$\mathscr{C} = \{C\colon \varnothing \neq C \subseteq D, \text{ for any } c, d \in C \text{ there exists } a \in L \text{ such that}$$

$$\{c, d\} \subseteq [a\varphi_0, a\varphi_1], \text{ and } C \text{ is maximal with respect to this property}\}.$$

Now, by making use of Theorem 2, we show that $\mathscr{C} = \mathscr{C}_\varrho (= D/\varrho)$ for some tolerance ϱ on D.

To check (C1) suppose $x, y \in C \in \mathscr{C}$. For an arbitrary $z \in C$ there exist $a, b \in L$ such that $x, z \in [a\varphi_0, a\varphi_1]$ and $y, z \in [b\varphi_0, b\varphi_1]$. Since φ_0 preserves joins and φ_1 is monotone, we obtain $x \vee y, z \in [a\varphi_0 \vee b\varphi_0, a\varphi_1 \vee b\varphi_1] \subseteq [(a \vee b)\varphi_0, (a \vee b)\varphi_1]$. From the maximality of C we obtain $x \vee y \in C$, showing that C is a sublattice. Let $c, d \in C, x \in D$ and $c < x < d$. Suppose that, e.g., $x \in D_0$, and let z be an arbitrary element of C. Then $c, z \in [a\varphi_0, a\varphi_1]$ for some $a \in L$. But $a\varphi_1 \in D_1$ implies $x < a\varphi_1$, whence $x, z \in [a\varphi_0, a\varphi_1]$. The maximality of C yields $x \in C$, i.e. C is a convex sublattice. By the maximality of C, $1\varphi_0 \in C$, so C is not empty.

From $[0\varphi_0, 0\varphi_1] \cup [1\varphi_0, 1\varphi_1] = L$ and Zorn Lemma (C2) follows.

Now suppose that, in contrary to (C3), $[C) = [E)$ and $(C] \neq (E]$ for $C, E \in \mathscr{C}$. Then one of $(C] \setminus (E]$ and $(E] \setminus (C]$, say $(C] \setminus (E]$ is not empty. Fix an element d from $C \setminus (E]$ and let x be an arbitrary element of E. Since $d \wedge x \in (C] \wedge [E) = [C) = [E)$, Lemma 3 yields $d, d \wedge x \in C$ and $x, d \wedge x \in E$. Hence $a\varphi_0 \leqq d \wedge x \leqq d \leqq a\varphi_1$ and $b\varphi_0 \leqq d \wedge x \leqq x \leqq b\varphi_1$ for some $a, b \in L$. By forming join we obtain $(a \vee b)\varphi_0 = a\varphi_0 \vee b\varphi_0 \leqq d \wedge x \leqq d \vee x \leqq a\varphi_1 \vee b\varphi_1 \leqq (a \vee b)\varphi_1$. Thus $x, d \in [(a \vee b)\varphi_0, (a \vee b)\varphi_1]$, contradicting the maximality of E. The rest of (C3) follows from the Duality Principle.

To show (C4), let $C, E \in \mathscr{C}$ and define $X = \{c \vee e : c \in C, e \in E\}$. For any two elements in X, say $c_1 \vee e_1$ and $c_2 \vee e_2$ $(c_i \in C, e_i \in E)$, there exists an $u \in L$ such that $c_i \vee e_i \in [u\varphi_0, u\varphi_1]$ for $i = 1, 2$. Indeed, $c_i \in [a\varphi_0, a\varphi_1]$ and $e_i \in [b\varphi_0, b\varphi_1]$ ($i = 1, 2$ and $a, b \in L$), and u can be defined as $a \vee b$. From Zorn Lemma we obtain the existence of an $F \in \mathscr{C}$ such that $X \subseteq F$. Since $(C] \vee (E] = (X] \leqq (F]$ is evident, $[C) \vee [E) = [F)$ has to be shown. If $x \in [C) \vee [E) = [C) \cap [E)$ then $x \geqq c$ and $x \geqq e$ for $c \in C, e \in E$. Hence $x \geqq c \vee e \in F$ implies $x \in [F)$, showing that $[C) \vee [E) \subseteq [F)$. Suppose that $[C) \vee [E) \subset [F)$. Then $F \setminus ([C) \cap [E))$ and so, e.g., $F \setminus [C)$ are not empty. Fix elements d, c, and e in $F \setminus [C), C$, and E, respectively. For an arbitrary $x \in C$ we have $x \wedge c, x \vee c \in C$ and $d, ((x \vee c) \vee e) \vee d \in F$. Therefore $a\varphi_0 \leqq x \wedge c \leqq x \vee c \leqq a\varphi_1$ and $b\varphi_0 \leqq d \leqq x \vee c \vee e \vee d \leqq b\varphi_1$ for some $a, b \in L$. By meeting we obtain $(a \wedge b)\varphi_0 \leqq a\varphi_0 \wedge b\varphi_0 \leqq x \wedge c \wedge d \leqq x \vee c \leqq a\varphi_1 \wedge b\varphi_1 = (a \wedge b)\varphi_1$. Now $c \wedge d \notin C$ and $x, c \wedge d \in [(a \wedge b)\varphi_0, (a \wedge b)\varphi_1]$ contradicts the maximality of C. The rest of (C4) is settled by the Duality Principle.

Before going on we show that

$$(*) \qquad [u\varphi_0, u\varphi_1] \in \mathscr{C} \quad \text{for any} \quad u \in L.$$

Only the maximality of $[u\varphi_0, u\varphi_1]$ has to be shown. Suppose $[u\varphi_0, u\varphi_1]$ is not maximal, then $[u\varphi_0, u\varphi_1] \subset C$ for some $C \in \mathscr{C}$. Fix an element c in $C \setminus [u\varphi_0, u\varphi_1]$. Since C is a sublattice, $c_0 = c \wedge u\varphi_0$ and $c_1 = c \vee u\varphi_1$ are in C, and either $c_0 < u\varphi_0$ or $c_1 > u\varphi_1$. If, e.g., $c_0 < u\varphi_0$, then $c_0, u\varphi_1 \in C$ implies $a\varphi_0 \leqq c_0 < u\varphi_0 < u\varphi_1 \leqq a\varphi_1$ for some $a \in L$. Hence $a\varphi_0 \neq u\varphi_0, (a \vee u)\varphi_0 = a\varphi_0 \vee u\varphi_0 = u\varphi_0$, and $u\varphi_1 = a\varphi_1 \wedge u\varphi_1 = (a \wedge u)\varphi_1$. The injectivity of φ_0 and φ_1 yields $a \neq u, a \vee u = u$, and $a \wedge u = u$, a contradiction.

Now suppose $x\in L$, $d\in C\in\mathscr{C}$ and for any $e\in C\cap(d]$ there exists $C_e\in\mathscr{C}$ such that $\{e,x\}\subseteq C_e$. Then for any $e\in C\cap(d]$ there exists $a_e\in L$ such that $e,x\in[a_e\varphi_0, a_e\varphi_1]$. Set $u=\bigwedge(a_e: e\in C\cap(d])$ and $h=\bigwedge(e: e\in C\cap(d])$. Since φ_1 preserves arbitrary meets and φ_0 is monotone, we obtain $u\varphi_0\leqq\bigwedge(a_e\varphi_0: e\in C\cap(d])\leqq h$ and $x\leqq\bigwedge(a_e\varphi_1: e\in C\cap(d])=u\varphi_1$, i.e., $h,x\in[u\varphi_0,u\varphi_1]=E$. From $(*)$ we conclude that $E\in\mathscr{C}$. Since $u\varphi_0\leqq h\leqq y$ holds for any $y\in C$ (indeed, $h\leqq y\wedge d\in C\cap(d]$), $[E)\leqq[C)$. Now (C3) and (C4) imply $(E]\leqq(C]$ (cf. the proof of Corollary 1). Therefore $x\in(C]$ follows from $x\in E\subseteq(E]\subseteq(C]$. The rest of (C5) follows from the Duality Principle.

Now let U be a convex sublattice of D and suppose that for any $a,b\in U$ there exists $E\in\mathscr{C}$ containing both a and b. Then $a,b\in[u\varphi_0,u\varphi_1]$ for some $u\in L$, and Zorn Lemma implies (C6).

We have shown that $\mathscr{C}$ is associated with a tolerance ϱ on D. Let $D/\varrho=\mathscr{C}$ denote the corresponding factor lattice. For $u\in L$ let $u\psi$ denote $[u\varphi_0,u\varphi_1]$. Then, by $(*)$, ψ is a map from L into D/ϱ. If $u,v\in L$ then $[(u\vee v)\psi)=[(u\vee v)\varphi_0)=$ $=[u\varphi_0\vee v\varphi_0)=[u\varphi_0)\vee[v\varphi_0)=[u\psi)\vee[v\varphi)$. Lemma 4 and the dual of Proposition 1 imply $(u\vee v)\psi=u\psi\vee v\psi$, showing that ψ is a homomorphism. Since φ_0 is injective, so is ψ. Therefore $L\in\mathbf{IST}\{D\}$.

In case L is finite, so is D. Then any convex sublattice and, in particular, any element of $\mathscr{C}$ is an interval. Hence ψ is surjective, and $L\in\mathbf{IT}\{D\}$. Q. e. d.

References

[1] I. Chajda, I. Niederle and B. Zelinka, On existence condition for compatible tolerances, *Czech. Math. J.*, **26** (1976), 304—311.

[2] I. Chajda and B. Zelinka, Tolerance relations on lattices, *Časop. Pěstov. Mat.*, **99** (1974), 394—399.

[3] G. Grätzer, *General lattice theory*, Akademie-Verlag, Berlin (1978).

[4] M. Kindermann, Über die Äquivalenz von Ordnungspolynomvollständigkeit und Toleranzeinfachheit endlicher Verbände, *Contributions to General Algebra* (Proceedings of the Klagenfurt Conference 1978, edited by H. Kautschitsch, W. B. Müller, and W. Nöbauer), 145—149.

BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

Acta Sci. Math., 44 (1982), 43—51

On the duality of interpolation spaces of several Banach spaces

DICESAR LASS FERNANDEZ

Introduction

Since the work by ARONSZAJN—GAGLIARDO ([1]) appeared, the problem of the duality of interpolation spaces of two Banach spaces has attracted the interest of many authors. See for instance LIONS [10] for the trace method, LIONS—PEETRE [12] for the mean methods, SCHERER [14], LACROIX—SONRIER [9], PEETRE [13] for the J- and K-methods, and CALDERÓN [4] for the complex method.

Although the study of interpolation spaces has been mainly restricted to couples of Banach spaces, many papers concerning interpolation spaces of several Banach spaces have appeared. See for instance LIONS [11], YOSHIKAWA [16], FAVINI [5], SPARR [15], FERNANDEZ [6], [7] and [8]. Thus, it is natural to pose the question of duality for the theories of interpolation of several Banach spaces. The purpose of this paper is to study the duality between the J- and K- interpolation methods for several Banach spaces introduced in FERNANDEZ [6]. The distinguishing feature of the J- and K- methods studied in [6] is that they deal with 2^n spaces and n-parameters. This permits us to show that the two methods are equivalent, in the sense that they generate the same interpolation spaces. The equivalence of the two methods is fundamental to the study of the duality problem. Also, the idea used in the proof of the equivalence is the same one used to prove a density theorem, which is another crucial point in the duality theory. In this way we have the tools to show that the J- and K- methods for 2^n spaces "are in duality" as is the case for $n=1$.

For the duality of the complex method for 2^n Banach spaces see BERTOLO [3].

Through this paper we shall use the following notations: (A) if $a=(a_1, \ldots, a_d)$, $b=(b_1, \ldots, b_d)\in \mathbf{R}^d$ then we set (i) $a\leqq b$ iff $a_j\leqq b_j$, $j=1,2,\ldots,d$; (ii) $a\cdot b=$ $=a_1b_1+\ldots a_db_d$; (iii) $a\circ b=(a_1b_1, \ldots, a_db_d)$; (iv) $|a|=a_1+\ldots+a_d$; (v) $a^b=a_1^{b_1}\ldots a_d^{b_d}$; (vi) $2^b=2^{b_1}\ldots 2^{b_d}$; (B) $\mathbf{1}=(1, \ldots, 1)$, (C) $L_*^Q=L_*^Q(\mathbf{R}^d)$ stands for the L^Q spaces with mixed norms of BENEDEK—PANZONE [2] with respect to the measure $d_*t=$ $=d_*t_1 \ldots d_*t_d=dt_1/t_1 \ldots dt_d/t_d$.

Received February 11, and in revised form, May 12, 1981.

1. Interpolation of 2^d Banach spaces

We shall first give a summary of facts on the theory of interpolation of 2^d Banach spaces. Also, we give the discretization of the methods here considered and a density theorem which has not appeared before.

1.1. Generalities. 1.1.1. The set of $k=(k_1, ..., k_d)\in \mathbf{R}^d$ such that $k_j=0$ or 1 will be denoted by $\square$. We have $\square=\{0, 1\}$ when $d=1$ and $\square=\{(0,0), (1,0), (0,1), (1,1)\}$ when $d=2$. The families of objects we shall consider will depend on indices in $\square$.

1.1.2. We shall consider families of 2^d Banach spaces $\mathbf{E}=(E_k|k\in\square)$ embedded, algebraically and continuously, in one and the same linear Hausdorff space V. Such a family will be called an admissible family of Banach spaces (in V).

1.1.3. If $\mathbf{E}=(E_k|k\in\square)$ is an admissible family of Banach spaces, the linear hull $\Sigma\mathbf{E}$ and the intersection $\cap\mathbf{E}$ are defined in the usual way. They are Banach spaces under the norms

(1) $$\|x\|_{\Sigma\mathbf{E}} = \inf\{\Sigma_k\|x_k\|_{E_k} \mid x = \Sigma_k x_k;\ x_k\in E_k,\ k\in\square\}$$

and

(2) $$\|x\|_{\cap\mathbf{E}} = \max\{\|x\|_{E_k} \mid k\in\square\}.$$

The spaces $\cap\mathbf{E}$ and $\Sigma\mathbf{E}$ are continuously embedded in V.

1.1.4. A Banach space E which satisfies

(1) $$\cap\mathbf{E}\subset E\subset\Sigma\mathbf{E}$$

will be called an intermediate space (with respect to $\mathbf{E}$). (Hereafter $\subset$ will denote a continuous embedding.)

1.2. Intermediate spaces. 1.2.1. Let $\mathbf{E}=(E_k|k\in\square)$ be an admissible family of Banach spaces. Suppose $t=(t_1, ..., t_d)>0$ and $t^k=t_1^{k_1}...t_d^{k_d}$. For $x\in\Sigma\mathbf{E}$, we set

(1) $$K(t; x) = K(t; x; \mathbf{E}) = \inf\{\Sigma_k t^k\|x_k\|_{E_k} \mid x = \Sigma_k x_k,\ x_k\in E_k,\ k\in\square\}$$

and for $x\in\cap\mathbf{E}$

(2) $$J(t; x) = J(t; x; \mathbf{E}) = \max\{t^k\|x\|_{E_k} \mid k\in\square\}.$$

Now, assume $0<\Theta=(\theta_1, ..., \theta_d)<1$ and $1\leqq Q=(q_1, ..., q_d)\leqq\infty$.

1.2.2. Definition. We define $\mathbf{E}_{\Theta;Q;K}=(E_k|k\in\square)_{\Theta;Q;K}$ to be the space of all elements $x\in\Sigma\mathbf{E}$ for which

(1) $$t^{-\Theta}K(t; x)\in L_*^Q,$$

and $\mathbf{E}_{\Theta;Q;J}=(E_k|k\in\Box)_{\Theta;Q;J}$ to be the space of all elements $x\in\Sigma\mathbf{E}$ for which there exists a strongly measurable function $u:\mathbf{R}_+^d\to\cap\mathbf{E}$ such that

(2) $$x=\int_{\mathbf{R}_+^d}u(t)\,d_*t\quad(\text{in }\Sigma\,\mathbf{E}),$$

and

(3) $$t^{-\Theta}J(t;u(t))\in L_*^Q.$$

1.2.3. Proposition. *The spaces* $\mathbf{E}_{\Theta;Q;K}$ *and* $\mathbf{E}_{\Theta;Q;J}$ *are Banach spaces under the norms*

(1) $$\|x\|_{\Theta;Q;K}=\|t^{-\Theta}K(t;x)\|_{L^Q},$$

and

(2) $$\|x\|_{\Theta;Q;J}=\inf\left\{\left\|t^{-\Theta}J(t;u(t))\right\|_{L_*^Q}\mid x=\int u(t)\,d_*t\right\},$$

respectively. Furthermore, the spaces $\mathbf{E}_{\Theta;Q;K}$ *and* $\mathbf{E}_{\Theta;Q;J}$ *are intermediate spaces with respect to* $\mathbf{E}$.

1.2.4. We shall say the spaces $\mathbf{E}_{\Theta;Q;K}$ and $\mathbf{E}_{\Theta;Q;J}$ are generated by the K- and J-methods, respectively.

The following result gives a connection between the spaces generated by the K- and J- method and says that those are actually equivalent.

1.2.5. Proposition. *If* $0<\Theta=(\theta_1,\dots,\theta_d)<1$ *and* $1\leqq Q=(q_1,\dots,q_d)<\infty$ *we have*

(1) $$\mathbf{E}_{\Theta;Q;K}=\mathbf{E}_{\Theta;Q;J}.$$

1.2.6. When we have no need to specify which interpolation method has generated the intermediate space we shall write simply $\mathbf{E}_{\Theta,Q}$ for the spaces $\mathbf{E}_{\Theta;Q;K}$ and $\mathbf{E}_{\Theta;Q;J}$.

For the proofs of the above results see FERNANDEZ [6].

1.3. The discretization on the K- and J-method. Let $\mathbf{E}=(E_k|k\in\Box)$ be an admissible family of Banach spaces.

1.3.1. Proposition. *Let* $x\in\Sigma\mathbf{E}$. *Then* $x\in(E_k|k\in\Box)_{\Theta;Q;K}$ *iff*

(1) $$\left(e^{-N\cdot\Theta}K(e^N;x)\right)_{N\in\mathbf{Z}^d}\in l^Q(\mathbf{Z}^d).$$

Moreover

(2) $$\|x\|_{\Theta;Q;K}\cong\left\|\left(e^{-N\cdot\Theta}K(e^N;x)\right)_{N\in\mathbf{Z}^d}\right\|_{l^Q(\mathbf{Z}^d)}.$$

Proof. If $t^{-\Theta}K(t;x)=t_1^{-\theta_1}\dots t_d^{-\theta_d}K(t_1,\dots,t_d;x)$, we have

$$\|x\|_{\Theta;Q;K}=\left(\sum_{m_d=-\infty}^{\infty}\int_{e^{m_d}}^{e^{m_d+1}}\dots\left(\sum_{m_1=-\infty}^{\infty}\int_{e^{m_1}}^{e^{m_1+1}}\left(t^{-\Theta}K(t;x)\right)^{q_1}d_*t_1\right)^{q_2/q_1}\dots d_*t_d\right)^{1/q_d}.$$

On the other hand, if $e^{m_j} \leqq t_j \leqq e^{m_j+1}$, $j=1, 2, \ldots, d$ we have

$$K(e^{m_1}, \ldots, e^{m_d}; x) \leqq K(t_m, \ldots, t_d; x) \leqq e^N K(e^{m_1}, \ldots, e^{m_d}; x)$$

and

(3) $$e^{-\Theta \cdot M} K(e^M; x) \leqq t^{-\Theta} K(t; x) \leqq e^d e^{-\Theta} K(e^M; x).$$

These inequalities imply (2) at once and prove the assertion.

1.3.2. Proposition. *Let* $x \in \Sigma \mathbf{E}$. *Then,* $x \in (E_k \mid k \in \square)_{\Theta; Q; J}$ *iff there is* $u_M \in \cap \mathbf{E}$, $M \in \mathbf{Z}^d$, *such that*

(1) $$x = \sum_{M \in \mathbf{Z}^d} u_M \quad (in\ \Sigma \mathbf{E})$$

and

(2) $$e^{-M \cdot \Theta} J(e^M; u_M)_{M \in \mathbf{Z}^d} \in l^Q(\mathbf{Z}^d).$$

Moreover

(3) $$\|x\|_{\Theta; Q; J} \cong \inf \left\{ \|(e^{-M \cdot \Theta} J(e^M; u_M))_{M \in \mathbf{Z}^d}\|_{l^Q(\mathbf{Z}^d)} \mid x = \Sigma_M u_M \right\}.$$

Proof. Let $x \in (E_k \mid k \in \square)_{\Theta; Q; J}$ and $u = u(t)$ be as in 1.2.2. If $M = (m_1, \ldots, m_d)$, let us set

$$u_m = u_{m_1 \cdots m_d} = \int_{e^{m_d}}^{e^{m_d+1}} \cdots \int_{e^{m_1}}^{e^{m_1+1}} u(t_1, \ldots, t_d)\, d_* t_1 \ldots d_* t_d.$$

Then we have

$$x = \int_{\mathbf{R}_+^d} u(t)\, d_* t = \sum_{M \in \mathbf{Z}^d} u_M$$

and

(4) $$\left\|(e^{-M \cdot \Theta} J(e^M; u_M))_{M \in \mathbf{Z}^d}\right\|_{l^Q(\mathbf{Z}^d)} \leqq C \left\| t^{-\Theta} J(t; u) \right\|_{L_*^Q}.$$

Taking the infimum in the above inequality we get one half of (3).

We proceed similarly to obtain the converse inequality in (4), which will imply the other half of (3). The proof is complete.

1.4. Density theorems. Let $\mathbf{E} = (E_k \mid k \in \square)$ be an admissible family and let us denote by $\overline{\cap \mathbf{E}}^K$ and $\overline{\cap \mathbf{E}}^J$ the closure of $\cap \mathbf{E}$ in $\mathbf{E}_{\Theta; Q; K}$ and $\mathbf{E}_{\Theta; Q; J}$ respectively. Of course, we have $\overline{\cap \mathbf{E}}^K = \overline{\cap \mathbf{E}}^J = \overline{\cap \mathbf{E}}$.

1.4.1. Proposition. *If* $0 < \Theta < 1$ *and* $1 \leqq Q < \infty$ *we have*

(1) $\overline{\cap \mathbf{E}}^K \subset \mathbf{E}_{\Theta; Q; K}$; (2) $\mathbf{E}_{\Theta; Q; J} \subset \overline{\cap \mathbf{E}}^J$.

Proof. The inclusion (1) is obvious. To prove (2), let $x \in \mathbf{E}_{\Theta; Q; J}$ and let $u = u(t)$ be as in 1.2.2(2)—(3). Let us set

$$x_M = x_{m_1 \ldots m_d} = \int_{1/m_d}^{m_d} \cdots \int_{1/m_1}^{m_1} u(t_1, \ldots, t_d)\, d_* t_1 \ldots d_* t_d.$$

Then

$$x - x_M = \int_{\mathbf{R}_+^d} Y_M(t) u(t)\, d_* t,$$

where $Y_M(t) = Y_{m_1 \dots m_d}(t) = 0$, if $1/m_j < t_j < m_j$ $(j = 1, 2, \dots, d)$ and $=1$ otherwise. Consequently

$$\|x - x_M\|_{\Theta;Q;J} \leqq \left\|t^{-\Theta} J(t; Y_M(t) u(t))\right\|_{L_*^Q} = \left\|t^{-\Theta} Y_M(t) J(t; u(t))\right\|_{L_*^Q}.$$

Finally, since $Y_M(t) \to 0$ as $M \to \infty$, the result follows.

1.4.2. Corollary. *We have* $\overline{\bigcap \mathbf{E}} = \mathbf{E}_{\Theta;Q}$.

Proof. It follows at once from 1.4.1(1)—(2) and the equivalence theorem.

2. Duality

2.1. Dual families. For a given admissible family $\mathbf{E} = (E_k \mid k \in \square)$ of Banach spaces there is a natural duality between $\bigcap \mathbf{E}$ and $\Sigma \mathbf{E}$, and $\mathbf{E}_{\Theta;Q;K}$ and $\mathbf{E}_{\Theta;Q;J}$. In order to examine this duality let us set the following hypothesis (H) on the admissible family $\mathbf{E}$:

(H) $\bigcap \mathbf{E}$ is dense in each E_k, $k \in \square$.

Let $\mathbf{E}' = (E_k' \mid k \in \square)$ be the family given by the duals of the elements of the family $\mathbf{E} = (E_k \mid k \in \square)$.

Since $\bigcap \mathbf{E} \subset E_k$, the spaces E_k', $k \in \square$, can be canonically embedded in $(\bigcap \mathbf{E})'$. The density hypothesis assures that this embedding does not identify distinct elements in E_k' with the same element in $(\bigcap \mathbf{E})'$. In this way, the family $\mathbf{E}' = (E_k' \mid k \in \square)$ of dual spaces is an admissible family of Banach spaces.

2.1.1. Proposition. *Let* $\mathbf{E} = (E_k \mid k \in \square)$ *be an admissible family which satisfies the hypothesis* (H) *and let* $\mathbf{E}' = (E_k' \mid k \in \square)$ *be its dual family. Then*

(1) $$(\bigcap \mathbf{E})' = \Sigma \mathbf{E}'$$

and

(2) $$\|x'\|_{\Sigma \mathbf{E}'} = \sup \{ |\langle x, x' \rangle_\cap| / \|x\|_{\cap \mathbf{E}} \mid x \in \bigcap \mathbf{E} \};$$

(3) $$(\Sigma \mathbf{E})' = \bigcap \mathbf{E}'$$

and

(4) $$\|x'\|_{\cap \mathbf{E}'} = \sup \{ |\langle x, x' \rangle_\Sigma| / \|x\|_{\cap \mathbf{E}} \mid x \in \Sigma \mathbf{E} \};$$

where $\langle\, , \rangle_\cap$ *denotes the duality between* $\bigcap \mathbf{E}$ *and* $(\bigcap \mathbf{E})'$ *and* $\langle\, , \rangle_\Sigma$ *between* $\Sigma \mathbf{E}$ *and* $(\Sigma \mathbf{E})'$.

Proof. Since $E'_k \subset (\bigcap \mathbf{E})'$, for each $k \in \square$, it follows that $\Sigma \mathbf{E}' \subset (\bigcap \mathbf{E})'$.

Conversely, if $\Phi \in (\bigcap \mathbf{E})'$, the linear form

$$\psi\colon (x_k \mid k \in \square) \to \Phi(2^{-d} \Sigma_k x_k)$$

is bounded in the diagonal subspace of $\oplus_k E_k$, with the norm $\max_k \|x_k\|_{E_k}$. By the Hahn—Banach theorem there is an $(x'_k \mid k \in \square) \in \oplus_k E'_k$ such that

$$\Sigma_k \langle x, x'_k \rangle_{E_k} = \psi(x)$$

for all $x \in \bigcap \mathbf{E}$, and

$$\Sigma_k \|x'_k\|_{E'_k} \leqq \|\Phi\|_{(\bigcap \mathbf{E})'}.$$

Now, if we take $x_k = x$, $k \in \square$, it follows that

$$\Phi(x) = \Sigma_k \langle x, x'_k \rangle_{E_k}, \quad x \in \bigcap \mathbf{E}.$$

Finally, by the density hypothesis (H), the linear forms $x'_k, k \in \square$, are determined by their values in $\bigcap \mathbf{E}$ and

$$\|\Phi\|_{(\bigcap \mathbf{E})'} \leqq \Sigma_k \|x'_k\|_{E'_k}.$$

Similarly we prove (3) and (4).

As a corollary of proposition 2.1.1 we get the following result on the K- and J-functional norms.

2.1.2. Proposition. *Let* $\mathbf{E} = (E_k \mid k \in \square)$ *be an admissible family of Banach spaces which satisfies the density hypothesis* (H) *and let* $\mathbf{E}' = (E'_k \mid k \in \square)$ *be its dual family. Then*

(1) $$K(t; x'; \mathbf{E}') = \sup \{|\langle x, x' \rangle| / J(t^{-1}; x; \mathbf{E}) \mid x \in \bigcap \mathbf{E}\}$$

and

(2) $$J(t; x'; \mathbf{E}') = \sup \{|\langle x, x' \rangle / K(t^{-1}; x; \mathbf{E}) \mid x \in \Sigma \mathbf{E}\}.$$

Proof. Let E be a normed space and $t > 0$. Let us denote space E with the norm $t\|\cdot\|_E$ by tE. Then we have $(tE)' = t^{-1}E'$.

Now, if we consider the family $(t^k E_k \mid k \in \square)$ we see that (1) and (2) follow at once from 2.1.1(2) and 2.1.1(4), respectively.

2.2. The duality of spaces $\mathbf{E}_{Q,\theta}$. Let $\mathbf{E}$ be an admissible family of Banach spaces which satisfies the density hypothesis (H). Then we can consider intermediate spaces with respect to the dual family $\mathbf{E}'$, and in particular the interpolation spaces $\mathbf{E}_{\theta,Q}$.

Let E be an intermediate space with respect to the admissible family $\mathbf{E}$. Then, a necessary and sufficient condition for E' to be an intermediate space with respect to the dual family $\mathbf{E}'$ is that $\bigcap \mathbf{E}$ be dense in E. Thus, if $E = \mathbf{E}_{\theta,Q}$ the density result of proposition 1.4.1 assures that $E' = (\mathbf{E}_{\theta,Q})'$ is an intermediate space with respect to the dual family $\mathbf{E}'$.

We shall now study the relationship between the spaces $\mathbf{E}_{\Theta,Q'}$ and $(\mathbf{E}_{\Theta,Q})'$. To this end we shall use again the notation $\mathbf{E}_{\Theta;Q;K}$ and $\mathbf{E}_{\Theta;Q;J}$ for the spaces generated by the K- and J-method, respectively.

2.2.1. Proposition. *Let* $\mathbf{E}=(E_k \mid k\in \square)$ *be an admissible family which satisfies the density hypothesis* (H) *and let* $\mathbf{E}'=(E_k' \mid k\in \square)$ *be its dual family. Suppose* $1\leqq Q=(q_1, \dots, q_d)<\infty$ *and* $0<\Theta=(\theta_1, \dots, \theta_d)<1$. *Then*

$$\mathbf{E}'_{\Theta;Q'} = (\mathbf{E}_{\Theta;Q})' \tag{1}$$

where $1/Q+1/Q'=1$ (i.e., $1/q_j+1/q_j'=1,\ j=1, 2, \dots, d$).

Proof. We shall prove that

$$\mathbf{E}'_{\Theta;Q';K} = (\mathbf{E}_{\Theta;Q;J})'. \tag{2}$$

By Prop. 2.1.1 it follows that

$$\mathbf{E}'_{\Theta;Q';K}\subset\Sigma\mathbf{E}' = (\cap\mathbf{E})'.$$

Now, if $x'\in\mathbf{E}'_{\Theta;Q;K}$ and $\langle\,,\rangle_\cap$ is the duality between $\cap\mathbf{E}$ and $(\cap\mathbf{E})'$, the relation $\langle x, x'\rangle_\cap$ makes sense for $x\in\mathbf{E}_{\Theta;Q;J}\cap(\cap\mathbf{E})$. Thus, by definition, there is a strongly measurable function $u\colon \mathbf{R}^d\to\cap\mathbf{E}$ such that $u\in L^1_*(\mathbf{R}^a_+;\cap\mathbf{E})$ and satisfies 1.2.2(2). From 2.1.2(1) it follows that

$$\int_{\mathbf{R}^d} |\langle u(t), x'\rangle|\, d_*t \leqq \int_{\mathbf{R}^d_+} J(t;\, u(t))K(t^{-1};\, x')\, d_*t = \tag{3}$$

$$= \int_{\mathbf{R}^d_+} t^{-\Theta}J(t;\, u(t))\, t^{\Theta}K(t^{-1}, x')\, d_*t \leqq \left\|t^{-\Theta}J(t;\, u(t))\right\|_{L^Q_*} \|t^{\Theta}K(t^{-1}, x')\|_{L^{Q'}_*}.$$

This shows that $x' o u\in L^1_*(\mathbf{R}^d_+)$ and thus

$$\int_{\mathbf{R}^d_+} \langle u(t), x'\rangle\, d_*\, t = \left\langle \int_{\mathbf{R}^d_+} u(t)\, d_*t;\, x' \right\rangle = \langle x, x'\rangle. \tag{4}$$

From (2) and (3) we get the following inequality of Hölder type

$$|\langle x, x'\rangle| \leqq \|x\|_{\Theta;Q;J}\|x'\|_{\Theta;Q';K}. \tag{5}$$

This Hölder inequality implies at once that $\langle\,,\rangle$ is a bounded linear form on a dense subspace of $\mathbf{E}_{\Theta;Q;J}$. Thus $\langle\,,\rangle$ can be extended boundedly to all $\mathbf{E}_{\Theta;Q;J}$. Hence, $x'\in(\mathbf{E}_{\Theta;Q;J})'$ and we have, for the dual norm

$$\|x'\|_{(\mathbf{E}_{\Theta;Q;J})'} = \sup\{|\langle x, x'\rangle|/\|x\|_{\Theta;Q;J} \mid x\in\mathbf{E}_{\Theta;Q;J}\} \leqq \|x'\|_{\Theta;Q';K}.$$

From this inequality we obtain one half of (1).

Conversely, let $x' \in (\mathbf{E}_{\theta;Q;J})'$. By 2.1.2(1), given $\varepsilon>0$ there is $Y_N=Y_{n_1\dots n_d}\in$ $\in \bigcap \mathbf{E}$ with $Y_N\neq 0$ and such that

$$\varepsilon K(e^N;\, x';\, \mathbf{E}') \leqq \langle y_N/J(e^N;\, y_N),\, x'\rangle.$$

Next, we denote by $l^{\theta,Q}(\mathbf{Z}^d)$ the space of all multiple sequences of real number $(x_N)_{N\in \mathbf{Z}^d}$ such that

$$\|(x_N)_{N\in \mathbf{Z}^d}\|_{l^{\theta,Q}} = \|(e^{N\cdot\theta}x_N)_{N\in \mathbf{Z}^d}\|_{l^Q} < \infty.$$

Now, if $\alpha=(\alpha_N)_N\in l^{\theta,Q}(\mathbf{Z}^d)$ and

$$x_\alpha = \sum_{N\in \mathbf{Z}^d} \alpha_N y_N/J(e^N;\, y_N),$$

it follows that

$$\|x_\alpha\|_{\theta;Q;J} \leqq \left\|\left(e^{-N\cdot\theta} J\left(e^N;\, \alpha_N y_N/J(e^N;\, y_N)\right)\right)_{N\in \mathbf{Z}^d}\right\|_{l^Q(\mathbf{Z}^d)} =$$
$$= \left\|\left(e^{-N\cdot\theta}|\alpha_N|\right)_{N\in \mathbf{Z}^d}\right\|_{l^Q(\mathbf{Z}^d)} = \|\alpha\|_{l^{\theta,Q}} < \infty.$$

Thus $x_\alpha\in \mathbf{E}_{\theta;Q;J}$. Also

$$\langle x_\alpha, x'\rangle = \langle \Sigma_N \alpha_N y_N/J(e^N, y_N),\, x'\rangle \geqq \varepsilon \Sigma_N \alpha_N K(e^{-N};\, x')$$

therefore

(6) $$\varepsilon \Sigma_N e^{-N}\alpha_N e^N K(e^{-N};\, x') \leqq \|\alpha\|_{\theta,Q}\|x'\|_{\theta;Q;J}.$$

Since $l^{\theta,Q}(\mathbf{Z}^d)$ and $l^{1-\theta,Q}(\mathbf{Z}^d)$ are in duality via the duality

$$\langle \alpha, \delta\rangle = \sum_{N\in \mathbf{Z}^d} e^{-|N|}\alpha_N\delta_N,$$

by taking the supremum over all $\alpha\in l^{\theta,Q}(\mathbf{Z}^d)$ with $\|\alpha\|_{l^{\theta,Q}}\leqq 1$ in (6) we obtain

$$\varepsilon\|e^N K(e^{-N};\, x')\|_{l^{\theta,Q'}} \leqq \|x'\|_{Q;\theta;J},$$

that is

$$\varepsilon\|x'\|_{\theta;Q';K} \leqq \|x'\|_{\theta;Q;J}.$$

Since ε is arbitrary we obtain the second half of (2).

From (2) and the equivalence theorem we obtain (1) and the proof is complete.

References

[1] N. Aronszajn—E. Gagliardo, Interpolation spaces and interpolation methods, *Ann. Mat. Pura Appl.*, **68** (1965), 51—118.

[2] A. Benedek—R. Panzone, The space L^P, with mixed norm, *Duke Math. J.*, **28** (1961), 301—324.

[3] J. I. Bertolo, Interpolation of 2^n Banach spaces: the complex method. *To appear.*

[4] A. P. Calderón, Intermediate spaces and interpolation, the complex method, *Studia Math.*, **24** (1964), 113—190.

[5] A. FAVINI, Su una estensione del metodo complesso d'interpolazione, *Rend. Sem. Mat. Univ. Padova*, **47** (1972), 244—298.

[6] D. L. FERNANDEZ, Interpolation of 2^n Banach spaces, *Studia Math.*, **65** (1979), 175—201.

[7] D. L. FERNANDEZ, On the interpolation of 2^n Banach spaces (1): The Lions—Peetre interpolation method, *Bul. Inst. Polytehn. Iaşi*, **26** (1980), 49—54.

[8] D. L. FERNANDEZ, An extension of the complex interpolation method, *Boll. Un. Mat. Ital. B*, **18** (1981), 721—732.

[9] M. T. LACROIX—SONRIER, Sur certains espaces d'interpolation, *C. R. Acad. Sci. Paris*, **272** (1971), A874—A877.

[10] J. L. LIONS, Sur les espaces d'interpolation: dualité, *Math. Scand.*, **9** (1961), 147—177.

[11] J. L. LIONS, Une construction d'espaces d'interpolation, *C. R. Acad. Sci. Paris*, **251** (1960), 1853—1855.

[12] J. L. LIONS—J. PEETRE, Sur une classe d'espaces d'interpolation, *Pub. Math. d'IHES*, **19** (1964), 5—68.

[13] J. PEETRE, Remark on the dual of an interpolation space, *Math. Scand.*, **34** (1974), 124—128.

[14] K. SCHERER, Über die Dualen von Banachräumen, die durch lineare Approximationsprozesse erzeugt werden, und Anwendugen für periodische Distributionen, *Acta Math. Acad. Sci. Hungar.*, **23** (1972), 343—365.

[15] G. SPARR, Interpolation of several Banach spaces, *Ann. Mat. Pura Appl.*, **99** (1974), 247—316.

[16] A. YOSHIKAWA, Sur la théorie d'espaces d'interpolation: les espaces de moyennes de plusieurs espaces de Banach, *J. Fac. Sci. Univ. Tokyo*, **16** (1970), 407—468.

INSTITUTO DE MATEMÁTICA
UNIVERSIDADE ESTADUAL DE CAMPINAS
CX. POSTAL 6155
13100 — CAMPINAS — S. P. — BRASIL

Acta Sci. Math., 44 (1982), 53—66

Amalgamated free product of lattices. I. The common refinement property

G. GRÄTZER and A. P. HUHN

1. Introduction. The common refinement property has been investigated for many algebraic constructions. Intuitively, we say that the common refinement property holds for the construction $*$ (e.g., direct product or free product) if, whenever A_0, A_1, B_0, B_1 are algebras for which $*$ is defined, $L=A_0 * A_1=B_0 * B_1$, and $A_0, A_1, B_0, B_1 \subseteq L$, then

(1) $A_i = (A_i \cap B_0) * (A_i \cap B_1), \quad i = 0, 1,$

(2) $B_j = (A_0 \cap B_j) * (A_1 \cap B_j), \quad j = 0, 1,$

(3) $L = (A_0 \cap B_0) * (A_0 \cap B_1) * (A_1 \cap B_0) * (A_1 \cap B_1).$

This is, of course, not a definition; we did not even specify what is meant by the right side of (3). In most concrete cases, however, the meaning of (1), (2), and (3) is clear: direct product of groups and rings, direct product of lattices with 0, free product of lattices (G. GRÄTZER and J. SICHLER [4]), and free product of algebras in a regular variety (B. JÓNSSON and E. NELSON [6]) are examples of algebraic constructions satisfying the common refinement property.

The present investigation was prompted by Problem VI. 2 in G. GRÄTZER [1], asking whether or not free {0, 1}-product of bounded lattices satisfies the common refinement property. We answer this question in the affirmative; the method of the proof, however, leads much farther. It will be shown that two free products amalgamated over the same finite lattice Q always have a common refinement. The Theorem gives, for an arbitrary lattice Q and any two representations of a lattice L as free Q-products, a necessary and sufficient condition for the existence of a common refinement.

2. Results. To define the concept of an amalgamated free product, let Q, A_0, A_1 be lattices ($Q=\varnothing$ is allowed), let Q be a sublattice of both A_0 and A_1, and let

Received May 28, 1981.
The research of the first author was supported by the NSERC of Canada.

$A_0 \cap A_1 = Q$. Then $A_0 \cup A_1$ is a partial lattice in a natural way (see Section 3 for a detailed definition). The free lattice generated by this partial lattice will be called the free product of A_0 and A_1 amalgamated over Q, or the Q-free product of A_0 and A_1; it will be denoted by $A_0 *_Q A_1$. In this paper, the formula $L = A_0 *_Q A_1$ always assumes that L is a lattice, A_0 and A_1 are sublattices of L, $Q = \emptyset$ or Q is a sublattice of both A_0 and A_1.

Our main theorem is as follows (for a more complete version see Section 4):

Theorem. *Let $L = A_0 *_Q A_1 = B_0 *_Q B_1$. These two decompositions of L have a common refinement, that is, conditions (1)—(3) of Section 1 hold for $*_Q$ if and only if for any $i, j \in \{0, 1\}$, $x \in A_i$, $y \in B_j$, the inequality $x \leqq y$ in L implies the existence of a $z \in A_i \cap B_j$ such that $x \leqq z$ in A_i and $z \leqq y$ in B_j.*

This theorem has several consequences.

Corollary 1. *If Q satisfies the Ascending Chain Condition or the Descending Chain Condition, then any two Q-free decompositions of a lattice have a common refinement.*

Clearly, the special case $Q = \{0, 1\}$ of Corollary 1 answers Problem VI. 2 of [1] in the affirmative.

Corollary 2. *Let $L = A_0 *_Q A_1 = B_0 *_Q B_1$. If, for any $i, j \in \{0, 1\}$, either A_i or B_j is convex in $A_i \cup B_j$, then the two decompositions have a common refinement.*

The most important open problem in this investigation is whether the condition given in the Theorem is a tautology or not; that is, whether Q-free products always have common refinements.

It follows easily from the main result of G. Grätzer and J. Sichler [4] that the free factors of a lattice L form a distributive lattice. This statement remains valid for Q-free factors ($Q \subseteq L$) if Q-free products always have common refinements (see Section 8). The next two corollaries establish distributivity like properties of the set of all Q-free factors for an arbitrary Q.

Corollary 3. *If $A_0 *_Q A_1 = A_0 *_Q A_2$ and $A_1 \subseteq A_2$, then $A_1 = A_2$.*

Corollary 4. *If $A_0 *_Q A_1 = A_0 *_Q A_2 = A_1 *_Q A_2$, then $Q = A_0 = A_1 = A_2$.*

3. Amalgamated free products. We need a lemma before we give the definition of an amalgamated free product.

Lemma 1. *Let A_0 and A_1 be lattices, let Q be a sublattice of both A_0 and A_1 or $Q = \emptyset$, and let $A_0 \cap A_1 = Q$. Then there exists a smallest partial lattice on the set $A_0 \cup A_1$ extending the operations of A_0 and A_1.*

Proof. Since the Amalgamation Property holds for lattices, there is an embedding of $A_0 \cup A_1$ into a lattice preserving the operations of A_0 and A_1. Restricting the operations of this lattice to $A_0 \cup A_1$, we get a partial lattice on the set $A_0 \cup A_1$. Therefore, the set of all partial lattices on the set $A_0 \cup A_1$ whose operations are extensions of the operations of A_0 and A_1 is nonempty. Now let $\langle A_0 \cup A_1; \wedge_\gamma, \vee_\gamma \rangle$, $\gamma \in \Gamma$, be partial lattices on the set $A_0 \cup A_1$. Let $\wedge$ and $\vee$ be the intersection of the $\wedge_\gamma$'s and $\vee_\gamma$'s, respectively ($\wedge_\gamma$ and $\vee_\gamma$ are sets, in fact, they are subsets of $(A_0 \cup A_1)^2 \times (A_0 \cup A_1)$). We shall prove that $\langle A_0 \cup A_1; \wedge, \vee \rangle$ is a partial lattice. This, will prove Lemma 1.

Here we need N. Funayama's characterization of partial lattices (see, e.g., G. GRÄTZER [1]): A partial algebra $\langle H; \wedge, \vee \rangle$ is a partial lattice if and only if, for arbitrary $a, b, c \in H$, the following five conditions and their duals hold.

(i) $a \wedge a$ exits and $a \wedge a = a$.
(ii) If $a \wedge b$ exists, then $b \wedge a$ exists and $a \wedge b = b \wedge a$.
(iii) If $a \wedge b$, $(a \wedge b) \wedge c$, $b \wedge c$ exist, then $a \wedge (b \wedge c)$ exists, and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$. If $b \wedge c, a \wedge (b \wedge c), a \wedge b$ exist, then $(a \wedge b) \wedge c$ exists and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.
(iv) If $a \wedge b$ exists, then $a \vee (a \wedge b)$ exists, and $a = a \vee (a \wedge b)$.
(v) If $[a) \vee [b) = [c)$ in $D_0(H)$, then $a \wedge b$ exists in H and equals c. (Here $D_0(H)$ denotes the lattice consisting of $\varnothing$ and all dual ideals of H. $D_0(H)$ is ordered by inclusion.)

Now we prove (v) for $\langle A_0 \cup A_1; \wedge, \vee \rangle$, the proof of the other four conditions is similar. Every $\langle A_0 \cup A_1; \wedge_\gamma, \vee_\gamma \rangle$, $\gamma \in \Gamma$, is a partial lattice, therefore, (v) holds for $\langle A_0 \cup A_1; \wedge_\gamma, \vee_\gamma \rangle$. Assume that $[a) \vee [b) = [c)$ in $D_0(\langle A_0 \cup A_1; \wedge, \vee \rangle)$. Then $[a) \vee [b) = [c)$ in $D_0(\langle A_0 \cup A_1; \wedge_\gamma, \vee_\gamma \rangle)$ for all $\gamma \in \Gamma$. In fact, $\wedge_\gamma$ is an extension of $\wedge$; therefore, the dual ideals generated by a and b relative to $\wedge_\gamma$ contain the dual ideals generated by a and b relative to $\wedge$, respectively. Thus $[a) \vee [b) \supseteq [c)$ in $D_0(\langle A_0 \cup A_1; \wedge_\gamma, \vee_\gamma \rangle)$. The reverse inclusion is trivial. Now, by (v), $a \wedge_\gamma b = c$ for all $\gamma \in \Gamma$. Hence $a \wedge b = c$. This completes the proof.

Definition 1. Let $Q, A_0; A_1$ be as in Lemma 1. Let $P(A_0, A_1, Q)$ denote the smallest partial lattice of Lemma 1. If $Q = A_0 \cap A_1$ is understood, we write $P(A_0, A_1)$. for $P(A_0, A_1, Q)$. Then the free lattice generated by $P(A_0, A_1, Q)$ will be called the *free product of A_0 and A_1 amalgamated over Q*, and it will be denoted by $A_0 *_Q A_1$.

A warning is in order here. We can partially order $A_0 \cup A_1$ by the smallest partial order containing the ordering of A_0 and the ordering of A_1. If we take $A_0 \cup A_1$ together with all the existing g.l.b.'s and l.u.b.'s relative to this ordering, then the resulting partial lattice is generally different from the one defined above.

Definition 1 can easily be extended to a definition of the Q-free product of an arbitrary finite number of lattices containing Q. In particular, if $L=A_0*_Q A_1= =B_0*_Q B_1$, then $(A_0\cap B_0)*_Q(A_0\cap B_1)*_Q(A_1\cap B_0)*_Q(A_1\cap B_1)$ is the free lattice generated by the smallest partial lattice on the set $(A_0\cap B_0)\cup(A_0\cap B_1)\cup(A_1\cap B_0)\cup \cup(A_1\cap B_1)$ whose operations extend the operations of all $A_i\cap B_j$, $i,j=0,1$.

We shall need a description of the ordering and of the ideals of $P(A_0, A_1)$.

Lemma 2. *Let $x\in A_0$ and $y\in A_1$. Then $x\leqq y$ in $P(A_0, A_1)$ if and only if there is a $z\in Q$ with $x\leqq z$ in A_0 and $z\leqq y$ in A_1.*

Proof. Define $\leqq$ on $A_0\cup A_1$ as follows: $\leqq$ retains its meaning on A_0 and A_1; for $x\in A_0$ and $y\in A_1$ (or $x\in A_1$ and $y\in A_0$) define $\leqq$ as in the lemma. It is obvious that $\leqq$ is a partial ordering on $A_0\cup A_1$. (This is used in the proof of the Amalgamation Property for lattices.) Consider the partial lattice $\langle A_0\cup A_1; \wedge, \vee\rangle$, where $a\wedge b=c$ iff c is the greatest lower bound of a and b with respect to $\leqq$; $a\vee b=c$ is defined dually.

Let $\leqq_1$ denote the ordering of $P(A_0, A_1)$. Since $P(A_0, A_1)$ is the smallest partial lattice on $A_0\cup A_1$, $\leqq_1$ must be contained in $\leqq$. To prove the converse, let $a\leqq b$, $a,b\in A_0\cup A_1$. If $a,b\in A_i$ for some i in $\{0,1\}$, then $a\leqq b$ in A_i. Hence, by the definition of $P(A_0, A_1)$, $a\leqq_1 b$. Therefore, and by symmetry, we may assume that $a\in A_0$ and $b\in A_1$. Thus there is an element c in $A_0\cap A_1$ such that $a\leqq c$ in A_0 and $c\leqq b$ in A_1. The same inequalities hold in $P(A_0, A_1)$, that is, $a\leqq_1 c\leqq_1 b$, as claimed.

Lemma 3. *Every ideal of $P(A_0, A_1)$ is the union of an ideal I_0 of A_0 and an ideal I_1 of A_1 satisfying $I_0\cap Q=I_1\cap Q$. Conversely, if I_0 is an ideal of A_0 and I_1 is an ideal of A_1 with $I_0\cap Q=I_1\cap Q$, then $I_0\cup I_1$ is an ideal of $P(A_0, A_1)$.*

Proof. Let I be an ideal of $P(A_0, A_1)$. Then $I_i=I\cap A_i$ is an ideal of A_i, $i=0,1$, and $I_0\cap Q=I\cap A_0\cap Q=I\cap A_0\cap A_1=I\cap A_1\cap Q=I_1\cap Q$, which proves the first statement.

To prove the converse, consider the partial algebra $\langle A_0\cup A_1; \vee, \wedge\rangle$, where $x\wedge y$ (resp., $x\vee y$) is defined if and only if x and y are in the same A_i and $x\wedge y$ (resp., $x\vee y$) is the meet (resp., join) of x and y in A_i. Call a set I an ideal of the partial algebra $\langle A_0\cup A_1; \wedge, \vee\rangle$ if, whenever $x,y\in I$ and $x\vee y$ is defined, then $x\vee y\in I$ and whenever $x\in I, y\in A_0\cup A_1$, and $y\leqq x$, then $y\in I$. (The partial order $\leqq$ was defined in Lemma 2.) Now let I_0 be an ideal of A_0 and let I_1 be an ideal of A_1 with $I_0\cap Q=I_1\cap Q$. The latter condition ensures that $I_0\cup I_1$ is an ideal of $\langle A_0\cup A_1; \wedge, \vee\rangle$. Now we prove that $I_0\cup I_1$ is an ideal of $P(A_0, A_1)$. In fact, $P(A_0, A_1)$ is the smallest partial lattice in which, besides the partial operations of $\langle A_0\cup A_1; \wedge, \vee\rangle$, all the meets and joins are defined that follow by iterated application of conditions

(i) to (v) and their duals. Therefore, it is sufficient to check that a single application of any one of (i) to (v) and their duals does not change the ideals; this is evident.

4. Smooth representations of ideals. The proofs in G. GRÄTZER and J. SICHLER [4] rely on two facts:

1. In a free product $L=A_0*A_1$ every element has a lower A_0-cover, which is an element of $(A_0)^b$ (that is, A_0 with a new 0 and 1 adjoined);

2. Forming lower A_0-covers is a homomorphism of L into $(A_0)^b$.

In general, these statements do not hold for amalgamated free products. In this section we find some statements that hold for amalgamated free products; these statements can be viewed as substitutes for the two facts mentioned above.

Throughout this section, let Q, A_0, A_1, L be lattices, let $L=A_0*_Q A_1$, and let $A=P(A_0, A_1, Q)$ as defined in Section 3. Let $I(A)$ (respectively, $I(A_i)$) denote the ideal lattice of A (respectively, of A_i). For any ideal I of L or of A define

$$(I)_i = I \cap A_i, \quad i = 0, 1$$

and for an ideal I of L define

$$I_A = I \cap A.$$

For a principal ideal I of L, the ideals $(I)_i$ and I_A correspond to the usual lower covers (see, e.g., [1]), however, $I \to (I)_i, I \in I(L)$, is not a homomorphism, that is,

$$(1) \qquad \big(p(I_0, \ldots, I_{n-1})\big)_i = p\big((I_0)_i, \ldots, (I_{n-1})_i\big)$$

does not hold for all polynomials p. For certain polynomials, however, (1) does hold (see Definition 2) and it will turn out (Lemma 8) that this happens often enough, making it possible to carry out some of the proofs of [4] under more general conditions.

Definition 2. Let $p=p(x_0, \ldots, x_{n-1})$ be an n-ary lattice polynomial, let $I, I_0, \ldots, I_{n-1}$ be ideals of L (of A, A_i, respectively), and let $I=p(I_0, \ldots, I_{n-1})$ in $I(L)$ (in $I(A), I(A_i)$, respectively). We say that $p(I_0, \ldots, I_{n-1})$ is a *smooth representation* of I (or that $p(I_0, \ldots, I_{n-1})$ *is smooth*) iff one of the following conditions holds:

a) $p=x_i$;

b) $p=p_0 \wedge p_1$ and both $p_0(I_0, \ldots, I_{n-1})$ and $p_1(I_0, \ldots, I_{n-1})$ are smooth;

c) $p=p_0 \vee p_1$, both $p_0(I_0, \ldots, I_{n-1})$ and $p_1(I_0, \ldots, I_{n-1})$ are smooth, and, for any $q \in Q$,

$$q \in p(I_0, \ldots, I_{n-1}) \text{ implies that } q \in p_0(I_0, \ldots, I_{n-1}) \text{ or}$$

$$q \in p_1(I_0, \ldots, I_{n-1}).$$

The following lemma shows that every representation of an element of L can be turned into a smooth representation.

Lemma 4. *Let* $a\in L, a_0, \dots, a_{n-1}\in A_0\cup A_1$, *and let* $a=p(a_0, \dots, a_{n-1})$ *where* p *is a lattice polynomial. Then there exist an integer* $m\geqq 0$, *a polynomial* $\tilde{p}$ *in* $n+m$ *variables, and subsets* $Q_0, \dots, Q_{m-1}$ *of* Q *such that*

$$(a] = \tilde{p}((a_0], \dots, (a_{n-1}], (Q_0], \dots, (Q_{m-1}])$$

is a smooth representation of $(a]$ *in* $I(L)$.

Proof. We prove this statement by induction on the rank of p.

If $p=x_i$, then we can choose $m=0, \tilde{p}=p$.

If $p=p_0\vee p_1$, then, by the induction hypothesis, there exist an $m\geqq 0$, polynomials $\tilde{p}_0$ and $\tilde{p}_1$ of $n+m-1$ variables, and subsets $Q_0, \dots, Q_{m-2}$ of Q such that

$$\tilde{p}_i((a_0], \dots, (a_{n-1}], (Q_0], \dots, (Q_{m-2}])$$

is a smooth representation of $p_i((a_0], \dots, (a_{n-1}])$ for $i=0$ and 1. Let $Q_{m-1}=(a]\cap Q$. We claim that

$$\tilde{p}_0((a_0], \dots, (a_{n-1}], (Q_0], \dots, (Q_{m-2}])\vee$$

$$\vee(\tilde{p}_1((a_0] \dots, (a_{n-1}], (Q_0], \dots, (Q_{m-2}])\vee(Q_{m-1}])$$

is a smooth representation of $(a]$. Indeed, by the definitions of $\tilde{p}_i$ and of Q_{m-1}, this ideal equals $(a]$. Moreover, $\tilde{p}_1((a_0], \dots)\vee(Q_{m-1}]$ is smooth because its components are smooth and if, for $q\in Q, q\in\tilde{p}_1((a_0], \dots)\vee(Q_{m-1}]$, then $q\in(a]$; thus, $q\in(Q_{m-1}]$ by the definition of Q_{m-1}. Similarly, $\tilde{p}_0((a], \dots)\vee(\tilde{p}_1((a_0], \dots)\vee(Q_{m-1}])$ is smooth.

Finally, if $p=p_0\wedge p_1$, then let $\tilde{p}_i((a_0], \dots, (a_{n-1}], (Q_0], \dots, (Q_{m-1}])$ be a smooth representation of $p_i((a_0], \dots, (a_{n-1}])$ for $i=0$ and 1. The meet of these two polynomials is obviously a smooth representation of $(a]$.

In the remainder of this section we have to compute polynomials in $L, I(L)$, $I(A)$, and $I(A_i)$, $i=0, 1$. We shall distinguish between the operations in $I(A)$ and $I(A_i)$ by superscripting them by A and i, respectively.

The following lemma is a consequence of the solution of the word problems for lattices freely generated by a partial lattice (see, e.g., G. Grätzer, A. Huhn, and H. Lakser [2]).

Lemma 5. *Let* $x, y\in L$. *Then*

$(x\vee y]\cap A = ((x]\cap A)\vee^A((y]\cap A)$, *and* $(x\wedge y]\cap A = ((x]\cap A)\wedge^A((y]\cap A)$.

Lemma 6. *Let* I *and* J *be ideals of* L. *Then*

$$(I\vee J)_A = (I)_A\vee^A(J)_A.$$

Furthermore, if $I\vee J$ *is smooth, then so is* $(I)_A\vee^A(J)_A$.

Proof. We prove that $(I \vee J)_A \subseteq (I)_A \vee^A (J)_A$ (the reverse inclusion is obvious). Let $a \in (I \vee J)_A$. Then $a \in A$ and there exist $i \in I$ and $j \in J$ such that $a \leqq i \vee j$. From Lemma 5, it follows that

$$a \in (i \vee j] \cap A \subseteq ((i] \cap A) \vee^A ((j] \cap A) \subseteq (I)_A \vee^A (J)_A.$$

This proves the first half of the lemma.

Assume now that $I \vee J$ is smooth. We have to prove that so is $(I)_A \vee^A (J)_A$. Let $q \in Q$ and let

$$q \in (I)_A \vee^A (J)_A.$$

Then $q \in I \vee J$; thus, $q \in I$ or $q \in J$, say $q \in I$. Since $q \in Q \subseteq A$, we have $q \in I \cap A = (I)_A$. This completes the proof.

Most of the results of this section are summarized in the following two lemmas that show that one can work with smooth representations as if forming lower covers were a homomorphism.

Lemma 7. *Let I and J be ideals of A and let us assume that $I \vee^A J$ is smooth. Then*

$$(I \vee^A J)_i = (I)_i \vee^i (J)_i \quad \textit{for} \quad i = 0, 1$$

and the right side of the equation is smooth.

Proof. We claim that

$$((I)_0 \vee^0 (J)_0) \cap Q = ((I)_1 \vee^1 (J)_1) \cap Q.$$

Indeed, let $q \in Q$ and let $q \in (I)_0 \vee^0 (J)_0$. Then $q \in I \vee^A J$; therefore, q is in I or J, say, $q \in I$. Then $q \in (I)_1 \subseteq (I)_1 \vee^1 (J)_1$, which verifies that the left side is contained in the right side. Repeating this argument starting with the right side, we verify the claim.

This claim, by Lemma 3, shows that

$$((I)_0 \vee^0 (J)_0) \cup ((I)_1 \vee^1 (J)_1)$$

is an ideal of $P(A_0, A_1)$; obviously, it is the smallest ideal containing both I and J, that is,

$$I \vee^A J = ((I)_0 \vee^0 (J)_0) \cup ((I)_1 \vee^1 (J)_1).$$

Now we compute (using the above claim again):

$$\begin{aligned}(I \vee^A J)_0 &= \\ &= (((I)_0 \vee^0 (J)_0) \cup ((I)_1 \vee^1 (J))_1) \cap A_0 = \\ &= ((I)_0 \vee^0 (J)_0) \cup (((I)_1 \vee^1 (J)_1) \cap Q) = \\ &= ((I)_0 \vee^0 (J)_0) \cup (((I)_0 \vee^0 (J)_0) \cap Q) = \\ &= (I)_0 \vee^0 (J)_0.\end{aligned}$$

Finally, we can see that $(I)_0 \vee^0 (J)_0$ is smooth arguing as we did in Lemma 6.

Lemma 8. *Let $p=p(x_0, \ldots, x_{n-1})$ be a lattice polynomial and let $I_0, \ldots, I_{n-1}$ be ideals of L, such that $p(I_0, \ldots, I_{n-1})$ is smooth. Then*

$$(p(I_0, \ldots, I_{n-1}))_i = p((I_0)_i, \ldots, (I_{n-1})_i)$$

is a smooth representation of $(p(I_0, \ldots, I_{n-1}))_i$.

Proof. By induction: if $p=x_i$ or $p=p_0 \wedge p_1$, then Lemma 8 is trivial; if $p=p_0 \vee p_1$, then Lemma 8 is a combination of Lemmas 6 and 7.

5. Amalgamated free products of sublattices. It was proved in B. Jónsson [5], that, if a variety V has the Amalgamation Property, then the following statement holds: for arbitrary algebras A_0 and A_1 in V and subalgebras A_0' of A_0 and A_1' of A_1 the set $A_0' \cup A_1'$ generates a subalgebra in the free product $A_0 * A_1$ canonically isomorphic to $A_0' * A_1'$. "Canonically" means that the isomorphism is the identity map on A_0' and on A_1'. Jónsson's proof is valid not only for varieties but also for classes closed under the formation of subalgebras and of direct products. Thus the proof works for Q-lattices, that is, lattices containing Q as a sublattice such that the elements of Q are regarded as nullary operations. This yields the following lemma.

Lemma 9. *Let $L=A_0 *_Q A_1$, let A_0' and A_1' be sublattices of A_0 and A_1, respectively, and let $Q \subseteq A_0'$ and $Q \subseteq A_1'$. Then the sublattice of $A_0 *_Q A_1$ generated by $A_0' \cup A_1'$ is canonically isomorphic to $A_0' *_Q A_1'$.*

There is an alternative proof by using the solution to the word problem for lattices generated by a partial lattice. For the case $Q=\varnothing$, such a proof appears in G. Grätzer, H. Lakser, and C. R. Platt [3]. (See also G. Grätzer [1].)

6. Proof of the Theorem. We introduce some new notation. For an ideal I of L, let I_{A_0} denote the ideal of L generated by $I \cap A_0$; we call I_{A_0} the lower A_0-cover of I. Similarly for I_{A_1}, I_{B_0}, and I_{B_1}. Note that Lemma 8 holds also for lower A_i (resp., B_j)-covers.

For arbitrary fixed $i, j \in \{0, 1\}$, we define $I_{ij}(L)$ as the set of principal ideals of L and the lower A_i-covers and lower B_j-covers of principal ideals of L.

We prove the main theorem in a stronger form:

Theorem. *Let $L=A_0 *_Q A_1 = B_0 *_Q B_1$. Then the following conditions are equivalent.*

(i) $L=(A_0 \cap B_0) *_Q (A_0 \cap B_1) *_Q (A_1 \cap B_0) *_Q (A_1 \cap B_1)$.

(ii) $A_i=(A_i \cap B_0) *_Q (A_i \cap B_1)$, *for* $i=0, 1$.

(iii) $B_j=(A_0 \cap B_j) *_Q (A_1 \cap B_j)$, *for* $j=0, 1$.

(iv) *For any* $i, j \in \{0, 1\}, x \in A_i$, *and* $y \in B_j$, $x \leqq y$ *in* L *implies the existence of a* $z \in A_i \cap B_j$ *such that* $x \leqq z$ *in* A_i *and* $z \leqq y$ *in* B_j.

(v) *For any* $i, j \in \{0, 1\}$ *and for any ideal* I *of* L, $I = (I \cap A_i] = (I \cap B_j]$ *implies that* $I = (I \cap A_i \cap B_j]$.

(iv) *For any* $i, j \in \{0, 1\}$ *and for any ideal* $I \in I_{ij}(L)$, $I = (I \cap A_i] = (I \cap B_j]$ *implies that* $I = (I \cap A_i \cap B_j]$.

Proof. We prove the theorem by the following scheme: (i)↔(ii), (i)↔(iii); (i), (ii), and (iii) jointly imply (iv); (iv)→(v)→(vi)→(ii).
(ii)→(i) is clear from the definition of the right side of (i) (given after Definition 1).
(i)→(ii). Let $a \in A_0$. Then, by (i), a can be expressed in the form

$$a = p(x_{00}, x_{00}', \dots, x_{01}, x_{01}', \dots, x_{10}, x_{10}', \dots, x_{11}, x_{11}', \dots)$$

where $x_{ij}, x_{ij}', \dots \in A_i \cap B_j, i, j \in \{0, 1\}$ and p is a lattice polynomial. By Lemma 4, $(a]$ has a smooth representation in $I(L)$:

$$(a] = \tilde{p}((x_{00}], \dots, (x_{01}], \dots, (x_{10}], \dots, (x_{11}], \dots, (Q_0], \dots),$$

where $Q_0, \dots \subseteqq Q$. Then, by Lemma 8,

$$(a] = (a]_{A_0} = \tilde{p}((x_{00}]_{A_0}, \dots, (x_{10}]_{A_0}, \dots, (x_{01}]_{A_0}, \dots, (x_{11}]_{A_0}, \dots, (Q_0], \dots).$$

We claim that, $(x_{10}]_{A_0}$, as well as $(x_{11}]_{A_0}$, is generated as an ideal by elements of Q. Indeed, let $(x_{10}]_{A_0}$ be generated by $x_\gamma, \gamma \in \Gamma$, in A_0. By Lemma 2, for every $\gamma \in \Gamma$, there is a $y_\gamma \in Q$ with $x_\gamma \leqq y_\gamma \leqq x_{10}$. Thus $\{y_\gamma | \gamma \in \Gamma\}$ generates $(x_{10}]_{A_0}$, and $\{y_\gamma | \gamma \in \Gamma\}$ is a subset of Q. Summarizing,

$$(a] = \tilde{p}((x_{00}], \dots, (x_{10}], \dots, (Q_{10}], \dots, (Q_{11}], \dots, (Q_0], \dots),$$

where $Q_{10}, \dots, Q_{11}, \dots, Q_0, \dots \subseteqq Q$. Hence a can be expressed by $x_{00}, \dots, x_{10}, \dots$, and elements of Q. Thus, a is in the sublattice generated by $(A_0 \cap B_0) \cup (A_0 \cap B_1)$. Therefore, $A_0 \cap B_0$ and $A_0 \cap B_1$ generate A_0. It follows from Lemma 9 that they generate A_0 freely over Q.
(i)↔(iii) follows by symmetry.
(i), (ii), and (iii) jointly imply (iv). By (ii) and (iii), the sublattice $[A_0 \cup B_0]$ generated by $A_0 \cup B_0$ is also generated by $A_0 \cap B_0, A_0 \cap B_1$, and $A_1 \cap B_0$. By (i), $A_0 \cap B_0$, $A_0 \cap B_1, A_1 \cap B_0$ freely generate over Q. Thus $[A_0 \cup B_0]$ is freely generated by $(A_0 \cap B_0) \cup (A_0 \cap B_1) \cup (A_1 \cap B_0)$. Hence, it is also freely generated by $[(A_0 \cap B_0) \cup \cup (A_0 \cap B_1)] \cup [(A_0 \cap B_0) \cup (A_1 \cap B_0)]$. By (ii) and (iii), this set is $A_0 \cup B_0$, and the relative sublattice of L on this subset is the partial lattice $P(A_0, B_0, A_0 \cap B_0)$. Therefore, $[A_0 \cup B_0]$ is the free $(A_0 \cap B_0)$-product of A_0 and B_0. Thus, Lemma 2 gives us (iv) for $i = j = 0$. Since (i), (ii), and (iii) are symmetric in i and j, condition (iv) now follows.

(iv)→(v). Let the ideal I be generated by $\{x_\gamma|\gamma\in\Gamma\}\subseteq A_i$ and by $\{y_\delta|\delta\in\Delta\}\subseteq B_j$; we can assume that $\{y_\delta|\delta\in\Delta\}$ is closed under finite joins. By (iv), for any $\gamma\in\Gamma$ we can choose a $\gamma'\in\Delta$ and a $z_{\gamma'}\in A_i\cap B_j$ satisfying $x_\gamma\leqq z_{\gamma'}\leqq y_{\gamma'}$. Obviously, $\{z_{\gamma'}|\gamma'\in\Gamma\}\subseteq \subseteq A_i\cap B_j$ generates I.
(v)→(vi) is obvious since (vi) is a special case of (v).
(vi)→(ii). By Lemma 9 and by symmetry, it suffices to prove that A_0 is generated by $(A_0\cap B_0)\cup(A_0\cap B_1)$.

For $a\in A_0$, there exist a polynomial p and elements $b_0, b_0', \ldots\in B_0, b_1, b_1', \ldots\in \in B_1$, such that $a=p(b_0, b_0', \ldots, b_1, b_1', \ldots)$. By Lemma 4, there exist a polynomial $\tilde{p}$ and $Q_0, Q_1, \ldots\subseteq Q$ such that

$$(a] = \tilde{p}((b_0], (b_0'], \ldots, (b_1], (b_1'], \ldots, (Q_0], (Q_1], \ldots)$$

is a smooth representation of $(a]$. Then, by Lemma 8,

$$(a] = (a]_{A_0} = \tilde{p}((b_0]_{A_0}, \ldots, (b_1]_{A_0}, \ldots, (Q_0]_{A_0}, \ldots).$$

In this expression, $(Q_0]_{A_0}=(Q_0], \ldots$. Furthermore, we shall prove the claim that $(b_0]_{A_0}, (b_0']_{A_0}, \ldots,$ and $(b_1]_{A_0}, (b_1']_{A_0}, \ldots$ are generated by elements of $A_0\cap B_0$ and $A_0\cap B_1$, respectively. Thus, each ideal occurring in the representation is generated by elements of $(A_0\cap B_0)\cup(A_0\cap B_1)$. Therefore, so is $(a]$. We conclude that $a\in \in[(A_0\cap B_0)\cup(A_0\cap B_1)]$, which was to be proved.

To verify the claim, it is sufficient to prove by symmetry that $(b_0]_{A_0}$ is generated by its elements in $A_0\cap B_0$.

First, we verify that $(b_0]_{A_0}$ is generated by its elements in B_0.

We start with a smooth representation

$$(b_0] = q((a_0], (a_0'], \ldots, (a_1], (a_1'], \ldots, (R_0], (R_1], \ldots),$$

where $a_0, a_0', \ldots\in A_0, a_1, a_1', \ldots\in A_1$ and $R_0, R_1, \ldots\subseteq Q$. Then

$$(b_0]_{A_0} = q((a_0]_{A_0}, \ldots, (a_1]_{A_0}, \ldots, (R_0]_{A_0}, \ldots) = q((a_0], \ldots, (a_1]_{A_0}, \ldots, (R_0], \ldots)$$

and, applying Lemma 8 twice, we obtain

$$(b_0]_{A_0} = ((b_0]_{B_0})_{A_0} = q(((a_0]_{B_0})_{A_0}, \ldots, ((a_1]_{B_0})_{A_0}, \ldots, ((R_0]_{B_0})_{A_0}, \ldots).$$

Hence,

$$(b_0]_{A_0} = q((a_0], \ldots, (a_1]_{A_0}, \ldots, (R_0], \ldots) \supseteqq q((a_0]_{B_0}, \ldots, (a_1]_{A_0}, \ldots, (R_0], \ldots) \supseteqq$$
$$\supseteqq q(((a_0]_{B_0})_{A_0}, \ldots, ((a_1]_{B_0})_{A_0}, \ldots, ((R_0]_{B_0})_{A_0}, \ldots) = (b_0]_{A_0}$$

therefore,

$$(b_0]_{A_0} = q((a_0]_{B_0}, \ldots, (a_1]_{A_0}, \ldots, (R_0], \ldots).$$

The ideals $(a_0]_{B_0}, \ldots$ are, by definition, generated by elements of B_0; the ideals $(a_1]_{A_0}, \ldots$ are generated by elements of Q by Lemma 2. Since $Q_0, \ldots\subseteq Q$, we conclude that $(b_0]_{A_0}$ is generated by elements of B_0.

Finally, since $(b_0]_{A_0}$ has been proved to be generated by its elements in B_0, and $(b_0]_{A_0}$ is by definition generated by its elements in A_0, and $(b_0]_{A_0} \in I_{00}(L)$, all the hypotheses of (vi) are satisfied. Condition (vi) yields that $(b_0]_{A_0}$ is generated by elements of $A_0 \cap B_0$, which completes the proof of the claim.

This finishes the proof of the implication (vi)→(ii) and of the Theorem.

7. Proof of Corollaries 1—4. Proof of Corollary 1. Let Q satisfy, for example, the Ascending Chain Condition, and let $L = A_0 *_Q A_1 = B_0 *_Q B_1$. We claim that, for any $i, j \in \{0, 1\}$, $I_{ij}(L)$ consists of all principal ideals of L. Indeed, let us take a smooth representation of the principal ideal $(x]$:

$$(x] = p\big((a_0], (a_0'], \ldots, (a_1], (a_1'], \ldots, (Q_0], (Q_1], \ldots\big),$$

$a_0, a_0', \ldots \in A_0$, $a_1, a_1', \ldots \in A_1$, and $Q_0, Q_1, \ldots \subseteq Q$. Then

$$(x]_{A_0} = p\big((a_0], (a_0'], \ldots, (a_1]_{A_0}, (a_1']_{A_0}, \ldots, (Q_0], (Q_1], \ldots\big).$$

It follows from Lemma 2 that the ideals $(a_1]_{A_0}, \ldots,$ are generated by elements of Q; thus, by the Ascending Chain Condition, these ideals and also $(Q]_0, \ldots$ are principal. Therefore, $(x]_{A_0}$ is a principal idéal. This proves the claim for $i = j = 0$. By symmetry, the claim is proved.

Using this claim, it is easy to establish condition (vi) of the Theorem: if the single generating element of an ideal in $I_{ij}(L)$ is both in A_i and in B_j, then it is in $A_i \cap B_j$. Thus the Theorem shows the existence of a common refinement.

Proof of Corollary 2. Let $L = A_0 *_Q A_1 = B_0 *_Q B_1$, and let us assume that the hypotheses of Corollary 2 hold, that is, for any i, j, A_i or B_j is convex in $A_i \cup B_j$. We are going to establish condition (v) of the Theorem. Let $i, j \in \{0, 1\}$, let, for instance, A_i be convex in $A_i \cup B_j$. Let $I \in I(L)$, such that $I = (I \cap A_i] = (I \cap B_j]$. Let G be a generating set of I in A_i and let H be a generating set of I in B_j. We can assume that both G and H are closed under finite joins. Then

$$I = \{x \mid x \leqq g \text{ for some } g \in G\} = \{x \mid x \leqq h \text{ for some } h \in H\}.$$

Thus, for any $g \in G$ there exists an $h_g \in H$ satisfying $g \leqq h_g$ and for $h_g \in H$ there exists an $g' \in G$ with $h_g \leqq g'$. Therefore, $g \leqq h_g \leqq g'$, so by the convexity of A_i in $A_i \cup B_j$, we conclude that $h_g \in A_i$; since $h_g \in G \subseteqq B_j$, $h_g \in A_i \cap B_j$. Now it is clear that $K = \{h_g \mid g \in G\}$ generates I and $K \subseteqq A_i \cap B_j$, verifying condition (v) of the Theorem.

Proof of Corollaries 3 and 4. Under the conditions of the Corollaries, $[A_1 \cup A_2]$ is the free product of A_1 and A_2 amalgamated over $A_1 \cap A_2$. Thus we may apply Lemma 2 to $A_1 \cup A_2$. Therefore, both corollaries follow from the following observation (due to E. FRIED):

Let $L=A_0*_QA_1=A_0*_QA_2$. If the conclusion of Lemma 2 holds for $A_1\cup A_2$ (that is, for $x\in A_1$ and $y\in A_2$, $x\leqq y$ iff $x\leqq z\leqq y$ for some $z\in A_1\cap A_2$ and symmetrically for $x\in A_2$ and $y\in A_1$), then $A_1=A_2$.

Indeed, under these conditions (iv) of the Theorem holds, hence there is a common refinement. Applying condition (ii) of the Theorem we obtain

$$A_1 = (A_0\cap A_1)*_Q(A_2\cap A_1) = Q*_Q(A_2\cap A_1).$$

Similarly, $A_2=Q*_Q(A_1\cap A_2)$, hence $A_1=A_2$.

8. Open problems. We repeat the question already mentioned in Section 2.

Problem 1. Is there a lattice Q such that Q-free products do not always have common refinements?

An equally important question arises in connection with Corollaries 3 and 4. In fact, they suggest, that some sort of a distributive law must be valid for Q-free factors.

Problem 2. Do Q-free factors of a lattice L form a distributive sublattice of the lattice of all sublattices of L? Does there exist some "natural" generalization of distributivity that holds for Q-free factors and implies Corollaries 3 and 4?

A negative answer to Problem 1 would answer both questions of Problem 2 in the affirmative; this can be seen from the following observations.

Let us assume that for a lattice Q, any two Q-free products of a lattice L have a common refinement. Let L be a lattice and let Q be a sublattice of L. Then $L=A*_QA'=B*_QB'$ implies that

$$L = (A\cap B)*_Q[A'\cup B'];$$

thus the Q-free factors form a sublattice of the lattice of all sublattices of L. Now let A, B, C be Q-free factors of L, that is, let

$$L = A*_QA' = B*_QB' = C*_QC'.$$

Then

$$A\cap[B\cup C] = [(A\cap B)\cup(A\cap C)],$$

since both sides are the Q-free products of $A\cap B\cap C$, $A\cap B\cap C'$, and $A\cap B'\cap C$.

9. Appendix: On the definition of amalgamated free products. In Section 3 we defined $A_0*_QA_1$ as the free lattice generated by the smallest partial lattice on the set $A_0\cup A_1$ $(A_0\cap A_1=Q)$ extending the operations of A_0 and A_1. We denoted this partial lattice by $P(A_0, A_1, Q)$. Here we prove the following characterization:

$P(A_0, A_1, Q)$ is the smallest *weak partial lattice* on the set $A_0\cup A_1$ extending the operations of A_0 and A_1.

By a weak partial lattice (see [1]) we mean a partial algebra $\langle H; \wedge, \vee\rangle$ satisfying conditions (i)—(iv) of Section 3 and their duals.

This result means the following: by definition, $P(A_0, A_1, Q)$ is formed by taking $A_0 \cup A_1$, and extending the $\wedge$ and $\vee$ of A_0 and A_1 by iterating (i)—(v) and their duals; according to the result of this appendix, condition (v) and its dual are not needed in this process.

Let $WP(A_0, A_1, Q) = WP$ be the smallest weak partial lattice on $A_0 \cup A_1$ extending the operations of A_0 and A_1. The existence of WP can be proved along the lines of the proof of Lemma 1. The proof of Lemma 2 shows that the partial ordering on WP is the same as the partial ordering on $P(A_0, A_1, Q)$. We are going to prove that WP is a partial lattice, that is, (v) and its dual hold. Then obviously $WP = P(A_0, A_1, Q)$.

By duality, it is sufficient to verify (v). To do that, let $a, b, c \in A_0 \cup A_1$ such that $(a] \vee (b] = (c]$ in the ideal lattice of WP. We have to show that $a \vee b$ exists and $a \vee b = c$ in WP.

If $a \vee b$ exists, then $(a \vee b]$ is obviously $(a] \vee (b]$, hence $(a \vee b] = (c]$. We conclude that $a \vee b = c$. Therefore, it is sufficient to show that if $(a] \vee (b] = (c]$, then $a \vee b$ exists.

If $a, b \in A_0$ or $a, b \in A_1$, then $a \vee b$ exists. Hence we can assume that $a \in A_0$ and $b \in A_1$. By symmetry, we can also assume that $c \in A_1$.

By the general description of join of ideals in a weak partial lattice (see Exercise 5.22 of [1]), $(a] \vee (b] = (c]$ implies the existence of a natural number n and elements

(1) $a = a_0 \leqq a_1 \leqq \ldots \leqq a_n$ in A_0,

(2) $b = b_0 \leqq b_1 \leqq \ldots \leqq b_n = c$ in A_1,

(3) $r_0 \leqq q_1 \leqq r_1 \leqq q_2 \leqq \ldots \leqq q_n \leqq q$ in Q

such that

(4) $r_i \leqq b_i, \quad 0 \leqq i \leqq n$,

(5) $q_i \leqq a_i, \quad 1 \leqq i \leqq n$,

(6) $b_{i+1} = b_i \vee q_{i+1}, \quad 0 \leqq i < n$,

(7) $a_{i+1} = a_i \vee r_{i+1}, \quad 0 \leqq i < n$,

(8) $a_n \leqq q \leqq b_n$.

(The symmetric case with $q_0 \leqq r_1 \leqq q_1 \leqq \ldots \leqq q_{n-1}$, $q_i \leqq a_i$, $0 \leqq i \leqq n$, $r_i \leqq b_i$, $1 \leqq i \leqq n$, $a_{i+1} = a_i \vee r_{i+1}$, $0 \leqq i < n$, $b_{i+1} = b_i \vee q_i$, $0 \leqq i < n$ is handled similarly.)

In the proof we shall utilize the following two properties of weak partial lattices:

(P1) If $x \vee y = z$ and $x \leqq u \leqq z$, then $u \vee y$ exists and $u \vee y = z$.

Indeed, by the associative identity,

$$u\vee(x\vee y) = (u\vee x)\vee y,$$

the left side exists and equals z; $u\vee x$ exist and equals u, hence $u\vee y$ exists and equals z, as claimed.

(P2) If $x\vee y=z$, $x=x_1\vee x_2$, and $x_2\leqq y$, then $x_1\vee y$ exists and $x_1\vee y=z$.

Indeed, by the associative identity,

$$(x_1\vee x_2)\vee y = x_1\vee(x_2\vee y),$$

the left side exists and equals z; in the right side $x_2\vee y$ exists and equals y, hence by (iii), $x_1\vee y$ exists and $x_1\vee y=z$, as claimed.

Now we prove $a\vee b=c$ by induction on n. Let $n=1$. Then we have the elements $a_0=a$, $b_0=b$, $b_1=c$, r_0, q_1, q, and $r_0\leqq q_1\leqq q$, $r_0\leqq b_0$, $r_0\leqq q_1\leqq a_1$, $a_1=a\vee r_0$, $b_1=b_0\vee q_1$, $a_1\leqq q\leqq b_1=c$.

By (P1), $q_1\vee b=c$ and $q_1\leqq a\leqq c$ implies that $a_1\vee b$ exist and $a_1\vee b=c$. Since $a_1=a\vee r_0$ and $r_0\leqq b$, by (P2), $a\vee b$ exists and $a\vee b=c$, as claimed.

Now let $n>1$. It is clear, that the elements $a_1\leqq\ldots\leqq a_n, b_1\leqq\ldots\leqq b_n, r_1\leqq q_2\leqq\ldots\leqq q_n\leqq q$ satisfy (1)—(8) with $n-1$. Therefore, $a_1\vee b_1$ exists and $a_1\vee b_1=c$. By (P2), $c=a_1\vee b_1=a_1\vee(q_1\vee b)=a_1\vee b$, since $q_1\leqq a_1$. Again by (P2), $c=a_1\vee b=(a\vee r_0)\vee b=a\vee b_1$ since $r_0\leqq b$. This proves the theorem.

References

[1] G. Grätzer, *General Lattice Theory. Pure and Applied Mathematics Series*, Academic Press (New York, N. Y.) Mathematische Reihe, Band 52, Birkhäuser Verlag, (Basel), Akademie Verlag (Berlin, 1978).

[2] G. Grätzer, A. P. Huhn and H. Lakser, On the structure of finitely presented lattices, *Canad. J. Math.*, **33** (1981), 404—411.

[3] G. Grätzer, H. Lakser and C. R. Platt, Free products of lattices, *Fund. Math.*, **69** (1970), 233—240.

[4] G. Grätzer and J. Sichler, Free decompositions of a lattice, *Canad. J. Math.*, **27** (1975), 276—285.

[5] B. Jónsson, Sublattices of a free lattice, *Canad. J. Math.*, **13** (1961), 256—264.

[6] B. Jónsson and E. Nelson, Relatively free products in regular varieties, *Algebra Universalis*, **4** (1974), 14—19.

(G. G.)
UNIVERSITY OF MANITOBA
WINNIPEG, MANITOBA
R3T 2N2 CANADA

(A. P. H.)
BOLYAI INTÉZET
6720 SZEGED
HUNGARY

Acta Sci. Math., 44 (1982), 67—75

A general moment inequality for the maximum of partial sums of single series

F. MÓRICZ

1. The main result

Let $(X, \mathcal{A}, \mu)$ be a (not necessarily finite or σ-finite) positive measure space. Let $\{\xi_k = \xi_k(x)\colon k=1, 2, \ldots\}$ be a given sequence of functions, defined on X, measurable with respect to $\mathcal{A}$, and such that $|\xi_k|^\gamma$ are integrable over X with respect to μ, where γ is a fixed real number, $\gamma \geqq 1$; i.e., our permanent assumption is that $\xi_k \in L^\gamma(X, \mathcal{A}, \mu)$ for each k. Set

$$S(b, l) = \sum_{k=b+1}^{b+l} \xi_k \quad \text{and} \quad M(b, m) = \max_{1 \leqq l \leqq m} |S(b, l)|,$$

where b is a nonnegative integer, l and m are positive integers.

In the following, $f(b, m)$ denotes a nonnegative function defined for integral $b \geqq 0$ and $m \geqq 1$, which possesses the 'superadditivity' property:

$$f(b, k) + f(b+k, l) \leqq f(b, k+l) \quad \text{for} \quad b \geqq 0, \quad k \geqq 1, \quad \text{and} \quad l \geqq 1. \tag{1.1}$$

We shortly explain the origin of the term 'superadditivity' in connection with the property expressed by (1.1). The fact is that $f(b, k)$ is actually a function of the interval $(b, b+k] = I_1$ with nonnegative integer endpoints. Considering the intervals $I_2 = (b+k, b+k+l]$ and $I = (b, b+k+l]$ too, we can see that the union $I_1 \cup I_2$ is a disjoint representation of I. Now (1.1) can be rewritten as follows

$$f(I_1) + f(I_2) \leqq f(I) \quad \text{where} \quad f(I_1) = f(b, k), \quad \text{etc.}$$

In the additive or subadditive case the relation '$\leqq$' should be replaced by '$=$' or '$\geqq$', respectively.

Received September 30, 1981.

This research was conducted while the author was on leave from Szeged University and a Visiting Professor at Ulm University, West Germany (supported by DAAD).

Further, by $\varphi(t, m)$ we denote a nonnegative function defined for real $t \geqq 0$ and integral $m \geqq 1$. We assume that $\varphi(t, m)$ is nondecreasing in both variables, i.e.,

$$\varphi(t_1, m_1) \leqq \varphi(t_2, m_2) \quad \text{whenever} \quad 0 \leqq t_1 \leqq t_2 \quad \text{and} \quad 1 \leqq m_1 \leqq m_2.$$

Our main result can be formulated as follows.

Theorem. *Let $\gamma \geqq 1$ be given. Suppose that there exist a nonnegative and superadditive function $f(b, m)$, and a nonnegative function $\varphi(t, m)$, nondecreasing in both variables, such that for every $b \geqq 0$ and $m \geqq 1$ we have*

$$\int |S(b, m)|^\gamma \, d\mu \leqq f(b, m)\, \varphi^\gamma(f(b, m), m). \tag{1.2}$$

Then for every $b \geqq 0$ and $m \geqq 2$ we have both the inequality

$$\int M^\gamma(b, m)\, d\mu \leqq 3^{\gamma-1} f(b, m) \left\{ \sum_{k=0}^{[\log m]-1} \varphi\left(\frac{f(b, m)}{2}, \left[\frac{m}{2^{k+1}}\right]\right)\right\}^\gamma \tag{1.3}$$

and the inequality

$$\int M^\gamma(b, m)\, d\mu \leqq \frac{5}{2} f(b, m) \left\{ \sum_{k=0}^{[\log m]} \varphi\left(\frac{f(b, m)}{2^k}, \left[\frac{m}{2^k}\right]\right)\right\}^\gamma. \tag{1.4}$$

Here and in the sequel the integrals are taken over the whole space X, $[t]$ denotes the integral part of t, and all logarithms are with base 2.

Remark 1. It is striking that the factor 5/2 in (1.4) does not depend on γ, in contrast to the factor $3^{\gamma-1}$ in (1.3). On the other hand, we have to take $[m/2^k]$ in the argument of φ on the right-hand side of (1.4), instead of $[m/2^{k+1}]$, which is the case in (1.3).

2. Special cases

We are going to present the riches of applicability of our Theorem, without aiming at completeness.

Let us take $\varphi(t, m) = t^{(\alpha-1)/\gamma}$ with an $\alpha > 1$. Then

$$\tilde{\Phi}(t, m) = \sum_{k=0}^{[\log m]} \varphi\left(\frac{t}{2^k}, \left[\frac{m}{2^k}\right]\right) \leqq (1 - 2^{(1-\alpha)/\gamma})^{-1} t^{(\alpha-1)/\gamma},$$

independently of m.

Corollary 1. *Let $\alpha > 1$ and $\gamma \geqq 1$ be given. Suppose that there exists a nonnegative and superadditive function $f(b, m)$ such that for every $b \geqq 0$ and $m \geqq 1$ we have*

$$\int |S(b, m)|^\gamma \, d\mu \leqq f^\alpha(b, m).$$

Then for every $b \geqq 0$ and $m \geqq 1$ we have

$$\int M^\gamma(b, m)\, d\mu \leqq \frac{5}{2}(1 - 2^{(1-\alpha)/\gamma})^{-\gamma} f^\alpha(b, m).$$

This result, apart from the factor 5/2 on the right-hand side, was proved by the present author in [3, Theorem 1], and somewhat later (with another constant) by LONGNECKER and SERFLING [2, Theorem 1].

Now take $\varphi(t, m)=t^{(\alpha-1)/\gamma} w(t)$, where again $\alpha>1$ and $w(t)$ is a (not necessarily nondecreasing, but positive) slowly varying function, i.e., $w(t)$ is defined and positive for real $t>0$, and for every fixed real $C>0$ we have

$$\frac{w(Ct)}{w(t)} \to 1 \quad \text{as} \quad t \to \infty.$$

For example, $w(t)=\{\log(1+t)\}^{\beta}\{\log\log(2+t)\}^{\delta}$ is such a function, where β and δ are arbitrary real numbers. It is not hard to check that we again have

$$\tilde{\Phi}(t, m) \leqq C(\alpha, \gamma, w)\, t^{(\alpha-1)/\gamma} w(t),$$

where $C(\alpha, \gamma, w)$ is a positive constant depending only on α, γ, and $w(t)$.

Corollary 2. *Let $\alpha>1$ and $\gamma\geqq 1$ be given. Suppose that there exist a nonnegative and superadditive function $f(b, m)$, and a slowly varying positive function $w(t)$, such that $t^{(\alpha-1)/\gamma} w(t)$ is nondecreasing and that for every $b\geqq 0$ and $m\geqq 1$ we have*

$$\int |S(b, m)|^{\gamma}\, d\mu \leqq f^{\alpha}(b, m)\, w^{\gamma}(f(b, m)).$$

Then for every $b\geqq 0$ and $m\geqq 1$ we have

$$\int M^{\gamma}(b, m)\, d\mu \leqq \frac{5}{2} C(\alpha, \gamma, w) f^{\alpha}(b, m)\, w^{\gamma}(f(b, m)).$$

Next take $\varphi(t, m)=\lambda(m)$, where $\{\lambda(m): m=1, 2, \ldots\}$ is a nondecreasing sequence of positive numbers.

Corollary 3. *Let $\gamma\geqq 1$ be given. Suppose that there exist a nonnegative and superadditive function $f(b, m)$, and a positive and nondecreasing sequence $\lambda(m)$ such that for every $b\geqq 0$ and $m\geqq 1$ we have*

$$\int |S(b, m)|^{\gamma}\, d\mu \leqq f(b, m)\, \lambda^{\gamma}(m).$$

Then for every $b\geqq 0$ and $m\geqq 1$ we have

$$\int M^{\gamma}(b, m)\, d\mu \leqq 3^{\gamma-1} f(b, m) \left\{\sum_{k=1}^{[\log m]} \lambda\left(\left[\frac{m}{2^k}\right]\right)\right\}^{\gamma}. \tag{2.1}$$

This moment inequality, apart from the factor $3^{\gamma-1}$ on the right-hand side, was already proved by the present author in a slightly different form in [3, Theorem 4].

Finally, it is quite obvious that in any case we can state the following

Corollary 4. *Under the conditions of the Theorem, for every* $b \geqq 0$ *and* $m \geqq 2$ *we have*

$$\int M^\gamma(b, m)\, d\mu \leqq 3^{\gamma-1} f(b, m)\, \varphi^\gamma\left(f(b, m), \left[\frac{m}{2}\right]\right)(\log m)^\gamma. \tag{2.2}$$

In the special case when $\varphi(t, m)=\lambda(m)$ is a slowly varying sequence, which is positive and nondecreasing, in particular, when $\varphi(t, m)=1$, the right-hand side of (2.2) is of the same order of magnitude as the right-hand side of (1.3) or (1.4). Thus, in this case the moment inequality (2.2) cannot be improved in the framework of our method.

Remark 2. Corollary 3 is proved in [3] by the socalled bisection technique with respect to the number m of the terms, which goes back to the proof of the well-known Rademacher—Menšov inequality (see, e.g. [4, p. 83]). The proof of Corollary 1 is based on the bisection technique with respect to the weight $f(b, m)$, which was firstly applied, it seems to us, by Erdős [1] concerning an upper estimation of the fourth moment of the partial sums of lacunary trigonometric series. Now, the proof of our Theorem presented in the next Section is based on an appropriate combination of these two bisection techniques. This combined technique was firstly used, as far as the author is aware, by Tandori [5] in order to obtain a special upper estimate for the second moment of the maximum of the partial sums of orthogonal series.

For a more detailed historical background of these moment inequalities see [3].

3. The proof of the theorem

Proof of (1.3). Setting

$$\Phi(t, 1) = \varphi(t, 1) \qquad (t \geqq 0)$$

and

$$\Phi(t, m) = \sum_{k=0}^{[\log m]-1} \varphi\left(\frac{t}{2^k}, \left[\frac{m}{2^{k+1}}\right]\right) \qquad (t \geqq 0,\ m \geqq 2),$$

it is clear that $\Phi(t, m)$ is also nondecreasing in both variables. This explicit expression for $\Phi(t, m)$ can be rewritten into the following recurrence one, which will be useful in the sequel:

$$\Phi(t, 1) = \Phi(t, 2) = \Phi(t, 3) = \varphi(t, 1) \quad (t \geqq 0) \tag{3.1}$$

and

$$\Phi(t, m) = \varphi\left(t, \left[\frac{m}{2}\right]\right) + \Phi\left(\frac{t}{2}, \left[\frac{m}{2}\right]\right) \qquad (t \geqq 0,\ m \geqq 4). \tag{3.2}$$

Now, statement (1.3) to be proved turns into

$$\int M^{\gamma}(b, m)\, d\mu \leqq 3^{\gamma-1} f(b, m)\, \Phi^{\gamma}(f(b, m), m). \tag{3.3}$$

The proof of (3.3) proceeds by induction on m. By (1.2) and (3.1), this is obvious for $m=1$ and for each b, even the factor $3^{\gamma-1}$ is superfluous on the right of (3.3) in this case.

In order to prove (3.3) for $m=2$ and 3 with arbitrary b, we use the trivial estimate

$$M(b, m) \leqq \sum_{k=b+1}^{b+m} |\xi_k|,$$

whence Minkowski's inequality and (1.2) provide that

$$\Big\{\int M^{\gamma}(b, m)\, d\mu\Big\}^{1/\gamma} \leqq \sum_{k=b+1}^{b+m} f^{1/\gamma}(k-1, 1)\, \varphi(f(k-1, 1), 1). \tag{3.4}$$

Taking into account the monotonicity of $\varphi(t, m)$ and making use of the elementary inequality

$$\sum_{k=b+1}^{b+m} t_k^{1/\gamma} \leqq m^{(\gamma-1)/\gamma} \Big(\sum_{k=b+1}^{b+m} t_k\Big)^{1/\gamma} \qquad (t_k \geqq 0, \gamma \geqq 1), \tag{3.5}$$

from (3.4) and (1.1) it follows that

$$\Big\{\int M^{\gamma}(b, m)\, d\mu\Big\}^{1/\gamma} \leqq \varphi(f(b, m), 1) \sum_{k=b+1}^{b+m} f^{1/\gamma}(k-1, 1) \leqq$$

$$\leqq m^{(\gamma-1)/\gamma} f^{1/\gamma}(b, m)\, \varphi(f(b, m), 1).$$

By (3.1) this is a sharpened form of (3.3) in case $m=2$, and (3.3) itself in case $m=3$.

Assume now as induction hypothesis that inequality (3.3) holds true for each nonnegative integer b and for each positive integer less than m, $m\geqq 4$, in the place of the second argument (we actually use that it is true for each positive integer not more than $[m/2]$). We will show that inequality (3.3) holds for m itself (and for arbitrary b).

We begin with an elementary observation. If $f(b, m)=0$ for some b and m, then, by (1.1), $f(b, k)=0$ and, by (1.2), $S(b, k)=0$ a.e. for each $k=1, 2, \ldots, m$, too. Consequently, $M(b, m)=0$ a.e. and thus (3.3) is obviously satisfied.

Henceforth we may and do assume that $f(b, m)\neq 0$. Then there exists an integer p, $1\leqq p\leqq m$, such that

$$f(b, p-1) \leqq \frac{1}{2} f(b, m) < f(b, p), \tag{3.6}$$

where we agree to set $f(b, 0)=0$ on the left of (3.6) in case $p=1$. It is also convenient to set $S(b,0)=M(b,0)=0$. Now (1.1) and (3.6) imply

$$f(b+p,\, m-p) \leqq f(b,m)-f(b,p) < \frac{1}{2} f(b,m). \tag{3.7}$$

We distinguish three cases according as $p=1$, $2\leqq p\leqq m-1$, and $p=m$.
Case (i): $2\leqq p\leqq m-1$. Set

$$p_1 = \left[\frac{p-1}{2}\right] \quad \text{and} \quad q_1 = \begin{cases} p_1 & \text{if } p-1 \text{ is even,} \\ p_1+1 & \text{if } p-1 \text{ is odd;} \end{cases}$$

$$p_2 = \left[\frac{m-p}{2}\right] \quad \text{and} \quad q_2 = \begin{cases} p_2 & \text{if } m-p \text{ is even,} \\ p_2+1 & \text{if } m-p \text{ is odd.} \end{cases}$$

It is clear that $p_1+q_1=p-1$ and $p_2+q_2=m-p$.

We are going to establish appropriate upper bounds for $|S(b,k)|$ under various values of k between 1 and m. It is easy to check that

$$|S(b,k)| \leqq \begin{cases} M(b,p_1) & \text{for } 1\leqq k\leqq p_1, \\ |S(b,q_1)|+M(b+q_1,p_1) & \text{for } q_1\leqq k\leqq p-1, \\ |S(b,p)|+M(b+p,p_2) & \text{for } p\leqq k\leqq p+p_2, \\ |S(b,p+q_2)|+M(b+p+q_2,p_2) & \text{for } p+q_2\leqq k\leqq m. \end{cases} \tag{3.8}$$

Hence we can derive a suitable upper estimate for $|S(b,k)|$ when k runs from 1 till m, which is independent of the value of k. Consequently, it will be an upper estimate for $M(b,m)$, as well:

$$M(b,m) \leqq |S(b,q_1)|+|S(b+q_1,p-q_1)|+|S(b+p,q_2)|+ \\ +\{M^\gamma(b,p_1)+M^\gamma(b+q_1,p_1)+M^\gamma(b+p,p_2)+M^\gamma(b+p+q_2,p_2)\}^{1/\gamma}. \tag{3.9}$$

Applying Minkowski's inequality, we find that

$$\Big\{\int M^\gamma(b,m)\,d\mu\Big\}^{1/\gamma} \leqq \Big\{\int |S(b,q_1)|^\gamma\,d\mu\Big\}^{1/\gamma}+\Big\{\int |S(b+q_1,p-q_1)|^\gamma\,d\mu\Big\}^{1/\gamma}+ \\ +\Big\{\int |S(b+p,q_2)|^\gamma\,d\mu\Big\}^{1/\gamma}+\Big\{\int M^\gamma(b,p_1)\,d\mu+\int M^\gamma(b+q_1,p_1)\,d\mu+ \\ +\int M^\gamma(b+p,p_2)\,d\mu+\int M^\gamma(b+p+q_2,p_2)\,d\mu\Big\}^{1/\gamma} = A+B, \tag{3.10}$$

where A denotes the sum of the first three terms and B denotes the fourth term on the right-hand side of (3.10).

Due to (1.2) and the facts that

$$q_1, q_2 \leqq \left[\frac{m-2}{2}\right]+1 = \left[\frac{m}{2}\right] \quad \text{and} \quad p-q_1 = p_1+1 \leqq \left[\frac{m-2}{2}\right]+1 = \left[\frac{m}{2}\right],$$

we have that

$$A \leqq f^{1/\gamma}(b, q_1)\,\varphi\big(f(b, q_1), q_1\big)+f^{1/\gamma}(b+q_1, p-q_1)\,\varphi\big(f(b+q_1, p-q_1), p-q_1\big)+$$
$$+f^{1/\gamma}(b+p, q_2)\,\varphi\big(f(b+p, q_2), q_2\big) \leqq$$
$$\leqq \varphi\left(f(b, m), \left[\frac{m}{2}\right]\right)\{f^{1/\gamma}(b, q_1)+f^{1/\gamma}(b+q_1, p-q_1)+f^{1/\gamma}(b+p, q_2)\}.$$

Using the elementary inequality (3.5) for $m=3$, by (1.1) we obtain

$$A \leqq 3^{(\gamma-1)/\gamma} f^{1/\gamma}(b, m)\,\varphi\left(f(b, m), \left[\frac{m}{2}\right]\right). \tag{3.11}$$

On the other hand, by the induction hypothesis,

$$B^\gamma \leqq 3^{\gamma-1}\big\{f(b, p_1)\,\Phi^\gamma\big(f(b, p_1), p_1\big)+ \tag{3.12}$$
$$+f(b+q_1, p_1)\,\Phi^\gamma\big(f(b+q_1, p_1), p_1\big)+f(b+p, p_2)\,\Phi^\gamma\big(f(b+p, p_2), p_2\big)+$$
$$+f(b+p+q_2, p_2)\,\Phi^\gamma\big(f(b+p+q_2, p_2), p_2\big)\big\} = 3^{\gamma-1}(B_1+B_2+B_3+B_4).$$

First consider B_1. Taking (3.6) into account, and that $p_1 \leqq p-1$ and $p_1 \leqq [m/2]$, it follows that

$$B_1 \leqq f(b, p_1)\,\Phi^\gamma\big(f(b, p-1), p_1\big) \leqq f(b, p_1)\,\Phi^\gamma\left(\frac{f(b, m)}{2}, \left[\frac{m}{2}\right]\right).$$

Similarly, by (3.6) and (3.7) we have in turn

$$B_2 \leqq f(b+q_1, p_1)\,\Phi^\gamma\left(\frac{f(b, m)}{2}, \left[\frac{m}{2}\right]\right),$$

$$B_3 \leqq f(b+p, p_2)\,\Phi^\gamma\left(\frac{f(b, m)}{2}, \left[\frac{m}{2}\right]\right),$$

$$B_4 \leqq f(b+p+q_2, p_2)\,\Phi^\gamma\left(\frac{f(b, m)}{2}, \left[\frac{m}{2}\right]\right).$$

To sum up, (3.12) and the estimates for B_i just obtained yield

$$B^\gamma \leqq 3^{\gamma-1}\Phi^\gamma\left(\frac{f(b, m)}{2}, \left[\frac{m}{2}\right]\right)\{f(b, p_1)+f(b+q_1, p_1)+f(b+p, p_2)+ \tag{3.13}$$
$$+f(b+p+q_2, p_2)\} \leqq 3^{\gamma-1} f(b, m)\,\Phi^\gamma\left(\frac{f(b, m)}{2}, \left[\frac{m}{2}\right]\right),$$

the last inequality following by (1.1).

Finally, putting (3.10), (3.11), and (3.13) together, we arrive at the inequality

$$\left\{\int M^\gamma(b, m)\,d\mu\right\}^{1/\gamma} \leqq 3^{(\gamma-1)/\gamma} f^{1/\gamma}(b, m)\left\{\varphi\left(f(b, m), \left[\frac{m}{2}\right]\right)+\Phi\left(\frac{f(b, m)}{2}, \left[\frac{m}{2}\right]\right)\right\},$$

which is equivalent to (3.3) owing to (3.2).

Case (ii): $p=1$. Now $f(b,1)>\frac{1}{2}f(b,m)$ and thus $f(b+1,m-1)<\frac{1}{2}f(b,m)$. Setting

$$p_2=\left[\frac{m-1}{2}\right] \quad \text{and} \quad q_2=\begin{cases} p_2 & \text{if } m-1 \text{ is even,}\\ p_2+1 & \text{if } m-1 \text{ is odd;}\end{cases}$$

we have, $q_2=[m/2]$. Now instead of (3.9) we can estimate in a simpler way:

$$M(b,m) \leqq |S(b,1)|+|S(b+1,q_2)|+\{M^\gamma(b+1,p_2)+M^\gamma(b+q_2+1,p_2)\}^{1/\gamma}. \tag{3.14}$$

The further reasonings are very similar, but somewhat shorter, to those in Case (i). We do not enter into details.

Case (iii): $p=m$. Now $f(b,m-1)\leqq\frac{1}{2}f(b,m)$ and

$$M(b,m) \leqq |S(b,q_2)|+|S(b+m-1,1)|+\{M^\gamma(b,p_2)+M^\gamma(b+q_2,p_2)\}^{1/\gamma}, \tag{3.15}$$

where p_2 and q_2 are the same as in Case (ii).

Thus inequality (1.3) has been completely proved.

Proof of (1.4). Setting

$$\tilde\Phi(t,m)=\sum_{k=0}^{[\log m]}\varphi\left(\frac{t}{2^k},\left[\frac{m}{2^k}\right]\right) \quad (t\geqq 0,\ m\geqq 1),$$

we have, instead of (3.1) and (3.2), the following recurrence relations:

$$\tilde\Phi(t,1)=\varphi(t,1)\quad (t\geqq 0) \quad\text{and}\quad \tilde\Phi(t,m)=\varphi(t,m)+\tilde\Phi\left(\frac{t}{2},\left[\frac{m}{2}\right]\right)\quad (t\geqq 0,\ m\geqq 2).$$

Statement (1.4) turns into

$$\int M^\gamma(b,m)\,d\mu \leqq \frac{5}{2}f(b,m)\,\Phi^\gamma(f(b,m),m). \tag{3.16}$$

This is obvious for $m=1$ even without the factor 5/2 on the right-hand side since $M(b,1)=S(b,1)$ for each b. In order to prove it for $m=2$ and for arbitrary b, we again use the trivial estimate

$$M(b,2)\leqq|\xi_{b+1}|+|\xi_{b+2}|,$$

whence Minkowski's inequality and (1.2) provide that

$$\left\{\int M^\gamma(b,2)\,d\mu\right\}^{1/\gamma}\leqq f^{1/\gamma}(b,1)\,\varphi(f(b,1),1)+f^{1/\gamma}(b+1,1)\,\varphi(f(b+1,1),1). \tag{3.17}$$

Making use of (1.1), we can conclude that either

$$f(b,1)\leqq\frac{1}{2}f(b,2) \quad\text{or}\quad f(b+1,1)\leqq\frac{1}{2}f(b,2).$$

Taking this and the monotonicity of $\varphi(t, m)$ into account, from (3.17) it follows that

$$\left\{\int M^\gamma(b, 2)\, d\mu\right\}^{1/\gamma} \leqq f^{1/\gamma}(b, 2)\left\{\varphi\big(f(b, 2), 1\big)+\varphi\left(\frac{f(b, 2)}{2}, 1\right)\right\} =$$
$$= f^{1/\gamma}(b, 2)\, \tilde{\Phi}\big(f(b, 2), 1\big),$$

which is a sharpened form of (3.16) for $m=2$.

The induction step is quite similar to that in the proof of (1.3), with the exception that this time one can start, instead of (3.9), from the following inequality, too:

$$M(b, m) \leqq \{|S(b, q_1)|^\gamma + |S(b, p)|^\gamma + |S(b, p+q_2)|^\gamma\}^{1/\gamma} + \tag{3.18}$$
$$+\{M^\gamma(b, p_1)+M^\gamma(b+q_1, p_1)+M^\gamma(b+p, p_2)+M^\gamma(b+p+q_2, p_2)\}^{1/\gamma}$$

(and analogous inequalities also instead of (3.14) and (3.15)). If one begins the calculations with (3.18), then one can avoid using inequality (3.5), as a result of which one gets the smaller factor 5/2. Indeed, now

$$\left\{\int M^\gamma(b, m)\, d\mu\right\}^{1/\gamma} \leqq \tilde{A}+B,$$

where

$$\tilde{A} = \left\{\int |S(b, q_1)|^\gamma\, d\mu + \int |S(b, p)|^\gamma\, d\mu + \int |S(b, p+q_2)|^\gamma\, d\mu\right\}^{1/\gamma}$$

and B is the same as in (3.10). Due to (1.2), the monotonicity of $\varphi(t, m)$ and (3.6), one can easily deduce:

$$\tilde{A} \leqq \{f(b, q_1)\,\varphi^\gamma\big(f(b, q_1), q_1\big)+f(b, p)\,\varphi^\gamma\big(f(b, p), p\big)+$$
$$+f(b, p+q_2)\,\varphi\big(f(b, p+q_2), p+q_2\big)\}^{1/\gamma} \leqq \varphi\big(f(b, m), m\big)\,\{f(b, q_1)+f(b, p)+$$
$$+f(b, p+q_2)\}^{1/\gamma} \leqq \left(\frac{5}{2}\right)^{1/\gamma} f^{1/\gamma}(b, m)\, \varphi\big(f(b, m), m\big).$$

The further reasoning runs along the same line as in the proof of (1.3).

Thus our Theorem has been completely proved.

References

[1] P. Erdős, On the convergence of trigonometric series, *J. Math. and Phys. Massachusetts Institute of Technology*, **22** (1943), 37—39.

[2] M. Longnecker and R. J. Serfling, General moment and probability inequalities for the maximum partial sums, *Acta Math. Acad. Sci. Hungar.*, **30** (1977), 129—133.

[3] F. Móricz Moment inequalities and the strong laws of large numbers, *Z. Wahrscheinlichkeitstheorie und verw. Gebiete*, **35** (1976), 299—314.

[4] P. Révész, *The Laws of Large Numbers*, Academic Press, (New York, 1968).

[5] K. Tandori, Bemerkung zur Konvergenz der Orthogonalreihen, *Acta Sci. Math.*, **26** (1965), 249—251.

BOLYAI INSTITUTE
UNIVERSITY OF SZEGED
ARADI VÉRTANÚK TERE 1
H—6720 SZEGED, HUNGARY

Acta Sci. Math., 44 (1982), 77—86

On the a.e. convergence of multiple orthogonal series. I (Square and spherical partial sums)

F. MÓRICZ

1. Notations. Let Z^d be the set of d-tuples $k=(k_1, \ldots, k_d)$ with nonnegative integral coordinates. Let $\varphi=\{\varphi_k(x): k\in Z^d\}$ be an orthonormal system (in abbreviation: ONS) on the unit cube $x=(x_1, \ldots, x_d)\in I^d$, where $I=[0, 1]$. Consider the d-multiple orthogonal series

(1) $$\sum_{k\in Z^d} a_k\varphi_k(x) = \sum_{k_1=0}^{\infty}\cdots\sum_{k_d=0}^{\infty} a_{k_1,\ldots,k_d}\varphi_{k_1,\ldots,k_d}(x_1, \ldots, x_d),$$

where $a=\{a_k: k\in Z^d\}$ is a system of coefficients, for which

(2) $$\sum_{k\in Z^d} a_k^2 < \infty.$$

Fixing a sequence $Q=\{Q_r: r=0, 1, \ldots\}$ of finite sets in Z^d with properties

$$Q_0\subset Q_1\subset Q_2\subset\ldots \quad\text{and}\quad \bigcup_{r=0}^{\infty} Q_r = Z^d,$$

our main goal is to study the convergence behaviour of the sums

(3) $$s_r(x) = \sum_{k\in Q_r} a_k\varphi_k(x) \quad (r = 0, 1, \ldots),$$

which can be regarded as a certain kind of partial sums of series (1). The case

$$Q_r^1 = \{k\in Z^d: \max_{1\le j\le d} k_j \le r\}$$

Received February 9, 1981.

Research completed while the author was on leave from Szeged University and a Visiting Professor at Ulm University, West Germany (supported by DAAD) and at The University of Western Ontario, Canada (supported by NSERC grant A—2983). The author gratefully acknowledges the support received at both places.

provides the square partial sums $s_r^1(x)$, while

$$Q_r^2 = \left\{k \in Z^d\colon |k| = \left(\sum_{j=1}^{d} k_j^2\right)^{1/2} \leqq r\right\}$$

provides the spherical partial sums $s_r^2(x)$ of (1).

2. A.e. convergence of $\{s_r(x)\colon r=0, 1, \dots\}$. Denote by $M(d, Q)$ the class of those systems $a=\{a_k\colon k\in Z^d\}$ of coefficients for which the sequence $\{s_r(x)\}$ defined by (3) converges a.e. for every ONS $\varphi=\{\varphi_k(x) : k\in Z^d\}$ on I^d. The set of measure zero of the divergence points may vary with each φ.

One can easily see that if $a\in M(d, Q)$, then (2) is necessarily satisfied. This follows from the obvious fact that the d-multiple Rademacher system

$$\{r_k(x)\} = \left\{\prod_{j=1}^{d} r_{k_j}(x_j)\colon k=(k_1, \dots, k_d)\in Z^d \quad\text{and}\quad x=(x_1, \dots, x_d)\in I^d\right\}$$

consists of stochastically independent functions and thus, for every choice of the sequence $Q=\{Q_r\colon r=0, 1, \dots\}$ of finite sets in Z^d, the sequence $\{s_r(x)\}$ defined by (3) for $\varphi=\{r_k(x)\}$ converges a.e. or diverges a.e. according as (2) is satisfied or not.

For a given system $a=\{a_k\colon k\in Z^d\}$ of coefficients we set

$$\mathscr{I}(a;\, d, Q, \varrho) = \sup \int_{I^d} \left(\max_{0\leqq r\leqq \varrho} |s_r(x)|\right)^2 dx,$$

where the supremum is taken over all ONS $\varphi=\{\varphi_k(x)\colon k\in Z^d\}$ on I^d and $dx= =dx_1 \dots dx_d$, further,

$$\|a;\, d, Q\| = \lim_{\varrho\to\infty} \mathscr{I}^{1/2}(a;\, d, Q, \varrho) \leqq \infty.$$

This limit exists since $\mathscr{I}(a; d, Q, \varrho)$ is nondecreasing in ϱ.

Theorem 1. (i) $a\in M(d, Q)$ *if and only if* $\|a; d, Q\| < \infty$;
(ii) $M(d, Q)$ *endowed with the norm* $\|\cdot\,; d, Q\|$ *is a separable Banach space.*

This theorem is essentially a reformulation of an earlier result of Tandori [11].

To this effect, let $\psi=\{\psi_{k_1}(x_1)\colon k=0, 1, \dots\}$ be a single ONS on I. Consider the ordinary orthogonal series

$$\sum_{k_1=0}^{\infty} c_{k_1}\psi_{k_1}(x_1), \tag{4}$$

where $c=\{c_{k_1}\colon k_1=0, 1, \dots\}$ is a sequence of coefficients for which

$$\sum_{k_1=0}^{\infty} c_{k_1}^2 < \infty. \tag{5}$$

Fixing a sequence $\nu=\{\nu_r: r=0, 1, \ldots\}$ of integers with the property $0\leqq\nu_0<\nu_1<<\nu_2<\ldots$, denote by $M(\nu)$ the class of those sequences $c=\{c_{k_1}\}$ for which the ν_rth partial sums of series (4) converge a.e. for every ONS $\psi=\{\psi_{k_1}(x_1)\}$ on I.

For a given sequence $c=\{c_{k_1}\}$ of coefficients we set

$$\mathcal{I}(c;\, \nu, \varrho) = \sup \int_I \left(\max_{0\leqq r\leqq \varrho} \left| \sum_{k_1=0}^{\nu_r} c_{k_1}\psi_{k_1}(x_1)\right|\right)^2 dx_1, \tag{6}$$

where the supremum is taken over all ONS $\psi=\{\psi_{k_1}(x_1)\}$ on I, and

$$\|c;\, \nu\| = \lim_{\varrho\to\infty} \mathcal{I}^{1/2}(c;\, \nu, \varrho) \leqq \infty.$$

It is not hard to see that

$$\mathcal{I}(c;\, \nu, \varrho) = \sup \int_I \left(\max_{0\leqq r\leqq \varrho} \left| \sum_{m_1=0}^{r} C_{m_1}\Psi_{m_1}(x_1)\right|\right)^2 dx_1,$$

where

$$C_m = \left(\sum_{k_1=\nu_{m-1}+1}^{\nu_m} c_{k_1}^2\right)^{1/2} \quad (m = 0, 1, \ldots;\ \nu_{-1} = -1)$$

and the supremum is taken over all ONS $\{\Psi_{m_1}(x_1)\}$ on I.

After these preliminaries the above-mentioned theorem of Tandori reads as follows.

Theorem A [11, Satz II]. (i) *$c\in M(\nu)$ if and only if $\|c;\nu\|<\infty$;*
(ii) *$M(\nu)$ endowed with the norm $\|\cdot\,;\nu\|$ is a separable Banach space.*

Now, it is a trivial observation that Theorem A remains valid if instead of the single ONS $\psi=\{\psi_{k_1}(x_1)\colon k_1=0, 1, \ldots\}$ on I we consider the d-multiple ONS $\varphi=\{\varphi_k(x)\colon k\in Z^d\}$ on I^d and take the integrals over I^d instead of I in (6). In fact, the sufficiency part in (i) is true over any measure space X (instead of $X=I$ or I^d), while the necessity part in (i) can be shown by the following simple observation: let $\nu_r=|Q_r|$, the number of the lattice points of Z^d contained in the set Q_r, and let $\varphi_k(x_1, \ldots, x_d)=\psi_{m_1}(x_1)$, where the mapping $k=k(m_1)$ is one-to-one for each pair $\nu_{r-1}<m_1\leqq\nu_r$ and $k\in Q_r\setminus Q_{r-1}$ ($r=0, 1, \ldots$; $\nu_{-1}=-1$ and $Q_{-1}=\emptyset$). Consequently, Theorem 1 is really a reformulation of Theorem A.

In the light of what has been said above, the result of [11, Satz III] can be reformulated as follows.

Theorem 2. *If two systems $a=\{a_k\colon k\in Z^d\}$ and $b=\{b_k\colon k\in Z^d\}$ of coefficients are such that*

$$B_r = \Big\{\sum_{k\in Q_r\setminus Q_{r-1}} b_k^2\Big\}^{1/2} \leqq \Big\{\sum_{k\in Q_r\setminus Q_{r-1}} a_k^2\Big\}^{1/2} = A_r \quad (r = 0, 1, \ldots),$$

then

$$\|b;\, d, Q\| \leqq \|a;\, d, Q\|;$$

consequently, if $a \in M(d, Q)$ *then* $b \in M(d, Q)$.

It is of interest to give an upper estimate for the norm $\|\cdot\,;\, d, Q\|$ which turns out to be exact in certain cases.

Theorem 3. *In each case we have*

$$\|a;\, d, Q\| \leqq C_1 \left\{ \sum_{r=0}^{\infty} \Big(\sum_{k \in Q_r \setminus Q_{r-1}} a_k^2 \Big) \log^2 (r+2) \right\}^{1/2}, \tag{7}$$

and in the special case when

$$A_r = \Big\{ \sum_{k \in Q_r \setminus Q_{r-1}} a_k^2 \Big\}^{1/2} \geqq \Big\{ \sum_{k \in Q_{r+1} \setminus Q_r} a_k^2 \Big\}^{1/2} = A_{r+1} \quad (r = 0, 1, \ldots)$$

an inequality opposite to (7) *holds also true:*

$$\|a;\, d, Q\| \geqq C_2 \left\{ \sum_{r=0}^{\infty} \Big(\sum_{k \in Q_r \setminus Q_{r-1}} a_k^2 \Big) \log^2 (r+2) \right\}^{1/2}.$$

Here C_1 *and* C_2 *are positive constants depending only on* d.

To prove Theorem 3 one has to start with the results of [7, Theorems 1 and 2] and to argue in a similar manner as it is done during the proof of [11, Satz VII].

We note that in the cases of the square and the spherical partial sums the right-hand sides in inequality (7) coincide, up to a constant:

$$\|a;\, d, Q^i\| \leqq C_1 \Big\{ \sum_{k \in Z^d} a_k^2 \log^2 (|k|+2) \Big\}^{1/2} \quad (i = 1, 2).$$

In spite of this fact, the norms $\|a; d, Q^1\|$ and $\|a; d, Q^2\|$ are not equivalent to each other in case $d \geqq 2$.

Theorem 4. *If* $d \geqq 2$, *then there exists a system* $a = \{a_k \colon k \in Z^d\}$ *of coefficients for which*

$$\|a;\, d, Q^1\| < \infty \quad \text{and} \quad \|a;\, d, Q^2\| = \infty,$$

and vice versa, there exists a system $a = \{a_k \colon k \in Z^d\}$ *of coefficients for which*

$$\|a;\, d, Q^1\| = \infty \quad \text{and} \quad \|a;\, d, Q^2\| < \infty.$$

This is an easy consequence of Theorem 1 and [7, Theorem 3].

We note that the result stated in [7, Theorem 3] can be strengthened in the following way:

Let T *be a regular method of summation* (see. e.g., [14, p. 74]). *Then there exists a double orthogonal series* (1) *such that* (2) *is satisfied, its square partial sums converge* *a.e., but its spherical partial sums are not summable by the method T a.e. on I^2; and vice versa.*

In the proof of the latter assertion one has to use a result of [4, p. 183]:

For every regular method T of summation there exists a strictly increasing sequence $\{\mu_r: r=0, 1, \dots\}$ of positive integers such that the a.e. T-summability of series (4) *under condition* (5) *involves the a.e. convergence of the μ_rth partial sums of* (4).

Keeping in mind the proof of [7, Theorem 3] one's task is essentially reduced to the construction of a single orthogonal series (4) with condition (5), the μ_rth partial sums of which diverge a.e., while the μ_{2r}th partial sums of which converge a.e. on I. This construction can be certainly done if the ratio μ_{r+1}/μ_r is large enough $(r=0, 1, \dots)$, and the last condition may be assumed without loss of generality.

3. A.e. $(C, \delta>0)$-summability of the spherical partial sums. Up to this point we studied the convergence properties of series (1) in the setting when $a=\{a_k: k\in Z^d\}$ is a fixed system of coefficients, while $\varphi=\{\varphi_k(x): k\in Z^d\}$ runs over all the ONS on I^d. From now on we consider an individual ONS $\varphi=\{\varphi_k\}$ on I^d with some nice properties and let $a=\{a_k\}$ run over all the systems of coefficients satisfying condition (2).

To this aim, we assume that $\varphi=\{\varphi_k(x): k\in Z^d\}$ is a product ONS on I^d in the sense that there exists a single ONS $\psi=\{\psi_{k_1}(x_1): k_1=0, 1, \dots\}$ on I such that

$$\varphi_k(x) = \prod_{j=1}^{d} \psi_{k_j}(x_j), \quad k = (k_1, \dots, k_d) \text{ and } x = (x_1, \dots, x_d); \tag{8}$$

furthermore, we assume that the system $\psi=\{\psi_{k_1}(x_1)\}$ is such that for every sequence $c=\{c_{k_1}: k_1=0, 1, \dots\}$ of coefficients we have

$$\int_I \left(\max_{0\le r\le \varrho} \left|\sum_{k_1=0}^{r} c_{k_1}\psi_{k_1}(x_1)\right|\right)^2 dx_1 \leqq C \sum_{k_1=0}^{\varrho} c_{k_1}^2 \quad (\varrho = 0, 1, \dots), \tag{9}$$

where C is a positive constant. Inequality (9) implies, among others, that series (4) converges a.e. under condition (5). The fact that inequality (9) is satisfied for the ordinary trigonometric system $\psi=\{1, \cos 2\pi k_1 x_1, \sin 2\pi k_1 x_1: k_1=1, 2, \dots\}$ is due to HUNT [3], while for the Walsh system $\psi=\{w_{k_1}(x_1): k_1=0, 1, \dots\}$ is due to SJÖLIN [8].

It is not hard to conclude from (9) the following upper estimate for the maximum of the square partial sums $s_r^1(x)$ of series (1):

$$\int_{I^d} \left(\max_{0\le r\le \varrho} |s_r^1(x)|\right)^2 dx \leqq 2^d C^d \sum_{k\in Q_\varrho} a_k^2 \quad (\varrho = 0, 1, \dots).$$

This means that the square partial sums $s_r^1(x)$ converge a.e. on I^d provided (2) is satisfied. (For more details, see [12] and [6].)

The question of a.e. convergence of the spherical partial sums $s_r^2(x)$ of series (1) under condition (2) seems to us to be an open problem for $d \geqq 2$. As to the multiple trigonometric system, we cite here two papers by Russian mathematicians. On the one hand, TEVZADZE [13] published in 1973 that he managed to prove that the spherical partial sums of the double Fourier expansion of a function $f(x_1, x_2)$ from $L^p(I^2)$ with $p>1$ converge a.e. on I^2, but the proof turned out to be false even in case $p=2$. On the other hand, BUADZE [2] announced in 1976 the existence of a continuous function $f(x_1, x_2)$ on I^2 such that the spherical partial sums of the double Fourier expansion of $f(x_1, x_2)$ diverge everywhere, but the construction has not yet appeared.

We are unable to decide this question. However, we can prove the a.e. $(C, \delta>0)$-summability of the spherical partial sums $s_r^2(x)$ of series (1) under the only conditions that $\varphi=\{\varphi_k(x)\}$ is an ONS with properties (8) and (9), and $a=\{a_k\}$ is a system of coefficients satisfying (2). To this end, we recall that the (C, δ)-means $\sigma_\varrho^\delta(x)$ in question are defined as follows:

$$\sigma_\varrho^\delta(x) = \frac{1}{A_\varrho^\delta} \sum_{r=0}^{\varrho} A_{\varrho-r}^{\delta-1} s_r^2(x) = $$

$$= \frac{1}{A_\varrho^\delta} \sum_{r=0}^{\varrho} A_{\varrho-r}^{\delta} \Big(\sum_{r-1<|k| \leqq r} a_k \varphi_k(x) \Big),$$

where

$$A_\varrho^\delta = \binom{\varrho+\delta}{\varrho} \quad (\varrho = 0, 1, \ldots; \ \delta > 0).$$

For a positive integer δ one can consider the following modified (C, δ)-means, too:

$$\tilde{\sigma}_\varrho^\delta(x) = \frac{1}{A_\varrho^\delta} \sum_{|k| \leqq \varrho} A_{\varrho-|k|}^\delta a_k \varphi_k(x),$$

in particular, for $\delta=1$,

$$\tilde{\sigma}_\varrho^1(x) = \sum_{|k| \leqq \varrho} \Big(1 - \frac{|k|}{\varrho+1}\Big) a_k \varphi_k(x).$$

Unfortunately, we can prove the statement that

$$\sigma_\varrho^\delta(x) - \tilde{\sigma}_\varrho^\delta(x) \to 0 \quad \text{as } \varrho \to \infty \quad \text{a.e. on } I$$

only in case $\delta=1$. In fact, writing

$$\sigma_\varrho^1(x) - \tilde{\sigma}_\varrho^1(x) = \frac{1}{\varrho+1} \sum_{r=0}^{\varrho} \Big(\sum_{r-1<|k| \leqq r} (r-|k|) a_k \varphi_k(x) \Big),$$

by virtue of the Kronecker lemma (see, e.g. [1, p. 72]) it is enough to show that the single orthogonal series

$$\sum_{r=0}^{\infty} \frac{1}{r+1} \Big(\sum_{r-1<|k| \leqq r} (r-|k|) a_k \varphi_k(x) \Big)$$

converges a.e. on I^d. But by the well-known Rademacher—Menšov theorem this is the case provided (2) is satisfied.

After these preliminaries we state the following

Theorem 5. *Assume that* $\varphi=\{\varphi_k(x)\}$ *is a product* ONS *on* I^d *given by* (8) *and satisfying condition* (9), $a=\{a_k\}$ *is a system of coefficients satisfying* (2), *and* δ *is a positive number. Then the spherical partial sums* $s_r^2(x)$ *of series* (1) *are* (C, δ)*-summable a.e. on* I^d.

Taking into account of what has been said above on the trigonometric and Walsh systems, hence it follows immediately the following

Corollary. *If* $\varphi=\{\varphi_k(x)\}$ *is the d-multiple trigonometric or Walsh system, then the spherical partial sums* $s_r^2(x)$ *of series* (1) *are* $(C, \delta>0)$*-summable a.e. on* I^d *provided* (2) *is satisfied.*

Remarks. (a) In the case when φ is the d-multiple trigonometric system, Stein [9] proved that the Bochner—Riesz means $\tilde{\sigma}_\varrho^\delta(x)$ of series (1) defined by

$$\tilde{\sigma}_\varrho^\delta(x) = \sum_{|k|<\varrho} \left(1-\frac{|k|^2}{\varrho^2}\right)^\delta a_k \varphi_k(x) \quad (\varrho, \delta > 0)$$

converge to $f(x)$ a.e. on I^d provided series (1) is the d-multiple Fourier expansion of a function $f(x)\in L^p(I^d)$, where

$$\delta > \frac{d-1}{2}\left(\frac{2}{p}-1\right) \quad \text{and} \quad 1 < p \leqq 2.$$

In particular, under condition (2) the means $\tilde{\sigma}_\varrho^\delta(x)$ converge a.e. on I^d again for every $\delta>0$.

(b) As to the multiple Haar system, Kemhadze [5] proved that the spherical partial sums of the expansion of a function $f(x)$ with respect to the d-multiple Haar system converge a.e. on I^d provided $f(x)\in L(\log^+ L)^{d-1}(I^d)$.

Proof of Theorem 5. Our starting point is that under the conditions of the theorem the square partial sums $s_r^1(x)$ of series (1) converge a.e. on I^d. We assume that $d\geqq 2$, since in case $d=1$ we have $s_r^2(x)\equiv s_r^1(x)$ $(r=0, 1, \ldots)$.

We will show that the subsequence $\{s_{d^m}^2(x)\colon m=0, 1, \ldots\}$ of the spherical partial sums of (1) also converges a.e. on I^d. This is an immediate consequence of Beppo Levi's theorem since

$$\sum_{m=0}^{\infty} \int_{I^d} (s_{d^m}^1(x)-s_{d^m}^2(x))^2\,dx = \sum_{m=0}^{\infty} \Big(\sum_{k\in Q_{d^m}^1 \setminus Q_{d^m}^2} a_k^2\Big) \leqq \sum_{k\in Z^d} a_k^2 < \infty.$$

Here we took into account that $\{Q_{d^m}^1 \setminus Q_{d^m}^2\colon m=0, 1, \ldots\}$ is a disjoint sequence of sets. In fact, if $k \in Q^1_{d^m} \setminus Q^2_{d^m}$ for a certain $m \geqq 1$, then $\max_{1 \leqq j \leqq d} k_j = d^m$ and hence

$$|k| \leqq d^{1/2} \max_{1 \leqq j \leqq d} k_j \leqq d^{m+1/2},$$

i.e., $k \notin Q^1_{d^n} \setminus Q^2_{d^n}$ for $n \geqq m+1$. On the other hand,

$$\max_{1 \leqq j \leqq d} k_j \geqq d^{-1/2}|k| > d^{m-1/2},$$

whence $k \notin Q^1_{d^n} \setminus Q^2_{d^n}$ follows for $n \leqq m-1$. We note that we should have taken the "thicker" subsequence $\{s^2_{[d^{m/2}]}(x)\}$ too, where $[\cdot]$ means the integral part.

In order to make the proof complete, we apply a result of Tandori [10] in a somewhat more general setting as stated originally and add some supplements. To this effect, let $\nu = \{\nu_r : r = 0, 1, \ldots\}$ be, as earlier, a sequence of integers, $0 \leqq \nu_0 < < \nu_1 < \nu_2 < \ldots$, and consider the ν_rth partial sums

$$\tilde{s}_{\nu_r}(x_1) = \sum_{k_1=1}^{\nu_r} c_{k_1} \psi_{k_1}(x_1)$$

of the orthogonal series (4) under condition (5). Now we form the $(C, \delta > 0)$-means $\sigma^\delta_\varrho(\nu; x_1)$ of the subsequence $\{\tilde{s}_{\nu_r}(x_1)\}$:

$$\sigma^\delta_\varrho(\nu; x_1) = \frac{1}{A^\delta_\varrho} \sum_{r=0}^{\varrho} A^{\delta-1}_{\varrho-r} \tilde{s}_{\nu_r}(x_1) = \tag{10}$$

$$= \frac{1}{A^\delta_\varrho} \sum_{r=0}^{\varrho} A^\delta_{\varrho-r} \left(\sum_{k_1=\nu_{r-1}+1}^{\nu_r} c_{k_1} \psi_{k_1}(x_1) \right)$$

$$(\varrho = 0, 1, \ldots; \; \nu_{-1} = -1).$$

Then the above-mentioned theorem of Tandori can be stated in a more general form as follows.

Theorem B ([10, Hilfssatz I]). *Let $\nu = \{\nu_r\}$ be a strictly increasing sequence of nonnegative integers, and let $\delta > 0$ and $q > 1$. Then, under condition* (5), *we have*

(i) $\tilde{s}_{\nu_{[q^m]}}(x_1) - \sigma^1_{[q^m]}(\nu; x_1) \to 0$ *as* $m \to \infty$, *and*

(ii) $\max_{[q^m] < r < [q^{m+1}]} \left(\sigma^1_r(\nu; x_1) - \sigma^1_{[q^m]}(\nu; x_1) \right) \to 0$ *as* $m \to \infty$

a.e. on I.

This theorem is proved in [10] for the special case $q = 2$, but the proof can be executed, without essential changes, for general $q > 1$, too.

Now, using the reasonings made in [4, pp. 186—187] for the special case $\nu_r \equiv r$, one can supplement (i)—(ii) as follows.

Theorem C. *Let* $\nu=\{\nu_r\}$ *be a strictly increasing sequence of nonnegative integers and let* $\delta>1/2$. *Then, under condition* (5), *we have*

$$\text{(iii)}\ \frac{1}{\varrho+1}\sum_{r=0}^{\varrho}\left(\sigma_r^{\delta-1}(\nu;x_1)-\sigma_r^{\delta}(\nu;x_1)\right)^2\to 0\quad as\ \varrho\to\infty$$

a.e. on I. *Consequently, if*

$$\sigma_r^{\delta}(\nu;x_1)\to f(x_1)\quad as\ r\to\infty$$

a.e. on I, *then*

$$\frac{1}{\varrho+1}\sum_{r=0}^{\varrho}\left(\sigma_r^{\delta-1}(\nu;x_1)-f(x_1)\right)^2\to 0\quad as\ \varrho\to\infty$$

a.e. on I.

Finally, we insert an elementary lemma which can be found e.g. in [4, p. 189]:

(iv) *If* $\delta>-1/2$ *and*

$$\frac{1}{\varrho+1}\sum_{r=0}^{\varrho}(\sigma_r^{\delta}-s)^2\to 0\quad as\ \varrho\to\infty,$$

where the σ_r^{δ} *are the* (C,δ)*-means of a numerical series, then, for every* $\varepsilon>0$, *we have*

$$\sigma_r^{\delta+1/2+\varepsilon}\to s\quad as\ r\to\infty.$$

Combining (i)—(iv) in such a manner as it is done in [4, pp. 189—190] for the case $\nu_r\equiv r$, one can conclude the following statement:

Under condition (5), *the a.e. convergence of the subsequence* $\{\tilde{s}_{\nu_{[q^m]}}(x_1): m=0,1,\ldots\}$ *of the partial sums of the orthogonal series* (4) *is equivalent to the a.e. convergence of the means* $\{\sigma_{\varrho}^{\delta}(\nu;x_1): \varrho=0,1,\ldots\}$ *defined by* (10), *where* $\delta>0$ *and* $q>1$ *are fixed numbers.*

On closing, one more remark: the latter statement clearly holds true if the interval I of orthogonality is replaced by any measure space X, in particular, by $X=I^d$.

This completes the proof of Theorem 5.

References

[1] G. Alexits, *Convergence properties of orthogonal series*, Pergamon (Oxford—New York, 1961).

[2] A. I. Buadze, On the divergence of the spherical partial sums of multiple Fourier series (Russian), *Soobšč. Akad. Nauk Gruzin. SSR*, **94** (1976), 561—563.

[3] R. A. Hunt, On the convergence of Fourier series, *Orthogonal expansions and their continuous analogues (Proc. Conf. Southern Illinois University, Edwardsville, 1967)*; 235—256, SIU (Carbondale, Illinois, 1968).

[4] S. Kaczmarz und H. Steinhaus, *Theorie der Orthogonalreihen*, Chelsea (New York, 1951).

[5] G. G. Kemhadze, On the convergence of the spherical partial sums of multiple Fourier-Haar series (Russian), *Trudy Tbilissk. Univ.*, **8** (1977), 27—38.

[6] F. Móricz, On the convergence of double orthogonal series, *Analysis Math.*, **2** (1976), 287—304.

[7] F. Móricz, On the square and the spherical partial sums of multiple orthogonal series, *Acta Math. Acad. Sci. Hungar.*, **37** (1981), 255—261.

[8] P. Sjölin, An inequality of Paley and convergence a.e. of Walsh—Fourier series, *Arkiv Mat.*, **7** (1967—69), 551—570.

[9] E. M. Stein, Localization and summability of multiple Fourier series, *Acta Math.*, **100** (1958), 93—147.

[10] K. Tandori, Über die orthogonalen Funktionen. VI (Eine genaue Bedingung für die starke Summation), *Acta Sci. Math.*, **20** (1959), 14—18.

[11] K. Tandori, Über die Konvergenz der Orthogonalreihen. II, *Acta Sci. Math.*, **25** (1964), 219—232.

[12] N. R. Tevzadze, On the convergence of the double Fourier series of quadratic summable functions (Russian), *Soobšč. Akad. Nauk Gruzin. SSR*, **58** (1970), 277—279.

[13] N. R. Tevzadze, Convergence of the spherical partial sums of a double Fourier series (Russian), *Soobšč. Akad. Nauk Gruzin. SSR*, **70** (1973), 277—280.

[14] A. Zygmund, *Trigonometric series*. I (Cambridge, 1959).

BOLYAI INSTITUTE
UNIVERSITY OF SZEGED
ARADI VÉRTANÚK TERE 1
H—6720 SZEGED, HUNGARY

Acta Sci. Math., 44 (1982), 87—93

Upper estimates for the eigenfunctions of the Schrödinger operator

I. JOÓ

For a series of questions concerning spectral theory of non-selfadjoint differential operators we need some estimates for the eigenfunctions.

In the present note we shall generalize the former results of IL'IN and Joó [3], [4], [5].

Let (a, b) be a finite interval and consider the formal differential operator

$$ly = -y'' + qy$$

with the complex potential $q \in L^1(a, b)$. A function u_i having absolutely continuous derivative on every closed subinterval of (a, b) is said to be an eigenfunction of order i of the operator l with the complex eigenvalue λ if there exist functions u_k $(k=1, 2, ..., i-1)$ with the same properties such that the equations

$$lu_k(x) = \lambda u_k(x) + u_{k-1}(x) \quad (k = 0, 1, 2, ..., i) \tag{1}$$

hold for almost all $x \in (a, b)$, with $u_{-1} \equiv 0$.

We prove the following

Theorem. *Every eigenfunction u_i of order i for the eigenvalue λ of the operator l has absolutely continuous derivatives on the closed interval $[a, b]$. Furthermore, setting for convenience $\lambda = \mu^2$ with $0 \leqq \arg \mu < \pi$, the following estimates hold:*

$$\|u_{k-1}\|_\infty \leqq C_k(1+|\mu|)(1+\operatorname{Im}\mu)\|u_k\|_\infty, \tag{2}$$

$$\|u_k\|_\infty \leqq C_k(1+\operatorname{Im}\mu)^{\frac{1}{p}}\|u_k\|_p \qquad (1 \leqq p \leqq \infty), \tag{3}$$

$$\|u_k'\|_\infty \leqq C_k(1+|\mu|)\|u_k\|_\infty \tag{4}$$

for $k=0, 1, ..., i$; the constants $C_k = C_k(b-a, \|q\|_1)$ do not depend on λ.

Remark. The estimates (2), (3), (4) strengthen and generalize the corresponding results of IL'IN [3] for the case of the Schrödinger operator with $q \in C^1[a, b]$. Our theorem was formulated in [5] and its proof is based only on the use of mean-

Received August 25, 1981.

value formulas, an essentially new idea, which is necessary if the potential q is not smooth. For fixed k, a, b and q the order of the estimates (2), (3), (4) in λ cannot be improved. (This will be established in a forthcoming paper [6]). Indeed, for numerous applications this is the most important aspect.

For the proof of the Theorem we need the following extensions of Titchmarsh classical formulae [2, p. 26].

Lemma. *We have*

(5)
$$u_k(x-t)+u_k(x+t) = 2u_k(x)\cos\mu t+ \int_{x-t}^{x+t} [q(\xi)u_k(\xi)-u_{k-1}(\xi)]\frac{\sin\mu(t-|x-\xi|)}{\mu}\,d\xi \quad \text{if} \quad \mu\neq 0,$$
$$u_k(x-t)+u_k(x+t) = 2u_k(x)+ \int_{x-t}^{x+t} [q(\xi)u_k(\xi)-u_{k-1}(\xi)](t-|x-\xi|)\,d\xi \quad \text{if} \quad \mu=0;$$

(6)
$$u_{k-1}(x)t\sin\mu t = \int_{x-t}^{x+t} u_{k-1}(\xi)\sin\mu(t-|x-\xi|)\,d\xi-$$
$$-\int_{x-t}^{x+t}[q(\xi)\,u_{k-1}(\xi)-u_{k-2}(\xi)]\int_{|x-\xi|}^{t}\frac{\sin\mu(\eta-|x-\xi|)}{\mu}\sin\mu(t-\eta)\,d\eta\,d\xi \quad \text{if} \quad \mu\neq 0,$$
$$u_{k-1}(x)\,t^2 = \int_{x-t}^{x+t} u_{k-1}(\xi)(t-|x-\xi|)\,d\xi-$$
$$-\int_{x-t}^{x+t}[q(\xi)\,u_{k-1}(\xi)-u_{k-2}(\xi)]\int_{|x-\xi|}^{t}(\eta-|x-\xi|)(t-\eta)\,d\eta\,d\xi \quad \text{if} \quad \mu=0.$$

Proof. (Only for $\mu\neq 0$; the case $\mu=0$ is similar.) We can write by (1)
$$\int_{x-t}^{x+t}[q(\xi)\,u_k(\xi)-u_{k-1}(\xi)]\frac{\sin\mu(t-|x-\xi|)}{\mu}\,d\xi = \int_{x-t}^{x+t}[u''(\xi)+\mu^2u_k(\xi)]\frac{\sin\mu(t-|x-\xi|)}{\mu}\,d\xi;$$
integrating by parts, we obtain (5).

On the other hand, in view of (5),
$$\int_{x-t}^{x+t} u_{k-1}(\xi)\sin\mu(t-|x-\xi|)\,d\xi = \int_0^t [u_{k-1}(x-\eta)+u_{k-1}(x+\eta)]\sin\mu(t-\eta)\,d\eta =$$
$$= \int_0^t 2u_{k-1}(x)\cos\mu\eta\sin\mu(t-\eta)\,d\eta+$$
$$+\int_0^t\int_{x-\xi}^{x+\xi}[q(\xi)u_{k-1}(\xi)-u_{k-2}(\xi)]\frac{\sin\mu(\eta-|x-\xi|)}{\mu}\,d\xi\sin\mu(t-\eta)\,d\eta;$$
applying the Fubini theorem, a short computation gives (6).

We shall also need the following elementary inequalities:

(7) $$|\sin z|,\ |\cos z| < 2;\ |\sin z| < 2|z| \quad \text{whenever} \quad |\operatorname{Im} z| \leqq 1;$$

(8) $$|\sin z| > \frac{1}{3}|z| \quad \text{if} \quad |z| \leqq 2;$$

(9) $$\sup_{1/2<\alpha<1} |\sin \alpha z| > \frac{1}{3} \quad \text{whenever} \quad |\operatorname{Im} z| \leqq 1 \quad \text{and} \quad |z| \geqq 2.$$

Proof of the Theorem. It is well known [1] that $u_k, u_k' \in L^\infty(a, b)$ and $u_k'' \in L^1(a, b)$. Next we show the auxiliary estimate

(10) $$\|u_k\|_\infty \leqq 10 \max_{[a+\delta,\, b-\delta]} |u_k| + 4\delta \min\left(2\delta, \frac{2}{|\mu|}\right)\|u_{k-1}\|_\infty \quad \text{for} \quad 0 \leqq \delta \leqq R,$$

where $R = \min\left\{\frac{b-a}{4}, \frac{1}{\operatorname{Im}\mu}, \frac{1}{4\|q\|_1}\right\}$. Indeed, for each $x \in \left[a, \frac{a+b}{2}\right]$ and $0 \leqq \delta \leqq R$ we obtain from (5) and (7)

(11) $$|u_k(x)| \leqq |u_k(x+2\delta)| + 4|u_k(x+\delta)| + 2\delta\|q\|_1\|u_k\|_\infty + 2\delta \min\left(2\delta, \frac{2}{|\mu|}\right)\|u_{k-1}\|_\infty.$$

An analogous estimate holds for $x \in \left[\frac{a+b}{2}, b\right]$, and hence

$$\|u_k\|_\infty \leqq 5 \max_{[a+\delta,\, b-\delta]} |u_k| + \frac{1}{2}\|u_k\|_\infty + 2\delta \min\left(2\delta, \frac{2}{|\mu|}\right)\|u_{k-1}\|_\infty.$$

Now we prove (2) by induction on k. The case $k=0$ is trivial (we set $C_0=1$). Suppose (2) holds with $k-1$ in place of k and consider the eigenfunction u_k. Comparing the expressions for the term $\int_{x-t}^{x+t} u_{k-1}(\xi) \sin \mu(t-|x-\xi|)\,d\xi$ in (5) and (6), respectively, and using (7) we obtain

$$|u_{k-1}(x)|\delta|\sin \delta\mu| \leqq (6+2\delta\|q\|_1)|\mu|\|u_k\|_\infty +$$

$$+2\delta^2 \min(2, 2\delta|\mu|)\|u_{k-1}\|_\infty + 2\delta^2 \min\left(\frac{4}{|\mu|}, 4\delta^2|\mu|\right)\|u_{k-2}\|_\infty$$

for all $x \in [a+b, b-\delta]$ and $0 \leqq \delta \leqq R$, thus (taking into account that $2\delta\|q\|_1 \leqq 1$)

$$\max_{[a+\delta,\, b-\delta]} |u_{k-1}|\, \delta|\sin \delta\mu| \leqq 7|\mu|\,\|u_k\|_\infty +$$

$$+2\delta \min(1, \delta|\mu|)\|u_{k-1}\|_\infty + 8\delta^2 \min\left(\frac{1}{|\mu|}, \delta^2|\mu|\right)\|u_{k-2}\|_\infty.$$

Applying (10) for u_{k-1} instead of u_k and expressing hence $\max_{[a+\delta,\, b-\delta]} |u_{k-1}|$ we get

$$\left\{\|u_{k-1}\|_\infty \cdot \frac{1}{10} - \frac{8}{10}\,\delta \min\left(\delta, \frac{1}{|\mu|}\right)\|u_{k-2}\|_\infty\right\}\delta\,|\sin \delta\mu| \leqq$$

$$\leqq 7\,|\mu|\,\|u_k\|_\infty + 4\delta^2\|q\|_1 \min(1, \delta|\mu|)\|u_{k-1}\|_\infty + 8\delta^2 \min\left(\frac{1}{|\mu|}, \delta^2|\mu|\right)\|u_{k-2}\|_\infty .$$

Using the induction hypothesis (i.e. $\|u_{k-2}\|_\infty \leqq C_{k-1}(1+|\mu|)(1+\operatorname{Im}\mu)\|u_{k-1}\|_\infty$)

$$\text{(12)}\qquad \frac{\delta^2}{7}\left\{\left|\frac{\sin\delta\mu}{\delta\mu}\right|\left[\frac{1}{10} - \delta\,\frac{C_{k-1}(1+|\mu|)(1+\operatorname{Im}\mu)}{|\mu|}\min(1, \delta|\mu|)\right] -\right.$$

$$\left. - \frac{4}{|\mu|}\|q\|_1 \min(1, \delta|\mu|) - \frac{8}{|\mu|}[\min(1, \delta|\mu|)]^2 \frac{C_{k-1}(1+|\mu|)(1+\operatorname{Im}\mu)}{|\mu|}\right\}\|u_{k-1}\|_\infty \leqq \|u_k\|_\infty .$$

Set $\delta_k = \min\left\{R, \left[960 C_{k-1}(1+\operatorname{Im}\mu)\left(1+\frac{b-a}{4}\right)\right]^{-1}, [480\|q\|_1]^{-1}\right\}$.

To examine (12), we distinguish two cases: a) $\delta_k|\mu| \leqq 2$, b) $\delta_k|\mu| > 2$.

Case a). In view of (8) and the fact $\delta(1+|\mu|) \leqq \delta + 1 \leqq 1 + \frac{b-a}{4}$, an application of (12) to $\delta = \frac{\delta_k}{2}$ yields

$$\frac{\delta^2}{7}\left\{\frac{1}{3}\left[\frac{1}{10} - \frac{1}{40}\right] - \frac{1}{120} - \frac{1}{120}\right\}\|u_{k-1}\|_\infty \leqq$$

$$\leqq \frac{\delta^2}{7}\left\{\frac{1}{3}\left[\frac{1}{10} - \delta^2 C_{k-1}(1+|\mu|)(1+\operatorname{Im}\mu)\right] - 4\delta\|q\|_1 -\right.$$

$$\left. - 8\delta^2 C_{k-1}(1+|\mu|)1 + \operatorname{Im}\mu)\right\}\|u_{k-1}\|_\infty \leqq \|u_k\|_\infty .$$

Thus, from the definition of δ we obtain

$$\text{(13)}\qquad \frac{\|u_{k-1}\|_\infty}{\|u_k\|_\infty} \leqq \frac{28\cdot 120}{\delta_k^2}.$$

Case b). According to (9) we may choose $\alpha \in \left(\frac{1}{2}, 1\right)$ such that $|\sin \alpha\delta_k\mu| > \frac{1}{3}$. Thus by setting $\delta = \alpha\delta_k$ in (12) we have

$$\frac{\delta^2}{7}\left\{\frac{1/30}{\delta|\mu|} - \frac{\delta(1+|\mu|)(1+\operatorname{Im}\mu)}{|\mu|}C_{k-1}\frac{1/3}{\delta|\mu|} - \frac{4}{|\mu|}\|q\|_1 -\right.$$

$$\left. - \frac{8}{|\mu|}\,\frac{C_{k-1}(1+|\mu|)(1+\operatorname{Im}\mu)}{|\mu|}\right\}\|u_{k-1}\|_\infty \leqq \|u_k\|_\infty .$$

Observe that

$$\frac{1+|\mu|}{|\mu|}=1+\frac{1}{|\mu|}\leqq 1+\frac{\delta_k}{2}\leqq 1+\frac{b-a}{4}.$$

Therefore

$$\frac{\delta}{|\mu|}\frac{1/7}{120}\|u_{k-1}\|_\infty \leqq \frac{\delta}{7|\mu|}\left\{\frac{1}{30}-\frac{\delta}{3}(1+\operatorname{Im}\mu)\left(1+\frac{b-a}{4}\right)C_{k-1}-\right.$$

$$\left.-4\delta\|q\|_1-8\delta C_{k-1}\left(1+\frac{b-a}{4}\right)(1+\operatorname{Im}\mu)\right\}\|u_{k-1}\|_\infty\leqq\|u_k\|_\infty,$$

i.e.

(14) $$\frac{\|u_{k-1}\|_\infty}{\|u_k\|_\infty}\leqq\frac{7\cdot 120|\mu|}{\delta}\leqq\frac{14\cdot 120|\mu|}{\delta_k}.$$

Summing up (13) and (14), and taking into account the definition of δ_k, estimate (2) follows with

$$C_k=28\cdot 120\left\{\left(\frac{4}{b-a}\right)^2+1+\left[960\left(1+\frac{b-a}{4}\right)C_{k-1}\right]^{-1}+[480\|q\|_1]^{-1}\right\}.$$

We prove (3) from (2). Integrating (11) by δ from 0 to δ_{k+1} we have for $x\in\left[a,\frac{a+b}{2}\right]$

$$\delta_{k+1}|u_k(x)|\leqq\int_0^{\delta_{k+1}}|u_k(x+2\delta)|\,d\delta+4\int_0^{\delta_{k+1}}|u_k(x+\delta)|\,d\delta+$$

$$+\delta_{k+1}^2\|q\|_1\|u_k\|_\infty+\min\left(\frac{4}{3}\delta_{k+1}^3,\frac{2\delta_{k+1}^2}{|\mu|}\right)\|u_{k-1}\|_\infty.$$

Applying Hölder's inequality and (2) it follows

$$\delta_{k+1}|u_k(x)|\leqq 5\delta_{k+1}^{1-1/p}\|u_k\|_p+\delta_{k+1}^2\|q\|_1\|u_k\|_\infty+$$

$$+\min\left(\frac{4}{3}\delta_{k+1}^3,\frac{2\delta_{k+1}^2}{|\mu|}\right)C_k(1+|\mu|)\,(1+\operatorname{Im}\mu)\|u_k\|_\infty,$$

whence (by considering the cases $|\mu|\leqq 1$ and $|\mu|>1$ separately)

$$|u_k(x)|\leqq 5\delta_{k+1}^{-1/p}\|u_k\|_p+\delta_{k+1}\|q\|_1\|u_k\|_\infty+4\delta_{k+1}C_k(1+\operatorname{Im}\mu)\|u_k\|_\infty.$$

An analogous inequality holds for $x\in\left[\frac{a+b}{2},b\right]$, and therefore

$$\|u_k\|_\infty\leqq 5\delta_{k+1}^{-1/p}\|u_k\|_p+\delta_{k+1}\|q\|_1\|u_k\|_\infty+4\delta_{k+1}C_k(1+\operatorname{Im}\mu)\|u_k\|_\infty,$$

i.e.

$$\|u_k\|_\infty\leqq 10\delta_{k+1}^{-1/p}\|u_k\|_p\leqq C_k(1+\operatorname{Im}\mu)^{1/p}\|u_k\|_p.$$

We turn to the proof of (4). In case of $x, x+t\in(a, b)$ we have

$$u_k(x+t) = u_k(x)\cos\mu t+u_k'(x)\frac{\sin\mu t}{\mu}+ \tag{15}$$

$$+\int_x^{x+t}[q(\xi)u_k(\xi)-u_{k-1}(\xi)]\frac{\sin\mu(x+t-\xi)}{\mu}d\xi \quad \text{if} \quad \mu\neq 0,$$

$$u_k(x+t) = u_k(x)+u_k'(x)\cdot t+$$

$$+\int_x^{x+t}[q(\xi)u_k(\xi)-u_{k-1}(\xi)](x+t-\xi)\,d\xi \quad \text{if} \quad \mu=0$$

((15) can be verified in a similar way as (5)). For each $x\in\left[a, \frac{a+b}{2}\right]$ and $t=\delta_{k+1}$ we obtain from (7) and (15)

$$|u_k'(x)|\left|\frac{\sin\mu t}{\mu}\right| \leqq (3+2\delta_{k+1}\|q\|_1)\|u_k\|_\infty+\delta_{k+1}\min\left(2\delta_{k+1}, \frac{2}{|\mu|}\right)\|u_{k-1}\|_\infty,$$

and therefore, applying (2) we get

$$|u_k'(x)|\left|\frac{\sin\mu t}{\mu}\right| \leqq (3+2\delta_{k+1}\|q\|_1)\|u_k\|_\infty+$$

$$+\delta_{k+1}\min\left(2\delta_{k+1}, \frac{2}{|\mu|}\right)C_k(1+|\mu|)(1+\operatorname{Im}\mu)\|u_k\|_\infty.$$

A similar estimate holds for $x\in\left[\frac{a+b}{2}, b\right]$. Hence by considering the cases $|\mu|\leqq 1$ and $|\mu|>1$ separately we conclude

$$\|u_k'\|_\infty\left|\frac{\sin\mu t}{\mu}\right| \leqq (3+2\delta_{k+1}\|q\|_1)\|u_k\|_\infty+4\delta_{k+1}C_k(1+\operatorname{Im}\mu)\|u_k\|_\infty.$$

If $\delta_{k+1}|\mu|\leqq 2$ then we get by (8)

$$\frac{1}{3}\|u_k'\| \leqq \delta_{k+1}^{-1}(3+2\delta_{k+1}\|q\|_1)\|u_k\|_\infty+4C_k(1+\operatorname{Im}\mu)\|u_k\|_\infty \leqq 5C_k\|u_k\|_\infty,$$

and if $\delta_{k+1}|\mu|>2$ then we have by (9) for $t=\alpha\delta_{k+1}$ instead of $t=\delta_{k+1}$ $\left(\alpha\in\left[\frac{1}{2}, 1\right]\right)$

$$\frac{1}{3}\frac{\|u_k'\|_\infty}{|\mu|} \leqq 5\|u_k\|_\infty+4\delta_{k+1}C_k(1+\operatorname{Im}\mu)\|u_k\|_\infty,$$

i.e.

$$\|u_k'\|_\infty \leqq 16(1+|\mu|)\|u_k\|_\infty.$$

The theorem is proved.

An important special case of (3) is

$$\|u_0\| \leqq 12\left\{\frac{1}{b-a}+\|q\|_1\right\}^{1/2}\|u_0\|_2 \qquad (\text{if } \lambda \geqq 0). \tag{16}$$

The author is indebted to Dr. V. Komornik and Dr. L. L. Stachó for their valuable remarks during the preparation of this paper.

References

[1] M. A. Neumark, *Lineare Differentialoperatoren,* Akademie-Verlag (Berlin, 1967).

[2] E. C. Titchmarsh, *Eigenfunction expansions associated with second order differential equations* (Oxford, 1946).

[3] В. А. Ильин, Необходимые и достаточные условия базисности и равносходимости с тригонометрическим рядом спектральных разложений. I—II, *Дифференциальные Уравнения,* **16** (1980), 771—794, 980—1009.

[4] В. А. Ильин, И. Йо, Равномерная оценка собственных функций и оценка сверху чусла собственных значений оператора Штурма-Лиувилля с потенциалом из класса L_p, *Дифференциальные Уравнения,* **15** (1979), 1164—1174.

[5] И. Йо, Некоторые вопросы спектральной теории для одномерного несамосопряженного оператора Шредингера с потенциалом из класса L^1_{loc}, *ДАН СССР,* **250** (1980), 29—31.

[6] V. Komornik, Lower estimates for the eigenfunctions of the Schrödinger operator, *Acta Sci. Math.,* **44** (1982), 95—98.

DEPARTMENT OF MATH. ANALYSIS. II
LORÁND EÖTVÖS UNIVERSITY
H—1445 BUDAPEST 8, PF. 323

Acta Sci. Math., 44 (1982), 95—98

Lower estimates for the eigenfunctions of the Schrödinger operator

V. KOMORNIK

Let $G=(a, b)$ be a bounded interval, $q \in L^1(G)$ a complex function and consider the formal differential operator

$$Lu = -u'' + q \cdot u.$$

A function $u_i: G \to \mathbf{C}, u_i \not\equiv 0$ $(i=0, 1, \ldots)$ is said to be an eigenfunction of order i (of the operator L) with the eigenvalue $\lambda \in \mathbf{C}$, if it is absolutely continuous together with its derivative on every compact subinterval of G, and for almost all $x \in G$ the equation

$$(1) \qquad -u_i''(x) + q(x) \cdot u_i(x) = \lambda \cdot u_i(x) + u_{i-1}(x)$$

holds, where $u_{i-1} \equiv 0$ for $i=0$ and u_{i-1} is an eigenfunction of order $i-1$, with the eigenvalue λ, for $i \geqq 1$.

It is known (see [1], pp. 167—169) that in this case u_i, together with its derivative, can be continuously extended to the closed interval $[a, b]$, and the extended functions are absolutely continuous on the whole interval $[a, b]$. Hence $u_i \in L^p(G)$ for all $1 \leqq p \leqq \infty$. For the sake of brevity, we shall use the notation $\|\cdot\|_p$ instead of $\|\cdot\|_{L^p(G)}$.

The aim of the present paper is to prove the following.

Theorem. *Let $G=(a, b)$ be a bounded interval and $q \in L^1(G)$ a complex function. Then, for an arbitrary eigenfunction u_i of order $i \geqq 0$ with the eigenvalue λ, and for any $1 \leqq p < q < \infty$, the following estimates hold:*

$$(2) \qquad \frac{\|u_i\|_\infty}{\|u_i\|_p} \geqq C_1 \cdot (1 + |\operatorname{Im} \sqrt{\lambda}|)^{1/p},$$

$$(3) \qquad C_2 \cdot (1 + |\operatorname{Im} \sqrt{\lambda}|)^{1/p - 1/q} \leqq \frac{\|u_i\|_q}{\|u_i\|_p} \leqq C_3 \cdot (1 + |\operatorname{Im} \sqrt{\lambda}|)^{1/p - 1/q},$$

Received April 22, and in revised form June 20, 1981.

where the positive constants C_1, C_2, C_3 depend on i, $b-a$, $\|q\|_1$, but do not depend on λ, p and q: $C_j=C_j(i, b-a, \|q\|_1), j=1, 2, 3$.

Remark. The estimate

$$\frac{\|u_i\|_\infty}{\|u_i\|_p} \leqq C_4(i, b-a, \|q\|_1)\cdot(1+|\mathrm{Im}\sqrt{\lambda}|)^{1/p} \tag{4}$$

is also true; this was proved by I. Joó [5]. Thus our result is exact from the view point of dependence on λ.

In the proof of the theorem we shall use the following result of [5]: If u_i is an arbitrary eigenfunction of order $i\geqq 1$ with the eigenvalue λ and $u_{i-1}\equiv Lu_i-\lambda\cdot u_i$ then

$$\|u_{i-1}\|_\infty \leqq C_0(i)\cdot(1+|\sqrt{\lambda}|)\cdot(1+|\mathrm{Im}\sqrt{\lambda}|)\cdot\|u_i\|_\infty, \tag{5}$$

where the constant $C_0(i)=C_0(i, b-a, \|q\|_1)$ does not depend on λ.

We recall the formula of E. C. Titchmarsh [2], having been extended for eigenfunctions of higher order in [5]: for any $x-t, x+t\in G$ and $i\in\{0, 1, \ldots\}$,

$$u_i(x+t)+u_i(x-t) = 2\cdot u_i(x)\cdot\cos(\sqrt{\lambda}\,t)+ \\ +\int_{x-t}^{x+t}(q(\xi)\cdot u_i(\xi)-u_{i-1}(\xi))\cdot\frac{\sin\sqrt{\lambda}(t-|x-\xi|)}{\sqrt{\lambda}}\,d\xi, \tag{6}$$

if $\lambda\neq 0$.

We mention also the simple inequalities:

$$\exp(|\mathrm{Im}\, z|)-1 \leqq |2\cdot\cos z|, \quad |2\cdot\sin z| \leqq \exp(|\mathrm{Im}\, z|)+1 \quad (z\in\mathbf{C}). \tag{7}$$

The proof of the theorem will be based on the following

Proposition. *Let u_i be an arbitrary eigenfunction of order $i\geqq 0$ with the eigenvalue λ. Then, setting for brevity $v=\mathrm{Im}\sqrt{\lambda}$ and $d_{a,b}(x)=\min(|x-a|, |x-b|)$, we have*

$$(m_i \equiv)\max_{x\in[a,b]}|u_i(x)|\cdot(1+|v|\cdot d_{a,b}(x))^{-i}\cdot\exp(|v|\cdot d_{a,b}(x)) \leqq M_i\cdot\|u_i\|_\infty, \tag{8}$$

where the constant $M_i=M_i(b-a, \|q\|_1)$ does not depend on λ.

Proof. We work by induction on i. For $i=-1$ (8) is formally true with $M_{-1}\equiv 0\ (u_{-1}\equiv 0)$. Let now $i\geqq 0$ be arbitrary and suppose (8) is true for $i-1$. In case $|\sqrt{\lambda}|\leqq 1+2^i\cdot\|q\|_1$ we obviously have

$$m_i \leqq \exp((1+2^i\cdot\|q\|_1)\cdot(b-a))\cdot\|u_i\|_\infty. \tag{9}$$

Consider now the case $|\sqrt{\lambda}|>1+2^i\cdot\|q\|_1$. Denote $y\in[a, b]$ such a point where the maximum on the left side of (8) is attained. Then

$$m_i = |u_i(y)|\cdot(1+|v|\cdot t)^{-i}\cdot\exp(|v|t) \quad (t\equiv d_{a,b}(y)). \tag{10}$$

By properties (10), (7), (6), (5) and the inductive hypothesis we can write the following chain of inequalities:

$$m_i\cdot(1+|\nu|t)^i-\|u_i\|_\infty \leqq m_i\cdot(1+|\nu|t)^i-|u_i(y)| = |u_i(y)|\cdot(\exp(|\nu|t)-1) \leqq$$

$$\leqq \left|2\cdot u_i(y)\cdot\cos(\sqrt{\lambda}\,t)\right| \leqq |u_i(y-t)+u_i(y+t)|+\left|\int_{y-t}^{y+t}(q(\xi)u_i(\xi)-\right.$$

$$\left.-u_{i-1}(\xi))\frac{\sin\sqrt{\lambda}\,(t-|y-\xi|)}{\sqrt{\lambda}}\,d\xi\right| \leqq 2\cdot\|u_i\|_\infty+\frac{\|q\|_1}{2\cdot|\sqrt{\lambda}|}\cdot\max_{|y-\xi|\leqq t}|u_i(\xi)|\cdot$$

$$\cdot(1+\exp(|\nu|(t-|y-\xi|)))+\frac{t}{|\sqrt{\lambda}|}\cdot\max_{|y-\xi|\leqq t}|u_{i-1}(\xi)|\cdot(1+\exp(|\nu|(t-|y-\xi|))) \leqq$$

$$\leqq 2\cdot\|u_i\|_\infty+2^{-i-1}\cdot(m_i\cdot(1+2\cdot|\nu|\,t)^i+\|u_i\|_\infty)+\frac{t}{|\sqrt{\lambda}|}\cdot(m_{i-1}(1+2\cdot|\nu|\,t)^{i-1}+\|u_{i-1}\|_\infty) \leqq$$

$$\leqq \frac{5}{2}\cdot\|u_i\|_\infty+\frac{1}{2}\,m_i\cdot(1+|\nu|\,t)^i+\frac{t}{|\sqrt{\lambda}|}\cdot(1+M_{i-1}\cdot(1+2\cdot|\nu|\,t)^{i-1})\cdot C_0(i)\cdot(1+|\sqrt{\lambda}|)\cdot$$

$$\cdot(1+|\nu|)\cdot\|u_i\|_\infty \leqq \frac{5}{2}\cdot\|u_i\|_\infty+\frac{1}{2}\,m_i\cdot(1+|\nu|\,t)^i+\frac{1+|\sqrt{\lambda}|}{|\sqrt{\lambda}|}(1+M_{i-1})\cdot(1+2\cdot|\nu|\,t)^{i-1}\cdot$$

$$\cdot C_0(i)\cdot(t+|\nu|\,t)\cdot\|u_i\|_\infty \leqq \frac{1}{2}\,m_i\cdot(1+|\nu|\,t)^i+\|u_i\|_\infty\cdot$$

$$\cdot\left(\frac{5}{2}+2^i\cdot(1+M_{i-1})\cdot C_0(i)\cdot(1+b-a)\cdot(1+|\nu|\,t)^i\right).$$

Hence

(11) $$m_i \leqq (7+2^{i+1}\cdot(1+M_{i-1})\cdot C_0(i)\cdot(1+b-a))\cdot\|u_i\|_\infty.$$

It follows from (9) and (11) that (8) is true for i if we put

$$M_i \equiv \max(\exp((1+2^i\cdot\|q\|_1)\cdot(b-a)),\ 7+2^{i+1}\cdot(1+M_{i-1})\cdot C_0(i)\cdot(1+b-a)).$$

The proposition is proved.

Corollary. *For any* $0<\alpha<1$ *there exists a constant* $M_i(\alpha)=M_i(\alpha, b-a, \|q\|_1)$ *independent of* λ *such that*

(12) $$\max_{x\in[a,b]}|u_i(x)|\exp(\alpha\cdot|\nu|\cdot d_{a,b}(x)) \leqq M_i(\alpha)\cdot\|u_i\|_\infty.$$

Let us turn to the proof of the theorem. Choosing for instance $\alpha=1/2$, we have by (12) for all $x\in G$:

$$|u_i(x)| \leqq M_i(1/2)\cdot\|u_i\|_\infty\cdot\exp\left(-\frac{1}{2}\cdot|\nu|\cdot d_{a,b}(x)\right).$$

Taking the $L^p(G)$ norm of both sides, we obtain for $|\nu|\geqq 1$

$$\|u_i\|_p \leqq M_i(1/2)\cdot\|u_i\|_\infty\cdot 4^{1/p}\cdot p^{-1/p}\cdot|\nu|^{-1/p}\leqq$$

$$\leqq 4\cdot M_i(1/2)\cdot\|u_i\|_\infty\cdot|\nu|^{-1/p}\leqq 8\cdot M_i(1/2)\cdot\|u_i\|_\infty\cdot(1+|\nu|)^{-1/p},$$

and hence,

$$\|u_i\|_\infty \geqq \big(8\cdot M_i(1/2)\big)^{-1}\cdot\|u_i\|_p\cdot(1+|\nu|)^{1/p}. \tag{13}$$

On the other hand, in the case $|\nu|<1$ we have obviously

$$\|u_i\|_p \leqq (b-a)^{1/p}\cdot\|u_i\|_\infty \leqq (1+b-a)\cdot\|u_i\|_\infty \leqq 2\cdot(1+b-a)\cdot\|u_i\|_\infty\cdot(1+|\nu|)^{-1/p},$$

and

$$\|u_i\|_\infty \geqq \big(2\cdot(1+b-a)\big)^{-1}\cdot\|u_i\|_p\cdot(1+|\nu|)^{1/p}; \tag{14}$$

and (13) and (14) yield the estimate (2) with

$$C_1\big(i,\, b-a,\, \|q\|_1\big) \equiv \min\big((8\cdot M_i(1/2))^{-1},\, (2\cdot(1+b-a))^{-1}\big).$$

The estimates (3) are easy consequences of (2) and (4). The theorem is proved.

The author is grateful to I. Joó for stimulating discussions.

References

[1] M. A. Neumark, *Lineare Differentialoperatoren*, Akademie-Verlag (Berlin, 1967), pp. 167—169.

[2] E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations*, Clarendon Press (Oxford, 1946).

[3] В. А. Ильин, Необходимые и достаточные условия базисности и равносходимости с тригонометрическим рядом спектральных разложений, часть 1, *Дифференциальные уравнения* **16**: 5 (1980), 771—794.

[4] В. А. Ильин, И. Йо, Равномерная оценка собственных функций и оценка сверху числа собственных значений оператора Штурма-Лиувилля с потенциалом из класса L^p *Дифференциальные уравнения*, **15**:7 (1979), 1164—1174.

[5] I. Joó, Upper estimates for the eigenfunctions of the Schrödinger operator, *Acta Sci. Math.*, **44** (1982), 87—93.

LORÁND EÖTVÖS UNIVERSITY
II. DEPARTMENT OF ANALYSIS
H—1445 BUDAPEST 8, PF. 323
HUNGARY

Acta Sci. Math. ,44 (1982), 99—124

A projection principle concerning biholomorphic automorphisms

L. L. STACHÓ

1. Introduction

Let E denote a Banach space and D be a bounded domain in E. A mapping F of D onto itself is called a biholomorphic automorphism of D if the Fréchet derivative of F exists at each point $x \in D$ and is a bounded invertible linear E-operator. Our basic motivation in this article is the problem of describing $\operatorname{Aut} B(E)$ the group of all biholomorphic automorphisms of the unit ball $B(E)$ of E. By recent results of W. KAUP [7] and J.-P. VIGUÉ [18], this problem stands in a close relationship with that of the classification of symmetric complex Banach manifolds which is solved since a long time in the finite dimensional case [2] but fairly not settled for infinite dimensions.

In 1979, E. VESENTINI [16] has shown that the unit ball of a nontrivial L^1-space admits only linear biholomorphic automorphisms. His proof goes back to investigations on Aut-invariant distances and a classical two dimensional result of M. KRITIKOS [9]. Using a characterization of polynomial vector fields tangent to $\partial B(E)$ (the boundary of $B(E)$) we found [11] an essentially two dimensional argument that enabled us to establish the sufficent and necessary condition for an L^p-space to have only linear unit ball automorphisms (for different approaches cf. also [1], [16]).

The purpose of Section 2 the general abstract part of this work is to clear up the deeper geometric background and connections of the seemingly different methods in treating L^p-spaces that occur in [16] and [11], respectively. Our main theorem provides a sufficent condition in terms of the Carathéodory (or Kobayashi) metric to reconstruct the biholomorphic automorphism group of Banach manifolds from those of its certain submanifolds via holomorphic projections. This result seems to be very well suited in calculating explicitly $\operatorname{Aut} B(E)$ in various Banach spaces E admitting a sufficiently large family of contractive linear projections. In Section 3 we illustrate the use of this projection principle by two typical examples where the con-

Received March 13, 1981.

clusion seems hardly available with other already published methods: After numerous partial solutions, recently T. FRANZONI [4] gave the complete description of $\operatorname{Aut} B(\mathscr{L}(H_1, H_2))$ where $\mathscr{L}(H_1, H_2) \equiv \{$bounded linear operators $H_1 \to H_2\}$ and H_1, H_2 are arbitrary Hilbert spaces. As we shall see, the projection principle makes it possible to obtain the exact description of $\operatorname{Aut} B(H_1 \otimes \ldots \otimes H_n)$ in an elementary way where $H_1 \otimes \ldots \otimes H_n \equiv \{$continuous n-linear functionals $H_1 \times \ldots \times H_n \to \mathbf{C}\}$. Note that $\mathscr{L}(H_1, H_2) \simeq H_1 \otimes H_2$ and for $n \geqq 3$, $H_1 \otimes \ldots \otimes H_n$ cannot be equipped with a suitable J^*-structure on which Franzoni's method is based. The key of the reduction by the projection principle is the fact that in finite dimensions the strong precompactness of $B(H_1 \otimes \ldots \otimes H_n)$ considerably simplifies the treatment of the space (Section 4). The second application concerns atomic Banach lattices. The unit balls of finite dimensional such spaces are exactly the convex Reinhardt domains. In 1974, T. SUNADA [13] characterized $\operatorname{Aut}_0 D$ for all the bounded Reinhardt domains D. However, his proofs depend on the Cartan theory of finite dimensional semisimple Lie algebras thus cannot be carried out in infinite dimensions. If the finite dimensional ideals form a dense submanifold, the projection principle reduces even the most general case to some straightforward 2 dimensional considerations. We remark that in this way also Sunada's proof can be simplified and the method applies in parts to other Banach lattices (cf. [12]).

2. Projection principle

Our main abstract result concerns with holomorphic vector fields on complex Banach manifolds (for basic definitions see [17], [7, § 2]). If M denotes a complex Banach manifold, a vector field $v: M \to TM$ is complete in M iff for every $x \in M$, there exists a mapping $e_x: \mathbf{R} \to M$ such that $e_x(0) = x$ and $\frac{d}{dt} e_x(t) = v(e_x(t))$ $\forall t \in \mathbf{R}$. In this case we define $\exp(tv)(x) \equiv e_x(t)$. A function $\delta: TM \to \mathbf{R}_+$ is called a differential Finsler metric on M if for any fixed $x \in M$, the functional $T_x M \ni w \mapsto \delta(x, w)$ is convex and positive-homogeneous and for each coordinate-map (U, Φ), the function $f_v^{(U, \Phi)}: \Phi U \ni e \mapsto \delta(\Phi^{-1} e, v(\Phi^{-1} e))$ is locally bounded and lower semicontinuous whenever v is a holomorphic vector field on M. We shall write d_M for the Carathéodory distance [3], [17] on M, i.e. $d_M(x, y) \equiv \sup \{\operatorname{areath} F(y): F$ is a holomorphic $M \to \triangle$ function, $F(x) = 0\}$ where $\triangle \equiv \{\zeta \in \mathbf{C}: |\zeta| < 1\}$. For a holomorphic mapping $F: M \to M$, we denote by F' its Fréchet derivative (recall that for any fixed $x \in M$, $F'(x)$ is a bounded linear $T_x M \to T_x M$ operator). For a Banach space E, we shall denote by E^*, $\| \ \|$, $^-$ and $B(E)$ its dual, norm, closure operation and open unit ball, respectively.

2.1. Theorem. *Let M be a complex Banach manifold, M' a (complex) submanifold of M and v a complete holomorphic vector field on M. Suppose P is a holomorphic mapping of M onto M' such that $P|_{M'}=\mathrm{id}_{M'}$ (the identity mapping on M').*

Suppose there exists a differential Finsler metric δ on M' such that

(i) *the vector field $P'v|_{M'}$ is δ-bounded (i.e. $\sup_{x\in M} \delta(x, P'(x)\, v(x)) <\infty$)*

and by writing d for the intrinsic distance generated by δ on M',

(ii) *the topology of the metric d is finer than that of M',*

(iii) *for any sequence $x_1, x_2, \ldots \in M'$ which is a Cauchy sequence with respect to d but which is not convergent in M' we have $d_{M'}(x_1, x_n)\to\infty$ $(n\to\infty)$.*

Then the vector field $P'v$ is complete in M'.

Proof. For the sake of simplicity, the proof will be divided into three steps.

1) From the definition of Carathéodory distance we see immediately that $d_{M'}(x, y)\geqq d_M(x, y)$ $\forall x, y\in M'$ since $M'\subset M$. It is also well-known [2] that the mapping P is a $d_M\to d_{M'}$ contraction. Hence the relation $P|_{M'}=\mathrm{id}_{M'}$ entails $d_{M'}(x, y)\leqq d_M(x, y)$. Thus we obtained $d_{M'}=d_M|_{M'}$.

In the sequel, we set $a_x(t)\equiv\exp(tv)(x)$ $(x\in M, t\in\mathbf{R})$ and b_x will denote the maximal solution of the initial value problem $\left\{\frac{d}{dt}y=P'(y)v(y); y(0)=x\right\}$.

We show that for arbitrarily fixed $z\in M'$,

$$d_{M'}(Pa_z(h), b_z(h)) = o(h) \quad (h\to 0). \tag{1}$$

Indeed: Consider any coordinate-map (U, Φ) from the atlas of M' for which $z\in U$. We may assume without loss of generality that Φ is a biholomorphism between U and the open unit ball of some Banach space E. Then for all $h\in\left\{t\in\mathrm{dom}\, b_z\colon b_z(t)\in\Phi^{-1}\left(\frac{1}{2}B(E)\right)\right\}$ we have

$$d_{M'}(Pa_z(h), b_z(h)) \leqq d(Pa_z(h), b_z(h)) = d_{B(E)}(\Phi Pa_z(h), \Phi b_z(h)) \leqq$$
$$\leqq \mu\|\Phi Pa_z(h)-\Phi b_z(h)\|$$

where $\mu\equiv\sup\left\{d_{B(E)}(f, g)/\|f-g\|: f, g\in\frac{1}{2}B(E)\right\}$. It is easily seen that $\mu\leqq$ $\leqq 2\sup\left\{d_{B(E)}(f, 0)/\|f\|: f\in\frac{1}{2}B(E)\right\}=2\sup\left\{\|f\|^{-1}\,\mathrm{areath}\,\|f\|: \|f\|\leqq\frac{1}{2}\right\}<\infty$.

The estimate $\|\Phi Pa_z(h)-\Phi b_z(h)\|=o(h)$ $(j\to 0)$ can be verified as follows: By definition, a is the solution of the initial value problem $\left\{\frac{d}{dt}y=v(y), y(0)=z\right\}$.

Therefore $\|\Phi a_z(h)-(\Phi z+h\Phi' v(z))\|=o(h)$. Thus $\left.\frac{d}{dh}\right|_0 [\Phi P a_z(h)-\Phi b_z(h)]=$
$=\left.\frac{d}{dh}\right|_0 \Phi P a_z(h)-\Phi' P' v(z)=\Phi' P' v(z)-\Phi' P' v(z)=0.$

An application of (1) directly yields that for any $x, y\in M'$,

$$\overline{\lim_{h\to 0}}\frac{1}{|h|}[d_{M'}(b_x(h), b_y(h))-d_{M'}(x, y)] = \overline{\lim_{h\to 0}}\frac{1}{|h|}[d_{M'}(Pa_x(h), Pa_y(h))-d_{M'}(x, y)] \leqq$$

$$\leqq \overline{\lim_{h\to 0}}\frac{1}{|h|}[d_M(a_x(h), a_y(h))-d_M(x, y)] = 0$$

(since P is a contraction $d_M\to d_{M'}$ and $d_{M'}=d_M|_{M'}$).

2) Henceforth we proceed by contradiction. Assume that the vector field $P'v$ is not complete in M'.

Now we may fix a point $x\in M'$ such that $\operatorname{dom} b_x\neq \mathbf{R}$. Let t_0 be a boundary point of the interval (or ray) dom b_x. Since $0\in \operatorname{dom} b_x$, we have $t_0\neq 0$. So (by passing to the vector field $\frac{1}{t_0}v$) we may assume $t_0=1$. Then consider the function

$$\varrho(t) \equiv d_{M'}\left(b_x(t), b_x\left(t+\frac{1}{2}\right)\right) \qquad \left(t\in\left[0, \frac{1}{2}\right)\right).$$

Since $b_x(t+h)=b_{b_x t}(h)$ and $b_x\left(t+\frac{1}{2}+h\right)=b_{b_x}\left(t+\frac{1}{2}\right)(h)$ whenever $t, t+h, t+\frac{1}{2}$, $t+\frac{1}{2}+h\in[0, 1)$, from step 3) it follows that

$$\overline{\lim_{h\to 0}}\frac{\varrho(t+h)-\varrho(t)}{|h|} \leqq 0 \quad \forall t\in\left[0, \frac{1}{2}\right).$$

We show that the function ϱ is locally Lipschitzian. Since the conclusion of the previous step can be interpreted as $\varrho'(t)=0$ for all such values t where $\varrho'(t)$ exists, hence we obtain that ϱ is constant i.e.

$$(2) \qquad d_{M'}\left(b_x(t), b_x\left(t+\frac{1}{2}\right)\right) = d_{M'}\left(x, b_x\left(\frac{1}{2}\right)\right) \quad \forall t\in\left[0, \frac{1}{2}\right).$$

Proof. By triangle inequality, it suffices to see that for any $z\in M'$, the mapping $t\mapsto b_z(t)$ is locally Lipschitzian with respect to the metric $d_{M'}$. Denote by $\delta_{M'}$ the Carathéodory differential Finsler metric of the manifold M' (for definition see [2], [17]). Then the function $\gamma\colon \tau\mapsto\delta_{M'}(b_z(\tau), P'b(b_z(\tau)))$ is locally bounded (cf.

[17]). Hence if $\mathcal{J}$ is a compact subinterval of dom b_z then $\sup_{t\in\mathcal{J}} \gamma(t)<\infty$ and therefore

$$d_{M'}(b_z(t'), b_z(t'')) \leqq \left|\int_{t'}^{t''} \delta_{M'}(b_z(t), b_z'(t))\,dt\right| = \left|\int_{t'}^{t''} \gamma(t)\,dt\right| \leqq$$
$$\leqq \sup_{t\in\mathcal{J}} \gamma(t)\cdot|t''-t'| \quad \text{whenever} \quad t', t''\in\mathcal{J}.$$

3) Write $K\equiv \sup_{x\in M'} \delta\,(x, P'v(x))$ and consider the sequence $t_n\equiv\frac{1}{2}-\frac{1}{2n}$ $(n=1, 2, \ldots)$. For $m\leqq n$ we have

$$d\left(b_x\left(t_m+\frac{1}{2}\right), b_x\left(t_n+\frac{1}{2}\right)\right) \leqq \int_{t_m}^{t_n} \delta(b_x(t), b_x'(t))\,dt =$$
$$= \int_{t_m}^{t_n} \delta(b_x(t), P'v(b_x(t)))\,dt \leqq \int_{t_m}^{t_n} K dt = \frac{K}{2}\left(\frac{1}{m}-\frac{1}{n}\right).$$

Thus $\left\{b_x\left(t_n+\frac{1}{2}\right)\right\}_{n\in\mathbf{N}}$ is a Cauchy sequence with respect to the metric d. Suppose $d\left(b_x\left(t_n+\frac{1}{2}\right), z\right)\to 0$ $(n\to\infty)$ for some point $z\in M'$. Then we would have $P'v(b_x(t_n))\to P'v(z)$ $(n\to\infty)$, as a consequence of (ii). However, in this case the function $\tilde{b}(t)\equiv\begin{cases} b_x(t) & \text{if } t\in\text{dom}\, b_x \\ b_z(t-1) & \text{if } 0\leqq(t-1)\in\text{dom}\, b_z\end{cases}$ is a solution of the initial value problem $\left\{\frac{d}{dt}y=P'v(y), y(0)=x\right\}$ with dom $\tilde{b}\supsetneqq$ dom b_x which is excluded by the maximality of b_x. Thus $\left\{b_x\left(t_n+\frac{1}{2}\right)\right\}$ does not converge in the metric d.

By condition (iii), $d_{M'}\left(b_x\left(\frac{1}{2}\right), b_x\left(1-\frac{1}{2n}\right)\right)=d_{M'}\left(b_x\left(t_1+\frac{1}{2}\right), b_x\left(t_n+\frac{1}{2}\right)\right)\to$ $\to\infty$ $(n\to\infty)$. From (2) we see

$$d_{M'}\left(b_x\left(\frac{1}{2}\right), b_x\left(\frac{1}{2}-\frac{1}{2n}\right)\right) \geqq d_{M'}\left(b_x\left(\frac{1}{2}\right), b_x\left(1-\frac{1}{2n}\right)\right)-$$
$$-d_{M'}\left(b_x\left(1-\frac{1}{2n}\right), b_x\left(\frac{1}{2}-\frac{1}{2n}\right)\right) = d_{M'}\left(b_x\left(\frac{1}{2}\right), b_x\left(1-\frac{1}{2n}\right)\right)-$$
$$-d_{M'}\left(x, b_x\left(\frac{1}{2}\right)\right) \to\infty \qquad (n\to\infty).$$

But this is impossible because the topology of a complex Banach manifold is always finer than that generated by its associated Carathéodory metric (cf. [17]) whence $d_{M'}\left(b_x\left(\frac{1}{2}\right), b_x\left(\frac{1}{2}-\frac{1}{2n}\right)\right)\to 0$ $(n\to\infty)$ since the mapping $t\mapsto b_x(t)$ is differentiable.

The obtained contradiction completes the proof.

2.2. Remark. From step 1) one immediately reads that in general we have

2.2a Lemma. *If $d^*: N \mapsto d^*_N$ is a metric valued functor on the category of complex Banach manifolds such that for all manifolds N, N',*

(iv) *d^*_N is a metric on N,*

(v) *each holomorphic map $N' \to N$ is a $d^*_N \to d^*_{N'}$ contraction,*

then $d^*_M|_{M'} = d^*_{M'}$, whenever M' is a submanifold of M and there can be found a holomorphic projection of M onto M'.

The proof of Theorem 2.1 can be carried out as well for any metric functor d^* with properties (iv), (v) and

(vi) $\sup\left\{d^*_{B(F)}(f, 0)/\|f\| : \|f\| \leq \frac{1}{2}\right\} < \infty$ for any Banach space E.

The Kobayashi invariant metric (def. see [17], [9]) also satisfies these requirements. Hence Theorem 2.1 holds when replacing Carathéodory distances by those of Kobayashi. Moreover we have the following important special case of Lemma 2.2a.

2.2b Lemma. *If E denotes a Banach space and P is a contractive linear projection $E \to E$ then $d_{B(E)}|_{B(PE)} = d_{B(PE)}$ and $d^k_{B(E)}|_{B(PE)} = d^k_{B(PE)}$ where d^k stands for the Kobayashi distance.*

Proof. Since $\|P\| = 1$ (otherwise we have the trivial case $P = 0$), PE is a closed subspace of E and $PB(E) = B(PE) \subset B(E)$. Thus Lemma 2.2a can be applied to $M \equiv B(E)$ and $M' \equiv B(PE)$.

This latter result can be further specialized as follows: Consider any unit vector $e \in E$. By the Hahn—Banach theorem, there exists $\Phi \in E^*$ with $\|\Phi\| = \langle e, \Phi \rangle = 1$. Then the mapping P: $f \mapsto \langle f, \Phi \rangle e$ is a contractive linear projection of E onto $\mathbf{C}e$. Thus Lemma 2.2b contains Vesentini's following observation.

2.2c Lemma (Vesentini [16]). *Let E be a Banach space, $e \in E$ a unit vector and $\zeta_1, \zeta_2 \in \Delta$. Then we have $d^k_{B(E)}(\zeta_1 e, \zeta_2 e) = d_{B(\mathbf{C}e)}(\zeta_1 e, \zeta_2 e) = d_\Delta(\zeta_1, \zeta_2) = \operatorname{areath}\left|\frac{\zeta_1 - \zeta_2}{1 - \bar{\zeta}_1 \zeta_2}\right|$, i.e. the curve $[\Delta \ni \zeta \mapsto \zeta e]$ is a complex geodesic with respect to both the Carathéodory and Kobayashi distances in $B(E)$.*

Later on, we restrict our attention to Banach space unit balls. Recall ([8], [18]) that in a Banach space E, the elements of $\operatorname{Aut}_0 B(E)$ (the connected component of $\operatorname{Aut} B(E)$ w.r.t. the topology $\mathcal{T}_a$ defined in [15]) are exactly the exponential images of the second degree polynomial vector fields being complete in $B(E)$ whose Lie-algebra will be denoted by $\log^* \operatorname{Aut} B(E)$. Moreover, the orbit $[\operatorname{Aut} B(E)]\{0\} \equiv \equiv \{F(0): F \in \operatorname{Aut} B(E)\}$ is the intersection of $B(E)$ with a subspace which, in the sequel, we shall denote by E_0 and we have $E_0 = [\log^* \operatorname{Aut} B(E)]\{0\}$.

2.3. Theorem. *If E is a Banach space and $P\colon E\to E$ is a contractive linear projection then* $P[\log^*\mathrm{Aut}\,B(E)]|_{PE}\subset\log^*\mathrm{Aut}\,B(PE)$.

Proof. Let $u\in\log^*\mathrm{Aut}\,B(E)$ be arbitrarily fixed. We have to show that the vector field $Pu|_{B(PE)}$ is complete in $B(PE)$. As in the proof of Lemma 2.2b, let us consider the manifolds $M\equiv B(E)$, $M'\equiv B(PE)$, the projection $P|_{B(E)}$ of M onto M' and the vector field $v\equiv u|_{B(E)}$ which is by definition complete in M. Take the differential Finsler metric $\delta(x,w)\equiv\|w\|$ $(x\in B(PE), w\in PE)$ on M' whose generated intrinsic distance is obviously $d(x,y)\equiv\|x-y\|$ $(x,y\in B(PE))$. To complete the proof, we need only to verify (i), (ii), (iii).

(i): For $x\in B(PE)$ we have $P'(x)v(x)=Pu(x)$ whence by a theorem of KAUP—UPMEIER [8],

$$\delta(x, P'v(x))=\|Pu(x)\|\leqq\|u(x)\|=\left\|u(0)+u'(0)x+\frac{1}{2}u''(0)(x,x)\right\|\leqq$$

$$\leqq\|u(0)\|+\|u'(0)\|_{\mathscr{L}(E,E)}+\left\|\frac{1}{2}u''(0)\right\|_{\{\text{bilin } E\times E\to E\}}.$$

(ii): Trivial.

(iii): Assume $x_1, x_2, \ldots$ is a Cauchy sequence with respect to the metric d without a limit in M'. Then for some unit vector $f\in PE$, $\|x_n-f\|\to 0$ $(n\to\infty)$ i.e. $\|x_n\|\to 1$. Therefore, by Lemma 2.2c, $d_{M'}(x_1,x_n)=d_{B(PE)}(x_1,x_n)\geqq d_{B(PE)}(x_n,0)-$ $-d_{B(PE)}(x_1,0)=\mathrm{areath}\,\|x_n\|=\mathrm{areath}\,\|x_1\|\to\infty$.

2.4. Corollary. *If E is a Banach space and $P\colon E\to E$ is a contractive linear projection then $P(E_0)\subset(PE)_0$. In particular, if $B(E)$ is a symmetric manifold then so is $B(PE)$, too.*

2.5. Corollary. *Let E be a Banach space. If one can find a family $\mathscr{P}$ of contractive linear projections $E\to E$ such that for every $P\in\mathscr{P}$,* $\mathrm{Aut}\,B(PE)$ *consists only of linear transformations and* $\bigcap_{P\in\mathscr{P}}\ker P=\{0\}$ *then all the elements of* $\mathrm{Aut}\,B(E)$ *are also linear.*

Proof. If $v\in\log^*\mathrm{Aut}\,B(E)$ then $Pv(0)=0$ $\forall P\in\mathscr{P}$ whence $v(0)=0$ i.e. the vector field v is linear. On the other hand $\mathrm{Aut}\,B(E)=\mathrm{Aut}^0B(E)\,\mathrm{Aut}_0B(E)=\mathrm{Aut}^0B(E)\cdot$ $\cdot\exp\log^*\mathrm{Aut}\,B(E)$, where $\mathrm{Aut}^0\equiv\{E\text{-unitarities}\}$.

3. Applications

Let (X,μ) denote a measure space. In [1], [11] it is proved

3.1. Theorem. *The unit ball of $E\equiv L^p(X,\mu)$ admits only linear biholomoprhic automorphisms unless* $\dim E=1$ *or* $p=2,\infty$.

As the first illustration of the projection principle, we show how can this result be reobtained from Thullen's classical 2 dimensional theorem [14].

Proof. Suppose $p\in[1,\infty]\setminus\{2\}$ and $\dim E>1$. If g_1, g_2 are functions in E with norm 1 having disjoint supports then it is easily seen that the mapping P_{g_1,g_2}: $E\ni f\mapsto\sum_{j=1}^{2}\int f\overline{g_j}|g_j|^{p-2}d\mu\cdot g_j$ is a contractive linear projection of E onto the subspace $E_{g_1,g_2}\equiv\sum_{j=1}^{2}\mathbf{C}g_j$. Now $B(E_{g_1,g_2})=\{\zeta_1 g_1+\zeta_2 g_2\colon|\zeta_1|^p+|\zeta_2|^p<1\}$ is a Reinhardt domain whose biholomorphic automorphisms are all linear by Thullen's theorem. Furthermore we have $\ker P_{g_1,g_2}=\{f\in E\colon\int f\overline{g_j}|g_j|^{p-2}d\mu=0\ \ (j=1,2)\}$. Thus $\bigcap_{g_1,g_2}\ker P_{g_1,g_2}=$ $=\{f\in E\colon\forall g\in E[\exists h\in E\ \ \min(|g|,|h|)=0]\Rightarrow\int f\bar{g}|g|^{p-2}d\mu=0\}\subset\{f\in E\colon\forall X_1\subset X[\exists X_2\subset$ $\subset X\setminus X_1\ \ 0<\mu(X_1),\mu(X_2)<\infty]\Rightarrow\int_{X_1}df\mu=0\}=\{0\}$. Hence Corollary 2.5 establishes the linearity of $\mathrm{Aut}\,B(E)$.

To the next application, let $H_1,\dots,H_n$ be arbitrarily fixed Hilbert spaces[1] of at least 2 dimensions and consider the biholomorphic automorphism group of the unit ball $B\equiv B(E)$ of the space $E\equiv H_1\otimes\dots\otimes H_n$, the Banach space of n-linear functionals endowed with the usual norm $\|F\|\equiv\sup\{|F(h_1,\dots,h_n)|\colon h_j\in H_j,\ \|h_j\|=1\ (j=1,\dots,n)\}$ for $F\in E$. For $n=1,2$, the description of $\mathrm{Aut}\,B$ is completely settled [5], [4]. It is worth to remark that, in the light of the Kaup Vigué theory, the difficulties in this case can be concentraded to the description of linear E-unitary operators: If $n=1, E$ can be identified with H_1 and for any fixed $c\in H_1$, the quadratic vector field $q\equiv[H_1\ni f\mapsto-(f|c)f]$ satisfies [11, (1)] i.e. tangent to the boundary of B.

Similarly, if $n=2$, E can be identified with $\mathscr{L}(H_1,H_2)$ and for fixed $C\in\mathscr{L}(E_1,E_2)$, the vector field $[\mathscr{L}(H_1,H_2)\ni F\mapsto-FC^*F]$ is quadratic and satisfies [11, (1)]. It is easily seen, in both cases that, we have $\{[\exp(tq)](0)\colon t\in\mathbf{R}\}=(-1,1)C$, thus B is symmetric and $\mathrm{Aut}\,B=(\mathrm{Aut}^0B)\exp\{q_c\colon c\in E\}$. Here we turn our attention first of all to the case $n\geq3$ which seems heavily treatable with other methods and is not touched by the literature.

3.2. Lemma. Span $\{UC\colon U$ linear$\in\mathrm{Aut}_0B\}=E$ *whenever* $C\in E\setminus\{0\}$ *and* $\dim H_j<\infty\ \ (j=1,\dots,n)$.

Proof. If $C\neq0$ then we may fix unit vectors $e_j\in H_j\ (j=1,\dots,n)$ such that $\gamma\equiv C(e_1,\dots,e_n)\neq0$. Then let P_j denote the orthogonal projection of H_j onto $\mathbf{C}e_j$ and set $U_j^\vartheta\equiv\exp(i\vartheta_jP_j), C(\vartheta_1,\dots,\vartheta_n)\equiv(U_j^\vartheta\otimes\dots\otimes U_j^\vartheta)C\ \ (\vartheta_j\in\mathbf{R}; j=1,\dots,n)$. Since the operators U_j^ϑ are H_j-unitary, $U_1^\vartheta\otimes\dots\otimes U_n^\vartheta\in\mathrm{Aut}_0B$, therefore $e_1\otimes\dots\otimes e_n=$

[1] Without danger of confusion, we write simply $(.|.)$ for the inner product in any of $H_1,\dots,H_n$. For $A_j\in\mathscr{L}(H_j,H_j)$ and $e_j\in H_j\ (j=1,\dots,n)$, we define $A_1\otimes\dots\otimes A_n\equiv[H_1\otimes\dots\otimes H_n\ni$ $\ni F\mapsto F(A_1f_1,\dots,A_nf_n)]$, $e_1\otimes\dots\otimes e_n\equiv[(f_1,\dots,f_n)\mapsto(f_1|e_1)\dots(f_n|e_n)]$ and $\delta_{e_1,\dots,e_n}\equiv[F\mapsto F(e_1,\dots,e_n)$, respectively.

$=\frac{i}{\gamma}\frac{\partial^n}{\partial\vartheta_1\dots\partial\vartheta_n}\Big|_0 C\in S\equiv\mathrm{Span}\,\{UC\colon U\ \text{linear}\in\mathrm{Aut}_0 B\}$. Thus for all H_j-unitary operators V_j, $(V_1e_1)\otimes\dots\otimes(V_ne_n)=(V_1\otimes\dots\otimes V_n)(e_1\otimes\dots\otimes e_n)\in S$ i.e. $f_1\otimes\dots\otimes f_n\in S$ whenever $f_1\in H_1,\dots,f_n\in H_n$, whence $S=E$ (since $\dim E<\infty$).

3.3. Proposition. *For* $n>2$, *all the elements of* $\mathrm{Aut}\,B(H_1\otimes\dots\otimes H_n)$ *are linear.*

Proof. Observe that the family $\mathscr{P}\equiv\{P_1\otimes\dots\otimes P_n\colon$ all P_j-s are orthogonal H_j-projections with $\dim P_jH_j=[2$ if $j\leqq 3$ and 1 if $j>3]\}$ consists of contractive E-projections and $\bigcap_{P\in\mathscr{P}}\ker P=\{0\}$. Since for arbitrary $P\in\mathscr{P}$, the subspace PE is isometrically isomorphic to $\mathbf{C}^2\otimes\mathbf{C}^2\otimes\mathbf{C}^2$ ($\mathbf{C}$ is endowed with its usual euclidean norm), by Corollary 2.5 it suffices to see only that the elements of the group $\mathrm{Aut}\,B(\mathbf{C}^2\otimes\mathbf{C}^2\otimes\mathbf{C}^2)$ are linear. Thus we may assume $n=3$ and $H_j=\mathbf{C}$ $(j=1,2,3)$. Assume now that $E_0\neq 0$. Now Lemma 3.2 establishes $E_0=E$ i.e. symmetry of B. We show that this is impossible.

Denote by e_1, e_2 the vectors $(1,0)$ and $(0,1)$ in $\mathbf{C}^2$, respectively, and consider the elements $C\equiv e_1\otimes e_1\otimes e_1$ and $F\equiv e_2\otimes e_1\otimes e_1+e_1\otimes e_2\otimes e_1+e_1\otimes e_1\otimes e_2$ of E. Since the space E is finite dimensional, for every $A\in E$ we can find $f_1,f_2,f_3\in\partial B(\mathbf{C}^2)$ with $\|A\|=A(f_1,f_2,f_3)$. In particular, for arbitrarily given $\lambda\in\left(0,\frac{1}{3}\right)$ we can fix unit vectors $f_j(\lambda)$ such that $\|C+\lambda F\|=\langle C+\lambda F,\delta_{f_1(\gamma),f_2(\lambda),f_3(\lambda)}\rangle$. Since $C,F\geqq 0$ (i.e. $C(g_1,g_2,g_3), F(g_1,g_2,g_3)\geqq 0$ $\forall g_1,g_2,g_3\geqq 0$) and since $\langle C+\lambda F,\delta_{e_2,e_2,e_2}\rangle=$ $=\lambda F(e_2,e_2,e_2)<1$, for some $r_j(\lambda)\geqq 0$ we can write $f_j(\lambda)=\dfrac{e_1+r_j(\lambda)e_2}{(1+r_j(\lambda))^{1/2}}$ $(j=1,2,3)$. Thus introducing the function $\Phi_\lambda(\varrho_1,\varrho_2,\varrho_3)\equiv\left\langle C+\lambda F,\delta_{\frac{e_1+\varrho_1e_2}{(1+\varrho_1^2)^{1/2}},\dots,\frac{e_1+\varrho_3e_2}{(1+\varrho_3^2)^{1/2}}}\right\rangle$ $=[1+\lambda(\varrho_1+\varrho_2+\varrho_3)]\sum_{k=1}^{3}(1+\varrho_k^2)^{-1/2}$, we have $\dfrac{\partial}{\partial\varrho_j}\Big|_{(r_1(\lambda),r_2(\lambda),r_3(\lambda))}\Phi_\lambda=0$ $(j=1,2,3)$.

So $\{\lambda(1+r_j^2)-[1+\lambda(r_1+r_2+r_3)]\}\cdot\sum_{k=1}^{3}(1+r_k^2)^{-3/2}=0\ (j=1,2,3)$ and hence $\lambda=\dfrac{r_1}{1-r_1(r_2+r_3)}=\dfrac{r_3}{1-r_2(r_1+r_3)}=\dfrac{r_3}{1-r_3(r_1+r_2)}$. Therefore $r_j\neq 0$ $(j=1,2,3)$ and $\dfrac{1}{r_1}+r_1=\dfrac{1}{r_2}+r_2=\dfrac{1}{r_3}+r_3\left(=\dfrac{1}{\lambda}+\sum_{j=1}^{3}r_j\right)$. Observe that from this and from the assumption $\lambda\in\left(0,\frac{1}{3}\right)$ it follows that $r_1=r_2=r_3$. (Otherwise there would be $r>0$ such that two of the numbers r_1,r_2,r_3 coincided with r and the third with $1/r$, respectively. But then $\lambda=\dfrac{1/r}{1-(1/r)(r+r)}<0$.) Thus the relation $\lambda=\dfrac{r}{1-2r}$ holds where $r(\lambda)\equiv r_1(\lambda)=r_2(\lambda)=r_3(\lambda)$. This fact can be so interpreted that for sufficiently small values of $r>0$ $\left(\text{namely for } \lambda>\frac{1}{3} \text{ i.e. } r<\frac{\sqrt{17}-3}{4}\right)$, $F_r\equiv C+\frac{r}{1-2r^2}F$, $\Phi_r\equiv\delta_{e_1+re_2,\, e_1+re_2,\, e_1+re_2}$ fulfill $\|F_r\|\cdot\|\Phi_r\|=\langle F_r, \Phi_r\rangle$. Then by [11, Lemma]

$$(2) \qquad \|F_r\|^2\overline{\langle C, \Phi_r\rangle}+\langle q(F_r, F_r), \Phi_r\rangle = 0 \qquad \left(0<r<\frac{\sqrt{17}-3}{4}\right),$$

for some symmetric bilinear map $q{:}\,E\times E\to E$. Here $\langle C, \Phi_r\rangle=1, \|F_r\|=\|\Phi_r\|^{-1}\langle F_r, \Phi_r\rangle=$ $=(1+r^2)^{-3/2}\left(1+3r\frac{r}{1-2r^2}\right)=(1+r^2)^{-1/2}(1-2r^2)^{-1}$ and $\langle q(F_r, F_r), \Phi_r\rangle=\langle q(C, C), \Phi_r\rangle+$ $+2\frac{r}{1-2r^2}\langle q(C, F), \Phi_r\rangle+\left(\frac{r}{1-2r}\right)^2\langle q(F, F), \Phi\rangle$. Taking into consideration that for fixed $V\in E$, the function $r\mapsto\langle V, \Phi_r\rangle$ is a polynomial of 3^{rd} degree in r, from (2) we obtain

$$(2') \qquad (1+r^2)^{-1}(1-2r^2)^{-2}+p_1(r)+p_2(r)(1-2r^2)^{-1}+p_3(r)(1-2r^2)^{-2} = 0$$

for some polynomial-triplet p_1, p_2, p_3. However, (2′) immediately implies the contradictory fact that the function $r\mapsto(1+r^2)^{-1}$ is a polynomial.

3.4. Theorem. *The linear $H_1\otimes\ldots\otimes H_n$-unitary operators are exactly those operators F for which there exists a permutation π of the index set $\{1, \ldots, n\}$ and there are surjective linear isometries $U_k{:}\, H_k\to H_{\pi(k)}$ $(k=1, \ldots, n)$ such that*

$$(3) \qquad F(L) = [(f_1, \ldots, f_n)\mapsto L(U_1^{-1}f_{\pi(1)}, \ldots, U_n^{-1}f_{\pi(n)})].$$

A linear vector field V belongs to $\log^*\mathrm{Aut}\, B$ *if and only if it is of the form*

$$(3') \qquad V = i\cdot\sum_{k=1}^{n}\mathrm{id}_{H_1}\otimes\ldots\otimes\mathrm{id}_{H_{k-1}}\otimes A_k\otimes\mathrm{id}_{H_{k+1}}\otimes\ldots\otimes\mathrm{id}_{H_n}$$

where the A_k-s are arbitrary self-adjoint H_k-operators.

Proof. Based on some compactness arguments, in the next section we shall establish independently the validity of (3′) if the spaces H_k are all finite dimensional. Our starting point here is (3′) for finite dimensional E. First we extend it to infinite dimensions.

Let V linear$\in\log^*\mathrm{Aut}\, B$ and $e_1^*\in\partial B(H_1), \ldots, e_n^*\in\partial B(H_n)$ be arbitrarily fixed and define the operator $\tilde{V}\equiv V-\langle V(e_1^*\otimes\ldots\otimes e_n^*), \delta_{e_1^*, \ldots, e_n^*}\rangle\mathrm{id}_E$. Since $i\cdot\mathrm{id}_E\in$ $\in\log^*\mathrm{Aut}\, B$, we have $\tilde{V}\in\log^*\mathrm{Aut}\, B$. Remark that $\tilde{V}(e_1^*\otimes\ldots\otimes e_n^*)=0$. Then consider the family of mappings $\mathcal{P}\equiv\{P_1\otimes\ldots\otimes P_n{:}\ P_k$ is an orthogonal H_k-projection, $\dim P_kH_k<\infty, e_k\in P_kH_k$ $(k=1, \ldots, n)\}$. Any element $P\equiv P_1\otimes\ldots\otimes P_n$ of $\mathcal{P}$ is a contractive linear projection of the space E onto its subspace $(P_1H_1)\otimes\ldots\otimes(P_nH_n)$. Thus by the projection principle, $P\tilde{V}|_{PE}\in\log^*\mathrm{Aut}\, B(PE)$ $\forall P\in\mathcal{P}$. Hence (applying (3′) to the *finite dimensional* $(P_1H_1)\otimes\ldots\otimes(P_nH_n)$) for each $P\in\mathcal{P}$, there exists a unique choice of $A_1^P \in$ {self-adj. H_1-op.-s}, ..., $A_n^P \in$ {self-adj. H_n-op. -s} such that

$$A_k^P H_k \subset P_k H_k (\text{i.e. } P_k A_k^P P_k = A_k^P) \quad \text{and} \quad (A_k^P e_k^* | e_k^*) = 0 \quad (k = 1, \dots, n),$$

$$P\tilde{V}P = \sum_{k=1}^{n} i \cdot \mathrm{id}_{H_1} \otimes \dots \otimes \mathrm{id}_{H_{k-1}} \otimes A_k \otimes \mathrm{id}_{H_{k+1}} \otimes \dots \otimes \mathrm{id}_{H_n}.$$

Introduce the following partial ordering $\leqq$ in $\mathscr{P}$: If $P = P_1 \otimes \dots \otimes P_n$ and $Q = Q_1 \otimes \dots \otimes Q_n$ then let $P \leqq Q \overset{\text{def}}{\Longleftrightarrow} P_k H_k \subset Q_k H_k$ (i.e. $P_k \leqq Q_k$) $k = 1, \dots, n$. From the relation $P \leqq Q \Rightarrow P\tilde{V}P = PQ\tilde{V}QP$ we immediately see

(4) $$A_k^P = P_k A_k^Q P_k \quad (k = 1, \dots, n) \quad \text{whenever} \quad P \leqq Q.$$

Observe that for any fixed $P \in \mathscr{P}$ and index k,

$$|(A_k^P e | f)| = |\langle (P\tilde{V})(e_1^* \otimes \dots \otimes e_{k-1}^* \otimes e \otimes e_{k+1}^* \otimes \dots \otimes e_n^*), \delta_{e_1^*, \dots, e_{k-1}^*, \dots, f, e_{k+1}^*, \dots, e_n^*} \rangle| \leqq$$

$$\leqq \|P\tilde{V}\| \cdot \|e_1^* \otimes \dots \otimes e \otimes \dots \otimes e_n^*\| \cdot \|\delta_{e_1^*, \dots, f, \dots, e_n^*}\| = \|P\tilde{V}\| \leqq \|\tilde{V}\| \qquad \forall e, f \in \partial B(H_k),$$

that is

(5) $$\|A_k^P\| \leqq \|\tilde{V}\| \quad (k = 1, \dots, n) \quad \forall P \in \mathscr{P}.$$

Since obviously $\forall P, Q \in \mathscr{P}\ \exists R \in \mathscr{P}\ P, Q \leqq R$ and since by (4), (5) the relation $P \leqq Q$ entails $|(A_k^Q e | f) - (A_k^P e | f)| = |(A_k^Q (e - P_k e) | f) + (A_k^Q P_k e | f - P_k f)| \leqq \|\tilde{V}\| (\|e - P_k e\| + \|f - P_k f\|)$ $\forall e, f \in \partial B(H_k)$, $k = 1, \dots, n$, the definitions

$$a_k(e, f) \equiv \lim_{P \in \mathscr{P}} (A_k^P e | f) \quad (e, f \in H_k, \quad k = 1, \dots, n)$$

make sense and determine bounded sesquilinear functionals. Therefore there exist self-adjoint operators $A_1 \colon H_1 \to H_1, \dots, A_n \colon H_k \to H_n$ such that $a_k(e, f) = (A_k e | f)$ and hence $(A_k^P e | f) = (A_k^P (P_k e) | P_k f) = (A_k P_k e | P_k f) = (A_k P_k e | P_k f) = (P_k A_k P_k e | f)$ $\forall e, f \in H_k$ i.e. $A_k^P = P_k A_k P_k$ $(P \in \mathscr{P}, k = 1, \dots, n)$. Now for arbitrary $L \in E$, $e_1 \in H_1, \dots, e_n \in H_n$ the projections $P_k \equiv \mathrm{proj}_{\mathrm{Span}\{e_k, A_k e_k, e_k\}}$ $(k = 1, \dots, n)$ satisfy

$$[\tilde{V}L](e_1, \dots, e_n) = [\tilde{V}L](P_1 e_1, \dots, P_n e_n) = [P\tilde{V}L](e_1, \dots, e_n) =$$

$$= \sum_{k=1}^{n} L(e_1, \dots, P_k A_k e_k, \dots, e_n) = \sum_{k=1}^{n} L(e_1, \dots, A_k e_k, \dots, e_n).$$

Thus we can write $VL(e_1, \dots, e_n) = \sum_{k=1}^{n} L(e_1, \dots, B_k e_k, \dots, e_n)$ where $B_j \equiv A_j$ for $j = 1, \dots, n-1$ and $B_n \equiv A_n + \langle V(e_1^*, \dots, e_n^*), \delta_{e_1^*, \dots, e_n^*} \rangle \, \mathrm{id}_E$, proving (3′) in general.

To prove (3), let F be an arbitrarily given linear E-unitary operator and introduce the families $\mathscr{P}_k \equiv \{P_1 \otimes \dots \otimes P_n \colon P_k$ is an orthogonal H_k-projection, $P_j = \mathrm{id}_{H_j}$ for $j \neq k\}$ $(k = 1, \dots, n)$. From (3′) we see $i\mathscr{P}_k \subset \log^* \mathrm{Aut}\, B$ and hence for every $P \in \mathscr{P}_k$, the mapping $Q \equiv FPF^{-1}$ also has the properties $iQ \in \log^* \mathrm{Aut}\, B$ and $Q^2 = Q$

(since $P^2=P$) which is possible (by (3')) only if $Q\in\mathscr{P}_{\ell_k(P)}$ for some index $\ell_k(P)$ $(k=1, ..., n)$.

Let $k\in\{1, ..., n\}$ be fixed. We show that $\ell_k(P_1)=\ell_k(P_2)$ $\forall P_1, P_2\in\mathscr{P}_k\setminus\{\mathrm{id}_E\}$. Indeed, if $\ell_k(R_1)\neq\ell_k(R_2)$ then the operators $Q_j\equiv FR_jF^{-1}(j=1,2)$ commute (i.e. $[Q_1, Q_2]\equiv Q_1Q_2-Q_2Q_1=0$) whence we would have $[R_1, R_2]=0$. Observe that $\forall P_1, P_2\in\mathscr{P}_k\setminus\{\mathrm{id}_E\}$ $\exists P_3\in\mathscr{P}_k$ $[P_1, P_3], [P_2, P_3]\neq 0$, thus (by taking $R_1\equiv P_j$ and $R_2\equiv P_3$ $j=1,2$) $\ell_k(P_j)=\ell_k(P_3)$ holds for $j=1,2$.

Therefore there exists a permutation π with

$$F\mathscr{P}_kF^{-1}=\mathscr{P}_{\pi(k)} \quad (k=1, ..., n). \tag{6}$$

Since the finite linear combinations of orthogonal projections form a dense submanifold of the algebra of linear operators in any Hilbert space, it directly follows the existence of surjective linear isometries $S_k\colon \mathscr{L}(H_k, H_k)\to\mathscr{L}(H_{\pi(k)}, H_{\pi(k)})$ such that

$$F(\mathrm{id}_{H_1}\otimes\ldots\otimes\mathrm{id}_{H_{k-1}}\otimes A_k\otimes\mathrm{id}_{H_{k+1}}\otimes\ldots\otimes\mathrm{id}_{H_n})F^{-1}=$$
$$=\mathrm{id}_{H_1}\otimes\ldots\otimes\mathrm{id}_{H_{\pi(k)-1}}\otimes S_k(A_k)\otimes\mathrm{id}_{H_{\pi(k)+1}}\otimes\ldots\otimes\mathrm{id}_{H_n}$$
$$(A_k\in\mathscr{L}(H_k, H_k);\ k=1, ..., n).$$

As a consequence of the relations (6), the mappings S_k send orthogonal projections into orthogonal projections and therefore they constitute *-isomorphisms between the C*-algebras $\mathscr{L}(H_k, H_k)$ and $\mathscr{L}(H_{\pi(k)}, H_{\pi(k)})$. It is well-known that now we can write

$$S_k\colon A_k\mapsto U_kA_kU_k^{-1} \quad (k=1, ..., n)$$

for some surjective linear isometries $U_k\colon H_k\mapsto H_{\pi(k)}$. Thus if we denote by σ the inverse of the permutation π, for any linear E-operator A of the form $A\equiv A_1\otimes\ldots\otimes A_n$ (where $A_k\in\mathscr{L}(H_k, H_k)$ $k=1, ..., n$) we have

$$(FAF^{-1})L=[(f_1, ..., f_n)\mapsto L(U_{\sigma(1)}A_{\sigma(1)}U_{\sigma(1)}^{-1}f, ..., U_{\sigma(n)}A_{\sigma(n)}U_{\sigma(n)}^{-1}f_n)] \quad \forall L\in E.$$

This means that $FAF^{-1}=UAU^{-1}$ $\forall A\in\mathscr{L}(E, E)$ holds for the E-unitary operator U defined by

$$U(L)\equiv[(f_1, ..., f_n)\mapsto L(U_1^{-1}f_{\pi(1)}, ..., U_n^{-1}f_{\pi(n)})] \quad (L\in E).$$

It is easily seen that this is possible only if $F=e^{i\vartheta}U$ for some $\vartheta\in\mathbf{R}$ which completes the proof.

In the remainder part of this section, by making use of the projection principle, we shall examine the structure of biholomorphic unit ball automorphisms in case of minimal atomic Banach lattices (abbr. by min. B-lattices).

A Banach lattice E is called a min. B-lattice if it is norm-spanned by its 1 dimensional ideals. Henceforth we reserve the symbol E to designate a fixed min. B-lattice.

According to a well-known representation lemma [10. p. 143, Ex. 7 (b)], we may assume that for a fixed set X, E is a sublattice of $\{X\to\mathbf{C}$ functions$\}$ such that

$$(7)\qquad 1_x\in E \quad\text{and}\quad \|1_x\|=1 \quad \forall x\in X,$$

(8) Span $\{1_x\colon x\in X\}=E$. (1_x stand for $[X\ni y\mapsto 1$ if $y=x$ and 0 elsewhere]).

Remark that then

$$(8')\qquad wf\in E \quad\text{and}\quad wf=\lim_{Y\text{ finite}\subset X} w1_Y f \quad\text{whenever}\quad f\in E,\quad \sup_{x\in X}|w(x)|\leq 1.^2$$

For the sake of simplicity we write $B\equiv B(E)$ and the functional $[E\ni f\mapsto f(x)]$ will be denoted by 1_x^*.

First we describe the linear part of Aut B.

3.5 Definition. For $x,y\in X$, let $x\sim y$ if $\langle \ell(1_x), 1_y\rangle\neq 0$ for some linear element ℓ of $\log^*\mathrm{Aut}\, B$.

3.6. Lemma. (i) *$x\sim y$ if and only if for all $f,g\in E$, $f-g\in 1_{\{x,y\}}E$ and $\sum_{z=x,y}|f(z)|^2=\sum_{z=x,y}|g(z)|^2$ entail $\|f\|=\|g\|$.*

(ii) *The relation $\sim$ is an equivalence. Moreover, in case of $x_1\sim\ldots\sim x_n$,*

$$f-g\in 1_{\{x,\ldots,x_n\}} \quad \textit{and} \quad \sum_{j=1}^{n}|f(x_j)|^2=\sum_{j=1}^{n}|g(x_j)|^2 \quad \textit{imply} \quad \|f\|=\|g\|$$

for all $f,g\in E$ whenever $x_1,\ldots,x_n$ are distinct points.

Proof. (i) Let $Y\equiv\{y_1,\ldots,y_n\}$ be an arbitrary finite subset of X and ℓ linear$\in$ $\in\log^*\mathrm{Aut}\, B$. Set $\alpha_{jk}\equiv\langle \ell(l_{y_j}), 1_{y_k}\rangle$ and assume $\alpha_{12}\neq 0$ (i.e. $y_1\sim y_2$). Since the mapping $P\colon f\mapsto 1_Y f$ is a band projection of E onto $\sum_{j=1}^{n}\mathbf{C}1_{y_j}$, the projection principle establishes $\tilde\ell\in\log^*\mathrm{Aut}\, PB$ where $\tilde\ell\equiv P\ell|_{PE}$. Thus by [11, Lemma][3]

$$(9)\qquad \mathrm{Re}\,\langle\tilde\ell(f),\Phi\rangle=0\Leftarrow\langle f,\Phi\rangle=\|f\|\|\Phi\|\quad\forall f\in PE,\ \Phi\in(PE)^*.$$

[2] Proof: Given $\varepsilon>0$, by (8), there are Z finite $\subset X$, $g\in 1_Z f$ with $\|f-g\|<\varepsilon/2$. Now $Z\subset Y_1, Y_2$ finite $\subset X$ implies $\|f-g\|\geq|f-1_Z f|\geq w|(f-1_Z f)|\geq|w(1_{Y_1\cup Y_2}f-1_{Y_j}f)|$ $(j=1,2)$ i.e. by triangle inequality $\varepsilon\geq\|w1_{Y_1}f-w1_{Y_2}f\|$. Thus $\{w1_Y f\}_{Y\text{ finite}}$ is a Cauchy net in E. Hence for some $h\in E^2$, $w1_Y f\to h$. But $h(x)=\langle h,1_x^*\rangle=\lim_Y\langle w1_Y f,1_x\rangle=w(x)f(x)$ $\forall x$.

[3] In the same way as in [11, Lemma], one can see that if a linear vector field ℓ on Banach space F belongs to $\log^*\mathrm{Aut}\, B(F)$ then $\mathrm{Re}\langle\ell(f),\Phi\rangle=0\Leftarrow\langle f,\Phi\rangle=\|f\|\,\|\Phi\|$ $\forall f\in F,\Phi\in F^*$.

Proof: Since ℓ is tangent to $\partial B(F)$, we have $\ell(f)\in(H-f)$ whenever $\|f\|=1$ and H is a real hyperplane in F supporting $B(f)$ at f. But the general form of such a supporting hyperplane is $H=\{h\in F\colon \mathrm{Re}\langle h,\Phi\rangle=1\}$ where $\Phi\in F^*$ with $\|\Phi\|=\langle f,\Phi\rangle=1$.

Introduce the function $p(\varrho_1, \ldots, \varrho_n) \equiv \sum_{j=1}^{n} \varrho_j 1_{y_j}$ on $\mathbf{R}_+^n$ and set $C \equiv \{\varrho \in \mathbf{R}_+^n$: $\mathrm{grad}|_\varrho p$ does not exist$\}$. Since p is an increasing positively homogenenous convex function, C is a cone of Lebesgue measure 0. Let us fix arbitrary vectors $\varrho \in \mathbf{R}_+^n \setminus C$, $\vartheta \in \mathbf{R}^n$ and set $\pi \equiv \mathrm{grad}|_\varrho p$, $f_0 \equiv \sum_{j=1}^{n} \varrho_j e^{i\vartheta_j} 1_{y_j}$, $\Phi \equiv \sum_{j=1}^{n} \varrho_j e^{-i\vartheta_j} 1_{y_j}^*$. Since the function p is increasing, $\pi, \ldots, \pi_n \geqq 0$. Since π is positive homogeneous and convex, $\sum_{j=1}^{n} \pi_j \varrho_j = p(\varrho_1, \ldots, \varrho_n)$ i.e. $\langle f_0, \Phi \rangle = \|f_0\|$. On the other hand, for any $f \in PE$

$$|\langle f, \Phi \rangle| = \left| \sum_{j=1}^{n} \pi_j e^{-i\vartheta_j} f(y_j) \right| \leqq \sum_{j=1}^{n} \pi_j |f(y_j)| \leqq p(|f(y_1)|, \ldots, |f(y_n)|) = \|f\|$$

i.e. $\|\Phi\| = 1$. Hence (9) can be applied to f_0 and Φ. Thus

$$\text{(9')} \qquad \mathrm{Re} \left\langle \ell \left(\sum_{j=1}^{n} \varrho_j e^{i\vartheta_j} 1_{y_j} \right), \sum_{j=1}^{n} \pi_j e^{-i\vartheta_j} 1_{y_j}^* \right\rangle = 0.$$

By the arbitrary choice of $\vartheta \in \mathbf{R}^n$, an equivalent form to (9′) is

$$\text{(9'')} \qquad \mathrm{Re} \left[\sum_j \varrho_j \pi_j \alpha_{jj} + \sum_{j \neq k} (\varrho_j \pi_k \alpha_{jk} + \varrho_k \pi_j \overline{\alpha_{kj}}) z_j z_k^{-1} \right] = 0$$

$$\text{whenever} \quad |z_1| = \ldots = |z_n| = 1.$$

This is possible only if the rational expression (w.r.t. $z_1, \ldots, z_n$) in the argument of the Re operation vanishes. Thus in particular $\varrho_1 \pi_2 \alpha_{12} + \varrho_2 \pi_1 \overline{\alpha_{21}} = 0$. I.e. we obtained the following partial differential equation

$$\text{(10)} \qquad \varrho_1 \frac{\partial p}{\partial \varrho_2} \alpha_{12} + \varrho_2 \frac{\partial p}{\partial \varrho_1} \overline{\alpha_{21}} = 0 \quad (\varrho \in \mathbf{R}_+^n \setminus C).$$

Since $\varrho_2 = \|\varrho_2 1_{y_2}\| \leqq \left\| \sum_j \varrho_j 1_{y_j} \right\| = p(\varrho)$ $\forall \varrho \in \mathbf{R}_+^n$, there exists $\varrho \in \mathbf{R}_+^n \setminus C$ with $\dfrac{\partial \varrho}{\partial p_2} > 0$. Therefore $\alpha_{21} \neq 0$, moreover $\overline{\alpha_{21}}/\alpha_{12} < 0$, i.e. $\overline{\alpha_{21}}/\alpha_{12} = -|\alpha_{21}|/|\alpha_{12}|$.

For $(\varrho_3, \ldots, \varrho_n) \in \mathbf{R}_+^{n-2}$, define $\varphi_{\varrho_3, \ldots, \varrho_n}: \mathbf{R} \to \mathbf{R}$ by $\varphi_{\varrho_3, \ldots, \varrho_n}(t) \equiv p(|\alpha_{12}| \cos t, |\alpha_{21}| \sin t, \varrho_3, \ldots, \varrho_n)$. Since C is a cone of measure 0 in $\mathbf{R}_+^n$, (10) implies

$$\text{(11)} \quad \varphi_{\varrho_3, \ldots, \varrho_n}(t) = 0 \quad \text{for almost every} \quad t \in (0, \pi/2) \quad \text{and} \quad (\varrho_3, \ldots, \varrho_n) \in \mathbf{R}_n^{n-2}.$$

From the convexity of p it follows that it is locally Lipschitzian in the interior of $\mathbf{R}_+^n$. Hence, by (11),

$$\text{(11')} \qquad \varphi_{\varrho_3, \ldots, \varrho_n}(t) = \varphi_{\varrho_3, \ldots, \varrho_n}(0) \quad \forall t \in [0, \pi/2], (\varrho_3, \ldots, \varrho_n) \in \mathbf{R}_+^{n-2}.$$

But then $|\alpha_{12}|=\varphi_{0,\dots,0}(\pi/2)=|\alpha_{21}|$ whence

$$p|\alpha_{12}|^{-1}(\varrho_1^2+\varrho_2^2)^{1/2}\cdot\varphi_{\varrho_3,\dots,\varrho_n}\left(\arccos\frac{\varrho_1}{(\varrho_1+\varrho_2)^{1/2}}\right)=|\alpha_{12}|^{-1}(\varrho_1^2+\varrho_2^2)^{1/2}\varphi_{\varrho_3,\dots,\varrho_n}(0)=$$

$$=p(\sqrt{\varrho_1^2+\varrho_2^2},0,\varrho_3,\dots,\varrho_n).$$

Let now $f,g\in E$ be functions such that $f-g\in 1_{\{y_1,y_2\}}E$ and $\sum_{j=1}^{2}|f(y_j)|^2=\sum_{j=1}^{2}|g(y_j)|^2$. Then $\|1_Yf\|=p\left(\left(\sum_{j=1}^{2}|f(y_j)|^2\right)^{1/2},0,|f(y_3)|,\dots,|f(y_n)|\right)=\|1_Yg\|$. Taking into consideration the fact that Y may be any finite subset of X, from (8′) we obtain $\|f\|=\|g\|$.

Conversely: Assume that $f-g\in 1_{\{y_1,y_2\}}E$ and $\sum_{j=1}^{2}|f(y_j)|^2=\sum_{j=1}^{2}|g(y_j)|^2$ imply $\|f\|=\|g\|$ for all $f,g\in E$. Then the mappings $U^t\equiv[f\mapsto 1_{X\setminus\{y_1,y_2\}}f+((\cos t)\cdot f(y_1)+(\sin t)\cdot f(y_2))1_y+((-\sin t)\cdot f(y_1)+c+(\cos t)\cdot f(y_2))1_y]$ $(t\in\mathbf{R})$ form a one-parameter E-unitary operator group. Hence the linear field $\frac{d}{dt}\Big|_0 U^t=[f\mapsto f(y_2)1_{y_1}-f(y_1)1_{y_2}]$ belongs to $\log^*\mathrm{Aut}\,B$.

Proof of (ii): Say that $f\sim^Y g$ if Y finite$\subset X$, $f,g\in E$, $f-g\in 1_YE$ and $\sum_{y\in Y}|f(y)|^2=\sum_{y\in Y}|g(y)|^2$. Obviously, the relations $\sim^Y$ are all equivalences. Consider the set $N\equiv\{m:\exists x_1\sim\dots\sim x_m\ \exists f,g\in E\ f\sim^{\{x_1,\dots,x_m\}}g,\ \|f\|\neq\|g\|\}$. Suppose $N\neq 0$ and set $n\equiv\min N$. From (i) it follows $n>2$. Fix a set $Y\equiv\{y_1,\dots,y_n\}$ and functions $f_1,f_2\in E$ such that $f_1\sim^Y f_2$, $y_1\sim\dots\sim y_n$ but $\|f_1\|\neq\|f_2\|$. Consider the functions $g_j\equiv 1_{(X\setminus Y)\cup\{y_1\}}f_j+\left(\sum_{k=2}^{n}f_j(y_k)^2\right)^{1/2}1_{y_2}$ $(j=1,2)$. Observe that $f_j\sim^{\{y_2,\dots,y_n\}}g_j$ whence $\|f_j\|=\|g_j\|$ $(j=1,2)$. However, $g_1\sim^{\{y_1,y_2\}}g$ and therefore by (i) we have $\|g_1\|=\|g_2\|$ contradicting the assumption $\|f_1\|\neq\|f_2\|$. Thus $N=\emptyset$. Hence if $y_1\sim y_2\sim y_3$ then $\forall f,g\in E$ $f\sim^{\{y_1,y_2,y_3\}}g\Rightarrow f\sim^{\{y_1,y_3\}}g$ i.e. by (i), $y_1\sim y_3$ holds.

3.7. Corollary. *The proof of* (i) *shows that* $\langle\ell(1_{y_1}),1_{y_2}^*\rangle=-\langle\ell(1_{y_2}),1_{y_1}^*\rangle$ *whenever* $y_1,y_2\in X$ *and* ℓ linear$\in\log^*\mathrm{Aut}\,B$.

3.8. Definition. From now on we reserve the notation $\{S_i: i\in\mathscr{I}\}$ to denote the partition of X formed by the equivalence classes of the relation $\sim$. For each $i\in\mathscr{I}$, we shall denote the projection band $1_{S_i}E$ of E by H_i.

3.9. Proposition. (i) *If* $f,g\in E$ *are functions with finite support and* $\|f|_{S_i}\|_{\ell^2}=\|g|_{S_i}\|_{\ell^2}$ $(\equiv(\sum_{x\in S_i}|g(x)|^2)^{1/2})$ $\forall i\in\mathscr{I}$ *then* $\|f\|=\|g\|$.

(ii) *For any* $i\in\mathcal{I}$, H_i *is a Hilbert space (i.e. the norm* $\|\cdot\|$ *restricted to* H_i *satisfies parallelogram identity). Namely, a function* $h\colon X\to\mathbf{C}$ *belongs to* H_i *iff* $\operatorname{supp}(h)\subset S^i$, $\sum_{x\in S_i}|h(x)|^2<\infty$, *furthermore we have* $\|f\|=\|f\|_{\ell^2}$ $\forall f\in H_i$.

(iii) *If* $f,g\in E$ *and* $\|f|_{S_i}\|=\|g|_{S_i}\|$ $\forall i\in\mathcal{I}$ *then* $\|f\|=\|g\|$.

(iv) *If* $g\colon X\to\mathbf{C}, f\in E$ *and* $\|f|_{S_i}\|_{\ell^2}=\|g|_{S_i}\|_{\ell^2}$ $\forall i\in\mathcal{I}$ *then* $g\in E$.

(v) *Assume* $\ell\in\mathcal{L}(E,E)$. *Then* $\ell\in\log^*\operatorname{Aut}B$ *if and only if there exists a family of linear mappings* $\{\ell_j\colon j\in\mathcal{I}\}$ *such that* $i\cdot\ell_j$ *is a self-adjoint* H_j*-operator for each* $j\in\mathcal{I}$, $\sup_{j\in\mathcal{I}}\|\ell_j\|<\infty$ *and* $\ell=\bigotimes_{j\in\mathcal{I}}\ell_j$.

Proof. (i) is a directe consequence of Lemma 3.6 (i).

(ii): Let $f\in H$ and $x_0\in E$ be arbitrarily fixed. By (i), $\|1_Yf\|=\|(\sum_{y\in Y}|f(y)|^2)^{1/2}1_{x_0}\|=(\sum_{y\in Y}|f(y)|^2)^{1/2}$ for all Y finite$\subset X$. Hence by (8′), $\infty>\|f\|=\|f\|_{\ell^2}$. Furthermore, if g is a function $X\to\mathbf{C}$ having support in S_i and $\|g\|_{\ell^2}<\infty$ then (i) ensures $\forall Y_1, Y_2$ finite$\subset X$, $\|1_{Y_1}f-1_{Y_2}f\|=\|1_{Y_1}f-1_{Y_2}f\|_{\ell_2}=\|1_{Y_1\Delta Y_2}f\|$ i.e. the net $\{1_Yf\}_Y$ is a Cauchy net whence $f\in E$.

(iii): Let $\varepsilon>0$ be fixed. According to (8′), one can find Y finite$\subset X$ with $\|f-1_Zf\|, \|g-1_Zg\|<\varepsilon$ $\forall Z\subset Y$. Since the index set $J\equiv\{i\in\mathcal{I}\colon Y\cap S_i=\emptyset\}$ is finite, there exists a family of sets $\{Z_i\colon i\in J\}$ such that $Y\cap S_i\subset Z_i$ finite$\subset S_i$ $(i\in J)$ and $\sum_{i\in J}\|1_{S_i}f-1_{Z_i}f\|_{\ell_2}<\varepsilon$. Consider now the functions $f_\varepsilon\equiv\sum_{i\in J}\|1_{Z_i}f\|_{\ell^2}\cdot 1_{x_i}$ and $g_\varepsilon\equiv\equiv\sum_{i\in J}\|1_{Z_i}g\|_{\ell^2}\cdot 1_{x_i}$ where x_i denotes an arbitrarily fixed point of S_i $(i\in J)$. By writing $Z\equiv\bigcup_{i\in J}Z_i$, we can see $\|f_\varepsilon\|=\|1_Zf\|$, $\|g_\varepsilon\|=\|1_Zg\|$ and $\|f-1_Zf\|, \|g-1_Zg\|<\varepsilon$. Using the triangle inequality, $\|f_\varepsilon-g_\varepsilon\|\leqq\sum_{i\in J}|\|1_{Z_i}f\|_{\ell^2}-\|1_{Z_i}g\|_{\ell^2}|=$(since $\|1_{S_i}f\|_{\ell^2}=\|1_{S_i}g\|_{\ell^2}$ for all i)$=\sum_{i\in J}|\|1_{Z_i}f\|_{\ell^2}-\|1_{S_i}f\|_{\ell^2}+\|1_{S_i}g\|_{\ell^2}-\|1_{Z_i}g\|_{\ell^2}|\leqq(\sum_{i\in J}(\|1_{S_i}f-1_{Z_i}f\|_{\ell^2}=\|1_{S_i}g-1_{Z_i}g\|_{\ell^2})<2\varepsilon$. Thus $|\|f\|=\|g\||\leqq\|f-1_Zf\|+|\|1_Zf\|=\|1_Zg\||+\|g-1_Zg\|\leqq 4\varepsilon$.

(iv): By (8′), to every number $n\in\mathbf{N}$, we can choose Z_n finite$\subset X$ such that $\|f-1_{Z_n}f\|<\frac{1}{n}$. We may assume without loss of generality $Z_1\subset Z_2\subset\dots$. Then set $\mathcal{I}_n\equiv\{i\in\mathcal{I}\colon Z_n\cap S_i\neq\emptyset\}$, $g_n\equiv\sum_{i\in\mathcal{I}_n}1_{S_i}g$. By (ii) and the finiteness of the sets $\mathcal{I}_n$, $g_n\in E$ $\forall n\in\mathbf{N}$. If $n>m$ then $\|g_n-g_m\|=\|\sum_{i\in\mathcal{I}_n}1_{S_i}g\|=$(by (iii))$=\|\sum_{i\in\mathcal{I}_n\setminus\mathcal{I}_m}1_{S_i}f\|\leqq$(since $|\sum_{i\in\mathcal{I}_n\setminus\mathcal{I}_m}1_{S_i}f|\leqq|f-1_{Z_m}f|)\leqq\|f-1_{Z_m}f\|<\frac{1}{m}$. Thus $\{g_n\}_n$ is a Cauchy sequence in E. For all $x\in X$, $\lim_{n\to\infty}g_n(x)=g(x)$ whence $g=\lim_{n\to\infty}g_n$.

(v) First let $\ell\in\log^*\operatorname{Aut}B$. If $j,k\in\mathcal{I}, j\neq k, x\in S_j, y\in S_k$ then by the definition of the classes S_i and by Lemma 3.6 (i), $\langle\ell(1_x), 1_y^*\rangle=0$. This fact shows $\ell(H_j)\subset H_j$

$\forall j\in\mathcal{I}$. Thus by setting $\ell_j\equiv\ell|_{H_j}$ we obviously have $\|\ell_j\|\leq\|\ell\|$ and $\ell=\bigoplus_{j\in\mathcal{I}}\ell_j$. Furthermore, [11, Lemma] establishes $\ell_j\in i\cdot\{\text{self-adj. } H_j\text{-op.-s}\}$ $\forall j\in\mathcal{I}$.

The converse statement is immediate from (ii) since then we have $\exp(\ell)=\bigoplus_{j\in\mathcal{I}}\exp(\ell_j)$ and, by assumption, all the operators $\exp(\ell_j)$ are H_j-unitary here.

3.10. Corollary. *For some subset $\mathcal{I}_0\subset\mathcal{I}$, by writing $X_0\equiv\bigcup_{i\in\mathcal{I}_0}S_i$, we have $E_0=1_{X_0}E$ (where $E_0\equiv\mathbf{C}\cdot[\text{Aut}\,B]\{0\}$ cf. Introduction).*

Proof. Set $Z\equiv\{x\in X: \exists c\in E_0\ c(x)\neq0\}$. Clearly $E_0\subset 1_Z E$. On the other hand, if $x\in Z, c\in E_0$ and $c(x)\neq0$ then, by (v), the linear field $\ell\equiv[f\mapsto i\cdot f(x)1_x]$ satisfies $1_{X\setminus\{x\}}c+e^{it}c(x)1_x=\exp(t\ell)\in E_0$ $\forall t\in\mathbf{R}$ whence $E_0\supset\text{Span}\{1_x:x\in Z\}=1_Z E$ i.e. $E_0=1_Z E$. Suppose now $x\in Z, c\in E_0, c(x)\neq0$ and $x\in S_i$. Let $y\in S_i\setminus\{x\}$ and $\ell_1\equiv[f\mapsto if(x)1_y+if(y)1_x]$. As in the previous case, $c_1\equiv\ell_1(c)=\frac{d}{dt}\Big|_0\exp(t\ell_1)c\in E_0$ since by (v), $\ell_1\in\log^*\text{Aut}\,B$. However, $c_1(y)=ic(x)\neq0$ i.e. $y\in S_i$. Thus $S_i\subset Z$.

Next we turn our attention to the quadratic part of $\log^*\text{Aut}\,B$.

In the sequel we shall use the notations $\mathcal{I}_0, X_0$ introduced in Corollary 3.10. Recall that for any $c\in E_0$, there is a unique symmetric bilinear form q_c: $E\times E\to E$ with $[f\mapsto c+q_c(f,f)]\in\log^*\text{Aut}\,B$ and that the mapping $c\mapsto q_c$ is conjugate-linear and continuous. Since the finitely supported functions are dense in E, to get the complete description of $\log^*\text{Aut}\,B$ it is enough to determine only the values $\langle q_{1_{x_1}}(1_{x_2},1_{x_3}),1_{x_4}\rangle$ $(x_1\in X_0, x_2,x_3,x_4\in X)$. To this task, the projection principle provides an essential reduction.

3.11. Lemma. *Let $x_1,\dots,x_n\in X, x_1\in X_0$ and $\beta^l_{jk}\equiv\langle q_{1_{x_l}}(1_{x_j},1_{x_k}),1^*_{x_l}\rangle$. Then*

(i) *$\beta^l_{jk}=0$ if $\{1,\ell\}\neq\{j,k\}$,*

(ii) *$\beta^1_{11}=-1$,*

(iii) *$\beta^2_{12}\in[-1,0]$ and $1_{\{x_1,x_2\}}B=\{\zeta_1 1_{x_1}+\zeta_2 1_{x_2}: |\zeta_1|^2+|\zeta_2|^{-1/\beta}<1\}$ if $\beta^2_{12}=0$ or $1_{\{x_1,x_2\}}B=\{\zeta_1 1_{x_1}+\zeta_2 1_{x_2}: \max(|\zeta_1|,|\zeta_2|)<1\}$ in case of $\beta^2_{12}=0$,*

(iv) *$\beta^2_{12}=-1/2$ if $x_1\sim x_2\neq x_1$ and $\beta^2_{12}=0$ if $x_1\not\sim x_2\in X_0$;*

(v) *if $x_1,\dots,x_n\in X_0$ and $x_i\not\sim x_j$ for $i\neq j$ then $\|\zeta_1 1_{x_1}+\dots+\zeta_n 1_{x_n}\|=\max(|\zeta_1|,\dots,|\zeta_n|)$ for all $\zeta_1,\dots,\zeta_n\in\mathbf{C}$.*

Proof. (i) Consider the band projection $P: f\mapsto 1_{\{x_1,\dots,x_n\}}f$. By the projection principle, $[f\mapsto 1_{x_1}+Pq_{1_x}(f,f)]\in\log^*\text{Aut}\,PB$. Applying [11, Lemma] to PB, we obtain

$$0=\|f\|^2\overline{\langle 1_{x_1},\Phi\rangle}+\langle Pq_{1_x}(f,f),\Phi\rangle\Leftarrow\|f\|\cdot\|\Phi\|=\langle f,\Phi\rangle\quad\forall f\in PE, \Phi\in(PE)^*.$$

Introducing the same function $p\colon \mathbf{R}^n_+ \to \mathbf{R}_+$ and set $C \subset \mathbf{R}^n_+$ as in the proof of Lemma 3.6,

$$(12)\qquad 0 = p(\varrho_1, \dots, \varrho_n)^2 \Big\langle 1_{x_1}, \sum_{j=1}^n \frac{\partial p}{\partial \varrho_j} e^{-i\vartheta_j} 1^*_{x_j} \Big\rangle +$$

$$+ \Big\langle q_{1_x}\Big(\sum_{j=1}^n \varrho_j e^{i\vartheta_j} 1_{x_j}, \sum_{k=1}^n \varrho_k e^{i\vartheta_k} 1_{x_k}\Big), \sum_{\ell=1}^n \frac{\partial p}{\partial \varrho_\ell} e^{-i\vartheta_\ell} 1^*_{x_\ell} \Big\rangle$$

for all $\varrho \in \mathbf{R}^n_+ \setminus C$ and $\vartheta \in \mathbf{R}^n$. Thus

$$(12')\qquad p^2 \frac{\partial p}{\partial \varrho_1} e^{i\vartheta_1} + \Big(\sum_{j,k,\ell=1}^n \beta^\ell_{jk} \varrho_j \varrho_k \frac{\partial p}{\partial \varrho_\ell} e^{i(\vartheta_j + \vartheta_k - \vartheta_\ell)}\Big) = 0 \qquad (\varrho \notin C,\ \vartheta \in \mathbf{R}^n).$$

Therefore (for fixed $\varrho \in \mathbf{R}^n_+ \setminus C$) the rational expression $p^2 \frac{\partial p}{\partial \varrho_1} z_1 + \sum_{j,k,\ell=1}^n \beta_{jk} \varrho_j \varrho_k \cdot \frac{\partial p}{\partial \varrho_\ell} z_j z_k z_\ell^{-1}$ vanishes on $\partial_0 \Delta^n$ i.e. its homogeneous parts are 0-s. Hence only the coefficients of the form $\beta^1_{1k} (= \beta^1_{k1})$ may differ from 0.

(ii) is immediate from (12′) if we take $n=1$ because then $p(\varrho_1) = \varrho_1$.

For the proof of (iii) and (iv), consider the case $n=2$. From (12′) and (ii) we then see

$$(12'')\qquad (p^2 - \varrho_1^2) \frac{\partial p}{\partial \varrho_1} + 2\varrho_1 \varrho_2 \frac{\partial p}{\partial \varrho_2} \beta^2_{12} = 0 \quad (\varrho \in \mathbf{R}^n_+ \setminus C).$$

Since $p(0, \varrho) = p(\varrho, 0)$ and since the function p is increasing and convex, $\forall \varrho \in [0,1)\ \exists! t \geqq 0\ \ p(\varrho, t) = 1$. Thus the function $t\colon [0,1) \to \mathbf{R}_+$ is welldefined by $p(\varrho, t(\varrho)) = 1$. Observe that now t is a decreasing concave function and $t(0) = 0$. By the implicite function theorem, $t'(\varrho_1) = -\dfrac{\partial p/\partial \varrho_1}{\partial p/\partial \varrho_2}$ whenever $(\varrho_1, t(\varrho_1)) \notin C$. Thus, since C is a cone with measure 0 in $\mathbf{R}^2_+$, (12″) implies

$$(12''')\qquad t'(\varrho)(1 - \varrho^2) = 2\varrho t(\varrho) \beta^2_{12} \quad \text{for almost every} \quad \varrho \in (0, 1).$$

Since $t' \leqq 0$, we have $\beta^2_{12} \leqq 0$. If $\beta^2_{12} = 0$ then $t(\varrho) = t(0) = 1\ \forall \varrho \in [0,1)$. In this case, $p(\varrho_1, \varrho_2) \leqq 1$ if $\varrho_1 < 1$ and $\varrho_2 \leqq t(\varrho_1) = 1$ or $\varrho_1 = 1$ and $\varrho_2 \leqq 1$, i.e. $p(\varrho_1, \varrho_2) = \max(\varrho_1, \varrho_2)$. If $\beta^2_{12} < 0$ then the solution of (12‴) with initial value $t(0) = 1$ is $t(\varrho) = (1 - \varrho^2)^{-\beta^2_{12}}$. Thus by setting $K \equiv \{(\varrho_1, \varrho_2)\colon p(\varrho_1, \varrho_2) \leqq 1\}$,

$$(13)\qquad K = \{(\varrho_1, \varrho_2)\colon \varrho_1^2 + \varrho_2^{-1/\beta^2_{12}} \leqq 1\}.$$

The convexity of the function p entails that K is convex whence $\beta^2_{12} \geqq -1$ yielding (iii).

(iv): If $x_1 \sim x_2 \neq x_1$ then $p(\varrho_1, \varrho_2)=(\varrho_1^2+\varrho_2^2)^{1/2}$ (cf. Proposition 3.9 (ii)), that is, by (13), we have $\beta_{12}^2=-\frac{1}{2}$.

On the other hand, suppose $x_1 \not\sim x_2 \in X_0$ and $\beta_{12}^2 \neq 0$. Since $x_2 \in X_0$, all the previous considerations can be carried out by interchanging x_1 and x_2. Thus by (iii),

$$1_{\{x_1,x_2\}}B = \{\zeta_1 1_{x_1}+\zeta_2 1_{x_2}: |\zeta_1|^2+|\zeta_2|^{-1/\langle q_{1_{x_1}}(1_{x_1},1_{x_2}),1_{x_2}^*\rangle} \leqq 1\} =$$

$$= \{\zeta_1 1_{x_1}+\zeta_2 1_{x_2}: |\zeta_2|^2+|\zeta_1|^{-1/\langle q_{1_{x_1}}(1_{x_2},1_{x_1}),1_{x_2}\rangle} \leqq 1\}.$$

This is possible only if $\beta_{12}^2=-\frac{1}{2}=\langle q_{1_{x_1}}(1_{x_2}, 1_{x_1}), 1_{x_1}^*\rangle$ thus $p(\varrho_1, \varrho_2)=(\varrho_1^2+\varrho_2^2)^{-1/2}$. If S_i denotes the equivalence class (w.r.t. $\sim$) of x_1 then by Proposition 3.9 (iii), $\|f+1_{x_2}\|=\|\|f\|_{\ell^2}\cdot 1_{x_1}+\varrho 1_{x_2}\|=p(\|f\|_{\ell^2}, \varrho)=\|f+\varrho 1_{x_2}\|_{\ell^2}$ for arbitrary $f\in H_i$ whence it follows $x_2\in S_i$ i.e. $x_1\sim x_2$. The obtained contradiction proves (iv).

(v): Let $y_1, \dots, y_n \in X_0$ be pairwise non-$\sim$-equivalent. Now for arbitrarily fixed $f, c\in 1_{\{y_1,\dots,y_n\}}E$,

$$q_c(f,f) = \sum_{m=1}^{n} \overline{c(y_m)}\, q_{1_{y_m}}(f,f) = \sum_{m=1}^{n} \overline{c(y_m)} \sum_{j,k,l=1}^{n} f(y_j)f(y_k)\langle q_{1_{y_m}}(1_{y_j}, 1_{y_j}), 1_{y_\ell}^*\rangle\, 1_{y_\ell}.$$

Applying (i) and (iii) to $x_1\equiv y_m, x_k\equiv y_k$ and $x_j\equiv y_j$, hence we obtain

$$q_c(f,f) = -\sum_{m=1}^{n} \overline{c(y_m)}\, f(y_m)^2 1_{y_m} = -\bar{c}\cdot f^2.$$

Therefore the solution of the initial value problem $\left\{\frac{d}{dt} f_t=c-q_c(f_t,f_t), f_0=0\right\}$ is $f_t=\tanh(tc)$. Hence $\left\{\sum_{m=1}^{n} \varrho_m 1_{y_m}: \varrho_1, \dots, \varrho_n\in[0, 1)\right\}\subset\{\exp[f\mapsto c+q_c(f,f)](0): c\in 1_{\{y_1,\dots,y_n\}}E\}\subset[\operatorname{Aut} B]\{0\}\subset B$. Then $\max_{m=1}^{n} \varrho_m \leqq \left\|\sum_{m=1}^{n} \varrho_m 1_{y_m}\right\|\leqq 1$ whenever $\varrho_1, \dots, \varrho_n\in[0, 1]$. Consequently $\left\|\sum_{m=1}^{n} \varrho_m 1_{y_m}\right\|=1$ whenever $\max_{m=1}^{n} |\varrho_m|=1$ whence $\left\|\sum_{j=1}^{n}\zeta_j 1_{y_j}\right\|=\max_{m=1}^{n} |\zeta_m|$. The proof is complete.

From Lemma 3.11 (i) and the symmetry of the bilinear mappings q_c follows directly that introducing the functions

$$w_{x_1}(x_2) \equiv \begin{cases} -1/2 & \text{if } x_1 = x_2 \\ \langle q_{1_x}(1_{x_1}, 1_{x_2}), 1_{x_2}^{*}\rangle & \text{if } x_1 \neq x_2 \end{cases} \quad (x_1\in X_0, x_2\in X),$$

we have

$$q_{1_x}(1_x, 1_x) = 2w_x(x)1_x \quad \text{for all} \quad x \in X_0,$$

$$q_{1_x}(1_x, 1_y) = w_x(y)1_y \quad \text{if} \quad x \in X_0,\ y \in X \setminus \{x\},$$

$$q_{1_x}(1_y, 1_z) = 0 \quad \text{if} \quad x \notin \{y, z\},\ x \in X_0.$$

Hence

(14) $$q_{1_x}(f, g) = f(x)\, w_x g + g(x) w_x f \quad (x \in X_0)$$

whenever the function $f \in E$ is finitely supported. Moreover by (8′) and Lemma 3.11 (iii), (14) holds for every $f \in E$.

For sake of brevity, in what follows we shall write $f^{(i)}$ instead of the function $1_{S_i} f$.

3.12. Lemma. (i) $w_x^{(i)} = -\frac{1}{2} 1_{S_i}$ *whenever* $x \in S_i$ $(i \in \mathcal{I}_0)$,

(ii) $w_x^{(i)} = 0$ *whenever* $x \notin S_i$ $(i \in \mathcal{I}_0)$,

(iii) *There exists a unique matrix* $(\gamma_{ij})_{i \in \mathcal{I}_0, j \in \mathcal{I} \setminus \mathcal{I}_0}$ *consisting of numbers belonging to* $[0, 1]$ *such that* $w_x^{(j)} = -\gamma_{ij} 1_{S_j}$ *whenever* $x \in S_i \subset X_0$ *and* $j \in \mathcal{I} \setminus \mathcal{I}_0$.

Proof. (i) and (ii) are contained in Lemma 3.11 (iv).

(iii): Let $x, x' \in S_i$ and $y, y' \in S_j$ where $i \in \mathcal{I}_0$, $j \notin \mathcal{I}_0$. From Proposition 3.9 (v) it follows the existence of an E-unitary operator U such that $1_{x'} = U 1_x$ and $1_{y'} = U 1_{y_1}$. From the elementary theory of Lie-groups it is well-known that $UvU^{-1} \in \log^* \mathrm{Aut}\, B$ for every $v \in \log^* \mathrm{Aut}\, B$. In particular, $[f \mapsto U(1_x + q_{1_x}(U^{-1}f, U^{-1}f))] \in \log^* \mathrm{Aut}\, B$ whence $q_{1_{x'}}(f, f) = q_{U1_x}(f, f) = q_{1_x}(U^{-1}f, U^{-1}f)$. Therefore $\langle q_{1_{x'}}(1_{x'}, 1_{y'}), 1_{y'}^* \rangle = \langle U q_{1_x}(U^{-1}1_{x'}, U^{-1}1_{y'}), 1_{y'}^* \rangle = \langle U q_{1_x}(1_x, 1_y), 1_{y'}^* \rangle\rangle = \langle q_{1_x}(1_x, 1_y), 1_y^* \rangle$ since if $U = \bigoplus_{i \in \mathcal{I}} U_i$ is the directe decomposition of U provided by Proposition 3.9 (v) and $f \in E$ then $\langle Uf, 1_{x'}^* \rangle = (U_i f^{(i)} | 1_x) = (f^{(i)} | U_i^{-1} 1_x) = (f^{(i)} | U_i^{-1} 1_{x'}) = (f^{(i)} | 1_x)$.

Henceforth we reserve the notation $(\gamma_{ij})_{i \in \mathcal{I}_0, j \in \mathcal{I} \setminus \mathcal{I}_0}$ for the matrix introduced in Lemma 3.12 (iii).

3.13. Corollary. *For arbitrary finitely supported* $c \in E_0$ *and* $f \in E$,

(15) $$q_c(f, f) = -\sum_{i \in \mathcal{I}_0} (f^{(i)} | c^{(i)}) f^{(i)} - 2 \sum_{j \in \mathcal{I} \setminus \mathcal{I}_0} \Big[\sum_{i \in \mathcal{I}_0} \gamma_{ij} (f^{(i)} | c^{(i)}) \Big] f^{(j)}.$$

Proof. Applying Lemma 3.12. and (14), we can see that if $c \in E_0$ and $f \in E$ have finite supports then $q_c(f, f) = -\sum_{x \in X_0} \overline{c(x)}\, q_{1_x}(f, f) \sum_{i \in \mathcal{I}_0} \sum_{x \in S_i} 2\overline{c(x)} f(x) \cdot \cdot \Big[-\frac{1}{2} f^{(i)} - \sum_{j \notin \mathcal{I}_0} \gamma_{ij} f^{(j)} \Big]$.

In order to extend (15) to every $c \in E_0$ and $f \in E$, we need the following observations.

3.14. Lemma. (i) $E_0=\bigoplus_{i\in\mathscr{I}_0}^{c_0} H_i$ *i.e. a function* $c\colon X\to\mathbf{C}$ *belongs to* E_0 *if and only if* $\forall i\in\mathscr{I}$ $\|c^{(i)}\|_{\ell^2}<\infty$ *and* $\forall\varepsilon>0$ $\{i\in\mathscr{I}_0\colon \|c^{(i)}\|_{\ell^2}\geqq\varepsilon\}$ finite$\subset\mathscr{I}_0$ (*in the latter case* $\|c\|=\sup_{i\in\mathscr{I}_0}\|c^{(i)}\|_{\ell^2}$).

(ii) $\sup_{j\in\mathscr{I}\setminus\mathscr{I}_0}\sum_{i\in\mathscr{I}_0}\gamma_{ij}\leqq 4\|q\|(\equiv 4\sup_{c\in B\cap E_0}\|q_c\|=4\sup_{\substack{c\in B\cap E_0\\ f,g\in B}}\|q_c(f,g)\|)$.

Proof. (i): Trivial from Proposition 3.9 (v), Lemma 3.11 (v) and the fact that the finitely supported functions are dense in E.

(ii): Let $j\in\mathscr{I}\setminus\mathscr{I}_0$, $i_1,\dots,i_n\in\mathscr{I}_0$, $y\in S_j$ and $x_1\in S_{i_1},\dots,x_n\in S_{i_n}$. Consider the functions $c\equiv\sum_{m=1}^{n}1_{x_m}$ and $f\equiv 1_y+\sum_{m=1}^{n}1_{x_m}$. By (i) we have $\|c\|=1$ and $\|f\|\leqq 2$. By (15), $\langle q_c(f,f),1_y^*\rangle=\sum_{m=1}^{n}\gamma_{i_m j}$. At the same time, $|\langle q_c(f,f),1_y^*\rangle|\leqq\|q\|\cdot\|c\|\cdot\|f\|^2\cdot\|1_y^*\|\leqq$ $\leqq 4\|q\|$.

3.15. Corollary. (15) *holds for each* $c\in E_0$ *and* $f\in E$.

Proof. The previous lemma shows that the right hand side of (15) makes always sense. Observe that the mapping $Q\colon E_0\times E\ni(c,f)\mapsto\{\text{right hand side of (15)}\}$ is real-linear in c and real-quadratic in f. For $\|c\|,\|f\|\leqq 1$ we have $\|Q(c,f)\|\leqq$ $\|\sum_{i\in\mathscr{I}_0}(f^{(i)}|c^{(i)})f^{(i)}\|+2\|\sum_{j\notin\mathscr{I}_0}(\sup_{k\notin\mathscr{I}_0}\sum_{i\in\mathscr{I}_0}\gamma_{ik}\|f^{(i)}\|_{\ell_2}\cdot\|c^{(i)}\|_{\ell_2})f^{(j)}\|\leqq\|f\|^2\cdot\|c\|+4\|q\|\cdot\|c\|\cdot$ $\cdot\|f\|^2$. Thus Q is a continuous map. On the other hand, the relation $Q(c,f)=$ $=+q_c(f,f)$ is already established for a dense submanifold of $E_0\times E$ by Corollary 3.13.

In this way we completely know log*Aut B. The mappings $\exp[B\ni f\mapsto$ $\mapsto c+q_c(f,f)]$ are easy to describe: By (15), the equation $\frac{d}{dt}f_t=c+q_c(f_t,f_t)$ is equivalent with

(16′) $$\frac{d}{dt}f_t^{(i)}=c^{(i)}-(f_t^{(i)}|c^{(i)})f_t^{(i)}\quad(i\in\mathscr{I}_0)$$

(16″) $$\frac{d}{dt}f_t^{(j)}=-2\sum_{i\in\mathscr{I}_0}\gamma_{ij}(f_t^{(i)}|c^{(i)})f_t^{(j)}\quad(j\in\mathscr{I}\setminus\mathscr{I}_0).$$

If we represent $c^{(i)}$ in the form $c^{(i)}\equiv\varrho_i c_0^{(i)}$ where $\varrho_i\geqq 0$, $\|c_0^{(i)}\|=1$ and if $f_0^{(i)}=$ $=\zeta_i c_0^{(i)}+f_\perp^{(i)}$ where $f_\perp^{(i)}$ lying orthogonally to $c_0^{(i)}$, one then cheks immediately that for arbitrarily given $f_0\in B$, the solution of (16′) is

(17′) $$f_t^{(i)}=M_{\varrho_i t}(\zeta_i)c_0^{(i)}+M_{\varrho_i t}^{\perp}(\zeta_i)f_\perp^{(i)}\quad(i\in\mathscr{I}_0)$$

where M_τ and $M_\tau^\perp$ are the Moebius- and co-Moebius transformations

(18) $$M_\tau(\zeta) \equiv \frac{\zeta+\tanh(\tau)}{1+\zeta\tanh(\tau)}, \; M_\tau^\perp(\zeta) \equiv \frac{\{1-(\tanh(\tau))^2\}^{1/2}}{1+\zeta\tanh(\tau)} \qquad (\tau\in\mathbf{R}, |\zeta|<1).$$

Substituting (17′) into (16″), we obtain

$$\frac{d}{dt} f_t^{(j)} = [-2\sum_{i\in\mathscr{I}_0} \gamma_{ij}\varrho_i M_{\varrho_i t}(\zeta_i)] f_t^{(j)} \quad (j\in\mathscr{I}\setminus\mathscr{I}_0)$$

whose solution is given by

(17″) $$f_t^{(j)} = \exp\left[-2\sum_{i\in\mathscr{I}_0}\gamma_{ij}\varrho_i \int_0^1 M_{\varrho_i\tau}(\zeta_i)\,d\tau\right] f_0^{(j)} =$$
$$= [\prod_{i\in\mathscr{I}_0} M_{\varrho_i t}^\perp(\zeta_i)^{2\gamma_{ij}}] f_0^{(j)} \quad (j\in\mathscr{I}\setminus\mathscr{I}_0).$$

The fact that the right hand side in (17″) makes sense, is guaranteed by Lemma 3.14 (ii). Fortunately, by Lemma 3.14 (i) and (17′),

$$[\mathrm{Aut}\, B]\{0\} = B\cap E_0 = \{\sum_{i\in\mathscr{I}_0}\lambda_i c_i:\ 0\le\lambda_i\le 1,\ c_i\in\partial B(H_i) \quad i\in\mathscr{I}_0 \quad \text{and}$$
$$[i\mapsto\lambda_i]\in c_0(\mathscr{I}_0)\} = \{\sum_{i\in\mathscr{I}_0} M_{\varrho_i}(0)c_i:\ \varrho_i\in\mathbf{R}_+,\ c_i\in\partial B(H_i) \quad \forall i\in\mathscr{I}_0 \quad \text{and}$$
$$[i\mapsto\lambda_i]\in c_0(\mathscr{I}_0)\} = \{\exp[f\mapsto c+q_c(f,f)](0): c\in E_0\}$$

where $c_0(\mathscr{I}_0)\equiv\{\mathscr{I}_0\to\mathbf{C}$ functions vanishing at infinity$\}$. A classical theorem of Cartan asserts that the relations $U\in\mathrm{Aut}\,B$ and $U(0)=0$ entail the linearity of U. Thus given $F\in\mathrm{Aut}\,B$, if we choose the vector $c\in E_0$ so that the automorphism $G\equiv\exp[B\ni f\mapsto -c+q_{(-c)}(f,f)]$ satisfies $G(0)=F^{-1}(0)$ then the automorphism $U\equiv F\circ G$ is necessarily linear, i.e. we have $F\in U\cdot\exp[f\mapsto c+q_c(f,f)]$ for suitable $c\in E_0$ and linear E-unitary U. Hence we arrive at the following characterization of $\mathrm{Aut}\,B$:

3.16. Theorem. *Let E denote a minimal atomic Banach lattice. The space E is spanned by a family $\{H_i: i\in\mathscr{I}\}$ of its pairwise lattice-orthogonal Hilbertian projection bands such that*

(i) *the linear members of* $\mathrm{Aut}_0\, B(E)$ *map* $B(H_i)$ *onto themselve* $(\forall i\in\mathscr{I})$,

(ii) *conversely, if for any index $i\in\mathscr{I}$, U_i is an H_i-unitary operator then* $\bigoplus_{i\in\mathscr{I}} U_i|_{B(E)}\in$ $\in\mathrm{Aut}_0\, B(E)$.

Furthermore there exists a matrix $(\gamma_{ij})_{i,j}\in\mathscr{I}$ and an index subfamily $\mathscr{I}_0\subset\mathscr{I}$ such that

(iii) $E_0(\equiv\mathbf{C}[\mathrm{Aut}\, B(E)]\{0\}) = \bigoplus_{i\in\mathscr{I}_0}^{c_0} H_i$,

(iv) $0 \leqq \gamma_{ij} \leqq 1$ *for all* $i, j \in \mathscr{I}$; $\gamma_{ii} = \frac{1}{2}$ *for all* $i \in \mathscr{I}_0$; $\gamma_{ij} = 0$ *whenever* $i, j \notin \mathscr{I}_0$ *or* i *and* j *are distinct elements of* $\mathscr{I}_0$.

(v) *A mapping* $F: B(E) \to E$ *belongs to* $\mathrm{Aut}_0 B(E)$ *if and only if, by denoting the band projection onto* H_i *by* P_i, *we have*

$$P_i F(f) = U_i\{M_{\varrho_i}((P_i f|c_i^0))\, c_i^0 + M_{\varrho_i}^{\perp}((P_i f|c_i^0))[P_i f - (P_i f|c_i^0)\, c_i^0]\} \quad (i \in \mathscr{I}_0),$$

$$P_j F(f) = \left\{\exp \int_0^1 \sum_{i \in \mathscr{I}_0} \gamma_{ij}\, \varrho_i M_{\varrho_i \tau}((P_i f|c_i^0))\, d\tau\right\} U_j P_j f \quad (j \in \mathscr{I} \setminus \mathscr{I}_0)$$

for suitable H_j*-unitary operators* U_j $(j \in \mathscr{I})$, *unit vectors* $c_i^0 \in H_i$ $(i \in \mathscr{I}_0)$ *and a function* $[\mathscr{I}_0 \ni i \mapsto \varrho_i]$ *assuming values in* $\mathbf{R}_+$ *and vanishing at infinity, respectively (the transformations* M_{ϱ_i}, $M_{\varrho_i}^{\top}$ *are those defined in* (18)*).*

4. Appendix
Linear finite dimensional tensor unit ball automorphisms

Throughout this section $H_1, \ldots, H_n$ are fixed finite dimensional Hilbert spaces. We are aimed to describe the structure of the linear unitary operators in the space $E \equiv H_1 \otimes \ldots \otimes H_n$.

We shall use the notations $B \equiv B(E), B^* \equiv B(E^*)$,

$$K \equiv \{F \in \partial B : \exists!\, \Phi \in \partial B^* \quad \langle F, \Phi \rangle = 1\},$$

$$K^* \equiv \{\Phi \in \partial B^* : \exists F \in K \quad \langle F, \Phi \rangle = 1\}.$$

4.1. Lemma. $K^* = \{\delta_{e_1, \ldots, e_n} : e_1 \in \partial B(H_1), \ldots, e_n \in \partial B(H_n)\}$.

Proof. Since $\dim E < \infty$, $\bar{B}$ is compact, thus for any n-linear functional $F \in \partial B$, one can find $e_1 \in \partial B(H_1), \ldots, e_n \in \partial B(H_n)$ with $F(e_1, \ldots, e_n) = 1$. Hence $K^* \subset \{\delta_{e_1, \ldots, e_n} : e_j \in \partial B(H_j)\}$. On the other hand, every E-unitary operator maps K onto itself and therefore also

(19) $$U^* K^* = K^* \quad \text{for all } E\text{-unitary operators.}$$

From the compactness of B it follows $K \neq \emptyset$ (indeed: for any smooth norm $\|\cdot\|_1$ on E, $\emptyset \neq \{F \in \partial B : \|F\|_1 \leqq \|G\|_1 \ \forall G \in \partial B\} \subset K$) whence $K^* \neq \emptyset$. That is, for some unit vectors $e_1^0 \in H_1, \ldots, e_n^0 \in H_n$ we have $\delta_{e_1^0, \ldots, e_n^0} \in K^*$. Now from (19) we obtain $\delta_{U_1 e_1^0, \ldots, U_n e_n^0} = (U_1 \otimes \ldots \otimes U_n)^* \delta_{e_1^0, \ldots, e_n^0} \in K^*$ whenever the U_j-s are H_j-unitary operators. Thus $\{\delta_{e_1, \ldots, e_n} : e_j \in \partial B(H_j)\} \supset K^*$.

4.2. Lemma. *Let* $\Phi \equiv \delta_{f_1, \ldots, f_n}$, $\psi \equiv \delta_{g_1, \ldots, g_n}$ *and* $\Theta \equiv \delta_{h_1, \ldots, h_n}$ *where* $0 \neq f_j, g_j, h_j \in H_j$ $(j = 1, \ldots, n)$ *and assume* $\Phi + \Psi = \Theta$. *Then there exists* k *such that for each* $j \neq k$ *we have* $f_j \| g_j$ *(i.e.* f_j *and* g_j *are linearly dependent).*

Proof. The statement holds obviously if for some index m, $f_j \| h_j$ for all $j \neq m$ or $f_j \| g_j$ for all $j \neq m$. In the contrary case $f_k \nparallel g_k$ and $f_m \nparallel h_m$ for some pair of indices $k \neq m$. We may then suppose $k=1$ and $m=2$. First we show that in this case we have $h_1 \nparallel f_1$. Indeed: from $h_1 \nparallel f_1$ it follows that introducing the tensor $\tilde{E} \equiv \tilde{g}_1 \otimes g_2 \otimes \ldots \otimes g_n$ where $\tilde{g}_1 \equiv g_1 - \|f_1\|^{-2}(g_1|f_1)f_1$ the relations $\langle \tilde{E}, \Phi \rangle = \langle \tilde{E}, \Theta \rangle = 0 \neq \langle \tilde{E}, \Psi \rangle$ hold. One can see in the same manner that $h_2 \nparallel g_2$. Since $h_1 \nparallel f_1$, there exists $u_1 \in H_1$ with $f_1 \perp u_1 \not\perp h_1$ and since $h_2 \nparallel g_2$ one can find $u_2 \in H_2$ with $g_2 \perp u_2 \not\perp h_2$. But then the tensor $T \equiv u_1 \otimes u_2 \otimes h_3 \otimes \ldots \otimes h_n$ satisfies $\langle T, \Phi \rangle = \langle T, \Psi \rangle = 0 \neq \langle T, \Theta \rangle$ which is impossible.

4.3. Proposition. *Set $r_j \equiv \dim H_j$ $(j=1, \ldots, n)$ and let $U \in \mathcal{L}(E, E)$ be fixed so that $U|_B \in \mathrm{Aut}_0 B$. Then one can choose H_j-unitary operators U_j such that $U = U_1 \otimes \ldots \otimes U_n$.*

Proof. It is enough to prove the statement only for E-unitary operators lying in a suitable neighbourhood of id_E as it is well-known (see e.g. [6]).

To do this, fix $\varepsilon > 0$ such that the functionals $\Phi \equiv \delta_{e_1, \ldots, e_n}$, $\tilde{\varphi} \equiv \delta_{\tilde{e}_1, \ldots, \tilde{e}_n}$, $\Psi \equiv \delta_{f_1, \ldots, f_n}$, $\tilde{\Psi} \equiv \delta_{\tilde{f}_1, \ldots, \tilde{f}_n}$ $(\in E^*)$ fulfil

$$\exists k \quad e_k \perp \tilde{e}_k, f_k \perp \tilde{f}_k \quad \text{and} \quad \forall j \neq k \quad e_j \| \tilde{e}_j,\ f_j \| \tilde{f}_j \tag{20}$$

whenever we have

$$\Phi - \tilde{\Phi},\ \Psi - \tilde{\Psi} \in K^*,\ \|\Phi - \tilde{\Phi}\| = \|\Psi - \tilde{\Psi}\| = \sqrt{2} \quad \text{and} \quad \|\Phi - \Psi\|,\ \|\tilde{\Phi} - \tilde{\Psi}\| < \varepsilon, \tag{21}$$

$$\|e_j\| = \|\tilde{e}_j\| = \|f_j\| = \|\tilde{f}_j\| = 1 \quad (j = 1, \ldots, n). \tag{22}$$

A value $\varepsilon > 0$ with the above properties in fact exists: Otherwise there would be a sequence $\Phi_m \equiv \delta_{e_1^m, \ldots, e_n^m}$, $\tilde{\Phi}_m \equiv \delta_{\tilde{e}_1^m, \ldots, \tilde{e}_n^m}$, $\Psi_m \equiv \delta_{f_1^m, \ldots, f_n^m}$, $\tilde{\Psi}_m \equiv \delta_{\tilde{f}_1^m, \ldots, \tilde{f}_n^m}$ $(m = 1, 2, \ldots)$ satisfying (21), (22) for $\varepsilon = \dfrac{1}{m}$ but without property (20). For a suitable index subsequence $\{m_s\}_s$ and for some unit vectors $e_j, \tilde{e}_j, f_j, \tilde{f}_j$ we have $e_j^{m_s} \to e_j$, $\tilde{e}_j^{m_s} \to \tilde{e}_j, f_j^{m_s} \to f_j, \tilde{f}_j^{m_s} \to \tilde{f}_j$ $(s \to \infty, j = 1, \ldots, n)$. Then the limits $\Phi, \tilde{\Phi}, \Psi, \tilde{\Psi}$ satisfy $\Phi = \Psi$, $\tilde{\Phi} = \tilde{\Psi}$, $\|\Phi - \tilde{\Phi}\| = \|\Psi - \tilde{\Psi}\| = \sqrt{2}$ and the contrary of (20). At the same time we also have $\Phi - \tilde{\Phi}, \Psi - \tilde{\Psi} \in K^*$ because of the closedness of K^*. Thus by Lemma 4.2; $\exists! k_0\ \forall j \neq k_0\ e_j \| \tilde{e}_j$. Since $\|\Phi - \tilde{\Phi}\| = \sqrt{2}$, hence $\|e_{k_0} - \tilde{e}_{k_0}\| = \sqrt{2}$ i.e. $e_{k_0} \perp \tilde{e}_{k_0}$. Similarly $\exists! \ell_0\ f_{\ell_0} \perp \tilde{f}_{\ell_0}$ and $\forall j \neq \ell_0\ f_j \| \tilde{f}_j$. Since (20) does not hold, necessarily $k_0 \neq \ell_0$. However the relations $\Phi = \Psi$, $\tilde{\Phi} = \tilde{\Psi}$ entail $k_0 = \ell_0$.

Now assume $\|U - \mathrm{id}_E\| < \varepsilon$. Fix an orthonormed basis $\{e_j^k : j = 1, \ldots, r_k\}$ in H_k $(k = 1, \ldots, n)$, respectively and let us write the functional $U^* \delta_{e_1^1, \ldots, e_1^n}$ in the form $U^* \delta_{e_1^1, \ldots, e_1^n} = \delta_{f_1^1, \ldots, f_1^n}$ (cf. Lemma 4.1.) where f_1^k is a fixed unit vector in H_k $(k = 1, \ldots, n)$. It follows from the choice of ε that for arbitrary index k, the singleton $\{f_1^k\}$ can be continued to an orthonormed basis $\{f_j^k : j = 1, \ldots, r_k\}$ of H_k in a unique way so that we have

$$U^*\delta_{e_1^1,\dots,e_1^{k-1}e_j^k,e_1^{k+1},\dots,e_1^n} = \delta_{f_1^1,\dots,f_1^{k-1}\,f_j^k,f_1^{k+1},\dots,f_1^n} \quad (j=1,\dots,r_k).$$

Set $I_0\equiv\{(1,\dots,1,\overset{k}{j},1,\dots,1): k=1,\dots,n; j=1,\dots,r_k\}$, $I_1\equiv\bigtimes_{k=1}^{n}\{1,\dots,r_k\}$ and let a family $I\subset I_1$ of multiindices be called *thick* if $\forall i\in I, \forall i'\in I_1 \quad i'\leqq i\Rightarrow i'\in I$.

Observe that for any multiindex $i\equiv(i_1,\dots,i_n)\in I_1$ there exists a unique complex number which we shall denote by $\varkappa_i$ such that $|\varkappa_i|=1$ and

$$(23) \qquad U^*\delta_{e_{i_1}^1,\dots,e_{i_n}^n} = \varkappa_i\delta_{f_{i_1}^1,\dots,f_{i_n}^n}.$$

Indeed: If not, we can find a minimal (w.r.t. $\leqq$) $i\in I_1$ not satisfying (23). Now $U^*\delta_{e_{i_1}^1,\dots,e_{i_n}^n}=\delta_{h_1,\dots,h_n}$ for some vectors $h_k\in\partial B(H_k)$ $(k=1,\dots,n)$. Since obviously $i\notin I_0$, for arbitrarily fixed k, there is $\tilde{k}\neq k$ with $i_{\tilde{k}}\neq 1$. Consider the multiindex j defined by $j_\ell\equiv[i_\ell \text{ if } \ell\neq k, 1 \text{ if } \ell=k]$ $(\ell=1,\dots,n)$. By the minimality of i, $U^*\delta_{e_{j_1}^1,\dots,e_{j_n}^n}= =\varkappa_j\delta_{f_{j_1}^1,\dots,f_{j_n}^n}$. Since $U^*\left(\frac{1}{\sqrt{2}}\delta_{e_{i_1}^1,\dots,e_{i_n}^n}+\frac{1}{\sqrt{2}}\delta_{e_{j_1}^1,\dots,e_{j_n}^n}\right)\in K^*$, using Lemma 4.2 we can see $h_k\| f_{i_k}^k$ i.e. $h_k=\alpha_k f_{i_k}^k$ for suitable $\alpha_j\in\partial\Delta$ $(k=1,\dots,n)$.

Then let I be a maximal thick subset of I_1 such that $I_1\supset I_0$ and $\varkappa_i=1$ $\forall i\in I$. (Remark: $\varkappa_i=1$ $\forall i\in I_0$.) We shall show that necessarily $I=I_1$. Hence and from the linearity of the mapping U, (23) immediately yields the statement of the lemma.

Assume $I_1\setminus I\neq\emptyset$. Let j be a minimal element of $I_1\setminus I$. Observation: $\forall i\in I_1$ $j\neq i\leqq j\Rightarrow i\in I$. I.e. the family $I'\equiv I\cup\{j\}$ is thick. Therefore it suffices to prove $\varkappa_j=1$ (which contradicts our assumption). By writing $J\equiv\{1,j_1\}\times\dots\times\{1,j_n\}$,

$$U^*\delta_{e_1^1+e_{j_1}^1,\dots,e_1^n+e_{j_n}^n} = \sum_{i\in J} U^*\delta_{e_{i_1}^1,\dots,e_{i_n}^n} = \sum_{i\in J}\varkappa_i\delta_{f_{i_1}^1,\dots,f_{i_n}^n} =$$

$$= \varkappa_j\delta_{f_{j_1}^1,\dots,f_{j_n}^n}+\sum_{i\in J\setminus\{j\}}\delta_{f_{i_1}^1,\dots,f_{i_n}^n} = (\varkappa_j-1)\delta_{f_{j_1}^1,\dots,f_{j_n}^n}+\delta_{f_1^1+f_{j_1}^1,\dots,f_1^n+f_{j_n}^n}.$$

However, the function $U^*\delta_{e_1^1+e_{j_1}^1,\dots,e_1^n+e_{j_n}^n}$ has the form $\delta_{h_1,\dots,h_n}$ whence directly $\varkappa_j=1$.

4.4. Corollary. *The vector fields V being tangent to $\partial B(E)$ are exactly those of the form*

$$V = i\cdot\sum_{j=1}^{n}\mathrm{id}_{H_1}\otimes\dots\otimes\mathrm{id}_{H_{j-1}}\otimes A_j\otimes\mathrm{id}_{H_{j+1}}\otimes\dots\otimes\mathrm{id}_{H_n}$$

where each A_j is a self-adjoint H_j-operator.

Proof. For every H_j-operator U_j there is a self-adjoint A_j with $U_j=\exp(i\cdot A_j)$. Thus by Proposition 4.3, V has the form $V=\left.\frac{d}{dt}\right|_0 \exp(it\cdot A_1)\otimes\dots\otimes\exp(it\cdot A_n)$.

References

[1] R. Braun—W. Kaup—H. Upmeier, On the automorphisms of circular and Reinhardt domains in complex Banach spaces, *Manuscripta Math.*, **25** (1978), 97—133.

[2] E. Cartan, Sur les domaines bornés homogènes de l'espace de n variables complexes, *Abh. Math. Sem. Univ. Hamburg*, **11** (1935), 116—162.

[3] C. J. Earl—R. S. Hamilton, A fixed point theorem for holomorphic mappings, *Proc. Sympos. Pure Math.*, **16**, Amer. Math. Soc., (Providence R. I., 1970), pp. 61—65.

[4] T. Franzoni, The group of biholomorphic automorphisms in certain J^*-algebras, *to appear.*

[5] T. L. Hayden—T. J. Suffridge, Biholomorphic maps in Hilbert space have fixed points, *Pacific J. Math.*, **38** (1971), 419—422.

[6] G. Hochschild, *The Structure of Lie Groups*, Holden Day (San Francisco, 1966).

[7] W. Kaup, Algebraic characterization of symmetric Banach manifolds, *Math. Ann.*, **228** (1979), 39—64.

[8] W. Kaup—H. Upmeier, Banach spaces with biholomorphically equivalent unit balls are isomorphic, *Proc. Amer. Math. Soc.*, **58** (1976), 129—133.

[9] S. Kobayashi, *Hyperbolic Manifolds and Holomorphic Mappings*, Marcel Dekker (New York, 1970).

[10] H. Schaeffer, *Banach Lattices and Positive Operators*, Springer Verlag (Berlin, 1974).

[11] L. L. Stachó, A short proof that the biholomorphic automorphisms of the unit ball in certain L^p-spaces are linear, *Acta Sci. Math.*, **41** (1979), 381—383.

[12] L. L. Stachó, *Holomorphic Maps and Fixed Points*, Dissertation, Scuola Normale Superiore, Pisa, 1979.

[13] T. Sunada, On bounded Reinhardt domains, *Proc. Japan Acad.*, **50** (1974), 119—123.

[14] P. Thullen, Zu den Abbildungen durch analytische Funktionen mehrerer komplexer Veränderlichen. Die Invarianz des Mittelpunktes von Kreiskörpern, *Math. Ann.*, **104** (1931), 244—259.

[15] H. Upmeier, *Über die Automorphismen-Gruppen beschränkter Gebiete in Banachräumen*, Dissertation, Tübingen, 1975.

[16] E. Vesentini, Variations on a theme of Carathéodory, *Ann. Scuola Norm. Sup. Pisa* (4), **6** (1979), 39—68.

[17] E. Vesentini—T. Franzoni, *Holomorphic maps and invariant distances*, North Holland (Amsterdam—New York, 1980).

[18] J. P. Vigué, Sur le groupe des automorphismes analytiques d'un domaine borné d'un espace de Banach complexe, *C. R. Acad. Sci. Paris*, A **282** (1976), 111—114; 211—213.

BOLYAI INSTITUTE
UNIVERSITY SZEGED
ARADI VÉRTANÚK TERE 1
H—6720 SZEGED, HUNGARY

Acta Sci. Math., 44 (1982), 125—131

Generalized resolvents of contractions

H. LANGER and B. TEXTORIUS

1. Let T be a contraction (that is a linear operator of norm $\leqq 1$), defined on a closed subspace $\mathfrak{D}(T)(\neq\mathfrak{H})$ of some Hilbert space $\mathfrak{H}$ and with values in $\mathfrak{H}$. By a contraction extension (c.e.) of T we mean an extension $\tilde{T}$ of T to some Hilbert space $\tilde{\mathfrak{H}}\supset\mathfrak{H}$, which is also a contraction. If $\tilde{\mathfrak{H}}=\mathfrak{H}$, the c.e. $\tilde{T}$ is called canonical.

Let $\tilde{T}$ on $\tilde{\mathfrak{H}}$ be a c.e. of T, and denote by $\tilde{P}$ the orthogonal projector of $\tilde{\mathfrak{H}}$ onto $\mathfrak{H}$. The function

(1) $$z\to R_z:=\tilde{P}(z\tilde{T}-I)^{-1}|_{\mathfrak{H}}\quad(|z|<1)$$

which is defined and holomorphic on the open unit disc $\mathbf{D}:=\{|z|<1\}$ and whose values are bounded linear operators in $\mathfrak{H}$, is called a generalized resolvent of T (generated by $\tilde{T}$). The generalized resolvent R_z is called canonical if $\tilde{T}=T$.

It is the aim of this note to give a description of all generalized resolvents of a nondensely defined contraction T in a Hilbert space $\mathfrak{H}$. This result is an analogue of the formula for the generalized resolvents of an isometric operator, proved in [1] for equal and in [2] (see also [3]) for arbitrary defect numbers.* In their turn these results have their origin in the classical formula of M. G. KREIN on the generalized resolvents of an hermitian operator with equal defect numbers ([4], [5]).

2. Let T be as above. By $\mathring{T}$ we denote the c.e. of T given by

$$\mathring{T}x:=\begin{cases}Tx & x\in\mathfrak{D}(T),\\ 0 & x\in\mathfrak{D}(T)^{\perp},\end{cases}$$

and set

$$D:=(I-\mathring{T}^*\mathring{T})^{1/2},\ D_*:=(I-\mathring{T}\mathring{T}^*)^{1/2},\ \mathscr{D}:=\overline{\mathfrak{R}(D)},\ \mathscr{D}_*:=\overline{\mathfrak{R}(D_*)}.$$

The characteristic function of $\mathring{T}^*$ is denoted by $X(z)$ (see [6, Chap. VI]):

$$X(z):=(-\mathring{T}^*-zD\mathring{R}_zD_*)|_{\mathscr{D}_*},\ \mathring{R}_z:=(z\mathring{T}-I)^{-1},\ z\in\mathbf{D}.$$

Received October 20, 1980.

* In these papers the more general case of an isometric operator in a $\pi_\varkappa$-space (Pontrjagin space with index $\varkappa$) has been considered.

It is defined and holomorphic on the open unit disc $\mathbf{D}$ and its values are contractions, mapping $\mathfrak{D}_*$ into $\mathfrak{D}$, see [6, chap. VI]. By $\mathcal{K}$ (or, sometimes, more explicitly by $\mathcal{K}(\mathfrak{D}(T)^\perp, \mathfrak{D}_*)$) we denote the set of all functions $G(z)$, defined and holomorphic on $\mathbf{D}$ and whose values are contractions from $\mathfrak{D}(T)^\perp$ into $\mathfrak{D}_*$, by $\mathcal{K}_0$ (or $\mathcal{K}_0(\mathfrak{D}(T)^\perp, \mathfrak{D}_*)$) the subset of $\mathcal{K}$, consisting of all $G\in\mathcal{K}$ which are independent of z. Finally, Γ is the orthogonal projector of $\mathfrak{H}$ onto $\mathfrak{D}(T)^\perp$.

Theorem. *Let T be a contraction in the Hilbert space $\mathfrak{H}$ with a closed domain $\mathfrak{D}(T)\neq\mathfrak{H}$. The formula*

$$R_z = \mathring{R}_z - z\mathring{R}_z D_* G(z)\big(I-\Gamma X(z)G(z)\big)^{-1}\Gamma D\mathring{R}_z \quad (|z|<1) \tag{2}$$

establishes a 1,1-*correspondence between the set of all generalized resolvents R_z of T and all $G\in\mathcal{K}$. The generalized resolvent R_z is canonical if and only if $G\in\mathcal{K}_0$.*

Proof. a) Let $\tilde{T}$ be a canonical c.e. of T. We define an operator F from $\mathfrak{D}(T)^\perp$ into $\mathfrak{H}$ by the formula $Fx:=\tilde{T}x$ $(x\in\mathfrak{D}(T)^\perp)$. Then we have

$$\mathring{T}\mathring{T}^*+FF^*\leqq I \quad\text{or}\quad FF^*\leqq I-\mathring{T}\mathring{T}^*=D_*^2.$$

Therefore the operator $F_1:=F^*D_*^{-1}$ is a contraction, which is densely defined on $\mathfrak{D}_*$ and with values in $\mathfrak{D}(T)^\perp$. The adjoint of its closure $G:=(\bar{F}_1)^*$ belongs to $\mathcal{K}_0$. Observing $\mathring{T}\Gamma=0$ we find with $R_z:=(z\tilde{T}-I)^{-1}$:

$$R_z-\mathring{R}_z = z\mathring{R}_z(\mathring{T}-\tilde{T})R_z = z\mathring{R}_z(\mathring{T}-\tilde{T})\Gamma R_z = -z\mathring{R}_z F\Gamma R_z. \tag{3}$$

It follows

$$R_z=(I+z\mathring{R}_z F\Gamma)^{-1}\mathring{R}_z,\ \Gamma R_z=(I+z\Gamma\mathring{R}_z F)^{-1}\Gamma\mathring{R}_z,$$

and (3) can be written as

$$R_z-\mathring{R}_z=-z\mathring{R}_z F(I+z\Gamma\mathring{R}_z F)^{-1}\Gamma\mathring{R}_z=-z\mathring{R}_z D_* G(I+z\Gamma\mathring{R}_z D_* G)^{-1}\Gamma\mathring{R}_z.$$

Furthermore,

$$\Gamma D=\Gamma,\quad \Gamma\mathring{T}^*=0 \tag{4}$$

and we get

$$\begin{aligned} R_z-\mathring{R}_z &= -z\mathring{R}_z D_* G(I+z\Gamma D\mathring{R}_z D_* G)^{-1}\Gamma\mathring{R}_z = \\ &= -z\mathring{R}_z D_* G\big(I-\Gamma(X(z)-\mathring{T}^*)G\big)^{-1}\Gamma\mathring{R}_z = \\ &= -z\mathring{R}_z D_* G\big(I-\Gamma X(z)G\big)^{-1}\Gamma\mathring{R}_z = -z\mathring{R}_z D_* G\big(I-\Gamma X(z)G\big)^{-1}\Gamma D\mathring{R}_z. \end{aligned} \tag{5}$$

b) Let now $\tilde{T}$ be an arbitrary (not necessarily canonical) c.e. of T in $\tilde{\mathfrak{H}}\supset\mathfrak{H}$, R_z the corresponding generalized resolvent. We shall prove the following statement:

(i) *If z is fixed in* $\mathbf{D}$, *then the operator* R_z^{-1} *exists and*

$$T_z := \frac{1}{z}(R_z^{-1}+I) \tag{6}$$

is a canonical c.e. of T.

Indeed, $R_z x=0$ for some $x\in\mathfrak{H}, x\neq 0$, implies $((z\tilde{T}-I)^{-1}x, x)=0$ and with $\tilde{u}:=(z\tilde{T}-I)^{-1}x$ we get

$$0=((z\tilde{T}-I)\tilde{u}, \tilde{u}) \quad \text{or} \quad \|\tilde{u}\|^2=z(\tilde{T}\tilde{u}, \tilde{u}),$$

hence $\tilde{u}=0$ as $|z|<1$ and $\|\tilde{T}\|\leqq 1$, a contradiction. In the same way it follows that the inverse of R_z^* exists, therefore the range of R_z is dense in $\mathfrak{H}$.

In order to see that T_z is a contraction we first show that the operator $S_z:=$ $=R_z^{-1}+I$ $(|z|<1)$ is a contraction, that is,

$$\|R_z^{-1}x+x\|^2\leqq\|x\|^2 \quad \text{or} \quad \|R_z^{-1}x\|^2+2\mathrm{Re}(R_z^{-1}x, x)\leqq 0 \tag{7}$$

holds for arbitrary $x\in\mathfrak{R}(R_z)$. Putting $R_z^{-1}x=y$, $(z\tilde{T}-I)^{-1}y=\tilde{v}$ we have

$$\|R_z^{-1}x\|^2+2\mathrm{Re}(R_z^{-1}x, x)=\|y\|^2+2\mathrm{Re}(y, (z\tilde{T}-I)^{-1}y)=$$
$$=\|(z\tilde{T}-I)\tilde{v}\|^2+2\mathrm{Re}((z\tilde{T}-I)\tilde{v}, \tilde{v})=\|z\tilde{T}\tilde{v}\|^2-\|\tilde{v}\|^2\leqq 0,$$

and (7) follows. Further, for an arbitrary pair $x, y\in\mathfrak{H}$, $\|x\|=\|y\|=1$, the function

$$f(z):=(S_z x, y) \quad (|z|<1)$$

is a holomorphic function of modulus $\leqq 1$, which vanishes at $z=0$. By Schwarz' lemma, $\frac{1}{z}f(z)$ is of modulus $\leqq 1$ in $\mathbf{D}$, hence also $T_z=\frac{1}{z}S_z$ is a contraction. Finally, if $x\in\mathfrak{D}(T)$ we find

$$(T_z-T)x=\frac{1}{z}R_z^{-1}(I+R_z-zR_zT)x=\frac{1}{z}R_z^{-1}\tilde{P}(z\tilde{T}-I)^{-1}(z\tilde{T}-zT)x=0,$$

therefore T_z is an extension of T. The statement (i) is proved.

Now the results of a) can be applied to the canonical c.e. T_z of T. Observing the relation $(zT_z-I)^{-1}=R_z$, the representation (5) gives

$$R_z-\mathring{R}_z=-z\mathring{R}_z D_* G(z)(I-\Gamma X(z)G(z))^{-1}\Gamma D\mathring{R}_z,$$

where $G(z):=\overline{(F_1(z))}^*$, $F_1(z):=F(z)^*D_*^{-1}$ and $F(z):=T_z|_{\mathfrak{D}(T)^\perp}$. As T_z is holomorphic in $\mathbf{D}$, the function $G(z)$ belongs to $\mathscr{K}$. Therefore, an arbitrary generalized resolvent of T admits a representation (2) with some $G\in\mathscr{K}$.

c) Let now, conversely, a function $G\in\mathscr{K}$ be given. According to [6, Chap. V, Prop. 2.1] its domain $\mathfrak{D}(T)^\perp$ and range $\mathscr{D}_*$ decompose as

$$\mathfrak{D}(T)^\perp=\mathfrak{D}'\oplus\mathscr{D}^0, \mathscr{D}_*=\mathscr{D}_*'\oplus\mathscr{D}_*^0 \quad \text{resp.,}$$

such that $G^0(z):=G(z)|_{\mathfrak{D}^0}$ is a purely contractive holomorphic function (see [6, Chap. V, 2.2], whose values are operators from $\mathfrak{D}^0$ into $\mathscr{D}_*^0$, and $G'(z):=G(z)|_{\mathfrak{D}'}$ is a unitary operator from $\mathfrak{D}'$ onto $\mathscr{D}_*'$, independent of z, $|z|<1$.

The purely contractive holomorphic function $G^0(z)$ is the characteristic function of some contraction S in a Hilbert space $\mathfrak{H}_1$, that is, $\mathfrak{D}^0$ and $\mathscr{D}_*^0$ can be identified with the subspaces $\mathscr{D}_s=\overline{\mathfrak{R}(D_s)}$ and $\mathscr{D}_{s*}=\overline{\mathfrak{R}(D_{s*})}$ resp. of $\mathfrak{H}_1$, and we have

$$G^0(z) = \big(-S-zD_{s*}(zS^*-I)^{-1}D_S\big)|_{\mathscr{D}_s} \quad (|z|<1).$$

Thus, $\mathfrak{D}^0$ and $\mathscr{D}_*^0$ can be considered as subspaces of $\mathfrak{H}$ as well as of $\mathfrak{H}_1$. Besides Γ, projecting $\mathfrak{H}$ orthogonally onto $\mathfrak{D}(T)^\perp$, we introduce the orthogonal projectors $\Gamma^0, \Gamma', \Gamma_*^0$ and Γ_*' in $\mathfrak{H}$ onto $\mathfrak{D}^0, \mathfrak{D}', \mathscr{D}_*^0$ and $\mathscr{D}_*'$ respectively and the orthogonal projectors P and P_* onto $\mathscr{D}_s$ and $\mathscr{D}_{S*}$ in $\mathfrak{H}_1$.

Now an extension $\tilde{T}$ of T, acting in the space $\mathfrak{H}\oplus\mathfrak{H}_1$, will be defined as follows: With respect to the decomposition

$$\mathfrak{H}\oplus\mathfrak{H}_1 = \mathfrak{D}(T)\oplus\mathfrak{D}'\oplus\mathfrak{D}^0\oplus\mathfrak{H}_1$$

of the initial space it has the matrix representation

$$\tilde{T} = \begin{pmatrix} \mathring{T}(1-\Gamma) & D_*\Gamma_*'G' & -D_*P_*S & D_*\Gamma_*^0D_{S*} \\ 0 & 0 & D_S & S^* \end{pmatrix}. \tag{8}$$

Clearly, $\tilde{T}$ is an extension of T. In order to see that $\tilde{T}$ is contractive we consider the operator $\tilde{T}\tilde{T}^*=(\tau_{ij})_{i,j=1,2}$ in $\mathfrak{H}\oplus\mathfrak{H}_1$. Observing

$$\tilde{T}^* = \begin{pmatrix} (1-\Gamma)\mathring{T}^* & 0 \\ G'^*\Gamma_*'D_* & 0 \\ -PS^*\Gamma_*^0D_* & D_S \\ D_{S*}\Gamma_*^0D_* & S \end{pmatrix}$$

and the fact that G'^* maps $\mathscr{D}_*'$ unitarily onto $\mathfrak{D}'$: $G'G_*'=I|_{\mathfrak{D}_*'}$, we find

$$\begin{aligned}
\tau_{11} &= \mathring{T}(I-\Gamma)\mathring{T}^*+D_*\Gamma_*'D_*+D_*P_*SPS^*\Gamma_*^0D_*+D_*\Gamma_*^0D_{S*}^2\Gamma_*^0D_* \leqq \\
&\leqq \mathring{T}(I-\Gamma)\mathring{T}^*+D_*\Gamma_*'D_*+D_*P_*SS^*\Gamma_*^0D_*+D_*\Gamma_*^0D_{S*}^2\Gamma_*^0D_* = \\
&= \mathring{T}(I-\Gamma)\mathring{T}^*+D_*\Gamma_*'D_*+D_*\Gamma_*^0D_* \leqq \mathring{T}\mathring{T}^*+D_*^2 = I, \\
\tau_{12} &= -D_*P_*SD_S+D_*\Gamma_*^0D_{S*}S = D_*P_*(-SD_S+D_{S*}S) = 0, \\
\tau_{22} &= D_S^2+S^*S = I.
\end{aligned}$$

Therefore, $\tilde{T}$ is a c.e. of T. Next we have to calculate the generalized resolvent of T, generated by $\tilde{T}$. In order to do this we observe the following proposition, whose simple proof will be left to the reader.

(ii) *If the c.e.* $\tilde{T}$ *of* T, *acting in* $\tilde{\mathfrak{H}}=\mathfrak{H}\oplus\mathfrak{H}_1$ *has the matrix form*

$$\tilde{T}=\begin{pmatrix}\hat{T} & C\\ B & A\end{pmatrix},$$

then we have

$$\tilde{P}(z\tilde{T}-I)^{-1}|_{\mathfrak{H}}=\left(z\hat{T}-I-z^2C(zA-I)^{-1}B\right)^{-1}.$$

We apply this proposition to the operator $\tilde{T}$ in (8). With respect to the decomposition $\mathfrak{H}\oplus\mathfrak{H}_1$ of initial and range space $\tilde{T}$ can be written as

$$\tilde{T}=\begin{pmatrix}\mathring{T}+D_*\Gamma'_*G'\Gamma'-D_*P_*S\Gamma^0 & D_*\Gamma^0_*D_{S^*}\\ D_S\Gamma^0 & S^*\end{pmatrix}$$

and we get for the corresponding generalized resolvent

(9) $$R_z=\left(z\mathring{T}-I+zD_*\Gamma'_*G'\Gamma'-zD_*P_*S\Gamma^0-z^2D_*\Gamma^0_*D_{S^*}(zS^*-I)^{-1}D_S\Gamma^0\right)^{-1}=$$
$$=\left(z\mathring{T}-I+zD_*(\Gamma'_*G'\Gamma'-P_*S\Gamma^0-z\Gamma^0_*D_{S^*}(zS^*-I)^{-1}D_{S^*}\Gamma^0\right)^{-1}=$$
$$=\left(z\mathring{T}-I+zD_*(\Gamma'_*G'\Gamma'+\Gamma^0_*G^0(z)\Gamma^0)\right)^{-1}=\left(z\mathring{T}-I+zD_*G(z)\Gamma\right)^{-1}=$$
$$=\mathring{R}_z\left(I+zD_*G(z)\Gamma\mathring{R}_z\right)^{-1}=\mathring{R}_z-z\mathring{R}_zD_*G(z)\left(I-\Gamma X(z)G(z)\right)^{-1}\Gamma\mathring{R}_z.$$

The last equality follows easily if one observes the relation (4) and

$$\Gamma X(x)G(z)=-z\Gamma D\mathring{R}_zD_*G(z).$$

As the formula (2) can be written as

$$R_z=\left(z\mathring{T}-I+zD_*G(z)\Gamma\right)^{-1}$$

(see (9)), the correspondence between the generalized resolvents R_z of T and the functions $G\in\mathcal{K}$ is bijective.

d) It remains to prove the last statement of the theorem. If $G_0\in\mathcal{K}_0$ is given we consider the operator $\tilde{T}$:

(10) $$\tilde{T}x:=\begin{cases}Tx & x\in\mathfrak{D}(T)\\ D_*G_0x & x\in\mathfrak{D}(T)^\perp.\end{cases}$$

Then $\tilde{T}$ is an extension of T and, moreover, we have

$$\tilde{T}\tilde{T}^*=\mathring{T}\Gamma\mathring{T}^*+D_*G_0G_0^*D\leqq\mathring{T}\mathring{T}^*+D_*^2=I,$$

hence $\tilde{T}$ is a c.e. of T. With the notation of part a) of the proof we find

$$F=D_*G_0,\quad F_1=G_0^*\quad\text{and}\quad G=G_0.$$

That is, the generalized resolvent of T, generated by $\tilde{T}$ from (10), is given by (2) with $G=G_0$. The theorem is proved.

Remark 1. In the case of an isometric operator T, the unitary extension $\tilde{T}$, generating a given generalized resolvent of T, is uniquely determined (up to isomorphisms) if some minimality condition is imposed on $\tilde{T}$. This is not true in the situation considered here. E.g., if $G\in\mathscr{K}_0$ and G is not a unitary constant, in the proof of the Theorem (parts c) and d)) two different extensions of T, which generate the same generalized resolvent, have been given.

Remark 2. With the notation in the proof of the theorem, the operator

$$T' := (\mathring{T}+D_* G'\Gamma')|_{\mathfrak{D}(T)\oplus\mathfrak{D}},$$

is a contraction in $\mathfrak{H}$ which extends T. Evidently, the operator $\tilde{T}$ in (8) is a c.e. of T'. Thus the generalized resolvent R_z of T in (9) is also a generalized resolvent of T'. That is, there exists a function $H\in\mathscr{K}(\mathfrak{D}^0, \mathscr{D}_{1,*})$, such that we have

$$R_z = \mathring{R}_{1,z} - z\mathring{R}_{1,z} D_{1,*} H(z)\big(I-X_1(z)H(z)\big)^{-1}\Gamma^0 \mathring{R}_{1,z}$$

$\big(\mathring{R}_{1,z} := (z\mathring{T}'-I)^{-1}, D_{1,*} := (I-\mathring{T}'(\mathring{T}')^*)^{1/2}, \mathscr{D}_{1,*} := \overline{\mathscr{R}(D_{1,*})}$ and X_1 is the corresponding characteristic function$\big)$. It is not hard to see that the functions G^0 and H are connected by the relation

$$D_{1,*}H(z) = D_* G^0(z) \quad (|z|<1).$$

We mention that the construction of the operator $\tilde{T}$ in part c) of the proof can also be used if G decomposes as

$$G(z) = G_1 \oplus G_2(z) \quad (|z|<1)$$

where $G_1\in\mathscr{K}_0(\mathfrak{D}_1, \mathscr{D}_{*,1})$, $G_2\in\mathscr{K}(\mathfrak{D}_2, \mathscr{D}_{*,2})$ $(\mathfrak{D}(T)^\perp = \mathfrak{D}_1\oplus\mathfrak{D}_2, \mathscr{D}=\mathscr{D}_{*,1}\oplus\mathscr{D}_{*,2})$ and G_2 is purely contractive. Then the above remark holds also true for the operator

$$T_1 := (\mathring{T}+D_* G_1\Gamma_1)|_{\mathfrak{D}(T)\oplus\mathfrak{D}_1}$$

(Γ_1 orthogonal projector onto $\mathfrak{D}_1$) instead of T'.

Remark 3. If $\tilde{T}$ is a noncanonical c.e. of T in some Hilbert space $\tilde{\mathfrak{H}}\supset\mathfrak{H}$, and if

$$\tilde{T} = \begin{pmatrix} \hat{T} & C \\ B & A \end{pmatrix}$$

with respect to the decomposition $\tilde{\mathfrak{H}} = \mathfrak{H}\oplus\mathfrak{L}$, then the operator function T_z in (6) can be written as

$$T_z = \hat{T} + zC(I-zA)^{-1}B.$$

Hence $T_{1/z}$ is the transfer function of the node $(A, B, C, \hat{T}, \mathfrak{L}, \mathfrak{H})$ in the sense of [7].

An analogue of the theorem above can be formulated for a dissipative operator. In this form it has applications to the spectral theory of canonical differential operators, which will be considered elsewhere.

Added in proof. An extension of the Theorem to dual pairs of contractions will appear in: Proceedings of the 6^{th} Conference on Operator Theory in Timisoara and Herculane, 1981, Birkhäuser (Basel—Boston—Stuttgart, 1982).

References

[1] M. G. Krein and H. Langer, Über die verallgemeinerten Resolventen und die charakteristische Funktion eines isometrischen Operators im Raume Π_κ, *Colloquia Mathematica Societatis János Bolyai*, **5**: *Hilbert space operators and operator algebras* (Tihany 1970), North Holland (Amsterdam—London, 1972), 353—399.

[2] H. Langer, Generalized coresolvents of a π-isometric operator with unequal defect numbers, *Funkcional. Anal. i Priložen.*, **5**; **4** (1971), 73—75 [Russian; English translation in *Functional Anal. Appl.*, 5 (1971/72), 329—331].

[3] H. Langer and P. Sorjonen, Verallgemeinerte Resolventen Hermitescher und isometrischer Operatoren im Pontrjaginraum, *Ann. Acad. Sci. Fenn.*, Ser. A I Math., **561** (1974), 3—45.

[4] M. G. Krein, On the resolvents of an hermitian operator with defect index (m, m), *Dokl. Akad. Nauk SSSR*, **52**, no. 8 (1946), 657—660. [Russian]

[5] N. I. Ahiezer and I. M. Glazman, *The theory of linear operators in Hilbert space*, Ungar (New York, 1961).

[6] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators in Hilbert space*, North Holland (Amsterdam—London, 1970).

[7] H. Bart, I. Gohberg and M. A. Kaashoek, *Minimal factorization of matrix and operator functions*, Birkhäuser (Basel—Boston—Stuttgart, 1979).

(H. L.)
SEKTION MATHEMATIK
TECHNISCHE UNIVERSITÄT
MOMMSENSTRASSE 13
DDR—8027 DRESDEN

(B. T.)
DEPARTMENT OF MATHEMATICS
LINKÖPING UNIVERSITY
S—58183 LINKÖPING

Acta Sci. Math., 44 (1982), 133—135

A note on hereditary radicals

EDMUND R. PUCZYŁOWSKI

All rings in this paper are associative. Fundamental definitions and properties of radicals may be found in [4]. It is known ([3]) that to any radical S there exist a unique maximal hereditary radical h_S and a unique maximal left hereditary radical lh_S contained in S. Of course $h_S \subseteqq \bar{S} = \{A|$ any ideal of A is in $S\}$ and $lh_S \subseteqq \tilde{S} = \{A|$ any left ideal of A is in $S\}$. It is easy to see that $\bar{S}$ and $\tilde{S}$ are radicals and S is hereditary (left hereditary) if and only if $S = \bar{S}$ $(S = \tilde{S})$. The radicals $\bar{S}$ and $\tilde{S}$ were introduced in [2] to investigate hereditariness of strong and similar radicals. Obviously $h_S \subseteqq \bar{\bar{S}} \subseteqq \bar{S}$ and $lh_S \subseteqq \tilde{\tilde{S}} \subseteqq \tilde{S}$. In the note we prove that $h_S = \bar{\bar{S}}$ and $lh_S = \tilde{\tilde{S}}$, and that there exists a radical S such that $h_S \neq \bar{S}$ and $lh_S \neq \tilde{S}$.

To denote that I is an ideal (left ideal) of a ring A we will write $I \triangleleft A$ $(I < A)$.

Lemma 1. *If A is an S-radical ring and for some integer n, $A^{n+2} = 0$ then $A^{n+1} \in S$.*

Proof. It is easy to see that for any $a \in A^n$ the mapping $f_a : A \to A^{n+1}$ defined by $f_a(x) = ax$ is a ring homomorphism. But $f_a(A) = aA \triangleleft A^{n+1}$. Thus $aA \in S$ and $A^{n+1} = \sum_{a \in A^n} aA \in S$.

Proposition 1. *If S is a radical such that any zero-S-ring is in $\bar{S}$ then $\bar{S} = h_S$.*

Proof. Let $J \triangleleft I \triangleleft A$. If J^* is the ideal of A generated by J and $A \in \bar{S}$ then J^* and $(J^*)^3$ are in S. Thus by Lemma 1 $(J^*)^2 \in S$. Now the assumption implies $J + (J^*)^2 \in S$. Since, by Andrunakievich lemma, $(J^*)^3 \subseteqq J$, similarly, we obtain that $J \cap (J^*)^2 \in S$. This and the fact that $(J + (J^*)^2)/(J^*)^2 \approx J/((J^*)^2 \cap J)$ implies $J \in S$. Thus if $A \in \bar{S}$ then $A \in \bar{\bar{S}}$, so $\bar{\bar{S}} = \bar{S} = h_S$.

Of course for any radical S the radical $\bar{S}$ satisfies the assumption of Proposition 1, so we have

Received July 22, 1980, and in revised form, May 5, 1981.

Corollary 1. *For any radical* S, $h_S=\bar{\bar{S}}$.

Remark. It is easy to check ([1]) that for any radical S, $h_S=aS=\{A|$ every accessible subring of A is in $S\}$. Thus, by Corollary 1, $aS=\bar{\bar{S}}$ for any radical S.

Now we prove

Proposition 2. *For any radical* S, $lh_S=\tilde{\tilde{S}}$.

Proof. If $K<L<A$ then $LK<A$ and $LK\lhd K$. Thus if $A\in\tilde{\tilde{S}}$ then $I=LK\in\tilde{S}$. Now if $R<K$ then $R+I<L$ and, since $A\in\tilde{\tilde{S}}$, $R+I\in S$. Also $R\cap I\in S$ as $R\cap I<I$ and $I\in\tilde{S}$. These and the fact that $(R+I)/I\approx R/(R\cap I)$ imply $R\in S$. Hence if $A\in\tilde{\tilde{S}}$ then $L\in\tilde{\tilde{S}}$, so the radical $\tilde{\tilde{S}}$ is left hereditary. This and the fact that $lh_S\subseteqq\tilde{\tilde{S}}$ ends the proof.

Example. Let Q be the field of rational numbers, Z the ring of integers and R the ring of all 2×2-matrices of the form $\begin{pmatrix}a, b\\ 0, 0\end{pmatrix}$ where $a, b\in Q$. Then $I=\left\{\begin{pmatrix}0, b\\ 0, 0\end{pmatrix}\middle| b\in Q\right\}$ is an ideal of R and $J=\left\{\begin{pmatrix}0, z\\ 0, 0\end{pmatrix}\middle| z\in Z\right\}$ an ideal of I. Let S be the lower radical determined by $\{R, I\}$. Since R and I are divisible rings, all S-rings are divisible. Thus $J\notin S$ and $R\notin\bar{S}$. Since $R\in\tilde{S}'$ and $\tilde{S}\subseteqq\bar{S}$ therefore $\tilde{S}\neq\tilde{\tilde{S}}$ and $\bar{S}\neq\bar{\bar{S}}$.

The above example shows that generally $lh_S\neq\tilde{S}$. In the following proposition we will describe some radicals for which $lh_S=\tilde{S}$.

Proposition 3. *For a radical* S *we have* $lh_S=\tilde{S}$, *provided* a) S *contains all zero-rings, or* b) $L<A$ *and* $A\in S$ *imply* $L=AL$.

Proof. Let $A\in\tilde{S}$ and $K<L<A$. Since $LK<A$, we have $LK\in S$. But $LK\lhd K$ and $(K/LK)^2=0$, so if S satisfies a) then $K\in S$. If S satisfies b) then $K=LK\in S$ as $K<L$ and $L\in S$. Thus in both cases $K\in S$. In consequence $L\in\tilde{S}$ and $\tilde{S}$ is left hereditary. Hence $lh_S=\tilde{S}$.

Now we will show that there exist non-hereditary radicals satisfying condition b) of the proposition above. Let us define for any class M of rings the class $M_1=\{A\in M|$ if $L<A$ then $AL=L\}$. We have

Proposition 4. *If a class* S *is radical then so is* S_1.

Proof. Certainly the class S_1 is homomorphically closed and any ring which is the sum of a chain of S_1-ideals is in S_1. So it suffices to prove that if $I\lhd A$ and $I, A/I$ are in S_1 then A is in S_1. Let $L<A$. Then $I(L\cap I)=L\cap I$. Also $(A/I)((L+I)/I)=(L+I)/I$, so $AL+I=L+I$. Thus if $l\in L$ then $l=m+i$ for some

$m \in AL$, $i \in I$. But since $AL \subseteqq L$, $i \in L \cap I$. Thus the equality $I(L \cap I) = L \cap I$ implies $i \in AL$ and $l \in AL$. Hence $L = AL$ and the result follows.

Corollary 2. *Let S be the lower radical determined by a class M. If $M = M_1$ then $S = S_1$.*

Proof. Since $M \subseteqq S$ therefore $M = M_1 \subseteqq S_1$. Now by Proposition 4, S_1 is a radical class containing M, so $S \subseteqq S_1$.

Let $M = \{R\}$, where R is the ring of the Example. Then $M = M_1$ and by Corollary 2 the lower radical S determined by M satisfies condition b) of Proposition 3. It is easy to see that any non-zero S-ring contains a non-zero idempotent element. Thus S is not hereditary as R contains a non-zero nilpotent ideal.

References

[1] L. C. A. van Leeuwen and R. Wiegandt, Constructions and compositions of radical and semisimple classes, *Ann. Univ. Sci. Budapest,* **21** (1978), 65—75.

[2] E. R. Puczyłowski, Hereditariness of strong and stable radicals, *Glasgow Math. J.,* **23** (1982), 85—90.

[3] F. Szász and R. Wiegandt, On hereditary radicals, *Period. Math. Hungar.,* **3** (1973), 235—241.

[4] R. Wiegandt, Radical and semisimple classes of rings, *Queen's Papers in Pure and Applied Mathematics,* No. 37 (Kingston, Ontario, 1974).

INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY, PKiN
00—901 WARSAW, POLAND

Acta Sci. Math., 44 (1982), 137—149

Approximate decompositions of certain contractions

PEI YUAN WU

In this paper we obtain an approximate decomposition for contractions the outer factors of whose characteristic functions admit scalar multiples. We show that such a contraction is quasi-similar to the direct sum of its $C_{\cdot 1}$ and $C_{\cdot 0}$ parts. This class of operators includes, among other things, weak contractions and $C_{1\cdot}$ contractions with at least one defect index finite. In particular, our result generalizes the C_0-C_{11} decomposition for weak contractions. Applying this to $C_{1\cdot}$ contractions, we obtain that any $C_{1\cdot}$ contraction with at least one defect index finite is completely injection-similar to an isometry. As consequences, we are able to characterize, among $C_{1\cdot}$ contractions, those which are cyclic, have commutative commutants or satisfy the double commutant property.

In Section 1 below we first fix the notation and review some basic facts needed in the subsequent discussions. Then in Section 2 we prove the approximate decomposition and some of its consequences. In Section 3 we restrict ourselves to $C_{1\cdot}$ contractions.

1. Preliminaries. In this paper all the operators are acting on complex, separable Hilbert spaces. We will use extensively the contraction theory of SZ.-NAGY and FOIAŞ. The main reference is their book [8].

Let T be a contraction on the Hilbert space H. Denote by $\mathfrak{D}_T=\overline{\operatorname{ran}(I-T^*T)^{1/2}}$ and $\mathfrak{D}_{T*}=\overline{\operatorname{ran}(I-TT^*)^{1/2}}$ the *defect spaces* and $d_T=\operatorname{rank}(I-T^*T)^{1/2}$ and $d_{T*}=\operatorname{rank}(I-TT^*)^{1/2}$ the *defect indices* of T. T is *completely non-unitary (c.n.u.)* if there exists no non-trivial reducing subspace on which T is unitary. T is *of class* $C_{1\cdot}$ (resp. $C_{\cdot 1}$) if $T^n x\not\to 0$ (resp. $T^{*n}x\not\to 0$) for any $x\neq 0$; T is *of class* $C_{0\cdot}$ (resp. $C_{\cdot 0}$) if $T^n x\to 0$ (resp. $T^{*n}x\to 0$) for any x. $C_{\alpha\beta}=C_{\alpha\cdot}\cap C_{\cdot\beta}$ for $\alpha,\beta=0,1$. Let $T=\begin{bmatrix} T_1 & X \\ 0 & T_2\end{bmatrix}$ be the canonical triangulation of type $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0}\end{bmatrix}$ on $H=H_1\oplus H_2$. If T is c.n.u., then this triangulation corresponds to the canonical factorization $\Theta_T=$

Received February 25, 1981.

This research was partially supported by National Science Council of Taiwan, China.

$=\Theta_2\Theta_1$ of the characteristic function $\{\mathfrak{D}_T, \mathfrak{D}_{T*}, \Theta_T(\lambda)\}$ of T, where $\{\mathfrak{D}_T, \mathfrak{F}, \Theta_1(\lambda)\}$ and $\{\mathfrak{F}, \mathfrak{D}_{T*}, \Theta_2(\lambda)\}$ are the outer and inner factors of Θ_T, respectively. Moreover, the characteristic functions of T_1 and T_2 are the purely contractive parts of Θ_1 and Θ_2, respectively. For c.n.u. T, we will consider its *functional model*, that is, consider T being defined on the space $H=[H^2(\mathfrak{D}_{T*})\oplus\overline{\Delta_T L^2(\mathfrak{D}_T)}]\ominus\{\Theta_T w\oplus\Delta_T w: w\in H^2(\mathfrak{D}_T)\}$ by $T(f\oplus g)=P(e^{it}f\oplus e^{it}g)$, where $\Delta_T=(I-\Theta_T^*\Theta_T)^{1/2}$ and P denotes the (orthogonal) projection onto H. Then H_1 and H_2 can be represented as

$$H_1=\{\Theta_2 u\oplus v: u\in H^2(\mathfrak{F}), v\in\overline{\Delta_T L^2(\mathfrak{D}_T)}\}\ominus\{\Theta_T w\oplus\Delta_T w: w\in H^2(\mathfrak{D}_T)\}$$

and

$$H_2=[H^2(\mathfrak{D}_{T*})\ominus\Theta_2 H^2(\mathfrak{F})]\oplus\{0\}.$$

A contractive analytic function $\{\mathfrak{D}, \mathfrak{D}_*, \Theta(\lambda)\}$ is said to admit the *scalar multiple* $\delta(\lambda)$ if $\delta(\lambda)\not\equiv 0$ is a scalar-valued analytic function and there exists a contractive analytic function $\{\mathfrak{D}_*, \mathfrak{D}, \Omega(\lambda)\}$ such that $\Omega(\lambda)\Theta(\lambda)=\delta(\lambda)I_{\mathfrak{D}}$ and $\Theta(\lambda)\Omega(\lambda)=\delta(\lambda)I_{\mathfrak{D}_*}$ for all λ in $D=\{\lambda: |\lambda|<1\}$.

For an arbitrary operator T on H, let $\{T\}'$, $\{T\}''$ and Alg T denote its commutant, double commutant and the weakly closed algebra generated by T and I. Let Lat T, Lat $''T$ and Hyperlat T denote the lattices of invariant subspaces, bi-invariant subspaces and hyperinvariant subspaces of T, respectively. Let μ_T denote the *multiplicity* of T, that is, the least cardinal number of a subset K of H for which $H=\bigvee_{n\geqq 0} T^n K$. T is *cyclic* if $\mu_T=1$. For operators T_1 and T_2 on H_1 and H_2, respectively, $T_1\overset{i}{\prec}T_2$ (resp. $T_1\prec T_2$) denotes that there exists an injection $X: H_1\to H_2$ (resp. an injection $X: H_1\to H_2$ with dense range, called *quasi-affinity*) such that $T_2X=XT_1$. $T_1\overset{ci}{\prec}T_2$ denotes that there exists a family $\{X_\alpha\}$ of injections $X_\alpha: H_1\to H_2$ such that $H_2=\bigvee_\alpha X_\alpha H_1$ and $T_2X_\alpha=X_\alpha T_1$ for each α. T_1 and T_2 are *quasi-similar* ($T_1\sim T_2$) if $T_1\prec T_2$ and $T_2\prec T_1$; they are *injection-similar* ($T_1\overset{i}{\sim}T_2$) if $T_1\overset{i}{\prec}T_2$ and $T_2\overset{i}{\prec}T_1$; they are *completely injection-similar* ($T_1\overset{ci}{\sim}T_2$) if $T_1\overset{ci}{\prec}T_2$ and $T_2\overset{ci}{\prec}T_1$. Note that $T_1\prec T_2$ implies that $\mu_{T_1}\geqq\mu_{T_2}$.

2. Approximate decomposition. We start with the following major result.

Theorem 2.1. *Let T be a contraction on H and let $T=\begin{bmatrix}T_1 & X\\ 0 & T_2\end{bmatrix}$ on $H=H_1\oplus H_2$ be the canonical triangulation of type $\begin{bmatrix}C_{\cdot 1} & *\\ 0 & C_{\cdot 0}\end{bmatrix}$. If the characteristic function of T_1 admits a scalar multiple, then $T\sim T_1\oplus T_2$. Moreover, if T is c.n.u., then there exist quasi-affinities $Y: H\to H_1\oplus H_2$ and $Z: H_1\oplus H_2\to H$ which intertwine T and $T_1\oplus T_2$ and such that $YZ=\delta(T_1\oplus T_2)$ and $ZY=\delta(T)$ for some outer function δ.*

Proof. Let $T=U\oplus T'$ be decomposed as the direct sum of a unitary operator U and a c.n.u. contraction T'. Let $T'=\begin{bmatrix} T_1' & * \\ 0 & T_2' \end{bmatrix}$ be of type $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$. Then

$$T=\begin{bmatrix} U & 0 & 0 \\ 0 & T_1' & * \\ 0 & 0 & T_2' \end{bmatrix},$$

where $\begin{bmatrix} U & 0 \\ 0 & T_1' \end{bmatrix}$ is of class $C_{\cdot 1}$ and T_2' is of class $C_{\cdot 0}$. Hence by the uniqueness of the canonical triangulation, we have $T_1=U\oplus T_1'$ and $T_2=T_2'$ (cf. [8], p. 73). Note that the characteristic functions of T_1 and T_1' coincide. Therefore the characteristic function of T_1' also admits a scalar multiple. If we can show that $T'\sim T_1'\oplus T_2'$, then $T==U\oplus T'\sim U\oplus T_1'\oplus T_2'=T_1\oplus T_2$. Hence without loss of generality, we may assume that T is c.n.u. As remarked before, we can consider the functional model of T. Let δ be an outer scalar multiple of Θ_1 (cf. [8], p. 217) and let $\{\mathfrak{F}, \mathfrak{D}_T, \Omega(\lambda)\}$ be a contractive analytic function such that $\Omega\Theta_1=\delta I_{\mathfrak{D}_T}$ and $\Theta_1\Omega=\delta I_{\mathfrak{F}}$. Define the operator S: $H_2\to H_1$ by $S(u\oplus 0)=P(0\oplus(-\Delta_T\Omega\Theta_2^*u))$ for $u\oplus 0\in H_2$. Note that $0\oplus(-\Delta_T\Omega\Theta_2^*u)$ is orthogonal to H_2 and therefore $P(0\oplus(-\Delta_T\Omega\Theta_2^*u))$ is indeed in H_1.

We first check that $T_1S-ST_2=\delta(T_1)X$. Note that for $u\oplus 0\in H_2$, we have

$$\begin{aligned} T_2(u\oplus 0) &= (e^{it}u\oplus 0)-(\Theta_T w\oplus\Delta_T w)-(\Theta_2 u'\oplus v') = \\ &= (e^{it}u-\Theta_T w-\Theta_2 u')\oplus(-\Delta_T w-v') =(e^{it}u-\Theta_T w-\Theta_2 u')\oplus 0 \end{aligned}$$

for some $w\in H^2(\mathfrak{D}_T)$ and $\Theta_2u'\oplus v'\in H_1$, where the last equality follows from the fact that $T_2(u\oplus 0)\in H_2$. Moreover, $X(u\oplus 0)=\Theta_2u'\oplus v'$. Hence

$$\begin{aligned} &(T_1S-ST_2)(u\oplus 0) = \\ &= T_1P(0\oplus(-\Delta_T\Omega\Theta_2^*u))-S((e^{it}u-\Theta_T w-\Theta_2u')\oplus 0) = \\ &= P(0\oplus(-e^{it}\Delta_T\Omega\Theta_2^*u))-P(0\oplus(-\Delta_T\Omega\Theta_2^*(e^{it}u-\Theta_T w-\Theta_2u'))) = \\ &= P(0\oplus(-\Delta_T\Omega\Theta_2^*\Theta_T w-\Delta_T\Omega\Theta_2^*\Theta_2u')) = P(0\oplus(-\Delta_T\delta w-\Delta_T\Omega u')). \end{aligned}$$

On the other hand,

$$\begin{aligned} \delta(T_1)X(u\oplus 0) &= \delta(T_1)(\Theta_2u'\oplus v') = P(\delta\Theta_2u'\oplus\delta v') = P(\Theta_T\Omega u'\oplus\delta v') = \\ &= P(0\oplus(\delta v'-\Delta_T\Omega u')). \end{aligned}$$

Since $-\Delta_T w-v'=0$, we obtain that $T_1S-ST_2=\delta(T_1)X$ as asserted.

Let $Y=\begin{bmatrix} \delta(T_1) & S \\ 0 & I \end{bmatrix}: H\to H_1\oplus H_2$ and $Z=\begin{bmatrix} I & V-S \\ 0 & \delta(T_2) \end{bmatrix}: H_1\oplus H_2\to H$, where V is the operator which appears in the triangulation of $\delta(T)$ with respect to $H_1\oplus H_2$:

$\delta(T)=\begin{bmatrix}\delta(T_1) & V\\ 0 & \delta(T_2)\end{bmatrix}$. We complete the proof in several steps. In each step the first statement is proved.

(i) $YT=(T_1\oplus T_2)Y$.

$$YT=\begin{bmatrix}\delta(T_1) & S\\ 0 & I\end{bmatrix}\begin{bmatrix}T_1 & X\\ 0 & T_2\end{bmatrix}=\begin{bmatrix}\delta(T_1)T_1 & \delta(T_1)X+ST_2\\ 0 & T_2\end{bmatrix}=$$

$$=\begin{bmatrix}T_1\delta(T_1) & T_1S\\ 0 & T_2\end{bmatrix}=\begin{bmatrix}T_1 & 0\\ 0 & T_2\end{bmatrix}\begin{bmatrix}\delta(T_1) & S\\ 0 & I\end{bmatrix}=(T_1\oplus T_2)Y.$$

(ii) $Z(T_1\oplus T_2)=TZ$. Since

$$\delta(T)T=\begin{bmatrix}\delta(T_1) & V\\ 0 & \delta(T_2)\end{bmatrix}\begin{bmatrix}T_1 & X\\ 0 & T_2\end{bmatrix}=\begin{bmatrix}\delta(T_1)T_1 & \delta(T_1)X+VT_2\\ 0 & \delta(T_2)T_2\end{bmatrix}=$$

$$=T\delta(T)=\begin{bmatrix}T_1 & X\\ 0 & T_2\end{bmatrix}\begin{bmatrix}\delta(T_1) & V\\ 0 & \delta(T_2)\end{bmatrix}=\begin{bmatrix}T_1\delta(T_1) & T_1V+X\delta(T_2)\\ 0 & T_2\delta(T_2)\end{bmatrix},$$

we have $\delta(T_1)X+VT_2=T_1V+X\delta(T_2)$. From $T_1S-ST_2=\delta(T_1)X$ we obtain that $T_1S-ST_2+VT_2=T_1V+X\delta(T_2)$. A simple computation using this relation shows that $Z(T_1\oplus T_2)=TZ$.

(iii) $ZY=\delta(T)$.

$$ZY=\begin{bmatrix}I & V-S\\ 0 & \delta(T_2)\end{bmatrix}\begin{bmatrix}\delta(T_1) & S\\ 0 & I\end{bmatrix}=\begin{bmatrix}\delta(T_1) & S+V-S\\ 0 & \delta(T_2)\end{bmatrix}=\delta(T).$$

(iv) $YZ=\delta(T_1\oplus T_2)$. Since

$$YZ=\begin{bmatrix}\delta(T_1) & S\\ 0 & I\end{bmatrix}\begin{bmatrix}I & V-S\\ 0 & \delta(T_2)\end{bmatrix}=\begin{bmatrix}\delta(T_1) & \delta(T_1)(V-S)+S\delta(T_2)\\ 0 & \delta(T_2)\end{bmatrix},$$

to complete the proof, it suffices to show that $\delta(T_1)(V-S)+S\delta(T_2)=0$. Note that $YT=(T_1\oplus T_2)Y$ implies that $Y\delta(T)=\delta(T_1\oplus T_2)Y$. But

$$Y\delta(T)=\begin{bmatrix}\delta(T_1) & S\\ 0 & I\end{bmatrix}\begin{bmatrix}\delta(T_1) & V\\ 0 & \delta(T_2)\end{bmatrix}=\begin{bmatrix}\delta(T_1)^2 & \delta(T_1)V+S\delta(T_2)\\ 0 & \delta(T_2)\end{bmatrix}$$

and

$$\delta(T_1\oplus T_2)Y=\begin{bmatrix}\delta(T_1) & 0\\ 0 & \delta(T_2)\end{bmatrix}\begin{bmatrix}\delta(T_1) & S\\ 0 & I\end{bmatrix}=\begin{bmatrix}\delta(T_1)^2 & \delta(T_1)S\\ 0 & \delta(T_2)\end{bmatrix}.$$

We conclude that $\delta(T_1)V+S\delta(T_2)=\delta(T_1)S$ as asserted.

(v) Y and Z are quasi-affinities. Since δ is outer, $\delta(T_1)$ and $\delta(T_2)$ are quasi-affinities (cf. [8], p. 118). It can be easily checked that Y and Z are also quasi-affinities.

It is interesting to contrast the preceding result with [14], Theorem 1, where the problem when T is similar to $T_1\oplus T_2$ was considered. Here we make a weaker assumption to obtain a (necessarily) weaker conclusion. Indeed, the intertwining operators Y and Z constructed here are closely related to the invertible intertwining operator appearing in the proof of [14], Theorem 1.

Corollary 2.2. *Let* $T=\begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$ *be as in Theorem 2.1. Assume that* T *is c.n.u. Then* Lat $T \cong$ Lat $(T_1 \oplus T_2)$, Lat$''T \cong$ Lat$''(T_1 \oplus T_2)$ *and* Hyperlat $T \cong$ Hyperlat $(T_1 \oplus T_2)$.

Proof. Let Y and Z be the operators constructed in the proof of Theorem 2.1. For $K \in$ Lat T and $L \in$ Lat $(T_1 \oplus T_2)$, consider the mappings $K \to \overline{YK}$ and $L \to \overline{ZL}$. It is easily checked that they are inverses to each other and preserve the lattice operations. Hence Lat $T \cong$ Lat $(T_1 \oplus T_2)$. To complete the proof, it suffices to show that (i) $K \in$ Lat$''T$ implies that $\overline{YK} \in$ Lat$''(T_1 \oplus T_2)$ and (ii) $K \in$ Hyperlat T implies that $\overline{YK} \in$ Hyperlat $(T_1 \oplus T_2)$. Then by a symmetric argument we also obtain that $L \in$ Lat$''(T_1 \oplus T_2)$ and $L \in$ Hyperlat $(T_1 \oplus T_2)$ imply that $\overline{ZL} \in$ Lat$''T$ and $\overline{ZL} \in$ Hyperlat T, respectively.

To prove (i), let $S \in \{T_1 \oplus T_2\}''$. We first check that $ZSY \in \{T\}''$. Indeed, $YVZ \in \{T_1 \oplus T_2\}'$ for any $V \in \{T\}'$. Hence $ZSYVZ = ZYVZS = \delta(T)VZS = V\delta(T)ZS = VZ\delta(T_1 \oplus T_2)S = VZS\delta(T_1 \oplus T_2) = VZSYZ$. It follows that $ZSYV = VZSY$, and therefore $ZSY \in \{T\}''$ as asserted. Since $K \in$ Lat$''T$, we have $\overline{ZSYK} \subseteq K$. Hence $\overline{YZSYK} \subseteq \overline{YK}$. But $\overline{YZSYK} = \overline{\delta(T_1 \oplus T_2)SYK} = \overline{SY\delta(T)K} = \overline{SYK}$. We conclude that $\overline{SYK} \subseteq \overline{YK}$ which shows that $\overline{YK} \in$ Lat$''(T_1 \oplus T_2)$. An analogous but easier argument than above shows that (ii) is also true. This completes the proof.

Corollary 2.3. *Let* $T=\begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$ *be as in Theorem 2.1. Then there exist bi-invariant subspaces* K_1 *and* K_2 *of* T *such that* $K_1 \vee K_2 = H$, $K_1 \cap K_2 = \{0\}$, $T|K_1$ *is of class* C_{11} *and* $T|K_2$ *is of class* $C_{\cdot 0}$. *Moreover,* K_1 *and* K_2 *can be chosen such that* $K_1 = H_1$ *and* $T|K_2 \sim T_2$.

Proof. As in the proof of Theorem 2.1, we may assume that T is c.n.u. Let Y and Z be the operators constructed there, and let $K_1 = \overline{Z(H_1 \oplus 0)}$ and $K_2 = \overline{Z(0 \oplus H_2)}$. Then $K_1, K_2 \in$ Lat$''T$, $K_1 \vee K_2 = H$ and $K_1 \cap K_2 = \{0\}$ by Corollary 2.2. From the definition of Z, it is easily seen that $K_1 = H_1$. On the other hand, since $Z|0 \oplus H_2$: $0 \oplus H_2 \to K_2$ and $Y|K_2 : K_2 \to 0 \oplus H_2$ are quasi-affinities which intertwine $0 \oplus T_2$ and $T|K_2$, we have $T|K_2 \sim T_2$. Moreover, it is easy to check that in this case $T|K_2$ must also be of class $C_{\cdot 0}$, completing the proof.

We remark that if $T=\begin{bmatrix} T_1' & X' \\ 0 & T_2' \end{bmatrix}$ is the type $\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix}$ canonical triangulation of the contraction T and if the characteristic function of T_2' admits a scalar multiple, then, by considering T^*, we obtain results analogous to Theorem 2.1 and Corollaries 2.2. and 2.3. Also note that weak contractions and $C_1.$ contractions with $d_T<\infty$ (cf. Lemma 3.2. below) are among the operators satisfying the assumption of Theorem 2.1. When applied to weak contractions, Theorem 2.1 yields the following result which has been obtained before in [15].

Corollary 2.4. *Let T be a c.n.u. weak contraction and let T_1 and T_1' be its C_{11} and C_0 parts. Then $T_1\sim T\oplus T_1'$.*

Proof. Let $T=\begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$ and $T=\begin{bmatrix} T_1' & X \\ 0 & T_2' \end{bmatrix}$ be the triangulations of types $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$ and $\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix}$, respectively. Since the characteristic functions of T_1 and T_2' admit scalar multiples (cf. [8], p. 325 and p. 217), by Theorem 2.1 and the remark above we have $T_1\oplus T_2\sim T\sim T_1'\oplus T_2'$. Note that T_1 and T_2' are of class C_{11} and T_2 and T_1' are of class C_0, it is routine to check that $T_1\sim T_2'$ and $T_2\sim T_1'$ (cf. proof of [15], Theorem 1). Hence $T\sim T_1\oplus T_1'$ as asserted.

Note that Corollary 2.2. generalizes the corresponding results for Lat$''T$ and Hyperlat T when T is a c.n.u. weak contraction with finite defect indices (cf. [18], Corollary 4.2. and [17], Theorem 3). Indeed, in this case $\mathrm{Lat}''T\cong\mathrm{Lat}''(T_1\oplus T_2)=\mathrm{Lat}''T_1\oplus\mathrm{Lat}''T_2\cong\mathrm{Lat}''T_1\oplus\mathrm{Lat}''T_1'=\mathrm{Lat}''(T_1\oplus T_1')$ and similarly for Hyperlat T, where T_1' denotes the C_0 part of T.

As for Corollary 2.3, it generalizes the C_0-C_{11} decomposition for c.n.u. weak contractions (cf. [8], pp. 331—332). To verify this, we have to show that, in the context of Corollary 2.3, if T is a c.n.u. weak contraction, then $T|K_2$ is the C_0 part of T. Since $T|K_2\sim T_2$ is of class C_0, we have $K_2\subseteq H_1'\equiv\{x\in H: T^nx\to 0 \text{ as } n\to\infty\}$. On the other hand, since $T_2\sim T|H_1'\equiv T_1'$ (cf. proof of Corollary 2.4), we have $T|K_2\sim\sim T_1'$. Note that $\sigma(T_1')\subseteq\sigma(T)$ (cf. [8], p. 332). Hence T_1' is a weak C_0 contraction. Let $W: H_1'\to K_2$ be a quasi-affinity intertwining T_1' and $T|K_2$ and let $V: K_2\to H_1'$ be the restriction of the identity operator. Then VW is an injection in $\{T_1'\}'$. We infer from [1], Corollary 2.8 that VW is a quasi-affinity. It follows that $K_2=H_1'$ whence $T|K_2$ is the C_0 part of T.

3. $C_1.$ contractions. In this section we restrict ourselves to $C_1.$ contractions with at least one defect index finite. We will show that they are completely injection-similar to isometries and characterize various algebras of operators associated with them. We start with the following lemma.

Lemma 3.1. *Let T be a c.n.u. $C_1.$ contraction with $d_T=d_{T*}<\infty$. Then T is of class C_{11}.*

Proof. Since T is of class $C_1.$, its characteristic function $\{\mathfrak{D}_T, \mathfrak{D}_{T*}, \Theta_T(\lambda)\}$ is a $*$-outer function. Hence $\Theta_T(\lambda)^*: \mathfrak{D}_{T*}\to\mathfrak{D}_T$ has dense range for all λ in D (cf.

[8], p. 191). We conclude from the assumption $d_T=d_{T*}<\infty$ that $\det \Theta_T\not\equiv 0$. By [8], Theorem VII. 6. 3 we infer that T is of class C_{11}.

Lemma 3.2. *Let T be a $C_{1\cdot}$ contraction with $d_T<\infty$ and let $T=\begin{bmatrix} T_1 & * \\ 0 & T_2\end{bmatrix}$ be of type $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0}\end{bmatrix}$. Then T_1 and T_2 are of classes C_{11} and C_{10}, respectively.*

Proof. Obviously, T_1 is of class C_{11}. As in the proof of Theorem 2.1, we may assume that T is c.n.u.. Let $T_2=\begin{bmatrix} T_3 & * \\ 0 & T_4\end{bmatrix}$ be the triangulation of type $\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot}\end{bmatrix}$. Note that T_3 is of class C_{00}. Indeed, since T_2 is of class $C_{\cdot 0}$, we have $T_2^{*n}=\begin{bmatrix} T_3^{*n} & 0 \\ * & T_4^{*n}\end{bmatrix}\to 0$ strongly. It follows that $T_3^{*n}\to 0$ strongly. Hence T_3 is of class $C_{\cdot 0}$ and thus of class C_{00}. We have

$$T=\begin{bmatrix} T_1 & * & * \\ 0 & T_3 & * \\ 0 & 0 & T_4\end{bmatrix}.$$

Let $T'=\begin{bmatrix} T_1 & * \\ 0 & T_3\end{bmatrix}$ with the corresponding regular factorization $\Theta_{T'}=\Theta_3\Theta_1$, where $\{\mathfrak{D}_{T'}, \mathfrak{D}_{T'*}, \Theta_{T'}(\lambda)\}$ is factored as the product of $\{\mathfrak{D}_{T'}, \mathfrak{F}, \Theta_1(\lambda)\}$ and $\{\mathfrak{F}, \mathfrak{D}_{T'*}, \Theta_3(\lambda)\}$. Since T_1 and T_3 are of classes C_{11} and C_{00}, the purely contractive parts of Θ_1 and Θ_3 are outer and inner from both sides, respectively (cf. [8], p. 257). We deduce that $\dim \mathfrak{D}_{T'}=\dim \mathfrak{F}$ and $\dim \mathfrak{F}=\dim \mathfrak{D}_{T'*}$ (cf. [8], p. 192). It follows that $\dim \mathfrak{D}_{T'}=\dim \mathfrak{D}_{T'*}$, that is, $d_{T'}=d_{T'*}$. Note that T' is of class $C_{1\cdot}$ and $d_{T'}\leqq d_T<\infty$. Hence by Lemma 3.1, T' is of class C_{11}. This implies that T_3 is of class $C_{\cdot 1}$, contradicting the fact that T_3 is of class C_{00}. We conclude that T_2 itself must be of class $C_{1\cdot}$ and therefore of class C_{10}.

If T is a $C_{1\cdot}$ contraction with $d_T<\infty$, then as shown above T_1 is of class C_{11} and has finite defect indices. Hence its characteristic function admits a scalar multiple (cf. [8], p. 318) and therefore Theorem 2.1 is applicable. In particular, we have the following corollary.

Corollary 3.3. *Let T and S be $C_{1\cdot}$ contractions with finite defect indices and let $T=\begin{bmatrix} T_1 & * \\ 0 & T_2\end{bmatrix}$ and $S=\begin{bmatrix} S_1 & * \\ 0 & S_2\end{bmatrix}$ be the triangulations of type $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0}\end{bmatrix}$. Then $T\sim S$ if and only if $T_1\sim S_1$ and $T_2\sim S_2$.*

Proof. The conclusion follows easily from the preceding remark and [22], Theorem 6.

Lemma 3.4. *Let* $T=U_1\oplus\ldots\oplus U_p\oplus S_q$ *on* $H=L^2(E_1)\oplus\ldots\oplus L^2(E_p)\oplus H_q^2$, *where* $0\leqq p, q\leqq\infty$, E_j*'s are Borel subsets of the unit circle satisfying* $E_1\supseteqq E_2\supseteqq\ldots\supseteqq E_p\neq\emptyset$, U_j *denotes the operator of multiplication by* e^{it} *on* $L^2(E_j)$, $j=1, \ldots, p$, *and* S_q *denotes the unilateral shift on* H_q^2. *Then* $\mu_T=p+q$.

Proof. Let $U=U_1\oplus\ldots\oplus U_p$. It is well known that $\mu_U=p$ and $\mu_{S_q}=q$. Hence $\mu_T\leqq\mu_U+\mu_{S_q}=p+q$. On the other hand, for almost all e^{it} in E_p, consider $H_t=\{h(e^{it}): h\in H\}$. Obviously, $H_t=\mathbf{C}^{p+q}$. We assume that $N\equiv\mu_T<\infty$ for otherwise the assertion is trivial. Let $K=\{h_1, \ldots, h_N\}$ be a set of vectors in H such that $H=\bigvee_{k=0}^{\infty} T^k K$. Then $H=\{p_1(T)h_1+\ldots+p_N(T)h_N: p_1, \ldots, p_N \text{ polynomials}\}^-$. We deduce that $H_t=\{p_1(e^{it})\, h_1(e^{it})+\ldots+p_N(e^{it})h_N(e^{it}): p_1, \ldots, p_N \text{ polynomials}\}^-$ for almost all e^{it} in E_p, that is, H_t is spanned by the set of N vectors $\{h_1(e^{it}), \ldots, h_N(e^{it})\}$. Hence we must have $p+q\leqq N$, and thus $\mu_T=N=p+q$.

Now we are ready to show the complete injection-similarity of $C_{1\cdot}$ contractions with isometries. The next theorem not only generalizes [20], Theorem 2.1 but the proof is much simpler.

Theorem 3.5. *Let T be a $C_{1\cdot}$ contraction with* $d_T<\infty$. *Then T is completely injection-similar to an isometry. If T is c.n.u., then* $U\oplus S_{m-n}\overset{\text{ci}}{\prec}T\prec U\oplus S_{m-n}$, *where* $m=d_{T^*}$, $n=d_T$, U *denotes the operator of multiplication by* e^{it} *on* $\overline{\Delta_T L_n^2}$ *and* S_{m-n} *denotes the unilateral shift on* H_{m-n}^2. *In particular,* $p+m-n\leqq\mu_T\leqq p+2(m-n)$, *where* $p=\mu_U$.

Proof. We may assume that T is c.n.u.. Let $T=\begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$ be the triangulation of type $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$ with the corresponding factorization $\Theta_T=\Theta_2\Theta_1$. By the remark before Corollary 3.3, we have $T\sim T_1\oplus T_2$. Note that T_1, being of class C_{11}, is quasi-similar to U on $\overline{\Delta_1 L_n^2}=\overline{\Delta_T L_n^2}$, where $\Delta_1=(I-\Theta_1^*\Theta_1)^{1/2}$ (cf. [8], pp. 71—72). On the other hand, since the characteristic function of T_2 is the purely contractive part of Θ_2, we infer that $d_{T_2}=n-r$ and $d_{T_2^*}=m-r$ for some r with $0\leqq r\leqq n$. Hence for the C_{10} contraction T_2 we have $S_{m-n}\overset{\text{ci}}{\prec}T_2\prec S_{m-n}$ (cf. [7], Theorem 3). We conclude that $U\oplus S_{m-n}\overset{\text{ci}}{\prec}T\prec U\oplus S_{m-n}$. Finally we verify the assertion concerning μ_T. Note that $T\prec U\oplus S_{m-n}$ implies that $\mu_T\geqq\mu_{U\oplus S_{m-n}}=p+m-n$ by Lemma 3.4. On the other hand, we have $\mu_T=\mu_{T_1\oplus T_2}\leqq\mu_{T_1}+\mu_{T_2}\leqq p+2(m-n)$ (cf. [10], Theorem 2). This completes the proof.

Unfortunately, we are yet unable to show the uniqueness of the isometry completely injection-similar to T although its unitary part is indeed unique. This follows from the following lemma.

Lemma 3.6. *For $j=1,2$, let $V_j=U_j\oplus S_j$ be an isometry, where U_j is a unitary operator and S_j is a unilateral shift. If $V_1 \overset{i}{\sim} V_2$, then $U_1 \cong U_2$.*

Proof. Assume that $V_j=U_j\oplus S_j$ is acting on $H_j=K_j\oplus L_j, j=1,2$. Let X: $H_1\to H_2$ and Y: $H_2\to H_1$ be the injections which intertwine V_1 and V_2. We claim that $XK_1\subseteq K_2$. Indeed, for any x in K_1 and $n\geqq 0, x=U_1^n y_n$ for some $y_n\in K_1$. Hence $Xx=XU_1^n y_n=XV_1^n y_n=V_2^n Xy_n\subseteq V_2^n H_2$ for any $n\geqq 0$. It follows that $Xx\in \bigcap_{n=0}^{\infty} V_2^n H_2 = K_2$, as asserted. Similarly, we have $YK_2\subseteq K_1$. Thus $U_1 \overset{i}{\sim} U_2$. We conclude that U_1 and U_2 are unitarily equivalent to direct summands of each other (cf. [3], Lemma 4.1). By the third test problem in [5], this implies that $U_1\cong U_2$.

We conjecture that if $V_1 \overset{i}{\sim} V_2$ and $\mu_{U_1}<\infty$ then $V_1\cong V_2$.

The next two theorems characterize those $C_{1\cdot}$ contractions which are cyclic or have commutative commutants. Analogous results have been obtained before for $C_{\cdot 0}$ contractions (cf. [23], Theorems 1.3 and 1.5).

Theorem 3.7. *Let T be a c.n.u. $C_{1\cdot}$ contraction with $d_T<\infty$. Then the following statements are equivalent:*

(1) T is cyclic;

(2) T is of class C_{11} and $T\sim M_E$ or T is of class C_{10} and $T\sim S$, where M_E denotes the operator of multiplication by e^{it} on $L^2(E)$, E being a Borel subset of the unit circle, and S denotes the simple unilateral shift.

The proof is the same as the one for [20], Theorem 3.2.

Corollary 3.8. *Let T be a c.n.u. $C_{1\cdot}$ contraction with $d_T<\infty$. If T is cyclic, so is T^* but not conversely.*

Proof. If T is cyclic, then $T\sim M_E$ or $T\sim S$. Hence $T^*\sim M_E^*$ or $T^*\sim S^*$. In either case, T^* is cyclic. The converse example is given by $T=S\oplus S$ (cf. [4], Problem 126).

Theorem 3.9. *Let T be a c.n.u. $C_{1\cdot}$ contraction with $d_T<\infty$. Then the following statements are equivalent:*

(1) $\{T\}'=\{T\}''$;

(2) T is of class C_{11} and $T\sim M_E$ or T is of class C_{10} and $d_{T^*}-d_T=1$.

Proof. (2)$\Rightarrow$(1). If T is of class C_{11} and $T\sim M_E$, then obviously T is cyclic. Hence (1) follows from [9], Theorem 1. On the other hand, if T is of class C_{10} and $d_{T^*}-d_T=1$, then (1) follows from [23], Theorem 1.5.

(1)$\Rightarrow$(2). Let $T=\begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$ on $H=H_1\oplus H_2$ be the triangulation of type

$\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$. As proved in Theorem 3.5, $T_1 \sim U$, the operator of multiplication by e^{it} on $\overline{\Delta_T L_n^2}$, and $T_2 \prec S_{m-n}$, where $m=d_{T*}$ and $n=d_T$. We consider the following two cases:

(i) If $m=n$, then $T=T_1$ is of class C_{11} by Lemma 3.1. Note that there are quasi-affinities $Y: H \to \overline{\Delta_T L_n^2}$ and $Z: \overline{\Delta_T L_n^2} \to H$ which intertwine T and U and such that $YZ=\delta(U)$ and $ZY=\delta(T)$ for some outer function δ (cf. [21], Lemma 2.1). It is easily verified that $\{T\}'=\{T\}''$ implies that $\{U\}'=\{U\}''$. Therefore U is cyclic (cf. [6], § 3) and so $T \sim M_E$ for some Borel subset E.

(ii) If $m \neq n$, then there exist finitely many operators $Z_i: H^2_{m-n} \to \overline{\Delta_T L_n^2}$ which intertwine S_{m-n} and U and such that $\bigvee_i \operatorname{ran} Z_i = \overline{\Delta_T L_n^2}$ (cf. [2], pp. 299—300). Hence there exist $Y_i: H_2 \to H_1$ which intertwine T_2 and T_1 and such that $\bigvee_i \operatorname{ran} Y_i = H_1$. On the other hand, using Theorem 2.1 and the assumption $\{T\}'=\{T\}''$ we infer that $\{T_1 \oplus T_2\}'=\{T_1 \oplus T_2\}''$. Thus any operator $Y: H_2 \to H_1$ which intertwines T_2 and T_1 must be 0. We conclude from above that $H_1=\{0\}$, that is, T is of class C_{10}. Moreover, $\{T\}'=\{T\}''$ implies that $m-n=1$ (cf. [23], Theorem 1.5).

Corollary 3.10. *Let T be a c.n.u. $C_{1\cdot}$ contraction with $d_T<\infty$. If T is cyclic, then $\{T\}'=\{T\}''$ but not conversely.*

Proof. The converse example is given in [10], pp. 321—322.

We remark that Corollaries 3.8 and 3.10 have been obtained before by Sz.-Nagy and Foiaş [9], Theorem 1 and [6].

In the final part of this paper, we determine when a $C_{1\cdot}$ contraction satisfies the double commutant property. Since a c.n.u. $C_{1\cdot}$ contraction T with $d_T<\infty$ is completely injection-similar to an isometry with an absolutely continuous unitary part, to motivate we first consider for such isometries. The next lemma partially generalizes [12], Theorem 3.3.

Lemma 3.11. *Let $V=U \oplus S$ be an isometry on $H=H_1 \oplus H_2$, where U is a unitary operator and S is a unilateral shift. Assume that U is absolutely continuous. Then the following statements are equivalent:*

(1) $S \neq 0$;

(2) *V is not unitary;*

(3) $\{V\}''=\{\varphi(V): \varphi \in H^\infty\}$.

Proof. (1)⇔(2). Trivial.

(1)⇒(3). Let $T \in \{V\}''$. Then $T=T_1 \oplus T_2$ where $T_1 \in \{U\}''$ and $T_2 \in \{S\}''$. Since $S \neq 0$, there exists $\varphi \in H^\infty$ such that $T_2=\varphi(S)$. As before, there are (possibly infinitely many) operators $Z_i: H_2 \to H_1$ which intertwine S and U and such that

$\bigvee_i \operatorname{ran} Z_i = H_1$ (cf. [2], pp. 299—300). Hence $\varphi(U)Z_i = Z_i\varphi(S) = Z_iT_2$ for all i. On the other hand, since $Y_i \equiv \begin{bmatrix} 0 & Z_i \\ 0 & 0 \end{bmatrix} \in \{V\}'$, we have $TY_i = Y_iT$. A simple computation shows that $T_1Z_i = Z_iT_2$. Thus $T_1Z_i = \varphi(U)Z_i$ for all i. We conclude that $T_1 = \varphi(U)$ and hence $T = \varphi(V)$.

(3)$\Rightarrow$(1). If $S=0$, then $V=U$ is a unitary operator. Hence $\{V\}'' = \{\psi(V): \psi \in L^\infty\}$, which is certainly not equal to $\{\varphi(V): \varphi \in H^\infty\}$.

Next we show that $C_1.$ contractions share similar properties. We need the following lemma.

Lemma 3.12. *Let T be a contraction on H and let* $T = \begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$ *on* $H = H_1 \oplus H_2$ *be the triangulation of type* $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$. *Then H_1 is hyperinvariant for T.*

Proof. Note that $H_2 = \{x \in H: T^{*n}x \to 0\}$ (cf. [8], p. 73). For $S \in \{T\}'$, we have $T^{*n}S^*x = S^*T^{*n}x \to 0$ as $n \to \infty$ for any $x \in H_2$. This shows that $S^*H_2 \subseteq H_2$. It follows that $SH_1 \subseteq H_1$, whence H_1 is hyperinvariant for T.

Theorem 3.13. *Let T be a c.n.u. $C_1.$ contraction with $d_T < \infty$. Let $m = d_{T^*}$ and $n = d_T$. Then the following statements are equivalent:*

(1) $m \neq n$;

(2) *T is not of class C_{11};*

(3) $\{T\}'' = \{\varphi(T): \varphi \in H^\infty\}$.

Proof. (1)$\Leftrightarrow$(2). This follows from Lemma 3.1 and the fact that C_{11} contractions have equal defect indices.

(1)$\Rightarrow$(3). As in the proof of Theorem 3.9, if $m \neq n$ then there exist finitely many operators $Y_i: H_2 \to H_1$ which intertwine T_2 and T_1 and such that $\bigvee_i \operatorname{ran} Y_i = H_1$. Let $W \in \{T\}''$. By Lemma 3.12, $W = \begin{bmatrix} W_1 & * \\ 0 & W_2 \end{bmatrix}$ on $H = H_1 \oplus H_2$. Obviously, $W_2 \in \{T_2\}'$. We check that actually $W_2 \in \{T_2\}''$. Let $R \in \{T_2\}'$, and let Y and Z be the operators constructed in the proof of Theorem 2.1. It is easily seen that $Z(I \oplus R)Y \in \{T\}'$. Hence $Z(I \oplus R)YW = WZ(I \oplus R)Y$. A simple computation shows that $\delta(T_2)RW_2 = W_2\delta(T_2)R = \delta(T_2)W_2R$. Since $\delta(T_2)$ is an injection, we have $RW_2 = W_2R$ whence $W_2 \in \{T_2\}''$ as asserted. Thus there exists $\varphi \in H^\infty$ such that $W_2 = \varphi(T_2)$ (cf. [13], Theorem 1). We have $\varphi(T_1)Y_i = Y_i\varphi(T_2) = Y_iW_2$ for all i. On the other hand, since $X_i \equiv \begin{bmatrix} 0 & Y_i \\ 0 & 0 \end{bmatrix} \in \{T\}'$, we have $WX_i = X_iW$. It follows that $W_1Y_i = Y_iW_2$ whence $W_1Y_i = \varphi(T_1)Y_i$ for all i. We conclude that $W_1 = \varphi(T_1)$. Thus W is triangulated as $\begin{bmatrix} \varphi(T_1) & * \\ 0 & \varphi(T_2) \end{bmatrix}$. But we also have $\varphi(T) = \begin{bmatrix} \varphi(T_1) & * \\ 0 & \varphi(T_2) \end{bmatrix}$. Hence

$W-\varphi(T)=\begin{bmatrix}0 & Q\\ 0 & 0\end{bmatrix}\in\{T\}''$, say. To complete the proof, it suffices to show that $Q=0$. To this end, let $S\colon H_2\to H_1$ be the operator defined in the proof of Theorem 2.1 and let $A=\begin{bmatrix}\delta(T_1) & S\\ 0 & 0\end{bmatrix}$. It is clear that $A\in\{T\}'$. Hence $A(W-\varphi(T))=(W-\varphi(T))A$. A simple computation shows that $\delta(T_1)Q=0$. Since $\delta(T_1)$ is an injection, we conclude that $Q=0$, completing the proof.

(3)$\Rightarrow$(2). If T is of class C_{11}, then $\{T\}''$ has been given in [19], Lemma 2. We will show that it is not the same as $\{\varphi(T)\colon \varphi\in H^\infty\}$. Note that T is quasi-similar to the operator $U=U_1\oplus\ldots\oplus U_p$ on $K=L^2(E_1)\oplus\ldots\oplus L^2(E_p)$, where $0\leqq p\leqq n$, $E_j=\{e^{it}\colon \operatorname{rank}\Delta_T(e^{it})\geqq j\}$ are Borel subsets of the unit circle satisfying $E_1\supseteqq E_2\supseteqq\ldots\supseteqq E_p\neq\emptyset$ and U_j denotes the operator of multiplication by e^{it} on $L^2(E_j)$, $j=1,2,\ldots,p$ (cf. [16], Theorem 2). Let $\delta=\det\Theta_T$ and Ω be the algebraic adjoint of Θ_T. Since $\delta\not\equiv 0$, there exists some $\varepsilon>0$ such that $F=\{e^{it}\in E_1\colon |\delta(e^{it})|\geqq\varepsilon\}$ has positive Lebesgue measure. Let $G\subseteqq F$ be such that G and $F\setminus G$ both have positive Lebesgue measure. Let

$$V=P\begin{bmatrix}0 & 0\\ -\chi_G\frac{1}{\delta}\Delta_T\Omega & \chi_G\end{bmatrix}.$$

It is easily checked that $V\in\{T\}''$ (cf. [19], Lemma 2). If $V=\varphi(T)$ for some $\varphi\in H^\infty$, then $\chi_G=\varphi$ on $\overline{\Delta_T L_n^2}$. In particular, $\chi_G=\varphi$ a.e. on E_1. This is certainly impossible. We conclude that $\{T\}''\neq\{\varphi(T)\colon\varphi\in H^\infty\}$.

Corollary 3.14. *Let T be a c.n.u. $C_{1\cdot}$ contraction with $d_T<d_{T*}\leqq\infty$. If T is cyclic, then $\{T\}'=\{\varphi(T)\colon\varphi\in H^\infty\}$.*

Proof. This follows from Corollary 3.10 and Theorem 3.13.

The preceding corollary has been obtained before in [11], Lemma 1.

Corollary 3.15. *Let T be a c.n.u. $C_{1\cdot}$ contraction with $d_T<\infty$. Then the following statements are equivalent:*

(1) $\{T\}''=\operatorname{Alg} T$;

(2) *either $d_T\neq d_{T*}$ or $d_T=d_{T*}$ and $\Theta_T(e^{it})$ is isometric for e^{it} in a set of positive Lebesgue measure.*

Proof. The assertion follows from Theorem 3.13 and [18], Theorem 3.8.

References

[1] H. Bercovici, C_0-Fredholm operators. I, *Acta Sci. Math.*, **41** (1979), 15—31.
[2] R. G. Douglas, On the hyperinvariant subspaces for isometries, *Math. Z.*, **107** (1968), 297—300.
[3] R. G. Douglas, On the operator equation $S^*XT=X$ and related topics, *Acta Sci. Math.*, **30** (1969), 19—32.
[4] P. R. Halmos, *A Hilbert space problem book*, van Nostrand (Princeton, New Jersey, 1967).
[5] R. V. Kadison and I. M. Singer, Three test problems in operator theory, *Pacific J. Math.*, **7** (1957), 1101—1106.
[6] B. Sz.-Nagy, Cyclic vectors and commutants, *Linear operator and approximation*, Birkhäuser (Basel—Stuttgart, 1972), 62—67.
[7] B. Sz.-Nagy, Diagonalization of matrices over H^∞, *Acta Sci. Math.*, **38** (1976), 223—238.
[8] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*, North Holland — Akadémiai Kiadó (Amsterdam—Budapest, 1970).
[9] B. Sz.-Nagy and C. Foiaş, Vecteurs cycliques et commutativité des commutants, *Acta Sci. Math.*, **32** (1971), 177—183.
[10] B. Sz.-Nagy and C. Foiaş, Jordan model for contractions of class $C_{\cdot 0}$, *Acta Sci. Math.*, **36** (1974), 305—322.
[11] B. Sz.-Nagy and C. Foiaş, Vecteurs cycliques et commutativité des commutants. II, *Acta Sci. Math.*, **39** (1977), 169—174.
[12] T. R. Turner, Double commutants of isometries, *Tôhoku Math. J.*, **24** (1972), 547—549.
[13] M. Uchiyama, Double commutants of $C_{\cdot 0}$ contractions. II, *Proc. Amer. Math. Soc.*, **74** (1979), 271—277.
[14] P. Y. Wu, On nonorthogonal decompositions of certain contractions, *Acta Sci. Math.*, **37** (1975), 301—306.
[15] P. Y. Wu, Quasi-similarity of weak contractions, *Proc. Amer. Math. Soc.*, **69** (1978), 277—282.
[16] P. Y. Wu, Jordan model for weak contractions, *Acta Sci. Math.*, **40** (1978), 189—196.
[17] P. Y. Wu, Hyperinvariant subspaces of weak contractions, *Acta Sci. Math.*, **41** (1979), 259—266.
[18] P. Y. Wu, Bi-invariant subspaces of weak contractions, *J. Operator Theory*, **1** (1979), 261—272.
[19] P. Y. Wu, C_{11} contractions are reflexive, *Proc. Amer. Math. Soc.*, **77** (1979), 68—72.
[20] P. Y. Wu, On contractions of class $C_{1\cdot}$, *Acta Sci. Math.*, **42** (1980), 205—210.
[21] P. Y. Wu, On a conjecture of Sz.-Nagy and Foiaş, *Acta Sci. Math.*, **42** (1980), 331—338.
[22] P. Y. Wu, On the quasi-similarity of hyponormal contractions, *Illinois J. Math.*, **25** (1981), 498—503.
[23] P. Y. Wu, $C_{\cdot 0}$ contractions: cyclic vectors, commutants and Jordan models, *J. Operator Theory*, **5** (1981), 53—62.

DEPARTMENT OF APPLIED MATHEMATICS
NATIONAL CHIAO TUNG UNIVERSITY
HSINCHU, TAIWAN, CHINA

Acta Sci. Math., 44 (1982), 151—155

When is a contraction quasi-similar to an isometry?

PEI YUAN WU

In this paper we answer the question in the title for contractions with finite defect indices. More precisely, we show that if T is a contraction with finite defect indices then T is quasi-similar to an isometry if and only if T is of class $C_{1.}$ and there exists a bounded analytic function Ω such that $\Omega\Theta_T=\delta I$ for some outer function δ, where Θ_T denotes the characteristic function of T. This condition is analogous to the one for a contraction similar to an isometry (cf. [3], Theorem 2.4.). We will also derive some related results.

In the following all the operators are acting on complex, separable Hilbert spaces. The main reference is the book of Sz.-Nagy and Foiaş [2]. Recall that for operators T_1 and T_2 on H_1 and H_2, respectively, $T_1\prec T_2$ denotes that T_1 is a *quasi-affine transform* of T_2, that is, there exists a one-to-one operator $X\colon H_1\to H_2$ with dense range (called *quasi-affinity*) such that $T_2X=XT_1$. T_1 and T_2 are *quasi-similar* ($T_1\sim T_2$) if $T_1\prec T_2$ and $T_2\prec T_1$.

For a contraction T, let $d_T=\operatorname{rank}(I-T^*T)^{1/2}$ and $d_{T^*}=\operatorname{rank}(I-TT^*)^{1/2}$ denote its *defect indices* and let Θ_T denote its characteristic function. For any $n\geqq 1$, let S_n denote the unilateral shift on H^2_n. The next lemma characterizes those contractions which are quasi-similar to a unilateral shift.

Lemma 1. *Let T be a contraction with finite defect indices. Then the following statements are equivalent:*

(1) T is quasi-similar to a unilateral shift;

(2) T is of class C_{10} and there exists a bounded analytic function Ω such that $\Omega\Theta_T=\delta I$ for some outer function δ.

Proof. Let $n=d_T$ and $m=d_{T^*}$.

(1)$\Rightarrow$(2). That T is of class C_{10} follows from [8], Lemma 1. Consider the functional model of T, that is, consider T being acting on $\mathfrak{H}\equiv H^2_m\ominus\Theta_T H^2_n$ by $Tf=P(e^{it}f)$

Received April 10, 1981.

This research was partially supported by National Science Council of Taiwan, China.

for $f\in\mathfrak{H}$, where P denotes the (orthogonal) projection onto $\mathfrak{H}$. Note that T must be quasi-similar to S_{m-n}. Indeed, this follows from the uniqueness of the Jordan model of T (cf. [4], Theorem 4). Let $Y: H^2_{m-n}\to\mathfrak{H}$ be the quasi-affinity intertwining S_{m-n} and T. Then Y is given by $Yg=P(\Phi g)$ for $g\in H^2_{m-n}$, where Φ is an $m\times(m-n)$ matrix valued bounded analytic function. Note that $\overline{\operatorname{ran} Y}=\mathfrak{H}$ if and only if $\Phi H^2_{m-n}+\Theta_T H^2_n$ is dense in H^2_m. Let Ψ denote the $m\times n$ matrix valued function $[\Phi, \Theta_T]$. Since $\Phi H^2_{m-n}+\Theta_T H^2_n=\Psi H^2_m$, we conclude from above that Ψ is an outer function. Let Ψ^A denote the algebraic adjoint of the matrix of Ψ. Say, $\psi^A=\begin{bmatrix}\Omega'\\ \Omega\end{bmatrix}$, where Ω' is $(m-n)\times m$ matrix valued and Ω is $n\times m$ matrix valued. Since $\Psi^A\Psi=\delta I$, where $\delta=\det\Psi$ is an outer function, we infer that $\Omega\Theta_T=\delta I$ as asserted.

(2)⇒(1). Consider the functional model of T and consider Ω as a multiplication operator from H^2_m to H^2_n. Let $\mathfrak{K}=\ker\Omega$. Define $X:\mathfrak{H}\to\mathfrak{K}$ by $Xf=\delta f-\Theta_T\Omega f$ for $f\in\mathfrak{H}$ and $Y:\mathfrak{K}\to\mathfrak{H}$ by $Yg=Pg$ for $g\in\mathfrak{K}$. Note that $\Omega Xf=\Omega\delta f-\Omega\Theta_T\Omega f=$ $=\Omega\delta f-\delta\Omega f=0$ for any $f\in\mathfrak{H}$. Hence X indeed maps $\mathfrak{H}$ to $\mathfrak{K}$. Let $S=S_m|\mathfrak{K}$. It is easily verified that X and Y intertwine T and S. Moreover, we have $XYg=XPg=$ $=X(g-\Theta_T w)=\delta(g-\Theta_T w)-\Theta_T\Omega(g-\Theta_T w)=\delta g-\Theta_T\Omega g=\delta g=\delta(S)g$ for any $g\in\mathfrak{K}$, where $w\in H^2_n$, and $YXf=Y(\delta f-\Theta_T\Omega f)=P(\delta f)-0=\delta(T)f$ for any $f\in\mathfrak{H}$. Since $\delta(S)$ and $\delta(T)$ are quasi-affinities, so are X and Y. This shows that T is quasi-similar to S, a unilateral shift, completing the proof.

We remark that the proof of (2)⇒(1) in the preceding lemma holds even without the finiteness assumption on the defect indices of T. Also note that Lemma 1 partially generalizes [4], Proposition 2 (for the case $d_T=1$ and $d_{T^*}=2$) and [6], Theorem 3.1 (for the case $d_{T^*}-d_T=1$). Next we consider contractions quasi-similar to isometries. We need the following lemma.

Lemma 2. *Let T be a contraction with finite defect indices. Then the following statements are equivalent:*

(1) *T is quasi-similar to an isometry;*

(2) *the completely non-unitary (c.n.u.) part of T is quasi-similar to an isometry.*

Proof. We have only to show (1)⇒(2). Assume that T is quasi-similar to the isometry V. By [8], Lemma 1, T is of class $C_{1\cdot}$. Let $V=U\oplus S$, where U is unitary and S is a unilateral shift, and let $T=T_1\oplus T_2$, where T_1 is unitary and T_2 is c.n.u. Let $T_2=\begin{bmatrix}T_3 & *\\ 0 & T_4\end{bmatrix}$ be the triangulation of type $\begin{bmatrix}C_{\cdot 1} & *\\ 0 & C_{\cdot 0}\end{bmatrix}$. Then T_3 is of class C_{11} and has finite defect indices. By [9], Theorem 2.1, $T_2\sim T_3\oplus T_4$. Hence $U\oplus S\sim T_1\oplus T_2$ $\sim T_1\oplus T_3\oplus T_4$. Note that U and $T_1\oplus T_3$ are of class C_{11}, S and T_4 are of class C_{10} (cf. [9], Lemma 3.2) and the defect indices of T_4 are finite. It follows from the proof of [8], Theorem 6 that $T_1\oplus T_3\sim U$ and $T_4\prec S$. Hence S must be the Jordan model of T_4 (cf. [8], Lemma 3), that is, $S=S_{m-n}$, where $m=d_{T_4^*}$ and $n=d_{T_4}$. Thus S has finite defect indices and we infer from [8], Theorem 6 again that $T_4 \sim S$. On the other hand, the C_{11} contraction T_3 is quasi-similar to a unitary operator (cf. [2], p. 72). We conclude from above that T_2 is quasi-similar to an isometry, completing the proof.

Theorem 3. *Let T be a contraction with finite defect indices and let* $T=\begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ *be the triangulation of type* $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$. *Then the following statements are equivalent:*

(1) *T is quasi-similar to an isometry;*

(2) *T_1 is quasi-similar to a unitary operator and T_2 is quasi-similar to a unilateral shift;*

(3) *T is of class $C_{1\cdot}$ and there exists a bounded analytic function Ω such that $\Omega\Theta_T=\delta I$ for some outer function δ.*

Proof. By Lemma 2, it suffices to consider c.n.u. T.

(1)⇒(2) is proved in Lemma 2.

(2)⇒(3). By [8], Lemma 1, both T_1 and T_2 are of class $C_{1\cdot}$. A simple calculation shows that T must also be of class $C_{1\cdot}$. Let $\Theta_T=\Theta_2\Theta_1$ be the canonical factorization corresponding to the triangulation $T=\begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$. Then the characteristic functions of T_1 and T_2 are the purely contractive parts of Θ_1 and Θ_2, respectively. Lemma 1 implies that there exists a bounded analytic function Ω_2 such that $\Omega_2\Theta_2=\delta_2 I$ for some outer function δ_2. On the other hand, T_1 is of class C_{11} implies that Θ_1 is outer (from both sides). Let Ω_1 be the algebraic adjoint of the matirx of Θ_1 and let $\Omega=\Omega_1\Omega_2$ and $\delta=\delta_2 \det\Theta_1$. Then $\Omega\Theta_T=\Omega_1\Omega_2\Theta_2\Theta_1=\Omega_1\delta_2\Theta_1=\delta I$, where δ is outer.

(3)⇒(1). As above, let $\Theta_T=\Theta_2\Theta_1$ be the factorization corresponding to $T=\begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$. From $\Omega\Theta_T=\delta I$ we have $\Theta_1\Omega\Theta_T\Omega_1=\Theta_1\delta\,\Omega_1=\delta(\det\Theta_1)I$, where Ω_1 is the algebraic adjoint of Θ_1. It follows that $(\Theta_1\Omega)\Theta_2=\delta I$. Since T_2 is of class C_{10} (cf. [9], Lemma 3.2), we infer from Lemma 1 that T_2 is quasi-similar to a unilateral shift. On the other hand, T_1 is quasi-similar to a unitary operator and $T\sim T_1\oplus T_2$ (cf. [9], Theorem 2.1). We conclude that T is quasi-similar to an isometry as asserted.

Note that the isometry quasi-similar to T is unique up to unitary equivalence (cf. [1], Theorem 3.1). It also follows from the preceding proof that if T is c.n.u., then the isometry quasi-similar to T has an absolutely continuous unitary part. We may contrast Theorem 3 with the corresponding results for contractions similar to isometries: a contraction T is similar to an isometry if and only if there is a bounded analytic function Ω such that $\Omega\Theta_T=I$ (cf. [3], Theorem 2.4); a c.n.u. T is similar to an isometry if and only if T_1 is similar to a unitary operator and T_2 is similar to a unilateral shift (cf. [5], Theorem 2).

Corollary 4. *Let T be a c.n.u. contraction with finite defect indices and let $\mathfrak{H}_1$ be an invariant subspace for T.*

(1) *If T is quasi-similar to an isometry, so is $T|\mathfrak{H}_1$.*

(2) *If T is quasi-similar to a unilateral shift, so is $T|\mathfrak{H}_1$.*

Proof. (1) By [8], Lemma 1, T is of class $C_{1\cdot}$. Hence $T|\mathfrak{H}_1$ is also of class $C_{1\cdot}$. Let $\Theta_T=\Theta_2\Theta_1$ be the corresponding regular factorization and let Ω be such that $\Omega\Theta_T=\delta I$ for some outer δ. Then $(\Omega\Theta_2)\Theta_1=\delta I$ and by Theorem 3 we conclude that $T|\mathfrak{H}_1$ is quasi-similar to an isometry.

(2) By [8], Lemma 1, T is of class C_{10}. It is easy to check that $T|\mathfrak{H}_1$ is also of class C_{10}. Similar arguments as above finish the proof.

Corollary 5. *Let T be a c.n.u. contraction on $\mathfrak{H}$ with finite defect indices. If T is quasi-similar to an isometry V on $\mathfrak{K}$, then there exist quasi-affinities $X:\mathfrak{H}\to\mathfrak{K}$ and $Y:\mathfrak{K}\to\mathfrak{H}$ which intertwine T and V and such that $XY=\delta(V)$ and $YX=\delta(T)$ for some outer function δ.*

Proof. Let $T=\begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ be the triangulation of type $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$. As before, since T_1 is of class C_{11} with finite defect indices, we have $T\sim T_1\oplus T_2$. Let $V=U\oplus S$ be the isometry quasi-similar to T, where U is unitary and S is a unilateral shift. As shown in the proof of Lemma 2, $T_1\sim U$ nad $T_2\sim S$. Note that all these three quasi-similarities can be implemented by quasi-affinities satisfying the corresponding properties in the conclusion of our assertion (cf. [9], Theorem 2.1, [7], Lemma 2.1 and proof of Lemma 1). Hence the same holds for the quasi-similarity of T and V.

For an operator T, let Lat T, Lat$''T$ and Hyperlat T denote, respectively, the lattices of invariant subspaces, bi-invariant subspaces and hyperinvariant subspaces of T. The next lemma will be needed in the proof of Theorem 7. It can be proved in the same fashion as [7], Lemma 2.3.

Lemma 6. *Let V be an isometry with an absolutely continuous unitary part and let $\mathfrak{N}\in\mathrm{Lat}\,V$. If δ is an outer function, then $\delta(V|\mathfrak{N})$ is a quasi-affinity on $\mathfrak{N}$.*

Theorem 7. *Let T be a c.n.u. contraction with finite defect indices. If T is quasi-similar to an isometry V, then* $\mathrm{Lat}\,T\cong\mathrm{Lat}\,V$, $\mathrm{Lat}''\,T\cong\mathrm{Lat}''\,V$ *and* $\mathrm{Hyperlat}\,T\cong \mathrm{Hyperlat}\,V$.

Proof. Note that T is of class $C_{1\cdot}$ by [8], Lemma 1. We may assume that T is not of class C_{11}, for otherwise the conclusion has already been proved in [7], Theorem 2.2.

Let X and Y be the quasi-affinities as in Corollary 5. For $\mathfrak{M}\in\mathrm{Lat}\,T$ and $\mathfrak{N}\in\mathrm{Lat}\,V$, consider the mappings $\mathfrak{M}\to\overline{X\mathfrak{M}}$ and $\mathfrak{N}\to\overline{Y\mathfrak{N}}$. Using Lemma 6, we can easily verify that they implement the lattice isomorphisms between Lat T and Lat V.

From [9], Theorem 3.13 and Lemma 3.11, we infer that Lat $T \cong$ Lat$''$ T and Lat $V \cong$ $\cong$ Lat$''$ V. Hence to complete the proof, it suffices to show that (i) $\mathfrak{M} \in$ Hyperlat T implies $\overline{X\mathfrak{M}} \in$ Hyperlat V and (ii) $\mathfrak{N} \in$ Hyperlat V implies $\overline{Y\mathfrak{N}} \in$ Hyperlat T. We only verify (i) and leave the verification of (ii) to the readers. Let $\mathfrak{M} \in$ Hyperlat T and $W \in \{V\}'$. Then $YWX \in \{T\}'$ and hence $\overline{YWX\mathfrak{M}} \subseteq \mathfrak{M}$. Applying X on both sides, we obtain $\overline{\delta(V)WX\mathfrak{M}} = \overline{XYWX\mathfrak{M}} \subseteq \overline{X\mathfrak{M}}$. Since $\delta(V)|\overline{WX\mathfrak{M}}$ is a quasi-affinity on $\overline{WX\mathfrak{M}}$ (by Lemma 6), we conclude that $\overline{WX\mathfrak{M}} \subseteq \overline{X\mathfrak{M}}$. This shows that $\overline{X\mathfrak{M}} \in$ Hyperlat V, completing the proof.

Corollary 8. *Let T be a c.n.u. contraction with finite defect indices. If T is quasi-similar to a unilateral shift, then* Lat T=Lat$''T = \{\overline{\text{ran } W} : W \in \{T\}'\}$, *where* $\{T\}'$ *denotes the commutant of T.*

Proof. This follows easily from Theorem 7 and the fact that a unilateral shift S satisfies Lat S=Lat$''$ $S = \{\overline{\text{ran } Z} : Z \in \{S\}'\}$.

References

[1] T. B. Hoover, Quasi-similarity of operators, *Illinois J. Math.*, **16** (1972), 678—686.

[2] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*, North Holland — Akadémiai Kiadó (Amsterdam—Budapest, 1970).

[3] B. Sz.-Nagy and C. Foiaş, On the structure of intertwining operators, *Acta Sci. Math.*, **35** (1973), 225—254.

[4] B. Sz.-Nagy and C. Foiaş, Jordan model for contractions of class $C_{\cdot 0}$, *Acta Sci. Math.*, **36** (1974), 305—322.

[5] P. B. Wu, On nonorthogonal decompositions of certain contractions, *Acta Sci. Math.*, **37** (1975), 301—306.

[6] P. Y. Wu. On contractions of class $C_{1\cdot}$, *Acta Sci. Math.*, **42** (1980), 205—210.

[7] P. Y. Wu. On a conjecture of Sz.-Nagy and Foiaş, *Acta Sci. Math.*, **42** (1980), 331—338.

[8] P. Y. Wu. On the quasi-similarity of hyponormal contractions, *Illinois J. Math.*, **25** (1981), 498—503.

[9] P. Y. Wu. Approximate decompositions of certain contractions, *Acta Sci. Math.*, **44** (1982), 137—149.

DEPARTMENT OF APPLIED MATHEMATICS
NATIONAL CHIAO TUNG UNIVERSITY
HSINCHU, TAIWAN, CHINA

Acta Sci. Math., 44 (1982), 157—163

Injection-similar isometries

L. KÉRCHY

1. To construct canonical models for contractions of classes C_{11} and C_0 on complex separable Hilbert spaces B. SZ.-NAGY and C. FOIAŞ generalized the notion of similarity (cf. [3, ch. II, sec. 3] and [4]). They called an operator $T_1 \in \mathscr{L}(\mathfrak{H}_1)$ a *quasi-affine transform* of the operator $T_2 \in \mathscr{L}(\mathfrak{H}_2)$, $T_1 \prec T_2$, if there exists a quasi-affinity (an injection with dense range) $X \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ which intertwines these operators, that is, $XT_1 = T_2X$. T_1 and T_2 are said to be *quasi-similar,* $T_1 \sim T_2$, if they are quasi-affine transforms of each other, $T_1 \prec T_2$ and $T_2 \prec T_1$. Finding Jordan-models for contractions of class $C_{\cdot 0}$ even quasi-similarity proved to be insufficient. Therefore SZ.-NAGY and FOIAŞ [5] introduced the notion of injection-similarity. Operators $T_1 \in \mathscr{L}(\mathfrak{H}_1)$ and $T_2 \in \mathscr{L}(\mathfrak{H}_2)$ are *injection-similar,* $T_1 \overset{\text{i}}{\sim} T_2$, if they can be injected into each other, $T_1 \overset{\text{i}}{\prec} T_2$ and $T_2 \overset{\text{i}}{\prec} T_1$, that is, there are injections $X \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ and $Y \in \mathscr{L}(\mathfrak{H}_2, \mathfrak{H}_1)$ such that $XT_1 = T_2X$ and $YT_2 = T_1Y$. T_1 and T_2 are *completely injection-similar,* $T_1 \overset{\text{c.i}}{\sim} T_2$, if they can be completely injected into each other, $T_1 \overset{\text{c.i}}{\prec} T_2$ and $T_2 \overset{\text{c.i}}{\prec} T_1$, that is, there exist families of intertwining injections $\{X_\alpha\}_\alpha \subseteq \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ and $\{Y_\beta\}_\beta \subseteq \mathscr{L}(\mathfrak{H}_2, \mathfrak{H}_1)$ such that $\bigvee_\alpha \operatorname{ran} X_\alpha = \mathfrak{H}_2$ and $\bigvee_\beta \operatorname{ran} Y_\beta = \mathfrak{H}_1$.

Recently P. Y. WU [1] has shown that every contraction T of class $C_{1\cdot}$, with at least one defect index finite, $d_T < \infty$, is completely injection-similar to an isometry. More precisely he proved that

$$U \oplus S^{(\alpha)} \overset{\text{c.i}}{\prec} T \prec U \oplus S^{(\alpha)}.$$

Here U is a unitary operator of the form $U = U_1 \oplus U_2$, where U_1 is the unitary part of the contraction T (cf. [3, Th. I.3.2]), and U_2 denotes the operator of multiplication by e^{it} on the space $(\Delta_T L^2(\mathfrak{D}_T))^-$ $(\Delta_T(e^{it}) = (I - \Theta_T(e^{it})^* \Theta_T(e^{it}))^{1/2}$, where Θ_T is the characteristic function of T). On the other hand $S^{(\alpha)}$ is the unilateral shift of multiplicity $\alpha = d_{T^*} - d_T$.

Received July 1, 1981.

As for uniqueness of this isometry, Wu has shown that the unitary parts of injection-similar isometries are unitarily equivalent. Moreover he made the conjecture that injection-similar isometries are really unitarily equivalent, at least in the case, when their unitary parts have finite multiplicities. (HOOVER [7] proved that quasi-similarity even implies unitary equivalence between isometries.)

In the present paper we give a negative answer to this conjecture and describe the isometries being completely injection-similar to the contraction T above. We follow the notation and terminology of [3]. For arbitrary operators $T_1 \in \mathscr{L}(\mathfrak{H}_1)$ and $T_2 \in \mathscr{L}(\mathfrak{H}_2)$, $\mathscr{I}(T_1, T_2)$ will denote the set of intertwining operators, that is, $\mathscr{I}(T_1, T_2) = \{X \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2) \mid T_2 X = X T_1\}$.

2. We recall that every isometry V has a unique decomposition $V = U \oplus S^{(\alpha)}$ such that U is a unitary operator and $S^{(\alpha)}$ denotes the direct sum of α copies of the simple unilateral shift S. ($S^{(\alpha)}$ is a completely non-unitary (c. n. u.) isometry with multiplicity α.) (Cf. [3, Th. I.1.1.]) The following proposition shows that Wu's conjecture has an affirmative answer, if V is a c. n. u. isometry or U is a singular unitary (s. u.) operator (the spectral measure of U is singular with respect to Lebesgue measure).

Proposition 1. *Let V_1 and V_2 be injection-similar isometries, $V_1 \overset{i}{\sim} V_2$. Let us assume that V_1 is c. n. u. or its unitary part is a s. u. operator. Then these operators are unitarily equivalent, $V_1 \cong V_2$.*

Proof. Let V_1 and V_2 act on the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. Let us consider the canonical decompositions of these operators: $V_1 = U_1 \oplus S^{(\alpha)}$, $V_2 = U_2 \oplus S^{(\beta)}$ on the spaces $\mathfrak{H}_1 = \mathfrak{R}_1 \oplus \mathfrak{L}_1$ and $\mathfrak{H}_2 = \mathfrak{R}_2 \oplus \mathfrak{L}_2$. We know by [1, Lemma 3.6] that $U_1 \cong U_2$. If V_1 is c. n. u., then $\mathfrak{R}_1 = \{0\}$, and so we obtain that $S^{(\alpha)} = V_1 \overset{i}{\sim} V_2 = S^{(\beta)}$. Now [5, Th. 5/6] results that $S^{(\alpha)} \cong S^{(\beta)}$. Consequently in this case we have that $V_1 \cong V_2$.

Let us assume now that $\mathfrak{R}_1 \neq \{0\}$ and U_1 is a s. u. operator. Let us suppose further that for instance $\mathfrak{L}_1 \neq \{0\}$. (The case $\mathfrak{L}_1 = \mathfrak{L}_2 = \{0\}$ is trivial.) Let $X \in \mathscr{I}(V_1, V_2)$ be an injection, and consider the matrix $\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ of X with respect to the decompositions above. It follows easily that $X_{12} \in \mathscr{I}(S^{(\alpha)}, U_2)$. Having denoted by $S_b^{(\alpha)}$ the minimal unitary dilation of $S^{(\alpha)}$, we define an operator $Y \in \mathscr{I}(S_b^{(\alpha)}, U_2)$ by the equation $Y(S_b^{(\alpha)})^{-n} f := U_2^{-n} X_{12} f$ ($f \in \mathfrak{L}_1, n \geqq 0$) and by taking bounded closure. Since, being a bilateral shift, $S_b^{(\alpha)}$ is an absolutely continuous unitary (a. c. u.) operator, we infer by [8, Theorem 3] that $Y = 0$. Taking into account that $X_{12} = Y | \mathfrak{L}_1$, it follows that $X_{12} = 0$. We conclude that $X_{22} \in \mathscr{I}(S^{(\alpha)}, S^{(\beta)})$ is an injection. In particular we infer that $\mathfrak{L}_2 \neq \{0\}$, and so a similar argument shows that we have $S^{(\beta)} \overset{i}{\prec} S^{(\alpha)}$

also. Therefore $S^{(\alpha)} \overset{i}{\sim} S^{(\beta)}$, and [5, Th. 5/6] implies again $S^{(\alpha)} \cong S^{(\beta)}$. The proof is completed.

3. In this section we shall see that the setting is contrary to the one in section 2, if the isometry V is not c. n. u. and its unitary part is not a s. u. operator. The following lemma plays an essential role in the sequel.

Lemma 2. *Let E be a measurable set on the unit circle $C=\{z\in\mathbf{C}\,|\,|z|=1\}$, and let M_E denote the operator of multiplication by e^{it} on the space $L^2(E)$. (We consider the normalized Lebesgue measure m on C.) If $m(E)>0$, then we have*

$$M_E \oplus S \prec M_E.$$

Proof. Let $\varphi_1 \in L^\infty(E)$ be a function such that $\varphi_1(e^{it}) \neq 0$ a. e. and $\int_E \log|\varphi_1(e^{it})|dm = -\infty$. On the other hand let $\varphi_2 \in L^\infty(E)$ be a function such that $|\varphi_2(e^{it})|=1$ a.e.. We consider S as the operator of multiplication by e^{it} on the Hardy space H^2. Now let us define the operator X as follows: $X\colon L^2(E)\oplus H^2 \to L^2(E)$, $X\colon f\oplus g \mapsto \varphi_1 f + \varphi_2 (g|E)$. It is obvious that $X \in \mathcal{I}(M_E \oplus S, M_E)$ is a quasi-surjection.

Let us assume now that $X(f\oplus g)=0$. Let us suppose further that $g\neq 0$. Then we have $g(e^{it})\neq 0$ a. e., and so $f(e^{it})\neq 0$ a. e. on E. From the assumption it immediately follows that $|\varphi_1(e^{it})|\cdot|f(e^{it})|=|g(e^{it})|$ a. e. on E. But this implies

$$\log|\varphi_1(e^{it})| = \log|g(e^{it})| - \log|f(e^{it})| \geqq \log|g(e^{it})| + 1 - |f(e^{it})|,$$

and so we infer that

$$-\infty = \int_E \log|\varphi_1(e^{it})|\,dm \geqq \int_E \log|g(e^{it})|\,dm + m(E) - \int_E |f(e^{it})|\,dm > -\infty$$

(cf. [3, ch. III]). This being a contradiction we conclude that $g=0$ and this results $f=0$. Therefore X is a quasi-affinity, and so $M_E\oplus S \prec M_E$.

Corollary 3. *Let M_E be as before. Then for any $\alpha=1,2,\ldots,\infty$ we have*

$$M_E \oplus S^{(\alpha)} \prec M_E.$$

Proof. By induction we immediately infer that the statement holds for every natural number. Let us now assume that $\alpha=\infty$. Let $\{E_n\}_{n=1}^\infty$ be a sequence of pairwise disjoint measurable subsets of E such that $\bigcup_{n=1}^\infty E_n = E$ and $m(E_n)>0$ for every n. Then we have $M_E\oplus S^{(\infty)} \cong \bigoplus_{n=1}^\infty (M_{E_n}\oplus S) \prec \bigoplus_{n=1}^\infty M_{E_n} \cong M_E$ by Lemma 2, and the proof is finished.

Corollary 4. *Let $V\in\mathcal{L}(\mathfrak{H})$ be a non-c. n. u. isometry, and let us assume that its unitary part $U\in\mathcal{L}(\mathfrak{K})$ $(\mathfrak{K}\neq\{0\})$ is not a s. u. operator. Then we have:*

(i) $V\overset{i}{\sim}U$, *more precisely* $U\overset{i}{\prec}V\prec U$;

(ii) *if even* $\mathfrak{H}\ominus\mathfrak{K}\neq\{0\}$ *holds, then* $V\overset{c.i}{\sim}U\oplus S$, *more precisely* $U\oplus S\overset{c.i}{\prec}V\prec\prec U\oplus S$.

Proof. After decomposing U into the direct sum of its singular and its absolutely continuous parts, $U=U_s\oplus U_a$, and considering the functional model of U_a (cf. [9]), we conclude these statements by Corollary 3.

On account of Corollary 4 we can state:

Proposition 5. *Let V_1 and V_2 be isometries, and let U_1, U_2 denote their unitary parts, respectively. Let us assume that V_1 is not c. n. u., and U_1 is not a s. u. operator. Then we have:*

(i) $V_1\overset{i}{\sim}V_2$ *if and only if* $U_1\cong U_2$;

(ii) $V_1\overset{c.i}{\sim}V_2$ *if and only if* $U_1\cong U_2$ *and* V_1, V_2 *are unitaries in the same time.*

Proof. These statements follow immediately by [1, Lemma 3.6] and the preceding corollary. We have only to note that for any operator $X\in\mathcal{I}(V_1, V_2)$ we have $(X\mathfrak{K}_1)^-\subseteq\mathfrak{K}_2$, where $\mathfrak{K}_i\in\operatorname{Lat}V_i$ is the subspace corresponding to U_i $(i=1, 2)$. (Cf. the proof of [1, Lemma 3.6].)

4. Now let T be a contraction of class $C_{1\cdot}$, with at least one finite defect index, $d_T<\infty$. Consider the triangulation $\begin{bmatrix}T_1 & *\\ 0 & T_2\end{bmatrix}$ of the type $\begin{bmatrix}C_{\cdot 1} & *\\ 0 & C_{\cdot 0}\end{bmatrix}$ of T. We know from [1] that $T_1\in C_{11}, T_2\in C_{10}$ and $T\sim T_1\oplus T_2$ (cf. [1, Th. 2.1 and Lemma 3.2]). Now it follows easily by [3, Prop. II.3.5] and [6, Th. 3] that

$$U\oplus S^{(\alpha)}\overset{c.i}{\prec}T\prec U\oplus S^{(\alpha)},$$

where U is a unitary operator and $S^{(\alpha)}$ is the unilateral shift of multiplicity $\alpha=d_{T^*}-d_T$. (Cf. [1, Th. 3.5].) Moreover we know by [1, Lemma 3.6] that the unitary part of every isometry, being injection-similar to T, is unitarily equivalent to U.

We shall say that T is *mixed with absolutely continuous part (m. w. a. c. p.)*, if $T\notin C_{11}\cup C_{10}$ and T_1 is not a s. u. operator in the previous triangulation. Now we obtain immediately by Proposition 1:

Theorem 6. *If $T\in C_{1\cdot}, d_T<\infty$ and T is not m. w. a. c. p., then $V=U\oplus S^{(\alpha)}$, $\alpha=d_{T^*}-d_T$, is the unique isometry which is completely injection-similar to T.*

On the other hand, in the contrary case we can state:

Theorem 7. *If* $T\in C_{1\cdot}, d_T<\infty$ *and* T *is m. w. a. c. p., then*

$$U\oplus S^{(\alpha)}\overset{\text{c.i}}{\prec} T \prec U\oplus S^{(\alpha)}$$

holds, if and only if $1\leqq\alpha\leqq d_{T*}-d_T$.

To prove this theorem we need:

Lemma 8. *If* T *is a contraction of class* C_{10} *and* $d_T<\infty$, *then* $\dim\ker T^*=d_{T*}-d_T$.

Proof. We can assume that T is given by its functional model. That is, T is the compression of the unilateral shift U_+ on the vector-valued Hardy space $H^2(\mathfrak{E}_*)$ to the subspace $\mathfrak{H}=H^2(\mathfrak{E}_*)\ominus\Theta_T H^2(\mathfrak{E})$ $(\in\mathrm{Lat}\, U_+^*)$, where $\dim\mathfrak{E}_*=d_{T*}$, $\dim\mathfrak{E}=d_T$ and Θ_T denotes the characteristic function of T. T being of class C_{10}, its characteristic function Θ_T is inner and $*$-outer (cf. [3, Prop. VI. 3.5]).

Since $T^*=U_+^*|\mathfrak{H}$, we infer that $\ker T^*=\mathfrak{H}\cap\ker U_+^*=\mathfrak{H}\cap\mathfrak{E}_*$. Let $v\in\mathfrak{E}_*$ be an arbitrary vector. We have that $v\in\mathfrak{H}$, if and only if v is orthogonal to $\Theta_T H^2(\mathfrak{E})$. But this is the case, if and only if v is orthogonal to $\Theta_T H^2(\mathfrak{E})\ominus\lambda\Theta_T H^2(\mathfrak{E})=\Theta_T\big(H^2(\mathfrak{E})\ominus\lambda H^2(\mathfrak{E})\big)=\Theta_T\mathfrak{E}$. (We have used that Θ_T is an isometry.) Now, for any vector $w\in\mathfrak{E}$, we have $\langle v, \Theta_T w\rangle=\int_C\langle v, \Theta_T(e^{it})w\rangle\, dm=\int_C\langle\Theta_T(e^{-it})^*v, w\rangle\, dm=\langle\Theta_T^{\sim}v, w\rangle=\langle P_{\mathfrak{E}}\Theta_T^{\sim}v, w\rangle$, where $P_{\mathfrak{E}}$ denotes the orthogonal projection of $H^2(\mathfrak{E})$ to the subspace $\mathfrak{E}$. Therefore, we conclude that $\ker T^*=\ker\,(P_{\mathfrak{E}}\Theta_T^{\sim}|\mathfrak{E}_*)$.

On the other hand, since $\Theta_T^{\sim}$ is an outer function, it follows that $H^2(\mathfrak{E})=(\Theta_T^{\sim}H^2(\mathfrak{E}_*))^-=(\Theta_T^{\sim}\mathfrak{E}_*)\vee\lambda\Theta_T^{\sim}H^2(\mathfrak{E}_*)\subseteqq(\Theta_T^{\sim}\mathfrak{E}_*)\vee(\lambda H^2(\mathfrak{E}))=(P_{\mathfrak{E}}\Theta_T^{\sim}\mathfrak{E}_*)^-\oplus\lambda H^2(\mathfrak{E})$. Therefore the operator $P_{\mathfrak{E}}\Theta_T^{\sim}|\mathfrak{E}_*\in\mathscr{L}(\mathfrak{E}^*, \mathfrak{E})$ is quasi-surjective, and so, taking into account that $\dim\mathfrak{E}<\infty$, we infer that $\dim\ker\,(P_{\mathfrak{E}}\Theta_T^{\sim}|\mathfrak{E}_*)=\dim\mathfrak{E}_*-\dim\mathfrak{E}=d_{T*}-d_T$. The proof is completed.

Now we can prove Theorem 7.

Proof of Theorem 7. Let T_1, T_2 and U be the operators as at the begining of this section. Since T is m.w.a.c.p., it follows that the space of U is not trivial (is not $\{0\}$), and that U is not a s. u. operator. Applying Corollary 3 we can easily infer that $U\oplus S^{(\alpha)}\overset{\text{c.i}}{\prec}U\oplus S^{(d_{T*}-d_T)}\prec U\oplus S^{(\alpha)}$, for every $1\leqq\alpha\leqq d_{T*}-d_T$. Therefore, it is enough to prove that $T\prec U\oplus S^{(\alpha)}$ implies $\alpha\leqq d_{T*}-d_T$.

So, let us assume that $T\prec U\oplus S^{(\alpha)}$. Then we have $U\oplus T_2\prec T_1\oplus T_2\prec T\prec U\oplus S^{(\alpha)}$. Let $X\in\mathscr{I}(U\oplus T_2, U\oplus S^{(\alpha)})$ be a quasi-affinity. Since then $X^*\in\mathscr{I}(U^*\oplus S^{*(\alpha)}, U^*\oplus T_2^*)$ is also a quasi-affinity it follows that $X^*|\ker S^{*(\alpha)}\colon\ker S^{*(\alpha)}\to\ker T_2^*$ is an injection. Therefore we get that $\alpha=\dim\ker S^{*(\alpha)}\leqq\dim\ker T_2^*$. Taking into account that $d_{T*}-d_T=d_{T_2^*}-d_{T_2}$, we conclude by Lemma 8 that $\alpha\leqq d_{T*}-d_T$. The proof is finished.

Corollary 9. *If T is a contraction as in Theorem 7, then for the multiplicity of T^* we have:* $\mu_{T^*}=\mu_U$.

Proof. We infer by Theorem 7 and Lemma 2 that $T\prec U\oplus S\prec U$. It follows that $U^*\prec T^*$, and so $\mu_{T^*}\leqq\mu_{U^*}=\mu_U$. On the other hand $T^*\sim T_1^*\oplus T_2^*\sim U^*\oplus T_2^*$ implies $\mu_{T^*}\geqq\mu_{U^*}=\mu_U$.

5. Finally we show that if $T\in C_{1\cdot}$, $d_T<\infty$ and T is m. w. a. c. p., then there always exists an isometry V such that $V\prec T$. It can be easily seen that this is not the case, if T is not m. w. a. c. p. (cf. [5, Th. 5 and Prop. 2]).

Theorem 10. *If $T\in C_{1\cdot}$, $d_T<\infty$, is a contraction m. w. a. c. p., then $U\oplus S^{(\alpha)}\prec\prec T$, where $\alpha=d_{T^*}$.*

Proof. Let T_1, T_2 and U be the operators as in the begining of section 4. Since T is m. w. a. c. p., it follows that these operators act on non-zero spaces, and that U is not a s. u. operator. Therefore there exists a reducing subspace $\mathfrak{L}$ of U such that $U|\mathfrak{L}\cong M_E$ for some measurable set E $(m(E)>0)$. Taking into account that $T\sim T_1\oplus T_2\sim U\oplus T_2$, it is enough to prove that $M_E\oplus S^{(\alpha)}\prec M_E\oplus T_2$, where $\alpha=d_{T^*}$.

Let us consider the minimal isometric dilation $W\in\mathcal{L}(\mathfrak{K}_+)$ of the contraction $T_2\in\mathcal{L}(\mathfrak{H})$. Since $T_2\in C_{\cdot 0}$, it follows that W is a unilateral shift of multiplicity $\alpha=d_{T^*}$ (cf. [3, Th. II.1.2 and II.2.1]). Therefore we infer by the proof of Corollary 3 that there exists an injection $Y\in\mathcal{I}(M_E\oplus W, M_E\oplus T_2)$ such that $(Y(L^2(E)\oplus\{0\}))^-= =(\mathrm{ran}\, Y)^-=L^2(E)\oplus\{0\}$. Let P denote the orthogonal projection of the space $L^2(E)\oplus\mathfrak{K}_+$ onto its subspace $\{0\}\oplus\mathfrak{H}$. Then the operator $X=Y+P\in \in\mathcal{L}(L^2(E)\oplus\mathfrak{K}_+, L^2(E)\oplus\mathfrak{H})$ is obviously a quasi-affinity.

On the other hand, for any vector $f\oplus g\in L^2(E)\oplus\mathfrak{K}_+$ we have

$$(M_E\oplus T_2)X(f\oplus g)=(M_E\oplus T_2)Y(f\oplus g)+(M_E\oplus T_2)P(f\oplus g)=$$

$$=Y(M_E\oplus W)(f\oplus g)+(0\oplus T_2Pg)=Y(M_E\oplus W)(f\oplus g)+(0\oplus PWg)=$$

$$=X(M_E\oplus W)(f\oplus g).$$

Consequently we obtained that $M_E\oplus W\prec M_E\oplus T_2$, and so the proof is completed.

By Theorems 7 and 10 it follows immediately:

Corollary 11. *If $T\in C_{1\cdot}$, $d_T<\infty$, is a contraction m. w. a. c. p. and $d_{T^*}=\infty$, then we have*

$$T\sim U\oplus S^{(\infty)}.$$

If both defect indices of T are finite, then it is in general not true that $T\sim \sim U\oplus S^{(\alpha)}$, where $\alpha=d_{T^*}-d_T$. Indeed, contractions T with finite defect indices and quasi-similar to an isometry V, were characterized by P. Y. WU [2]. We note that if $T\in C_{1\cdot}$, $d_T<\infty$ and T is quasi-similar to an isometry V, then V is necessarily unitarily equivalent to the operator $U\oplus S^{(\alpha)}$, where $\alpha=d_{T*}-d_T$. This follows easily by Theorems 6 and 7.

References

[1] P. Y. WU, Approximate decompositions of certain contractions, *Acta Sci. Math.*, **44** (1982), 137—149.

[2] P. Y. WU, When is a contraction quasi-similar to an isometry?, *Acta Sci. Math.*, **44** (1982), 151—155.

[3] B. SZ.-NAGY and C. FOIAŞ, *Harmonic analysis of operators on Hilbert space*, North Holland — Akadémiai Kiadó (Amsterdam—Budapest, 1970).

[4] B. SZ.-NAGY and C. FOIAŞ, Modèles de Jordan pour une classe d'opérateurs de l'espace de Hilbert, *Acta Sci. Math.*, **31** (1970), 91—115.

[5] B. SZ.-NAGY and C. FOIAŞ, Jordan model for contractions of class $C_{\cdot 0}$, *Acta Sci. Math.*, **36** (1974), 305—322.

[6] B. SZ.-NAGY, Diagonalization of matrices over H^∞, *Acta Sci. Math.*, **38** (1976), 223—238.

[7] T. B. HOOVER, Quasi-similarity of operators, *Illinois J. Math.*, **16** (1972), 678—686.

[8] R. G. DOUGLAS, On the operator equation $S^*XT=X$ and related topics, *Acta Sci. Math.*, **30** (1969), 19—32.

[9] N. DUNFORD and J. SCHWARTZ, *Linear operators*, Part II (New York, 1963).

BOLYAI INSTITUTE
UNIVERSITY SZEGED
ARADI VÉRTANÚK TERE 1
H—6720 SZEGED, HUNGARY

Acta Sci. Math., 44 (1982), 165—171

Moment theorems for operators on Hilbert space

ZOLTÁN SEBESTYÉN

Introduction

The present note raises and solves moment like problems on the existence of a contraction, a subnormal operator and of a continuous semigroup of contractions, respectively, on a (complex) Hilbert space:

(A) Given a sequence $\{h_n\}_{n\geqq 0}$ of elements of the Hilbert space H, under what condition does there exist a contraction or a subnormal operator T on H such that

(1) $$h_n = T^n h_0 \quad \text{holds for} \quad n = 1, 2, \ldots .$$

(B) Given a continuous family $\{h_t\}_{t\geqq 0}$ of elements of the Hilbert space H, under what condition does there exist a continuous semigroup $\{T_t\}_{t\geqq 0}$ of contractions on H such that

(2) $$h_t = T_t h_0 \quad \text{holds for} \quad t \geqq 0.$$

The key to the solution (and of the source of these questions) is the theory of unitary and normal dilations.

The author is indepted to Professor B. Sz.-Nagy for his valuable advices, for his personal stimulation.

For normal extension of subnormal operators we refer to BRAM [1], HALMOS [2] and SZ.-NAGY [3].

Results

THEOREM A. *Let $\{h_n\}_{n\geqq 0}$ be a sequence of elements of the Hilbert space H. There exists a contraction T on H satisfying* (1) *if and only if*

(i) $$\Big\|\sum_{n,n'} c_{n,n'} h_{n+n'}\Big\|^2 \leqq \sum_{\substack{m\geqq n\\ m',n'}} c_{m,m'} \bar{c}_{n,n'} (h_{m-n+m'}, h_{n'}) + \sum_{\substack{m<n\\ m',n'}} c_{m,m'} \bar{c}_{n,n'} (h_{m'}, h_{n-m+n'})$$

holds for any finite double sequence $\{c_{n,n'}\}_{n\geqq 0,\, n'\geqq 0}$ of complex numbers.

Received April 22, 1981.

Theorem B. *Let* $\{h_t\}_{t\geqq 0}$ *be a continuous family of elements of a Hilbert space* H. *There exists a continuous semigroup* $\{T_t\}_{t\geqq 0}$ *of contractions in* H *satisfying* (2) *if and only if*

(ii) $$\Big\|\sum_{t,t'} c_{t,t'}\, h_{t+t'}\Big\|^2 \leqq \sum_{\substack{s\geqq t\\ s',t'}} c_{s,s'}\,\bar c_{t,t'}\,(h_{s-t+s'},\, h_{t'}) + \sum_{\substack{s<t\\ s',t'}} c_{s,s'}\,\bar c_{t,t'}\,(h_{s'},\, h_{t-s+t'})$$

holds for any finite double sequence $\{c_{t,t'}\}_{t\geqq 0,\, t'\geqq 0}$ *of complex numbers.*

Theorem C. *Let* $\{h_n\}_{n\geqq 0}$ *be a sequence of elements of the Hilbert space* H *such that*

(iii) $\{h_n\}$ *spans the space* H,

(iv) $\|h_n\|\leqq \mathscr{K}^n$ $(n=0,1,2,\ldots)$ *for some constant* $\mathscr{K}$.

There exists a subnormal operator T *on* H *satisfying* (1) *if and only if there exists a double sequence* $\{h_n^{n'}\}_{n,\, n'\geqq 0}$ *of elements of* H *such that*

(v) $h_n^0=h_n$ *for* $n=0,1,2,\ldots$,

(vi) $(h_n^{n'}, h_m)=(h_n, h_{m+n'})$ *for* $m,n,n'\geqq 0$, *and that*

(vii) $$\Big\|\sum_{n,n'} c_{n,n'}\, h_n^{n'}\Big\|^2 \leqq \sum_{m,m',n,n'} c_{m,m'}\,\bar c_{n,n'}\,(h_{m+n'},\, h_{m'+n})$$

holds for all finite double sequence $\{c_{n,n'}\}_{n,n'\geqq 0}$ *of complex numbers.*

Necessity

(A) Let U be a unitary dilation of the contraction T on the Hilbert space K containing H such that

(3) $$PU^n h = T^n h \quad (h\in H;\ n=1,2,\ldots)$$

holds with the orthogonal projection P of K onto H. Let further $\{c_{n,n'}\}_{n\geqq 0,\, n'\geqq 0}$ be a finite double sequence of complex numbers. We have then by (1) and (3)

$$\Big\|\sum_{n,n'} c_{n,n'}\, h_{n+n'}\Big\|^2 = \Big\|\sum_{n,n'} c_{n,n'}\, T^n h_{n'}\Big\|^2 = \Big\|\sum_{n,n'} c_{n,n'}\, PU^n h_{n'}\Big\|^2 \leqq$$

$$\leqq \Big\|\sum_{n,n'} c_{n,n'}\, U^n h_{n'}\Big\|^2 = \sum_{m,m',n,n'} c_{m,m'}\,\bar c_{n,n'}\,(U^m h_{m'},\, U^n h_{n'}) =$$

$$= \sum_{\substack{m\geqq n\\ m',n'}} c_{m,m'}\,\bar c_{n,n'}\,(U^{m-n} h_{m'},\, h_{n'}) + \sum_{\substack{m<n\\ m',n'}} c_{m,m'}\,\bar c_{n,n'}\,(h_{m'},\, U^{n-m} h_{n'}) =$$

$$= \sum_{\substack{m\geqq n\\ m',n'}} c_{m,m'}\,\bar c_{n,n'}\,(T^{m-n} h_{m'},\, h_{n'}) + \sum_{\substack{m<n\\ m',n'}} c_{m,m'}\,\bar c_{n,n'}\,(h_{m'},\, T^{n-m} h_{n'}) =$$

$$= \sum_{\substack{m\geqq n\\ m',n'}} c_{m,m'}\,\bar c_{n,n'}\,(h_{m-n+m'},\, h_{n'}) + \sum_{\substack{m<n\\ m',n'}} c_{m,m'}\,\bar c_{n,n'}\,(h_{m'},\, h_{n-m+n'}).$$

(B) The unitary dilation of a continuous semigroup $\{T_t\}_{t\geqq 0}$ of contractions is a continuous semigroup $\{U_t\}_{t\geqq 0}$ of unitaries on the dilations space K, such that

(4) $$PU_t h = T_t h \quad (h\in H,\ t\geqq 0)$$

holds, where P is the orthogonal projection of K onto H. Assume further $\{c_{t,t'}\}_{t\geqq 0,\, t'\geqq 0}$ is a finite double sequence of complex numbers indexed by nonnegative real numbers. (2) and (4) imply (ii) exactly in the same manner as before.

(C) Suppose N is a normal extension of T acting on a Hilbert space K containing H, and such that

(5) $$PN^{*n'}N^n h = T^{*n'}T^n h \quad (h\in H;\ n, n'\geqq 0)$$

holds with the orthogonal projection P of K onto H. Let further

(6) $$h_n^{n'} = T^{*n'}T^n h_0 \quad (n, n' = 0, 1, 2, \ldots).$$

Assuming (1) we have then $h_n^0 = T^n h_0 = h_n$ for $n = 0, 1, 2, \ldots$; and we have by (6) also that

$$(h_n^{n'}, h_m) = (T^{*n'}T^n h_0, T^m h_0) = (T^n h_0, T^{m+n'} h_0) =$$

$$= (h_n, h_{m+n'}) \quad (m, n, n' = 0, 1, 2, \ldots)$$

and, finally, that

$$\|\sum_{n,n'} c_{n,n'} h_n^{n'}\|^2 = \|\sum_{n,n'} c_{n,n'} T^{*n'}T^n h_0\|^2 = \|P\sum_{n,n'} c_{n,n'} N^{*n'}N^n h_0\|^2 \leqq$$

$$\leqq \|\sum_{n,n'} c_{n,n'} N^{*n'}N^n h_0\|^2 = \sum_{m,m',n,n'} c_{m,m'}\bar{c}_{n,n'}(N^{m+n'}h_0, N^{m'+n}h_0) =$$

$$= \sum_{m,m',n,n'} c_{m,m'}\bar{c}_{n,n'}(T^{m+n'}h_0, T^{m'+n}h_0) = \sum_{m,m',n,n'} c_{m,m'}\bar{c}_{n,n'}(h_{m+n'}, h_{m'+n})$$

holds for any finite double sequence $\{c_{n,n'}\}_{n,n'\geqq 0}$ of complex numbers.

Sufficiency

(A) Let F_0 be the (complex) linear space of all finite double sequences $\{c_{n,n'}\}_{n\geqq 0,\, n'\geqq 0}$ of complex numbers with the shift operation

$$U_0\{c_{n,n'}\} = \{c'_{n,n'}\}, \quad \text{where} \quad c'_{n,n'} = c_{n-1,n'}\ (n\geqq 1) \quad \text{and} \quad c'_{0,n'} = 0.$$

Let us introduce a semi-inner product in F_0 (in view of (i)) by

(7) $$\langle\{c_{m,m'}\}, \{d_{n,n'}\}\rangle = \sum_{\substack{m\geqq n\\ m',n'}} c_{m,m'}\bar{d}_{n,n'}(h_{m-n+m'}, h_{n'}) + \sum_{\substack{m<n\\ m',n'}} c_{m,m'}\bar{d}_{n,n'}(h_{m'}, h_{n-m+n'}).$$

U_0 is an isometry with respect to this semi-inner product. Defining

$$V_0\{c_{n,n'}\} = \sum_{n,n'} c_{n,n'} h_{n+n'} \quad \text{for} \quad \{c_{n,n'}\} \in F_0$$

we obtain a contraction V_0 from F_0 into H.

Let F be the Hilbert space resulting from F_0 by factoring with respect to the null space of $\langle \cdot, \cdot \rangle$ and by completing. At the same time U_0 induces an isometry U on F and V_0 induces a contraction V from F into H. In what follows the equivalence class represented by $\{c_{n,n'}\}$ is also denoted shortly by $\{c_{n,n'}\}$. We show that

(8) $$V^* h_k = \{d_{n,n'}\}, \quad \text{where} \quad d_{n,n'} = \begin{cases} 1 & \text{if} \quad n = 0, \quad \text{and} \quad n' = k \quad (k = 0, 1, \ldots), \\ 0 & \text{otherwise.} \end{cases}$$

To show this let $k \geqq 0$, $\{c_{m,m'}\} \in F$ so that (7) gives

$$\langle \{c_{m,m'}\}, V^* h_k \rangle = \langle V\{c_{m,m'}\}, h_k \rangle = \sum_{m,m'} c_{m,m'} (h_{m+m'}, h_k) = \langle \{c_{m,m'}\}, \{d_{n,n'}\} \rangle$$

as desired. Because of (8) we get

$$UV^* h_k = \{d_{n,n'}\}, \quad \text{where} \quad d_{n,n'} = \begin{cases} 1 & \text{if} \quad n = 1 \quad \text{and} \quad n' = k \quad (k = 0, 1, \ldots), \\ 0 & \text{otherwise.} \end{cases}$$

Defining

$$T = VUV^*$$

we have $Th_k = VUV^* h_k = h_{k+1}$ for all $k = 0, 1, \ldots$, but this is actually identical with (1).

(B) Let F_0 be, similarly as before, the linear space of all double sequences $\{c_{s,s'}\}_{s \geqq 0, s' \geqq 0}$ of complex numbers indexed by nonnegative real numbers. Define, for all $t \geqq 0$, by

$$U_t\{c_{s,s'}\} = \{c_{s-t,s'}\} \quad \text{for} \quad \{c_{s,s'}\} \in F_0$$

a shift operation and a semi-inner product (in view of (i)) by

$$\langle \{c_{r,r'}\}, \{d_{s,s'}\} \rangle = \sum_{\substack{r \geqq s \\ r,s'}} c_{r,r'} \bar{d}_{s,s'} (h_{r-s+r'}, h_{s'}) + \sum_{\substack{r < s \\ r,s'}} c_{r,r'} \bar{d}_{s,s'} (h_{r'}, h_{s-r+s'});$$

$\{U_t\}_{t \geqq 0}$ is then a continuous semigroup of isometries of the Hilbert space F derived from F_0 as before. By defining

$$V\{c_{s,s'}\} = \sum_{s,s'} c_{s,s'} h_{s+s'} \quad \text{for} \quad \{c_{s,s'}\} \in F_0$$

we get a contraction operator from F into H. The proof that $T_t = VU_tV^*$ $(t \geqq 0)$ is a continuous semigroup of contractions satisfying (2) only needs a slight modification of the argument used above, so we omit it.

(C) Let $\{h_n^{n'}\}_{n,n'\geqq 0}$ be in H such that conditions (iii—iv) are satisfied. Take the (complex) linear space F_0 of all finite double sequences $\{c_{n,n'}\}_{n,n'\geqq 0}$ of complex numbers with a shift operation

$$N_0\{c_{n,n'}\} = \{c'_{n,n'}\}, \quad \text{where} \quad c'_{n,n'} = c_{n-1,n'} \, (n \geqq 1), \quad \text{and} \quad c'_{0,n'} = 0;$$

and (in view of (vii)) with a semi-inner product in F_0 defined by

(9)
$$\langle \{c_{m,m'}\}, \{d_{n,n'}\} \rangle = \sum_{m,m',n,n'} c_{m,m'} \bar{d}_{n,n'} (h_{m+n'}, h_{m'+n}).$$

We are going to prove that

(*)
$$\|N_0\| \leqq \mathscr{K}$$

with the same $\mathscr{K}$ as that in (iv). First of all, for any $\{c_{n,n'}\}\in F_0$ and $i,j=0,1,2,\ldots$ we define

$$c_{n,n'}^{(i,j)} = \begin{cases} c_{n-i,n'-j}, & \text{if} \quad n \geqq i, n' \geqq j, \\ 0 & \text{otherwise.} \end{cases}$$

Now, by (9) we have

$$\|\{c_{n,n'}^{(i,j)}\}\|^2 = \sum_{m,m',n,n'} c_{m,m'} \bar{c}_{n,n'} (h_{m+n'+i+j}, h_{m'+n+i+j}) =$$

$$= \langle \{c_{n,n'}^{(i+j,i+j)}\}, \{c_{n,n'}\} \rangle \leqq \|\{c_{n,n'}^{(i+j,i+j)}\}\| \cdot \|\{c_{n,n'}\}\|.$$

So by induction we can derive

$$\|\{c_{n,n'}^{(1,0)}\}\|^{2^{k+1}} \leqq \|\{c_{n,n'}^{(2^k,2^k)}\}\| \cdot \|\{c_{n,n'}\}\|^{1+2+\ldots+2^k} \quad \text{for} \quad k = 0, 1, \ldots .$$

The definition of N_0 shows that $N_0\{c_{n,n'}\} = \{c_{n,n'}^{(1,0)}\}$ and so the above inequality, (9) and (iv) imply that

$$\|N_0\{c_{n,n'}\}\|^{2^{k+1}} \leqq \|\{c_{n,n'}^{(2^k,2^k)}\}\| \cdot \|\{c_{n,n'}\}\|^{1+2+\ldots+2^k} =$$

$$= \Big\{ \sum_{m,m',n,n'} c_{m,m'} \bar{c}_{n,n'} (h_{m+n'+2^{k+1}}, h_{m'+n+2^{k+1}}) \Big\}^{1/2} \|\{c_{n,n'}\}\|^{2^{k+1}-1} \leqq$$

$$\leqq \Big\{ \sum_{m,m',n,n'} |c_{m,m'}| |\bar{c}_{n,n'}| \|h_{m+n'+2^{k+1}}\| \cdot \|h_{m'+n+2^{k+1}}\| \Big\}^{1/2} \|\{c_{n,n'}\}\|^{2^{k+1}-1} \leqq$$

$$\leqq \|\{c_{n,n'}\}\|^{2^{k+1}-1} \sum_{n,n'} |c_{n,n'}| \, \mathscr{K}^{n+n'+2^{k+1}}.$$

This gives

$$\|N_0\{c_{n,n'}\}\| \leqq \|\{c_{n,n'}\}\|^{1-2^{-k-1}} \cdot \mathscr{K} \Big\{ \sum_{n,n'} |c_{n,n'}| \, \mathscr{K}^{n+n'} \Big\}^{2^{-k-1}}.$$

Let $k \to \infty$, so we obtain (*).

Defining

$$V_0\{c_{n,n'}\} = \sum_{n,n'} c_{n,n'} h_n^{n'} \quad \text{for} \quad \{c_{n,n'}\} \in F_0,$$

(vii) shows that V_0 is a contraction from F_0 into H. We obtain a Hilbert space F from F_0 by factoring with respect to the null space of $\langle \cdot, \cdot \rangle$ and then by completing.

At the same time, V_0 induces a contraction V from F into H and N_0 induces a bounded linear operator N on F.

Finally define the operator

$$T = VNV^* \tag{10}$$

on H. We are going to show that this operator is the desired one. First of all, for any $k \geqq 0$

$$V^* h_k = \{d_{n,n'}\}, \quad \text{where} \quad d_{n,n'} = \begin{cases} 1, & \text{if} \quad n = k \quad \text{and} \quad n' = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed,

$$\langle \{c_{m,m'}\}, V^* h_k \rangle = \langle V\{c_{m,m'}\}, h_k \rangle = \sum_{m,m'} c_{m,m'} (h_m^{m'}, h_k) =$$

$$= \sum_{m,m'} c_{m,m'} (h_m, h_{m'+k}) = \langle \{c_{m,m'}\}, \{d_{n,n'}\} \rangle.$$

Thus

$$T h_k = VNV^* h_k = V\{d_{n-1,n'}\} = \sum_{n,n'} d_{n,n'} h_{n+1}^{n'} = h_{k+1}^0 = h_{k+1}$$

holds for all $k = 0, 1, 2, \ldots$. We have (1) also as was desired. We have only to show that T in (10) is subnormal, that is,

$$\sum_{m,n} (T^m g_n, T^n g_m) \geqq 0 \tag{11}$$

holds for all finite sequence $\{g_n\}_{n \geqq 0}$ in H. We have (11) for elements of the form $g_n = \sum_{n'} \bar{c}_{n,n'} h_{n'}$ (where $\{c_{n,n'}\} \in F$) as a consequence of (vii). Indeed,

$$\sum_{m,n} (T^m g_n, T^n g_m) = \sum_{m,n} \Big(\sum_{n'} \bar{c}_{n,n'} T^m h_{n'}, \sum_{m'} \bar{c}_{m,m'} T^n h_{m'} \Big) =$$

$$= \sum_{m,m',n,n'} c_{m,m'} \bar{c}_{n,n'} (T^m h_{n'}, T^n h_{m'}) = \sum_{m,m',n,n'} c_{m,m'} \bar{c}_{n,n'} (h_{m+n'}, h_{m'+n}) \geqq$$

$$\geqq \Big\| \sum_{n,n'} c_{n,n'} h_n^{n'} \Big\|^2 \geqq 0,$$

which implies (11) in general by (iii). The theorem is proved.

Note that the proof of the theorem yields the following

Proposition. *Let $\{h_n^{n'}\}_{n,n' \geqq 0}$ be a double sequence in H which spans H. There exists a normal operator T on H such that*

$$T^{*n'} T^n h_0^0 = h_n^{n'} \quad (n, n' = 0, 1, 2, \ldots) \tag{12}$$

holds if and only if

$$\|h_n^{n'}\| \leqq \mathscr{K}^{n+n'} \quad \textit{for some constant} \quad \mathscr{K} > 0 \quad (n, n' \geqq 0) \tag{13}$$

and

$$(h_m^{m'}, h_n^{n'}) = (h_{m+n'}^0, h_{m'+n}^0) \quad (m, m', n, n' \geqq 0). \tag{14}$$

Proof. Assume (12), then (13) is trivial and (14) is elementary

$$(h_m^{m'}, h_n^{n'}) = (T^{*m'}T^m h_0^0, T^{*n'}T^n h_0) = (T^{m+n'}h_0^0, T^{m'+n}h_0) = (h_{m+n'}^0, h_{m'+n}^0).$$

Assume now (13) and (14) and denote h_n^0 by h_n ($n=0, 1, 2, \ldots$). We have then (v—vii) with equality in (vii), consequently the operator V, appearing in the proof of Theorem C, is a unitary operator from F onto H. Simple calculation shows that

$$N^*\{c_{n,n'}\} = \{c_{n,n'-1}\} \quad \text{for} \quad \{c_{n,n'}\}\in F \tag{15}$$

holds which yields $NN^*=N^*N$, that is, N is a normal operator. Since V is unitary, $T=VNV^*$ is normal, too. We have finally to show (12). T satisfies (1), and, by similar argument as in the proof of Theorem C, (15) implies that

$$VN^{*n'}V^*h_n = h_n^{n'}.$$

So we have

$$T^{*n'}T^n h_0^0 = (VN^*V^*)^{n'}h_n = VN^{*n'}V^*h_n = h_n^{n'} \quad \text{for} \quad n, n' = 0, 1, \ldots.$$

The proof is complete.

References

[1] J. Bram, Subnormal operators, *Duke Math. J.*, **22** (1955), 75—94.

[2] P. R. Halmos, Normal dilations and extensions of operators, *Summa Brasil. Math.*, **2** (1950), 153—156.

[3] B. Sz.-Nagy, *Extensions of linear transformations in Hilbert space with extend beyond this space* (Appendix to F. Riesz, B. Sz.-Nagy, Functional Analysis), Ungar (New York, 1960).

II. CHAIR OF ANALYSIS
EÖTVÖS LORÁND UNIVERSITY
MÚZEUM KÖRÚT 6—8
1088 BUDAPEST, HUNGARY

Bibliographie

R. E. Edwards, Fourier Series. A Modern Introduction, Vol. 1 (Graduate Texts in Mathematics, 64), xii+224 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1979.

This is the second edition of a book appeared first in 1967. There are numerous minor corrections. In addition, the author made a few substantial changes and supplements to the exposition.

The main aim of this book is to provide an introduction to some aspects of Fourier series and related topics, in which a liberal use is made of modern techniques. It may serve as a useful preparation for Rudin's "Harmonic Analysis on Groups" and for the second volume of Hewitt and Ross' "Abstract Harmonic Analysis".

The emphasis on modern techniques effects not only the type of arguments, but also to a considerable extent the choice of material. Above all, it leads to a minimal treatment of pointwise convergence and summability. The famous treatises by Zygmund and Bary on trigonometric series cover these aspects in great detail. On the other hand, a considerable attention is paid to matters that have not yet received a detailed treatment in a book form. Among such material, there appear comments on capacity, spectral synthesis sets, Helson sets and so forth, as well as remarks on extensions of results to more general groups. Katznelson's book "Introduction to Harmonic Analysis" can be read as a companion text.

The table of contents is the following: 1. Trigonometric series and Fourier series, 2. Group structure and Fourier series, 3. Convolutions of functions, 4. Homomorphisms of convolution algebras, 5. The Dirichlet and Fejér kernels, Cesàro summability, 6. Cesàro summability of Fourier series and its consequences, 7. Some special series and their applications, 8. Fourier series in L^2, 9. Positive definite series and Bochner's theorem, 10. Pointwise convergence of Fourier series.

The reader is supposed only to be familiar with Lebesgue integration. What is needed from functional analysis (Baire's category theorem, uniform boundedness principles, the closed graph, open mapping and Hahn—Banach theorems) is dealt with in Appendices A and B. The basic terminology of linear algebra is used, but no result of any depth is assumed.

Each chapter ends with exercises, the more difficult ones being provided with hints to their solutions. The bibliography contains many suggestions for further reading. The treatment is supplemented by a list of Symbols and an Index.

The present volume is an excellent introduction. It is addressed to undergraduate students and warmly recommended to everyone who wants to make a quick acquaintance with Fourier Analysis.

F. Móricz (Szeged)

Euclidean Harmonic Analysis, Proceedings of Seminars Held at the University of Maryland, 1979, edited by J. J. Benedetto (Lecture Notes in Mathematics, 779), iv + 177 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980.

During the spring semester of 1979 a program in Euclidean harmonic analysis was presented at the University of Maryland. This volume comprises six lecture series of them. The table of contents reads as follows.

1. L. Carleson, Some analytic problems related to statistical mechanics.

This is addressed to two main problems of classical statistical mechanics: (i) the verification of expected equilibrium thermodynamic properties, and (ii) the validity of the Gibbs theory for dynamical systems.

2. Y. Domar, On spectral synthesis in $\mathbf{R}^n$, $n\geqq 2$.

3. L. Hedberg, Spectral synthesis and stability in Sobolev spaces.

The following problem is discussed in these two lecture series: Let X be a class of distributions with support contained in a fixed subset of E of $\mathbf{R}^n$; determine whether or not a given element $\mu\in X$ is the limit in some designated topology of bounded measures contained in X. In Domar's case, the Fourier transform of X is a subset of $L^\infty(\mathbf{R}^n)$ with the weak* topology. In Hedberg's case, X is a Sobolev space with the norm topology.

4. R. Coifman and Y. Meyer, Fourier analysis of multilinear convolutions, Calderón's theorem, and analysis on Lipschitz curves.

5. R. Coifman, M. Cwikel, R. Rochberg, Y. Sagher and G. Weiss, The complex method for interpolation of operators acting on families of Banach spaces.

6. A. Córdoba, (i) Maximal functions: A problem of A. Zygmund, and (ii) Multipliers of $\mathscr{F}(L^p)$.

These three lecture series deal with the harmonic analysis of operators of L^p spaces. The problems studied have emerged mainly from the research of Zygmund, Calderón and Stein. In order to verify various L^p estimates for the Hilbert transform and related operators, R. Coifman and Y. Meyer present a range of real and complex methods. Next, G. Weiss, in a joint work with several others, set forth a theory of interpolation, which includes the Riesz—Thorin theorem and Stein's theorem for analytic families of operators. Finally, A. Córdoba solved several specific problems involving a thorough mix of many of the real methods.

The present book gives excellent accounts on the fast-growing development of Euclidean harmonic analysis, which has maintained a vital relationship with several other areas of mathematics for over 150 years. It will certainly stimulate some of the readers to attack the rather difficult problems of this important and fascinating field. We warmly recommend the book to everybody who wants to keep pace with up-to-date developments in Harmonic Analysis.

F. Móricz (Szeged)

T. W. Gamelin, Uniform Algebras and Jensen Measures (London Mathematical Society Lecture Note Series, 32), VIII + 162 pages, Cambridge University Press, Cambridge—London—New York—Melbourne, 1978.

These notes are based on various courses given by the author. The unifying theme is the notion of subharmonicity with respect to a uniform algebra. Dual to the generalized subharmonic functions are the Jensen measures.

The book consists of nine chapters. Chapter 1 provides an abstract treatment of R-measures, including the basic ideas of the Choquet theory. Chapters 2 and 3 show three natural choices for R-measures: the representing measures, Arens—Singer measures and Jensen measures.

Chapter 4 is based on some unpublished work of B. Cole, in which an open Riemann surface is constructed for which the corona problem has a negative answer. Chapter 5 introduces and treats various classes of quasi-subharmonic functions, algebras generated by Hartogs series, and the abstract Dirichlet problem for function algebras. The abstract development is applied in Chapter 6 to algebras of analytic functions of several complex variables. The key to applications is a theorem of H. Bremermann asserting that the abstract subharmonic functions essentially coincide with the plurisubharmonic functions.

Chapters 7 and 8 are devoted to the theory of the conjugation operation in the setting of uniform algebras. The M. Riesz and Zygmund inequalities turn out to be valid for Jensen measures, and the constants are the same as those that arise in the case of the disc algebra. On the other hand, they fail to extend to arbitrary representing measures. In Chapter 9 the problem of characterizing the moduli of the functions in $H^2(\sigma)$ is considered. The discussion is based on Cole's proof of a theorem of Helson.

Each chapter ends with references. The book is supplemented by a List of notation and an (author and subject) Index.

The presentation is self-contained and unified. Some of the results are published here for the first time. The book may serve as a starting point for research in an area of current interest. It is highly recommended for every graduate student who wishes to continue studies in Abstract Harmonic Analysis or Functional Analysis.

F. Móricz (Szeged)

Herman H. Goldstine, A History of the Calculus of Variations from the 17th through the 19th Century (Studies in the History of Mathematics and Physical Sciences, 5), XVIII+410 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1980.

The beginning of the calculus of variations can perhaps be dated from Fermat's elegant principle of least time, formulated in 1662 to show how a light ray was refracted at the interface between two optical media of different densities. He used the methods of the calculus to minimize the time of passage of a light ray through the two media. (By the way, Greek mathematicians were already aware of isoperimetric problems and their results were preserved for us by Pappus (c. 300 A. D.), but their methods were, of course, geometrical and not analytical).

The author attempts to trace the development of the calculus of variations during the period, in which the foundations of the modern theory were being laid. He chooses the most famous mathematicians of the period in question and concentrates on their major works.

The book is divided into seven chapters, and ends with a rich Bibliography containing about 200 items and a detailed Index.

Chapter 1 is entitled "Fermat, Newton, Leibniz, and the Bernoullis". During the 17th century mathematical notation began to improve quite markedly and the reasonable symbolisms contributed greatly to the development of mathematics. Fermat's work mentioned above seems to be clearly the first real contribution to the field. His method was adapted by John Bernoulli in 1696/97 to solve the brachystochrone problem (from *brachystos*, shortest, and *chronos*, time). The first genuine problem of the calculus of variations was formulated and solved by Newton in 1685. He investigated the motions of bodies moving through a fluid and led up to the general problem of motion in a resisting medium.

Chapters 2 and 3 ("Euler" and "Lagrange and Legendre") present the main achievements in the calculus of variations during the 18th century. In his book Euler treated 100 special problems and not only solved them but also set up the beginnings of a real general theory. His systematic investigations also served to influence the young Lagrange to seek and find a very elegant apparatus for solving problems. Lagrange explicitly formulated the famous multiplier rule, the so-called Euler—Lagrange rule, which became a sovereign tool in his hands for discussing analytical mechanics. This new tool caused Euler to name the subject appropriately the *calculus of variations*. In 1786 Legendre broke new ground by extending the calculus of variations from a study of the first variation to a study of the second variation as well.

Chapter 4 ("Jacobi and His School") is devoted to the works made in the first half of the 19th century. Legendre's analysis was not error-free, but Jacobi in 1836 wrote a remarkable paper on the second variation, in which the root of the matter was recognized. Among other things, he showed that the partial derivatives with respect to each parameter of a family of extremals satisfy the so-called Jacobi differential equation. However, none of Jacobi's results was proved in his paper. As a result a large number of commentaries were published, mainly to establish an elegant result of his on exact differentials. The celebrated Hamilton—Jacobi equation underlies some of the most profound and elegant results not only of the calculus of variations but also of mechanics, both classical and modern.

In the second half of 19th century two quite different directions were taken. On the one hand, Weierstrass went back to the first principles and not only placed the subject on a rigorous basis using the techniques of complex-variable theory, but discovered the so-called Weierstrass condition, fields of extremals, sufficient conditions for weak and strong minima, etc. On the other hand, Clebsch tentatively and A. Mayer decisively moved on quite another route. They succeeded in establishing the usual conditions for ever more general classes of problems. E.g., Mayer gave an elegant treatment of isoperimetric problems, in which he formulated his well-known reciprocity theorem. Details of their researches are presented in Chapter 5 ("Weierstrass") and Chapter 6 ("Clebsch, Mayer, and Others").

At the international mathematical congress of 1900 Hilbert gave a beautiful discussion of the calculus of variations. His greatest contributions were perhaps the discovery of his invariant integral together with the results that stem from it, the perception of the second variation as a quadratic functional with a complete set of eigenvalues and eigenfunctions, and his examination of existence theorems. Osgood, Bolza, Kneser, Carathéodory, etc., were also outstanding mathematicians at the turn of the century, whose major results are contained in Chapter 7 entitled "Hilbert, Kneser, and Others". Upon this point the present volume ends.

The above listing of the contents could hardly give a right impression of the richness of the book. It is written with a brilliant style and the text is illuminated by 66 illustrations. The book will certainly be a very instructive and profitable reading for everyone interested in the Calculus of Variations.

F. Móricz (Szeged)

G. Iooss and D. D. Joseph, Elementary stability and bifurcation theory (Undergraduate Texts in Mathematics), XV+286 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1980.

The nonlinear differential equations governing evolution problems generally contain some parameters. Therefore, the equilibrium solutions of such an equation depend on these parameters. Bifurcating sulutions are equilibrium solutions which form intersecting branches in a suitable space of functions. One of the central problems in bifurcation theory is: how do stability properties of equilibrium solutions change at bifurcation points?

The book is a very good text for teaching the principles of bifurcation. It gives a general theory abstracted from the detailed theory required for particular applications, and providing the reader with a "skeleton on which detailed structures of the applications must rest."

The following types of equilibrium are treated: steady solutions of autonomous problems, periodic solutions of nonautonomous problems, periodic solutions of steady problems, subharmonic solutions of periodic problems, subharmonic bifurcating solutions of periodic solutions of autonomous problems. Bifurcation of periodic solutions of autonomous and nonautonomous problems into "asymptotically quasi-periodic" solution is considered as well.

The book follows the simplest way of teaching the subject, starting with the analysis of one and two-dimensional problems and later demonstrating how the lower-dimensional problems relate to high-dimensional problems. Instead of the Center Manifold Theorem, the Implicit Function Theorem and the Fredholm Alternative are used for the computation of power series solutions and for the determination of qualitative properties of the bifurcating solutions.

Owing to its simplicity and generality, the book should be very useful to persons working in fields as diverse as biology, chemistry, engeneering, mathematics, and physics.

L. Hatvani (Szeged)

J. E. Marsden and M. McCracken, The Hopf bifurcation and its applications (Applied Mathematical Sciences, 19), XIII+408 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1976.

The Hopf bifurcation occurs in connection with dynamical systems containing some parameters and refers to the development of periodic orbits ("self-oscillations") from a stable fixed point, as a parameter crosses a critical value. This phenomenon can be illustrated by the following example. A rigid, hollow sphere with a small ball inside hangs from the ceiling and rotates about a vertical axis through its center. For small rotation frequences the bottom of the sphere is a stable point. But if the frequency exceeds a critical value then this equilibrium becomes unstable, the ball moves up the side of the sphere to a new fixed point. For each value of the frequency greater than the critical one there is a stable, invariant circle of fixed points.

The applications necessitate examination of Hopf bifurcation for vector fields and diffeomorphisms given on manifolds. The book originated at a seminar given in Berkeley in 1973—74 and contains contributions of many authors. It offers an excellent discussion of the theoretical results and applications of this topic. The basic tool is the "Center Manifold Theorem" which enables the infinite-dimensional problems to be reduced to finite dimensional ones. The authors give a survey on the necessary preliminaries from functional analysis, thus their book is readable for a wide circle of readers interested in this theory and its applications.

The book treats not only the new directions of research but also the classical results. For example, a translation of Hopf's original and generally unavailable paper is included. In Hopf's original approach, the determination of the stability of the resulting periodic orbits is, in concrete problems, an unpleasant calculation. The authors give explicit algorithms for this calculation which are easy to apply in examples. The method of averaging also is used for reducing the problem and establishing stability properties.

Chapters are devoted to partial differential equations, where the key assumption is that the semi-flow defined by the equations is smooth in all variables for $t>0$.

The importance of bifurcation theory is in its very close connections with applications. The reader can find interesting problems arising in fluid dynamics, population dynamics, celluar biology etc.

To sum up, we can warmly recommend this book for mathematicians, users of mathematics as well as science students.

L. Hatvani—J. Terjéki (Szeged)

R.O. Wells, Differential Analysis on Complex Manifolds (Graduate Texts in Mathematics, 65), x+260 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1979.

This book is the second edition of a succesful work which was first published by Prentice-Hall, Inc. (1973). The main program of the author is to give a very elegant development of Hodge's theory of harmonic integrals and Kodaira's characterization of projective algebraic manifolds.

The first four chapters discuss four somewhat different areas of mathematics.

Firstly differentiable manifolds and vector bundles are studied. Besides summarizing some of the basic definitions and results, this chapter contains some nontrivial embedding theorems, the continuous and C^∞ classification of vector bundles. Almost-complex structures and calculus of differentiable forms are also introduced.

Roughly speaking, sheaf theory gives techniques for passage from local information to global information. This theory is desribed in chapter 2.

Chapter 3 is an exposition of the basic ideas of Hermitian differential geometry with applications to Chern classes and holomorphic line bundles. The general theory of elliptic differential operators on compact differentiable manifolds can be found in the following chapter. The decomposition theorem of Hodge is proved here, asserting that for a self-adjoint differential operator the vector space of the sections is the orthogonal direct sum of the finite-dimensional null space and of the range of the operator. The Hodge's representation of the de Rahm cohomology by harmonic forms is also described.

The following chapter 5 is a main chapter of the book. Compact complex manifolds are studied here with the application of the previous discussions. Many basic theorems of this field are proved, for example the Lefschetz decomposition theorem, the Hodge decomposition theorem, Hodge's generalization of the Riemannian period relations for integrals of harmonic forms on Kähler manifolds, the Kadaira—Spencer upper semicontinuity theorem, etc. This chapter contains also a new section in addition to the first edition of the book. This is the classical finite dimensional representation theory for $sl(2C)$ which is then used for giving a natural proof of the Lefschetz decomposition theorem.

In the last chapter the famous Kodaire Embedding Theorem is proved, which asserts that a compact complex manifold admits an algebraic embedding into a complex projective space iff it is a Hodge manifold.

The book should be suitable for a graduate level course on the general topic of complex manifolds. The text is relatively self-contained but assumes familiarity with the usual first year graduate courses.

Z. I. Szabó (Szeged)

H. Werner und R. Schaback, Praktische Mathematik II (Methoden der Analysis), Hochschultext; Zweite, neubearbeitete und erweiterte Auflage, VIII+388 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1979.

The aim of this textbook is to provide a rigorous background of certain results widely used in Numerical Analysis. The treatment is self-contained, it requires the knowledge of calculus only.

The present volume consists of four chapters. Chapter 1 treats the theory of interpolation, involving multiple dimensional interpolation and fast Fourier transform. Chapter 2 is devoted to approximation theory, among others, to the Remes algorithm, the Fourier and Čebyšev expansions of continuous functions. Chapter 3 begins with spline functions, including cubic splines, B-splines etc. These results are then applied to the problem of representation of linear functionals, in particular, to numerical differentiation and integration. Chapter 4 deals with numerical methods for the initial value problem of ordinary differential equations. Both one-step methods, especially the classical Runge—Kutta methods, and predictor-corrector methods are presented in details. The notions of consistency, stability and convergence of a method plays central role in the treatment. This chapter ends with the presentation of stability theorems of Dahlquist.

Throughout the text there are various examples and figures (altogether 36) illuminating the material presented, and giving hints to further results found in the literature. The orientation of the reader is helped by a notational index as well as an author and subject index. Among the references one finds references to more than 40 textbooks.

The material presented in this well-readable book belongs to the main body of up-to-date Numerical Analysis. It will certainly be useful as a textbook for both science and engineering undergraduate students.

F. Móricz (Szeged)

A. Weron (ed.), Probability on Vector Spaces II, Proceedings, Błażejewko, Poland, 1979. (Lecture Notes in Mathematics, 628), XIII+324 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980.

From the editor's foreword: "This volume contains 30 contributions — the written and often extended versions of most lectures given at the Conference. A great majority of papers present new results in the field and the rest are expository in nature. The material in this volume complements the material in the earliner volume Probability Theory on Vector Spaces, Proceedings Lecture Notes in Math. vol. 656, 1978, Springer-Verlag".

Lajos Horváth (Szeged)

George W. Whitehead, Elements of Homotopy Theory (Graduate Texts in Mathematics, 61), XXI+744 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1979.

Homotopy theory is one of the most essential field of topology, which had its inception in the work of L. E. J. Brouwer. The book is concerned with the basic ideas and results of this theory in a modern treatment.

The fundamental notions and problems of the theory such as homotopy classes of mappings, fibrations, CW-complexes, the H- and H'-spaces, the Hurewitz map of homotopy group into homology group etc. are introduced in the first four chapters. The Hurewitz theorem is also proved, asserting that the Hurewitz homomorphism is an isomorphism if the basic space is $(n-1)$-connected.

The fifth chapter is devoted to the study of CW-approximations of spaces and of the extension problem of maps from a relative CW-complex onto the CW-complex. In the following chapter a new homology group is introduced, with the help of which results parallel to those of obstruction theory can then be proved.

The relationships among the homotopy groups of spaces arising from a fibration are expressed by an exact sequence. But the behaviour of the homology groups is much more complicated, and this can be examined only in certain cases. These problems are discussed in chapter 7, while the following chapter is devoted to the study of several cohomology operations.

For a 0-connected space X and positive integer N, one can embed X in a space X^N such that (X^N, X) is a $(N+1)$ -connected relative CW-complex with $\pi_q(X^N)=0$ for all $q>N$. The space X^{N+1} can be constructed from X^N with the help of a certain cohomology class $k^{N+1}\in H^{N+2}(X^N, \pi_{N+1}(X))$, and X is determined up to weak homotopy type by the so-called Postnikov system $\{X^N, k^{N+2}\}$ of X. In Chapter 9 the Postnikov systems are used to give an alternative treatment of obstruction theory for maps into X.

In the last three chapters the author turns to the detailed study of H-spaces, homotopy operations and homology theories without the dimension axiom.

The book is a very careful and clear work. It is a very good introduction to the field, at the same time it can be considered as a high level survey of the subject. It is assumed that the reader is familar with fundamental group theory and singular homology theory, including the universal coefficients and Künneth theorems.

Z.I. Szabó (Szeged)

Livres reçus par la rédaction

B. Angéniol, Familles de cycles algébriques —Schéma de Chow (Lecture Notes in Mathematics, Vol. 896), VI+140 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 18,—.

J. P. Bickel—N. El Karoui—M. Yor, Ecole d'été de probabilités de Saint-Flour IX—1979 (Lecture Notes in Mathematics, Vol. 876), IX+280 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. —DM 29,—.

J. Bourgain, New classes of $\mathcal{L}^p$-spaces (Lecture Notes in Mathematics, Vol. 889), V+143 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 18,—.

W. Buchholz—S. Feferman—W. Pohlers—W. Sieg, Iteratived inductive definitions and subsystems of analysis: Recent proof-theoretical studies (Lecture Notes in Mathematics, Vol. 897), V+383 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 39,—.

S. Burris—H. P. Sankappanavar, A course in universal algebra (Graduate Texts in Mathematics, Vol. 78), XVI+276 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 72,—.

H. Cajar, Billingsley dimension in probability spaces (Lecture Notes in Mathematics, Vol. 892), III+106 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 18,—.

C. T. Cannon—S. Dostrovsky, The evolution of dynamics: Vibration theory from 1687 to 1742 (Studies in the History of Mathematics and Physical Sciences, Vol. 6), IX+184 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 98,—.

Combinatorial Mathematics VIII, Proceedings, Geelong, Australia 1980. Edited by K. L. McAvaney (Lecture Notes in Mathematics, Vol. 884), XIV+359 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 39,—.

Combinatorics and Graph Theory, Proceedings, Calcutta 1980. Edited S. B. Rao (Lecture Notes in Mathematics, Vol. 885), VII+500 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 53,50.

The correspondence between A. A. Markov and A. A. Chuprov on the theory of probability and mathematical statistics. Edited by K. O. Ondar, XVII+181 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 74,—.

Cylindric Set Algebras and Related Structures. By L. Henkin, J. D. Monk and A. Tarski. On Cylindric-Relativized Set Algebras. By H. Andréka and I. Németi (Lecture Notes in Mathematics, Vol. 883), VII+323 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 34,50.

P. Deligne—J. S. Milne—A. Ogus—Kuang-yen Shih, Hodge cycles, motives, and Shimura varieties (Lecture Notes in Mathematics, Vol. 900), V+414 pages, Springer-Verlag, Berlin—Heidelberg —New York, 1981. — DM 43,50.

A. J. Dodd, The Core Model (London Mathematical Society Lecture Note Series, 61), XXXVIII+229 pages, Cambridge University Press, Cambridge—London—New York—Melbourne, 1982. — £ 12,50.

F. Dumortier—P. R. Rodrigues—R. Roussarie, Gems of diffeomorphisms in the plane (Lecture Notes in Mathematics, Vol. 902), IV+197 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 25,—.

Dynamical Systems and Turbulence, Warwick 1980, Proceedings. Edited by D. A. Rand and L.-S. Young (Lecture Notes in Mathematics, Vol. 898), VI+390 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 39,—.

S. Erlander, Optimal spatial interaction and the gravity model (Lecture Notes in Economics and Mathematical Systems, Vol. 173), VI+107 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980. — DM 22,—.

J. Erven—B.-J. Falkowski, Low order cohomology and applications (Lecture Notes in Mathematics, Vol. 877), VI+126 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 18,—.

R. Färe, Laws of diminishing returns (Lecture Notes in Economics and Mathematical Systems, Vol. 176), VIII+97 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980. — DM 22,—.

Fixed Point Theory. Proceedings of a Conference Held at Sherbrooke, Québec, Canada, June 2—21, 1980. Edited by E. Fadell and G. Fournier (Lecture Notes in Mathematics, Vol. 886), XII+511 pages, J pringer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 53,50.

O. Forster, Lectures on Riemann surfaces (Graduate Texts in Mathematics, Vol. 81), VIII+254 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 88,—.

L. R. Foulds, Optimization techniques. An introduction (Undergraduate Texts in Mathematics), XI+502 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 84,—.

Geometries and Gıoups. Proceedings of a Colloquium held at the Freie Universität Berlin, May 1981. Edited by M. Aigner and D. Jungnickel (Lecture Notes in Mathematics, Vol. 893), X+250 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 29,—.

Geometry and Analysis. Papers Dedicated to the Memory of V. K. Patodi. (Published for the Indian Academy of Sciences, Bangalore and Tata Institute of Fundamental Research, Bombay), IX+166 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981.

Geometry Symposium Utrecht 1980. Proceedings. Edited by E. Looijenga, D. Siersma and F. Takens (Lecture Notes in Mathematics, Vol. 894), V+153 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 21,50.

R. K. Guy, Unsolved problems in number theory (Problems Books in Mathematics. Unsolved Problems in Intuitive Mathematics, Vol. 1), XVIII+161 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 42,—.

E. Hecke, Lectures on the theory of algebraic numbers (Graduate Texts in Mathematics, Vol. 77), XII+239 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 79,—.

J. A. Hillman, Alexander ideals of links (Lecture Notes in Mathematics, Vol. 895), V+178 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 21,50.

F. C. Hoppensteadt, Mathematical methods of population biology (Cambridge Studies in Mathematical Biology), VII+149 pages, Cambridge University Press, Cambridge—London—New York—Melbourne, 1982. — £ 15.00

Loo-Keng Hua, Starting with the unit circle background to higher analysis (Translated from the Chinese by Kuniko Weltin), XI+179 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 69,—.

D. L. Iglehart—G. S. Shedler, Regenerative simulation of response times in networks of queues (Lecture Notes in Control and Information Sciences, Vol. 26), XII+204 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980. — DM 28,—.

Integral Representations and Applications, Proceedings, Oberwolfach, Germany 1980. Edited by K. W. Roggenkamp (Lecture Notes in Mathematics, Vol. 882), XII+479 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 48,50.

S. Itaka, Algebraic geometry. An introduction birational geometry of algebraic varieties (Graduate Texts in Mathematics, Vol. 76), X+357 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982. — DM 69,—.

A. Kock, Synthetic differential geometry (London Mathematical Society Lecture Notes Series, 51), VIII+311 pages, Cambridge University Press, Cambridge—London—New York—Melbourne, 1981. — £ 13.00.

D. S. Kubert—S. Lang, Modular units (Grundlehren der mathematischen Wissenschaften — A Series of Comprehensive Studies in Mathematics, 244), XIII+358 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1981. — DM 98,—.

Logic Symposia Hakone 1979, 1980. Proceedings of Conference held in Hakone, Japan, March 21—24, 1979 and February 4—7, 1980. Edited by G. H. Müller, G. Takeuti, and T. Tugué (Lecture

Notes in Mathematics, Vol. 891), XI+394 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 43,50.

R. Lutz—M. Goze, Nonstandard analysis. A practical guide with applications (Lecture Notes in Mathematics, Vol. 881), XIV+261 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 29,—.

J. Macki—A. Strauss, Introduction to optimal control theory (Undergraduate Texts in Mathematics), XIII+165 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982. — DM 58,—.

R. S. Millman—G. D. Parker, Geometry: A metric approach with models (Undergraduate Texts in Mathematics), X+355 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 78,—.

Model Theory and Arithmetic. Proceedings Paris 1979/80. Edited by C. Berline, K. McAloon, and J.-P. Ressayre (Lecture Notes in Mathematics, Vol. 890), VI+306 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 34,50.

Non Commutative Harmonic Analysis and Lie Groups. Proceedings, Marseille—Luminy 1980. Edited by J. Carmona and M. Vergne (Lecture Notes in Mathematics, Vol. 880), IV+553 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 59,—.

Numerical Solution of Nonlinear Equations. Proceedings, Bremen 1980. Edited by E. L. Allgower, K. Glashoff and H.-O. Peitgen (Lecture Notes in Mathematics, Vol. 878), XIV+440 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 48,50.

F. M. J. van Oystaeyen—A. H. M. J. Verschoren, Non-commutative algebraic geometry (Lecture Notes in Mathematics, Vol. 887), VI+404 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 43,50.

Padé Approximation and its Applications Amsterdam 1980. Proceedings, Edited by M. G. de Bruin and H. van Rossum (Lecture Notes in Mathematics, Vol. 888), VI+383 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 39,—.

R. Péter, Recursive functions in computer science, 179 pages, Akadémiai Kiadó, Budapest, 1981.

Representations of Algebras. Proceedings, Puebla, Mexico 1980. Edited by M. Auslander and E. Lluis (Lecture Notes in Mathematics, Vol. 903), XV+371 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 39,—.

R. D. Richtmyer, Principles of advanced mathematical physics, Vol. II (Texts and Monographs in Physics), XI+332 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 72,—.

C. P. Rourke—B. J. Sanderson, Introduction to piecewise-linear topology (Springer Study Edition), VIII+123 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982. — DM 24,—.

S. H. Saperstone, Semidynamical systems in infinite dimensional spaces (Applied Mathematical Sciences, Vol. 37), XIII+474 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982. — DM 68,—.

V. V. Sazonov, Normal approximation — Some recent advances (Lecture Notes in Mathematics, Vol. 879), VII+105 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 18,—.

Séminaire Bourbaki Vol. 1980/81. Exposés 561—578. Avec table par noms d'autors de 1967/68 à 1980/81 (Lecture Notes in Mathematics, Vol. 901), III+299 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 34,50.

J. K. Sengupta, Optimal decisions under uncertainty (Lecture Notes in Economics and Mathematical Systems, Vol. 193), VII+156 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 28,—.

R. W. Shephard, Cost and production functions (Lecture Notes in Economics and Mathematical Systems, Vol. 194), XI+104 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 22,—.

E. H. Spanier, Algebraic topology, XVI+528 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 69,—.

S. K. Srinivasan—R. Subramanian, Probabilistic analysis of redundant systems (Lecture Notes in Economics and Mathematical Systems, Vol. 175), VII+356 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980. — DM 52,—.

Statistische Nachrichtentheorie und ihre Anwendungen. Vorträge, gehalten auf dem 2. Internationalen Seminar — SNT 80 — in Schnett (Thüringen), DDR April 12.—16., 1980 (Abhandlungen der Akademie der Wissenschaften der DDR, Abteilung Mathematik, Naturwissenschaften, Technik, *N* 6(1981)), 219 Seiten, Akademie-Verlag, Berlin, 1981. — 35,— M.

M. Suzuki, Group theory I (Grundlehren der mathematischen Wissenschaften — A Series of Comprehensive Studies in Mathematics, 247), XIV+434 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982. — DM 118,—.

O. Teichmüller, Gesammelte Abhandlungen — Collected Papers, herausgegeben von L. V. Ahlfors und F. W. Gehring, VIII+751 Seiten, Springer-Verlag, Berlin—Heidelberg—New York, 1982. — DM 96,—.

H. W. Ursprung, Die elementare Katastrophentheorie: Eine Darstellung aus der Sicht der Ökonomie (Lecture Notes in Economics and Mathematical Systems, Vol. 195), VII+332 Seiten, Springer-Verlag, Berlin—Heidelberg—New York, 1982. — DM 46,—.

P. Walters, An introduction to ergodic theory (Graduate Texts in Mathematics, Vol. 79), IX+250 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982. — DM 69,50.

INDEX

ACTA SCIENTIARUM MATHEMATICARUM
SZEGED (HUNGARIA), ARADI VÉRTANÚK TERE 1
On peut s'abonner à l'entreprise de commerce des livres et journaux „Kultúra" (1061 Budapest, I., Fő utca 32)

ISSN 0324-6523 Acta Univ. Szeged
ISSN 0001-6969 Acta Sci. Math.

INDEX: 26024

81-5495 — Szegedi Nyomda — F. v.: Dobó József igazgató

Felelős szerkesztő és kiadó: Leindler László
A kézirat a nyomdába érkezett: 1981. december 21
Megjelenés: 1982. június 30

Példányszám: 1200. Terjedelem: 16 (A/5) ív
Készült monószedéssel, íves magasnyomással,
az MSZ 5601-24 és az MSZ 5602-55 szabvány szerint