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INSTITUTUM BOLYAIANUM UNIVERSITATIS SZEGEDIENSIS

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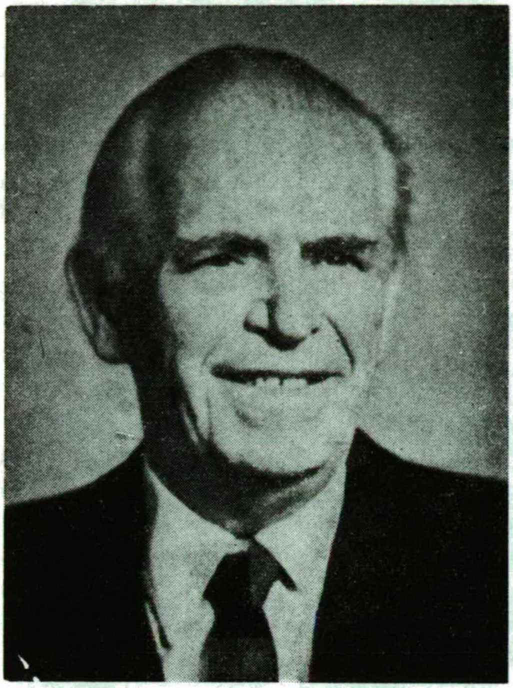

LÁSZLÓ RÉDEI

(1900—1980)

On November 21, 1980, six days after completing his eightieth year, passed away László Rédei, member of the editorial board of our *Acta* since its eleventh volume, professor emeritus and doctor honoris causa of our University, Nestor of the Hungarian algebraists.

He was born near Budapest and went to secondary school in the capital. His mathematical abilities presented themselves these years already; in 1918 he was a prize-winner of the Eötvös Competition. After having graduated from the University of Budapest, he became a secondary school teacher of mathematics and physics first in the town Miskolc, and then in Mezőtúr. In 1940, he was appointed professor in the University of Szeged, where he spent 27 years: the most fruitful years of his life. In 1967 he moved to Budapest, where he lead the Algebra Department of the Mathematical Institute of the Hungarian Academy of Sciences until his retirement in 1971.

The creative activity of László Rédei started in the twenties with algebraic number theory. He achieved deep results in the study of class groups of quadratic number fields, he determined among others the number of those invariants which are divisible by 2^k (the case $k=1$ was already settled by Gauss). Since 1940 his interests were more and more shifted to abstract algebra, and his enthusiastic works and lectures attracted many of his pupils to this subject also. Almost all algebraists working now in Hungary are his direct or indirect disciples. Particularly productive was his collaboration with the untimely deceased young mathematician Tibor Szele; among others, they were the first to recognize the functional completeness of finite fields. Rédei's further work on finite fields has also a growing interest in view of its connections with combinatorial theory.

In abstract algebra, group theory was Rédei's most favourite topic; even in his last year of life he studied the subdirect irreducibles in the variety of nilpotent groups of class 2. Today, when the old problem of finite simple groups is completely solved, we have to remind of Rédei's pioneering paper "Ein Satz über die endlichen einfachen Gruppen" (*Acta Math.*, **84** (1950), 129—153), which was a forerunner for the development of this area in the last three decades.

Besides about one and a half hundred papers, László Rédei was also the author of the books "Algebra", "The theory of finitely generated commutative semigroups", "Foundation of the Euclidean and non-Euclidean geometries", and "Lacunary polynomials over finite fields". The first and the third of these are advanced text-books with rich contents while the other two books summarize Rédei's own results in monographical form.

The merits of Professor Rédei were widely acknowledged by the competent communities. He was elected a corresponding member of the Hungarian Academy of Sciences in 1949, and an ordinary member in 1955. He was awarded the Kossuth prize two times and also several further high decorations.

László Rédei was one of the great Masters of Hungarian mathematics. He was a teacher and advisor for generations, and also an interesting personality, hero of numerous anecdotes and legends. He was an attractive lecturer with a fine sense of humour and self-irony. Last but not least, he had the rare endowment of being fully devoted to his beloved profession. His decease is a heavy loss for this journal also. We shall cherish his memory.

The Editors

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Reflexive and hyper-reflexive operators of class C_0

H. BERCOVICI, C. FOIAŞ, B. SZ.-NAGY

Dedicated to P. R. Halmos on his 65th birthday

The Jordan model of a finite matrix was used for the first time in the study of reflexive operators (on finite dimensional spaces) by DEDDENS and FILLMORE [5]. Their result was extended in [1] to the class of algebraic operators on Hilbert space, using the quasi-similar Jordan model (in fact in [1] the notion of para-reflexivity is studied, but one can easily see that reflexivity and para-reflexivity are equivalent for algebraic operators). The possibility of extending these results to the entire class C_0 was then indicated in [6] for the separable case and [2] (where a sketch of proof is done) for the nonseparable case. It appeared that the reflexivity of an operator of class C_0 is equivalent to the reflexivity of a single "Jordan block" $S(m)$ (cf. § 1 below for the precise statement).

In this note we give a simplified version of the proofs of [6] and [2]. We further study the related notion of hyper-reflexivity (stronger than reflexivity for the class C_0) and prove an analogous characterization of hyper-reflexive operators of class C_0.

1. Notations and results

We shall denote by $\mathfrak{H}$ a complex Hilbert space and by $\mathscr{B}(\mathfrak{H})$ the algebra of linear and bounded operators acting on $\mathfrak{H}$. For an algebra $\mathscr{A}\subset\mathscr{B}(\mathfrak{H})$, Lat $\mathscr{A}$ will stand for the set of closed linear subspaces $\mathfrak{M}\subset\mathfrak{H}$ invariant with respect to all elements of $\mathscr{A}$: $X\mathfrak{M}\subset\mathfrak{M}$, $X\in\mathscr{A}$. For a family $\mathscr{L}$ of closed linear subspaces of $\mathfrak{H}$, Alg $\mathscr{L}$ will denote the algebra of operators $X\in\mathscr{B}(\mathfrak{H})$ for which $X\mathfrak{M}\subset\mathfrak{M}$ whenever $\mathfrak{M}\in\mathscr{L}$. The algebra $\mathscr{A}\subset\mathscr{B}(\mathfrak{H})$ is called *reflexive* if $\mathscr{A}=\mathrm{Alg\,Lat}\,\mathscr{A}$. An operator $T\in\mathscr{B}(\mathfrak{H})$ is *reflexive* if the weakly closed algebra $\mathscr{A}_T$ generated by T and $I_{\mathfrak{H}}$ is a reflexive algebra. An operator $T\in\mathscr{B}(\mathfrak{H})$ will be called *hyper-reflexive* if its commutant $\{T\}'=(\mathscr{A}_T)'$ is a reflexive algebra.

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Recall that a completely nonunitary contraction $T\in\mathcal{B}(\mathfrak{H})$ is an operator of class C_0 if $u(T)=0$ for some $u\in H^\infty$, $u\neq 0$ (cf. [10], ch. V). The simplest operators of class C_0 are the "Jordan blocks" $S(m)$, with $m\in H^\infty$ an inner function, defined by

$$S(m)u = P_{\mathfrak{H}(m)}(z\,u(z)),\, u\in\mathfrak{H}(m) = H^2\ominus mH^2. \tag{1.1}$$

By the results of [11], [4] and [3], every operator T of class C_0 is quasi-similar to a unique Jordan operator, that is to an operator of the form

$$S = \bigoplus_{\alpha} S(m_\alpha) \tag{1.2}$$

where the values of α are ordinal numbers and the inner functions m_α are subject to the conditions

$$m_\alpha = 1 \quad \text{for some} \quad \alpha \geqq 0; \tag{1.3}$$

$$m_\alpha \text{ divides } m_\beta \text{ whenever } \alpha \geqq \beta; \tag{1.4}$$

$$m_\alpha = m_\beta \quad \text{whenever} \quad \operatorname{card}(\alpha) = \operatorname{card}(\beta). \tag{1.5}$$

Let us note that m_0 coincides with the minimal function m_T of T. The operators quasi-similar to some $S(m)$ are precisely the cyclic operators of class C_0 (multiplicity-free operators). For multiplicity-free T it follows from [12] that $\operatorname{Lat} T = \operatorname{Lat}\{T\}'$ and $\mathcal{A}_T = (\mathcal{A}_T)'$ so for such operators reflexivity and hyper-reflexivity are equivalent.

We are now able to state the main results of this note.

Theorem A. *An operator T of class C_0 with Jordan model $S=\bigoplus_\alpha S(m_\alpha)$ is reflexive if and only if $S(m_0/m_1)$ is reflexive.*

Theorem B. *Let T and S be as in Theorem* A. *Then T is hyper-reflexive if and only if $S(m_0)$ is reflexive.*

Recently P. Y. Wu [15] published a proof of Theorem A for the particular case of operators of class C_0 with finite defect indices.

2. Preliminary results

The following theorem plays an important role in the study of reflexive operators of class C_0 (cf. [13] and [14] for the proof).

Theorem 2.1. *For every operator T of class C_0 we have*

$$\mathcal{A}_T = \{T\}'' = \{T\}' \cap \operatorname{Alg}\operatorname{Lat} T. \tag{2.1}$$

Corollary 2.2. *An operator T of class C_0 is reflexive if and only if* Alg Lat $T\subset\subset\{T\}'$. — Obvious from relation (2.1).

Corollary 2.3. *Let $T\in\mathscr{B}(\mathfrak{H})$ be an operator of class C_0 and let $\mathfrak{M}_j\in\mathrm{Lat}\,T$ $(j\in J)$ be such that $T|\mathfrak{M}_j$ is reflexive for each j. If $\mathfrak{H}=\bigvee_{j\in J}\mathfrak{M}_j$ then T is reflexive.*

Proof. It follows from Corollary 2.2 that it is enough to show that every $X\in\mathrm{Alg\,Lat}\,T$ commutes with T. But it is obvious that for $X\in\mathrm{Alg\,Lat}\,T$ we have $X|\mathfrak{M}_j\in\mathrm{Alg\,Lat}\,(T|\mathfrak{M}_j)$ so that $X|\mathfrak{M}_j\in\{T|\mathfrak{M}_j\}'$ by the hypothesis. Therefore,

$$\ker(XT-TX)\supset\bigvee_{j\in J}\mathfrak{M}_j=\mathfrak{H},\quad\text{that is}\quad X\in\{T\}'.$$

Corollary 2.4. *Let $T\in\mathscr{B}(\mathfrak{H})$ be a reflexive operator of class C_0. For every $X\in\mathscr{A}_T$ the operator $T|(X\mathfrak{H})^-$ is reflexive.*

Proof. Let us take $Y\in\mathrm{Alg\,Lat}\,(T|(X\mathfrak{H})^-)$. Since $X\in\mathrm{Alg\,Lat}\,T$ we infer $YX\in$ Alg Lat T and therefore $YX\in\{T\}'$, by the reflexivity of T and Corollary 2.2. As X and T commute, we have $YT\cdot X=YX\cdot T=TY\cdot X$ such that $Y\in\{T|(X\mathfrak{H})^-\}'$ and the conclusion follows again by Corollary 2.2.

We shall introduce now an auxiliary property.

Definition 2.5. A completely nonunitary contraction T has *property* $(*)$ if for any quasi-affinity $X\in\{T\}'$ there exists a quasi-affinity $Y\in\{T\}'$ such that

$$XY=YX=u(T)\quad\text{for some } u\in H^\infty \tag{2.3}$$

for some $u\in H^\infty$.

Lemma 2.6. *Let T and T' be two quasi-similar completely nonunitary contractions. If T has property $(*)$ then T' does also. Moreover, if T has property $(*)$ then there exist quasi-affinities A, B such that $T'B=BT$, $TA=AT'$ and*

$$AB=u(T),\quad BA=u(T')\quad\textit{for some}\quad u\in H^\infty. \tag{2.4}$$

Proof. Let us assume that T has property $(*)$ and A, B' are two quasi-affinities such that $T'B'=B'T$ and $TA=AT'$. For any quasi-affinity $X\in\{T'\}'$ we have $AXB'\in\{T\}'$ so that, by the assumption, we have $AXB'\cdot Y'=Y'\cdot AXB'=u(T)$ for some quasi-affinity $Y'\in\{T\}'$ and $u\in H^\infty$. We obviously have

$$A(X\cdot B'Y'A-u(T'))=AXB'Y'\cdot A-Au(T')=u(T)A-Au(T')=0,$$

$$(B'Y'A\cdot X-u(T'))B'=B'\cdot Y'AXB'-u(T')B'=B'u(T)-u(T')B'=0$$

so that $X\cdot B'Y'A=u(T')$ by the injectivity of A, and $B'Y'A\cdot X=u(T')$ by the quasi-surjectivity of B'. So we have $XY=YX=u(T')$ for $Y=B'Y'A$ and therefore T' has property $(*)$. For the last assertion of the Lemma it is enough to set $B=B'Y'$ where Y' is obtained from the preceding proof for $X=I$. The Lemma follows.

Lemma 2.7. *Every Jordan operator of the form* $S=S(m_0)\oplus S(m_1)$ *has property* $(*)$.

Proof. Let $X\in\{S\}'$ be a quasi-affinity. By the Lifting Theorem ([10], sec. II. 2.3) there exists a 2×2 matrix $A=\begin{bmatrix} a & b \\ c & d\end{bmatrix}$ with entries in H^∞, such that

(2.5) $P_{\mathfrak{H}}A(I-P_{\mathfrak{H}})=0$ on $H^2\oplus H^2$, and $X=P_{\mathfrak{H}}A$ on $\mathfrak{H}=\mathfrak{H}(m_0)\oplus\mathfrak{H}(m_1)$.

Let us remark that

$$a\wedge b\wedge m_0 = 1. \tag{2.6}$$

Indeed, if $q=a\wedge b\wedge m_0\neq 1$ it follows that $\hat{q}=(1-\overline{q(0)}q)\oplus 0$ is a non-zero vector in $\mathfrak{H}$ such that for every vector of the form Xh $(h=h_0\oplus h_1\in\mathfrak{H})$ we have

$$(Xh,\hat{q})=(P_{\mathfrak{H}}Ah,\hat{q})=(Ah,\hat{q})=\int\big((a/q)h_0+(b/q)h_1\big)\big(q-q(0)\big)=0,$$

and this is impossible since X has dense range. Moreover, we have

$$\det A\wedge m_1 = 1. \tag{2.7}$$

Indeed, let us set $p=\det A\wedge m_1$ and denote $h=-b(m_1/p)\oplus a(m_1/p)$. Then we have, by (2.5),

$$XP_{\mathfrak{H}}h=P_{\mathfrak{H}}AP_{\mathfrak{H}}h=P_{\mathfrak{H}}Ah=P_{\mathfrak{H}}\big(0\oplus m_1\cdot(\det A)/p\big)=0$$

and therefore $P_{\mathfrak{H}}h=0$ by the injectivity of X. Hence, $h\in m_0H^2\oplus m_1H^2$, which implies that p divides b and a; taking account of the definition ot p we infer that p divides $a\wedge b\wedge m_1\wedge\det A$ also. Then (2.6) forces p to equal 1, concluding the proof of (2.7). From (2.6—7) it obviously follows that

$$\det A\wedge m_1 a\wedge m_1 b\wedge m_0 = 1$$

so that [7] (cf. also [9]) implies the existence of $c',d',e'\in H^\infty$ (even constans) such that $(\det A+m_1(ad'-be')+m_0e')\wedge m_0=1$ or, equivalently,

$$(\det A+m_1(ad'-bc'))\wedge m_0 = 1. \tag{2.8}$$

Let us remark now that the matrix $A'=\begin{bmatrix} a & b \\ c+m_1c' & d+m_1d'\end{bmatrix}$ satisfies the relations analogous to (2.5) and moreover $\det A'\wedge m_0=1$ by (2.8). Let us define $Yh=P_{\mathfrak{H}}Bh$ for $h\in\mathfrak{H}=\mathfrak{H}(m_0)\oplus\mathfrak{H}(m_1)$, where $B=\begin{bmatrix} d+m_1d' & -b \\ -c-m_1c' & a\end{bmatrix}$. It follows by direct computation that $Y\in\{S\}'$ and $XY=YX=u(S)$ with $u=\det A'$. Now $u(S)$ is a quasi-affinity because $u\wedge m_0=1$ (cf. [10], Prop. III. 4.7b) and therefore Y is also a quasi-affinity. The Lemma follows.

Remark 2.8. Lemma 2.7 also applies to operators of the form $S=S(m)$ (take $m_1=1$). By the celebrated theorem of SARASON [8] we have then, in fact, $X=u(S)$ with some $u\in H^\infty$, for *every* $X\in\{S\}'$.

3. Reflexive operators

The role of property $(*)$ in the study of reflexive operators is underlined by the following result.

Lemma 3.1. *Let T and T' be two quasi-similar operators of class C_0 having property $(*)$. Then T is reflexive if and only if T' is reflexive.*

Proof. By Lemma 2.6 there exist quasi-affinities A, B such that $T'B=BT$, $TA=AT'$ and $AB=u(T)$, $BA=u(T')$ for some $u\in H^\infty$. Assume T is reflexive. For any $X\in \operatorname{Alg\,Lat} T'$ and $\mathfrak{M}\in \operatorname{Lat} T$ we have $AXB\mathfrak{M}\subset A(B\mathfrak{M})^-\subset(AB\mathfrak{M})^-=$ $=(u(T)\mathfrak{M})^-\subset\mathfrak{M}$ because $(B\mathfrak{M})^-\in \operatorname{Lat} T'$ and $u(T)\in \operatorname{Alg\,Lat} T$. By the reflexivity of T we have $AXB\in\{T\}'$ and from the relations

$$A\cdot XT'\cdot B=AXB\cdot T=T\cdot AXB=A\cdot T'X\cdot B$$

it follows that $X\in\{T'\}'$. The reflexivity of T' follows then by Corollary 2.2, and Lemma 3.1 is proved.

For easier reference, let us formulate the following:

Lemma 3.2. *For two ('comparable') inner functions, say p and q, the operator V_{pq}: $\mathfrak{H}(p)\to\mathfrak{H}(q)$, defined by*

$$V_{pq}h=\begin{cases}P_{\mathfrak{H}(q)}h & \text{if } q \text{ divides } p \quad (3.1)\\ (q/p)h & \text{if } p \text{ divides } q \quad (3.2)\end{cases}\qquad (h\in\mathfrak{H}(p)),$$

intertwines $S(p)$ and $S(q)$.

Proof. If q divides p, we have for $h\in\mathfrak{H}(p)$, using (1.1) and (3.1),

$$(S(q)V_{pq}-V_{pq}S(p))h=P_{\mathfrak{H}(q)}\{zP_{\mathfrak{H}(q)}h-P_{\mathfrak{H}(p)}zh\}=0$$

because $zP_{\mathfrak{H}(q)}h=z(h+qw)=zh+qw'$, $P_{\mathfrak{H}(p)}zh=zh+pw'=zh+qw''$ with some $w, w', w''\in H^2$, and hence $\{...\}\in qH^2$.

If, conversely, p divides q, then we use the relation $P_{\mathfrak{H}(m)}u=u-m[\bar{m}u]_+$, valid for any inner m and for any $u\in H^2$, $[...]_+$ denoting here the natural projection $L^2\to H^2$. We get by (1.1) and (3.2)

$$(S(q)V_{p,q}-V_{p,q}S(p))h=P_{\mathfrak{H}(q)}z\frac{q}{p}h-\frac{q}{p}P_{\mathfrak{H}(p)}(zh)=$$

$$=\left(z\frac{q}{p}h-q\left[\bar{q}z\frac{q}{p}h\right]_+\right)-\frac{q}{p}(zh-p[\bar{p}zh]_+)=0$$

because $\bar{q}q=1, \dfrac{1}{p}=\bar{p}$ on the circle $\{z\colon |z|=1\}$.

Lemma 3.3. *Let* $S=S(m_0)\oplus S(m_1)$ *be a Jordan operator. Then for every* $X\in \operatorname{Alg\,Lat} S$ *there exists* $Y\in\mathscr{A}_S$ *such that* $X-Y=Z\oplus 0$ *with some operator* Z *on* $\mathfrak{H}(m_0)$ *and the zero operator on* $\mathfrak{H}(m_1)$.

Proof. The subspaces $\mathfrak{H}(m_0)\oplus\{0\}$ and $\{0\}\oplus\mathfrak{H}(m_1)$ are invariant for S so the assumption $X\in\operatorname{Alg\,Lat} S$ implies

$$X=X_0\oplus X_1,\quad X_j\in\operatorname{Alg\,Lat} S(m_j)\quad (j=1,2).$$

Consider the (obviously isometric) operator $V=V_{m_0,m_1}$ defined by (3.2), and the subspaces

$$\{Vh\oplus h\colon h\in\mathfrak{H}\{m_1)\}\quad\text{and}\quad\{VS(m_1)h\oplus h\colon h\in\mathfrak{H}(m_1)\}.$$

By Lemma 3.2, both are invariant for S, and hence for X also. So we infer

$$X_0Vh=VX_1h\quad\text{and}\quad X_0VS(m_1)h=VS(m_1)X_1h\quad\text{for}\quad h\in\mathfrak{H}\{m_1).$$

Apply the first equation for $S(m_1)h$ in place of h and compare the results to obtain $VX_1S(m_1)h=VS(m_1)X_1h$ for all $h\in\mathfrak{H}(m_1)$. Hence, $X_1S(m_1)=S(m_1)X_1$. By a well-known theorem of SARASON [8] this implies that $X_1=u(S(m_1))$ for some $u\in H^\infty$. Hence, $Y=u(S)=u(S(m_0))\oplus u(S(m_1))$ has the property we needed.

Lemma 3.4. *Let* $S=S(m_0)\oplus S(m_1)$ *be a Jordan operator and let* Z *be an operator on* $\mathfrak{H}(m_0)$ *such that* $Z\oplus 0\in\operatorname{Alg\,Lat} S$. *Then*

$$Z(qH^2\ominus m_0H^2)\subset qm_1H^2\ominus m_0H^2 \tag{3.3}$$

for every inner divisor q *of* m_0/m_1.

Proof. As m_1 is a divisor of m_0/q, which, in turn, is a divisor of m_0, we can consider the operators $V_0=V_{m_0/q,m_0}$ and $V_1=V_{m_0/q,m_1}$ defined by (3.2) and (3.1), respectively, and observe that $\{V_0h\oplus V_1h\colon h\in\mathfrak{H}(m_0/q)\}$ is a subspace invariant for S (closure follows from the fact that V_0 is an isometry, namely multiplication by the inner function q). Then it is invariant for $Z\oplus 0$ also. Hence we infer that for every $h\in\mathfrak{H}(m_0/q)$ there exists $h'\in\mathfrak{H}(m_0/q)$ such that $ZV_0h=V_0h'$ and $0=V_1h'$. As $V_1h'=P_{\mathfrak{H}(m_1)}h'$ by (3.1), we must have $h'\in\left(H^2\ominus\frac{m_0}{q}H^2\right)\ominus(H^2\ominus m_1H^2)$ i.e. $h'\in m_1H^2\ominus\frac{m_0}{q}H^2$. We conclude that $Zq\mathfrak{H}\left(\frac{m_0}{q}\right)\subset q\left(m_1H^2\ominus\frac{m_0}{q}H^2\right)$, and this obviously implies (3.3).

Remark. In the particular cases $q=1$ and $q=\frac{m_0}{m_1}$ (3.3) implies

$$\operatorname{ran} Z\subset m_1H^2\ominus m_0H^2\quad\text{and}\quad\ker Z\supset(m_0/m_1)H^2\ominus m_0H^2. \tag{3.4}$$

In the proof of the following result we shall use the *unitary* operator

$$R\colon m_1H^2\ominus m_0H^2 \to \mathfrak{H}(m_0/m_1) \quad \text{defined by} \quad Rh = h/m_1, \tag{3.5}$$

which satisfies the relation

$$RS(m_0)|(m_1H^2\ominus m_0H^2) = S(m_0/m_1)R = P_{\mathfrak{H}(m_0/m_1)}S(m_0)R. \tag{3.6}$$

Proposition 3.5. *The Jordan operator* $S=S(m_0)\oplus S(m_1)$ *is reflexive whenever* $S(m_0/m_1)$ *is reflexive.*

Proof. By Lemmas 3.3, 3.4, and Corollary 2.2 it suffices to show that every operator $Z\in\operatorname{Alg\,Lat} S(m_0)$ satisfying (3.3) commutes with $S(m_0)$. We claim that for such a Z we have $RZ|\mathfrak{H}(m_0/m_1)\in\operatorname{Alg\,Lat} S(m_0/m_1)$. Indeed, the general form of the subspaces in $\operatorname{Lat} S(m_0/m_1)$ is $qH^2\ominus(m_0/m_1)H^2$ for q a divisor of m_0/m_1. By (3.3—4) we have $RZ(qH^2\ominus(m_0/m_1)H^2)\subset RZ(qH^2\ominus m_0H^2)\subset R(qm_1H^2\ominus m_0H^2)=$ $=qH^2\ominus(m_0/m_1)H^2$. The reflexivity of $S(m_0/m_1)$ implies $RZ|\mathfrak{H}(m_0/m_1)\in\{S(m_0/m_1)\}'$. Therefore,

$$R(ZS(m_0)-S(m_0)Z)|\mathfrak{H}(m_0/m_1) = ((RZ)S(m_0)-RS(m_0)Z)|\mathfrak{H}(m_0/m_1) =$$
$$= ((RZ)P_{\mathfrak{H}(m_0/m_1)}S(m_0)-S(m_0/m_1)RZ)|\mathfrak{H}(m_0/m_1) = 0$$

so that Z commutes with $S(m_0)$ on $\mathfrak{H}(m_0/m_1)$. Because by (3.4) we have $ZS(m_0)=$ $=S(m_0)Z=0$ on $(m_0/m_1)H^2\ominus m_0H^2$ it follows that $Z\in\{S(m_0)\}'$. The Proposition is proved.

Proof of Theorem A. Let $T\in\mathscr{B}(\mathfrak{H})$ be of class C_0, with Jordan model $S=\bigoplus_\alpha S(m_\alpha)$ on $\mathfrak{H}=\bigoplus_\alpha \mathfrak{H}(m_\alpha)$. If T is reflexive we infer by Corollary 2.4 that $T|(m_1(T)\mathfrak{H})^-$ is reflexive. But $T|(m_1(T)\mathfrak{H})^-$ is quasi-similar to $S(m_0/m_1)$ and the reflexivity of $S(m_0/m_1)$ follows by Lemma 3.1 and Remark 2.8.

Conversely, let us assume that $S(m_0/m_1)$ is reflexive. Let X be any quasi-affinity such that $TX=XS$. Let us consider the spaces $\mathfrak{H}_\alpha=(X\mathfrak{H}(m_\alpha))^-$ and $\mathfrak{K}_\alpha=$ $=(X\ker m_\alpha(S|\mathfrak{H}(m_0)))^-$ for every ordinal number α. Then the restriction $T|\mathfrak{H}_0\vee\mathfrak{H}_1$ is quasi-similar to $S(m_0)\oplus S(m_1)$ and $T|\mathfrak{K}_\alpha\vee\mathfrak{H}_\alpha(\alpha\geqq 1)$ is quasi-similar to $S(m_\alpha)\oplus S(m_\alpha)$. All these restrictions are reflexive by Lemmas 2.7, 3.1 and Proposition 3.5 so that the reflexivity of T follows by Corollary 2.3 because $(\mathfrak{H}_0\vee\mathfrak{H}_1)\vee$ $\vee(\bigvee_{\alpha\geqq 1}(\mathfrak{H}_\alpha\vee\mathfrak{K}_\alpha))=\bigvee_{\alpha\geqq 0}\mathfrak{H}_\alpha=\mathfrak{H}$.

Corollary 3.6. *Let T and T' be two quasi-similar operators of class C_0. Then T is reflexive if and only if T' is reflexive.*

Proof. Two operators of class C_0 are quasi-similar if and only if they have he same Jordan model. Corollary obviously follows from Theorem A.

4. Hyper-reflexive operators

Proposition 4.1. *If the operators T and T' are quasi-similar and one of them is hyper-reflexive then so is the other.*

Proof. Let X and Y be two quasi-affinities such that $T'X=XT$ and $TY=YT'$ and let $A\in\operatorname{Alg\,Lat}\{T\}'$. Then $XAY\in\operatorname{Alg\,Lat}\{T'\}'$; indeed, for each $\mathfrak{M}\in\operatorname{Lat}\{T'\}'$ we have

$$\mathfrak{N}=\bigvee_{Z\in\{T\}'} ZY\mathfrak{M}\in\operatorname{Lat}\{T\}' \tag{4.1}$$

and $X\mathfrak{N}\subset\bigvee_{Z\in\{T\}'} XZY\mathfrak{M}\subset\bigvee_{Z'\in\{T'\}'} Z'\mathfrak{M}=\mathfrak{M}$. In particular, $XAY\mathfrak{M}\subset XA\mathfrak{N}\subset X\mathfrak{N}\subset\mathfrak{M}$ and $XAY\in\operatorname{Alg\,Lat}\{T'\}'$ because $\mathfrak{M}\in\operatorname{Lat}\{T'\}'$ is arbitrary.

If T' is hyper-reflexive it follows that $XAY\in\{T'\}'$ so that $X\cdot AT\cdot Y=XAY\cdot T'=$ $=T'\cdot XAY=X\cdot TA\cdot Y$ and $A\in\{T\}'$ because X and Y are quasi-affinities. It follows that T is hyper-reflexive. The Proposition is proved.

Proof of Theorem B. By the preceding proposition it is enough to consider the case $T=S$. Let us assume that S is hyper-reflexive and take $A\in\operatorname{Alg\,Lat}S(m_0)$. Then the operator $B=\bigoplus_\alpha A_\alpha$, where $A_0=A$ and $A_\alpha=0$ for $\alpha\geqq 1$, belongs to $\operatorname{Alg\,Lat}\{S\}'$. Indeed, since each $\mathfrak{K}\in\operatorname{Lat}\{S\}'$ has the form $\bigoplus_\alpha\mathfrak{K}_\alpha$ where $\mathfrak{K}_\alpha\in\operatorname{Lat}S(m_\alpha)$, we have $B\mathfrak{K}\subset\mathfrak{K}$. It follows that $B\in\{S\}'$ and this implies $A\in\{S(m_0)\}'$. The reflexivity of $S(m_0)$ follows by Corollary 2.2.

Conversely, let us assume that $S(m_0)$ is reflexive. Because $S(m_\alpha)$ is unitarily equivalent to $S(m_0)|(\operatorname{ran}u_\alpha(S(m_0))^-$ $(u_\alpha=m_0/m_\alpha)$ it follows by Corollary 2.4 that $S(m_\alpha)$ is reflexive for every α. We consider the operators $R_{\alpha\beta}\in\{S\}'$ defined by $R_{\alpha\beta}(\bigoplus_\gamma h_\gamma)=$ $=\bigoplus_\gamma k_\gamma$ where $k_\gamma=0$ for $\gamma\neq\alpha$ and

$$k_\alpha=V_{m_\beta,m_\alpha}h_\beta=\begin{cases}P_{\mathfrak{H}(m_\alpha)}h_\beta & \text{whenever}\quad \alpha>\beta,\\ (m_\alpha/m_\beta)h_\beta & \text{whenever}\quad \alpha\leqq\beta.\end{cases} \tag{4.2}$$

Cf. (3.1—2). Obviously, $P_\alpha=R_{\alpha\alpha}$ coincides with the orthogonal projection of $\bigoplus_\gamma\mathfrak{H}(m_\gamma)$ α-component space.

Let $A\in\operatorname{Alg\,Lat}\{S\}'$; we have $P_\alpha AP_\beta\in\operatorname{Alg\,Lat}\{S\}'$ and $A=\sum_{\alpha,\beta}P_\alpha AP_\beta$ in the strong operator topology. To conclude the proof it is enough to show that $P_\alpha AP_\beta\in\{S\}'$. Let us note that the operators $R_{\beta\alpha}P_\alpha AP_\beta$ and $P_\alpha AP_\beta R_{\beta\alpha}$ belong to $\operatorname{Alg\,Lat}\{S\}'$ and are of the form $\bigoplus_\gamma T_\gamma$ with $T_\gamma=0$ for $\gamma\neq\beta$ and $\gamma\neq\alpha$, respectively. Considering the spaces of the form $\ker m(S)\in\operatorname{Lat}\{S\}'$ for m a divisor of m_0, it is easily seen that necessarily $T_\gamma\in\operatorname{Alg\,Lat}S(m_\gamma)$ so that $T_\gamma\in\{S(m_\gamma)\}'$ by the reflexivity of $S(m_\gamma)$. It follows that $R_{\beta\alpha}P_\alpha AP_\beta$ and $P_\alpha AP_\beta R_{\beta\alpha}$ commute with S and therefore

$$R_{\beta\alpha}(P_\alpha AP_\beta S-SP_\alpha AP_\beta)=(P_\alpha AP_\beta S-SP_\alpha AP_\beta)R_{\beta\alpha}=0.$$

If the range of $R_{\beta\alpha}$ does not contain ran P_β it follows that $\beta<\alpha$ and therefore $R_{\beta\alpha}$ is one-to-one on ran P_α; therefore in both cases we infer $P_\alpha AP_\beta\in\{S\}'$. The Theorem is proved.

Remark 4.2. It follows from Theorems A and B that each hyper-reflexive operator of class C_0 is also reflexive. This fact can be proved directly also, by using Theorem 2.1.

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On the commutant of C_{11}-contractions

LÁSZLÓ KÉRCHY

1. We say that a Hilbert space operator T has property (P), or belongs to the operator class $\mathscr{P}$, if every injection $X \in \{T\}'$ is a quasi-affinity. B. SZ.-NAGY and C. FOIAŞ [1] proved that the operators of class C_0 and of finite multiplicity have property (P). H. BERCOVICI [2] characterized the class of all C_0-operators having property (P). Recently P. Y. WU [3] showed that every completely non-unitary (c. n. u.) C_{11}-contraction with finite defect indices belongs to the class $\mathscr{P}$. (Actually, he proved more.) The main purpose of this note is to characterize the class of all C_{11}-contractions having property (P).

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2. Only bounded linear operators on complex separable Hilbert spaces will be considered. Separability does not mean a restriction of generality, as it will turn out in section 5. We follow the notation and the terminology used in [4].

It is well-known that every contraction T of class C_{11} is quasi-similar to a unitary operator U (cf. [4], II.3.5). Moreover, since quasi-similar unitary operators are unitarily equivalent (cf. [4], II.3.4), the operator U is uniquely determined up to unitary equivalence.

If T is, moreover, a *c. n. u.* contraction of class C_{11}, then T is quasi-similar to the operator U of multiplication by e^{it} on the Hilbert space $\overline{\Delta L^2(\mathfrak{E})}$. (Cf. [4], VI.2.3.) Here Δ is the operator-valued function defined by $\Delta(e^{it})=[I-\Theta(e^{it})^*\Theta(e^{it})]^{1/2}$, where Θ denotes the characteristic function of T. This operator U has absolutely continuous spectral measure on the unit circle (i.e., is an a. c. u. operator). So U is unitarily equivalent to an operator M of the form $M=M_{E_1}\oplus M_{E_2}\oplus\dots$, where $\{E_n\}_n$ is a decreasing sequence of measurable subsets of the unit circle C of $\mathbf{C}$, and M_{E_n} denotes the operator of multiplication by e^{it} on the space $L^2(E_n)$. (We consider the normalized Lebesgue measure m on C.) For every measurable subset F of C let $F^=$ denote the closed support of the measure $m|F$, the restriction of m on the set F.

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If it is assumed that $E_n = E_n^=$ for every n, then the operator M is uniquely determined (cf. [5]). M will be called the *canonical functional model* of the a. c. u. operator U, and the *Jordan model* of the c. n. u. C_{11}-contraction T (cf. [6]).

Now we can state our main result:

Theorem 1. *Let T be a c. n. u. contraction of class C_{11} on the separable Hilbert space $\mathfrak{H}$, and let $M = M_{E_1} \oplus M_{E_2} \oplus \dots$ be its Jordan model. Then T has property (P) if and only if $m(\bigcap_{n \geqq 1} E_n) = 0$.*

Sufficiency and necessity of this condition will be proved in sec. 3 and sec. 4, respectively. In sec. 5 some corollaries are treated, while in sec. 6 we consider arbitrary C_{11}-contractions.

We shall use the following notation. For an operator-valued function N let $d_N(e^{it})$ denote the rank of the operator $N(e^{it})$. If T is a c. n. u. C_{11}-contraction, then let d_T be the function defined by $d_T(e^{it}) = d_\Delta(e^{it})$, where $\Delta = \Delta(e^{it})$ is the operator-valued function derived from the characteristic function $\Theta(e^{it})$ of T.

For two operators, T_1 and T_2, we denote by $\mathscr{I}(T_1, T_2)$ the set of intertwining operators $\mathscr{I}(T_1, T_2) = \{X | XT_1 = T_2 X\}$. Let Hyp lat (T) denote the lattice of hyperinvariant subspaces of T.

A system $\{\mathfrak{H}_n\}_{n \geqq 1}$ of subspaces of $\mathfrak{H}$ will be called *basic* if, for any n, the subspaces $\mathfrak{H}_n$, $\bigvee_{k \neq n} \mathfrak{H}_k$ are complementary and $\bigcap_{n \geqq 1} (\bigvee_{k \geqq n} \mathfrak{H}_k) = \{0\}$ (cf. [7]).

3. We shall need some lemmas. The first one should be contrasted with [4], VI. Th.6.1.

Lemma 1. *Let $N(e^{it})$ $(0 \leqq t \leqq 2\pi)$ be a function with values operators on a (separable) Hilbert space $\mathfrak{E}$, and measurable. Let us denote by U the restriction of the operator of multiplication by e^{it} on its reducing subspace $\mathfrak{N} = \overline{NL^2(\mathfrak{E})}$; and let $M = M_{E_1} \oplus M_{E_2} \oplus \dots$ be its canonical functional model. Then $d_N(e^{it}) = \operatorname{rank} N(e^{it})$ is a measurable function and for every $n \geqq 1$ we have*

$$E_n = \{e^{it} | d_N(e^{it}) \geqq n\}^=.$$

Proof. Let $\{e_j\}_j$ be an orthonormal basis of $\mathfrak{E}$. We denote by f_j the bounded measurable functions $f_j(e^{it}) = N(e^{it}) e_j$. Obviously the set $\{f_j(e^{it})\}_j$ generates $(N(e^{it})\mathfrak{E})^-$ for every $e^{it} \in C$, and therefore by [8], Ch. II, Prop. 9 it follows that the family $\mathfrak{H}(e^{it}) = (N(e^{it})\mathfrak{E})^-$, supplied with the notion of measurability induced by the constant field $\mathfrak{K}(e^{it}) = \mathfrak{E}$, is a measurable field of Hilbert spaces. Now we infer by [8], Ch. II, Prop. 1 that the function d_N is measurable. Moreover, by [8], Ch. II, Prop. 7 we have $\mathfrak{N} = \int_C^\oplus \mathfrak{H}(e^{it}) \, dm$, and so U is the diagonal operator $\int_C^\oplus e^{it} \, dm$. Denoting by F_m the measurable sets $F_m = \{e^{it} | d_N(e^{it}) = m\}$ $(m = 1, 2, \dots; \aleph_0)$

and applying [8], Ch. II, Prop. 3, we get that

$$U \cong \int_C^{\oplus} e^{it}\,dm \cong \bigoplus_m \Big(\int_{F_m}^{\oplus} e^{it}\,dm \Big) \cong \bigoplus_m M_{F_m}^{(m)} \cong \bigoplus_n M_{E_n},$$

where $E_n = \{e^{it} | d_N(e^{it}) \geqq n\}^{=}$. (For an arbitrary operator S, $S^{(m)}$ denotes the direct sum of m copies of S.)

Taking into account this Lemma we get a characterization for the measurable subsets in the Jordan model of a c. n. u. C_{11}-contraction. Namely, we have

Corollary 1. *If T is a c. n. u. contraction of class C_{11} on a (separable) Hilbert space $\mathfrak{H}$ and $M = M_{E_1} \oplus M_{E_2} \oplus \dots$ is its Jordan model, then $d_T(e^{it})$ is a measurable function, and for every natural number n we have*

$$E_n = \{e^{it} | d_T(e^{it}) \geqq n\}^{=}.$$

We shall frequently use the following:

Lemma 2. *If $T \in \mathcal{L}(\mathfrak{H})$ and $\mathfrak{H}_n \in \operatorname{Hyp\,lat} T$ $(n = 1, 2, \dots)$ are such that $\mathfrak{H} = \bigvee_{n \geqq 1} \mathfrak{H}_n$ and $T|\mathfrak{H}_n$ has property (P) for every n, then T has property (P).*

Lemma 3. *Let U be an a. c. u. operator on the separable Hilbert space $\mathfrak{H}$, let $M = M_{E_1} \oplus M_{E_2} \oplus \dots \in \mathcal{L}(\mathfrak{K})$ be its canonical functional model, and let E be the set defined by $E = \bigcap_{n \geqq 1} E_n$. Then the following conditions are equivalent:*

$$\text{(i)}\quad U \in \mathcal{P}; \qquad \text{(ii)}\quad m(E) = 0.$$

Proof.

a) Let us assume that $m(E) > 0$. Then

$$\begin{aligned} U &\cong M_{E_1} \oplus M_{E_2} \oplus \dots \cong (M_{E_1 \setminus E} \oplus M_{E \setminus E_2} \oplus \dots) \oplus (M_E \oplus M_E \oplus \dots) \cong \\ &\cong (M_{E_1 \setminus E} \oplus M_{E_2 \setminus E} \oplus \dots) \oplus (M_E \oplus M_E \oplus \dots) \oplus (M_E \oplus M_E \oplus \dots) \cong \\ &\cong (M_{E_1} \oplus M_{E_2} \oplus \dots) \oplus (M_E \oplus M_E \oplus \dots) = M \oplus M_E^{(\aleph_0)}. \end{aligned}$$

It is evident that $M_E^{(\aleph_0)} \notin \mathcal{P}$. Therefore $M \oplus M_E^{(\aleph_0)} \notin \mathcal{P}$, and so $U \notin \mathcal{P}$.

b) Let us assume that $m(E) = 0$. For every n let $\mathfrak{H}_n$ and $\mathfrak{K}_n$ be the subspaces defined by $\mathfrak{H}_n = \chi_{CE_n}(U)\mathfrak{H}$ and $\mathfrak{K}_n = \chi_{CE_n}(M)\mathfrak{K} = L^2(E_1 \setminus E_n) \oplus \dots \oplus L^2(E_{n-1} \setminus E_n)$. Since $M|\mathfrak{K}_n$ has finite multiplicity, and $U|\mathfrak{H}_n$ is unitary equivalent to $M|\mathfrak{K}_n$, we infer by [3], Lemma 2.5 that $U|\mathfrak{H}_n$ belongs to $\mathcal{P}$ for every n. On the other hand $\mathfrak{H}_n$ is a hyperinvariant subspace of U for every n, and in virtue of the assumption $\bigvee_{n \geqq 1} \mathfrak{H}_n = \mathfrak{H}$. The Proposition follows by Lemma 2.

We shall need yet the following:

Lemma 4. *If T is a c. n. u. contraction of class C_{11} on a separable Hilbert space $\mathfrak{H}$ and $\Theta_T(e^{it})^* \Theta_T(e^{it}) \geqq \delta$ holds a.e. for some constant $\delta > 0$, then T is similar to a unitary operator. (Here Θ_T denotes the characteristic function of T.)*

Proof. We infer by Propositions [4], V.7.1 and V.4.1 that Θ_T has an outer function scalar multiple u such that $|u(e^{it})|\geqq\delta^{1/2}$ a.e. Then $\|\Theta_T(\lambda)^{-1}\|$ has a bound independent of λ, and this implies by [4], Theorem IX.1.2 that T is similar to a unitary operator.

We are now able to prove the sufficiency.

Proposition 1. *Let T be a c. n. u. contraction of class C_{11} on a (separable) Hilbert space $\mathfrak{H}$, and let $M=M_{E_1}\oplus M_{E_2}\oplus\ldots$ be its Jordan model. If $m(\bigcap_{n\geqq 1} E_n)=0$, then $T\in\mathscr{P}$.*

Proof. Let $\Theta\in H^\infty(\mathscr{L}(\mathfrak{E}))$ coincide with the characteristic function of T. Let $N\in L^\infty(\mathscr{L}(\mathfrak{E}))$ be the function defined by $N(e^{it})=[\Theta(e^{it})^*\Theta(e^{it})]^{1/2}=[I-\Delta^2(e^{it})]^{1/2}$. In virtue of Corollary 1 we infer by the assumption that

(1) $$d_T(e^{it})=d_{\Delta^2}(e^{it})<\infty \quad \text{a.e.}$$

On the other hand since $T\in C_{11}$, it follows that Θ is outer from both sides, therefore $N(e^{it})$ is a quasi-affinity a.e. (Cf. [4], VI.3.5 and V.2.4.) Now we infer easily from these facts that $N(e^{it})$ is invertible a.e. Therefore its lower bound function $m(e^{it})=\inf\{\langle N(e^{it})e, e\rangle | e\in\mathfrak{E}, \|e\|=1\}$ is positive

(2) $$m(e^{it})>0 \quad \text{a.e.}$$

For every natural number n let α_n be the measurable set defined by

(3) $$\alpha_n=\left\{e^{it}\,|\,m(e^{it})>\frac{1}{n}\right\}.$$

It is evident that $\{\alpha_n\}_n$ is increasing:

(4) $$\alpha_1\subseteqq\alpha_2\subseteqq\cdots.$$

Moreover, in virtue of (2) we have

(5) $$m\big(C\setminus(\bigcup_{n\geqq 1}\alpha_n)\big)=0.$$

By the proof of Theorem VII.5.2 of [4], T has hyperinvariant subspaces $\mathfrak{H}_n$, such that

(6) $$\mathfrak{H}_1\subseteqq\mathfrak{H}_2\subseteqq\cdots,\quad \bigvee_{n\geqq 1}\mathfrak{H}_n=\mathfrak{H};$$

(7) $$\Theta_n(e^{it})^*\Theta_n(e^{it})=\begin{cases}\Theta(e^{it})^*\Theta(e^{it}) & \text{a.e. on } \alpha_n\\ I & \text{a.e. on } C\alpha_n,\ ^{1)}\end{cases}$$

1) Here and in the sequel we also use the notation $C\alpha$ for the set $C\setminus\alpha$, where α is any subset of C.

where Θ_n denotes a contractive analytic function such that the purely contractive part of Θ_n coincides with the characteristic function of $T|\mathfrak{H}_n$;

(8) $$T_n = T|\mathfrak{H}_n \in C_{11} \quad \text{for every } n.$$

We infer by Lemma 4 that, for every n, T_n is similar to a unitary operator.

Quasi-similar unitary operators being unitarily equivalent T_n is similar to its Jordan model $M_n = M_{E_1^{(n)}} \oplus M_{E_2^{(n)}} \oplus \ldots$ We infer by (7) that

(9) $$d_{T_n}(e^{it}) \leqq d_T(e^{it}) \quad \text{a.e.},$$

and it follows by (1) that

(10) $$d_{T_n}(e^{it}) < \infty \quad \text{a.e.}$$

By Corollary 1 and Lemma 3 we see that $M_n \in \mathscr{P}$. Since similarity preserves property (P), so for every n

(11) $$T_n \in \mathscr{P}.$$

Taking into account (6) and (11), we infer by Lemma 2, that $T \in \mathscr{P}$. The proof is finished.

4. Preparing for the proof of necessity we consider some Lemmas concerning a. c. u. operators.

Lemma 5. *Let U_1 and U_2 be a. c. u. operators having property (P). Then the operator $U = U_1 \oplus U_2$ has also property (P).*

Proof. Let $M_1 = M_{E_1} \oplus M_{E_2} \oplus \ldots \in \mathscr{L}(\mathfrak{H}')$ and $M_2 = M_{F_1} \oplus M_{F_2} \oplus \ldots \in \mathscr{L}(\mathfrak{H}'')$ be the canonical functional models of the operators U_1 and U_2 respectively. It is enough to prove that the operator $M = M_1 \oplus M_2 \in \mathscr{L}(\mathfrak{H} = \mathfrak{H}' \oplus \mathfrak{H}'')$ has the property (P).

Taking into account that the sequences $\{E_n\}_n$ and $\{F_n\}_n$ are decreasing we infer by Lemma 3 that $m(\bigcap_{n\geqq 1} (E_n \cup F_n)) = 0$. Therefore the hyperinvariant subspaces $\mathfrak{H}_n$, defined by $\mathfrak{H}_n = \chi_{C(E_n \cup F_n)} \mathfrak{H}$ $(n = 1, 2, \ldots)$, span the space $\mathfrak{H}$. Moreover $M|\mathfrak{H}_n$ has finite multiplicity, and so it belongs to $\mathscr{P}$ by Lemma 3. It follows that the operator M also has the property (P).

Lemma 6. *Let $U_1, U_2, \ldots$ be a. c. u. operators. If $m(\bigcap_{n\geqq 1} \sigma(\bigoplus_{k \geqq n} U_k)) > 0$, then there exists a strictly increasing sequence $\{n_k\}_k$ of natural numbers such that $n_1 = 0$ and $m\left(\bigcap_{k\geqq 1} \sigma\left(\bigoplus_{l=n_k+1}^{n_{k+1}} U_l\right)\right) > 0$.* ($\sigma(T)$ denotes the spectrum of T.)

Proof.

a) First of all we show that $\lim_{n\to\infty} m\left(\sigma\left(\bigoplus_{k=1}^{n} U_k\right)\right) = m\left(\sigma\left(\bigoplus_{k=1}^{\infty} U_k\right)\right)$. If $E_n(\cdot)$ denotes the spectral measure of U_n for every n, then $E(\cdot)=\bigoplus_{n=1}^{\infty} E_n(\cdot)$ will be the spectral measure of the a. c. u. operator $U=\bigoplus_{n=1}^{\infty} U_n$. Therefore $E\left[\sigma\left(\bigoplus_{k=1}^{\infty} U_k\right)\setminus\left(\bigcup_{n=1}^{\infty}\sigma\left(\bigoplus_{k=1}^{n} U_k\right)\right)\right]=0$, and so $m\left[\sigma\left(\bigoplus_{k=1}^{\infty} U_k\right)\setminus\left(\bigcup_{n=1}^{\infty}\sigma\left(\bigoplus_{k=1}^{n} U_k\right)\right)\right]=0$. Since $\sigma\left(\bigoplus_{k=1}^{\infty} U_k\right)=\left[\bigcup_{n=1}^{\infty}\sigma\left(\bigoplus_{k=1}^{n} U_k\right)\right]^{-}$, we have $\lim_{n\to\infty} m\left[\sigma\left(\bigoplus_{k=1}^{n} U_k\right)\right]=m\left[\sigma\left(\bigoplus_{k=1}^{\infty} U_k\right)\right]$.

b) Let σ denote the set $\sigma=\bigcap_{n\geqq 1}\sigma(\bigoplus_{k\geqq n} U_k)$. Let us assume that we have defined $0=n_1<n_2<\ldots<n_r$ such that for every $1\leqq k\leqq r-1$ we have $m\left[\sigma\setminus\sigma\left(\bigoplus_{l=n_k+1}^{n_{k+1}} U_l\right)\right]<\frac{m(\sigma)}{4^k}$. Applying the result of a) we infer that the sequence $\left\{m\left[\sigma\setminus\sigma\left(\bigoplus_{l=n_r+1}^{n} U_l\right)\right]\right\}_n$ tends to zero. Therefore there exists an index $n_{r+1}>n_r$ such that $m\left[\sigma\setminus\sigma\left(\bigoplus_{l=n_r+1}^{n_{r+1}} U_l\right)\right]<\frac{m(\sigma)}{4^r}$. The sequence defined by recursion in this way has the property that $m\left[\sigma\setminus\left(\bigcap_{k=1}^{\infty}\sigma\left(\bigoplus_{l=n_k+1}^{n_{k+1}} U_l\right)\right)\right]<\sum_{k=1}^{\infty}\frac{m(\sigma)}{4^k}<m(\sigma)$. Therefore $m\left[\bigcap_{k=1}^{\infty}\sigma\left(\bigoplus_{l=n_k+1}^{n_{k+1}} U_l\right)\right]>0$, and the proof is finished.

Lemma 7. *Let $U_1\in\mathcal{L}(\mathfrak{H}_1)$, $U_2\in\mathcal{L}(\mathfrak{H}_2)$, ... be a. c. u. operators having property (P). Then the a. c. u. operator $U=\bigoplus_{n=1}^{\infty} U_n\in\mathcal{L}(\mathfrak{H})$ has property (P) if and only if $m(\bigcap_{n\geqq 1}\sigma(\bigoplus_{k\geqq n} U_k))=0$.*

Proof.

a) Let us assume that $m(\bigcap_{n\geqq 1}\sigma(\bigoplus_{k\geqq n} U_k))>0$. In virtue of Lemma 6 there exists a sequence $\{n_k\}_k$, $(n_1=0)$, such that $m(\sigma)>0$, where $\sigma=\bigcap_{k\geqq 1}\sigma(V_k)$ and $V_k=\bigoplus_{l=n_k+1}^{n_{k+1}} U_l$ for every natural number k. Then for every k we can decompose V_k into the direct sum $V_k=V_k'\oplus V_k''$ such that V_k' is unitary equivalent to M_σ. Let $X_k'\in\mathcal{I}(V_k', V_{k+1}')$ be a unitary operator, and $X_k''\in\{V_k''\}'$ be the identity operator $(k=1, 2, \ldots)$. In this way we get an injection $X\in\{U\}'$ which is not a quasi-surjection. Therefore $U\notin\mathcal{P}$.

b) Let us assume now that $m(\bigcap_{n\geqq 1} F_n)=0$, where $F_n=\sigma\left(\bigoplus_{k=n}^{\infty} U_k\right)$. Then the hyperinvariant subspaces $\mathfrak{M}_n=\chi_{CF_n}(U)\mathfrak{H}$ $(n=1, 2, \ldots)$ of U span the space $\mathfrak{H}$: $\bigvee_{n\geqq 1}\mathfrak{M}_n=\mathfrak{H}$. On the other hand, for every natural number k, $\chi_{CF_n}(U_k)\mathfrak{H}_k$ reduces

U_k, and so $U_k|\chi_{CF_n}(U_k)\mathfrak{H}_k \in \mathscr{P}$. Since $U|\mathfrak{M}_n \cong \bigoplus_{k=1}^{n-1} U_k|\chi_{CF_n}(U_k)\mathfrak{H}_k$, we infer by Lemma 5 that $U|\mathfrak{M}_n \in \mathscr{P}$ for every n. Therefore $U \in \mathscr{P}$, and this completes the proof.

Now we are ready to prove:

Proposition 2. *Let T be a c. n. u. contraction of class C_{11} on a (separable) Hilbert space $\mathfrak{H}$, and let $M = M_{E_1} \oplus M_{E_2} \oplus \dots$ be its Jordan model on the Hilbert space $\mathfrak{K}$. If $m(\bigcap_{n \geq 1} E_n) > 0$, then $T \notin \mathscr{P}$.*

Proof.

a) Since T is quasi-similar to the unitary operator M, we infer by [7] that there exist a basic system $\{\mathfrak{H}_n\}_n$ of invariant subspaces of T, and a reducing decomposition $\mathfrak{K} = \bigoplus_n \mathfrak{K}_n$ of $\mathfrak{K}$ such that for every n $T_n = T|\mathfrak{H}_n$ is similar to the a. c. u. operator $U_n = M|\mathfrak{K}_n$. For every n let $C_n \in \mathscr{I}(U_n, T_n)$ be an affinity, and let P_n denote the canonical projection of $\mathfrak{H}$ onto $\mathfrak{H}_n$ determined by the decomposition $\mathfrak{H} = \mathfrak{H}_n \dotplus (\bigvee_{k \neq n} \mathfrak{H}_k)$.

b) We can reduce the proof to the following two special cases:

(i) There exists an n such that $U_n \notin \mathscr{P}$.

(ii) $m(\bigcap_{n \geq 1} \sigma(U_n)) > 0$.

Indeed, assuming that $U_n \in \mathscr{P}$ for every n, and taking into account that $M = \bigoplus_{n \geq 1} U_n \notin \mathscr{P}$ (cf. Lemma 3), we infer by Lemmas 7 and 6 that there exists a sequence $\{n_k\}_k$, $(n_1 = 0)$, such that $m\left(\bigcap_{k \geq 1} \sigma\left(\bigoplus_{l = n_k + 1}^{n_{k+1}} U_l\right)\right) > 0$. Replacing the basic system $\{\mathfrak{H}_n\}_n$ by $\{\mathfrak{H}'_k\}_k$, where $\mathfrak{H}'_k = \mathfrak{H}_{n_k + 1} \dotplus \dots \dotplus \mathfrak{H}_{n_{k+1}}$, and the affinities C_n $(n = 1, 2, \dots)$ by $C'_k = C_{n_k + 1} \oplus \dots \oplus C_{n_{k+1}}$ $(k = 1, 2, \dots)$, we gain the case (ii). (It can be easily seen that for every finite index-set N_1 the linear manifolds $\dotplus_{k \in N_1} \mathfrak{H}_k$ and $(\bigvee_{k \in N_1} \mathfrak{H}_k) \dotplus (\bigvee_{k \notin N_1} \mathfrak{H}_k)$ are closed. Therefore the operators $C'_k = \bigoplus_{l = n_k + 1}^{n_{k+1}} C_l$ $(k = 1, 2, \dots)$ will be affinities, and $\{\mathfrak{H}'_k\}_k$ will be a basic system.)

c) Let us assume that there exists an n such that $U_n \notin \mathscr{P}$. It can be supposed that $n = 1$. Since similarity preserves the property (P), we infer that $T_1 \notin \mathscr{P}$. Therefore there exists an injection $X_1 \in \{T_1\}'$ which is not a quasi-surjection. Let $\{\alpha_n\}_n$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \alpha_n \|P_n\| < \infty$, and let $X \in \{T\}'$ be the operator defined by $Xf = \alpha_1 X_1 P_1 f + \sum_{k=2}^{\infty} \alpha_k P_k f$ $(f \in \mathfrak{H})$. If $Xf = 0$ $(f \in \mathfrak{H})$, then for every n $P_n f = 0$, and we can prove by induction that $f \in (\bigvee_{k \geq n} \mathfrak{H}_k)$ for every n. Therefore $f = 0$, and so X is an injection. On the other hand, $(\operatorname{ran} X)^- =$

$=(\operatorname{ran} X_1)^- \dotplus (\bigvee_{n\geq 2} \mathfrak{H}_n) \neq \mathfrak{H}_1 \dotplus (\bigvee_{n\geq 2} \mathfrak{H}_n) = \mathfrak{H}$, that is X is not a quasi-surjection. Therefore T does not belong to the class $\mathcal{P}$.

d) Let us now suppose that $m(\sigma)>0$, where $\sigma=\bigcap_{n\geq 1}\sigma(U_n)$. Then for every n there exists a reducing decomposition $\mathfrak{R}_n=\mathfrak{R}_n'\oplus\mathfrak{R}_n''$ such that $U_n|\mathfrak{R}_n'$ is unitary equivalent to the operator M_σ. Let $\mathfrak{H}_n'$ and $\mathfrak{H}_n''$ denote the subspaces defined by $\mathfrak{H}_n'=C_n\mathfrak{R}_n'$, $\mathfrak{H}_n''=C_n\mathfrak{R}_n''$. Then $\mathfrak{H}_n'\dotplus\mathfrak{H}_n''=\mathfrak{H}_n$ and $T_n'=T_n|\mathfrak{H}_n'$ is similar to $T_{n+1}'=T_{n+1}|\mathfrak{H}_{n+1}'$ for every n.

Let $X_n\in\mathcal{I}(T_n', T_{n+1}')$ be an affinity, and let P_n' denote the canonical projection of $\mathfrak{H}_n$ onto $\mathfrak{H}_n'$ determined by the decomposition $\mathfrak{H}_n=\mathfrak{H}_n'\dotplus\mathfrak{H}_n''$, moreover let P_n'' be the projection: $P_n''=I_{\mathfrak{H}_n}-P_n'$. Let $\{\alpha_n\}_n$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty}\alpha_n(\|X_n\|\|P_n'\|+\|P_n''\|)\|P_n\|<\infty$, and let $X\in\{T\}'$ denote the operator defined by $Xf=\sum_{n=1}^{\infty}\alpha_n(X_nP_n'+P_n'')P_nf$ $(f\in\mathfrak{H})$. As in the preceding point, it can be easily seen that X is an injection. On the other hand $(\operatorname{ran} X)^-=\mathfrak{H}_1''\dotplus(\bigvee_{n\geq 2}\mathfrak{H}_n)\neq\mathfrak{H}_1\dotplus(\bigvee_{n\geq 2}\mathfrak{H}_n)=\mathfrak{H}$, that is X is not a quasi-surjection. Therefore T does not have property (P), and the proof is completed.

5. In this section we consider some corollaries of Theorem 1.

Corollary 2. *Let T be a c. n. u. contraction of class C_{11}. Then T belongs to $\mathcal{P}$ if and only if its Jordan model $M=M_{E_1}\oplus M_{E_2}\oplus\ldots$ does.*

Proof. Cf. Theorem 1 and Lemma 3.

Corollary 3. *Property (P) is a quasi-similarity invariant for c. n. u. C_{11}-contractions.*

Corollary 4. *If T is a c. n. u. C_{11}-contraction having property (P), then its adjoint T^* also has property (P).*

Proof. We have only to note that the adjoint of an operator of the form M_E is unitary equivalent to the operator $M_{E^\sim}$, where $E^\sim=\{e^{it}|e^{-it}\in E\}$.

Corollary 5. *Let T be a c. n. u. contraction of class C_{11} on the non-necessarily separable Hilbert space $\mathfrak{H}$. If T has property (P), then the space $\mathfrak{H}$ is separable.*

Proof. Let us assume that T has property (P) and the space $\mathfrak{H}$ is non-separable. Then there exists a decomposition $\mathfrak{H}=\bigoplus_{\alpha<\beta}\mathfrak{H}_\alpha$ reducing for T, such that for every ordinal α less than the ordinal β the space $\mathfrak{H}_\alpha$ is separable. Let $M_\alpha=\bigoplus_{n\geq 1}M_{E_{\alpha,n}}$ be the Jordan model of the operator $T_\alpha=T|\mathfrak{H}_\alpha$. Since $m(E_{\alpha,1})>0$ for every $\alpha>\beta$, and β is non-denumerable, there exist a positive number $\varepsilon>0$ and a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of ordinals less than β, such that for every n we have $m(E_{\alpha_n,1})>\varepsilon$.

Let T' be the operator defined by $T'=\bigoplus_{n=1}^{\infty} T_{\alpha_n}$ on the separable Hilbert space $\mathfrak{H}'=\bigoplus_{n=1}^{\infty} \mathfrak{H}_{\alpha_n}$. Taking into account that $T\in\mathscr{P}$, we infer that $T'\in\mathscr{P}$, and $T_\alpha\in\mathscr{P}$ for every $\alpha<\beta$. T' being quasi-similar to the unitary operator $\bigoplus_{n=1}^{\infty} M_{\alpha_n}$, it follows that $\bigoplus_{n=1}^{\infty} M_{\alpha_n}$ is unitary equivalent to the Jordan model of T'. By Corollary 2 we infer that $\bigoplus_{n=1}^{\infty} M_{\alpha_n}\in\mathscr{P}$, and $M_{\alpha_n}\in\mathscr{P}$ for every n. Now it follows by Lemma 7 that $\lim_{n\to\infty} m(\sigma(\bigoplus_{k\geqq n} M_{\alpha_k}))=0$.

On the other hand for every n we have $m(\sigma(\bigoplus_{k\geqq n} M_{\alpha_k}))\geqq m(\sigma(M_{\alpha_n}))=m(E_{\alpha_n,1})>\varepsilon$, what is a contradiction. Therefore the space $\mathfrak{H}$ can't be separable, and the proof is completed.

Corollary 6. *Let T be a c. n. u. contraction of class C_{11}. If T has property (P) and $\mathfrak{L}$ is an invariant subspace of T such that $T|\mathfrak{L}\in C_{11}$, then $T|\mathfrak{L}$ has property (P) also.*

Proof. We infer by [4], VI.2.3, VII.1.1, VII.2.1 and VII.3.3 that $d_{T|\mathfrak{L}}(e^{it})\leqq \leqq d_T(e^{it})$ a.e. Now it follows by Corollary 1 and Theorem 1 that $T|\mathfrak{L}$ has property (P).

Corollary 7. *Let T_1 and T_2 be c. n. u. contractions of class C_{11}. If T_1 and T_2 belong to the class $\mathscr{P}$, then the direct sum $T_1\oplus T_2$ has property (P) also.*

Proof. We have only to refer to Corollary 2 and Lemma 5.

Corollary 8. *Let $T_1, T_2, \ldots$ be c. n. u. contractions of class C_{11} having property (P). Then the contraction $T=\bigoplus_{n=1}^{\infty} T_n$ belongs to the class $\mathscr{P}$ if and only if the series $\sum_{n=1}^{\infty} d_{T_n}(e^{it})$ converges a.e.*

Proof. Since $T_n\in\mathscr{P}$, it follows that $d_{T_n}(e^{it})<\infty$ a.e., and the Jordan model M_n of T_n has property (P). (Cf. Theorem 1, Corollary 1 and Lemma 3.) On the other hand we infer by Corollary 2 that the condition $T\in\mathscr{P}$ is equivalent to the condition $\bigoplus_{n=1}^{\infty} M_n\in\mathscr{P}$. But this latter is equivalent to $m(\bigcap_{n\geqq 1}\sigma(\bigoplus_{k\geqq n} M_k))=0$ by Lemma 7. On account of Corollary 1 and the proof of Lemma 6 we see that $m(\bigcap_{n\geqq 1}\sigma(\bigoplus_{k\geqq n} M_k))=0$ holds if and only if $\sum_{n=1}^{\infty} d_{T_n}(e^{it})<\infty$ a.e., and this completes the proof.

6. Finally we intend to characterize the non-necessarily c.n.u. contractions of class C_{11} having property (P). First of all we prove the following:

Lemma 8. *Let $T\in\mathscr{L}(\mathfrak{H})$ be a c. n. u. contraction of class C_{11}, and let $U\in\mathscr{L}(\mathfrak{N})$ be an a. c. u. operator. If both T and U have property (P), then their direct sum $S=T\oplus U\in\mathscr{L}(\mathfrak{E})$ belongs to $\mathscr{P}$ also.*

Proof. Let $M\in\mathscr{L}(\mathfrak{K})$ denote the Jordan model of T. By [7] there exist a basic system $\{\mathfrak{L}_k\}_k$ of invariant subspaces of T, and a decomposition $\mathfrak{K}=\bigoplus_{k\geq 1}\mathfrak{B}_k$ of $\mathfrak{K}$ reducing for M, such that for every k $T|\mathfrak{L}_k$ is similar to $M_k'=M|\mathfrak{B}_k$. Let $C_k\in\mathscr{I}(M_k', T|\mathfrak{L}_k)$ be an affinity $(k=1, 2, \ldots)$.

Since $T\in\mathscr{P}$, we infer by Corollary 2 that $M\in\mathscr{P}$ also. Now by Lemma 7 it follows that $m(\bigcap_{n\geq 1} C\sigma_n)=0$, where $\sigma_n=C\sigma(\bigoplus_{k>n} M_k')$ $(n=1, 2, \ldots)$. For every n let $\mathfrak{K}_n$, $\mathfrak{K}_n'$, $\mathfrak{H}_n$, $\mathfrak{H}_n'$ be defined by $\mathfrak{K}_n=\chi_{\sigma_n}(M)\mathfrak{K}=\bigoplus_{k\geq 1}\mathfrak{K}_{n,k}$, $\mathfrak{K}_n'=\chi_{C\sigma_n}(M)\mathfrak{K}=\bigoplus_{k\geq 1}\mathfrak{K}_{n,k}'$, where $\mathfrak{K}_{n,k}=\chi_{\sigma_n}(M_k')\mathfrak{B}_k$, $\mathfrak{K}_{n,k}'=\chi_{C\sigma_n}(M_k')\mathfrak{B}_k$ $(k=1, 2, \ldots)$, and $\mathfrak{H}_n=\bigvee_{k\geq 1}\mathfrak{H}_{n,k}$, $\mathfrak{H}_n'=$ $=\bigvee_{k\geq 1}\mathfrak{H}_{n,k}'$, where $\mathfrak{H}_{n,k}=C_k\mathfrak{K}_{n,k}$, $\mathfrak{H}_{n,k}'=C_k\mathfrak{K}_{n,k}'$ $(k=1, 2, \ldots)$. It is clear that for every n $\mathfrak{K}_{n,k}=\{0\}$ if $k>n$, and so $\mathfrak{K}_n=\bigoplus_{k=1}^{n}\mathfrak{K}_{n,k}$. It follows that $\mathfrak{H}_{n,k}=\{0\}$ if $k>n$, that is $\mathfrak{H}_n=\mathfrak{H}_{n,1}\dotplus\ldots\dotplus\mathfrak{H}_{n,n}$. Therefore the subspaces $\mathfrak{H}_n$ and $\mathfrak{H}_n'$ are complementary: $\mathfrak{H}_n\dotplus\mathfrak{H}_n'=\mathfrak{H}$, and $T_n=T|\mathfrak{H}_n$ is similar to $M_n=M|\mathfrak{K}_n$. Moreover $T_n'=T|\mathfrak{H}_n'$ is quasi-similar to $M_n'=M|\mathfrak{K}_n'$, and $m(\sigma(M_n)\Delta\sigma_n)=m(\sigma(M_n')\Delta C\sigma_n)=0$ for every n.

For every n let the subspaces $\mathfrak{N}_n$, $\mathfrak{N}_n'$, $\mathfrak{E}_n$, $\mathfrak{E}_n'$ be defined by $\mathfrak{N}_n=\chi_{\sigma_n}(U)\mathfrak{N}$, $\mathfrak{N}_n'=\chi_{C\sigma_n}(U)\mathfrak{N}$, $\mathfrak{E}_n=\mathfrak{H}_n\oplus\mathfrak{N}_n$ and $\mathfrak{E}_n'=\mathfrak{H}_n'\oplus\mathfrak{N}_n'$. Then the decomposition $\mathfrak{E}=\mathfrak{E}_n\dotplus\mathfrak{E}_n'$ reduces S, moreover the restriction $S_n=S|\mathfrak{E}_n=T_n\oplus U_n$ $(U_n=U|\mathfrak{N}_n)$ of S onto $\mathfrak{E}_n$ is similar to $M_n\oplus U_n$, and the restriction $S_n'=S|\mathfrak{E}_n'=T_n'\oplus U_n'$ $(U_n'=U|\mathfrak{N}_n')$ of S onto $\mathfrak{E}_n'$ is quasi-similar to $M_n'\oplus U_n'$.

Let $X\in\{S\}'$ be an arbitrary operator, and let n be a natural number. Let

$$\begin{bmatrix} X_{11}^{(n)} & X_{12}^{(n)} \\ X_{21}^{(n)} & X_{22}^{(n)} \end{bmatrix}$$

be the matrix of X in the decomposition $\mathfrak{E}=\mathfrak{E}_n\dotplus\mathfrak{E}_n'$. On account of $X\in\{S\}'$ we infer that $X_{21}^{(n)}\in\mathscr{I}(S_n, S_n')$. Let $Y_n\in\mathscr{I}(M_n\oplus U_n, S_n)$ and $Z_n\in\mathscr{I}(S_n', M_n'\oplus U_n')$ be quasi-affinities. Then the operator $X_n'=Z_nX_{21}^{(n)}Y_n$ belongs to $\mathscr{I}(M_n\oplus U_n, M_n'\oplus U_n')$ and we infer by [9], Lemma 4.1 that $(\ker X_n')^{\perp}$ and $(\operatorname{ran} X_n')^{-}$ are reducing subspaces of $M_n\oplus U_n$ and $M_n'\oplus U_n'$ respectively, and $(M_n\oplus U_n)|(\ker X_n')^{\perp}$ is unitary equivalent to $(M_n'\oplus U_n')|(\operatorname{ran} X_n')^{-}$. Since $m(\sigma(M_n\oplus U_n)\Delta\sigma_n)=m(\sigma(M_n'\oplus U_n')\Delta C\sigma_n)=0$, and $M_n\oplus U_n$, $M_n'\oplus U_n'$ are a. c. u. operators, it follows that $X_n'=0$, and so $X_{21}^{(n)}=0$. Therefore $\mathfrak{E}_n\in\operatorname{Hyp\,lat}(S)$.

On the other hand we infer by the equation $m(\bigcap_{n\geq 1} C\sigma_n)=0$ that $\bigvee_{n\geq 1} \mathfrak{E}_n=\mathfrak{E}$. Since we have that $S_n\in\mathscr{P}$ for every n (cf. Lemma 5), an argument similar to the end of the proof of Proposition 1 completes the proof.

It is well-known that for every contraction T of class C_{11} on the Hilbert space $\mathfrak{H}$ there exists a (unique) "canonical" decomposition $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2\oplus\mathfrak{H}_3$ of $\mathfrak{H}$ reducing for T, such that $T_1=T|\mathfrak{H}_1$ is a c. n. u. contraction of class C_{11}, $T_2=T|\mathfrak{H}_2$ is an a. c. u. operator and $T_3=T|\mathfrak{H}_3$ is a singular unitary operator. (Cf. [4], I.3.2.)

Theorem 2. *Let T be a contraction of class C_{11}, and let $T=T_1\oplus T_2\oplus T_3$ be its "canonical" decomposition. Then T has property (P) if and only if T_i belongs to $\mathscr{P}$ for $i=1, 2, 3$.*

Proof. Let us assume that $T_i\in\mathscr{P}$ for $i=1, 2, 3$. (The other part of the proof is trivial.)

Let $X\in\{T\}'$ be an arbitrary operator, and let $\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22}\end{bmatrix}$ be its matrix in the decomposition $\mathfrak{H}=(\mathfrak{H}_1\oplus\mathfrak{H}_2)\oplus\mathfrak{H}_3$. Then $X_{21}Z\in\mathscr{I}(M_1\oplus T_2, T_3)$, where M_1 is the Jordan model of T_1, and $Z\in\mathscr{I}(M_1\oplus T_2, T_1\oplus T_2)$ is a quasi-affinity. Since $M_1\oplus T_2$ is an a. c. u. operator and T_3 is a singular unitary operator, we infer by [9], Lemma 4.1 that $X_{21}Z=0$, and so $X_{21}=0$. A similar argument shows that $X_{12}=0$ also holds, therefore $\mathfrak{H}_1\oplus\mathfrak{H}_2$ and $\mathfrak{H}_3$ belong to Hyp lat (T). Applying Lemma 8 the Theorem follows.

It can be given a "canonical" functional model for an arbitrary singular unitary operator also. Now a singular measure μ plays the role of the Lebesgue measure, and the form of the space of the functional model is $L^2_\mu(E_1)\oplus L^2_\mu(E_2)\oplus\ldots$, $(E_1\supseteq E_2\supseteq\ldots)$. Lemma 3 also holds its validity if condition (ii) is replaced by $\mu(E)=0$. Taking into account the previous theorems, it can be easily seen that Corollaries 2—8 hold for arbitrary contractions of class C_{11} also.

*

In a subsequent paper shall continue the study of the class $\mathscr{P}\cap C_{11}$. Among others we shall show that, for quasi-similar $\mathscr{P}\cap C_{11}$-contractions, the lattices of C_{11}-invariant subspaces are isomorphic. (An invariant subspace $\mathfrak{L}$ for T is called C_{11}-invariant if $T|\mathfrak{L}\in C_{11}$.)

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Beitrag zur Theorie der starken Summierbarkeit mit einer Anwendung auf Orthonormalreihen

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§ 1. Einleitung

1. Der Begriff der starken Summierbarkeit — ursprünglich eingeführt von HARDY und LITTLEWOOD im Zusammenhang mit Fourierreihen — findet sich in der Literatur in verschiedenen Formen. Wir legen hier die Definition nach BORWEIN zugrunde, der in einer grundlegenden Arbeit [2] insbesondere die starke Summierbarkeit mittels Hausdorff-Verfahren untersucht hatte.

Definition 1. Es sei $P=(p_{n\nu})$ eine positive Matrix $(p_{n\nu}\geqq 0)$, $Q=(q_{n\nu})$ eine beliebige Matrix und $k>0$. Eine Folge $\{s_n\}_0^\infty$ heißt bezüglich P, Q *stark limitierbar von der Ordnung k gegen s:*

$$s_n \to s([P, Q]^k)$$

wenn die folgenden Reihen alle existieren und wenn gilt:

$$\sum_{\nu=0}^{\infty} p_{n\nu} \left| \sum_{\lambda=0}^{\infty} q_{\nu\lambda} s_\lambda - s \right|^k = o(1) \quad \text{für} \quad n \to \infty.$$

Wir werden uns hier besonders mit der $[C_\beta, C_\alpha]^k$-Summierbarkeit $(\beta>0, \alpha>-1)$ beschäftigen, die in der Vergangenheit schon sehr oft untersucht wurde. Ohne Anspruch auf Vollständigkeit seien etwa erwähnt: HARDY—LITTLEWOOD [3] $(\beta=1,$ $\alpha=0,\ k=1, 2)$, ZYGMUND [9] $\left(\beta=1,\ \alpha>-\frac{1}{2},\ k=2\right)$; ALEXITS [1] $(\beta=1,\ \alpha>-1,$ $k=2$; $[C_1, C_{\alpha-1}]^2$ heißt dort die starke C_α-Summierbarkeit); SUNOUCHI [7] $\left(\beta>\frac{1}{2},\right.$ $\left.\alpha>-\frac{1}{2},\ k=1\right)$; LEINDLER [6] (allgemeine Parameter).

2. Aus der starken Summierbarkeit läßt sich auf gewöhnliche Summierbarkeit schließen. So gilt allgemein, wenn P regulär und $k\geqq 1$ ist: $[P, Q]^k \Rightarrow Q$. (BOR-

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WEIN [2], Theorem 3.) In unserem Fall gilt also $[C_\beta, C_\alpha]^k \Rightarrow C_\beta C_\alpha \sim C_{\beta+\alpha}$ $(k \geqq 1)$. Für $k>1$ läßt sich sogar auf kleinere Ordnungen schließen.

Satz 1. *Es sei* $\beta>0$, $k>1$. *Dann gilt*

$$\text{(1)} \qquad \textit{für} \quad \alpha > -1+\frac{1}{k}: \quad [C_\beta, C_\alpha]^k \Rightarrow \begin{cases} C_{(\beta/k)+\alpha+\varepsilon} & (\varepsilon \geqq 0) \\ \sigma_n^{(\beta/k)+\alpha} = o\big((\ln n)^{1-(1/k)}\big), \end{cases}$$

$$\text{(2)} \qquad \textit{für} \quad \alpha = -1+\frac{1}{k}: \quad [C_\beta, C_\alpha]^k \Rightarrow \sigma_n^{(\beta/k)+\alpha+\varepsilon} = o\big((\ln n)^{1-(1/k)}\big) \quad (\varepsilon \geqq 0).$$

Der erste Teil der Aussage (1) wurde in Spezialfällen bewiesen von ZYGMUND [9] ($\beta=1$, $k=2$), KUTTNER [5] ($\beta=1$, $\alpha=0$, $k>1$), HYSLOP [4] ($\beta=1$, $k>1$).

Bemerkung. Wählen wir für ein $k>1$ $\beta=k-1$, so ist $\frac{\beta}{k}=1-\frac{1}{k}$ d. h. für $\alpha=-1+\frac{1}{k}$ ist $\frac{\beta}{k}+\alpha=0$ und wir erhalten

$$[C_{k-1}, C_{-1+(1/k)}]^k \Rightarrow \sigma_n^0 = s_n = o\big((\ln n)^{1-(1/k)}\big).$$

Für $k=2$ liefert dies einen Satz von ZYGMUND [9]:

$$[C_1, C_{-1/2}]^2 s_n = o\big(\sqrt{\ln n}\big).$$

3. Während man, wie oben erwähnt, für $\beta>0$, $k\geqq 1$ aus der starken Summierbarkeit $[C_\beta, C_\alpha]^k$ immer auf die gewöhnliche Summierbarkeit $C_\beta C_\alpha$ schließen kann, ist das Umgekehrte nicht der Fall. So hat schon WINN [8] ein Beispiel einer Folge gegeben, die $C_1 C_\alpha$-summierbar ist, aber nicht $[C_1, C_\alpha]^1$-summierbar. Man benötigt, um von der gewöhnlichen Summierbarkeit auf starke Summierbarkeit schließen zu können, noch eine starke Tauberbedingung. Da diese Bedingung öfters vorkommt, ist es zweckmäßig, eine Abkürzung einzuführen.

Definition 2. Eine Reihe $\sum_0^\infty a_n$ genügt der *Tauberbedingung* $T([P, Q]^k)$ falls

$$n a_n \rightarrow O([P, Q]^k).$$

HYSLOP [4] zeigte nun für $k \geqq 1$:

$$\{C_1 C_\alpha,\ T([C_1, C_1 C_\alpha]^k)\} \Leftrightarrow [C_1, C_\alpha]^k.$$

Dieses Resultat wurde verallgemeinert von BORWEIN [2]. Er zeigte für eine beliebige Hausdorffmatrix H und $k \geqq 1$

$$\{C_1 H,\ T([C_1, C_1 H]^k)\} \Leftrightarrow [C_1, H]^k.$$

Wir werden zeigen:

Satz 2. *Es sei* $0<\beta\leqq 1$, $\alpha>-1$, $k\geqq 1$. *Dann gilt*

$$\{C_\beta C_\alpha,\ T([C_\beta, C_1 C_\alpha]^k)\} \Leftrightarrow [C_\beta, C_\alpha]^k.$$

Bemerkungen. 1) In der Richtung $\Rightarrow$ läßt sich der Satz verschärfen. Es genügt, $C_1 C_\alpha$ vorauszusetzen.

2) Ist P regulär, so folgt für beliebige Matrizen Q und $k>0$: $Q \Rightarrow [P, Q]^k$ (vgl. Borwein [2], Theorem 3). In unserem Fall lautet die Aussage: $C_\alpha \Rightarrow [C_\beta, C_\alpha]^k$. Mit der Verschärfung unseres letzten Satzes können wir zeigen, daß wir den Index α in der Folgerung erniedrigen können. Ersetzen wir nämlich dort α durch $\alpha-1$, so lautet die Aussage

$$\{C_\alpha,\ T([C_\beta, C_\alpha]^k)\} \Leftrightarrow [C_\beta, C_{\alpha-1}]^k.$$

4. Als nächstes zeigen wir, daß Orthonormalreihen unter einer gewissen Koeffizientenbedingung eine starke Tauberbedingung erfüllen.

Satz 3. *Es sei* $0<\beta\leqq 1$ *und* $\sum c_n \varphi_n(x)$ *eine beliebige Orthonormalreihe. Dann folgt aus*

$$\text{(1)} \qquad \sum_1^\infty n^{1-\beta} c_n^2 < \infty \quad \textit{für} \quad \alpha > \frac{1}{2},$$

$$\text{(2)} \qquad \sum_1^\infty n^{1-\beta} \ln n c_n^2 < \infty \quad \textit{für} \quad \alpha = \frac{1}{2},$$

die Tauberbedingung $T([C_\beta, C_\alpha]^2)$ *f. ü.*

Aus den beiden vorangehenden Sätzen folgt schließlich aus Bemerkung 2 zu Satz 2 und Satz 3:

Satz 4. *Es sei* $0<\beta\leqq 1$ *und* $\sum c_n \varphi_n(x)$ *eine beliebige Orthonormalreihe. Dann folgt aus der* C_α*-Summierbarkeit f. ü. und*

$$\text{(1)} \qquad \sum_1^\infty n^{1-\beta} c_n^2 < \infty \quad \textit{für} \quad \alpha > \frac{1}{2}$$

$$\text{(2)} \qquad \sum_1^\infty n^{1-\beta} \ln n c_n^2 < \infty \quad \textit{für} \quad \alpha = \frac{1}{2}$$

die $[C_\beta, C_{\alpha-1}]^2$*-Summierbarkeit f. ü.*

§ 2. Beweis von Satz 1

Wir benutzen öfters, mit der üblichen Bezeichnung $A_\nu^\gamma = \binom{\nu+\gamma}{\nu}$, die Beziehung $\sum_{\nu=1}^{n-1} \frac{A_\nu^\gamma}{n-\nu} \sim n^\gamma \ln n$ $(\gamma > -1)$. Ferner definieren wir wie üblich für eine Reihe $\sum_{\nu=0}^\infty a_\nu$

mit Partialsummenfolge $\{s_n\}_0^\infty$ die n-ten Cesàromittel der Ordnung α:

$$\sigma_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^\alpha a_\nu = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_\nu = \frac{S_n^\alpha}{A_n^\alpha}.$$

1) Es genügt, den Satz unter der Annahme $\sum_0^\infty a_n = O([C_\beta, C_\alpha]^k)$ zu beweisen. Ist nämlich $\sum_0^\infty a_n = s\ ([C_\beta, C_\alpha]^k)$, so folgt für die Reihe $\sum_0^\infty \bar{a}_n = (a_0 - s) + a_1 + \dots$ wegen $\bar{\sigma}_n^\alpha = \sigma_n^\alpha - s$: $\sum_0^\infty \bar{a}_n = o([C_\beta, C_\alpha]^k)$. Hieraus folgt dann $\bar{\sigma}_n^{(\beta/k)+\alpha+\varepsilon} \to 0$ bzw. $\bar{\sigma}_n^{(\beta/k)+\alpha+\varepsilon} = o\left((\ln n)^{1-(1/k)}\right)$ und damit $\sigma_n^{(\beta/k)+\alpha+\varepsilon} \to s$ bzw. $\sigma_n^{(\beta/k)+\alpha+\varepsilon} = o\left((\ln n)^{1-(1/k)}\right)$.

2) Es sei also $\sum_0^\infty a_n = O([C_\beta, C_\alpha]^k)$, d. h. $\sum_{\nu=0}^{n} A_{n-\nu}^{\beta-1} |\sigma_\nu^\alpha|^k = o(n^\beta)$. Dann folgt mit der Hölder'schen Ungleichung für $\varepsilon \geqq 0$:

$$|s_n^{(\beta/k)+\alpha+\varepsilon}| = \left| \sum_{\nu=0}^{n} A_{n-\nu}^{(\beta/k)+\varepsilon-1} A_\nu^\alpha \sigma_\nu^\alpha \right| =$$

$$= \left| \sum_{\nu=0}^{n} (A_{n-\nu}^{\beta-1})^{1/k} \sigma_\nu^\alpha \cdot A_{n-\nu}^{(\beta/k)+\varepsilon-1} A_\nu^\alpha (A_{n-\nu}^{\beta-1})^{-1/k} \right| \leqq \left\{ \sum_{\nu=0}^{n} A_{n-\nu}^{\beta-1} |\sigma_\nu^\alpha|^k \right\}^{1/k} \{S_n\}^{(k-1)/k},$$

mit

$$S_n := \sum_{\nu=0}^{n} \langle A_{n-\nu}^{(\beta/k)+\varepsilon-1} A_\nu^\alpha (A_{n-\nu}^{\beta-1})^{-1/k} \rangle^{k/(k-1)}.$$

Nun ist

$$S_n = O(1) \left\{ \langle n^{-1+(1/k)+\varepsilon} \rangle^{k/(k-1)} + \sum_{\nu=1}^{n-1} \langle (n-\nu)^{-1+(1/k)+\varepsilon} \nu^\alpha \rangle^{k/(k-1)} + \langle n^\alpha \rangle^{k/(k-1)} \right\} =$$

$$= O(1) \left\{ n^{-1+\varepsilon(k/k-1)} + \sum_{\nu=1}^{n-1} (n-\nu)^{-1+\varepsilon(k/k-1)} \nu^{\alpha(k/k-1)} + n^{\alpha(k/k-1)} \right\}.$$

a) für $\alpha > -1 + \frac{1}{k}$, d. h. $\alpha \frac{k}{k-1} > -1$ und $\varepsilon > 0$ ergibt sich

$$S_n = O(1) \left\{ n^{-1+\varepsilon(k/k-1)} + \sum_{\nu=1}^{n-1} A_{n-\nu}^{-1+\varepsilon(k/k-1)} A_\nu^{\alpha(k/k-1)} + n^{\alpha(k/k-1)} \right\} =$$

$$= O(1) \sum_{\nu=0}^{n} A_{n-\nu}^{-1+\varepsilon(k/k-1)} A_\nu^{\alpha(k/k-1)} = O(1) A_n^{\varepsilon(k/k-1)+\alpha(k/k-1)}.$$

Es folgt $s_n^{(\beta/k)+\alpha+\varepsilon} = o(n^{\beta/k}) O(n^{\alpha+\varepsilon})$ order $\sigma_n^{(\beta/k)+\alpha+\varepsilon} = o(1)$.

b) Für $\alpha > -1 + \frac{1}{k}$ und $\varepsilon = 0$ ergibt sich

$$S_n = O(1) \left\{ n^{-1} + \sum_{\nu=1}^{n-1} \frac{A_\nu^{\alpha(k/k-1)}}{n-\nu} + n^{\alpha(k/k-1)} \right\} =$$

$$= O(1) \{ n^{-1} + n^{\alpha(k/k-1)} \ln n + n^{\alpha(k/k-1)} \} = O(n^{\alpha(k/k-1)} \ln n).$$

Es folgt $s_n^{(\beta/k)+\alpha} = o(n^{\beta/k}) \cdot O(n^\alpha (\ln n)^{(k-1)/k})$ oder $\sigma_n^{(\beta/k)+\alpha} = o\left((\ln n)^{(k-1)/k}\right)$.

c) Für $\alpha=-1+\frac{1}{k}$ und $\varepsilon>0$ ergibt sich

$$S_n=O(1)\left\{n^{-1+\varepsilon(k/k-1)}+\sum_{\nu=1}^{n-1}\frac{A_{n-\nu}^{-1+\varepsilon(k/k-1)}}{\nu}+n^{-1}\right\}=$$

$$=O(1)\{n^{-1+\varepsilon(k/k-1)}+n^{-1+\varepsilon(k/k-1)}\ln n+n^{-1}\}=O(n^{-1+\varepsilon(k/k-1)}\ln n.$$

Es folgt $s_n^{(\beta/k)+\alpha+\varepsilon}=o(n^{\beta/k})\cdot O\left(n^{-(k-1/k)+\varepsilon}(\ln n)^{(k-1)/k}\right)$ oder

$$\sigma_n^{(\beta/k)+\alpha+\varepsilon}=o\left((\ln n)^{(k-1)/k}\right).$$

d) Für $\alpha=-1+\frac{1}{k}$ und $\varepsilon=0$ ergibt sich

$$S_n=O(1)\left\{n^{-1}+\sum_{\nu=1}^{n-1}\frac{1}{(n-\nu)\nu}+n^{-1}\right\}=$$

$$=O(1)\left\{2n^{-1}+\frac{1}{n}\sum_{\nu=1}^{n-1}\left(\frac{1}{\nu}+\frac{1}{n-\nu}\right)\right\}=O(1)\left\{2n^{-1}+\frac{2}{n}\ln n\right\}=O(n^{-1}\cdot\ln n).$$

Es folgt $s_n^{(\beta/k)+\alpha}=o(n^{\beta/k})\cdot O\left(n^{-(k-1)/k}(\ln n)^{(k-1)/k}\right)$ oder $\sigma_n^{(\beta/k)+\alpha}=o\left((\ln n)^{(k-1)/k}\right)$.

§ 3. Beweis von Satz 2

1) Es sei $\sum a_n=s$ $([C_\beta, C_\alpha]^k)$. Dann folgt zuerst $\sum a_n=s$ (C_β, C_α). Ferner mit der bekannten Beziehung

$$\tau_n^\alpha:=\frac{1}{A_n^\alpha}\sum_{\nu=1}^{n}A_{n-\nu}^{\alpha-1}\nu a_\nu=-\alpha(\sigma_n^\alpha-\sigma_n^{\alpha-1}): \tag{3.1}$$

$$\frac{1}{A_n^\beta}\sum_{\nu=1}^{n}A_{n-\nu}^{\beta-1}|\tau_\nu^{\alpha+1}|^k=O(1)\cdot\frac{1}{A_n^\beta}\sum_{\nu=1}^{n}A_{n-\nu}^{\beta-1}|\sigma_\nu^{\alpha+1}-\sigma_\nu^\alpha|^k=$$

$$=O(1)\left\{\frac{1}{A_n^\beta}\sum_{\nu=0}^{n}A_{n-\nu}^{\beta-1}|\sigma_\nu^{\alpha+1}-s|^k+\frac{1}{A_n^\beta}\sum_{\nu=0}^{n}A_{n-\nu}^{\beta-1}|\sigma_\nu^\alpha-s|\right\}=O(1)\{(1)+(2)\}.$$

Da aus der $C_\beta C_\alpha$-Summierbarkeit die $C_1 C_\alpha\sim C_{1+\alpha}$-Summierbarkeit folgt, gilt $\sigma_n^{\alpha+1}\to s$ und wegen der Regularität von C_β folgt $(1)=o(1)$. Die Voraussetzung besagt gerade $(2)=o(1)$. Damit ist gezeigt, daß $na_n\to 0$ $([C_\beta, C_{1+\alpha}]^k)$ d. h. es gilt $T([C_\beta, C_{1+\alpha}]^k)$.

2) Es sei $\sum_{0}^{\infty}a_n=s$ $(C_1 C_\alpha)$ und es gelte $T([C_\beta, C_{1+\alpha}]^k)$. Mit $(\alpha+1)\sigma_\nu^\alpha=\tau_\nu^{\alpha+1}+$

$+(\alpha+1)\sigma_\nu^{\alpha+1}$ und $(\alpha+1)(\sigma_\nu^\alpha-s)=\tau_\nu^{\alpha+1}+(\alpha+1)(\sigma_\nu^{\alpha+1}-s)$ folgt

$$\frac{1}{A_n^\beta}\sum_{\nu=0}^n A_{n-\nu}^{\beta-1}|\sigma_\nu^\alpha-s|^k = O(1)\left\{\frac{1}{A_n^\beta}\sum_{\nu=0}^n A_{n-\nu}^{\beta-1}|\tau_\nu^{\alpha+1}|^k+\frac{1}{A_n^\beta}\sum_{\nu=0}^n A_{n-\nu}^{\beta-1}|\sigma_\nu^{\alpha+1}-s|^k\right\} =$$
$$= O(1)\{(3)+(4)\}.$$

Die Voraussetzung $\sum_0^\infty a_n=s$ (C_1C_α) besagt $\sigma_\nu^{\alpha+1}\to s$. Hieraus folgt wegen der Regularität von C_β $(4)=o(1)$. Die Voraussetzung $T([C_\beta, C_{1+\alpha}]^k)$ bedeutet gerade $(4)=o(1)$. Damit ist die $[C_\beta, C_\alpha]^k$-Summierbarkeit gezeigt.

§ 4. Beweis von Satz 3

Die zu beweisende Tauberbedingung $T([C_\beta, C_\alpha]^k)$ lautet

$$\frac{1}{A_n^\beta}\sum_{\nu=0}^n A_{n-\nu}^{\beta-1}|\tau_\nu^\alpha|^2 = o(1).$$

Wir setzen

$$\delta_n(\beta,\alpha;2;x):=\frac{1}{A_n^\beta}\sum_{\nu=0}^n A_{n-\nu}^{\beta-1}|\sigma_\nu^\alpha(x)-\sigma_\nu^{\alpha-1}(x)|^2$$

und haben also wegen (3.1) zu zeigen: $\delta_n(\beta,\alpha;2;x)=o_X(1)$ f. ü.

1) Wir behandeln zuerst den Fall $\alpha>\frac{1}{2}$

a) Für $\nu\geqq 1$ gilt (vgl. etwa Alexits [1] S. 67):

$$\int_a^b(\sigma_\nu^\alpha-\sigma_\nu^{\alpha-1})^2dx=\frac{1}{\alpha^2(A_\nu^\alpha)^2}\sum_{j=1}^\nu j^2(A_{\nu-j}^{\alpha-1})^2c_j^2.$$

Hieraus folgt

$$\int_a^b\delta_n(\beta,\alpha;2;x)dx=\frac{1}{\alpha^2A_n^\beta}\sum_{\nu-1}^n A_{n-\nu}^{\beta-1}\frac{1}{(A_\nu^\alpha)^2}\sum_{j=1}^\nu j^2(A_{\nu-j}^{\alpha-1})^2c_j^2=$$

$$=\frac{1}{\alpha^2A_n^\beta}\sum_{j=1}^n j^2c_j^2\sum_{\nu=j}^n\frac{A_{n-\nu}^{\beta-1}(A_{\nu-j}^{\alpha-1})^2}{(A_\nu^\alpha)^2}.$$

Wegen $-1<\beta-1<0$ ist $A_k^{\beta-1}$ mit k monoton abnehmend, also $A_{n-\nu}^{\beta-1}\leqq 1$. Wegen

$$R_j^\alpha=\sum_{\nu=j}^\infty\frac{(A_{\nu-j}^{\alpha-1})^2}{(A_\nu^\alpha)^2}=O\left(\frac{1}{j}\right)$$

(vgl. etwa ALEXITS [1], S. 102) ergibt sich

$$\int_a^b \delta_n(\beta,\alpha;2;x)\,dx \leqq \frac{1}{\alpha^2 A_n^\beta}\sum_{j=1}^{n} j^2 c_j^2 \sum_{\nu=j}^{\infty}\frac{(A_{\nu-j}^{\alpha-1})^2}{(A_\nu^\alpha)^2} = O(1)\frac{1}{A_n^\beta}\sum_{j=1}^{n} j c_j^2.$$

Für eine beliebige Folge $\{n_i\}_1^\infty$ natürlicher Zahlen mit $1 \leqq n_1 < n_2 < \dots$ folgt $(n_0=0)$:

$$\sum_{i=1}^{\infty}\int_a^b \delta_{n_i}(\beta,\alpha;2;x)\,dx \leqq O(1)\sum_{i=1}^{\infty}\frac{1}{A_{n_i}^\beta}\sum_{j=1}^{n_i} j c_j^2 =$$

$$= O(1)\sum_{i=1}^{\infty}\frac{1}{A_{n_i}^\beta}\left(\sum_{k=1}^{i}\sum_{j=n_{k-1}+1}^{n_k} j c_j^2\right) = O(1)\sum_{k=1}^{\infty}\sum_{i=k}^{\infty}\frac{1}{A_{n_i}^\beta}\sum_{j=n_{k-1}+1}^{n_k} j c_j^2 =$$

$$= O(1)\sum_{k=1}^{\infty}\sum_{j=n_{k-1}+1}^{n_k} j c_j^2 \sum_{i=k}^{\infty}\frac{1}{A_{n_i}^\beta}.$$

Wählen wir $n_i = 2^i$ $(i=1,2,\dots)$, so gilt

$$\sum_{i=k}^{\infty}\frac{1}{A_{n_i}^\beta} = O(1)\sum_{i=k}^{\infty}\frac{1}{n_i^\beta} = O(1)\sum_{i=k}^{\infty}\frac{1}{2^{\beta i}} = \frac{O(1)}{2^{\beta k}}\sum_{l=0}^{\infty}\frac{1}{2^{\beta l}} = O(1)\frac{1}{2^{\beta k}}.$$

Nun gilt für $2^{k-1}+1 \leqq j \leqq 2^k$: $j^\beta \leqq 2^{\beta k}$. Es folgt

$$\sum_{i=1}^{\infty}\int_a^b \delta_{2^i}(\beta,\alpha;2;x)\,dx \leqq O(1)\sum_{k=1}^{\infty}\sum_{j=2^{k-1}+1}^{2^k} j^{1-\beta}c_j^2 j^\beta \frac{1}{2^{\beta k}} \leqq O(1)\sum_{j=0}^{\infty} j^{1-\beta}c_j^2 < \infty.$$

Hieraus folgt $\delta_{2^n}(\beta,\alpha;2;x) \to 0$ f. ü.

b) Für $2^n < k < 2^{n+1}$ folgt mit $\sigma_\nu(x) := (\sigma_\nu^\alpha(x) - \sigma_\nu^{\alpha-1}(x))^2$ aus

$$\delta_k(\beta,\alpha;2;x) - \delta_{2^n}(\beta,\alpha;2;x) = \sum_{\nu=2^n+1}^{k}\{\delta_\nu(\beta,\alpha;2;x) - \delta_{\nu-1}(\beta,\alpha;2;x)\}:$$

$$(1) = |\delta_k(\beta,\alpha;2;x) - \delta_{2^n}(\beta,\alpha;2;x)| \leqq \sum_{\nu=2^n+1}^{2^{n+1}} |\delta_\nu(\beta,\alpha;2;x) - \delta_{\nu-1}(\beta,\alpha;2;x)| \leqq$$

$$\leqq \sum_{\nu=2^n+1}^{2^{n+1}}\left|\sum_{k=0}^{\nu-1}\left(\frac{A_{\nu-k}^{\beta-1}}{A_\nu^\beta} - \frac{A_{\nu-1-k}^{\beta-1}}{A_{\nu-1}^\beta}\right)\sigma_k(x) + \frac{A_0^{\beta-1}}{A_\nu^\beta}\sigma_\nu(x)\right|.$$

Aus $0 \leqq k \leqq \nu-1 < \nu$ folgt $k\beta < \nu$ und hieraus

$$\frac{A_{\nu-k}^{\beta-1}}{A_\nu^\beta} - \frac{A_{\nu-1-k}^{\beta-1}}{A_{\nu-1}^\beta} = \frac{A_{\nu-1-k}^{\beta-1}}{A_{\nu-1}^\beta}\left(\frac{\nu-k+\beta-1}{\nu-k}\cdot\frac{\nu}{\nu+\beta} - 1\right) = \frac{A_{\nu-1-k}^{\beta-1}}{A_{\nu-1}^\beta}\cdot\frac{k\beta-\nu}{(\nu-k)(\nu+\beta)} < 0.$$

Hieraus ergibt sich unter Berücksichtigung von $A_0^{\beta-1}=1$:

$$(1) \leqq \sum_{\nu=2^n+1}^{2^{n+1}} \sum_{k=0}^{\nu-1} \left(\frac{A_{\nu-1-k}^{\beta-1}}{A_{\nu-1}^{\beta}} - \frac{A_{\nu-k}^{\beta-1}}{A_{\nu}^{\beta}} \right) \sigma_k(x) + \sum_{\nu=2^n+1}^{2^{n+1}} \frac{\sigma_\nu(x)}{A_\nu^\beta} =$$

$$= \sum_{\nu=2^n+1}^{2^{n+1}} \sum_{k=0}^{\nu-2} \frac{A_{\nu-1-k}^{\beta-1}}{A_{\nu-1}^{\beta}} \sigma_k(x) + \sum_{\nu=2^n+1}^{2^{n+1}} \frac{\sigma_{\nu-1}(x)}{A_{\nu-1}^{\beta}} -$$

$$- \sum_{\nu=2^n+1}^{2^{n+1}} \sum_{k=0}^{\nu-1} \frac{A_{\nu-k}^{\beta-1}}{A_{\nu}^{\beta}} \sigma_k(x) + \sum_{\nu=2^n+1}^{2^{n+1}} \frac{\sigma_\nu(x)}{A_\nu^\beta} =$$

$$= \sum_{\mu=2^n}^{2^{n+1}-1} \sum_{k=0}^{\mu-1} \frac{A_{\mu-k}^{\beta-1}}{A_{\mu}^{\beta}} \sigma_k(x) + \sum_{\nu=2^n+1}^{2^{n+1}} \frac{\sigma_{\nu-1}(x)}{A_{\nu-1}^{\beta}} -$$

$$- \sum_{\nu=2^n+1}^{2^{n+1}} \sum_{k=0}^{\nu-1} \frac{A_{\nu-k}^{\beta-1}}{A_{\nu}^{\beta}} \sigma_k(x) + \sum_{\nu=2^n+1}^{2^{n+1}} \frac{\sigma_\nu(x)}{A_\nu^\beta} =$$

$$= \sum_{k=0}^{2^n-1} \frac{A_{2^n-k}^{\beta-1}}{A_{2^n}^{\beta}} \sigma_k(x) - \sum_{k=0}^{2^{n+1}-1} \frac{A_{2^{n+1}-k}^{\beta-1}}{A_{2^{n+1}}^{\beta}} \sigma_k(x) + 2 \sum_{\nu=2^n}^{2^{n+1}} \frac{\sigma_\nu(x)}{A_\nu^\beta} \leqq$$

$$\leqq \sum_{k=0}^{2^n} \frac{A_{2^n-k}^{\beta-1}}{A_{2^n}^{\beta}} \sigma_k(x) + \sum_{k=0}^{2^{n+1}} \frac{A_{2^{n+1}-k}^{\beta-1}}{A_{2^{n+1}}^{\beta}} \sigma_k(x) + 2 \sum_{\nu=2^n}^{2^{n+1}} \frac{\sigma_\nu(x)}{A_\nu^\beta} =$$

$$= \delta_{2^n}(\beta, \alpha; 2; x) + \delta_{2^{n+1}}(\beta, \alpha; 2; x) + 2 \sum_{\nu=2^n}^{2^{n+1}} \frac{\sigma_\nu(x)}{A_\nu^\beta}.$$

Da die letzte Summe ein Cauchy-Abschnitt der Reihe $\sum_1^\infty \frac{\sigma_\nu(x)}{A_\nu^\beta}$ ist, folgt mit 1) die Aussage $\lim_{k\to\infty} \delta_k(\beta, \alpha; 2; x) = 0$ f. ü. wenn wir noch die Konvergenz f. ü. dieser Reihe zeigen. Dies folgt aber aus

$$\sum_{\nu=1}^\infty \frac{1}{A_\nu^\beta} \int_0^1 \sigma_\nu(x)\, dx = \sum_{\nu=1}^\infty \frac{1}{A_\nu^\beta} \frac{1}{\alpha^2 (A_\nu^\alpha)^2} \sum_{j=1}^\nu j^2 c_j^2 (A_{\nu-j}^{\alpha-1})^2 =$$

$$= \frac{1}{\alpha^2} \sum_{j=1}^\infty j^2 c_j^2 \sum_{\nu=j}^\infty \frac{(A_{\nu-j}^{\alpha-1})^2}{A_\nu^\beta (A_\nu^\alpha)^2} \leqq \frac{1}{\alpha^2} \sum_{j=1}^\infty j^2 c_j^2 \frac{1}{A_j^\beta} \sum_{\nu=j}^\infty \frac{(A_{\nu-j}^{\alpha-1})^2}{(A_\nu^\alpha)^2} = O(1) \sum_{j=1}^\infty j^{1-\beta} c_j^2.$$

2) Wir betrachten nun den Fall $\alpha = \frac{1}{2}$.

a) Hier ergibt sich im Beweisschritt 1a) eine andere Abschätzung für $R_j^{1/2}$:

$$R_j^{1/2} = \sum_{\nu=j}^\infty \frac{(A_{\nu-j}^{-1/2})^2}{(A_\nu^{1/2})^2} = \sum_{\nu=j}^{2j} + \sum_{\nu=2j+1}^\infty .$$

Es ergibt sich

$$\sum_{\nu=j}^{2j} \leqq \frac{1}{(A_j^{1/2})^2} \sum_{\nu=j}^{2j} (A_{\nu-j}^{-1/2})^2 = O(1)\frac{1}{j}\left(1+\sum_{l=1}^{j}\frac{1}{l}\right) = O\left(\frac{\ln j}{j}\right),$$

$$\sum_{\nu=2j+1}^{\infty} = O(1) \sum_{\nu=2j+1}^{\infty} \frac{(\nu-j)^{-1}}{\nu} = O(1) \sum_{\nu=2j+1}^{\infty} \frac{1}{\nu^2} = O\left(\frac{1}{j}\right).$$

Es ergibt sich also $R_j^{1/2} = O\left(\frac{\ln j}{j}\right)$. Hieraus folgt schließlich $\delta_{2^n}(\beta, \alpha; 2; x) = o_x(1)$ f. ü., wenn wir die Voraussetzung der Aussage (2) heranziehen.

b) Der zweite Teil des Beweises verläuft analog wie 1b) bis zum Schluß, wo wieder die obige Abschätzung für $R_j^{1/2}$ und dann die Voraussetzung der Aussage (2) herangezogen werden muß.

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The boundedness of closed linear maps in C^*-algebras

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The domain of a closed *-derivation in a C^*-algebra has many properties. In particular, ŌTA [6] studied such domains by using Lorentz representation and obtained some interesting results on the boundedness of closed *-derivations. Especially, he showed that a closed *-derivation, which is bounded on the unitary group of the domain, is bounded.

Now in connection with strongly continuous one-parameter semi-groups of positive maps on C^*-algebras, we are interested in the boundedness of more general closed linear maps. One of the crucial points in [6] is that the domain of a closed *-derivation becomes a semi-simple Banach *-algebra under the graph norm. Although such fact is not valid in our general situation, we have some generalizations of results in [6] by virtue of a simple lemma on Banach algebras.

Let A and A_0 be respectively a unital C^*-algebra and a *-subalgebra of A which contains the identity e of A. The following lemma is elementary, but it is essential in what follows.

Lemma. *Suppose that there exists a closed linear map Φ of A_0 into a Banach space. Then A_0 is a semi-simple Banach algebra with an isometric involution under some norm $\|\cdot\|'$ which is equivalent to the graph norm $\|\cdot\|_\Phi=\|\cdot\|+\|\Phi(\cdot)\|$.*

Proof. Since $(A_0, \|\cdot\|_\Phi)$ is a Banach space, by the closed graph theorem, the product in A_0 is separately continuous with respect to $\|\cdot\|_\Phi$, and hence A_0 is a Banach algebra under some norm which is equivalent to $\|\cdot\|_\Phi$ (see [8, p. 5]). Since A_0 is semi-simple by the proof of [8, Theorem 4.4.10], JOHNSON's theorem [5] implies that the involution is continuous in $\|\cdot\|_\Phi$, and hence we have the desired norm $\|\cdot\|'$ by another equivalent renorming. The proof is complete.

By the above lemma and [8, Theorem 4.1.5], it follows that a *-subalgebra A_0, which is the domain of a closed linear map, has sufficiently many unitary elements, more precisely, every element of A_0 is a linear combination of unitary elements of A_0.

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An involutive Banach algebra is said to be C^*-equivalent if it is *-isomorphic to some C^*-algebra. B. RUSSO and H. A. DYE [9] showed that a linear map on a unital C^*-algebra, which is bounded on the unitary group, is bounded. This result and the above mentioned remark suggest the following:

Theorem 1. *Let Φ be a closed linear map of A_0 into a Banach space. If Φ is norm bounded on the unitary group of A_0, then A_0 is a C^*-algebra and Φ is bounded.*

Proof. Since the norm $\|\cdot\|'$ in the Lemma is equivalent to the graph norm $\|\cdot\|_\Phi$, there exists a constant $N>0$ such that $\|a\|'\leqq N\|a\|_\Phi$ for all $a\in A_0$. Then we have

$$\sup\{\|u\|' : u \text{ is unitary in } A_0\}\leqq N\sup\{1+\|\Phi(u)\| : u \text{ is unitary in } A_0\}\leqq$$
$$\leqq N+N\sup\{\|\Phi(u)\| : u \text{ is unitary in } A_0\}<+\infty.$$

Hence from [7, Corollary 12] A_0 is C^*-equivalent, which implies that A_0 is a C^*-algebra. Hence by the closed graph theorem or by Corollary 1 in [9] Φ is bounded.

Theorem 1 implies that any closed *-homomorphism of A_0 into A is automatically bounded. Moreover, this assertion is true for a more general class of maps. More precisely, let Φ be a 2-positive map from A_0 into another C^*-algebra B, that is, for all pairs $\{x_1, x_2\}$ in A_0, the matrices $(\Phi(x_i^* x_j))$ are positive in the C^*-algebra of all 2×2 matrices over B. Then the Schwarz inequality $\Phi(a^*)\Phi(a)\leqq\|\Phi(e)\|\,\Phi(a^*a)$ $(a\in A_0)$ follows easily ([1], [4]), and hence Φ is bounded if it is closed.

It is natural to ask if every closed positive linear map Φ from A_0 into another C^*-algebra B is automatically bounded, where positivity of Φ means that $\Phi(a^*a)$ is positive in B for all $a\in A_0$. We have however no answer to this question.

Now let Φ be a completely positive linear map on A and put $L_\Phi(x)=\Phi(x)-$ $-\frac{1}{2}\{\Phi(e)x+x\Phi(e)\}$ for $x\in A$. Then the generator of a uniformly continuous semi-group of unital completely positive maps on A is essentially determined by two classes of operators, that is, *-derivations on A and maps of the form L_Φ for Φ ([2]). In this connection, the following corollary is interesting.

Corollary. *Suppose that A_0 is strongly dense in A. Let Φ be a completely positive map from A_0 into A. If L_Φ generates a strongly continuous semi-group of linear maps on A, then $A_0=A$, that is, Φ is everywhere defined.*

A linear map δ from A_0 into A is called a Jordan derivation if $\delta(h^2)=h\delta(h)+$ $+\delta(h)h$ for all $h=h^*$ in A_0. Then we have the following theorem, which is a generalization of Theorem 2.4 in [6].

Theorem 2. *Suppose that A_0 is strongly dense in A. Let δ be a closed Jordan derivation from A_0 into A. If A_0 is closed under the square root operation of positive* *elements $A_0 \cap A^+$ where A^+ denotes the positive part of A, then δ is everywhere defined and is bounded.*

Proof. Since the norm $\|\cdot\|'$ in the Lemma is equivalent to the graph norm $\|\cdot\|_\delta$, $\lim_{n\to\infty} \|x^n\|_\delta^{1/n}$ exists and is equal to $\lim_{n\to\infty} \|x^n\|'^{1/n}$ for $x\in A_0$. Hence, for $h=h^*\in A_0$ we have

$$\lim_{n\to\infty}\|h^n\|_\delta^{1/n} = \lim_{n\to\infty} (\|h\|^{2^n}+\|\delta(h^{2^n})\|)^{1/2^n} \leqq$$
$$\leqq \lim_{n\to\infty} \|h\|\{1+(2^n\|\delta(h)\|)/\|h\|\}^{1/2^n} = \|h\|$$

because $\|\delta(h^{2^n})\| \leqq 2^n\|h\|^{2^n-1}\|\delta(h)\|$ $(n=1, 2, 3, \ldots)$ where $\|\cdot\|$ is the norm of A. Hence

$$\lim_{n\to\infty} \|h^n\|'^{1/n} \leqq \|h\| = \inf\{\sum|\lambda_i|:\ h = \sum \lambda_i u_i,\ u_i\text{'s are unitaries in } A\} \leqq$$
$$\leqq \inf\{\sum|\lambda_i|:\ h = \sum \lambda_i u_i,\ u_i\text{'s are unitaries in } A_0\}$$

which implies that the semi-simple involutive Banach algebra A_0 is hermitian from [7, Corollary 5 and 9]. Denote the spectrum of an element x of A_0 in A (resp. A_0) by sp (x) (resp. $\mathrm{sp}_0(x)$). Now let h be a hermitian element of A_0. If $\mathrm{sp}_0(h)\geqq 0$, then $\mathrm{sp}(h)\geqq 0$, and hence there exists a hermitian element k in A_0 such that $k^4=h$ from our assumption. Hence $\mathrm{sp}_0(k^2)=\{\lambda^2:\ \lambda\in \mathrm{sp}_0(k)\}\geqq 0$ since A_0 is hermitian. Therefore, A_0 is C^*-equivalent from [3, Corollary], which implies that $A_0=A$, and hence δ is bounded from the closed graph theorem. The proof is completed.

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On the Korovkin closure

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Let X be a topological space and denote by $C(X)$ the vector lattice of all continuous real functions defined on X. Given a linear subspace $\mathfrak{H}\subset C(X)$, we denote by $\mathfrak{H}_0$, as usual, the set of all $\mathfrak{H}$-bounded, continuous functions:

$$\mathfrak{H}_0 \equiv \{f\in C(X)\colon\ \exists h_1, h_2\in\mathfrak{H} \text{ with } h_1 \leqq f\leqq h_2\}.$$

We define the Korovkin closure of $\mathfrak{H}$ as the set of all functions $f\in\mathfrak{H}_0$ having the following property: "For every net $(L_i)_{i\in I}$ of positive linear maps $L_i\colon \mathfrak{H}_0\to\mathfrak{H}_0$ such that L_ih converges to h pointwise on X for all $h\in\mathfrak{H}$, L_if also converges to f pointwise on X."

We denote the Korovkin closure of the linear subspace $\mathfrak{H}$ by Kor $(\mathfrak{H})$. The following inclusions are obvious:

$$\mathfrak{H} \subset \operatorname{Kor}(\mathfrak{H}) \subset \mathfrak{H}_0 \subset C(X).$$

This paper is devoted to the characterization of Kor $(\mathfrak{H})$ in some general cases. We shall extend some results of H. BAUER [1] and K. DONNER [2].

To formulate our theorems, we recall the definition due to H. Bauer, of the space of $\mathfrak{H}$-affine functions. This space, denoted by $\mathfrak{H}$, consists of all $f\in\mathfrak{H}_0$ satisfying the equality

$$\sup\{h\in\mathfrak{H}\colon\ h \leqq f\} = \inf\{h\in\mathfrak{H}\colon\ h \geqq f\}.$$

We shall prove:

Theorem 1. *If X is locally compact and Hausdorff, then for all linear subspaces $\mathfrak{H}$ of $C(X)$ the following identity holds:*

$$\operatorname{Kor}(\mathfrak{H}) = \hat{\mathfrak{H}}.$$

Remark 1. This identity was proved by H. BAUER [1], Theorem 3.3, in the special case when the linear subspace $\mathfrak{H}$ is adapted, i.e. satisfies the following three conditions:

(i) $\mathfrak{H} = \mathfrak{H}^+ - \mathfrak{H}^+$ where $\mathfrak{H}^+\equiv\{h\in\mathfrak{H}\colon\ h \geqq 0\}$,

(ii) $\forall x\in X\ \exists h_x\in\mathfrak{H}\colon\ h_x(x) \neq 0$,

(iii) $\forall h\in\mathfrak{H}\ \exists h_1\in\mathfrak{H}\ \forall\varepsilon > 0$: the closure of $\{t\in X\colon\ |h(t)| > \varepsilon\cdot|h_1(t)|\}$ is compact.

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Remark 2. Recently K. Donner [2] proved a general theorem which can be applied to our situation when $\mathfrak{H}_0$ is a vector lattice. But $\mathfrak{H}_0$ is a vector lattice if and only if $\mathfrak{H}=\mathfrak{H}^+-\mathfrak{H}^+$. Thus Donner's result yields that special case of our theorem when the linear subspace $\mathfrak{H}$ satisfies the condition $\mathfrak{H}=\mathfrak{H}^+-\mathfrak{H}^+$.

Theorem 1 will be got as a special case of the following more general one:

Theorem 2. *If X is a topological space and $\mathfrak{H}$ is a linear subspace of $C(X)$, then each of the following five conditions implies the identity* $\operatorname{Kor}(\mathfrak{H})=\hat{\mathfrak{H}}$:

(a) *X is locally compact and totally regular,*

(b) $\mathfrak{H}=\mathfrak{H}^+-\mathfrak{H}^+$,

(c) $\dim \mathfrak{H}<\infty$,

(d) *all the functions in $\mathfrak{H}$ are bounded,*

(e) *each point of X has a neighbourhood in the weak topology, induced by $C(X)$, where all the functions from $\mathfrak{H}$ are bounded.*

In the proof we shall use the following lemma, essentially proved by H. Bauer:

Lemma. *For any $g\in\mathfrak{H}_0$, $x\in X$ and $c\in\boldsymbol{R}$ such that*

$$\sup\{h(x)\colon\ g\geqq h\in\mathfrak{H}\}\leqq c\leqq\inf\{h(x)\colon\ g\leqq h\in\mathfrak{H}\},$$

there exists a positive linear functional $\mu\colon\ \mathfrak{H}_0\to\boldsymbol{R}$ with

(A) $\mu(g)=c$, *and*

(B) $\mu(h)=h(x)$ *for all* $h\in\mathfrak{H}$.

Proof (compare with [1; 2.2 Lemma]). On $\mathfrak{H}_0$ the map $f\mapsto\inf\{h(x)\colon f\leqq h\in\mathfrak{H}\}$ is a sublinear functional p. This functional majorizes the linear form $\lambda\cdot g\mapsto\lambda\cdot c$ defined on the linear subspace of $\mathfrak{H}_0$ generated by g. The Hahn—Banach theorem hence implies the existence of a linear form μ on $\mathfrak{H}_0$ satisfying (A) and the relation $\mu\leqq p$. (B) and the positivity of μ follow from this latter inequality.

Proof of Theorem 2. The relation $\hat{\mathfrak{H}}\subset\operatorname{Kor}(\mathfrak{H})$ is well-known (see [1], Corollary 1.3). Conversely, we shall show that given any $g\in\mathfrak{H}_0\setminus\hat{\mathfrak{H}}$, g does not belong to $\operatorname{Kor}(\mathfrak{H})$.

As condition (e) is weaker than conditions (a), (c), (d), we treat only cases (b) and (e).

Because of $g\notin\hat{\mathfrak{H}}$ there is a point $x\in X$ and a number $c\in\boldsymbol{R}$ such that

$$\sup\{h(x)\colon\ g\geqq h\in\mathfrak{H}\}<c<\inf\{h(x)\colon\ g\leqq h\in\mathfrak{H}\},\quad c\neq g(x).$$

Let us fix by the above Lemma a positive linear functional μ satisfying (A) and (B). By the relation $g\in\mathfrak{H}_0\setminus\hat{\mathfrak{H}}$ we can choose a function h_0 with

(C) $h_0\in\mathfrak{H},\quad h_0\geqq 0,\quad h_0(x)>1.$

(Indeed, for any functions $h_1,h_2\in\mathfrak{H}$, $h_1\leqq g\leqq h_2$ we have $h_2-h_1\geqq 0$ and $h_2(x)-h_1(x)>0$.)

If condition (b) is satisfied, fix a neighbourhood base $\mathscr{B}$ of x in the weak topology induced by $C(X)$ so as to satisfy

(D) $h_0(t) > 1$ for any $t \in U \in \mathscr{B}$.

If condition (e) is satisfied, fix a neighbourhood base $\mathscr{B}$ of x in the weak topology induced by $C(X)$, satisfying over and above (D) also the following condition:

(E) Each function from $\mathfrak{H}$ is bounded on each element of $\mathscr{B}$.

Assign to every $U \in \mathscr{B}$ a function $q_U \in C(X)$ such that

(F) $0 \leqq q_U \leqq 1,\quad q_U(x) = 1,\quad q_U(t) = 0$ for all $t \in X \setminus U$.

(This is possible because the weak topology is totally regular.)

For $U \in \mathscr{B}$ and $f \in \mathfrak{H}_0$ we define

$$L_U f \equiv \mu(f) \cdot q_U + f - f \cdot q_U.$$

Obviously, L_U: $\mathfrak{H}_0 \to C(X)$ is a positive linear map. Moreover, L_U: $\mathfrak{H}_0 \to \mathfrak{H}_0$ is also true: being $\mathfrak{H}_0$ a linear subspace, this will follows from the two relations $q_U \in \mathfrak{H}_0$ and $f \cdot q_U \in \mathfrak{H}_0$ (for all $U \in \mathscr{B}$ and $f \in \mathfrak{H}_0$). The first relation follows from (C), (D) and (F): $0 \leqq q_U \leqq h_0$. If condition (b) is satisfied, then there is an $h \in \mathfrak{H}$ with $-h \leqq f \leqq h$ from which we get $-h \leqq f \cdot q_U \leqq h$, proving the second relation. If condition (e) is satisfied, then $f \cdot q_U$ is bounded by (E), (F) and vanishes on $X \setminus U$ then there is therefore by (C) and (D) a real number d with $-d \cdot h_0 \leqq f \cdot q_U \leqq d \cdot h_0$. Hence again $f \cdot q_U \in \mathfrak{H}_0$.

Finally, take the net $(L_U)_{U \in \mathscr{B}}$ of positive linear maps L_U: $\mathfrak{H}_0 \to \mathfrak{H}_0$. An easy computation shows that the net $(L_U h)_{U \in \mathscr{B}}$ converges to h pointwise (moreover uniformly) on X for all $h \in \mathfrak{H}$ $(\mu(h) = h(x)$ by (B)), but the net $(L_U g)_{U \in \mathscr{B}}$ does not converge to g pointwise on X because $((L_U g)(x))_{U \in \mathscr{B}}$ is a constant net with the constant $\mu(g) = c \neq g(x)$ by (A). Thus g does not belong to the Korovkin closure of $\mathfrak{H}$ and the theorem is proved.

Remark. The results of this paper (and the proofs) remain valid if we replace pointwise convergence by uniform convergence on the compact subsets of X in the definition of the Korovkin closure. The author wishes to thank Dr. Z. Sebestyén for having followed with attention these investigations.

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On the convergence of solutions of functional differential equations

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1. Introduction

The application of Ljapunov functions and functionals has proved to be useful in the study of the stability of solutions of functional differential equations. Such investigations were initiated by N. N. KRASOVSKIĬ [9] and B. S. RAZUMIKHIN [10]. The Ljapunov functions and functionals are usable for studying other properties, too. For instance, S. R. BERNFELD and J. R. HADDOCK ([1], [2], [4]) examined the existence of the limit of solutions as $t\to\infty$ by the aid of Ljapunov functions. But their method was not applicable when the right-hand side of the equation is the sum of an ordinary and a functional part of the same order. But such equations have occurred in the applications, for example in the investigation of biological populations [3]. In this case the problem was solved for certain autonomous and periodic equations only [5], [6]. In this paper we give a sufficient condition for the existence of the limit of solutions in case of non-periodic equations. Our main result guarantees the existence of the limit of a Ljapunov function along the solutions as $t\to\infty$. We present several applications in which we show that the solutions or their norm tend to a constant as $t\to\infty$. Among these, we study a stability example of N. N. KRASOVSKIĬ proving that his assumptions imply the existence of the limit of solutions in addition to the stability of the zero solution.

The main theorems are valid results for functional differential equations in any Banach space X. But they also yield new results for the special case $X=R$ (Section 4).

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2. Notations and definitions

Let R be the set of real numbers and R^+ the set of nonnegative real numbers. Let X be a Banach space with norm $|\cdot|$ and let $C=C([-r, 0], X)$ denote the space of continuous functions which map the interval $[-r, 0]$ into X, where $r>0$. For $\varphi\in C$ define $\|\varphi\|=\max_{-r\leqq s\leqq 0}|\varphi(s)|$. If x: $[t_0-r, t_0+A)\to X$ is a continuous function ($t_0\in R^+$, $0<A\leqq\infty$), then for $t\in[t_0, t_0+A)$ the function $x_t\in C$ is defined by $x_t(s)=x(t+s)$, $-r\leqq s\leqq 0$.

We consider the *nonlinear, non-autonomous* functional differential equation

$$\dot{x}(t) = F(t, x_t), \tag{2.1}$$

where F: $R^+\times C_F\to X$, $C_F\subset C$.

Let $t_0\in R^+$ and $\varphi_0\in C_F$ be given. A function $x(\cdot)=x(t_0, \varphi_0)(\cdot)$ is said to be a solution of (2.1) (with the initial function φ_0 at t_0) if there exists a number A ($0<A\leqq\infty$) such that $x(\cdot)$ is defined and continuous on $[t_0-r, t_0+A)$, absolutely continuous on the bounded intervals of $[t_0, t_0+A)$, $x_{t_0}=\varphi_0$, $x_t\in C_F$ for $t\in[t_0, t_0+A)$ and $\dot{x}(t)=F(t, x_t)$ almost everywhere on $[t_0, t_0+A)$. In this paper we suppose $A=\infty$, i.e. the solutions of (2.1) exist for $t\geqq t_0$ (see, for example, [7], [8]).

By a *Ljapunov function* we mean a continuous function V: $[-r, \infty)\times X\to R$. The upper *right-hand derivative* $D^+_{(2.1)}V$ of a Ljapunov function V with respect to system (2.1) is defined by

$$D^+_{(2.1)}V(t, \varphi) = \varlimsup_{h\to 0+}\frac{1}{h}\left[V\big(t+h, \varphi(0)+hF(t, \varphi)\big)-V\big(t, \varphi(0)\big)\right] \quad \big((t, \varphi)\in R^+\times C_F\big).$$

If V is a Ljapunov function and $(t, \varphi)\in R^+\times C_F$, then let

$$\bar{V}(t, \varphi) = \sup_{-r\leqq s\leqq 0} V\big(t+s, \varphi(s)\big), \quad \underline{V}(t, \varphi) = \inf_{-r\leqq s\leqq 0} V\big(t+s, \varphi(s)\big).$$

Finally, for a Ljapunov function V and given numbers $0<\eta\leqq\varepsilon$ define

$$\bar{S}(V, \eta, \varepsilon) = \big\{(t, \varphi)\in R^+\times C_F\colon V\big(t, \varphi(0)\big)\geqq \varepsilon,\ \bar{V}(t, \varphi)\leqq 2\varepsilon,\ \bar{V}(t, \varphi)-V\big(t, \varphi(0)\big)<\eta\big\},$$

$$\underline{S}(V, \eta, \varepsilon) = \big\{(t, \varphi)\in R^+\times C_F\colon V\big(t, \varphi(0)\big)\leqq -\varepsilon,$$
$$\underline{V}(t, \varphi)\geqq -2\varepsilon,\ V\big(t, \varphi(0)\big)-\underline{V}(t, \varphi)<\eta\big\}.$$

3. The main result

The main result guarantees the existence of the limit of a Ljapunov function along the solutions of (2.1) as $t\to\infty$.

Theorem 3.1. *Suppose that for a nonnegative Ljapunov function V there exists a functional W: $R^+\times C_F\to R$ with the following property: for every $\varepsilon>0$ there exist $\eta=\eta(\varepsilon)>0$ and $\xi=\xi(\varepsilon)>0$ such that*

(i) *if* $(t, \varphi)\in \bar{S}(V, \eta, \varepsilon)$, *then*

$$W(t, \varphi) \leqq \xi[\bar{V}(t, \varphi)-V(t, \varphi(0))], \tag{3.1}$$

(ii) *if* $x(\cdot)$ *is a solution of* (2.1) *and* $(t, x_t)\in \bar{S}(V, \eta, \varepsilon)$ *for every* $t\in[t_1, t_2]$ $(t_0\leqq t_1\leqq t_2)$, *then*

$$V(t_2, x(t_2))-V(t_1, x(t_1)) \leqq \int_{t_1}^{t_2} W(t, x_t)\, dt. \tag{3.2}$$

Then for each solution $x(\cdot)$ *of* (2.1) $\lim_{t\to\infty} V(t, x(t))$ *exists.*

We first prove the following lemma.

Lemma. *If the conditions of Theorem* 3.1 *are satisfied, then for each solution* $x(\cdot)$ *of* (2.1) *the function* $\bar{V}(\cdot, x.)$ *is non-increasing.*

Proof. Assume that (2.1) has a solution $x(\cdot)$ such that $\bar{V}(\cdot, x.)$ is not a non-increasing function. Then there exists a $t_1\geqq t_0$ and in any right-hand side neighbourhood of t_1 there exists a t such that $\bar{V}(t, x_t)>\bar{V}(t_1, x_{t_1})=V(t_1, x(t_1))>0$. Let ε be chosen such that $0<\varepsilon<V(t_1, x(t_1))<2\varepsilon$ and choose $\eta=\eta(\varepsilon)$, $\xi=\xi(\varepsilon)$ according to assumptions of the lemma. Obviously there exist t_2, t_3 such that $t_3>t_2\geqq t_1$, $t_3-t_2<\frac{1}{\xi}$, $V(t_2, x(t_2))=V(t_1, x(t_1))<V(t_3, x(t_3))\leqq 2\varepsilon$, $V(t_3, x(t_3))-V(t_2, x(t_2))<\eta$ and if $t\in[t_2, t_3]$, then $V(t_2, x(t_2))\leqq V(t, x(t))$ and $\bar{V}(t, x_t)\leqq V(t_3, x(t_3))$. For such t_2, t_3 we have $(t, x_t)\in \bar{S}(V, \eta, \varepsilon)$ provided $t\in[t_2, t_3]$. Also

$$\bar{V}(t, x_t)-V(t, x(t)) \leqq V(t_3, x(t_3))-V(t_2, x(t_2)) \quad (t\in[t_2, t_3]). \tag{3.3}$$

It follows from (3.1), (3.2) and (3.3) that

$$V(t_3, x(t_3))-V(t_2, x(t_2)) \leqq \int_{t_2}^{t_3} W(t, x_t)dt \leqq$$

$$\leqq \int_{t_2}^{t_3} \xi[\bar{V}(t, x_t)-V(t, x(t))]\, dt \leqq \int_{t_2}^{t_3} \xi[V(t_3, x(t_3))-V(t_2, x(t_2))]\, dt =$$

$$= (t_3-t_2)\xi[V(t_3, x(t_3))-V(t_2, x(t_2))].$$

Hence $t_3-t_2\geqq\frac{1}{\xi}$. This is a contradiction. The lemma is proved.

Proof of Theorem 3.1. Suppose that (2.1) has a solution $x(\cdot)$ such that the limit $\lim_{t\to\infty} V(t, x(t))$ does not exist. Then $\lim_{t\to\infty} \bar{V}(t, x_t)=\alpha>0$ (this limit exists by Lemma). Let ε be chosen so that $0<\varepsilon<\alpha<2\varepsilon$ and choose $\eta=\eta(\varepsilon)$ and $\xi=\xi(\varepsilon)$ according to the assumptions of the theorem. Then we can find a constant β

$\left(0<\beta<\min\left\{\frac{\eta}{3}, \frac{\alpha-\varepsilon}{3}\right\}\right)$ and numbers t_1, t_2 such that

$$t_2 > t_1 \geqq t_0, \quad t_2-t_1 \leqq r, \tag{3.4}$$

$$V(t_2, x(t_2))-V(t_1, x(t_1)) = \beta, \tag{3.5}$$

$$\bar{V}(t_1, x_{t_1})-\alpha < \frac{\eta}{3}, \quad |\alpha-V(t_2, x(t_2))| < \frac{\eta}{3}, \tag{3.6}$$

$$0 \leqq V(t_2, x(t_2))-V(t, x(t)) \leqq \beta \quad (t\in[t_1, t_2]),$$

$$V(t_1, x(t_1)) \geqq \varepsilon, \quad \bar{V}(t_1, x_{t_1}) \leqq 2\varepsilon, \tag{3.7}$$

$$\bar{V}(t_1, x_{t_1})-V(t_2, x(t_2)) \leqq \gamma, \tag{3.8}$$

where $\gamma>0$ and

$$\frac{1}{\xi}\sum_{k=1}^{\left[\frac{\beta}{\gamma}\right]} \frac{1}{k+1} > r. \tag{3.9}$$

From (3.8) and the monotonicity of $\bar{V}(\cdot, x.)$ it follows that

$$\bar{V}(t, x_t)-V(t_2, x(t_2)) \leqq \gamma \quad (t\in[t_1, t_2]). \tag{3.10}$$

Since $\beta<\frac{\eta}{3}$, by (3.6) we have

$$\bar{V}(t, x_t)-V(t, x(t)) \leqq \bar{V}(t, x_t)-\alpha+|\alpha-V(t_2, x(t_2))|+V(t_2, x(t_2))-V(t, x(t)) < \tag{3.11}$$

$$< \frac{\eta}{3}+\frac{\eta}{3}+\frac{\eta}{3} = \eta \quad (t\in[t_1, t_2]).$$

From (3.5), (3.6), (3.7) and (3.11) we obtain $(t, x_t)\in\bar{S}(V, \eta, \varepsilon)$ for $t\in[t_1, t_2]$. Thus, (3.1) and (3.2) hold for (t, x_t) as $t\in[t_1, t_2]$. Let $\tau_k\in[t_1, t_2]$ be the greatest number for which

$$V(t_2, x(t_2))-V(\tau_k, x(\tau_k)) = k\gamma \quad \left(k = 1, 2, \dots, \left[\frac{\beta}{\gamma}\right]\right). \tag{3.12}$$

From (3.10) and the choice of τ_k it follows that

$$\bar{V}(t, x_t)-V(t, x(t)) \leqq (k+1)\gamma \quad \left(t\in[\tau_k, t_2];\ k = 1, 2, \dots, \left[\frac{\beta}{\gamma}\right]\right). \tag{3.13}$$

By (3.1), (3.2), (3.12) and (3.13) we have

$$\gamma = V(\tau_{k-1}, x(\tau_{k-1}))-V(\tau_k, x(\tau_k)) \leqq \int_{\tau_k}^{\tau_{k-1}} W(t, x_t)\,dt \leqq$$

$$\leqq \int_{\tau_k}^{\tau_{k-1}} \xi[\bar{V}(t, x_t)-V(t, x(t))]\,dt \leqq \int_{\tau_k}^{\tau_{k-1}} \xi(k+1)\gamma\,dt =$$

$$= (\tau_{k-1}-\tau_k)\xi(k+1)\gamma \quad \left(k = 1, 2, \dots, \left[\frac{\beta}{\gamma}\right];\ \tau_0 = t_2\right).$$

Hence $\tau_{k-1}-\tau_k \geqq \frac{1}{\xi}\frac{1}{k+1}$. Thus, by (3.9)

$$t_2-t_1 \geqq t_2-\tau_{\left[\frac{\beta}{\gamma}\right]} = (t_2-\tau_1)+(\tau_1-\tau_2)+\ldots+\left(\tau_{\left[\frac{\beta}{\gamma}\right]-1}-\tau_{\left[\frac{\beta}{\gamma}\right]}\right) \geqq \frac{1}{\xi}\sum_{k=1}^{\left[\frac{\beta}{\gamma}\right]}\frac{1}{k+1} > r,$$

which contradicts (3.4). This completes the proof.

Remark 3.1. If the Ljapunov function V in Theorem 3.1 is locally Lipschitzian and $W=D^+_{(2.1)}V$, then for each solution $x(\cdot)$ of (2.1) the assumption (3.2) is satisfied and even

$$V(t_2, x(t_2))-V(t_1, x(t_1)) = \int_{t_1}^{t_2} D^+_{(2.1)}V(t, x_t)\,dt \quad (t_0 \leqq t_1 \leqq t_2).$$

This can be shown as follows. If $\dot{x}(t)=F(t, x_t)$, then $x(t+h)=x(t)+hF(t, x_t)+ +o(h)$ $(h\to 0+)$. From the Lipschitz condition for V we obtain

$$V(t+h, x(t+h))-V(t, x(t)) \leqq V(t+h, x(t)+hF(t, x_t))+L|o(h)|-V(t, x(t))$$
$$(h \to 0+),$$

where L is the Lipschitz constant for V on a neighbourhood of $(t, x(t))$. Hence $D^+V(t, x(t)) \leqq D^+_{(2.1)}V(t, x_t)$, where $D^+V(t, x(t))$ is the upper right-hand derivative of V along the solution $x(t)$ of (2.1), that is

$$D^+V(t, x(t)) = \overline{\lim_{h\to 0+}}\frac{1}{h}\left[V(t+h, x(t+h))-V(t, x(t))\right].$$

Likewise we can prove $D^+V(t, x(t)) \geqq D^+_{(2.1)}V(t, x_t)$ and we obtain

(3.14) $$D^+V(t, x(t)) = D^+_{(2.1)}V(t, x_t).$$

(3.14) was proved by T. Yoshizawa [11] for ordinary differential equations in the case $X=R^m$. Since V is locally Lipschitzian, $V(\cdot, x(\cdot))$ is absolutely continuous on every bounded interval of $[t_0, \infty)$ and thus

(3.15) $$V(t_2, x(t_2))-V(t_1, x(t_1)) = \int_{t_1}^{t_2} D^+V(t, x(t))\,dt \quad (t_0 \leqq t_1 \leqq t_2).$$

From (3.14) and (3.15) it follows that our statement holds.

Corollary 3.1. *If for every $\varepsilon>0$ there exists an $\eta=\eta(\varepsilon)>0$ such that $(t, \varphi)\in \bar{S}(|\varphi(0)|, \eta, \varepsilon)$ implies $D^+_{(2.1)}|\varphi| \leqq 0$, then for each solution $x(\cdot)$ of (2.1) $\lim_{t\to\infty}|x(t)|$ exists.*

Proof. We apply Theorem 3.1. Let $V(t, x)=|x|$ and $W(t, \varphi)=D^+_{(2.1)}|\varphi|$. Since the condition of Corollary 3.1 is stronger than condition (i) of Theorem 3.1 and the function V is locally Lipschitzian, from Remark 3.1 it is obvious that the limit exists.

Corollary 3.1 is due to J. R. HADDOCK [4].

In the next corollary of Theorem 3.1 we do not assume that the Ljapunov function is nonnegative.

Corollary 3.2. *Suppose that for a Ljapunov function V there exist functionals W_1, W_2: $R^+ \times C_F \to R$ with the following property: for every $\varepsilon > 0$ there exist $\eta = \eta(\varepsilon) > 0$ and $\xi = \xi(\varepsilon) > 0$ such that*

(i) $(t, \varphi) \in \bar{S}(V, \eta, \varepsilon)$ *implies* $W_1(t, \varphi) \leqq \xi[\bar{V}(t, \varphi) - V(t, \varphi(0))]$,

(ii) $(t, \varphi) \in \underline{S}(V, \eta, \varepsilon)$ *implies* $W_2(t, \varphi) \leqq \xi[V(t, \varphi(0)) - \underline{V}(t, \varphi)]$,

(iii) *if* $(t, x_t) \in \bar{S}(V, \eta, \varepsilon)$ $(t \in [t_1, t_2],\ t_0 \leqq t_1 \leqq t_2)$, *then*

$$V(t_2, x(t_2)) - V(t_1, x(t_1)) \leqq \int_{t_1}^{t_2} W_1(t, x_t)\, dt,$$

(iv) *if* $(t, x_t) \in \underline{S}(V, \eta, \varepsilon)$ $(t \in [t_1, t_2],\ t_0 \leqq t_1 \leqq t_2)$, *then*

$$V(t_1, x(t_1)) - V(t_2, x(t_2)) \leqq \int_{t_1}^{t_2} W_2(t, x_t)\, dt,$$

where $x(\cdot)$ is a solution of (2.1).

Then for each solution $x(\cdot)$ of (2.1) $\lim_{t\to\infty} V(t, x(t))$ *exists.*

Proof. Let $V_1(t, x) = \max\{V(t, x), 0\}$, $V_2(t, x) = -\min\{V(t, x), 0\}$. From conditions (i), (ii), (iii), (iv) of Corollary 3.2 it follows that V_1, W_1 and V_2, W_2 satisfy conditions (i), (ii) of Theorem 3.1. This implies that for every solution $x(\cdot)$ of (2.1) the limits $\lim_{t\to\infty} V_1(t, x(t))$ and $\lim_{t\to\infty} V_2(t, x(t))$ exist. Thus the corollary is proved.

4. Applications and examples

I. Consider the equation

$$\dot{x}(t) = f(t, x(t)) + g(t, x_t), \tag{4.1}$$

where f: $R^+ \times X_f \to X$, $X_f \subset X$, g: $R^+ \times C_g \to X$, $C_g \subset C$. (4.1) is the special case of the equation (2.1), when

$$F(t, \varphi) = f(t, \varphi(0)) + g(t, \varphi).$$

Theorem 4.1. *Suppose that for a nonnegative, locally Lipschitzian Ljapunov function V there exist functions α, p: $R^+ \to R^+$ with the following properties:*

(i) $\alpha(t)$ *is bounded for* $t \geqq t_0$,

(ii) *the function* p *is locally Lipschitzian on* $(0, \infty)$,

(iii) $V(t, x+y) \leqq V(t, x) + V(t, y)$ *for all* $(t, x), (t, y) \in R^+ \times X$,

(iv) $\overline{\lim}_{h\to 0+} \frac{1}{h}[V(t+h, x+hf(t, x)) - V(t, x)] \leqq -\alpha(t)p(V(t, x))$

for all $(t, x) \in R^+ \times X_f$,

(v) *for every* $\varepsilon > 0$ *there exists an* $\eta = \eta(\varepsilon) > 0$ *such that* $(t, \varphi) \in \bar{S}(V, \eta, \varepsilon)$ *implies* $\overline{\lim}_{h\to 0+} \frac{1}{h} V(t+h, hg(t, \varphi)) \leqq \alpha(t)p(\bar{V}(t, \varphi))$.

Then for each solution $x(\cdot)$ *of* (4.1) $\lim_{t\to\infty} V(t, x(t))$ *exists.*

Proof. We apply Theorem 3.1. Let $W = D^+_{(4.1)}V$. By Remark 3.1 it is sufficient to prove that condition (i) of Theorem 3.1 is satisfied. Let $\varepsilon > 0$ be given and choose $\eta = \eta(\varepsilon)$ according to assumption (v). If $(t, \varphi) \in \bar{S}(V, \eta, \varepsilon)$, then from conditions (i)—(v) we obtain

$$D^+_{(4.1)}V(t, \varphi) = \overline{\lim}_{h\to 0+} \frac{1}{h}[V(t+h, \varphi(0) + hf(t, \varphi(0)) + hg(t, \varphi)) - V(t, \varphi(0))] \leqq$$

$$\leqq \overline{\lim}_{h\to 0+} \frac{1}{h}[V(t+h, \varphi(0) + hf(t, \varphi(0))) - V(t, \varphi(0))] + \overline{\lim}_{h\to 0+} \frac{1}{h} V(t+h, hg(t, \varphi)) \leqq$$

$$\leqq \alpha(t)[p(\bar{V}(t, \varphi)) - p(V(t, \varphi(0)))] \leqq KL[\bar{V}(t, \varphi) - V(t, \varphi(0))] =$$

$$= \xi[\bar{V}(t, \varphi) - V(t, \varphi(0))],$$

where L is the Lipschitz constant of p on $[\varepsilon, 2\varepsilon]$ and K is an upper bound for α on $[t_0, \infty)$. This completes the proof.

II. We now apply Theorem 4.1 to obtain a result for equation (4.1) in the case $X = R$.

Theorem 4.2. *Let* $X = R$. *If* $f(t, 0) \equiv 0$, $xf(t, x) \leqq -a(t)x^2$ *for all* $(t, x) \in R^+ \times X_f$, $|g(t, \varphi)| \leqq a(t)\|\varphi\|$ *for all* $(t, \varphi) \in R^+ \times C_g$ *and* $a(t)$ *is bounded for* $t \geqq t_0$, *then for each solution* $x(\cdot)$ *of* (4.1) $\lim_{t\to\infty} x(t)$ *exists.*

Proof. In Theorem 4.1 let $V(t, x) = |x|$, $\alpha(t) = a(t)$, $p(u) \equiv 1$, $\eta(\varepsilon) = \varepsilon$. Thus, V is a Lipschitzian function and conditions (i), (ii), (iii) and in the case $x = 0$ condition (iv) in Theorem 4.1 are obviously satisfied. We have

$$\overline{\lim}_{h\to 0+} \frac{1}{h}(|x + hf(t, x)| - |x|) \leqq \overline{\lim}_{h\to 0+} \frac{1}{h}|x|\left(1 + h\frac{f(t, x)}{x} - 1\right) \leqq -a(t)|x|,$$

if $x \neq 0$ and

$$\overline{\lim}_{h\to 0+} \frac{1}{h}|hg(t, \varphi)| = |g(t, \varphi)| \leqq a(t)\|\varphi\|,$$

that assures also conditions (iv), (v) in Theorem 4.1 to be satisfied. This completes the proof.

Example 4.1. Let us consider the scalar equation

$$\dot{x}(t) = -ax(t)+b(t)x(t-\tau(t)), \tag{4.2}$$

where $a>0$, $b(t)$ and $\tau(t)$ are continuous for $t \geqq t_0$, $|b(t)| \leqq a$, $0 \leqq \tau(t) \leqq r$.

For equation (4.2) in this case N. N. KRASOVSKIĬ [9] proved that the zero solution is uniformly stable. Applying Theorem 4.2 we obtain that $x(t)$ tends to a constant as $t \to \infty$, where $x(\cdot)$ is a solution of (4.2).

III. Let us consider the following special form of equation (4.1):

$$\dot{x}(t) = -a(t)x(t)+\sum_{k=1}^{n} b_k(t)x(t-\tau_k(t)), \tag{4.3}$$

where $a, b_k, \tau_k \colon R^+ \to R$ are continuous functions and $0 \leqq \tau_k(t) \leqq r$ $(k=1, 2, \ldots, n)$.

Theorem 4.3. *Let $k \colon [-r, \infty) \to (0, \infty)$ be a continuous and locally Lipschitzian function. If there exists a $K \in R^+$ such that*

$$k(t)\sum_{k=1}^{n}\frac{|b_k(t)|}{k(t-\tau_k(t))} \leqq a(t)-\frac{D^+k(t)}{k(t)} \leqq K \quad (t \in R^+), \tag{4.4}$$

then for each solution $x(\cdot)$ of (4.3) $\lim_{t\to\infty} |k(t)x(t)|$ exists.

Proof. Apply Theorem 4.1 setting $V(t, x)=|k(t)x|$, $\alpha(t)=a(t)-\dfrac{D^+k(t)}{k(t)}$, $p(u)\equiv 1$, $\eta(\varepsilon)=\varepsilon$. It is clear that conditions (i), (ii), (iii) in Theorem 4.1 are satisfied. Using (4.4) we can check conditions (iv), (v) in Theorem 4.1 as follows

$$\overline{\lim_{h\to 0+}}\frac{1}{h}\big(|k(t+h)(x-ha(t)x)|-|k(t)x|\big) = |k(t)x|\overline{\lim_{h\to 0+}}\frac{1}{h}\frac{k(t+h)(1-ha(t))-k(t)}{k(t)} =$$

$$= |k(t)x|\left(\frac{D^+k(t)}{k(t)}-a(t)\right) = -\alpha(t)V(t, x),$$

$$\overline{\lim_{h\to 0+}}\frac{1}{h}\left|k(t+h)h\sum_{k=1}^{n}b_k(t)x(t-\tau_k(t))\right| = k(t)\left|\sum_{k=1}^{n}b_k(t)x(t-\tau_k(t))\right| \leqq$$

$$\leqq k(t)\sum_{k=1}^{n}\frac{|b_k(t)|}{k(t-\tau_k(t))}\bar{V}(t, x_t) \leqq \left(a(t)-\frac{D^+k(t)}{k(t)}\right)\bar{V}(t, x_t) = \alpha(t)\bar{V}(t, x_t).$$

This completes the proof.

Remark 4.1. If for equation (4.3) the inequality $\sum_{k=1}^{n}|b_k(t)|\leqq a(t)\leqq K$ holds, then $k(t)\equiv 1$ satisfies (4.4) and by Theorem 4.3 $\lim_{t\to\infty}|x(t)|$ exists. But Theorem 4.3 can be used even if this inequality does not hold, as the following example shows.

Example 4.2. Let us consider the equation

$$\dot{x}(t) = -a(t)x(t)+b(t)x(t-\tau(t)), \tag{4.5}$$

where $a(t)$, $b(t)$ and $\tau(t)$ are continuous for $t\geqq t_0$, $0\leqq\tau(t)\leqq r$ and there exists a $K\in R^+$ such that $a(t)\leqq|b(t)|\leqq K$ for $t\geqq t_0$. Let $k(t)=\exp\left(\int_0^t (a(s)-|b(s)|)ds\right)$. We have

$$\frac{D^+k(t)}{k(t)} = a(t)-|b(t)|,$$

$$\exp\left(\int_0^t (a(s)-|b(s)|)\,ds\right)\frac{|b(t)|}{\exp\left(\int_0^{t-\tau(t)} (a(s)-|b(s)|)ds\right)} \leqq |b(t)| \leqq K.$$

Thus, from Theorem 4.3 it follows that for each solution $x(\cdot)$ of (4.5)

$$\lim_{t\to\infty}\left|x(t)\exp\left(\int_0^t (a(s)-|b(s)|)\,ds\right)\right|$$

exists.

IV. Let us consider the equation

$$\dot{x}(t) = -h(x(t))+h(x(t-\tau(t))), \tag{4.6}$$

where $\tau(t)$ is continuous for $t\geqq t_0$, $0\leqq\tau(t)\leqq r$ and $h(s)$ is continuous for $s\in R$.

Theorem 4.4. *If the function h is increasing and locally Lipschitzian on $(-\infty, 0)$ and $(0, \infty)$, then for each solution $x(\cdot)$ of (4.6) $\lim_{t\to\infty} x(t)$ exists.*

Proof. Apply Corollary 3.2 setting $V(t, x)=x$, $W_1(t, \varphi)=-W_2(t, \varphi)=D^+_{(4.6)}V(t, \varphi)$. Let $\varepsilon>0$ be given, $\eta(\varepsilon)=\varepsilon$ and $\xi(\varepsilon)=\max\{L_1, L_2\}$, where L_1, L_2 are the Lipschitz constants of h on $[\varepsilon, 2\varepsilon]$, $[-2\varepsilon, -\varepsilon]$, respectively. Since $W_1(t, x_t)=-W_2(t, x_t)=\dot{x}(t)$ it is obvious that conditions (iii), (iv) in Corollary 3.2 are satisfied. If $(t, x_t)\in\bar{S}(x(t), \eta, \varepsilon)$ then

$$\dot{x}(t) = -h(x(t))+h(x(t-\tau(t))) \leqq -h(x(t))+h(\bar{x}_t) \leqq \xi(\bar{x}_t-x(t)).$$

If $(t, x_t)\in\underline{S}(x(t), \eta, \varepsilon)$, then

$$-\dot{x}(t) = h(x(t))-h(x(t-\tau(t))) \leqq h(x(t))-h(\underline{x}_t) \leqq \xi(x(t)-\underline{x}_t).$$

Thus conditions (i), (ii) in Corollary 3.2 are satisfied and the theorem is proved.

Remark 4.2. Applying Theorem 4.4 to case $h(u)=u^{1/3}$, $\tau(t)\equiv r$ we get a new proof for the following conjecture of S. R. BERNFELD and J. R. HADDOCK [1], which was solved by C. JEHU [5]: each solution of the scalar equation $\dot{x}(t)=-x^{1/3}(t)+$ $+x^{1/3}(t-r)$ tends to a constant as $t\to\infty$.

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Imbedding theorems and strong approximation

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1. Several recent papers, e.g. [2], [3], [4], [8], investigate problems of the following type: Under what conditions can a certain class of functions be imbedded in another class, where at least one of these classes is determined by certain properties of the strong approximation of Fourier series.

Our aim is also to prove two theorems of this type.

Before formulating them we give the definitions and notations used in the paper and draw some background.

Let $f(x)$ be a continuous and 2π-periodic function and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

be its Fourier series. Denote by $s_n = s_n(x) = s_n(f, x)$ the n-th partial sum of (1) and by $f^{(r)}$ the r-th derivative of f.

Let $\omega(\delta)$ be a non-decreasing continuous function on the interval $[0, 2\pi]$ having the properties: $\omega(0)=0$, $\omega(\delta_1+\delta_2) \leqq \omega(\delta_1)+\omega(\delta_2)$ for any $0 \leqq \delta_1 \leqq \delta_2 \leqq \delta_1+\delta_2 \leqq 2\pi$. Such a function will be called a *modulus of continuity*.

Let $E_n(f)$ denote the best approximation of f by trigonometric polynomials of order at most n.

We define the following class of functions:

$$W^r H^\omega := \{f : \omega(f^{(r)}; \delta)\} = O(\omega(\delta)),$$

where $\omega(f; \delta)$ is the modulus of continuity of f. In the case $\omega(\delta)=\delta^\alpha$ we write $W^r H^\alpha$ instead of $W^r H^{\delta^\alpha}$; and if $r=0$ H^ω stands for $W^0 H^\omega$, and in many cases Lip 1 will denote the class H^1.

Generalizing a theorem of SZABADOS [7] we proved in ([6]) the following result:

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If $0<\alpha\leqq 1$, $p>0$ *and* r *is a nonnegative integer, then*

$$\left\|\sum_{n=1}^{\infty} n^{(r+\alpha)p-1}|s_n-f|^p\right\|<\infty \tag{2}$$

implies that

$$\omega(f^{(r)};\delta)=\begin{cases} O\left(\delta\log\frac{1}{\delta}\right) & \textit{if} \quad \alpha=1,\\ O(\delta^{\alpha}) & \textit{if} \quad 0<\alpha<1,\end{cases}$$

where $\|\cdot\|$ *denotes the usual maximum norm. These estimations are best possible.*

On account of this result it is clear that condition (2) with $\alpha=1$ does not imply that $f^{(r)}\in\mathrm{Lip}\,1$. But the following condition

$$\sum_{m=0}^{\infty}\left\|\left\{\sum_{n=2^m+1}^{2^{m+1}} n^{(r+1)p-1}|s_n-f|^p\right\}^{1/p}\right\|<\infty, \tag{3}$$

which claims just a little bit more than (2) with $\alpha=1$ does, is already sufficient for $f^{(r)}$ to belong to the class Lip 1 (see [5], Theorem 5). Thus it is natural to ask whether condition (3) is also necessary for $f^{(r)}\in\mathrm{Lip}\,1$. We shall prove that the answer to this question is negative, but condition (3) cannot be weakened in general. Indeed, the following more general theorem also holds.

Theorem 1. *Let* $\varepsilon=\{\varepsilon_n\}$ *be a given monotone sequence. Then for any positive* p *the condition*

$$\varepsilon_n\geqq c>0 \quad (n=1,2,\ldots) \tag{4}$$

is necessary and sufficient that

$$S_p(\varepsilon,r):=\left\{f\colon \sum_{m=0}^{\infty}\varepsilon_m\left\|\left\{\sum_{n=2^m+1}^{2^{m+1}} n^{(r+1)p-1}|s_n-f|^p\right\}^{1/p}\right\|<\infty\right\}\subset W^rH^1. \tag{5}$$

Furthermore, for any sequence ε *satisfying condition* (4), *there exists a function* f_0 *such that* $f_0\in W^rH^1$ *but* $f_0\notin S_p(\varepsilon,r)$. *Thus the imbedding* (5) *is proper.*

As a special case of Lemma 6 of [6], it is also proved that (3) is equivalent to

$$\sum_{n=1}^{\infty} n^r E_n(f)<\infty.$$

Moreover, in [6] we verified that for any $p>0$ and for any positive monotone sequence $\mu=\{\mu_n\}$ with the property $0<k\leqq\mu_{2n}/\mu_n\leqq K<\infty$ the conditions

$$\sum_{m=0}^{\infty}\left\|\sum_{n=2^m+1}^{2^{m+1}}\mu_n|s_n-f|^p\right\|<\infty$$

and

(6) $$\sum_{n=1}^{\infty} \mu_n E_n^p(f) < \infty$$

are equivalent.

Thus it is obvious that (6) implies

(7) $$\left\| \sum_{n=1}^{\infty} \mu_n |s_n - f|^p \right\| < \infty;$$

in other words, (6) is a sufficient condition for (7), but presumably not a necessary one. We shall prove that this is the case, indeed, but we shall also verify that (6) cannot be weakened generally.

We define two further classes of functions:

$$S_p(\mu) := \left\{ f: \left\| \sum_{n=1}^{\infty} \mu_n |s_n - f|^p \right\| < \infty \right\} \quad \text{and} \quad E(\varepsilon) := \{f: E_n(f) = O(\varepsilon_n)\},$$

where $\mu = \{\mu_n\}$ and $\varepsilon = \{\varepsilon_n\}$ are given positive monotone sequences and $p > 0$.

Using these notations we can formulate our statement as follows:

Theorem 2. *Let $p > 0$ and let $\varepsilon = \{\varepsilon_n\}$ and $\mu = \{\mu_n\}$ be given positive monotone sequences such that $0 < k \leqq \mu_{2n}/\mu_n \leqq K < \infty$. In order that*

(8) $$E(\varepsilon) \subset S_p(\mu)$$

it is necessary and sufficient that

(9) $$\sum_{n=1}^{\infty} \mu_n \varepsilon_n^p < \infty.$$

If $\mu_n = n^\gamma$, $\gamma > -1$, then inclusion (8) is proper for any $\varepsilon = \{\varepsilon_n\}$ satisfying (9), that is, there exists a function F such that $F \in S_p(\mu)$ but $F \notin E(\varepsilon)$.

From Theorem 2, using a result of Krotov and Leindler [3] (see our Lemma 1), we immediately obtain

Corollary. *If there exists a positive monotone sequence $\mu = \{\mu_n\}$ such that $0 < k \leqq \mu_{2n}/\mu_n \leqq K < \infty$,*

(10) $$\sum_{n=1}^{\infty} \mu_n \varepsilon_n^p < \infty \quad \textit{and} \quad \sum_{n=1}^{m} (n\mu_n)^{-1/p} = O\left(m\omega\left(\frac{1}{m}\right)\right),$$

then $E(\varepsilon) \subset H^\omega$.

2. We require some lemmas to prove our theorems.

Lemma 1 ([3, Theorem]). *If* $0<p<\infty$ *and* $\lambda=\{\lambda_n\}$ *is a monotone sequence such that* $\{n^\theta\lambda_n\}$ *with a certain* $0<\theta<1$ *increases, then condition*

$$\sum_{\nu=1}^{n}(\nu\lambda_\nu)^{-1/p}=O\left(n\omega\left(\frac{1}{n}\right)\right) \tag{2.1}$$

is necessary and sufficient for

$$S_p(\lambda)\subset H^\omega. \tag{2.2}$$

(We mention that the assumption $n^\theta\lambda_n\uparrow$ is not needed to the proof of the implication (2.1)⇒(2.2).)

Lemma 2 ([1, see the proof of Theorem 1]). *The function*

$$g(x)=\sum_{k=1}^{\infty}\frac{(-1)^k}{2^k}\sum_{l=2^{k-1}+1}^{2^k}\left(\frac{\sin(5\cdot 2^k-l)x}{l}-\frac{\sin(5\cdot 2^k+l)x}{l}\right)$$

belongs to the class Lip 1.

Lemma 3 ([6, Theorem 3 with $r=0$]). *Let* $p>0$. *Suppose that* $\{\lambda_k\}$ *is a monotonesequence satisfying the following conditions. Setting* $\Lambda_n=\sum_{k=1}^{n}\lambda_k$, $\{n^{-1}\Lambda_n\}$ *is monotone,* $\{n^{\eta-1}\Lambda_n\}$ *is non-decreasing for a certain* $\eta<1$, *and* $n\lambda_n\leqq K\Lambda_n$. *Then the function*

$$F(x)=\sum_{n=1}^{\infty}\frac{\sin nx}{n\Lambda_n^{1/p}}$$

has the following properties:

$$\left\|\sum_{n=1}^{\infty}\lambda_n|s_n(F)-F|^p\right\|<\infty \quad \text{and} \quad \omega\left(F;\frac{1}{n}\right)\geqq C\frac{1}{n}\sum_{\nu=1}^{n}\Lambda_\nu^{-1/p},$$

where C is a positive constant.

Lemma 4. *If* $a_n\geqq 0$ *and* $\sum_{n=1}^{\infty}a_n=\infty$ *then for any sequence* $\{\varepsilon_n\}$ *tending to zero there exists a monotone sequence* $\{b_n\}$ *such that* $b_n\to 0$,

$$\sum_{n=1}^{\infty}a_nb_n=\infty \tag{2.3}$$

and

$$\sum_{n=1}^{\infty}a_nb_n\varepsilon_n<\infty. \tag{2.4}$$

Proof. Since $\sum a_n=\infty$ we can define a sequence $\{\nu_m\}$ such that $\nu_1=2$, $\nu_2=4$ and if $m\geqq 3$ then

$$\sum_{n=\nu_{m-1}+1}^{\nu_m}a_n>m+\sum_{n=\nu_{m-2}+1}^{\nu_{m-1}}a_n$$

and for any $k \geqq \nu_m$

$$\varepsilon_k < \frac{1}{m^2}.$$

From this sequence $\{\nu_m\}$ we deduce the required sequence $\{b_n\}$ as follows: Let $b_1 = b_2 = 1$ and if $\nu_{m-1} < n \leqq \nu_m$ $(m \geqq 2)$ then let $b_n = \left(\sum_{i=\nu_{m-1}+1}^{\nu_m} a_i\right)^{-1}$.

Hence an elementary calculation shows that (2.3) and (2.4) hold, and this ends the proof.

3. Proof of Theorem 1. First we prove the implication (4)$\Rightarrow$(5). If $f \in S_p(\varepsilon, r)$ and (4) holds then condition (3) is also fulfilled whence $f^{(r)} \in \mathrm{Lip}\, 1$ follows (see Theorem 5 of [5]), i.e. imbedding (5) holds.

To prove the necessity of (4) we give functions showing that (5) does not hold if $\varepsilon_n \to 0$.

First we define a new sequence $\{\varepsilon_n^*\}$ as follows: Let $\varepsilon_n^* = \varepsilon_m$ if $2^m < n \leqq 2^{m+1}$ $(m \geqq 0)$ and $\varepsilon_1^* = 1$. By Lemma 4 for the sequence $\{\varepsilon_n^*\}$ there exists a monotone sequence $\{\lambda_n\}$ such that $\lambda_n \to \infty$ and

$$\sum_{n=1}^{\infty} \frac{1}{n\lambda_n} = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\varepsilon_n^*}{n\lambda_n} < \infty. \tag{3.1}$$

Using this sequence we define the following function

$$F(x) := \sum_{n=1}^{\infty} \lambda_n^{-1} n^{-r-2} \operatorname{cvs} nx$$

where cvs x means $\cos x$ or $\sin x$ according as r is an odd or even integer, resp. It is clear that

$$F^{(r)}(x) = \pm \sum_{n=1}^{\infty} \lambda_n^{-1} n^{-2} \sin nx,$$

and by (3.1) $F^{(r)}$ does not belong to the class Lip 1.

On the other hand

$$|s_n(F) - F| \leqq \sum_{k=n}^{\infty} \lambda_k^{-1} k^{-r-2} \leqq K\lambda_n^{-1} n^{-r-1},$$

whence we get that

$$\left\| \left\{ \sum_{n=2^m+1}^{2^{m+1}} n^{(r+1)p-1} |s_n(F) - F|^p \right\}^{1/p} \right\| \leqq K_0 \left\{ \sum_{n=2^m+1}^{2^{m+1}} \lambda_n^{-p} n^{-1} \right\}^{1/p} \leqq \frac{K_1}{\lambda_{2^m}}.$$

Hence, on account of (3.1), we already obtain immediately that $F \in S_p(\varepsilon, r)$. As we have seen $F \notin W^r H^1$, thus (5) cannot be valid, and this proves the necessity of (4).

In order to prove that imbedding (5) is proper we consider the following functions:

If r is even then let

$$f_0(x)=\sum_{n=1}^{\infty}\frac{\cos nx}{n^{r+2}},$$

and if r is odd then let

$$f_1(x)=\sum_{m=1}^{\infty}\frac{(-1)^m}{2^m}\sum_{l=2^{m-1}+1}^{2^m}\left(\frac{\cos(5\cdot 2^m-l)x}{(5\cdot 2^m-l)^r l}-\frac{\cos(5\cdot 2^m+l)x}{(5\cdot 2^m+l)^r l}\right).$$

It is well known that $f_0^{(r)}\in \mathrm{Lip}\,1$, and on the other hand, by Lemma 2, $f_1^{(r)}\in \mathrm{Lip}\,1$ is also proved.

Thus it remains to prove that f_0 and f_1 do not belong to the class $S_p(\varepsilon, r)$ for any ε satisfying (4).

A standard calculation shows that

$$|f_0(0)-s_n(0)|\geqq cn^{-r-1},\quad (c>0),$$

whence

$$\left\|\left\{\sum_{n=2^m+1}^{2^{m+1}} n^{(r+1)p-1}|s_n-f_0|^p\right\}^{1/p}\right\|\geqq\frac{c}{2}$$

follows, consequently $f_0\notin S_p(\varepsilon, r)$.

The proof of the statement $f_1\notin S_p(\varepsilon, r)$ needs a longer calculation. First we give a lower estimation for the difference $|f_1(0)-s_n(f_1, 0)|$ if n satisfies the inequalities $22\cdot 2^{m-2}<n\leqq 23\cdot 2^{m-2}$ $(m\geqq 4)$. Such an n can be written in the form:

$$n=5\cdot 2^m+l\quad\text{with}\quad 2^{m-1}<l\leqq 3\cdot 2^{m-2}.$$

Therefore, by the definition of f_1 it is easy to see that

$$|s_n(f_1;0)-f_1(0)|\geqq\left|\frac{1}{2^m}\sum_{l=n-5\cdot 2^m+1}^{2^m}\frac{-(-1)^m}{(5\cdot 2^m+l)^r l}+\sum_{\mu=m+1}^{\infty}\frac{(-1)^\mu}{2^\mu}R_\mu\right|,\tag{3.2}$$

where

$$R_\mu=\sum_{l=2^{\mu-1}+1}^{2^\mu}\left(\frac{1}{(5\cdot 2^\mu-l)^r l}-\frac{1}{(5\cdot 2^\mu+l)^r l}\right).$$

If we show that $R_\mu\geqq R_{\mu+1}(\geqq 0)$ then by (3.2) we obtain that $|s_n(0)-f_1(0)|$ is not less than the absolute value of the first sum in (3.2), namely the sums in (3.2) have the same sign. Since $r\geqq 1$

$$\frac{1}{(5\cdot 2^{\mu+1}-2i)^r 2i}-\frac{1}{(5\cdot 2^{\mu+1}+2i)^r 2i}+\frac{1}{(5\cdot 2^{\mu+1}-2i+1)^r(2i-1)}-$$

$$-\frac{1}{(5\cdot 2^{\mu+1}+2i-1)^r(2i-1)}\leqq\left(\frac{1}{2i}+\frac{1}{2i-1}\right)\left(\frac{1}{(5\cdot 2^{\mu+1}-2i)^r}-\frac{1}{(5\cdot 2^{\mu+1}+2i)^r}\right)\leqq$$

$$\leqq\frac{2}{2^r(2i-1)}\left(\frac{1}{(5\cdot 2^\mu-i)^r}-\frac{1}{(5\cdot 2^\mu+i)^r}\right)\leqq\frac{1}{i}\left(\frac{1}{(5\cdot 2^\mu-i)^r}-\frac{1}{(5\cdot 2^\mu+i)^r}\right),$$

whence

$$R_{\mu+1} = \sum_{l=2^{\mu}+1}^{2^{\mu+1}} \frac{1}{l}\left(\frac{1}{(5\cdot 2^{\mu+1}-l)^r} - \frac{1}{(5\cdot 2^{\mu+1}-l)^r}\right) \leqq$$

$$\leqq \sum_{l=2^{\mu-1}+1}^{2^{\mu}} \frac{1}{l}\left(\frac{1}{(5\cdot 2^{\mu}-l)^r} - \frac{1}{(5\cdot 2^{\mu}+l)^r}\right) = R_\mu$$

follows obviously.

Continuing the estimate of (3.2) we have

$$|s_n(f_1, 0)-f_1(0)| \geqq \frac{1}{2^m} \sum_{l=n-5\cdot 2^m+1}^{2^m} l^{-1}(5\cdot 2^m+l)^{-r} \geqq$$

$$\geqq 6^{-r}2^{-m(r+1)} \sum_{l=n-5\cdot 2^m+1}^{2^m} l^{-1}.$$

Using this we obtain that $(m\geqq 4)$

$$\sum_{n=2^{m+2}+1}^{2^{m+3}} n^{(r+1)p-1}|s_n(0)-f_1(0)|^p \geqq \sum_{n=22\cdot 2^{m-2}+1}^{23\cdot 2^{m-2}} n^{(r+1)p-1}|s_n(0)-f_1(0)|^p \geqq$$

$$\geqq \sum_{n=22\cdot 2^{m-2}+1}^{23\cdot 2^{m-2}} n^{(r+1)p-1}6^{-rp}2^{-m(r+1)p}\left(\sum_{l=3\cdot 2^{m-2}+1}^{2^m} l^{-1}\right)^p \geqq C_{r,p} > 0,$$

where $C_{r,p}$ is independent of n.

Hence, as before, the statement $f_1 \notin S_p(\varepsilon, r)$ follows clearly, and this completes the proof of Theorem 1.

Proof of Theorem 2. *Sufficiency of condition* (9). It is clear that if $f\in E(\varepsilon)$ then (9) implies

$$\sum_{n=1}^{\infty} \mu_n E_n^p(f) < \infty,$$

and this is equivalent to

$$\sum_{m=0}^{\infty} \left\| \sum_{n=2^m+1}^{2^{m+1}} \mu_n |s_n - f|^p \right\| < \infty$$

(see Theorem 4 of [6], where the restriction on the rate of μ_{2n}/μ_n is required), whence $f\in S_p(\mu)$ follows obviously. Thus (8) holds if (9) is fulfilled.

Necessity of condition (9) will be proved indirectly. If we assume that (8) holds and

$$\sum_{n=1}^{\infty} \mu_n \varepsilon_n^p = \infty, \tag{3.3}$$

then the function

$$f_0(x) = \sum_{n=2}^{\infty} (\varepsilon_n - \varepsilon_{n+1}) \cos (n+1)\, x$$

leads to a contradiction. Namely,

$$E_n(f_0) \leqq \|s_n(f_0, x) - f_0(x)\| \leqq \varepsilon_n,$$

i.e. $f_0 \in E(\varepsilon)$, but

$$|s_n(f_0, 0) - f_0(0)| = \sum_{k=n}^{\infty} (\varepsilon_k - \varepsilon_{k+1}) = \varepsilon_n,$$

whence

$$\left\| \sum_{n=2}^{N} \mu_n |s_n(f_0) - f_0|^p \right\| \geqq \sum_{n=2}^{N} \mu_n \varepsilon_n^p;$$

by (3.3) this shows that f_0 does not belong to the class $S_p(\mu)$, and this contradicts (8).

Hereby the necessity of (9) is proved.

In order to prove that inclusion (8) is strict let us consider the function

$$F(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^{1+(1/p)+(\gamma/p)}}.$$

An elementary calculation shows that if $\lambda_n = n^\gamma$ $(\gamma > -1)$ then all conditions of Lemma 3 are satisfied and thus

$$\left\| \sum_{n=1}^{\infty} n^\gamma |s_n(F) - F|^p \right\| < \infty,$$

i.e. $F \in S_p(n^\gamma)$, moreover,

$$\frac{1}{n} \sum_{\nu=1}^{n} \nu^{-(1/p)-(\gamma/p)} \leqq K\omega\left(F, \frac{1}{n}\right). \tag{3.4}$$

We show that $F \notin E(\varepsilon)$ for any ε satisfying (9). Namely, if we assume that $F \in E(\varepsilon)$ and

$$\sum_{n=1}^{\infty} n^\gamma \varepsilon_n^p < \infty \quad \text{with} \quad -1 < \gamma \leqq p-1$$

hold then (3.4) leads us to a contradiction. Indeed, these assumptions imply

$$\sum_{n=1}^{\infty} n^\gamma E_n^p(F) < \infty;$$

whence, considering the block $(n, 2n)$ in this series we infer that

$$n^{\gamma+1} E_n^p(F) \to 0. \tag{3.5}$$

Consequently, the well-known inequality

$$\omega\left(F, \frac{1}{n}\right) \leqq K \frac{1}{n} \sum_{\nu=0}^{n} E_\nu(F),$$

and (3.5) contradict (3.4) if $\gamma \leqq p-1$.

If $\gamma > p-1$, we can only give a somewhat longer proof of the statement $F \notin E(\varepsilon)$.

First we show that if $m \geqq 2$ and $2^m \leqq \nu \leqq 2^{m+1}$ then

$$|s_\nu(F, h_m) - F(h_m)| \geqq \frac{1}{4^{1+(1/p)+(\gamma/p)}} \nu^{-(1/p)-(\gamma/p)} \tag{3.6}$$

holds, where $h_m = \frac{\pi}{2^{m+4}}$.

Let $N_m = 2^{m+4}$ and $\alpha = 1 + \frac{1}{p} + \frac{\gamma}{p}$. Then

$$F(h_m) - s_\nu(F, h_m) = \sum_{n=\nu+1}^{\infty} \frac{\sin nh_m}{n^\alpha} = \left(\sum_{n=\nu+1}^{\frac{N_m}{4}} + \sum_{n=\frac{N_m}{4}+1}^{N_m} + \sum_{n=N_m+1}^{2N_m} + \sum_{k=2}^{\infty} \sum_{n=kN_m+1}^{(k+1)N_m} \right) \frac{\sin nh_m}{n^\alpha}.$$

It is clear that for any $l \geqq 1$

$$\sum_{n=2lN_m+1}^{(2l+1)N_m} \frac{\sin nh_m}{n^\alpha} > \left| \sum_{n=(2l+1)N_m+1}^{(2l+2)N_m} \frac{\sin nh_m}{n^\alpha} \right|,$$

and thus the sum

$$\sum_{k=2}^{\infty} \sum_{n=kN_m+1}^{(k+1)N_m} \frac{\sin nh_m}{n^\alpha}$$

is positive. Furthermore we show that

$$\sum_{n=\frac{N_m}{4}+1}^{N_m} \frac{\sin nh_m}{n^\alpha} > \left| \sum_{n=N_m+1}^{2N_m} \frac{\sin nh_m}{n^\alpha} \right|.$$

It is clear that

$$\sum_{n=\frac{N_m}{2}+1}^{N_m} \frac{\sin n h_m}{n^\alpha} > \left| \sum_{n=N_m+1}^{\frac{3}{2}N_m-1} \frac{\sin nh_m}{n^\alpha} \right|,$$

and an easy calculation verifies that on account of $\alpha \geqq 1$

$$\sum_{n=\frac{N_m}{4}+1}^{N_m/2} \frac{\sin nh_m}{n^\alpha} \geqq \sum_{n=2^{m+2}+1}^{2^{m+3}} \frac{\sqrt{2}}{2n^\alpha} \geqq \frac{\sqrt{2}}{4} 2^{m+3}(2^{m+3})^{-\alpha} \geqq$$

$$\geqq (2^{m+3}+1) \frac{1}{3} (2^{m+3})^{-\alpha} \geqq \left| \sum_{n=3\cdot 2^{m+3}}^{2^{m+5}} \frac{\sin nh_m}{n^\alpha} \right| = \left| \sum_{n=\frac{3}{2}N_m}^{2N_m} \frac{\sin nh_m}{n^\alpha} \right|.$$

Collecting the results we obtain that

$$F(h_m)-s_\nu(F,h_m) \geqq \sum_{n=\nu+1}^{\frac{N_m}{4}} \frac{\sin nh_m}{n^\alpha} \geqq$$

$$\geqq \sum_{n=2^{m+1}+1}^{2^{m+2}} \frac{\sin nh_m}{n^\alpha} \geqq \frac{2}{\pi} h_m \sum_{n=2^{m+1}+1}^{2^{m+2}} n^{1-\alpha} >$$

$$> 2^{-m-3}\cdot 2^{m+1}\cdot 2^{(m+2)(1-\alpha)} \geqq 4^{-\alpha}\nu^{-(1/p)-(\gamma/p)},$$

which proves (3.6).

By (3.6) we obviously obtain

$$\left\| \sum_{n=2^m+1}^{2^{m+1}} n^\gamma |s_n(F)-F|^p \right\| \geqq C > 0, \quad \text{whence} \quad \sum_{m=0}^{\infty} \left\| \sum_{n=2^m+1}^{2^{m+1}} n^\gamma |s_n-F|^p \right\| = \infty$$

and, by the mentioned equivalence theorem (see Theorem 4 of [6]),

$$\sum_{n=1}^{\infty} n^\gamma E_n^p(F) = \infty$$

follow, i.e. $F \in E(\varepsilon)$ does not hold.

Thus Theorem 2 is proved.

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A Rees matrix semigroup over a semigroup and its maximal right quotient semigroup

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As with the Rees matrix semigroup over a group with zero, one can construct a semigroup $M^0(I, W, \Lambda; P)$ where W is a semigroup with zero [3]. Under certain conditions, we show that the maximal right quotient semigroup $Q(S)$ of a Rees matrix semigroup $S=M^0(W, n; P)$ is isomorphic to the endomorphism monoid of $S=M^0(Q(W), n; P)$ as a right S-system.

1. Preliminaries. Although much of the basic notations and definitions are given in this section, we assume the reader is familiar with the basic terminology and results on algebraic semigroups as presented in CLIFFORD and PRESTON [2]. Those wishing a more indepth view of S-systems and semigroups of quotients should read the survey article by WEINERT [6].

A *right S-system with zero M_S* is a semigroup S with zero, a set M, and a function $M\times S\to M$ with $(m, s)\to ms$ for which the following properties hold:

(i) $(ms)t=m(st)$ for $m\in M$ and $s, t\in S$;
(ii) M contains an element ϑ (necessarily unique) such that $\vartheta s=\vartheta$ for all $s\in S$;
(iii) for all $m\in M$, $m0=\vartheta$ where 0 is the zero of S.

An *S-subsystem* N of M_S is a subset N of M such that $NS\subseteqq N$; this will be denoted by $N_S\subseteqq M_S$. Let M_S and N_S be S-systems with $f\colon M_S\to N_S$ a mapping such that $f(ms)=f(m)s$ for all $m\in M$ and $s\in S$, then f is called an *S-homomorphism.* The set of all S-homomorphisms from M_S to N_S is denoted by $\mathrm{Hom}_S(M, N)$. Let $N_S\subseteqq M_S$, then N_S is *intersection large* (*$\cap$-large*) in M_S if for $\{\vartheta\}\neq X_S\subseteqq M_S$, $X\cap N\neq\neq\{\vartheta\}$. We denote this by $N_S\subseteqq' M_S$. Note that this is equivalent to saying that for all $\vartheta\neq m\in M$ there exists $s\in S^1$ such that $\vartheta\neq ms\in N$. The *singular congruence* ψ_M on M_S is a right congruence such that $a\psi_M b$ if and only if $ax=bx$ for all x in an $\cap$-large right ideal of S. If every nonzero S-subsystem of M is $\cap$-large, then M_S

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is said to be *intersection uniform* ($\cap$-*uniform*). An S-subsystem N of M_S is *dense* in M_S if for each $m_1, m_2, m_3 \in M$ where $m_1 \neq m_2$ there exists an $s \in S^1$ such that $m_1 s \neq m_2 s$ and $m_3 s \in N$. It is easy to see that when N_S is dense in M_S then $N_S \subseteq' M_S$.

The construction of the maximal right quotient semigroup is due to McMorris [4] and will not be repeated here. You will recall that every $\cap$-large right ideal of S is dense if and only if $\psi_S = \iota_S$ the identity congruence. In view of this result, we define S as being *right nonsingular* if every $\cap$-large right ideal of S is dense.

2. The Maximal Right Quotient Semigroup. Let $S = M^0(I, G, \Lambda; P)$ be a Rees matrix semigroup over a group G. In studying these structures, Botero de Meza [1] showed that if each row of P has a nonzero entry and S is nonsingular then the maximal right quotient semigroup of S, denoted $Q(S)$, is isomorphic to $\mathrm{Hom}_S(S, S)$. In general, this is not the case for $M^0(I, W, \Lambda; P)$ where W is a semigroup. What restrictions must we place on $M^0(I, W, \Lambda; P)$ to obtain similar results?

First let us consider the fact that the only $\cap$-large right ideal of S is S itself. This is not true for $M^0(I, W, \Lambda; P)$ as the next theorem and example will illustrate.

Theorem 1. *Let $S = M^0(I, W, \Lambda; P)$ where W is a semigroup with 0 and 1, T is a unitary subgroup of W, and P has an entry from T in each row. If L is an $\cap$-large right ideal of W then $\mathcal{L} = \{(i, l, \lambda) | l \in L, i \in I, \lambda \in \Lambda\}$ is an $\cap$-large right ideal of S.*

Proof. Let $0 \neq (i, a, \lambda) \in S$. Since P has an entry from T in each row then there exists $p_{\lambda j} \in T$ for some $j \in I$. For $0 \neq a \in W$, there exists $x \in W$ such that $0 \neq ax \in L$ since $L_W \subseteq' W_W$. Hence we let $b = p_{\lambda j}^{-1} x$ and choose $\mu \in \Lambda$ then $(i, a, \lambda) * (j, b, \mu) = (i, a p_{\lambda j} b, \mu) = (i, a p_{\lambda j} p_{\lambda j}^{-1} x, \mu) = (i, ax, \mu) \in \mathcal{L}$. It is easy to see that $\mathcal{L}$ is a right ideal of S and hence an $\cap$-large right ideal of S.

Example 2. Let $W = \{0, e, 1\}$, the semilattice $0 < e < 1$ and consider $M^0(I, W, I; P)$ where $P = (p_{ij})$ and $p_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. The right ideal $\{0, e\}$ is $\cap$-large in W and so by Theorem 1, $\mathcal{L} = \{(i, a, j) | a \in \{0, e\}\ i, j \in I\}$ is an $\cap$-large right ideal of $M^0(I, W, I; P)$; however, it is clear that $\mathcal{L} \neq M^0(I, W, I; P)$.

If $\mathcal{L}$ is an $\cap$-large right ideal of $M^0(I, W, \Lambda; P)$ what is its relation to the $\cap$-large right ideals of W, if any? If we let $I = \Lambda$, we are able to obtain some results to this question.

Theorem 3. *Let $S = M^0(I, W, I; P)$ where W is a semigroup with 0 and 1, T is a unitary subgroup of W, and P has an entry from T in each row. If $\mathcal{L}$ is an $\cap$-large right ideal of S, then for each $k \in I$, $M_k = \{s \in W | (k, s, k) \in \mathcal{L}\}$ is an $\cap$-large right ideal of W.*

Proof. Let $m \in M_k$ where $k \in I$ arbitrary but fixed, and let $s \in W$. Since each row of P has an entry from T there exists $p_{kj} \in T$ for some $j \in I$. Since $\mathcal{L}$ is a right ideal of S, then $(k, m, k)*(j, p_{kj}^{-1}s, k)\in\mathscr{L}$. But then $(k, ms, k)\in\mathscr{L}$ and so $ms\in M_k$. Thus M_k is a right ideal of W. To see that M_k is $\cap$-large, we let $0\neq z\in W$ and consider $(k, z, k)\in S$. Since $\mathscr{L}_S\subseteq' S_S$ then either $(k, z, k)\in\mathscr{L}$ or there exists $(n, s, m)\in S$ such that $0\neq(k, z, k)*(n, s, m)\in\mathscr{L}$. In the former case, $z\in M_k$ and there is nothing to prove. In the latter case, we have $0\neq(k, zp_{kn}s, m)\in\mathscr{L}$. Hence there exist $p_{mj}\in T$ and $0\neq(k, zp_{kn}s, m)*(j, 1, k)\in\mathscr{L}$ and so $0\neq zp_{kn}sp_{mj}\in M_k$. Thus M_k is $\cap$-large in W.

If we restrict our Rees matrix semigroup over a semigroup further to $|I|=n<\infty$, then $\bigcap_{k\in I} M_k$ is also an $\cap$-large right ideal of W and we develop the following results for $M^0(W, n; P)$.

Theorem 4. *Let $S=M^0(W, n; P)$ where W is a semigroup with 0 and 1, T is a unitary subgroup of W, and P has an entry from T in each row. If $\mathscr{L}$ is an $\cap$-large right ideal of S then $\mathscr{R}=\{(i, t, j)|t\in\bigcap_{k\in I} M_k,\ i, j\in I\}$ is an $\cap$-large right ideal of S contained in $\mathscr{L}$.*

Proof. Since $\bigcap_{k\in I} M_k$ is $\cap$-large in W then by Theorem 1, $\mathscr{R}$ is an $\cap$-large right ideal of S. Let $(i, t, j)\in\mathscr{R}$. Since $t\in\bigcap_{k\in I} M_k$ then $t\in M_i$ a right ideal of S and so $tp_{ih}^{-1}\in M_i$ for some $h\in I$ with $p_{ih}\in T$. Consequently, $(i, tp_{ih}^{-1}, i)\in\mathscr{L}$ and since $\mathscr{L}$ is a right ideal of S then $(i, tp_{ih}^{-1}, i)*(h, 1, j)\in\mathscr{L}$. But this says that $(i, t, j)\in\mathscr{L}$ since $(i, tp_{ih}^{-1}, i)*(h, 1, j)=(i, tp_{ih}^{-1}p_{ih}1, j)=(i, t, j)$.

Theorem 5. *Let $S=M^0(W, n; P)$ where W is a semigroup with 0 and 1, T is a unitary subgroup of W, and P has an entry from T in each row. If S is right nonsingular then W is right nonsingular.*

Proof. Let $k, j\in I$ such that $p_{kj}\in T$ and suppose $a\,\psi_W b$. Then since ψ_W is a right congruence $(ap_{kh})\psi_W(bp_{kh})$ for all $h\in I$. Hence there exists an $\cap$-large right ideal L of W such that for $x\in L$, $h\in I$ we have $ap_{kh}=bp_{kh}x$. By Theorem 1, L induces an $\cap$-large right ideal $\mathscr{L}=\{(i, x, j)|x\in L$ and $i, j\in I\}$ on S. Hence for (j, a, k), $(j, b, k)\in S$ and $(i, x, m)\in\mathscr{L}$ we have $(j, a, k)*(i, x, m)=(j, ap_{ki}x, m)$ and $(j, b, k)*(i, x, m)=(j, bp_{ki}x, m)$. Thus $(j, a, k)\psi_S(j, b, k)$ and since S is right nonsingular then $(j, a, k)=(j, b, k)$ and so $a=b$.

The converse to this result is in general false since if G is a group with zero adjoined and $S=M^0(I, G, \Lambda; P)$ is regular then S is right reductive if and only if no two rows of P are left proportional; that is, for any two rows μ and λ of P there does not exist $c\in G$ such that $p_{\mu i}=cp_{\lambda i}$ for all $i\in I$ [5, p. 156]. Hence if S is not right reductive then S is not right nonsingular.

McMorris [4] showed that a semigroup W with 0 and 1 can be embedded in $Q(W)$ by $\xi\colon W\to Q(W)$ defined by $x\to[\lambda_x]$ where $\lambda_x\in\mathrm{Hom}_W(W, W)$ defined by $t \mapsto xt$. The zero and identity of W are the zero and identity of $Q(W)$. If P is a sandwich matrix defined on W, we can define a new sandwich matrix on $Q(W)$ by allowing the entries of P to be operated on by ξ; that is, $p_{ij} \in P$ would become $[p_{pij}]_{ij}$. For convenience, we simply write p_{ij} and let $P = \xi(P)$. It is not difficult to see that $M^0(W, n; P)$ can be embedded into $M^0(Q(W), n; P)$.

We are interested in obtaining a characterization of $Q(S)$, the maximal right quotient semigroup of $S = M^0(W, n; P)$, where W is a semigroup with 0 and 1, T is a unitary subgroup of W and P has an entry from T in each row.

Theorem 6. *Let $S = M^0(W, n; P)$ where W is an $\cap$-uniform semigroup with 0 and 1, T is a unitary subgroup of W and P has an entry from T in each row. If $f\colon \mathscr{L} \to S$ is an S-homomorphism where $\mathscr{L}$ is an $\cap$-large right ideal of S then there exists an indexing function $i\colon I \to I$, and for each $h \in I$ a W-homomorphism $\hat{f}_h\colon \bigcap_{k \in I} M_k \to W$ such that $f|_{\mathscr{R}}((m, x, t)) = (i(m), \hat{f}_m(x), t)$ where $\mathscr{R}$ is defined in Theorem* 4.

Proof. Let $f\colon \mathscr{L} \to S$ be an S-homomorphism and $\mathscr{L}$ an $\cap$-large right ideal of S. By Theorem 3, for each $k \in I$, $M_k = \{s \in W \mid (k, s, k) \in \mathscr{L}\}$ is an $\cap$-large right ideal of W and so is $\bigcap_{k \in I} M_k$. By Theorem 4, we can construct $\mathscr{R} = \{(d, z, g) \mid z \in \bigcap_{k \in I} M_k$ and $d, g \in I\}$ an $\cap$-large right ideal of S contained in $\mathscr{L}$. Let ∂ be a fixed element of I. For $m \in I$ and $x \in \bigcap_{k \in I} M_k$ we have $f((m, x, \partial)) = f((m, x, \partial) * (j, p_{\partial j}^{-1}, \partial)) = f((m, x, \partial)) * (j, p_{\partial j}^{-1}, \partial)$ for some $j \in I$ with $p_{\partial j} \in T$, since f is an S-homomorphism. Thus $f((m, x, \partial)) = (i, y, \partial)$ for some $i \in I$ and $y \in W$. Now let $s \in I$ and $a, b \in \bigcap_{k \in I} M_k$ and suppose $f((s, a, \partial)) = (i, y, \partial)$ and $f((s, b, \partial)) = (h, z, \partial)$. Since W is $\cap$-uniform then $aW \cap bW \neq 0$ and so there exists $0 \neq x \in aW \cap bW$ such that $x = aw$ and $x = bu$ for some $w, u \in W$. Since $(s, aw, \partial), (s, bu, \partial) \in \mathscr{R}$ then $f((s, aw, \partial)) = (i, y, \partial) * (j, p_{\partial j}^{-1} w, \partial)$ and similarly $f((s, bu, \partial)) = (h, z, \partial) * (j, p_{\partial j}^{-1} u, \partial)$ for some $j \in I$. But $f((s, aw, \partial)) = f((s, bu, \partial))$ and so $(i, yw, \partial) = (h, zu, \partial)$. Hence $i = h$ and we can consider the first index as a function of s, denoted $i(s)$. Now for each $h \in I$, we define $\hat{f}_h\colon \bigcap_{k \in I} M_k \to W$ by $x \mapsto y$ where $f((h, x, \partial)) = (i(h), y, \partial)$. Each $\hat{f}_h$ is a W-homomorphism since for $s \in W$ and $x \in \bigcap_{k \in I} M_k$ we have

$$f((h, xs, \partial)) = f((h, xp_{\partial j} p_{\partial j}^{-1} s, \partial)) = f((h, x, \partial)) * (j, p_{\partial j}^{-1} s, \partial) =$$

$$= (i(h), y, \partial) * (j, p_{\partial j}^{-1} s, \partial) = (i(h), y p_{\partial j} p_{\partial j}^{-1} s, \partial) = (i(h), ys, \partial)$$

and so $\hat{f}_h(x) s = ys = \hat{f}_h(xs)$. The remainder of the theorem now follows from the fact that for $(m, x, t) \in \mathscr{R}$, $f((m, x, t)) = f((m, x, \partial) * (j, p_{\partial j}^{-1}, t)) = f((m, x, \partial)) * (j, p_{\partial j}^{-1}, t) = (i(m), \hat{f}_m(x), \partial) * (j, p_{\partial j}^{-1}, t) = (i(m), \hat{f}_m(x), t)$.

We now prove the main result of this paper.

Theorem 7. *Let* $S=M^0(W,n;P)$ *where* W *is an* $\cap$*-uniform semigroup with* 0 *and* 1, T *is a unitary subgroup of* W *and* P *has an entry from* T *in each row. If* S *is right nonsingular then* $Q(S)\approx \mathrm{Hom}_S(\hat{S},\hat{S})$ *where* $\hat{S}=M^0(Q(W),n;P)$.

Proof. Let $[f]\in Q(S)$ and define $\mu_{[f]}\colon \hat{S}\to\hat{S}$ by $(s,q,t)\to(i(s),q_sq,t)$ where $q_s=[\hat{f}_s]$ and $i(s)$ are defined in Theorem 6. We should note that by Theorem 5, S being right nonsingular implies that W is right nonsingular and so $[\hat{f}_s]\in Q(W)$. To see that $\mu_{[f]}$ is an $\hat{S}$-homomorphism we consider $\mu_{[f]}((s,q,t))*(r,g,u)= =(i(s),q_sq,t)*(r,g,u)=(i(s),q_sqp_{tr}g,u)$ and

$$\mu_{[f]}((s,q,t)*(r,g,u))=\mu_{[f]}((s,qp_{tr}g,u))=(i(s),q_sqp_{tr}g,u).$$

Thus $\mu_{[f]}$ is an $\hat{S}$-homomorphism. We now define $\varphi\colon Q(S)\to\mathrm{Hom}_S(\hat{S},\hat{S})$ by $[f]\to\mu_{[f]}$. To see that φ is well defined suppose $[f]=[g]$. Then f and g agree on some dense right ideal $\mathscr{L}$ of S. We must show that $[\hat{f}_j]=[\hat{g}_j]$ for all $j\in I$. Since $\mathscr{L}$ is a dense right ideal of S then it is also $\cap$-large. By Theorem 3, for each $k\in I$, $M_k=\{s\in W|(k,s,k)\in\mathscr{L}\}$ is an $\cap$-large right ideal of W and so is $\bigcap_{k\in I}M_k$. By Theorem 4, $\mathscr{R}=\{(d,z,h)|z\in\bigcap_{k\in I}M_k \text{ and } d,h\in I\}$ is an $\cap$-large right ideal of S contained in $\mathscr{L}$ so $f|_{\mathscr{R}}=g|_{\mathscr{R}}$ agree. Thus for $j\in I$, $\hat{f}_j\colon\bigcap_{k\in I}M_k\to W$ and $\hat{g}_j\colon\bigcap_{k\in I}M_k\to W$ agree on their domains and by Theorem 5 $\bigcap_{k\in I}M_k$ is a dense right ideal of W so $[\hat{f}_j],[\hat{g}_j]\in Q(W)$ and $[\hat{f}_j]=[\hat{g}_j]$. We now show that φ is one-to-one by supposing that $\mu_{[f]}=\mu_{[g]}$. Since for each $k\in I$ $(k,1,\partial)\in\hat{S}$, $\mu_{[f]}((k,1,\partial))=(i(k),[\hat{f}_k],\partial)$ and $\mu_{[g]}((k,1,\partial))=(f(k),[\hat{g}_k],\partial)$ then $[\hat{f}_k]=[\hat{g}_k]$ and $i(k)=j(k)$ for all $k\in I$. Thus for each $k\in I$, $\hat{f}_k$ and $\hat{g}_k$ agree on some $\cap$-large right ideal of W call it L_k. Let $L=\bigcap_{k\in I}L_k$. Since L is an $\cap$-large right ideal of W then by Theorem 1 $\mathscr{L}=\{(d,z,h)|z\in L \text{ and } d,h\in I\}$ is an $\cap$-large right ideal of S. We claim that f and g agree on $\mathscr{L}$. Let $(d,z,h)\in\mathscr{L}$ then there exists $m\in I$ such that $p_{\partial m}\in T$ in P and so $f((d,z,h))=f((d,z,\partial))*(m,p_{\partial m}^{-1},h)$ and $g((d,z,h))=g((d,z,\partial))*(m,p_{\partial m}^{-1},h)$. But $f((d,z,\partial))=g((d,z,\partial))$ since $\hat{f}_d$ and $\hat{g}_d$ agree on $L_d\subseteq L$. Consequently, the claim is established and $[f]=[g]$ in $Q(S)$; furthermore φ is one-to-one. To show that φ is onto let $\sigma\in\mathrm{Hom}_S(\hat{S},\hat{S})$ and consider $\sigma^{-1}S=\{x\in\hat{S}|\sigma(x)\in S\}$. Since $S\subseteq'\hat{S}$ then $\sigma^{-1}S\subseteq'\hat{S}$ and $S\cap\sigma^{-1}S\subseteq'S$. Next define $\tau\colon S\cap\sigma^{-1}S\to S$ by $x\mapsto\sigma(x)$. Clearly, τ is an S-homomorphism so by Theorem 6, there exists $\mathscr{D}$ an $\cap$-large right ideal of S, an indexing function $i\colon I\to I$, and for each $h\in I$ a W-homomorphism $\hat{\tau}\colon\bigcap_{k\in I}D_k\to W$ such that $\tau|_{\mathscr{D}}((m,x,t))=(i(m),\hat{\tau}_m(x),t)$. Note that $\mathscr{D}$ is a dense right ideal of S since S is right nonsingular and that $\tau\mathscr{D}\subseteq S$. Let $\bar{\tau}=\tau|_{\mathscr{D}}$ and consider $[\bar{\tau}]\in Q(S)$. Since S_S is dense in $\hat{S}_S$ then $\mu_{[\bar{\tau}]}=\sigma$ and so φ is onto.

What remains to be shown is that φ is a semigroup homomorphism; that is, $\mu_{[fg]}=\mu_{[f]}\mu_{[g]}$ where the operation on the right is composition of functions. Let $(s, q, t)\in\hat{S}$ then $\mu_{[f]}\mu_{[g]}((s, q, t))=\mu_{[f]}(\mu_{[g]}((s, q, t)))=\mu_{[f]}((j(s), [\hat{g}_s]q, t))=(i(j(s)), [\hat{f}_{j(s)}][\hat{g}_s]q, t)=(i(j(s)), [\hat{f}_{j(s)}\hat{g}_s]q, t)=\mu_{[fg]}((s, q, t))$ since $fg((s, x, t))=f(g((s, x, t)))=f((j(s), \hat{g}_s(x), t))=(i(j(s)), \hat{f}_{j(s)}(\hat{g}_s(x)), t)=(i(j(s)), \hat{f}_{j(s)}\hat{g}_s(x), t)$.

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Models for operators with bounded characteristic function

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1. Introduction. In the theory of B. SZ.-NAGY and C. FOIAŞ [15], the characteristic function Θ_T of a completely non-unitary contraction T is used to generate a functional model for T. In addition, if Θ is an arbitrary purely contractive analytic function, then Θ can be used to generate a contraction that has Θ as its characteristic function. The SZ.-NAGY and FOIAŞ theory provides in fact a model of the minimal unitary dilation U of the contraction: U is represented as a shift acting on a subspace of the direct sum of two vector valued L^2 spaces, and the characteristic function is identified as a projection on the dilation space. (See [15, Chapter VI].)

Now suppose that T is any bounded operator on a Hilbert space $\mathfrak{H}$. The characteristic function Θ_T of T is the operator valued analytic function

$$\Theta_T(\lambda) = [-TJ_T + \lambda J_{T^*}Q_{T^*}(I-\lambda T^*)^{-1}J_TQ_T]|\mathfrak{D}_T,$$

where $J_T = \operatorname{sgn}(I-T^*T)$, $J_{T^*} = \operatorname{sgn}(I-TT^*)$, $Q_T = |I-T^*T|^{1/2}$, $Q_{T^*} = |I-TT^*|^{1/2}$, and $\mathfrak{D}_T = J_T\mathfrak{H}$. $\Theta_T(\lambda)$ is defined for those complex numbers λ for which $I-\lambda T^*$ is boundedly invertible, and takes values which are continuous operators from $\mathfrak{D}_T$ to the space $\mathfrak{D}_{T^*} = J_{T^*}\mathfrak{H}$. (See [11]; cf. [1], [3], [4], [5], [6], [8], [10], [13], [15].)

It was shown in [11] that if $\Theta(\lambda)$ is an operator valued analytic function (defined for $\lambda \in D$, the open unit disk in the complex plane), then $\Theta(\lambda)$ coincides with the characteristic function of some operator if and only if it is purely contractive and fundamentally reducible (see Sec. 2 below). This result was obtained by using a model of BALL [1], which is much less geometric than the type constructed by SZ.-NAGY and FOIAŞ. In particular, the model in [1] does not provide the interpretation of the characteristic function as a projection.

In this paper, we restrict our attention to bounded operator valued analytic functions $\Theta(\lambda)$, i.e., for which $\sup_{\lambda\in D} \|\Theta(\lambda)\| < \infty$. We are then able to obtain a functional model of the SZ.-NAGY and FOIAŞ type, which provides the extension of

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their theory that was promised in the concluding section of [11]. In Sec. 14 we describe the relationship between this model and the model of BALL [1].

Remark. Other authors [3], [4], [5], [6] have also considered the problem of representing an arbitrary $\Theta(\lambda)$ (satisfying certain conditions) as a characteristic function, but have not used a SZ.-NAGY and FOIAŞ type model. In [9], however, a model of this type is used to represent dissipative operators, with the unit disk in the SZ.-NAGY and FOIAŞ theory replaced by the upper half plane.

2. Krein spaces. Purely contractive analytic functions. A *Krein space* is a space $\mathfrak{K}$ with an indefinite inner product $[.,.]$ (i.e., $[x, x]$ can be negative for some $x \in \mathfrak{K}$) on which is defined a *fundamental symmetry* J: $J^2 = I$, $[Jx, y] = [x, Jy]$, and the J-inner product $[J.,.]$ makes $\mathfrak{K}$ a Hilbert space. The topology on $\mathfrak{K}$ is that defined by the J-norm $\|x\|_J = [Jx, x]^{1/2}$. For an operator A on $\mathfrak{K}$, we denote by A^* the adjoint of A with respect to the indefinite inner product $[.,.]$. (See [2], [11].)

If $\mathfrak{A}$ and $\mathfrak{B}$ are subsets of $\mathfrak{K}$, then we write $\mathfrak{A} \perp \mathfrak{B}$ if $[a, b] = 0$ for all $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$. We define $\mathfrak{A}^\perp = \{x \in \mathfrak{K}: [a, x] = 0 \text{ for all } a \in \mathfrak{A}\}$ and $\mathfrak{A} \ominus \mathfrak{B} = \mathfrak{A} \cap \mathfrak{B}^\perp$. A *projection* on $\mathfrak{K}$ is a continuous operator P such that $P = P^* = P^2$. A *regular* subspace of $\mathfrak{K}$ is a subspace $\mathfrak{L}$ such that $\mathfrak{L} \oplus \mathfrak{L}^\perp = \mathfrak{K}$. The regular subspaces are precisely those that are the ranges of projections (cf. [12]).

An *operator valued analytic function* is a function Θ which is defined and analytic in D, the open unit disk in the complex plane, and which takes values that are continuous operators from a Krein space $\mathfrak{D}$ to a Krein space $\mathfrak{D}_*$. Θ is said to be *purely contractive* if, for each $\lambda \in D$,

$$[\Theta(\lambda)a, \Theta(\lambda)a] < [a, a] \quad (a \in \mathfrak{D},\ a \neq 0)$$

and

$$[\Theta(\lambda)^* a_*, \Theta(\lambda)^* a_*] < [a_*, a_*] \quad (a_* \in \mathfrak{D}_*,\ a_* \neq 0).$$

Let $\Theta_0 = \Theta(0)$. We call Θ *fundamentally reducible* if there are fundamental symmetries on $\mathfrak{D}$ and $\mathfrak{D}_*$ that commute with $\Theta_0^*\Theta_0$ and $\Theta_0\Theta_0^*$, respectively [11, Sec. 3].

The spaces $\mathfrak{D}_T$ and $\mathfrak{D}_{T^*}$, defined in Sec. 1, are Krein spaces with the indefinite inner products

$$[x, y] = (J_T x, y) \quad (x, y \in \mathfrak{D}_T) \quad \text{and} \quad [x, y] = (J_{T^*} x, y) \quad (x, y \in \mathfrak{D}_{T^*}).$$

The characteristic function Θ_T is an operator valued analytic function that is purely contractive and fundamentally reducible [11, Sec. 4].

3. Coincidence of characteristic functions. If $\mathfrak{D}$ and $\mathfrak{D}'$ are two Krein spaces, then an operator τ: $\mathfrak{D} \to \mathfrak{D}'$ is said to be *unitary* if it is continuous and invertible, and if $[\tau x, \tau x] = [x, x]$ for all $x \in \mathfrak{D}$. Two operator valued analytic functions $\Theta(\lambda)$: $\mathfrak{D} \to \mathfrak{D}_*$ and $\Theta'(\lambda)$: $\mathfrak{D}' \to \mathfrak{D}'_*$ are said to *coincide* if there are unitary operators τ: $\mathfrak{D} \to \mathfrak{D}'$ and τ_*: $\mathfrak{D}_* \to \mathfrak{D}'_*$ such that $\Theta'(\lambda) = \tau_* \Theta(\lambda) \tau^{-1}$ for all $\lambda \in D$.

As in [15, Sec. VI.1.2], we have the following result.

Proposition 3.1. *The characteristic functions of unitarily equivalent operators coincide.*

Proof. Let T_1 and T_2 be bounded operators on Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively, and suppose that for some unitary operator $\sigma\colon \mathfrak{H}_1 \to \mathfrak{H}_2$ we have $T_2 = \sigma T_1 \sigma^{-1}$. Then, if we define $\tau = \sigma | \mathfrak{D}_{T_1}$ and $\tau_* = \sigma | \mathfrak{D}_{T_1^*}$, it is clear that

$$(3.1)\quad \mathfrak{D}_{T_2} = \tau \mathfrak{D}_{T_1}, \quad \mathfrak{D}_{T_2^*} = \tau_* \mathfrak{D}_{T_1^*}, \quad \text{and} \quad J_{T_2} = \tau J_{T_1} \tau^{-1}, \quad J_{T_2^*} = \tau_* J_{T_1^*} \tau_*^{-1},$$

$$\Theta_{T_2}(\lambda) = \tau_* \Theta_{T_1}(\lambda) \tau^{-1}.$$

It follows from (3.1) that τ and τ_* are unitary operators, and thus Θ_{T_1} and Θ_{T_2} coincide. □

For any bounded operator T on a Hilbert space $\mathfrak{H}$ there is a unique maximal subspace $\mathfrak{H}_0$ in $\mathfrak{H}$ reducing T to a unitary operator (see, for example, [7, Sec. 4]). If $\mathfrak{H}_1 = \mathfrak{H} \ominus \mathfrak{H}_0$, then $T | \mathfrak{H}_1$ is completely non-unitary, i.e. there is no non-zero subspace of $\mathfrak{H}_1$ which reduces T to a unitary operator.

Proposition 3.2. *The characteristic functions of a bounded operator and its completely non-unitary part coincide.*

Proof. Formally the same as [15, Sec. VI.1.2]. □

In Sec. 6 we will deduce (Theorem 6.1) that, for completely non-unitary operators with bounded characteristic functions, coincidence of the characteristic functions implies unitary equivalence of the operators.

4. Dilations. Fourier representations. Let T be a completely non-unitary operator on a separable Hilbert space $\mathfrak{H}$, and suppose that T has bounded characteristic function $\Theta_T(\lambda)$, i.e. $\sup\limits_{\lambda \in D} \|\Theta_T(\lambda)\| < \infty$.

We can construct (see [7]) a Krein space $\mathfrak{K}$ containing $\mathfrak{H}$ as a subspace (with the indefinite inner product $[.,.]$ of $\mathfrak{K}$ restricting to the Hilbert space inner product $(.,.)$ on $\mathfrak{H}$) and an operator U on $\mathfrak{K}$ which is a *minimal unitary dilation* of T, i.e. U is unitary and satisfies

$$[U^n h, k] = (T^n h, k) \quad (h, k \in \mathfrak{H},\ n = 1, 2, \ldots) \quad \text{and} \quad \bigvee_{n=-\infty}^{\infty} U^n \mathfrak{H} = \mathfrak{K}.$$

(The symbol $\vee$ denotes closed linear span.)

The following subspaces of $\mathfrak{K}$ are important in the study of the geometry of the dilation space (see [13]; cf. [15]):

$$\mathfrak{L} = \overline{(U-T)\mathfrak{H}}, \quad \mathfrak{L}_* = \overline{(I-UT^*)\mathfrak{H}}, \quad M(\mathfrak{L}) = \bigvee_{n=-\infty}^{\infty} U^n \mathfrak{L}, \quad M(\mathfrak{L}_*) = \bigvee_{n=-\infty}^{\infty} U^n \mathfrak{L}_*,$$

$$M_+(\mathfrak{L}) = \bigvee_{n=0}^{\infty} U^n \mathfrak{L}, \quad M_+(\mathfrak{L}_*) = \bigvee_{n=0}^{\infty} U^n \mathfrak{L}_*, \quad \mathfrak{R} = M(\mathfrak{L}_*)^{\perp}, \quad \mathfrak{K}_+ = \bigvee_{n=0}^{\infty} U^n \mathfrak{H}.$$

We are assuming that T is completely non-unitary and has bounded characteristic function. Therefore, by [13, Sec. 6], $M(\mathfrak{L}_*)$ is regular and, by [13, Corollary 3.2],

$$M(\mathfrak{L}) \vee M(\mathfrak{L}_*) = \mathfrak{K}.$$

Hence, if P denotes the projection of $\mathfrak{K}$ onto $M(\mathfrak{L}_*)$ (i.e., the projection with range $M(\mathfrak{L}_*)$ and null space $\mathfrak{R}$), then we have

$$\overline{(I-P)M(\mathfrak{L})} = \mathfrak{R} \tag{4.1}$$

(cf. [15, Sec. VI.2.1]).

It follows from the construction of the dilation in [7] that there are unitary operators φ: $\mathfrak{L} \to \mathfrak{D}_T$ and φ_*: $\mathfrak{L}_* \to \mathfrak{D}_{T^*}$ and a fundamental symmetry J on $\mathfrak{K}$ such that

$$\varphi(U-T)h = Q_T h, \quad \varphi_*(I-UT^*)h = J_{T^*}Q_{T^*}h \quad (h \in \mathfrak{H});$$

$$\varphi J|\mathfrak{L} = J_T \varphi, \quad \varphi_* UJU^*|\mathfrak{L}_* = J_{T^*}\varphi_*;$$

$$\|\varphi l\| = \|l\|, \quad \|\varphi_* l_*\| = \|l_*\| \quad (l \in \mathfrak{L},\ l_* \in \mathfrak{L}_*).$$

(See [13, Sec. 3].)

Let $P_{\mathfrak{L}}$ denote the projection of $\mathfrak{K}$ onto $\mathfrak{L}$. If $h \in M(\mathfrak{L})$, then the *Fourier coefficients of h in $M(\mathfrak{L})$* are

$$l_n = P_{\mathfrak{L}} U^{*n} h \quad (-\infty < n < \infty).$$

When Θ_T is bounded, we have $\sum_{n=-\infty}^{\infty} \|l_n\|^2 < \infty$ (see [13, Sec. 6]; cf. [8, Sec. III.1]), and thus we can define $\Phi_{\mathfrak{L}}$, the *Fourier representation of* $M(\mathfrak{L})$, by

$$(\Phi_{\mathfrak{L}} h)(t) = \sum_{n=-\infty}^{\infty} e^{int} \varphi l_n.$$

$\Phi_{\mathfrak{L}}$ is a unitary operator from $M(\mathfrak{L})$ to $L^2(\mathfrak{D}_T)$, the Krein space of square integrable $\mathfrak{D}_T$-valued functions with inner product

$$[u, v] = 1/2\pi \int_0^{2\pi} [u(t), v(t)]\, dt \quad (u, v \in L^2(\mathfrak{D}_T)).$$

Similarly, if $h \in M(\mathfrak{L}_*)$ and $l_n = P_{\mathfrak{L}_*} U^{*n} h$ are the Fourier coefficients of h in $M(\mathfrak{L}_*)$, then we define $\Phi_{\mathfrak{L}_*}$, the *Fourier representation of* $M(\mathfrak{L}_*)$, by

$$(\Phi_{\mathfrak{L}^*} h)(t) = \sum_{n=-\infty}^{\infty} e^{int} \varphi_* l_n.$$

$\Phi_{\mathfrak{L}_*}$ is a unitary operator from $M(\mathfrak{L}_*)$ to $L^2(\mathfrak{D}_{T^*})$. (See [13, Sec. 6]; cf. [15, Chapter V].)

5. Functional models for a given operator. If $\mathfrak{D}$ is a Krein space with fundamental symmetry J, then we also denote by J the fundamental symmetry induced on $L^2(\mathfrak{D})$ by $(Jv)(t)=J\cdot v(t)$. Thus we have on $L^2(\mathfrak{D}_T)$ and $L^2(\mathfrak{D}_{T*})$ the fundamental symmetries J_T and J_{T*}, respectively. As in [15, Sec. V.2] we have the operator $\Theta_T\colon L^2(\mathfrak{D}_T)\to L^2(\mathfrak{D}_{T*})$ defined by

$$(\Theta_T v)(t) = \Theta_T(e^{it})v(t) \quad \text{a.e.} \quad (v\in L^2(\mathfrak{D}_T)),$$

where $\Theta_T(e^{it})=\lim\limits_{r\to 1-}\Theta_T(re^{it})$. Since Θ_T is a purely contractive analytic function, it satisfies $[(I-\Theta_T^*\Theta_T)v, v]\geqq 0$ for all $v\in L^2(\mathfrak{D}_T)$, or in terms of the Hilbert space inner product on $L^2(\mathfrak{D}_T)$, $(J_T(I-\Theta_T^*\Theta_T)v, v)\geqq 0$. We can therefore define $\Delta_T= =(J_T(I-\Theta_T^*\Theta_T))^{1/2}$, an operator on $L^2(\mathfrak{D}_T)$ that satisfies the relation $\|\Delta_T v\|^2= =[(I-\Theta_T^*\Theta_T)v, v]$, for all $v\in L^2(\mathfrak{D}_T)$.

For $f\in M(\mathfrak{L})$ we have, using the fact that the Fourier representations are unitary and the relation $\Theta_T\Phi_{\mathfrak{L}}=\Phi_{\mathfrak{L}_*}P|M(\mathfrak{L})$ [13, equation (6.4)],

$$\begin{aligned}[(I-P)f,(I-P)f] &= [f,f]-[Pf,Pf] = [\Phi_{\mathfrak{L}}f, \Phi_{\mathfrak{L}}f]-[\Phi_{\mathfrak{L}*}Pf, \Phi_{\mathfrak{L}*}Pf] = \\ &= [\Phi_{\mathfrak{L}}f, \Phi_{\mathfrak{L}}f]-[\Theta_T\Phi_{\mathfrak{L}}f, \Theta_T\Phi_{\mathfrak{L}}f] = [(I-\Theta_T^*\Theta_T)\Phi_{\mathfrak{L}}f, \Phi_{\mathfrak{L}}f] = \|\Delta_T\Phi_{\mathfrak{L}}f\|^2\end{aligned}$$

(cf. [15, Sec. VI.2.1]). Hence, by (4.1), there is a unitary operator $\Phi_{\mathfrak{R}}\colon \mathfrak{R}\to\overline{\Delta_T L^2(\mathfrak{D}_T)}$ such that

$$\Phi_{\mathfrak{R}}(I-P)f = \Delta_T\Phi_{\mathfrak{L}}f \quad (f\in M(\mathfrak{L})).$$

Here we are considering $\mathfrak{R}$ as a Hilbert space with the inner product [., .] [13, Theorem 7.1], and $\overline{\Delta_T L^2(\mathfrak{D}_T)}$ as a Hilbert space with the usual inner product on $L^2(\mathfrak{D}_T)$. *(In the sequel, $\overline{\Delta_T L^2(\mathfrak{D}_T)}$ will always be considered as a Hilbert space.)*

Since $M(\mathfrak{L}_*)$ is regular [13, Sec. 6] we can write

$$\mathfrak{K} = M(\mathfrak{L}_*)\oplus\mathfrak{R}.$$

If we make the definition

$$\mathbf{K} = L^2(\mathfrak{D}_{T*})\oplus\overline{\Delta_T L^2(\mathfrak{D}_T)},$$

then we can deduce that the operator $\Phi=\Phi_{\mathfrak{L}_*}\oplus\Phi_{\mathfrak{R}}$ is a unitary operator from $\mathfrak{K}$ to $\mathbf{K}$. Φ is known as the *Fourier representation* of $\mathfrak{K}$.

If we let e^{it} also denote multiplication by the function e^{it} then $e^{it}\Theta_T=\Theta_T e^{it}$ and $e^{it}J_T=J_Te^{it}$, and hence $e^{it}\Delta_T=\Delta_Te^{it}$. We also have $UP=PU$ and $\Phi_{\mathfrak{L}}U= =e^{it}\Phi_{\mathfrak{L}}$, and so (cf. [15, Sec. VI.2.1])

$$\Phi_{\mathfrak{R}}U(I-P)f = e^{it}\Phi_{\mathfrak{R}}(I-P)f \quad (f\in M(\mathfrak{L})).$$

By continuity, it follows that $\Phi_{\mathfrak{R}}Uh=e^{it}\Phi_{\mathfrak{R}}h$ for all $h\in\mathfrak{R}$.

Let $\mathbf{U}$ denote multiplication by e^{it} on $\mathbf{K}$, i.e.

$$\mathbf{U}(u\oplus v) = e^{it}u\oplus e^{it}v \quad (u\in L^2(\mathfrak{D}_{T*}),\ v\in\overline{\Delta_T L^2(\mathfrak{D}_T)}).$$

Then, since $\Phi_{\mathfrak{R}}U=e^{it}\Phi_{\mathfrak{R}}$ and $\Phi_{\mathfrak{L}_*}U=e^{it}\Phi_{\mathfrak{L}_*}$, we have $\Phi U=\mathbf{U}\Phi$.

The subspace $M_+(\mathfrak{L}_*)$ is regular [8, Sec. III. 2], and thus, by [13, Sec. 4], we have

$$\mathfrak{K}_+ = M_+(\mathfrak{L}_*) \oplus \mathfrak{R}$$

(recall the definitions of these subspaces in Sec. 4). Consequently, Φ maps $\mathfrak{K}_+$ onto the space

$$\mathbf{K}_+ = H^2(\mathfrak{D}_{T^*}) \oplus \overline{\Delta_T L^2(\mathfrak{D}_T)},$$

where $H^2(\mathfrak{D}_{T^*})$ (the space of $\mathfrak{D}_{T^*}$-valued analytic functions with square summable Taylor coefficients) is identified with a subspace of $L^2(\mathfrak{D}_{T^*})$ in the usual manner cf. [15, Sec. V. 1.1]). Then if $U_+ = U|\mathfrak{K}_+$ and $\mathbf{U}_+ = \mathbf{U}|\mathbf{K}_+$, we have $\Phi U_+ = \mathbf{U}_+ \Phi$.

For $u \in H^2(\mathfrak{D}_{T^*})$ and $v \in \overline{\Delta_T L^2(\mathfrak{D}_T)}$ we have then

$$\mathbf{U}_+(u \oplus v) = e^{it} u \oplus e^{it} v \quad \text{and} \quad \mathbf{U}_+^*(u \oplus v) = e^{-it}(u - u_0) \oplus e^{-it} v.$$

Remark on notation. Here (and in the sequel) it is assumed that, for each n, u_n denotes the nth coefficient in the Fourier series of the function u. Thus, for $u \in H^2(\mathfrak{D}_{T^*})$, $u_0 = u(0)$. Also, we will not be distinguishing between a vector and the constant function whose range is that vector.

Let us make the definition $\mathbf{H} = \Phi\mathfrak{H}$. Since $\mathfrak{H}$ is a Hilbert space with the inner product $[.,.]$, and since Φ is unitary, $\mathbf{H}$ is also a Hilbert space. We know by [13, equation (3.3)] that $\mathfrak{K}_+ = \mathfrak{H} \oplus M_+(\mathfrak{L})$, and therefore we deduce $\mathfrak{H} = \mathfrak{K}_+ \ominus M_+(\mathfrak{L})$. Hence,

$$\mathbf{H} = \mathbf{K}_+ \ominus \Phi M_+(\mathfrak{L}).$$

We can obtain an explicit description of $\Phi M_+(\mathfrak{L})$ by making the observation that, for $g \in M_+(\mathfrak{L})$,

$$\Phi g = \Phi[Pg + (I-P)g] = \Phi_{\mathfrak{L}^*} Pg \oplus \Phi_{\mathfrak{R}}(I-P)g = \Theta_T \Phi_{\mathfrak{L}} g \oplus \Delta_T \Phi_{\mathfrak{L}} g$$

(using [13, equation (6.4)]). Hence $\Phi M_+(\mathfrak{L}) = \{\Theta_T u \oplus \Delta_T u \colon u \in H^2(\mathfrak{D}_T)\}$. Consequently we obtain

$$\mathbf{H} = \mathbf{K}_+ \ominus \{\Theta_T u \oplus \Delta_T u \colon u \in H^2(\mathfrak{D}_T)\}.$$

If we denote by $\mathbf{T}$ the operator $\Phi T \Phi^{-1}$ on $\mathbf{H}$, then we have $\mathbf{T}^* = \mathbf{U}_+^*|\mathbf{H}$, and thus we obtain the following functional model.

Theorem 5.1. (cf. [15, Theorem VI.2.3]) *Let T be a completely non-unitary operator on a separable Hilbert space $\mathfrak{H}$, with bounded characteristic function Θ_T. Then the Krein space*

$$\mathbf{H} = [H^2(\mathfrak{D}_{T^*}) \oplus \overline{\Delta_T L^2(\mathfrak{D}_T)}] \ominus \{\Theta_T u \oplus \Delta_T u \colon u \in H^2(\mathfrak{D}_T)\}$$

is a Hilbert space and T is unitarily equivalent to the operator $\mathbf{T}$ on $\mathbf{H}$ defined by

$$\mathbf{T}^*(u \oplus v) = e^{-it}(u - u_0) \oplus e^{-it} v \quad (u \oplus v \in \mathbf{H}).$$

The unitary dilation U of T constructed in [7] *is unitarily equivalent to the operator* $\mathbf{U}$ *defined on the Krein space*

$$\mathbf{K} = L^2(\mathfrak{D}_{T^*}) \oplus \overline{\Delta_T L^2(\mathfrak{D}_T)} \quad by \quad \mathbf{U}(u \oplus v) = e^{it}u \oplus e^{it}v \quad (u \oplus v \in \mathbf{K}). \quad \square$$

6. Unitary equivalence of completely non-unitary operators.

Theorem 6.1. *Let T_1 and T_2 be completely non-unitary operators with bounded characteristic functions. Then T_1 and T_2 are unitarily equivalent if and only if their characteristic functions coincide.*

Proof. By Proposition 3.1, if T_1 and T_2 are unitarily equivalent, then their characteristic functions coincide. Conversely, suppose that $\tau\colon \mathfrak{D}_{T_1} \to \mathfrak{D}_{T_2}$ and $\tau_*\colon \mathfrak{D}_{T_1^*} \to \mathfrak{D}_{T_2^*}$ are unitary operators such that $\Theta_{T_2}(\lambda) = \tau_* \Theta_{T_1}(\lambda)\tau^{-1}$ ($\lambda \in D$). Then we obtain (since $\Theta_T(0) = -TJ_T$)

$$I - T_2^* T_2 = I - \Theta_{T_2}(0)^* \Theta_{T_2}(0) = \tau(I - \Theta_{T_1}(0)^* \Theta_{T_1}(0))\tau^{-1} = \tau(I - T_1^* T_1)\tau^{-1},$$

and hence $J_{T_2} = \tau J_{T_1} \tau^{-1}$. We similarly deduce that $J_{T_2^*} = \tau_* J_{T_1^*} \tau_*^{-1}$, and thus τ and τ_* are unitary with respect to the Hilbert space inner products as well as the indefinite inner products.

We can regard τ as mapping $L^2(\mathfrak{D}_{T_1})$ to $L^2(\mathfrak{D}_{T_2})$ (and similarly for τ_*), and then we have $\Delta_{T_2} = \tau \Delta_{T_1} \tau^{-1}$.

Let $\mathbf{T}_1$ and $\mathbf{T}_2$ be the operators (on $\mathbf{H}_1$ and $\mathbf{H}_2$) defined in Theorem 5.1, unitarily equivalent to T_1 and T_2, respectively. Then, as in [15, Sec. VI. 2.3], we can deduce that the operator $\hat{\tau}$, taking $u \oplus v$ to $\tau_* u \oplus \tau v$ ($u \oplus v \in \mathbf{H}_1$), is a unitary operator from $\mathbf{H}_1$ to $\mathbf{H}_2$ such that $\mathbf{T}_2 = \hat{\tau}\mathbf{T}_1\hat{\tau}^{-1}$. It then follows that T_1 and T_2 are unitarily equivalent. $\square$

7. Notes on functional models. When Θ_T is bounded and $\lim_{n\to\infty} T^{*^n} = 0$, then we have $\mathfrak{R} = \{0\}$ [13, Theorem 5.5] and the model of Theorem 5.1 can also be described as follows:

Let $\mathbf{K}_+$ be the space of sequences $\{h_n\}_{n\geq 0}$ with $h_n \in \mathfrak{D}_{T^*}$ ($n = 0, 1, 2, \ldots$) and $\sum_{n=0}^{\infty} \|h_n\|^2 < \infty$. The inner product on $\mathbf{K}_+$ is defined by

$$[\{h_n\}_{n\geq 0}, \{k_n\}_{n\geq 0}] = \sum_{n=0}^{\infty} [h_n, k_n] = \sum_{n=0}^{\infty} (J_{T^*} h_n, k_n).$$

Clearly, $\mathbf{K}_+$ is a Krein space, with the fundamental symmetry $J\{h_n\}_{n\geq 0} = \{J_{T^*} h_n\}_{n\geq 0}$.

We consider $\mathfrak{H}$ as a subspace of $\mathbf{K}_+$ by identifying the vector $h \in \mathfrak{H}$ and the sequence

$$\mathbf{h} = \{J_{T^*} Q_{T^*} T^{*n} h\}_{n\geq 0}.$$

By [13, Corollary 8.3], $\mathbf{h}$ is in $\mathbf{K}_+$, and we have (since $\lim_{n\to\infty} T^{*^n}=0$)

$$[\mathbf{h}, \mathbf{h}] = \sum_{n=0}^{\infty} (Q_{T^*}T^{*n}h, J_{T^*}Q_{T^*}T^{*n}h) =$$

$$= \sum_{n=0}^{\infty} (T^n(I-TT^*)T^{*n}h, h) = (h - \lim_{n\to\infty} T^nT^{*n}h, h) = \|h\|^2.$$

Thus the identification of $\mathfrak{H}$ as a subspace of $\mathbf{K}_+$ is valid.

If V is the unilateral shift on $\mathbf{K}_+$, mapping $(h_0, h_1, h_2, \ldots)$ to $(0, h_0, h_1, \ldots)$, then we have $T^*=V^*|\mathfrak{H}$. If we identify $\mathbf{K}_+$ with the space $H^2(\mathfrak{D}_{T^*})$, in the obvious manner, then V is identified with multiplication by e^{it} (thinking of H^2 as a subspace of L^2). The above model then coincides with the model of Theorem 5.1, which in the case $\lim_{n\to\infty} T^{*^n}=0$ identifies $\mathfrak{H}$ with the space

$$H^2(\mathfrak{D}_{T^*}) \ominus \Theta_T H^2(\mathfrak{D}_T)$$

(since $\mathfrak{R}=\{0\}$). (Cf. [15, p. 277].)

In [14], Rota obtains a model for operators with spectrum in the open unit disk, and this case is obviously included in the case considered above (namely, Θ_T bounded and $\lim_{n\to\infty} T^{*^n}=0$). Rota's model, however, differs somewhat from the model described above, and gives only a similarity model for T.

In the remaining sections of this paper we will be considering an arbitrary purely contractive analytic function $\Theta(\lambda)$. We will prove, by constructing a suitable functional model (based on that of Sz.-Nagy and Foiaş [15, Chapter VI]), that if Θ is bounded and fundamentally reducible then it is the characteristic function of some completely non-unitary operator (cf. [11]).

8. The functional model for a bounded purely contractive analytic function. Let $\Theta(\lambda)$: $\mathfrak{D}\to\mathfrak{D}_*$ be a bounded purely contractive analytic function. We will assume that Θ is fundamentally reducible, so that there are fundamental symmetries on $\mathfrak{D}$ and $\mathfrak{D}_*$ commuting with $\Theta_0^*\Theta_0$ and $\Theta_0\Theta_0^*$, respectively. As in [11, Sec. 5] we define the fundamental symmetries $J=\operatorname{sgn}(I-\Theta_0^*\Theta_0)$ on $\mathfrak{D}$ and $J_*=\operatorname{sgn}(I-\Theta_0\Theta_0^*)$ on $\mathfrak{D}_*$. The Hilbert space inner products and norms that we will use on $\mathfrak{D}$ and $\mathfrak{D}_*$ (and on $L^2(\mathfrak{D})$ and $L^2(\mathfrak{D}_*)$) will be the J- and J_*-inner products and norms obtained from these fundamental symmetries.

We also define the operators $Q=|I-\Theta_0^*\Theta_0|^{1/2}$ and $Q_*=|I-\Theta_0\Theta_0^*|^{1/2}$. They satisfy the relations (see [7, Sec. 2])

$$JQ^2 = I-\Theta_0{}^*\Theta_0, \quad J_*Q_*^2 = I-\Theta_0\Theta_0{}^*, \quad \Theta_0 J=J_*\Theta_0, \quad \Theta_0 Q = Q_*\Theta_0,$$
$$\Theta_0{}^*J_* = J\Theta_0{}^*, \quad \Theta_0{}^*Q_* = Q\Theta_0{}^*.$$

Since Θ is bounded and purely contractive, it can be considered as an operator from $L^2(\mathfrak{D})$ to $L^2(\mathfrak{D}_*)$, and we can define the operator $\Delta=(J(I-\Theta^*\Theta))^{1/2}$ on

$L^2(\mathfrak{D})$. The space $\overline{\Delta L^2(\mathfrak{D})}$ will always be considered as a Hilbert space (with the J-inner product), and we have $\|\Delta v\|^2=[(I-\Theta^*\Theta)v, v]$ for $v\in L^2(\mathfrak{D})$ (cf. Sec. 5).

Consider the Krein spaces

$$\mathbf{K}=L^2(\mathfrak{D}_*)\oplus\overline{\Delta L^2(\mathfrak{D})} \quad \text{and} \quad \mathbf{K}_+=H^2(\mathfrak{D}_*)\oplus\overline{\Delta L^2(\mathfrak{D})}\subset\mathbf{K},$$

and let

$$\mathbf{G}=\{\Theta w\oplus\Delta w\colon\ w\in H^2(\mathfrak{D})\}\subset\mathbf{K}_+.$$

For v and w in $H^2(\mathfrak{D})$ we have

$$[\Theta v, \Theta w]+(\Delta v, \Delta w)=[\Theta^*\Theta v, w]+[(I-\Theta^*\Theta)v, w]=[v, w].$$

Hence, since Θ and Δ are continuous, the operator $\Theta\oplus\Delta$, mapping v to $\Theta v\oplus\Delta v$, is an isometry from $H^2(\mathfrak{D})$ to $\mathbf{K}$. Therefore $\mathbf{G}$, which is the range of $\Theta\oplus\Delta$, is a regular subspace, of both $\mathbf{K}$ and $\mathbf{K}_+$ [12, Theorem 5.2].

If we define $\mathbf{H}=\mathbf{K}_+\ominus\mathbf{G}$, then $\mathbf{H}$ is a regular subspace of both $\mathbf{K}$ and $\mathbf{K}_+$.

Let $\mathbf{U}$ be multiplication by e^{it} on $\mathbf{K}$. Then $\mathbf{U}$ is a unitary operator and $\mathbf{K}_+$ is invariant for $\mathbf{U}$; we define $\mathbf{U}_+=\mathbf{U}|\mathbf{K}_+$. Since $e^{it}\Theta=\Theta e^{it}$ and $e^{it}\Delta=\Delta e^{it}$, $\mathbf{G}$ is invariant for $\mathbf{U}_+$, and thus $\mathbf{H}$ is invariant for $\mathbf{U}_+^*$. We can therefore define an operator $\mathbf{T}$ on $\mathbf{H}$ by $\mathbf{T}^*=\mathbf{U}_+^*|\mathbf{H}$. If we denote by P the projection of $\mathbf{K}$ onto $\mathbf{H}$ then we have, as in [15, Sec. VI.3.1],

$$\mathbf{T}^n=P\mathbf{U}^n|\mathbf{H} \quad (n\geqq 0). \tag{8.1}$$

We also have

$$\mathbf{T}^*(u\oplus v)=e^{-it}(u-u_0)\oplus e^{-it}v \quad (u\oplus v\in\mathbf{H}). \tag{8.2}$$

It should be noted that, since the spectrum of $\mathbf{U}$ is in the unit circle, $\mathbf{T}$ has spectrum in the closed unit disk.

9. Basic properties of the model. A vector $u\oplus v$ $\left(u\in H^2(\mathfrak{D}_*),\ v\in\overline{\Delta L^2(\mathfrak{D})}\right)$ is in $\mathbf{H}$ if and only if $u\oplus v\perp\Theta w\oplus\Delta w$ for all $w\in H^2(\mathfrak{D})$. Since we have the equations

$$[u\oplus v, \Theta w\oplus\Delta w]=[u, \Theta w]+(v, \Delta w)=[\Theta^*u, w]+(\Delta v, w)=[\Theta^*u+J\Delta v, w],$$

we conclude that $u\oplus v\in\mathbf{H}$ if and only if $\Theta^*u+J\Delta v\perp H^2(\mathfrak{D})$. In this case

$$\Theta^*u+J\Delta v=\sum_{n=1}^{\infty}e^{-int}f_n, \tag{9.1}$$

with f_n given by

$$f_n=1/2\pi\int_0^{2\pi}e^{int}(\Theta^*u+J\Delta v)(t)\,dt. \tag{9.2}$$

From (8.1) we deduce that, for $u\oplus v\in\mathbf{H}$,

$$\mathbf{T}(u\oplus v)=(e^{it}u-\Theta f_1)\oplus(e^{it}v-\Delta f_1) \quad \text{(cf. [15, Sec. VI. 3.5])}. \tag{9.3}$$

Proposition 9.1. *For* $u\oplus v\in\mathbf{H}$ *we have*

$$(I-\mathbf{T}^*\mathbf{T})(u\oplus v)=e^{-it}(\Theta-\Theta_0)f_1\oplus e^{-it}\Delta f_1$$

and

$$(I-\mathbf{T}\mathbf{T}^*)(u\oplus v)=(I-\Theta\Theta_0^*)u_0\oplus-\Delta\Theta_0^*u_0.$$

Proof. The first formula follows immediately from (9.3) and (8.2). For the second formula, we need to obtain the vector f_1 corresponding to $\mathbf{T}^*(u\oplus v)$, which (by (8.2)) is done by considering

$$\Theta^*[e^{-it}(u-u_0)]+J\Delta[e^{-it}v]=e^{-it}(\Theta^*u+J\Delta v)-e^{-it}\Theta^*u_0.$$

Since $\Theta^*u+J\Delta v\perp H^2(\mathfrak{D})$, we deduce that the required vector is $-\Theta_0^*u_0$ and hence, applying (9.3), we obtain

$$\mathbf{T}\mathbf{T}^*(u\oplus v)=\big((u-u_0)+\Theta\Theta_0^*u_0\big)\oplus(v+\Delta\Theta_0^*u_0)=u\oplus v-[(I-\Theta\Theta_0^*)u_0\oplus-\Delta\Theta_0^*u_0].\ \Box$$

Lemma 9.2. *If* $u\oplus v$ *is given by*

$$u\oplus v=e^{-it}(\Theta-\Theta_0)f\oplus e^{-it}\Delta f,$$

where $f\in\mathfrak{D}$, *then* $u\oplus v\in\mathbf{H}$, *and the vector* f_1 *defined by* (9.2) *is* $f_1=(I-\Theta_0^*\Theta_0)f$.

Proof. (Cf. [15, Sec. VI.3.5].) Since $J\Delta^2=I-\Theta^*\Theta$, we have

$$\Theta^*u+J\Delta v=e^{-it}\Theta^*(\Theta-\Theta_0)f+e^{-it}(I-\Theta^*\Theta)f=e^{-it}(I-\Theta^*\Theta_0)f.$$

Therefore, $\Theta^*u+J\Delta v\perp H^2(\mathfrak{D})$, and so $u\oplus v\in\mathbf{H}$. We also have

$$f_1=1/2\pi\int_0^{2\pi}(I-\Theta(e^{it})^*\Theta_0)f\,dt=(I-\Theta_0^*\Theta_0)f.\qquad\Box$$

Let us define the subset $\mathfrak{D}_1$ in $\mathfrak{D}$ by

$$\mathfrak{D}_1=\Big\{f_1=1/2\pi\int_0^{2\pi}e^{it}(\Theta^*u+J\Delta v)(t)\,dt:\ u\oplus v\in\mathbf{H}\Big\}.$$

Proposition 9.3. $\mathfrak{D}_1$ *is dense in* $\mathfrak{D}$.

Proof. Since Θ is purely contractive, the set $\{(I-\Theta_0^*\Theta_0)g:g\in\mathfrak{D}\}$ is dense in $\mathfrak{D}$. But Lemma 9.2 shows that $(I-\Theta_0^*\Theta_0)g$ is the vector f_1 for $u\oplus v=e^{-it}(\Theta-\Theta_0)g\oplus\oplus e^{-it}\Delta g$, and therefore $(I-\Theta_0^*\Theta_0)g\in\mathfrak{D}_1$ for all $g\in\mathfrak{D}$. $\Box$

Proposition 9.4. *The set* $\{u_0\colon u\oplus v\in\mathbf{H}\}$ *is dense in* $\mathfrak{D}_*$.

Proof. Since Θ is purely contractive, the set $\{(I-\Theta_0\Theta_0^*)g\colon g\in\mathfrak{D}_*\}$ is dense in $\mathfrak{D}_*$. If $u\oplus v=(I-\Theta\Theta_0^*)g\oplus-\Delta\Theta_0^*g$, where $g\in\mathfrak{D}_*$, then we have

$$\Theta^*u+J\Delta v=\Theta^*(I-\Theta\Theta_0^*)g-(I-\Theta^*\Theta)\Theta_0^*g=(\Theta^*-\Theta_0^*)g.$$

Hence, $\Theta^* u + J\Delta v \perp H^2(\mathfrak{D})$, and so $u \oplus v \in \mathbf{H}$. The proof is completed by noting that $u_0 = (I - \Theta_0 \Theta_0^*)g$. $\square$

10. The spaces $\mathfrak{D}_T$ and $\mathfrak{D}_{T^*}$. Since Θ is a purely contractive analytic function, the operators Q and Q_* (defined in Sec. 8) are injective. Thus, for $f \in \mathfrak{D}_1$ and $u \oplus v \in \mathbf{H}$, we can define

$$\varphi(JQf) = e^{-it}(\Theta - \Theta_0)f \oplus e^{-it}\Delta f \quad \text{and} \quad \varphi_*(Q_* u_0) = (I - \Theta\Theta_0^*)u_0 \oplus -\Delta\Theta_0^* u_0.$$

JQ and Q_* have dense range, and hence (by Propositions 9.3 and 9.4) φ and φ_* are densely defined on $\mathfrak{D}$ and $\mathfrak{D}_*$, respectively. If we define

$$\mathfrak{D}_T = \overline{(I - \mathbf{T}^*\mathbf{T})\mathbf{H}} \quad \text{and} \quad \mathfrak{D}_{T^*} = \overline{(I - \mathbf{T}\mathbf{T}^*)\mathbf{H}},$$

then Proposition 9.1 shows that the range of φ is dense in $\mathfrak{D}_T$ and the range of φ_* is dense in $\mathfrak{D}_{T^*}$.

Using the fact that $[\Theta f, \Theta_0 f] = [\Theta_0 f, \Theta_0 f]$ for $f \in \mathfrak{D}_1$, we have

$$\begin{aligned}(10.1)\qquad & [\varphi JQf, \varphi JQf] = [(\Theta - \Theta_0)f, (\Theta - \Theta_0)f] + \|\Delta f\|^2 = \\ & = [\Theta f, \Theta f] - [\Theta_0 f, \Theta_0 f] + [(I - \Theta^*\Theta)f, f] = [(I - \Theta_0^*\Theta_0)f, f] = \|JQf\|^2.\end{aligned}$$

Also, since $[\Theta^* u_0, \Theta_0^* u_0] = [\Theta_0^* u_0, \Theta_0^* u_0]$, we have

$$\begin{aligned}(10.2)\qquad & [\varphi_* Q_* u_0, \varphi_* Q_* u_0] = [(I - \Theta\Theta_0^*)u_0, (I - \Theta\Theta_0^*)u_0] + \|\Delta\Theta_0^* u_0\|^2 = \\ & = [u_0, u_0] - 2[\Theta_0^* u_0, \Theta_0^* u_0] + [\Theta\Theta_0^* u_0, \Theta\Theta_0^* u_0] + [(I - \Theta^*\Theta)\Theta_0^* u_0, \Theta_0^* u_0] = \\ & = [(I - \Theta_0\Theta_0^*)u_0, u_0] = \|Q_* u_0\|^2.\end{aligned}$$

If we put on $\mathbf{K}$ the norm obtained from the fundamental symmetry $J_* \oplus I$, then we have, using the Cauchy—Schwarz inequality [2, Lemma II.11.4],

$$\|JQf\|^2 = [\varphi JQf, \varphi JQf] \leqq \|\varphi JQf\|^2 \quad (f \in \mathfrak{D}_1)$$

and

$$\|Q_* u_0\|^2 = [\varphi_* Q_* u_0, \varphi_* Q_* u_0] \leqq \|\varphi_* Q_* u_0\|^2 \quad (u \oplus v \in \mathbf{H}).$$

Therefore φ^{-1}, defined on a dense subset of $\mathfrak{D}_T$, is continuous and has a unique continuous extension to all of $\mathfrak{D}_T$. Similarly, φ_*^{-1} has a unique continuous extension to all of $\mathfrak{D}_{T^*}$. By (10.1) and (10.2), these extensions are unitary, with $\mathfrak{D}$ and $\mathfrak{D}_*$ being considered as Hilbert spaces with the J- and J_*-inner products. The adjoints of these unitary maps are then unitary extensions of φ and φ_*, and these extensions will also be denoted by φ and φ_*.

We can now assert that

$$(10.3)\qquad \varphi(JQf) = e^{-it}(\Theta - \Theta_0)f \oplus e^{-it}\Delta f \quad \text{for all} \quad f \in \mathfrak{D},$$

and

$$(10.4)\qquad \varphi_*(Q_* g) = (I - \Theta\Theta_0^*)g \oplus -\Delta\Theta_0^* g \quad \text{for all} \quad g \in \mathfrak{D}_*.$$

Note that φ and φ_* are unitary with the inner product $[.,.]$ on $\mathfrak{D}_T$ and $\mathfrak{D}_{T*}$; and with the Hilbert space J- and J_*-inner products on $\mathfrak{D}$ and $\mathfrak{D}_*$, respectively. We conclude from this that $\mathfrak{D}_T$ and $\mathfrak{D}_{T*}$ are Hilbert spaces. Since they are the ranges of isometries, $\mathfrak{D}_T$ and $\mathfrak{D}_{T*}$ are regular subspaces of $\mathbf{K}$ [12, Theorem 5.2], and hence, by [2, Theorem V.5.2], the intrinsic topologies on $\mathfrak{D}_T$ and $\mathfrak{D}_{T*}$ (i.e., the topologies obtained from the norms $[h, h]^{1/2}$ on $\mathfrak{D}_T$ and $\mathfrak{D}_{T*}$) coincide with the strong topologies inherited from $\mathbf{K}$.

Theorem 10.1. $(I-\mathbf{T}^*\mathbf{T})\varphi=\varphi(I-\Theta_0^*\Theta_0)$ *and* $(I-\mathbf{T}\mathbf{T}^*)\varphi_*=\varphi_*(I-\Theta_0\Theta_0^*)$.

Proof. If f is in $\mathfrak{D}$, then the vector f_1 corresponding to $u\oplus v=\varphi JQf$ (given by (9.2)) is $f_1=(I-\Theta_0^*\Theta_0)f$. (This follows immediately from (10.3) and Lemma 9.2.) Hence by Proposition 9.1,

$$(I-\mathbf{T}^*\mathbf{T})\varphi JQf= e^{-it}(\Theta-\Theta_0)f_1\oplus e^{-it}\Delta f_1 = \varphi JQf_1 = \varphi JQ(I-\Theta_0^*\Theta_0)f =$$
$$= \varphi(I-\Theta_0^*\Theta_0)JQf.$$

The first assertion of the theorem then follows.

If g is in $\mathfrak{D}_*$, and if $u\oplus v=\varphi_*Q_*g$, then (10.4) shows that $u_0=(I-\Theta_0\Theta_0^*)g$. Hence, by Proposition 9.1,

$$(I-\mathbf{T}\mathbf{T}^*)\varphi_*Q_*g = (I-\Theta\Theta_0^*)u_0\oplus -\Delta\Theta_0^*u_0 = \varphi_*Q_*u_0 =$$
$$= \varphi_*Q_*(I-\Theta_0\Theta_0^*)g = \varphi_*(I-\Theta_0\Theta_0^*)Q_*g,$$

and the second assertion follows. □

Since $\mathfrak{D}_T$ and $\mathfrak{D}_{T*}$ are Hilbert spaces, we can define $J_T=\operatorname{sgn}(I-\mathbf{T}^*\mathbf{T})$ and $Q_T=|I-\mathbf{T}^*\mathbf{T}|^{1/2}$ as operators on $\mathfrak{D}_T$, and $J_{T*}=\operatorname{sgn}(I-\mathbf{T}\mathbf{T}^*)$ and $Q_{T*}=|I-\mathbf{T}\mathbf{T}^*|^{1/2}$ as operators on $\mathfrak{D}_{T*}$.

Corollary 10.2. $J_T\varphi=\varphi J$, $Q_T\varphi=\varphi Q$, $J_{T*}\varphi_*=\varphi_*J_*$, *and* $Q_{T*}\varphi_*=\varphi_*Q_*$. □

We have shown that the inner product $[.,.]$ is positive definite on $\mathfrak{D}_T$ and $\mathfrak{D}_{T*}$. With the inner products $[J_T.,.]$ and $[J_{T*}.,.]$, $\mathfrak{D}_T$ and $\mathfrak{D}_{T*}$ are Krein spaces having fundamental symmetries J_T and J_{T*}, respectively. Corollary 10.2 shows that φ and φ_* are Krein space isomorphisms intertwining the fundamental symmetries J and J_T, and J_* and J_{T*}.

11. The characteristic function. $\mathfrak{D}_T$ is regular, and so we can extend J_T and Q_T to operators on $\mathbf{H}$ by defining them to be zero on $\mathbf{H}\ominus\mathfrak{D}_T$. We similarly extend J_{T*} and Q_{T*} to operators defined on $\mathbf{H}$. It is clear that these extensions are self-adjoint, and that $J_TQ_T^2=I-\mathbf{T}^*\mathbf{T}$ and $J_{T*}Q_{T*}^2=I-\mathbf{T}\mathbf{T}^*$. We define

$$\Theta_T(\lambda) = -\mathbf{T}J_T+\lambda J_{T*}Q_{T*}(I-\lambda\mathbf{T}^*)^{-1}J_TQ_T|\mathfrak{D}_T$$

for those complex numbers λ for which $(I-\lambda\mathbf{T}^*)^{-1}$ exists. It will be shown in the next section that $\mathbf{H}$ is a Hilbert space, so that Θ_T is in fact the characteristic function of $\mathbf{T}$.

Theorem 11.1. *For* $\lambda\in D$, $\Theta_T(\lambda)\varphi=\varphi_*\Theta(\lambda)$.

Proof. It suffices to show that $-\mathbf{T}J_T\varphi=\varphi_*\Theta_0$ and, for $n=1,2,3,\dots$, $J_{T*}Q_{T*}\mathbf{T}^{*^{n-1}}J_TQ_T\varphi=\varphi_*\Theta_n$. By (10.3) and Corollary 10.2, we have for all $f\in\mathfrak{D}$,

$$-\mathbf{T}J_T\varphi(Qf)=-\mathbf{T}\varphi(JQf)=-\mathbf{T}\{e^{-it}(\Theta-\Theta_0)f\oplus e^{-it}\Delta f\}.$$

Lemma 9.2 and (9.3) then give us

$$-\mathbf{T}J_T\varphi(Qf)=-\{[(\Theta-\Theta_0)f-\Theta(I-\Theta_0^*\Theta_0)f]\oplus[\Delta f-\Delta(I-\Theta_0^*\Theta_0)f]\}=$$
$$=(I-\Theta\Theta_0^*)\Theta_0f\oplus-\Delta\Theta_0^*\Theta_0f=\varphi_*(Q_*\Theta_0f)=\varphi_*\Theta_0(Qf).$$

Since vectors of the form Qf, with $f\in\mathfrak{D}$, are dense in $\mathfrak{D}$, we conclude that $-\mathbf{T}J_T\varphi=$ $=\varphi_*\Theta_0$.

Now let us assume that for all $f\in\mathfrak{D}$ and for some $n\geqq 1$ we have

$$\mathbf{T}^{*n-1}J_TQ_T\varphi f=e^{-int}\left(\Theta-\sum_{k=0}^{n-1}e^{ikt}\Theta_k\right)f\oplus e^{-int}\Delta f. \tag{11.1}$$

By (10.3) and Corollary 10.2, (11.1) is true for $n=1$. If we let

$$u=e^{-int}(\Theta-\sum_{k=0}^{n-1}e^{ikt}\Theta_k)f,$$

then $u_0=\Theta_nf$, and we obtain from (8.2) (assuming (11.1))

$$\mathbf{T}^{*n}J_TQ_T\varphi f=e^{-it}\left[e^{-int}\left(\Theta-\sum_{k=0}^{n-1}e^{ikt}\Theta_k\right)f-\Theta_nf\right]\oplus e^{-i(n+1)t}\Delta f=$$
$$=e^{-i(n+1)t}\left(\Theta-\sum_{k=0}^{n}e^{ikt}\Theta_k\right)f\oplus e^{-i(n+1)t}\Delta f.$$

Hence, by induction, (11.1) is true for all $n\geqq 1$.

It follows from (11.1) and Proposition 9.1 that, for $n=1,2,3,\dots$ and $f\in\mathfrak{D}$,

$$Q_{T^*}(J_{T^*}Q_{T^*}\mathbf{T}^{*n-1}J_TQ_T\varphi f)=(I-\mathbf{T}\mathbf{T}^*)\left[e^{-int}\left(\Theta-\sum_{k=0}^{n-1}e^{ikt}\Theta_k\right)f\oplus e^{-int}\Delta f\right]=$$
$$=(I-\Theta\Theta_0^*)\Theta_nf\oplus-\Delta\Theta_0^*\Theta_nf=\varphi_*(Q_*\Theta_nf)=Q_{T^*}\varphi_*\Theta_nf.$$

(The last two steps used (10.4) and Corollary 10.2.) Since Q_{T*} is injective on $\mathfrak{D}_{T*}$, we conclude that $J_{T*}Q_{T*}\mathbf{T}^{*^{n-1}}J_TQ_T\varphi=\varphi_*\Theta_n$ for $n=1,2,3,\dots$ and the theorem is proved. □

12. Positivity of H. In this section we prove that, with the inner product $[.,.]$, $\mathbf{H}$ is a Hilbert space. We will need the following results.

Lemma 12.1. (cf. [15, Sec. VI.3.2]) *Suppose that the vector* $h\in\mathbf{H}$ *satisfies*

$$(12.1)\qquad (I-\mathbf{T}^*\mathbf{T})\mathbf{T}^n h = 0 = (I-\mathbf{T}\mathbf{T}^*)\mathbf{T}^{*n} h$$

for all $n=0, 1, 2, \ldots$. *Then* $h=0$.

Proof. We can write h in the form $h=u\oplus v$. Take $n\geqq 0$, and assume that $u_k=0$ for all $k<n$; when $n=0$, this is assuming nothing about u. Then (8.2) shows that

$$\mathbf{T}^{*n} h = e^{-int}u\oplus e^{-int}v.$$

By (12.1) and Proposition 9.1, we deduce

$$0=(I-\mathbf{T}\mathbf{T}^*)\mathbf{T}^{*n}h = (I-\Theta\Theta_0^*)u_n\oplus -\Delta\Theta_0^* u_n.$$

In particular, $(I-\Theta_0\Theta_0^*)u_n=0$, and since Θ is purely contractive, we have $u_n=0$. Therefore, by induction, $u=0$ and $h=0\oplus v$.

Since $h\in\mathbf{H}$, v must satisfy $J\Delta v=\sum_{k=1}^{\infty} e^{-ikt}f_k$ for some vectors $f_k\in\mathfrak{D}$ $(k=1, 2, \ldots)$. Take $n\geqq 0$, and assume that $f_k=0$ for all $k\leqq n$; again, this is a null assumption when $n=0$. Then clearly we have, using (9.3), $\mathbf{T}^n h=0\oplus e^{int}v$, and also

$$J\Delta(e^{int}v)=\sum_{k=1}^{\infty} e^{-ikt}f_{n+k}.$$

By (12.1) and Proposition 9.1, we deduce

$$0=(I-\mathbf{T}^*\mathbf{T})\mathbf{T}^n h = e^{-it}(\Theta-\Theta_0)f_{n+1}\oplus e^{-it}\Delta f_{n+1}.$$

Therefore we have $(\Theta-\Theta_0)f_{n+1}=0=\Delta f_{n+1}$ and hence

$$0=\Theta^*(\Theta-\Theta_0)f_{n+1}+J\Delta^2 f_{n+1}=(I-\Theta^*\Theta_0)f_{n+1}.$$

In particular, $(I-\Theta_0^*\Theta_0)f_{n+1}=0$, and since Θ is purely contractive, we have $f_{n+1}=0$. We conclude (by induction) that $J\Delta v=0$, and thus $v=0$ $\left(v\in\overline{\Delta L^2(\mathfrak{D})}\right)$. Therefore $h=0$. □

Theorem 12.2. *Let* $\mathfrak{U}$ *be a neighborhood of zero contained in the unit disk* D. *Then* $\mathbf{H}$ *is the closed linear span of vectors of the form* $(I-\mu\mathbf{T}^*)^{-1}J_T Q_T\varphi f$ *and* $(I-\mu\mathbf{T})^{-1}Q_{T*}\varphi_* g$, *where* $\mu\in\mathfrak{U}$, $f\in\mathfrak{D}$, *and* $g\in\mathfrak{D}_*$.

Proof. Since $\mathbf{T}$ has spectrum in the closed unit disk (Sec. 8), both $(I-\mu\mathbf{T}^*)^{-1}$ and $(I-\mu\mathbf{T})^{-1}$ are defined for $\mu\in\mathfrak{U}$.

Suppose that the vector $h\in\mathbf{H}$ is orthogonal to $(I-\mu\mathbf{T}^*)^{-1}J_T Q_T\varphi f$ and $(I-\mu\mathbf{T})^{-1}Q_{T*}\varphi_* g$, for all $\mu\in\mathfrak{U}$, $f\in\mathfrak{D}$, and $g\in\mathfrak{D}_*$. The theorem will be proved once we show that $h=0$.

We have, for all $f\in\mathfrak{D}$ and $\mu\in\mathfrak{U}$,

$$0=[h, (I-\mu\mathbf{T}^*)^{-1}J_T Q_T\varphi f]=[J_T Q_T(I-\bar{\mu}\mathbf{T})^{-1}h, \varphi f],$$

and thus, since $\mathfrak{D}_T$ is a Hilbert space, $J_T Q_T(I-\bar{\mu}\mathbf{T})^{-1}h=0$ for all $\mu\in\mathfrak{U}$. This is true only if $J_T Q_T \mathbf{T}^n h=0$ for $n=0, 1, 2, \dots$, and hence

$$(12.2) \qquad (I-\mathbf{T}^*\mathbf{T})\mathbf{T}^n h = 0 \quad \text{for} \quad n = 0, 1, 2, \dots .$$

Also, for $g\in\mathfrak{D}_*$ and $\mu\in\mathfrak{U}$, we have

$$0 = [h, (I-\mu\mathbf{T})^{-1}Q_{T^*}\varphi_* g] = [Q_{T^*}(I-\bar{\mu}\mathbf{T}^*)^{-1}h, \varphi_* g],$$

and so it follows, as above, that

$$(12.3) \qquad (I-\mathbf{T}\mathbf{T}^*)\mathbf{T}^{*n} h = 0 \quad \text{for} \quad n = 0, 1, 2, \dots .$$

(12.2) and (12.3), together with Lemma 12.1, imply that $h=0$. □

H is known to be regular, and thus (by [2, Theorem V.3.4]) **H** is a Krein space. Therefore, in order to prove **H** is a Hilbert space it suffices to show that it is positive. Obviously we need only show that $[h, h]\geqq 0$ for a set of vectors h dense in **H**, and in particular (by Theorem 12.2) for vectors of the form

$$(12.4) \qquad h = \sum_{i=1}^{n} \{(I-\mu_i\mathbf{T}^*)^{-1}J_T Q_T \varphi f_i + (I-\mu_i\mathbf{T})^{-1}Q_{T^*}\varphi_* g_i\},$$

where $n\geqq 1$ and, for $i=1, 2, \dots, n$, $f_i\in\mathfrak{D}$, $g_i\in\mathfrak{D}_*$, and $\mu_i\in\mathfrak{U}$, some neighborhood of zero in the unit disk.

For the vector h defined by (12.4) we have

$$\begin{aligned}
[h, h] = \sum_{i=1}^{n}\sum_{j=1}^{n} \{&[(I-\mu_i\mathbf{T}^*)^{-1}J_T Q_T\varphi f_i, (I-\mu_j\mathbf{T}^*)^{-1}J_T Q_T\varphi f_j]+\\
&+[(I-\mu_i\mathbf{T}^*)^{-1}J_T Q_T\varphi f_i, (I-\mu_j\mathbf{T})^{-1}Q_{T^*}\varphi_* g_j]+\\
&+[(I-\mu_i\mathbf{T})^{-1}Q_{T^*}\varphi_* g_i, (I-\mu_j\mathbf{T}^*)^{-1}J_T Q_T\varphi f_j]+\\
&+[(I-\mu_i\mathbf{T})^{-1}Q_{T^*}\varphi_* g_i, (I-\mu_j\mathbf{T})^{-1}Q_{T^*}\varphi_* g_j]\} =\\
= \sum_{i=1}^{n}\sum_{j=1}^{n} \{&[\varphi^{-1}Q_T(I-\bar{\mu}_j\mathbf{T})^{-1}(I-\mu_i\mathbf{T}^*)^{-1}J_T Q_T\varphi f_i, f_j]+\\
&+[\varphi_*^{-1}J_{T^*}Q_{T^*}(I-\bar{\mu}_j\mathbf{T}^*)^{-1}(I-\mu_i\mathbf{T}^*)^{-1}J_T Q_T\varphi f_i, g_j]+\\
&+[\varphi^{-1}Q_T(I-\bar{\mu}_j\mathbf{T})^{-1}(I-\mu_i\mathbf{T})^{-1}Q_{T^*}\varphi_* g_i, f_j]+\\
&+[\varphi_*^{-1}J_T Q_{T^*}(I-\bar{\mu}_j\mathbf{T}^*)^{-1}(I-\mu_i\mathbf{T})^{-1}Q_{T^*}\varphi_* g_i, g_j]\}.
\end{aligned}$$

In the above calculation it should be recalled that φ is unitary from the Krein space $\mathfrak{D}$ to the Krein space $\mathfrak{D}_T$, with the inner product $[J_T.,.]$. A similar observation applies to φ_*.

It can be shown ([10, Sec. IV.5]; cf. [11, Sec. 4] and [15, Sec. VI.1.1]) that, for $\lambda, \mu \in D$,

$$I-\Theta_T(\mu)^*\Theta_T(\lambda) = (1-\lambda\bar{\mu})Q_T(I-\bar{\mu}\mathbf{T})^{-1}(I-\lambda\mathbf{T}^*)^{-1}J_TQ_T,$$

$$I-\Theta_T(\bar{\mu})\Theta_T(\bar{\lambda})^* = (1-\lambda\bar{\mu})J_{T^*}Q_{T^*}(I-\bar{\mu}\mathbf{T}^*)^{-1}(I-\lambda\mathbf{T})^{-1}Q_{T^*},$$

$$\Theta_T(\lambda)-\Theta_T(\bar{\mu}) = (\lambda-\bar{\mu})J_{T^*}Q_{T^*}(I-\bar{\mu}\mathbf{T}^*)^{-1}(I-\lambda\mathbf{T}^*)^{-1}J_TQ_T,$$

and

$$\Theta_T(\bar{\lambda})^*-\Theta_T(\mu)^* = (\lambda-\bar{\mu})Q_T(I-\bar{\mu}\mathbf{T})^{-1}(I-\lambda\mathbf{T})^{-1}Q_{T^*}.$$

Hence, using Theorem 11.1, we have

$$\begin{aligned}
(12.5) \qquad [h, h] = \sum_{i=1}^{n}\sum_{j=1}^{n}\{&[(1-\mu_i\bar{\mu}_j)^{-1}(I-\Theta(\mu_j)^*\Theta(\mu_i))f_i, f_j]+\\
&+[(\mu_i-\bar{\mu}_j)^{-1}(\Theta(\mu_i)-\Theta(\bar{\mu}_j))f_i, g_j]+\\
&+[(\mu_i-\bar{\mu}_j)^{-1}(\Theta(\bar{\mu}_i)^*-\Theta(\mu_j)^*)g_i, f_j]+\\
&+[(1-\mu_i\bar{\mu}_j)^{-1}(I-\Theta(\bar{\mu}_j)\Theta(\bar{\mu}_i)^*)g_i, g_j]\}.
\end{aligned}$$

Equation (12.5) can be rewritten in the form

$$(12.6) \qquad [h, h] = \sum_{i=1}^{n}\sum_{j=1}^{n}[k(\mu_j, \mu_i)(f_i\oplus g_i), (f_j\oplus g_j)],$$

where $k(\mu, \lambda)$ is the operator matrix given by [11, Equation (6.1)]. By [11, Theorem 3], $k(\mu, \lambda)$ is positive definite when Θ is purely contractive and fundamentally reducible, and it therefore follows that $[h, h]\geqq 0$. Thus $\mathbf{H}$ is a Hilbert space.

13. The functional model for a bounded purely contractive analytic function: the main theorem.

Theorem 13.1. *Let $\Theta(\lambda)$: $\mathfrak{D}\to\mathfrak{D}_*$ be a bounded purely contractive fundamentally reducible analytic function. Then the Krein space*

$$\mathbf{H} = [H^2(\mathfrak{D}_*)\oplus\overline{\Delta L^2(\mathfrak{D})}]\ominus\{\Theta w\oplus\Delta w\colon w\in H^2(\mathfrak{D})\}$$

is a Hilbert space and the operator $\mathbf{T}$ on $\mathbf{H}$ defined by

$$\mathbf{T}^*(u\oplus v) = e^{-it}(u-u_0)\oplus e^{-it}v \quad (u\oplus v\in\mathbf{H})$$

is completely non-unitary. The function Θ coincides with the characteristic function of $\mathbf{T}$. The operator $\mathbf{U}$ on the Krein space $\mathbf{K}=L^2(\mathfrak{D}_*)\oplus\overline{\Delta L^2(\mathfrak{D})}$ defined by $\mathbf{U}(u\oplus v)= =e^{it}u\oplus e^{it}v$ $(u\oplus v\in\mathbf{K})$ is unitarily equivalent to the unitary dilation of $\mathbf{T}$ given by the construction in [7].

Proof. It was shown in Sec. 12 that $\mathbf{H}$ is a Hilbert space, and Lemma 12.1 shows that $\mathbf{T}$ is completely non-unitary. Θ coincides with Θ_T by virtue of

Theorem 11.1. Finally, Theorem 5.1 shows that $\mathbf{U}$ is unitarily equivalent to the dilation of $\mathbf{T}$ given in [7]. □

The construction of the dilation in [7] defines, in a natural way, a fundamental symmetry on the dilation space (referred to in Sec. 4 of this paper). For the space $\mathbf{K}$ above, this fundamental symmetry is not the obvious one $(J_*\oplus I)$, but the one defined as follows:

Let $\mathbf{M}=\{u\oplus 0\colon u\perp H^2(\mathfrak{D}_*)\}$. Then we have

$$\mathbf{K}=\mathbf{M}\oplus\mathbf{K}_+=\mathbf{M}\oplus\mathbf{H}\oplus\mathbf{G}$$

(see Sec. 8). We can therefore define a fundamental symmetry $\mathbf{J}$ on $\mathbf{K}$ by

$$\mathbf{J}(u\oplus 0)=J_*u\oplus 0\quad (u\oplus 0\in\mathbf{M}),\quad \mathbf{J}(u\oplus v)=u\oplus v\quad (u\oplus v\in\mathbf{H}),$$
$$\mathbf{J}(\Theta w\oplus\Delta w)=\Theta Jw\oplus\Delta Jw\quad (\Theta w\oplus\Delta w\in\mathbf{G}).$$

$\mathbf{J}$ is a fundamental symmetry since $\mathbf{H}$ is a Hilbert space and, for $w\in H^2(\mathfrak{D})$, we have

$$[\Theta Jw\oplus\Delta Jw,\Theta w\oplus\Delta w]=[\Theta^*\Theta Jw+(I-\Theta^*\Theta)Jw,w]=[Jw,w]\geqq 0.$$

14. Comparison with the model of Ball. In this section we determine the relationship between the model of BALL [1] and the model described in Theorem 13.1.

Assume that Θ satisfies the conditions of Theorem 13.1 and let $k(\mu,\lambda)$ be the operator matrix given by [11, Equation (6.1)]. Then the matrix

$$k'(\mu,\lambda)=(I\oplus J_*)k(\bar\lambda,\bar\mu)(J\oplus I) \tag{14.1}$$

coincides with the kernel matrix defined in [1, Theorem 2] (cf. [11, Sec. 6]). Also, as in [11], we will define $\tilde\Theta(\lambda)=\Theta(\bar\lambda)^*$.

Let us now consider an element $u\oplus v$ in $\mathbf{H}$, so that $u\in H^2(\mathfrak{D}_*)$ and $v\in\overline{\Delta L^2(\mathfrak{D})}$, with $\Theta^*u+J\Delta v\perp H^2(\mathfrak{D})$. Therefore, if w is defined by

$$w(e^{it})=e^{-it}[\Theta^*u+J\Delta v](-t),$$

then $w\in H^2(\mathfrak{D})$. Thus we can define a map Γ from $\mathbf{H}$ to $H^2(\mathfrak{D})\oplus H^2(\mathfrak{D}_*)$ by

$$\Gamma(u\oplus v)=w\oplus J_*u.$$

We will prove that $\Gamma\mathbf{H}$, normed so that Γ is unitary, is the Hilbert space $\mathfrak{D}(B)$ considered by BALL [1, Sec. 3.1].

Let us take $f\in\mathfrak{D}$ and $\mu\in D$. We denote by f^μ and f_μ the functions $f^\mu(\lambda)=(1-\lambda\mu)^{-1}f$ $(\lambda\in D)$ (cf. [15, Sec. V.8]) and $f_\mu(t)=(e^{it}-\mu)^{-1}f$ $(t\in[0,2\pi])$. It is clear that $f^\mu\in H^2(\mathfrak{D})$ and $f_\mu\in L^2(\mathfrak{D})$. From the boundedness of Θ, and the fact that $(\lambda-\mu)^{-1}(\Theta(\lambda)-\Theta(\mu))f$ is analytic for $\lambda\in D$, we conclude that the function

$$u=(\Theta-\Theta(\mu))f_\mu \tag{14.2}$$

s in $H^2(\mathfrak{D}_*)$, and the function

$$w = \big(I - \bar{\Theta}\Theta(\mu)\big) f^{\mu} \tag{14.3}$$

is in $H^2(\mathfrak{D})$. It is immediate from the definitions of k, $\bar{\Theta}$, f^{μ}, and f_{μ}, that $w(\lambda)\oplus \oplus u(\lambda) = k(\bar{\lambda}, \mu)(f\oplus 0)$, for all $\lambda \in D$.

Let us also consider the function

$$v = \Delta f_{\mu} \tag{14.4}$$

in $\overline{\Delta L^2(\mathfrak{D})}$. Then we have, using (14.2) and (14.4),

$$\Theta^* u + J\Delta v = \Theta^*\big(\Theta - \Theta(\mu)\big) f_{\mu} + (I - \Theta^*\Theta) f_{\mu} = \big(I - \Theta^*\Theta(\mu)\big) f_{\mu},$$

and hence

$$e^{-it}[\Theta^* u + J\Delta v](-t) = e^{-it}\big(I - \Theta(e^{-it})^*\Theta(\mu)\big)(e^{-it} - \mu)^{-1} f =$$
$$= \big(I - \bar{\Theta}(e^{it})\Theta(\mu)\big)(1 - e^{it}\mu)^{-1} f = w(e^{it}),$$

where w is given by (14.3). Therefore $u\oplus v \in \mathbf{H}$ and $\Gamma(u\oplus v) = w \oplus J_* u$, i.e.,

$$\Gamma(u\oplus v)(\lambda) = (I\oplus J_*) k(\bar{\lambda}, \mu)(f\oplus 0).$$

By using (8.2), (14.2), and (14.4), we obtain

$$(I - \mu \mathbf{T}^*)(u\oplus v) = [u - \mu e^{-it}(u - u_0)] \oplus [v - \mu e^{-it} v] =$$
$$= e^{-it}[(e^{it} - \mu) u + \mu u_0] \oplus e^{-it}(e^{it} - \mu) v = e^{-it}\big[\big(\Theta - \Theta(\mu)\big) f - \big(\Theta_0 - \Theta(\mu)\big) f\big] \oplus e^{-it}\Delta f =$$
$$= e^{-it}(\Theta - \Theta_0) f \oplus e^{-it}\Delta f.$$

It therefore follows, using (10.3) and Corollary 10.2, that

$$(I - \mu\mathbf{T}^*)(u\oplus v) = \varphi J Q f = J_T Q_T \varphi f,$$

and thus we obtain

$$\Gamma\big((I - \mu\mathbf{T}^*)^{-1} J_T Q_T \varphi f\big)(\lambda) = (I\oplus J_*) k(\bar{\lambda}, \mu)(f\oplus 0). \tag{14.5}$$

Let us take $g \in \mathfrak{D}_*$ and $\mu \in D$, and consider the functions $u' = \big(I - \Theta\Theta(\bar{\mu})^*\big) g^{\mu}$, $v' = -\Delta\Theta(\bar{\mu})^* g^{\mu}$, and $w' = \big(\bar{\Theta} - \bar{\Theta}(\mu)\big) g_{\mu}$. Then we obtain, in a manner similar to that used in deriving (14.5), the formula

$$\Gamma\big((I - \mu\mathbf{T})^{-1} Q_{T^*}\varphi_* g\big)(\lambda) = (I\oplus J_*) k(\bar{\lambda}, \mu)(0\oplus g). \tag{14.6}$$

By Theorem 12.2, $\mathbf{H}$ is the closed linear span of vectors of the form $(I - \mu\mathbf{T}^*)^{-1} J_T Q_T \varphi f$ and $(I - \mu\mathbf{T})^{-1} Q_{T^*}\varphi_* g$, where $f \in \mathfrak{D}$, $g \in \mathfrak{D}_*$, and $\mu \in D$. The space $\mathfrak{D}(B)$ in [1] is defined so that a dense subset is that spanned by vectors which are pairs of functions (in λ) of the form $(I\oplus J_*) k(\bar{\lambda}, \mu)(f\oplus 0)$ and $(I\oplus J_*) k(\bar{\lambda}, \mu)(0\oplus g)$, where $f \in \mathfrak{D}$, $g \in \mathfrak{D}_*$, and $\mu \in D$. (Recall that the kernel matrix in [1] is given by (14.1).) Thus we will have $\Gamma\mathbf{H} = \mathfrak{D}(B)$, with Γ unitary, once we have checked that the norm induced by Γ, on the dense subset of $\mathfrak{D}(B)$ described above, is the same as that defined in [1].

Consider the vector $h\in\mathbf{H}$ defined by (12.4). Then, by (14.5) and (14.6),

$$(\Gamma h)(\lambda)=(I\oplus J_*)\sum_{i=1}^{n}k(\bar\lambda,\mu_i)(f_i\oplus g_i)=\sum_{i=1}^{n}k'(\bar\mu_i,\lambda)(Jf_i\oplus g_i)$$

(using (14.1)), and it follows from the definition of the inner product in [1] that

$$\|\Gamma h\|^2=\sum_{i=1}^{n}\sum_{j=1}^{n}\big(k'(\bar\mu_i,\bar\mu_j)(Jf_i\oplus g_i),(Jf_j\oplus g_j)\big)=$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{n}\big((I\oplus J_*)k(\mu_j,\mu_i)(f_i\oplus g_i),(Jf_j\oplus g_j)\big)=$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{n}\big((J\oplus J_*)k(\mu_j,\mu_i)(f_i\oplus g_i),(f_j\oplus g_j)\big).$$

But the inner product $(.,.)$ on $\mathfrak{D}\oplus\mathfrak{D}_*$ is the $J\oplus J_*$-inner product, and hence

$$\|\Gamma h\|^2=\sum_{i=1}^{n}\sum_{j=1}^{n}[k(\mu_j,\mu_i)(f_i\oplus g_i),(f_j\oplus g_j)].$$

Consequently, we have (by (12.6)) $\|\Gamma h\|^2=[h,h]$, and so $\Gamma\mathbf{H}=\mathfrak{D}(B)$ with Γ unitary.

In [1] the characteristic function B of an operator T is taken to be $B=\bar\Theta_T$ (cf. [11, Sec. 6]), and so in comparing the two models we should take $B=\bar\Theta$. In Ball's model, B is shown to be the characteristic function of the operator R on $\mathfrak{D}(B)$ defined by

$$R(w\oplus u)=e^{-it}(w-w_0)\oplus(e^{it}u-\bar BJw_0).$$

We show now that $R\Gamma=\Gamma\mathbf{T}$.

For $u\oplus v\in\mathbf{H}$, we have defined $\Gamma(u\oplus v)=w\oplus J_*u$, where $w(e^{it})=$ $=e^{-it}[\Theta^*u+J\Delta v](-t)$. Thus w_0 is the vector f_1 given by (9.2) and, using (9.3), we have

$$\mathbf{T}(u\oplus v)=(e^{it}u-\Theta w_0)\oplus(e^{it}v-\Delta w_0).$$

Note that

$$\Theta^*(e^{it}u-\Theta w_0)+J\Delta(e^{it}v-\Delta w_0)=e^{it}(\Theta^*u+J\Delta v)-w_0,$$

and hence

$$\Gamma\mathbf{T}(u\oplus v)=e^{-it}(w-w_0)\oplus J_*(e^{it}u-\Theta w_0).$$

Since $B=\bar\Theta$, we have $\bar B=J_*\Theta J$, and thus we conclude that

$$R\Gamma(u\oplus v)=R(w\oplus J_*u)=e^{-it}(w-w_0)\oplus(e^{it}J_*u-J_*\Theta w_0)=\Gamma\mathbf{T}(u\oplus v).$$

Theorem 14.1. *Suppose Θ satisfies the conditions of Theorem* 13.1, *and let* $\mathbf{T}$ *be the operator, defined in that theorem, having Θ as its characteristic function. Then*

T *is unitarily equivalent to the operator* R *defined in* [1], *with* $B=\bar{\Theta}$. *The equivalence is implemented by the unitary operator* Γ: $\mathbf{H}\to\mathfrak{D}(B)$ *given by* $\Gamma(u\oplus v)=w\oplus J_* u$ $(u\oplus v\in \mathbf{H})$, *where*

$$w(e^{it}) = e^{-it}[\Theta^* u+J\Delta v](-t). \quad \square$$

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Derivations on l-semigroups and formal languages

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Introduction

Transformations of lattices resp. lattice-ordered semigroups satisfying formal properties which correspond to the differentiation rules of the sum, product and composite of real functions have been studied in [8], [9]. However it was shown in [8] that for the most important l-semigroups only the identity mapping resp. the zero mapping (if defined) satisfy all these formal rules. Thus the abstraction in this sense turned out to be "not very" useful; besides the motivation for the investigation of such transformations was still missing. In this paper we give a motivation studying the derivations of formal languages (see [2], [3]).

Since the set $P(X^*)$ of all formal languages on an alphabet X forms a *lattice ordered semigroup* (briefly: *l-semigroup*) with respect to set theoretical intersection, union and complex product, in general we consider such l-semigroups (see [5], [6]). By this we mean a set S with three binary operations $\wedge$, $\vee$, $\cdot$, such that

1) $(S, \wedge, \vee)$ is a lattice,

2) $(S, \cdot)$ is a semigroup,

3) $a(b\vee c)=(ab)\vee(ac), \quad (a\vee b)c=(ac)\vee(bc) \quad (\forall a, b, c\in S)$.

We note that every such l-semigroup is a *partially ordered semigroup* with respect to its lattice order, i.e. $a\leqq b\Rightarrow ac\leqq bc$ and $ca\leqq cb$ $(\forall c\in S)$. A *dual l-semigroup* satisfies 1), 2), 3) and

4) $a(b\wedge c)=(ab)\wedge(ac), \quad (a\wedge b)c=(ac)\wedge(bc) \quad (\forall a, b, c\in S)$.

A *right lattice ordered semigroup* (briefly: *rl-semigroup*, see [7]) is defined by 1) and 2) and

3') $(a\vee b)c=(ac)\vee(bc), \quad (a\wedge b)c=(ac)\wedge(bc) \quad (\forall a, b, c\in S)$.

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The aim of this paper is to investigate transformations of $(r)l$-semigroups S which satisfy the most important properties of the derivations of formal languages (see [2], [3]). Essentially they are lattice homomorphisms which are at the same time semigroup translations. First we prove some general properties of such "derivations" and state their explicit form on special $(r)l$-semigroups. Next we suppose S to have *right quotients*, a condition which corresponds to an important property of the l-semigroup $(P(X^*), \cap, \cup, \cdot)$. This motivates the study of mappings of the form $\varphi(x)=x:a$ ($\forall x\in S$, $a\in S$ fixed) which reflect the definition of derivation in $P(X^*)$. We investigate such transformations on general l-semigroups and show that they satisfy all the properties of a "derivation". Conversely it turns out that every "derivation" on certain l-semigroups has the form $\varphi(x)=x:a$ for a certain $a\in S$. These results become more apparent by the theorem that under certain conditions every such l-semigroup is isomorphic to the l-semigroup $P(X^*)$ of all formal languages on an alphabet X. In particular we obtain that it is impossible to define transformations on $P(X^*)$ satisfying the three essential properties of a derivation, but different from the mappings of the form $\varphi(A)=A:a$ $(\forall A\in P(X^*))$.

1. Derivations with dual chain-rule

Let $X\neq\emptyset$ be an alphabet and $(X^*, \cdot)$ the free semigroup of all words $w=a_1 \dots a_n$ with letters a_i in X with respect to concatenation "$\cdot$", the empty word λ being the identity element of $(X^*, \cdot)$. Then every subset A of X^* is called a *formal language on X*. The set theoretical union, intersection and the complex product of two formal languages A, B on X $(A\cdot B=\{ab|a\in A, b\in B\}$, $A\cdot\emptyset=\emptyset\cdot A=\emptyset)$ are again formal languages on X. With respect to the operations $\cap$, $\cup$ and $\cdot$ the power set $P(X^*)$ forms a lattice ordered semigroup with identity $\{\lambda\}$.

The *derivative of a formal language $A\in P(X^*)$ with respect to a letter $a\in X$* $(\{a\}\in P(X^*))$ is defined as the subset $D_a(A)=\{x\in X^*|ax\in A\}$ of X^* (see [3]). Thus for a fixed $a\in X$ to every $A\in P(X^*)$ there corresponds a $B\in P(X^*)$ defined by $B=D_a(A)$; consequently D_a can be interpreted as a mapping $D_a\colon P(X^*)\to P(X^*)$ satisfying the following three properties which are of most importance for the application to regular expressions and the construction of finite automatas, i.e. acceptors (see [3]):

1) $D_a(A\cup B)=D_a(A)\cup D_a(B) \quad (\forall A, B\in P(X^*))$
2) $D_a(A\cap B)=D_a(A)\cap D_a(B) \quad (\forall a\in X)$
3) $D_a(A\cdot B)\ =D_a(A)\cdot B\cup\delta(A)\cdot D_a(B)$,

where $\delta(A)=\lambda$ if $\lambda\in A$ and $\delta(A)=\emptyset$ if $\lambda\notin A$. Furthermore we have $D_a(\emptyset)=\emptyset$, $D_a(X^*)=X^*$ and $D_a(M)=\{\lambda\}$ for at least one $M\in P(X^*)$. Since $(P(X^*), \cap, \cup)$

is also uniquely complemented, we have also $D_a(A')=(D_a(A))'$ $(\forall A\in P(X^*))$. Thus every mapping D_a is a (0, 1)-lattice homomorphism satisfying a certain "dual chain rule" (for $\delta(A)=\lambda$) resp. the property of a semigroup translation (for $\delta(A)=\emptyset$). This motivates the definition of the concept of *derivation* on general $(r)l$-semigroups and the study of such transformations.

Definition. Let S be an $(r)l$-semigroup with identity e. A transformation $\varphi\colon S\to S$ is called a *derivation of* S, if for all $x, y\in S$

$$\varphi(x\vee y) = \varphi(x)\vee\varphi(y), \tag{1}$$

$$\varphi(x\wedge y) = \varphi(x)\wedge\varphi(y), \tag{2}$$

$$\varphi(xy) = \begin{cases}\varphi(x)y\vee\varphi(y) & \text{if } x\geqq e,\\ \varphi(x)y & \text{if } x\not\geqq e.\end{cases} \tag{3}$$

Remarks. 1. Let $\varphi\colon S\to S$ be a mapping satisfying (1) or (2); then for $x\geqq e$ we get $xy\geqq y$ $(\forall y\in S)$, thus $\varphi(xy)\geqq\varphi(y)$ (since φ is order preserving by (1) or (2)). Consequently, if we can show the equation $\varphi(xy)=\varphi(x)y$ for *all* $x, y\in S$, then $\varphi(xy)=\varphi(xy)\vee\varphi(y)=\varphi(x)y\vee\varphi(y)$ for $x\geqq e$ and for all $y\in S$.

2. Let $(S, \wedge, \vee)$ be uniquely complemented and let $\varphi\colon S\to S$ satisfy (1) and (2); then $\varphi(x')=[\varphi(x)]'$ $(\forall x\in S)$ iff $\varphi(o)=o$ and $\varphi(i)=i$ (o is the least, i is the greatest element of S), because $\varphi(x)\vee\varphi(x')=\varphi(x\vee x')=\varphi(i)$, $\varphi(x)\wedge\varphi(x')==\varphi(x\wedge x')=\varphi(o)$ $(\forall x\in S)$.

3. Let $(S, \wedge, \vee)$ be a Boolean lattice and let $\varphi\colon S\to S$ satisfy (1) [resp. (2)] and $\varphi(x')=[\varphi(x)]'$ $(\forall x\in S)$; then φ satisfies (2) [resp. (1)], for $x\wedge y=(x'\vee y')'$ $(\forall x, y\in S)$ implies $\varphi(x\wedge y)=[\varphi(x'\vee y')]'=([\varphi(x)]'\vee[\varphi(y)]')'=\varphi(x)\wedge\varphi(y)$ $(\forall x, y\in S)$.

Examples. Let S be an $(r)l$-semigroup with identity e.

1. The identity mapping $\mathrm{id}(x)=x$ $(\forall x\in S)$ is a derivation on S.

2. If S has a least element o with $o=ox$ $(\forall x\in S)$, then $\varphi(x)=o$ $(\forall x\in S)$ is a derivation of S.

3. If S is an l-semigroup, then $\varphi(x)=a$ $(\forall x\in S,\ a\in S$ fixed) is a derivation of S iff a is a left zero of $(S, \cdot)$. (If φ is a derivation and there is an $x\in S$ with $x\not\geqq e$, then $a=\varphi(xy)=\varphi(x)y=ay$ for every $y\in S$; if $x\geqq e$ $(\forall x\in S)$, then $a=\varphi(xy)==\varphi(x)y\vee\varphi(y)=ay\vee a=ay$ for all $y\in S$; in both cases a is a left zero of S. Conversely, if $a=ay$ $(\forall y\in S)$ and $\varphi(x)=a$ $(\forall x\in S)$, then $\varphi(xy)=a=ay=\varphi(x)y$ $(\forall x, y\in S)$ and we can apply Remark 1.)

4. The transformation $\varphi(x)=ax$ $(\forall x\in S,\ a\in S$ fixed) is a derivation of S iff $a\in S$ is left distributive with respect to $\wedge$ (and $\vee$). (Since $\varphi(xy)=a(xy)=(ax)y==\varphi(x)y$ $(\forall x, y\in S)$, we can apply Remark 1; the properties (1) and (2) are clear iff $a(x\wedge y)=(ax)\wedge(ay)$ resp. $a(x\vee y)=(ax)\vee(ay)$ for all $x, y\in S$.)

Remark. In $S=P(X^*)$ the only left zero is $\varnothing$; but $\varphi(A)=\varnothing$ $(\forall A\in P(X^*))$ is not a derivation D_a in the sense of formal languages, since $D_a(X^*)\neq\varnothing$ for every $a\in X$.

Properties. Let S be an $(r)l$-semigroup with identity e and φ a derivation of S.

1.1. If S has a least element o with $o=xo$ $(\forall x\in S)$, then $\varphi(o)=o$. (If there exists $x\ngeq e$, then $\varphi(o)=\varphi(xo)=\varphi(x)o=o$; if $x\geqq e$ $(\forall x\in S)$, then $e=o$ and $o=xo=xe=x$ $(\forall x\in S)$; but for $S=\{o\}$, trivially $\varphi(o)=o$.)

1.2. If S is an l-semigroup, then $\varphi(x^n)=\varphi(x)x^{n-1}$ for all $x\in S$ and for all integers $n\geqq 2$. (Case $n=2$: for $x\geqq e$ we have $\varphi(x^2)=\varphi(xx)=\varphi(x)x\vee\varphi(x)=\varphi(x)x$, and for $x\ngeq e$ we have $\varphi(x^2)=\varphi(x)x$; induction: for $x\geqq e$ we have $\varphi(x^n)=\varphi(xx^{n-1})=\varphi(x)x^{n-1}\vee\varphi(x^{n-1})=\varphi(x)(x^{n-1}\vee x^{n-2})=\varphi(x)x^{n-1}$ and for $x\ngeq e$ we have $\varphi(x^n)=\varphi(x)x^{n-1}$.)

1.3. If S has o with $e\neq o$, then $\varphi(c)=c$ for all right zeros $c\in S$. (Since $e\neq o$, there exists $x<e$ and $\varphi(c)=\varphi(xc)=\varphi(x)c=c$.)

1.4. $\varphi(c)\geqq\varphi(x)$ $(\forall x\in S)$ for every left zero $c\geqq e$. ($\varphi(c)=\varphi(cx)=\varphi(c)x\vee\varphi(x)\geqq\varphi(x)$ for all $x\in S$.)

1.5. If S has a greatest element i, then $c\geqq\varphi(x)$ $(\forall x\in S)$ and $c=\varphi(i)$ for a two-sided zero $c\geqq e$. ($c=\varphi(c)\geqq\varphi(x)$ for all $x\in S$ by 1.3 and 1.4; since $c\leqq i$ implies $c=\varphi(c)\leqq\varphi(i)$, we get $c=\varphi(i)$.)

1.6. If $(S,\wedge,\vee)$ is uniquely complemented with $e=i$ and $\varphi(e)=e$, then $\varphi=\mathrm{id}$. (By Theorem 1.2, Ch. II of [5] we have $xy=x\wedge y$ $(\forall x,y\in S)$; thus for $x<e$, $\varphi(x)\wedge y=\varphi(x)y=\varphi(xy)=\varphi(x\wedge y)=\varphi(x)\wedge\varphi(y)$ $(\forall y\in S)$; for $x=y$ we get $\varphi(x)\wedge x=\varphi(x)$, thus $\varphi(x)\leqq x$; but $\varphi(e)=e$ implies $\varphi(x)=\varphi(ex)=\varphi(e)x\vee\varphi(x)=x\vee\varphi(x)\geqq x$, and $\varphi(x)=x$ for all $x\in S$. If $i=e=o$, then $S=\{o\}$ and $\varphi(x)=x$ holds trivially.)

1.7. If $(S,\wedge,\vee)$ is Boolean and $\varphi(e)=e$, $\varphi(o)=o$, then $\varphi=\mathrm{id}$. (Since $\varphi(e)=e$, we have again $\varphi(x)\geqq x$ $(\forall x\in S)$; $x=i$ implies $\varphi(i)=i$, so that $[\varphi(x)]'=\varphi(x')$ $(\forall x\in S)$ by Remark 2; since S is Boolean, $\varphi(x)\geqq x$ implies $\varphi(x')=[\varphi(x)]'\leqq x'$ $(\forall x\in S)$, hence $\varphi(x)=x$ $(\forall x\in S)$.)

Remark. By Property 1.3 every right zero is a fixed point of S with $e\neq o$. In $S=P(X^*)$ the only right zero is $\varnothing$, which is indeed a fixed point for every derivation D_a. By a Theorem in [4], every order preserving transformation (in particular every derivation) has at least one fixed point iff $(S,\wedge,\vee)$ is complete; for example $D_a(M)=M$ for $M=\{\lambda, a^n;\ n=1,2,3,\ldots\}$.

In the following we prove three lemmas on the explicit form of derivations on special l-semigroups S. Although the conditions imposed on S are not satisfied for $P(X^*)$, these results seem to have some interest on their own. We note, that in every $(r)l$-semigroup S with identity e: $\varphi(x)=\varphi(ex)=\varphi(e)x\vee\varphi(x)\geqq\varphi(e)x$ for all $x\in S$ and for every derivation φ.

Lemma 1.1. *Let S be an $(r)l$-semigroup with identity e. If there is a right invertible element $a\neq e$ in S, then every derivation φ of S has the form $\varphi(x)=\varphi(e)x$ $(\forall x\in S)$.*

Proof. We have $ab=e$ for some $b\in S$. If $a>e$, then $b<e$ (since $a>e$ implies $e=ab\geqq b$, but $b=e$ implies $a=e$); consequently we get $\varphi(x)=\varphi(ex)=$ $=\varphi(abx)=\varphi(a)bx\vee\varphi(b)x=[\varphi(a)b\vee\varphi(b)]x=\varphi(ab)x=\varphi(e)x$ for all $x\in S$. If $a\not\geqq e$, then we conclude for every $x\in S$ that

$$\varphi(x)=\varphi(abx)=\varphi(a)(bx)=[\varphi(a)b]x=\varphi(ab)x=\varphi(e)x.$$

Corollary. *Let S be a dual l-semigroup with identity e and right invertible element $a\neq e$. Then the derivations on S are exactly the inner left translations $\varphi(x)=cx$ $(\forall x\in S,\ c\in S$ arbitrary).*

Remark. If we suppose that for every derivation $\varphi\neq\mathrm{id}$ there exists at least one $b\neq e$ in S with $\varphi(b)=e$, then also the converse of Lemma 1.1 is true: if $\varphi\neq\mathrm{id}$, then $\varphi(e)\neq e$, since $\varphi(e)=e$ implies $\varphi(x)=\varphi(ex)=\varphi(e)x=ex=x$, thus $\varphi=\mathrm{id}$; but $\varphi(b)=e$ for some $b\neq e$, hence $e=\varphi(b)=\varphi(e)b$ and $a=\varphi(e)\neq e$ is a right invertible element.

Example. Let $(L,\wedge,\vee)$ be an arbitrary lattice and $S=F(L)$ the set of all transformations of L. With pointwise intersection, union and with the composition of functions $(f\circ g)(x)=f[g(x)]$ $(\forall x\in L)$, $(S,\wedge,\vee,\circ)$ forms an rl-semigroup with identity id (see [7]). Since every permutation of L is (right)invertible with respect to "$\circ$", every derivation φ of S has the form $\varphi(f)=\varphi(\mathrm{id})\circ f$ $(\forall f\in S)$. Choosing $\varphi(\mathrm{id})$ as a constant function f_a defined by $f_a(x)=a$ $(\forall x\in L)$, $\varphi(\mathrm{id})$ is left distributive with respect to "$\wedge$" and "$\vee$", so that $\varphi(f)=f_a$ $(\forall f\in S)$ (see Example 4).

Lemma 1.2. *Let S be a uniquely complemented rl-semigroup with identity e. If $ox=o$, $ix=i$ $(\forall x\in S)$, then every derivation φ of S with $\varphi(o)=o$, $\varphi(i)=i$ has the form $\varphi(x)=\varphi(e)x$ $(\forall x\in S)$.*

Proof. If $e=o$, then $x=ex=e$ for all $x\in S$ and $S=\{e\}$; thus $\varphi(e)=e$ and $\varphi(x)=\varphi(e)x$ $(\forall x\in S)$. Let $e\neq o$; then $e'\not\geqq e$. Furthermore $(xy)'=x'y$ $(\forall x,y\in S)$, since $(xy)\vee(x'y)=iy=i$, $(xy)\wedge(x'y)=oy=o$. Consequently, for all $x\in S$, $\varphi(x)=\varphi(ex)=\varphi[(e'x)']=[\varphi(e'x)]'=[\varphi(e')x]'=[\varphi(e')]'x=\varphi(e)x$ (see Remark 2).

Lemma 1.3. *Let S be a Boolean rl-semigroup with $e\neq o$ and $ab=o$ iff $a=o$ or $b=o$. Then every derivation of S with $\varphi(i)=i$ has the form*

$$\varphi(x)=\begin{cases} i & \text{for each} \quad x\geqq e \\ x & \text{for each} \quad x\ngeqq e. \end{cases}$$

Proof. Since $\varphi(o)=o$ by Property 1.1 and $\varphi(i)=i$, by Remark 2 we have $\varphi(x')=[\varphi(x)]'$ $(\forall x\in S)$. Let first $x\geqq e$; then $x'\ngeqq e$ (otherwise $o=x\wedge x'\geqq x\wedge e=e$); consequently, $o=\varphi(o)=\varphi(oy)=\varphi[(x\wedge x')y]=\varphi(xy)\wedge\varphi(x'y)=[\varphi(x)y\vee\varphi(y)]\wedge\varphi(x')y=$ $=o\vee[\varphi(y)\wedge\varphi(x')y]$ $(\forall y\in S)$. For $y=i$ we get $o=\varphi(x')i$; since $i\neq o$, we conclude $\varphi(x')=o$, thus $\varphi(x)=i$ $(\forall x\in S$ with $x\geqq e)$. In particular we have $\varphi(e)=i$, hence $\varphi(x)=\varphi(ex)=\varphi(e)x\vee\varphi(x)\geqq ix\geqq x$, i.e. $\varphi(x)\geqq x$ $(\forall x\in S)$.

Next let $x\| e$; if $x'\geqq e$, then $x\ngeqq e$ and similarly to the case $x\geqq e$, $x'\ngeqq e$, we get $\varphi(x)=o$; but then $\varphi(x)\geqq x$ $(\forall x\in S)$ implies $x=o$, a contradiction. If $x'\ngeqq e$ then $o=\varphi(xy)\wedge\varphi(x'y)=\varphi(x)y\wedge\varphi(x')y=\varphi(x)y\wedge[\varphi(x')y\vee x'y]=o\vee[\varphi(x)y\wedge x'y]=$ $=[\varphi(x)\wedge x']y$ $(\forall y\in S)$. For $y=i$ we get $o=[\varphi(x)\wedge x']i$ and thus $\varphi(x)\wedge x'=o$; that means $\varphi(x)\leqq x$. Together with $\varphi(x)\geqq x$ $(\forall x\in S)$ we conclude $\varphi(x)=x$ $(\forall x\in S, x\| e)$.

Finally let $x<e$; if $x=o$, then $\varphi(o)=o$; if $x\neq o$, then $x'\ngeqq e$ (otherwise $x'\geqq e\geqq x$, which implies $x=o$). Similarly to the case $x\| e$, $x'\ngeqq e$, we obtain $\varphi(x)=x$ for all $x<e$ in S and the proof is complete.

Corollary. *Let S be a Boolean rl-semigroup with $e\neq o$ and $ab=o$ iff $a=o$ or $b=o$. Then every derivation φ on S with $\varphi(i)=i$ is the identity mapping.*

Proof. Since by Example 1 the identity transformation $\mathrm{id}(x)=x$ is always a derivation with $\varphi(i)=i$, by Lemma 1.3 we have $x=\mathrm{id}(x)=i$ for all $x\geqq e$. In particular $e=i$; thus every derivation φ of S with $\varphi(i)=i$ satisfies $\varphi(e)=e$, too. But then $\varphi=\mathrm{id}$ by Property 1.7.

2. Quotient mappings

Returning to the concept of derivation of a formal language $A\in P(X^*)$ with respect to a letter $a\in X$, defined by

$$D_a(A)=\{x\in X^*|ax\in A\},$$

we may interprete $D_a(A)$ as the greatest set $B\subseteqq X^*$ with respect to set inclusion such that $aB\subseteqq A$. More generally, the derivation of a formal language $A\in P(X^*)$ *with respect to a set* $T\subseteqq X^*$ is defined by

$$D_T(A)=\{x\in X^*|\,Tx\subseteqq A\},$$

i.e. as the greatest set $C\subseteqq X^*$ such that $T\cdot C\subseteqq A$ (see [3]).

The l-semigroup $(P(X^*), \cap, \cup, \cdot)$ has the property that for each ordered pair $A, B \in P(X^*)$ in the set of all $Y \in P(X^*)$ with $A \cdot Y \subseteq B$ there is a greatest element with respect to set inclusion, denoted by $B:A$ and called the *right quotient of B with respect to A.* Therefore a derivation D_T with respect to $T \subseteq X^*$ can be viewed as a mapping D_T: $P(X^*) \to P(X^*)$ which associates to every $A \in P(X^*)$ the right quotient $A:T = D_T(A)$, the greatest element $C \in P(X^*)$ such that $T \cdot C \subseteq A$. If T consists of a single letter $a \in X$, then $D_a(A) = A:a$, the greatest element $B \in P(X^*)$ with $aB \subseteq A$.

Consequently, for this section we suppose S to be an *l-semigroup with right quotients* (see [5], [6]); by this we mean an l-semigroup S such that for every ordered pair $a, b \in S$, in the set of all $x \in S$ with $ax \leqq b$ there exists a greatest element with respect to "$\leqq$", denoted by $b:a$ and called the *right quotient of b with respect to a.* We recall [5] that a *complete l-semigroup* is defined as an l-semigroup S which is a complete lattice satisfying the identity: $a(\vee b_i) = \vee (ab_i)$ for all $a, b \in S$, where the join is taken for an arbitrary set of indices. In particular it is easy to see, that in a complete l-semigroup S with $xo = o$ $(\forall x \in S)$ the right quotients $b:a$ exists for all pairs $a, b \in S$.

Motivated by the (complete) l-semigroup $(P(X^*), \cap, \cup, \cdot)$ with right quotients and its derivations $D_T(A) = A:T$ $(\forall A \in P(X^*))$ we give the following

Definition. Let S be an l-semigroup with right quotients; then every transformation $\varphi_\alpha(x) = x:\alpha$ $((\forall x \in S),\ \alpha \in S$ fixed$)$ is called a *quotient mapping of S with respect to* $\alpha \in S$.

First we show that every quotient mapping can be reduced to quotient mappings with respect to atoms of S:

Lemma 2.1. *Let S be a complete, atomic, Boolean l-semigroup with $xo = o$, $(\forall x \in X)$; then every φ_α, $\alpha \neq o$, is an intersection of quotient mappings φ_a on S with respect to atoms $a \in S$.*

Proof. Since $(S, \wedge, \vee)$ is an atomic Boolean lattice, every element $\alpha \in S$, $\alpha \neq o$, is a join of atoms $a_i \in S$, that is $\alpha = \bigvee a_i$ (Theorem III.1.5 in [1]). But $(S, \wedge, \vee, \cdot)$ is also complete, so that right quotients exist and by [5], p. 156, $x:\alpha = x:(\bigvee a_i) = \bigwedge (x:a_i)$. Consequently, $\varphi_\alpha(x) = x:\alpha = \bigwedge(x:a_i) = \bigwedge \varphi_{a_i}(x)$ $(\forall x \in S)$, i.e. $\varphi_\alpha = \bigwedge \varphi_{a_i}$.

Remarks. 1. For formal languages the same result is valid: for arbitrary $T \subseteq X^*$ we have $D_T = \{x \in X^* \mid Tx \subseteq A\} = \{x \in X^* \mid tx \in A\ (\forall t \in T)\} = \bigcap_{t \in T} \{x \in X^* \mid tx \in A\} = \bigcap_{t \in T} D_t(A)$, where $t \in T$ is a word over X, that is an atom in the power set $(P(X^*), \cap, \cup)$.

2. By the definition of the derivation of a formal language $A \in P(X^*)$ with respect to a word $a = a_1 \dots a_n$ of length $n > 1$ (see [3]), D_a is decomposed into derivations with respect to words of length 1, i.e. with respect to single letters $a_i \in X$, by the following rule:

$$D_{a_1 a_2}(A) = D_{a_2}[D_{a_1}(A)], \quad D_{a_1 \dots a_n}(A) = D_{a_n}[D_{a_1 \dots a_{n-1}}(A)] \quad (\forall A \in P(X^*)),$$

where $a_1, \dots, a_n$ are single letters in X. Therefore D_a is the composite of the D_{a_i} $(i = 1, \dots, n)$, that is

$$D_a = D_{a_1 \dots a_n} = D_{a_n} \circ D_{a_{n-1}} \circ \dots \circ D_{a_1}.$$

We prove the same result for derivations on certain l-semigroups S with identity e. We use the concept of *irreducible elements* (see [5]): we call an element $a \in S$ irreducible if $a = bc$ $(b, c \in S)$ implies $b = a$, $c = e$ or $b = e$, $c = a$. In $(P(X^*), \cap, \cup, \cdot)$ the irreducible elements are exactly the letters of X, i.e. the words of length 1, and every word $w \in X^*$ is a product of (irreducible) letters of X.

Lemma 2.2. *Let S be an l-semigroup with identity e and with right quotients, such that every atom $(\neq e)$ is a finite product of irreducible atoms. Then every φ_a $(a \in S$ is an atom$)$ can be splitted up into derivations with respect to irreducible atoms of S.*

Proof. If $a = a_1 \dots a_n$, and $a_i \in S$ are irreducible atoms, then we have by [5], p. 155

$$x : a = x : (a_1 \dots a_n) = (x : a_1) : (a_2 \dots a_n) = (((x : a_1) : a_2) : \dots : a_n),$$

so that

$$\varphi_a(x) = x : (a_1 \dots a_n) = [(\varphi_{a_1}(x) : a_2) : \dots : a_n] =$$
$$= \varphi_{a_n}(\varphi_{a_{n-1}}(\varphi_{a_{n-2}}(\dots \varphi_{a_1}(x)))),$$

i.e.

$$\varphi_a(x) = (\varphi_{a_n} \circ \varphi_{a_{n-1}} \circ \dots \circ \varphi_{a_1})(x) \quad (\forall x \in S).$$

Thus every mapping φ_α $(\alpha \in S)$ on a complete, atomic, Boolean l-semigroup with identity and $xo = o$ $(\forall x \in S)$ can be decomposed into a product and (or) an intersection of quotient mappings with respect to irreducible atoms. Therefore we can restrict our investigations to mappings φ_a with respect to irreducible atoms $a \in S$.

Properties. Let S be an l-semigroup with identity e $(\neq o$ if o exists$)$ and with right quotients and let φ_a be an arbitrary quotient mapping on S with a an (irreducible) atom of S.

2.1. If S has i and o with $ox = o$ $(\forall x \in S)$, then $\varphi_0(x) = i$ $(\forall x \in S)$. $(\varphi_0(x) = x : o = i$ by definition.)

2.2. If $\alpha \in S$ with $\alpha \geqq e$, then $\varphi_\alpha(x) \leqq x$ $(\forall x \in S)$. (In fact, $\varphi_\alpha(x) = x : \alpha$ and $\alpha(x : \alpha) \leqq x$; but $\alpha \geqq e$ implies $\alpha(x : \alpha) \geqq x : \alpha$, thus $x : \alpha \leqq x$ for all $x \in S$.)

2.3. If S has i and $ab=o$ implies $b=o$, then $\varphi_a(o)=o$, $\varphi_a(i)=i$. (By definition $\varphi_a(o)=o:a$ and $a(o:a)\leqq o$ implies $a(o:a)=o$; since a is an atom ($\neq o$) it follows $o:a=o$; moreover $\varphi_a(i)=i:a=i$ since $ax\leqq i$ for all $x\in S$.)

2.4. If $e\in S$ is an irreducible atom and if $ab=o$ implies $b=o$, then $\varphi_a(e)=o$ or e. ($\varphi_a(e)=e:a$ and $a(e:a)\leqq e$; if $a(e:a)=o$, then $e:a=o$; if $a(e:a)=e$, then $a=e$, $e:a=e$.)

2.5. $\varphi_e(x)=x$ for all $x\in S$, i.e. φ_e is the identity mapping.

2.6. If $ab=a$ implies that b is an atom, then $\varphi_a(a)=e$. ($\varphi_a(a)=a:a$ and $a(a:a)\leqq a$; since $ae\leqq a$, by definition $e\leqq a:a$; but $e<a:a$ would imply $a\leqq a(a:a)\leqq a$ and $a(a:a)=a$, so that $a:a$ would be an atom of S, which is impossible for $e<a:a$.)

Lemma 2.3. *Let S be an l-semigroup with identity e and with right quotients, such that $ax=a$ is an atom only if x is an atom. Then $\varphi_a=\varphi_b$ iff $a=b$ (i.e. quotient mappings with respect to different atoms are distinct).*

Proof. Let $\varphi_a=\varphi_b$, i.e. $x:a=x:b$ for all $x\in S$. For $x=a$ we obtain $e=a:a=$ $=a:b$ by Property 2.6; consequently, $b=be=b(a:b)\leqq a$. Similarly, for $x=b$ we get $a\leqq b$, so that $a=b$.

Lemma 2.4. *Let S be an l-semigroup with right quotients such that* 1) *$ax=o$ implies $x=o$,* 2) *$ab=c$ is an atom only if b is one. Then for all atoms a, c of S with $c\neq a$ we have:*

$$\varphi_a(c)=\begin{cases} b & \textit{if} \quad c=ab \\ o & \textit{otherwise.}\end{cases}$$

Proof. If $c\in S$ is an arbitrary atom of S which can be written in the form $c=ab$, then $b\in S$ is uniquely determined: let $c=ab=ad$, then $c=a(b\vee d)$ which implies that $b\vee d$ is an atom with $b\vee d\geqq b, d$; consequently $b\vee d=b=d$. From $c=ab$ we conclude $b=c:a$, since if there is an $m\in S$ with $am\leqq c$ but $m>b$, then $am\geqq ab=c$, so that $am=c=ab$ and thus $m=b$, a contradiction. Consequently; $\varphi_a(c)=b$. If $c\in S$ is not a product with $a\in S$ as a left factor, then also $a(c:a)\neq c$ since $a(c:a)\leqq c$ and $c\in S$ is an atom, we have $a(c:a)=o$, thus $c:a=o$, i.e. $\varphi_a(c)=o$.

Corollary. *Let S be a complete, atomic, Boolean l-semigroup such that $xo=o$ ($\forall x\in S$) and $ab\neq o$ for arbitrary atoms $a, b\in S$. Then for every atom $c\in S$ with $c\neq a$ we have*

$$\varphi_a(c)=\begin{cases} b & \textit{if} \quad c=ab \\ o & \textit{otherwise.}\end{cases}$$

Proof. We have to consider only the second case when $a(c:a)=o$. If $c:a\neq o$, then by Theorem III.1.5 of [1] we have $c:a=\bigvee a_i$ with $a_i\in S$ atoms. Hence $o=a(\bigvee a_i)=\bigvee(aa_i)\geqq aa_i$, thus $aa_i=o$, a contradiction.

3. Quotient mappings are derivations

In this section we shall show, that every quotient mapping φ_a with respect to an irreducible atom $a\in S$ satisfies the properties of a general derivation. Though its proof will be easy by Theorem 4.2, we give a direct proof by a series of lemmas, since the conditions imposed on the l-semigroup under consideration are often very much weaker than those of Theorem 4.2.

Theorem 3.1. *Let S be a complete, atomic, Boolean l-semigroup with identity e such that* 1) $ox=xo=o$ $(\forall x\in S)$, 2) *ab is an atom iff a, b are atoms,* 3) *every atom can be uniquely represented as a product of irreducible atoms. Then every quotient mapping φ_a with respect to an irreducible atom $a\in S$ is a derivation.*

Lemma 3.2. *Let S be a complete, atomic, Boolean l-semigroup in which $xo=o$ $(\forall x\in S)$ and the product of any two atoms is again an atom. Then every φ_a ($a\in S$ is an atom) satisfies*

$$\varphi_a\big(\bigvee_I x_i\big)=\bigvee_I \varphi_a(x_i) \quad (\forall x_i\in S \text{ and any set } I \text{ of indices}).$$

Proof. We have to show that $\big(\bigvee_I x_i\big):a=\bigvee_I (x_i:a)$. Let $x_i:a=\alpha_i$ and let $m\in S$ be such that $m>\bigvee_I \alpha_i$ but $am\leqq\bigvee_I x_i$. Since $m\neq o$, by Theorem III.1.5 of [1], $m=\bigvee_J b_j$ and $am=\bigvee_J (ab_j)\leqq\bigvee_I x_i$. But a, b_j are atoms, hence ab_j are atoms, too; therefore $ab_j\leqq am\leqq\bigvee_I x_i$ implies $ab_j\leqq x_i$ for at least one index $i\in I$ (see [1]). Denote by J_i the set of all indices $j\in J$ with $ab_j\leqq x_i$; then $\bigvee_{J_i}(ab_j)=a\big(\bigvee_{J_i} b_j\big)\leqq$ $\leqq x_i$, hence $\bigvee_{J_i} b_j\leqq x_i:a=\alpha_i$. Thus $m=\bigvee_J b_j=\bigvee_I\big(\bigvee_{J_i} b_j\big)\leqq\bigvee_I \alpha_i$, a contradiction. Since $a\big(\bigvee_I \alpha_i\big)=a\big(\bigvee_I (x_i:a)\big)=\bigvee_I \big(a(x_i:a)\big)\leqq\bigvee_I x_i$, the proof is complete.

Lemma 3.3. *Let S be an l-semigroup with right quotients; then for every φ_a ($a\in S$ is an arbitrary atom) we have $\varphi_a(x\wedge y)=\varphi_a(x)\wedge\varphi_a(y)$ for all $x, y\in S$.*

Proof. By [5], p. 155, we have $(x\wedge y):a=(x:a)\wedge(y:a)$ $(\forall a, x, y\in S)$.

Lemma 3.4. *Let S be a complete, atomic, Boolean l-semigroup, in which $xo=o$ for all $x\in S$ and the product of any two atoms is again an atom. Then every φ_a ($a\in S$ is an atom) satisfies $\varphi_a(o)=o$, $\varphi_a(i)=i$ and thus $[\varphi_a(x)]'=\varphi_a(x')$ $(\forall x\in S)$.*

Proof. We show, that $ab=o$ implies $b=o$. Suppose $b\neq o$; then by Theorem III.1.5 of [1], $b=\bigvee_I b_i$ with atoms $b_i\in S$; thus $o=ab=\bigvee_I (ab_i)$, hence $ab_i=o$, but $o\in S$ is not an atom, a contradiction. By Property 2.3 we obtain $\varphi_a(o)=o$, $\varphi_a(i)=i$ and by Lemmas 3.2 and 3.3 we get $\varphi_a(x)\vee\varphi_a(x')=\varphi_a(x\vee x')=\varphi_a(i)=i$ and $\varphi_a(x)\wedge\varphi_a(x')=\varphi_a(o)=o$. Therefore $\varphi_a(x')=[\varphi_a(x)]'$ $(\forall x\in S)$.

Lemma 3.5. *Let S be a complete, atomic, Boolean l-semigroup such that $xo=ox=o$ ($\forall x\in S$) and ab is an atom iff a, b are atoms of S; then every atom of S is left and right cancellable.*

Proof. Let $ax=ay$ for some atom $a\in S$, $x, y\in S$ arbitrary. If $x=o$, then $ax=o=ay$ and by the proof of Lemma 3.4 we get $y=o$. Let $x\neq o$ and $y\neq o$; then again $x=\bigvee_I x_i$, $y=\bigvee_J y_j$ with $x_i, y_j\in S$ atoms; thus $\bigvee_I (ax_i)=\bigvee_J (ay_j)\geqq ay_j$ ($\forall j\in J$). Hence we conclude again $ay_j\leqq ax_i$ for at least one $i\in I$ (since ay_j is an atom); but ax_i is an atom, too, so that $ay_j=ax_i$. This implies $a(x_i\vee y_j)=ax_i$ where ax_i is an atom; hence $x_i\vee y_j$ has to be an atom, which implies $x_i\vee y_j=x_i=y_j$. Consequently, every x_i ($i\in I$) is equal to at least one y_j ($j\in J$); analogously we obtain that every y_j ($j\in J$) is equal to at least one x_i ($i\in I$). Hence the sets $\{x_i|i\in I\}$ and $\{y_j|j\in J\}$ are equal and so $x=y$.

Lemma 3.6. *Let S be a complete, atomic, Boolean l-semigroup with identity e, which is an atom of S. If S satisfies the properties* 1) *a, b are atoms iff ab is an atom,* 2) *$ox=xo=o$ ($\forall x\in S$),* 3) *every atom $\neq e$ is a unique finite product of irreducible atoms, then every φ_a ($a\in S$ is an irreducible atom) satisfies*

$$\varphi_a(xy)=\begin{cases}\varphi_a(x)y\vee\varphi_a(y) & \textit{if}\quad x\geqq e\\ \varphi_a(x)y & \textit{otherwise.}\end{cases}$$

Proof. Let first $x\geqq e$; then we have to show $q=(xy):a=(x:a)y=r$ ($\forall y\in S$). By definition $ar=a(x:a)y\leqq xy$, thus $r\leqq(xy):a=q$. For the converse we prove first that $ab=cd$ with a an irreducible atom and b, c, d arbitrary atoms such that $c\neq e$ implies $ab=awd$ for an appropriate atom $w\in S$. By Condition 3), $ab=cd$ implies $ab_1\dots b_m=c_1\dots c_k d_1\dots d_n$, from which it follows that $a=c_1$ and $ab=awd$ for an atom $w\in S$ (Condition 1), which may be equal to $e\in S$. More generally, the equation $az=xy$ with an irreducible atom $a\in S$, and with $x, y, z\in S$ satisfying $x\geqq e$, $x\neq o$, $y\neq o$, $z\neq o$, implies $az=awy$ for some $w\in S$. Since $x=\bigvee_I x_i$, $y=\bigvee_J y_j$, $z=\bigvee_K z_k$, we obtain from $az=xy$ that $\bigvee_K (az_k)=(\bigvee_I x_i)(\bigvee_J y_j)=\bigvee_I(\bigvee_J x_iy_j)=\bigvee_{I\times J}(x_iy_j)\geqq x_iy_j$ for all $(i, j)\in I\times J$. Since x_iy_j is an atom, we conclude again that $x_iy_j\leqq az_k$ for at least one $k\in K$, so that $x_iy_j=az_k=aw_iy_j$ for some atoms $w_i\in S$ (since $x=\bigvee x_i\ngeqq e$ implies $x_i\ngeqq e$, $\forall i\in I$). In particular we get by Lemma 3.5 that $x_i=aw_i$ ($\forall i\in I$). Thus $az=\bigvee_{I\times J}(x_iy_j)=\bigvee_{I\times J}(aw_iy_j)=a(\bigvee_I w_i)(\bigvee_J y_j)=awy$.

In order to show $q\leqq r$, we proceed as follows. By definition we have $aq=a[(xy):a]\leqq xy$; then there exist $x_1\leqq x$, $y_1\leqq y$ such that $aq=x_1y_1$. Indeed, if $aq=o$, then $x_1=o\leqq x$, $y_1=o\leqq y$ satisfy the equation; if $aq\neq o$, then by Condition 2), $x\neq o$, $y\neq o$ and again $aq=\bigvee_L a_l\leqq\bigvee_{I\times J}(x_iy_j)$ implies $a_l\leqq x_iy_j$ for at least one $(i,j)\in I\times J$ and every $l\in L$; hence $a_l=x_iy_j$ and $aq=\bigvee_L a_l=\bigvee_{I\times J} x_iy_j=$

$=(\bigvee_I x_i)(\bigvee_J y_j)=x_1 y_1$ with $x_1 \leqq x$, $y_1 \leqq y$; in particular $x_1 \neq o$, $y_1 \neq o$, since otherwise $aq=o$ and therefore by the proof of Lemma 3.4, $q=o\leqq r$ holds trivially. Consequently $aq=x_1 y_1$ implies $aq=awy_1$ (since $x_1 \nleqq e$ for $x \nleqq e$ and $x \geqq x_1$); by Lemma 3.5 we get $q=wy_1$ and $aw=a(\bigvee_I w_i)=(\bigvee_I aw_i)=\bigvee_I x_i=x$, thus by definition $w\leqq x:a$. Hence we conclude that $q=wy_1\leqq wy\leqq (x:a)y=r$, i.e. $q\leqq r$.

Finally let $x\geqq e$; then $x\neq o$ because e is an atom. We have to show $\varphi_a(xy)=$ $=\varphi_a(x)y\vee\varphi_a(y)$ for all $y\in S$. We have $x=\bar{x}\vee e$ with $\bar{x}\nleqq e$, since $x\neq o$ implies $x=\bigvee_I x_i$ with some atoms $x_i\in S$, hence $e\leqq\bigvee_I x_i$ and $e\leqq x_i$, i.e. $e=x_i$ for at least one $i\in I$; if J denotes the set of all $i\in I$ such that $x_i\neq e$, then $\bar{x}=\bigvee_J x_i\nleqq e$ (otherwise $x_j=e$ for at least one $j\in J$) and $x=(\bigvee_J x_i)\vee(\bigvee_{I-J} x_i)=\bar{x}\vee e$. Applying Lemma 3.2 we get

$$\varphi_a(xy) = \varphi_a[(\bar{x}\vee e)y] = \varphi_a(\bar{x}y)\vee\varphi_a(y) = \varphi_a(\bar{x})y\vee\varphi_a(y) \quad (\forall y\in S),$$
$$\varphi_a(x) = \varphi_a(\bar{x}\vee e) = \varphi_a(\bar{x})\vee\varphi_a(e).$$

We have to distinguish two cases: (i) e is not a product with a as a left factor: (ii) $e=ab$ with some $b\in S$.

In case (i), $\varphi_a(e)=o$ for $a\neq e$, since — applying the first part of the proof of Lemma 3.4 — Condition 1 of Lemma 2.4 is satisfied, and $a=e$ gives $\varphi_a=\mathrm{id}$ (Property 2.5), which is a derivation; consequently $\varphi_a(x)=\varphi_a(\bar{x})$. In case (ii), by Lemma 2.4, $\varphi_a(e)=b$ (if $a=e$, then $\varphi_a=\mathrm{id}$); but $\varphi_a(e)y=\varphi_a(y)$ $(\forall y\in S)$, since $e=ab$ implies $y=aby$ for all $y\in S$; if $y=o$, then $\varphi_a(e)o=\varphi_a(o)=o$ by Property 2.3 and Lemma 3.4 (for $y=a$ we get by Property 2.6 that $\varphi_a(a)=e$ and $\varphi_a(e)a=$ $=ba=e$, since $a=aba$ implies $e=ba$ by Lemma 3.5); if $y\neq o$, then $y=\bigvee_J y_j$ with certain atoms $y_j\in S$ and thus by Lemma 3.2 $\varphi_a(y)=\bigvee_J\varphi_a(y_j)=\bigvee_J[\varphi_a(e)y_j]=$ $=\varphi_a(e)(\bigvee_J y_j)=\varphi_a(e)y$ $(\forall y\in S)$. Consequently, in this case $\varphi_a(x)=\varphi_a(\bar{x})\vee\varphi_a(e)$ implies

$$\varphi_a(x)y\vee\varphi_a(y) = \varphi_a(\bar{x})y\vee\varphi_a(e)y\vee\varphi_a(y) = \varphi_a(\bar{x})y\vee\varphi_a(y) = \varphi_a(xy)$$

for every $y\in S$ and the proof is complete.

4. The converse

The next theorem establishes the converse of Theorem 3.1, that is that every derivation on S is a quotient mapping of S. We note that any quotient mapping φ_a of S with respect to an atom $a\in S$ such that $ab=a$ implies that b is an atom has a further property: there exists $b\in S$ with $\varphi_a(b)=e$, for example $\varphi_a(a)=e$ by Property 2.6. Supposing this as a supplementary property of a derivation we get

Theorem 4.1. *Let S be a complete, atomic, Boolean l-semigroup with identity e, in which following conditions are satisfied: e is an atom; $ox=xo=o$ for all $x\in S$;*

a and b are atoms iff ab is an atom. Then every transformation $\varphi : S \to S$ satisfying the following four properties is a quotient mapping with respect to a uniquely determined atom:

1) $\varphi(\bigvee_I x_i) = \bigvee_I \varphi(x_I) \quad (\forall x_i \in S);$
2) $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y) \quad (\forall x, y \in S);$
3) $\varphi(xy) = \begin{cases} \varphi(x) y \vee \varphi(y) & (\forall x \geqq e,\ y \in S) \\ \varphi(x) y & (\forall x \not\geqq e,\ y \in S); \end{cases}$
4) $\varphi(x) = e$ *for at least one* $x \in S$.

Proof. By 4) there is an $x \in S$ with $\varphi(x) = e$; $x \neq o$, since $\varphi(o) = o \neq e$ by Property 1.1. If $x = e$, then $\varphi(e) = e$ implies by Property 1.7 that $\varphi(x) = x$ for all $x \in S$; but by Property 2.5, $\varphi_e(x) = x$ $(\forall x \in S)$, hence $\varphi = \varphi_e$ (φ_e is uniquely determined by Lemma 2.3).

Now, if $x \neq o, x \neq e$, then $x = \bigvee_I x_i$, where x_i are atoms of S (by Theorem III.1.5 of [1]); hence by 1), $\varphi(x) = \bigvee_I \varphi(x_i) = e$ and $\varphi(x_i) \leqq e$ $(i \in I)$. If $\varphi(x_i) = o$ $(\forall i \in I)$, then $\varphi(x) = o = e$, a contradiction; consequently $\varphi(x_i) = e$ for at least one $i \in I$; let $a = x_i$ be that atom (it is determined uniquely, since $\varphi(b) = e$ for an other atom $b \in S$ implies $e = \varphi(a \wedge b) = \varphi(o) = o$, a contradiction). Thus we have $\varphi(a) = e$, $\varphi(ax) = \varphi(a)x = ex = x$ $(\forall x \in S)$.

Next we show for every atom $c \in S$ that $\varphi(c) = c_1$ iff $c = ac_1$ (c_1 is an atom) and $\varphi(c) = o$ in the other cases. If $\varphi(c) \neq o$, then $\varphi(c) = \bigvee_J c_j$ for some atoms $c_j \in S$; thus $\varphi(c) \wedge c_j = c_j$ $(\forall j \in J)$. Since $\varphi(ac_j) = c_j$ $(\forall j \in J)$, too, we get by 2) that $\varphi(c \wedge ac_j) = c_j$ $(\forall j \in J)$. If $c \neq ac_j$, then $c \wedge ac_j = o$ and $o = \varphi(o) = c_j$, a contrandiction; hence $c = ac_j$ and $\varphi(c) = \varphi(ac_j) = c_j$. Conversely let $\varphi(c) = c_1$ with an atom c_1 of S; then $\varphi(ac_1) = c_1$ implies $\varphi(c \wedge ac_1) = c_1$; but $c \neq ac_i$ would imply $o = \varphi(o) = c_1$, which is impossible; hence $c = ac_1$.

Thus we can conclude by Lemma 2.4, that $\varphi(c) = \varphi_a(c)$ for all atoms $c \in S$ (if $c = a$, then $\varphi(a) = e$ and also $\varphi_a(a) = e$ by Property 2.6). Consequently, for $x = o$ we get $\varphi(o) = o$ and also $\varphi_a(o) = o$ by Property 1.1 and Property 2.3; for $x \neq o$ we have $x = \bigvee_I x_i$ with some atoms $x_i \in S$, and by Lemma 3.2, $\varphi(x) = \bigvee_I \varphi(x_i) = \bigvee_I \varphi_a(x_i) = \varphi_a(\bigvee_I x_i) = \varphi_a(x)$ $(\forall x \in S)$, i.e. $\varphi = \varphi_a$.

Theorems 3.1 and 4.1 state that on certain l-semigroups the derivations are exactly the quotient mappings, like in $(P(X^*), \cap, \cup, \cdot)$. Thus the three (four) defining properties of a derivation are characteristic for the concept of the derivative of formal languages. Consequently, it is impossible to find other mappings on the l-semigroup $(P(X^*), \cap, \cup, \cdot)$ having the essential properties of derivations, that the quotient mappings. The equivalence of these two concepts becomes still more apparent by the following

Theorem 4.2. *An l-semigroup S is isomorphic to the l-semigroup $P(X^*)$ of all formal languages on an alphabet X iff* 1) *S is a complete, atomic, Boolean l-semigroup;* 2) *ab is an atom iff a, b are atoms;* 3) *every atom is a unique finite product of irreducible atoms;* 4) *$ox=xo=o$ for all $x\in S$.*

Proof. By [10] the complete, atomic, Boolean lattice $(S, \wedge, \vee)$ is isomorphic to the power set $P(M)$ of the set M of all atoms of S. A lattice isomorphism is given by $\varphi(x)=M_x$ $(\forall x\in S)$ with $M_x=\{a\in S | a$ is an atom with $a\leqq x\}$. It is easy to see that φ is also a semigroup homomorphism from $(S, \cdot)$ to $(P(M), \cdot)$. Indeed, if $z\in M_x M_y=\{\alpha\beta | \alpha, \beta$ are atoms with $\alpha\leqq x$, $\beta\leqq y\}$ then $z=\alpha\beta\leqq xy$ and z is an atom (by 2)), so that $z\in M_{xy}$. Conversely, if $z\in M_{xy}$ then $z\leqq xy$ and z is an atom, which implies that $z=x_1 y_1$ with $x_1\leqq x$, $y_1\leqq y$ (similarly to the proof of Lemma 3.6); therefore x_1, y_1 are atoms (by 2)), so that $z\in M_x M_y$. The cases $x=o$ resp. $y=o$ imply $M_x=\emptyset$ resp. $M_y=\emptyset$, thus $M_x M_y=\emptyset$, and $xy=o$ (by 4)), thus $M_{xy}=\emptyset$, too. Consequently, $\varphi(xy)=M_{xy}=M_x M_y=\varphi(x)\varphi(y)$ for all $x, y\in S$.

Finally let X be the set of all irreducible atoms of S (see 3)). Then $M=X^*$, the free semigroup generated by X. In fact $a\in X^*$ implies $a=a_1\dots a_n$, $a_i\in X$, so that a is an atom of S (by 2)) and $a\in M$. Conversely $b\in M$ implies $b=b_1\dots b_k$ where the b_j are irreducible atoms of S (by 3)), i.e. $b_j\in X$ and therefore $b\in X^*$. Thus S is isomorphic to $P(M)=P(X^*)$. The converse is trivial.

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Korrektur und Ergänzung zu meiner Arbeit „Über Schreiersche Gruppenerweiterungen und ihre Kommutatorgruppen"

R. QUINKERT

Wir zeigen, daß entgegen einem fehlerhaften Beispiel in § 4 von [1] die im Zusammenhang mit Satz 4.2 in [1] auftretenden Bedingungen IV* und IV** (bei Gültigkeit einer ohnehin benötigten Bedingung I*) äquivalent sind. Da IV* eine Abschwächung von IV** ist. ergibt sich daraus eine Verschärfung des Satzes 4.2.

Es sei M eine beliebige Gruppe und $\mathscr{G}=\times\mathscr{G}_\lambda$ $(\lambda\in\Lambda)$ ein (diskretes) direktes Produkt zyklischer Gruppen $\mathscr{G}_\lambda=\langle X_\lambda\rangle$ der Charakteristik n_λ. Dann hat nach [1], § 4 jede Schreiersche Erweiterung G von M mit $\mathscr{G}$ ein Parametersystem mit folgenden Bestimmungsstücken:

a) Die Automorphismen $\mathscr{A}_\lambda$ und, für $n_\lambda\neq 0$, die Faktoren $[X_\lambda, X_\lambda^{n_\lambda-1}]=v_\lambda\in M$ der in G enthaltenen Schreierschen Erweiterungen von M mit $\mathscr{G}_\lambda$, $\lambda\in\Lambda$. Sie erfüllen

$$(*)\qquad v_\lambda^{\mathscr{A}_\lambda}=v_\lambda \quad \text{und} \quad \mathscr{A}_\lambda^{n_\lambda}=\mathscr{I}(v_\lambda) \quad \text{mit} \quad \alpha\to\alpha^{\mathscr{I}(v_\lambda)}=v_\lambda^{-1}\alpha v_\lambda.$$

b) Bezüglich einer willkürlich gewählten linearen Ordnung $<$ von Λ für jedes Paar $i<j$ aus $\Lambda\times\Lambda$ ein Element $[X_j, X_i]=\gamma_{ji}\in M$.

Diese Elemente genügen für alle $\sigma, \tau, \varrho\in\{1, -1\}$ mit $\sigma^*=-1$ für $\sigma=-1$, sonst $\sigma^*=0$ $(\tau^*, \varrho^*$ entsprechend) den Gleichungen

$$\mathrm{I}^{**}\qquad \mathscr{A}_j^\tau\mathscr{A}_i^\sigma=\mathscr{A}_i^\sigma\mathscr{A}_j^\tau\mathscr{I}\big(\gamma_{ji}^{\tau\sigma\mathscr{A}_j^{\tau^*}\mathscr{A}_i^{\sigma^*}}\big),\quad i<j,$$

weiteren Gleichungen II** und III** für alle $i<j$ und, falls $|\Lambda|\geqq 3$, noch

$$\mathrm{IV}^{**}\qquad \gamma_{kj}^{\varrho\tau\mathscr{A}_k^{\varrho^*}\mathscr{A}_j^{\tau^*}}\gamma_{ki}^{\varrho\sigma\mathscr{A}_k^{\varrho^*}\mathscr{A}_i^{\sigma^*}\mathscr{A}_j^{\tau}}\gamma_{ji}^{\tau\sigma\mathscr{A}_j^{\tau^*}\mathscr{A}_i^{\sigma^*}}=\gamma_{ji}^{\tau\sigma\mathscr{A}_j^{\tau^*}\mathscr{A}_i^{\sigma^*}\mathscr{A}_k^{\varrho}}\gamma_{ki}^{\varrho\sigma\mathscr{A}_k^{\varrho^*}\mathscr{A}_i^{\sigma^*}}\gamma_{kj}^{\varrho\tau\mathscr{A}_k^{\varrho^*}\mathscr{A}_j^{\tau^*}\mathscr{A}_i^{\sigma}},$$
$$i<j<k.$$

Dabei tritt z. B. $\sigma=-1$ und damit $\sigma^*=-1$ nur auf, wenn $\mathscr{G}_i$ unendliche Ordnung hat. Für eine Erweiterung von M mit einem direkten Produkt $\mathscr{G}=\times\mathscr{G}_\lambda$ endlicher zyklischer Gruppen vereinfachen sich also I**—IV** erheblich, nämlich zu den aus ihnen für $\sigma=\tau=\varrho=1$ entstehenden Gleichungen I*—IV*.

Für den allgemeinen Fall zeigten wir im Beweis von Satz 4.2, daß auch *umgekehrt zu jedem System von Automorphismen $\mathscr{A}_\lambda$ von M, von Elementen $v_\lambda\in M$ für $n_\lambda\neq 0$ und von Elementen $\gamma_{ji}\in M$ $(\lambda, i, j\in\Lambda, i<j)$, welche den Bedingungen* $(*)$ *und* I**—IV** *genügen, eine Schreiersche Erweiterung von M mit $\mathscr{G}=\times\mathscr{G}_\lambda$ gehört.* Andererseits

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zeigten wir die Äquivalenzen I*⇔I**, II*⇔II** und III*⇔III** (die letzten beiden falls auch I* bzw. I** erfüllt ist) und gaben ein Beispiel, für welches $(*)$ und I*—IV*, aber nicht IV** erfüllt sein sollte. Daraus ergab sich unsere Formulierung der Kennzeichnung der Erweiterungen von M mit $\mathscr{G}=\times\mathscr{G}_\lambda$ in Satz 4.2 mit den Bedingungen $(*)$, I*—III* und IV**.

Dieses Beispiel verletzt jedoch, wie wir übersehen haben, auch die Bedingung I*, und in der Tat gilt nach der folgenden Proposition, daß bereits I* und IV* stets auch IV** implizieren. Damit läßt sich Satz 4.2 von [1] einfacher mit $(*)$, I*—IV* formulieren. In dieser Form entspricht er nach den in [1] durchgeführten Vergleichen (Seite 344 einschließlich i)) dem von O. Schreier in [2] angegebenen, jedoch nur für endliche $\mathscr{G}_\lambda$ bewiesenen Satz III. Die in [1] folgende Behauptung ii), daß im allgemeinen Fall eine Verschärfung der Bedingungen gemäß $(*)$, I*—III* und IV** erforderlich wäre, ist also falsch.

Proposition. *Automorphismen $\mathscr{A}_\lambda$ und Elemente γ_{ji} von M $(\lambda, i, j \in \Lambda,\ i<j)$, welche den Gleichungen* I* *und* IV* *genügen, erfüllen auch die Gleichungen* IV**.

Beweis. Die Anwendung von $\mathscr{A}_k^{-1}$ auf IV* ergibt

$$\gamma_{kj}^{\mathscr{A}_k^{-1}}\gamma_{ki}^{\mathscr{A}_j\mathscr{A}_k^{-1}}\gamma_{ji}^{\mathscr{A}_k^{-1}}=\gamma_{ji}\gamma_{ki}^{\mathscr{A}_k^{-1}}\gamma_{kj}^{\mathscr{A}_i\mathscr{A}_k^{-1}},\quad i<j<k.$$

Unter Verwendung der mit I* gleichwertigen Formel I** können wir hier

$$\mathscr{A}_j\mathscr{A}_k^{-1}=\mathscr{A}_k^{-1}\mathscr{A}_j\mathscr{I}(\gamma_{kj}^{\mathscr{A}_k^{-1}})\quad\text{und}\quad\mathscr{A}_i\mathscr{A}_k^{-1}=\mathscr{A}_k^{-1}\mathscr{A}_i\mathscr{I}(\gamma_{ki}^{\mathscr{A}_k^{-1}})$$

einsetzen und erhalten

$$\gamma_{kj}^{\mathscr{A}_k^{-1}}\gamma_{ki}^{\mathscr{A}_k^{-1}\mathscr{A}_j\mathscr{I}(\gamma_{kj}^{\mathscr{A}_k^{-1}})}\gamma_{ji}^{\mathscr{A}_k^{-1}}=\gamma_{ji}\gamma_{ki}^{\mathscr{A}_k^{-1}}\gamma_{kj}^{\mathscr{A}_k^{-1}\mathscr{A}_i\mathscr{I}(\gamma_{ki}^{\mathscr{A}_k^{-1}})}.$$

Das ergibt

$$\gamma_{ki}^{\mathscr{A}_k^{-1}\mathscr{A}_j}\gamma_{kj}^{\mathscr{A}_k^{-1}}\gamma_{ji}^{\mathscr{A}_k^{-1}}=\gamma_{ji}\gamma_{kj}^{\mathscr{A}_k^{-1}\mathscr{A}_i}\gamma_{ki}^{\mathscr{A}_k^{-1}},$$

woraus schließlich folgt

$$\gamma_{kj}^{-1\mathscr{A}_k^{-1}}\gamma_{ki}^{-1\mathscr{A}_k^{-1}\mathscr{A}_j}\gamma_{ji}=\gamma_{ji}^{\mathscr{A}_k^{-1}}\gamma_{ki}^{-1\mathscr{A}_k^{-1}}\gamma_{kj}^{-1\mathscr{A}_k^{-1}\mathscr{A}_i},$$

also IV** für $\varrho=-1$, $\sigma=\tau=1$. Auf die damit für $\varrho\in\{1,-1\}$, $\sigma=\tau=1$ gezeigte Formel IV** wendet man nun $\mathscr{A}_j^{-1}$ an und erhält nach ähnlicher Umformung IV** für $\varrho\in\{1,-1\}$, $\tau=-1$, $\sigma=1$. Damit gilt IV** für $\varrho,\tau\in\{1,-1\}$, $\sigma=1$, worauf man in einem dritten Schritt $\mathscr{A}_i^{-1}$ anzuwenden hat.

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A characterization of compactness

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The aim of this paper is to prove that if X is a compact Hausdorff space and $f\colon X\to f(X)$ is a continuous function, then $f(\operatorname{Lim\,sup} A_n)=\operatorname{Lim\,sup} f(A_n)$ for each net A_n of subsets of X (recall that for a net A_n $(n\in D)$ of subsets of a Hausdorff topological space X, $\operatorname{Lim\,sup} A_n$ is the set of all points $x\in X$, for which the set $\{n,\ A_n\cap U\neq\emptyset\}$ is cofinal in D for every neighbourhood U of X). This condition is used to obtain a characterization of compact spaces.

From now on, a *space* always means a Hausdorff topological space. The reader is referred to [2] for general results concerning nets of subsets of a space.

DUDA [1], p. 23, has proved the following fact: if X is a compact metric space and a function $f\colon X\to f(X)$ is continuous, then $f(\operatorname{Lim\,sup} A_n)=\operatorname{Lim\,sup} f(A_n)$ for each sequence $A_1, A_2, \ldots$ of subsets of X. We prove much stronger results.

Theorem 1. *A function $f\colon X\to f(X)$ is continuous if and only if $f(\operatorname{Lim\,sup} A_n)\subset \subset\operatorname{Lim\,sup} f(A_n)$ for each net $\{A_n,\ n\in D\}$ of subsets of X.*

Proof. Putting $A_n=A$ we obtain $f(\operatorname{cl} A)=f(\operatorname{Lim\,sup} A_n)\subset\operatorname{Lim\,sup} f(A_n)= \operatorname{cl} f(A)$, i.e., the continuity of f (because $A_n=A$ implies $\operatorname{Lim\,sup} A_n=\operatorname{cl} A$). Conversely, take a net $\{A_n, n\in D\}$ and a point $y\in f(\operatorname{Lim\,sup} A_n)$. There is a point $x\in\operatorname{Lim\,sup} A_n$ such that $f(x)=y$. By [2], Proposition 2.2, p. 170, there exist a net $\{x_i,\ i\in E\}$ of points of X and a function $p\colon E\to D$ such that the following conditions hold:

(*) for each element $n\in D$ there is an element $i_0\in E$ such that for each element $i\in E$ satisfying $i_0\leqq i$ we have $p(i)\geqq n$, and

$$x_i\in A_{p(i)} \quad \text{und} \quad x=\lim x_i.$$

The function f is continuous, hence $y=f(x)=f(\lim x_i)=\lim f(x_i)$, and therefore $y\in\operatorname{Lim\,sup} f(A_n)$ ([2], ib.).

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Theorem 2. *If a space X is compact and a function f: $X \to f(X)$ is continuous, then* $f(\operatorname{Lim\,sup} A_n) = \operatorname{Lim\,sup} f(A_n)$ *for each net* $\{A_n, n \in D\}$ *of subsets of* X.

Proof. It is sufficient to prove that $\operatorname{Lim\,sup} f(A_n) \subset f(\operatorname{Lim\,sup} A_n)$. For, take a point $y \in \operatorname{Lim\,sup} f(A_n)$. By [2], ib., there exist a net $\{y_i, i \in E\}$ of points of $f(X)$ and a function p: $E \to D$ satisfying $(*)$ and the following condition:

$$y_i \in f(A_{p(i)}) \quad \text{and} \quad y = \lim y_i.$$

There are points $x_i \in A_{p(i)}$ such that $f(x_i) = y_i$. Since the space is compact, the net $\{x_i, i \in E\}$ has a convergent subnet, say $\{x_k, k \in F\}$. With $x = \lim x_k$ we obtain $x \in \operatorname{Lim\,sup} A_n$. Since the function is continuous we infer that the net $\{f(x_k), k \in F\}$ converges to $f(x) = y$. This implies that $y \in f(\operatorname{Lim\,sup} A_n)$.

Te following theorem gives a characterization of compact spaces.

Theorem 3. *A space X is compact if and only if there is a continuous function f: $X \to Y$ onto a compact space Y such that* $f(\operatorname{Lim\,sup} A_n) = \operatorname{Lim\,sup} f(A_n)$ *for each net* $\{A_n, n \in D\}$ *of subsets of* X.

Proof. Consider a net $\{x_n, n \in D\}$ of points of X. Since the space Y is compact, the set $\operatorname{Lim\,sup} f(\{x_n\})$ is non-void. It follows that the set $\operatorname{Lim\,sup} \{x_n\}$ is non-void, and consequently, the net $\{x_n, n \in D\}$ has a convergent subnet [2], ib. The converse implication follows from Theorem 2.

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C_p-minimal positive approximants

DONALD D. ROGERS and JOSEPH D. WARD

§ 1. Introduction

In [8], P. R. HALMOS initiated the study of positive operator approximation. Among other things he established the proximinality of the convex set of positive operators on Hilbert space by producing a canonical best positive approximant. This approximant, hereafter referred to as the Halmos approximant was later shown by R. H. BOULDIN [2] to be maximal, in the sense of order, among all positive approximants to a given operator.

This paper originated in the attempt to find a canonical minimal approximant since canonical approximants shed much light on the structure of the set of best approximants [3], [4], [5]. As will be shown in 4, there need not be a positive approximant minimal in the sense of order. Nevertheless, we construct a positive approximant P_m that is minimal in a sense given by the following theorem, in which $\|\cdot\|_p$ denotes the usual C_p norm on finite matrices.

Theorem 1.1. *Each operator $A=B+iC$ on a finite dimensional complex Hilbert space $\mathfrak{H}$ has a positive approximant P_m such that $A-P_m$ is a normal operator and such that for each positive operator $Q\neq P_m$ it follows that $\|A-Q\|_p>\|A-P_m\|_p$ for all finite p sufficiently large. This operator P_m will be referred to as the C_p-minimal positive approximant of A.*

In section 2, relevant background information is given along with needed notation. Section 3 contains the proof of the main theorem, the heart of which involves an inductive construction. There are many open questions related to our result, and these questions along with some examples comprise section 4.

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§ 2. Preliminaries

The term operator shall mean a bounded linear operator on a complex Hilbert space, and the operator norm of an operator X is denoted by $\|X\|=\sup\{\|Xf\|: f\in\mathfrak{H}, \|f\|=1\}$. If $\mathfrak{M}$ is a set of operators, then an operator $Y_0\in\mathfrak{M}$ is an $\mathfrak{M}$-approximant of X if $\|X-Y_0\|=\inf\{\|X-Y\|: Y\in\mathfrak{M}\}$; approximants using other norms are defined similarly. We shall follow Halmos's convention of using "positive operator" as synonymous with "nonnegative operator" and "approximant" in place of "best approximant". For the reader's convenience we restate the following results proved by Halmos in [8].

Theorem 2.1. *If $B+iC$ is the usual Cartesian representation for the operator A, then*

$$\inf\{A-P: P\geqq 0\}=\inf\{r: r\geqq\|C\|,\ B+(r^2-C^2)^{1/2}\geqq 0\}.$$

The first infimum shall henceforth be denoted $\delta(A)$.

Theorem 2.2. *If $B+iC$ is the usual Cartesian representation for the operator A and if $P_H=B+((\delta(A))^2-C^2)^{1/2}$, then P_H is a positive approximant of A.*

The operator P_H is the Halmos approximant referred to in the introduction.

Theorem 2.3. *Any operator A has a representation of the form $P+U\delta(A)$ where $P\geqq 0$ and U is unitary with negative real part. If A is not a positive operator, then the above representation is unique.*

In another direction, the notion of a *strict approximant* was introduced by J. R. Rice [10] in the course of his investigations into l_∞ approximation as a method of selecting one approximant among many. A full discussion of strict approximants would lead us too far astray but, roughly speaking, to find a strict approximant one minimizes as much as one can. The following example will serve to illustrate.

Example. Consider the vector $v\equiv(2i, i, 0)$ viewed as an element of $l_\infty(3)$. The distance of v to the set of positive functions is 2, and there are clearly an infinite number of positive approximants. The vector $(0, 0, 0)$, however, is the unique strict approximant since 0 is the nearest nonnegative number to $2i$, i and 0.

It was later shown by B. Mitiagin [9] and J. Descloux [6] that the strict approximants have an additional approximation property.

Theorem 2.4. *Let $l_p(n)$ denote n-dimensional complex Cartesian space endowed with the l_p norm and M a subspace of $l_p(n)$. If $x\in l_p(n)\setminus M$, let y_p denote an approximant from M. Then $y=\lim_{p\to\infty} y_p$ exists, and y is the strict approximant of x in $l_\infty(n)$.*

The construction in the next section is modelled after the construction of the strict approximant, although the fact that the space of $n\times n$ matrices is not a commutative algebra introduces some new twists into the construction.

§ 3. The Main Result

In this section the proof of Theorem 1.1 is given. The first lemma is stated in more generality than is needed, but it seems of interest in its own right.

Lemma 3.1. *Let $\mathfrak{C}$ be a norm-closed convex set of compact operators on a uniformly convex Banach space $\mathfrak{B}\neq 0$. Define $d=\inf\{\|X\|:\ X\in\mathfrak{C}\}$ and*

$$\mathfrak{D}=\{X\in\mathfrak{C}:\ \|X\|=d\}.$$

If $\mathfrak{D}$ is separable, then there exist unit vectors $y, z\in\mathfrak{B}$ such that for every $X\in\mathfrak{D}$ it follows that $Xy=dz$. In particular, if D is in $\mathfrak{D}$, then

$$\bigcap_{x \text{ in } \mathfrak{D}} \ker(X-D)\neq\{0\}.$$

Proof. Let $\{X_1, X_2, \ldots\}$ be dense in $\mathfrak{D}$; define operators $Y_n\in\mathfrak{D}$ to be the corresponding Cesaro means, i.e. $Y_n=(X_1+\ldots+X_n)/n$. Because each Y_n is a compact operator, there exists a unit vector $y_n\in\mathfrak{B}$ such that $\|Y_n y_n\|=d$; define the unit vector $z_n=Y_n y_n/d$. Since $\mathfrak{B}$ is reflexive, the sequences $\{y_n\}$ and $\{z_n\}$ have weak cluster points in the unit ball of $\mathfrak{B}$. Thus it is possible to find vectors y, z and subsequences $\{y_{n,j}\}$ and $\{z_{n,j}\}$ that converge weakly to y and to z. Fix $k\geqq 1$. Because X_k is compact it follows that $X_k(y_{n,j})$ converges to $X_k(y)$ in norm, as $j\to\infty$. But $X_k(y_{n,j})=dz_{n,j}$ for all j sufficiently large, by the definition of $Y_{n,j}$ and the fact that $\mathfrak{B}$ is uniformly convex. Thus $dz_{n,j}$ converges to $X_k(y)$ in norm. Hence $\|X_k(y)\|=d$, which implies $\|y\|=1$ since $\|X_k\|=d$. Also, $X_k y=dz$ since $dz_{n,j}$ converges weakly to dz; thus $\|z\|=1$. Since $\{X_k\}$ is dense in $\mathfrak{D}$, it follows that $Xy=dz$ for each $X\in\mathfrak{D}$.

The next lemma is crucial in what follows. It is a slight generalization of a lemma appearing in [2].

Lemma 3.2. *If $X=X^*$, $Y=Y^*$, $P=P^*$, and $d=\|X+iY-P\|$, then $P\leqq X+\sqrt{d^2I-Y^2}$.*

Proof. As in [1], [8] it follows that $(P-X)^2+Y^2\leqq d^2I$. Because the square root function is order-preserving, it follows that $P-X\leqq\sqrt{(P-X)^2}\leqq\sqrt{d^2I-Y^2}$.

Proof of Theorem 1.1. We proceed with constructing the operator P_m by defining numbers $\{\delta_k\}$ and subspaces $\{M_k\}$ that reduce C. If $C(k)$ denotes the part of C on M_k and $I(k)$ denotes the orthogonal projection from H onto M_k, then $P_m=B+\Sigma\sqrt{\delta_k^2 I(k)-C^2(k)}$. The construction of the sequences $\{\delta_k\}$ and $\{M_k\}$ is by induction.

Define $\delta_1=\delta(A)$ (recall the definition immediately following Theorem 2.1) and $M_1=\bigcap_Q \ker(B+\sqrt{\delta_1^2-C^2}-Q)$ where this intersection is taken over all positive approximants Q. Lemma 3.1 can be applied to the convex sets $\mathfrak{C}_1=\{A-P\colon P\geqq 0\}$ and $\mathfrak{D}_1=\{A-Q\colon Q$ is a positive approximant of $A\}$ using $d=\delta_1$ and $D=A-(B+\sqrt{\delta_1^2-C^2})$ to show $M_1\neq\{0\}$.

The fact that M_1 reduces C is shown in [1, proof of Lemma 4.1]; a different proof is given here. Let f be a unit vector in M_1 and let Q be a positive approximant of A. Then $(B-Q)f=-\sqrt{\delta_1^2-C^2}f$ by the definition of M_1; thus both $A-Q$ and $(A-Q)^*$ attain their norm at f. Hence $|A-Q|^2f=|(A-Q)^*|^2f$, and this implies that $(B-Q)Cf=C(B-Q)f$. Thus $(B-Q)Cf=C(B-Q)f=C(-\sqrt{\delta_1^2-C^2})f=-\sqrt{\delta_1^2-C^2}(Cf)$. Hence $(B+\sqrt{\delta_1^2-C^2}-Q)(Cf)=0$, so that $Cf\in M_1$.

Thus M_1 reduces C, and it also reduces $A-Q$ for each approximant Q. Clearly A has a unique approximant if and only if $M_1=H$. Define the subspace $H_1=H$ and the projection $E_1=I$.

Let H_1, E_1, $\mathfrak{C}_1$, δ_1, $\mathfrak{D}_1$, M_1 be as defined above. Define $H_2=H\ominus M_1$ with orthogonal projection $E_2\colon H\to H_2$. Put $\mathfrak{C}_2=\{(A-Q)E_2\colon Q\geqq 0$ and $(A-Q)E_1\in\mathfrak{D}_1\}$; this set $\mathfrak{C}_2$ is convex because $\mathfrak{D}_1$ is convex. Define $\delta_2=\min\{\|X\|\colon X\in\mathfrak{C}_2\}$ and $\mathfrak{D}_2=\{X\in\mathfrak{C}_2\colon \|X\|=\delta_2\}$; this set $\mathfrak{D}_2$ is convex because $\mathfrak{C}_2$ is convex.

The construction of M_2 is as follows. For an arbitrary operator X on H let $X_2=E_2XE_2$; clearly $M_1\subseteq\ker X_2$ and M_1 reduces X_2. Choose $Q\geqq 0$ such that $(A-Q)E_2\in\mathfrak{D}_2$. Then $0\leqq Q_2\leqq B_2+\sqrt{\delta_2^2E_2-C_2^2}$ because M_1 reduces $A-Q$; this inequality follows from Lemma 3.2 with $X=B_2$, $Y=C_2$, $P=Q_2$ and $d=\delta_2$. Notice that for each such Q it follows that $Q|M_1=(B+\sqrt{\delta_1^2I(1)-C(1)^2})|M_1$ by the definition of M_1. Hence the operator $Z=B+\sqrt{\delta_1^2I(1)-C(1)^2}+\sqrt{\delta_2^2E_2-C_2^2}$ satisfies $Z\geqq Q\geqq 0$ for each such Q. Thus the operator $D_2=iC_2-\sqrt{\delta_2^2E_2-C_2^2}$ is $\in\mathfrak{D}_2$ because the operator $Z=B+\sqrt{\delta_1^2I(1)-C^2(1)}+\sqrt{\delta_2^2E_2-C_2^2}$ is a positive operator such that $(A-Z)E_2\in\mathfrak{C}_2$. Define $M_2=\bigcap_X\ker(X-D_2)\cap H_2$ where the intersection is over all $X\in\mathfrak{D}_2$. From Lemma 3.1 with $\mathfrak{C}=\mathfrak{C}_2$, $d=\delta_2$, $\mathfrak{D}=\mathfrak{D}_2$ and $D=D_2$, considered as operators from H_2 to itself, it follows that $M_2\neq\{0\}$ if $H_2\neq\{0\}$. If $H_2=\{0\}$, then $M_2=\{0\}$.

The fact that M_2 reduces the operator $C_2=C|H_2$ is shown by a proof similar to that used for M_1. Let f be a unit vector in M_2 and let $Q\geqq 0$ be such that $(A-Q)E_2\in\mathfrak{D}_2$. Then H_2 reduces $(A-Q)E_2$ and $(B-Q)f=-\sqrt{\delta_2^2E_2-C_2^2}f$ by the definition of M_2; thus both $(A-Q)E_2$ and $(A-Q)^*E_2$ attain their norm at f. Hence $|A_2-Q_2|^2f=|(A_2-Q_2)^*|^2f=\delta_2^2f$; this implies $(B_2+\sqrt{\delta_2^2E_2-C_2^2}-Q)(C_2f)=0$ as before. In other words, $C_2f\in M_2$, and thus M_2 reduces C_2.

In general, once H_k, E_k, $\mathfrak{C}_k$, δ_k, $\mathfrak{D}_k$, M_k have been defined, put $H_{k+1}=H\ominus(M_1\oplus\ldots\oplus M_k)$ with orthogonal projection $E_{k+1}\colon H\to H_{k+1}$. Let $\mathfrak{C}_{k+1}=\{(A-Q)E_{k+1}\colon Q\geqq 0$ and $(A-Q)E_k\in\mathfrak{D}_k\}$; this set $\mathfrak{C}_{k+1}$ is convex because $\mathfrak{D}_k$ is convex. Define $\delta_{k+1}=\min\{\|X\|\colon X\in\mathfrak{C}_{k+1}\}$ and $\mathfrak{D}_{k+1}=\{X\in\mathfrak{C}_{k+1}\colon \|X\|=\delta_{k+1}\}$; this set $\mathfrak{D}_{k+1}$ is convex because $\mathfrak{C}_{k+1}$ is convex.

To define M_{k+1}, write $X_{k+1}=E_{k+1}XE_{k+1}$ for each X; clearly $M_j\subset\ker X_{k+1}$ for $1\leqq j\leqq k$. The operator $D_{k+1}=iC_{k+1}-\sqrt{\delta_{k+1}^2E_{k+1}-C_{k+1}^2}$ is in $\mathfrak{D}_{k+1}$ because the operator $Z=B+\sqrt{\delta_1^2I(1)-C(1)^2}+\ldots+\sqrt{\delta_k^2I(k)-C(k)^2}+\sqrt{\delta_{k+1}^2E_{k+1}-C_{k+1}^2}$ is a positive operator such that $(A-Z)E_{k+1}$ is in $\mathfrak{C}_{k+1}$. Define the subspace M_{k+1} by $M_{k+1}=\bigcap_X \ker(X-D_{k+1})\cap H_{k+1}$; this intersection is taken over all $X\in\mathfrak{D}_{k+1}$. Lemma 3.1 shows that $M_{k+1}\neq\{0\}$ if $H_{k+1}\neq\{0\}$, and the operator D_{k+1} can be used to show M_{k+1} reduces C_{k+1}. This completes the inductive definition.

Thus for each integer k it is possible to define M_k and δ_k. Because H is finite-dimensional, the subspaces H_{k+1} will be $\{0\}$ for all k sufficiently large. Thus it is possible to define the positive operator P_m by

$$P_m=B+\Sigma\sqrt{\delta_k^2I(k)-C(k)^2}.$$

Clearly $A-P_m$ is a normal operator.

It remains to establish the minimality of P_m. If Q is a positive operator different from P_m, then there exists a least integer $k\geqq 1$ such that $(A-Q)E_k\notin\mathfrak{D}_k$. If $k=1$, then $\|A-Q\|>\delta_1$. Hence if h denotes the dimension of H, then for all p sufficiently large it follows that $\|A-Q\|_p^p\geqq\|A-Q\|^p>h\delta_1^p\geqq\|A-P_m\|_p^p$. If $k>1$, then let $\Delta_k=\|(A-Q)E_k\|$. Then $\Delta_k>\delta_k$ because $(A-Q)E_j$ is in $\mathfrak{D}_j$ for each $j\leqq k-1$. For each $j\leqq k-1$ the subspace M_j reduces $A-Q$, and the part of $A-Q$ on M_j is equal to the part of $A-P_m$ on M_j, which is $iC(j)-\sqrt{\delta_j^2I(j)-C(j)^2}$ and is δ_j times a unitary operator. Thus for all p sufficiently large and m_j=dimension of M_j, it follows that

$$\|A-Q\|_p^p\geqq m_1\delta_1^p+\ldots+m_{k-1}\delta_{k-1}^p+\Delta_k^p>m_1\delta_1^p+\ldots+m_{k-1}\delta_{k-1}^p+$$
$$+(h-m_1-\ldots-m_{k-1})\delta_k^p\geqq\|A-P_m\|_p^p.$$

This proves Theorem 1.1.∎

§ 4. Examples and Open Questions

Example 4.1. There does not always exist a positive approximant that is minimal in the sense of order.

Let A be the self-adjoint 3×3 matrix given by $A=\begin{pmatrix}-1&0&0\\0&0&0\\0&0&2\end{pmatrix}$. It is easily seen that $\delta(A)=1$, and that no positive approximant is smaller than $P_0=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$. For if there were such an approximant P_1, then $P_0-P_1\geqq 0$ and P_1

necessarily would have the form $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix}$, $\alpha<1$. But then P_1 would no longer be an approximant of A, so P_0 is the only candidate to be minimal. On the other hand, it is easily checked that $P_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/8 & 1/2 \\ 0 & 1/2 & 5/2 \end{pmatrix}$ is an approximant of A and clearly $P_2-P_0\geqq 0$.

For a given matrix $A=B+iC$ let $\|A\|_p$ denote the C_p norm of A. It is well known (and follows easily from [7, p. 94]) that B is a self-adjoint approximant of A in the C_p norm for all p, and it is unique in case $1<p<\infty$. Thus if S_p denotes the self-adjoint C_p approximant to A, then $S_p=B$ so $\lim_{p\to\infty}\|S_p-B\|=0$. Let R_p denote a positive approximant to A in the C_p norm which again is unique if $1<p<\infty$.

Q1. For a given matrix A and corresponding C_p minimal positive approximant P_m, does $\lim_{p\to\infty}\|R_p-P_m\|=0$?

A weaker question is:

Q2. For a given A, does the corresponding net $\{R_p\}$ have a limit in the uniform norm as $p\to\infty$?

Note that the C_p-minimal positive approximant P_m seems to be the operator analogue of the strict approximant mentioned in section 2. Since the strict approximant of Rice is a limit of l_p approximants by Theorem 2.4, the answer to Q1 could likewise be yes. Moreover Q1 and Q2 both have affirmative answers in the case A is a 2×2 matrix. This follows from the fact that for a given 2×2 matrix A and any positive approximant P, $A-P$ is normal; each convergent subnet of $\{R_p\}$ must converge to a uniform positive approximant, which can in this case be shown to be P_m by using the minimality condition defining P_m. To establish that $A-P$ is normal, note that one of two cases occurs:

i) P_H is the unique approximant so that $A-P_H$ is a multiple of a unitary by Theorem 2.3.

ii) The subspace M_1 mentioned in the proof of Theorem 1.1 is 1-dimensional. In this case for any approximant P the errors $A-P$ and $A-P_H$ can differ only in the (2, 2) entry (when viewed as matrices with respect to the subspaces M_1 and $M_1^{\perp}$). Thus $A-P$ is normal.

Questions analogous to Q1 and Q2 may be asked for $p\to 1$:

Q3. Does $\lim_{p\to 1} R_p$ exist?

If the answer to Q3 is yes, then

Q4. Can the limit in Q3 be identified by any characteristics?

An affirmative answer to Q4 would yield a canonical approximant for positive approximation in the trace norm.

Finally it seems as if Theorem 1.1 must have some extension at least to the compact operator case. Relevant to this problem is the following

Example 4.2. There exists a compact operator with no compact positive operator approximant.

Indeed, let $\{e_1, e_2, \ldots\}$ denote an orthonormal basis and let f be the vector $f \equiv \Sigma e_k/k$. Define Q to be the rank one orthogonal projection onto sp $\{f\}$, C the compact operator given by $C(e_k)=e_k/k$, $B\equiv(1-Q)-\sqrt{1-C^2}$, and finally set $A=B+iC$. Then A is a compact operator and has a unique positive approximant P_H [1, p. 282]. Now P_H is not compact since $A-P_H$ is a multiple of a unitary.

Q5. Which compact operators admit compact positive approximants; is there a "minimal" approximant in this case?

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Compact and Hilbert—Schmidt composition operators on weighted sequence spaces

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1. Preliminaries. If $(X, \mathcal{S}, \lambda)$ is a σ-finite measure space, then every non-singular measurable transformation T from X into itself induces the composition transformation C_T from $L^p(\lambda)$ into the linear space of all complex valued functions on X defined by $C_T f = f \circ T$ for every $f \in L^p(\lambda)$. If C_T turns out to be a continuous linear transformation from $L^p(\lambda)$ into itself, then we designate it as a composition operator on $L^p(\lambda)$.

If $w = \{w_n\}$ is a sequence of strictly positive real numbers, then we define the measure λ on the measurable space $(N, \mathcal{P}(N))$ as

$$\lambda(E) = \sum_{n \in E} w_n \quad \text{for every} \quad E \in \mathcal{P}(N),$$

the power set of the set N of positive integers. Thus $(N, \mathcal{P}(N), \lambda)$ becomes a σ-finite measure space. The L^p-space of this measure space is known as a weighted sequence space and w is called the sequence of weights. We denote this weighted sequence space by l^p_w. It is a well established fact that l^p_w (more generally $L^p(\lambda)$) is a Banach space. If $p=2$, then l^p_w is a Hilbert space under pointwise addition and scalar multiplication with the inner product defined as

$$\langle f, g \rangle = \int_N f \bar{g} \, d\lambda = \sum_{n=1}^{\infty} w_n f(n) \bar{g}(n)$$

for every f and g in l^2_w. It is also interesting to note that the space l^2_w is a functional Hilbert space. By $B(l^2_w)$ we denote the Banach algebra of all bounded operators on l^2_w.

The main purpose of this note is to characterise compact, finite rank and Hilbert—Schmidt composition operators on l^2_w.

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2. Compact composition operators. If $(X, \mathcal{S}, \lambda)$ is a non-atomic measure space, then it has been shown in [6] that no composition operator on $L^2(\lambda)$ is compact. If the sequence w of weights is a constant sequence, then it can be easily established that l^2_w does not admit any compact composition operator. Thus in particular no composition operator on l^2 is compact, though it is an L^2-space of an atomic measure space. But if the sequence w is a non-constant suitably chosen sequence, then there are compact composition operators on l^2_w. This fact makes this study a little interesting. Before the characterisation of compact composition operators on l^2_w we shall need the following easy lemma.

Lemma 2.1. *Every weakly convergent sequence in l^2_w is pointwise convergent.*

Proof. Let $\{f_n\}$ be a sequence in l^2_w converging to zero weakly. Then, since $\{w_j f_n(j)\} = \{\langle f_n, e_j \rangle\}$ converges to zero, where $e_j(i) = \delta_{ij}$ (the Kronecker delta), it follows that $\{f_n\}$ is pointwise convergent.

Remark. The above lemma is true in any functional Hilbert space.

Let $T\colon N \to N$ be a mapping and let $\varepsilon > 0$. Then the set M_ε is defined as

$$M_\varepsilon = \{n\colon n \in N \text{ and } \lambda T^{-1}(\{n\}) > \varepsilon \lambda(\{n\})\}.$$

The following theorem characterises compact composition operators on l^2_w in terms of the cardinality of M_ε.

Theorem 2.2. *Let $C_T \in B(l^2_w)$. Then C_T is compact if and only if M_ε, for every $\varepsilon > 0$, contains finitely many elements.*

Proof. Let $\varepsilon > 0$ be given and let $\{f_n\}$ be a sequence in l^2_w converging weakly to zero. Suppose M_ε contains k elements. Then, since $\lambda T^{-1}(\{n\}) \leqq \varepsilon \lambda(\{n\})$ for every $n \in N \setminus M_\varepsilon$ and $\lambda T^{-1}(\{n\}) \leqq M \lambda(\{n\})$ for every $n \in N$ and for some finite $M > 0$ [7],

$$\|C_T f_n\|^2 = \int_N |f_n|^2 \, d\lambda T^{-1} = \int_{M_\varepsilon} |f_n|^2 \, d\lambda T^{-1} + \int_{N/M_\varepsilon} |f_n|^2 d\lambda T^{-1} =$$

$$\leqq M \cdot k |f_n(m_r)|^2 \lambda(\{m_s\}) + \varepsilon \|f_n\|^2,$$

where $|f_n(m_r)| = \max\{|f_n(m_t)|\colon m_t \in M_\varepsilon\}$ and $\lambda(\{m_s\}) = \max\{\lambda(\{m_t\})\colon m_t \in M_\varepsilon\}$. Since by the above lemma $\{f_n\}$ converges to zero pointwise, we can find $m \in N$ such that for every $n > m$,

$$\|C_T f_n\|^2 \leqq \varepsilon_1 M k \lambda(\{m_s\}) + \varepsilon \|f_n\|^2.$$

Since every weakly convergent sequence is norm bounded [1, p. 145] and ε_1 and ε are arbitrary, we conclude that the sequence $\{\|C_T f_n\|\}$ converges to zero. Hence C_T is compact.

Conversely, suppose M_ε contains infinitely many elements for some $\varepsilon>0$. Let M_ε^e be the closure of span $\{e_n: n\in M_\varepsilon\}$. Then for $f\in M_\varepsilon^e$,

$$\|C_T f\|^2=\int_N |f|^2\, d\lambda T^{-1}>\int_{M_\varepsilon}|f|^2\, d\lambda=\varepsilon\|f\|^2.$$

Thus C_T is bounded away from zero on M_ε^e. This shows that the range of $C_T|M_\varepsilon^e$, the restriction of C_T to the subspace M_ε^e, is a closed infinite dimensional subspace contained in the range of C_T. Hence by problem 141 of [3] C_T is not compact.

Corollary 1. *Let $C_T\in B(l_w^2)$. Then C_T is compact if and only if $\lambda(T^{-1}(\{n\}))/\lambda(\{n\})$ tends to zero as n tends to ∞.*

Corollary 2. *No composition operator on l^2 is compact.*

Proof. If $C_T\in B(l^2)$, then the range of T contains infinitely many elements [8]. Hence $M_\varepsilon=T(N)$ whenever $\varepsilon<1$. Thus C_T is not compact.

Let a be a strictly positive real number and let $w=\{w_n\}$ be the sequence defined as $w_n=a^n$ for $n\in N$. Then the corresponding l_w^2 is denoted by l_a^2. In the light of the following two theorems it is comparatively easier to locate compact composition operators on l_a^2.

Theorem 2.3. *Let $C_T\in B(l_a^2)$, where $0<a<1$. Then C_T is compact if and only if the sequence $\{n-T(n)\}$ tends to ∞ as n tends to ∞.*

Proof. Suppose the sequence $\{n-T(n)\}$ tends to ∞ as n tends to ∞. Let m be in the range of T and let $T^{-1}(\{m\})=\{m_1, m_2, m_3, \dots\}$ be the arrangement of $T^{-1}(\{m\})$ in the ascending order. Then

$$\frac{\lambda(T^{-1}(\{m\}))}{\lambda(\{m\})}=\sum_i a^{m_i-m}<\sum_{i=0}^{\infty} a^{m_1-m+i}=\frac{a^{m_1-m}}{1-a}=\frac{a^{m_1-T(m_1)}}{1-a}.$$

Since $a<1$, we can conclude from the hypothesis that $\lim_{m\to\infty}\frac{\lambda(T^{-1}(\{m\}))}{\lambda(\{m\})}=0$. Hence by the Corollary 1 of Theorem 2.2 C_T is compact.

Conversely, suppose the sequence $\{n-T(n)\}$ does not tend to ∞ as n tends to ∞. By Theorem 1 of [7] the sequence is bounded from below. Hence the sequence $\{n-T(n)\}$ has bounded subsequences. Let $\{n_k-T(n_k)\}$ be a bounded subsequence with a bound M. Then

$$\frac{\lambda(T^{-1}(\{T(n_k)\}))}{\lambda(\{T(n_k)\})}>a^{n_k-T(n_k)}>a^M>0.$$

Hence again by the Corollary 1 of Theorem 2.2 C_T is not compact. This completes the proof of the theorem.

Example. Let $T\colon N \to N$ be defined as $T(m) = n/3$ if $n-2 \leqq m \leqq n$, where n is a multiple of 3. Then C_T is a composition operator on l_a^2, $0<a<1$. Since $\lambda T^{-1}(\{n\})/\lambda(\{n\})\,(=a^{2n}(1+a^{-1}+a^{-2}))$ tends to zero as n tends to ∞, we can conclude that C_T is compact.

Theorem 2.4. *Let $C_T \in B(l_a^2)$, where $a>1$. Then C_T is compact if and only if $\{T(n)-n\}$ tends to ∞ as n tends to ∞.*

Proof. The proof is dual to the proof of Theorem 2.3.

Example. Let $T\colon N \to N$ be the mapping defined by $T(m) = n^2$ if $n-2 \leqq m \leqq n$, where n is a multiple of 3. Then, since $\lambda T^{-1}(\{n\})/\lambda(\{n\}) = 0$ for $n \in N \setminus T(N)$ and $(a^{-2}+a^{-1}+1)/a^{\sqrt{n}(\sqrt{n}-1)}$ for $n \in T(N)$, C_T is a compact composition operator.

We now give several sufficient conditions for non-compactness of composition operators on l_a^2.

Theorem 2.5. *Let $T\colon N \to N$ be an injection and $C_T \in B(l_a^2)$, where $0<a<1$. Then C_T is not compact.*

Proof. Suppose C_T is compact. We infer from Theorem 2.3 that $\{n-T(n)\} \to \infty$. Therefore there exists a number $n_0 \in N$ such that for every $n>n_0$, we have $T(n)<n$. Let $n_1' = \max\{T(i) \mid 1 \leqq i \leqq n_0\}$, $n_1 = \max\{n_1', n_0\}$, $N_1 = \{1, 2, \ldots, n_1\}$ and $N_2 = N_1 \cup \{n_1+1\}$. Since $T(n)<n$ for $n>n_0$ and T is injective, $T(N_1) = N_1 = T(N_2)$ which contradicts the injectivity of T.

The following is an example of a function T which is not an injection, but it induces a compact composition operator.

Example. Let $E_n = \{2^{n-1}(2k-1) \mid k \in N\}$. Then $\bigcup_n E_n = N$. Let $T\colon N \to N$ be defined as $T(m) = n$ for every $m \in E_n$. Then C_T is a composition operator on l_a^2, $0<a<1$. Since

$$\lambda T^{-1}(\{n\})/\lambda(\{n\}) = a^{2^{n-1}}/a^n(1-a^{2^n}),$$

C_T is compact.

Theorem 2.6. *Let $T\colon N \to N$ be a surjection and $C_T \in B(l_a^2)$, where $a>1$. Then C_T is not a compact composition operator.*

Proof. Suppose C_T is compact. We infer from Theorem 2.4 that $\{T(n)-n\}$ tends to ∞. Therefore there exists a number $n_0 \in N$ such that for every $n>n_0$ we have $T(n)>n$. Let $n_1' = \max\{T(i) \mid 1 \leqq i \leqq n_0\}$, $n_1 = \max\{n_1'+1, n_0+1\}$ and $N_1 = \{1, 2, \ldots, n_1\}$. Then $T(N \setminus N_1) \subset N \setminus N_1$ and so $T(N) \cap N_1 = T(N_1) \cap N_1$. Since $T(n_1) > n_1$, we get $\operatorname{Card}(T(N) \cap N_1) = \operatorname{Card}(T(N_1) \cap N_1) < \operatorname{Card} N_1$. This means T is not surjective. This proves the Theorem.

The following is an example of a compact composition operator induced by a non-surjective mapping.

Example. Let T: $N \to N$ be defined as $T(n)=2n$. Then C_T is a composition operator on l_a^2, $a>1$. Since

$$\lambda T^{-1}(\{n\})/\lambda(\{n\}) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1/a^{n/2}, & \text{if } n \text{ is even,} \end{cases}$$

C_T is compact.

The following Theorem characterises finite rank composition operators on l_w^2, where $\Sigma w_i < \infty$.

Theorem 2.7. *Let $C_T \in B(l_w^2)$, where $\Sigma w_i < \infty$. Then C_T is a finite rank operator if and only if the range of T contains finitely many elements.*

Proof. Since the range of C_T is dense in $l_w^2(N, T^{-1}(\mathscr{P}(N)), \lambda)$, $\lambda(E) = \sum_{n \in E} w_n$ [10, Lemma 2.4], the proof follows trivially.

3. Hilbert—Schmidt composition operators

Definition. A bounded linear operator A on an infinite dimensional separable Hilbert space H is said to be a Hilbert—Schmidt operator if there exists an orthonormal basis $\{e_n : n \in N\}$ in H such that $\Sigma \|Ae_n\|^2 < \infty$. It is well known that the definition is independent of the choice of the orthonormal basis.

Let T: $N \to N$ be a mapping and let $y = \{y(m)\}$ be the sequence defined by $y(m) = \|K_{T(m)}\|$ for every $m \in N$, where K_m is the kernel function for l_w^2 defined by $K_m = e_m/w_m$. Then we prove the following Theorem.

Theorem 3.1. *Let $C_T \in B(l_w^2)$. Then C_T is a Hilbert—Schmidt operator if and only if $y \in l_w^2$.*

Proof. Since the family $\{f_n\}$ defined by $f_n = e_n/\sqrt{w_n}$ forms an orthonormal basis for l_w^2, C_T is a Hilbert—Schmidt operator if and only if

$$\sum_n \|C_T f_n\|^2 = \sum_n \sum_m w_m \left| \frac{e_n(T(m))}{\sqrt{w_n}} \right|^2 = \sum_n \sum_{m \in T^{-1}(\{n\})} w_m \cdot \frac{1}{w_n} = \sum_n \sum_{m \in T^{-1}(\{n\})} w_n \cdot w_m \frac{1}{w_n^2} =$$

$$= \sum_n \sum_m w_n \cdot w_m \left| \frac{e_{T(m)}(n)}{w_{T(m)}} \right|^2 = \sum_m w_m \sum_n w_n \left| \frac{e_{T(m)}(n)}{w_{T(m)}} \right|^2 = \sum_m w_m \|K_{T(m)}\|^2 = \|y\|^2 < \infty.$$

This finishes the proof of the Theorem.

Example. Let the sequence $\{w_n\}$ of weights be the sequence $\{n\}$ and let T: $N \to N$ be defined as $T(n) = n^3$. Then C_T is a composition operator on l_w^2. Since

$$\|y\|^2 = \sum_n w_n/w_{T(n)} = \sum_n n/n^3 = \sum_n 1/n^2 < \infty,$$

C_T is Hilbert—Schmidt.

The following exampale shows that the set of Hilbert—Schmidt composition operators is properly contained in the set of all compact composition operators on l^2_w.

Example. Let $\{w_n\}=\{n\}$ and let C_T be the composition operator on l^2_w induced by the mapping $T(n)=n^2$. Then C_T is compact but is not Hilbert—Schmidt.

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On the converse of the Fuglede—Putnam theorem

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1. Let $\mathfrak{K}$ and $\mathfrak{H}$ be Hilbert spaces, and let $\mathcal{L}(\mathfrak{K}, \mathfrak{H})$ denote the space of all bounded linear operators from $\mathfrak{K}$ to $\mathfrak{H}$. (We also write $\mathcal{L}(\mathfrak{H})=\mathcal{L}(\mathfrak{H}, \mathfrak{H})$.) The well-known Fuglede—Putnam theorem [2] asserts that if $A\in\mathcal{L}(\mathfrak{H})$ and $B\in\mathcal{L}(\mathfrak{K})$ are normal, then the pair (A, B) of operators has the following property:

(FP) *If* $AX=XB$ *where* $X\in\mathcal{L}(\mathfrak{K}, \mathfrak{H})$, *then* $A^*X=XB^*$.

In this note we shall show that the normality of A and B in the above theorem is essential.

2. We say that an ordered pair (A, B) of operators $\big(A\in\mathcal{L}(\mathfrak{K})$ and $B\in\mathcal{L}(\mathfrak{H})\big)$ is *disjoint* if the only operator $X\in\mathcal{L}(\mathfrak{K}, \mathfrak{H})$ satisfying $AX=XB$ is $X=0$. Let $A=A_1\oplus A_2$ and $B=B_1\oplus B_2$. Then it is easy to see that (A, B) is disjoint if and only if (A_i, B_j) $(i, j=1, 2)$ is disjoint. Also, if (A, B) is disjoint, then it trivially satisfies the property (FP). We recall the fact that each operator A can be written uniquely $A=A_{(n)}\oplus A_{(c.n.)}$ where $A_{(n)}$ is normal and $A_{(c.n.)}$ is *completely nonnormal*, that is, no nontrivial direct summand of $A_{(c.n.)}$ is normal (see e.g. [1]).

Theorem. *Let* $A\in\mathcal{L}(\mathfrak{H})$ *and* $B\in\mathcal{L}(\mathfrak{K})$. *The following statements are equivalent*:

(i) *The pair* (A, B) *has the property* (FP).

(ii) *If* $AY=YB$ *where* $Y\in\mathcal{L}(\mathfrak{K}, \mathfrak{H})$, *then* $(\operatorname{ran} Y)^-$ *reduces* A, $(\ker Y)^\perp$ *reduces* B, *and the restrictions* $A|(\operatorname{ran} Y)^-$ *and* $B|(\ker Y)^\perp$ *are normal operators, where* ran *and* ker *denote the range and the kernel, respectively.*

(iii) *The pairs* $(A, B_{(c.n.)})$ *and* $(A_{(c.n.)}, B)$ *are disjoint.*

(iv) *A and B can be decomposed as follows:*

$A=A_1\oplus A_2$ *and* $B=B_1\oplus B_2$, *where* A_1 *and* B_1 *are normal, and the pairs* (A, B_2) *and* (A_2, B) *are disjoint.*

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Proof. (i)$\Rightarrow$(ii): Since $AY=YB$ and (A, B) satisfies (FP), $A^*Y=YB^*$ and so $(\operatorname{ran} Y)^-$ and $(\ker Y)^\perp$ are reducing subspaces for A and B, respectively. Since $A(AY)=(AY)B$, we obtain $A^*(AY)=(AY)B^*$ by (FP), and the identity $A^*Y= =YB^*$ implies $A^*AY=AA^*Y$. Thus we see that $A|(\operatorname{ran} Y)^-$ is normal. Clearly (B^*, A^*) satisfies (FP), and $B^*Y^*=Y^*A^*$. Therefore it follows from the above argument that $B^*|(\operatorname{ran} Y^*)^-=(B|(\ker Y)^\perp)^*$ is normal.

(ii)$\Rightarrow$(iii): Let us write $A=A_{(n)}\oplus A_{(c.n.)}$ on $\mathfrak{H}=\mathfrak{H}_{(n)}\oplus\mathfrak{H}_{(c.n.)}$. Suppose that $A_{(c.n.)}X=XB$ where $X\in\mathscr{L}(\mathfrak{K}, \mathfrak{H}_{(c.n.)})$. We define $\tilde{X}\in\mathscr{L}(\mathfrak{K}, \mathfrak{H})$ by setting $\tilde{X}x=Xx$ for $x\in\mathfrak{K}$. Then $A\tilde{X}=\tilde{X}B$, and by the condition (ii) $(\operatorname{ran}\tilde{X})^-$ reduces A and $A|(\operatorname{ran}\tilde{X})^-$ is normal, that is, $(\operatorname{ran} X)^-$ reduces $A_{(c.n.)}$ and $A_{(c.n.)}|(\operatorname{ran} X)^-$ is normal. But since $A_{(c.n.)}$ has no normal direct summand, $(\operatorname{ran} X)^-=\{0\}$, that is, $X=0$. Thus $(A_{(c.n.)}, B)$ is disjoint. Similarly, we see that $(A, B_{(c.n.)})$ is disjoint.

(iii)$\Rightarrow$(iv) is trivial.

(iv)$\Rightarrow$(i): Let $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2$ and $\mathfrak{K}=\mathfrak{K}_1\oplus\mathfrak{K}_2$ be the decompositions corresponding to $A=A_1\oplus A_2$ and $B=B_1\oplus B_2$, respectively. Suppose $AX=XB$. Then by the condition (iv) X has the form $X=\begin{pmatrix} X_1 & 0\\ 0 & 0\end{pmatrix}$ with respect to the decompositions $\mathfrak{K}=\mathfrak{K}_1\oplus\mathfrak{K}_2$ and $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2$. Therefore for the proof of the equation $A^*X=XB^*$ it suffices to show $A_1^*X_1=X_1B_1^*$, but this follows from the Fuglede—Putnam theorem since A_1 and B_1 are normal.

3. The following fact is known as a corollary of the Fuglede—Putnam theorem (see [2, Theorem 1.6.4] and its proof). Let $A\in\mathscr{L}(\mathfrak{H})$ and $B\in\mathscr{L}(\mathfrak{K})$ be normal. If there exists a *quasi-affinity* $X\in\mathscr{L}(\mathfrak{K}, \mathfrak{H})$ (i.e., X is one-to-one and has dense range) such that $AX=XB$, then A and B are unitarily equivalent.

An immediate corollary of our theorem is the following.

Corollary 1. *Suppose that (A, B) has the property* (FP). *If there exists a quasi-affinity X such that $AX=XB$, then A and B are unitarily equivalent normal operators.*

An operator A is called *hyponormal* or *cohyponormal* according as $A^*A-AA^*\geqq 0$ or $\leqq 0$. Radjabalipour [3], Stampfli and Wadhwa [4] proved the following theorem (indeed, they obtained more general results there); if A is hyponormal and B is cohyponormal, and if there exists a quasi-affinity X such that $AX=XB$, then A and B are normal operators.

We can rephrase their theorem as follows;

Corollary 2. *If A is hyponormal and B is cohyponormal then the pair (A, B) has the property* (FP).

Proof. It is easy to see that every invariant subspace for a hyponormal operator T on which T is normal is reducing. From this fact and the theorem of Radjabalipour, Stampfli and Wadhwa, we see that (A, B) satisfies the condition (ii) in Theorem. Therefore (A, B) has the property (FP).

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Beitrag zur Konvergenz der Orthogonalreihen

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1. Es sei $r=\{r_k(t)\}_1^\infty$ $(r_k(t)=\operatorname{sign}\sin 2^k\pi t)$ das Rademachersche Funktionensystem. Der folgende Satz ist bekannt (vgl. [4], S. 47).

Es sei $\varphi=\{\varphi_k(t)\}_1^\infty$ ein orthonormiertes System im Intervall (0, 1). Für $a=\{a_k\}_1^\infty\in l^2$ konvergiert die Reihe

$$\sum_{k=1}^\infty r_k(t)a_k\varphi_k(x)$$

bei fast jedem festen Punkt $x\in(0, 1)$ für fast jede $t\in(0, 1)$; m. a. W. bei fast jedem Punkt x konvergiert die Reihe

$$\sum_{k=1}^\infty \pm a_k\varphi_k(x)$$

mit der Wahrscheinlichkeit 1.

In dieser Note werden wir zeigen, daß unter einer stärkeren Voraussetzung über die Koeffizientenfolge a, ähnliche Behauptungen auch im Falle gelten, wenn die Vorzeichenverteilung durch ein beliebiges *vorzeichenartiges Funktionensystem* $\psi=\{\psi_k(t)\}_1^\infty$ gegeben ist, d. h. durch meßbare Funktionen $\psi_k(t)$, mit $|\psi_k(t)|=1$ f. ü. für $t\in(0, 1)$, $k=1, 2, \ldots$.

2. Es sei Ω die Klasse aller orthonormierten Systeme $\varphi=\{\varphi_k(x)\}_1^\infty$ im Intervall (0, 1). Weiterhin sei M die Klasse der Koeffizientenfolgen $a=\{a_k\}_1^\infty$, für die die Reihe

$$\sum_{k=1}^\infty a_k\varphi_k(x) \tag{1}$$

bei jedem System $\varphi\in\Omega$ im Intervall (0, 1) fast überall konvergiert. Es ist bekannt (vgl. [2], [5]), daß

$$\sum_{k=2}^\infty a_k^2\log^2 k<\infty\Rightarrow a\in M.$$

Weitere hinreichende Bedingungen für $a\in M$ sind z. B. in der Arbeit [6] gegeben.

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Für eine Folge a setzen wir

$$\|a\| = \sup_{\varphi \in \Omega} \left\{ \int_0^1 \sup_{1 \leq i \leq j} \left(\sum_{k=i}^{j} a_k \varphi_k(x) \right)^2 dx \right\}^{1/2}.$$

In [7] haben wir gezeigt, daß $a \in M$ mit $\|a\| < \infty$ äquivalent ist.

Es seien $a \in M$, $\varphi \in \Omega$, und sei ψ ein System von vorzeichenartigen Funktionen in $(0, 1)$. Dann ist $\{\psi_k(t)\varphi_k\}_1^\infty \in \Omega$ für jedes $t \in (0, 1)$, und so folgt, daß die Reihe

$$\sum_{k=1}^{\infty} \psi_k(t)\, a_k \varphi_k(x) \tag{2}$$

bei jedem $t \in (0, 1)$ in $(0, 1)$ fast überall konvergiert. Durch Anwendung des Fubinischen Satzes erhalten wir:

Satz I. *Es sei* $a \in M$, $\varphi \in \Omega$ *und* ψ *ein beliebiges vorzeichenartiges System. Dann konvergiert die Reihe* (2) *bei fast jedem* $x \in (0, 1)$ *für fast jedes* $t \in (0, 1)$.

3. Es sei $\lambda = \{\lambda_k\}_1^\infty$ eine nichtabnehmende Folge von positiven Zahlen. Mit $\Omega(\lambda)$ bezeichnen wir die Klasse der orthonormierten Systeme $\varphi = \{\varphi_k(x)\}_1^\infty$ im Intervall $(0, 1)$, für die die Lebesgueschen Funktionen

$$L_n(\varphi;\, x) = \int_0^1 \left| \sum_{k=1}^{n} \varphi_k(x)\varphi_k(t) \right| dt$$

f. ü. mit λ_n beschränkt sind; d. h. für die

$$\sup_n \frac{L_n(\varphi;\, x)}{\lambda_n} < \infty \tag{3}$$

f. ü. in $(0, 1)$ besteht.

Mit $M(\lambda)$ bezeichnen wir die Klasse der Folgen a, für die die Reihe (1) bei jedem System $\varphi \in \Omega(\lambda)$ in $(0, 1)$ fast überall konvergiert. Es ist bekannt, daß die Folgen a mit

$$\sum_{k=1}^{\infty} a_k^2 \lambda_k < \infty$$

zu $M(\lambda)$ gehören [1]. Für jede Folge a setzen wir

$$\|a;\, \lambda\| = \sup_{\varphi} \int_0^1 \sup_{1 \leq i \leq j} \left| \sum_{k=i}^{j} a_k \varphi_k(x) \right| dx,$$

wobei das Supremum über alle in $(0, 1)$ orthonormierte Systeme φ mit

$$\int_0^1 \sup_n \frac{L_n(\varphi;\, x)}{\lambda_n}\, dx \leq 1$$

gebildet wird. In [8] haben wir gezeigt, daß die Bedingungen $a\in M(\lambda)$ und $\|a;\lambda\|<\infty$ äquivalent sind.

Es sei $\varphi\in\Omega(\lambda)$ und ψ sei ein beliebiges vorzeichenartiges System. Dann gilt $\{\psi_k(t)\varphi_k\}_1^\infty\in\Omega(\lambda)$ für jedes $t\in(0, 1)$. Ist $a\in M(\lambda)$, dann folgt, daß die Reihe (2) bei jedem $t\in(0, 1)$ in $(0, 1)$ fast überall konvergiert. Durch Anwendung des Fubinischen Satzes ergibt sich:

Satz II. *Es sei $a\in M(\lambda)$, $\varphi\in\Omega(\lambda)$ und ψ ein beliebiges vorzeichenartiges System. Dann konvergiert die Reihe* (2) *bei fast jedem $x\in(0, 1)$ für fast alle $t\in(0, 1)$.*

Im Falle $\lambda_n=O(1)$ gilt also die folgende Behauptung.

Gilt $a\in l^2$, und für das orthonormierte System φ besteht $\sup L_n(\varphi; x)<\infty$ f. ü. in $(0, 1)$, dann konvergiert die Reihe (2) bei fast jedem $x\in(0, 1)$ fast überall in $(0, 1)$.

4. Bemerkungen. 1) Ohne die Bedingung $a\in M$, bzw. $a\in M(\lambda)$ gelten die Sätzen nicht. Ist nämlich $a\notin M$, bzw. $a\notin M(\lambda)$, dann gibt es ein System $\varphi\in\Omega$, bzw. $\varphi\in\Omega(\lambda)$, derart, daß die Reihe (1) in $(0, 1)$ fast überall divergiert, also für das System $\psi_k(t)\equiv 1$ $(t\in(0, 1); k=1, 2, \dots)$ divergiert die Reihe (2) bei fast jedem $x\in(0, 1)$ für jedes $t\in(0, 1)$.

2) In unseren Sätzen läßt sich Orthonormalität des Systems φ durch die schwächere Voraussetzung ersetzen, daß das System ein Konvergenzsystem für l^2 dem Maß nach ist. Nach einem Satz von NIKISCHIN [3] ist nämlich das System φ in diesem Falle fast orthonormiert, d. h. für jede positive Zahl ε gibt es eine meßbare Menge $F_\varepsilon(\subseteq(0, 1))$, eine positive Zahl M_ε, und ein orthonormiertes System $\Phi(\varepsilon)=\{\Phi_k(\varepsilon; x)\}_1^\infty$ in $(0, 1)$ mit $\operatorname{mes} F_\varepsilon\geqq 1-\varepsilon$, $\varphi_k(x)=M_\varepsilon\Phi_k(\varepsilon; x)$ $(x\in F_\varepsilon; k=1, 2, \dots)$.

So ist die folgende Behauptung klar:

Satz I. a. *Es seien $a\in M$, φ ein Konvergenzsystem für l^2 dem Maß nach in* $(0, 1)$, *und ψ ein beliebiges vorzeichenartiges System. Dann konvergiert die Reihe* (2) *bei fast jedem $x\in(0, 1)$ für fast alle $t\in(0, 1)$.*

Es gilt auch:

Satz II. a. *Es seien $a\in M(\lambda)$, φ ein Konvergenzsystem für l^2 dem Maß nach in* $(0, 1)$ *mit* (3), *und ψ ein beliebiges vorzeichenartiges System. Dann konvergiert die Reihe* (2) *bei fast jedem $x\in(0, 1)$ für fast alle $t\in(0, 1)$.*

Beweis des Satzes II. a. Mit $\Omega^*(\lambda)$ bezeichnen wir die Klasse der Konvergenzsysteme φ für l^2 dem Maß nach in $(0, 1)$, für die (3) f. ü. in $(0, 1)$ besteht. Weiterhin sei $M^*(\lambda)$ die Klasse der Koeffizientenfolgen a, für die Reihe (1) bei jedem $\varphi\in\Omega^*(\lambda)$ fast überall konvergiert. Nach einem Satz in [9] ist $M^*(\lambda)=M(\lambda)$, woher Behauptung folgt.

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On the radical classes determined by regularities

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1. Introduction

All rings considered in this paper are associative. For a given class $\mathscr{C}$ of rings each ring $A\in\mathscr{C}$ is called a *$\mathscr{C}$-ring*, and an ideal B of a ring A is called *ideal* if B (as ring) is a $\mathscr{C}$-ring.

It is well known that a non-empty subclass $\mathscr{C}$ of rings is a *radical class* or briefly a *radical* (relative to the class of all associative rings) in the sense of KUROŠ [13] and AMITSUR [1] if it satisfies the following conditions:

(i) $\mathscr{C}$ is homomorphically closed, that is, every homomorphic image of a $\mathscr{C}$-ring is a $\mathscr{C}$-ring.

(ii) The sum of all $\mathscr{C}$-ideas of a ring A is a $\mathscr{C}$-ideal.

(iii) $\mathscr{C}$ is closed under extensions, that is, if both B and A/B are $\mathscr{C}$-rings, then A is also a $\mathscr{C}$-ring.

In ring theory many so-called *regularities* determine radical classes, for instance the von Neumann regularity [17], quasi-regularity [18], G-regularity [6], strong regularity [3], and so on.

The aim of this paper is to give the definition of regularity of associative rings in the common terminology of polynomials and formal powerserieses, and to show the radical characteristic of regularities in this sense. At the same time we shall get a diagram to define radicals by regularities. In view of our results it become clear that well known regularities and ring properties considered in [14], [17], [22] and [25], are radicals.

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2. Regularities determined by polynomials

$Z[x_0, x_1, ..., x_n]$ denotes the set of polynomials in non-commutative indeterminates $x_0, x_1, x_2, ..., x_n$ with integer coefficients.

Definition 1. Suppose $f(x_0, x_1, ..., x_n)$ is in $Z[x_0, x_1, ..., x_n]$. An element a_0 of a ring A is said to be *f regular* if there exist elements $a_1, a_2, ..., a_n$ in A such that the equality

$$f(a_0, a_1, ..., a_n) = 0$$

is valid in A.

A ring A is said to be f-regular if every element of A is f-regular. An ideal B of a ring A is an f-regular ideal if B is an f-regular ring.

The following theorem characterizes the radical property for f-regularities.

Theorem 1. *Suppose $f(x_0, x_1, ..., x_n) \in Z[x_0, x_1, ..., x_n]$, then the class of all f-regular rings is a radical class if and only if the following conditions are satisfied:*

1) $f(x_0, x_1, ..., x_n)$ *has no constant term.*

2) *If B is an f-regular ideal of a ring A, and for every $a_0 \in A$ there exist elements $a_1, a_2, ..., a_n$ in A such that $f(a_0, a_1, ..., a_n) \in B$, then A is an f-regular ring.*

Proof. Assume that the class $\mathscr{C}$ of all f-regular rings is a radical class. Since the zero ideal is a $\mathscr{C}$-ideal in every ring, the first condition is always satisfied.

Now suppose that B is an f-regular ideal of a ring and for every $a_0 \in A$ there exist elements $a_1, a_2, ..., a_n$ such that $f(a_0, a_1, ..., a_n) \in B$. We have to show that the ring A is f-regular. Let us consider the factor ring A/B. Take any element $\bar{a} \in A/B$. Let an element a_0 be in the coset $\bar{a}$. By hypothesis there exist elements $a_1, a_2, ..., a_n$ such that $f(a_0, a_1, ..., a_n) \in B$. So in the factor ring A/B the equality

$$f(\bar{a}, \bar{a}_1, ..., \bar{a}_n) = 0$$

holds. Hence the element $\bar{a}$ is f-regular. This implies the f-regularity of A/B. Since radicals are closed under extensions, A is f-regular. Thus the second condition is valid.

Conversely, assume that $f(x_0, x_1, ..., x_n)$ satisfies the conditions of the theorem. Clearly, $\mathscr{C}$ is homomorphically closed. Now, suppose that for an ideal J of a ring A, both J and A/J are $\mathscr{C}$-rings. Since A/J is f-regular, therefore for every element $a_0 \in A$ there exist elements $a_1, a_2, ..., a_n$ in A such that the cosets $\bar{a}_0, \bar{a}_1, ..., \bar{a}_n$ satisfy the equality

$$f(\bar{a}_0, \bar{a}_1, \bar{a}_2, ..., \bar{a}_n) = 0$$

in the factor ring A/J. This implies $f(a_0, a_1, ..., a_n) \in J$. By the second condition of the theorem, the ring A is f-regular. Hence the class $\mathscr{C}$ is closed under extensions.

Suppose both B_1 and B_2 be $\mathscr{C}$-ideals of a ring A. By the second isomorphism theorem we have

$$\frac{B_1+B_2}{B_2} \cong \frac{B_1}{B_1 \cap B_2}.$$

Since the class $\mathscr{C}$ is homomorphically closed and closed under extensions the above isomorphism implies that B_1+B_2 is a $\mathscr{C}$-ring. By a simple induction we can prove that the sum of any finite number of $\mathscr{C}$-ideals of a ring A is again a $\mathscr{C}$-ideal.

Finally, it is easy to see that the sum $\mathscr{C}(A)$ of all $\mathscr{C}$-ideals of a ring A is a $\mathscr{C}$-ideal. This completes the proof of the theorem.

As a radical criterion of f-regularities we have the following

Corollary 1. *For a polynomial $f(x_0, x_1, \ldots, x_n)$ in $Z[x_0, x_1, \ldots, x_n]$ without constant term, the class of all f-regular rings is a radical class if one of the following two conditions is satisfied.*

(A) *For arbitrary elements $a_0, a_1, \ldots, a_n$ in a ring A, if the element $f(a_0, a_1, \ldots, a_n)$ is f-regular then the element a_0 is also f-regular.*

(B) *Let B be an f-regular ideal of a ring A; if the coset $\bar{a}_0$ containing $a_0 \in A$ is f-regular in the factor ring A/B, then the element a_0 is f-regular in the ring A.*

Proof. The assertion is an immediate consequence of Theorem 1. It is easy to check that the conditions of Theorem 1 are satisfied.

Remark. By Corollary 1, the conditions (A) and (B) are sufficient for an f-regularity to be a radical. It is not known whether the converse is true.

3. Regularities determined by formal power serieses

We shall use the following notations: $X_i = (x_{i1}, x_{i2}, \ldots, x_{ik}, \ldots)$ for $i = 1, 2, \ldots, n$; $Z\{X_1, X_2, \ldots, X_n\}$ denotes the set of all formal power series in infinite number of non-commutative indeterminates $x_{i1}, x_{i2}, \ldots$; $i = 1, 2, \ldots, n$, and with integer coefficients; that is, every $f[X_1, X_2, \ldots, X_n] \in \tilde{Z}\{X_1, X_2, \ldots, X_n\}$ may be written in the form

$$f[X_1, X_2, \ldots, X_n] = \sum_{\alpha} m_\alpha \prod_{k=1}^{n_\alpha} x_{i_k l_k}^{\alpha_{i_k l_k}}$$

where $m_\alpha \in Z$, and $x_{ik} x_{jl} \neq x_{jl} x_{ik}$ if $(i, k) \neq (j, l)$.

For arbitrary natural numbers α_i, $i = 1, 2, \ldots, n$, $f|_{\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle}$ denotes the expression which is obtained from $f[X_1, X_2, \ldots, X_n]$ by putting $x_{ik} = 0$ for $k > \alpha_i$; $i = 1, 2, \ldots, n$.

Definition 2. The formal power series $f[X_1, X_2, ..., X_n]\in\tilde{Z}\{X_1, X_2, ..., X_n\}$ is said to be *admissible* if for arbitrary natural numbers α_i, $i=1, 2, ..., n$, we always have

$$f|_{\langle\alpha_1,\alpha_2,...,\alpha_n\rangle}\in Z[x_{11}, ..., x_{1\alpha_1}, ..., x_{n_1}, ..., x_{n\alpha_n}].$$

Examples. Let us consider the simple case $n=2$. Let

$$X_1 = X = (x_0, x_1, x_2, ...), \quad X_2 = Y = (y_0, y_1, y_2, ...).$$

a) Consider

$$f_1(X, Y) = \sum_{i=0}^{\infty}(a_i x_i y_i + b_i y_i x_i).$$

For arbitrary natural numbers α_i, $i=1, 2$, we have

$$f_1|_{\langle\alpha_1,\alpha_2\rangle} = \sum_{i=0}^{\alpha} a_i x_i y_i + b_i y_i x_i$$

where $\alpha=\min\{\alpha_1, \alpha_2\}$. Therefore, $f_1|_{\langle\alpha_1,\alpha_2\rangle}$ is in $Z[x_0, ..., x_{\alpha_1}, y_0, ..., y_{\alpha_2}]$ and $f_1[X, Y]$ is admissible.

b) Let

$$f_2[X, Y] = \sum_{k=0}^{\infty} x_0^k + y_0^k + x_k y_k.$$

For any natural numbers α_i, $i=1, 2$, we have

$$f_2|_{\langle\alpha_1,\alpha_2\rangle} = \sum_{k=0}^{\infty} x_0^k + y_0^k + \sum_{i=0}^{\alpha} x_i y_i$$

where $\alpha=\min\{\alpha_1, \alpha_2\}$. Clearly, $f_2|_{\langle\alpha_1,\alpha_2\rangle}$ is not in $Z[x_0, ..., x_{\alpha_1}, y_0, ..., y_{\alpha_2}]$. Therefore, $f_2[X, Y]$ is not admissible.

Definition 3. Suppose that the formal power series

$$f[X_1, X_2, ..., X_n]\in\tilde{Z}\{X_1, X_2, ..., X_n]$$

is admissible. An element a_0 of a ring A is called *f-regular* in A if there exist natural numbers $\alpha_1, \alpha_2, ..., \alpha_n$ such that the polynomial $f|_{\langle\alpha_1,\alpha_2,...,\alpha_n\rangle}$ has at least one solution in A with $x_{i1}=a_0$.

A ring A is said to be f-regular if every element of A is f-regular. An ideal B of a ring A is f-regular if B is an f-regular ring.

The following assertions are analogous to the corresponding assertions in section 2. The proof of the following theorem is a minor modification of the above proof of Theorem 1 therefore, we omit it.

Theorem 1′. *Suppose that the formal power series*

$$f[X_1, X_2, \ldots, X_n] \in \tilde{Z}\{X_1, X_2, \ldots, X_n\}$$

is admissible. Then the class of all f-regular rings is a radical class if and only if the following conditions are satisfied.

1)′ $f[X_1, X_2, \ldots, X_n]$ *has no constant term.*

2)′ *If B is an f-regular ideal of A, and for every element* $a_{11} \in A$ *there exist natural numbers* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *and elements* $a_{i1}, \ldots, a_{i\alpha_i}$, $i=1, 2, \ldots, n$, *in A such that* $f|_{\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle}(a_{11}, \ldots, a_{n\alpha_n}) \in B$, *then the ring A is f-regular.*

Corollary 1′. *Suppose that the formal power series*

$$f[X_1, X_2, \ldots, X_n] \in \tilde{Z}\{X_1, X_2, \ldots, X_n\}$$

without constant terms is admissible. Then the class of all f-regular rings is a radical class if one of the following two conditions is satisfied.

(A)′ *For arbitrary elements* $a_{i1}, a_{i2}, \ldots, a_{i\alpha_i}$, $i=1, 2, \ldots, n$, *in a ring A, if the element* $f|_{\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle}(a_{11}, \ldots, a_{n\alpha_n})$ *is f-regular, then the element* a_{11} *is also f-regular.*

(B)′ *Let J be an f-regular ideal of a ring A; if the coset* $\bar{a}_0$ *containing* $a_0 \in A$ *is f-regular in the factor ring* A/J, *then the element* a_0 *is f-regular in A.*

4. Applications

For the sake of brevity we shall call a polynomial or an admissible formal power series f a *radical expression* if the class of all f-regular rings is a radical class. Next we shall give some radical expressions.

Proposition 2. *The following formal power series are radical expressions.*

a) $$G(m_1, m_2, m_3, m_4) = x_0 + m_1 x_1 + m_2 x_0 x_1 + \sum_{i=1}^{\infty} m_3 y_i x_0 Z_i + m_4 y_i Z_i$$

where m_i, $i=1, \ldots, 4$, *are integers satisfying the condition* $m_1 m_3 = m_2 m_4$.

b) $$F(m_1, m_2, m_3) = x_0 + m_1 x_1 x_0 + m_2 x_0 x_2 + \sum_{i=1}^{\infty} m_s y_i x_0 Z_i$$

where m_i, $i=1, 2, 3$, *are integers satisfying the condition* $m_1 m_2 = 0$ *or* $m_1 m_2 = m_3$.

c) $$H(n, k) = x_0 + \sum_{i=0}^{\infty} k y_{1i} x_0 y_{2i} \ldots x_0 y_{ni}$$

d) $$P_n[p_1(x_0), p_2(x_0)] = x_0 + \sum_{i=1}^{\infty} p_1(x_0) y_{1i} p_1(x_0) \ldots y_{n_i} p_2(x_0)$$

where $p_i(x) \in Z[x]$, $i=1, 2$.

Proof. In order to prove that $G(m_1, m_2, m_3, m_4)$,

$$F(m_1, m_2, m_3), \quad P_n[p_1(x_0), p_2(x_0)] \quad \text{and} \quad H(n, k)$$

are radical expressions, we shall show that each of them satisfies one of the conditions of Corollary 1′.

First we prove that $G(m_1, m_2, m_3, m_4)$ satisfies condition (A)′ of Corollary 1′. Suppose $a_0, \ldots, a_{\alpha_1}, b_0, \ldots, b_{\alpha_2}, c_0, \ldots, c_{\alpha_3}$ are elements of a ring A such that the element

$$G(m_1, m_2, m_3, m_4)|_{\langle\alpha_1, \alpha_2, \alpha_3\rangle}(a_0, \ldots, c_{\alpha_3}) = a_0 + m_1 a_1 + m_2 a_0 a_1 + \sum_{i=1}^{\alpha} m_3 b_i a_0 c_i + m_4 b_i c_i,$$

where $\alpha = \min\{\alpha_1, \alpha_2\}$, is $G(m_1, m_2, m_3, m_4)$-regular. By Definition 3, there exist elements $a_1', b_0', b_1', \ldots, b_{\alpha_2'}', c_0', c_1', \ldots, c_{\alpha_3'}'$ in A such that the following equality is satisfied:

$$\begin{aligned}
&\left[a_0 + m_1 a_1 + m_2 a_0 a_1 + \sum_{i=1}^{\alpha} m_3 b_i a_0 c_i + m_4 b_i c_i\right] + m_1 a_1' + \\
&+ m_2 \left[a_0 + m_1 a_1 + m_2 a_0 a_1 + \sum_{i=1}^{\alpha} m_3 b_i a_0 c_i + m_4 b_i c_i\right] a_1' + \\
&+ \sum_{j=1}^{\alpha'} \left[m_3 b_j' \left(a_0 + m_1 a_1 + m_2 a_0 a_1 + \sum_{i=1}^{\alpha} m_3 b_i a_0 c_i + m_4 b_i c_i\right) c_j' + m_4 b_j' c_j'\right] = 0
\end{aligned}$$

where $\alpha' = \min\{\alpha_2', \alpha_3'\}$.

A straightforward calculation shows that

$$a_0 + m_1 a_1'' + m_2 a_0 a_1'' + \sum_{k=1}^{\alpha''} m_3 b_k'' a_0 c_k'' + m_4 b_k'' c_k'' = 0$$

where

$$\alpha'' = 2(\alpha + \alpha') + \alpha\alpha',$$

$$a_1'' = a_1 + a_1' + m_2 a_1 a_1',$$

$$b_k'' = \begin{cases} b_k & \text{if } 0 < k \leqq \alpha, \\ b_i & \text{if } \alpha < k = \alpha + i \leqq 2\alpha, \\ b_i' & \text{if } 2\alpha < k = 2\alpha + i \leqq 2\alpha + \alpha', \\ m_2 b_i' & \text{if } 2\alpha + \alpha' < k = 2\alpha + \alpha' + i \leqq 2(\alpha + \alpha'), \\ m_3 b_j' b_i & \text{if } 2(\alpha+\alpha') + (j-1)\alpha < k = 2(\alpha+\alpha') + (j-1)\alpha + i \leqq \\ & \qquad \leqq 2(\alpha+\alpha') + j\alpha \quad \text{for } j = 1, 2, \ldots, \alpha', \end{cases}$$

$$c_k'' = \begin{cases} c_k & \text{if } 0 < k \leqq \alpha, \\ c_i a_1' & \text{if } \alpha < k = \alpha + i \leqq 2\alpha, \\ c_i' & \text{if } 2\alpha < k = 2\alpha + i \leqq 2\alpha + \alpha', \\ a_1 c_i' & \text{if } 2\alpha + \alpha' < k = 2\alpha + \alpha' + i \leqq 2(\alpha + \alpha'), \\ c_i c_j' & \text{if } 2(\alpha+\alpha') + (j-1)\alpha < k = 2(\alpha+\alpha') + (j-1)\alpha + i \leqq 2(\alpha+\alpha') + j\alpha \\ & \quad \text{for } j = 1, 2, \ldots, \alpha'. \end{cases}$$

Hence the element a_0 is $G(m_1, m_2, m_3, m_4)$-regular, and condition (A)′ is satisfied. Thus $G(m_1, m_2, m_3, m_4)$ is a radical expression. The remaining assertions are proved similarly.

Now let us survey some well-known regularities and ring properties which have already been shown to be radicals.

1) An element a_0 of a ring A is said to be regular in the sense of VON NEUMANN [17], if $a_0 \in a_0 A a_0$. If in a) in Proposition 2 we take $n=1$, $p_i(x_0)=x_0$, $i=1, 2$, then, clearly, $P_1(x_0, x_0)$-regularity coincides with the regularity in the sense of von Neumann. Therefore, the class of all von Neumann regular rings is a radical class.

2) An element a_0 of a ring A is said to be right quasi-regular, as defined by PERLIS [18] and later studied by BAER [4] and JACOBSON [11], if $a_0+a_1+a_0a_1=0$, for some element a_1 of A. By a) and d) in Proposition 2 we have

$$G(1, 1, 0, 0) = x_0+x_1+x_0x_1.$$

Hence right quasi-regularity is nothing else than $G(1, 1, 0, 0)$-regularity. Thus, right quasi-regularity is a radical property, namely, the Jacobson radical.

3) BROWN and MCCOY [6] have introduced the notion of G-regularity. An element a_0 of a ring A is said to be G-regular if the element a_0 is in $G(a_0)$, where

$$G(a_0) = A(1+a_0)+A(1+a_0)A.$$

By a) in Proposition 2 it is clear that $G(1, 1, 1, 1)$-regularity coincides with G-regularity. Thus the Brown—McCoy radical may be determined by the radical expression $G(1, 1, 1, 1)$.

4) The notion of strongly regular rings had been introduced by ARENS and KAPLANSKY [3] and was later studied by KANDÔ [12], LAJOS and SZÁSZ [14] and others. A ring A is strongly regular if $a \in a^2 A$ for every $a \in A$. If in d) in Proposition 2 we take $n=1$, $p_1(x_0)=x_0^2$, $p_2(x_0)=1$, then it is clear that $P_1(x_0^2, 1)$-regularity is the same as strong regularity. Thus, strong regularity is a radical property.

5) DE LA ROSE [19] has introduced the notion of λ-regularity. An element a_0 of a ring A is λ-regular if $a_0 \in A a_0 A$. By a) and d) in Proposition 2 we have

$$G(0, 0, 1, 0) = F(0, 0, 1) = x_0+\sum_{i=1}^{\infty} y_i x_i Z_i.$$

Clearly, λ-regularity can be defined by the radical expression $G(0, 0, 1, 0)$. Thus the class of λ-regular rings is a radical class.

6) DIVINSKY [8] has introduced left pseudo-regularity. An element a_0 of a ring A is left pseudo-regular if $a_0+a_1a_0+a_1a_0^2=0$ for some element $a_1 \in A$. If in d) in

Proposition 2 we take $n=1$, $p_1(x_0)=1$, $p_2(x_0)=x_0+x_0^2$, then it is easy to see that $P_1(1, x_0+x_0^2)$-regularity coincides with left pseudo-regularity. Therefore left pseudo-regularity is a radical property.

7) Following SZÁSZ [22] a ring A is called an E_5-ring if every homomorphic image of A has no non-zero left annihilators. As is proved in [22], a ring A is an E_5-ring if and only if $a\in Aa+AaA$ holds for every $a\in A$. By b) in Proposition 2, the class of E_5-rings is the class of all $F(1, 0, 1)$-regular rings, so it is a radical class.

8) Following SZÁSZ [25] a ring A is called an E_6-ring if every homomorphic image of A has no non-zero two-sided annihilators. A ring A is an E_6-ring if and only if $a\in aA+Aa+AaA$ holds for every $a\in A$. By b) in Proposition 2 the class of E_6-rings coincides with the class of $F(1, 1, 1)$-regular rings. Thus, it is a radical class.

9) BLAIR [5] introduced the notion of f-regularity, which was later studied by ANDRUNAKIEVIČ [2]. An element a of a ring A is said to be f-regular (in the sense of Blair) if $a\in(a)^2$, where (a) denotes the principal ideal of A generated by a. Blair has shown that an element a in a ring A is f-regular if and only if there exist elements u_i, v_i and w_i in A such that $a=\sum_{i=1}^{n} u_i a_0 v_i a w_i$. Hence, by c) in Proposition 2, f-regularity in the sense of Blair is the same as $H(3, 1)$-regularity. Thus it is a radical property.

10) By b) in Proposition 2 we have

$$F(1,0,0) = x_0+x_1x_0.$$

Therefore $F(1, 0, 0)$-regularity is D-regularity of DIVINSKY [9].

11) If in d) in Proposition 2 we take $n=1$, $p_1(x_0)=q(x_0)$, $p_2(x_0)=q(x_0)$, then we see that $P_1(p(x_0), q(x_0))$-regularity is nothing else than (p, q)-regularity, introduced by MCKNIGHT [15] and also studied by others [10], [16].

Remark. By means of Proposition 2, we can get a great variety of radical classes. In order to show that consider for instance the radical expressions $G(p, -p, k, -k)$, where p is a prime number. Denote by $S_{(p,k)}$ the class of $G(p, -p, k, -k)$-regular rings. Take a fixed set of symbols $M=\{\alpha, \beta, \ldots\}$. Let A_p be the (associative and noncommutative) ring on the set M over the field Z_p of integers modulo p with the relation: $\alpha\beta=\alpha$, $\alpha, \beta\in M$.

One can prove easily that A_p is not in $S_{(p,k)}$ but it belongs to $S_{(p',k)}$ for every prime $p'>p$. Hence $S_{(p,k)}\neq S_{(p',k)}$ if $p\neq p'$.

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The hyperbolic M. Riesz theorem

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1. Introduction. We shall prove the non-Euclidean hyperbolic versions of the theorems of M. RIESZ [7] and of L. FEJÉR and F. RIESZ [4] (see [8, Theorem VIII.45, p. 339 and Theorem VIII.46, p. 340], [3, p. 46]) and show a property of conformal mappings in terms of the non-Euclidean geometry in the unit disk.

Let $D=\{|z|<1\}$, $T=[0, 2\pi)$, and $K=\{e^{it}|t\in T\}$. Let

$$\sigma(z, w) = \tanh^{-1}(|z-w|/|1-\bar{z}w|)$$

be the non-Euclidean hyperbolic distance between z and w in D, and let

$$\sigma(z) \equiv \sigma(z, 0) = (1/2) \log [(1+|z|)/(1-|z|)], \quad z\in D.$$

Let B be the family of functions f, holomorphic and bounded, $|f|<1$, in D. Then $\sigma(f)$ for $f\in B$, like $|f|$, has the property that $\log \sigma(f)$ is subharmonic in D, so that $\sigma(f)^p=\exp [p \log \sigma(f)]$ is subharmonic in D for all $p>0$; see [10]. Let H^p_σ be the family of $f\in B$ such that

$$\int_T \sigma(f)^p (re^{it})\, dt$$

is bounded for $0\leqq r<1$ $(0<p<\infty)$. The class H^p_σ is the hyperbolic counterpart of the (parabolic) Hardy class H^p in D [3, p. 2], and is called the hyperbolic Hardy class. (Recently, it is observed that an "elliptic" analogue of H^p $(0<p<\infty)$, namely, a meromorphic Hardy class yields no new family [11, Theorem 1].) Each $f\in B$ has the radial limit $f^*(t)=\lim_{r\to 1-0} f(re^{it})$ at e^{it} for a.e. $t\in T$, and as will be seen, $\sigma(f^*)$ is of class $L^p(T)$ for all $f\in H^p_\sigma$ $(0<p<\infty)$. The hyperbolic M. Riesz theorem is

Theorem 1. *Let C be a rectifiable curve with the initial point a and the terminal point b (possibly, $a=b$) in the complex plane. Suppose that*

$$C\subset D\cup K \quad \text{and} \quad C\cap K\subset \{a, b\}.$$

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Set for $t \in T$, *with* $e^{it} \notin C \cap K$,

$$V(t) = \int_C |d \arg (e^{it} - w)| \quad (w \in C).$$

Then, for each $f \in H_\sigma^p$ $(0 < p < \infty)$,

$$\int_C \sigma(f)^p(z)\, |dz| \leqq \pi^{-1} \int_T \sigma(f^*)^p(t) V(t)\, dt.$$

If C is the diameter $\{xe^{is} | -1 \leqq x \leqq 1\}$ $(s \in T)$, then $V(t) \equiv \pi/2$, so that the hyperbolic Fejér and F. Riesz theorem is

Theorem 2. *For each* $f \in H_\sigma^p$ $(0 < p < \infty)$ *and each* $s \in T$,

$$\int_{-1}^{1} \sigma(f)^p(xe^{is})\, dx \leqq (1/2) \int_T \sigma(f^*)^p(t)\, dt.$$

The Fejér and F. Riesz theorem has the obvious application to conformal mappings from D onto a Jordan domain with the rectifiable boundary [4, Satz IV]; see [8, Corollary, p. 341] and [3, Corollary, p. 47]. The hyperbolic version is not so apparent as in the cited case; namely, the following theorem does not appear to be a direct consequence of Theorem 2. There is no relation between $\sigma(f)$ and $|f'|/(1-|f|^2)$ like that between $|f|$ and $|f'|$.

Theorem 3. *Let* γ *be a Jordan curve in* D *with finite non-Euclidean length* L. *Let* f *be a one-to-one conformal mapping from* D *onto the interior of* γ. *Then the non-Euclidean length of the image of each diameter by* f *is not greater than* $L/2$. *The constant* 2 *in* $L/2$ *cannot be replaced by any larger constant.*

For the proofs of Theorems 1 and 3, the principal idea is to obtain the M. Riesz theorem for subharmonic functions of class PL in the sense of E. F. Beckenbach and T. Radó [2] (see also [6, p. 9]); see Theorem 4 in Section 2.

2. Subharmonic functions of class PL. A function u defined in D is called of class PL in D if $u \geqq 0$ (possibly, $u \equiv 0$) and $\log u$ is subharmonic in D; we regard $-\infty$ as a subharmonic function. The family of all functions of class PL in D is denoted by PL again. All members of PL are subharmonic in D, and if $u \in PL$, then $u^p \in PL$ for each $p > 0$. If f is holomorphic in D, then $|f| \in PL$, and further, if $f \in B$, then $\sigma(f) \in PL$. Let PL^p be the family of all $u \in PL$ such that u^p has a harmonic majorant in D $(0 < p < \infty)$. Here, a function v subharmonic in D is said to have a harmonic majorant h in D if h is harmonic and $v \leqq h$ in D. The class H^p is the family of f holomorphic in D such that $|f| \in PL^p$, while H_σ^p is the family of $f \in B$ such that $\sigma(f) \in PL^p$ $(0 < p < \infty)$.

Theorem 4. *Let C and V be as in Theorem* 1. *Then, each* $u \in PL^p$ $(0<p<\infty)$ *has the radial limit* $u^*(t)$ *at* e^{it} *for a.e.* $t \in T$, *and*

$$\int_C u^p(z)\,|dz| \leqq \pi^{-1} \int_T (u^*)^p(t) V(t)\,dt.$$

Earlier, a special case of Theorem 4, where $p=1$, C is an arbitrary diameter, u is continuous on $D \cup K$, and $u \in PL$, was established by Beckenbach [1, Theorem 2]. It is now an easy exercise to extend a geometric theorem of Beckenbach [1, Theorem 3] with the aid of Theorem 4.

Theorem 1 (and consequently, Theorem 2) now follows from Theorem 4, applied to $\sigma(f) \in PL^p$; note that $\sigma(f)^* = \sigma(f^*)$. The theory of subharmonic functions of class PL thus serves for the differential geometry, as originated by Beckenbach and Radó, as well as for the hyperbolic Hardy classes.

For the proof of Theorem 4 we shall make use of

Lemma 1 [5, Theorem]. *Let* $\varphi \geqq 0$ *be a function convex and increasing on* $(-\infty, +\infty)$, *and suppose that*

$$\varphi(t)/t \to +\infty \quad as \quad t \to +\infty.$$

Set $\varphi(-\infty) = \lim_{t \to -\infty} \varphi(t)$, *and let v be a subharmonic function in D such that* $\varphi(v)$, *again subharmonic, has a harmonic majorant in D. Then the radial limit* $v^*(t)$ *exists at* e^{it} *for a.e.* $t \in T$, *and is of* $L^1(T)$, *such that*

$$v(z) \leqq (2\pi)^{-1} \int_T \frac{1-|z|^2}{|e^{it}-z|^2} v^*(t)\,dt.$$

Furthermore, $\varphi(v^*) \in L^1(T)$.

In effect, v admits a positive harmonic majorant in D (see [9, p. 65]), so that $v = v^\wedge - q$, where $q \geqq 0$ is a Green's potential and $v^\wedge$ is the least harmonic majorant of v in D, expressed by the Poisson integral of a signed measure

$$d\mu(t) = v^*(t)\,dt + d\mu_S(t) \quad \text{on } T,$$

where $d\mu_S$ is singular with respect to dt. Now, [5, Theorem] asserts that $d\mu_S(t) \leqq 0$ on T and $\varphi(v^*) \in L^1(T)$.

Proof of Theorem 4. Since $u^p \in PL^1$ with $(u^p)^* = (u^*)^p$, it suffices to prove the theorem in the case $p=1$. Set $\varphi(t) = e^t$ and $v = \log u$ and consider Lemma 1. Since $\varphi(v)$ has a harmonic majorant, v has the harmonic majorant

$$h(z) = (2\pi)^{-1} \int_T \frac{1-|z|^2}{|e^{it}-z|^2} v^*(t)\,dt \quad (z \in D).$$

Furthermore, $h^* = v^* = \log u^*$. Since $u^* = \varphi(v^*) \in L^1(T)$ by Lemma 1, it follows from Jensen's inequality that $e^h \leqq g$, where g is the Poisson integral of $\varphi(v^*)=u^*$. Thus,

$$f \equiv e^{h+ik} \in H^1,$$

where k is a harmonic conjugate of h in D. Therefore $|f^*|=|f|^*=e^{h^*}=u^*$ and

$$u = e^v \leqq e^h = |f| \quad \text{in } D.$$

We now apply M. Riesz's cited theorem to f of Hardy class H^1 to obtain the following chain of estimates:

$$\int_C u(z)\,|dz| \leqq \int_C |f(z)|\,|dz| \leqq \pi^{-1} \int_T |f^*(t)|\, V(t)\, dt =$$

$$= \pi^{-1} \int_T u^*(t) V(t)\, dt,$$

whence follows Theorem 4.

3. Conformal mappings. We remember that if f is holomorphic in D and if $f' \in H^1$, then f is continuous on $D \cup K$ and $f(e^{it})$ is absolutely continuous as a function of $t \in T$ with

$$\frac{d}{dt} f(e^{it}) = ie^{it}(f')^*(t) \quad \text{for a.e.} \quad t \in T, \tag{3.1}$$

where $(f')^*(t)$ is again the radial limit of f' at e^{it}; see [3, Theorem 3.11, p. 42]. For $f \in B$ we denote

$$f^\#(z) = |f'(z)|/(1-|f(z)|^2), \quad z \in D,$$

and for the proof of Theorem 3 we shall make use of

Lemma 2. *Let $f \in B$ and $f' \in H^1$, and assume that $|f(e^{it})|<1$ for all $t \in T$. Then $f^\# \in PL^1$ and*

$$(f^\#)^*(t) = \left|\frac{d}{dt} f(e^{it})\right| / (1-|f(e^{it})|^2)$$

for a.e. $t \in T$.

Proof. A calculation yields that $\Delta \log f^\# = 4(f^\#)^2 > 0$ except for the zeros of f', so that $f^\# \in PL$. Since $|f|$ is bounded by a constant $c<1$ in D, it follows from $f^\# \leqq |f'|/(1-c^2)$ in D with $f' \in H^1$ that $f^\# \in PL^1$. Since $(f^\#)^* = |(f')^*|/(1-|f|^2)$ a.e. on T, the second assertion follows from (3.1).

Proof of Theorem 3. Since γ is rectifiable (in the Euclidean sense), it follows from [3, Theorem 3.12, p. 44] that $f' \in H^1$. By Lemma 2, $f^\# \in PL^1$. Since

$$L = \int_T \frac{|df(e^{it})|}{1-|f(e^{it})|^2} = \int_T \frac{\left|\frac{d}{dt} f(e^{it})\right|}{1-|f(e^{it})|^2}\, dt,$$

it further follows from Lemma 2 that

$$L = \int_T (f^{\#})^*(t)\, dt.$$

The first assertion in Theorem 3 now follows from Theorem 4 applied to each diameter C and $f^{\#} \in PL^1$.

It remains to prove the sharpness of 2 in $L/2$. For simplicity we consider the half-plane $R=\{w|\mathrm{Re}\, w>0\}$ with the non-Euclidean metric $|dw|/[2\,\mathrm{Re}\, w]$ in the differential form. The non-Euclidean length of a curve Γ in R is denoted by $\lambda(\Gamma)$. Let $\varepsilon>0$ and let $0<a<b$. Consider the Euclidean rectangle Q with the vertices $z_1=a+\varepsilon i$, $z_2=a-\varepsilon i$, $z_3=b-\varepsilon i$, and $z_4=b+\varepsilon i$. Let f_ε be a one-to-one conformal mapping from D onto Q such that f_ε maps the diameter $[-1, 1]$ onto the segment ab on the real axis. If we show that

$$\lambda(f_\varepsilon(K))/\lambda(ab) \to 2 \quad \text{as} \quad \varepsilon \to 0, \tag{3.2}$$

then the function $(f_\varepsilon-1)/(f_\varepsilon+1)$ serves as an example for the sharpness. Let z_1z_2, z_2z_3, z_3z_4, and z_4z_1 be the four sides of $f_\varepsilon(K)$. A calculation yields that

$$\lambda(z_4z_1) = \lambda(z_2z_3) = (1/2)\log(b/a) = \lambda(ab),$$

and as $\varepsilon \to 0$,

$$\lambda(z_1z_2) = \varepsilon/a \to 0 \quad \text{and} \quad \lambda(z_3z_4) = \varepsilon/b \to 0.$$

Therefore (3.2) holds.

Appendix. Tsuji's proof of M. Riesz's theorem contains an obscure point. There is a gap between (5) and (6) in [8, p. 341]; the meanings of $\partial/\partial x$ in (5) and (6) are different. Since M. Riesz did not raise his result explicitly as in [8, Theorem VIII.46, p. 340], we must avoid this difficulty. The principal point is to prove that, for f holomorphic on $D \cup K$,

$$\int_C |f(w)|\,|dw| \leqq \pi^{-1} \int_T |f(e^{it})|\, V(t)\, dt, \tag{A}$$

where C and V are the same as in Theorem 1. Choose points $w_0=a, w_1, \ldots, w_{n-1}, w_n=b$ on C in this order. Then

$$V(t) = \lim \sum_{k=1}^{n} |\arg(e^{it}-w_k) - \arg(e^{it}-w_{k-1})|$$

as $\max\limits_{1\leqq k\leqq n} |w_k-w_{k-1}| \to 0$, where $\arg(e^{it}-w)$ is a fixed branch in D; $V(t)$ is Lebesgue measurable on T. Now it follows from [7, (3), p. 54] (a careful reading shows that the cited point is true even if A or B lies on K) that the following estimate of the integral on the rectilinear segment $w_{k-1}w_k$ holds:

$$\int_{w_{k-1}w_k} |f(w)|\,|dw| \leqq \pi^{-1} \int_T |f(e^{it})|\,|\arg(e^{it}-w_k)-\arg(e^{it}-w_{k-1})|\,dt,$$

$1 \leqq k \leqq n$. Summing up both sides from $k=1$ to n, and letting $\max_{1\leqq k\leqq n} |w_k - w_{k-1}| \to 0$, we obtain (A).

Remark. It might be more appropriate to call [8, Theorem VIII.46] the F. Carlson and M. Riesz theorem.

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A maximum principle for interpolation in H^∞

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There are several contexts in which it is desirable to calculate norms in quotient algebras of H^∞, the algebra of all bounded analytic functions in the open unit disc U, with supremum norm. For example, one version of the Nevanlinna—Pick problem (see [3] or [7]) is to find a function $f \in H^\infty$ which takes prescribed values at given points of U and minimises $\|f\|_{H^\infty}$: the minimum value of $\|f\|_{H^\infty}$ can be expressed as the quotient norm $\|g+\varphi H^\infty\|_{H^\infty/\varphi H^\infty}$ where g is some function and φ is a Blaschke product, and, once this quotient norm is known, an algorithm due to Schur and Nevanlinna enables the construction of the desired minimising function f [7]. It is less immediate that quotient norms of H^∞ are also important in an extremal problem for matrices raised by V. PTÁK [4]: to find the maximum value of $\|A^m\|$ over all contractions A on n-dimensional Hilbert space, $n \leqq m$, subject to $|A|_\sigma \leqq r < 1$ ($|A|_\sigma$ is the spectral radius of A). An account of this problem is given in [5].

The second example motivates the study of the quantity $\|\psi+\varphi H^\infty\|_{H^\infty/\varphi H^\infty}$ as a function of the zeros of the Blaschke product φ, for fixed ψ. It is a plausible guess that some sort of maximum principle should hold for this quantity: the closer the zeros of φ are allowed to approach the unit circle, the larger should be $\|\psi+\varphi H^\infty\|$. This is in fact true in the case $\psi(z)=z^n$, where n is the degree of φ, as was established by PTÁK [4] in 1968. The purpose of this paper is to prove a generalization of Pták's result: a maximum principle is true whenever ψ is a Blaschke product of the same degree as φ. The method is quite different from Pták's, and offers some hope of further generalization.

We shall say that *the maximum principle holds for* $f\colon \Omega \subseteqq \mathbf{C} \to \mathbf{R}$ if, for any compact set $K \subseteqq \Omega$, the supremum of f on K is attained at some point of the boundary of K relative to Ω. And, if Ω is a subset of a complex vector space V, we say that the maximum principle holds for $f\colon \Omega \to \mathbf{R}$ if, for any plane π in V, the maximum principle holds for the restriction of f to $\pi \cap \Omega$; or, to put it more precisely, if, for all $a \in \Omega$ and $b \in V$, then maximum principle holds for the function

$$g\colon \{\lambda \in \mathbf{C}\colon a+\lambda b \in \Omega\} \to \mathbf{R} \quad \text{defined by} \quad g(\lambda) = f(a+\lambda b).$$

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Theorem: *Let n be a natural number and let*

$$F(\alpha, \beta) = \|\psi+\varphi H^{\infty}\|_{H^{\infty}/\varphi H^{\infty}}$$

where $\alpha=(\alpha_1, \dots, \alpha_n)\in U^n$, $\beta=(\beta_1, \dots, \beta_n)\in U_n$,

$$\varphi(z) = \prod_{i=1}^{n} \frac{z-\alpha_i}{1-\bar{\alpha}_i z} \quad \text{and} \quad \psi(z) = \prod_{i=1}^{n} \frac{z-\beta_i}{1-\bar{\beta}_i z}.$$

Then $F(\alpha, \beta)=F(\beta, \alpha)$ *and, for any* $\alpha, \beta\in U^n$, *the maximum principle holds for the functions* $F(\alpha, \cdot)$, $F(\cdot, \beta)$ *on* U^n.

We shall show that F is the composition of a strictly increasing and a separately plurisubharmonic function. This will follow from an interesting formula which expresses F in terms of the norm of an analytic operator-valued function. This formula also makes it easy to see the surprising symmetry property of F, which was earlier established by a different method in [5, Sec. 7].

The whole is based the well known theorem of SARASON [6] which interprets $\|\psi + \varphi H^{\infty}\|$ operator-theoretically. Let T denote the unilateral backward shift on l^2: that is,

$$T(x_0, x_1, x_2, \dots) = (x_1, x_2, x_3, \dots).$$

Sarason's theorem asserts (among other things) that, for any $\psi\in H^{\infty}$ and inner function φ,

$$\|\psi+\varphi H^{\infty}\|_{H^{\infty}/\varphi H^{\infty}} = \|\psi(T)|\operatorname{Ker}\varphi(T)\|, \tag{1}$$

the symbol $\|\cdot\|$ denoting the operator norm on l^2 and the vertical bar denoting restriction.

Let P^*: $\operatorname{Ker}\varphi(T)\to l^2$, Q^*: $\operatorname{Ker}\psi(T)\to l^2$ be the natural injection mappings. Note that PP^*, QQ^* are the identity operators on $\operatorname{Ker}\varphi(T)$, $\operatorname{Ker}\psi(T)$ respectively, while P^*P, Q^*Q are the Hermitian projections with ranges $\operatorname{Ker}\varphi(T)$, $\operatorname{Ker}\psi(T)$.

Lemma 1. *QP^*: $\operatorname{Ker}\varphi(T)\to\operatorname{Ker}\psi(T)$ is invertible.*

Proof. Since both kernels have dimension n it suffices to show that QP^* is injective, or equivalently, that $\operatorname{Ker}\varphi(T)\cap\operatorname{Ker}\psi(T)^{\perp}=\{0\}$. For this purpose it is convenient to identify l^2 with the Hardy space H^2 of analytic functions in U in the usual way [2]. T acts on H^2 by

$$Th(z) = \frac{1}{z}\{h(z)-h(0)\}$$

while T^* acts as multiplication by z. Thus

$$\operatorname{Ker}\psi(T)^{\perp} = \operatorname{Range}\psi(T)^* = \psi^* H^2,$$

where $\psi^*(z)=\psi(\bar{z})^-$. Thus every element of $\operatorname{Ker}\psi(T)^{\perp}$ has at least n zeros in U. It is not hard to show that $\operatorname{Ker}\varphi(T)$ consists of all functions of the form g/k where

g is a polynomial of degree less than n and $k(z)=(1-a_1z)\dots(1-a_nz)$ (this can be done by induction on n). It follows that any element of $\operatorname{Ker}\varphi(T)$ which does not vanish identically has at most $n-1$ zeros in U, and therefore does not belong to $\operatorname{Ker}\psi(T)^\perp$.

Lemma 2. *Let*

$$p(z)=(z-\alpha_1)\dots(z-\alpha_n),\quad p_0(z)=(1-\bar\alpha_1 z)\dots(1-\bar\alpha_n z),$$
$$q(z)=(z-\beta_1)\dots(z-\beta_n),\quad q_0(z)=(1-\bar\beta_1 z)\dots(1-\bar\beta_n z).$$

Then

$$F(\alpha,\beta)^2=1-\|I-q_0(T)^{*-1}p(T)^*q(T)p_0(T)^{-1}\|^{-2}.$$

Proof. $\psi(T)^*$ is the operation of multiplication by the Blaschke product ψ^* and is therefore an isometry, so that $\psi(T)\psi(T)^*=I$, Hence $I-\psi(T)^*\psi(T)$ is a Hermitian projection, and its range is easily seen to be $\operatorname{Ker}\psi(T)$, Thus

$$I-\psi(T)^*\psi(T)=Q^*Q.$$

Equation (1) now gives

$$F(\alpha,\beta)^2=\|\psi(T)|\operatorname{Ker}\varphi(T)\|^2=\|\psi(T)P^*\|^2=\|P\psi(T)^*\psi(T)P^*\|=$$
$$=\|P(I-Q^*Q)P^*\|=|I_{\operatorname{Ker}\varphi(T)}-PQ^*QP^*|_\sigma.$$

Since $0\leqq PQ^*QP^*\leqq I$, this implies

$$F(\alpha,\beta)^2=1-\inf\sigma(PQ^*QP^*),$$

and since PQ^*, QP^* are invertible (by Lemma 1) this can be written

$$F(\alpha,\beta)^2=1-|(QP^*)^{-1}(PQ^*)^{-1}|_\sigma^{-1}. \tag{2}$$

We can make further progress through the use of non-orthogonal projections. Recall that if l^2 is the direct sum of subspaces E and F then the *projection of l^2 on E along F* is defined to be the operator $R\colon l^2\to l^2$ given by $R(x+y)=x$ $(x\in E, y\in F)$. If is easy to see that R is characterized by the three properties (a) $R^2=R$ (b) Range $R\subseteqq E$ (c) Range $R^*\subseteqq F^\perp$. Thus, in particular, the projection on $\operatorname{Ker}\psi(T)$ along $\operatorname{Ker}\varphi(T)^\perp$ is characterized by the three properties

$$\text{(a)}\ R^2=R,\qquad \text{(b)}\ \psi(T)R=0,\qquad \text{(c)}\ \varphi(T)R^*=0. \tag{3}$$

Lemma 3. *The following are equivalent for an operator R on l^2:*
(i) *R is the projection on* $\operatorname{Ker}\psi(T)$ *along* $\operatorname{Ker}\varphi(T)^\perp$;
(ii) $R=Q^*(PQ^*)^{-1}P$;
(iii) $R=I-q_0(T)^{*-1}p(T)^*q(T)p_0(T)^{-1}$.

Proof. It is clear that $R=Q^*(PQ^*)^{-1}P$ satisfies conditions (a)—(c) of (3). These conditions may also be verified for R in (iii) with the aid of the identities

$$q(T)p(T)^*=p_0(T)q_0(T)^*,\quad p(T)q(T)^*=q_0(T)p_0(T)^*. \tag{4}$$

To prove the former, note that, since $TT^*=I$,

$$(T-\beta_j I)(T-\alpha_j I)^* = (I-\bar{\alpha}_j T)(I-\bar{\beta}_j T)^*.$$

Induction on n now establishes (4).

We return to the proof of Lemma 2. Let R be the projection on $\operatorname{Ker}\psi(T)$ along $\operatorname{Ker}\varphi(T)^{\perp}$. By Lemma 3,

$$(PQ^*)^{-1} = QQ^*(PQ^*)^{-1}PP^* = QRP^*$$

and hence (2) becomes

$$F(\alpha,\beta)^2 = 1-|PR^*Q^*QRP^*|_\sigma^{-1}. \tag{5}$$

From the definition of R and the fact that Q^*Q, P^*P are the orthogonal projections onto $\operatorname{Ker}\psi(T)$, $\operatorname{Ker}\varphi(T)$ respectively, it follows that $Q^*QR=R=RP^*P$. In view of the fact that $|AB|_\sigma=|BA|_\sigma$ whenever AB and BA are both defined, (5) yields

$$F(\alpha,\beta)^2 = 1-|PR^*RP^*|_\sigma^{-1} = 1-|RP^*PR^*|_\sigma^{-1} = 1-|RR^*|_\sigma^{-1} =$$
$$= 1-\|R\|^{-2} = 1-\|I-q_0(T)^{*-1}p(T)^*q(T)p_0(T)^{-1}\|^{-2}.$$

Lemma 2 is proved.

Lemma 4. *Let*

$$G(\alpha,\beta) = \|I-q_0(T)^{*-1}p(T)^*q(T)p_0(T)^{-1}\|.$$

For any $\alpha,\beta\in U^n$ *the functions* $G(\alpha,\cdot)$, $G(\cdot,\beta)$ *are plurisubharmonic on* U^n, *and* $G(\alpha,\beta)=G(\beta,\alpha)$.

Proof. If R has the same meaning as above then $G(\alpha,\beta)=\|R\|$ and

$$G(\beta,\alpha) = \|I-p_0(T)^{*-1}q(T)^*p(T)q_0(T)^{-1}\| = \|R^*\| = \|R\| = G(\alpha,\beta).$$

A function on U^n is plurisubharmonic provided it is upper semi-continuous and its restriction to the intersection with U^n of any plane is subharmonic. Now

$$q(T) = T^n-(\beta_1+\ldots+\beta_n)T^{n-1}+\ldots+(-1)^n\beta_1\ldots\beta_n I,$$
$$q_0(T)^* = I-(\beta_1+\ldots+\beta_n)T^*+\ldots+(-1)^n\beta_1\ldots\beta_n T^{*n}.$$

Hence $q(T)$ and $q_0(T)^*$ are analytic operator-valued functions of β in U^n, and the same is therefore true of R (for fixed α). Certainly $G(\alpha,\cdot)$ is continuous on U^n. Furthermore, for any analytic function f defined on an open set $\Omega\subseteqq\mathbf{C}$ and taking values in any Banach space, the real-valued function $z\to\|f(z)\|$ is subharmonic on Ω [1, Thm. 3.12.1]. It follows that, for a fixed α, $G(\alpha,\cdot)$ is plurisubharmonic on U^n, and so, by symmetry, is $G(\cdot,\beta)$ for fixed β.

We can now conclude the proof of the theorem. Since the maximum principle holds for any subharmonic function defined in an open subset of the plane [1], the maximum principle also holds for any plurisubharmonic function on U^n, hence for $G(\alpha, \cdot)$. By Lemma 2 $F(\alpha, \cdot)=h \circ G(\alpha, \cdot)$ where $h(t)=(1-t^{-2})^{1/2}$. Since h is a strictly increasing function on $[1, \infty)$, the maximum principle holds for $F(\alpha, \cdot)$. And since G is symmetric in α and β, so is F.

Let us consider the implications of the theorem for Pták's problem. Suppose we wish to find an operator A on n-dimensional Hilbert space which maximises $\|\psi(A)\|$ subject to the constraints $\|A\| \leqq 1$, $|A|_\sigma \leqq r$. The search can be split into two steps:

I. For each polynomial p having all its zeros in the disc $\{z: |z| \leqq r\}$ find an $n \times n$ matrix which maximises $\|\psi(A)\|$ subject to $\|A\| \leqq 1$, $p(A)=0$.

II. Among all such polynomials p find the one for which the corresponding maximum of $\|\psi(A)\|$ is the largest.

Problem I has been completely solved — see [5, Sec. 3]. The solution is made simpler than might be expected by the fact that an extremal matrix can be given which is independent of ψ.

Problem II, called in [5, Sec. 2] the "problem of the worst polynomial", is difficult. It has been solved only in the case that $\psi(z)=z^n$, when a worst polynomial is $p(z)=(z-\varepsilon r)^n$, for any ε, $|\varepsilon|=1$. One can hardly doubt that the same polynomial will be extremal for higher powers of z also, and it is even conceivable that $(z-\varepsilon r)^n$ is the worst polynomial for *all* $\psi \in H^\infty$. However, twelve years have elapsed since Pták solved the case $\psi(z)=z^n$ by a very special method, and attempts by several mathematicians to extend Pták's conclusion to other functions have had no success. It is not even known whether the worst polynomial has its zeros on the circle $|z|=r$ in general. The present paper shows that, if ψ is a Blaschke product of degree n, then there is a worst polynomial with all its zeros on $|z|=r$. The question remains as to whether these zeros can all be taken to coincide, as in the known case. It would also be desirable to extend the theorem to higher powers of z or, more generally, to Blaschke products ψ of arbitrary degree.

I will indicate the difficulties involved in extending the above method to the case that ψ is a Blaschke product of degree m, $m>n$. Exactly as before we have

$$\|\psi+\varphi H^\infty\|^2 = 1-\inf \sigma(PQ^*QP^*),$$

but it is no longer true that QP^* is invertible (it acts between spaces of different dimension). To get round this we can introduce the space $E=\text{Range } Q^*QP^*$, the orthogonal projection of Ker $\varphi(T)$ on Ker $\psi(T)$, and let Q_1^* be the natural injection of E into l^2. Then Q_1P^* is invertible and

$$\|\psi+\varphi H^\infty\|^2 = 1-\inf \sigma(PQ_1^*Q_1P^*),$$

and if we denote by R_1 the projection on E along Ker $\varphi(T)^\perp$ then we have as before

$$\|\psi+\varphi H^\infty\|^2 = 1-\|R_1\|^{-2}.$$

It is not hard to show that

$$E = \operatorname{Ker} \psi(T) \cap (\operatorname{Ker} \varphi(T) + \psi^* H^2)$$

and that R_1 can be expressed by the formula

$$R_1^* = I - X[\varphi(T)X]^{-1}\varphi(T), \quad \text{where} \quad X = pq_0^{-1}(T)^* F + q(T)^* T^{m-n},$$

$\psi = q/q_0$ and F is the Hermitian projection with range Ker T^{m-n}. The problem is that R_1 is no longer an analytic function of α. It is of course still possible that $\|R_1\|$ is plurisubharmonic, but I have not been able to prove it.

Note that the formula in Lemma 2 can be used to give an upper bound for $\|\psi + \varphi H^\infty\|$ which is strictly less than one.

I conclude with an observation about the intriguing fact that $\|\psi + \varphi H^\infty\| = \|\varphi + \psi H^\infty\|$ when φ, ψ are Blaschke products of the same degree. In fact a stronger statement is known to folklore: if u is a continuous unimodular function on the circle having winding number zero then u and $1/u$ are equally distant from H^∞. Paul Koosis has provided a neat and simple proof of this fact in a personal communication. His proof is based on the observation that, if $|1-w| \leqq d \leqq 1$, $w \in \mathbf{C}$, then $\left|1 - \frac{1-d^2}{w}\right| \leqq d$. If $g \in H^\infty$ and $\|u-g\| \leqq d < 1$ then an application of Rouché's theorem establishes that $1/g \in H^\infty$, and

$$\left\| \frac{1}{u} - \frac{1-d^2}{g} \right\| \leqq d.$$

The symmetry of $F(\alpha, \beta)$ is deduced by putting $u = \psi/\varphi$.

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The ideal lattice of a distributive lattice with 0 is the congruence lattice of a lattice

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The congruence lattice of an arbitrary lattice is a distributive algebraic lattice, i.e. the ideal lattice of a distributive semilattice with 0. The converse of this statement is a long-standing conjecture of lattice theory. We prove the following:

Theorem. *Let L be the lattice of all ideals of a distributive lattice with* 0. *Then there exists a lattice K such that L is isomorphic to the congruence lattice of K.*

The conjecture was first established for finite distributive lattices by R. P. Dilworth. Later, it was solved for the ideal lattice of relatively pseudo-complemented join-semilattices (E. T. Schmidt [4], [5]).

The first section of this paper reviews the definitions and gives the outline of the proof. The basic notion is the so-called distributive homomorphism of a semilattice (see [4]). The second section proves that for every distributive lattice F with 0 there exists a generalized Boolean algebra B — considered as a semilattice — and a distributive homomorphism of B onto F. In the third section we prove the main result and in the last section we give some generalizations.

1. Preliminaries

Semilattice always means a join-semilattice in this paper. The compact elements of an algebraic lattice L form a semilattice L^c with 0, and L is isomorphic to the ideal lattice of L^c. We denote by $\mathrm{Con}(K)$ the congruence lattice of the lattice K. The compact elements of $\mathrm{Con}(K)$ are called compact congruence relations, these form the semilattice $\mathrm{Con}^c(K)$.

Let B be a sublattice of a lattice K. The connection between $\mathrm{Con}^c(B)$ and $\mathrm{Con}^c(K)$ is of course very loose. Let θ be a congruence relation of B.

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Then there exists a smallest congruence relation $\theta^0 \in \mathrm{Con}(K)$ such that $\theta^0|_B \geqq \theta$. It is easy to see that $\theta_1^0 \vee \theta_2^0 = (\theta_1 \vee \theta_2)^0$, i.e. the correspondence $\theta \to \theta^0$ is a homomorphism of $\mathrm{Con}^c(B)$ into the semilattice $\mathrm{Con}^c(K)$. If this homomorphism is onto we call K a *strong extension of B* [1]; or we say that B is a strongly large sublattice. It is an important case if $\theta^0|_B = \theta$ holds, then we write $\bar{\theta}$ instead of θ^0. $\bar{\theta}$ is called the extension of θ.

It is well known that in generalized Boolean lattices (i.e. relatively complemented distributive lattices with zero) there is a one-to-one correspondence between congruence relations and ideals and therefore if B denotes a generalized Boolean lattice then $\mathrm{Con}^c(B) \cong B$. Let F be a distributive semilattice with 0. We would like to get a lattice K such that $\mathrm{Con}^c(K) = F$ holds. Therefore we start with a generalized Boolean lattice B which has a join-homomorphism onto F and we construct a strong extension K of B such that $\theta \to \theta^0$ is the given join-homomorphism. The construction of a strong extension of this kind was developed in [4].

We will make a further assumption that B is a convex sublattice of K. In this case the homomorphism $\theta \to \theta^0$ has an additional property, formulated in the next proposition.

Proposition 1. *Let B be a convex sublattice of K and let $\theta^0 = \Phi^0 \vee \Psi^0$ where $\theta, \Phi, \Psi \in \mathrm{Con}^c(B)$. Then there exist $\Phi_1, \Psi_1 \in \mathrm{Con}^c(B)$ such that $\Phi_1 \vee \Psi_1 = \theta$ and $\Phi_1^0 \leqq \Phi^0$, $\Psi_1^0 \leqq \Psi^0$.*

Proof. θ is a compact congruence relation of B, hence $\theta = \bigvee_{i=1}^{n} \theta(a_i, b_i)$, where $a_i < b_i$, $a_i b_i \in B$. From $\theta^0 = \Phi^0 \vee \Psi^0$ we get $a_i \equiv b_i(\Phi^0 \vee \Psi^0)$, $i = 1, 2, \ldots, n$. We have therefore for every i a finite chain $a_i = c_{0,i} < c_{1,i} < \ldots < c_{n,i} = b_i$ such that $c_{j,i} \equiv \equiv c_{j+1,i}(\Phi^0)$ or $c_{j,i} \equiv c_{j+1,i}(\Psi^0)$. By the assumption, B is a convex sublattice, i.e $c_{j,i} \in B$. Let Φ_1 be the join of all principal congruences $\theta(c_{j,i}, c_{j+1,i}) \in \mathrm{Con}^c(B)$ with $c_{j,i} \equiv c_{j+1,i}(\Phi^0)$. In a similar way we get Ψ_1. Then $a_i \equiv b_i(\Phi_1 \vee \Psi_1)$ for every i, i.e. $\theta = \Phi_1 \vee \Psi_1$, and $\Phi_1^0 \leqq \Phi^0$, $\Psi_1^0 \leqq \Psi^0$.

This Proposition suggests the following

Definition 1. Let S, T be two distributive semilattices. A homomorphism φ of S into T is called *weak-distributive* if $\varphi(u) = \varphi(x \vee y)$ implies the existence of $x_1, y_1 \in S$ such that $x_1 \vee y_1 = u$, $\varphi(x_1) \leqq \varphi(x)$, $\varphi(y)_1 \leqq \varphi(y)$ (see Figure 1).

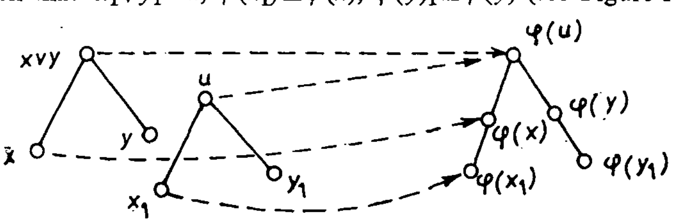

Figure 1.

The congruence relation induced by a weak-distributive homomorphism is called a *weak-distributive congruence.*

Let φ be a homomorphism of the semilattice S into the semilattice T. The congruence relation of S induced by φ is denoted by θ_φ.

Proposition 2. *Let S be a distributive semilattice. $\varphi\colon S\to T$ is a weak-distributive homomorphism if and only if $a\equiv b\vee c\ (\theta_\varphi)$, $a\geqq b\vee c$ imply the existence of elements $b_1\geqq b$, $c_1\geqq c$ such that $b\equiv b_1\ (\theta_\varphi)$, $c\equiv c_1\ (\theta_\varphi)$ and $b_1\vee c_1=a$* (Figure 2).

Proof. Let us assume that φ is a weak-distributive homomorphism and let $a\geqq b\vee c$, $\varphi(a)=\varphi(b\vee c)=\varphi(b)\vee\varphi(c)$, i.e. $a\equiv b\vee c\ (\theta_\varphi)$. φ is weak-distributive, hence we have elements $b_0, c_0\in S$ such that $b_0\vee c_0=a$, $\varphi(b_0)\leqq\varphi(b)$, $\varphi(c_0)\leqq\varphi(c)$. Let $b_1=b\vee b_0$, $c_1=c\vee c_0$ then $b_1\vee c_1=b\vee c\vee b_0\vee c_0=b\vee c\vee a=a$ and $\varphi(b_1)=\varphi(b\vee b_0)=$ $=\varphi(b)\vee\varphi(b_0)=\varphi(b)$, i.e. $b_1\equiv b\ (\theta_\varphi)$. Similarly we get $c_1\equiv c\ (\theta_\varphi)$ which proves that θ_φ satisfies the given property.

Let θ_φ be a congruence relation with the property formulated in the Proposition. Let $a[\theta_\varphi]=x[\theta_\varphi]\vee y[\theta_\varphi]$, i.e. $a\equiv x\vee y\ (\theta_\varphi)$. Then $a\vee x\vee y\equiv x\vee y\ (\theta_\varphi)$ and there exist $x_1, y_1\in S$ satisfying $x_1\vee y_1=x\vee y\vee a$, $x\equiv x_1\ (\theta_\varphi)$, $y\equiv y_1\ (\theta_\varphi)$. Therefore $x_1\vee y_1\geqq a$, hence by the distributivity of S we get elements x_2, y_2 for which $x_2\leqq x_1$, $y_2\leqq y_1$ and $x_2\vee y_2=a$. These elements satisfy $\varphi(x_2)\leqq\varphi(x_1)\leqq\varphi(x)$, i.e. φ is weak-distributive.

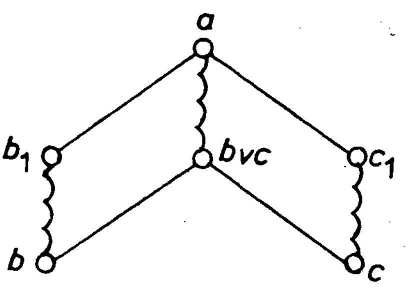

Figure 2.

It is easy to give an example for a semilattice S and $a, b\in S$ such that there is no smallest weak-distributive congruence satisfying $a\equiv b\ (\theta)$, i.e. the principal weak-distributive congruence does not exist. We follow another way to define a special weak-distributive congruence which plays the role of the principal congruence. The principal congruences of a semilattice have the property that every congruence class contains a maximal element.

Definition 2. [4] A congruence relation θ of a semilattice is called *monomial* if every θ-class has a maximal element.

The monomial congruence are special meet-representable congruences. Every congruence relation of a semilattice is the join of principal congruence relations therefore it is natural to introduce the following notion.

Definition 3. [4] A congruence relation θ of a semilattice is called *distributive* if θ is the join of weak-distributive monomial congruences. A homomorphism $\varphi\colon S \to T$ is distributive iff the congruence relation θ induced by φ is distributive.

Remark. It is easy to prove that the join of weak-distributive congruences is weak-distributive. The basic properties of distributive congruences are listed in [6].

If $(B; \vee, \wedge)$ is a generalized Boolean lattice, then the semilattice $(B; \vee)$ will be called a generalized Boolean semilattice.

For the solution of the characterization problem of congruence lattices of attices it is enough to solve the following two problems.

Problem 1. Let B be a generalized Boolean semilattice and let θ be a distributive congruence of B. Does there exist a lattice K satisfying $\mathrm{Con}^c(K) \cong B/\theta$? Does there exist a strong extension of B satisfying the same property?

This problem was solved positively in [4]. In section 3 we give the sketch of the proof.

Problem 2. Let F be a distributive semilattice with 0. Does there exist a generalized Boolean semilattice B and a distributive congruence θ of B such that F is isomorphic to B/θ?

This problem is open. We solve this problem if F is a lattice, i.e. we prove the following.

Theorem 1. *Let F be a distributive lattice with 0. Then there exist a generalized Boolean semilattice B and a distributive congruence θ of B such that $F \cong B/\theta$.*

The proof of this theorem will be given in the next sections. We present here the basic idea of the proof.

Let F be a semilattice, $a, b \in F$. The pseudocomplement $a*b$ of a relative to b is an element $a*b \in F$ satisfying $a \vee x \geqq b$ iff $x \leqq a*b$. If $a*b$ exists for all $a, b \in F$ then F is a relatively pseudocomplemented semilattice. (In the literature the pseudocomplement is usually defined in meet-semilattices.)

Let F be a relatively pseudocomplemented lattice (i.e. the join-semilattice $F^{\vee}$ is relatively pseudocomplemented). The proof of Theorem 1 in this case is quite easy. Let B be the Boolean lattice R-generated by F. (See [2], p. 87.) Then for every $x \in B$ there exists a smallest $\bar{x} \in F$ satisfying $x \leqq \bar{x}$. The mapping $x \to \bar{x}$ is a distributive homomorphism of B onto F. The congruence relation induced by this mapping is monomial. The converse of this statement is true: if θ is a monomial distributive congruence of B then B/θ is a relatively pseudocomplemented lattice.

If F is a relatively pseudocomplemented *semilattice* then this construction does not work. In this case we consider for every $a \in F$, $a \neq 0$ the *skeleton* of $(a]$, i.e. $S(a)=\{x*a;\ x \leqq a\}$ ([2], p. 112). $S(a)$ is a Boolean lattice. Consider the lower discrete direct product $\prod_d (S(a);\ a \in F, a \neq 0)$, i.e. the sublattice of the direct product $\prod S(a)$ of those sequences t for which $t(a)=0$ for all but finitely many $a \in F$. This is a generalized Boolean lattice B, and it is easy to show that B has a distributive congruence θ satisfying $B/\theta \cong F$ (see [4]).

To prove Theorem 1 we generalize the notion of the skeleton. Let φ be the identity $\varphi\colon S(1) \to F$. If B denotes $S(1)$ and $0, I \in B$ then this φ obviously has the following properties:

(1) φ is a $\{0, 1\}$-homomorphism of the Boolean semilattice B into the semilattice F,

(2) if $\varphi(I)=x \vee y$ in F then there exist $x_1, y_1 \in B$ such that $x_1 \vee y_1 = I$, $\varphi(x_1) \leqq x$, $\varphi(y_1) \leqq y$.

(1) follows from the property that $S(a)$ is a subsemilattice of F, and (2) is obvious if we take $x_1 = y*1$, $y_1 = x_1 * 1$.

Definition 4. Let F be a distributive semilattice with 0, $1 \in F$ and let B be a Boolean semilattice with unit element I and zero element 0. B is called a *pre-skeleton* of F if there exists a mapping φ of B into F such that conditions (1) and (2) are satisfied.

Condition (2) is related to the distributivity of φ; if (2) is satisfied for every $a \in B$ (instead of I) and φ is onto then we get that φ is distributive.

2. The pre-skeleton

To prove Theorem 1 we shall show that every bounded distributive lattice has a pre-skeleton. First we verify some simple well-known properties of free Boolean algebras. The free Boolean algebra B generated by the set G is denoted by $F(G)$. If $|G|=m$ we shall write $F(m)$ for $F(G)$. 1 denotes the unit element of $F(G)$. Let $G' = \{x' | x \in G\}$ (x' denotes the complement of x) and $G_1 = G \cup G'$. For $g \in G$, g^e is either g or g'. Let k be a natural number. We consider the subset G_k of B defined by $G_0=\{1\}$ and $G_k=\{x | x \in B,\ x \neq 0,\ x = g_1^e \wedge \ldots \wedge g_k^e$, where $g_1, \ldots, g_k$ are different elements of $G\}$. From these sets G_k we get $\mathscr{H} = \bigcup_{i=0}^{\infty} G_i$. If $|G|=n$ is a natural number then G_n is the set of atoms of $F(n)$ and each $a \in F(n)$, $a \neq 0$ has a unique representation as a join of elements of G_n. If G is infinite we have no atoms, therefore we must take the whole set $\mathscr{H}$, which is of course a relative sublattice of B.

The most important properties of $\mathscr{H}$ are collected in the following definition.

Definition 5. A relative sublattice $\mathscr{H}$ of a Boolean algebra B is called a *join-base* iff the following conditions are satisfied:

(i) $0 \notin \mathscr{H}$ and $1 \in \mathscr{H}$.
(ii) Each $a \in B$, $a \neq 0$ has a representation as a join of elements of $\mathscr{H}$.
(iii) There is a dimension function δ from $\mathscr{H}$ onto an ideal of the chain of non-negative integers such that $\delta(1)=0$ and $x \prec y$ in $\mathscr{H}$ if and only if $x \leqq y$ and $\delta(x)=\delta(y)+1$. The set of all $x \in \mathscr{H}$ with $\delta(x)=i$ is denoted by $\mathscr{H}_i$.
(iv) For every finite subset $U=\{u_1, \ldots, u_n\}$ of B there exists an $i \in \mathrm{N}$ such that each $\mathscr{H}_k$ $(k \geqq i)$ has a finite subset $A_k(U)$ with the property that each $u \in U$ has a unique join representation as a join of elements of $A_k(U)$.
(v) If $a \wedge b \neq 0$ in B, $a, b \in \mathscr{H}$ then $a \wedge b \in \mathscr{H}$; if $a \vee b$ exists in $\mathscr{H}$ and a, b are incomparable then $a, b \in \mathscr{H}_i$, $a \vee b \in \mathscr{H}_{i-1}$ for some $i \in \mathrm{N}$. Assume, that there exists an $a_0 \in \mathscr{H}_{i-1}$, $a_0 \neq a \vee b$, $a_0 > a$, then there is a $b_0 \in \mathscr{H}_{i-1}$ such that $a_0 \vee b_0$ exists and $a_0 \wedge (a \vee b)=a$, $b_0 \wedge (a \vee b)=b$.

Let $\mathscr{H}$ be a join-base of a Boolean semilattice B and let $f\colon \mathscr{H} \to L$ be a homomorphism into a distributive lattice (i.e. $f(a \wedge b)=f(a) \wedge f(b)$ whenever $a \wedge b$ exists, and the same for $\vee$). We want to extend f to a homomorphism $\varphi\colon B \to L$ (i.e., φ will be a join-homomorphism of the Boolean algebra B). Let $a=h_1 \vee \ldots \vee h_n$ where $h_i \in \mathscr{H}$. The only way to define φ is the following: $\varphi(a)=f(h_1) \vee \ldots \vee f(h_n)$. Condition (iv) yields that this definition is unique and (ii) implies that φ maps B into L.

Definition 6. The homomorphism φ of the Boolean semilattice into L is called an *L-valued homomorphism of B induced by f.*

To prove Theorem 1 we need the definition of free $\{0, 1\}$-distributive product (see G. Grätzer [2], p. 106).

Definition 7. Let D be the class of all bounded distributive lattices and let L_i, $i \in I$ be lattices in D. A lattice L in D is called a *free* $\{0, 1\}$*-distributive product* of the L_i, $i \in I$, iff every L_i has an embedding ε_i into L such that

(i) L is generated by $\bigcup(\varepsilon_i L; i \in I)$.

(ii) If K is any lattice in D and φ_i is a $\{0, 1\}$-homomorphism of L_i into K for $i \in I$, then there exists a $\{0, 1\}$-homomorphism φ of L into K satisfying $\varphi_i=\varphi\varepsilon_i$ for all i.

The free $\{0, 1\}$-distributive product is denoted by $\Pi^*(A_i; i \in I)$ or by $A * B$. The lower discrete direct product is denoted by $\Pi_d(A_i; i \in I)$ and finally if A_i are lattices with unit element then $\Pi^d(A_i; i \in I)$ is the upper discrete direct product, i.e. the sublattice of the direct product ΠA_i of those sequences t for which $t(a)=1$ for all but finitely many a.

Lemma 1. *Let L be a bounded distributive lattice and let A_i $(i\in I)$ be Boolean semilattices. If φ_i: $A_i\to L$ $(i\in I)$ are L-valued $\{0, 1\}$-homomorphisms generated by f_i: $\mathscr{H}^i\to L$ then the free $\{0, 1\}$-distributive product $\Pi^* A_i$ has a join-base $\mathscr{H}$ and a homomorphism f: $\mathscr{H}\to L$ such that $\mathscr{H}\cap A_i=\mathscr{H}^i$ for each $i\in I$. There exists an L-valued homomorphism φ of $\Pi^* A_i$ generated by f satisfying $\varphi_i=\varphi\varepsilon_i$.*

Proof. Let $\mathscr{H}$ be the set of all those elements $h\neq 0$ of Π^*A_i which have a finite meet-representation as a meet of elements from $\bigvee\mathscr{H}^i$. (Then $\mathscr{H}$ is isomorphic to the upper direct product $\Pi^d\mathscr{H}^i$.) Obviously $\mathscr{H}^i\subseteqq\mathscr{H}$, $\mathscr{H}^i=\mathscr{H}\cap A_i$. Let $u=h_1\wedge \wedge h_2\wedge\dots\wedge h_n$ where the $h_i\in\mathscr{H}^i$ belong to different components, then this representation is unique. We have by (iii) the functions δ_i: $\mathscr{H}^i\to\mathrm{N}$. Now let δ: $\mathscr{H}\to\mathrm{N}$ be defined by $\delta(u)=\delta_1(h_1)+\dots+\delta_n(h_n)$. It is easy to verify (iv) and (v). Assume that f_i: $\mathscr{H}^i\to L$ are homomorphisms, then we can extend them as follows: $f(u)= =f_1(h_1)\wedge\dots\wedge f_n(h_n)$. Hence $x\geqq y$ $(x, y\in\Pi^*A_i)$ implies $f(x)\geqq f(y)$. Let us assume that for incomparable $b, c\in\mathscr{H}$, $b\vee c$ exists, i.e. $b\vee c\in\mathscr{H}$. Then by (v) there exist an i and $b_0, c_0\in\mathscr{H}_i$ such that $b=b_0\wedge(b\vee c)$ and $c=c_0\wedge(b\vee c)$. Thus we get by the distributivity of L that $f(b)\vee f(c)=[f_i(b_0)\wedge f(b\vee c)]\vee[f_i(c_0)\wedge f(b\vee c)]=(f_i(b_0)\vee f_i(c_0))\wedge \wedge f(b\vee c)$. But f_i: $\mathscr{H}^i\to L$ is a homomorphism, hence $f_i(b_0\vee c_0)=f_i(b_0)\vee f_i(c_0)$. Obviously $b_0\vee c_0\geqq b\vee c$, i.e. $f(b_0\vee c_0)\geqq f(b\vee c)$. This yields $f(b)\vee f(c)=f(b\vee c)$, i.e. f is a homomorphism of $\mathscr{H}$ into L.

The free Boolean algebra on m generators is the free $\{0, 1\}$-distributive product of m copies of the free Boolean algebra on one generator, i.e. if $B_i\cong F(1)$, $i\in I$ then $F(m)\cong\Pi^* B_i$.

Corollary. *If each $B_i\cong F(1)$ has a $\{0, 1\}$-homomorphism φ_i into the distributive lattice L, then there exists an L-valued homomorphism φ of $F(m)$ into L such that $\varphi_i=\varphi\varepsilon_i$.*

Lemma 2. *Let L be a bounded distributive lattice. Then there exists a pre-skeleton B of L.*

Proof. First assume that B is a pre-skeleton and ψ: $B_1\to B$ is a lattice homomorphism of the Boolean lattice B_1 onto B. Then it is easy to see that B_1 is again a pre-skeleton and the corresponding join-homomorphism is $\varphi\psi(x)$. Therefore to prove our Lemma it is enough to take a free Boolean algebra generated by a "big" set.

We start with the set G_1 of all pairs (a, b) satisfying $a, b\in L$, $a\vee b=1$, $a, b\neq 1$. Let G be a subset of G_1 which is maximal with respect to the property: $(a, b)\in G$ iff $(b, a)\notin G$.

In the free Boolean algebra $F(G)$ we define $(a, b)'=(b, a)$, i.e. the complement of (a, b) is (b, a). The mapping $\varphi\colon F(G)\to L$ is defined as follows. For $(a, b)\in G_1$ we set $\varphi((a, b))=a$ and let $\varphi(0)=0$. Then $\varphi((a, b))\vee\varphi((b, a))=a\vee b=1$, i.e. φ is a $\{0, 1\}$-homomorphism of the semilattice $F((a, b))$ into L. Then by the Corollary to Lemma 1 there exists an extension φ of these homomorphisms. Let $x\vee y=1=\varphi(I)$, $x, y\neq 1$, where I denotes the unit element of $F(G)$. Take $x_1=(x, y)$, $y_1=(y, x)\in F(G)$. By the definition of φ we have $\varphi(x_1)=x$, $\varphi(y_1)=y$, i.e. $F(G)$ is a pre-skeleton of L.

Example 1. As an illustration consider the lattice L represented by Figure 3.

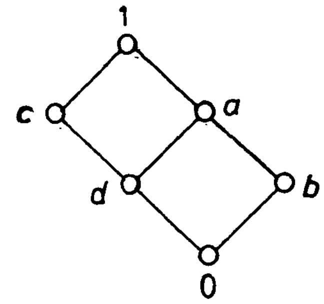

Figure 3.

The set G_1 contains the pairs (a, c), (b, c), (c, a), (c, b) and for a generating set we can choose $G=\{(a, c), (b, c)\}$; then B is the free Boolean algebra generated by two elements, i.e. $B\cong 2^4$. Figure 4 gives the join-homomorphism φ, in which the wavy line indicates congruence modulo $\theta=\operatorname{Ker}\varphi$.

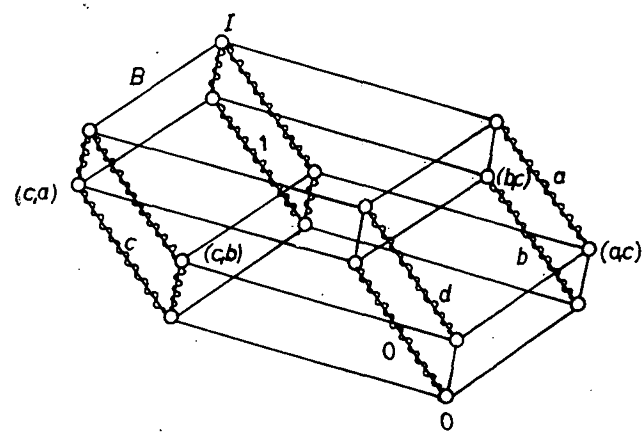

Figure 4.

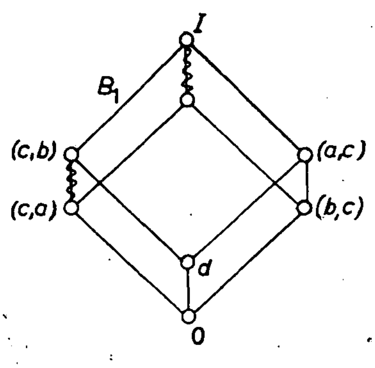

Figure 5.

Remark. The set G_1 can be made into a poset as follows: $(x, y) \leqq (u, v)$ iff $x \leqq u$ and $y \geqq v$. We adjoin 0 and I and we take the Boolean algebra B_1 freely generated by this poset. B_1 is of course the homomorphic image of B defined above. Sometimes it is easier to work with this "smaller" Boolean algebra (see Figure 5).

Example 2. Let L be the lattice shown in Figure 6.

Let $\mathbf{N}=\{0, 1, 2, \ldots\}$ be the set of all natural numbers. B is the Boolean-algebra containing all finite and cofinite subsets of $\mathbf{N}$. We define $(a_i, b)=\{x_i;\ x \geqq i\}$, $(b, a_i)= =\{0, 1, \ldots, i-1\}$. Then $G=\{(a_i, b), (b, a_i);\ i=0, 1, \ldots\}$ is a generating set. The corresponding join homomorphism is the following. Let A be a subset of $\mathbf{N}$ with the smallest element $f(A)$. If A is finite then $\varphi(A)$ is b if $f(A)=0$ and $\varphi(A)=c_{f(A)}$ if $f(A)>0$. For an infinite A we have $\varphi(A)=1$ if $f(A)=0$ and $\varphi(A)=a_{f(A)}$ if $f(A)>0$. It is easy to see that φ is a distributive homomorphism of B onto L, which proves that $I(L) \cong L$ is the congruence lattice of a lattice. This is the simplest example to show that $\mathrm{Con}^c(K)$ need not to be relatively pseudocomplemented.

Lemma 3. *Let A_1, A_2 be Boolean semilattices and let φ_i: $A_i \to L$ be L-valued $\{0\}$-homomorphisms generated by the homomorphisms f_i: $\mathscr{H}_i \to L$ of the join-bases $\mathscr{H}_i \subseteq A_i$ $(i=1, 2)$. Then $\mathscr{H}=\mathscr{H}_1 \cup \mathscr{H}_2 \cup \{1\}$ is a join-base of $A_1 \times A_2$ and if φ is the homomorphism generated by f: $\mathscr{H} \to L$ then $\varphi_i = \varphi \varepsilon_i$*

Proof. The proof is obvious.

Remark. Lemma 3 is true for lower discrete direct product. In the infinite case this is a generalized Boolean algebra.

The basic idea of the proof of Theorem 1 can be illustrated by the following lattice (Figure 7).

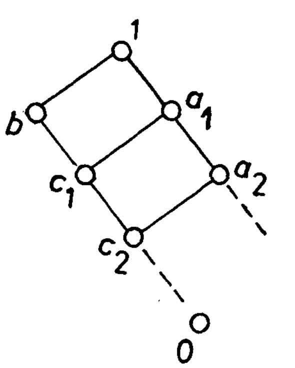

Figure 6.

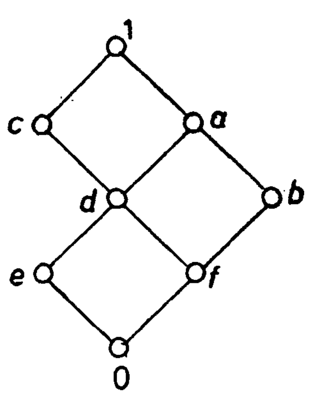

Figure 7.

Let a be an element of L. Then $(a]$ is a bounded distributive lattice. If B is a pre-skeleton of $(a]$ then we write $B=B(a)$; $B(1)$ is a pre-skeleton of L.

By Lemma 2 we have a homomorphism φ_1 of the pre-skeleton $B(1)$ onto the semilattice containing the elements $\{1, a, b, c, d, 0\}$. Applying again Lemma 2 for the principal ideal $(a]$ we get the mapping φ_a of the pre-skeleton $B(a)$ of $(a]$ onto $\{a, d, e, b, f, 0\}$. Let x be an element of $B(1)$ for which $\varphi_1(x)=a$. $B(1)$ is the direct product $(x]\times(x']$ where x' denotes the complement of x. Take the free $\{0, 1\}$-distributive product C of $(x]$ and $B(a)$. Let B be the Boolean semilattice $C\times(x']$ then by Lemmas 1 and 3 φ_1 and φ_a can be extended to a homomorphism $\varphi\colon B\to L$ which is a distributive homomorphism onto L.

We need the following

Definition 8. Let B be a Boolean semilattice and let L be a distributive lattice with 0. Let $\varphi\colon B\to L$ be a 0-preserving distributive homomorphism. (B, φ, L) is called a *saturated triple* if $\varphi(u)=x\vee y$ implies the existence of $x_1, y_1\in B$ such that $x_1\vee y_1=u$, $\varphi(x_1)\leqq x$, $\varphi(y_1)\leqq y$.

Lemma 4. *If (C, f, L), (D, g, L) are saturated triples then there exists a distributive homomorphism $h\colon C\times D\to L$ such that $h|_C=f$, $h|_D=g$ and $(C\times D, h, L)$ is saturated.*

Proof. For $(c, d)\in C\times D$ we define $h((c, d))=f(c)\vee g(d)$. Then $h((c, 0))=$ $=f(c)\vee 0=f(c)$, $h|_C=f$. Similarly $h|_D=g$. Now

$$h((a, b)\vee(c, d)) = h((a\vee c, b\vee d)) = f(a\vee c)\vee g(b\vee d) = (f(a)\vee f(c))\vee$$
$$\vee(g(b)\vee g(d)) = (f(a)\vee g(b))\vee(f(c)\vee g(d)) = h((a, b))\vee h((c, d))$$

which means that h is a homomorphism. We prove that h is distributive.

Let $h(c, d)=f(c)\vee g(d)=x\vee y$ in L. By the distributivity of L we get elements $x_1, x_2, y_1, y_2\in L$ such that $x_1\vee y_1=f(c)$, $x_2\vee y_2=g(d)$, $x_1, x_2\leqq x$, $y_1, y_2\leqq y$. Since (C, f, L) is saturated, therefore we have $c_1, c_2\in C$ such that $c_1\vee c_2=c$ and $f(c_1)\leqq x_1$, $f(c_2)\leqq y_1$. Similarly we get elements $d_1, d_2\in D$ with $d_1\vee d_2=d$, $g(d_1)\leqq x_2$, $g(d_2)\leqq y_2$. Set $\bar{x}=(c_1, d_1)$, $\bar{y}=(c_2, d_2)$. Then $\bar{x}\vee\bar{y}=(c_1\vee c_2, d_1\vee d_2)=(c, d)$, $h((c_1, d_1))=f(c_1)\vee$ $\vee g(d_1)\leqq x$, $h(c_2, d_2)\leqq y$. This proves that h is weak-distributive. Let $\theta=\operatorname{Ker} f$, $\Phi=\operatorname{Ker} g$. Then $\theta=\vee\theta_j$, $\Phi=\vee\Phi_j$; θ_j, Φ_j are monomial distributive congruences. θ_i resp. Φ_j can be extended to $C\times D$, $\bar{\theta}_i\cup\bar{\Phi}_j$ which are again monomial. It is easy to see that $\operatorname{Ker} h=\vee(\bar{\theta}_i\vee\bar{\Phi}_j)$.

Corollary. *Let C, D be two Boolean semilattices and f resp. g distributive homomorphisms of these Boolean semilattices into the distributive lattice L. If $f(C)$ resp. $g(D)$ are ideals of L then there exists a distributive homomorphism $h\colon C\times D\to L$ such that $h|_C=f$, $h|_D=g$.*

Remark. In Lemma 4 f and g are not necessarily L-valuations induced by some join-bases.

Let L be an arbitrary distributive lattice with 0. If $a \in L$, $a \neq 0$ the principal ideal $(a]$ is a bounded distributive lattice. Assume that for every $(a]$ we have a Boolean semilattice B_a and a distributive homomorphism φ_a of B_a onto $(a]$. Consider the lower discrete direct product $B = \Pi_d(B_a | a \in L,\ a \neq 0)$. B is a generalized Boolean semilattice. By Lemma 4 we have a distributive homomorphism $\varphi\colon B \to L$ which is onto. Consequently to prove Theorem 1 we can assume that L is a bounded distributive lattice. By Lemma 2 we have a pre-skeleton $B(1)$ with a homomorphism $\varphi_1\colon B(1) \to L$ which satisfies (2). Let u be an arbitrary non-zero element of the join-basis $H \subseteq B(1)$, $a = \varphi_1(u)$. The principal ideal $(a]$ of L is a bounded distributive lattice, therefore we can apply again Lemma 2 to get a pre-skeleton $B(a)$ and a homomorphism $\varphi_a\colon B(a) \to (a]$ into $(a]$. If u' denotes the complement of u in $B(1)$ then $B = B(1)$ is the direct product $(u'] \times (u]$. Take the free $\{0, 1\}$-distributive product $(u] * B(a)$ and finally the Boolean semilattice

$$B[I, u] = \big((u] * B(a)\big) \times (u'].$$

By Lemmas 1 and 3 we have a homomorphism $\varphi\colon B[I, u] \to L$, satisfying the following condition:

$(*)$ if $r \in T = \{I, u\}$, $\varphi(r) = x \vee y$ then there exist $x_1, y_1 \in B[I, u]$ with $x_1 \vee y_1 = r$, $\varphi(x_1) \leqq x$, $\varphi(y_1) \leqq y$.

Using the same method for an element $v \in B \subset B[I, u]$ we get from $B[I, u]$ a Boolean algebra $B[I, u, v]$ satisfying $(*)$ for the set $T = \{I, u, v\}$.

Lemma 5. *Let* $u, v \in B$, *then* $B[I, u, v] \cong B[I, v, u]$.

Proof. If H denotes a join-base of B and $x \in H$ then we shall write $H(x)$ for $H \cap (x]$. It is easy to show that $H(x) \cup H(x')$ is again a join-base and L-valuations generated by these join-bases coincide. If $u, v \in B$ then we have therefore a join-base $H(u \wedge v) \vee H(u \wedge v') \vee H(u' \wedge v) \vee H(u' \wedge v')$. Hence we get for $B[I, u, v]$ resp. $B[I, v, u]$ the following. Let H_u resp. H_v be a join base of $B(\varphi_1(u))$ resp. $B(\varphi_1(v))$; then $(H_u^1 \times H_v^1 \times H^1(u \wedge v)) \cup (H_u^1 \times H^1(u \wedge v')) \cup (H_v^1 \times H^1(u' \wedge v)) \cup H^1(u' \wedge v')$ which proves the isomorphism.

Continuing this construction we get for arbitrary $u_1, u_2, \ldots, u_n \in B$ a Boolean semilattice $B[I, u_1, \ldots, u_n]$ and a homomorphism of this Boolean semilattice into L such that condition $(*)$ is satisfied for $T = \{I, u_1, \ldots, u_n\}$.

All these Boolean semilattices form a direct family. Let C_1 be the direct limit Then $B(1) = C_0$ is a Boolean subalgebra of C_1 and we have $\varphi\colon C_1 \to L$ which satisfies $(*)$ for all $x \in T = B(1)$. Then we start with C_1 and in the same way we get a Boolean semilattice C_2. Then C_1 is a Boolean subalgebra of C_2. Similarly, we get

C_i $(i=3, 4, \ldots)$. These algebras C_i form again a direct family. Let $\bar{B}$ be the direct limit. Let $\varphi\colon \bar{B}\to L$ be the corresponding homomorphism. Then (B, φ, L) is saturated, hence φ is a weak-distributive homomorphism into L.

Lemma 6. *$\bar{B}$ has a join-base.*

Proof. This is a trivial consequence of Lemmas 1 and 3.

Lemma 7. *Let $\varphi\colon B\to L$ be a weak-distributive homomorphism of a Boolean semilattice B generated by a homomorphism $f\colon H\to L$ of a join-base H. Then φ is distributive.*

Proof. Let θ be the congruence relation induced by φ. H_k denotes the set of all $x\in H$ of dimension k. Take two elements $a, b\in B$, $a>b$ satisfying $a\equiv b$ (θ). Then a and b have join-representations as joins of elements from some H_k, say $a=h_1\vee\ldots\vee h_n\vee h_{n+1}$ and $b=h_1\vee\ldots\vee h_n$. If $c=h_1\vee\ldots\vee h_k$, $k<n$ and $d=h_i\vee\ldots\vee h_n$, $i\leqq k$ then $c\vee d=b$. By condition (iv) of Definition 5 we can assume that these representations of a, b, c, d are unique. By the weak distributivity of θ we have elements $\bar{c}\geqq c$, $\bar{d}\geqq d$ such that $\bar{c}\vee\bar{d}=a$ and $c\equiv\bar{c}$ (θ), $d\equiv\bar{d}$ (θ). For $\bar{c}$, $\bar{d}$ we have the following possibilities: (i) $\bar{c}=c\vee h_{n+1}$, $\bar{d}=d$; (ii) $\bar{c}=c$, $\bar{d}=d\vee h_{n+1}$; (iii) $\bar{c}=c\vee h_{n+1}$, $\bar{d}=d\vee h_{n+1}$.

We define a binary relation θ_{ab} on B as follows: $x\equiv y$ (θ_{ab}), $x>y$ iff $x\equiv y$ (θ) and $y\leqq b$, $x\vee b=a$. Then the assumption that θ is induced by the join-base H we get that each θ_{ab}-class contains a maximal element. Let $\theta_{ab}^{\vee}$ be the smallest join congruence of B satisfying $\theta_{ab}^{\vee}\geqq\theta_{ab}$. Then $u\equiv v$ $(\theta_{ab}^{\vee})$, $u\geqq v$ iff there exist $x\geqq y$, $x\equiv y$ (θ_{ab}) such that $y\leqq v$ and $x\vee v=u$. Obviously $\theta_{ab}^{\vee}\leqq\theta$, $\bigvee\theta_{ab}^{\vee}=\theta$. The first part of the proof yields that $\theta_{ab}^{\vee}$ is distributive.

An element $a\in L$ is of finite order if there exists a sequence $a=x_0, x_1, x_2, \ldots, x_n$ such that $a<a\vee x_1<a\vee x_1\vee x_2<a\vee x_1\vee\ldots\vee x_{n-1}<a\vee x_1\vee\ldots\vee x_n=1$ and $a\vee x_1\vee\vee x_2\vee\ldots\vee x_{i-1}$ is incomparable with x_i $(i=1, \ldots, n)$. By the construction of $\varphi\colon\bar{B}\to L$ the image of each $u\in\bar{B}$, $u\neq 0$ is the meet of elements of finite order. Now we have for every $a\in L$ a Boolean semilattice $B(a)$ and a distributive homomorphism $\varphi_a\colon B(a)\to(a]$ which maps $B(a)$ onto the set of all elements having a meet representation of elements of finite order in the lattice $(a]$. Then the triple $(B(a), \varphi_a, (a])$ is saturated. The lower discrete product of these Boolean semilattices B has by Lemma 4 a distributive homomorphism onto L which proves Theorem 1.

3. Construction of a strong extension

In this section we give the outline of the proof of the following theorem, which was proved in [4]. Combining Theorems 1 and 2 we get our main theorem.

Theorem 2. *Let θ be a distributive congruence of a generalized Boolean semilattice B. The lattice of all ideals of B/θ is the congruence lattice of a lattice.*

We denote the five element modular non-distributive lattice by M_3; M_3 with an additional atom is called M_4, etc. If α is an arbitrary cardinal number then M_α is the modular lattice of length 2 with α atoms.

Let $M=\{0<a, b, c<1\}$ be a lattice isomorphic to M_3 and let D be a bounded distributive lattice with zero element o, and unit element i. Identifying a with i and 0 with o, we get a partial lattice ${}_DM_3=D\cup M_3$ (Fig. 8), $D\cap M_3=\{0, a\}$ and D, M_3 are sublattices; $d\vee b$ resp. $d\vee c$ $(d\in D)$ is defined iff $d\in\{0, a\}$ (see MITSCHKE & WILLE [3]). There exists a modular lattice $M_3[D]$ generated by ${}_DM_3$ such that ${}_DM_3$ is a relative sublattice of $M_3[D]$. In [3] it was proved that there exists only one modular lattice with these properties, the modular lattice $FM({}_DM_3)$ freely generated by ${}_DM_3$. This lattice was introduced in [4] and has the following description.

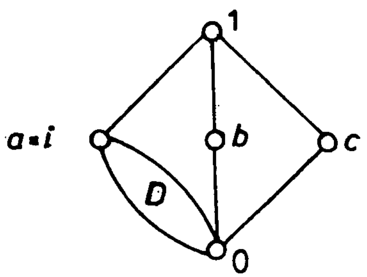

Figure 8.

An element $(x, y, z)\in D\times D\times D$ is called *normal* if $x\wedge y=x\wedge z=y\wedge z$. Let $M_3[D]$ be the poset of all normal elements, then $M_3[D]$ is a modular lattice. Let $a=(i, 0, 0)$, $b=(0, i, 0)$, $c=(0, 0, i)$, $1=(i, i, i)$, $0=(0, 0, 0)$. Then these elements form a sublattice isomorphic to M_3. The set of all elements $(x, 0, 0)$, $(x\in D)$ form a sublattice isomorphic to D. D is a strongly large sublattice of $M_3[D]$, and every congruence relation $\theta\in\mathrm{Con}(D)$ can be extended to $M_3[D]$, i.e. $\mathrm{Con}(D)\cong\mathrm{Con}(M_3[D])$. We can use the same construction for distributive lattices without unit element.

We prove Theorem 2 first for monomial congruences of Boolean semilattices i.e. for relatively pseudocomplemented lattices.

Lemma 8. *Let θ be a monomial distributive congruence of a generalized Boolean semilattice B. Then there exists a lattice N such that* $\mathrm{Con}^c(N)\cong B/\theta$.

Sketch of the proof. Consider $D=B$ and the corresponding lattice $M_3[B]$. We define a subset N of $M_3[B]$ as follows

$(**)$ $(x, y, z)\in M_3[B]$ belongs to N iff x is a maximal element of a θ-class.

Then N is a lattice and $(x, 0, 0)\in N$ iff x is a maximal element of θ-class, i.e., the ideal I generated by $(i, 0, 0)$ is isomorphic to B/θ. N is a strong extension of I, a congruence relation of I has an extension to N iff it has the form $\theta(I')$, where I' is an ideal of N. Thus $\mathrm{Con}^c(N)\cong B/\theta$, i.e. $\mathrm{Con}(N)\cong I(B/\theta)$.

The ideal J of N, generated by $(0, 0, i)$ is isomorphic to B. By the definition of I and J we have $I\cap J=0$ (Fig. 9).

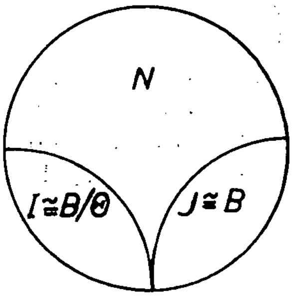

Figure 9.

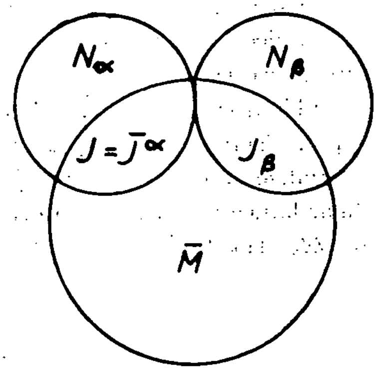

Figure 10.

Let θ be an arbitrary distributive congruence relation of the generalized Boolean semilattice B. Then θ is the join of monomial distributive congruence relations, say $\theta=\bigvee(\theta_\alpha|\alpha\in\Omega)$. We take first for every α the lattice N_α defined before. This N_α has two ideals $I_\alpha\cong B/\theta_\alpha$ and $J_\alpha\cong B$. Moreover $\mathrm{Con}^c(N_\alpha)\cong B/\theta_\alpha$.

On the other hand we consider the direct product $\Pi(B_\alpha|\alpha\in\Omega)$. M denotes the sublattice of the direct product of those normal sequences t for which $\{t(\alpha)|\alpha\in\Omega\}$ is finite, i.e. the weak direct product is normal if $\alpha, \beta, \gamma\in\Omega$, $\alpha\neq\beta$, $\alpha\neq\gamma$, $\beta\neq\gamma$ imply $t(\alpha)\wedge t(\beta)=t(\alpha)\wedge t(\gamma)=t(\beta)\wedge t(\gamma)$. Let J^α be the ideal of M consisting of all t for which $t(\beta)=0$ if $\beta\neq\alpha$. Then $J^\alpha\cong B$. M is a strong extension of J^α and $\mathrm{Con}^c(M)\cong\mathrm{Con}^c(J^\alpha)\cong\mathrm{Con}^c(B)$. Let $\overline{M}$ be the dual lattice of M. Then $\bar{J}^\alpha$ is a dual of $\overline{M}$. $\bar{J}^\alpha$ is a Boolean algebra, therefore we have a natural isomorphism $\bar{J}^\alpha\cong J^\alpha$ $(x\to x')$. We use the Hall—Dilworth gluing construction for $\overline{M}$ and N_α $(\alpha\in\Omega)$, we identify for every α the dual ideal $\bar{J}^\alpha$ and the ideal J_α. In this way we get a partial lattice P (see Figure 10).

$\overline{M}$ and N_α are sublattices of P, and P is a meet-semilattice. Let $F(P)$ be the free lattice generated by P. Then $\mathrm{Con}^c(F(P))\cong B/\theta$. This proves Theorem 2.

4. Some remarks on the characterization problem

The key problem of the characterization of congruence lattices of lattices is to prove the existence of a pre-skeleton of a bounded distributive semilattice. We reformulate this problem.

Let L be a bounded distributive semilattice. Let $F(G)$ be denote the free Boolean algebra generated by the set G. If $g_i \in G$ then the elements $0, g_i, g_i', I$ form a Boolean subalgebra which is the free Boolean algebra $F(g_i)$ generated by g_i. We have remarked that $F(G)$ is the free $\{0, 1\}$-distributive product of the Boolean algebras $F(g_i)$, $g_i \in G$. Let us assume that every $F(g_i)$ has a $\{0, 1\}$-homomorphism φ_i into L. Does there exist a $\{0, 1\}$-homomorphism φ: $F(G) \to L$ such that $\varphi|_{F(g_i)} = \varphi_i$? For finite G the answer is yes, we have

Proposition 3. *Let B be a finite Boolean algebra. If φ_1: $B \to L$ and φ_2: $F(g) \to L$ are $\{0, 1\}$-homomorphisms into L then there exists a $\{0, 1\}$-homomorphism φ of the free $\{0, 1\}$-distributive product $B * F(g)$ into L such that $\varphi|B = \varphi_1$, $\varphi|_{F(g)} = \varphi_2$.*

Proof. Let $p_1, p_2, \ldots, p_n$ denote the atoms of B. The atoms of the free product are $p_1 \wedge g, \ldots, p_n \wedge g,\ p_1 \wedge g', \ldots, p_n \wedge g'$. Then $g < p_1 \vee \ldots \vee p_r = I$ yields $\varphi_2(g) < \varphi_1(p_1) \vee \ldots \vee \varphi_1(p_n) = 1 \in F$. But F is a distributive semilattice hence we have elements $a_1, a_2, \ldots, a_n \in F$ such that $\varphi_2(g) = a_1 \vee \ldots \vee a_n$, $a_i \leqq \varphi_1(p_i)$ $(i = 1, 2, \ldots, n)$. Similarly $g' < p_1 \vee \ldots \vee p_n$ therefore we have elements $b_1, \ldots, b_n \in L$ satisfying $\varphi_2(g') = b_1 \vee \ldots \vee b_n$, $b_i \leqq \varphi_1(p_i)$. On the other hand $p_i \leqq g \vee g'$ hence $\varphi_1(p_i) \leqq \varphi_2(g) \vee \varphi_2(g')$. Thus we get elements u_i, v_i such that $\varphi_1(p_i) = u_i \vee v_i$, $u_i \leqq \varphi_2(g)$, $v_i \leqq \varphi_2(g')$. Define $\varphi(p_i \wedge g) = a_i \vee u_i$, $\varphi(p_i \wedge g') = b_i \vee v_i$. Every u of $B * F(g)$ has a unique representation as a join of atoms, say $u = \vee g_i$. We define $\varphi(u) = \vee \varphi(g_i)$. This φ is obviously a homomorphism. From $p_i = (p_i \wedge g) \vee (p_i \wedge g')$ we get $\varphi(p_i) = (p_i \wedge g)(p_i \wedge g') = (a_i \vee u_i) \vee (b_i \vee v_i) = a_i \vee b_i \vee \varphi_1(p_i) = \varphi_1(p_i)$. Similarly $g = \bigvee_{i=1}^{n} (p_i \wedge g) = \bigvee_i (a_i \vee u_i) = \bigvee_{i=1}^{n} a_i \vee \bigvee_{i=1}^{n} u_i = \varphi_2(g)$. (I.e. $\varphi|_B = \varphi_1$, $\varphi|_{F(g)} = \varphi_2$).

It is necessary to generalize Lemma 1 for distributive semilattice. Let B be the free Boolean algebra $F(G)$. Then the join-base is $H = \bigcup_{i=0}^{\infty} H_i \cup \{1\}$.

We have for every $g_i \in G$ a $\{0, 1\}$-homomorphism φ_i: $F(g_i) = \{0, g_i, g_i', I\} \to L$, i.e. we have a mapping $H_1 \to L$ and we want to get a $\{0, 1\}$-homomorphism φ: $B \to L$ which is a common extension of each φ_i. To define such a φ it is natural to use induction on k. If $x \in H_1$ then $x = g_i$ or $x = g_i'$ for some $g_1 \in G$ and we have $\varphi(x) = \varphi_i(x)$. Using the method of Proposition 3 it is easy to define $\varphi(x)$ for all $x \in H_2$. How can we define $\varphi(x)$ for $x \in H_3$?

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A parameter for subdirectly irreducible modular lattices with four generators

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BIRKHOFF [1; Problem 43] suggested to study modular lattices with four generators by imposing relations, first — e.g. — the relations expressing that the generators split into two complemented pairs. Basing on more special results of DAY, HERRMANN, and WILLE [2] and SAUER, SEIBERT, and WILLE [9] Birkhoff's problem has been solved in [6]. Remarkably enough, the subdirectly irreducible factors can be given by diagrams (including infinite ones) — these factors are the lattices M_4, $S(n, 4)$, R_∞ and its dual defined in § 1. In [7] there have been constructed lattice polynomials s_n (and their duals s_n^* — see § 2) such that a subdirectly irreducible modular lattice M (with more than 5 elements) is one of the above if and only if $s_n=1$ and $s_n^*=0$ holds in M for all n. In the present note we want to provide a basis for the study of subdirectly irreducible four generated modular lattices not being one of the above. In particular, we show that an inductive approach is possible using the polynomials s_n.

Theorem. *Let M be a subdirectly irreducible modular lattice with four generators a, b, c, d not being isomorphic to any of the lattices M_4, $S(n, 4)$ $(n<\infty)$, R_∞ or its dual. Then there is an n such that either*

(i) $s_n(a, b, c, d) = 0 = ab = ac = ad = bc = bd = cd$

or

(ii) $s_n^*(a, b, c, d) = 1 = a+b = a+c = a+d = b+c = b+d = c+d.$

Examples of such lattices are the rational projective geometries of finite dimension (GELFAND and PONOMAREV [4; § 8]) and, more generally, all subdirectly irreducible modular lattices generated by a frame ([5] and [7]). The use of the s_n in the analysis these examples has been pointed out in [7]. Clearly, such lattices can be visualized by diagrams in the most trivial cases, only.

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Corollary. *The* M_4, $S(n, 4)$ $(n<\infty)$, R_∞ *and its dual are the only subdirectly irreducible modular lattices generated by* a, b, c, d *such that* $a+b=c+d=1$ *and* $ab=cd=0$ [6]. M_4 *and* R_∞ *are the only ones for which, in addition,* $ac=ad=bc=bd=0$ (SAUER, SEIBERT, and WILLE [9]). R_∞ *is the modular lattice freely generated by the partial lattice* J_1^4 (DAY, HERRMANN, and WILLE [2]).

Also, it follows that the lattices listed in the Corollary are the only four generated subdirectly irreducible modular lattices of breadth ≤ 2 (FREESE [3]) or, more generally, satisfying the 2-distributive law ([6]).

The proofs do not depend on [2] nor [9]. From [6] we need only § 2 and 3 and from [7] § 1 and 5. The basic tool is the neutral element method from [6] — see § 3.

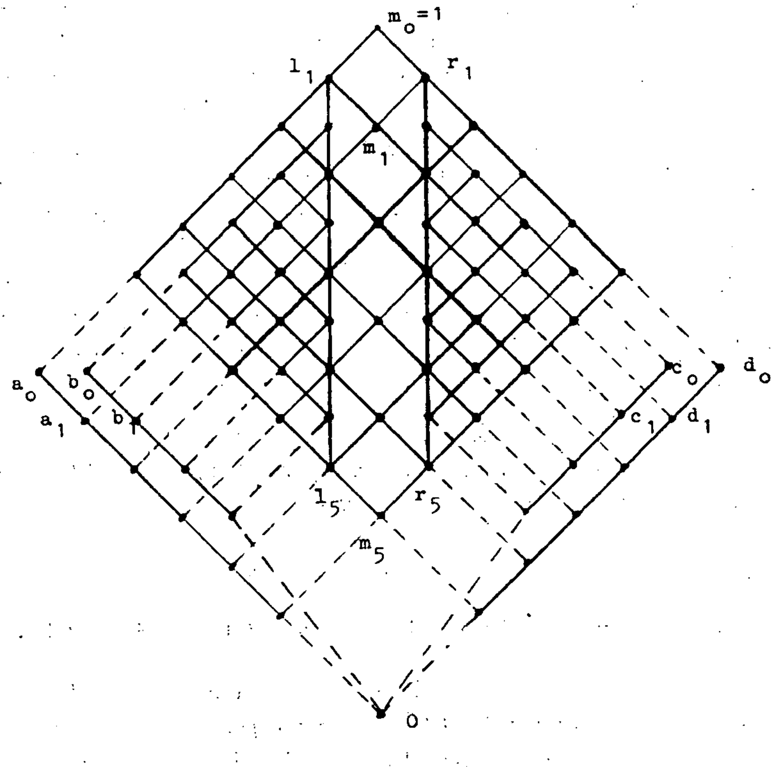

Figure 1

Replace $a_i, b_i, c_i, d_i, m_i, l_i, r_i$; 0, 1 respectively

a) by $\hat{a}_i, \hat{b}_i, \hat{c}_i, \hat{d}_i, \hat{m}_i, \hat{l}_i, \hat{r}_i, \hat{0}, \hat{1}$, b) by $\bar{a}_i, \bar{c}_i, \bar{b}_i, \bar{d}_i, \bar{m}_i, \bar{l}_i, \bar{r}_i, \bar{0}\ \bar{1}$,

c) by $\tilde{a}_i, \tilde{d}_i, \tilde{b}_i, \tilde{c}_i, \tilde{m}_i, \tilde{l}_i, \tilde{r}_i, \tilde{0}, \tilde{1}$.

§ 1. The breadth two models

First, let us introduce the lattices referred to in the main theorem. M_n is the length two lattice with n atoms. Let A_∞ (cf. Fig. 1) consist of the elements $x(i,j)$ $0 \leqq i \leqq j \leqq \infty$, $x \in E = \{a, b, c, d\}$ with the equalities $a(i,i) = b(i,i) = c(i,i) = d(i,i) =: m_i$ $(0 \leqq i \leqq \infty)$, $a(i-1, i) = b(i-1, i) =: l_i$ and $c(i-1, i) = d(i-1, i) =: r_i$ $(1 \leqq i < \infty)$ and no others. The relation $\leqq$ on A_∞ is defined in the following way (with $x \neq y$ in E, $0 \leqq i \leqq j \leqq \infty$, and $0 \leqq k \leqq l \leqq \infty$)

$$x(i,j) \leqq x(k,l) \quad \text{if and only if} \quad k \geqq i \text{ and } l \geqq j,$$

$$x(i,j) \leqq y(k,l) \quad \text{if and only if} \quad \begin{cases} l \leqq i & \text{for} \quad \{x,y\} \neq \{a,b\}, \{c,d\} \\ l \leqq i+1 & \text{and} \quad k \leqq i \text{ else.} \end{cases}$$

This yields a modular lattice order on A_∞ such that

$$x(i,j) + x(k,l) = x(s,t) \quad \text{with} \quad s = \min(i,k), \quad t = \min(j,l)$$

$$x(i,j) \cdot x(k,l) = x(s,t) \quad \text{with} \quad s = \max(i,k), \quad t = \max(j,l)$$

$$\left.\begin{array}{l} x(i,j) + y(k,l) = x(i,s) \text{ for } i \leqq k \text{ and } s = \min(j,k) \\ x(i,j) \cdot y(k,l) = x(s,j) \text{ for } j \geqq l \text{ and } s = \max(i,l) \end{array}\right\} \text{ if } \{x,y\} \neq \{a,b\}, \{c,d\}$$

$$\left.\begin{array}{l} x(i,j) + y(k,l) = x(i,s) \text{ for } i \leqq k \text{ and } s = \min(k+1, j, l) \\ x(i,j) \cdot y(k,l) = x(s,j) \text{ for } j \geqq l \text{ and } s = \max(i,l) \end{array}\right\} \text{ else.}$$

Put $x_i = x(i, \infty)$. Then every element of A_∞ has a unique representation $m_i (0 \leqq i \leqq \infty)$, l_i, r_i $(1 \leqq i < \infty)$, x_i $(0 \leqq i < \infty)$, or $x_i + m_n$ $(0 \leqq i \leqq n-2)$ with x in E. A_∞ is generated by the x_0 $(x \in E)$ as one derives from the relations $m_0 = 1$, $m_\infty = 0$, $l_{n+1} = a_n + b_n$, $r_{n+1} = c_n + d_n$, $m_{n+1} = r_{n+1} l_{n+1}$, and $x_{n+1} = x_0 m_{n+1}$.

Observe that every proper quotient of A_∞ contains a prime quotient $x(i,j)/x(k,l)$ with $l = j$ and $k = i+1$ or $k = i$ and $l = j+1$. Moreover, $x(i,j)/x(i+1,j)$ is transposed upward to $y(k,l)/y(s,t)$ if and only if $x = y$, $i+1 = s = k+1$, and $j \geqq l = t$ or $x \neq y$, $\{x,y\} \neq \{a,b\}$, $\{c,d\}$, $k = s$, and $i+1 \geqq t = l+1$ or, finally, $\{x,y\} \in \{\{a,b\}, \{c,d\}\}$ and $l = s = t = i+1 = k+1$ or $k = s \leqq i$, $t \leqq i+2$, and $t = l+1$. On the other hand $x(i,j)/x(i,j+1)$ is transposed upward to $y(k,l)/y(s,t)$ if and only if $x = y$, $k = s \leqq i$, $l = j$, and $t = j+1$ respectively $\{x,y\} \in \{\{a,b\}, \{c,d\}\}$ and $i = j = l$, $k = i-1 = s$, $t = i+1$ or $i = j$, $k = s \leqq i-2$, $l = i = t-1$. Thus, every prime quotient is projective to one of $1/l_1$ and $1/r_1$. Let Q consist of all quotients $x(i,n)/x(i+1,n)$ with i even and $x = c, d$ or i odd and $x = a, b$ as well as the quotients $x(i,n)/x(i,n+1)$ with n even and $x = a, b$ or n odd and $x = c, d$ and, finally, the r_i/r_{i+1} with i odd and l_i/l_{i+1} with i even. Then $1/l_1$ is in Q and Q describes a minimal congruence θ. Let R_∞ be the homomorphic image A_∞/θ. Its operation table can be derived easily from that of A_∞. (Actually, R_∞ is the lattice

$FM(J_1^4)$ from [2] where its diagram is given.) Let φ be defined as θ interchanging "odd" with "even". By symmetry, A_∞/φ is isomorphic to R_∞. The intersection $\theta\cap\varphi$ is the identity and every proper congruence of A_∞ contains θ or φ. Thus, R_∞ is subdirectly irreducible. Since $A_\infty/\theta\vee\varphi$ is the simple lattice M_4 there are no other homomorphic images of A_∞.

The section $[m_n, 1]$ of A_∞ is called A_n. It is generated by the $x(0, n)$ (x in E). The restrictions of the congruences θ and φ to A_n yield a subdirect decomposition into two isomorphic simple factors called $S(n, 4)$ — use the same arguments as above! Clearly, $S(n, 4)$ is isomorphic to the section $[[m_n]\theta, 1]$ of R_∞.

§ 2. Some lattice polynomials

We have to recall some definitions and results from [7]. Let F be the modular lattice with 0 and 1 freely generated by four elements $a=e_1$, $b=e_2$, $c=e_3$, $d=e_4$. Write $E=\{a, b, c, d\}$ and $\mathbf{n}=\{1, \ldots, n\}$. Put $q_1=(a+b)(c+d)$, $q_2=(a+c)(b+d)$, $q_3=(a+d)(b+c)$. Let $x\mapsto x^i=x(aq_i, bq_i, cq_i, dq_i)$ denote the endomorphism of F with $1\mapsto q_i$, $0\mapsto 0$, and $e\mapsto eq_i$ for $e\in E$. Define by induction

$$s_0=1,\quad s_1=a+b+c+d,\qquad s_{n+1}=\sum(s_n^i|i\in\mathbf{3})$$

$$t_0=1,\quad t_1=(a+b+c)(a+b+d)(a+c+d)(b+c+d),\qquad t_{n+1}=\sum(t_n^i|i\in\mathbf{3}).$$

Let x^* be the dual of x. Then 1.1, 1.3, 1.2, and 5.1 of [7] yield

Lemma 2.1. *For $n\geqq 0$ and $i\neq j$ in $\mathbf{3}$ one has*

(1) $q_iq_j=q_j^i$ *and* $(x^i)^j=(x^j)^i$ *for all x in F.*

(2) $s_{n+1}=s_n^i+s_n^j$ *and* $t_{n+1}=t_n^i+t_n^j$ *for* $n\geqq 1$.

(3) $q_is_{n+1}=s_n^i$ *and* $q_it_{n+1}=t_n^i$.

(4) $s_m^*\leqq s_{n+1}\leqq t_n\leqq s_n$ *and* $ef\leqq s_n$ *for all m and $e\neq f$ in E.*

(5) $q_i(e_l+e_k)=q_ie_l+q_ie_k$ *for* $k\neq l$ *in $\mathbf{4}$ with* $|\{i, i+1, k, l\}|=3$.

Lemma 2.2. *s_1, t_1, s_2, and t_2 are neutral elements of F. For $i\neq j$ in $\mathbf{3}$ and e in E one has $s_2q_i+s_2q_j=s_2$ and $et_2=et_2q_i+et_2q_j$.*

Lemma 2.3. *Let u be s_n or t_n ($n\geqq 1$), i in $\mathbf{3}$, and e, f, g distinct elements of E. Then the sublattices generated by $e, f+g, u$ and e, q_i, u and e, f, u, respectively, are distributive. Moreover*

$$q_i(a+u, b+u, c+u, d+u)=q_i+u \quad\text{and}\quad u(a+u, b+u, c+u, d+u)=u,$$

$$q_i(au, bu, cu, du)=q_iu \qquad\text{and}\quad u(au, bu, cu, du)=u.$$

Proof. For $n\leqq 2$ anything follows by neutrality (Lemma 2.2). The distributivity of $\langle e, f+g, u\rangle$ and $\langle e, q_i, u\rangle$ and $u=u(a+u, b+u, c+u, d+u)$ have been shown in [7; 5.3]. Thus, $e+h, f+g, u$ is distributive, too. Assuming $t_1=1$ we have $(e+h)u+(f+g)u=eu+(f+g)u=u$. We prove the remaining claims by induction. For $n\geqq 2$ we get by 2.1 and the inductive hypothesis $as_{n+1}+bs_{n+1}\geqq a^2s_n^2+b^2s_n^2+$ $+a^3s_n^3+b^3s_n^3=(a^2+b^2)s_n^2+(a^3+b^3)s_n^3=(a+b)q_2s_{n+1}+(a^3+b^3)s_n^3=(a+b)s_{n+1}(q_2+$ $+(a^3+b^3)s_n^3)$. Now $q_2+(a^3+b^3)s_n^3\geqq q_2+q_2^3+q_1^3s_n^3\geqq q_2+s_n^3\geqq s_{n+1}$ by 2.1 (2) whence $as_{n+1}+bs_{n+1}=(a+b)s_{n+1}$. By symmetry, $es_{n+1}+fs_{n+1}=(e+f)s_{n+1}$ for all $e\neq f$ in E. Thus, $(es_{n+1}+hs_{n+1})(fs_{n+1}+gs_{n+1})=(e+h)(f+g)s_{n+1}$.

By the inductive hypothesis we have $(q_2s_n)^1=(q_2(as_n, bs_n, cs_n, ds_n))^1=((as_n+$ $+cs_n)(bs_n+ds_n))^1=(a^1s_n^1+c^1s_n^1)(b^1s_n^1+d^1s_n^1)=(q_1as_{n+1}+q_1cs_{n+1})(q_1bs_{n+1}+q_1ds_{n+1})=$ $=q_2^1(as_{n+1}, bs_{n+1}, cs_{n+1}, ds_{n+1})\leqq q_1(as_{n+1}, bs_{n+1}, cs_{n+1}, ds_{n+1})$ using 2.1 (3) and (1). Similarly, $(q_3s_n)^1\leqq q_1(as_{n+1}, bs_{n+1}, cs_{n+1}, ds_{n+1})$ whence $q_1s_{n+1}=s_n^1=(q_2s_n+q_3s_n)^1=$ $=(q_2s_n)^1+(q_3s_n)^1\leqq q_1(as_{n+1}, bs_{n+1}, cs_{n+1}, ds_{n+1})$ by 2.1 (2) and (3). The converse inclusion holds due to monotony. By symmetry we get $q_is_{n+1}=q_i(as_{n+1}, bs_{n+1}, cs_{n+1}, ds_{n+1})$ for all $i\in\mathbf{3}$. Finally, with the inductive hypothesis and 2.1 (3) it follows

$$\begin{aligned} s_{n+1}(as_{n+1}, bs_{n+1}, cs_{n+1}, ds_{n+1}) &= \sum s_n^i(as_{n+1}, bs_{n+1}, cs_{n+1}, ds_{n+1}) = \\ = \sum s_n(q_ias_{n+1}, q_ibs_{n+1}, q_ics_{n+1}, q_ids_{n+1}) &= \sum s_n(a^is_n^i, b^is_n^i, c^is_n^i, d^is_n^i) = \\ = \sum s_n(as_n, bs_n, cs_n, ds_n)^i &= \sum s_n^i = s_{n+1}. \end{aligned}$$

For t_n the proof is quite analogous.

Corollary 2.4. *Let u and v be any of the s_n, t_n $(n\geqq 0)$ such that $u\geqq v$. Then $u(au+v, bu+v, cu+v, du+v)=u$, $v(au+v, bu+v, cu+v, du+v)=v$, and $q_j(au+$ $+v, bu+v, cu+v, du+v)=q_ju+v$ for j in $\mathbf{3}$.*

Define by induction $q_{0i}=1$ and $q_{n+1,i}=q_i(aq_{ni}, bq_{ni}, cq_{ni}, dq_{ni})$. Write $\varrho_ix=x^i$ and $\varrho_i^0x=x$.

Lemma 2.5. *$\varrho_i^n1=q_{ni}$, and $\varrho_i^ne=eq_{ni}$ for i in $\mathbf{3}$ and e in E.*

Proof. The first claim is 1.5 in [7]. The other follows by induction on n: $\varrho_i^{n+1}e=$ $=\varrho_i^nq_{1i}e=\varrho_i^nq_{1i}\varrho_i^ne=\varrho_i^{n+1}1\,eq_{ni}=eq_{n+1,i}$.

§ 3. The neutral element method revisited

An element of a modular lattice M is *neutral*, if for all a and b in M the sublattice generated by u, a, and b is distributive. Then the map $x\mapsto(ux, u+x)$ yields a subdirect representation of M. In [6] we proved

Proposition 3.1. *Let u be an element of a modular lattice M. Let S be a lattice and α an order preserving map of S in M such that $x\mapsto u+\alpha x$ preserves meets and*

$x \mapsto u\alpha x$ preserves joins. Moreover, let M be generated by the union of all intervals $[u\alpha x, \alpha x]$ and $[u\alpha x, u]$ with x in S. Then u is a neutral element of M.

Here, we need a more sophisticated version.

Proposition 3.2. *Let M be a finitely generated subdirectly irreducible modular lattice and u_n $(n \geqq 0)$ a descending chain of elements of M. Let S be a lattice and γ a meet homomorphism of S into M such that M is generated by the image of γ. Assume that for all x and y in S and $n \geqq 0$ there is an $m \geqq n$ with $u_m\gamma x + u_m\gamma y = u_m\gamma(x+y)$. Then either M is a homomorphic image of S or there is an n such that u_n is the smallest element of M.*

Proof. Let $\mathscr{F}(M)$ denote the lattice of all filters on M with partial order dual to set inclusion. Then $\mathscr{F}(M)$ is a dually algebraic lattice having M as a sublattice. Write $\prod$ for the meets in $\mathscr{F}(M)$. In particular, let $u = \prod u_n$ be the filter generated by the u_n $(n \geqq 0)$. Let M' be the sublattice generated by M and u. By lower continuity and the hypothesis we have for any x, y in S: $u\gamma x + u\gamma y = \prod u_n\gamma x + \prod u_n\gamma y = \prod (u_n\gamma x + u_n\gamma y) = \prod u_n\gamma(x+y) = u\gamma(x+y) \geqq u(\gamma x + \gamma y)$. Thus, $x \mapsto u\gamma x$ is a join homomorphism of S into M' and the sublattice generated by u, γx, and γy is distributive for all x, y in S. Consequently, $(u+\gamma x)(u+\gamma y) = u + \gamma x\gamma y = u + \gamma xy$ and Prop. 3.1 applies to conclude that u is neutral in M'.

Therefore, the map $x \mapsto (ux, u+x)$ yields a subdirect representation of M'. M being subdirectly irreducible the induced subdirect representation of M has to be trivial, i.e. one of the maps $x \mapsto ux$ $(x \in M)$ and $x \mapsto u+x$ $(x \in M)$ has to be an embedding. In the first case we get $x = ux$ i.e. $x \leqq u$ for all x in M. Then, $x \mapsto u\gamma x = \gamma x$ is a homomorphism of S onto M.

In the second case we have $x = u+x$ i.e. $x \geqq u$ for all x in M. Then, $u \leqq 0_M$, the smallest element of M. Since 0_M is the smallest element of $\mathscr{F}(M)$, too, it follows $u = 0_M$. The filter u being generated by the descending chain u_n $(n \geqq 0)$ there has to be an n such that $u_n = 0_M$.

§ 4. Proof of the Theorem

Let M be as in the Theorem. The Lemma in [6] states that either

$$\text{(i}'\text{)} \quad ab = ac = ad = bc = bd = cd = \prod (q_{n1}q_{n2}q_{n3} | n < \infty)$$

or the dual of (i′) takes place. Thus, let us assume (i′). For any map ε of $\{a_0, b_0, c_0, d_0\}$ onto $\{a, b, c, d\}$ we define a map $\gamma = \gamma^\varepsilon$ of A_∞ into M recursively:

$$\gamma m_0 = 1$$

$$\gamma l_{n+1} = \varepsilon a_0 \gamma m_n + \varepsilon b_0 \gamma m_n, \quad \gamma r_{n+1} = \varepsilon c_0 \gamma m_n + \varepsilon d_0 \gamma m_n$$

$$\gamma(m_{n+1}+x_0) = \varepsilon x_0 + \gamma l_{n+1} \text{ for } x = a, b, \quad \gamma(m_{n+1}+x_0) = \varepsilon x_0 + \gamma r_{n+1} \text{ for } x = c, d,$$

and for $1 \leqq i \leqq n-1$

$$\gamma(m_{n+1}+x_i) = \gamma(m_{n+1}+x_0)\gamma(m_n+x_i); \quad \gamma m_{n+1} = \gamma l_{n+1}\gamma r_{n+1};$$

$$\gamma x_k = \varepsilon x_0 \gamma m_n \quad \text{for} \quad x = a, b, c, d; \quad \gamma m_\infty = 0.$$

Claim 1. γ^ε is a meet homomorphism of A_∞ into M.

Proof. In section 2 of [6] it has been shown that γ^ε restricted to A_n is a meet homomorphism for every n. Due to (i′) and the definition of γ^ε the claim follows, immediately.

Proposition 3.2 will be applied with L being a subdirect product of three copies of A_∞. We use the notation $\hat{x}=x$ for elements in the first, $\bar{a}_i=a_i$, $\bar{b}_i=c_i$, $\bar{d}_i=d_i$, $\bar{m}_i=m_i$ for elements in the second, and $\tilde{a}_i=a_i$, $\tilde{b}_i=c_i$, $\tilde{c}_i=d_i$, $\tilde{d}_i=b_i$, $\tilde{m}_i=m_i$ for elements in the third copy — see Fig. 1. In analogy, we write $\hat{\gamma}=\gamma^\varepsilon$ with $\varepsilon\hat{e}_0=e$, $\bar{\gamma}=\gamma^\varepsilon$ with $\varepsilon\bar{e}_0=e$, and $\tilde{\gamma}=\gamma^\varepsilon$ with $\varepsilon\tilde{e}_0=e$ for $e\in E$. Observe (by induction) that $\hat{\gamma}\hat{m}_n=q_{n1}=:\hat{q}_n$, $\bar{\gamma}\bar{m}_n=q_{n2}=:\bar{q}_n$, and $\tilde{\gamma}\tilde{m}_n=q_{n3}=:\tilde{q}_n$. Define $L=\{(0,0,0)\}\cup\bigcup\left([(m_i, m_j, m_k), (1,1,1)]\cup\{(e_i, e_j, e_k)|e\in E\}\right)|i,j,k<\infty)$.

Claim 2. L is the sublattice of $A_\infty\times A_\infty\times A_\infty$ generated by the elements $\check{e}=(\hat{e}_0, \bar{e}_0, \tilde{e}_0)$ with $e\in E$.

Proof. Component wise calculation yields the sublattice property, easily. We show by induction on i that the union of the intervals $[(m_i, 1, 1), (1, 1, 1)]$ and $[\hat{e}_i, \hat{e}_0]\,(e\in E)$ belongs to the sublattice S generated by the $\check{e}$. Namely, with $g=(\hat{m}_i, 1, 1)$ we have $(\hat{m}_{i+1}, 1, 1)=(\check{a}g+\check{b}g)(\check{c}g+\check{d}g)$ in S whence $(\hat{e}_j+\hat{m}_{i+1}, 1, 1)$ for $j\leqq i$ and $(\hat{e}_{i+1}, 1, 1)=(\hat{e}_0, 1, 1)(\hat{m}_{i+1}, 1, 1)$ are in S, too. Using symmetry and forming meets we get that S contains L. Trivially one obtains

Claim 3. $\gamma(\hat{x}, \bar{y}, \tilde{z})=\hat{\gamma}\hat{x}\bar{\gamma}\bar{y}\tilde{\gamma}\tilde{z}$ defines a meet homomorphism of L into M with $\gamma\check{e}=e$, $\gamma(\hat{m}_i, \bar{m}_j, \tilde{m}_k)=\hat{q}_i\bar{q}_j\tilde{q}_k$, and $\gamma(\hat{e}_i, \bar{e}_j, \tilde{e}_k)=e\hat{q}_i\bar{q}_j\tilde{q}_k$.

For $m\geqq 0$ define the map $\sigma_m\colon L\to M$ by $\sigma_m x=s_m\gamma x$. For $n\geqq 0$ define

$$S_n = [(\hat{m}_n, \bar{m}_n, \tilde{m}_n), (1,1,1)]\cup\{(\hat{e}_i, \bar{e}_j, \tilde{e}_k)|e\in E,\ i,j,k<n\}.$$

Claim 4. S_n is a join subsemilattice of L and $\sigma_m|S_n$ a join homomorphism if $m>3n$.

Proof. Let us write $1=(1,1,1)$. Observe that for $i=n-1$ and $e\neq f$ in E $(\hat{e}_i, \bar{e}_i, \tilde{e}_i)+(\hat{f}_i, \bar{f}_i, \tilde{f}_i)\geqq(\hat{m}_n, \bar{m}_n, \tilde{m}_n)$. Since $\{(\hat{e}_i, \bar{e}_j, \tilde{e}_k)|i,j,k<n\}=[(\hat{e}_{n-1}, \bar{e}_{n-1}, \tilde{e}_{n-1}), (\hat{e}_0, \bar{e}_0, \tilde{e}_0)]$ and $[(\hat{m}_n, \bar{m}_n, \tilde{m}_n), 1]$ are intervals this suffices to prove that S_n is closed under joins.

The second claim will be shown by induction on n. The modular lattice identities (a)—(f) we refer to shall be proved at the end of the section. The case $n=0$ is trivial. Let be $n\geqq 1$, $m>3n$, and assume that $\sigma_m|S_{n-1}$ is a join homomorphism.

Step 1. $\sigma_m|[(\hat{l}_n, 1, 1), 1]$ and $\sigma_m|[(\hat{r}_n, 1, 1), 1]$ preserve joins. Since $[(\hat{l}_n, 1, 1), 1]$ is the union of $[(\hat{m}_{n-1}, 1, 1), 1]$, $\{(\hat{l}_n, 1, 1)\}$, and the chains $[(\hat{e}_{n-2}+\hat{m}_n, 1, 1), (\hat{e}_0+ +\hat{m}_n, 1, 1)]$ $(e=a, b)$ it suffices to show $\sigma_m(\hat{a}_{n-2}, 1, 1)+\sigma_m(\hat{b}_{n-2}, 1, 1)\geqq\sigma_m(\hat{m}_{n-1}, 1, 1)]$ i.e.

(a) $s_m a\hat{q}_{n-2}+s_m b\hat{q}_{n-2} \geqq s_m\hat{q}_{n-1}$

and $\sigma_m(\hat{e}_i, 1, 1)+\sigma_m(\hat{m}_{n-1}, 1, 1)=\sigma_m(\hat{e}_i+\hat{m}_{n-1}, 1, 1)$, i.e.

(b) $s_m e\hat{q}_i+s_m\hat{q}_{n-1} = s_m(e+f\hat{q}_{n-2})\hat{q}_i$ for $\{e,f\} = \{a, b\}$ and $i \leqq n-2$.

(We have $\hat{\gamma}(\hat{e}_i+\hat{m}_{n-1})=\hat{\gamma}(\hat{e}_0+\hat{m}_{n-1})\hat{\gamma}\hat{m}_i$ since $\hat{\gamma}$ is a meet homomorphism.) The second claim follows by symmetry.

Step 2. $\sigma_m|[(\hat{m}_n, 1, 1), 1]$ is a join homomorphism. Since $[(\hat{m}_n, 1, 1), 1]$ is the union of $[(\hat{l}_n, 1, 1), 1]$, $[(\hat{r}_n, 1, 1), 1]$ and $\{(\hat{m}_n, 1, 1)\}$ and because of $(\hat{l}_n, 1, 1)+ +(\hat{r}_n, 1, 1)=(\hat{m}_{n-1}, 1, 1)$ it suffices to show $\sigma_m(\hat{l}_n, 1, 1)+\sigma_m(\hat{r}_n, 1, 1)=\sigma_m\{\hat{m}_{n-1}, 1, 1)$, i.e.

(c) $s_m(a\hat{q}_{n-1}+b\hat{q}_{n-1})+s_m(c\hat{q}_{n-1}+d\hat{q}_{n-1}) = s_m\hat{q}_{n-1}$.

Step 3. $\sigma_m|[(\hat{m}_n, \bar{m}_n, \tilde{m}_n), 1]$ is a join homomorphism. By symmetry, the restriction of σ_m to any of $[(\hat{m}_n, 1, 1), 1]$, $[(1, \bar{m}_n, 1), 1]$, and $[(1, 1, \tilde{m}_n), 1]$ is a join homomorphism. In view of

(i) $s_m\bar{q}_n+s_m\tilde{q}_n = s_m$ and $s_m\hat{q}_n+s_m\bar{q}_n\tilde{q}_n = s_m$

the $\sigma_m(\hat{m}_n, 1, 1)$, $\sigma_m(1, \bar{m}_n, 1)$, and $\sigma_m(1, 1, \tilde{m}_n)$ are dually independent in $[0, s_m]$. $\sigma_m|[(\hat{m}_n, \bar{m}_n, \tilde{m}_n), 1]$ being the product of the above three restrictions it is a join homomorphism, too.

Step 4. $\sigma_m|\{(\hat{e}_i, \bar{e}_j, \tilde{e}_k)|i, j, k<n\}$ is a join homomosphism for $e\in E$. This means for $i, j, k, r, s, t<n$, $u=\min(i, r)$, $v=\min(j, s)$, $w=\min(k, t)$

(d) $s_m e\hat{q}_i\bar{q}_j\tilde{q}_k+s_m e\hat{q}_r\bar{q}_s\tilde{q}_t = s_m e\hat{q}_u\bar{q}_v\tilde{q}_w$.

Step 5. $\sigma_m|S_n$ is a join homomorphism. Since S_n is the union of the intervals $[(\hat{m}_n, \bar{m}_n, \tilde{m}_n), 1]$ and $[(\hat{e}_i, \bar{e}_i, \tilde{e}_i), (\hat{e}_0, \bar{e}_0, \tilde{e}_0)]$ $(i=n-1,\ e\in E)$ it suffices to check $\sigma_m(\hat{e}_i, \bar{e}_i, \tilde{e}_i)+\sigma_m(\hat{f}_i, \bar{f}_i, \tilde{f}_i)\geqq\sigma_m\{\hat{m}_n, \bar{m}_n, \tilde{m}_n)$, i.e.

(e) $s_m e\hat{q}_i\bar{q}_i\tilde{q}_i+s_m f\hat{q}_i\bar{q}_i\tilde{q}_i \geqq s_m\hat{q}_n\bar{q}_n\tilde{q}_n$ for $i = n-1$, $e \neq f$ in E

and $\sigma_m(\hat{e}_i, \bar{e}_j, \tilde{e}_k)+\sigma_m(\hat{m}_n, \bar{m}_n, \tilde{m}_n)=\sigma_m(\hat{e}_i+\hat{m}_n, \bar{e}_j+\bar{m}_n, \tilde{e}_k+\tilde{m}_n)$ for $i, j, k<n$ and e in E. Due to symmetry and Step 3 the latter is satisfied if $\sigma_m(\hat{e}_i, \bar{e}_n, \tilde{e}_n)+$ $+\sigma_m(\hat{m}_n, \bar{m}_n, \tilde{m}_n)=\sigma_m(\hat{e}_i+\hat{m}_n, \bar{m}_n, \tilde{m}_n)$, i.e.

(f) $\quad s_m e\hat{q}_i\bar{q}_n\tilde{q}_n+s_m\hat{q}_n\bar{q}_n\tilde{q}_n = s_m(e+f\hat{q}_{n-1})\hat{q}_i\bar{q}_n\tilde{q}_n$ for $i<n$ and $\{e,f\}=\{a, b\}$.

Now, we are ready to prove the Theorem. Observe that M_4 and R_∞ are the only subdirectly irreducible homomorphic images of L. Namely, L is a subdirect product of six copies of R_∞ having M_4 as its only proper homomorphic image. Thus, the subdirectly irreducible lattice M cannot be a homomorphic image of L. Due to Claims 3 and 4 we may apply Proposition 3.2 and conclude that there is an n such that $s_n=\sigma_n 1=0$.

To prove the Corollary observe that induction yields $s_n=1$ and $s_n^*=0$ for all n and all lattices listed there. Namely, $q_1=1$ whence by Lemma 2.1 $s_{n+1}\geqq$ $\geqq q_1 s_n=s_n=1$. For the additional results recall that according to A. HUHN [8] in a 2-distributive lattice frames may have order at most 2. In view of Corollary 1.4 and 2.1, 3.2, and 3.3 from [7] this implies that $t_n=s_{n+1}$ for $n\geqq 1$ and $t_n=s_n$ for $n\geqq 3$. Thus, by Lemma 2.2 the only subdirectly irreducibles with $s_n=0$ for an n may be D_2 and M_3.

Before we come to the proof of the formulas (a)—(f) we need a Lemma.

Lemma 4.1. *For all* $m\geqq n$ *and* $i\in\mathbf{3}$ *one has* $s_m q_{ni}=\varrho_i^n s_{m-n}$. *Also,* e, q_{ni}, *and* s_m *generate a distributive sublattice for all* e *in* E.

Proof. By induction on n. For $n=1$ this is Lemma 2.1 (3) and 2.3. For $n>1$ one has by 2.5 $s_m q_{ni}=s_m q_i q_{ni}=\varrho_i s_{m-1}\varrho_i q_{n-1,i}=\varrho_i\varrho_i^{n-1}s_{m-n}=\varrho_i^n s_{m-n}$. Show $\varrho_i^n(e+s_k)=q_{ni}(e+s_{k+n})$ for all k. Indeed $\varrho_i^{n+1}(e+s_k)=\varrho_i^n\varrho_i(e+s_k)=\varrho_i^n(q_i e+$ $+q_i s_{k+1})=\varrho_i^n q_i(e+s_{k+1})=\varrho_i^n q_i\varrho_i^n(e+s_{k+1})=q_{n+1,i}q_{ni}(e+s_{k+1+n})=q_{n+1,i}(e+s_{k+n+1})$ by the hypothesis, and 2.5. Thus, $eq_{ni}+s_m q_{ni}=\varrho_i^n e+\varrho_i^n s_{m-n}=\varrho_i^n(e+s_{m-n})=q_{ni}(e+s_m)$ and the distributivity follows.

Proof of (a). $s_m a\hat{q}_{l-1}+s_m b\hat{q}_{l-1}=\varrho_1^{l-1}(as_{m-l+1}+bs_{m-l+1})=\varrho_1^{l-1}(a+b)s_{m-l+1}\geqq$ $\geqq\varrho_1^{l-1}q_1 s_{m-l+1}\geqq\hat{q}_l s_m$ for $l\leqq m+1$ by 2.5 and 4.1, 2.3 and 2.5, and 4.1 again.

Proof of (c). By 2.3 one has $s_k(a+b)+s_k(c+d)=s_k$ for $k\geqq 1$. (c) follows immediately applying the homomorphism ϱ_1^{n-1} in the case $k=m-n+1$ and appealing to 2.5 and 4.1.

Proof of (b). By 4.1 one has $s_k a+s_k\hat{q}_j=s_k(a+\hat{q}_j)$ for $k\geqq j$. Apply the homomorphism ϱ_1^i in the case $j=l-i$ and $k=m-i$ (for $i\leqq l<m$) to obtain $s_m a\hat{q}_i+s_m\hat{q}_l=s_m\hat{q}_i(a\hat{q}_i+\hat{q}_l)=s_m\hat{q}_i(a+\hat{q}_l)$. Now $a+\hat{q}_l=a+(a+b\hat{q}_{l-1})(c\hat{q}_{l-1}+d\hat{q}_{l-1})=$ $=(a+b\hat{q}_{l-1})(a+c\hat{q}_{l-1}+d\hat{q}_{l-1})$ by modularity and $a+c+d\geqq t_1\geqq s_{m-e+1}$ whence $a+c\hat{q}_{l-1}+d\hat{q}_{l-1}\geqq s_m$ (applying ϱ_1^{l-1}) and $s_m aq_i+s_m\hat{q}_l\geqq s_m\hat{q}_i(a+q_{l-1})$. Due to

$s_m\hat{q}_n\bar{q}_i\tilde{q}_i \geqq s_m\hat{q}_n\bar{q}_n\tilde{q}_n$ and the following Lemma (e) may be obtained from the formula proved under (a) (with $l-1=i=n-1$ and $m>3n-2i>l$) by application of the homomorphism $\varrho_2^i\varrho_3^i$.

Lemma 4.2. *$\varrho_j^m q_{ni}=q_{mj}q_{ni}$ for all $i\neq j$ in* **3** *and* $m, n\geqq 0$.

Proof. We show $\varrho_j q_{ni}=q_j q_{ni}$ by induction over n: $\varrho_j q_{n+1,i}=\varrho_j\varrho_i q_{ni}=$ $=\varrho_i\varrho_j q_{ni}=\varrho_i(q_j q_{ni})=\varrho_i q_j \varrho_i q_{ni}=q_i q_j q_{n+1,i}=q_j q_{n+1,i}$ by 2.1 (1) and 2.5. Now we induce over m: $\varrho_j^{m+1}q_{ni}=\varrho_j\varrho_j^m q_{ni}=\varrho_j(q_{jm}q_{ni})=\varrho_j q_{mj}\varrho_j q_{ni}=q_{m+1,j}q_{mj}q_{ni}=q_{m+1,j}q_{ni}$.

Next, observe that (f) and (d) are consequences of the following formula

(g) $\bar{q}_j\tilde{q}_k s_m e+\hat{q}_l s_m \geqq \bar{q}_j\tilde{q}_k s_m(e+\hat{q}_l)$ for $j+k+l<m$ and e in E.

Namely, for (f) put $j=k=l=n$, multiply both sides with $\hat{q}_i\bar{q}_n\tilde{q}_n$ and observe $a+\hat{q}_l\geqq$ $\geqq s_m(a+b\hat{q}_{l-1})$ as proved under (b).

For (d) assume w.l.o.g. $j\geqq s$, $k\geqq t$, and $i\leqq r=l$ and multiply both sides of (g) with $e\hat{q}_i\bar{q}_s\tilde{q}_t$.

In the proof of (g) assume w.l.o.g. $e=a$. Fisst, we show that q_1, as_h, and $q_3 s_h$ distribute for $h\geqq 3$: By 2.1 and 2.3 we have $q_1 s_h a+q_1 q_3 s_h=(s_{h-1}a+q_3 s_{h-1})^1=$ $=(s_{h-1}(a+q_3))^1=(s_{h-1}(a+b+c)(a+d))^1=(s_{h-1}(a+d))^1=s_h(q_1 a+q_1 d)=s_h q_1(a(c+$ $+d)+d)=s_h q_1(a+d)\geqq q_1(s_h a+s_h q_3)$.

Now, $\varrho_1^l(s_h a+s_h q_3)=q_{l1}(s_{h+l}a+s_{h+l}q_3)$ for $h\geqq 2$ follows by induction: $\varrho_1^{l+1}(s_h a+s_h q_3)=\varrho_1^l\varrho_1(s_h a+s_h q_3)=\varrho_1^l(q_1 s_{h+1}a+q_1 s_{h+1}q_3)=\varrho_1^l q_1(s_{h+1}a+s_{h+1}q_3)=$ $=\varrho_1^l q_1\varrho_1^l(s_{h+1}a+s_{h+1}q_3)=q_{l+1,1}q_{l1}(s_{h+1+l}a+s_{h+1+l}q_3)=q_{l+1,1}(s_{h+l+1}a+s_{h+l+1}q_3)$ using 2.1 and 2.5. Thus, for $h-l\geqq 2$ q_{l1}, $s_h a$, and $s_h q_3$ distribute: $q_{l1}s_h a+q_{l1}s_h q_3=$ $=\varrho_1^l s_{h-l}a+\varrho_1^l s_{h-l}q_3=\varrho_1^l(s_{h-1}a+s_{h-1}q_3)=q_{l1}(s_h a+s_h q_3)$ by 4.1 and 4.2.

Induction on $j+k$ yields $\varrho_2^j\varrho_3^k(s_h a+s_h q_{l1})=q_{j2}q_{k3}(as_{h+j+k}+q_{l1}s_{h+j+k})$ for $h>l$: $\varrho_2^j\varrho_3^k(s_h a+s_h q_{l1})=\varrho_2^j\varrho_3^{k-1}\varrho_3(s_h a+s_h q_{l1})=\varrho_2^j\varrho_3^{k-1}(aq_3 s_{h+1}+q_3 q_{l1}s_{h+1})=$ $=\varrho_2^j\varrho_3^{k-1}q_3(as_{h+1}+q_{l1}s_{h+1})=\varrho_2^j\varrho_3^{k-1}q_3\varrho_2^j\varrho_3^{k-1}(as_{h+1}+q_{l1}s_{h+1})=q_{j2}q_{k3}(as_{h+j+k}+q_{l1}s_{h+j+k})$ assuming $k>0$ w.l.o.g. (since $\varrho_2^j\varrho_3^k=\varrho_3^k\varrho_2^j$ by 2.1 (1)), and using 2.3 and 4.2. Finally, we get $\bar{q}_j\tilde{q}_k s_m a+\hat{q}_l s_m\geqq\bar{q}_j\tilde{q}_k s_m a+\bar{q}_j\tilde{q}_k\hat{q}_l s_m=\varrho_2^j\varrho_3^k s_{m-j-k}a+\varrho_2^j\varrho_3^k\hat{q}_l s_{m-j-k}=$ $=\varrho_2^j\varrho_3^k(as_{m-j-k}+\hat{q}_l s_{m-j-k})=\bar{q}_j\tilde{q}_k(as_m+\hat{q}_l s_m)=\bar{q}_j\tilde{q}_k s_m(a+\hat{q}_l)$ applying the above, 4.2 and 4.1.

Finally, to prove (i) we show by induction on m:

(j) $s_m\hat{q}_j+s_m\bar{q}_k\tilde{q}_l=s_m$ for $j+k+l\leqq m$.

The cases $m\leqq 1$, $j=0$, or $k=l=0$ being trivial, let $m\geqq 2$, $j\geqq 1$, $k\geqq 1$. Then

$$s_m\hat{q}_j+s_m\bar{q}_k\tilde{q}_l=s_m\hat{q}_j+s_m\hat{q}_1\bar{q}_k\tilde{q}_l+s_m\hat{q}_j\bar{q}_1+s_m\bar{q}_k\tilde{q}_l=\hat{\varrho}(s_{m-1}\hat{q}_{j-1}+s_{m-1}\bar{q}_k\tilde{q}_l)+$$
$$+\bar{\varrho}(s_{m-1}\hat{q}_j+s_{m-1}\bar{q}_{k-1}\tilde{q}_l)=\hat{\varrho}s_{m-1}+\bar{\varrho}s_{m-1}=s_m.$$

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Some propositions on analytic matrix functions related to the theory of operators in the space $\Pi_\varkappa$

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It is well known that certain classes of analytic functions play a useful role in the theory of hermitian and selfadjoint operators in Hilbert space. On the other hand, sometimes, general propositions from the spectral theory of operators yield simple solutions of problems in complex function theory. This is especially true for the theory of selfadjoint and unitary operators in spaces with indefinite metric.

In this note we prove some consequences of the theory of Q-functions and characteristic functions of hermitian and isometric operators in the space $\Pi_\varkappa$, as developed in [1] and [2], for scalar and matrix valued analytic functions of a complex variable. It seems rather unexpected to us that in this way we get new results [1]) also for the so-called Nevanlinna or R-functions (mappings of the upper half-plane into itself) so well studied in different contexts during the last 50 years.

There are now several papers (see, e.g., [5]) which generalize the well known theorem of Rouché to matrix or operator functions. In these papers, however, it is assumed that the boundary of the domain considered consists of regular points only. Here we show that our methods permit a generalization of Rouché's theorem to the case of matrix functions of the class $\mathcal{D}^{n\times n}$ (see § 4 below) over the unit disc. Instead of the unit disc more general domains with sufficiently smooth Jordan boundaries may be considered. For the case of scalar functions this generalization was proved [2]) by V. M. ADAMJAN, D. Z. AROV and M. G. KREIN in [6] and has found essential applications in the theory of Hankel operators with scalar kernel. Theorem 4.2 below can be used in the investigation of Hankel operators with matrix kernel.

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[1]) Some of the results were proved by a different method in [3] (cf. also [4]).

[2]) However in a less complete form, without counting the generalized poles of negative type (see the remark after Theorem 4.2).

§ 1. Basic propositions

1. An $(n\times n)$-matrix function K, defined on a nonempty set $Z\times Z$, is said to have $\varkappa$ negative squares (on Z) if it has the following two properties:
1) $K(z,\zeta)=K(\zeta,z)^*$ $(z,\zeta\in Z)$,
2) for any positive integer k, any $z_1, \ldots, z_k\in Z$ and n-vectors $\xi_1, \ldots, \xi_k\in\mathfrak{C}^n$ [1]) the matrix

$$(K(z_\nu, z_\mu)\xi_\nu, \xi_\mu)_{\nu,\mu=1,2,\ldots,k}$$

has at most $\varkappa$ negative eigenvalues and for at least one choice of k, $z_1, \ldots, z_k$, and $\xi_1, \ldots, \xi_k$, it has exactly $\varkappa$ negative eigenvalues.

In this note the following three classes of analytic $(n\times n)$-matrix functions will play an important role.

a) $\mathbf{N}_\varkappa^{n\times n}$ is the set of all $(n\times n)$-matrix functions Q which are meromorphic on $\mathfrak{C}_+$ and such that the kernel N_Q:

$$N_Q(z,\zeta):=\frac{Q(z)-Q(\zeta)^*}{z-\zeta^*}\quad (z,\zeta\in\mathfrak{D}_Q)$$

has $\varkappa$ negative squares ($\mathfrak{D}_Q\subset\mathfrak{C}_+$ denotes the domain of holomorphy of Q).

b) $\mathbf{C}_\varkappa^{n\times n}$ is the set of all $(n\times n)$-matrix functions F which are meromorphic on $\mathfrak{D}$ and such that the kernel C_F:

$$C_F(z,\zeta):=\frac{F(z)+F(\zeta)^*}{1-z\zeta^*}\quad (z,\zeta\in\mathfrak{D}_F)$$

has $\varkappa$ negative squares.

c) $\mathbf{S}_\varkappa^{n\times n}$ is the set of all $(n\times n)$-matrix functions θ which are meromorphic on $\mathfrak{D}$ and such that the kernel S_θ:

$$S_\theta(z,\zeta):=\frac{I-\theta(\zeta)^*\theta(z)}{1-z\zeta^*}\quad (z,\zeta\in\mathfrak{D}_\theta)$$

has $\varkappa$ negative squares.

In the special case $n=1$ these classes (of scalar valued functions) were studied in [7]. In the more general case where the values of the functions Q and θ are bounded linear operators on a Hilbert space, the corresponding classes were introduced in [1] and [2].

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[1]) $\mathfrak{C}$ is the set of all complex numbers, $\mathfrak{C}_+(\mathfrak{C}_-)$ the open upper (lower) half-plane, $\overline{\mathfrak{C}}_+$ the closure of $\mathfrak{C}_+$ in $\mathfrak{C}$. Furthermore, $\mathfrak{D}$ $(\overline{\mathfrak{D}})$ denotes the open (closed) unit disc and $\partial\mathfrak{D}$ the boundary of $\mathfrak{D}$. The usual scalar product and norm in $\mathfrak{C}^n$ are denoted by $(.,.)$ and $\|.\|$. If A is an $n\times n$ matrix, $\|A\|$ denotes the norm of the operator induced by A on $\mathfrak{C}^n$. If $z\in\mathfrak{C}$, then z^* denotes the complex conjugate of z.

We mention that these classes can be defined in a different way (cf. [1] and [2]). For instance, an $(n\times n)$-matrix function Q_0 which is defined and continuous on some open set $\mathfrak{D}'\subset\mathfrak{C}_+$ and for which the kernel N_{Q_0} has $\varkappa$ negative squares on $\mathfrak{D}'$ can be extrapolated in a unique way to a function $Q\in\mathbf{N}_\varkappa^{n\times n}$. Further, a function $Q\in\mathbf{N}_\varkappa^{n\times n}$ can be extrapolated to a function $\tilde{Q}$ locally meromorphic on $\mathfrak{C}_+\cup\mathfrak{C}_-$ by the formula

$$\tilde{Q}(z):=\begin{cases}Q(z), & z\in\mathfrak{D}_Q,\\ Q(z^*)^*, & z^*\in\mathfrak{D}_Q.\end{cases}$$

Then the kernel $N_{\tilde{Q}}$ has $\varkappa$ negative squares on $\mathfrak{D}_Q\cup\mathfrak{D}_Q^*$. In a similar way, $F\in\mathbf{C}_\varkappa^{n\times n}$ can be extrapolated to the complement of the closed unit disc by setting $\tilde{F}(z^{-1}):=-F(z^*)^*$ $(z^*\in\mathfrak{D}_F)$.

The classes $\mathbf{N}_\varkappa^{n\times n}$ and $\mathbf{C}_\varkappa^{n\times n}$ are very closely related. Namely, if φ is a linear fractional mapping of $\mathfrak{D}$ onto $\mathfrak{C}_+$, then the formula $F=iQ\circ\varphi$ $(Q\in\mathbf{N}_\varkappa^{n\times n})$ establishes a one-to-one correspondence between $\mathbf{N}_\varkappa^{n\times n}$ and $\mathbf{C}_\varkappa^{n\times n}$. Hence the statements about the class $\mathbf{C}_\varkappa^{n\times n}$ given below can easily be transferred to the class $\mathbf{N}_\varkappa^{n\times n}$.

Proposition 1.1. *Let* $F\in\mathbf{C}_\varkappa^{n\times n}$ *and* $\alpha\in\mathfrak{C}$, $\operatorname{Re}\alpha>0$. *Then the function* θ *defined by*

$$\theta(z):=\big(F(z)-\alpha^*I\big)\big(F(z)+\alpha I\big)^{-1} \tag{1.1}$$

belongs to the class $\mathbf{S}_\varkappa^{n\times n}$.

Proof. First we show that for each α, $\operatorname{Re}\alpha>0$, we can find a $z_0\in\mathfrak{D}_F$ such that $\big(F(z_0)+\alpha I\big)^{-1}$ exists. Otherwise for some fixed α, $\operatorname{Re}\alpha>0$, and each $z\in\mathfrak{D}_F$ there would exist an n-vector $\xi(z)\neq0$ such that $F(z)\xi(z)=-\alpha\xi(z)$. It follows

$$(1-z\zeta^*)^{-1}\big((F(z)+F(\zeta)^*)\xi(z),\xi(\zeta)\big)=-2\operatorname{Re}\alpha(1-z\zeta^*)^{-1}\big(\xi(z),\xi(\zeta)\big)= \tag{1.2}$$

$$=-\operatorname{Re}\alpha(2\pi)^{-1}\int_0^{2\pi}(e^{i\vartheta}-z)^{-1}(e^{-i\vartheta}-\zeta^*)^{-1}d\vartheta\big(\xi(z),\xi(\zeta)\big).$$

If $z_1, z_2, \dots, z_k\in\mathfrak{D}$, then the $k\times k$ matrix

$$\Big(\int_0^{2\pi}(e^{i\vartheta}-z_\nu)^{-1}(e^{-i\vartheta}-z_\mu^*)^{-1}d\vartheta\big(\xi(z_\nu),\xi(z_\mu)\big)\Big)_{\nu,\mu=1,2,\dots,k}$$

has k positive eigenvalues. This follows from the fact that for arbitrary $\alpha_1, \alpha_2, \dots, \alpha_k\in\mathfrak{C}$, not all equal to zero, we have

$$\int_0^{2\pi}\Big\|\sum_{\nu=1}^{k}\frac{\xi(z_\nu)\alpha_\nu}{e^{i\vartheta}-z_\nu}\Big\|^2 d\vartheta>0.$$

If we choose $k>\varkappa$, from (1.2) we get a contradiction to the assumption $F\in\mathbf{C}_\varkappa^{n\times n}$.

Thus $\det(F(z)+\alpha I)\not\equiv 0$. Hence the meromorphic function $\det(F(z)+\alpha I)$ can vanish only on a set σ_α of isolated points of $\mathfrak{D}$. For $z,\zeta\notin\sigma_\alpha$ it follows that $I-\theta(\zeta)^*\theta(z)=2(\operatorname{Re}\alpha)(F(\zeta)^*+\alpha^*I)^{-1}(F(z)+F(\zeta)^*)(F(z)+\alpha I)^{-1}$. Therefore the kernel S_θ has $\varkappa$ negative squares on σ_α.

2. Let $\Pi_\varkappa$ be a $\pi_\varkappa$-space with indefinite scalar product $[.,.]$.[1]) A bounded linear operator T in $\Pi_\varkappa$ is called *contractive* if $\mathfrak{D}(T)=\Pi_\varkappa$ and $[Tx,Tx]\leqq[x,x]$ $(x\in\Pi_\varkappa)$, *isometric* if $[Tx,Tx]=[x,x]$ $(x\in\mathfrak{D}(T))$, and *unitary* if it is isometric and $\mathfrak{D}(T)=$ $=\mathfrak{R}(T)=\Pi_\varkappa$. An isometric operator T with $\mathfrak{D}(T)=\Pi_\varkappa$ or $\mathfrak{R}(T)=\Pi_\varkappa$ is called *maximal isometric.*

Proposition 1.2. *A contractive operator T in a $\pi_\varkappa$-space $\Pi_\varkappa$ has a $\varkappa$-dimensional nonpositive invariant subspace $\mathscr{L}$ such that $|\sigma(T|\mathscr{L})|\geqq 1$. If $\mathscr{L}$ is not uniquely determined, the points of $\sigma(T|\mathscr{L})$ and their algebraic multiplicities do not depend on the choice of $\mathscr{L}$.*

We shall write $\sigma_0(T):=\sigma(T|\mathscr{L})$ if T and $\mathscr{L}$ are as in Proposition 1.2. For $\lambda\in\sigma_0(T)$ the algebraic multiplicity of λ with respect to $T|\mathscr{L}$ will be called the index of λ with respect to T and denoted by $\varkappa_\lambda(T)$. Evidently, it is the dimension of the intersection $\mathscr{L}\cap\mathscr{S}_\lambda(T)$, where $\mathscr{S}_\lambda(T):=\{x:\ (T-\lambda I)^k x=0$ for some $k=1,2,\dots\}$. If $\mathfrak{U}\subset\{z:\ |z|\geqq 1\}$, the index $\varkappa_{\mathfrak{U}}(T)$ of $\mathfrak{U}$ is defined by

$$\varkappa_{\mathfrak{U}}(T):=\sum_{\lambda\in\sigma_0(T)\cap\mathfrak{U}}\varkappa_\lambda(T).$$

The first statement of Proposition 1.2, and the second statement for points $\lambda\in\sigma(T|\mathscr{L})$, $|\lambda|>1$, follow from [9, Theorem 11.2]. For a unitary operator T the second statement was completely proved in [10]; this result is also an immediate consequence of the spectral theorem [11]. In the following only these conclusions of Proposition 1.2 will be used.

However, for the sake of completeness, we prove the second statement for an abirtrary contractive operator T in $\Pi_\varkappa$. To this end, observe first that T has a unitary dilation $\tilde U$ in some larger $\pi_\varkappa$-space $\tilde\Pi_\varkappa\supset\Pi_\varkappa$, that is

$$T^n x=\tilde P\tilde U^n x\quad(x\in\Pi_\varkappa,\ n=0,1,2,\dots),\tag{1.3}$$

where $\tilde P$ denotes the π-orthogonal projector of $\tilde\Pi_\varkappa$ onto $\Pi_\varkappa$ (see [12]), A relation between certain invariant subspaces of T and $\tilde U$ is established by the following lemma.

Lemma 1.3. *If T and $\tilde U$ are as above and $\mathscr{L}_0$ is a nonpositive subspace of $\Pi_\varkappa$ such that $T\mathscr{L}_0\subset\mathscr{L}_0$ and $|\sigma(T|\mathscr{L}_0)|=1$, then $\tilde Ux=Tx$ $(x\in\mathscr{L}_0)$.*

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[1]) Here we use the notation of [8]. For the properties of $\pi_\varkappa$-spaces and their bounded linear operators see [9].

Proof. The operator T has the property

$$[Tx, Ty] = [x, y] \quad (x, y \in \mathcal{L}_0). \tag{1.4}$$

Indeed, consider $V := (T|\mathcal{L}_0)^{-1}$. Then $|\sigma(V)| = 1$. On the other hand, if $\mathcal{L}_0$ is equipped with the nonnegative scalar product $-[x, y]$ $(x, y \in \mathcal{L}_0)$, then V induces a contraction $\hat{V}$ in the factor space $\hat{\mathcal{L}} := \mathcal{L}_0/\mathcal{L}_{00}$, where $\mathcal{L}_{00} := \{x \in \mathcal{L}_0 \colon [x, x] = 0\}$. Since $\sigma(\hat{V}) \subset \sigma(V)$, we have $|\sigma(\hat{V})| = 1$, and by a well known result on contractive operators in a unitary space, $\hat{V}$ is unitary. Therefore $[Vx, Vy] = [x, y]$ $(x, y \in \mathcal{L}_0)$ and (1.4) follows. Using (1.4) and (1.3), for $x \in \mathcal{L}_0$ we find

$$[x, x] = [Tx, Tx] = [\tilde{P}\tilde{U}x, \tilde{P}\tilde{U}x] \leqq [\tilde{U}x, \tilde{U}x] = [x, x];$$

hence $Tx = \tilde{P}\tilde{U}x = \tilde{U}x$.

Now we continue the proof of Proposition 1.2. The Lemma 1.3 implies that every $\lambda \in \sigma(T|\mathcal{L})$, $|\lambda| = 1$ belongs to $\sigma_0(\tilde{U})$. As the subspace $\mathcal{L}_0$ of Lemma 1.3 can always be extended to a $\varkappa$-dimensional nonpositive invariant subspace of U (see [16, Theorem VIII. 2.1]) we have for these λ

$$\varkappa_\lambda(T|\mathcal{L}) \leqq \varkappa_\lambda(\tilde{U}), \tag{1.5}$$

where $\varkappa_\lambda(T|\mathcal{L})$ denotes the dimension of $\mathcal{L} \cap \mathcal{S}_\lambda$. The same inequality (1.5) holds if $\lambda \in \sigma(T|\mathcal{L})$, $|\lambda| > 1$. Indeed, (1.3) implies that

$$(T - zI)^{-1} = \tilde{P}(\tilde{U} - z\tilde{T})^{-1} \quad (|z| > 1,\ z \notin \sigma(T) \cap \sigma(\tilde{U})),$$

and it follows that the dimension of the Riesz projector corresponding to λ and T is not greater than the dimension of the Riesz projector corresponding to λ and $\tilde{U}$. Now (1.5) yields

$$\varkappa = \sum_{\lambda \in \sigma(T|\mathcal{L})} \varkappa_\lambda(T|\mathcal{L}) \leqq \sum_{\lambda \in \sigma_0(\tilde{U})} \varkappa_\lambda(\tilde{U}) = \varkappa,$$

that is, in (1.5) the sign $=$ must hold. But the right hand side of (1.5) is independent of $\mathcal{L}$, and the statement follows.

The following proposition can be proved in the same way as Satz 1.2 in [1].

Proposition 1.4. *Let (T_n) be a sequence of contractive operators in $\Pi_\varkappa$, $\|T_n - T_0\| \to 0$ $(n \to \infty)$, and $\lambda_0 \in \sigma_0(T_0)$. Then for each sufficiently small neighbourhood $\mathfrak{U}$ of λ_0 there exists an $n(\mathfrak{U}) > 0$ such that for $n \geqq n(\mathfrak{U})$ we have $\varkappa_{\mathfrak{U}}(T_n) = \varkappa_{\lambda_0}(T_0)$.*

Because of the relation

$$\sum_{\lambda \in \sigma_0(T_0)} \varkappa_\lambda(T_0) = \sum_{\lambda \in \sigma_0(T_n)} \varkappa_\lambda(T_n) = \varkappa,$$

under the conditions of Proposition 1.9 the points of $\sigma_0(T_0)$ are the only "accumulation points" of $\sigma_0(T_n)$, $n = 1, 2, \ldots$.

3. A close connection between functions $F \in \mathbf{C}_\varkappa^{n \times n}$ and isometric operators in a $\pi_\varkappa$-space $\Pi_\varkappa$ is given by the following proposition ([2, Satz 2.2]):

a) *Let V be a maximal isometric $(\mathfrak{R}(V)=\Pi_\varkappa)$ operator in a $\pi_\varkappa$-space $\Pi_\varkappa$, S a hermitian $n\times n$ matrix and Γ a linear mapping from $\mathfrak{C}^n$ into $\Pi_\varkappa$. Then the function F:*

$$(1.6)\qquad F(z)=iS+\Gamma^*(V+zI)(V-zI)^{-1}\Gamma \quad \left(z^{-1}\notin\sigma(V^{-1}),\ |z|<1\right)$$

belongs to the class $\mathbf{C}_{\varkappa'}^{n\times n}$ for some $\varkappa'$, $0\leqq\varkappa'\leqq\varkappa$. If the operators V and Γ are closely i-connected then $\varkappa'=\varkappa$.

b) *If $F\in\mathbf{C}_\varkappa^{n\times n}$ and $0\in\mathfrak{D}_F$, then there exist a $\pi_\varkappa$-space $\Pi_\varkappa$, a maximal isometric $(\mathfrak{R}(V)=\Pi_\varkappa)$ operator V in $\Pi_\varkappa$ and a linear mapping Γ from $\mathfrak{C}^n$ into $\Pi_\varkappa$, closely i-connected with V, so that the representation* (1.6) *holds with* $S=\operatorname{Im} F(0)$.

We remind the reader that an operator Γ from $\mathfrak{C}^n$ into $\Pi_\varkappa$ is said to be *closely i-connected* with the maximal isometric $(\mathfrak{R}(V)=\Pi_\varkappa)$ operator V in $\Pi_\varkappa$ if $\Pi_\varkappa$ is the closed linear span of all elements $(V-zI)^{-1}\Gamma\xi$, $\xi\in\mathfrak{C}^n$, $z\in\varrho(V)$, $|z|<1$. Here, of course, $(V-zI)^{-1}$ is always to be understood as $V^{-1}(I-zV^{-1})^{-1}$ with the isometric operator $V^{-1}=V^+$ defined on all of $\Pi_\varkappa$, $z^{-1}\in\varrho(V^{-1})$.

The function $F\in\mathbf{C}_\varkappa^{n\times n}$, $0\in\mathfrak{D}_F$, admits also a representation (1.6) with a unitary operator V in $\Pi_\varkappa$. Consider this operator V, and let $\mathscr{L}$ be a $\varkappa$-dimensional nonpositive invariant subspace of V such that $|\sigma(V|\mathscr{L})|\geqq 1$. Denote the characteristic polynomial of $V|\mathscr{L}$, which does not depend on the choice of $\mathscr{L}$, by p and put $g(z)=p^*(z^{-1})p(z)$. Then we have $[g(V)x, x]\geqq 0$ $(x\in\Pi_\varkappa)$ and it follows that

$$\operatorname{Re}\Gamma^* g(V)(V+zI)(V-zI)^{-1}\Gamma\geqq 0 \quad (z\in\mathfrak{D}).$$

Hence there exists a nondecreasing bounded $(n\times n)$-matrix function Σ on $[0, 2\pi)$, such that

$$(1.7)\qquad \Gamma^* g(V)(V+zI)(V-zI)^{-1}\Gamma=\int_0^{2\pi}(e^{i\vartheta}+z)(e^{i\vartheta}-z)^{-1}\,d\Sigma(\vartheta).$$

Introducing the $(n\times n)$-matrix function G:

$$G(z):=\Gamma^*\big(g(z)I-g(V)\big)(V+zI)(V-zI)^{-1}\Gamma$$

we get from (1.6) and (1.7)

$$(1.8)\qquad F(z)=iS+\frac{1}{g(z)}\int_0^{2\pi}(e^{i\vartheta}+z)(e^{i\vartheta}-z)^{-1}\,d\Sigma(\vartheta)+\frac{1}{g(z)}G(z) \quad (z\in\mathfrak{D}).$$

As a consequence of b) we prove the following

Proposition 1.5. *The function $\theta\in\mathbf{S}_\varkappa^{n\times n}$, $0\in\mathfrak{D}_\theta$, admits the representation*

$$(1.9)\qquad \theta(z)=U_{22}+zU_{21}(I-zT)^{-1}U_{12} \quad (z\in\mathfrak{D}_\theta),$$

where T is a contractive operator in a space $\Pi_\varkappa$, which has no eigenvalues on the unit circle, and U_{12}, U_{21}, U_{22} are such mappings that the matrix

$$U=\begin{pmatrix} T & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \tag{1.10}$$

defines an isometric operator in the space $\Pi_\varkappa \oplus \mathfrak{C}^n$. The space $\Pi_\varkappa$ and the operator U can be chosen so that

$$\Pi_\varkappa = \text{c.l.s.}\ \{(I-zT)^{-1}U_{12}\xi\colon\ \xi\in\mathfrak{C}^n,\ z^{-1}\in\varrho(T)\};$$

then they are uniquely determined up to unitary equivalence.

Here, if $u, v\in\Pi_\varkappa$ and $\xi,\eta\in\mathfrak{C}^n$, the scalar product of $\{u,\xi\}$, $\{v,\eta\}\in\Pi_\varkappa\oplus\mathfrak{C}^n$ is defined by

$$[\{u,\xi\},\ \{v,\eta\}] = [u,v]+(\xi,\eta).$$

The operator U_{12} maps $\mathfrak{C}^n$ into $\Pi_\varkappa$, U_{21} maps $\Pi_\varkappa$ into $\mathfrak{C}^n$, and U_{22} maps $\mathfrak{C}^n$ into itself.

Proof. We may suppose that $\det(I-\theta(0))\neq 0$. Indeed, if this relation does not hold we consider θ_γ: $\theta_\gamma(z):=\gamma\theta(z)$ instead of θ for some γ: $|\gamma|=1$, $\det(I-\theta_\gamma(0))\neq 0$. Having found the representation of θ_γ with some operator U_γ, the representation of θ follows with an operator U, which is obtained from U_γ by multiplication of the second row by γ^{-1}.

Consider for $\alpha\in\mathfrak{C}$, $\operatorname{Re}\alpha>0$, the function F:

$$F(z):=\big(\alpha^* I+\alpha\theta(z)\big)\big(I-\theta(z)\big)^{-1}. \tag{1.11}$$

Then

$$F(z)+F(\zeta)^* = 2(\operatorname{Re}\alpha)\big(I-\theta(\zeta)^*\big)^{-1}\big(I-\theta(\zeta)^*\theta(z)\big)\big(I-\theta(z)\big)^{-1},$$

and it follows that $F\in\mathfrak{C}_\varkappa^{n\times n}$. From the relations (1.11), $F(0)=iS+\Gamma^*\Gamma$ and (1.6) we find

$$\begin{aligned}
\theta(z) &= I-2\operatorname{Re}\alpha\big(F(z)+\alpha I\big)^{-1} = I-2\operatorname{Re}\alpha\big(F(0)+\alpha I+2z\Gamma^*V^{-1}(I-zV^{-1})^{-1}\Gamma\big)^{-1} = \\
&= I-2(\operatorname{Re}\alpha)\big(F(0)+\alpha I\big)^{-1}+4(\operatorname{Re}\alpha)z\big(F(0)+\alpha I\big)^{-1}\Gamma^*V^{-1}\times \\
&\qquad \times\big(I-zV^{-1}+2z\Gamma(F(0)+\alpha I)^{-1}\Gamma^*V^{-1}\big)^{-1}\Gamma(F(0)+\alpha I)^{-1} = \\
&= \big(F(0)-\alpha I\big)\big(F(0)+\alpha I\big)^{-1}+ \\
&\qquad +4(\operatorname{Re}\alpha)z\big(F(0)+\alpha I\big)^{-1}\Gamma^*V^{-1}(I-zW_\alpha V^{-1})\Gamma\big(F(0)+\alpha I\big)^{-1}
\end{aligned}$$

with $W_\alpha:=I-2\Gamma\big(F(0)+\alpha I\big)^{-1}\Gamma^*$. Setting

$$T:=W_\alpha V^{-1},\quad U_{12}:=2\sqrt{\operatorname{Re}\alpha}\big(F(0)+\alpha I\big)^{-1},$$

$$U_{21}:=2\sqrt{\operatorname{Re}\alpha}\big(F(0)+\alpha I\big)^{-1}\Gamma^*V^{-1},\quad U_{22}:=\big(F(0)-\alpha^* I\big)\big(F(0)+\alpha I\big)^{-1},$$

and using the relation $\Gamma^*\Gamma=2^{-1}(F(0)+F(0)^*)$, it is not hard to verify that the matrix U satisfies $U^*U=I$.

The operator T is contractive in $\Pi_\varkappa$. Indeed, we have for $u\in\Pi_\varkappa$, $v:=V^{-1}u$:

$$\begin{aligned}[Tu,Tu] &= [v,v]-2\left[\Gamma\left(F(0)+\alpha I\right)^{-1}\Gamma^*v,v\right]-2\left[\Gamma\left(F(0)^*+\alpha^*I\right)^{-1}\Gamma^*v,v\right]+\\ &\quad +4\left(\Gamma^*\Gamma(F(0)+\alpha I)^{-1}\Gamma^*v,(F(0)+\alpha I)^{-1}\Gamma^*v\right)=\\ &= [v,v]-4\operatorname{Re}\alpha\left\|\left(F(0)+\alpha I\right)^{-1}\Gamma^*v\right\|^2\leqq[u,u].\end{aligned}\tag{1.12}$$

Assume that $Tu_0=\lambda_0u_0$, $|\lambda_0|=1$. Then, by (1.12), $\Gamma^*V^{-1}u_0=0$ and $W_\alpha V^{-1}u_0=V^{-1}u_0=\lambda_0u_0$. Hence $\Gamma^*u_0=0$, $(V^{-1})^*u_0=Vu_0=\lambda_0^{-1}u_0$, and for arbitrary $\xi\in\mathbf{C}^n$, $z^{-1}\in\varrho(V^{-1})$, $|z|<1$, we get

$$[V^{-1}(I-zV^{-1})^{-1}\Gamma\xi,u_0]=(\xi,\Gamma^*u_0)(\lambda_0-z)^{-1}=0.$$

As Γ and V are closely i-connected, this implies $u_0=0$. The proof of the uniqueness of U is left to the reader.

Remark. The function $F\in\mathbf{C}_\varkappa^{n\times n}$ in the proof of Proposition 1.5 admits also a representation (1.6) with a unitary operator V in a $\pi_\varkappa$-space $\Pi_\varkappa$. This implies a representation (1.9) of the function θ, where the operator (1.10) is unitary in the space $\Pi_\varkappa\oplus\mathfrak{C}^n$. Then θ is the characteristic function of the operator T^*, see (1.10) (the case $n=1$ was considered in [7]).

We mention that Proposition 1.5 is an immediate generalization of [7, Satz 6.5]. It can be reversed and generalized to functions θ with values in $[\mathfrak{H}]$, the Banach algebra of all bounded linear operators mapping the Hilbert space $\mathfrak{H}$ into itself.

4. In [2, Satz 3.2] it was shown that a function $\theta\in\mathbf{S}_\varkappa^{n\times n}$ admits also the representation

$$\theta(z)=B_0(z)^{-1}\theta_0(z)\quad(z\in\mathfrak{D}_\theta)\tag{1.13}$$

with a Blaschke—Potapov product B_0,

$$B_0(z)=U_0\overset{\curvearrowleft}{\prod_{j=1}^{l}}B_j(z),\quad B_j(z)=\overset{\curvearrowleft}{\prod_{k=1}^{k_j}}\left(\frac{z-\alpha_j}{1-z\alpha_j^*}P_{jk}+Q_{jk}\right),\tag{1.14}$$

and a function $\theta_0\in\mathbf{S}_0^{n\times n}$. Here $\alpha_j\in\mathfrak{D}$, $\alpha_j\neq\alpha_{j'}$, for $j\neq j'$; P_{jk} and Q_{jk} are idempotent hermitian matrices with $P_{jk}+Q_{jk}=I$ for $k=1,2,\dots,k_j$ and $j=1,2,\dots,l$; U_0 is a unitary matrix and $\theta_0\in\mathbf{S}_0^{n\times n}$.

The Blaschke—Potapov product B_0 is called *regular* if

$$P_{j1}\geqq P_{j2}\geqq\dots\geqq P_{jk_j},\quad j=1,2,\dots,l.$$

The representation (1.13) is called *regular* if B_0 is regular and

$$\mathfrak{R}\left(P_{j1}Y_{j+1}(\alpha_j)\right)=\mathfrak{R}(P_{j1}),\quad j=1,2,\dots,l\tag{1.15}$$

holds; here

$$(1.16)\quad Y_j(z) := \left(\overset{\curvearrowright}{\prod_{\nu=j}^{l}} B_\nu(z)^{-1}\right) U_0^{-1}\theta_0(z), \quad j=1,2,\dots,l, \quad Y_{l+1}(z) := U_0^{-1}\theta_0(z).$$

The *order* of the Blaschke—Potapov product B_0 in (1.14) is defined as

$$\sum_{j=1}^{l}\sum_{k=1}^{k_j} \dim P_{jk};$$

according to [2, Satz 3.2] it is equal to $\varkappa$, if the representation (1.13) is regular.

§ 2. Zeros and poles in $\mathfrak{D}$

1. The multiplicity of zeros and poles of a meromorphic matrix or operator function was defined e.g. in [5]. Here we use the following characterization of the pole multiplicity (see [1, Lemma 4.1]): If $A(z)$ is a meromorphic function whose values are bounded linear operators in a Banach space $\mathfrak{B}$ and which has a pole α with Laurent expansion

$$(2.1)\qquad A(z) = (z-\alpha)^{-k}A_{-k} + \dots + (z-\alpha)^{-1}A_{-1} + A_0 + \dots$$

for z near α, $z \neq \alpha$, then the *pole multiplicity* of α with respect to $A(z)$ is the dimension of the range of the operator

$$\mathfrak{A} := \begin{pmatrix} A_{-k} & 0 & \dots 0 & 0 \\ A_{-k+1} & A_{-k} & \dots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_{-2} & A_{-3} & \dots A_{-k} & 0 \\ A_{-1} & A_{-2} & \dots A_{-k+1} & A_{-k} \end{pmatrix}$$

in the space $\mathfrak{B}^k$. The matrix A will be called *associated to the singular part* of the expansion (2.1).

In the following we need two simple properties of the pole multiplicity, which are easy consequences of the characterization given above.

a) If $A(z)$ is as above, Γ_1 is a bounded linear mapping from a Banach space $\mathfrak{B}_1$ into $\mathfrak{B}$, and Γ_2 is a bounded linear mapping from $\mathfrak{B}$ into $\mathfrak{B}_1$, then the pole multiplicity of α with respect to $\Gamma_2 A(z)\Gamma_1$ is not greater than the pole multiplicity of α with respect to $A(z)$.

b) If α is an isolated eigenvalue of the operator T in $\mathfrak{B}$ and a pole of the resolvent of T, then its pole multiplicity with respect to this resolvent is equal to the algebraic multiplicity of the eigenvalue α.

Lemma 2.1. *Let $A(z)$ be a meromorphic $(n\times n)$-matrix function with a pole α of multiplicity $\varkappa$ and Laurent expansion* (2.1), *and let $Y(z)$ be an $(n\times n)$-matrix func-*

tion, holomorphic at $z=\alpha$. *If there exists a subspace* $\mathcal{L}\subset\mathfrak{R}(Y(\alpha))$ *such that*

$$A_{-j}\mathcal{L}\subset\mathcal{L},\quad A_{-j}\mathcal{L}^{\perp}=\{0\},\quad j=1,2,\dots,k, \tag{2.2}$$

then $A(z)Y(z)$ *has at* $z=\alpha$ *a pole of multiplicity* $\varkappa$.

Proof. The singular part of the Laurent expansion of $A(z)Y(z)$ at $z=\alpha$ has the associated matrix $\mathfrak{A}\mathfrak{Y}$, where $\mathfrak{Y}=(Y_{ij})_{i,j=1,2,\dots,k}$, $Y_{ij}:=\dfrac{1}{(i-j)!}Y^{(i-j)}(\alpha)$ if $i\geqq j$, $Y_{ij}:=0$ if $i<j$, $i,j=1,2,\dots,k$. Put $\mathfrak{Y}_0:=(P_0Y_{ij})_{i,j=1,2,\dots,k}$, where P_0 is the orthogonal projector onto $\mathcal{L}$. According to (2.2), the range of $\mathfrak{A}\mathfrak{Y}$ coincides with the range of $\mathfrak{A}\mathfrak{Y}_0$, and the range of $\mathfrak{Y}_0$ is $\mathcal{L}^k=\mathcal{L}\dotplus\mathcal{L}\dotplus\dots\dotplus\mathcal{L}$. On the other hand, the full range of $\mathfrak{A}$ is obtained if $\mathfrak{A}$ is applied to $\mathcal{L}^k$. The lemma is proved.

2. Consider now a function $\theta\in\mathbf{S}^{n\times n}_{\varkappa}$. If $\alpha\in\mathfrak{D}$ is a pole of θ, we denote its multiplicity by $\pi(\alpha)$. For some j, $1\leqq j\leqq l$, α coincides with α_j in a regular representation (1.13). We denote by $\varkappa(\alpha)$ the order of the corresponding factor B_j of the Blaschke—Potapov product in (1.13), that is

$$\varkappa(\alpha)=\sum_{k=1}^{k_j}\dim P_{jk}.$$

According to [2, § 3.4], $\varkappa(\alpha)$ coincides with the number of negative squares of the kernel S_{B_j}, and the number of negative squares of S_{Y_j} is $\varkappa(\alpha)$ plus the number of negative squares of $S_{Y_{j+1}}$, where Y_j is given by (1.16) and the representation (1.13) is again supposed to be regular.

If $0\in\mathfrak{D}_\theta$, then we denote by $\nu(\alpha)$ the dimension of the algebraic eigenspace, corresponding to α^{-1}, of a contractive operator T in $\Pi_\varkappa$ in a representation (1.9) of θ. This notation is correct because of [9, Theorem 11.2] and the following theorem.

Theorem 2.2. *If* $\theta\in\mathbf{S}^{n\times n}_{\varkappa}$ *and* $\alpha\in\mathfrak{D}$ *is a pole of* θ, *then* $\pi(\alpha)=\varkappa(\alpha)$. *If, additionally,* $0\in\mathfrak{D}_\theta$, *then* $\pi(\alpha)=\varkappa(\alpha)=\nu(\alpha)$.

Proof. First we show that the multiplicity of the pole α_j of B_j^{-1} in (1.14) is equal to $\sum_{k=1}^{k_j}\dim P_{jk}$. As the pole multiplicity is invariant under a fractional linear transformation of the independent variable, we may here suppose $\alpha_j=0$. Instead of P_{jk} we shall briefly write P_k, $k=1,2,\dots,k_j$. Then the matrix associated with the singular part of the expansion of B_j^{-1} at $z=0$ is

$$\begin{pmatrix} P_{k_j} & 0 & \dots & 0 & 0\\ P_{k_j-1}-P_{k_j} & P_{k_j} & \dots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ P_2-P_3 & P_3-P_4 & \dots & P_{k_j} & 0\\ P_1-P_2 & P_2-P_3 & \dots & P_{k_j-1}-P_{k_j} & P_{k_j}\end{pmatrix}.$$

Evidently, its range is $\mathfrak{R}(P_{k_j})+\mathfrak{R}(P_{k_j-1})+\ldots+\mathfrak{R}(P_1)$; therefore the dimension of this range is $\sum_{k=1}^{k_j}\dim P_k$. Thus the pole multiplicity of α_j coincides with the order of B_j.

Furthermore, we have

$$\theta(z)=B_1(z)^{-1}B_2(z)^{-1}\ldots B_{j-1}(z)^{-1}B_j(z)^{-1}Y_{j+1}(z).$$

From (1.15) it follows that Lemma 2.1 can be applied to $A=B_j^{-1}$, $Y=Y_{j+1}$ and $\mathscr{L}=\mathfrak{R}(P_{j1})$. Hence $B_j(z)^{-1}\,Y_{j+1}(z)$ has at $z=\alpha_j$ a pole of multiplicity $\sum_{k=1}^{k_j}\dim P_{jk}$. Finally

$$B_1(z)^{-1}B_2(z)^{-1}\ldots B_{j-1}(z)^{-1}$$

is holomorphic and boundedly invertible at $z=\alpha_j$. Therefore the pole multiplicity of $\theta(z)$ at $z=\alpha_j$ is $\sum_{k=1}^{k_j}\dim P_{jk}$, that is $\pi(\alpha_j)=\varkappa(\alpha_j)$.

To prove the second statement, consider a representation (1.9) of θ. According to the statements a) and b) in § 2.1, we have $\varkappa(\alpha_j)\leqq\nu(\alpha_j)$. On the other hand, the spectrum of T outside the unit disc consists of eigenvalues of total multiplicity $\varkappa$ (Propositions 1.2 and 1.5). Hence

$$\varkappa=\sum_{j=1}^{l}\varkappa(\alpha_j)\leqq\sum_{j=1}^{l}\nu(\alpha_j)=\varkappa,$$

and $\varkappa(\alpha_j)=\nu(\alpha_j)$, $j=1,2,\ldots,l$, follows. The theorem is proved.

We mention that for a fractional linear transformation $z\to\zeta(z):=\dfrac{z-\beta}{1-z\bar\beta}$, $|\beta|<1$, $\beta\in\mathfrak{D}_\theta$ of $\mathfrak{D}$ onto $\mathfrak{D}$ the function θ_1: $\theta_1(\zeta):=\theta(z)$ always has the property $0\in\mathfrak{D}_{\theta_1}$. Also, it is easy to check that $\theta\in S_\varkappa^{n\times n}$ implies $\theta_1\in S_\varkappa^{n\times n}$.

Corollary 1. *$\theta\in\mathbf{S}_\varkappa^{n\times n}$ has poles in $\mathfrak{D}$ of total multiplicity $\varkappa$.*

Let now $F\in\mathbf{C}_\varkappa^{n\times n}$ be given. Choose α, $\operatorname{Re}\alpha>0$. Then by Proposition 1.1 the function θ:

$$\theta(z)=I-2\operatorname{Re}\alpha\big(F(z)+\alpha I\big)^{-1}$$

belongs to $\mathbf{S}_\varkappa^{n\times n}$, and (see [5]) the poles of θ coincide, including multiplicities, with the zeros of $F(z)+\alpha I$.

Corollary 2. *If $F\in\mathbf{C}_\varkappa^{n\times n}$ and $\operatorname{Re}\alpha>0$, then the function $F(z)+\alpha I$ has in $\mathfrak{D}$ zeros of total multiplicity $\varkappa$.*

The corresponding conclusion for a function $Q\in\mathbf{N}_\varkappa^{n\times n}$ reads as follows.

Corollary 3. *If* $Q \in \mathbf{N}_\varkappa^{n\times n}$ *and* $\operatorname{Im}\beta > 0$, *then the function* $Q(z)+\beta I$ *has in* $\mathfrak{C}_+$ *zeros of total multiplicity* $\varkappa$.

As an application, consider the function Q:

$$Q(z) = Q_0(z) + \sum_{j=1}^{l} \sum_{k=1}^{k_j} \left((z-\alpha_j)^k B_{jk} + (z-\alpha_j^*)^k B_{jk}^*\right),$$

where $Q_0 \in \mathbf{N}_0^{n\times n}$, B_{jk} are arbitrary $(n\times n)$-matrices and $\alpha_j \in \mathfrak{C}_+$, $k=1, 2, \ldots, k_j$; $j=1, 2, \ldots, l$. It follows as in [1, Satz 4.5] that $Q \in \mathbf{N}_\varkappa^{n\times n}$, where $\varkappa = \sum_{j=1}^{l} \varkappa_j$,

$$\varkappa_j = \dim \begin{pmatrix} B_{jk_j} & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ B_{j2} & B_{j3} & & 0 \\ B_{j1} & B_{j2} & \ldots & B_{jk_j} \end{pmatrix}.$$

Hence Corollary 3 implies that for each β, $\operatorname{Im}\beta > 0$, the function $Q(z)+\beta I$ has zeros of total multiplicity $\varkappa$ in $\mathfrak{C}_+$.

§ 3. Generalized zeros and poles of negative type on the boundary

1. Definition. Let $F \in \mathbf{C}_\varkappa^{n\times n}$. The point $z_0 \in \partial\mathfrak{D}$ is called a *generalized pole (or zero) of negative type and multiplicity* $\pi(z_0)$ *for* F, if for each sufficiently small neighbourhood $\mathfrak{U}$ of z_0 there exists an $n(\mathfrak{U}) > 0$ such that for $\alpha > n(\mathfrak{U})$ (or $0 < \alpha < n(\mathfrak{U})$, resp.) the function $F(z)+\alpha I$ has zeros of total multiplicity $\pi(z_0)$ in $\mathfrak{U} \cap \mathfrak{D}$.

To explain this definition e.g. in the case of a generalized pole, let us take a scalar function F. Instead of F we consider its continuation $\tilde{F}$ to $\{z\colon |z| \neq 1\}$ (see § 1.1) and assume that it has been continued analytically also to arcs of the unit circle $|z|=1$ if possible, that is if the boundary values of $\tilde{F}$ at the points of this arc exist and are purely imaginary. Suppose this continuation $\tilde{F}$ has a pole at $z_0 \in \partial\mathfrak{D}$.

If $\varkappa = 0$, that is $F \in \mathbf{C}_0$ [1]), then $\operatorname{Re} F(z) \geqq 0$ for all $z \in \mathfrak{D}_F$. What is more, for each ϑ, $0 < \vartheta < \pi/2$, there exists a ϑ_1, $0 < \vartheta_1 < \pi/2$, such that the relations $z_n \in \mathfrak{D}$, $-\vartheta < \arg(z_n - z_0) < \vartheta$ and $z_n \to z_0$ imply that $F(z_n)$ tends to infinity and $-\vartheta_1 < \arg F(z_n) < \vartheta_1$.

On the other hand, if $\varkappa > 0$, there may be poles z_0 on $\partial\mathfrak{D}$ with the property that there exists a sequence $(z_n) \subset \mathfrak{D}$, $z_n \to z_0$, such that $F(z_n)$ tends to infinity along the negative real half-axis. Moreover, it turns out that there may be a finite number of points z_0 on $\partial\mathfrak{D}$ which are no poles but which also do have the property $F(z_n) \to -\infty$ for some sequence $(z_n) \subset \mathfrak{D}$, $z_n \to z_0$. These two kinds of points z_0 are the generalized poles

[1]) We write $\mathbf{C}_\varkappa$ etc. instead of $\mathbf{C}_\varkappa^{1\times 1}$.

of negative type. We mention already here that, for each point $\hat{z}\in\partial\mathfrak{D}$ which is not a generalized pole of negative type, there exists a neighbourhood $\hat{\mathfrak{U}}$ of $\hat{z}$ and a $\hat{\gamma}>0$ such that $\operatorname{Re} F(z)\geqq-\hat{\gamma}$ for all $z\in\hat{\mathfrak{U}}\cap\mathfrak{D}$ (see the Corollary subsequent to Theorem 3.5).

We show that the poles in $\mathfrak{D}$ of $F\in\mathbf{C}_\varkappa^{n\times n}$ have the same property as generalized poles on $\partial\mathfrak{D}$.

Proposition 3.1. *Let $F\in\mathbf{C}_\varkappa^{n\times n}$. If $z_0\in\mathfrak{D}$ is a pole of multiplicity $\pi(z_0)$ of F, then for each sufficiently small neighbourhood $\mathfrak{U}$ of z_0 there exists an $n(\mathfrak{U})>0$ such that $\alpha>n(\mathfrak{U})$ implies that the function $F(z)+\alpha I$ has zeros of total multiplicity $\pi(z_0)$ in $\mathfrak{U}$.*

Proof. For all $\alpha>0$, the point z_0 is also a pole of multiplicity $\pi(z_0)$ of $F(z)+\alpha I$. We choose a disc $\mathfrak{C}_0\subset\mathfrak{D}$ with centre z_0 such that z_0 is the only pole of F in $\mathfrak{C}_0$. Then F is holomorphic on $\mathfrak{C}_0\setminus\{z_0\}$ and we consider, for sufficiently large $\alpha>0$, the logarithmic residuum (see [5])

$$\frac{1}{2\pi i}\operatorname{trace}\int_{\partial\mathfrak{C}_0}F'(z)\big(F(z)+\alpha I\big)^{-1}dz=\frac{1}{2\pi i\alpha}\operatorname{trace}\int_{\partial\mathfrak{C}_0}F'(z)\big(\alpha^{-1}F(z)+I\big)^{-1}dz.$$

If α is large, this value is zero; hence for these α the total multiplicity of the zeros of $F(z)+\alpha I$ in $\mathfrak{C}_0$ is equal to $\pi(z_0)$.

For the zeros of $F\in\mathbf{C}_\varkappa^{n\times n}$ another simple application of the logarithmic residuum theorem gives the following result, the proof of which is left to the reader.

Proposition 3.2. *Let $F\in\mathbf{C}_\varkappa^{n\times n}$, $\det F(z)\not\equiv0$. If z_0 is a zero of multiplicity $\mu(z_0)$ of F, then for each sufficiently small neighbourhood $\mathfrak{U}$ of z_0 there exists an $n(\mathfrak{U})>0$ such that $0<\alpha<n(\mathfrak{U})$ implies that the function $F(z)+\alpha I$ has zeros of total multiplicity $\mu(z_0)$ in $\mathfrak{U}$.*

Proposition 3.3. *Let $F\in\mathbf{C}_\varkappa^{n\times n}$, $\det F(z)\not\equiv0$. Then the following statements are true:*

a) $F^{-1}\in\mathbf{C}_\varkappa^{n\times n}$ $(F^{-1}(z):=F(z)^{-1})$;

b) *the zeros (poles) of F in $\mathfrak{D}$ coincide, multiplicities counted, with the poles (zeros) of F^{-1} in $\mathfrak{D}$;*

c) *the generalized zeros (poles) of F of negative type on $\partial\mathfrak{D}$ coincide, multiplicities counted, with the generalized poles (zeros) of F^{-1} of negative type.*

The proof of a) follows immediately from the definition of the class $\mathbf{C}_\varkappa^{n\times n}$, while b) is a general property of zeros and poles of matrix functions. To prove c), consider e.g. a generalized zero $z_0\in\partial\mathfrak{D}$ of F of negative type and choose a neighbourhood $\mathfrak{U}$ of z_0 such that $\mathfrak{U}\cap\mathfrak{D}$ does not contain any zero or pole of F. Then the statement follows easily from the identity

$$\alpha F(z)\big(F(z)^{-1}+\alpha^{-1}I\big)=F(z)+\alpha I\quad(z\in\mathfrak{U}\cap\mathfrak{D}).$$

2. Proposition 3.1 and Corollary 2 of Theorem 2.2 imply that the total multiplicity of poles in $\mathfrak{D}$ and generalized poles of negative type on $\partial\mathfrak{D}$ of a function $F\in \mathbf{C}_{\varkappa}^{n\times n}$ is at most $\varkappa$. We shall show that this multiplicity is exactly $\varkappa$ (Theorem 3.5). To this end, we consider the operator V of (1.6) which is maximal isometric in $\Pi_{\varkappa}$.

Proposition 3.4. *Let $F\in \mathbf{C}_{\varkappa}^{n\times n}$, $0\in\mathfrak{D}_F$. The point $z_0\in\overline{\mathfrak{D}}$ is a pole in $\mathfrak{D}$, or a generalized pole of negative type on $\partial\mathfrak{D}$, of F if and only if z_0^{-1} belongs to $\sigma_0(V^{-1})$; in this case $\pi(z_0)=\varkappa_{z_0^{-1}}(V^{-1})$.*

Proof. Let z_0 be, say, a generalized pole of negative type and multiplicity $\pi(z_0)$ of F. Then for each sufficiently small neighbourhood $\mathfrak{U}$ of z_0 there exists an $n(\mathfrak{U})>0$ such that for $\alpha>n(\mathfrak{U})$ the function $F(z)+\alpha I$ has zeros of total multiplicity $\pi(z_0)$ in $\mathfrak{U}\cap\mathfrak{D}$.

The function θ given by (1.1) and its contractive operator T in representation (1.9) will now be denoted by θ_α and T_α, respectively. Then, by (1.1), θ_α has poles of total multiplicity $\pi(z_0)$ in $\mathfrak{U}\cap\mathfrak{D}$. Theorem 2.2 implies that $T_\alpha|\mathscr{L}_\alpha$ has eigenvalues of total multiplicity $\pi(z_0)$ in $(\mathfrak{D}\cap\mathfrak{U})^{-1}$, where $\mathscr{L}_\alpha$ denotes a $\varkappa$-dimensional nonpositive invariant subspace of T_α with $|\sigma(T_\alpha|\mathscr{L}_\alpha)|\geqq 1$. If $\alpha\uparrow\infty$ then $\|T_\alpha-V^{-1}\|\to 0$ (see the proof of Proposition 1.5) and Proposition 1.4 implies that z_0^{-1} is an eigenvalue of algebraic multiplicity $\pi(z_0)$ of $V^{-1}|\mathscr{L}$. This reasoning can be reversed, and the statement follows.

We can now state the main result of this section.

Theorem 3.5. *Let $F\in \mathbf{C}_{\varkappa}^{n\times n}$. Then F has poles in $\mathfrak{D}$ and generalized poles of negative type on $\partial\mathfrak{D}$ of total multiplicity $\varkappa$. If, moreover, $\det F(z)\not\equiv 0$, then F has zeros in $\mathfrak{D}$ and generalized zeros of negative type on $\partial\mathfrak{D}$ of total multiplicity $\varkappa$.*

This follows immediately from Propositions 3.4 and 3.3 if we only observe that the condition $0\in\mathfrak{D}_F$ can always be fulfilled at the expense of a fractional linear transformation of $\mathfrak{D}$ onto itself.

By Proposition 3.4 and the definition of g appearing in the representation (1.8), the generalized poles of negative type of F are the zeros on $\partial\mathfrak{D}$ of the function g. Suppose the point $\hat{z}\in\partial\mathfrak{D}$ is not a generalized pole of negative type of $F\in \mathbf{C}_{\varkappa}^{n\times n}$, and choose an open arc $\hat{\Delta}\subset\partial\mathfrak{D}$ which contains $\hat{z}$ and has a positive distance from all generalized poles of negative type of F. Consider the decomposition

$$\Pi_{\varkappa}=\hat{\Pi}\oplus\Pi'_{\varkappa},\quad \hat{\Pi}:=E(\hat{\Delta})\Pi_{\varkappa},$$

where $\Pi_{\varkappa}$ is the space that plays a role in the representation (1.6) of F by a unitary operator V, and E denotes the spectral function of V (see [11]). Let the corresponding decomposition of V be $V=\hat{V}\oplus V'$. Then

$$F(z)=\hat{F}(z)+F'(z),$$

where $\hat{F}(z):=\Gamma^* E(\hat{\Delta})(\hat{V}+z\hat{I})(\hat{V}-z\hat{I})^{-1}E(\hat{\Delta})\Gamma$. As $\hat{\Pi}$ is a Hilbert space (see [11]), we have $\hat{F}\in \mathbf{C}_\varkappa^{n\times n}$. Moreover, $\widetilde{F'}$ (see the beginning of § 3) is holomorphic on $\hat{\Delta}$. This implies the following

Corollary. *Let $F\in\mathbf{C}_\varkappa^{n\times n}$. Then for each point $\hat{z}\in\partial\mathfrak{D}$ which is not a generalized pole of negative type of F, there exists a neighbourhood $\hat{\mathfrak{U}}$ of $\hat{z}$ and a number $\hat{\gamma}$ such that* $\operatorname{Re} F(z)\geqq -\hat{\gamma}I$ *for all* $z\in\hat{\mathfrak{U}}\cap\mathfrak{D}$.

§ 4. A generalization of Rouché's theorem

1. We denote by $\mathscr{D}^{n\times n}$ the set of all $(n\times n)$-matrix functions F which are defined and holomorphic in $\mathfrak{D}$ and admit a representation $F=y^{-1}Y$ with a bounded outer function y and a bounded holomorphic $(n\times n)$-matrix function Y in $\mathfrak{D}$ (equivalent definitions are given e.g. in [13]). Then the function

$$\det F(z) = y(z)^{-n}\det Y(z)$$

belongs to the class $\mathscr{D}(=\mathscr{D}^{1\times 1})$, hence it has, almost everywhere on $\partial\mathfrak{D}$, finite nontangential limits which are, almost everywhere, different from zero. Therefore the nontangential boundary values $F(\zeta)$ of F, which exist almost everywhere on $\partial\mathfrak{D}$, have an inverse $F(\zeta)^{-1}$ almost everywhere.

The function $F_0\in\mathscr{D}^{n\times n}$ is called *outer* if $\det F_0(z)$ is an outer function. In this case we have $\det F_0(z)\neq 0$ $(z\in\mathfrak{D})$, hence $F_0(z)^{-1}$ exists for all $z\in\mathfrak{D}$ and the function F_0^{-1} belongs again to $\mathscr{D}^{n\times n}$.

The function $F\in\mathscr{D}^{n\times n}$ is said to have an *inner factor of order* $\varkappa$ if it admits a representation

$$F(z) = U_0\left(\overset{\curvearrowleft}{\prod_{j=1}^{l}} B_j(z)\right)F_0(z), \tag{4.1}$$

where $F_0\in\mathscr{D}^{n\times n}$ is an outer function and $U_0\overset{\curvearrowleft}{\prod\limits_{j=1}^{l}} B_j(z)$ is a regular Blaschke—Potapov product of order $\varkappa$ (see § 1).

Lemma 4.1. *Let f be a complex function which is holomorphic in $\mathfrak{D}$ and has no zeros there, and denote by* $\operatorname{Arg} f$ *a continuous branch of the function* $\arg f$. *If* $\gamma:=\sup\{|\operatorname{Arg} f(z)|:\ z\in\mathfrak{D}\}<\infty$, *then f is an outer function.*

Proof. Choose an integer n such that $n>\dfrac{2\gamma}{\pi}$. Then the function f_1: $f_1(z):=(f(z))^{1/n}=|f(z)|^{1/n}\exp\left(\dfrac{i}{n}\operatorname{Arg} f(z)\right)$ has the property $\operatorname{Re} f_1(z)>0$ $(z\in\mathfrak{D})$. By [14, p. 51, Exercise 1], f_1 is an outer function; thus f is an outer function.

2. Now we prove the following generalization of Rouché's theorem.

Theorem 4.2 *Suppose* $F, G \in \mathscr{D}^{n\times n}$, $\det(F(z)-G(z)) \not\equiv 0$ *in* $\mathfrak{D}$ *and*

$$\|G(\zeta)F(\zeta)^{-1}\| \leqq 1 \quad \text{a.e. on } \partial\mathfrak{D}. \tag{4.2}$$

If F *has an inner factor of order* $\varkappa_F(<\infty)$, *then* $F-G$ *has an inner factor of order* $\varkappa_{F-G} \leqq \varkappa_F$. *If, additionally,* $F(F-G)^{-1}|_{\partial\mathfrak{D}} \in L_1^{n\times n}(\partial\mathfrak{D})$, [1]) *then* $\varkappa_{F-G} = \varkappa_F$.

Remark. From the proof it will follow that the difference $\varkappa_F - \varkappa_{F-G}$ is the total multiplicity of generalized poles of negative type on $\partial\mathfrak{D}$ of the function $(F+G)(F-G)^{-1}(=-I+2F(F-G)^{-1})$, which belongs to $\mathbf{C}_{\varkappa'}^{n\times n}$ for some $\varkappa' \leqq \varkappa_F$.

Proof of Theorem 3. We write the representation (4.1) of F in the form $F = BF_0$. Then $F-G=(B-GF_0^{-1})F_0$, $GF_0^{-1} \in \mathscr{D}^{n\times n}$, $F(F-G)^{-1} = B(B-GF_0^{-1})^{-1}$ and

$$\|G(\zeta)F(\zeta)^{-1}\| = \|G(\zeta)F_0(\zeta)^{-1}B(\zeta)^{-1}\| \leqq 1 \quad \text{a.e. on} \quad \partial\mathfrak{D}.$$

As F_0 is outer, the order of the inner factor of $F-G$ coincides with the order of the inner factor of $B-GF_0^{-1}$. Therefore, in the proof of the theorem we may suppose that $F=B$.

The matrix $B(\zeta)$, $|\zeta|=1$, is unitary, hence (4.2) implies $\|G(\zeta)\| \leqq 1$ a.e. on $\partial\mathfrak{D}$. Applying [13, Lemma 1.1] it follows that $\|G(z)\| \leqq 1$ for all $z \in \mathfrak{D}$. This is equivvalent (see [2, Lemma 3.1]) to $G \in \mathbf{S}_0^{n\times n}$ and $G^* \in \mathbf{S}_0^{n\times n}$, where G^* is the $(n\times n)$-matrix function $G^*(z) := G(z^*)^*$ $(z \in \mathfrak{D})$.

Consider now the function $B^{*-1}G^*$. According to [2, Lemma 3.5] it belongs to some class $\mathbf{S}_{\varkappa'}^{n\times n}$, where $\varkappa' \leqq \varkappa_F$. Then the same is true for GB^{-1} [2, Folgerung 3.3], and both functions have poles in $\mathfrak{D}$ of total multiplicity $\varkappa'$ (Corollary 1 of Theorem 2.2).

The condition $\det(B(z)-G(z)) \not\equiv 0$ implies that $(I-GB^{-1})^{-1}$ exists. Moreover, it is easy to check that the function C:

$$C(z) = (I+G(z)B(z)^{-1})(I-G(z)B(z)^{-1})^{-1} = -I + 2(I-G(z)B(z)^{-1})^{-1} \tag{4.3}$$

belongs to $\mathbf{C}_{\varkappa'}^{n\times n}$. According to Theorem 3.5 it has poles of total multiplicity $\varkappa''(\leqq \varkappa')$ in $\mathfrak{D}$, and the difference $\varkappa' - \varkappa''$ is the total multiplicity of its generalized poles of negative type on $\partial\mathfrak{D}$. In view of (4.3), the function $I - G(z)B(z)^{-1}$ has zeros of total multiplicity $\varkappa''$ in $\mathfrak{D}$. By [5, (1.3)], for a meromorphic $(n\times n)$-matrix function the difference of the total zero and total pole multiplicities in $\mathfrak{D}$ coincides with

[1]) $L_1^{n\times n}(\partial\mathfrak{D})$ denotes the class of $(n\times n)$-matrix functions defined a.e. on $\partial\mathfrak{D}$ with entries in $L_1(\partial\mathfrak{D})$.

the corresponding difference for its determinant function. Therefore, using the relation

$$\det\big(B(z)-G(z)\big) = \det\big(I-G(z)B(z)^{-1}\big)\det B(z)$$

we find that

$$\varkappa''' = \varkappa_F+\varkappa''-\varkappa',$$

where $\varkappa'''$ denotes the total zero multiplicity of $B-G$ in $\mathfrak{D}$. Observing that $\varkappa''\leqq\varkappa'$, the inequality $\varkappa'''\leqq\varkappa_F$ follows.

Therefore, the difference $B-G$ admits a regular representation $B-G=B_0H$ with a Blaschke—Potapov product B_0 of order $\varkappa'''$ and an $(n\times n)$-matrix function H which is holomorphic in $\mathfrak{D}$ and does not have any zeros there (cf. the proof of [2, Satz 3.2]). The first statement of the theorem follows if we show that H is an outer function. But this is true if and only if $\det H(z)^{-1}$ is an outer function.

We have $H=2B_0^{-1}(I+C)^{-1}B$ (see (4.3)). According to (1.8), C admits an integral representation

$$C(z) = iS+\frac{1}{g(z)}\int_0^{2\pi}\frac{e^{i\vartheta}+z}{e^{i\vartheta}-z}\,d\Sigma(\vartheta)+\frac{1}{g(z)}D(z), \tag{4.4}$$

where S, Σ, D and g have the properties mentioned in § 1.3. It follows that

$$H(z)^{-1} = \frac{1}{2}B(z)^{-1}\big((I+iS)g(z)+D(z)+C_0(z)\big)B_0(z)g(z)^{-1},$$

where C_0 denotes the integral in (4.4). As $\operatorname{Re} C_0(z)\geqq 0$ $(z\in\mathfrak{D})$ and g, D are polynomials of z and z^{-1}, for each entry of the matrix $H(z)^{-1}$ the argument is bounded on $\mathfrak{D}$. Then the same is true for $\det H(z)^{-1}$. Hence Lemma 4.1 implies that $\det H(z)^{-1}$ is an outer function.

Let now $B(\zeta)\big(B(\zeta)-G(\zeta)\big)^{-1}\in L_1^{n\times n}(\partial\mathfrak{D})$. Then according to (4.3) $C\in L_1^{n\times n}(\partial\mathfrak{D})$, and it remains to show that C does not have generalized poles of negative type on $\partial\mathfrak{D}$.

The function $C\in\mathbf{C}_{\varkappa'}^{n\times n}$ can be written as the sum of a rational function $C_1\in\mathbf{C}_{\varkappa_1}^{n\times n}$ with poles in $\mathfrak{D}$ and a function $C_2\in\mathbf{C}_{\varkappa_2}^{n\times n}$, which is holomorphic in $\mathfrak{D}$, $\varkappa'=\varkappa_1+\varkappa_2$. The function $C_2(\zeta)$, $\zeta\in\partial\mathfrak{D}$, also belongs to $L_1^{n\times n}(\partial\mathfrak{D})$, and it is sufficient to show that $\varkappa_2=0$. This will be accomplished if we prove the following two statements[1]):

a) $C_2\in H_1^{n\times n}$;

b) If for some $\varkappa<\infty$ we have $H_1^{n\times n}\cap\mathbf{C}_\varkappa^{n\times n}\neq\emptyset$, then $\varkappa=0$.

To prove a) we first observe that $E\in\mathbf{C}_\varkappa^{n\times n}$ implies $E\in H_\delta^{n\times n}$ for $\delta<(1+2\varkappa)^{-1}$. Indeed, as

$$\|E(z)\|^\delta \leqq \big(\sqrt{n}\max_{i,j}|e_{ij}(z)|\big)^\delta,$$

[1]) For the definition of the Hardy classes $H_\delta^{n\times n}$ see e.g. [13].

it is sufficient to show that the entries e_{ij} of E belong to H_δ. According to (1.8), every e_{ij} is of the form $g_1(z)^{-1}h(z)$, where $h \in H_\delta$ for all $\delta < 1$ (see [15, II. 4.5]) and g_1 is a polynomial of degree $\leqq 2\varkappa$ with zeros on $\partial\mathfrak{D}$. If $\delta < (1+2\varkappa)^{-1}$, we choose $\delta_1 < (2\varkappa)^{-1}$ and $\delta_2 < 1$ so that $\delta = \delta_1\delta_2(\delta_1+\delta_2)^{-1}$. Setting $p = \delta_2^{-1}(\delta_1+\delta_2)$ and $q = \delta_1^{-1}(\delta_1+\delta_2)$ we obtain

$$\int_0^{2\pi} |g_1(re^{i\vartheta})^{-1}h(re^{i\vartheta})|^\delta \, d\vartheta \leqq \Big(\int_0^{2\pi} |g_1(re^{i\vartheta})|^{-\delta p} \, d\vartheta\Big)^{1/p} \Big(\int_0^{2\pi} |h(re^{i\vartheta})|^{\delta q} \, d\vartheta\Big)^{1/q} =$$

$$= \Big(\int_0^{2\pi} |g_1(re^{i\vartheta})|^{-\delta_1} \, d\vartheta\Big)^{1/p} \Big(\int_0^{2\pi} |h(re^{i\vartheta})|^{\delta_2} \, d\vartheta\Big)^{1/q} \leqq K < \infty$$

for all $0 < r < 1$. Thus $E \in H_\delta^{n\times n}$. In particular, $C_2 \in H_\delta^{n\times n}$. As $C_2(\zeta) \in L_1^{n\times n}(\partial\mathfrak{D})$, by a theorem of V. I. Smirnov (cf. [15, II. 6]) we have $C_2 \in H_1^{n\times n}$.

To prove b), we use the representation

$$E(z) = i \operatorname{Im} E(0) + \frac{1}{2\pi} \int_0^{2\pi} (e^{i\vartheta}+z)(e^{i\vartheta}-z)^{-1} \operatorname{Re} E(e^{i\vartheta}) \, d\vartheta \quad (z \in \mathfrak{D}),$$

which holds for arbitrary functions $E \in H_1^{n\times n}$, and the representation (1.8):

$$E(z) = iS + \frac{1}{g(z)} \int_0^{2\pi} (e^{i\vartheta}+z)(e^{i\vartheta}-z)^{-1} \, d\Sigma(\vartheta) + \frac{1}{g(z)} G(z),$$

valid for $E \in C_\varkappa^{n\times n}$, $0 \in \mathfrak{D}_E$. Making the right-hand sides equal, multiplying by $g(z)$ and using Stieltjes—Livšic inversion formula it follows that

$$\int_{\vartheta_1}^{\vartheta_2} g(e^{i\vartheta}) \operatorname{Re} E(e^{i\vartheta}) \, d\vartheta = \int_{\vartheta_1}^{\vartheta_2} d\Sigma(\vartheta) \geqq 0$$

whenever $0 \leqq \vartheta_1 < \vartheta_2 \leqq 2\pi$. Therefore $\operatorname{Re} E(e^{i\vartheta}) \geqq 0$ almost everywhere on $[0, 2\pi]$. Hence $\operatorname{Re} E(z) > 0$ $(z \in \mathfrak{D})$, and $\varkappa = 0$.

The theorem is proved.

§ 5. Further examples

1. In this section we consider two examples of functions of the class $\mathbf{N}_\varkappa^{n\times n}$. By the connection between the classes $\mathbf{N}_\varkappa^{n\times n}$ and $\mathbf{C}_\varkappa^{n\times n}$ mentioned in § 1.1, the notions of generalized zeros and poles of negative type carry over to functions $Q \in \mathbf{N}_\varkappa^{n\times n}$ in the following way. Let φ be a linear fractional mapping from $\mathfrak{D}$ onto $\mathfrak{C}_+$. The point $t_0 \in R_1 \cup \{\infty\}$, $t_0 = \varphi(\zeta_0)$ $(|\zeta_0| = 1)$, is said to be a *generalized pole (zero) of negative type and multiplicity* $\pi(t_0)$ *of* Q if ζ_0 is a generalized pole (or zero, resp.) of negative type and multiplicity $\pi(t_0)$ of $F = iQ\circ\varphi$, or equivalently, if for each

sufficiently small neighbourhood $\mathfrak{U}$ of t_0 (we admit the case $t_0=\infty$) in the closed complex plane there exists an $n(\mathfrak{U})>0$ such that for $\alpha>n(\mathfrak{U})$ (or $0<\alpha<n(\mathfrak{U})$, resp.) the function $Q(z)+i\alpha I$ has zeros of total multiplicity $\pi(t_0)$ in $\mathfrak{U}\cap\mathfrak{C}_+$.

Let $Q_0\in\mathbf{N}_0^{n\times n}$. Then Q_0 has a representation

$$Q_0(z) = A_0+zB_0+\int_{-\infty}^{\infty}\left((t-z)^{-1}-t(1+z^2)^{-1}\right)d\Sigma(t)$$

with hermitian $(n\times n)$-matrices A_0, B_0; $B_0\geqq 0$, and a nondecreasing $(n\times n)$-matrix function Σ on R_1, $\int_{-\infty}^{\infty}(1+t^2)^{-1}d\Sigma(t)<\infty$. Now let B_1 be a hermitian $(n\times n)$-matrix and let us consider the function Q_1:

(5.1) $$Q_1(z):=Q_0(z)-zB_1.$$

Then

$$N_Q(z,\zeta)=\int_{-\infty}^{\infty}(t-z)^{-1}(t-\zeta^*)^{-1}d\Sigma(t)+B_0-B_1,$$

and, considering $N_{Q_1}(z,z)$ for $|z|$ sufficiently large, it follows that $Q_1\in\mathbf{N}_\varkappa^{n\times n}$, where $\varkappa$ denotes the number of negative eigenvalues of the matrix B_0-B_1. It is easy to see that Q_1 has a generalized pole of negative type and multiplicity $\varkappa$ at ∞. Moreover, Theorem 3.5 implies:

Proposition 5.1. *If* $\det Q_1(z)\not\equiv 0$, *then the function* Q_1 *in* (5.1) *has zeros in* $\mathfrak{C}_+$ *and generalized zeros of negative type in* $R_1\cup\{\infty\}$ *of total multiplicity* $\varkappa$, *where* $\varkappa$ *denotes the number of negative eigenvalues of the matrix* B_0-B_1.

In special cases this result can be given a more explicit formulation. Here we consider the case where $n=1$ and

(5.2) $$Q_1(z)=\int_{-\infty}^{\infty}(t-z)^{-1}d\sigma(t)+\alpha-z$$

with α a real number and σ a nondecreasing function on R_1 such that

$$\int_{-\infty}^{\infty}(1+|t|)^{-1}d\sigma(t)<\infty;$$

without loss of generality, the coefficient of z has been chosen -1.

Proposition 5.2. *The function* Q_1 *in* (5.2) *has either exactly one zero in* $\mathfrak{C}_+$ *or one generalized zero of negative type in* $R_1\cup\{\infty\}$. *This (possibly, generalized)* *zero z_α is $\neq\infty$ and of multiplicity* 1. *It can be characterized among the points of $\overline{C}_+$ by the following two properties:*

a) $$\int_{-\infty}^{\infty} |t-z_\alpha|^{-2}\, d\sigma(t) \leqq 1,$$

b) $$\int_{-\infty}^{\infty} (t-z_\alpha)^{-1}\, d\sigma(t)+\alpha-z_\alpha = 0.$$

Proof. The first statement including the claim about the multiplicity follows from Proposition 5.1.

Next we show that a zero or generalized zero $z_\alpha\in\overline{\mathfrak{C}}_+$ of Q_1 has the properties a) and b). If z_α is a zero $(z_\alpha\in\mathfrak{C}_+)$, this is obvious if we observe that

$$0 = \operatorname{Im} Q_1(z_\alpha) = \operatorname{Im} z_\alpha\Big(\int_{-\infty}^{\infty}\frac{d\sigma(t)}{|t-z_\alpha|^2}-1\Big).$$

Let now $z_\alpha\in R_1$. Then there exists a sequence $(z_n)\subset\mathfrak{C}_+$, $z_n\to z_\alpha$, $\operatorname{Im} z_n\downarrow 0$ $(n\to\infty)$ such that $Q_1(z_n)\to 0$, $\operatorname{Im} Q_1(z_n)<0$. It follows that $(\operatorname{Im} z_n)\Big(\int_{-\infty}^{\infty}|t-z_n|^{-2}d\sigma(t)-1\Big)<0$, or

$$\int_{-\infty}^{\infty} |t-z_n|^{-2}\, d\sigma(t) < 1, \quad n = 1, 2, \ldots .$$

Applying Fatou's lemma we get $\int_{-\infty}^{\infty}(t-z_\alpha)^{-2}d\sigma(t)\leqq 1$, and Lebesgue's theorem gives

$$0 = \lim_{n\to\infty} Q_1(z_n) = \int_{-\infty}^{\infty}(t-z_\alpha)^{-1}\, d\sigma(t)+\alpha-z_\alpha.$$

It remains to show that a) and b) have at most one solution z_α in $\overline{\mathfrak{C}}_+$. To this end we introduce the π_1-space $\Pi_1 := \mathfrak{C}\oplus L_2(\sigma)$ of all pairs $\{\xi, x\}$, $\xi\in\mathfrak{C}$, $x\in L_2(\sigma)$ with indefinite scalar product

$$[\{\xi, x\}, \{\eta, y\}] = -\xi\eta^* + \int_{-\infty}^{\infty} x(t)y(t)^*\, d\sigma(t) \quad (\xi,\eta\in\mathfrak{C};\ x, y\in L_2(\sigma)).$$

It is easy to check that the operator A:

$$A\{\xi, x\} := \Big\{\alpha\xi - \int_{-\infty}^{\infty} x(t)\, d\sigma(t),\ tx(t)+\xi\Big\}, \tag{5.3}$$

which is defined for every $\{\xi, x\}\in\Pi_1$ such that the function $t\to tx(t)+\xi$ belongs to $L_2(\sigma)$, is selfadjoint in Π_1. In order to find its eigenvalues λ we have to solve the equation

$$(A-\lambda I)\{\xi, x\}=0. \tag{5.4}$$

From (5.3) and (5.4) it follows that $x(t)=-\xi(t-\lambda)^{-1}(\sigma-a.e)$. In particular, $\int_{-\infty}^{\infty}|t-\lambda|^{-2}\,d\sigma(t)<\infty$. Moreover, the first component in (5.4) gives

$$\alpha-\lambda+\int_{-\infty}^{\infty}(t-\lambda)^{-1}\,d\sigma(t)=0. \tag{5.5}$$

Conversely, it is easy to see that any solution λ of (5.5) with $\int_{-\infty}^{\infty}|t-\lambda|^{-2}d\sigma(t)<\infty$ is an eigenvalue of A with corresponding eigenelement $\{\xi, -\xi(t-\lambda)^{-1}\}$ $(\xi\neq 0)$.

Since A is a selfadjoint operator in Π_1, it has exactly one eigenvalue $\lambda_0\in\overline{\mathfrak{C}}_+$ such that the corresponding eigenvector is nonpositive:

$$-1+\int_{-\infty}^{\infty}|t-\lambda_0|^{-2}\,d\sigma(t)\leqq 0.$$

Therefore, $z_\alpha=\lambda_0$ is the only solution of the system a)—b) in $\overline{\mathfrak{C}}_+$. The proposition is proved.

Remark 1. The zeros of the function Q_1 are the fixed points of the function Q_0: $Q_0(z):=\int_{-\infty}^{\infty}(t-z)^{-1}d\sigma(t)+\alpha$, and by a fractional linear transformation of $\mathfrak{C}_+$ onto $\mathfrak{D}$ the equation $Q_0(z)=z$ is transformed into $G_0(\zeta)=\zeta$, where G_0 maps $\mathfrak{D}$ holomorphically into itself ($G_0\in S_0$). However, the usual fixed point argument does not seem to be applicable in this case, as the boundary values of G_0 on $\partial\mathfrak{D}$ are, in general, discontinuous.

Remark 2. Suppose that the function σ in (5.2) satisfies the additional condition

$$\int_{-\infty}^{\infty}(t-x)^{-2}\,d\sigma(t)=\infty \quad \text{for all } x\in\mathfrak{E}_\sigma,$$

where $\mathfrak{E}_\sigma$ denotes the set of all points of increase of σ. Then for every real α the function Q_1 in (5.2) has one and only one zero $z_\alpha\in\overline{\mathfrak{C}}_+\setminus\mathfrak{E}_\sigma$ and $\int_{-\infty}^{\infty}|t-z_\alpha|^{-2}d\sigma(t)\leqq 1$.

Remark 3. Besides the zero z_α, the function Q_1 can have an arbitrary number ($\leqq\infty$) of real zeros which do not satisfy condition a).

To see this, we suppose that the function σ in (5.2) is constant on some interval (a, b) which is *special* in the sense that

$$\lim_{x\downarrow a}\int_{-\infty}^{a}(t-x)^{-1}\,d\sigma(t)=-\infty,\quad \lim_{x\uparrow b}\int_{b}^{\infty}(t-x)^{-1}\,d\sigma(t)=\infty.$$

Then Q_1 is holomorphic in (a, b) and $Q_1(a+0)=-\infty$, $Q_1(b-0)=\infty$. Therefore it has at least one zero in (a, b), more exactly, it has an odd number of zeros in (a, b), counted with multiplicities. Denote these zeros by $x_1\leqq x_2\leqq\ldots\leqq x_{2k+1}$. It is easy to see that $Q_1'(x_{2j})\leqq 0$, $j=1, 2, \ldots, k$, that is $x_{2j}=z_\alpha$, $j=1, 2, \ldots, k$. Hence the function Q_1 has in (a, b) either one simple zero, either zeros of total multiplicity three; in the second case the zero x_2 coincides with z_α.

Consequently, if σ has $N\,(\leqq\infty)$ special intervals (a_j, b_j), $j=1, 2, \ldots, N$, then the corresponding function Q_1 has in no more than one of these intervals zeros of total multiplicity three, in each of the remaining intervals it has exactly one (simple) zero. The case $N=\infty$ occurs, for example, if Q_1 is a meromorphic function with an infinite number of poles.

We mention that any simple selfadjoint operator $\tilde{A}$ in a π_1-space $\tilde{\Pi}_1$ is unitarily equivalent to the operator A appearing in the proof of Proposition 5.2. Here $\tilde{A}$ is called simple if there exists an $e\in\mathfrak{D}(\tilde{A})$, $[e, e]<0$, such that

$$\tilde{\Pi}_1=\text{c.l.s.}\,\{(\tilde{A}-\zeta\tilde{I})^{-1}e:\pm\zeta\in\mathbb{C}_+\cap\varrho(\tilde{A})\}.$$

Indeed, suppose $[e, e]=-1$ and consider the decomposition

$$\Pi_1=\mathscr{L}_0\dot{+}\mathscr{L}_1,\ \mathscr{L}_0=\text{l.s.}\,\{e\},\ \mathscr{L}_1=\mathscr{L}_0^{[\perp]}. \tag{5.6}$$

Then $\mathscr{L}_1$ is a Hilbert space with scalar product $[.,.]$, and $\mathscr{L}_0$ can be identified with $\mathbb{C}$ by writing $e=\{1, 0\}$ with respect to the decomposition (5.6).

If $\tilde{A}e=\{\alpha, h\}$, $\alpha\in\mathbb{C}$, $h\in\mathscr{L}_1$, then the matrix representation of $\tilde{A}$ is

$$\tilde{A}=\begin{pmatrix}\alpha & -[.,h]\\ h & A_{11}\end{pmatrix} \tag{5.7}$$

with some selfadjoint operator A_{11} in the Hilbert space $\mathscr{L}_1$. Now an easy calculation gives

$$(\tilde{A}-\zeta\tilde{I})^{-1}e=\xi\{1, -(A_{11}-\zeta I)^{-1}h\},\quad \xi=(\alpha-\zeta+[(A_{11}-\zeta I)^{-1}h, h])^{-1}.$$

It follows that $\mathscr{L}_1=\text{c.l.s.}\,\{(A_{11}-\zeta I)^{-1}h:\pm\zeta\in\mathbb{C}_+\}$. Hence A_{11} is unitarily equivalent to the operator of multiplication by the independent variable in the space $L_2(\sigma)$, $\sigma(t):=[E_t h, h]$, where E_t is the spectral function of A_{11} and h corresponds to the function $h(t)\equiv 1$ belonging to $L_2(\sigma)$. With this realization of $\mathscr{L}_1$ and A_{11}, the matrix in (5.7) defines the operator A in (5.3).

This model of an arbitrary simple selfadjoint operator in a π_1-space (or, more generally, in a $\pi_\varkappa$-space) and the characterization of its eigenvalues by conditions a) and b) were first given in [4; III, § 6].

2. In this section we consider an $(n\times n)$-matrix function $Q_0 \in \mathbf{N}_0^{n\times n}$ of the form

$$Q_0(z) = -\sum_{j=1}^{l} \frac{B_j}{t_j+z} + \int_0^\infty (t-z)^{-1}\, d\Sigma(t),$$

where B_j are nonnegative hermitian matrices, $0<t_1<t_2<\dots<t_l$, and Σ is a nondecreasing $(n\times n)$-matrix function on $[0,\infty)$ with the properties $\Sigma(0+)=\Sigma(0)=0$, $\int_0^\infty (1+t)^{-1} d\Sigma(t)<\infty$. Then the function Q: $Q(z)=zQ_0(z^2)$ has the representation

$$Q(z) = \frac{1}{2}\sum_{j=1}^{l} B_j\left\{\left(i\sqrt{t_j}-z\right)^{-1} - \left(i\sqrt{t_j}+z\right)^{-1}\right\} + \int_{-\infty}^{\infty} (z-s)^{-1}\, d\tilde{\Sigma}(s),$$

$$\tilde{\Sigma}(s) := \begin{cases} 2^{-1}\Sigma(s^2) & s \geqq 0, \\ -2^{-1}\Sigma(s^2) & s \leqq 0. \end{cases}$$

According to the example at the end of § 2.2 we have $Q\in \mathbf{N}_\varkappa^{n\times n}$, where $\sum_{j=1}^{l} \dim B_j = \varkappa$ Evidently, Q is antisymmetric with respect to the imaginary axis: $Q(-z^*) = -Q(z)^*$.

Proposition 5.3. *The function $Q(z)+iI$ has zeros of total multiplicity $\varkappa$ in $\mathbf{C}_+$. These zeros z_j, $j=1,2,\dots,m\ (\leqq\varkappa)$, are on the imaginary axis and $0<|z_j|<t_1^2$.*

Proof. The first statement follows immediately from Corollary 3 of Theorem 2.2. To prove the second statement, we first consider the case $n=1$. To find the solutions of the equation $Q_0(z^2)=-iz^{-1}$, $\operatorname{Im} z>0$, we put $z=is$. Then it takes the form $Q_0(-s^2)=-s^{-1}$, and a simple consideration of the graphs of $Q_0(-s^2)$ and $-s^{-1}$ shows that this equation has $\varkappa$ zeros in $(0,\infty)$ and that these zeros are smaller than t_1. By the first statement of the proposition, these $\varkappa$ zeros give the only zeros of $Q(z)+iI$ in $\mathbf{C}_+$.

Let now n be arbitrary and consider a zero $z_0\in\mathbf{C}_+$ of $Q(z)+iI$. If z_0 is outside the interval $(0, it_1^2)$ of the imaginary axis, then Q is holomorphic at z_0. Hence there exists a vector $\xi\neq 0$ such that

$$z_0\left(Q_0(z_0^2)\xi,\xi\right) + i(\xi,\xi) = 0.$$

But we have shown (case $n=1$) that this is impossible.

As an application of Proposition 5.3 we consider the Schrödinger equation

$$\frac{d^2\psi(r)}{dr^2} - V(r)\psi(r) + k^2\psi(r) = 0, \quad \psi(0)=0, \tag{5.8}$$

with a short range potential V: $V(r)=0$ if $r>a$ for some $a<\infty$, $V\in L_1$. To find the nonreal resonances k of the problem (5.8) we observe that for $r>a$ the solution ψ of (5.8) is $\psi(r; k^2)=Ce^{ikr}$ and, considering $r=a$, we get $\psi'(a; k^2)\psi(a; k^2)^{-1}=ik$. This equation can be written as $k\psi(a; k^2)\psi'(a; k^2)^{-1}+i=0$. But $\psi(a; z)\psi'(a; z)^{-1}$ is a function of class $\mathbf{N}_0$. Indeed, $\psi(r; z)$ satisfies the equation

$$\psi''(r; z)-V(r)\psi(r; z)+z\psi(r; z)=0, \quad \psi(0; z)=0,$$

and it follows that

$$\psi(r; z)\psi'(r; z)^{-1}-\psi'(r; z^*)^{-1}\psi(r; z^*)=$$

$$=(z-z^*)\psi'(r; z^*)^{-1}\int_0^r |\psi(s; z^*)|^2\,ds\cdot\psi'(r; z)^{-1}.$$

Evidently, the number of negative poles of $\psi(a; z)\,\psi'(a; z)^{-1}$ is equal to the number of negative zeros of $\psi'(a; z)$, that is the number of negative eigenvalues λ of the boundary problem

(5.9) $$\psi''(r)-V(r)\psi(r)+\lambda\psi(r)=0, \quad \psi(0)=0, \quad \psi'(a)=0.$$

Proposition 5.3 now implies the following statement:

The number of nonreal resonances of (5.8) *in* $\mathfrak{C}_+$ *is equal to the number of negative eigenvalues* λ *of the boundary problem* (5.9).

Without going into details we mention that Proposition 5.3 can be used to prove a similar statement in the case of a vector equation (5.8).

Note. We use this opportunity to mention that in our paper [7] the statement of Satz 3.4 is incorrect. To make it correct, in formula (3.10) one has to replace ϱ_0 (R_0) by $\hat{\varrho}_0$ $(\hat{R}_0$, resp.) and to define

$$\varrho_0:=\hat{\varrho}_0-1,\ R_0(z):=\hat{R}_0(z)-z(1-z^2)^{\varrho_0}\int_{\mathfrak{U}_0}\prod_{j=1}^{r}\frac{(1+t^2)^{\varrho_j}}{(t-\alpha_j)^{2\varrho_j}}\,d\sigma(t).$$

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Carleman and Korotkov operators on Banach spaces

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This paper is an attempt to use basic theory of vector measures as an approach to study certain types of classical integral operators and their generalizations. The first section begins with a look at the classical Carleman integral operators from L_2 to L_2. In the course a connection, which seems to us to be heretofore unnoticed, is established with operators whose truncates on large sets is compact into L_∞. This is then generalized to the consideration of operators from any Banach space into the spaces $L_p(\mu)$ $0 \leqq p \leqq \infty$. The second section is devoted to an application to a "folklore theorem" about weak compactness in $L_\infty(\mu)$. Next specialization is made of the general results of section 1 to integral operators from one L_p space to another in section 3. A class of operators we call Korotkov operators are studied in section 4. The last section is devoted to extensions to general function spaces.

1. Carleman operators

We start with a description of the classical situation as motivation for the current work. The book of Halmos and Sunder [6] may be consulted for more details. Let (Ω, Σ, μ) be a finite separable measure space and let T be a linear operator from $L_2(\mu)$ to $L_2(\mu)$. The operator T is called an *integral operator* if there is a $\mu \times \mu$-measurable function k such that

(i) $k(s, \cdot)f(\cdot) \in L_1(\mu)$ for μ-almost all s and all $f \in L_2(\mu)$,

(ii) $\int_\Omega k(\cdot, t)f(t)\,d\mu(t) \in L_2(\mu)$ for all f in $L_2(\mu)$, and

(iii) $Tf(s) = \int_\Omega k(s, t)f(t)\,d\mu(t)$ almost everywhere for all f in $L_2(\mu)$.

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It is an old theorem of BANACH [1] that such an operator is automatically continuous. An integral operator is called a *Carleman integral operator* if its kernel additionally satisfies

(iv) $k(s, \cdot) \in L_2(\mu)$ for μ-almost all s.

Following Halmos—Sunder, Korotkov and probably many others, note that the last condition can be interpreted to mean that there exists a function $g\colon \Omega \to L_2(\mu)$ such that $g(s)(\cdot) = k(s, \cdot)$ for almost all s in Ω. By the Riesz representation theorem we may regard g as taking values in $(L_2(\mu))^* (= L_2(\mu))$ and read condition (ii) as saying $\langle g(\cdot), f\rangle \in L_2(\mu)$ for all f in $L_2(\mu)$ and read (iii) as saying that $Tf(\cdot) = \langle g(\cdot), f\rangle$ for all f in $L_2(\mu)$. Moreover since $\langle g(\cdot), f\rangle \in L_2(\mu)$ for all f in $L_2(\mu)$ and since $L_2(\mu)^* = L_2(\mu)$ we see that g is a weakly measurable function into $L_2(\mu)$. Since $L_2(\mu)$ is separable here, PETTIS's measurability theorem [3, II. 2.2] shows that g is (strongly) measurable.

Conversely if $g\colon \Omega \to L_2(\mu)$ is measurable and if $\langle g(\cdot), f\rangle \in L_2(\mu)$ for all f in $L_2(\mu)$, then a theorem of DUNFORD and PETTIS [4, III. 11.17] produces a $\mu \times \mu$-measurable function k such that $k(s, \cdot) = g(s)(\cdot)$ for almost all s in Ω with the property that

$$\langle g(\cdot), f\rangle = \int_{\Omega} k(\cdot, t) f(t)\, d\mu(t)$$

almost everywhere. Thus the operator on $L_2(\mu)$ to $L_2(\mu)$ defined by $f \to \langle g(\cdot), f\rangle$ is a Carleman integral operator.

This proves the first theorem and sets the perspective of most of this paper.

Theorem 1. *Let (Ω, Σ, μ) be a finite separable measure space. A linear operator $T\colon L_2(\mu) \to L_2(\mu)$ is a Carleman integral operator if and only if there exists a measurable $g\colon \Omega \to L_2(\mu)$ such that for each f in $L_2(\mu)$ the equality $(Tf)(\cdot) = \langle g(\cdot), f\rangle$ obtains almost everywhere.*

Throughout the remainder of the paper let (Ω, Σ, μ) be an arbitrary finite measure space; let $L_p(\mu)$, $0 < p \leqq \infty$ be the usual Lebesgue spaces and let $L_0(\mu)$ be the space of all measurable (equivalence classes of) functions on Ω under the topology of convergence in measure.

Definition 2. Let $0 \leqq p \leqq \infty$ and let X be a Banach space. A linear operator $T\colon X \to L_p(\mu)$ is called a *Carleman operator* if there is a measurable function $g\colon \Omega \to X^*$ such that $(Tx)(s) = \langle g(s), x\rangle$ almost everywhere for all x in X. In this case g is called the *kernel* of T.

A quick application of the closed graph theorem shows that a Carleman operator is necessarily continuous.

Recall that [3, Chap. III] a Banach space X has the *Radon—Nikodým property* (RNP) if for any finite measure space (Ω, Σ, μ) and for any bounded linear operator $T\colon L_1(\mu)\to X$ there is a bounded measurable function $g\colon \Omega\to X$ such that

$$Tf=\int_\Omega fg\,d\mu,$$

as a Bochner integral, for all f in $L_1(\mu)$.

The first theorem is a generalization of a theorem of KOROTKOV [10, 11] and is closely related to a theorem of WONG [17].

Theorem 3. *Let $0\leqq p\leqq\infty$ and let X be a Banach space whose dual X^* has* RNP. *A bounded linear operator $T\colon X\to L_p(\mu)$ is a Carleman operator if and only if there is a measurable function Φ on Ω such that*

$$|(Tx)(s)|\leqq\|x\|\,\Phi(s)$$

almost everywhere for each x in X.

Proof. Suppose T is a Carleman operator with kernel g. Then, for s in Ω, we have

$$|(Tx)(s)|=|g(s)x|\leqq\|g(s)\|_{X^*}\|x\|_X.$$

Since g is measurable, so is $\|g(\cdot)\|_{X^*}$. Taking $\Phi(s)=\|g(s)\|_{X^*}$ proves the necessity.

Conversely, suppose there is a measurable function $\Phi\colon \Omega\to\mathbf{R}$ such that $|(Tx)(s)|\leqq\|x\|\,\Phi(s)$ almost everywhere for all $x\in X$. Without loss of generality, we may assume that $\Phi(w)$ is everywhere finite. For $n\geqq1$, define the sets $E_n=[n-1\leqq|\Phi|<n]$ and note that Φ is bounded on each E_n. Define the operator E_n on every L_p by

$$E_nf=f\chi_{E_n}.$$

By hypothesis, $E_n\circ T$ has its range in $L_\infty(\mu)$; moreover an application of the closed graph theorem shows that

$$E_n\circ T\colon X\to L_\infty(\mu)$$

is continuous. Define $S_n\colon L_\infty(\mu)^*\to X^*$ to be the adjoint of $E_n\circ T$ and consider its restriction to $L_1(\mu)$. Since X^* has RNP, there is a measurable $g_n\colon \Omega\to X^*$ such that

$$S_nf=\int_\Omega fg_n\,d\mu\quad\text{for all}\quad f\in L_1(\mu).$$

Next observe that $((E_n\circ T)x)(\cdot)=\langle g_n(\cdot),x\rangle$ a.e. by computing for f in $L_1(\mu)$ and x in X the integral

$$\int_\Omega(E_nT)xf\,d\mu=\langle S_n(f),x\rangle=\Big\langle\int_\Omega fg_n\,d\mu,x\Big\rangle=\int_\Omega f\langle g_n,x\rangle\,d\mu$$

where the last equality follows from the "commuting" of the Bochner integral with bounded linear operators [3, II. 2.9].

In particular, $\langle g_n(\cdot), x\rangle$ vanishes almost everywhere outside E_n for each x in X. Without loss of generality, we may take $g_n(\omega)=0$ for $\omega \notin E_n$. Then g_n is still measurable and $((E_n \circ T)x)(\cdot)=\langle g_n(\cdot), x\rangle$ for all x in X.

Now define g: $\Omega \to X^*$ by $g(s)=g_n(s)$ for s in E_n (recall that the E_n are disjoint and exhaustive). Then g is measurable and it remains only to show that $(Tx)(s)=\langle g(s), x\rangle$ almost everywhere for x in X.

To this end, note that if $h \in L_p(\mu)$, then

$$h = \lim_m h\chi_{F_m} \quad \text{in measure, where} \quad F_m = \bigcup_{n=1}^m E_n.$$

Hence if $x \in X$, then

$$Tx = \lim_m (Tx)\chi_{F_m} = \lim_m \sum_{n=1}^m (E_n \circ T)x = \lim_m \sum_{n=1}^m \langle g_n(\cdot), x\rangle =$$

$$= \lim_m \langle g(\cdot), x\rangle \chi_{F_m} = \langle g(\cdot), x\rangle,$$

where all limits are taken in measure. This completes the proof.

The first corollary is implicit in STEGALL [16]. Its converse is also true, but we shall not prove it here because it is the main theme of Stegall's paper.

Corollary 4. *If X^* has* RNP, *then every continuous linear operator from X into $L_\infty(\mu)$ is a Carleman operator.*

Proof. If T: $X \to L_\infty(\mu)$ is a continuous linear operator, then $|Tx|(\cdot) \leqq \|T\| \|x\|$ a.e.; apply Theorem 3.

Corollary 5. *A weakly compact operator from an arbitrary Banach space into $L_\infty(\mu)$ is a Carleman operator.*

Proof. Let X be an arbitrary Banach space and let T: $X \to L_\infty(\mu)$ be a weakly compact operator. Then T^*: $L_\infty(\mu) \to X^*$ is weakly compact as is its restriction to $L_1(\mu)$. By a classical theorem of DUNFORD, PETTIS, and PHILLIPS [3, III. 2.12], there exists a measurable g: $\Omega \to X^*$ such that

$$T^*f = \int fg\,d\mu, \quad f \in L_1(\mu).$$

Now, by the proof of Theorem 3, we see that

$$(Tx)(\cdot) = \langle g(\cdot), x\rangle$$

for all $x \in X$, so that T is indeed a Carleman operator.

The next theorem characterizes Carleman operators from X to $L_p(\mu)$ in terms of compactness of the operator into $L_\infty(\mu)$. No RNP assumptions need be made on X or its dual. This theorem appears to be new even in the classical case of Carleman integral operators from $L_2(\mu)$ to $L_2(\mu)$. Recall that to each set E in Σ there is the associated operator E: $L_p(\mu)\to L_p(\mu)$ defined by $Ef=f\chi_E$. Call a linear operator T: $X\to L_p(\mu)$ *almost weakly (or norm) compact into* $L_\infty(\mu)$ if for each $\varepsilon>0$ there is a set $E\in\Sigma$ with $\mu(\Omega\setminus E)<\varepsilon$ such that $E\circ T$ is a weakly (or norm) compact operator X into $L_\infty(\mu)$.

Theorem 6. *Let $0\leqq p\leqq\infty$ and let X be a Banach space. If a continuous linear operator T: $X\to L_p(\mu)$ is almost weakly compact into $L_\infty(\mu)$, then T is a Carleman operator. Conversely, if T is a Carleman operator, then T is almost norm compact into $L_\infty(\mu)$.*

Proof. Suppose T: $X\to L_p(\mu)$ is almost weakly compact into $L_\infty(\mu)$. Then there is a disjoint sequence (E_n) in Σ with $\bigcup_{n=1}^{\infty} E_n=\Omega$ such that $E_n\circ T$: $X\to L_\infty(\mu)$ is weakly compact. By Corollary 5 each $E_n\circ T$ is a Carleman operator and thus is given by $(E_n\circ T)x=g_n(\cdot)x$ where g_n: $\Omega\to X^*$ is a measurable function supported on E_n. Define $g(s)=g_n(s)$ for $s\in E_n$ and proceed as in the proof of Theorem 3 to prove $Tx=g(\cdot)x$. Thus T is a Carleman operator.

On the other hand, suppose T: $X\to L_p(\mu)$ is a Carleman operator with kernel g. Since g is measurable, there is a sequence (g_n) of measurable simple functions from Ω to X^* such that $\lim_m \|g_n(\cdot)-g(\cdot)\|_X=0$ almost everywhere. Fix $\varepsilon>0$ and use Egorov's theorem to obtain a set $E\in\Sigma$ such that $\mu(\Omega\setminus E)<\varepsilon$ and $\lim_m \|g_n(\cdot)-$ $-g(\cdot)\|_X=0$ uniformly on E. Since g must be bounded on E, $E\circ T$ maps X into $L_\infty(\cdot)$. Define the finite rank operators T_n: $X\to L_\infty(\mu)$ by $T_n x=\chi_E g_n(\cdot)x$. Then

$$\lim_n \sup_{\|x\|\leqq 1} \|E\circ T(x)-E\circ T_n(x)\|_\infty=\lim_n \sup_{\|x\|\leqq 1} \|\chi_E g(\cdot)x-\chi_E g_n(\cdot)x\|_\infty\leqq$$

$$\leqq \lim_n \sup_{\|x\|\leqq 1} \sup_{s\in E} |g(s)x-g_n(s)x| = \lim_n \sup_{s\in E} \|g(s)-g_n(s)\|_{X^*}=0.$$

Thus, $E\circ T$: $X\to L_\infty(\mu)$ is the uniform limit of finite rank operators and is consequently a compact operator.

At this point, we are again getting close to the theme of STEGALL [16]. In his recent study of the RNP in dual spaces, Stegall effectively works with Carleman operators from X to $L_\infty(\mu)$ and proves that X^* has RNP if and only if all operators from X to $L_\infty(\mu)$ are Carleman operators. The connection is as follows: An operator T: $X\to L_\infty(\mu)$ is a Carleman operator if and only if the restriction of the adjoint T^* on $L_1(\mu)$ is representable in the sense of [3, Chap. II].

2. Weakly compact sets in $L_\infty(\mu)$

This section is a bit of a digression to some well-established but not so well-known facts about weakly compact sets in $L_\infty(\mu)$. Probably this section should have been included in [3, Chap. VIII]; it certainly could have been. The subject of this section is a folklore theorem which explains the interchange between norm and weak compactness in $L_\infty(\mu)$ and thus explains the interchange in Theorem 6. Probably all of this section should be attributed to GROTHENDIECK [5], but unfortunately the theorem which we are about to prove does not seem to be generally known. In keeping with the terminology above, a subset W of $L_\infty(\mu)$ will be called (relatively) *almost norm compact* if for each $\varepsilon>0$ there exists E in Σ with $\mu(\Omega\setminus E)<\varepsilon$ such that $\chi_E W$ is (relatively) norm compact in $L_\infty(\mu)$.

Theorem 7. (Folklore) *If W is a relatively weakly compact subset of $L_\infty(\mu)$, then W is relatively almost norm compact in $L_\infty(\mu)$.*

Proof. Let W be a relatively weakly compact subset of $L_\infty(\mu)$. By the factorization theorem of DAVIS, FIGIEL, JOHNSON, and PEŁCZYŃSKI [2] there is a reflexive Banach space R and a bounded linear operator $T\colon R\to L_\infty(\mu)$ such that $W\subseteqq T(B_R)$, where B_R is the closed unit ball of R. By Corollary 5, the weakly compact operator T is Carleman. By Theorem 6, given $\varepsilon<0$ there is $E\in\Sigma$ with $\mu(\Omega\setminus E)<\varepsilon$ such that $E\ T\colon R\to L_\infty(\mu)$ is a compact operator. Thus $\chi_E W\subseteqq E\circ T(B_R)$ is relatively norm compact. This completes the proof.

Note that the converse of Theorem 7 is not true. Indeed, if X^* has the RNP, then any operator $T\colon X\to L_\infty(\mu)$ is a Carleman operator by Corollary 4. Thus by Theorem 6, the operator T is almost norm compact into $L_\infty(\mu)$; i.e. $T(B_X)$ is relatively almost norm compact in $L_\infty(\mu)$ as in the conclusion of Theorem 7. But, if X is not reflexive, then T need not be a weakly compact operator; so, $T(B_X)$ need not be weakly compact in $L_\infty(\mu)$.

Theorem 7 has an easy corollary which seems to have been known to Grothendieck. To our best knowledge it was first stated explicitly by PERESSINI [14, Prop. 5]. Weaker versions of it were proved by ZOLEZZI [18] and KHURANA [8]. (See KHURANA [9] for an interesting generalization to the vector-valued case.)

Corollary 8. *If (Ω, Σ, μ) is a finite measure space, then a weakly convergent sequence in $L_\infty(\mu)$ converges almost everywhere.*

3. Carleman operators as classical integral operators

The definition of a Carleman operator was motivated by that of a Carleman integral operator [6] from $L_2(\mu)$ to $L_2(\mu)$. The definition of Carleman integral operator has a natural extension to the other $L_p(\mu)$ spaces and it will be shown here that the natural extension coincides with the definition of Carleman operators on $L_p(\mu)$. Throughout this section $(S, \mathcal{F}, \lambda)$ and (Ω, Σ, μ) will be finite measure spaces. In addition p and r will be numbers such that $0 \leqq p \leqq \infty$ and $1 < r < \infty$. The number s will be conjugate to r in the sense that $r^{-1} + s^{-1} = 1$.

Theorem 9. *A continuous linear operator* $T\colon L_r(\lambda) \to L_p(\mu)$ *is a Carleman operator if and only if there exists a* $\lambda \times \mu$*-measurable function* $k\colon S \times \Omega \to \mathbf{R}$ *such that* $k(\cdot, y)$ *in* $L_s(\lambda)$ *for* μ*-almost all* y *in* Ω *and such that*

$$(Tf)(\cdot) = \int_S f(x)\, k(x, \cdot)\, d\lambda(x)$$

for all f *in* $L_r(\lambda)$ μ*-almost everywhere.*

Proof. Suppose $T\colon L_r(\lambda) \to L_p(\mu)$ is a Carleman operator with kernel $g\colon \Omega \to (L_r(\lambda))^* = L_s(\lambda)$ so that $Tf(\cdot) = \langle g(\cdot), f \rangle$. In order to produce a $\lambda \times \mu$-measurable function k such that $k(x, y) = (g(y))(x)$ a.e., consider $g(\cdot)/\|g(\cdot)\|_s$. This is a bounded $L_s(\lambda)$-valued measurable function and is therefore μ-Bochner integrable. An appeal to a theorem of Dunford and Pettis [4, III. 11.17] produces a $\lambda \times \mu$-measurable function k_1 such that

$$k_1(x, y) = ((g(y))(x))/\|g(y)\|_s$$

for μ-almost all y in Ω. Now set $k(x, y) = \|g(y)\|_s k_1(x, y)$, note that k is $\lambda \times \mu$-measurable. Now observe that for f in $L_r(\lambda)$ and μ-almost all y in Ω, we have

$$T(f)(y) = \langle g(y), f \rangle = \int_S (g(y))(x) f(x)\, d\lambda(x) = \int_S f(x)\, k(x, y)\, d\lambda(x).$$

Conversely, suppose there is a jointly measurable function $k\colon S \times \Omega \to \mathbf{R}$ such that $k(\cdot, y) \in L_s(\lambda)$ for μ-almost all y in Ω and such that, for f in $L_r(\lambda)$, we have

$$(Tf)(\cdot) = \int_S f(x)\, k(x, \cdot)\, d\lambda(x)$$

almost everywhere. Without loss of generality we assume that $k(\cdot, y) \in L_s(\lambda)$ for all y in Ω. Define $g_1\colon \Omega \to L_s(\lambda)$ by $g_1(y) = k(\cdot, y)$. Since

$$\langle g_1(y), f \rangle = \int f(x)(g_1(y))(x)\, d\lambda(x) = \int f(x)\, k(x, y)\, d\lambda(x) = (Tf)(y)$$

for all $f \in L_r(\lambda)$ for μ-almost all y, we see that g_1 is weakly measurable. But any weakly measurable function with values in a reflexive space (or WCG space, for that matter) is equivalent to a (strongly) measurable function [3, p. 88]; viz. there is a measurable $g\colon \Omega\to L_s(\lambda)$ such that for all $x^*\in L_r(\lambda)^*$ we have $\langle x^*, g\rangle=\langle x^*, g_1\rangle$ μ-almost everywhere. Thus $\langle g_1(y), f\rangle=\langle g(y), f\rangle$ for almost all $y\in\Omega$ and all $f\in L_s(\lambda)$. Consequently, for each f in $L_r(\lambda)$, we have

$$(Tf)(y)=\int_S f(x)\,k(x,y)\,d\lambda(x)=\langle g_1(y), f\rangle=\langle g(y), f\rangle$$

for μ-almost all y. This proves that T is a Carleman operator.

The sufficiency proof of Theorem 9 also shows how to replace kernels that are not jointly measurable by ones that are, since joint measurability of k is not used in that argument. The requirement of joint measurability in the definition of integral operator is not necessary for most of the work but is usually assumed so as to guarantee that each integral operator has a unique kernel. The question of whether every operator determined by a nonjointly measurable kernel can be induced from a jointly measurable kernel appears to be open (see [6, § 8] for a discussion), but is easily settled for the Carleman operators.

Corollary 10. *Let $T\colon L_r(\lambda)\to L_p(\mu)$ be a continuous linear operator. If there exists a (possibly) nonjointly measurable function $k\colon S\times\Omega\to\mathbf{R}$ such that $k(\cdot, y)\in L_s(\lambda)$ for μ-almost all y in Ω and such that*

$$(Tf)(y)=\int_S f(x)\,k(x,y)\,d\lambda(x)$$

for all $f\in L_r(\lambda)$ and μ-almost all y in Ω, then T is a Carleman operator (and hence given also by a jointly measurable kernel).

4. Compactness and Korotkov operators

It follows directly that if $T\colon X\to L_\infty(\mu)$ is a Carleman operator, then $T(B_X)$ is compact in every $L_p(\mu)$ for $0\leqq p<\infty$; indeed Theorem 6 guarantees that for a set E of large measure $E\circ T\colon X\to L_\infty(\mu)$ is compact. Thus, if $T\colon X\to L_\infty(\mu)$ is Carleman and $\lim\limits_{\mu(E)\to 0}\int_E |Tx|^p\,d\mu=0$ uniformly in $\|x\|\leqq 1$, then T maps bounded sets into relatively compact subsets of $L_p(\mu)$. Moreover, since $0<r<p<\infty$ implies that $\lim\limits_{\mu(E)\to 0}\int_E |f|^r\,d\mu=0$ uniformly for any bounded set of $L_p^\bullet(\mu)$, it follows that if $T\colon X\to L_p(\mu)$ is a Carleman operator, then T maps bounded subsets of X into $L_r(\mu)$-relatively compact sets. This line of reasoning holds up for a class of operators which strictly includes the Carleman operators.

Definition 11. A bounded linear operator $T\colon X\to L_p(\mu)$ is called a *Korotkov operator* if there is a measurable real function Φ on Ω such that $|(Tx)(\cdot)|\leqq\|x\|\,\Phi(\cdot)$ for all x in X μ-almost everywhere (cf. Korotkov [10, 11]).

Recall that Theorem 3 shows that every Carleman operator is a Korotkov operator while the converse evidently requires that X^* have the RNP. It is, therefore, not surprising that a representation for the Korotkov operators can be found which differs only in its measurability requirements.

Lemma 12. *Let X be a separable Banach space and let $0\leqq p\leqq\infty$. If $T\colon X\to L_p(\mu)$ is a Korotkov operator then there exists a weak*-measurable function $g\colon \Omega\to X^*$ such that $(Tx)(\cdot)=\langle g(\cdot), x\rangle$ for all $x\in X$ almost everywhere.*

Proof. We refer to the sufficiency part of the proof of Theorem 3, where we have the bounded linear operators $(E_m\circ T)^*\colon L_1(\mu)\to X^*$. Replace the hypothesis that X^* has RNP by the hypothesis that X is separable. Standard arguments ([4, VI. 8.6], [3, p. 79]) for representing such operators yield $g_n\colon \Omega\to X^*$ vanishing off E_n such that

$$((E_m\circ T)^*f)(x) = \int_{E_n} g_m(\cdot)x\, f\, d\mu$$

for all $x\in X$ and $f\in L_1(\ )$. Piece g together as in that proof to get that $g(\cdot)x$ is measurable for each x and that $Tx=\langle g, x\rangle$ for all $x\in X$.

It is possible to drop the separability assumption in Lemma 12. This, however, depends on much deeper arguments than those used — viz. the existence of liftings. With this tool it is possible to prove [7] that if X is an arbitrary Banach space and $S\colon L_1(\mu)\to X^*$ is any continuous linear operator, then there exists a bounded weak*-measurable function $h\colon \Omega\to X^*$ such that $((Sf)x)(\cdot)=\int_\Omega h(\cdot)xf\,d\mu$ for all $x\in X$ and $f\in L_1(\mu)$. This would generalize Lemma 12 to non separable spaces; since, however, we can manage without this deep theorem we shall not use it.

Definition 13. Let $0<p<\infty$. A bounded subset K of $L_p(\mu)$ is called *equi-integrable* if

$$\lim_{\mu(E)\to 0}\int_E |f|^p\, d\mu = 0$$

uniformly in $f\in K$. A subset of $L_0(\mu)$ is called *equi-integrable* if it is relatively compact.

Recall that a subset M of a Banach space it is called *weakly conditionally compact* if every sequence in M has a weak Cauchy subsequence.

Theorem 14. *Let $0<p<\infty$ and X be an arbitrary Banach space. A Korotkov operator $T\colon X\to L_p(\mu)$ with the property that $T(B_X)$ is equi-integrable in $L_p(\mu)$ maps weakly conditionally compact sets into norm compact sets.*

Proof. Let W be a weakly conditionally compact subset of X and let (y_n) be a sequence in W. It must be shown that $(T(y_n))$ has an $L_p(\mu)$-convergent subsequence. Let y be the (separable) closed subspace of X determined by $\{y_n\}_{n=1}^{\infty}$. The restriction of T to y, still denoted by T, is a Korotkov operator from y to $L_p(\mu)$. According to Lemma 12 there is a function g: $\Omega \to y^*$ such that $Ty = \langle g, y \rangle$ for all y in Y. Since W is weakly conditionally compact, the sequence (y_n) has a weak Cauchy subsequence (y_{n_j}). Since g has B values in y^*, it follows that $(T(y_{n_j})) = (\langle g, y_{n_j} \rangle)$ is a pointwise convergent sequence in an equi-integrable set. Vitali's convergence theorem guarantees that $(T(y_{n_j}))$ is $L_p(\mu)$ convergent. This completes the proof.

Corollary 15. *Let $0 \leqq r < p \leqq \infty$ and X be an arbitrary Banach space. A Korotkov operator from X into $L_p(\mu)$ maps weakly conditionally compact subsets of X into relatively compact subsets of $L_r(\mu)$. A Korotkov operator from X into $L_0(\mu)$ maps weakly conditionally compact subsets of X into relatively compact subsets of $L_0(\mu)$.*

Proof. For $p > 0$, the Hölder inequality can be used to show that $L_p(\mu)$-bounded sets are equi-integrable in $L_r(\mu)$ so that Theorem 14 applies.

For $p = 0$, glance at the proof of Theorem 14 and remember that pointwise convergence implies convergence in measure for sequences.

Rosenthal's characterization [15] of Banach spaces containing copies of l_1 and Corollary 15 give the next corollary.

Corollary 16. *Let $0 \leqq p > \infty$. Let X be a Banach space containing no copy of l_1. A Korotkov operator from X to L_p that maps bounded sets into equi-integrable sets maps bounded sets into relatively compact sets. Consequently, if $0 \leqq r < p \leqq \infty$, then a Korotkov operator from X to $L_p(\mu)$ maps bounded sets into $L_r(\mu)$-relatively compact sets; and a Korotkov operator from X into $L_0(\mu)$ maps bounded sets into relatively compact sets.*

5. Extensions to Banach function spaces and other function spaces

Let (Ω, Σ, μ) be any finite measure space and $Y(\mu)$ be any linear topological space (not necessarily locally convex) of (equivalence classes of μ-measurable functions on Ω. For a Banach space X, say that a continuous linear operator T: $X \to Y(\mu)$ is a Carleman operator if there is a measurable g: $\Omega \to X^*$ such that

$$Tx = \langle g, x \rangle$$

for all x in X. If $Y(\mu)$ has the property that φ in $Y(\mu)$ implies $\varphi\chi_E \in Y(\mu)$ for all E in Σ, then a check of the proofs of the theorems of Section 1 shows that they remain true if $L_p(\mu)$ is replaced by $Y(\mu)$.

Theorem 9 generalizes readily to a wide class of Banach function spaces. (For the basic definitions and results used here see LUXEMBURG & ZAANEN [13] and LUXEMBURG [12].) We give a brief summary. Start with a measure space (Ω, Σ, μ), which for simplicity we assume is a finite measure space. Let M be the set of all measurable scalar functions and M^+ the nonnegative members of M. The order on M is pointwise and functions differing only on a null set are identified. A *function norm* is a function $\varrho\colon M^+ \to [0, \infty]$ that is positive homogeneous, subadditive, takes the value zero if and only if the function is zero almost everywhere, and preserves order (viz. $u \leqq v$; $v \in M^+ \Rightarrow \varrho(u) \leqq \varrho(v)$). The function norm is extended to all of M by $\varrho(f) = \varrho(|f|)$. We denote by L_ϱ the set of all $f \in M$ satisfying $e(f) < \infty$. The result is an ordered normed vector space. We assume that $L_\varrho(\mu)$ is norm complete. Without loss of generality we also assume that ϱ is *saturated* i.e. there are no ϱ-unfriendly sets (a set $E \subset \Omega$ such that $e(\chi_F) = \infty$ for every $F \subset E$ with $\mu(F) > 0$). The associate norm is defined by

$$e'(g) = \sup\left\{\left|\int fg\, d\mu\right| e(f) \leqq 1\right\};$$

and, of course, Hölder's inequality $\left|\int fg\, d\mu\right| \leqq \varrho(f)\varrho'(g)$ obtains.

This last step gives a function norm whose corresponding $L_{\varrho'}(\mu)$ is a Banach space. Finally a function in L_ϱ is of *absolutely continuous norm* whenever $\varrho(f_n) \downarrow 0$ for every $(f_n) \subseteqq L_\varrho$ such that $|f| \geqq f_1 \geqq f_2 \geqq \ldots \downarrow 0$. We call the collection of all such functions L_ϱ^α. At this point the extension of Theorem 9 to the context of Banach function spaces goes right through. Let $(\mathcal{S}, f, \lambda)$ and (Ω, Σ, μ) be finite measure spaces. Let $Y(\mu)$ be as above; let ϱ_1 be a function norm such that L_{ϱ_1} is reflexive (this is equivalent to $L_\varrho = L_\varrho^\alpha$, $L_{\varrho'} = L_\varrho^\alpha$, and $\varrho(f_n) \uparrow \varrho(f)$ whenever $0 \leqq f_n \uparrow f$). A continuous linear operator $T\colon L_{\varrho_1}(\lambda) \to Y(\mu)$ is a Carleman operator if and only if there exists a $\lambda \times \mu$-measurable function $k\colon \mathcal{S} \times \Omega \to \mathbf{R}$ such that $k(\cdot, y) \in L_{\varrho_1'}(\lambda)$ for μ-almost all $y \in \Omega$ with

$$(Tf)(y) = \int_{\mathcal{S}} f(x) k(x, y)\, d\lambda(x) \quad \text{g.e.}$$

for all $f \in L_{\varrho_1}(\lambda)$.

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On center-valued states of von Neumann algebras

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Center-valued states are projections of norm one onto the centre of the algebra. This concept is the natural extension of the notion of the (scalar-valued) state. The space of normal states is sequentially complete and the same can be said about the space of normal center-valued states with respect to the pointwise weak convergence.

We remark that center-valued states are central-linear maps. Central-linear maps (or module homomorphisms) onto the centre were studied extensively also in [4] and in [8].

On a von Neumann algebra $\mathscr{A}$ each normal state φ has the representation $\varphi(A)=\sum_{i=1}^{\infty}\langle Ax_i, x_i\rangle$ where $\sum_{i=1}^{\infty}\|x_i\|^2=1$. In section 2 we prove a similar formula for center-valued states: if $\int\oplus\mathscr{A}(z)\,d\mu(z)$ is the central decomposition of $\mathscr{A}$ in the Hilbert space $\mathfrak{H}$, then any central-valued state τ has the form

$$\tau(A)=\int\oplus\sum_{i=1}^{\infty}\langle A(z)x_i(z), x_i(z)\rangle I(z)\,d\mu(z)\quad \left(A=\int\oplus A(z)\,d\mu(z)\right)$$

where $x_i\in\mathfrak{H}$ $(i\in\mathbf{N})$.

In the last section we use the above representation theorem to obtain an alternative proof of a result of H. HALPERN [5] and S. STRĂTILĂ—L. ZSIDÓ [8] concerning central ranges for elements of von Neumann algebras (here on separable spaces).

0. Preliminaries. We only consider separable Hilbert spaces $\mathfrak{H}$. $\mathscr{A}$ will always denote a von Neumann algebra on $\mathfrak{H}$, and $\mathscr{A}_1$ its closed unit ball.

For the reduction theory of von Neumann algebras we refer to [3] and [7].

In this paper Z always means a separable metric space and μ a positive Borel measure on Z. If

$$\mathfrak{H}=\int_Z\oplus\mathfrak{H}(z)\,d\mu(z)$$

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(cf. [3], chap. II, § 1, def. 3) then $\{x_i\}_{i=1}^{\infty}$ will be a dense sequence in $\mathfrak{H}$, for which we may assume that, for all $z\in Z$, $\{x_i(z)\}_{i=1}^{\infty}$ is dense in $\mathfrak{H}(z)$ and the map $z\mapsto\|x_i(z)\|$ is bounded.

If $\mathscr{B}\subset\mathscr{A}$ is bounded and $B\in\mathscr{B}$ then

$$V(n, m)=\left\{T\in\mathscr{B}\colon |\langle(T-B)x_i, x_j\rangle|\leqq\frac{1}{m},\ i, j\leqq n\right\}\quad(n, m\in\mathbf{N})$$

is a neighbourhood base of B in $\mathscr{B}$, endowed with the weak operator topology Consequently, $\mathscr{A}_1$ endowed with the weak operator topology can be metrized with the metric ϱ defined by

$$\varrho(A, B)=\sum_{i,j\in\mathbf{N}}|\langle(A-B)x_i, x_j\rangle|\cdot 2^{-i-j}.$$

1. Center-valued states. In this section we introduce the notion of center-valued state and establish some properties. (See also [4] and [5].)

1.1. Definition. Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{C}$. By a *center-valued state* we mean a linear mapping τ from $\mathscr{A}$ into $\mathscr{C}$ such that

(i) $\tau(C\cdot A)=C\tau(A)$ $(A\in\mathscr{A},\ C\in\mathscr{C})$

(ii) $\tau(I)=I$

(iii) if $A\geqq 0$ then $\tau(A)\geqq 0$ $(A\in\mathscr{A})$.

1.2. Proposition. *Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{C}$. The linear mapping $\tau\colon\mathscr{A}\to\mathscr{C}$ is a center-valued state if and only if the following conditions are fulfilled:*

(a) $\|\tau\|=1$, (b) $\tau(C)=C$ $(C\in\mathscr{C})$.

Proof. Let τ be a center-valued state. If $A\geqq 0$ then $0\leqq\tau(A)\leqq\tau(\|A\|\cdot I)=$ $=\|A\|\cdot I$ so $\|\tau(A)\|\leqq\|A\|$. For an arbitrary $A\in\mathscr{A}$ the Schwarz-inequality gives that $\|\tau(A)\|^2=\|\tau(A)^*\tau(A)\|=\|\tau(A^*)\tau(A)\|\leqq\|\tau(A^*A)\|\leqq\|A^*A\|=\|A\|^2$. Hence $\|\tau\|\leqq 1$, and (a) and (b) follow.

The converse is a special case of a well-known result of Tomiyama [10] on projections of norm one.

1.3. Definition. If $\mathscr{A}$ is a von Neumann algebra then the set of all center-valued states on $\mathscr{A}$ will be denoted by $\Sigma(\mathscr{A})$ and by Σ if $\mathscr{A}$ is fixed. We endow Σ with the topology of pointwise weak convergence.

1.4. Proposition. *$\Sigma(\mathscr{A})$ is compact.*

Proof. Let $X=\Pi\{X_A\colon A\in\mathscr{A}\}$ where X_A is $\|A\|\cdot\mathscr{C}_1$ with the compact weak operator topology. So X is compact. Define $e\colon\Sigma\to X$ by the formula $pr_A e(\tau)=\tau(A)$.

e is a topological embedding and we want to show that the range of e is closed. Set

$$\begin{aligned} H_1(A,B) &= \{\tau\in X:\ pr_{A+B}\tau = pr_A\tau + pr_B\tau\},\\ H_2(A,\lambda) &= \{\tau\in X:\ pr_{\lambda A}\tau = \lambda pr_A\tau\},\\ H_3(C) &= \{\tau\in X:\ pr_C\tau = C\}. \end{aligned}$$

These sets are closed for any $A, B\in\mathscr{A}$, $C\in\mathscr{C}$ and $\lambda\in\mathbf{C}$. Since $\|pr_A\tau\|\leqq\|A\|$ for any $A\in\mathscr{A}$ and $\tau\in X$, according to point 1.2,

$$e(\Sigma) = \bigcap_{A,B\in\mathscr{A}} H_1(A,B)\cap\bigcap_{\substack{A\in\mathscr{A}\\ \lambda\in\mathbf{C}}} H_2(A,\lambda)\cap\bigcap_{C\in\mathscr{C}} H_3(C)$$

that is the range of e is closed.

1.5. Proposition. *For a center-valued state τ on the von Neumann algebra $\mathscr{A}$ the following conditions are equivalent:*

(i) *τ is σ-weakly continuous,*
(ii) *τ is weakly continuous on the unit ball,*
(iii) *τ is strongly continuous on the unit ball,*
(iv) *$\tau^{-1}(0)$ is σ-weakly closed,*
(v) *τ is normal.*

Proof. We obtain the assertion by applying a theorem of Tomiyama [10] for the case of projections of norm onto the center.

1.6. Example. Assume that the von Neumann algebra $\mathscr{A}$ in the Hilbert space $\mathfrak{H}$ is expressed as a direct integral of factors, $\mathscr{A}=\int_Z\oplus\,\mathscr{A}(z)\,d\mu(z)$, and let $\mathfrak{H}=\int_Z\oplus\,\mathfrak{H}(z)\,d\mu(z)$ be the corresponding decomposition of $\mathfrak{H}$. If $x\in\mathfrak{H}$ such that $\|x(z)\|=1$ for μ-a.e. on Z, then

$$\tau\colon A\mapsto\int_Z\oplus\langle A(z)x(z),x(z)\rangle I(z)\,d\mu(z)\quad\Big(A=\int_Z\oplus A(z)\,d\mu(z)\Big)$$

is a normal center-valued state. (Here $I(z)$ stands for the identity operator on the space $\mathfrak{H}(z)$.)

The center of $\mathscr{A}$ consists of the diagonal operators and the verifications of (a) and (b) in 1.2 is easy. By Prop. 1.5 it remains only to prove the strong operator continuity of τ on the unit ball of $\mathscr{A}$.

Assume that $A_n\in\mathscr{A}$, $\|A_n\|\leqq 1$ and $A_n\xrightarrow{\text{so}}0$. In order to prove that $\tau(A_n)\xrightarrow{\text{so}}0$ it suffices to show that $\|\tau(A_n)u\|\to 0$ for every $u\in\mathfrak{H}$ such that $\|u(z)\|$ is bounded on Z (cf. [3], chap. II, § 1, prop. 7). But, setting $K=\sup\{\|u(z)\|:\ z\in Z\}$ we have by the Schwarz inequality

$$\|\tau(A_n)u\|^2 = \langle \tau(A_n)^* \tau(A_n)u, u\rangle \leqq \langle \tau(A_n^* A_n)u, u\rangle =$$
$$= \int \langle A_n(z)^* A_n(z), x(z), x(z)\rangle \langle u(z), u(z)\rangle \, d\mu(z) \leqq$$
$$\leqq K^2 \int \langle A_n(z)^* A_n(z) x(z), x(z)\rangle \, d\mu(z) = K^2 \|A_n x\|^2 \to 0.$$

1.7. Definition. $\Sigma^n(\mathscr{A})$ denotes the set of all normal center-valued states on the von Neumann algebra $\mathscr{A}$ endowed with the topology of pointwise convergence in the weak operator topology.

1.8. Proposition. *$\Sigma^n(\mathscr{A})$ is sequentially complete.*

Proof. It is sufficient to see that Σ^n is sequentially closed in Σ. Suppose that $\tau_n \to \tau$ and $\tau_n \in \Sigma^n$, $\tau \in \Sigma$. Let f be a normal linear functional on $\mathscr{C}$. Then $f \circ \tau_n$ is normal linear functional on $\mathscr{A}$. $f \circ \tau_n(A) \to f \circ \tau(A)$ for every $A \in \mathscr{A}$ and so $f \circ \tau$ is normal (see [1] Cor. III.3). Since $f \circ \tau$ is normal for every normal f on $\mathscr{C}$, τ is also normal.

2. Decomposition of center-valued states. In this section we show that if the von Neumann algebra $\mathscr{A}$ is expressed as a direct integral of von Neumann algebras then any normal center-valued state of $\mathscr{A}$ is decomposable concerning the integral.

2.1. Lemma. *Assume that $\mathscr{A} = \int_Z \oplus \mathscr{A}(z) \, d\mu(z)$. Then there exists a countable family $\mathscr{T}$ in $\mathscr{A}_1$ such that*

(i) *$\mathscr{T}$ is strongly dense in $\mathscr{A}_1$,*

(ii) *$\mathscr{T}(z) = \{T(z) \colon T \in \mathscr{T}\}$ is strongly dense in $\mathscr{A}(z)_1$, μ-a.e. on Z.*

(Here $T = \int_Z \oplus T(z) \, d\mu(z)$.)

Proof. By the definition of the direct integral of von Neumann algebras there is a sequence $A_n = \int_Z \oplus A_n(z) \, d\mu(z)$ $(n \in \mathrm{N})$ such that $\mathscr{A}(z)$ is the von Neumann algebra generated by $\{A_n(z) \colon n \in \mathrm{N}\}$ μ-a.e. on Z and we may assume that $\mathscr{A}$ is generated by $\{A_n \colon n \in \mathrm{N}\}$. Let $\mathscr{K}$ be the *-algebra over the complex rationals generated by $\{A_n \colon n \in \mathrm{N}\}$. Take

$$\mathscr{T} = \Big\{ \int_Z \oplus \tilde{T}(z) \, d\mu(z) \colon T = \int_Z \oplus T(z) \, d\mu(z) \in \mathscr{K} \Big\}, \quad \tilde{A} = \begin{cases} A & \text{if } \|A\| \leqq 1, \\ A \cdot \|A\|^{-1} & \text{if } \|A\| > 1. \end{cases}$$

$\mathscr{T}$ is countable and by Kaplansky's density theorem it satisfies (i)—(ii).

2.2. Theorem. *Let $\mathscr{A} = \int_Z \oplus \mathscr{A}(z) \, d\mu(z)$ and τ be a normal center-valued state on $\mathscr{A}$. Then for almost every $z \in Z$ there is a normal center-valued state τ_z on $\mathscr{A}(z)$ such that for every $A = \int_Z \oplus A(z) \, d\mu(z) \in \mathscr{A}$ the operator field $z \mapsto \tau_z A(z)$ is μ-meas-*

urable and

$$\tau(A) = \int_Z \oplus \tau_z A(z)\, d\mu(z).$$

Proof. Using the lemma we have two countable families $\mathscr{S}$ and $\mathscr{T}$ such that

(i) $\mathscr{T}(z) \subset \mathscr{A}(z)_1$ and $\zeta(z) \subset \mathscr{C}(z_1)$ μ-a.e. on Z,

(ii) $\mathscr{T}\big(\mathscr{T}(z)\big)$ is strongly dense in $\mathscr{A}_1$ $\big($in $\mathscr{A}(z)_1$ μ-a.e. on $Z\big)$,

(iii) $\mathscr{S}$ $\big(\mathscr{S}(z)\big)$ is strongly dense in $\mathscr{C}_1$ $\big($in $\mathscr{C}(z)_1$ μ-a.e. on $Z\big)$.

Let

$$\mathscr{R} = \left\{\sum_{i=1}^{k} \alpha_i S_i T_i \colon\ k \in \mathbf{N};\ S_i \in \mathscr{S},\ T_i \in \zeta,\ \alpha_i \text{ is complex rational } (i \geqq k)\right\}.$$

If τ is a normal center-valued state then for $z \in Z$ we define $\hat{\tau}_z$ by the formula

$$\hat{\tau}_z\left(\sum_{i=1}^{k} \alpha_i S_i(z) T_i(z)\right) = \sum_{i=1}^{k} \alpha_i S_i z) \tau(T_i)(z)$$

where $\sum_{i=1}^{k} \alpha_i S_i T_i \in \mathscr{R}$. We will show that $\hat{\tau}_z$ is well-defined μ-a.e. on Z.

Take $R_1, R_2 \in \mathscr{R}$ $\left(R_1 = \sum_{i=1}^{k} \alpha_i S_i T_i,\ R_2 = \sum_{j=1}^{l} \beta_j S_j T_j\right)$

and put

$$H(R_1, R_2) = \left\{z \in Z\colon\ R_1(z) = R_2(z),\ \sum_{i=1}^{k} \alpha_i S_i(z) \tau(T_i)(z) \neq \sum_{j=1}^{l} \beta_j S_j(z) \tau(T_j)(z)\right\}.$$

This set is measurable and its characteristic function χ belongs to $\mathscr{C}$. Hence

$$\chi\tau(R_1) = \tau(\chi R_1) = \tau(\chi R_2) = \chi\tau(R_2).$$

So $\tau(R_1)(z) = \tau(R_2)(z)$ for μ-a.e. $z \in H(R_1, R_2)$. Since $\sum_{i=1}^{k} \alpha_i S_i(z)\, \tau(T_i)(z) = \tau(R_1)(z)$ and $\sum_{j=1}^{l} \beta_j S_j(z) \tau(T_j)(z) = \tau(R_2)(z)$ μ-a.e., we have obtained that $\mu\big(H(R_1, R_2)\big) = 0$. Since $\mathscr{R}$ is countable, if follows

$$\mu\Big(\bigcup_{R_1, R_2 \in \mathscr{R}} H(R_1, R_2)\Big) = 0.$$

Let

$$S = \{z \in Z\colon\ \hat{\tau}_z | \mathscr{R}(z)_1 \text{ is not weak operator continuous at } 0\},$$

where $\mathscr{R}(z)_1 = \{R(z)\colon\ R \in \mathscr{R},\ \|R(z)\| \leqq 1\}$.

We claim that $\mu(S) = 0$. For $A, B \in \mathscr{A}(z)$ define

$$\varrho_z(A, B) = \sum_{i,j \in \mathbf{N}} |\langle (A-B) x_j(z), x_i(z) \rangle| 2^{-i-j}.$$

So ϱ_z is a measurable field of metrics metrizing the unit ball of $\mathscr{A}(z)$ endowed with the weak operator topology (see 0). The set

$$H(k, l, \varepsilon, \delta) = \{z \in Z: \text{ there is } R \in \mathscr{R} \text{ such that } \|R(z)\| < 1, \\ \varrho_z(R(z), 0) < \delta \text{ and } |\langle \hat{\tau}_z(R(z)) x_l(z), x_k(z)\rangle| > \varepsilon\}$$

is measurable and

$$S = \bigcup_{l=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcap_{j=0}^{\infty} H(k, l, \varepsilon_i, \delta_j)$$

provided that $\varepsilon_i \searrow 0$ and $\delta_j \searrow 0$. Hence S is measurable.

Suppose that $\mu(S) > 0$. Then we have $K \subset S$, $\varepsilon > 0$, and $k, l \in \mathbf{N}$ such that

(iv) $\mu(K) > 0$

and for $z \in K$ and $j \in \mathbf{N}$ there is an $R_z^j \in \mathscr{R}$ with the properties

(v) $\|R_z^j(z)\| < 1$,

(vi) $\varrho_z(R_z^j(z), 0) < \delta_j$,

(vii) $|\langle \hat{\tau}_z(R_z^j(z)) x_l(z),\ x_k(z)\rangle| > \varepsilon$.

By Lusin's lemma we may assume that K is compact and the functions

(viii) $z \mapsto \|R(z)\|$,

(ix) $z \mapsto \varrho_z(R(z), 0)$,

(x) $z \mapsto \langle (\tau R)(z) x_l(z), x_k(z)\rangle$

are continuous on K for any $R \in \mathscr{R}$. In this case the inequalities (v)—(vii) are fulfilled on an open set in K. For any $j \in \mathbf{N}$ a compactness argument gives a measurable partition $\{H_i^j: i \leqq p(j)\}$ of K and operators $R_i^j \in \mathscr{R}$ $(i \leqq p(j))$ such that for $z \in H_i^j$ $R_i^j(z)$ satisfies (v)—(vii). Let χ_i^j be the characteristic function of H_i^j $(j \in \mathbf{N}, i \leqq p(j))$ and define

$$R^j(z) = \sum_{i=1}^{P(j)} \chi_i^j(z) R_i^j(z) e_i^j(z) \quad \text{where} \quad e_i^j(z) = \overline{\operatorname{Arg} \langle \hat{\tau}_z R_i^j(z) x_l(z), x_k(z)\rangle},$$

and for $0 \neq \lambda \in \mathbf{C}$ set $\operatorname{Arg} \lambda = \lambda \cdot |\lambda|^{-1}$.

Taking $R^j = \int_Z \oplus R^j(z)\, d\mu(z)$ we have $R^j \in \mathscr{A}_1$ and

$$\varrho(R^j, 0) \leqq \int_Z \varrho_z(R^j(z), 0)\, d\mu(z) = \sum_{i=1}^{P(j)} \int_{H_i^j} \varrho_z(R_i^j(z), 0)\, d\mu(z) \leqq \mu(K)\delta_j;$$

moreover,

$$\langle \tau(R^j) x_l, x_k\rangle = \int_Z \langle \tau R^j(z) x_l(z), x_k(z)\rangle\, d\mu(z) = \\ = \sum_{i=1}^{P(j)} \int_{H_i^j} |\langle \hat{\tau}_z R_i^j(z) x_l(z), x_k(z)\rangle|\, d\mu(z) \geqq \mu(K)\varepsilon.$$

This contradicts the continuity of τ. Hence $\mu(S)=0$ so $\hat{\tau}_z|\mathscr{R}(z)_1$ is weak operator continuous μ-a.e. on Z. It is then also uniformly continuous with respect to the uniformity defined by the metric ϱ_z.

Now extend $\hat{\tau}_z|\mathscr{R}(z)_1$ by uniform continuity with respect to the compact metrizable weak operator topology to $\mathscr{A}(z)_1$ and then by the homogeneity to $\mathscr{A}(z)$. So we get a linear τ_z such that $\|\tau_z\|\leq 1$ and $\tau_z|\mathscr{C}(z)$ is the identity. Hence τ_z is a center-valued state μ-a.e. on Z.

We want to check that $\tau(A)=\int_Z\oplus\,\tau_z A(z)\,d\mu$ if $A=\int_Z\oplus\,A(z)\,d\mu(z)$. We may assume that $\|A\|\leq 1$. In this case there is a sequence $T_n\in\mathscr{T}$ $(n\in\mathbf{N})$ such that $T_n\xrightarrow{s}A$. Then for a subsequence T_{n_k} we have $T_{n_k}(z)\xrightarrow{s}A(z)$ for μ-a.e. $z\in Z$ (cf. [3] chap. II. § 2. prop. 4). According to the weak continuity of τ_z now ${}_zT_{n_k}(z)\xrightarrow{w}\tau_z A(z)$. Consequently $\tau(T_{n_k})=\int_Z\oplus\,\tau_z T_{n_k}(z)\,d\mu(z)\xrightarrow{w}\int_Z\oplus\,\tau_z A(z)\,d\mu(z)$, On the other hand $\tau(T_{n_k})\xrightarrow{w}\tau_z(A)$ so $\tau(A)=\int_Z\oplus\,\tau_z A(z)\,d\mu(z)$.

2.3. Theorem. *Let $\mathscr{A}$ be a von Neumann algebra acting on the Hilbert space $\mathfrak{H}=\int\oplus\,\mathfrak{H}(z)\,d\mu(z)$ and suppose that $\mathscr{A}$ is decomposable as a direct integral of factors $\int\oplus\,\mathscr{A}(z)\,d\mu(z)$. Then $\tau\colon\mathscr{A}\to\mathscr{C}$ is a normal center-valued state if and only if it has the form*

$$\tau(A)=\int\oplus\sum_{i=1}^{\infty}\langle A(z)u_i(z),u_i(z)\rangle I(z)\,d\mu(z)\quad\left(A=\int\oplus A(z)\,d\mu(z)\right)$$

where $u_i\in\mathfrak{H}$ $(i\in\mathbf{N})$ and $\sum_{i=1}^{\infty}\|u_i(z)\|^2=1$ μ-a.e. on Z.

Proof. If $\tau\colon\mathscr{A}\to\mathscr{C}$ has the form described above then τ is a center-valued state since $\mathscr{C}$ consists of the diagonal operators and it follows from 1.6 that τ is normal.

Now assume that τ is a normal center-valued state. By Theorem 2.2, $\tau=\int_Z\oplus\,\tau_z\,d\mu(z)$. Let $H_n=\{z\in Z\colon \dim\mathfrak{H}(z)=n\}$ $(n=1,2,\dots,\infty)$ and put $\tau_n=$ $=\int_{H_n}\oplus\,\tau_z\,d\mu(z)$. So $\tau=\oplus\tau_n$ and it suffices to prove the theorem for τ_n $(n=1,2,\dots,\infty)$. Hence we may identify each $\mathfrak{H}(z)$ with a fixed Hilbert space $\mathfrak{H}_0$.

Let Y be the unit ball of $\mathfrak{H}_0\oplus\mathfrak{H}_0\oplus\dots$ endowed with the weak topology. So Y is a compact metrizable space.

Let $\mathscr{T}\subset\mathscr{A}$ be a countable family with the properties (i)—(ii) in 2.1. Define

$$H(T)=\left\{(z,y_1,y_2,\dots)\in Z\times Y\colon \tau_z T(z)=\sum_{i=0}^{\infty}\langle T(z)y_i,y_i\rangle I(z)\right\}$$

$H(T)$ is a Borel set in $Z\times Y$ and so is $H=\bigcap\{H(T)\colon T\in\mathscr{T}\}$.

We will use the principle of measurable choice (see [7] p. 35). H is analytic and for $z\in Z$ there is a normal state φ_z on $\mathscr{A}(z)$ such that $\tau_z=\varphi_z\cdot I(z)$. Hence,

$$\tau_z(S) = \sum_{i=1}^{\infty} \langle Su_z^i, u_z^i\rangle I(z)$$

for every $S\in\mathscr{A}(z)$ and for some $u_z^i\in\mathfrak{H}_0$ $(i\in\mathbf{N})$. Consequently $(z, u_z^1, u_z^2, \dots)\in H$. By the principle of measurable choice there exists a μ-measurable function $\Phi\colon Z\to Y$ such that $\Phi(z)\in H$ μ-a.e. on Z. If $\Phi(z)=(u_1(z), u_2(z), \dots)$ then $u_i\in\mathfrak{H}$ $(i\in\mathbf{N})$. We have obtained that

$$\tau_z T(z) = \sum_{i=1}^{\infty} \langle T(z)u_i(z), u_i(z)\rangle I(z)$$

for any $T\in\mathscr{T}$, μ-a.e. on Z. Since $\mathscr{T}$ is dense in $\mathscr{A}_1$ and τ is continuous we have

$$\tau(A) = \int_Z \oplus \sum_{i=1}^{\infty} \langle A(z)u_i(z), u_i(z)\rangle I(z)\, d\mu(z)$$

for every $A\in\mathscr{A}$. Moreover, $\sum_{i=1}^{\infty}\|u_i(z)\|^2=1$ μ-a.e. on Z because $\tau(I)=I$. This completes the proof.

3. An application. In this section we use 2.2 in order to give an alternative proof for an extension, given in [8] and [5], of a result of J. B. Conway.

3.1. We introduce some notations. If A belongs to the algebra $\mathscr{A}$ then let $C_0(A)$ be the convex hull of the set $\{UAU^*\colon U \text{ is a unitary in } \mathscr{A}\}$. Moreover, let $C(A)=\overline{C_0(A)}^w\cap\mathscr{C}$ and $\overline{W}(A)$ be the closed numerical range of A. The following proposition was proved by J. B. Conway [2].

3.2. Proposition. *If $\mathscr{A}$ is a type III factor and $A\in\mathscr{A}$ then $C(A)=\overline{W}(A)=$ $=\Sigma(A)$.*

3.3. Proposition. *If $\mathscr{A}=\int_Z\oplus\mathscr{A}(z)\,d\mu(z)$ and $A=\int_Z\oplus A(z)\,d\mu(z)$ then $C(A)=\int_Z\oplus C(A(z))\,d\mu(z)$.*

The latter assertion was proved by S. Komlósi [5] and it means that $B=\int_Z\oplus B(z)\,d\mu(z)\in C(A)$ if and only if $B(z)\in C(A(z))$ μ-a.e. on Z.

3.4. Lemma. *Let $\mathscr{A}=\int_Z\oplus\mathscr{A}(z)\,d\mu(z)$ and $A=\int_Z\oplus A(z)\,d\mu(z)\in\mathscr{A}$. Assume that U is a weak operator neighbourhood of the diagonal operator $B=\int_Z\oplus f(z)I(z)\,d\mu(z)$*

and $f(z)\in\overline{W}(A(z))$ *for* $z\in Z$. *Then there is a* $u\in\mathfrak{H}$ *such that*

(i) $\int_Z \oplus \langle A(z)u(z), u(z)\rangle d\mu(z)\in U,$

(ii) $\|u(z)\|=1$ μ-a.e. on Z.

Proof. Take a sequence $\{e_n\}\subset\mathfrak{H}$ such that $\|e_n(z)\|=1$ and $\{e_n(z)\}$ is dense in $\{s\in\mathfrak{H}(z): \|s\|=1\}$ μ-a.e. on Z. Suppose that U is determined by $\varepsilon>0$ and $y_i\in\mathfrak{H}$ $(i\leq m)$ that is

$$U=\{T\in\mathscr{A}: |\langle (T-B)y_i: y_j\rangle|<\varepsilon\ \ (i,j\leq m)\}.$$

Choose a compact $K\subset Z$ with the properties

(a) $z\mapsto\langle A(z)e_i(z), e_j(z)\rangle$ is continuous on K $(i,j\in\mathbf{N})$,

(b) $\mu(Z\setminus K)<\delta$,

(c) $\int_{Z\setminus K} |\langle y_i(z), y_j(z)\rangle|\, d\mu(z)<\delta\quad (i,j\leq m)$,

(d) f is continuous on K.

(δ is arbitrary but fixed). We can find $x_z\in\{e_n\}$ and an open G_z containing z such that

$$|f(v)-\langle A(v)x_z(v), x_z(v)\rangle|<\delta\quad (v\in G_z)$$

for $z\in K$. Using compactness one has a measurable partition $\{H_i: i\leq k\}$ of K such that $H_i\subset G_{z_i}$ for some $z_i\in K$ $(i\leq k)$. Let χ_i (χ) be the characteristic function of H_i $(Z\setminus K)$ and define u by

$$u(z)=\chi(z)e_1(z)+\sum_{i=1}^{k}\chi_i(z)x_{z_i}(z)\quad (z\in Z).$$

$\|u(z)\|=1$ is fulfilled evidently for μ-a.e. $z\in Z$. An easy estimation gives

$$\Big|\Big\langle\Big(\int_Z\oplus\langle A(z)u(z), u(z)\rangle I(z)\, d\mu(z)-B\Big) y_i,\ y_j\Big\rangle\Big|\leq\delta(\|A\|+\|B\|+\|y_i\|\cdot\|y_j\|).$$

So if δ is small enough then (i) is satisfied.

3.5. Theorem. *If* $\mathscr{A}$ *is a type* III *von Neumann algebra and* $A\in\mathscr{A}$ *then*

$$C(A)=\overline{\Sigma^n(A)}^w.$$

Proof. We express $\mathscr{A}$ as a direct integral of type III factors: $\int_Z\oplus\mathscr{A}(z)\, d\mu(z)$. If $B=\int_Z\oplus B(z)d\mu(z)\in C(A)$ then $B(z)\in C(A(z))$ μ-a.e. on Z by 3.3. According to 3.2 $B(z)\in\overline{W}(A(z))$ and we can use 3.4. For every weak neighbourhood U of B

there is a $u\in\mathfrak{H}$ such that

$$\int_Z \oplus A\langle(z)u(z),\, u(z)\rangle\, I(z)\, d\mu(z)\in U.$$

However, $T\mapsto \int_Z \oplus\, \langle T(z)u(z),\, u(z)\rangle I(z)\, d\mu(z)$ defines a normal center-valued state (cf. 1.6) hence $C(A)\subset \overline{\Sigma^n(A)}^w$.

Conversely, for any $\tau\in\Sigma^n$, $\tau=\int_Z \oplus\tau_z\, d\mu(z)$ follows from 2.2 and $\tau_z(A(z))\in C(A(z))$ from 3.2. By 3.3 we have $\int_Z \oplus\tau_z(A(z))\, d\mu(z)\in \int_Z \oplus C(A(z))\, d\mu(z)=C(A)$. So $\tau(A)\in C(A)$ and we have obtained that $\Sigma^n(A)\subset C(A)$. Since $C(A)$ is closed, it follows $\overline{\Sigma^n(A)}^w\subset C(A)$.

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Bibliographie

P. J. Cameron and J. H. van Lint, Graphs, codes and designs, London Math. Soc. Lecture Note Series 43, VII+147 pages, Cambridge University Press, Cambridge—London—New York—New Rochelle—Melbourne—Sidney, 1980.

The study of highly symmetric finite structures (block designs, finite geometries, latin squares, error-correcting codes, strongly regular graphs etc.) has been a fascinating field of combinatorial research for centuries. Characteristic of the subject is the abundance of extremely difficult problems. Is there a projective plane of order 10? Is there a 6-transitive non-trivial permutation group? One feels that such questions should be answerable by straightforward arguments; but in fact they are unsolved in spite of the efforts of a sizeable group of outstanding people working on them. As a matter of fact, after a few trivial steps virtually any progress assumes the introduction of surprising and ingenious methods, quite often involving modern advanced mathematics. Just a fraction of such methods would mean a breakthrough in most other fields of combinatorics.

This book is a considerably revised, updated and extended version of a previous work by the authors (LMS Lecture Notes Series 19). A chapter containing a brief introduction to design theory has been added, making the book more self-contained and easier accessible for non-specialists. After discussing strongly regular graphs, partial geometries, and other symmetric structures, the main weight is put on the study of codes. The authors discuss or sketch recent important results such as the inequalities of Ray-Chandhuri and Wilson, the linear programming bound by Delsarte, the Krein—Scott bound, and others. Cross-references betwen chapters, and references to further relevant publications make this excellent book a very interesting, informative and enjoyable reading for combinatorialists.

L. Lovász (Szeged)

R. E. Edwards, A Formal Background to Mathematics 2: A Critical Approach to Elementary Analysis, Universitext, XLVII+1170 pages, two volumes, Springer Verlag, Berlin—Heidelberg—New York, 1980.

The aim of the book is to face and clear up the most problematical points in elementary analysis in a half-formal framework and to treat some more advanced topics. Thus, it is "not intended for readers totally new to convergence, calculus, etc. but rather for those who have some informal working acquaintance with these matters and who wish to review their understanding and see links and contrasts with the formal background."

However, the necessity of this "link...with the formal background" is not quite convincing. Generally the study of elementary analysis precedes that of mathematical logic, furthermore a good informal treatment may be just as useful as a formal approach. These formal parts, comments, cross references etc. break the line of the book into pieces. By my count 60—70% of its text-part is an almost ordinary (and quite good!) presentation of analysis and the remaining part consists of remarks and comments which quite often do not help to understand the material.

The mentioned relation with formalism forced the author to decline from ordinary notation; to mention just one example, it hardly causes any problem in an elementary analysis book if the restriction of the function $x-a$ to the interval I is denoted also by $x-a$ and not by $j-a_I$.

The selected topics do not cover the usual college part of analysis, e.g. the multidimensional case is completely omitted. The way of the introduction of exp z for complex z is interesting; and we can only greet such themes as the irrationality of π, transcendental numbers, etc.

Volume 2 contains a number of more or less advanced exercises (on more than 300 pages) but many of them belong rather to logic than to calculus. Elements of advanced analysis are also incorporated in these exercises.

The book may be recommended to teachers or future reseachers of elementary analysis.

V. Totik (Szeged)

P. Hájek, T. Havránek, Mechanizing Hypothesis Formation Mathematical Foundations for a General Theory, Springer-Verlag, Berlin, Heidelberg, New York, 1978, XV+396 p.

The volume is devoted to a systematic formalization and development of one important aspect of scientific reasoning, the hypothesis formation. The authors themselves declare their aim as being a response to the challenging question: "Can computers formulate and justify scientific hypotheses?".

In organization the book follows a proposal by Gordon Plotkin. After an introductory chapter, the authors develop and investigate in details the so called "logic of induction" and "logic of suggestion", which are, according to the proposal, determined as "studying the notion of justification" and "studying methods of suggesting reasonable hypotheses", respectively.

Logic of induction presented in this volume (Part A) is a generalized version of Suppes's predicate calculus. The modifications are motivated by statistical considerations, using particular "quantifiers" such as

"for sufficiently many x, $P(x)$" and "the property $Q(x)$ is associated with $R(x)$",

can be viewed as an illustrative example.

Dealing with logic of suggestions (Part B), the authors define and study some GUHA (General Unary Hypotheses Automaton) methods.

Presentation of the matter is clear and completely logically oriented with an extreme stress on Tarskian semantics and computability (mechanizability).

The authors recommend their text to mathematical logicians and statisticians with computer scientific interest, and to students interested in interrelations among mathematical logic, statistics, computer science, and artificial intelligence.

P. Ecsedi-Tóth (Szeged)

Peter G. Hinman, Recursion-Theoretic Hierarchies, Springer-Verlag, Berlin, Heidelberg New York, 1978. XII+480 p.

Notions of definability are among the most popular, and of course, among the most importan topics of current research in mathematical logic. The book under review, published as the second volume of the noteworthy sequence "Perspectives in Mathematical Logic", is concerned with one aspect of definability. Central to the discussion in the book is "the classification of mathematical objects according to various measures of their complexity" into levels, that is, into a hierarchy in such a way that objects being in a higher level are more complex than those being in a lower one. In fact, this topic is common in different mathematical theories such as Descriptive Set Theory and (Generalized) Recursion Theory. Generally speaking, Descriptive Set Theory deals with explicit and constructive means (i.e. that do not require the Axiom of Choice) in analysis and is developed as the analytical counterpart of Cantorian set theory. Recursion Theory deals also with a kind of constructivity, the mechanical computability of certain mathematical objects. These two areas of "constructive mathematics" developed parallelly for decades until it was realized that they had a common generalization. This book presents the generalized theory in a clear and attractive manner pointing out how earlier results follow.

The volume is divided into three parts, each of them containing several chapters and sections, dealing with the basic notions of definability (background, ordinary recursion theory, hierarchies and definability with arithmetical and analytical hierarchies, inductive and implicit definability, arithmetical forcing etc.), the analytical and projective hierarchies (wellorderings, boundedness principle, Borel hierarchy, hyperarithmetical hierarchy, pre-wellordering, constructibility, projective determinancy etc.) and generalized recursion theories (recursion in Type-2 and Type-3 functionals, higher types, recursion on ordinals etc.) respectively.

The book is intended for a "student with some general background in abstract mathematics — at least a smattering of topology, measure theory, and set theory — who has finished a course in logic covering the completeness and incompleteness theorems". In fact, this excellently written volume can be recommended to everyone who is interested in definability or constructivity and, in particular, in (Generalized) Recursion Theory and Descriptive Set Theory.

P. Ecsedi-Tóth (Szeged)

D. L. Johnson, Topics in the Theory of Group Presentations (London Mathematical Society Lecture Note Series, 42) VII+311 pages, Cambridge University Press, Cambridge—London—New York—Melbourne, 1980.

This book is an extended version of the author's earlier contribution to this series. As before, the emphasis is on concrete examples of groups, demonstrating the pervasive connection between group theory and other branches of mathematics and bringing, this way, the material closer to the graduate and post-graduate students.

Chapter I starts with a proof of the Nielsen—Schreier theorem (every subgroup of a free group is free) and works out the elements of group presentations. In Chapter II presentations of some popular groups (such as the dihedral group, the generalized quaternion group and the symmetric group) and the presentations of Abelian groups are considered. Chapter III deals with groups with few relations, introducing some new examples of groups, which are already less well-known than those in the previous chapter, but are, on the other hand, most useful for the specialist of this theory. Chapter IV concerns presentations of subgroups. A geometric application is found in Chapter V, where tessalations of a plane by a triangle are described in each of the three different cases, the Euclidean, the elliptic, and the hyperbolic one. The main purpose of Chapter VI is to give a proof of the celebrated theorem of Golod and Šafarevič stating that, if a finite p-group G is minimally generated by d elements, then the minimal number of relations needed to define G is greater than $d^2/4$. Cohomology of groups plays an essential part in the proof. Chapter VII deals with small cancellation groups. Finally, Chapter VIII reflects the intimate connection between group theory and topology. This chapter deals with the classification theory of surfaces and the theory of knots.

"Such is the current rate of progress in combinatorial group theory that no attempt at completeness in feasible, though it is hoped to bring the reader to within hailing distance of the frontiers of research in one or two places."

This lecture notes can serve as a text for beginning research students and as an introduction for specialists in other fields.

A. P. Huhn (Szeged)

P. Koosis, Introduction to H_p spaces, London Math. Soc. Lecture Notes Series, 40, XV+376 pages, Cambridge, University Press, 1980.

This book contains the lectures on the elementary theory of H_p spaces held at the Stockholm Institute of Technology in 1977—78. It is a good introduction to the theory of H_p spaces. The details and long explanations help the reader in the understanding of the material very much. Much effort is made to enlighten the role of the theorem of the brothers Riesz. Beyond classical results quite recent developments are also treated: Marshall's theorem about the uniform approximation of H_∞ functions by Blaschke products; the maximal function characterization of RH_1, etc.

Chapter X contains a brief introduction of BMO. Finally, in the appendix Wolff's simple(?) proof of the corona theorem is given.

V. Totik (Szeged)

W. Rudin, Function Theory in the Unit Ball of C^n (Grundlehren der mathematischen Wissenschaften 241), XIII+436 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980.

This book contains a quite new and rapidly developing theory for functions on the unit ball in C^n. The subject is presented in an extraordinarily clear way. Domains different from the unit ball are almost completely omitted and this enabled the author to show the main ideals withouth the hard and very technical "side-dish" of the general theory.

Two further values of the book: 1. it is self-contained, 2. it is up to date, many results being dated from the late seventies. The publisher did also his best by the quick, well-ordered, and very accurate printing.

The main topics are the following: integral representations, boundary behaviour of Poisson and Cauchy integrals, measures related to the ball algebra, interpolation sets, invariant function spaces, analytic varieties, proper holomorphic maps, the $\bar{\partial}$-problem, the Henkin—Skoda theorem, tangential Cauchy—Riemann operators.

The book concludes with a series of open problems which will certainly stimulate further investigations.

It is highly recommended to anyone who wants to get acquainted with this beautiful part of complex function theory.

V. Totik (Szeged)

Livres reçus par la rédaction

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