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C_0-Fredholm operators. II

HARI BERCOVICI

SZ.-NAGY and FOIAŞ [16] proved that the operators T of class C_0 and of finite multiplicity have the following property:

(P) *any injection* $X \in \{T\}'$ *is a quasi-affiniti.*

In [3] we showed that property (P) also holds for weak contractions of class C_0. In sec. 4 of the present note we shall characterize the class $\mathcal{P}$ of C_0 operators having property (P).

UCHIYAMA [18] has shown that some quasi-affinities intertwining two contractions of class $C_0(N)$ induce isomorphisms between the corresponding lattices of hyper-invariant subspaces. This is not verified for arbitrary operators of class C_0 (cf. Example 2.10 below). For operators of the class $\mathcal{P}$ we show (cf. sec. 4) that any intertwining quasi-affinity induces isomorphisms between the corresponding lattices of invariant and hyper-invariant subspaces. However the other results proved in [18] for operators of the class $C_0(N)$ hold for arbitrary operators of class C_0; this is shown in sec. 2 of this note. In sec. 2 we also show which is the connection between the lattice of hyper-invariant subspaces of a C_0 operator and the corresponding lattice of the Jordan model.

In sec. 3 of this note we prove a continuity property of the Jordan model. This is useful when dealing with operators of class $\mathcal{P}$.

In [16] B. SZ.-NAGY and C. FOIAŞ made the conjecture that any operator T of class C_0 and of finite multiplicity has the property:

(Q) $T|\ker X$ *and* $T_{\ker X^*}$ *are quasisimilar for any* $X \in \{T\}'$.

This conjecture was infirmed in [3], Proposition 3.2, but was proved under the stronger assumption $X \in \{T\}''$ for any operator T of class C_0 (cf. also UCHIYAMA [19]).

Uchiyama began the study of the class of operators satisfying the property (Q) showing in particular that there exist operators of class $C_0(N)$ and multiplicity 2 wich have this property (cf. [19], Example 2). In sec. 5 of this note we characterise in terms of the Jordan model the class $\mathcal{Q}$ of C_0 operators having property (Q).

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In [3] the determinant function of a weak contraction was used for proving various index results. In sec. 6 of this note we extend the notion of inner function in order to find a substitute of the determinant function for the case of operators of class $\mathscr{P}$. In sec. 7 it is shown that the class of generalised inner functions (defined in sec. 6) naturally appears in the study of index problems. In sec. 8 we generalise the notion of C_0-fredholmness defined in [3]. All results of [3] are extended to this more general setting.

1. Notation and preliminaries

Let us recall that Lat (T) and $\mathrm{Lat}_{\frac{1}{2}}(T)$ stand for the lattice of all invariant, respectively semi-invariant subspaces of the operator T. We shall denote by Hyp Lat (T) the lattice of hyper-invariant subspaces of T. If $\mathfrak{M}\in\mathrm{Lat}_{\frac{1}{2}}(T)$, $T_{\mathfrak{M}}$ stands for the compression of T to the subspace $\mathfrak{M}$ and $\mu_T(\mathfrak{M})$ stands for the multiplicity of $T_{\mathfrak{M}}$. The notations $T\prec T'$, $T\overset{i}{\prec}T'$ mean that T is a quasi-affine transform of T', respectively that T can be injected into T' (cf. e.g. [15]).

The following result will be frequently used in the sequel.

Lemma 1.1. *If T and T' are operators of class C_0 and $T\prec T'$ then T and T' are quasisimilar.*

Proof. Cf. [16], Theorem 1 or [4], Corollary 2.10.

Lemma 1.2. *Let $\{m_i\}_{i=0}^{\infty}$ be a sequence of pairwise relatively prime inner functions. If the operator $T=\bigoplus_{i=0}^{\infty} S(m_i)$ is of class C_0, the Jordan model of T is $S(m)$, $m=m_T$.*

Proof. If T is of class C_0 it follows that T is a weak contraction (cf. the proof of [6], Lemma 8.4) and from the assumption we easily infer $d_T=m_T$. The conclusion follows by [6], Theorem 8.7.

For two operators T and T' we denote by $\mathscr{I}(T',T)$ the set of intertwining operators

$$\mathscr{I}(T',T)=\{X\colon T'X=XT\}. \tag{1.1}$$

Let us recall (cf. [3], Definition 2.1) that $X\in\mathscr{I}(T',T)$ is a lattice-isomorphism if the mapping $\mathfrak{M}\mapsto(X\mathfrak{M})^-$ is an isomorphism of Lat (T) onto Lat (T').

Definition 1.3. An operator T has *property* (P) if any injection $A\in\{T\}'$ is a quasi-affinity.

We introduce the property (Q) as in [19]:

Definition 1.4. An operator T has *property* (Q) if for any $A\in\{T\}'$, $T|\ker A$ and $T_{\ker A^*}$ are quasisimilar.

Obviously (P) is implied by (Q).

Lemma 1.5. *The operator T of class C_0 acting on the Hilbert space $\mathfrak{H}$ has the property* (P) *if and only if there does not exist $\mathfrak{H}'\in\operatorname{Lat}(T)$, $\mathfrak{H}'\neq\mathfrak{H}$, such that T and $T|\mathfrak{H}'$ are quasisimilar.*

Proof. Let T be quasisimilar to $T|\mathfrak{H}'$, $\mathfrak{H}'\in\operatorname{Lat}(T)$ and let $X\colon \mathfrak{H}\to\mathfrak{H}'$ be a quasi-affinity such that $(T|\mathfrak{H}')X=XT$. Then $A=JX$ (where J denotes the inclusion of $\mathfrak{H}'$ into $\mathfrak{H}$) commutes with T and $\ker A=\{0\}$. If T has the property (P) we infer $\mathfrak{H}'=(A\mathfrak{H})^-=\mathfrak{H}$. Conversely, if $A\in\{T\}'$ is an injection, T and $T|(A\mathfrak{H})^-$ are quasisimilar by Lemma 1.1.

We shall denote by H_i^∞ the set of inner functions in H^∞. The set H_i^∞ is (pre)-ordered by the relation

$$m \leqq m' \quad \text{if and only if} \quad |m(z)| \geqq |m'(z)|, \quad |z|<1. \tag{1.2}$$

Obviously $m\leqq m'$ if and only if m divides m'. The relations $m\leqq m'$ and $m'\leqq m$ imply that m and m' differ by a complex multiplicative constant of modulus one; we shall not distinguish between the functions m and m' in this case.

Let us recall (cf. [4]) that a Jordan operator is an operator of the form

$$S(M)=\bigoplus_{\alpha} S(m_\alpha), \quad m_\alpha = M(\alpha) \tag{1.3}$$

where M is a model function, that is M is an inner function valued mapping defined on the class of ordinal numbers and

$$\begin{cases} m_\alpha \leqq m_\beta & \text{whenever} \quad \alpha \geqq \beta; \\ m_\alpha = m_\beta & \text{whenever} \quad \bar{\alpha} = \bar{\beta}; \end{cases} \tag{1.4}$$

$$m_\alpha = 1 \quad \text{for some } \alpha, \tag{1.5}$$

where $\bar{\alpha}$ denotes the cardinal number associated with the ordinal number α.

The Jordan model $S(M)$ is acting on a separable space if and only if $m_\omega=1$, where ω denotes the first transfinite ordinal number. In this case the Jordan operator is determined by the sequence $\{m_j\}_{j=0}^\infty$. If $m_n=1$ for some $n<\omega$, we shall also use the notation $S(m_0, m_1, \ldots, m_{n-1})$ for $S(M)$ (cf. [13]). If $S(M)$ is the Jordan model of the operator T of class C_0, we shall use the notation $m_\alpha[T]=M(\alpha)$ (cf. [4]).

2. Hyper-invariant subspaces of operators of class C_0

In this section we continue the study of hyper-invariant subspaces for the class C_0 begun by UCHIYAMA [18] (for the case of operators of class $C_0(N)$). The following Proposition extends [18], Theorem 3 and Corollaries 4 and 5 to the class of general Jordan operators.

Proposition 2.1. *Let $T=S(M)$ be a Jordan operator acting on the Hilbert space*

$$\mathfrak{H}(M) = \bigoplus_{\alpha} \mathfrak{H}(m_\alpha), \quad m_\alpha = M(\alpha). \tag{2.1}$$

(i) *A subspace $\mathfrak{M}\subset\mathfrak{H}(M)$ is hyper-invariant for T if and only if it is of the form*

$$\mathfrak{M} = \bigoplus_{\alpha} (m''_\alpha H^2 \ominus m_\alpha H^2), \quad m''_\alpha \leqq m_\alpha, \tag{2.2}$$

and the functions M' and M'' given by $M''(\alpha)=m''_\alpha$ and $M'(\alpha)=m_\alpha/m''_\alpha$ are model functions.

(ii) *If $\mathfrak{M}$ is a subspace of the form* (2.2) *then $T'=T|\mathfrak{M}$ is unitarily equivalent to $S(M')$ and $T''=T_{\mathfrak{M}^\perp}$ is unitarily equivalent to $S(M'')$. In particular,*

$$m_T = m_{T'} m_{T''} \tag{2.3}$$

if $\mathfrak{M}$ is hyper-invariant.

(iii) *If $\mathfrak{M}_1, \mathfrak{M}_2 \in \operatorname{Hyp\,Lat}(T)$ are such that $T|\mathfrak{M}_1$ and $T|\mathfrak{M}_2$ are quasisimilar, we have $\mathfrak{M}_1=\mathfrak{M}_2$.*

Proof. We shall denote by $P_{\mathfrak{H}(m_\alpha)}$ the projection of H^2 onto $\mathfrak{H}(m_\alpha)$, by $\tilde{P}_{\mathfrak{H}(m_\alpha)}$ the projection of $\mathfrak{H}(M)$ onto $\mathfrak{H}(m_\alpha)$ and by J_α the inclusion of $\mathfrak{H}(m_\alpha)$ into $\mathfrak{H}(M)$. By the lifting Theorem (cf. [12], Theorem II.2.3) $\{T\}'$ is strongly generated by the operators $\psi(T)$, where $\psi\in H^\infty$, and the operators $A_{\beta\alpha}$ given by

$$\begin{cases} A_{\beta\alpha} = J_\beta P_{\mathfrak{H}(m_\beta)} \tilde{P}_{\mathfrak{H}(m_\alpha)} & \text{if} \quad \alpha \leqq \beta; \\ A_{\beta\alpha} = J_\beta (m_\beta/m_\alpha) \tilde{P}_{\mathfrak{H}(m_\alpha)} & \text{if} \quad \alpha > \beta, \end{cases} \tag{2.4}$$

and therefore the subspace $\mathfrak{M}\subset\mathfrak{H}(M)$ is a hyper-invariant subspace if and only it is invariant and $A_{\alpha\beta}\mathfrak{M}\subset\mathfrak{M}$ for each α and β. Let us assume that $\mathfrak{M}$ is hyper-invariant. Because $A_{\alpha\alpha}\mathfrak{M}=\tilde{P}_{\mathfrak{H}(m_\alpha)}\mathfrak{M}\subset\mathfrak{M}$ we have

$$\mathfrak{M} = \bigoplus_{\alpha} \mathfrak{M}_\alpha \tag{2.5}$$

where $\mathfrak{M}_\alpha\in\operatorname{Lat}(S(m_\alpha))$, say $\mathfrak{M}_\alpha=m''_\alpha H^2\ominus m_\alpha H^2$; therefore $\mathfrak{M}$ is of the form (2.2). Now let α and β be ordinal numbers such that $\alpha<\beta$; the conditions $A_{\alpha\beta}\mathfrak{M}\subset\mathfrak{M}$ and $A_{\beta\alpha}\mathfrak{M}\subset\mathfrak{M}$ are equivalent to $P_{\mathfrak{H}(m_\beta)}\mathfrak{M}_\alpha\subset\mathfrak{M}_\beta$ and $(m_\alpha/m_\beta)\mathfrak{M}_\beta\subset\mathfrak{M}_\alpha$. We infer $m''_\alpha\in m''_\beta H^2$ and $(m_\alpha/m_\beta)m''_\beta\in m''_\alpha H^2$ so that $m''_\alpha\geqq m''_\beta$ and $m_\alpha/m''_\alpha\geqq m_\beta/m''_\beta$, respectively; therefore M' and M'' are model functions.

Conversely, let $\mathfrak{M}$ be given by (2.2) and assume M' and M'' are model functions. It easily follows that $P_{\mathfrak{H}(m_\beta)}\mathfrak{M}_\alpha \subset \mathfrak{M}_\beta$ and $(m_\alpha/m_\beta)\mathfrak{M}_\beta \subset \mathfrak{M}_\alpha$ whenever $\alpha<\beta$. Thus $A_{\alpha\beta}\mathfrak{M}\subset\mathfrak{M}$ for each α and β so that $\mathfrak{M}\in \mathrm{Hyp\ Lat}\,(T)$ and (i) follows.

To prove (ii) let us remark that, if $\mathfrak{M}$ is given by (2.2), we have $T|\mathfrak{M} = \bigoplus_\alpha S(m_\alpha)|\mathfrak{M}_\alpha$ and $T_{\mathfrak{M}^\perp} = \bigoplus_\alpha S(m_\alpha)_{\mathfrak{M}_\alpha^\perp}$, where $\mathfrak{M}_\alpha = m''_\alpha H^2 \ominus m_\alpha H^2$ and $S(m_\alpha)|\mathfrak{M}_\alpha$ is unitarily equivalent to $S(m'_\alpha)$ while $S(m_\alpha)_{\mathfrak{M}_\alpha^\perp}$ is unitarily equivalent to $S(m''_\alpha)$. If $\mathfrak{M}$ is hyper-invariant then $S(M')$ and $S(M'')$ are Jordan operators and therefore they are the Jordan models of T' and T'', respectively. In particular $m_{T'} = m'_0 = m_0/m''_0 = m_T/m_{T''}$ and (2.3) follows.

Finally, if $\mathfrak{M}_1, \mathfrak{M}_2 \in \mathrm{Hyp\ Lat}\,(T)$ and $T|\mathfrak{M}_1$, $T|\mathfrak{M}_2$ are quasisimilar it follows that $T|\mathfrak{M}_1$ and $T|\mathfrak{M}_2$ have the same Jordan model. By (ii) $\mathfrak{M}_1$ is determined by the Jordan model of $T|\mathfrak{M}_1$. Therefore $\mathfrak{M}_1 = \mathfrak{M}_2$ and (iii) follows.

Remark 2.2. The proof of Proposition 2.1 can be applied with minor changes to the description of Hyp Lat (T) when $T = \bigoplus_{j\in J} S(m_j)$ and $\{m_j\}_{j\in J}$ is a totally ordered subset of H_i^∞.

For further use let us note that the general form of a subspace $\mathfrak{M}\in \mathrm{Hyp\ Lat}\,(T)$ is

$$\mathfrak{M} = \bigoplus_{j\in J} (m''_j H^2 \ominus m_j H^2), \quad m''_j \leqq m_j \quad \text{for} \quad j\in J \tag{2.5}$$

where $m''_j \leqq m''_k$ and $m_j/m''_j \leqq m_k/m''_k$ whenever $m_j \leqq m_k$.

Remark 2.3. Let the subspaces $\mathfrak{M}_j$ be given by

$$\mathfrak{M}_j = \bigoplus_\alpha (m_j(\alpha) H^2 \ominus m_\alpha H^2), \quad j = 1, 2. \tag{2.6}$$

Then

$$\begin{cases} \mathfrak{M}_1 \cap \mathfrak{M}_2 = \bigoplus_\alpha (m_1(\alpha) \vee m_2(\alpha) H^2 \ominus m_\alpha H^2), \\ \mathfrak{M}_1 \vee \mathfrak{M}_2 = \bigoplus_\alpha (m_1(\alpha) \wedge m_2(\alpha) H^2 \ominus m_\alpha H^2); \end{cases} \tag{2.7}$$

in particular $\mathfrak{M}_1 \subset \mathfrak{M}_2$ if and only if $m_1(\alpha) \geqq m_2(\alpha)$ for each α.

We shall now characterize the Jordan operators having a totally ordered lattice of hyper-invariant subspaces thus extending [18], Theorem 6.

Proposition 2.4. *The lattice* Hyp Lat (T), $T = S(M)$, *is totally ordered if and only if one of the following situations* (i), (ii) *occurs:*

(i) $m_0 = \left(\frac{z-a}{1-\bar{a}z}\right)^n$ *and* $m_\alpha \in \left\{1, \left(\frac{z-a}{1-\bar{a}z}\right)^{n-1}, \left(\frac{z-a}{1-\bar{a}z}\right)^n\right\}$ *for each* α, *with* $|a|<1$ *and a natural number n.*

(ii) $m_0 = \exp\left(t\frac{z+a}{z-a}\right)$ *with* $|a|=1$, $t>0$, *and* $m_\alpha = m_0$ *whenever* $m_\alpha \neq 1$.

Proof. For two inner divisors m, m' of m_T we have $(\operatorname{ran} m(T))^- \subset (\operatorname{ran} m'(T))^-$ if and only if $m \geqq m'$ (cf. [4], Lemma 1.7). If Hyp Lat (T) is totally ordered it follows that the lattice of divisors of $m_T = m_0$ is also totally ordered. Therefore we have *either* $m_0 = \left(\frac{z-a}{1-\bar{a}z}\right)^n$ ($|a|<1$, n a natural number) *or* $m_0 = \exp\left(t\frac{z+a}{z-a}\right)$ ($|a|=1$, $t>0$).

Let us consider the first situation. Then $m_\alpha = \left(\frac{z-a}{1-\bar{a}z}\right)^{n(\alpha)}$ where $n(\alpha)$ is a decreasing function of α. By Proposition 2.1 and Remark 2.3, Hyp Lat (T) is isomorphic to the lattice of natural number valued decreasing functions $k(\alpha)$ such that $k(\alpha) \leqq n(\alpha)$ and $n(\alpha)-k(\alpha)$ is also decreasing. Assume there exists α_0 such that $m = n(\alpha_0) \notin \{n, n-1, 0\}$ and define $k_1(\alpha) = \max\{n(\alpha)-1, 0\}$ and $k_2(\alpha) = \min\{m, n(\alpha)\}$. Then we have $k_1(0) = n-1 > k_2(0) = m$ and $k_1(\alpha_0) = m-1) < k_2(\alpha_0) = m$ so that k_1 and k_2 are incomparable. Thus we necessarily have $n(\alpha) \in \{n, n-1, 0\}$. Conversely, if $n(\alpha) \in \{n, n-1, 0\}$ for every α, let us take two functions k_1, k_2 of the type considered before. If k_1 and k_2 would not be comparable there would exist $\alpha < \beta$ such that $n(\beta) \neq 0$ and, by example, $k_1(\alpha) < k_2(\alpha)$, $k_1(\beta) > k_2(\beta)$. From the assumption it follows that $n(\alpha) \leqq n(\beta)+1$ so that $n(\beta) - k_2(\beta) \leqq n(\alpha) - k_2(\alpha) \leqq n(\beta)+1-k_2(\alpha)$ and therefore $k_2(\alpha)-1 \leqq k_2(\beta)$. Now $k_1(\beta) \leqq k_1(\alpha) \leqq k_2(\alpha)-1 \leqq k_2(\beta)$, a contradiction. This shows that Hyp Lat (T) is totally ordered in this case.

Now let us consider the case $m_0(z) = \exp\left(t\frac{z+a}{z-a}\right)$. Then $m_\alpha(z) = \exp\left(t(\alpha)\frac{z+a}{z-a}\right)$, where $t(\alpha)$ is a positive number valued decreasing function. Again by Proposition 2.1 and Remark 2.3, Hyp Lat (T) is isomorphic to the lattice of positive number valued decreasing functions $s(\alpha)$ such that $s(\alpha) \leqq t(\alpha)$ and $t(\alpha)-s(\alpha)$ is also decreasing. Assume there exists α_0 such that $t(\alpha_0) \notin \{t, 0\}$ and let us take $0 < \varepsilon < \min\{t(\alpha_0), t-t(\alpha_0)\}$. Then the functions $s_1(\alpha) = \max\{t(\alpha)-\varepsilon, 0\}$ and $s_2(\alpha) = \min\{t(\alpha), t(\alpha_0)\}$ are such that $s_1(0) = t(0)-\varepsilon > s_2(0) = t(\alpha_0)$ and

$$s_1(\alpha_0) = t(\alpha_0)-\varepsilon < s_2(\alpha_0) = t(\alpha_0);$$

therefore s_1 and s_2 are incomparable. Thus we necessarily have $t(\alpha) \in \{t, 0\}$ if Hyp Lat (T) is totally ordered.

Conversely, let us assume $t(\alpha) \in \{t, 0\}$ for each α. If s is a function of the type considered above and $t(\alpha) \neq 0$, we have $s(0) \geqq s(\alpha)$ and $t-s(0) \geqq t(\alpha)-s(\alpha) = t-s(\alpha)$ so that $s(\alpha) = s(0)$. Thus $s(\alpha) = s(0)$ if $t(\alpha) \neq 0$ and $s(\alpha) = 0$ if $t(\alpha) = 0$. It is obvious that Hyp Lat (T) is totally ordered in this case also. The Proposition is proved.

Uchiyama [18] has shown that two quasisimilar operators of class $C_0(N)$ have isomorphic lattices of hyper-invariant subspaces. This result is also verified, as we shall see in sec. 4, for operators of class C_0 having property (P). The same thing is not true for arbitrary operators of class C_0 (cf. Example 2.10). However we can find a connection between Hyp Lat (T) and Hyp Lat (S) if S is the Jordan model of the C_0 operator T. This allows us to extend [18], Corollaries 2 and 5 to arbitrary operators of class C_0.

Theorem 2.5. *Let T be an operator of class C_0 acting on the Hilbert space $\mathfrak{H}$ and let $S=S(M)$ be the Jordan model of T. Let φ:* Hyp Lat $(S)\to$ Hyp Lat (T) *be defined by*

$$\varphi(\mathfrak{M}) = \bigvee_{X\in\mathscr{I}(T,S)} X\mathfrak{M} \tag{2.8}$$

and let ψ: Hyp Lat $(T)\to$ Hyp Lat (S),

ψ_*: Hyp Lat $(T^*)\to$ Hyp Lat (S^*)

be defined by analogous formulas.

(i) *There exist $Y\in\mathscr{I}(S,T)$ and $X\in\mathscr{I}(T,S)$ such that $\psi(\mathfrak{M})=(Y\mathfrak{M})^-=X^{-1}(\mathfrak{M})$, $\mathfrak{M}\in$* Hyp Lat (T). *In particular $S|\psi(\mathfrak{M})$ is unitarily equivalent to the Jordan model of $T|\mathfrak{M}$.*

(ii) $\psi\circ\varphi = \mathrm{id}_{\mathrm{Hyp\,Lat}\,(S)}$.

(iii) $\psi_*(\mathfrak{M}^\perp) = (\psi(\mathfrak{M}))^\perp$, $\mathfrak{M}\in$ Hyp Lat (T).

Proof. By [4], Theorem 3.4, there exists an almost-direct decomposition

$$\mathfrak{H} = \bigvee_{\alpha} \mathfrak{H}_\alpha, \quad \mathfrak{H}_\alpha\in\mathrm{Lat}_0(T), \tag{2.9}$$

such that $T|\mathfrak{H}_\alpha$ is quasisimilar to $S(m_\alpha)$ and $\mathfrak{H}_{\alpha+n}\perp\mathfrak{H}_{\beta+m}$ if α and β are different limit ordinals and m, $n<\omega$. If we put

$$\mathfrak{H}_\alpha^* = (\bigvee_{\beta\neq\alpha} \mathfrak{H}_\beta)^\perp\in\mathrm{Lat}(T^*) \tag{2.10}$$

we also have $\mathfrak{H}=\bigvee_\alpha \mathfrak{H}_\alpha^*$ by [4], Lemma 1.11; because

$$T_{\mathfrak{H}_\alpha^*}(P_{\mathfrak{H}_\alpha^*}|\mathfrak{H}_\alpha) = (P_{\mathfrak{H}_\alpha^*}|\mathfrak{H}_\alpha)(T|\mathfrak{H}_\alpha) \tag{2.11}$$

and obviously $P_{\mathfrak{H}_\alpha^*}|\mathfrak{H}_\alpha$ is a quasi-affinity, $T_{\mathfrak{H}_\alpha^*}$ is also quasisimilar to $S(m_\alpha)$. We choose quasi-affinities X_α: $\mathfrak{H}(m_\alpha)\to\mathfrak{H}_\alpha$, Y_α: $\mathfrak{H}_\alpha^*\to\mathfrak{H}(m_\alpha)$ such that $(T|\mathfrak{H}_\alpha)X_\alpha = X_\alpha S(m_\alpha)$ and $S(m_\alpha)Y_\alpha = Y_\alpha T_{\mathfrak{H}_\alpha^*}$ and moreover

$$\sum_{n<\omega} \|Y_{\alpha+n}\| \leqq 1, \quad \sum_{n<\omega} \|X_{\alpha+n}\| \leqq 1 \tag{2.12}$$

for each limit ordinal α. Then we can define quasi-affinities $X\in\mathscr{I}(T, S)$, $Y\in\mathscr{I}(S, T)$ by the formulas

(2.13)
$$Xh = \sum_{\alpha} X_\alpha h_\alpha,\; h = \bigoplus_{\alpha} h_\alpha \in \mathfrak{H}(M),$$
$$Yh = \bigoplus_{\alpha} J_\alpha Y_\alpha P_{\mathfrak{H}_\alpha^*} h, \quad h\in\mathfrak{H}.$$

Indeed, from (2.12) it follows that X and Y are bounded (of norm $\leqq 1$).

Let us remark that $Y_\alpha(P_{\mathfrak{H}_\alpha^*}|\mathfrak{H}_\alpha)X_\alpha\in\{S(m_\alpha)\}'$ is a quasi-affinity such that by Sarason's Theorem [10] we have

(2.14)
$$Y_\alpha(P_{\mathfrak{H}_\alpha^*}|\mathfrak{H}_\alpha)X_\alpha = u_\alpha(S(m_\alpha)), \quad u_\alpha\in H^\infty,\; u_\alpha\wedge m_\alpha = 1.$$

If $\mathfrak{M}\in \mathrm{Hyp\,Lat}\,(S)$ we obviously have $\psi(\varphi(\mathfrak{M}))\subset\mathfrak{M}$. Now, let $\mathfrak{M}$ be given by (2.2) and denote $\mathfrak{M}_\alpha = m_\alpha'' H^2\ominus m_\alpha H^2$. Then, by (2.14),

$$(YX\mathfrak{M})^- \supset (YX\mathfrak{M}_\alpha)^- = (YX_\alpha\mathfrak{M}_\alpha)^- = (Y_\alpha P_{\mathfrak{H}_\alpha^*} X_\alpha \mathfrak{M}_\alpha)^- =$$
$$= (u_\alpha(S(m_\alpha))\mathfrak{M}_\alpha)^- = \mathfrak{M}_\alpha \quad\text{and therefore}\quad \mathfrak{M} = (YX\mathfrak{M})^- \subset \psi(\varphi(\mathfrak{M}));$$

this proves (ii).

Let us consider the operators $R_{\beta\alpha}\in\{T\}'$ defined by

(2.15)
$$\begin{cases} R_{\beta\alpha} = X_\beta P_{\mathfrak{H}(m_\beta)} Y_\alpha P_{\mathfrak{H}_\alpha^*} & \text{if} \quad \alpha\leqq\beta, \\ R_{\beta\alpha} = X_\beta(m_\beta/m_\alpha) Y_\alpha P_{\mathfrak{H}_\alpha^*} & \text{if} \quad \alpha>\beta, \end{cases}$$

and let $A_{\beta\alpha}\in\{S\}'$ be defined by (2.4). Then, for $\alpha\leqq\beta$,

$$YR_{\beta\alpha} = J_\beta Y_\beta P_{\mathfrak{H}_\beta^*} X_\beta P_{\mathfrak{H}(m_\beta)} Y_\alpha P_{\mathfrak{H}_\alpha^*} =$$
$$= J_\beta u_\beta(S(m_\beta)) P_{\mathfrak{H}(m_\beta)} Y_\alpha P_{\mathfrak{H}_\alpha^*} =$$
$$= u_\beta(S) J_\beta P_{\mathfrak{H}(m_\beta)} \tilde{P}_{\mathfrak{H}(m_\alpha)} Y P_{\mathfrak{H}_\alpha^*} = u_\beta(S) A_{\beta\alpha} Y P_{\mathfrak{H}_\alpha^*}$$

and because $A_{\beta\alpha} Y P_{(\mathfrak{H}_\alpha^*)^\perp} = 0$ we obtain

(2.16)
$$YR_{\beta\alpha} = u_\beta(S) A_{\beta\alpha} Y$$

in this case. The relation (2.16) is proved analogously when $\alpha>\beta$. If $\mathfrak{N}\in\mathrm{Hyp\,Lat}\,(T)$ and $\mathfrak{M}=(Y\mathfrak{N})^-$ we infer from (2.16) $u_\beta(S)A_{\beta\alpha}\mathfrak{M}\subset\mathfrak{M}$. Because $u_\alpha\wedge m_\alpha = 1$ we infer by [3], Corollary 2.9, that $u_\alpha(S(m_\alpha))|(A_{\alpha\alpha}\mathfrak{M})^-$ is a quasi-affinity; therefore $\mathfrak{M}\supset(u_\alpha(S(m_\alpha))(A_{\alpha\alpha}\mathfrak{M})^-)^- = (A_{\alpha\alpha}\mathfrak{M})^- = (\tilde{P}_{\mathfrak{H}(m_\alpha)}\mathfrak{M})^-$. As in the proof of Proposition 2.1 it follows that $\mathfrak{M} = \bigoplus_\alpha \mathfrak{M}_\alpha$, $\mathfrak{M}_\alpha = m_\alpha'' H^2 \ominus m_\alpha H^2 \in \mathrm{Lat}\,(S(m_\alpha))$ and for $\alpha<\beta$, $u_\beta m_\alpha'' \in m_\beta'' H^2$ and $u_\alpha(m_\alpha/m_\beta) m_\beta'' \in m_\alpha'' H^2$. Because $u_\alpha\wedge m_\alpha = 1$, $u_\beta\wedge m_\beta = 1$ we also have $u_\alpha\wedge m_\alpha'' = 1$, $u_\beta\wedge m_\beta'' = 1$ so that from the preceding relations we infer $m_\alpha''\in m_\beta'' H^2$, respectively $(m_\alpha/m_\beta)m_\beta''\in m_\alpha'' H^2$. By Proposition 2.1 we proved

(2.17)
$$(Y\mathfrak{N})^-\in\mathrm{Hyp\,Lat}\,(S) \quad\text{whenever}\quad \mathfrak{N}\in\mathrm{Hyp\,Lat}\,(T).$$

Analogously we infer

(2.17)* $\quad (X^*\mathfrak{N})^- \in \operatorname{Hyp} \operatorname{Lat}(S^*)$ whenever $\mathfrak{N} \in \operatorname{Hyp} \operatorname{Lat}(T^*)$.

If $\mathfrak{N} \in \operatorname{Hyp} \operatorname{Lat}(T)$ we have $X^*(\mathfrak{N}^\perp) \subset (Y\mathfrak{N})^\perp$. Indeed, if $h \in \mathfrak{N}$, $g \in \mathfrak{N}^\perp$, we have $(Yh, X^*g) = (XYh, g) = 0$ because $XYh \in \mathfrak{N}$. An analogous argument shows that

(2.18) $\quad \psi_*(\mathfrak{N}^\perp) \subset (\psi(\mathfrak{N}))^\perp, \quad \mathfrak{N} \in \operatorname{Hyp} \operatorname{Lat}(T).$

In particular we have

$$T^*|\mathfrak{N}^\perp \prec S^*|(X^*\mathfrak{N}^\perp)^- \overset{i}{\prec} S^*|\psi_*(\mathfrak{N}^\perp) \overset{i}{\prec} S^*|(\psi(\mathfrak{N}))^\perp \overset{i}{\prec} S^*|(Y\mathfrak{N})^\perp.$$

Because $P_{(Y\mathfrak{N})^\perp} Y|\mathfrak{N}^\perp$ has dense range and $S_{(Y\mathfrak{N})^\perp}(P_{(Y\mathfrak{N})^\perp} Y|\mathfrak{N}^\perp) = (P_{(Y\mathfrak{N})^\perp} Y|\mathfrak{N}^\perp) T_{\mathfrak{N}^\perp}$ it follows that $S^*|(Y\mathfrak{N})^\perp \overset{i}{\prec} T^*|\mathfrak{N}^\perp$. By [16], Theorem 1 (cf. also [4], Corollary 2.10) the operators $T^*|\mathfrak{N}^\perp$, $S^*|(X^*\mathfrak{N}^\perp)^-$, $S^*|\psi_*(\mathfrak{N}^\perp)$, $S^*|(\psi(\mathfrak{N}))^\perp$ and $S^*|(Y\mathfrak{N})^\perp$ are pairwise quasisimilar. Because S^* is also (unitarily equivalent to) a Jordan operator it follows by Proposition 2.1 (iii) that $(X^*\mathfrak{N}^\perp)^- = \psi_*(\mathfrak{N}^\perp) = (\psi(\mathfrak{N}))^\perp = (Y\mathfrak{N})^\perp$. This proves the assertions (i) and (iii) of the Theorem.

The following Corollary extends [18], Corollary 5, to arbitrary operators of class C_0.

Corollary 2.6. *If T is an operator of class C_0 on $\mathfrak{H}$ and* $T = \begin{bmatrix} T' & X \\ 0 & T'' \end{bmatrix}$ *is the triangularization of T with respect to the decomposition $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{M}^\perp$, $\mathfrak{M} \in \operatorname{Hyp} \operatorname{Lat}(T)$, we have*

(2.19) $$m_T = m_{T'} m_{T''}.$$

Proof. If ψ is as in Theorem 2.5, T' is quasisimilar to $S|\psi(\mathfrak{M})$ and T'' is quasisimilar to $S_{(\psi(\mathfrak{M}))^\perp}$. The Corollary follows by Proposition 2.1 (ii).

Corollary 2.7. *Let T and T' be two quasisimilar operators of class C_0, let S be their Jordan model and let* η: Hyp Lat $(T) \to$ Hyp Lat (T'), ψ: Hyp Lat $(T) \to$ $\to$ Hyp Lat (S), ψ': Hyp Lat $(T') \to$ Hyp Lat (S) *be defined by formulas analogous to* (2.8).

(i) $\psi' \circ \eta = \psi$; *in particular* $T|\mathfrak{M}$ *and* $T'|\eta(\mathfrak{M})$ *are quasisimilar for* $\mathfrak{M} \in \operatorname{Hyp} \operatorname{Lat}(T)$.

(ii) *If* $\mathfrak{M} \in \operatorname{Hyp} \operatorname{Lat}(T)$, $\mathfrak{M}' \in \operatorname{Hyp} \operatorname{Lat}(T')$ *are such that* $T|\mathfrak{M}$ *and* $T'|\mathfrak{M}'$ *are quasisimilar, then* $T_{\mathfrak{M}^\perp}$ *and* $T'_{\mathfrak{M}'^\perp}$ *are also quasisimilar.*

Proof. The inclusion $(\psi' \circ \eta)(\mathfrak{M}) \subset \psi(\mathfrak{M})$ is obvious for $\mathfrak{M} \in \operatorname{Hyp} \operatorname{Lat}(T)$. Then by Theorem 2.5 (i) we infer $T|\mathfrak{M} \overset{i}{\prec} S|(\psi' \circ \eta)(\mathfrak{M}) \overset{i}{\prec} S|\psi(\mathfrak{M}) \prec T|\mathfrak{M}$. By [16], Theorem 1, $T|\mathfrak{M}$, $S|(\psi' \circ \eta)(\mathfrak{M})$, $S|\psi(\mathfrak{M})$ are pairwise quasisimilar and the equality $\psi' \circ \eta = \psi$ follows by Proposition 2.1 (iii). Now it is obvious by Theorem 2.5 (i) that $T|\mathfrak{M}$ and $T'|\eta(\mathfrak{M})$ are both quasisimilar to $S|\psi(\mathfrak{M})$; (i) follows.

To prove (ii) we remark that, by Theorem 2.5 (i), $S|\psi(\mathfrak{M})$ and $S|\psi'(\mathfrak{M}')$ are quasisimilar and therefore $\psi(\mathfrak{M})=\psi'(\mathfrak{M}')$ by Proposition 2.1 (iii). Again by Theorem 2.5 it follows that $T_{\mathfrak{M}^\perp}$ and $T'_{\mathfrak{M}'^\perp}$ are both quasisimilar to $S_{\mathfrak{N}^\perp}$ where $\mathfrak{N}=\psi(\mathfrak{M})=$ $=\psi'(\mathfrak{M}')$. Corollary follows.

Corollary 2.8. *Let T, S, φ, ψ be as in Theorem 2.5 and let* φ_*: Hyp Lat $(S^*)\to$ $\to$Hyp Lat (T^*) *be defined by a formula analogous to* (2.8). *Among the spaces* $\mathfrak{N}\in$Hyp Lat (T) *such that* $T|\mathfrak{N}$ *is quasisimilar to* $S|\mathfrak{M}$ *for a given* $\mathfrak{M}\in$Hyp Lat (S), $\varphi(\mathfrak{M})$ *is the least one and* $(\varphi^*(\mathfrak{M}^\perp))^\perp$ *is the greatest one.*

Proof. If $T|\mathfrak{N}$ is quasisimilar to $S|\mathfrak{M}$ we have $\psi(\mathfrak{N})=\mathfrak{M}$ by Theorem 2.5 (i) and Proposition 2.1 (iii) and therefore $\varphi(\mathfrak{M})=\varphi(\psi(\mathfrak{N}))\subset\mathfrak{N}$. Now, by Corollary 2.7, $T|\mathfrak{N}$ and $S|\mathfrak{M}$ are quasisimilar if and only if $T_{\mathfrak{N}^\perp}$ and $S_{\mathfrak{M}^\perp}$ are quasisimilar. Because $\varphi_*(\mathfrak{M}^\perp)$ is the least hyper-invariant subspace of T^* such that $T_{\varphi_*(\mathfrak{M}^\perp)}$ and $S_{\mathfrak{M}^\perp}$ are quasisimilar, the last assertion of the Corollary follows.

Corollary 2.9. *Let T, S, ψ, φ, φ_* be as before. The following assertions are equivalent:*

(i) *φ is a bijection;*

(ii) *φ_* is a bijection;*

(iii) $\varphi(\mathfrak{M})^\perp=\varphi_*(\mathfrak{M}^\perp)$ *for* $\mathfrak{M}\in$Hyp Lat (S);

(iv) *if* $\mathfrak{N}_1, \mathfrak{N}_2\in$Hyp Lat (T) *and* $T|\mathfrak{N}_1$, $T|\mathfrak{N}_2$ *are quasisimilar, we have* $\mathfrak{N}_1=\mathfrak{N}_2$.

Proof. By Theorem 2.5 (ii) φ is a bijection if and only if ψ is one-to-one. By Theorem 2.5 (i) and Proposition 2.1 (iii) ψ is one-to-one if and only (iv) holds. Thus the equivalence (i)$\Leftrightarrow$(iv) is established.

By Theorem 2.5 (iii) we have $\psi_*(\mathfrak{M}^\perp)=\psi(\mathfrak{M})^\perp$ so that ψ is one-to-one if and only if ψ_* is one-to-one. This establishes the equivalence (i)$\Leftrightarrow$(ii).

$T|\varphi(\mathfrak{M})$ and $T|(\varphi_*(\mathfrak{M}^\perp))^\perp$ are both quasisimilar to $S|\mathfrak{M}$ so that $\varphi(\mathfrak{M})=$ $=(\varphi_*(\mathfrak{M}^\perp))^\perp$ if (iv) holds. Conversely, if (iii) holds and $T|\mathfrak{N}_1$, $T|\mathfrak{N}_2$ are quasisimilar, by the preceding Corollary we have $\varphi(\mathfrak{M})\subset\mathfrak{N}_j\subset(\varphi_*(\mathfrak{M}^\perp))^\perp=\varphi(\mathfrak{M})$, $j=1, 2$, where $\mathfrak{M}=\psi(\mathfrak{N}_1)=\psi(\mathfrak{N}_2)$. Thus $\mathfrak{N}_1=\mathfrak{N}_2=\varphi(\mathfrak{M})$ and the Corollary is proved.

Example 2.10. Let $S=S(m^2)^{(\aleph_0)}$ and $T=S\oplus S(m)$, where $m\in H_i^\infty$ and $S(m^2)^{(\aleph_0)}$ denotes the direct sum of $\aleph_0$ copies of $S(m^2)$. By [2], Corollary 1, S is the Jordan model of T. The subspaces $\ker m(T)$, $\operatorname{ran} m(T)$ are hyper-invariant for T and $T|\ker m(T)$, $T|\operatorname{ran} m(T)$ are both quasisimilar to $S(m)^{(\aleph_0)}$. By Corollary 2.9 it follows that in this case φ is not onto, ψ is not one-to-one.

If we take in particular $m(z)=z^2$ ($|z|<1$) it is easily seen that card (Hyp Lat (T))$=9$ and card (Hyp Lat (S))$=5$. Thus Hyp Lat (T) and Hyp Lat (S) are not isomorphic. Moreover, one can verify, by the proof of Proposition 2.4, that Hyp Lat (T) is not totally ordered while Hyp Lat (S) is totally ordered.

3. A theorem on monotonic sequences of invariant subspaces

If T is an operator of class C_0 acting on $\mathfrak{H}$ and $\mathfrak{H}_j \in \mathrm{Lat}(T)$ are such that $\mathfrak{H}_j \subset \mathfrak{H}_{j+1}$, $j=0, 1, \dots$, and $\mathfrak{H} = \bigvee_{j \geq 0} \mathfrak{H}_j$, it is clear that m_T is the least common inner multiple of the functions $m_{T|\mathfrak{H}_j}$, $j=0, 1, \dots$. The following Theorem shows that the same thing is verified for all the functions appearing in the Jordan model of T.

Theorem 3.1. *Let T be an operator of class C_0 acting on the Hilbert space $\mathfrak{H}$ and let $\{\mathfrak{H}_j\}_{j=0}^{\infty} \subset \mathrm{Lat}(T)$ be such that $\mathfrak{H}_j \subset \mathfrak{H}_{j+1}$, $0 \leqq j < \infty$, and $\mathfrak{H} = \bigvee_{j \geq 0} \mathfrak{H}_j$. Then*

$$m_\alpha[T] = \bigvee_{j \geq 1} m_\alpha[T|\mathfrak{H}_j] \tag{3.1}$$

for each ordinal number α.

Proof. Because $T|\mathfrak{H}_j \overset{i}{\prec} T|\mathfrak{H}_{j+1} \overset{i}{\prec} T$ it follows that $m_\alpha[T|\mathfrak{H}_j] \leqq m_\alpha[T|\mathfrak{H}_{j+1}] \leqq m_\alpha[T]$ for each α (cf. [4], Corollary 2.9). Let us consider firstly the case $\alpha \geqq \omega$ and denote $m = \bigvee_{j \geq 0} m_\alpha[T|\mathfrak{H}_j]$; then m divides $m_\alpha[T]$. Because $m_\alpha[T|\mathfrak{H}_j]$ divides m we have $\mu_{T|(m(T)\mathfrak{H}_j)^-} = \mu_{T|\mathfrak{H}_j}(m) \leqq \bar{\alpha}$ (cf. [4], Remark 2.12). Because obviously $(m(T)\mathfrak{H})^- = \bigvee_{j \geq 0} m(T)\mathfrak{H}_j$ we infer $\mu_T(m) = \mu_{T|(m(T)\mathfrak{H})^-} \leqq \aleph_0 \cdot \bar{\alpha} = \bar{\alpha}$ and therefore $m_\alpha[T]$ divides m by [4], Definition 2.4. Thus $m_\alpha[T] = m$ and (3.1) is proved for $\alpha \geqq \omega$.

Now let us recall that by [4], Theorem 3.3, there exists an orthogonal decomposition

$$\mathfrak{H} = \bigoplus_{\alpha} \mathfrak{M}_\alpha, \quad \mathfrak{M}_\alpha \in \mathrm{Lat}(T), \tag{3.2}$$

such that $T|\mathfrak{M}_\alpha$ is quasisimilar to $\bigoplus_{j<\omega} S(m_{\alpha+j}[T])$ for each limit ordinal α. If we define $\mathfrak{K}_j = (P_{\mathfrak{M}_0} \mathfrak{H}_j)^-$ we obviously have $\mathfrak{M}_0 = \bigvee_{j \geq 0} \mathfrak{K}_j$ and $T^*_{\mathfrak{K}_j} \overset{i}{\prec} T^*_{\mathfrak{H}_j}$ so that $T|\mathfrak{K}_j \overset{i}{\prec} T|\mathfrak{H}_j$ by [4], Corollary 2.9. Again by [4], Corollary 2.9 we infer $m_\alpha[T|\mathfrak{K}_j] \leqq m_\alpha[T|\mathfrak{H}_j]$, $\alpha < \omega$, and therefore it will be enough to prove the relation (3.1) for $\mathfrak{H} = \mathfrak{M}_0$ and $\mathfrak{H}_j = \mathfrak{K}_j$, that is for T acting on a separable space.

We may assume that T is a functional model, that is

$$\mathfrak{H} = \mathfrak{H}(\Theta) = H^2(\mathfrak{U}) \ominus \Theta H^2(\mathfrak{U}) \tag{3.3}$$

where $\mathfrak{U}$ is a separable Hilbert space, Θ is a two-sided inner function, $\Theta \in H_i^\infty(\mathscr{L}(\mathfrak{U}))$, and

$$Th = S(\Theta)h = P_{\mathfrak{H}(\Theta)} \chi h, \ \chi(z) = z, \quad h \in \mathfrak{H}(\Theta). \tag{3.4}$$

With each subspace $\mathfrak{H}_j$ we can associate by [12], Theorem VII.1.1 a factorisation

$$\Theta = \Theta_j^{(2)} \Theta_j^{(1)} \tag{3.5}$$

such that the functions $\Theta_j^{(1)}$ and $\Theta_j^{(2)}$ are two-sided inner,

$$\mathfrak{H}_j = \Theta_j^{(2)} H^2(\mathfrak{U}) \ominus \Theta H^2(\mathfrak{U}), \tag{3.6}$$

and $T|\mathfrak{H}_j$ is unitarily equivalent to $S(\Theta_j^{(1)})$. The inclusion $\mathfrak{H}_j \subset \mathfrak{H}_{j+1}$ is equivalent to $\Theta_j^{(2)} H^2(\mathfrak{U}) \subset \Theta_{j+1}^{(2)} H^2(\mathfrak{U})$ and therefore

$$\Theta_j^{(2)} = \Theta_{j+1}^{(2)} \Omega_j \quad \text{for some} \quad \Omega_j \in H_i^\infty(\mathscr{L}(\mathfrak{U})). \tag{3.7}$$

The condition $\mathfrak{H} = \bigvee_{j \geq 0} \mathfrak{H}_j$ is equivalent to $H^2(\mathfrak{U}) = \bigvee_{j \geq 0} \Theta_j^{(2)} H^2(\mathfrak{U})$. In particular, if $u \in \mathfrak{U}$, we have $\lim_{j \to \infty} \|u - P_{\Theta_j^{(2)} H^2(\mathfrak{U})} u\| = 0$. It is easily seen that $P_{\Theta_j^{(2)} H^2(\mathfrak{U})} u = \Theta_j^{(2)} \Theta_j^{(2)}(0)^* u$. Indeed, it is enough to verify that the scalar product

$$\left(u - \Theta_j^{(2)}(z) \Theta_j^{(2)}(0)^* u, \ \Theta_j^{(2)}(z) z^n v\right)$$

vanishes for $v \in \mathfrak{U}$ and natural n; this is a simple computation. Thus we have $u = \lim_{j \to \infty} \Theta_j^{(2)} \Theta_j^{(2)}(0)^* u$, $u \in \mathfrak{U}$. Because the functions $\Theta_j^{(2)} \Theta_j^{(2)}(0)^* u$ are uniformly bounded we also have $u_1 \wedge u_2 \wedge \ldots \wedge u_n = \lim_{j \to \infty} (\Theta_j^{(2)})^{\wedge n} (\Theta_j^{(2)}(0)^*)^{\wedge n} (u_1 \wedge \ldots \wedge u_n)$, $u_1, u_2, \ldots, u_n \in \mathfrak{U}$, and therefore

$$\bigvee_{j \geq 0} (\Theta_j^{(2)})^{\wedge n} H^2(\mathfrak{U}^{\wedge n}) \supset \mathfrak{U}^{\wedge n}.$$

Because $\bigvee_{j \geq 0} (\Theta_j^{(2)})^{\wedge n} H^2(\mathfrak{U}^{\wedge n})$ is invariant with respect to the unilateral shift on $H^2(\mathfrak{U}^{\wedge n})$ we necessarily have

$$H^2(\mathfrak{U}^{\wedge n}) = \bigvee_{j \geq 0} (\Theta_j^{(2)})^{\wedge n} H^2(\mathfrak{U}^{\wedge n}). \tag{3.8}$$

The subspaces

$$\mathfrak{H}_j^n = (\Theta_j^{(2)})^{\wedge n} H^2(\mathfrak{U}^{\wedge n}) \ominus \Theta^{\wedge n} H^2(\mathfrak{U}^{\wedge n}) \tag{3.9}$$

are invariant with respect to $S(\Theta^{\wedge n})$ and because $\Theta^{\wedge n} = (\Theta_j^{(2)})^{\wedge n} (\Theta_j^{(1)})^{\wedge n}$ is a regular factorization, $S(\Theta^{\wedge n})|\mathfrak{H}_j^n$ is unitarily equivalent to $S((\Theta_j^{(1)})^{\wedge n})$. By (3.7) we have $(\Theta_j^{(2)})^{\wedge n} = (\Theta_{j+1}^{(2)})^{\wedge n} \Omega_j^{\wedge n}$ and therefore $\mathfrak{H}_j^n \subset \mathfrak{H}_{j+1}^n$ for $0 \leq j < \infty$. Finally, relation (3.8) shows that $\mathfrak{H}(\Theta^{\wedge n}) = \bigvee_{j \geq 0} \mathfrak{H}_j^n$ and therefore

$$m_0[S(\Theta^{\wedge n})] = \bigvee_{j \geq 0} m_0[S(\Theta^{\wedge n})|\mathfrak{H}_j^n]. \tag{3.10}$$

By [6], Corollary 3.3, and relation (2.5) we have $m_0[S(\Theta^{\wedge n})] = m_0[T] m_1[T] \ldots m_{n-1}[T]$ and $m_0[S(\Theta^{\wedge n})|\mathfrak{H}_j^n] = m_0[S((\Theta_j^{(1)})^{\wedge n})] = m_0[T|\mathfrak{H}_j] m_1[T|\mathfrak{H}_j] \ldots m_{n-1}[T|\mathfrak{H}_j]$. Let us put $m_k = \bigvee_{j \geq 0} m_k[T|\mathfrak{H}_j]$ for $k < \omega$; then m_k divides $m_k[T]$ and relation (3.10) shows that

$$m_0[T] m_1[T] \ldots m_{n-1}[T] = m_0 m_1 \ldots m_{n-1}, \quad 1 \leq n < \omega.$$

Therefore we have necessarily $m_k[T]=m_k$ and (3.1) is proved for $\alpha<\omega$. The Theorem follows.

Remark 3.2. The relation (3.1) is not verified if the sequence $\{\mathfrak{H}_j\}_{j=0}^{\infty}$ is replaced by an arbitrary totally ordered family of invariant subspaces. Indeed, let us take a Jordan operator $T=S(M)$ such that $m_\Omega=1$, where Ω denotes the first uncountable ordinal number. The subspaces $\mathfrak{H}_\alpha=\bigoplus_{\beta<\alpha}\mathfrak{H}(m_\beta)$ for $\alpha<\Omega$ are separable and $\mathfrak{H}(M)=\bigvee_{\alpha<\Omega}\mathfrak{H}_\alpha$. The relation (3.1) is not verified in this case because $m_\omega[T|\mathfrak{H}_\alpha]=1$ while it is possible to have $m_\omega[T]\neq 1$. However the relation (3.1) is verified for $\alpha<\omega$ and for any totally ordered family $\{\mathfrak{H}_j\}_{j\in J}$ of invariant subspaces such that $\mathfrak{H}=\bigvee_{j\in J}\mathfrak{H}_j$. Indeed, if $\mathfrak{H}$ is separable we can select an increasing sequence $\{\mathfrak{H}_{j_n}\}_{n=0}^{\infty}$ such that $\mathfrak{H}=\bigvee_{n\geqq 0}\mathfrak{H}_{j_n}$ and then apply Theorem 3.1. If $\mathfrak{H}$ is not separable, the proof of Theorem 3.1 shows how to reduce the problem of verifying (3.1) to the separable case.

Let us recall that for a contraction T of class C_0 and for a subspace $\mathfrak{M}\in\mathrm{Lat}_{\frac{1}{2}}(T)$ such that $T_{\mathfrak{M}}$ is a weak contraction, $d_T(\mathfrak{M})$ denotes the determinant function of $T_{\mathfrak{M}}$ (cf. [3], Definition 1.1).

Corollary 3.3. *Let T be a weak contraction of class C_0 acting on $\mathfrak{H}$ and let $\mathfrak{H}_j\in\mathrm{Lat}(T)$, $0\leqq j<\infty$.*

(i) *If $\mathfrak{H}_j\subset\mathfrak{H}_{j+1}$ and $\bigvee_{j\geqq 0}\mathfrak{H}_j=\mathfrak{H}$, we have $d_T=\bigvee_{j\geqq 0}d_T(\mathfrak{H}_j)$.*

(ii) *If $\mathfrak{H}_j\supset\mathfrak{H}_{j+1}$ and $\bigcap_{j\geqq 0}\mathfrak{H}_j=\{0\}$, we have $\bigwedge_{j\geqq 0}d_T(\mathfrak{H}_j)=1$.*

Proof. (i) Obviously $\bigvee_{j\geqq 0}d_T(\mathfrak{H}_j)$ divides d_T. Now, $m_0[T|\mathfrak{H}_j]m_1[T|\mathfrak{H}_j]\dots m_n[T|\mathfrak{H}_j]$ divides $\bigvee_{j\geqq 0}d_T(\mathfrak{H}_j)$ for every natural n; by Theorem 3.1 it follows that $m_0[T]m_1[T]\dots$ $\dots m_n[T]$ divides $\bigvee_{j\geqq 0}d_T(\mathfrak{H}_j)$ and therefore d_T divides $\bigvee_{j\geqq 0}d_T(\mathfrak{H}_j)$.

(ii) Since T^* is also a weak contraction we infer by (i) $d_T=\bigvee_{j\geqq 0}d_T(\mathfrak{H}_j^{\perp})$. Because $d_T=d_T(\mathfrak{H}_j)d_T(\mathfrak{H}_j^{\perp})$ (cf. [6], Proposition 8.2) we obtain

$$d_T=\Big(\bigwedge_{j\geqq 0}d_T(\mathfrak{H}_j)\Big)\cdot\Big(\bigvee_{j\geqq 0}d_T(\mathfrak{H}_j^{\perp})\Big)=\Big(\bigwedge_{j\geqq 0}d_T(\mathfrak{H}_j)\Big)\cdot d_T.$$

The Corollary follows.

Proposition 3.4. *Let T be an operator of class C_0 acting on the separable Hilbert space $\mathfrak{H}$. Then*

$$\bigwedge_{j<\omega}m_j[T]=1 \tag{3.11}$$

if and only if for any sequence $\{\mathfrak{H}_j\}_{j=0}^{\infty}\subset \mathrm{Lat}\,(T)$ *such that* $\mathfrak{H}_j\supset\mathfrak{H}_{j+1}$ *and* $\bigcap_{j\geq 0}\mathfrak{H}_j=\{0\}$, *we have*

$$\bigwedge_{j\geq 0} m_0[T|\mathfrak{H}_j]=1. \tag{3.12}$$

Proof. As shown in the proof of [5], Theorem 1, there exists a decreasing sequence $\{\mathfrak{H}_j\}_{j=0}^{\infty}\subset \mathrm{Lat}\,(T)$ such that $\bigcap_{j\geq 0}\mathfrak{H}_j=\{0\}$ and $m_0[T|\mathfrak{H}_j]=m_j[T]$ so that (3.11) follows from (3.12).

Conversely, let us assume (3.11) holds. For any natural number k we have the decomposition

$$\mathfrak{H}_j=(m_k(T)\mathfrak{H}_j)^-\oplus\mathfrak{N}_j^k=\mathfrak{M}_j^k\oplus\mathfrak{N}_j^k,\quad m_k=m_k[T].$$

Because obviously $m_0[T_{\mathfrak{N}_j^k}]$ divides m_k, it follows by [12], Proposition III.6.1, that

$$m_0[T|\mathfrak{H}_j]\ \text{ divides }\ m_0[T|\mathfrak{M}_j^k]\cdot m_k,\quad 0\leq j<\infty. \tag{3.13}$$

Now, $\mathfrak{M}_j^k\subset(m_k(T)\mathfrak{H})^-$ and $T|(m_k(T)\mathfrak{H})^-$ is an operator of finite multiplicity, in particular a weak contraction (cf. [6], Theorem 8.5). Because $\bigcap_{j\geq 0}\mathfrak{M}_j^k\subset\bigcap_{j\geq 0}\mathfrak{H}_j=\{0\}$ we infer by the preceding Corollary $\bigwedge_{j\geq 0} d_T(\mathfrak{M}_j^k)=1$, in particular $\bigwedge_{j\geq 0} m_0[T|\mathfrak{M}_j^k]=1$. By (3.13) $\bigwedge_{j\geq 0} m_0[T|\mathfrak{H}_j]$ necessarily divides m_k and the relation (3.12) follows from the assumption. The Proposition is proved.

4. Operators of class C_0 having property (P)

In [16], Theorem 2, the operators of class C_0 and of finite multiplicity were shown to have property (P). In [3], Corollary 2.8 we extended this result to the class of weak contractions of class C_0. We are now going to characterise the class of C_0 operators having property (P).

Theorem 4.1. *Let T be an operator of class C_0 acting on the Hilbert space $\mathfrak{H}$. Then T has property* (P) *if and only if*

$$\bigwedge_{j<\omega} m_j[T]=1. \tag{4.1}$$

In particular, if T has property (P), *$\mathfrak{H}$ is separable and T^* also has property* (P).

Proof. Let us assume (4.1) holds and denote $m_j=m_j[T]$. For each $j<\omega$ the subspace

$$\mathfrak{H}_j=(m_j(T)\mathfrak{H})^- \tag{4.2}$$

is hyper-invariant for T and $\mu_T(\mathfrak{H}_j)<\infty$ (cf. [4], Remark 2.12). If $A\in\{T\}'$ is an injection then $A|\mathfrak{H}_j\in\{T|\mathfrak{H}_j\}'$ is also an injection and by [16], Theorem 2,

(4.3) $$(A\mathfrak{H})^- \supset (A\mathfrak{H}_j)^- = \mathfrak{H}_j.$$

We have $(\bigvee_{j<\omega} \mathfrak{H}_j)^\perp = \bigcap_{j<\omega} \ker m_j^\sim(T^*) = \mathfrak{H}^0$ and the minimal function m^0 of $T^*|\mathfrak{H}^0$ divides $m_j^\sim$, $j<\omega$. By the assumption we infer $m^0=1$ so that $\mathfrak{H}^0=\{0\}$ and therefore $\bigvee_{j<\omega} \mathfrak{H}_j=\mathfrak{H}$. From (4.3) we infer

(4.4) $$(A\mathfrak{H})^- \supset \bigvee_{j<\omega} \mathfrak{H}_j = \mathfrak{H}$$

that is, A is a quasi-affinity. The injection A being arbitrary it follows that T has property (P).

Conversely, let us assume that (4.1) does not hold. We claim that there exist an inner function m such that T and $T\oplus S(m)$ are quasisimilar. If $\mathfrak{H}$ is separable we may take $m=\bigwedge_{j<\omega} m_j[T]$ and apply [1], Lemma 3. If $\mathfrak{H}$ is nonseparable we may take $m=m_\omega[T]$. Then $T\oplus S(m)$ and T have the same Jordan model so that they are quasisimilar. Let us take a quasi-affinity X such that

(4.5) $$(T\oplus S(m))X = XT.$$

Let us put

(4.6) $$\mathfrak{M} = (X^*(\{0\}\oplus\mathfrak{H}(m)))^-, \quad \mathfrak{N} = \mathfrak{H}\ominus\mathfrak{M}.$$

Then $\mathfrak{M}\in\operatorname{Lat}(T^*)$ and $T^*|\mathfrak{M}$ is quasisimilar to $S(m)^*$. If P_1 and P_2 denote the orthogonal projections of $\mathfrak{H}\oplus\mathfrak{H}(m)$ onto $\mathfrak{H}$, $\mathfrak{H}(m)$, respectively, the operator

(4.7) $$Y = P_1X|\mathfrak{N}$$

satisfies the relation

(4.8) $$TY = Y(T|\mathfrak{N}).$$

We claim that Y is a quasi-affinity. We show firstly that ran Y^* is dense in $\mathfrak{N}$. Indeed, because $P_\mathfrak{N}X^*|\{0\}\oplus\mathfrak{H}(m)=0$ (by the definition (4.6) of $\mathfrak{M}$ and $\mathfrak{N}$), we have

(4.9) $$\operatorname{ran} Y^* = P_\mathfrak{N}X^*(\mathfrak{H}\oplus\{0\}) = P_\mathfrak{N}X^*(\mathfrak{H}\oplus\mathfrak{H}(m))$$

which shows that

(4.10) $$(\operatorname{ran} Y^*)^- = (P_\mathfrak{N}(\operatorname{ran} X^*)^-)^- = P_\mathfrak{N}\mathfrak{H} = \mathfrak{N}.$$

Now let us show that $\ker Y^*=\{0\}$. To do this let us remark that the subspace

(4.11) $$\mathfrak{K} = \ker Y^*\oplus\mathfrak{H}(m) = \{u\in\mathfrak{H}\oplus\mathfrak{H}(m);\ X^*u\in\mathfrak{M}\}$$

is invariant with respect to $(T\oplus S(m))^*$, $(X^*\mathfrak{K})^-=\mathfrak{M}$ and $(T^*|\mathfrak{M})X^*= =X^*(T\oplus S(m))^*|\mathfrak{K}$ so that $T^*|\mathfrak{M}$ and $(T\oplus S(m))^*|\mathfrak{K}$ are quasisimilar. By the remark following relation (4.6), $(T\oplus S(m))^*|\mathfrak{K}$ is quasisimilar to $S(m)^*$. But

$(T\oplus S(m))^*|\{0\}\oplus\mathfrak{H}(m)$ is unitarily equivalent to $S(m)^*$ so that $\mathfrak{R}=\{0\}\oplus\mathfrak{H}(m)$ by [14], Theorem 2, and the injectivity of Y^* is proved. Relation (4.8) and Lemma 1.1 show that T and $T|\mathfrak{N}$ are quasisimilar. Because $\mathfrak{M}\neq\{0\}$, we have $\mathfrak{N}\neq\mathfrak{H}$ so that T does not have property (P) by Lemma 1.5.

Theorem is proved.

Corollary 4.2. *An operator T of class C_0 has property* (P) *if and only if there does not exist T' of class C_0 on a nontrivial Hilbert space such that T and $T\oplus T'$ are quasisimilar.*

Proof. Let T and $T\oplus T'$ be quasisimilar. Since T' acts on a nontrivial space, there exists a nonconstant inner function m such that $T\oplus S(m)\overset{i}{\prec}T$. Because obviously $T\overset{i}{\prec}T\oplus S(m)$, $T\oplus S(m)$ and T are quasisimilar by [16], Theorem 1. By the proof of Theorem 4.1 it follows that T does not have the property (P). The converse assertion of the Corollary follows from the proof of Theorem 4.1.

Corollary 4.3. *If T and T' are two quasisimilar operators of class C_0, then T has property* (P) *if and only if T' has property* (P).

Proof. Theorem 4.1 exprimes the property (P) in terms of the Jordan model so that the Corollary is obvious.

Proposition 4.4. *Let $T=\begin{bmatrix}T' & X\\ 0 & T''\end{bmatrix}$ be the triangularization of the operator T of class C_0 with respect to the decomposition $\mathfrak{H}=\mathfrak{H}'\oplus\mathfrak{H}''$, $\mathfrak{H}'\in\operatorname{Lat}(T)$. Then T has property* (P) *if and only if T' and T'' have property* (P).

Proof. Let $S(M)$, $S(M')$, $S(M'')$ be the Jordan models of T, T', T'', respectively. Let us assume that T has property (P). Because $S(M')\overset{i}{\prec}S(M)$ it follows that m'_α divides m_α for each α (cf. [4], Corollary 2.9), therefore by Theorem 4.1 we have $\bigwedge_{j<\omega} m'_j=1$ and T' has property (P). Analogously T''^* has property (P) because T^* has property (P) and it follows by Theorem 4.1 that T'' also has property (P).

Conversely, let us assume that T' and T'' have property (P) so that

$$\bigwedge_{j<\omega} m'_j=\bigwedge_{j<\omega} m''_j=1. \tag{4.12}$$

We consider firstly the case $\mu_{T'}<\infty$. In this case the space

$$\mathfrak{H}_j=(m''_j(T)\mathfrak{H})^-\in\operatorname{Hyp}\operatorname{Lat}(T),\quad j<\omega, \tag{4.13}$$

is contained in $\mathfrak{H}'\oplus(m''_j(T'')\mathfrak{H}'')^-$ so that $\mu_T(\mathfrak{H}_j)<\infty$ and by [16], Theorem 2, $T|\mathfrak{H}_j$ has property (P). Because $\bigwedge_{j<\omega} m''_j=1$ we have $\bigvee_{j<\omega}\mathfrak{H}_j=\mathfrak{H}$ (cf. the proof of

Theorem 4.1) and the first part of the proof of Theorem 4.1 shows that T has property (P).

Considering the operator T^* instead of T, it follows that T has property (P) in the case $\mu_{T''}<\infty$ also.

We are now considering the general case $\mu_{T'}=\mu_{T''}=\aleph_0$. Let us define the hyperinvariant subspaces $\mathfrak{H}_j$ by (4.13). The operator $T|\mathfrak{H}'\oplus(m_j''(T'')\mathfrak{H}'')^-$ has property (P) because $\mu_{T''|(m_j''(T'')\mathfrak{H}'')^-}<\infty$ and from the first part of the proof of our Proposition it follows that $T|\mathfrak{H}_j$ also has the property (P). Because $\bigvee_{j<\omega}\mathfrak{H}_j=\mathfrak{H}$ we infer as in the first part of the proof of Theorem 4.1 that T has property (P). The proposition is proved.

Corollary 4.5. *If T is an operator of class C_0 having property* (P) *and* $\mathfrak{M}\in\mathrm{Lat}_{\frac{1}{2}}(T)$, *then $T_{\mathfrak{M}}$ also has property* (P).

Proof. We have $\mathfrak{M}=\mathfrak{U}\ominus\mathfrak{V}$, $\mathfrak{U},\mathfrak{V}\in\mathrm{Lat}(T)$ and $T|\mathfrak{U}$ has property (P) by Proposition 4.4. Again by Proposition 4.4 and Theorem 4.1 it follows that $T_{\mathfrak{M}}$ has property (P) because $T_{\mathfrak{M}}^*=(T|\mathfrak{U})^*|\mathfrak{M}$.

Proposition 4.6. *Let T be an operator of class C_0 acting on $\mathfrak{H}$ and let $\mathfrak{H}_j\in\mathrm{Lat}(T)$ be such that $\mathfrak{H}_j\subset\mathfrak{H}_{j+1}$, $j<\omega$, $\mathfrak{H}_0=\{0\}$ and $\mathfrak{H}=\bigvee_{j<\omega}\mathfrak{H}_j$. Then T has property* (P) *if and only if $T_{\mathfrak{K}_j}$, $\mathfrak{K}_j=\mathfrak{H}_{j+1}\ominus\mathfrak{H}_j$ $(j<\omega)$ have property* (P) *and*

$$\bigwedge_{j<\omega} m_0[T_{\mathfrak{H}_j^\perp}]=1. \tag{4.14}$$

Proof. If T has property (P) then $T_{\mathfrak{K}_j}$ have property (P) by Corollary 4.5. By Theorem 4.1 and Proposition 3.4 we infer the necessity of (4.14).

Conversely let us assume that $T_{\mathfrak{K}_j}$ have property (P) and (4.14) holds; let us put $m_j=m_0[T_{\mathfrak{H}_j^\perp}]$. If we define

$$\mathfrak{L}_j=(m_j(T)\mathfrak{H})^-\in\mathrm{Hyp\,Lat}(T) \tag{4.15}$$

then, as in the proof of Theorem 4.1, from (4.14) we infer $\bigvee_{j<\omega}\mathfrak{L}_j=\mathfrak{H}$ and the first part of the proof of Theorem 4.1 shows us that it is enough to prove that $T|\mathfrak{L}_j$ have property (P). Now, obviously $\mathfrak{L}_j\subset\mathfrak{H}_j$ so that by Corollary 4.5 we have only to show that $T|\mathfrak{H}_j$ have property (P). This easily proved inductively since the triangularization of $T|\mathfrak{H}_{j+1}$ with respect to the decomposition $\mathfrak{H}_{j+1}=\mathfrak{H}_j\oplus\mathfrak{K}_j$ is of the form $T|\mathfrak{H}_{j+1}=\begin{bmatrix}T|\mathfrak{H}_j & X_j\\ 0 & T_{\mathfrak{K}_j}\end{bmatrix}$. The Proposition follows.

Corollary 4.7. *Let T be an operator of class C_0 acting on $\mathfrak{H}$ and let $\mathfrak{H}_j\in\mathrm{Lat}(T)$ be such that $\mathfrak{H}_{j+1}\subset\mathfrak{H}_j$, $j<\omega$, $\mathfrak{H}_0=\mathfrak{H}$ and $\bigcap_{j<\omega}\mathfrak{H}_j=\{0\}$. Then T has property* (P) *if and only if $T_{\mathfrak{K}_j}$, $\mathfrak{K}_j=\mathfrak{H}_j\ominus\mathfrak{H}_{j+1}$ $(j<\omega)$, have property* (P) *and*

$$\bigwedge_{j<\omega} m_0[T|\mathfrak{H}_j]=1. \tag{4.16}$$

Proof. By Theorem 4.1, T has property (P) if and only if T^* has property (P). Therefore we have only to replace T by T^*, $\mathfrak{H}_j$ by $\mathfrak{H}_j^\perp$ and apply the preceding Proposition.

We are now going to extend [18], Theorem 1, and [3], Corollaries 2.4, 2.8 and 2.9 to the case of C_0 contractions having property (P).

Proposition 4.8. *Let T and T' be two quasisimilar operators of class C_0 acting on $\mathfrak{H}$, $\mathfrak{H}'$, respectively, and having property* (P). *Let us define*

$$\xi\colon \mathrm{Hyp\,Lat}\,(T) \to \mathrm{Hyp\,Lat}\,(T') \quad \textit{and} \quad \eta\colon \mathrm{Hyp\,Lat}\,(T') \to \mathrm{Hyp\,Lat}\,(T)$$

by

$$\xi(\mathfrak{M}) = \bigvee_{X\in\mathscr{I}(T',T)} X\mathfrak{M}, \quad \eta(\mathfrak{N}) = \bigvee_{Y\in\mathscr{I}(T,T')} Y\mathfrak{N}. \tag{4.17}$$

(i) *Each injection $A\in\mathscr{I}(T',T)$ is a lattice-isomorphism.*

(ii) $\xi(\mathfrak{M})=(A\mathfrak{M})^-=B^{-1}\mathfrak{M}$, $\mathfrak{M}\in \mathrm{Hyp\,Lat}\,(T)$, *for any quasi-affinities $A\in\mathscr{I}(T',T)$, $B\in\mathscr{I}(T,T')$.*

(iii) *ξ is bijective and $\eta=\xi^{-1}$.*

Proof. (i) If $A\in\mathscr{I}(T',T)$ is an injection, T is quasisimilar to $T'|(A\mathfrak{H})^-$ so that T' and $T'|(A\mathfrak{H})^-$ are quasisimilar. Now T' has property (P) so that $(A\mathfrak{H})^-=\mathfrak{H}'$ by Lemma 1.5 and A is a quasi-affinity.

Let $\mathfrak{K}', \mathfrak{K}''\in \mathrm{Lat}\,(T)$ be such that $(A\mathfrak{K}')^-=(A\mathfrak{K}'')^-=\mathfrak{K}^*$; then we also have $(A\mathfrak{K})^-=\mathfrak{K}^*$ with $\mathfrak{K}=\mathfrak{K}'\vee\mathfrak{K}''$. The operators $T|\mathfrak{K}'$, $T|\mathfrak{K}''$ and $T|\mathfrak{K}$ are quasisimilar to $T'|\mathfrak{K}^*$. By Proposition 4.4 $T|\mathfrak{K}$ has the property (P) and therefore $\mathfrak{K}'=\mathfrak{K}''=\mathfrak{K}$ by Lemma 1.5. Thus we have shown that the mapping $\mathfrak{K}\to(A\mathfrak{K})^-$ is one-to-one on Lat (T). Because we have shown that A is a quasi-affinity, the same argument can be applied to T'^*, T^* and A^* thus proving, via [3], Lemma 1.4, that A is a lattice-isomorphism.

(ii) Let us take any quasi-affinities $A\in\mathscr{I}(T',T)$ and $B\in\mathscr{I}(T,T')$; by (i) A and B are lattice isomorphisms. For each $\mathfrak{M}\in \mathrm{Hyp\,Lat}\,(T)$, $BA\in\{T\}'$ so that $BA\mathfrak{M}\subset\mathfrak{M}$ and since $T|\mathfrak{M}$ also has property (P) by Proposition 4.4 and $BA|\mathfrak{M}\in\{T|\mathfrak{M}\}'$ is one-to-one, we infer by (i) $(BA\mathfrak{M})^-=\mathfrak{M}$. Now, B is a lattice-isomorphism so that we infer

$$B^{-1}(\mathfrak{M}) = (A\mathfrak{M})^-. \tag{4.18}$$

If $X\in\mathscr{I}(T',T)$, we have $BX\in\{T\}'$ so that $BX\mathfrak{M}\subset\mathfrak{M}$ and by (4.18) $X\mathfrak{M}\subset B^{-1}(\mathfrak{M})=(A\mathfrak{M})^-$; it follows that $\xi(\mathfrak{M})\subset(A\mathfrak{M})^-$. Because the inclusion $(A\mathfrak{M})^-\subset\xi(\mathfrak{M})$ is obvious, (ii) is proved.

(iii) If $A\in\mathscr{I}(T',T)$, $B\in\mathscr{I}(T,T')$ are quasi-affinities we have by (ii) $(BA\mathfrak{M})^-=\mathfrak{M}$ and $(AB\mathfrak{N})^-=\mathfrak{N}$ for any $\mathfrak{M}\in \mathrm{Hyp\,Lat}\,(T)$, $\mathfrak{N}\in \mathrm{Hyp\,Lat}\,(T')$. Because, again by (ii), $\xi(\mathfrak{M})=(A\mathfrak{M})^-$ and $\eta(\mathfrak{N})=(B\mathfrak{N})^-$, (iii) follows.

The Proposition is proved.

Corollary 4.9. *Let T, S, φ, ψ be as in Theorem 2.5. If T has property* (P), *φ is a bijection and $\psi=\varphi^{-1}$.*

Proof. Obviously follows from the preceding Proposition.

The following result extends [3], Proposition 2.3, to the class of C_0 operators having property (P).

Proposition 4.10. *Let T, T', T'' be operators of class C_0 acting on $\mathfrak{H}$, $\mathfrak{H}'$, $\mathfrak{H}''$, respectively, and let $A\in\mathcal{I}(T,T')$, $B\in\mathcal{I}(T,T'')$ be such that $A\mathfrak{H}'\subset(B\mathfrak{H}'')^-$. If T has property* (P) *then*

(i) $(A^{-1}(B\mathfrak{H}''))^- = \mathfrak{H}'$ *and* (ii) $(A\mathfrak{H}'\cap B\mathfrak{H}'')^- \supset A\mathfrak{H}'$.

Proof. Because (ii) easily follows from (i), we have only to prove (i). We may assume that A is one-to-one, B is a quasi-affinity and T has the property (P). Indeed, we have only to replace T, T', T'', A, B, by $T|(B\mathfrak{H}'')^-$, $T'_{(\ker A)^\perp}$, $T''_{(\ker B)^\perp}$, $A|(\ker A)^\perp$, $B|(\ker B)^\perp$, respectively. Now the operator T'' has property (P) being quasisimilar to T (cf. Corollary 4.3) and T' has property (P) being quasisimilar to $T|(A\mathfrak{H}')^-$ (cf. Proposition 4.4). Then the operators $T'\oplus T''$ and $T'\oplus T$ are quasisimilar and have property (P) by Proposition 4.4. The operator $X\colon \mathfrak{H}'\oplus\mathfrak{H}''\to\mathfrak{H}'\oplus\mathfrak{H}$ given by

$$X(h'\oplus h'') = h'\oplus(Ah'-Bh''),\quad h'\oplus h''\in\mathfrak{H}'\oplus\mathfrak{H}'', \tag{4.19}$$

is an injection. Indeed, $X(h'\oplus h'')=0$ implies $h'=0$ and $Bh''=Ah'=0$, thus $h''=0$ by the injectivity of B. Because $X\in\mathcal{I}(T'\oplus T, T'\oplus T'')$ it follows by Proposition 4.8 (i) that X is a lattice-isomorphism. In particular $X(X^{-1}(\mathfrak{H}'\oplus\{0\}))$ is dense in $\mathfrak{H}'\oplus\{0\}$. But

$$X(X^{-1}(\mathfrak{H}'\oplus\{0\})) = \{h'\oplus 0;\ h'\in\mathfrak{H}' \text{ and } Ah' = Bh'' \text{ for some } h''\}$$

so that (i) follows and the Proposition is proved.

Corollary 4.11. *Let T, T', T'', A and B be as in the preceding Proposition. If T' is multiplicity-free then $A^{-1}(B\mathfrak{H}'')$ contains cyclic vectors of T'.*

Proof. Let us denote by P the orthogonal projection of $\mathfrak{H}'\oplus\mathfrak{H}$ onto $\mathfrak{H}'$. From Proposition 4.10 it follows that $A^{-1}(B\mathfrak{H}'')=PX(X^{-1}(\mathfrak{H}'\oplus\{0\}))$ is dense in $\mathfrak{H}'$ (where X is defined by relation (4.19)). Let us denote $\mathfrak{H}_0=(X^{-1}(\mathfrak{H}'\oplus\{0\}))\ominus$ $\ominus\ker(X|X^{-1}(\mathfrak{H}'\oplus\{0\}))\in\mathrm{Lat}_{\frac{1}{2}}(T'\oplus T'')$. Then we have

$$T'(PX|\mathfrak{H}_0) = (PX|\mathfrak{H}_0)\,(T'\oplus T'')_{\mathfrak{H}_0}$$

and by Lemma 1.1 T' and $(T'\oplus T'')_{\mathfrak{H}_0}$ are quasisimilar; in particular $(T'\oplus T'')_{\mathfrak{H}_0}$

is also multiplicity-free. If h_0 is any cyclic vector of $(T'\oplus T'')_{\mathfrak{H}_0}$ then $PXh_0\in A^{-1}(B\mathfrak{H}'')$ is a cyclic vector of T'. Corollary follows.

Finally let us remark that the result of [4] concerning the quasi-direct decomposition of the space on which a weak contraction acts can be extended, via Proposition 4.8 (i), to the class of C_0 operators having property (P).

Corollary 4.12. *Let T be an operator of class C_0 having property* (P) *and acting on the (necessarily separable) Hilbert space $\mathfrak{H}$ and let $\bigoplus_{j<\omega} S(m_j)$ be the Jordan model of T. There exists a decomposition of $\mathfrak{H}$*

$$\mathfrak{H} = \bigvee_{j<\omega} \mathfrak{H}_j \tag{4.19}$$

into a quasi-direct sum of invariant subspaces of T such that $T|\mathfrak{H}_j$ is quasisimilar to $S(m_j)$.

Proof. Cf. the proof of [4], Proposition 3.5.

5. Operators of class C_0 having property (Q)

The following Lemma extends [19], Proposition 3, to the entire class of C_0 operators.

Lemma 5.1. *Let T and T' be two quasisimilar operators of class C_0. Then T has property* (Q) *if and only if T' has property* (Q).

Proof. Because (Q) implies (P), by Corollary 4.3 it is enough to prove the Lemma for T and T' having the property (P). Let $X\in\mathscr{I}(T,T')$, $Y\in\mathscr{I}(T',T)$ be two quasi-affinities. By Proposition 4.8 (i) X and Y are lattice-isomorphisms. Let us take $A\in\{T'\}'$; then $B=XAY\in\{T\}'$. Obviously $\ker B=Y^{-1}(\ker A)$, X being an injection. Because Y is a lattice-isomorphism we have $(Y(\ker B))^-=\ker A$ so that $Y|\ker B$ is a quasi-affinity from $\ker B$ into $\ker A$. Because

$$Y|\ker B\in\mathscr{I}(T'|\ker A, T|\ker B)$$

it follows by Lemma 1.1 that $T|\ker B$ and $T'|\ker A$ are quasisimilar. Analogously $T_{\ker B^*}$ and $T'_{\ker A^*}$ are quasisimilar. If T has the property (Q), the operators $T|\ker B$ and $T_{\ker B^*}$ are quasisimilar and it follows from the preceding considerations that $T'|\ker A$ and $T'_{\ker A^*}$ are quasisimilar. Since $A\in\{T'\}'$ is arbitrary it follows that T' has the property (Q). The Lemma is proved.

Lemma 5.2. *For any inner function m and natural number k the operator $T=S(\underbrace{m,m,\dots,m}_{k\text{ times}})$ has the property* (Q).

Proof. By the lifting Theorem (cf. [12], Theorem II.2.3) any operator $X\in\{T\}'$ is given by

$$Xh = P_{\mathfrak{H}} Ah,\ h\in\mathfrak{H} = \underbrace{\mathfrak{H}(m)\oplus\mathfrak{H}(m)\oplus\ldots\oplus\mathfrak{H}(m)}_{k\text{ times}} \tag{5.1}$$

where $A=[a_{ij}]_{1\le i,j\le k}$ is an arbitrary matrix over H^∞. As shown by NORDGREN [9] (cf. also SZŰCS [17] and SZ.-NAGY [11]) there exist matrices B, U, V which determine by formulas analogous to (5.1) operators Y, K, L in $\{T\}'$ such that

$$(\det U)(\det V)\wedge m = 1; \tag{5.2}$$

$$AU = VB, \tag{5.3}$$

$$B = [b_{ij}]_{1\le i,j\le k}, \quad b_{ij} = 0 \quad \text{for} \quad i\ne j. \tag{5.4}$$

From (5.2) we infer as in [8] that K and L are quasi-affinities and therefore lattice-isomorphisms by Proposition 4.8 (i). From (5.3) we infer

$$XK = LY \tag{5.5}$$

so that $K(\ker Y)\subset\ker X$ and $K^{-1}(\ker X)\subset\ker Y$; because K is a lattice-isomorphism it follows that $(K(\ker Y))^- = \ker X$ and therefore $T|\ker X$ and $T|\ker Y$ are quasisimilar. Analogously $T_{\ker X^*}$ and $T_{\ker Y^*}$ are quasisimilar. We have $Y=\bigoplus_{j=1}^{k} b_{jj}(S(m))$ and $\ker Y=\bigoplus_{j=1}^{k} (\ker b_{jj}(S(m)))$ so that $T|\ker Y$ is unitarily equivalent (cf. [15], p. 315) to $\bigoplus_{j=1}^{k} S(m_j)$, where $m_j=m\wedge b_{jj}$. Analogously we can show that $T_{\ker Y^*}$ is unitarily equivalent to $\bigoplus_{j=1}^{k} S(m_j)$. We have shown $T|\ker Y$ and $T_{\ker Y^*}$ are unitarily equivalent; we infer that $T|\ker X$ and $T_{\ker X^*}$ are quasisimilar. Because X is arbitrary in $\{T\}'$, the Lemma follows.

Lemma 5.3. *If $T\oplus S$ has the property* (Q) *then T and S also have the property* (Q).

Proof. It is obvious since $\{T\oplus S\}'\supset\{T\}'\oplus I\cup I\oplus\{S\}'$.

The following Theorem characterizes the class of C_0 operators having the property (Q) in terms of the Jordan model.

Theorem 5.4. *An operator T of class C_0 has property* (Q) *if and only if*

(i) $\bigwedge_{j<\omega} m_j = 1, \quad m_j = m_j[T],$ and

(ii) *the functions $m_0/m_1, m_1/m_2, \ldots$ are pairwise relatively prime.*

In particular, if T has property (Q), *then T acts on a separable Hilbert space and T^* also has property* (Q).

Proof. Let T have property (Q). Then T also has property (P) so that the necessity of (i) follows by Theorem 4.1. By Lemma 5.1 the Jordan model $S(M)$ of T also has the property (Q) so that $S^j = S(m_j) \oplus S(m_{j+1}), j < \omega$, must have property (Q) by Lemma 5.3. The matrix

$$A = \begin{bmatrix} 0 & m_j/m_{j+1} \\ 0 & 0 \end{bmatrix} \tag{5.6}$$

determines an operator $X \in \{S^j\}'$ by the formula

$$Xh = P_{\mathfrak{H}_j} Ah, \quad h \in \mathfrak{H}_j = \mathfrak{H}(m_j) \oplus \mathfrak{H}(m_{j+1}). \tag{5.7}$$

Obviously

$$\ker X = \mathfrak{H}(m_j) \oplus \{0\}$$

so that $S^j|\ker X$ is unitarily equivalent to $S(m_j)$. Now

$$\operatorname{ran} X = ((m_j/m_{j+1})H^2 \ominus m_j H^2) \oplus \{0\}$$

so that $\ker X^* = \mathfrak{H}(m_j/m_{j+1}) \oplus \mathfrak{H}(m_{j+1})$ and it follows that $S^j_{\ker X^*}$ is unitarily equivalent to $S(m_j/m_{j+1}) \oplus S(m_{j+1})$. The Jordan model of $S(m_j/m_{j+1}) \oplus S(m_{j+1})$ is

$$S((m_j/m_{j+1}) \vee m_{j+1}) \oplus S((m_j/m_{j+1}) \wedge m_{j+1})$$

by [2], Lemma 4. Because S^j has the property (Q) this Jordan model must coincide with $S(m_j)$ so that $(m_j/m_{j+1}) \wedge m_{j+1} = 1$. In particular m_j/m_{j+1} and m_k/m_{k+1} are relatively prime for $k > j$; (ii) is proved.

Conversely, let us assume that conditions (i) and (ii) are satisfied. Let us denote

$$u_j = m_j/m_{j+1}, \quad j < \omega. \tag{5.8}$$

Then by Lemma 1.2, $S(m_0)$ is quasisimilar to $\bigoplus_{j<\omega} S(u_j)$, $S(m_1)$ is quasisimilar to $\bigoplus_{1 \leq j < \omega} S(u_j), \ldots, S(m_k)$ is quasisimilar to $\bigoplus_{k \leq j < \omega} S(u_j)$ so that T is quasisimilar to

$$S = \bigoplus_{j<\omega} T^j, \quad T^j = \underbrace{S(u_j, u_j, \ldots, u_j)}_{j+1 \text{ times}}. \tag{5.9}$$

Because the functions $u_0, u_1, \ldots$ are pairwise relatively prime we have $(m_0/u_j) \wedge u_j = 1$ so that $(m_0/u_j)(T^k) = 0$, $k \neq j$, and $(m_0/u_j)(T^j)$ is a quasi-affinity. This implies that

$$\mathfrak{H}^j = \underbrace{\mathfrak{H}(u_j) \oplus \mathfrak{H}(u_j) \oplus \ldots \oplus \mathfrak{H}(u_j)}_{j+1 \text{ times}} = (\operatorname{ran}(m_0/u_j)(S))^-$$

is a hyper-invariant subspace of S. We are now able to prove that S, and therefore T, has property (Q). Any operator $X \in \{S\}'$ has the property $X\mathfrak{H}^j \subset \mathfrak{H}^j$, $j < \omega$, so that $X = \bigoplus_{j<\omega} X^j$, $X^j \in \{T^j\}'$. By Lemma 5.2, $T^j|\ker X^j$ and $T^j_{\ker X^{j*}}$ are quasisimilar. But obviously $\ker X = \bigoplus_{j<\omega} \ker X^j$, $\ker X^* = \bigoplus_{j<\omega} \ker X^{j*}$ so that $S|\ker X = \bigoplus_{j<\omega} T^j|\ker X^j$ and $S_{\ker X^*} = \bigoplus_{j<\omega} T^j_{\ker X^{j*}}$; it follows that $S|\ker X$ and $S_{\ker X^*}$ are quasisimilar. The Theorem is proved.

We are now able to give a complete description of the lattice of hyper-invariant subspaces of an operator of class C_0 having property (Q).

Proposition 5.5. *An operator of class C_0 having property* (P) *has property* (Q) *if and only if*

$$\text{Hyp Lat}(T) = \{(\operatorname{ran} m(T))^- : m \in H_i^\infty,\ m \leqq m_0[T]\}. \tag{5.10}$$

Proof. As usual $S(M)$ denotes the Jordan model of T. Assume (5.10) holds; by Proposition 4.8 (iii), (5.10) also holds for $S(M)$. In particular,

$$\ker m_{j+1}(S(M)) = \bigoplus_{i\leqq j} ((m_i/m_{j+1})H^2 \ominus m_i H^2) \oplus \bigoplus_{j+1\leqq i<\omega} \mathfrak{H}(m_i)$$

is of the form $(\operatorname{ran} u(S(M)))^-$ for some inner divisor u of m_0. Because $\operatorname{ran} u(S(m_0)) = (m_0/m_{j+1})H^2 \ominus m_0 H^2$ we must have $u = m_0/m_{j+1}$. We have also

$$(m_0/m_{j+1}) \wedge m_{j+1} = 1 \tag{5.11}$$

because $u(S(m_{j+1}))$ must have dense range. From (5.11) we infer $(m_j/m_{j+1}) \wedge m_{j+1} = 1$, $j<\omega$. By Theorem 5.4 it follows that T has property (Q).

Conversely, let us assume that T has property (Q). By the proof of Theorem 5.4, T is quasisimilar to

$$S = \bigoplus_{j<\omega} S^j \quad \text{on} \quad \mathfrak{H} = \bigoplus_{j<\omega} \mathfrak{H}^j, \tag{5.12}$$

where

$$S^j = S(\underbrace{u_j, u_j, \ldots, u_j}_{j+1 \text{ times}}), \quad \mathfrak{H}^j = \underbrace{\mathfrak{H}(u_j) \oplus \mathfrak{H}(u_j) \oplus \ldots \oplus \mathfrak{H}(u_j)}_{j+1 \text{ times}}, \tag{5.13}$$

$$u_j = m_j/m_{j+1}, \tag{5.14}$$

and

$$\mathfrak{H}^j = ((m_0/u_j)(S)\mathfrak{H})^- \in \text{Hyp Lat}(S). \tag{5.15}$$

Let us take $\mathfrak{M} \in \text{Lat}(S)$ and denote $\mathfrak{M}_j = ((m_0/u_j)(S)\mathfrak{M})^-$. We claim that

$$\mathfrak{M} = \bigoplus_{j<\omega} \mathfrak{M}_j \quad \text{and} \quad \mathfrak{M}_j = \mathfrak{M} \cap \mathfrak{H}^j. \tag{5.16}$$

The inclusion $\mathfrak{M} \supset \bigoplus_{j<\omega} \mathfrak{M}_j$ is obvious. Now, the minimal function m of $S_{\mathfrak{N}}$, $\mathfrak{N} = \mathfrak{M} \ominus (\bigoplus_{j<\omega} \mathfrak{M}_j) = \bigcap_{j<\omega} \ker (m_0/u_j)^\sim((S|\mathfrak{M})^*)$ divides m_0/u_j, $j<\omega$, so that $m \wedge u_j = 1$. It follows that $m=1$, $\mathfrak{N} = \{0\}$ and (5.16) is proved.

Moreover, by (5.16), $\mathfrak{M}_j$ is a hyper-invariant subspace of S^j if $\mathfrak{M}\in\text{Hyp Lat}(S)$. By Proposition 2.1 (i) we have $\mathfrak{M}_j=\underbrace{\mathfrak{M}_j^0\oplus\mathfrak{M}_j^0\oplus\ldots\oplus\mathfrak{M}_j^0}_{j+1\text{ times}}$ where $\mathfrak{M}_j^0=u_j'H^2\ominus u_jH^2$ so that $\mathfrak{M}_j=u_j'(S^j)\mathfrak{H}^j$. Let us denote by m the limit of an arbitrary converging subsequence of $\{u_0'u_1'\ldots u_k'\}_{k<\omega}$; we shall have $(m/u_j')\wedge u_j=1$ so that $\mathfrak{M}_j=(m(S^j)\mathfrak{H}^j)^-$. Using (5.16) we infer $\mathfrak{M}=(m(S)\mathfrak{H})^-$ and by Proposition 4.8 (iii) the proof is done.

Let us denote by $\mathscr{L}_m^k$ the lattice $\text{Lat}\,(S(\underbrace{m, m, \ldots, m}_{k\text{ times}}))$ $(m\in H_i^\infty,\ 1\le k<\omega)$. The preceding proof also characterizes Lat (T) for T having property (Q).

Corollary 5.6. *Let T be an operator of class C_0 having the property* (Q). *Then* Lat (T) *is isomorphic to* $\prod_{j<\omega}\mathscr{L}_{u_j}^{j+1}$, *where* $u_j=m_j[T]/m_{j+1}[T],\ j<\omega$.

Proof. The decomposition (5.16) was proved for any $\mathfrak{M}\in\text{Lat}\,(S)$. The Corollary follows by Proposition 4.8 (i).

Example 5.7. There are operators T of class C_0 for which (5.10) holds without property (P). In fact it can be shown that *a Jordan operator $S(M)$ satisfies the condition* (5.10) *if and only if* $(m_0/m_\alpha)\wedge m_\alpha=1$ *for each ordinal number* α.

Proof. The necessity of the condition $(m_0/m_\alpha)\wedge m_\alpha=1$ is proved analogously with the proof of (5.11). Conversely, let us assume $(m_0/m_\alpha)\wedge m_\alpha=1$ and let $\mathfrak{M}\in\text{Hyp Lat}\,(S(M))$ be given by (2.2). Then m_α/m_α'' divides m_0/m_0'' so that m_0''/m_α'' divides m_0/m_α and therefore $(m_0''/m_\alpha'')\wedge m_\alpha=1$. We infer $(m_0''(S(m_\alpha))\mathfrak{H}(m_\alpha))^-=$ $=(m_\alpha''(S(m_\alpha))(m_0''/m_\alpha'')(S(m_\alpha))\mathfrak{H}(m_\alpha))^-=m_\alpha''H^2\ominus m_\alpha H^2$ because $(m_0''/m_\alpha'')(S(m_\alpha))$ is a quasi-affinity (cf. [12], Proposition III.4.7). We infer

$$\mathfrak{M}=(\text{ran}\, m_0''(S(M)))^-.$$

Remark 5.8. As shown by Example 2.10, property (5.10) is not stable with respect to quasisimilarities.

6. Generalized inner functions

Let us recall (cf. [7]) that a function $m\in H_i^\infty$ has a factorization

$$m = cbs \tag{6.1}$$

where c is a complex constant of modulus one, b is a Blaschke product

$$b(z)=\prod_k\frac{\bar a_k}{|a_k|}\cdot\frac{a_k-z}{1-\bar a_k z},\quad |a_k|<1,\quad \sum_k(1-|a_k|)<\infty \tag{6.2}$$

and s is a singular inner function, that is

$$s(z) = \exp\left(-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t)\right) \tag{6.3}$$

where μ is a finite Borel measure on $[0, 2\pi]$, singular with respect to Lebesgue measure. Let us denote by $\sigma(z)$ the multiplicity of the zero z in the Blaschke product (6.2), that is,

$$\sigma(z) = \operatorname{card}\{k\colon\ a_k = z\}. \tag{6.4}$$

The convergence condition in (6.2) is equivalent to

$$\sum_{|z|<1} \sigma(z)(1-|z|) < \infty. \tag{6.5}$$

We shall denote by Γ the set of pairs $\gamma=(\sigma, \mu)$, where μ is a finite Borel measure singular with respect Lebesgue's measure on $[0, 2\pi]$, $\sigma(z)$ is a natural number for $|z|<1$ and the condition (6.5) is satisfied. With respect to the adition $(\sigma, \mu)+(\sigma', \mu')=(\sigma+\sigma', \mu+\mu')$, Γ becomes a commutative monoid. The set Γ is ordered by the relation $(\sigma, \mu)\leqq(\sigma', \mu')$ if and only if $\sigma\leqq\sigma'$ and $\mu\leqq\mu'$. Moreover, in Γ are defined the lattice operations:

$$(\sigma, \mu)\vee(\sigma', \mu') = (\sigma\vee\sigma',\ \mu\vee\mu'),$$
$$(\sigma, \mu)\wedge(\sigma', \mu') = (\sigma\wedge\sigma', \mu\wedge\mu')$$

where $\mu\vee\mu'$, $\mu\wedge\mu'$ have the usual sense and $\sigma\vee\sigma'=\max\{\sigma, \sigma'\}$, $\sigma\wedge\sigma'=\min\{\sigma, \sigma'\}$. A mapping $\gamma\colon H_i^\infty\to\Gamma$ is defined by $\gamma(m)=(\sigma, \mu)$, where σ is given by (6.4) and μ by (6.3) if m has the decomposition (6.1). We have also a mapping $\delta\colon \Gamma\to H_i^\infty$ defined by

$$(\delta(\gamma))(z) = \prod_{|z|<1}\left(\frac{\bar a}{|a|}\cdot\frac{a-z}{1-\bar a z}\right)^{\sigma(a)}\cdot\exp\left(-\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\, d\mu(t)\right) \tag{6.6}$$

where $\gamma=(\sigma, \mu)$. Then $\gamma\circ\delta=\mathrm{id}$ and $\delta(\gamma(m))=cm$ with c a complex constant of modulus one.

Let us recall that, for a function $f\in H^\infty$, the function $f^\sim$ is defined by $f^\sim(z)=\overline{f(\bar z)}$. For $\gamma=(\sigma, \mu)\in\Gamma$ we shall define the element $\gamma^\sim=(\sigma^\sim, \mu^\sim)\in\Gamma$ by $\sigma^\sim(z)=\sigma(\bar z)$ and $\mu^\sim=\mu\circ j$ where $j\colon [0, 2\pi]\to[0, 2\pi]$ is given by $j(t)=2\pi-t$.

Let us list some properties of the mapping γ.

Lemma 6.1. (i) $\gamma(m_1m_2)=\gamma(m_1)+\gamma(m_2)$, $m_1, m_2\in H_i^\infty$.

(ii) $\gamma(m_1)\leqq\gamma(m_2)$ *if and only if* $m_1\leqq m_2$; $\gamma(m_1)=\gamma(m_2)$ *if and only if* m_1 *and* m_2 *differ by a complex multiplicative constant of modulus one.*

(iii) $\gamma(m^\sim)=\gamma(m)^\sim$, $m\in H_i^\infty$.

(iv) *If* $\{m_j\}_{j=0}^{\infty}\subset H_i^{\infty}$, *then the family* $\{m_0 m_1 \dots m_j\}_{j=0}^{\infty}$ *has a least inner multiple* m *if and only if* $\sum_{j=0}^{\infty}\gamma(m_j)\in\Gamma$ *and in this case* $\gamma(m)=\sum_{j=0}^{\infty}\gamma(m_j)$.

Proof. (i), (ii) and (iii) are obvious. To prove (iv) let us assume firstly that $\{m_0 m_1 \dots m_j\}_{j=0}^{\infty}$ has a least inner multiple m. Then obviously $\gamma\geqq\gamma(m)$ if and only if $\gamma\geqq\sum_{j\leqq n}\gamma(m_j)$ for each natural n. Consequently $\sum_{j=0}^{\infty}\gamma(m_j)\in\Gamma$ and $\gamma(m)=\sum_{j=0}^{\infty}\gamma(m_j)$. Conversely if $\gamma=\sum_{j=0}^{\infty}\gamma(m_j)\in\Gamma$ then $\delta(\gamma)\geqq m_0 m_1 m_2 \dots m_j$ for each j so that the family $\{m_0 m_1 \dots m_j\}_{j=0}^{\infty}$ has a least inner multiple. The Lemma is proved.

We shall now introduce the class $\mathcal{M}$ of (not necessarily finite) Borel measures μ on $[0, 2\pi]$ for which there exists a finite Borel measure ν singular with respect to Lebesgue measure such that $\mu\prec\nu$, where the absolute continuity $\mu\prec\nu$ is understood as

$$\mu = \bigvee_n (\mu\wedge n\nu). \tag{6.7}$$

We shall denote by $\mathcal{M}_0$ the class of σ-finite measures $\mu\in\mathcal{M}$ and by $\mathcal{M}_\infty$ the class of measures $\mu\in\mathcal{M}$ which take the values 0 and ∞ only.

Lemma 6.2. (i) *If* $\mu\in\mathcal{M}$ *and* ν *is a finite measure such that* $\mu\prec\nu$, *we have a decomposition*

$$d\mu = f d\nu \tag{6.8}$$

where $f\colon [0, 2\pi]\to[0, +\infty]$ *is a Borel function.*

(ii) *Every* $\mu\in\mathcal{M}$ *admits a unique decomposition* $\mu=\mu_0+\mu_\infty$, *where* $\mu_0\in\mathcal{M}_0$, $\mu_\infty\in\mathcal{M}_\infty$ *and* μ_0 *and* μ_∞ *are mutually singular.*

(iii) *If* $\{\mu_j\}_{j=0}^{\infty}\subset\mathcal{M}$ *then* $\sum_{j=0}^{\infty}\mu_j\in\mathcal{M}$.

Proof. (i) The measure $\mu_n=\mu\wedge n\nu$ is finite, $\mu_n\prec\nu$, and by the Radon—Nikodym theorem we have $d\mu_n=f_n d\nu$, where $f_n\colon [0, 2\pi]\to[0, n]$ is a Borel function. Because $\mu_n\leqq\mu_{n+1}$ we have $f_n\leqq f_{n+1}$ $d\nu$-a.e.; replacing f_n by $f_n'=f_1\vee f_2\vee\dots\vee f_n$ we may assume $f_n\leqq f_{n+1}$. Now it is clear that the function $f=\lim_{n\to\infty} f_n$ satisfies the relation (6.8).

(ii) Let ν and f be as before; let us denote $A=\{t; f(t)=+\infty\}$ and $f_\infty=f\chi_A$, $f_0=f(1-\chi_A)$. Then we may take $d\mu_0=f_0\cdot d\nu$, $d\mu_\infty=f_\infty d\nu$.

(iii) Let us take finite measures ν_j such that $\mu_j\prec\nu_j$; then $\sum_{j=0}^{\infty}\mu_j\prec\nu$, where ν is defined by

$$\nu = \sum_{j=0}^{\infty} 2^{-j}\nu_j/\nu_j([0, 2\pi]).$$

Remark 6.3. Obviously, every measure μ of the form (6.8) belongs to $\mathcal{M}$ if ν is a finite singular measure on $[0, 2\pi]$.

Lemma 6.4. *If* $\mu_j, \nu_j \in \mathcal{M}, j=0, 1, \dots,$ *are such that* $\sum_{j=0}^{\infty} \mu_j = \sum_{j=0}^{\infty} \nu_j$ *then there exist* $\mu_{ij} \in \mathcal{M},\ i, j=0, 1, \dots,$ *such that* $\sum_{j=0}^{\infty} \mu_{ij} = \mu_i,\ \sum_{i=0}^{\infty} \mu_{ij} = \nu_j,\ i, j=0, 1, \dots$.

Proof. Let us take a finite singular measure α such that $\mu_j \prec \alpha,\ \nu_j \prec \alpha, j=0, 1, \dots$. By Lemma 6.2 we have

$$d\mu_j = f_j d\alpha, \quad d\nu_j = g_j d\alpha, \quad 0 \leqq j < \infty. \tag{6.9}$$

By the hypothesis we have

$$\sum_{j=0}^{\infty} f_j = \sum_{j=0}^{\infty} g_j \quad d\alpha\text{-a.e.} \tag{6.10}$$

It will be enough to find Borel functions h_{ij} such that

$$\sum_{j=0}^{\infty} h_{ij} = f_i, \quad \sum_{i=0}^{\infty} h_{ij} = g_j \quad d\alpha\text{-a.e.}, \quad 0 \leqq i, j < \infty, \tag{6.11}$$

and then to define $d\mu_{ij} = h_{ij} d\alpha$.

If the sum (6.10) is $d\alpha$-a.e. finite we may define h_{ij} inductively by

$$\begin{cases} h_{00} = f_0 \wedge g_0, \quad h_{0j} = \left(f_0 - \sum_{k=0}^{j-1} h_{0k}\right) \wedge g_j, \quad 1 \leqq j < \infty; \\ h_{i0} = f_i \wedge \left(g_0 - \sum_{k=0}^{i-1} h_{k0}\right), \quad 1 \leqq i < \infty; \\ h_{ij} = \left(f_i - \sum_{r=0}^{j-1} h_{ir}\right) \wedge \left(g_j - \sum_{k=0}^{i-1} h_{kj}\right), \quad 1 \leqq i, j < \infty. \end{cases} \tag{6.12}$$

If the sum (6.10) is not $d\alpha$-a.e. finite we can find increasing sequences $\{f_i^{(n)}\}_{n=0}^{\infty}$, $\{g_j^{(n)}\}_{n=0}^{\infty}$ such that $f_i = \lim_{n \to \infty} f_i^{(n)},\ g_j = \lim_{n \to \infty} g_j^{(n)}$ $d\alpha$-a.e., $0 \leqq i, j < \infty$, and $\sum_{i=0}^{\infty} f_i^{(n)} = \sum_{j=0}^{\infty} g_j^{(n)} < \infty$ $d\alpha$-a.e., $0 \leqq n < \infty$.

Let $h_{ij}^{(n)}$ be defined by (6.12) with f_i, g_j replaced by $f_i^{(1)}$, $g_j^{(1)}$ in case $n=0$, and by $f_i^{(n+1)} - f_i^{(n)}$, $g_j^{(n+1)} - g_j^{(n)}$ in case $n \geqq 1$. We can take $h_{ij} = \sum_{n=0}^{\infty} h_{ij}^{(n)}$ and the Lemma follows.

We shall now introduce the class $\tilde{\Gamma}$ of "generalized inner functions". An element γ of $\tilde{\Gamma}$ is a pair $\gamma = (\sigma, \mu)$ where $\mu \in \mathcal{M}$ and σ is a natural number valued function defined on $\{z;\ |z| < 1\}$ such that

$$\sum_{\sigma(z) \neq 0} (1 - |z|) < \infty. \tag{6.13}$$

The subclass $\tilde{\Gamma}_0 \subset \tilde{\Gamma}$ consists of the pairs $\gamma=(\sigma, \mu)\in\tilde{\Gamma}$ such that $\mu\in\mathcal{M}_0$. Analogously with Γ, $\tilde{\Gamma}$ is a commutative monoid and an ordered set in which the lattice operations are defined. For $\gamma=(\sigma, \mu)\in\tilde{\Gamma}$ we define $\gamma^{\sim}=(\sigma^{\sim}, \mu^{\sim})\in\tilde{\Gamma}$ as in the case $\gamma\in\Gamma$. Any $\gamma=(\sigma, \mu)\in\tilde{\Gamma}$ has a decomposition

(6.14) $$\gamma = \gamma_0+\gamma_\infty, \quad \gamma_0 = (\sigma, \mu_0)\in\tilde{\Gamma}_0, \quad \gamma_\infty = (0, \mu_\infty)$$

where $\mu=\mu_0+\mu_\infty$ is the decomposition of μ given by Lemma 6.2 (ii).

Lemma 6.5. (i) *$\tilde{\Gamma}_0$ is the set of simplifiable elements of $\tilde{\Gamma}$, that is $\gamma\in\tilde{\Gamma}_0$ if and only if $\gamma'+\gamma=\gamma''+\gamma$ implies $\gamma'=\gamma''$ for $\gamma', \gamma''\in\tilde{\Gamma}$.*

(ii) *$\gamma'+\gamma=\gamma''+\gamma$ implies $\gamma'=\gamma''$ whenever $\gamma_\infty\leqq\gamma'\wedge\gamma''$.*

Proof. (i) It is obvious that $\gamma'+\gamma=\gamma''+\gamma$ implies $\gamma'=\gamma''$ whenever $\gamma\in\tilde{\Gamma}_0$. Conversely, if $\gamma\notin\tilde{\Gamma}_0$, we have $0\neq\gamma_\infty$ and $0+\gamma=\gamma_\infty+\gamma$.

(ii) By (i) we can simplify γ_0 from the equality $\gamma'+\gamma=\gamma''+\gamma$ and we obtain $\gamma'+\gamma_\infty=\gamma''+\gamma_\infty$. Now the assumption implies $\gamma'+\gamma_\infty=\gamma'$ and $\gamma''+\gamma_\infty=\gamma''$; the Lemma follows.

We shall consider the cartesian product $\mathcal{K}=\tilde{\Gamma}\times\tilde{\Gamma}$ and on $\mathcal{K}$ we define the relation "$\sim$" by

(6.15) $$(\gamma, \gamma_1) \sim (\gamma', \gamma_1') \quad \text{if and only if} \quad \gamma+\gamma_1' = \gamma'+\gamma_1.$$

The relation "$\sim$" is not an equivalence relation; however, as shown by Lemma 6.5 (i) the restriction of "$\sim$" on $\mathcal{K}_0=\tilde{\Gamma}_0\times\tilde{\Gamma}_0$ is an equivalence relation. The quotient $\mathcal{G}_0=\mathcal{K}_0/\sim$ is a group- the group of formal differences $\gamma-\gamma'$, $\gamma, \gamma'\in\tilde{\Gamma}_0$. We may assume $\tilde{\Gamma}_0\subset\mathcal{G}_0$ identifying the element $\gamma\in\tilde{\Gamma}_0$ with the class of $(\gamma, 0)$ in $\mathcal{K}_0/\sim$.

We shall now describe the connection of $\tilde{\Gamma}$ and $\tilde{\Gamma}_0$ with Γ.

Proposition 6.6. (i) *If $\{\gamma_j\}_{j=0}^\infty\subset\Gamma$ are such that*

(6.16) $$\gamma_j \geqq \gamma_{j+1}, \quad 0\leqq j<\infty, \quad \bigwedge_{j\geqq 0} \gamma_j = 0,$$

then

(6.17) $$\gamma = \sum_{j=0}^{\infty} \gamma_j\in\tilde{\Gamma}.$$

Conversely, each $\gamma\in\tilde{\Gamma}$ has a representation of the form (6.17) *such that* (6.16) *is satisfied.*

(ii) *If $\{\gamma_j\}_{j=0}^\infty\subset\Gamma$ satisfy* (6.16) *and, moreover,*

(6.18) $$(\gamma_j-\gamma_{j+1})\wedge(\gamma_k-\gamma_{k+1}) = 0, \quad j\neq k,$$

then the element γ defined by (6.17) *belongs to $\tilde{\Gamma}_0$. Conversely, each $\gamma\in\tilde{\Gamma}_0$ has a representation of the form* (6.17) *such that* (6.16) *and* (6.18) *are verified.*

Proof. (i) If $\gamma_j=(\sigma_j, \mu_j)$, $0\leqq j<\infty$, we have $\mu=\sum_{j=0}^{\infty}\mu_j\in\mathcal{M}$ by Lemma 6.2 (iii); it remains to show that $\sigma=\sum_{j=0}^{\infty}\sigma_j$ is finite and the condition (6.13) is satisfied. But $\bigwedge_{j\geqq 0}\sigma_j=0$ imply that for each z, $\sigma_j(z)=0$ for some j and the finiteness of σ is obvious. The condition (6.13) is satisfied because $\sigma(z)\neq 0$ implies $\sigma_0(z)\neq 0$ and therefore

$$\sum_{\sigma(z)\neq 0}(1-|z|)\leqq\sum_{|z|<1}\sigma_0(z)(1-|z|)<\infty.$$

Conversely, if $\gamma=(\sigma, \mu)$ we define

$$\text{(6.19)}\qquad \begin{cases}\sigma_j(z)=0 & \text{if}\quad \sigma(z)\leqq j\\ \quad\ \ =1 & \text{if}\quad \sigma(z)>j,\ 0\leqq j<\infty.\end{cases}$$

To define μ_j let us write $d\mu=f\cdot d\nu$ for some finite measure ν and put $d\mu_j=f_j\cdot d\nu$, where

$$\text{(6.20)}\qquad f_0=f\wedge 1,\quad f_j=\Big(f-\sum_{k=0}^{j-1}f_k\Big)\wedge 1/(j+1),\quad 1\leqq j<\infty.$$

It is obvious that $\gamma_j=(\sigma_j, \mu_j)$ satisfy (6.16—17).

(ii) Let us put $\gamma_j=(\sigma_j, \mu_j)$; from (6.18) we infer the existence of a sequence of pairwise disjoint Borel subsets $A_j\subset[0, 2\pi]$ such that $[0, 2\pi]=\bigcup_{j=0}^{\infty}A_j$ and $\mu_j(\bigcup_{k<j}A_k)=0$. If $\mu=\sum_{j=0}^{\infty}\mu_j$, we have $\mu(A_j)=(\mu_0+\mu_1+\ldots+\mu_j)(A_j)<\infty$; thus μ is σ-finite. Conversely, let us take $\gamma=(\sigma, \mu)\in\tilde{\Gamma}_0$ and define σ_j by (6.19). If $d\mu=f\cdot d\nu$ and ν is finite, f is $d\nu$-a.e. finite so that $[0, 2\pi]=\bigcup_{j=0}^{\infty}A_j$ where $A_j=\{x; f(x)\in[j, j+1)\}$. We define

$$f_j=\sum_{k=j}^{\infty}(k+1)^{-1}f\chi_{A_k}$$

and $d\mu_j=f_j\cdot d\nu$. It is clear that $\gamma_j=(\sigma_j, \mu_j)$ satisfy the conditions (6.16—18). Proposition 6.6 is proved.

Proposition 6.7. *If $\{\gamma_j\}_{j=0}^{\infty}, \{\gamma_j'\}_{j=0}^{\infty}\subset\tilde{\Gamma}$ are such that $\sum_{j=0}^{\infty}\gamma_j=\sum_{j=0}^{\infty}\gamma_j'\in\tilde{\Gamma}$ then there exist $\{\gamma_{ij}\}_{0\leqq i,j<\infty}\subset\tilde{\Gamma}$ such that $\sum_{j=0}^{\infty}\gamma_{ij}=\gamma_i$, $\sum_{i=0}^{\infty}\gamma_{ij}=\gamma_j$, $0\leqq i,j<\infty$.*

Proof. If $\gamma_j=(\sigma_j, \mu_j)$, $\gamma_j'=(\sigma_j', \mu_j')$, $0\leqq j<\infty$, we shall define $\gamma_{ij}=(\sigma_{ij}, \mu_{ij})$, where μ_{ij} are given by Lemma 6.4 and σ_{ij} are defined by formulas analogous to (6.12) with f_j and g_j replaced by σ_j and σ_j', respectively. The Proposition follows.

7. C_0-dimension of a subspace

We shall denote by $\mathscr{P}$ the class of C_0 operators having the property (P). If $T\in\mathscr{P}$ and $S(M)$ is the Jordan model of T we have $\bigwedge_{j<\infty}\gamma(m_j)=0$, $m_j=m_j[T]$, by Theorem 4.1 and Lemma 6.1. This fact and Proposition 6.6 suggest the following Definition.

Definition 7.1. The *dimension* γ_T of the operator $T\in\mathscr{P}$ is defined as

$$\gamma_T = \sum_{j=0}^{\infty}\gamma(m_j), \quad m_j = m_j[T]. \tag{7.1}$$

If T is an operator of class C_0 and $\mathfrak{M}\in\mathrm{Lat}_{\frac{1}{2}}(T)$ is such that $T_{\mathfrak{M}}\in\mathscr{P}$, then the *T-dimension* $\gamma_T(\mathfrak{M})$ is defined as

$$\gamma_T(\mathfrak{M}) = \gamma(\mathfrak{M}) = \gamma_{T_{\mathfrak{M}}}. \tag{7.2}$$

Remark 7.2. (i) Because $m_j[T^*]=m_j[T]^{\sim}$ (cf. [4], Corollary 2.8) we have $\gamma_{T^*}=\gamma_T^{\sim}$, $T\in\mathscr{P}$. Moreover, if T is of class C_0 and $\mathfrak{M}\in\mathrm{Lat}_{\frac{1}{2}}(T)$ is such that $T_{\mathfrak{M}}\in\mathscr{P}$, then

$$\gamma_{T^*}(\mathfrak{M}) = \gamma_T(\mathfrak{M})^{\sim}. \tag{7.3}$$

(ii) It is clear that $\gamma_T=0$ *if and only if T acts on the trivial space* $\{0\}$.

(iii) *The dimension γ_T is a quasisimilarity invariant of T.* Indeed, γ_T is defined in terms of the Jordan model.

We shall say C_0-dimension instead of T-dimension if no confusion is possible. The usual dimension is a particular case of the C_0-dimension. Indeed, the operator $T=0\in\mathscr{L}(\mathfrak{H})$ is a C_0 operator and each subspace $\mathfrak{M}\subset\mathfrak{H}$ is invariant for T. By Theorem 4.1, $T|\mathfrak{M}$ has the property (P) if and only if $\dim\mathfrak{M}<\infty$ and in this case $\gamma_T(\mathfrak{M})=(\sigma, 0)$ where $\sigma(0)=\dim\mathfrak{M}$ and $\sigma(z)=0$ otherwise.

Lemma 7.3. *An operator $T\in\mathscr{P}$ is a weak contraction if and only if $\gamma_T\in\Gamma$ and in this case*

$$\gamma_T = \gamma(d_T). \tag{7.4}$$

Proof. Obviously follows from Lemma 6.1 (iv), [6], Theorem 8.5 and [3], Definition 1.1.

By Proposition 6.6, Theorems 4.1 and 5.4, we have $\{\gamma_T;\ T\in\mathscr{P}\}=\tilde{\Gamma}$ and $\{\gamma_T;\ T$ has the property (Q)$\}=\tilde{\Gamma}_0$. It is natural to define $\mathscr{P}_0$ by

$$T\in\mathscr{P}_0 \text{ if and only if } T\in\mathscr{P} \text{ and } \gamma_T\in\tilde{\Gamma}_0. \tag{7.5}$$

Lemma 7.4. *If $T\in\mathscr{P}$ is acting on $\mathfrak{H}$ and $\mathfrak{H}_j\in\mathrm{Lat}(T)$ are such that $\mathfrak{H}_j\subset\mathfrak{H}_{j+1}$, $0\leq j<\infty$, and $\bigvee_{j\geq 0}\mathfrak{H}_j=\mathfrak{H}$, we have*

$$\gamma_T = \bigvee_{j\geq 0}\gamma_T(\mathfrak{H}_j). \tag{7.6}$$

Proof. Because $T|\mathfrak{H}_j \overset{i}{\prec} T$, we have $m_k[T|\mathfrak{H}_j] \leqq m_k[T]$ for each natural number k; therefore $\gamma(m_k[T|\mathfrak{H}_j]) \leqq \gamma(m_k[T])$ and the inequality $\gamma_T \geqq \bigvee_{j\geqq 0} \gamma_T(\mathfrak{H}_j)$ follows. Now, by Lemma 6.1 we shall have $\bigvee_{j\geqq 0} \gamma_T(\mathfrak{H}_j) \geqq \sum_{k=0}^{n} \gamma\left(\bigvee_{j\geqq 0} m_k[T|\mathfrak{H}_j]\right)$ for each natural number n; by Theorem 3.1 we infer $\bigvee_{j\geqq 0} \gamma_T(\mathfrak{H}_j) \geqq \sum_{k=0}^{n} \gamma(m_k[T])$. Since n is arbitrary the inequality $\bigvee_{j\geqq 0} \gamma_T(\mathfrak{H}_j) \geqq \gamma_T$ follows. Lemma 7.4 is proved.

Remark 7.5. From (7.3) it follows that Lemma 7.4 also holds under the assumption $\mathfrak{H}_j \in \operatorname{Lat}(T^*)$ instead of $\mathfrak{H}_j \in \operatorname{Lat}(T)$, $0 \leqq j < \infty$.

Corollary 7.6. *If $T, T' \in \mathscr{P}$, we have $\gamma_{T\oplus T'} = \gamma_T + \gamma_{T'}$.*

Proof. By Remark 7.2 (iii) it is enough to prove the Corollary for $T = S(M)$, $T' = S(M')$. For each j the space $\mathfrak{K}_j = \mathfrak{H}_j \oplus \mathfrak{H}'_j \in \operatorname{Lat}(T \oplus T')$, where $\mathfrak{H}_j = \mathfrak{H}(m_0) \oplus \oplus \mathfrak{H}(m_1) \oplus \ldots \oplus \mathfrak{H}(m_j)$, $\mathfrak{H}'_j = \mathfrak{H}(m'_0) \oplus \mathfrak{H}(m'_1) \oplus \ldots \oplus \mathfrak{H}(m'_j)$ and $\mathfrak{H}(M) = \bigvee_{j\geqq 0} \mathfrak{H}_j$, $\mathfrak{H}(M') = \bigvee_{j\geqq 0} \mathfrak{H}'_j$. By Lemma 7.4 we have $\gamma_{T\oplus T'} = \bigvee_{j\geqq 0} \gamma_{T\oplus T'}(\mathfrak{K}_j)$, $\gamma_T = \bigvee_{j\geqq 0} \gamma_T(\mathfrak{H}_j)$, $\gamma_{T'} = \bigvee_{j\geqq 0} \gamma_{T'}(\mathfrak{H}'_j)$. By Lemma 7.3 and [3], Theorem 1.3, the Corollary follows.

We shall now introduce a relation ϱ on the class $\mathscr{P}$, connected to index problems.

Definition 7.7. For $T_1, T_2 \in \mathscr{P}$ we write $T_1 \varrho T_2$ if there exist $T \in \mathscr{P}$ and $X \in \{T\}'$ such that T_1 and T_2 are quasisimilar to $T|\ker X$ and $T_{\ker X^*}$, respectively.

Lemma 7.8. *If $T \in \mathscr{P}$ and $\mathfrak{H} \in \operatorname{Lat}(T)$ then $T \varrho (T_{\mathfrak{H}} \oplus T_{\mathfrak{H}^\perp})$.*

Proof. The operator $S = T \oplus T_{\mathfrak{H}} \in \mathscr{P}$ by Proposition 4.4 and the operator X defined by $X(u \oplus v) = v \oplus 0$ commutes with S. It is easy to see that $S|\ker X$ is unitarily equivalent to T and $S_{\ker X^*}$ is unitarily equivalent to $T_{\mathfrak{H}} \oplus T_{\mathfrak{H}^\perp}$; Lemma 7.8 follows.

By Theorem 4.1 and Remark 7.2 (iii), $\gamma_{T_1} = 0$ if and only if $\gamma_{T_2} = 0$ if $T_1 \varrho T_2$. The connection between ϱ and γ is stronger than that, as it will be shown in the following propositions.

Theorem 7.9. *If $T_1, T_2 \in \mathscr{P}$ and $T_1 \varrho T_2$ then $\gamma_{T_1} = \gamma_{T_2}$.*

Proof. It is enough to show that for $T \in \mathscr{P}$ and $X \in \{T\}'$ we have $\gamma_T(\ker X) = \gamma_T(\ker X^*)$. Let T be acting on $\mathfrak{H}$ and let $S(M)$ be the Jordan model of T. As shown in the proof of Theorem 4.1 we have

$$\mathfrak{H} = \bigvee_{j\geqq 0} \mathfrak{H}_j, \quad \mathfrak{H}_j = \left(m_j(T)\mathfrak{H}\right)^- \in \operatorname{Hyp}\operatorname{Lat}(T). \tag{7.7}$$

For each natural j we have $X\mathfrak{H}_j \subset \mathfrak{H}_j$ and $X_j = X|\mathfrak{H}_j \in \{T|\mathfrak{H}_j\}'$. Because $T|\mathfrak{H}_j$ is of finite multiplicity, we infer by [3], Corollary 2.6, and Lemma 7.3,

$$\gamma(\ker X_j) = \gamma(\ker X_j^*). \tag{7.8}$$

Because obviously $Xm_j(T)|\ker X=0$, we have $\ker X_j\supset(m_j(T)\ker X)^-$ and, as in the proof of Theorem 4.1, we infer $\ker X=\bigvee_{j\geq 0}\ker X_j$. Therefore, by Lemma 7.4 applied to $T|\ker X$ it follows that

$$\gamma(\ker X)=\bigvee_{j\geq 0}\gamma(\ker X_j). \tag{7.9}$$

We have $X_j^*P_{\mathfrak{H}_j}|\ker X^*=P_{\mathfrak{H}_j}X^*P_{\mathfrak{H}_j}|\ker X^*=P_{\mathfrak{H}_j}X^*|\ker X^*=0$ so that $P_{\mathfrak{H}_j}(\ker X^*)\subset\ker X_j^*$. Because $P_{\mathfrak{H}_j}T^*=T^*_{\mathfrak{H}_j}P_{\mathfrak{H}_j}$ we shall have $P_{\mathfrak{H}_j}T^*|\ker X^*= =(T^*_{\mathfrak{H}_j}|\ker X_j^*)P_{\mathfrak{H}_j}|\ker X^*$. This relation implies that $(T^*|\ker X^*)_{\mathfrak{K}_j}$, where

$$\mathfrak{K}_j=(\ker(P_{\mathfrak{H}_j}|\ker X^*))^\perp=\ker X^*\ominus(\ker X^*\cap\mathfrak{H}_j^\perp)\in\operatorname{Lat}(T_{\ker X^*}),$$

is quasisimilar to some restriction of $T^*_{\mathfrak{H}_j}|\ker X_j^*$ and therefore

$$\gamma(\mathfrak{K}_j)\leqq\gamma(\ker X_j^*). \tag{7.10}$$

Now $\bigvee_{j\geq 0}\mathfrak{K}_j=\ker X^*\ominus(\ker X^*\cap(\bigcap_{j\geq 0}\mathfrak{H}_j^\perp))=\ker X^*$ so that from (7.8—10) and Lemma 7.4 applied to $T_{\ker X^*}$ we infer $\gamma(\ker X^*)=\bigvee_{j\geq 0}\gamma(\mathfrak{K}_j)\leqq\bigvee_{j\geq 0}\gamma(\ker X_j^*)= =\bigvee_{j\geq 0}\gamma(\ker X_j)=\gamma(\ker X)$.

By the same argument applied to T^* instead of T we infer $\gamma(\ker X)\leqq\gamma(\ker X^*)$. The Theorem follows.

Corollary 7.10. *If $T\in\mathscr{P}$ and $\mathfrak{H}\in\operatorname{Lat}(T)$ then $\gamma_T=\gamma_T(\mathfrak{H})+\gamma_T(\mathfrak{H}^\perp)$.*

Proof. Obviously follows from Corollary 7.6 and Theorem 7.9.

Corollary 7.11. *Let $T\in\mathscr{P}$ be acting on $\mathfrak{H}$ and let $\mathfrak{H}_j\in\operatorname{Lat}(T)$ be such that $\mathfrak{H}_0=\mathfrak{H}$, $\mathfrak{H}_j\supset\mathfrak{H}_{j+1}$ $(0\leqq j<\infty)$ and $\bigcap_{j\geq 0}\mathfrak{H}_j=\{0\}$. Then $\gamma_T=\sum_{j=0}^{\infty}\gamma_T(\mathfrak{K}_j)$, where $\mathfrak{K}_j= =\mathfrak{H}_j\ominus\mathfrak{H}_{j+1}$ $(0\leqq j<\infty)$.*

Proof. By Lemma 7.4 and Remark 7.5 we have $\gamma_T=\bigvee_{j\geq 0}\gamma_T(\mathfrak{H}_j^\perp)$. Because $\mathfrak{H}_{j+1}^\perp=\mathfrak{H}_j^\perp\oplus\mathfrak{K}_j$ and $\mathfrak{K}_j\in\operatorname{Lat}(T_{\mathfrak{H}_{j+1}^\perp})$ we have $\gamma_T(\mathfrak{H}_{j+1}^\perp)=\gamma_T(\mathfrak{H}_j^\perp)+\gamma_T(\mathfrak{K}_j)$ by the Corollary 7.10. By induction it follows that $\gamma_T(\mathfrak{H}_{j+1}^\perp)=\sum_{n=0}^{j}\gamma_T(\mathfrak{K}_n)$. Corollary 7.11 follows.

Corollary 7.12. *Let $T\in\mathscr{P}$ be acting on $\mathfrak{H}$. Then $T\in\mathscr{P}_0$ if and only if $\bigwedge_{j\geq 0}\gamma_T(\mathfrak{H}_j)=0$ for each decreasing sequence $\{\mathfrak{H}_m\}_{m=0}^\infty\subset\operatorname{Lat}(T)$ such that $\bigcap_{j\geq 0}\mathfrak{H}_j= =\{0\}$.*

Proof. Let us assume $T\in\mathscr{P}_0$. By Corollary 7.10 we have $\gamma_T=\gamma_T(\mathfrak{H}_j)+\gamma_T(\mathfrak{H}_j^\perp)$ so that by Lemma 7.4 we infer $\gamma_T=\gamma_T+\bigwedge_{j\geq 0}\gamma_T(\mathfrak{H}_j)$. Because $\gamma_T\in\tilde{\Gamma}_0$ it follows that $0=\bigwedge_{j\geq 0}\gamma_T(\mathfrak{H}_j)$.

Conversely, if $T\notin\mathcal{P}_0$, let $S(M)$ be the Jordan model of T. By the proof of [5], Theorem 1, there exist $\mathfrak{H}_j\in\text{Lat}(T)$ such that $\mathfrak{H}_{j+1}\subset\mathfrak{H}_j$, $\bigcap_{j\geq 0}\mathfrak{H}_j=0$ and the Jordan model of $T|\mathfrak{H}_j$ is $\bigoplus_{k\geq j} S(m_k)$. Because $\gamma_T(\mathfrak{H}_j^\perp)=\sum_{k<j}\gamma(m_k)\in\Gamma$, from the relation $\gamma_T=\gamma_T(\mathfrak{H}_j^\perp)+\gamma_T(\mathfrak{H}_j)$ we infer $(\gamma_T)_\infty=(\gamma_T(\mathfrak{H}_j))_\infty$ and therefore $\bigwedge_{j\geq 0}\gamma_T(\mathfrak{H}_j)\geq(\gamma_T)_\infty\neq 0$. Corollary 7.12 is proved.

We shall prove now a partial converse of Theorem 7.9.

Theorem 7.13. (i) *If* $T, T'\in\mathcal{P}$ *are weak contractions and* $\gamma_T=\gamma_{T'}$, *then* $T\varrho T'$.

(ii) *If* $T, T'\in\mathcal{P}$ *are such that* $\gamma_T=\gamma_{T'}$ *then there exists* $S\in\mathcal{P}$ *such that* $T\varrho S$ *and* $S\varrho T'$.

Proof. Let $S(M)$ and $S(M')$ be the Jordan models of T and T', respectively. The condition $\gamma_T=\gamma_{T'}$. is equivalent to $d_T=d_{T'}$; let us denote $d=d_T=d_{T'}$. If we denote $d_j=d/m_0m_1\dots m_{j-1}$, $d_{-j}=d/m_0'm_1'\dots m_{j-1}'$ for $1\leq j<\infty$ and $d_0=d$, we have $\bigwedge_{j\geq 0} d_j=\bigwedge_{j\geq 0} d_{-j}=1$ and by Theorem 4.1 and Proposition 4.4 the operator

$$K=\bigoplus_{j=-\infty}^{+\infty} S(d_j) \tag{7.11}$$

has property (P), that is, $K\in\mathcal{P}$. We define now an operator $X\in\{K\}'$ by $X(\bigoplus_{j=-\infty}^{+\infty} h_j)=$ $=\bigoplus_{j=-\infty}^{+\infty} k_j$ where

$$\begin{cases} k_j=P_{\mathfrak{H}(d_j)}h_{j-1} & \text{if } j\geq 1,\\ =(d_j/d_{j-1})h_{j-1} & \text{if } j\leq 0.\end{cases} \tag{7.12}$$

It is easy to see that $\ker X=\bigoplus_{j=0}^{+\infty}\ker(X|\mathfrak{H}(d_j))$ and $\ker X^*=\bigoplus_{j=0}^{-\infty}\ker(X^*|\mathfrak{H}(d_j))$. For $j\geq 0$

$$\ker(X|\mathfrak{H}(d_j))=d_{j+1}H^2\ominus d_jH^2$$

so that $S(d_j)|\ker(X|\mathfrak{H}(d_j))$ is unitarily equivalent to $S(d_j/d_{j+1})=S(m_j)$ and therefore $K|\ker X$ is unitarily equivalent to $S(M)$. We can analogously verify that $K_{\ker X^*}$ is unitarily equivalent to $S(M')$.

Let us remark that the minimal function of K coincides with the common determinant function of T and T'.

(ii) Let $S(M)$ and $S(M')$ be the Jordan models of T and T', respectively. The equality $\gamma_T=\gamma_{T'}$ is equivalent to $\sum_{j=0}^{\infty}\gamma(m_j)=\sum_{j=0}^{\infty}\gamma(m_j')$. By Proposition 6.7 we can find $\gamma_{ij}\in\tilde{\Gamma}$ such that $\sum_{j=0}^{\infty}\gamma_{ij}=\gamma(m_i)$ and $\sum_{i=0}^{\infty}\gamma_{ij}=\gamma(m_j')$, $0\leq i,j<\infty$. Because $\gamma_{ij}\leq$

$\leqq\gamma(m_i)$ we have $\gamma_{ij}\in\Gamma$ and therefore $\gamma_{ij}=\gamma(m_{ij})$ for $m_{ij}=\delta(\gamma_{ij})\in H_i^\infty$. We define the operator

$$S=\bigoplus_{i=0}^{\infty}\left(\bigoplus_{j=0}^{\infty}S(m_{ij})\right)=\bigoplus_{i=0}^{\infty}S_i,\quad S_i=\bigoplus_{j=0}^{\infty}S(m_{ij}),\quad 0\leqq i<\infty. \tag{7.13}$$

Because $\gamma(m_i)=\sum_{j=0}^{\infty}\gamma(m_{ij})$, the operator S_1 is a weak contraction and $\gamma_{S_i}=$ $=\gamma_{S(m_i)}$, $0\leqq i<\infty$ (cf. Lemma 7.3). By the proof of (i) we can find operators $K^i\in\mathscr{P}$ acting on $\mathfrak{H}_i$ and contractions $X_i\in\{K^i\}'$ such that

$$m_0[K^i]=m_i,\quad 0\leqq i<\infty, \tag{7.14}$$

$K^i|\ker X_i$ and $K^i_{\ker X_i^*}$ are unitarily equivalent to $S(m_i)$ and S_i, respectively. The operator $K=\bigoplus_{i=0}^{\infty}K^i$ is of class C_0, $X=\bigoplus_{i=0}^{\infty}X_i\in\{K\}'$ and $K|\ker X$, $K_{\ker X^*}$ are unitarily equivalent to $S(M)$, S, respectively.

Let us show that $K\in\mathscr{P}$. The spaces $\mathfrak{K}_i=\mathfrak{H}_0\oplus\mathfrak{H}_1\oplus\ldots\oplus\mathfrak{H}_i$ are invariant for T, $\bigvee_{i\geqq0}\mathfrak{K}_i=\bigoplus_{i=0}^{\infty}\mathfrak{H}_i$ and $m_0[K|\mathfrak{K}_i^\perp]=m_{i+1}$, $0\leqq i<\infty$. Because $T\in\mathscr{P}$ we have $\bigwedge_{i\geqq0}m_{i+1}=1$ and by Proposition 4.6 it follows that $K\in\mathscr{P}$. In particular S also has the property (P) by Proposition 4.4 and therefore we proved that $T\varrho S$. The relation $S\varrho T'$ is proved analogously. The Theorem follows.

Remark 7.14. If T and T' have finite multiplicities, then the operator K used for the proof of (i) also has finite multiplicity. Thus we obtain a new proof of Proposition 3.2 of [3].

8. C_0-Fredholm operators

The results of sec. 7 suggest the following generalization of [3], Definition 2.2.

Definition 8.1. Let T and T' be operators of class C_0 and let $X\in\mathscr{I}(T',T)$. Then X is called a *(T',T)-semi-Fredholm* operator if $X|(\ker X)^\perp$ is a $(T'|(\operatorname{ran}X)^-,$ $T_{(\ker X)^\perp})$*-lattice-isomorphism* and either $T|\ker X\in\mathscr{P}$ or $T'_{\ker X^*}\in\mathscr{P}$ holds. A (T',T)-semi-Fredholm operator X is *(T',T)-Fredholm* if both $T|\ker X$ and $T'_{\ker X^*}$ have property (P). If X is (T',T)-Fredholm, its index is defined as

$$\operatorname{ind}(X)=(\gamma_T(\ker X),\ \gamma_{T'}(\ker X^*))\in\tilde{\Gamma}\times\tilde{\Gamma}. \tag{8.1}$$

If X is (T',T)-semi-Fredholm but not (T',T)-Fredholm, we define

$$\operatorname{ind}(X)=+\infty\quad \textit{if}\quad T|\ker X\notin\mathscr{P}; \tag{8.2}$$

$$=-\infty\quad \textit{if}\quad T'_{\ker X^*}\notin\mathscr{P}.$$

Let us remark that for $T|\ker X\in\mathscr{P}_0$ and $T'_{\ker X^*}\in\mathscr{P}_0$, ind (X) is uniquely determined (modulo the relation "$\sim$") by the element $\gamma_T(\ker X)-\gamma_{T'}(\ker X^*)\in\mathscr{G}_0$ (cf. sec. 6).

In order to distinguish the operator introduced by Definition 8.1 from the operators considered in [3] we shall denote by $\Phi(T', T)$ and $\sigma\Phi(T', T)$ the set of (T', T)-Fredholm and (T', T)-semi-Fredholm operators, respectively. If $T'=T$ we write $\Phi(T)$, and $\sigma\Phi(T)$ instead of $\Phi(T, T)$, $\sigma\Phi(T, T)$, respectively.

Obviously $\mathscr{F}(T', T)\subset\Phi(T', T)$ and for $X\in\mathscr{F}(T', T)$ we have

(8.3) $$\operatorname{ind}(X)=\gamma(j(X))$$

if ind (X) is interpreted as an element of $\mathscr{G}_0$ and

$$\gamma(m/n)=\gamma(m)-\gamma(n)\quad\text{for}\quad m, n\in H_i^\infty.$$

The following Proposition extends [3], Corollary 2.6 and Remark 2.7.

Proposition 8.2. (i) *If $T, T'\in\mathscr{P}$ then $\Phi(T', T)=\mathscr{I}(T', T)$ and*

(8.4) $$\operatorname{ind}(X)\sim(\gamma_T, \gamma_{T'})\quad \textit{for}\quad X\in\mathscr{I}(T', T).$$

(ii) *If exactly one of the operators T and T' has property* (P) *then* $\Phi(T', T)=\emptyset$, $\sigma\Phi(T', T)=\mathscr{I}(T', T)$, *and for* $X\in\mathscr{I}(T', T)$,

$$\begin{aligned}\operatorname{ind}(X)&=+\infty\quad \textit{if}\quad T\notin\mathscr{P},\\ &=-\infty\quad \textit{if}\quad T'\notin\mathscr{P}.\end{aligned}$$

Proof. (i) because $T_{(\ker X)^\perp}$ and $T'|(\operatorname{ran} X)^-$ are quasisimilar and have the property (P) for any $X\in\mathscr{I}(T', T)$ (cf. Corollary 4.5 and Lemma 1.1) it follows that $X|(\ker X)^\perp$ is a lattice-isomorphism by Proposition 4.8 (i). In particular $\gamma_T((\ker X)^\perp)=\gamma_{T'}((\operatorname{ran} X)^-)$. By Corollary 7.10 it follows that $\gamma_T=\gamma_T(\ker X)+$ $+\gamma_T((\ker X)^\perp)$ and $\gamma_{T'}(\ker X^*)+\gamma_{T'}((\operatorname{ran} X)^-)=\gamma_{T'}$ so that

$$\gamma_T+\gamma_{T'}(\ker X^*)+\gamma=\gamma_{T'}+\gamma_T(\ker X)+\gamma$$

where $\gamma=\gamma_T((\ker X)^\perp)=\gamma_{T'}((\operatorname{ran} X)^-)$. Because

$$\gamma\leqq\gamma_T\wedge\gamma_{T'}$$

we infer by Lemma 6.5 (ii):

$$\gamma_T+\gamma_{T'}(\ker X^*)=\gamma_{T'}+\gamma_T(\ker X);$$

this means exactly ind $(X)\sim(\gamma_T, \gamma_{T'})$.

(ii) As in the preceding proof $T_{(\ker X)^\perp}$ and $T'|(\operatorname{ran} X)^-$ are quasisimilar and one of them must have the property (P) by Corollary 4.5. Then Corollary 4.3 and Proposition 4.8 (i) show that $X|(\ker X)^\perp$ is a lattice-isomorphism. To end the proof it is enough to show that $\Phi(T', T)=\emptyset$. Assume by example $T'\notin\mathscr{P}$; then for any $X\in\mathcal{I}(T', T)$, $T'|(\text{ran }X)^-\in\mathcal{P}$ so that $T'_{\ker X^*}\notin\mathcal{P}$ by Proposition 4.4. The case $T\notin\mathcal{P}$ is treated analogously. The Proposition is proved.

Example 8.3. The relation $\text{ind}(X)\sim(\gamma_T, \gamma_{T'})$ obtained in Proposition 8.2 cannot be improved. By example, if $\gamma_T=\gamma_{T'}$ it does not follow that $\gamma_T(\ker X)=$ $=\gamma_{T'}(\ker X^*)$ for each $X\in\mathcal{I}(T', T)$. Indeed, let us take $T'=S(M)\in\mathcal{P}$ such that $\gamma_{T'}=(0, \mu)$, $\mu\in\mathcal{M}_\infty$, and $T=\bigoplus_{j\geq 1} S(m_j)$. Then $\gamma_{T'}=\gamma_T+\gamma(m_0)$ so that $\gamma_T=\gamma_{T'}$ by the choice of γ_T. The inclusion $X\colon \bigoplus_{j\geq 1}\mathfrak{H}(m_j)\to\bigoplus_{j\geq 0}\mathfrak{H}(m_j)$ is one-to-one and $\gamma_{T'}(\ker X^*)=\gamma(m_0)\neq 0$.

Lemma 8.4. *For any two contractions T and T' of class C_0 we have $\sigma\Phi(T, T')^*=$ $=\sigma\Phi(T'^*, T^*)$, $\Phi(T, T')^*=\Phi(T'^*, T^*)$ and*

$$\text{ind}(X^*)=-\text{ind}(X)^\sim,\quad X\in\sigma\Phi(T, T') \tag{8.5}$$

(*here* $-(\gamma, \gamma')^\sim=(\gamma'^\sim, \gamma^\sim)$).

Proof. Cf. the proof of [3], Lemma 2.10.

The following Theorem extends [3], Theorem 2.11 to this more general setting.

Theorem 8.5. *Let T, T', T'' be operators of class C_0, $A\in\sigma\Phi(T', T)$, $B\in\sigma\Phi(T'', T')$. If* $\text{ind}(A)+\text{ind}(B)$ *makes sense we have $BA\in\sigma\Phi(T'', T')$ and*

$$\text{ind}(BA)\sim\text{ind}(A)+\text{ind}(B). \tag{8.6}$$

Proof. We have to follow the proof of [3], Theorem 2.11, replacing weak contractions by contractions having property (P) and using Proposition 4.10 instead of [3], Proposition 2.3. Only relation (8.6) needs some comments if A and B are C_0-Fredholm. With the notation of the proof of [3], Theorem 2.11 we have

$$\gamma_T(\ker BA)=\gamma_T(\ker A)+\gamma_{T'}(\mathfrak{H}_1)\quad \text{([3], relation (2.18))}, \tag{8.7}$$

$$\gamma_{T'}(\mathfrak{H}_2)=\gamma_{T'}(\mathfrak{H}_2^*)\quad \text{([3] relation (2.20))}, \tag{8.8}$$

$$\gamma_{T''}(\ker(BA)^*)=\gamma_{T''}(\ker B^*)+\gamma_{T'}(\mathfrak{H}_1^*)\quad \text{(relation (2.18)*)}, \tag{8.9}$$

and

$$\ker B=\mathfrak{H}_1\oplus\mathfrak{H}_2,\quad \ker A^*=\mathfrak{H}_1^*\oplus\mathfrak{H}_2^*\quad \text{(relation (2.19))}. \tag{8.10}$$

We infer, with the notation $\gamma=\gamma_{T'}(\mathfrak{H}_2)=\gamma_{T'}(\mathfrak{H}_2^*)$, that

$$\gamma_T(\ker BA)+\gamma=\gamma_T(\ker A)+\gamma_{T'}(\mathfrak{H}_1)+\gamma=\gamma_T(\ker A)+\gamma_{T'}(\ker B)$$

and

$$\gamma_{T''}(\ker(BA)^*)+\gamma=\gamma_{T''}(\ker B^*)+\gamma_{T'}(\mathfrak{H}_1^*)+\gamma=\gamma_{T'}(\ker A^*)+\gamma_{T''}(\ker B^*).$$

By addition we obtain

$$\gamma_T(\ker BA)+\gamma_{T'}(\ker A^*)+\gamma_{T''}(\ker B^*)+\gamma=$$
$$=\gamma_{T''}(\ker(BA)^*)+\gamma_T(\ker A)+\gamma_{T'}(\ker B)+\gamma$$

and since $\gamma \leqq \gamma_{T'}(\ker B) \wedge \gamma_{T'}(\ker A^*)$, Lemma 6.5 (ii) implies

$$\gamma_T(\ker BA) + \gamma_{T'}(\ker A^*) + \gamma_{T''}(\ker B^*) =$$
$$= \gamma_{T''}(\ker (BA)^*) + \gamma_T(\ker A) + \gamma_{T'}(\ker B).$$

The last relation is equivalent to (8.6). The Theorem follows.

The proof of [3], Theorem 2.12 is easily extended to the general setting.

Proposition 8.6. *Let T be an operator of class C_0 acting on the Hilbert space $\mathfrak{H}$ and let $X \in \{T\}'$ be such that $T|(X\mathfrak{H})^- \in \mathscr{P}$. Then $Y = I + X \in \Phi(T)$ and $(T|\ker Y) \varrho T_{\ker Y^*}$. In particular* $\operatorname{ind}(Y) \sim (0, 0)$.

Proof. We have shown in the proof of [3], Theorem 2.12 that $\ker Y = \ker (Y|\mathfrak{U})$, $\mathfrak{U} = (X\mathfrak{H})^-$, and that $(T|\mathfrak{U})_{\ker(Y|\mathfrak{U})^*}$ and $T_{\ker Y^*}$ are similar. This shows that $(T|\ker Y) \varrho T_{\ker Y^*}$.

In fact we shall prove a more general perturbation theorem.

Theorem 8.7. *Let T, T' be two operators of class C_0 acting on $\mathfrak{H}$, $\mathfrak{H}'$, respectively, and let us take $X \in \sigma\Phi(T', T)$, $Y \in \mathscr{I}(T', T)$. If $T'|(Y\mathfrak{H})^- \in \mathscr{P}$, we have $X + Y \in \sigma\Phi(T', T)$ and*

$$\operatorname{ind}(X+Y) \sim \operatorname{ind}(X) + (\gamma, \gamma), \quad \gamma = \gamma_{T'}((Y\mathfrak{H})^-). \tag{8.11}$$

Proof. We shall prove firstly that $(X+Y)(\mathfrak{H})$ is dense in each cyclic subspace of T' contained in $((X+Y)\mathfrak{H})^-$. The same argument applied to $(X+Y)^*$ will show, via [3], Lemma 1.4, that $(X+Y)|(\ker (X+Y))^\perp$ is a lattice-isomorphism.

In proving this we may assume that $\mathfrak{H}' = X\mathfrak{H} \vee Y\mathfrak{H}$ so that $\ker X^* = (P_{\ker X^*} Y\mathfrak{H})^-$; it follows that $T'_{\ker X^*} \overset{i}{\prec} T'|(Y\mathfrak{H})^-$ so that necessarily $T'_{\ker X^*} \in \mathscr{P}$ (cf. Corollary 4.5). Analogously we may assume that $T|\ker X \in \mathscr{P}$ so that X is C_0-Fredholm.

The injection J: $\ker Y \to \mathfrak{H}$ is C_0-Fredholm, $J \in \Phi(T, T|\ker Y)$ by the assumption of the Theorem, and therefore, by Theorem 8.5, $XJ \in \Phi(T', T|\ker Y)$; in particular $T'_{\ker(XJ)^*} = T'_{\mathfrak{U}} \in \mathscr{P}$ where $\mathfrak{U} = \ker (XJ)^* = (X(\ker Y))^\perp$.

Let us take $f \in ((X+Y)\mathfrak{H})^-$ and denote $\mathfrak{H}'_f = \bigvee_{j \geqq 0} T'^j f$. Because

$$P_{\mathfrak{U}}|\mathfrak{H}'_f \in \mathscr{I}(T'_{\mathfrak{U}}, T'|\mathfrak{H}'_f)$$

and $P_{\mathfrak{U}}(X+Y) \in \mathscr{I}(T'_{\mathfrak{U}}, T)$ are such that $\operatorname{ran}(P_{\mathfrak{U}}|\mathfrak{H}'_f) \subset (\operatorname{ran} P_{\mathfrak{U}}(X+Y))^-$ we infer by Corollary 4.11 the existence of a cyclic vector g of $T'|\mathfrak{H}'_f$ such that $P_{\mathfrak{U}} g = P_{\mathfrak{U}}(X+Y)h$ for some $h \in \mathfrak{H}$. Then the difference $g' = g - (X+Y)h \in (\operatorname{ran} XJ)^- = (X(\ker Y))^-$ and because XJ is a C_0-Fredholm operator we infer the existence of $h' \in \ker Y$ such that Xh' is cyclic for $T'|\mathfrak{H}'_{g'}$. Let us denote

$$\mathfrak{H}_0 = \mathfrak{H}_h \vee \mathfrak{H}_{h'} \quad \text{and} \quad Z = (X+Y)|\mathfrak{H}_0 \in \mathscr{I}(T', T|\mathfrak{H}_0).$$

Then $(Z\mathfrak{H}_0)^- \supset \mathfrak{H}'_f$; indeed, because $h' \in \ker Y$, we have $Zh' = Xh'$ and therefore $(Z\mathfrak{H}_0)^- \supset \mathfrak{H}'_{Xh'} = \mathfrak{H}'_{g'}$, in particular $g' \in (Z\mathfrak{H}_0)^-$. Now $g = g' + Zh \in (Z\mathfrak{H}_0)^-$ so that $(Z\mathfrak{H}_0)^- \supset \mathfrak{H}'_g = \mathfrak{H}'_f$. By Proposition 8.2 (ii) $Z \in \sigma\Phi(T', T|\mathfrak{H}_0)$ so that $\mathfrak{H}_f = (Z\mathfrak{K})^- = ((X+Y)\mathfrak{K})^-$ for some $\mathfrak{K} \in \operatorname{Lat}(T|\mathfrak{H}_0) \subset \operatorname{Lat}(T)$. The first part of the proof is done.

Let us assume that $T|\ker X \in \mathscr{P}$. Then $\ker(X+Y) \subset X^{-1}(Y\mathfrak{H})$ and

$$T|X^{-1}((Y\mathfrak{H})^-) = \begin{bmatrix} T|\ker X & * \\ 0 & T_1 \end{bmatrix}$$

where $T_1 \overset{i}{\prec} T'|(Y\mathfrak{H})^-$ so that T_1 has the property (P) (cf. Corollary 4.5). By Proposition 4.4, $T|X^{-1}((Y\mathfrak{H})^-) \in \mathscr{P}$ and therefore $T|\ker(X+Y) \in \mathscr{P}$. Analogously $T'_{\ker(X+Y)^*} \in \mathscr{P}$ if $T'_{\ker X^*} \in \mathscr{P}$ so that in any case $X+Y \in \sigma\Phi(T', T)$. Conversely, because $X = (X+Y) - Y$, $T|\ker X \in \mathscr{P}$ whenever $T|\ker(X+Y) \in \mathscr{P}$ and $T'_{\ker X^*} \in \mathscr{P}$ whenever $T'_{\ker(X+Y)^*} \in \mathscr{P}$. Therefore $\operatorname{ind}(X) \in \{+\infty, -\infty\}$ if and only if

$$\operatorname{ind}(X+Y) \in \{+\infty, -\infty\}$$

and in this case $\operatorname{ind}(X) = \operatorname{ind}(X+Y)$.

It remains to prove that (8.11) holds whenever $X \in \Phi(T', T)$. To do this let us remark that $P_{(Y\mathfrak{H})^\perp} \in \Phi(T'_{(Y\mathfrak{H})^\perp}, T')$ and $\operatorname{ind}(P_{(Y\mathfrak{H})^\perp}) = (\gamma, 0)$, where $\gamma = \gamma_{T'}((Y\mathfrak{H})^-)$. Because obviously $P_{(Y\mathfrak{H})^\perp}(X+Y) = P_{(Y\mathfrak{H})^\perp}X$ we infer by Theorem 8.5

$$\operatorname{ind}(X+Y) + (\gamma, 0) \sim \operatorname{ind}(P_{(Y\mathfrak{H})^\perp}X) \sim \operatorname{ind}(X) + (\gamma, 0) \tag{8.12}$$

so that

$$\gamma_T(\ker(X+Y)) + \gamma + \gamma_{T'}(\ker(P_{(Y\mathfrak{H})^\perp}X)^*) = \\ = \gamma_{T'}(\ker(X+Y)^*) + \gamma_T(\ker P_{(Y\mathfrak{H})^\perp}X)$$

and

$$\gamma_T(\ker P_{(Y\mathfrak{H})^\perp}X) + \gamma_{T'}(\ker X^*) = \\ = \gamma_{T'}(\ker(P_{(Y\mathfrak{H})^\perp}X)^*) + \gamma_T(\ker X) + \gamma.$$

By addition we obtain

$$\begin{cases} \gamma_T(\ker(X+Y)) + \gamma_{T'}(\ker X^*) + \gamma + \gamma_T(\ker P_{(Y\mathfrak{H})^\perp}X) + \gamma_{T'}(\ker(P_{(Y\mathfrak{H})^\perp}X)^*) = \\ = \gamma_{T'}(\ker(X+Y)^*) + \gamma_T(\ker X) + \gamma + \gamma_T(\ker P_{(Y\mathfrak{H})^\perp}X) + \gamma_{T'}(\ker(P_{(Y\mathfrak{H})^\perp}X)^*). \end{cases} \tag{8.13}$$

As shown in the proof of Theorem 8.5 (cf. relations (8.8—10)) we have

$$\gamma_T(\ker P_{(Y\mathfrak{H})^\perp}X) \leqq \gamma_T(\ker X) + \gamma_{T'}((Y\mathfrak{H})^-) = \gamma_T(\ker X) + \gamma$$

and

$$\gamma_{T'}(\ker(P_{(Y\mathfrak{H})^\perp}X)^*) \leqq \gamma_{T'}(\ker X^*) + \gamma.$$

Moreover, as shown in the first part of this proof, we have $\gamma_T(\ker(X+Y)) \leqq \leqq \gamma_T(X^{-1}((Y\mathfrak{H})^-)) \leqq \gamma_T(\ker X) + \gamma$ and analogously $\gamma_{T'}(\ker X^*) \leqq \gamma_{T'}(\ker(X+Y)^*) + \gamma$.

All these relations show, via Lemma 6.5 (ii), that from (8.13) we may infer

$$\gamma_T(\ker(X+Y))+\gamma_{T'}(\ker X^*)+\gamma = \gamma_{T'}(\ker(X+Y)^*)+\gamma_T(\ker X)+\gamma.$$

The last relation is equivalent to (8.11). Theorem 8.7 is proved.

We shall prove now a partial converse of Theorem 8.5. For simplifying notations we shall consider the case of a single operator T of class C_0.

Proposition 8.8. *Let T be an operator of class C_0 acting on $\mathfrak{H}$ and let $A\in\{T\}'$. If there exist $B, C\in\{T\}'$ such that $AB, CA\in\Phi(T)$, we have $A\in\Phi(T)$.*

Proof. Because $\ker A\subset\ker CA$ and $\ker A^*\subset\ker(AB)^*$ we obviously have $T|\ker A$, $T_{\ker A^*}\in\mathscr{P}$. We shall now prove that the mapping $\mathfrak{K}\to(A\mathfrak{K})^-$ is onto $\mathrm{Lat}\,(T|(A\mathfrak{H})^-)$. As in the first part of the proof of Theorem 8.7 we take $f\in(A\mathfrak{H})^-$ and remark that

$$P_{(A\mathfrak{H})^-\ominus(AB\mathfrak{H})^-}|\mathfrak{H}_f\in\mathscr{I}(T_{(A\mathfrak{H})^-\ominus(AB\mathfrak{H})^-}, T|\mathfrak{H}_f),$$

$$P_{(A\mathfrak{H})^-\ominus(AB\mathfrak{H})^-}A\in\mathscr{I}(T_{(A\mathfrak{H})^-\ominus(AB\mathfrak{H})^-}, T);$$

an application of Corollary 4.11 proves the existence of a cyclic $g\in\mathfrak{H}_f$ and of a vector $h\in\mathfrak{H}$ such that $g-Ah\in(AB\mathfrak{H})^-$. Because $AB\in\Phi(T)$ we find h' such that ABh' is cyclic for $T|\mathfrak{H}_{g-Ah}$. If $\mathfrak{H}_0=\mathfrak{H}_h\vee\mathfrak{H}_{Bh'}$ we obtain as in the proof of Theorem 8.7 $(A\mathfrak{H}_0)^-\supset\mathfrak{H}_f$ and therefore $\mathfrak{H}_f=(A\mathfrak{K})^-$ for some $\mathfrak{K}\in\mathrm{Lat}\,(T|\mathfrak{H}_0)\subset\mathrm{Lat}\,(T)$.

Analogously we can show, using the operator $A^*C^*\in\Phi(T^*)$, that the mapping $\mathfrak{K}\to(A^*\mathfrak{K})^-$ is onto $\mathrm{Lat}\,(T^*|(A^*\mathfrak{H})^-)$. By [3], Lemma 1.4, Proposition 8.8 follows.

Example 8.9. *For each pair $(\gamma,\gamma')\in\tilde{\Gamma}\times\tilde{\Gamma}$ there exist a C_0-operator T and $X\in\Phi(T)$ such that* $\mathrm{ind}\,(X)=(\gamma,\gamma')$.

Proof. As in the proof of [3], Proposition 3.1, we take operators $K, K'\in\mathscr{P}$ such that $\gamma_K=\gamma$, $\gamma_{K'}=\gamma'$ and we define $T=(K\otimes I)\oplus(K'\otimes I)$, where I denotes the identity on H^2. If U_+ denotes the unilateral shift on H^2, the required C_0-Fredholm operator is given by

$$X=(I\otimes U_+^*)\oplus(I\otimes U_+).$$

The proof of [3], Proposition 3.4, can be applied to obtain the following result.

Proposition 8.10. *For each operator T of class C_0 we have $\sigma\Phi(T)\cap\{T\}''=\Phi(T)\cap\{T\}''$ and* $\mathrm{ind}\,(X)\sim(0,0)$ *for $X\in\Phi(T)\cap\{T\}''$.*

The operators X_n, X defined in the proof of [3], Proposition 3.6, are such that $X_n\notin\sigma\Phi(T)$, $X\in\Phi(T)$, and $\lim_{n\to\infty}\|X_n-X\|=0$. Thus we have the following result.

Proposition 8.11. *The sets $\sigma\Phi(T)$, $\Phi(T)$ are not generally open subsets of $\{T\}'$, for T an operator of class C_0.*

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On the Jordan model of C_0 operators. II

HARI BERCOVICI

The existence of the Jordan model for operators of class C_0 was established in [9] and [10] for operators of finite multiplicity, in [4] for operators acting on separable Hilbert spaces and in [2] for operators acting on nonseparable spaces. In Sec. 2 of this note we give a common description of these three types of Jordan models. We also find a direct definition of the inner functions appearing in the Jordan model.

B. SZ.-NAGY and C. FOIAŞ have shown in [9], Sec. 7, that the space $\mathfrak{H}$ on which an operator T of class $C_0(N)$ is acting admits a decomposition into an approximate sum of invariant subspaces $\mathfrak{H}_j$ for T such that $T|\mathfrak{H}_j$ is multiplicity-free. In Sec. 3 of this note we extend this result to operators of class C_0 of arbitrary multiplicity. In fact we prove the existence of an almost-direct decomposition (cf. Theorem 3.4). Moreover, in the case of weak contractions (which contains the case discussed in [9]) we show that there exists a quasi-direct decomposition (cf. [7], ch. III). The main ingredient in Sec. 3 is a generalization of [4], Proposition 2.

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1. Preliminaries

We begin with some known facts about cardinal and ordinal numbers (cf. [12]). Here 0 is considered as ordinal number so that each ordinal α is the ordering type of the well-ordered set of ordinals $\{\beta: \beta<\alpha\}$. An ordinal number is a limit ordinal if it has no predecessor. Each ordinal number is of the form $\alpha+n$ with α a limit ordinal and $n<\omega$, where ω is the first transfinite ordinal. For each ordinal number α we denote by $\bar{\alpha}$ the associated cardinal number.

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Lemma 1.1. *For each cardinal number* $\aleph$ *we have* $\aleph=\text{card}\,\{\alpha:\ \bar{\alpha}<\aleph\}$.

Proof. Let us denote $A=\{\alpha:\ \bar{\alpha}<\aleph\}$ and let β be the ordinal number corresponding to A. Then $\bar{\beta}=\text{card}\,A$ and $\beta\notin A$ so that $\bar{\beta}=\text{card}\,A\geqq\aleph$. Now let γ be the first ordinal number such that $\bar{\gamma}=\aleph$; then $\gamma\notin A$ so that $\gamma\geqq\beta$ and therefore $\aleph=\bar{\gamma}\geqq\bar{\beta}=\text{card}\,A$. The Lemma follows by the Cantor—Bernstein theorem.

Remark 1.2. The preceding proof shows that $\beta=\gamma=$the first ordinal with $\bar{\beta}=\aleph$.

Corollary 1.3. *If* $\aleph_1<\aleph_2$ *are cardinal numbers and* $\aleph_2$ *is transfinite, we have* $\aleph_2=\text{card}\,\{\alpha:\ \aleph_1\leqq\bar{\alpha}<\aleph_2\}$.

Proof. By Lemma 1.1 we have $\aleph_2=\text{card}\,\{\alpha:\ \bar{\alpha}<\aleph_2\}=\text{card}\,\{\alpha:\ \bar{\alpha}<\aleph_1\}+$ $+\text{card}\,\{\alpha:\ \aleph_1\leqq\bar{\alpha}<\aleph_2\}=\aleph_1+\aleph$, where $\aleph=\text{card}\,\{\alpha:\ \aleph_1\leqq\bar{\alpha}<\aleph_2\}$. Because $\aleph_2$ is transfinite $\aleph_1$ or $\aleph$ must be transfinite and we have $\aleph_2=\max\,\{\aleph_1,\aleph\}=\aleph$ because $\aleph_1\neq\aleph_2$. The Corollary is proved.

Corollary 1.4. *If* $\aleph$ *is a transfinite cardinal number then* $\aleph'=\text{card}\,\{\alpha:\ \bar{\alpha}=\aleph\}$ *is the first cardinal greater than* $\aleph$.

Proof. We have only to apply the preceding Corollary for $\aleph_1=\aleph$ and $\aleph_2=$ $=$the successor of $\aleph$ in the series of cardinal numbers.

Now let us recall that the multiplicity μ_T of the operator T acting on the Hilbert space $\mathfrak{H}$ is the minimum dimension of a subspace $\mathfrak{M}\subset\mathfrak{H}$ such that $\mathfrak{H}=\bigvee_{n\geqq 0}T^n\mathfrak{M}$. It is obvious that

(1.1) $$\mu_T\leqq\dim\mathfrak{H}\leqq\aleph_0\cdot\mu_T$$

so that the equality

(1.2) $$\mu_T=\dim\mathfrak{H}$$

holds whenever $\dim\mathfrak{H}>\aleph_0$ or $\mu_T\geqq\aleph_0$.

Lemma 1.5. *We have* $\mu_T=\mu_{T^*}$ *for any operator* T *of class* C_0.

Proof. For $\mu_T<\aleph_0$ see [10], Theorem 3. Therefore if $\mu_T\geqq\aleph_0$ we also have $\mu_{T^*}\geqq\aleph_0$ and the equality $\mu_T=\mu_{T^*}$ follows from (1.2).

Let us recall that the operator T can be injected into T' ($T\overset{i}{\prec}T'$) if there exists an injection X such that $T'X=XT$. If there exists a quasi-affinity X such that $T'X=XT$ we say that T is a quasi-affine transform of T' ($T\prec T'$).

Lemma 1.6. *If* T *and* T' *are two operators of class* C_0 *and* $T\overset{i}{\prec}T'$, *we have* $\mu_T\leqq\mu_{T'}$. *If* $T\prec T'$ *then* $\mu_T=\mu_{T'}$.

Proof. Let T, T' be acting on $\mathfrak{H}$, $\mathfrak{H}'$, respectively, and let X be any injection such that $T'X=XT$. Then X^* has dense range; if $\mathfrak{M}\subset\mathfrak{H}'$ is such that $\bigvee_{n\geq 0} T'^{*n}\mathfrak{M}=\mathfrak{H}'$ we have $\bigvee_{n\geq 0} T^{*n}X^*\mathfrak{M}=\mathfrak{H}$ and obviously $\dim (X^*\mathfrak{M})^-\leq\dim\mathfrak{M}$. Therefore $\mu_{T*}\leq\mu_{T'*}$ so that $\mu_T\leq\mu_{T'}$ by Lemma 1.5. If $T\prec T'$, we may assume X has dense range so that $\mu_{T'}\leq\mu_T$ obviously also follows. The Lemma is proved.

If T is an operator of class C_0 we shall use the notation

$$\mu_T(m)=\mu_{T|(\operatorname{ran} m(T))^-},\quad m\in H_i^\infty \tag{1.3}$$

where H_i^∞ denotes the set of inner functions in H^∞. We shall consider the set H_i^∞ (pre)ordered as in [2]. Namely, we write $m_1\leq m_2$ if m_1 divides m_2 or, equivalently, if $|m_1(z)|\geq|m_2(z)|$ for $|z|<1$.

The following Lemma also follows from [8], Theorem III.6.3; we prove it for the sake of completeness.

Lemma 1.7. *If T is an operator of class C_0 and $m_1, m_2\leq m_T$, then $(\operatorname{ran} m_1(T))^-\subset(\operatorname{ran} m_2(T))^-$ if and only if $m_1\geq m_2$.*

Proof. If $m_1\geq m_2$, we have $m_1=m_2m_3$ so that obviously $\operatorname{ran} m_1(T)\subset\operatorname{ran} m_2(T)$. Conversely, if $(\operatorname{ran} m_1(T))^-\subset(\operatorname{ran} m_2(T))^-$, we have $(m_T/m_2)(T)m_1(T)=0$ and therefore $m_T\leq(m_T/m_2)m_1$. The Lemma follows.

Corollary 1.8. *The function μ_T is decreasing on H_i^∞.*

Proof. Obviously follows from Lemma 1.6 and the proof of Lemma 1.7.

Corollary 1.9. *If T and T' are operators of class C_0 and $T\overset{i}{\prec}T'$, we have $\mu_T(m)\leq\mu_{T'}(m)$, $m\in H_i^\infty$. If $T\prec T'$, we have $\mu_T(m)=\mu_{T'}(m)$, $m\in H_i^\infty$.*

Proof. If X is any injection such that $T'X=XT$, we also have $m(T')X=Xm(T)$, $m\in H_i^\infty$, and therefore $T|(\operatorname{ran} m(T))^-\overset{i}{\prec}T'|(\operatorname{ran} m(T'))^-$. If X is a quasi-affinity we have $(X\operatorname{ran} m(T))^-=(\operatorname{ran} m(T'))^-$ so that $T|(\operatorname{ran} m(T))^-\prec T'|(\operatorname{ran} m(T'))^-$. The Corollary follows by Lemma 1.6.

We shall see that the converse of Corollary 1.9 is also true.

Let us recall that for an operator T of class C_0 acting on $\mathfrak{H}$ and for $f\in\mathfrak{H}$, m_f stands for the minimal function of $T|\mathfrak{H}_f$, where

$$\mathfrak{H}_f=\bigvee_{n\geq 0} T^n f. \tag{1.4}$$

The following result is proved in [4], Proposition 1.

Proposition 1.10. *The set $\{f\colon m_f=m_T\}$ is dense in $\mathfrak{H}$.*

In fact, from the proof of [10], Theorem 1, it follows that $\{f\colon m_f = m_T\}$ is a dense G_δ.

Finally let us recall the definition of approximate sums and quasi-direct sums (cf. [6] and [5], ch. III). Let $\mathfrak{H}$ be a Hilbert space and $\{\mathfrak{H}_j\}_{j\in J}$ be a family of subspaces of $\mathfrak{H}$ such that

(1.5) $$\mathfrak{H} = \bigvee_{j\in J} \mathfrak{H}_j.$$

We say that $\mathfrak{H}$ is the *approximate sum* of $\{\mathfrak{H}_j\}_{j\in J}$ if for each subset $K\subset J$ we have

(1.6) $$\Big(\bigvee_{j\in K} \mathfrak{H}_j\Big)\cap\Big(\bigvee_{j\notin K} \mathfrak{H}_j\Big) = \{0\}.$$

We say that H is the *quasi-direct sum* of $\{\mathfrak{H}_j\}_{j\in J}$ if for each family $\{K_a\}_{a\in A}$ of subsets of J we have

(1.7) $$\bigcap_{a\in A}\Big(\bigvee_{j\in K_a} \mathfrak{H}_j\Big) = \bigvee_{j\in K} \mathfrak{H}_j, \quad K = \bigcap_{a\in A} K_a.$$

We shall introduce an intermediate notion. Namely, we shall say that $\mathfrak{H}$ is the *almost-direct sum* of $\{\mathfrak{H}_j\}_{j\in J}$ if the relation (1.7) holds whenever $K=\emptyset$.

Lemma 1.11. *Let $\{\mathfrak{H}_j\}_{j\in J}$ be a family of subspaces of $\mathfrak{H}$ such that* (1.5) *holds. $\mathfrak{H}$ is the almost-direct sum of $\{\mathfrak{H}_j\}_{j\in J}$ if and only if we have*

(1.8) $$\mathfrak{H} = \bigvee_{j\in J} \mathfrak{H}_j^*, \quad \textit{where} \quad \mathfrak{H}_j^* = \Big(\bigvee_{k\neq j} \mathfrak{H}_k\Big)^\perp, \quad j\in J.$$

Proof. If $\mathfrak{H}$ is the almost-direct sum of $\{\mathfrak{H}_j\}_{j\in J}$, we have

$$\bigvee_{j\in J} \mathfrak{H}_j^* = \bigvee_{j\in J}\Big(\bigvee_{k\neq j} \mathfrak{H}_k\Big)^\perp \supset \Big(\bigcap_{j\in J}\Big(\bigvee_{k\neq j} \mathfrak{H}_k\Big)\Big)^\perp = (\{0\})^\perp = \mathfrak{H}.$$

Conversely, if (1.8) holds and $\{K_a\}_{a\in A}$ are such that $\bigcap_{a\in A} K_a=\emptyset$, then

$$\Big(\bigcap_{a\in A}\Big(\bigvee_{j\in K_a} \mathfrak{H}_j\Big)\Big)^\perp \supset \bigvee_{a\in A}\Big(\bigvee_{j\in K_a} \mathfrak{H}_j\Big)^\perp \supset \bigvee_{a\in A}\Big(\bigvee_{j\notin K_a} \mathfrak{H}_j^*\Big)$$

and because $\bigcup_{a\in A}\{j\colon j\notin K_a\}=J$, we have $\bigvee_{a\in A}\Big(\bigvee_{j\notin K_a} \mathfrak{H}_j^*\Big)=\bigvee_{j\in J} \mathfrak{H}_j^*=\mathfrak{H}$. The Lemma follows.

2. Jordan models

Definition 2.1. A *model function* is a function M which associates with every ordinal number α an inner function $M(\alpha)$ such that

(i) $M(\beta) \leqq M(\alpha)$ whenever $\bar{\alpha} \leqq \bar{\beta}$;

(ii) $M(\alpha) = M(\beta)$ whenever $\bar{\alpha} = \bar{\beta}$;

(iii) $M(\alpha) = 1$ for some α.

If M is a model function, the *operator* $S(M)$ acting on $\mathfrak{H}(M)$ is defined as

$$S(M) = \bigoplus_{\alpha} S(m_\alpha), \quad m_\alpha = M(\alpha). \tag{2.1}$$

Lemma 2.2. *Let* $\{m_a\}_{a\in A}\subset H_i^\infty$ *be a totally ordered family of nonconstant functions. Then the multiplicity of* $T=\bigoplus_{a\in A} S(m_a)$ *equals* card A.

Proof. If A is finite, the assertion follows from [9]. If A is infinite, it follows from the inequality $\mu_{T'\oplus T''}\geqq\mu_{T'}$ that μ_T is also infinite so that $\mu_T=\dim\left(\bigoplus_{a\in A}\mathfrak{H}(m_a)\right)$ by (1.2). Therefore, card $A\leqq\mu_T\leqq$card $A\cdot\aleph_0=$card A. The Lemma follows.

Corollary 2.3. *If M is a model function, we have* $\mu_{S(M)}=\bar{\alpha}$, *where α is the first ordinal number such that* $m_\alpha=1$.

Proof. If α is the first ordinal number with $m_\alpha=1$, it follows from Definition 2.1 (ii) that $\{\beta\colon m_\beta\neq 1\}=\{\beta\colon \bar{\beta}<\bar{\alpha}\}$ so that the Corollary follows by Lemmas 1.1 and 2.2.

Definition 2.4. For any operator T of class C_0 we define

$$M_T(\alpha) = m_\alpha[T] = \wedge\{m\colon \mu_T(m)\leqq\bar{\alpha}\} \tag{2.2}$$

where "$\wedge$" stands for the greatest common inner divisor.

Let us remark that $M_T(0)=m_0[T]$ coincides with the minimal function of T. M_T is a model function. Indeed, the conditions (i) and (ii) of Definition 2.1 are obviously satisfied while (iii) is satisfied because $M_T(\alpha)=1$ whenever $\bar{\alpha}=\dim\mathfrak{H}$ $(\mu_T(1)=\mu_T\leqq\dim\mathfrak{H}$ by (1.1)). It is also clear by Corollary 1.9 that M_T is invariant with respect to quasi-affine transforms.

Proposition 2.5. *If M is a model function we have* $M_{S(M)}=M$.

Proof. Let us put $T=S(M)$, $M'=M_T$, $m_\alpha=M(\alpha)$ and $m'_\alpha=M'(\alpha)$. Let us assume $m\geqq m_\beta$. Because $m(S(m'))=0$ if and only if $m\geqq m'$ (moreover, $S(m')|(\operatorname{ran} m(S(m')))^-$ is quasisimilar to $S(m'/m\wedge m')$), by Lemma 2.2 we have

$$\mu_T(m)\leqq\mu_T(m_\beta)\leqq\text{card}\,\{\alpha;\ \bar{\alpha}<\beta\}=\bar{\beta}.$$

Conversely, let us assume m not $\geqq m_\beta$. Then $\mu_T(m)\geqq$card $\{\alpha\colon\bar{\alpha}\leqq\beta\}>\bar{\beta}$. By (2.2) we infer $m'_\beta=m_\beta$ and the Proposition is proved.

Now let us recall the definition of a Jordan operator (cf. [2]). If $\aleph$ is a cardinal number and T is an operator, $T^{(\aleph)}$ denotes the direct sum of $\aleph$ copies of T.

Definition 2.6. A *Jordan operator*, is an operator of the form

$$T=\bigoplus_{m\in H_i^\infty} S(m)^{(h(m))} \tag{2.3}$$

where h is a cardinal number valued function on H_i^∞ such that
(i) $A=\{m: h(m)\neq 0\}$ is a well anti-ordered set;
(ii) $\{m\in A: h(m)<\aleph_0\}$ is a decreasing (possibly finite or empty) sequence;
(iii) $h(m)>\sum_{m'>m} h(m')$ whenever $\sum_{m'>m} h(m')\geqq\aleph_0$.

Our condition (iii) slightly differs from condition (b) of [2], Definition 1. If we analyse the proof of [2], Theorem 1, we remark that the Jordan model obtained there satisfies the actual condition (iii). Indeed, if $h(m)=\sum_{m'>m} h(m')$ it is easy to see that (with the notation of [2]) m is not a saltus point for f.

Let us remark that, by Lemma 2.2, we have

$$\mu_T(u)=\sum_{u \text{ not} \geqq m} h(m), \quad u\in H_i^\infty \tag{2.4}$$

if T is the operator given by (2.3).

Theorem 2.7. *Each operator T of class C_0 is quasisimilar to $S(M_T)$.*

Proof. From Corollary 1.9 it follows that M_T is a quasisimilarity invariant Therefore, by [2], Theorem 1, it is enough to prove that for T a Jordan operator in the sense of Definition 2.6, T and $S(M_T)$ are unitarily equivalent. So, let T be given by (2.3) and denote $m_\alpha=M_T(\alpha)$. It is enough to prove that

$$\operatorname{card}\{\alpha;\ m_\alpha=m\}=h(m), \quad m\in H_i^\infty. \tag{2.5}$$

Let us assume firstly that $h(m)=0$. There exists a last $m^1\in A=\{m': h(m')\neq 0\}$ such that $m^1\geqq m\wedge m_T$. Thus for $m'\in A$ we have $m(S(m'))=0$ if and only if $m^1(S(m'))=0$. By Lemma 2.2 we infer $\mu_T(m)=\mu_T(m^1)$ so that by (2.2) there is no α such that $m_\alpha=m$ and (2.5) is proved in this case.

Now let us assume $0<h(m)<\aleph_0$. Then the sum

$$k=\sum_{m'>m} h(m') \tag{2.6}$$

is finite by Definition 2.6 (iii). It is clear that $\mu_T(u)\leqq k$ if and only if $u\geqq m$ and therefore if and only if $\mu_T(u)\leqq k+n-1$, $n=h(m)$. We obtain

$$m_k=m_{k+1}=\ldots=m_{k+n-1}=m.$$

Analogously we obtain $m_{k+n}=m'$ where m' is the predecessor of m in A; thus $\{\alpha: m_\alpha=m\}=\{k, k+1, \ldots, k+n-1\}$ and (2.5) is proved in this case also.

Finally let us assume $h(m)\geqq\aleph_0$. If $k\leqq\bar{\alpha}<h(m)$, where k is defined by (2.6), we have $\mu_T(u)\leqq\bar{\alpha}$ if and only if $u\geqq m$. Indeed, if u not $\geqq m$, we have $\mu_T(u)\geqq h(m)$ by Lemma 2.2. Therefore

$$m_\alpha=m \quad \text{whenever} \quad k\leqq\bar{\alpha}<h(m). \tag{2.7}$$

If $\tilde{\alpha} \geqq h(m)$ and m' is the predecessor of m in A (if m is the first element of A we take $m'=1$) then, again by Lemma 2.2, $\mu_T(m') = \sum_{m''>m'} h(m'') = \sum_{m''>m} h(m'') + h(m) = h(m)$ so that $m_\alpha \neq m$. Therefore

$$\{\alpha;\ m_\alpha = m\} = \{\alpha;\ k \leqq \tilde{\alpha} < h(m)\}$$

and (2.5) follows by Corollary 1.4 in this case. The Theorem is proved.

Let us recall that $f^\sim(z) = \overline{f(\bar{z})}$ for $f \in H^\infty$.

Corollary 2.8. *For each operator T of class C_0 we have $\mu_T(m) = \mu_{T^*}(m^\sim)$, $m \in H_i^\infty$ and $m_\alpha[T^*] = m_\alpha[T]^\sim$ for each ordinal number α.*

Proof. Since $\mu_T(m)$ is a quasisimilarity invariant it is enough to prove the Corollary for $T = S(M)$ and in this case the assertions of the Corollary become obvious.

We are now able to prove the converse of Corollary 1.9.

Corollary 2.9. *For two operators T, T' of class C_0 the following assertions are equivalent:*

(i) $T \overset{i}{\prec} T'$;

(i)$_*$ $T^* \overset{i}{\prec} T'^*$;

(ii) $\mu_T(m) \leqq \mu_{T'}(m)$, $m \in H_i^\infty$;

(iii) $m_\alpha[T] \leqq m_\alpha[T']$ *for each ordinal number* α.

Proof. (i)⇒(ii) by Corollary 1.9. (ii)⇒(iii) by Definition 2.4.

(iii)⇒(i). Let us denote $m_\alpha = m_\alpha[T]$, $m'_\alpha = m_\alpha[T']$. There exist (cf. [9]) isometries R_α: $\mathfrak{H}(m_\alpha) \to \mathfrak{H}(m'_\alpha)$ such that $S(m'_\alpha)R_\alpha = R_\alpha S(m_\alpha)$. If X and Y are two quasi-affinities such that $T'X = XS(M_{T'})$ and $S(M_T)Y = YT$, the operator $Z = X(\bigoplus_\alpha R_\alpha)Y$ is an injection and $T'Z = ZT$.

Finally, the condition $m_\alpha[T] \leqq m_\alpha[T']$ is equivalent to $m_\alpha[T^*] \leqq m_\alpha[T'^*]$ by Corollary 2.8; it follows that the condition (i)$_*$ is equivalent with (i)—(iii). The Corollary is proved.

The following Corollary gives in particular a new proof of [11], Theorem 1.

Corollary 2.10. *For two operators T, T' of class C_0 the following assertions are equivalent:*

(i) $T \prec T'$;

(ii) $T \overset{i}{\prec} T'$ *and* $T' \overset{i}{\prec} T$;

(iii) $\mu_T(m) = \mu_{T'}(m)$, $m \in H_i^\infty$;

(iv) *T and T' are quasisimilar.*

Proof. (i)⇒(iii) and (ii)⇒(iii) by Corollary 1.9. (iii)⇒(iv). By Definition 2.4 we infer $m_\alpha[T]=m_\alpha[T']$ so that T and T' are quasisimilar having the same Jordan model. (iv)⇒(i) and (iv)⇒(ii) are obvious.

Corollary 2.11. *If T is an operator of class C_0 on the Hilbert space $\mathfrak{H}$ then each invariant subspace $\mathfrak{M}$ of T is of the form $\mathfrak{M}=(X\mathfrak{H})^-=\ker Y$ for some $X, Y\in\{T\}'$.*

Proof. Let us denote by T' the restriction $T|\mathfrak{M}$ and by J the inclusion of $\mathfrak{M}$ into $\mathfrak{H}$. By Corollary 2.9 we have $T'^* \overset{i}{\prec} T^*$ so that there exists an injection $Z\colon \mathfrak{M}\to\mathfrak{H}$ such that $T^*Z=ZT'^*$. Then $X=JZ^*\in\{T\}'$ and $(X\mathfrak{H})^-=J(Z^*\mathfrak{H})^-= =J\mathfrak{M}=\mathfrak{M}$. Analogously $\mathfrak{M}^\perp=(Y^*\mathfrak{H})^-$ for some $Y^*\in\{T^*\}'$ so that $\mathfrak{M}=\ker Y$. The Corollary follows.

As shown by Proposition 2.5 and Theorem 2.7 the operators of the form $S(M)$ with M a model function form a complete system of representants for the class C_0 with respect to the relation of quasisimilarity. Sometimes it is more convenient to use Jordan operators as given by Definition 2.6.

Proposition 2.12. *If M is a model function and*

$$h(m) = \text{card}\,\{\alpha;\ m_\alpha = m\},\quad m\in H_i^\infty, \tag{2.8}$$

then the function h satisfies the conditions (i)—(iii) *of Definition* 2.6.

Proof. (i) $A=\{m\colon h(m)\neq 0\}$ is the range of the decreasing function M defined on a well-ordered set so that obviously A is well anti-ordered.

(ii) If $h(m)<\aleph_0$ we infer $m\neq m_\alpha$ for $\alpha\geqq\omega$. Therefore $\{m\colon 0<h(m)<\aleph_0\}$ is the range of the function M on a segment of the natural numbers.

(iii) Let us assume $h(m)\geqq\aleph_0$ and let α be the first ordinal number such that $m_\alpha=m$. By Lemma 1.1 $\bar\alpha=\sum_{m'>m} h(m')$. If α is a finite number, the relation $h(m)>\bar\alpha$ is obvious. If α is transfinite we infer by Corollary 1.4 and Definition 2.1 (ii)

$$h(m)\geqq \text{card}\,\{\beta;\ \bar\beta=\bar\alpha\}=\bar\alpha'>\bar\alpha=\sum_{m'>m} h(m'),$$

where $\bar\alpha'$ is the successor of $\bar\alpha$ in the series of cardinal numbers. The Proposition is proved.

From now on we shall call *Jordan operators* the operators $S(M)$ with M a model function and $S(M_T)$ will be called the *Jordan model of the operator T* of class C_0.

Remark 2.13. *For any operator T of class C_0 we have*

$$\mu_T(m_\alpha[T])\leqq\bar\alpha. \tag{2.9}$$

Indeed, we have only to verify (2.9) for $T=S(M)$ and in this case (2.9) is obvious.

3. Decomposition theorems

The following Lemma is essentially contained in [9], sec. 2. We prove it for the sake of completeness. Let us remark that Lemma 3.1 also follows from [11], Theorem 2.

Lemma 3.1. *Let T and T' be operators of class C_0, both quasisimilar to $S(m)$ ($m\in H_i^\infty$) and let A be such that $T'A=AT$. Then A is one-to-one if and only if it has dense range.*

Proof. Let X and Y be two quasi-affinities such that $TX=XS(m)$ and $S(m)Y=YT'$. The operator YAX commutes with $S(m)$ so that $YAX=u(S(m))$ for some $u\in H^\infty$ by Sarason's Theorem [7]. If A is one to one or has dense range then so does $u(S(m))$ and therefore $u\wedge m=1$. Now

$$XYAXY = Xu(S(m))Y = u(T)XY = XYu(T')$$

so that $XYA=u(T)$ and $AXY=u(T')$. $u(T)$ and $u(T')$ are quasi-affinities because $u\wedge m=1$ and $\operatorname{ran} A\supset\operatorname{ran} u(T')$, $\ker A\subset\ker u(T)$ so that A is a quasi-affinity in both cases.

The following result is a generalisation of [4], Proposition 2.

Proposition 3.2. *Let T and T' be two operators of class C_0 acting on $\mathfrak{H}$, $\mathfrak{H}'$, respectively, X be a quasi-affinity such that $T'X=XT$, $f\in\mathfrak{H}$ be such that $m_f=m_T$ and $\varepsilon>0$. Then there exist subspaces $\mathfrak{H}_1$, $\mathfrak{M}_1$ invariant for T and $\mathfrak{H}_1^*$, $\mathfrak{M}_1^*$ invariant for T'^* such that:*

(i) $\mathfrak{H}_1=\mathfrak{H}_f$;

(ii) $\|P_{\mathfrak{H}_1^*}Xf-Xf\|<\varepsilon$;

(iii) $\mathfrak{M}_1=(X^*\mathfrak{H}_1^*)^\perp$, $\mathfrak{M}_1^*=(X\mathfrak{H}_1)^\perp$;

(iv) $\mathfrak{H}_1\vee\mathfrak{M}_1=\mathfrak{H}$, $\mathfrak{H}_1\cap\mathfrak{M}_1=\{0\}$, $\mathfrak{H}_1^*\vee\mathfrak{M}_1^*=\mathfrak{H}'$, $\mathfrak{H}_1^*\cap\mathfrak{M}_1^*=\{0\}$;

(v) $P_{\mathfrak{H}_1^*}X|\mathfrak{H}_1$ *and* $P_{\mathfrak{M}_1^*}X|\mathfrak{M}_1$ *are quasi-affinities.*

Proof. The conditions (i)—(v) are not independent. Indeed, let us assume that (i) and (iii) are verified and $P_{\mathfrak{H}_1^*}X|\mathfrak{H}_1$ is a quasi-affinity. It follows that $T'|(X\mathfrak{H}_1)^-$ and $(T^*|\mathfrak{H}_1^*)^*$ are both quasisimilar to $S(m_T)$ and $P_{\mathfrak{H}_1^*}|(X\mathfrak{H}_1)^-$ has dense range; by Lemma 3.1 $P_{(X\mathfrak{H}_1)^-}|\mathfrak{H}_1^*$ also has dense range, that is $(X\mathfrak{H}_1)^-=(P_{(X\mathfrak{H}_1)^-}\mathfrak{H}_1^*)^-$. Then $\mathfrak{M}_1=\ker P_{\mathfrak{H}_1^*}X$ so that $\mathfrak{H}_1\cap\mathfrak{M}_1=\ker P_{\mathfrak{H}_1^*}X|\mathfrak{H}_1=\{0\}$. Analogously $\mathfrak{H}_1^*\cap\mathfrak{M}_1^*=\{0\}$. Now $\mathfrak{H}'=(X\mathfrak{H}_1)^-\oplus\mathfrak{M}_1^*=(P_{(X\mathfrak{H}_1)^-}\mathfrak{H}_1^*)\vee\mathfrak{M}_1^*=\mathfrak{H}_1^*\vee\mathfrak{M}_1^*$ and analogously $\mathfrak{H}_1\vee\mathfrak{M}_1=\mathfrak{H}$. Obviously $\mathfrak{M}_1^*=(P_{\mathfrak{M}_1^*}X\mathfrak{H})^-=(P_{\mathfrak{M}_1^*}X\mathfrak{M}_1)^-$ and $\mathfrak{M}_1=(P_{\mathfrak{M}_1}X^*\mathfrak{H}')^-=(P_{\mathfrak{M}_1}X^*\mathfrak{M}_1^*)^-$ and it follows that $P_{\mathfrak{M}_1^*}X|\mathfrak{M}_1$ is a quasi-affinity.

It follows by the preceding remark that it will be enough to define $\mathfrak{H}_1$ by (i), to find $\mathfrak{H}_1^*$ satisfying (ii) and such that $P_{\mathfrak{H}_1^*}X|\mathfrak{H}_1$ is a quasi-affinity and then to define $\mathfrak{M}_1$, $\mathfrak{M}_1^*$ by (iii).

The operator $T'|(X\mathfrak{H}_1)^-$ has the cyclic vector Xf so that by [10], Theorem 2, $(T'|(X\mathfrak{H}_1)^-)^*$ has a cyclic vector k. Moreover, by Proposition 1.10, the set of cyclic vectors of $(T'|(X\mathfrak{H}_1)^-)^*$ is dense in $(X\mathfrak{H}_1)^-$ so that we may assume

$$\|k-Xf\| < \varepsilon. \tag{3.1}$$

We define $\mathfrak{H}_1^* = \bigvee_{n\geqq 0} T'^{*n}k$ so that $k\in\mathfrak{H}_1^*$ and (ii) is verified by (3.1). Let us compute the minimal function m of $(T'^*|\mathfrak{H}_1^*)^*$. Obviously m divides $m_{T'}=m_T$. Now the operator $Y=P_{(X\mathfrak{H}_1)^-}|\mathfrak{H}_1^*$ satisfies the relation

$$(T'|(X\mathfrak{H}_1)^-)^*Y = YT'^*|\mathfrak{H}_1^* \tag{3.2}$$

and $\operatorname{ran} Y\ni k$; it follows that Y has dense range and from (3.2) we infer $m^\sim((T'|(X\mathfrak{H}_1)^-)^*)Y=Ym^\sim(T^*|\mathfrak{H}_1^*)=0$ so that $m_{T'|(X\mathfrak{H}_1)^-}=m_T$ divides m. Because $(T'|(X\mathfrak{H}_1)^-)^*$ and $T'^*|\mathfrak{H}_1^*$ are both quasisimilar to $S(m_T)$ we infer by Lemma 3.1 that Y is a quasi-affinity. In particular, $Y^*X|\mathfrak{H}_1=P_{\mathfrak{H}_1^*}X|\mathfrak{H}_1$ is a quasi-affinity. Proposition 3.3 follows.

Lemma 3.3. *Let T be an operator of class C_0 acting on $\mathfrak{H}$, let $S(M)$ be the Jordan model of T and let $\mathfrak{H}'(\subset\mathfrak{H})$ be a separable space. Then there exists a reducing subspace $\mathfrak{H}_0$ for T such that $T|\mathfrak{H}_0$ is quasisimilar to $\bigoplus_{j<\omega} S(m_j)$ $(m_j=M(j))$ and $\mathfrak{H}_0\supset\mathfrak{H}'$.*

Proof. Let X be any quasi-affinity such that

$$TX = XS(M). \tag{3.3}$$

We shall denote by $\mathfrak{H}_0$ the least reducing subspace of T containing $\mathfrak{H}'$ and $X(\bigoplus_{j<\omega}\mathfrak{H}(m_j))$. The space $\mathfrak{H}_0$ is separable; let $\bigoplus_{j<\omega} S(m_j')$ be the Jordan model of $T|\mathfrak{H}_0$. We have $m_j'\leqq m_j$ by Corollary 2.9. Because $\mathfrak{H}_0\supset(X(\bigoplus_{j<\omega}\mathfrak{H}(m_j)))^-$ we have:

$$\bigoplus_{j<\omega} S(m_j) \overset{i}{\prec} T|\mathfrak{H}_0 \tag{3.4}$$

and therefore $m_j\leqq m_j'$ again by Corollary 2.9. Therefore $m_j=m_j'$ and the Lemma follows.

Theorem 3.4. *Let T be an operator of class C_0 acting on $\mathfrak{H}$ and let $S(M)$ be the Jordan model of T. We can associate with each limit ordinal α a reducing subspace $\mathfrak{H}_\alpha$ for T such that:*

(i) $\mathfrak{H} = \bigoplus_\alpha \mathfrak{H}_\alpha$;

(ii) *$T|\mathfrak{H}_\alpha$ is quasisimilar to* $\bigoplus_{j<\omega} S(m_{\alpha+j})$.

Proof. Let X be as in the preceding proof. We shall construct by transfinite induction reducing subspaces $\mathfrak{H}_\alpha$ for each limit ordinal α such that:

(3.5) $$\bigoplus_{\alpha<\beta} H_\alpha \supset X(\bigoplus_{\alpha<\beta}\bigoplus_{j<\omega}\mathfrak{H}(m_{\alpha+j}));$$

(3.6) $$T|\mathfrak{H}_\alpha \text{ is quasisimilar to } \bigoplus_{j<\omega} S(m_{\alpha+j}).$$

Let $\mathfrak{H}_0$ be given by Lemma 3.3 (with $\mathfrak{H}'=(X(\bigoplus_{j<\omega}\mathfrak{H}(m_j))^-)$ and assume $\mathfrak{H}_\alpha$ are defined for $\alpha<\beta$. Let us denote:

(3.7) $$\mathfrak{L}=\bigoplus_{\alpha<\beta}\mathfrak{H}_\alpha,\quad \mathfrak{K}=\mathfrak{H}\ominus\mathfrak{L}.$$

Then $\mathfrak{K}$ reduces T; let us denote by $S(M')$ the Jordan model of $T|\mathfrak{K}$. From the condition (3.5) we infer $X^*(\mathfrak{K})\subset\bigoplus_{\gamma\geqq\beta}\mathfrak{H}(m_\gamma)$ and therefore:

(3.8) $$T^*|\mathfrak{K}\overset{i}{\prec}\bigoplus_{\gamma} S(m_{\beta+\gamma})^*.$$

By Corollary 2.9 we infer:

(3.9) $$M'(\gamma)\leqq m_{\beta+\gamma}.$$

By Theorem 2.7 and Definition 2.2 we have for any ordinal γ:

(3.10) $$m_{\beta+\gamma}=\bigwedge\{m\colon \mu_T(m)\leqq\overline{\beta+\gamma}\}=$$
$$=\bigwedge\{m\colon \mu_{(T|\mathfrak{K})\oplus(T|\mathfrak{L})}(m)\leqq\overline{\beta+\gamma}\}.$$

Now,

(3.11) $$\mu_{(T|\mathfrak{K})\oplus(T|\mathfrak{L})}(m)\leqq\mu_{T|\mathfrak{K}}(m)+\mu_{T|\mathfrak{L}}\leqq$$
$$\leqq\mu_{T|\mathfrak{K}}(m)+\bar\beta\cdot\aleph_0=\mu_{T|\mathfrak{K}}(m)+\bar\beta$$

since β is transfinite. Because: $\overline{\beta+\gamma}=\bar\beta+\bar\gamma$, we infer:

(3.12) $$m_{\beta+\gamma}\leqq\bigwedge\{m\colon \mu_{T|\mathfrak{K}}(m)\leqq\bar\gamma\}=M'(\gamma).$$

From (3.9) and (3.12) it follows that $M'(\gamma)=m_{\beta+\gamma}$. An application of Lemma 3.3 to $T|\mathfrak{K}$ shows the existence of a reducing subspace $\mathfrak{H}_\beta\subset\mathfrak{K}$ such that:

(3.13) $$T|\mathfrak{H}_\beta \text{ is quasisimilar to } \bigoplus_{j<\omega} S(m_{\beta+j})$$

and

(3.14) $$H_\beta\supset P_{\mathfrak{K}}X(\bigoplus_{j<\omega}\mathfrak{H}(m_{\beta+j})).$$

Conditions (3.5—6) are obviously conserved. Theorem 3.4 follows now because from (3.5) we infer $\mathfrak{H}=\bigoplus_\alpha\mathfrak{H}_\alpha$.

The proof of the following theorem is a refinement of the proof of [4], Theorem 1.

Theorem 3.5. *Let T be an operator of class C_0 acting on $\mathfrak{H}$ and let $S(M)$ be the Jordan model of T. There exists a decomposition of $\mathfrak{H}$ into an almost-direct sum*

$$\mathfrak{H} = \bigvee_{\alpha} \mathfrak{H}_\alpha \tag{3.15}$$

of invariant subspaces of T such that:

(i) *$T|\mathfrak{H}_\alpha$ is quasisimilar to $S(m_\alpha)$ for each ordinal α;*

(ii) *$\mathfrak{H}_{\alpha+n} \perp \mathfrak{H}_{\beta+m}$ if α, β are different limit ordinals and $m, n<\omega$.*

Proof. Theorem 3.4 allows us to consider only the case where $\mathfrak{H}$ is separable. Let $\{\psi_j\}_{j=0}^\infty$ be a sequence of vectors dense in $\mathfrak{H}$ and let $\{\varphi_j\}_{j=0}^\infty$ be a sequence in which each ψ_k appears infinitely many times. We shall construct inductively subspaces $\mathfrak{H}_0, \mathfrak{H}_1, \ldots, \mathfrak{H}_n$, $\mathfrak{M}_n$ invariant for T and $\mathfrak{H}_0^*, \mathfrak{H}_1^*, \ldots, \mathfrak{H}_n^*$, $\mathfrak{M}_n^*$ invariant for T^* such that

$$\mathfrak{H}_n = \mathfrak{H}_{f_n}, \quad f_n \in \mathfrak{M}_{n-1} \quad \text{and} \quad m_{f_n} = m_{T|\mathfrak{M}_{n-1}}; \quad \mathfrak{H}_n^* \subset \mathfrak{M}_{n-1}^*; \tag{3.16}$$

$$(\mathfrak{H}_0 \vee \mathfrak{H}_1 \vee \ldots \vee \mathfrak{H}_n)^\perp = \mathfrak{M}_n^*, \quad (\mathfrak{H}_0^* \vee \mathfrak{H}_1^* \vee \ldots \vee \mathfrak{H}_n^*)^\perp = \mathfrak{M}_n; \tag{3.17}$$

$$P_{\mathfrak{M}_n^*}|\mathfrak{M}_n \quad \text{is a quasi-affinity;} \tag{3.18}$$

$$\begin{cases} \|P_{\mathfrak{H}_0 \vee \mathfrak{H}_1 \vee \ldots \vee \mathfrak{H}_n} \varphi_k - \varphi_k\| < 2^{-n}, & k = n/2 \text{ if } n \text{ is even,} \\ \|P_{\mathfrak{H}_0^* \vee \mathfrak{H}_1^* \vee \ldots \vee \mathfrak{H}_n^*} \varphi_k - \varphi_k\| < 2^{-n}, & k = (n-1)/2 \text{ if } n \text{ is odd.} \end{cases} \tag{3.19}$$

To begin we put $\mathfrak{M}_{-1} = \mathfrak{M}_{-1}^* = \mathfrak{H}$; the conditions (3.16—19) are obviously satisfied for $n=-1$. Let us assume that the spaces $\mathfrak{H}_j$, $\mathfrak{H}_j^*$, $\mathfrak{M}_j$, $\mathfrak{M}_j^*$ have been constructed for $0 \leq j \leq n-1$. From (3.17) and (3.18) we infer

$$\mathfrak{H}_0 \vee \mathfrak{H}_1 \vee \ldots \vee \mathfrak{H}_{n-1} \vee \mathfrak{M}_{n-1} = (\mathfrak{H}_0 \vee \mathfrak{H}_1 \vee \ldots \vee \mathfrak{H}_{n-1}) \oplus (P_{\mathfrak{M}_{n-1}^*} \mathfrak{M}_{n-1})^- = \mathfrak{H}$$

and analogously $\mathfrak{H}_0^* \vee \mathfrak{H}_1^* \vee \ldots \vee \mathfrak{H}_{n-1}^* \vee \mathfrak{M}_{n-1}^* = \mathfrak{H}$. Therefore there exist $u \in \mathfrak{H}_0 \vee \mathfrak{H}_1 \vee \ldots \vee \mathfrak{H}_{n-1}$, $v \in \mathfrak{M}_{n-1}$ and $u^* \in \mathfrak{H}_0^* \vee \mathfrak{H}_1^* \vee \ldots \vee \mathfrak{H}_{n-1}^*$, $v^* \in \mathfrak{M}_{n-1}^*$ such that

$$\begin{cases} \|\varphi_k - u - v\| < 2^{-n-1}, & k = n/2 \text{ if } n \text{ is even,} \\ \|\varphi_k - u^* - v^*\| < 2^{-n-1}, & k = (n-1)/2 \text{ if } n \text{ is odd.} \end{cases} \tag{3.20}$$

By Proposition 1.10 we can choose $f_n \in \mathfrak{M}_{n-1}$ with $m_{f_n} = m_{T|\mathfrak{M}_{n-1}}$ and such that

$$\begin{cases} \|f_n - v\| < 2^{-n-1} & \text{if } n \text{ is even,} \\ \|P_{\mathfrak{M}_{n-1}^*} f_n - v^*\| < 2^{-n-2} & \text{if } n \text{ is odd.} \end{cases} \tag{3.21}$$

Proposition 3.2 allows us to construct the subspaces $\mathfrak{H}_n = \mathfrak{H}_{f_n}$, $\mathfrak{H}_n^*$, $\mathfrak{M}_n$ and $\mathfrak{M}_n^*$ such that

$$\|P_{\mathfrak{H}_n^*} P_{\mathfrak{M}_{n-1}^*} f_n - P_{\mathfrak{M}_{n-1}^*} f_n\| < 2^{-n-2}; \tag{3.22}$$

$$\mathfrak{M}_n^* = \mathfrak{M}_{n-1}^* \ominus (P_{\mathfrak{M}_{n-1}^*} \mathfrak{H}_n)^-, \quad \mathfrak{M}_n = \mathfrak{M}_{n-1} \ominus (P_{\mathfrak{M}_{n-1}} \mathfrak{H}_n^*)^-; \tag{3.23}$$

$$P_{\mathfrak{M}_n^*}|\mathfrak{M}_n \quad \text{is quasi-affinity.} \tag{3.24}$$

Let us show that the conditions (3.16—19) are verified. (3.16) is obvious and (3.18) coincides with (3.24). For (3.17) we have $(\mathfrak{H}_0 \vee \mathfrak{H}_1 \vee \ldots \vee \mathfrak{H}_n)^\perp = (\mathfrak{H}_0 \vee \mathfrak{H}_1 \vee \ldots \vee \mathfrak{H}_{n-1})^\perp \cap \mathfrak{H}_n^\perp = \mathfrak{M}_{n-1}^* \cap \mathfrak{H}_n^\perp = \mathfrak{M}_{n-1}^* \ominus (P_{\mathfrak{M}_{n-1}^*} \mathfrak{H}_n)^- = \mathfrak{M}_n^*$ by (3.23) and analogously $(\mathfrak{H}_0^* \vee \mathfrak{H}_1^* \vee \ldots \vee \mathfrak{H}_n^*)^\perp = \mathfrak{M}_n$. If n is even we have

$$\|P_{\mathfrak{H}_0 \vee \mathfrak{H}_1 \vee \ldots \vee \mathfrak{H}_n} \varphi_k - \varphi_k\| \leqq \|u + f_n - \varphi_k\| \leqq \|u + v - \varphi_k\| + \|v - f_n\| < 2^{-n},$$

by (3.20) and (3.21). If n is odd we have

$$\|P_{\mathfrak{H}_0^* \vee \mathfrak{H}_1^* \vee \ldots \vee \mathfrak{H}_n^*} \varphi_k - \varphi_k\| \leqq \|u^* + P_{\mathfrak{H}_n^*} P_{\mathfrak{M}_{n-1}^*} f_n - \varphi_k\| <$$

$$< \|u^* + v^* - \varphi_k\| + \|v^* - P_{\mathfrak{M}_{n-1}^*} f_n\| + \|P_{\mathfrak{M}_{n-1}^*} f_n - P_{\mathfrak{H}_n^*} P_{\mathfrak{M}_{n-1}^*} f_n\| < 2^{-n} \text{ by}$$

(3.20—22); thus (3.19) is also verified.

From (3.19) we infer

(3.25)
$$\mathfrak{H} = \bigvee_{j<\omega} \mathfrak{H}_j = \bigvee_{j<\omega} \mathfrak{H}_j^*.$$

If $i \neq j$ (say $i < j$ by example) we have $\mathfrak{H}_i \perp \mathfrak{M}_i^*$ and $\mathfrak{H}_j^* \subset \mathfrak{M}_i^*$ by (3.16), so that $\mathfrak{H}_i \perp \mathfrak{H}_j^*$. Therefore $\mathfrak{H}_j^* \subset (\bigvee_{i \neq j} \mathfrak{H}_i)^\perp$ and (3.25) shows, by Lemma 1.11, that the decomposition $\mathfrak{H} = \bigvee_{j<\omega} \mathfrak{H}_j$ is almost direct. To finish the proof let us remark that $\mathfrak{M}_{n+1} = (\mathfrak{H}_0^* \vee \mathfrak{H}_1^* \vee \ldots \vee \mathfrak{H}_{n+1}^*)^\perp \subset \mathfrak{M}_n$ by (3.17), so that $m_{f_{n+1}}$ divides m_{f_n}. As in [4], Theorem 1, it follows that the Jordan model of T is $\bigoplus_{j<\omega} S(m_j)$, where $m_j = m_{f_j}$. Theorem 3.5 is proved.

In the case of weak contractions the result of Theorem 3.5 can be improved.

Proposition 3.6. *Let T be a weak contraction of class C_0 acting on the (necessarily separable) Hilbert space $\mathfrak{H}$ and let $\bigoplus_{j<\omega} S(m_j)$ be the Jordan model of T. There exists a decomposition*

(3.26)
$$\mathfrak{H} = \bigvee_{j<\omega} \mathfrak{H}_j$$

of $\mathfrak{H}$ into a quasi-direct sum of invariant subspaces of T such that $T|\mathfrak{H}_j$ is quasi-similar to $S(m_j)$.

Proof. Let X be a quasi-affinity such that $TX = X(\bigoplus_{j<\omega} S(m_j))$ and define $\mathfrak{H}_j = (X\mathfrak{H}(m_j))^-$. Let $\{K_\alpha\}_{\alpha \in A}$ be a family of subsets of the natural numbers and denote $K = \bigcap_{\alpha \in A} K_\alpha$. Because the mapping $\mathfrak{M} \mapsto (X\mathfrak{M})^-$ is an isomorphism of the lattice of invariant subspaces of $\bigoplus_{j<\omega} S(m_j)$ onto the lattice of invariant subspaces of T (cf. [3], Corollary 2.4) we have

$$\bigcap_{\alpha \in A} (\bigvee_{j \in K_\alpha} \mathfrak{H}_j) = (X(\bigcap_{\alpha \in A} (\bigoplus_{j \in K_\alpha} \mathfrak{H}(m_j))))^- = (X(\bigoplus_{j \in K} \mathfrak{H}(m_j)))^- = \bigvee_{j \in K} \mathfrak{H}_j.$$

Proposition 3.6 follows.

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BUCHAREST
ROMANIA
Current address:
UNIVERSITY OF MICHIGAN
DEPT. OF MATHEMATICS
MICHIGAN, ANN ARBOR 48 109. U.S.A.

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Finite homogeneous algebras. I

BÉLA CSÁKÁNY and TAT'JANA GAVALCOVÁ

1. Preliminaries. Following MARCZEWSKI [7], an operation $f\colon A^k \to A$ is called *homogeneous* if $h(f(x_1, \ldots, x_k)) = f(h(x_1), \ldots, h(x_k))$ for every permutation h and any elements $x_1, \ldots, x_k$ of A. An algebra $\langle A; F\rangle$ is said to be *homogeneous* if each operation $f \in F$ is homogeneous.

In this paper, we shall describe all finite homogeneous algebras up to equivalence. This is the same as determining all clones of homogeneous operations on finite sets. In the present Part I we shall

(1) list all minimal clones consisting of homogeneous operations (it turns out that this list contains at most three items on any finite set, and the dual discriminator function d, introduced by E. Fried and A. F. Pixley, always generates such a minimal clone);

(2) determine all clones of homogeneous operations containing the minimal clone generated by the dual discriminator.

Let us start with notions and notations. The symbol $\mathbf{n}$ means the set $\{0, 1, \ldots, n-1\}$. For the sake of simplicity, we shall consider algebras of the form $\langle \mathbf{n}; F\rangle$ only. The following description of homogeneous operations was given by MARCZEWSKI [7]: for a homogeneous k-ary operation f on $\mathbf{n}$, $f(a_1, \ldots, a_k) = a_i$ where $1 \leqq i \leqq k$, or, possibly, $f(a_1, \ldots, a_k) = a_{k+1}$ if a_{k+1} is the unique element of $\mathbf{n}$ distinct from $a_1, \ldots, a_k$, in such a way that the index of the value of $f(a_1, \ldots, a_k)$ depends upon the pattern of equalities in the sequence $\langle a_1, \ldots, a_k\rangle$ only. A homogeneous operation f is called a *pattern function* provided $f(a_1, \ldots, a_k)$ always belongs to $\{a_1, \ldots, a_k\}$.

Several kinds of homogeneous operations will play an important role in the sequel: Pixley's ternary discriminator p, the dual discriminator d, the switching function s, the k-ary near-projection l_k where $k \geqq 3$ (they are defined on any set); further, the $(n-1)$-ary operation r_n, defined on $\mathbf{n}$ for $n \geqq 2$, and Świerczkowski's ternary function f_0, defined on $\mathbf{4}$. Let us recall their definitions:

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$p(a, b, c) = c$ if $a = b$, and $p(a, b, c) = a$ otherwise;
$d(a, b, c) = a$ if $a = b$, and $d(a, b, c) = c$ otherwise;
$s(a, b, c) = c$ if $a = b$, $s(a, b, c) = b$ if $a = c$ and $s(a, b, c) = a$ otherwise;
$l_k(a_1, \ldots, a_k) = a_1$ if $a_1, \ldots, a_k$ are pairwise distinct and $l_k(a_1, \ldots, a_k) = a_k$ otherwise;
$r_n(a_1, \ldots, a_{n-1}) = a_n$ if $\{a_1, \ldots, a_{n-1}, a_n\} = \mathbf{n}$ and $r_n(a_1, \ldots, a_{n-1}) = a_1$ otherwise;

finally,

$$f_0(1, 2, 3) = f_0(0, 1, 1) = f_0(1, 0, 1) = f_0(1, 1, 0) = f_0(0, 0, 0) = 0$$

(see [8], [7], [9], [3], [2]).

A set of operations on a set $\mathbf{n}$ is called a *clone* if it contains all trivial operations (i.e., all projections) and it is closed under superposition. For any set F of operations on $\mathbf{n}$, we say that F *produces* the operation g and we use the symbol $F \to g$ if g can be obtained from operations in F and the projections by superposition (in this case, one can also say that g is a term function of the algebra $\langle \mathbf{n}; F \rangle$). In the case $F = \{f\}$ we write $f \to g$. Obviously, the relation $\to$ is transitive. For the negation of $F \to g$ we write $F \not\to g$. An algebra $\langle \mathbf{n}; F \rangle$ is *functionally complete* if the set $F \cup \{0, 1, \ldots, n-1\}$ (i.e., F together with the constant nullary operations) produces each possible operation on $\mathbf{n}$. The *clone* $[F]$ *generated by* F is the set of all operations F produces. We write $[f_1, f_2, \ldots]$ instead of $[\{f_1, f_2, \ldots\}]$. The algebras $\langle \mathbf{n}; F \rangle$ and $\langle \mathbf{n}; G \rangle$ are said to be *equivalent* if $[F] = [G]$. A clone T is called *minimal* if the clone of all projections is the unique one which is contained in T properly; this means that T contains a non-projection, and any non-projection in T produces every other non-projection.

In the next lemma we collect the basic facts about how the above-mentioned homogeneous operations produce each other:

Lemma 1. *On a finite set* $\mathbf{n}$, *the following hold:*

(1) $p \to f$ *for any pattern function* f.
(2) $l_j \to l_k$ *for* $j \leqq k$.
(3) $r_n \to l_{n-1}$ *for* $n > 3$.
(4) $l_k \not\to d$ *for* $n > 1$.
(5) $d \not\to l_k$ *for* $n > 2$, $n \geqq k$.
(6) $l_j \not\to l_k$ *for* $j > k$, $n \geqq k$.

Proof. (1) is a result in [4].
(2). It is sufficient to establish $l_j \to l_{j+1}$, and this is given by the identity $l_{j+1}(x_1, \ldots, x_j, x_{j+1}) = l_j(l_j(x_1, x_3, \ldots, x_j, x_{j+1}), l_j(x_2, x_3, \ldots, x_j, x_{j+1}), x_4, \ldots, x_{j+1})$.
(3). $l_{n-1}(x_1, \ldots, x_{n-1}) = r_n(x_{n-1}, \ldots, x_3, x_2, r_n(x_{n-1}, \ldots, x_2, x_1))$.

To prove (4)—(6), we use the following fact. Let f, g be operations on $\mathbf{n}$ and $f\to g$; then, for any natural number t, the subalgebras of $\langle \mathbf{n};f\rangle^t$ are closed under the (componentwise performed) operation g.
(4). Observe that $\sigma=\{\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle\}$ is a subalgebra of $\langle \mathbf{n}; l_k\rangle^3$ but $d(\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle)=\langle 0,0,0\rangle\notin\sigma$. Hence $l_k\to d$ is impossible.

Concerning (5) and (6), we present the crucial subalgebras only:

(5) $$\{\langle k-1,0\rangle, \dots, \langle 2,0\rangle, \langle 1,0\rangle, \langle 0,0\rangle, \langle 0,1\rangle\}\subset\langle \mathbf{n};\ d\rangle^2,$$

(6) $$\{\langle j-2,0\rangle, \dots, \langle 2,0\rangle, \langle 1,0\rangle, \langle 0,0\rangle, \langle 0,1\rangle\}\subset\langle \mathbf{n};\ l_j\rangle^2.$$

2. Minimal clones of homogeneous operations. In this section, our main tool is the following fact:

Lemma 2. *For $n\geqq 3$, every non-trivial pattern function on $\mathbf{n}$ produces d or some l_k with $k\leqq n$.*

Proof. It was proved in [2] (see the proof of Lemma 5 there) that any non-trivial pattern function on $\mathbf{n}$ produces d or an l_k which is non-trivial; but l_k is trivial if $k>n$.

The clones in the title of this paragraph are given by

Theorem 1. *The minimal clones consisting of homogeneous operations on a finite set $\mathbf{n}$ $(n>1)$ are the following:*

$[l_n]$ and $[d]$, if $n\geqq 5$;
$[l_4]$, $[d]$ and $[f_0]$, if $n=4$;
$[l_3]$, $[d]$ and $[r_3]$, if $n=3$;
$[s]$, $[d]$ and $[r_2]$, if $n=2$.

Proof. First we prove that, for $n\geqq 3$, $[l_n]$ is minimal on $\mathbf{n}$. Take a non-trivial f with $l_n\to f$; it is sufficient to show $f\to l_n$. As pattern functions produce pattern functions only, by Lemma 2 we have $f\to d$ or $f\to l_k$ for a suitable $k\leqq n$. From $f\to d$ it follows $l_n\to d$, contradicting Lemma 1(4); therefore $f\to l_k$ holds. Now $k<n$ is impossible by Lemma 1(6), i.e., $f\to l_n$, which was needed.

For $n\geqq 3$, the minimality of $[d]$ can be proved by an analogous argument; here we have to apply Lemma 1(5) instead of (4).

For $n\geqq 5$, there is no other minimal clone of operations on $\mathbf{n}$. In order to show this, we shall verify that each non-trivial homogeneous operation g on $\mathbf{n}$ produces l_n or d. There are two possibilities:

a) $g\to r_n$. Then, by Lemma 1(3) and (2), we have $g\to l_n$.

b) $g\not\to r_n$. If, in addition, g is a pattern function, then Lemma 2 applies in the above manner. If g is not a pattern function, then we can identify variables of g (if necessary) so that we obtain an $(n-1)$-ary g' satisfying $g'(a_1, \dots, a_{n-1})=a_n$, whenever $\{a_1, \ldots, a_{n-1}, a_n\} = \mathbf{n}$, i.e., a_n is the unique element of $\mathbf{n}$ distinct from $a_1, \ldots, a_{n-1}$. Now, if there exist two variables of g' whose identification furnishes a non-trivial pattern function, then, applying Lemma 2 for g' again, our claim follows. Suppose that g' turns into a projection by identifying any two of its variables. By a result of Świerczkowski, g' always turns into the same projection ([8]; see also [5], pp. 206—207; note that g' is at least quaternary). Hence g' equals r_n up to permutation of variables, implying $g \to r_n$, contrary to the hypothesis.

Next we prove that $[f_0]$ is minimal on $\mathbf{4}$. Let $f_0 \to f$ and suppose $f \not\to f_0$. Then $\langle \mathbf{4}; f_0 \rangle$ and $\langle \mathbf{4}; f \rangle$ are not equivalent. A homogeneous non-trivial algebra $\langle \mathbf{4}; F \rangle$ is not functionally complete iff it is equivalent to $\langle \mathbf{4}; f_0 \rangle$ (see [2]); therefore, $\langle \mathbf{4}; f \rangle$ is functionally complete. Now, $\langle \mathbf{4}; f_0 \rangle$ is functionally complete a fortiori, a contradiction.

Similarly, a non-trivial homogeneous functionally incomplete algebra $\langle \mathbf{3}; F \rangle$ is equivalent to $\langle \mathbf{3}; r_3 \rangle$ (see [2]), hence the minimality of $[r_3]$ on $\mathbf{3}$ follows.

Furthermore, every non-trivial homogeneous operation g on $\mathbf{4}$ produces one of l_4, d and f_0, showing that there are no other minimal clones of homogeneous operations on $\mathbf{4}$. Indeed, if g is a pattern function, Lemma 2 applies. If g fails to be a pattern function, then an appropriate identification of variables of g leads to a ternary g' satisfying $g'(a_1, a_2, a_3) = a_4$, whenever $\{a_1, \ldots, a_4\} = \mathbf{4}$. As we have $g'(a_1, a_2, a_3) = a_i$ $(1 \leqq i \leqq 3)$ if card $\{a_1, a_2, a_3\} < 3$, and the pattern of equalities in $\langle a_1, a_2, a_3 \rangle$ determines the value of i, the operation g' is defined uniquely by the sequence $\langle g'(0,1,1), g'(1,0,1), g'(1,1,0) \rangle$ (of course, $g'(0,0,0) = 0$ always). Let us denote g' by f_k $(k = 0, 1, \ldots, 7)$ if this sequence is the dyadic form of k (i.e., $4g'(0,1,1) + 2g'(1,0,1) + g'(1,1,0) = k$). This notation is consistent with the original definition of f_0. We have to verify that every f_k produces one of l_4, d and f_0.

One can check the following identities:

(a) $f_3(x, y, z) = r_4(x, y, z)$;

(b) $f_5(y, x, z) = f_6(z, y, x) = f_3(x, y, z)$;

(c) $f_1(y, z, f_1(z, y, x)) = f_4(y, f_4(z, x, y), z) = p(x, y, z)$;

(d) $f_2(y, f_2(y, z, x), x) = f_7(y, f_7(y, z, x), x) = d(x, y, z)$.

From (a) and Lemma 1(3) and (2), it follows $f_3 \to l_4$. From (b), $f_5 \to l_4$ and $f_6 \to l_4$. Further, (c) together with Lemma 1(1) implies $f_1 \to d$ and $f_4 \to d$; finally, (d) shows $f_2 \to d$ and $f_7 \to d$. The case $n = 4$ is settled.

In the case $n = 3$ we can proceed similarly. Any non-trivial homogeneous function g on $\mathbf{3}$ is either a pattern function — then we use Lemma 2 — or not. In the latter case g produces a binary g' in the usual way such that $g'(a_1, a_2) = a_3$ whenever $\{a_1, a_2, a_3\} = \mathbf{3}$, and $g'(a, a) = a$. Clearly, $g' = r_3$, hence $g \to r_3$, as required.

All minimal clones we have found are distinct. This is implied by Lemma 1(4) and the fact that pattern functions produce merely pattern functions.

The case $n=2$ of Theorem 1 can be realized by casting a glance at the diagram of the lattice of all clones on **2**, due to POST (see, e.g., [6]; note that $r_2(x)=$ $=x+1 \bmod 2$ and $d(x, y, z)=xy+xz+yz \bmod 2$ on **2**).

3. Homogeneous dual discriminator algebras. After WERNER [9], an algebra $\langle \mathbf{n}; F\rangle$ is said to be a discriminator algebra (or quasi-primal algebra) if $p\in[F]$. Analogously, an algebra $\langle \mathbf{n}; F\rangle$ will be called a *dual discriminator algebra* if $d\in[F]$. In this paragraph we determine all homogeneous dual discriminator algebras up to equivalence, i.e., for any **n**, we determine all clones of homogeneous operations on **n** containing d. From now on, n is fixed and $n\geqq 3$.

Call a ternary operation m on **n** a *majority operation* if, for any $x, y\in\mathbf{n}$, $m(x, x, y)=m(x, y, x)=m(y, x, x)=x$ holds. The dual discriminator is a majority operation. The following theorem of BAKER and PIXLEY [1; Corollary 5.1] is basic for our considerations (see also [9]):

Let $\langle \mathbf{n}; F\rangle$ be a finite algebra such that F produces a majority operation and let g be an arbitrary operation on **n**. If every subalgebra of $\langle \mathbf{n}; F\rangle^2$ is closed under the (componentwise performed) operation g, then F produces g.

For a clone T on **n**, let ST stand for the set consisting of base sets of all subalgebras of $\langle \mathbf{n}; T\rangle^2$. Let $\mathcal{F}$ be the set of all clones on the set **n** containing d. We call a set P of subsets of $\mathbf{n}^2$ *complete* if there exists a clone $T\in\mathcal{F}$ such that $P=\mathrm{S}T$ (i.e., if there exists a dual discriminator algebra on **n** such that P is the set of all subalgebras of the direct square of this algebra). Denote by $\mathcal{S}$ the set of all complete sets.

Lemma 3. S *is an inclusion-reversing one-to-one mapping of* $\mathcal{F}$ *onto* $\mathcal{S}$.

Proof. The unique non-trivial part of this assertion is that S is one-to-one. Suppose $T_1, T_2\in\mathcal{F}$ and $\mathrm{S}T_1=\mathrm{S}T_2$. If $f\in T_2$ then every set in $\mathrm{S}T_1(=\mathrm{S}T_2)$ is closed under f, hence, by the Baker—Pixley theorem, $T_1\to f$ follows. This means $f\in T_1$ as T_1 is a clone. Therefore, $T_2\subseteq T_1$ (and by symmetry, $T_1\subseteq T_2$). We get $T_1=T_2$, which was needed.

By virtue of Lemma 3, we can investigate complete sets instead of clones. First we establish some properties of complete sets. Subsets of $\mathbf{n}^2$ may be considered as binary relations on **n**. The following lemma is familiar:

Lemma 4. *Any complete set contains the complete relation; furthermore, it is closed under relation product, intersection and forming the inverse relation.*

For convenience, several kinds of subsets of $\mathbf{n}^2$ will bear special names. A set of form $K\times L$ with $K, L\subseteq\mathbf{n}$, card $K=k$, card $L=l$ is a *block of size* (k, l). A set of form $\{\langle i_1, j_1\rangle, \ldots, \langle i_k, j_k\rangle\}$, where $i_1, \ldots, i_k$ are pairwise distinct as well as $j_1, \ldots, j_k$, is a *string of size* k. A set of form $\{\langle i_k, j_1\rangle, \ldots, \langle i_2, j_1\rangle, \langle i_1, j_1\rangle, \langle i_1, j_2\rangle, \ldots, \langle i_1, j_l\rangle\}$

$(k, l \geqq 2)$ is called a *cross of size* (k, l). Essentially, a string of size k is a partial permutation with a k-element domain and a cross of size (k, l) is the union of two blocks of size $(k, 1)$ and $(1, l)$ with a non-empty intersection. *Block of size m* means a block of size (m, l) or (k, m); similarly for crosses.

Lemma 5. *Any complete set consists of blocks, strings and crosses; in particular,* S$[d]$ *consists of all blocks, strings and crosses.*

Proof. A complete set consists of subsets of $\mathbf{n}^2$ preserved by d, and, by result of FRIED and PIXLEY [3; Theorem 2.4], d preserves a subset σ of $\mathbf{n}^2$ iff σ is *p-rectangular*, i.e.,

$$\langle i, j_1\rangle, \langle i, j_2\rangle, \langle k, l\rangle \in \sigma \quad \text{implies} \quad \langle i, l\rangle \in \sigma \quad \text{for} \quad j_1 \neq j_2$$

and

$$\langle i_1, j\rangle, \langle i_2, j\rangle, \langle k, l\rangle \in \sigma \quad \text{implies} \quad \langle k, j\rangle \in \sigma \quad \text{for} \quad i_1 \neq i_2.$$

Clearly, blocks, strings and crosses are p-rectangular and the converse can also be checked without trouble.

From now on, we shall use the following notations: B is the set of all blocks and B' is the set of all blocks of size (k, l) with $k, l \neq n-1$. The set of strings and crosses S, S' and C, C', resp., are defined analogously. Finally, let C_m be the set of all crosses of size (k, l) with $k, l \leqq m$. Now Lemma 5 can be reformulated as follows:

For any complete set P, the inclusion $P \subseteq B \cup S \cup C$ holds; in particular, S$[d] = B \cup S \cup C$.

Next we clear up the structure of several further complete sets:

Lemma 6. (1) S$[d, l_{m+1}] = B \cup S \cup C_m$ for $m = 2, \ldots, n-1$.

(2) S$[p] = B \cup S$.

(3) S$[d, l_{m+1}, r_n] = B' \cup S' \cup C_m$ for $m = 2, \ldots, n-2$.

(4) S$[p, r_n] = B' \cup S'$.

Proof. (1) The following inclusions are obvious: $B \cup S \cup C_m \subseteq$ S$[d, l_{m+1}] \subseteq$ S$[d] = B \cup S \cup C$. Take a set from $C \setminus C_m$, i.e., a cross of form $\{\langle i_k, j_1\rangle, \ldots, \langle i_1, j_1\rangle, \ldots, \langle i_1, j_l\rangle\}$ with $k > m$ (the case $l > m$ can be settled similarly). Then $l_{m+1}(\langle i_{m+1}, j_1\rangle, \ldots, \langle i_2, j_1\rangle, \langle i_1, j_2\rangle) = \langle i_{m+1}, j_2\rangle$ showing that our cross is not closed under l_{m+1}. Thus, the set of all subalgebras of $\langle \mathbf{n}; d, l_{m+1}\rangle^2$ is $B \cup S \cup C_m$, as asserted.

(2)—(4) can be verified in an analogous manner observing that no cross is closed under p, because we have $p(\langle i_2, j_1\rangle, \langle i_1, j_1\rangle, \langle i_1, j_2\rangle) = \langle i_2, j_2\rangle$; furthermore, no block, string and cross, each of size $n-1$, is closed under r_n. Indeed, take, e.g., a block $\{i_1, \ldots, i_{n-1}\} \times L$ of size $n-1$ and a $j \in L$; then $\langle i_1, j\rangle, \ldots, \langle i_{n-1}, j\rangle$ belong to this block but $r_n(\langle i_1, j\rangle, \ldots, \langle i_{n-1}, j\rangle)$ does not.

Lemma 7. *For the clone H of all homogeneous operations on* $\mathbf{n}$, $SH=B'\cup S'$.

Proof. By (4) of the previous lemma, $SH\subseteqq B'\cup S'$. On the other hand, SH contains all permutations of $\mathbf{n}$, i.e. all strings of size n, since for any operation f homogeneity means that each permutation is a subalgebra of $\langle\mathbf{n};f\rangle^2$. Now we can apply Lemma 4 in order to obtain all sets in $B'\cup S'$. Namely, every string of size less than $n-1$ is the intersection of two permutations, every block of size (k, n) is the (relation) product of a string of size k and the complete relation, every block of size (n, l) is the inverse of a block of size (l, n), and every block of size (k, l) is the intersection of blocks of size (k, n) and (n, l).

In view of Lemmas 5 and 7, our task is reduced to determining all complete sets between $B'\cup S'$ and $B\cup S\cup C$.

Lemma 8. *All complete sets containing* $B'\cup S'$ *and contained in* $B\cup S\cup C$ *are those listed in Lemma* 6.

Proof. It is sufficient to prove the following two propositions:

(a) If a complete set contains $B'\cup S'$ and a block, or a string, or a cross, any of them of size $n-1$, then it contains $B\cup S$.

(b) If a complete set contains $B'\cup S'$ and a cross of size m, then it contains C_m; moreover, if $m\geqq n-1$, it contains even $B\cup S$.

Indeed, suppose (a) and (b) are fulfilled, and let P be a complete set with $B'\cup S'\subseteqq P\subseteqq B\cup S\cup C$. If P contains no crosses, then (a) implies $P=B'\cup S'$ or $P=B\cup S$. Otherwise, let m be the maximum of the sizes of crosses in P. If there is a block or a string of size $n-1$ in P, then in virtue of (a), (b) and the maximality of m we have $P=B\cup S\cup C_m$. In the opposite case, $P=B'\cup S'\cup C_m$ by the same reason.

It remains to prove (a) and (b). As for (a), one can check easily that all blocks and strings of size $n-1$ can be obtained from sets in $B'\cup S'$ and an arbitrary *fixed* block or string or cross, any of them of size $n-1$, by product, intersection and formation of inverse relation. Applying Lemma 4, the assertion (a) follows.

(b) First let R be a complete set containing $B'\cup S'$ and an arbitrary cross ζ of size (m, l) where $2\leqq l<m\leqq n-1$. Then any cross of the same size (m, l) can be obtained in the form $\pi_1\zeta\pi_2$ with appropriate strings π_1, π_2 of size n; crosses of size (l, m) arise as inverses of the previous ones; crosses of size (m, m) can be represented as $\zeta_1\pi\zeta_2$ where ζ_1 and ζ_2 are crosses of size (m, l) and (l, m), respectively, and π is a string of size n; finally, an arbitrary cross of size (k_1, k_2) with $k_1, k_2\leqq m$ is the intersection of a cross of size (m, m) and an appropriate block of size (k_1, k_2). Thus, $C_m\subseteqq R$, as required. In the case $m=n-1$, the second part of (b) is a consequence of (a).

Secondly, let R be complete with $R\supseteqq B'\cup S'$ and let R contain a cross of size n. The preceding considerations show that we have two possibilities only, namely, $R=B'\cup S'\cup C'$ or $R=B\cup S\cup C$. The proof will be complete if we deduce that $B'\cup S'\cup C'$ is not a complete set. Assume $SF=B'\cup S'\cup C'$ for some homogeneous dual discriminator algebra $\langle \mathbf{n}; F\rangle$. As SF is closed under r_n, we have $F\to r_n$ by the Baker—Pixley theorem, hence, according to Lemma 1(3) and (2), $F\to l_n$ follows. However, as we have seen in the proof of Lemma 6(1), our cross of size n is not closed under l_n, a contradiction.

Now we are ready to formulate the main result of this paragraph.

Theorem 2. *The finite homogeneous dual discriminator algebras with more than one element are the following (up to equivalence):*

$\langle \mathbf{2};\, d\rangle$, $\langle \mathbf{2};\, p\rangle$, $\langle \mathbf{2};\, p, r_2\rangle$;

$\langle \mathbf{3};\, d\rangle$, $\langle \mathbf{3};\, p\rangle$, $\langle \mathbf{3};\, p, r_3\rangle$, $\langle \mathbf{3};\, d, l_3\rangle$;

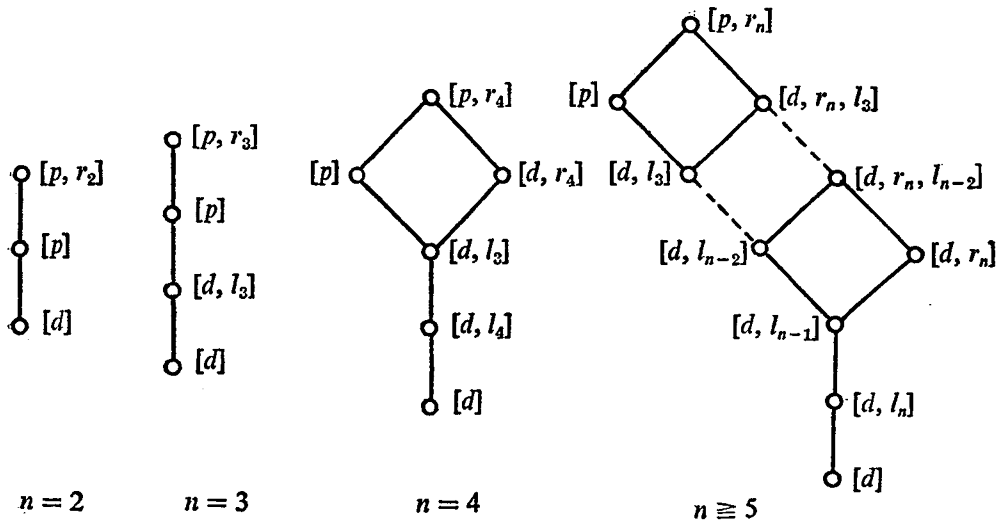
$\langle \mathbf{4};\, d\rangle$, $\langle \mathbf{4};\, p\rangle$, $\langle \mathbf{4};\, p, r_4\rangle$, $\langle \mathbf{4};\, d, l_3\rangle$, $\langle \mathbf{4};\, d, l_4\rangle$, $\langle \mathbf{4};\, d, r_4\rangle$

and for $n\geqq 5$

$\langle \mathbf{n};\, d\rangle$, $\langle \mathbf{n};\, p\rangle$, $\langle \mathbf{n};\, p, r_n\rangle$, $\langle \mathbf{n};\, d, l_k\rangle$ $(k=3, \ldots, n)$,

$\langle \mathbf{n};\, d, r_n\rangle$, $\langle \mathbf{n};\, d, r_n, l_k\rangle$ $(k=3, \ldots, n-2)$.

The interval of clones between $[d]$ and $H=[p, r_n]$ on $\mathbf{n}$ is the lattice with the diagram presented below:



Proof. For $n>2$, this follows immediately from Lemmas 6, 7 and 8. The case $n=2$ can be found in Post's work ([6], pp. 72—76).

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BOLYAI INSTITUTE
JÓZSEF ATTILA UNIVERSITY
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

KATEDRA MATEMATIKY
STROJNÍCKA FAKULTA VŠT
ŠVERMOVA 9
041 87 KOŠICE, CZECHOSLOVAKIA

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Unbounded operators with spectral decomposition properties

I. ERDÉLYI

The general spectral decomposition problem for bounded linear operators on a complex Banach space X has been formulated and studied in [4]. In this paper we extend the problem to the unbounded case and show that the single valued extension property remains valid for a class of closed linear operators on X.

While the theory of unbounded decomposable operators considered in [2, 3] relies heavily upon the concept of spectral capacity [1], here we make the theory independent of such an external constraint.

A short glossary of notations now follows. For a subset S of the complex plane C, $\bar{S}$ denotes the closure, S^c the complement, conv S the convex hull and $d(\lambda, S)$ the distance from a point λ to S. $\mathcal{G}$ denotes the collection of all open sets in C. For a linear operator T on X we use the following notations: the domain D_T, the spectrum $\sigma(T)$, the resolvent set $\varrho(T)$ and the resolvent operator $R(\cdot\,; T)$. A subspace (closed linear manifold) Y of X is invariant under T if $T(Y\cap D_T)\subset Y$. Inv (T) denotes the family of invariant subspaces under T. For $Y\in\operatorname{Inv}(T)$, we write $T|Y$ for the restriction of T to Y and we abbreviate $\lambda I-T$ by $\lambda-T$, where $\lambda\in C$ and I stands for the identity operator.

Let T: $D_T(\subset X)\to X$ be a closed linear operator.

1. Definition. A *spectral decomposition* of X by T is a finite system $\{(G_i, Y_i)\}\subset\subset\mathcal{G}\times\operatorname{Inv}(T)$ with the following properties:

(i) $$\sigma(T)\subset\bigcup_i G_i;$$

(ii) $$X=\sum_i Y_i;$$

(iii) $$\sigma(T|Y_i)\subset G_i \quad \text{for all } i.$$

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2. Definition. T is said to have the *spectral decomposition property* (abbrev. SDP) if for every finite open cover $\{G_i\}$ of $\sigma(T)$, there is a system $\{Y_i\}\subset \mathrm{Inv}(T)$ with the following properties:

(I) $\qquad Y_i \subset D_T$ if G_i is relatively compact;

(II) $\qquad \{(G_i, Y_i)\}$ is a spectral decomposition of X by T.

Our objective is to show that T with the SDP possesses the single valued extension property. For this we need a lemma.

3. Lemma. *Given T, let $f\colon D\to D_T$ be holomorphic on an open connected set $D\subset C$ and satisfy conditions:*

$$f(\lambda)\not\equiv 0 \quad \text{and} \quad (\lambda-T)f(\lambda)=0 \quad \text{on } D.$$

If $Y\in \mathrm{Inv}(T)$ is such that $\{f(\lambda)\colon \lambda\in G\}\subset Y$ for some $G\in\mathcal{G}$ then $D\subset\sigma(T|Y)$.

Proof. Define

$$H=\{\lambda\in D\colon f(\lambda), f'(\lambda), f''(\lambda), \dots, \in Y\}.$$

H has the following properties:

(a) $H\neq\emptyset$; (b) H is open; (c) H is closed in D; (d) $H\subset\sigma(T|Y)$.

(a): Let $\lambda_0\in G$. For $r>0$ sufficiently small, $\Gamma=\{\lambda\in C\colon |\lambda-\lambda_0|=r\}\subset G$ and then by hypothesis, $\{f(\lambda)\colon \lambda\in\Gamma\}\subset Y$. By Cauchy's formula

$$f^{(n)}(\lambda_0)=\frac{n!}{2\pi i}\int_\Gamma \frac{f(\lambda)\,d\lambda}{(\lambda-\lambda_0)^{n+1}}\in Y, \quad n=0,1,2,\dots$$

(b): Let $\lambda_0\in H$. Then $f(\lambda_0), f'(\lambda_0), f''(\lambda_0), \ldots\in Y$. Since $f, f', f'', \ldots$ are analytic, they admit Taylor series expansions in an open neighborhood $V(\lambda_0)$ of λ_0 and hence $f^{(n)}(\lambda)\in Y$ on $V(\lambda_0)$ for $n=0,1,2,\ldots$. Thus $V(\lambda_0)\subset H$.

(c):
$$H=\Big[\bigcap_{n=0}^{\infty}(f^{(n)})^{-1}(Y)\Big]\cap D.$$

(d): Let $\lambda\in H$. The vectors $f^{(n)}(\lambda)$ are not all zero because otherwise $f=0$. Let

$$m=\min\{n\colon f^{(n)}(\lambda)\neq 0\}.$$

If $m=0$ then $Tf(\lambda)=\lambda f(\lambda)$ and

$$Tf^{(m)}(\lambda)=\lambda f^{(m)}(\lambda) \quad \text{for} \quad m>0. \tag{1}$$

(1) holds because f is T-analytic (cf. [5, Lemma 2.1]) on D. In either case, $f^{(m)}(\lambda)$ is an eigenvector of $T|Y$ with respect to the eigenvalue λ.

By properties (a), (b), (c) $H=D$ and then property (d) concludes the proof.

4. Theorem. *Every T with the* SDP *has the single valued extension property.*

Proof. *Let f: $D \to D_T$ be locally holomorphic on an open $D \subset C$ and satisfy identity*

$$(\lambda - T)f(\lambda) = 0 \quad \text{on } D.$$

We shall adapt the proof of [4, Theorem 8] to the unbounded case. We may assume that D is connected and contained in $\sigma(T)$, for $D \cap \varrho(T) \neq \emptyset$ implies that $f=0$ on some open set and hence on all of D, by analytic continuation. Fix $\lambda_0 \in D$ and choose real numbers r_1 and r_2 such that $0 < r_2 < r_1 < d(\lambda_0, D^c)$. Let

$$G_1 = \{\lambda: |\lambda - \lambda_0| < r_1\}, \quad G_2 = \{\lambda: |\lambda - \lambda_0| > r_2\}.$$

Then G_1, G_2 cover $\sigma(T)$, $\bar{G}_1$ is both convex and compact, $D - \bar{G}_1 \neq \emptyset$ and

$$D \not\subset G_2. \tag{2}$$

By the SDP of T, there are $Y_1, Y_2 \in \operatorname{Inv}(T)$ verifying the following conditions:

$$X = Y_1 + Y_2 \quad \text{with} \quad Y_1 \subset D_T;$$

$$\sigma(T|Y_i) \subset G_i, \quad i = 1, 2. \tag{3}$$

There is an open $V \subset D - \bar{G}_1$ and there are functions f_i: $V \to Y_i$ $(i=1, 2)$ such that

$$f(\mu) = f_1(\mu) + f_2(\mu) \quad \text{on } V. \tag{4}$$

Since the ranges of both f and f_1 are contained in D_T, so is the range of f_2. There is a function g: $V \to Y_1 \cap Y_2$ defined by

$$g(\mu) = (\mu - T)f_1(\mu) = (T - \mu)f_2(\mu) \in Y_1 \cap Y_2, \quad \mu \in V.$$

Since $Y_1 \cap Y_2$ is invariant under $T|Y_1$ and G_1 is convex, we have

$$\sigma(T|Y_1 \cap Y_2) \subset \operatorname{conv} \sigma(T|Y_1) \subset G_1.$$

Consequently, $V \subset G_1^c \subset \varrho(T|Y_1 \cap Y_2)$. The function h: $V \to Y_1 \cap Y_2$, defined by

$$h(\mu) = R(\mu; T|Y_1 \cap Y_2)\, g(\mu) \in Y_1 \cap Y_2, \quad \mu \in V$$

has property

$$(\mu - T)[h(\mu) - f_1(\mu)] = 0.$$

Since both $h(V) \subset Y_1$, $f_1(V) \subset Y_1$ and $V \subset \varrho(T|Y_1)$, we have

$$f_1(\mu) = h(\mu) \in Y_1 \cap Y_2 \quad \text{on } V.$$

Then (4) implies that $f(\mu) \in Y_2$ on V and hence $f(\mu) \in Y_2$ on all of D, by analytic continuation. Now if f is not identically zero on D then Lemma 3 implies that $D \subset \sigma(T|Y_2)$. This, under hypothesis (2), contradicts the second inclusion of (3). □

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MATHEMATICS DEPARTMENT
TEMPLE UNIVERSITY
PHILADELPHIA, PA 19122
USA

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Weighted shifts quasisimilar to quasinilpotent operators

LAWRENCE A. FIALKOW

1. Introduction. The purpose of this note is to resolve certain questions raised in [8] and [9] concerning quasisimilarity and quasinilpotent operators. We prove that a weighted shift is quasisimilar to a quasinilpotent operator if and only if it is a direct sum of quasinilpotents (Theorems 2.7 and 2.8). As an application, we show that there exist operators T such that T and T^* are quasiaffine transforms of quasinilpotent operators but such that T is not quasisimilar to any quasinilpotent operator (Corollary 2.9). In section 3 we relate our results to several open problems concerning quasisimilarity and spectra.

Let $\mathfrak{H}$ denote a separable infinite dimensional complex Hilbert space and let $\mathscr{L}(\mathfrak{H})$ denote the algebra of all bounded linear operators on $\mathfrak{H}$. Let $\mathscr{N}$ and $\mathscr{Q}$ denote, respectively, the subsets of $\mathscr{L}(\mathfrak{H})$ consisting of all nilpotent and quasinilpotent operators. For T in $\mathscr{L}(\mathfrak{H})$, let $\mathfrak{M}(T)=\{x\in\mathfrak{H}\colon \|T^nx\|^{1/n}\to 0\}$. $\mathfrak{M}(T)$ is a linear manifold whose closure is hyperinvariant for T; moreover, T is quasinilpotent if and only if $\mathfrak{M}(T)=\mathfrak{H}$ [7, Lemma, page 28].

An operator X in $\mathscr{L}(\mathfrak{H})$ is a *quasiaffinity* if X is injective and has dense range. An operator B is a *quasiaffine transform* of an operator A if there exists a quasiaffinity X such that $AX=XB$. Operators A and B are *quasisimilar* if they are quasiaffine transforms of each other [18]. C. Apostol, R. G. Douglas, and C. Foiaş [4, Corollary, page 413] gave necessary and sufficient conditions for two nilpotent operators to be quasisimilar, but analogous results for quasinilpotent operators appear to be unknown. The present note concerns the quasisimilarity orbit of $\mathscr{Q}$. Let $\mathscr{Q}_{af}=\{T\in\mathscr{L}(\mathfrak{H})\colon$ T is a quasiaffine transform of some quasinilpotent operator$\}$, and let $\mathscr{Q}^*_{af}=\{T\in\mathscr{L}(\mathfrak{H})\colon T^*$ is in $\mathscr{Q}_{af}\}$. Let $\mathscr{Q}_{qs}$ denote the quasisimilarity orbit of $\mathscr{Q}$, i.e. $\mathscr{Q}_{qs}=\{T\in\mathscr{L}(\mathfrak{H})\colon T$ is quasisimilar to some quasinilpotent operator$\}$.

In [8] and [9] we obtained the following invariants for membership in $\mathscr{Q}_{qs}$. A compact subset $K\subset\mathbf{C}$ is the spectrum of an operator in $\mathscr{Q}_{qs}$ if and only if K

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is connected and contains 0 [8, Theorem 3.11]. If T is in $\mathcal{Q}_{qs'}$, then T satisfies the following properties:

(I) $\mathfrak{M}(T)$ and $\mathfrak{M}(T^*)$ both contain orthonormal bases for $\mathfrak{H}$; in particular, $\mathfrak{M}(T)$ and $\mathfrak{M}(T^*)$ are dense in $\mathfrak{H}$ [8, Proposition 3.13].

(II) If $\mathfrak{M}\neq\{0\}$ is an invariant subspace for T, then $\sigma(T|\mathfrak{M})$ is connected and contains 0; if additionally, $\mathfrak{M}\neq\mathfrak{H}$, then $\sigma((1-P_{\mathfrak{M}})T|(1-P_{\mathfrak{M}})\mathfrak{H})$ is connected and contains 0 [8, Theorem 3.1]. ($P_{\mathfrak{M}}$ denotes the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{M}$ and $\sigma(\cdot)$ denotes the spectrum of an operator.) Each operator satisfying (I) also satisfies (II) [8, Proposition 3.15]; several equivalent reformulations of (II) are given in [9, section 3].

Note that $\mathcal{Q}_{qs}\subset\mathcal{Q}_{af}\cap\mathcal{Q}_{af}^*$ and that if T is in $\mathcal{Q}_{af}$, then $\mathfrak{M}(T^*)$ is dense [8, Lemma 3.12]. C. Apostol [3] proved that $\mathfrak{M}(T^*)$ is dense if and only if T is a quasiaffine transform of a compact quasinilpotent operator. Thus an operator T satisfies (I) if and only if T is in $\mathcal{Q}_{af}\cap\mathcal{Q}_{af}^*$.

In [8] we studied whether (I) actually implies membership in $\mathcal{Q}_{qs}$, or equivalently (in view of Apostol's result), whether $\mathcal{Q}_{qs}=\mathcal{Q}_{af}\cap\mathcal{Q}_{af}^*$. In [8] we obtained an affirmative answer to this question for decomposable operators (including normal, spectral, compact, and Riesz operators) and for hyponormal operators. If T is decomposable and $\mathfrak{M}(T^*)$ is dense, then T is quasinilpotent [8, Corollary 3.4]; moreover, the only hyponormal operator satisfying $\mathfrak{M}(T)^-=\mathfrak{H}$ is the zero operator [8, Theorem 3.6]. In section 2 we show that despite these positive results, $\mathcal{Q}_{qs}$ is actually a proper subset of $\mathcal{Q}_{af}\cap\mathcal{Q}_{af}^*$, so that neither (I) nor (II) necessarily implies membership in $\mathcal{Q}_{qs}$.

2. Weighted shifts in $\mathcal{Q}_{qs}$. Let $I=\mathbf{Z}$ or $\mathbf{Z}^+$ and let $\alpha=\{\alpha_n\}_{n\in I}$ denote a bounded sequence of complex numbers. Let $\{e_n\}_{n\in I}$ denote an orthonormal basis for $\mathfrak{H}$. The *weighted shift with weight sequence* α, W_α, is defined by the relations $W_\alpha e_n=\alpha_n e_{n+1}$ $(n\in I)$. If $I=\mathbf{Z}^+$, W_α is a *unilateral* shift, while if $I=\mathbf{Z}$, W_α is a *bilateral* shift. T. B. Hoover [14] exhibited weight sequences α and β, both with infinitely many zero terms, such that W_α and W_β are quasisimilar, W_α is quasinilpotent, and the spectrum of W_β is the closed unit disk. In this section we characterize the weighted shifts in $\mathcal{Q}_{qs}$.

For T in $\mathcal{L}(\mathfrak{H})$ and $n\geq 0$, let $\mathfrak{M}_n(T)=\ker(T^{n+1})\ominus\ker(T^n)$. Let $\mathfrak{P}(T)=\bigvee_{n=1}^{\infty}\ker(T^n)=\sum_{n=0}^{\infty}\oplus\mathfrak{M}_n(T)$, and let $\mathfrak{M}_\infty(T)=\mathfrak{H}\ominus\mathfrak{P}(T)=\bigcap_{n=1}^{\infty}(\mathfrak{H}\ominus\ker(T^n))$. In the sequel, $\dim\mathfrak{M}$ refers to the orthogonal dimension of a closed subspace $\mathfrak{M}\subset\mathfrak{H}$.

Lemma 2.1. *If A and B are quasisimilar operators in $\mathcal{L}(\mathfrak{H})$, then A and B have the following properties:*

1) $\dim\mathfrak{M}_n(A)=\dim\mathfrak{M}_n(B)$ *for* $0\leq n\leq\infty$;

2) $\dim\ker(A^n)=\dim\ker(B^n)$ *for* $n>0$.

Proof. Let X and Y denote quasiaffinities such that $AX=XB$ and $YA=BY$. To prove 1) it suffices to show that $\dim \mathfrak{M}_n(B) \geqq \dim \mathfrak{M}_n(A)$ for $0 \leqq n \leqq \infty$, for then 1) follows by symmetry. Let $0 \leqq n < \infty$; we may assume that $\dim \mathfrak{M}_n(A) > 0$. Let $\{e_k\}_{0 \leqq k < p}$ $(0 < p \leqq \infty)$ denote an orthonormal basis for $\mathfrak{M}_n(A)$. Let $P_0 = 0$ and for $n > 0$, let P_n denote the orthogonal projection onto $\ker(B^n)$; note that $P_{n+1} - P_n$ is the projection onto $\mathfrak{M}_n(B)$.

We show that $\{(1-P_n)Ye_k\}_{0 \leqq k < p}$ is an independent sequence in $\mathfrak{M}_n(B)$. Since $A^{n+1}e_k = 0$, then $B^{n+1}Ye_k = YA^{n+1}e_k = 0$, so $(1-P_n)Ye_k = (P_{n+1}-P_n)Ye_k \in \mathfrak{M}_n(B)$. Suppose $0 \leqq j < p$, $c_0, \ldots, c_j \in \mathbf{C}$ and $\sum_{i=0}^{j} c_i(1-P_n)Ye_i = 0$. Then $\sum c_i Ye_i = P_n \sum c_i Ye_i \in \ker(B^n)$, and so $YA^n(\sum c_i e_i) = B^n(\sum c_i Ye_i) = 0$. Since Y is injective, $\sum c_i e_i \in \ker(A^n)$, and thus $0 = (\sum c_i e_i, e_m) = c_m$ for $0 \leqq m \leqq j$. Therefore $\{(1-P_n)Ye_k\}_{0 \leqq k < p}$ is independent, and it follows (via Gram—Schmidt) that $\dim \mathfrak{M}_n(B) \geqq p = \dim \mathfrak{M}_n(A)$. This completes the proof of 1) for $n < \infty$.

Note that if $y \in \mathfrak{M}_\infty(A)$, then $X^* y \in \mathfrak{M}_\infty(B)$. Indeed, if $z \in \mathfrak{H}$, $n > 0$, and $B^n z = 0$, then $(X^* y, z) = (y, Xz) = 0$ since $Xz \in \ker(A^n)$ and $y \in \mathfrak{M}_\infty(A)$. Since X^* is injective, it follows that $\dim \mathfrak{M}_\infty(B) \geqq \dim \mathfrak{M}_\infty(A)$; the reverse inequality follows by symmetry.

For 2), note that since $\ker(A^{n+1}) = \ker(A^n) \oplus \mathfrak{M}_n(A)$, $\mathfrak{M}_0(A) = \ker(A)$, $\ker(B^{n+1}) = \ker(B^n) \oplus \mathfrak{M}_n(B)$, and $\mathfrak{M}_0(B) = \ker(B)$, the result follows from 1) by induction on n.

Corollary 2.2. *Let A and B be quasisimilar operators in $\mathscr{L}(\mathfrak{H})$. Then there is an operator B' unitarily equivalent to B such that $\mathfrak{M}_n(A) = \mathfrak{M}_n(B')$ for $0 \leqq n \leqq \infty$.*

Proof. For $0 \leqq n \leqq \infty$, let P_n and Q_n denote, respectively, the orthogonal projections onto $\mathfrak{M}_n(A)$ and $\mathfrak{M}_n(B)$. Note that $\sum_{0 \leqq n \leqq \infty} P_n = \sum_{0 \leqq n \leqq \infty} Q_n = 1$ and $P_i P_j = Q_i Q_j = 0$ for $i \neq j$ $(0 \leqq i, j \leqq \infty)$. Lemma 2.1 implies that there exists an isometric operator V_n which maps $\mathfrak{M}_n(A)$ onto $\mathfrak{M}_n(B)$. Let $V = \sum_{0 \leqq n \leqq \infty} V_n P_n$ (strong convergence); then $V^* = \sum V_n^* Q_n$ and V is unitary. If $B' = V^* B V$, it follows that $\mathfrak{M}_n(A) = \mathfrak{M}_n(B')$ for each n.

Remark. An analogue of Corollary 2.2 for $n = 0$ is implicit in the proof of [19, Lemma 2].

For T in $\mathscr{L}(\mathfrak{H})$, let $(T)' = \{S \in \mathscr{L}(\mathfrak{H}) : TS = ST\}$ and let $(T)'' = \{R \in \mathscr{L}(\mathfrak{H}) : RS = SR$ for each S in $(T)'\}$. In the sequel $r(T)$ denotes the spectral radius of T.

Lemma 2.3. *Let A, B, X, and Y be operators such that $AX = XB$ and $YA = BY$. If $R \in (B)''$, then $XRY \in (A)'$ and $r(XRY) \leqq r(YX) r(R)$.*

Proof. The hypothesis implies that $XRYA = XRBY = XBRY = AXRY$, so XRY commutes with A. Since $R \in (B)''$ and $YX \in (B)'$, R commutes with YX, and thus $r(XRY) = r(YXR) \leqq r(YX) r(R)$.

Corollary 2.4. *If A is in $\mathcal{Q}_{qs}$, then A commutes with a nonzero quasinilpotent operator.*

Proof. Let $B\in\mathcal{Q}$ be quasisimilar to A and let X and Y denote quasiaffinities such that $AX=XB$ and $YA=BY$. Lemma 2.3 implies that XBY is a quasinilpotent operator commuting with A; moreover, since X is injective and Y has dense range, XBY is nonzero if B is nonzero. If $B=0$, then $A=0$, so the result is clear in this case also.

Lemma 2.5. *Let W be a noninvertible injective weighted shift such that $r(W)>0$. If S commutes with W, then $\sigma(S)$* (the spectrum of S) *has nonempty interior or S is a scalar multiple of the identity.*

Proof. The proof depends on several results from [15] to which we refer the reader for complete details. We consider first the case when W is a unilateral shift. In this case S may be represented as a multiplication operator M_Φ on a space of formal power series $H^2(\beta)$ [15, Theorem 3(b)]. The power series for the multiplier Φ is convergent in $D=\{z\in C: |z|<r(W)\}$ [15, Theorem 10(iii)], and thus represents an analytic function $\Phi(z)$ in D. Now $\sigma(M_\Phi)$ coincides with the spectrum of Φ in $H^\infty(\beta)$ [15, Proposition 20], and thus $\sigma(M_\Phi)$ contains $\Phi(D)$ [15, page 79]. If M_Φ is not a scalar multiple of the identity, then Φ is non-constant, and it follows that $\Phi(D)$, and thus also $\sigma(M_\Phi)$, has nonempty interior. The proof for the case when W is a non-invertible bilateral shift is analogous; the pertinent results are [15, Theorem 3(a)], [15, Theorem 10'(iii—b)], and the remarks of [15, page 83].

Remark. The conclusion of Lemma 2.5 may fail if W is invertible; consider the unweighted bilateral shift, whose spectrum is the unit circle. Note also that there exist noninjective, non-quasinilpotent weighted shifts which commute with nonzero quasinilpotent operators.

Corollary 2.6. *If W is a noninvertible injective weighted shift and $r(W)>0$, then W commutes with no nonzero quasinilpotent operator.*

Theorem 2.7. *Let $W=W_\alpha$ be a bilateral weighted shift. The following are equivalent.*

1) $W\in\mathcal{Q}_{qs}$;
2) *W is a direct sum of quasinilpotent operators, and if α has at most finitely many zero terms, then W is quasinilpotent.*

Proof. The implication 2)$\Rightarrow$1) follows from [8, Proposition 3.10]. For the converse, we assume that $W\in\mathcal{Q}_{qs}$ and we consider several cases depending on the number and location of the zero terms in the weight sequence α. Note that since $W\in\mathcal{Q}_{qs}$, then W is noninvertible [14], [12].

i) W is injective. Since $W \in \mathcal{Q}_{qs}$, Corollary 2.4 implies that W commutes with a nonzero quasinilpotent; thus Corollary 2.6 implies that W is quasinilpotent.

ii) For each integer N, there exist integers m and n, $n<N<m$, such that $\alpha_m=\alpha_n=0$. It is clear that in this case W is an infinite direct sum of finite dimensional nilpotent operators.

iii) There exist integers n and m, $n \leqq m$, such that $\alpha_n=0$, $\alpha_m=0$, and $\alpha_k \neq 0$ for $k<n$ or $k>m$. We consider only the case $n<m$; the case $n=m$ may be treated similarly. Let $\mathfrak{H}_1=\langle e_n, e_{n-1}, e_{n-2}, \ldots\rangle$, $\mathfrak{H}_2=\langle e_{n+1}, \ldots, e_m\rangle$, and $\mathfrak{H}_3= =\langle e_{m+1}, e_{m+2}, \ldots\rangle$. Relative to the decomposition $\mathfrak{H}=\mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \mathfrak{H}_3$, the operator matrix of W is of the form $W=W_\beta^* \oplus N \oplus W_\gamma$, where W_β and W_γ are injective unilateral weighted shifts on $\mathfrak{H}_1$ and $\mathfrak{H}_3$ respectively, and $N^{m-n}=0$.

Suppose that W is quasisimilar to a quasinilpotent operator Q. Let X and Y be quasiaffinities such that $WX=XQ$ and $YW=QY$. Note that $\mathfrak{P}(W)=\mathfrak{H}_1 \oplus \mathfrak{H}_2$ and $\mathfrak{M}_\infty(W)=\mathfrak{H}_3$. Corollary 2.2 implies that there is an operator Q' unitarily equivalent to Q such that $\mathfrak{M}_n(W)=\mathfrak{M}_n(Q')$ $(0 \leqq n \leqq \infty)$, and thus $\ker(W^n)=\ker(Q'^n)$ for $n \geqq 0$. Let U denote a unitary operator such that $Q'=U^*QU$. Let $X'=XU$ and $Y'=U^*Y$; clearly X' and Y' are quasiaffinities and (*) $W^n X'=X' Q'^n$ and $Y' W^n=Q'^n Y'$ for $n>0$. Since $\ker(W^n)=\ker(Q'^n)$, the preceding equations imply that $\mathfrak{M} \equiv \mathfrak{P}(W)=\mathfrak{P}(Q')=\mathfrak{H}_1 \oplus \mathfrak{H}_2$ is an invariant subspace for X' and Y'.

Relative to the decomposition $\mathfrak{H}=\mathfrak{M} \oplus \mathfrak{M}^\perp$, the operator matrices of X', Y', Q', and W are of the form

$$\begin{pmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{pmatrix}, \quad \begin{pmatrix} Y_{11} & Y_{12} \\ 0 & Y_{22} \end{pmatrix}, \quad \begin{pmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} Z & 0 \\ 0 & W_\gamma \end{pmatrix},$$

where X_{22} and Y_{22} have dense range, and $Q_{22} \in \mathcal{Q}$. The equations (*) imply that $W_\gamma X_{22}=X_{22} Q_{22}$ and $Y_{22} W_\gamma=Q_{22} Y_{22}$. Lemma 2.3 implies that $R=X_{22} Q_{22} Y_{22}$ is a quasinilpotent operator commuting with W_γ, and we assert that R is nonzero. For otherwise, since $Y_{22} X_{22}$ commutes with Q_{22}, it follows that $0=Y_{22} R=Y_{22} X_{22} Q_{22} Y_{22}= =Q_{22} Y_{22} X_{22} Y_{22}$. Since X_{22} and Y_{22} have dense range, it follows that $Q_{22}=0$. Now (*) implies that $W_\gamma=0$, which is a contradiction.

Thus R is a nonzero quasinilpotent commuting with W_γ, so Corollary 2.6 implies that W_γ is quasinilpotent. By applying the preceding method to W^*, we conclude that W_β is also quasinilpotent. (Note that $W^*=W_\beta \oplus N^* \oplus W_\gamma^*$, so that $\mathcal{P}(W^*)= =\mathfrak{H}_2 \oplus \mathfrak{H}_3$.) Since $W_\gamma, W_\beta \in \mathcal{Q}$ and $N \in \mathcal{N}$, it follows that W is quasinilpotent, which (together with case i)) completes the proof of the second part of 2).

iv) There exists a largest integer N such that $\alpha_N=0$, but there exists no smalles, such integer. Let $\mathfrak{M}=\langle e_N, e_{N-1}, \ldots\rangle$. Relative to the decomposition $\mathfrak{H}=\mathfrak{M} \oplus \mathfrak{M}^\perp$, $W=N \oplus W_\gamma$, where N is an infinite direct sum of finite dimensional nilpotentst and W_γ is an injective unilateral weighted shift. Since $\mathfrak{P}(W)=\mathfrak{M}$, the method of case iii) implies that W_γ is quasinilpotent, so W is a direct sum of quasinilpotents.

v) There is a smallest integer N such that $\alpha_N=0$, but there is no largest such integer. The desired conclusion that W is a direct sum of quasinilpotents follows by applying case iv) to W^*, which is a bilateral weighted shift relative to the basis $\{f_n\}_{n=-\infty}^{+\infty}$, where $f_n=e_{-n}$.

Theorem 2.8. *Let $W=W_\gamma$ be a unilateral weighted shift. The following are equivalent.*

1) $W\in\mathcal{Q}_{qs}$;
2) $W\in\mathcal{Q}$ *or* γ *has infinitely many zero terms;*
3) W *is a direct sum of quasinilpotent operators.*

Proof. The implication 2)⇒3) is obvious and 3)⇒1) follows from [8, Prop. 3.10]. Suppose that $W_\gamma\in\mathcal{Q}_{qs}$. Let W_β be a quasinilpotent injective unilateral weighted shift and let $W_\alpha=W_\beta^*\oplus W_\gamma$. Thus W_α is a bilateral weighted shift and W_α is in $\mathcal{Q}_{qs}$ [8]. If at most a finite number of the weights of W_α are zero, then Theorem 2.7 implies that $W_\alpha\in\mathcal{Q}$, from which it follows that $W_\gamma\in\mathcal{Q}$. In the remaining case, W_α has infinitely many zero weights, and since W_β is injective, these weights correspond to zero terms in γ.

Corollary 2.9. *$\mathcal{Q}_{qs}$ is a proper subset of $\mathcal{Q}_{af}\cap\mathcal{Q}_{af}^*$.*

Proof. According to [9, Example 3.2], there exists a non-quasinilpotent injective unilateral weighted shift W such that $\mathfrak{M}(W)$ and $\mathfrak{M}(W^*)$ both contain the orthonormal basis $\{e_n\}_{n=0}^{\infty}$. Thus $\mathfrak{M}(W)$ and $\mathfrak{M}(W^*)$ are both dense, and so [3] implies that $W\in\mathcal{Q}_{af}\cap\mathcal{Q}_{af}^*$. However, since W is injective and non-quasinilpotent, Theorem 2.8 implies that W is not in $\mathcal{Q}_{qs}$.

Remark. The shift W in the preceding proof satisfies properties (I) and (II) of section 1. It follows that in general neither property implies membership in $\mathcal{Q}_{qs}$. These results provide negative answers to Question 3.9, Question 3.14, and Question 3.16 of [8]. Theorem 2.7 and Theorem 2.8 answer [8, Question 3.7]. We note also that it is possible to prove Theorem 2.8 directly, without recourse to Theorem 2.7, by employing the same technique used to prove Theorem 2.7.

In [11] C. Foiaş and C. Pearcy proved that if Q is quasinilpotent, then Q and Q^* are quasiaffine transforms of compact operators (which are necessarily quasinilpotent). (This result also follows from [3].) In [11, Proposition 1.5] it is also proved that there exists a quasinilpotent operator that is not quasisimilar to any compact operator. The shift W of Corollary 2.9 is an example of a non-quasinilpotent operator such that W and W^* are quasiaffine transforms of compact operators but such that W is not quasisimilar to any compact operator. The fact that W and

W^* are quasiaffine transforms of compact operators follows from [3]. Now each nonzero operator quasisimilar to a compact operator commutes with a nonzero compact operator [11, Proposition 1.5]; since the spectrum of a compact operator is countable, Lemma 2.5 implies that W is not quasisimilar to any compact operator.

3. Conclusion. In this section we relate our results to a conjecture of [13], discuss some related questions. Let $\{\mathfrak{M}_n\}_{1\leqq n<k}$ $(2<k\leqq\infty)$ denote a sequence of closed subspaces of $\mathfrak{H}$. The sequence $\{\mathfrak{M}_n\}$ is said to be a *basic sequence* for an operator T in $\mathscr{L}(\mathfrak{H})$ if the following properties are satisfied: 1) for each n, $\mathfrak{M}_n$ is invariant for T, i.e. $T\mathfrak{M}_n\subset\mathfrak{M}_n$; 2) For each n, $\mathfrak{M}_n$ and $\bigvee_{m\neq n}\mathfrak{M}_m$ are complementary in $\mathfrak{H}$; 3) If $k=\infty$, then $\bigcap_{n=1}^{\infty}\left(\bigvee_{m\geqq n}\mathfrak{M}_m\right)=\{0\}$. The *trivial* basic sequence for any operator is the sequence $\mathfrak{M}_1=\{0\}$, $\mathfrak{M}_2=\mathfrak{H}$. The concept of a basic sequence is due to C. Apostol [1].

D. A. Herrero [13, Conjecture 1] stated the following

Conjecture H. [13] *If an operator T has no non-trivial basic sequence, then each operator S quasisimilar to T satisfies $\sigma(S)=\sigma(T)$.*

Theorem 2.8 can be interpreted as offering some (albeit limited) support to this conjecture. Indeed, an injective unilateral weighted shift T has no nontrivial pair of complementary invariant subspaces [15, Corollary 2, page 63]. Thus T has no nontrivial basic sequence, and Theorem 2.8 shows that if $r(T)>0$, then $r(S)>0$ for each operator S quasisimilar to T. We can show a bit more. Suppose T shifts the basis $\{e_n\}_{n=0}^{\infty}$. Let X and Y be operators with dense range and let S be an operator such that $TX=XS$ and $YT=SY$. Since XY commutes with T, [16, page 780] implies that the matrix of XY relative to $\{e_n\}$ is given by a formal power series $\sum_{n=0}^{\infty}a_nT^n$. Since XY has dense range, $|a_0|>0$. By method quite different than that used in section 2 it can be shown that if $\sum_{n=1}^{\infty}|a_n|\,\|S^n\|<|a_0|$, then $r(S)\geqq r(T)$; note that if $a_n=0$ for each $n\geqq 1$, then XY is invertible, so T and S are similar.

In a different direction, S. Clary [5] has studied subnormal operators quasisimilar to the unweighted unilateral shift U. It follows from [5] that there exists subnormal operators S such that S and U are quasisimilar but not similar; however, quasisimilar subnormal operators do have equal spectra [6, Theorem 2]. The preceding remarks suggest the following question.

Question 3.1. If T is an injective unilateral weighted shift and S is quasisimilar to T, does $\sigma(S)=\sigma(T)$?

C. Apostol [1] proved that an operator T is quasisimilar to a normal operator if and only if T has a basic sequence $\{\mathfrak{M}_n\}$ such that each restriction $T|\mathfrak{M}_n$ is similar to a normal operator. In [9, Theorem 5.5] it is proved that if an operator T has a basic sequence $\{\mathfrak{M}_n\}$ such that each restriction $T|\mathfrak{M}_n$ is a spectral operator, then T is quasisimilar to a spectral operator. The proof of this result also yields the following sufficient condition for membership in $\mathcal{Q}_{qs}$.

Proposition 3.2. *If $\{\mathfrak{M}_n\}$ is a basic sequence for an operator T such that each restriction $T|\mathfrak{M}_n$ is quasinilpotent, then T is in $\mathcal{Q}_{qs}$.*

Question 3.3. Is the converse of Proposition 3.2 true?

The results of [8] show that if T is in $\mathcal{Q}_{qs}$ and T is decomposable or hyponormal, then T is quasinilpotent, so Question 3.3 has an affirmative answer for operators in these classes. More generally, the answer is affirmative for each operator T satisfying property (C) in the sense of [17], since for each such operator, $\mathfrak{M}(T)$ is closed. The answer is also affirmative for weighted shifts; indeed Theorem 2.7 and Theorem 2.8 may be reformulated as follows.

Theorem 3.4. *A weighted shift W is in $\mathcal{Q}_{qs}$ if and only if there exists a basic sequence $\{\mathfrak{M}_n\}$ for W such that each restriction $W|\mathfrak{M}_n$ is quasinilpotent.*

Proof. Theorem 2.7 and Theorem 2.8 imply that if a weighted shift W is in $\mathcal{Q}_{qs}$, then W is a direct sum of quasinilpotents; this direct sum decomposition gives rise to the desired basic sequence. The converse follows from Proposition 3.2.

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DEPARTMENT OF MATHEMATICS
WESTERN MICHIGAN UNIVERSITY
KALAMOZOO, MICHIGAN 49008
USA

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On the admissibility of topological vector spaces

O. HADŽIĆ

1. Introduction. Let X be a Hausdorff topological vector space. A subset A of X is called *admissible* [7] if for every compact subset $K \subset A$ and for every neighbourhood U of zero in X there is some continuous mapping $h\colon K \to A$ such that

(i) $\dim(\operatorname{span} h(K)) < \infty$,

(ii) $x - hx \in U$, for all $x \in K$.

S. HAHN and K. F. PÖTTER [3] proved fixed point theorems for admissible subsets of Hausdorff topological vector spaces. NAGUMO proved that all convex subsets of a locally convex space are admissible [9] and the admissibility of many non-convex topological vector spaces has been proved by KLEE [6], RIEDRICH [14], [15], ICHII [4], PALLASCHKE [12] and KRAUTHAUSEN [7].

But the following questions remained open:

a) Which Hausdorff topological vector spaces are admissible?

b) Which convex subsets are admissible?

c) For which compact subsets K of a Hausdorff topological vector space X is the following valid:

(*) If U is an arbitrary neighbourhood of zero in X then there is a finite dimensional continuous mapping $h\colon K \to \operatorname{co} K$ such that $x - hx \in U$ for all $x \in K$.

Recently, MATUSOV [8] proved that every compact convex subset of a Hausdorff topological vector space has the fixed point property using an idea of SARIMSAKOV [10] and a result of KASAHARA [5].

Now we give Kasahara's definition of paranormed spaces [5].

A linear mapping Φ of a topological semifield E into another F is said to be *positive* if $\Phi(x) \geqq 0$ in F for every $x \in E$ with $x \geqq 0$. Let $\| \ \|$ be a mapping of a

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linear space X into a topological semifield E and let Φ be a continuous positive linear mapping of E into itself. The triple $(X, \| \|, \Phi)$ is called a *paranormed space over* E and $\| \|$ a *Φ-paranorm* on X over E if the following conditions are satisfied:
(P1) $\|x\| \geqq 0$, for every $x \in X$;
(P2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for every real λ and every $x \in X$;
(P3) $\|x+y\| \leqq \Phi(\|x\| + \|y\|)$ for every $x, y \in X$.

A set $K, K \subset X$ where X is a topological vector space, is said to be of type Φ iff $(X, \| \|, \Phi)$ is a paranormed space and for every $n \in N$, every $x_1, x_2, \ldots, x_n \in K-K$ and every $\lambda_i, 0 \leqq \lambda_i \leqq 1$ $(i=1, 2, \ldots, n)$ such that $\lambda_1+\lambda_2+\ldots+\lambda_n=1$, we have $\left\| \sum_{i=1}^{n} \lambda_i x_i \right\| \leqq \sum_{i=1}^{n} \lambda_i \Phi(\|x_i\|)$. If $K=X$, the space X is of type Φ.

In this paper we shall prove:

a′) Every Hausdorff topological vector space of type Φ is admissible.

b′) Every convex subset of type Φ of a Hausdorff topological vector space is admissible.

c′) For every compact subset K of type Φ of a Hausdorff topological vector space property (*) is valid.

As a Corollary we shall obtain an extension of Matusov's fixed point theorem.

2. The main result. We use the following theorem from KASAHARA's paper [5].

Let (X, τ) be a topological linear space. Then there exists a paranormed space $(X, \| \|, \Phi)$ over a Tihonov semifield E such that:

(1) *For every neighbourhood U of $0 \in X$ there are an $\varepsilon > 0$ and an indecomposable idempotent $\varrho \in E$ such that*

$$\{x \in X : \|x\| \cdot \varrho \leqq \varepsilon \varrho\} \subset U.$$

(2) *For every neighbourhood U of $0 \in E$ the set*

$$\{x \in X : \|x\| \in U\}$$

is a neighbourhood of $0 \in X$.

The Tihonov semifield E from the above Theorem is R_Δ, the set of all mappings from Δ into R where Δ is a set of paranorms generating the topology of X and satisfying the condition that for each $p \in \Delta$ there are $\alpha > 0$ and $q \in \Delta$ such that

$$p(x+y) \leqq \alpha\big(q(x)+q(y)\big), \quad \text{for all} \quad x, y \in X.$$

Now we are ready to formulate our main theorem.

Theorem. *For every compact subset K of type Φ of a topological vector space X and for every neighbourhood U of zero in X there exists a finite dimensional continuous mapping h: $K\to \mathrm{co}\, K$ such that $x-hx\in U$ for all $x\in K$.*

Proof. Let U be an arbitrary neighbourhood of zero in X and let $\mu=\{t_1, t_2, \dots, t_n\}\subset\Delta$ and $\varepsilon>0$ such that

$$\|x-y\|\in U_{\mu,\varepsilon}\Rightarrow x-y\in U,$$

where

$$U_{\mu,\varepsilon}=\{u: u\in R_\Delta,\ u(t_j)<\varepsilon,\ j=1,2,\dots,n\}.$$

Further, since the mapping Φ: $R_\Delta\to R_\Delta$ is a continuous linear mapping there exists a neighbourhood $V_1(\mu,\varepsilon)$ of zero in R_Δ such that

$$\|x-y\|\in V_1(\mu,\varepsilon)\Rightarrow \Phi(\|x-y\|)\in U_{\mu,\varepsilon}.$$

Suppose now that $V_2(\mu,\varepsilon)$ is a circled neighbourhood of zero in X such that

$$x-y\in V_2(\mu,\varepsilon)\Rightarrow \|x-y\|\in V_1(\mu,\varepsilon).$$

Since X is a Hausdorff topological vector space it is also a Hausdorff uniform space and let d be a pseudometric on X and $\delta>0$ such that

$$d(x,y)<\delta\Rightarrow x-y\in V_2(\mu,\varepsilon).$$

We shall use the notation

$$V_x(d,\delta)=\{y: y\in X,\ d(x,y)<\delta\}\quad(\delta>0).$$

Since the set K is compact there exists a finite set $\{x_1, x_2, \dots, x_m\}\subset K$ such that for every $x\in K$ there exists $i\in\{1,2,\dots,m\}$ such that

$$x\in V_{x_i}(d,\delta).$$

So if we define the functions f_i: $K\to R^+$ $(i=1,2,\dots,m)$ so that

$$f_i(x)=\max\{0,\delta-d(x,x_i)\}$$

for every $x\in K$ and $i\in\{1,2,\dots,m\}$ it follows that

$$f_i(x)\neq 0\Leftrightarrow d(x,x_i)<\delta.$$

Since for every $x\in K$ there exists at least one $i\in\{1,2,\dots,m\}$ such that $f_i(x)\neq 0$ we conclude that for every $x\in K$,

$$s(x)=\sum_{i=1}^{m} f_i(x)\neq 0$$

and that all mappings f_i $(i=1, 2, \ldots, m)$ are continuous since the mapping $x \mapsto d(x, x_i)$ is continuous for every $i \in \{1, 2, \ldots, m\}$. Now, let

$$h(x) = \frac{1}{s(x)} \sum_{i=1}^{m} f_i(x) x_i \quad \text{for all} \quad x \in K.$$

Then $h(K) \subset \operatorname{co} K$ and h is a continuous mapping from K into a finite dimensional subspace of X. Further we have

$$\|hx - x\| = \left\| \frac{1}{s(x)} \sum_{i=1}^{m} f_i(x) x_i - \frac{1}{s(x)} \sum_{i=1}^{m} f_i(x) x \right\| =$$

$$= \left\| \frac{1}{s(x)} f_i(x)(x - x_i) \right\| \leqq \frac{1}{s(x)} \sum_{i=1}^{m} f_i(x) \Phi(\|x - x_i\|).$$

Since $f_i(x) \neq 0 \Leftrightarrow d(x, x_i) < \delta$ it follows that for every $x \in K$ such that $f_i(x) \neq 0$ we have that

$$\Phi(\|x - x_i\|) \in U_{\mu, \varepsilon}$$

and so

$$\|hx - x\|(t) \leqq \frac{1}{s(x)} \sum_{i=1}^{m} f_i(x) \Phi(\|x - x_i\|)(t) < \frac{1}{s(x)} \sum_{i=1}^{m} f_i(x) \varepsilon = \varepsilon \quad \text{for every} \quad t \in \mu.$$

So we have $\|hx - x\| \in U_{\mu, \varepsilon}$, which implies $hx - x \in U$ and the proof is complete.

Corollary 1. *Every convex subset A of type Φ of a Hausdorff topological vector space is admissible.*

Proof. If K is a compact subset of A and U is an arbitrary neighbourhood of zero, the Theorem implies the existence of a finite dimensional continuous mapping $h\colon K \to \operatorname{co} K$ with the following property:

$$x - hx \in U \quad \text{for all} \quad x \in K.$$

Since A is convex it follows that $\operatorname{co} K \subset A$ and so A is admissible.

Corollary 2. *Every Hausdorff topological vector space of type Φ is admissible.*

Corollary 3. *Let A be a closed and convex subset of type Φ of a Hausdorff topological vector space E and $h\colon A \to A$ be a continuous mapping such that $\overline{h(A)}$ is compact. Then there exists at least one fixed point of the mapping h.*

Proof. Since A is admissible we can apply a fixed point theorem from [3] and so the set of fixed points of the mapping h is nonempty.

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DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
UNIVERSITY OF NOVI SAD
ULICA DR ILIJE DJURIČIĆA 4
21000 NOVI SAD
YUGOSLAVIA

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New generalizations of Banach's contraction principle

M. HEGEDŰS

Many research papers have appeared on different generalizations of Banach's contraction principle. A. MEIR and E. KEELER [2] studied mappings $f\colon X\to X$ of a metric space (X, ϱ) having the property that for every $\varepsilon>0$ there exists a $\delta>0$ such that $\varepsilon\leqq\varrho(x,y)<\varepsilon+\delta$ implies $\varrho(f(x),f(y))<\varepsilon$. In the present paper we consider the following generalization of a restriction of this definition. For $x,y\in X$ let $d_f(x,y)=\operatorname{diam}\{x,y,f(x),f(y),f^2(x),f^2(y),\ldots\}$. Here "diam" abbreviates diameter.

The mapping $f\colon X\to X$ is called a generalized Meir—Keeler contraction if $d_f(x,y)<\infty$ for $x,y\in X$ and if for every $\varepsilon>0$ there exist ε', ε'' such that $0<\varepsilon'<\varepsilon<\varepsilon''$ and $d_f(x,y)<\varepsilon''$ implies $\varrho(f(x),f(y))<\varepsilon'$.

Lj. B. ĆIRIĆ [1] studied mappings $f\colon X\to X$ for which $d_f(x,y)<\infty$ and there exists a constant α, $0\leqq\alpha<1$, such that

$$\varrho(f(x),f(y))\leqq\alpha\max\{\varrho(x,y),\ \varrho(x,f(x)),\ \varrho(y,f(y)),\ \varrho(x,f(y)),\ \varrho(y,f(x))\}$$

for $x,y\in X$. In the present paper we consider the following class of mappings wider than that considered by Ćirić.

The mapping $f\colon X\to X$ is called a generalized Banach contraction if $d_f(x,y)<\infty$ for $x,y\in X$ and if there exists a constant α, $0\leqq\alpha<1$ such that $\varrho(f(x),f(y))\leqq\alpha d_f(x,y)$ for all $x,y\in X$.

It is obvious that every generalized Banach contraction is a generalized Meir—Keeler contraction. The function $f(x)=\sin x$ on $X=[0,\pi/2]$ is a generalized Meir—Keeler contraction which is not a generalized Banach contraction. This may be seen in the following way. Firstly, if $\sin x$ were a generalized Banach contraction on $[0,\pi/2]$, then we would have $|\sin x|\leqq\alpha|x|$ for all $x\in[0,\pi/2]$ with some α, $0\leqq\alpha<1$. But this is impossible, since $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$. Secondly, for any given ε, $0<\varepsilon\leqq 1$ let ε' be a number such that $\sin\varepsilon<\varepsilon'<\varepsilon$. Denote $\varepsilon''=\arcsin\varepsilon'$. If $x,y\in\left[0,\frac{\pi}{2}\right]$, $y\leqq x$ and $|x-y|\leqq\varepsilon''$, then $|\sin x-\sin y|=\int_y^x\cos t\,dt=\int_0^{x-y}\cos(x+t)\,dt\leqq$

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$\leqq \int_0^{x-y} \cos t\,dt = \sin(x-y) \leqq \sin(\arcsin \varepsilon') = \varepsilon'$. Consequently, $\sin x$ is a generalized Meir—Keeler contraction on $\left[0, \frac{\pi}{2}\right]$.

Now we give an example of a generalized Banach contraction which is not of Ćirić type. In fact, let $X=\{1, 2, 3, 4\}$ and $\varrho(1, 2)=3.9$, $\varrho(1, 3)=3.7$, $\varrho(1, 4)=4.0$, $\varrho(2, 3)=3.9$, $\varrho(2, 4)=3.9$, $\varrho(3, 4)=3.0$. Furthermore, let f be defined on X by the equalities $f(1)=2$, $f(2)=3$, $f(3)=4$, $f(4)=4$. Then $\varrho(f(1), f(2)) = \max\{\varrho(1, 2), \varrho(1, f(1)), \varrho(2, f(2)), \varrho(1, f(2)), \varrho(2, f(1))\}$. However, it is easy to verify that in this case $\varrho(x, y) \leqq 0{,}99 d_f(x, y)$ for all $x, y \in X$.

The objective of the present paper is to prove the following theorems.

Theorem 1. *Let f: $X \to X$ be a generalized Meir—Keeler mapping. Then there exists at most one fixed point of f, and $\{f^n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence for every $x \in X$. If X is complete, then for every $x \in X$, $f^n(x)$ converges to the unique fixed point of f as $n \to \infty$.*

Theorem 2. *Let f: $X \to X$ be a generalized Banach contraction with constant α, let $x_0 \in X$ be fixed, and let $\delta_n = \operatorname{diam}\{f^n(x_0), f^{n+1}(x_0), \ldots\}$. Then*

$$\delta_n \leqq \frac{\alpha^n}{1-\alpha}\, \varrho(x_0, f(x_0)) \quad (n = 0, 1, \ldots),$$

$$\delta_n \leqq \frac{\alpha}{1-\alpha}\, \varrho(f^{n-1}(x_0), f^n(x_0)) \quad (n = 1, 2, \ldots).$$

If X is complete, then

$$\varrho(z, f^n(x_0)) \leqq \frac{\alpha^n}{1-\alpha}\, \varrho(x_0, f(x_0)) \quad (n = 0, 1, \ldots).$$

$$\varrho(z, f^n(x_0)) \leqq \frac{\alpha}{1-\alpha}\, \varrho(f^{n-1}(x_0), f^n(x_0)) \quad (n = 1, 2, \ldots),$$

where z denotes the unique fixed point of f.

The proofs will be based on the following

Lemma. *Let f: $X \to X$ be a generalized Meir—Keeler mapping, and let $x_0 \in X$, $\delta_n = \operatorname{diam}\{f^n(x_0), f^{n+1}(x_0), \ldots\}$. Then $\delta_n = \sup_{k>n} \varrho(f^n(x_0), f^k(x_0))$ $(n=0, 1, \ldots)$.*

Proof. It is sufficient to consider the case $n=0$. If $\delta_0=0$, then the statement of the lemma is obvious. If $\delta_0>0$, then choose δ_0', δ_0'' in such a way that we have $0<\delta_0'<\delta_0<\delta_0''$ and $\varrho(f(x), f(y))<\delta_0'$ if $d_f(x, y)<\delta_0''$. Now let $k, l \geqq 1$. Since $\delta_{\min\{k-1, l-1\}} \leqq \delta_0 < \delta_0''$, we have $\varrho(f^k(x_0), f^l(x_0)) < \delta_0' < \delta_0$. This immediately implies the assertion of our lemma.

Proof of Theorem 1. Let z' and z'' be fixed points of f. If $\varrho(z', z'')=\varepsilon>0$, then choose ε', ε'' such that $0<\varepsilon'<\varepsilon<\varepsilon''$ and $\varrho(f(x), f(y))<\varepsilon'$ if $d_f(x, y)<\varepsilon''$.

Since z', z'' are fixed points, $d_f(z', z'')=\varrho(z', z'')=\varepsilon<\varepsilon''$. Consequently, $\varrho(z', z'')=$ $=\varrho(f(z'), f(z''))<\varepsilon'<\varepsilon$, a contradiction. Therefore we must have $\varrho(z', z'')=0$, i.e., that $z'=z''$.

Now let $x_0\in X$ be fixed and use the notations of Lemma. We have to prove that $\delta_n\to 0$. It follows from the definition of δ_n that $\delta_0\geqq\delta_1\geqq\ldots\geqq 0$. Consequently, $\delta_n\to\varepsilon$ for some $\varepsilon\geqq 0$. Assume that $\varepsilon>0$, and choose $\varepsilon', \varepsilon''$ so that we have $0<\varepsilon'<$ $<\varepsilon<\varepsilon''$ and $\varrho(f(x), f(y))<\varepsilon'$ if $d_f(x, y)<\varepsilon''$. Let n_0 be so large that $\delta_{n_0}<\varepsilon''$. If $k, l>n_0$, then $\varrho(f^k(x_0), f(x_0))<\varepsilon'<\varepsilon$, since $d_f(f^{k-1}(x_0), f^{l-1}(x_0))=\delta_{\min\{k-1; l-1\}}\leqq$ $\leqq\delta_{n_0}<\varepsilon''$. Therefore, $\delta_{n_0+1}\leqq\varepsilon'$, a contradiction since $\delta_n\downarrow\varepsilon$. Hence $\varepsilon=0$, and $\{f^n(x_0)\}_{n=1}^{\infty}$ is a Cauchy sequence.

Now let X be complete. Then $f^n(x_0)$ converges to an element z of X. We have to prove that z is invariant under f. Let $\delta_n^*=d_f(f^n(z), f^n(z))$. We must prove that $\delta_0^*=0$. Assume the contrary, i.e., that $\delta_0^*>0$, and choose $\delta_0^{*\prime}, \delta_0^{*\prime\prime}$ so that $0<\delta_0^{*\prime}<$ $<\delta_0^*<\delta_n^{*\prime\prime}$ and $\varrho(f(x), f(y))<\delta_0^{*\prime}$ if $d_f(x, y)<\delta_0^{*\prime\prime}$. If $k\geqq 1$, then for all large enough n, $d_f(f^{k-1}(z), f^{n-1}(x_0))<\delta_0^{*\prime\prime}$ since $f^n(x_0)\to z$ as $n\to\infty$. For such n we have $\varrho(f^k(z), f^n(x_0))<\delta_0^{*\prime}$. If we let n tend to ∞, then we obtain that $\varrho(f^k(z), z)\leqq$ $\leqq\delta_0^{*\prime}$. Consequently, according to Lemma, $\delta_0^*\leqq\delta_0^{*\prime}$, a contradiction. Therefore z is a fixed point of f.

Proof of Theorem 2. The inequalities involving δ_n imply the other two inequalities. The second inequality concerning δ_n is an immediate consequence of the first. To prove the first, we observe that $\delta_n\leqq\alpha^n\delta_0$. This is so since for $k>1\geqq n$ we have $\varrho(f^k(x_0), f^l(x_0))\leqq\alpha\delta_{n-1}$. Consequently, $\delta_n\leqq\alpha\delta_{n-1}$. We obtain from this by recursion that $\delta_n\leqq\alpha^n\delta_0$. Now let $k=1, 2, \ldots$. Then

$$\varrho(x_0, f^k(x_0))\leqq\varrho(x_0, f(x_0))+\varrho(f(x_0), f^k(x_0))\leqq$$
$$\leqq\varrho(x_0, f(x_0))+\delta_1\leqq\varrho(x_0, f(x_0))+\alpha\delta_0$$

on the basis of what we have just observed. According to Lemma we therefore have $\delta_0\leqq\varrho(x_0, f(x_0))+\alpha\delta_0$, i.e., $\delta_0\leqq\dfrac{1}{1-\alpha}\varrho(x_0, f(x_0))$. This is the inequality to be proved for $n=0$. For $n=1, 2, \ldots$ we obtain from this and from what we have observed at the beginning of our proof that $\delta_n\leqq\alpha^n\delta_0\leqq\dfrac{\alpha^n}{1-\alpha}\varrho(x_0, f(x_0))$.

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DEPT. OF MATHEMATICS, K. MARX UNIV. OF ECONOMICS
DIMITROV TÉR 8
1093 BUDAPEST, HUNGARY

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A simple proof for von Neumann's minimax theorem

I. JOÓ

To the memory of F. Riesz (1880—1956)

1. The usual proofs of the von Neumann minimax theorem and its generalizations are based on deep results of Sperner or Brouwer (cf. [2], [4], [5]). Our proof is based on the simple lemma due to F. Riesz (cf. [3], p. 41) that if a system of compact subsets of a topological space has the finite intersection property (i.e. every finite set has non-empty intersection) then the whole system has non-empty intersection. This proof is a development of the ideas of the paper [1].

2. Theorem. *Let E and F be topological vector spaces, and let* $K_1\subset E$, $K_2\subset F$ *be convex compact sets. Let* $f(x, y)$ *be a real-valued continuous function on* $K_1\times K_2$, *which is concave in x for any fixed* $y\in K_2$, *and convex in y for any fixed* $x\in K_1$. *Then*

$$\min_{y\in K_2}\max_{x\in K_1} f(x, y) = \max_{x\in K_1}\min_{y\in K_2} f(x, y).$$

Proof. Let c be a (fixed) real number such that

$$H_y^{(c)} = H_y = \{x\colon f(x, y) \geqq c\} \neq \emptyset \quad \text{for every} \quad y\in K_2,$$

where $\emptyset$ denotes the empty set. The sets H_y are convex and compact. We assert that

$$\bigcap_{y\in K_2} H_y \neq \emptyset. \tag{1}$$

According to the lemma of Riesz it is enough to prove that for any finite set $\{y_1, \dots, y_n\}\subset K_2$ we have

$$\bigcap_{i=1}^{n} H_{y_i} \neq \emptyset.$$

We prove this by induction on n.

Consider the case $n=2$. Suppose there exist $y_1, y_2\in K_2$ for which

$$H_{y_1}\cap H_{y_2} = \emptyset \tag{2}$$

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and set $H(\lambda)=H_{y_1+(1-\lambda)y_2}$ for $\lambda\in[0,1]$; $H(\lambda)\neq\emptyset$ by the convexity of $f(x,y)$ in y. Next we show that

(3) $$H(\lambda)\subset H_{y_1}\cup H_{y_2}.$$

For every $x\in K_1$ and $x\notin H_{y_1}\cup H_{y_2}$ we have

$$f(x,\lambda y_1+(1-\lambda)y_2)\leqq f(x,y_1)+(1-\lambda)f(x,y_2)<c$$

since f is convex in y. Thus $x\notin H(\lambda)$. Therefore, (3) follows because of the definitions of H_{y_1}, H_{y_2}.

Using (2) and (3) we show that for arbitrary $\lambda\in[0,1]$

(4) $$\text{either } H(\lambda)\subset H_{y_1} \quad\text{or}\quad H(\lambda)\subset H_{y_2}.$$

Suppose the contrary:

$$H(\lambda^*)\cap H_{y_1}\neq\emptyset \quad\text{and}\quad H(\lambda^*)\cap H_{y_2}\neq\emptyset$$

for some $\lambda^*\in[0,1]$. Let $y_1^*\in H(\lambda_1^*)\cap H_{y_1}$ and $y_2^*\in H(\lambda^*)\cap H_{y_2}$ be arbitrarily chosen. Consider the closed interval

$$[y_1^*,y_2^*]=\{\lambda y_1^*+(1-\lambda)y_2^*:\ 0\leqq\lambda\leqq 1\}.$$

By the convexity of the sets H_y we have

$$[y_1^*,y_2^*]\subset H(\lambda^*).$$

From (2) and the compactness of H_{y_1} and H_{y_2} we see that there exists $y^*\in[y_1^*,y_2^*]$ such that

$$y^*\notin([y_1^*,y_2^*]\cap H_{y_1})\cup([y_1^*,y_2^*]\cap H_{y_2}),$$

and hence $y^*\notin H_{y_1}\cup H_{y_2}$. On the other hand, $y^*\in H(\lambda^*)$ which contradicts (3). So (4) is proved.

To complete the proof of (3), we need the following statement: If $H(\lambda_1)\cap\cap H_{y_1}\neq\emptyset$ for $\lambda_1\in[0,1]$, then there exists $\varepsilon_1=\varepsilon_1(y_1,y_2,\lambda_1)>0$ such that

(5) $$H(\lambda)\cap H_{y_1}\neq\emptyset \quad\text{for}\quad |\lambda-\lambda_1|<\varepsilon_1.$$

[Similarly: if $H(\lambda_2)\cap H_{y_2}\neq\emptyset$ for $\lambda_2\in[0,1]$, then there exists $\varepsilon_2=\varepsilon_2(y_1,y_2,\lambda_2)>0$ such that

(6) $$H(\lambda)\cap H_{y_2}\neq\emptyset \quad\text{for}\quad |\lambda-\lambda_2|<\varepsilon_2.]$$

We prove (5). If $H(\lambda_1)\cap H_{y_1}\neq\emptyset$ then according to (4), $H(\lambda_1)\cap H_{y_2}=\emptyset$, that is

(7) $$f(x,\lambda_1 y_1+(1-\lambda_1)y_2)<c \quad\text{for every}\quad x\in H_{y_2}.$$

Since $f(x,\lambda y_1+(1-\lambda)y_2)$ is a continuous function in (x,λ), it follows from (7) that for every $x\in H_{y_2}$ there exists a neighborhood U_x of x and $\varepsilon(x)>0$ such that

$$f(x,\lambda y_1+(1-\lambda)y_2)<c \quad\text{for}\quad (x,\lambda)\in U_x\times(\lambda_1-\varepsilon(x),\lambda_1+\varepsilon(x)).$$

Therefore,

$$H_{y_2} \subset \bigcup_{x \in H_{y_2}} U_x.$$

Since H_{y_2} is compact we can choose a finite system $\{U_{x_i}\}_{i=1}^n$ such that

$$H_{y_2} \subset \bigcup_{i=1}^{n} U_{x_i}.$$

Then for $\varepsilon_1 = \min \{\varepsilon(x_i): i=1, ..., n\}$ we have (5). The proof of (6) is similar.

From (4), (5), (6) it follows that the set $\{\lambda \in [0, 1]: H(\lambda) \subset H_{y_1}\}$ is open in [0, 1]. Similarly, the set $\{\lambda \in [0, 1]: H(\lambda) \subset H_{y_2}\}$ is also open in [0, 1]. Taking (4) into consideration, we arrive at a decomposition of the interval [0, 1] into two disjoint non-empty relatively open sets, which is impossible. Thus we proved that

$$H_{y_1} \cap H_{y_2} \neq \emptyset.$$

Suppose we know that for any subset $\{y_1, ..., y_k\}$ of $K_2 (\subset F)$ having at most n elements we have

$$\bigcap_{i=1}^{k} H_{y_i} \neq \emptyset$$

and then we prove the same for $n+1$ elements.

Suppose there exist $y_1, ..., y_{n+1}$ such that

$$\bigcap_{i=1}^{n+1} H_{y_i} = \emptyset \tag{8}$$

Then we have

$$(H_{y_1} \cap H_3) \cap (H_{y_2} \cap H_3) = \emptyset \quad \text{for} \quad H_3 = \bigcap_{i=3}^{n+1} H_{y_i}.$$

Now using the induction assumption and (8) we can apply the idea of the proof of $n=2$ for the sets

$$H_{y_i}^3 = H_{y_i} \cap H_3 \quad (i = 1, 2).$$

Thus we obtain

$$\bigcap_{i=1}^{n+1} H_{y_i} \neq \emptyset,$$

and so, according to the lemma of Riesz, (1) is proved.

Denote by $\mathscr{C}$ the set of real numbers c for which $H_y^{(c)} = H_y \neq \emptyset$ whenever $y \in K_2$. If $c_0 \in \mathscr{C}$, then $c \in \mathscr{C}$ for every $c \leqq c_0$. Since the function f is continuous, the set $\mathscr{C}$ is bounded from above. Denote by c^* its smallest upper bound. From the lemma of Riesz we deduce that $c^* \in \mathscr{C}$. We prove that

$$\min_{y \in K_2} \max_{x \in K_1} f(x, y) \leqq c^*. \tag{9}$$

Suppose

$$\min_{y\in K_2}\max_{x\in K_1} f(x, y) > c^*,$$

then there exists $\tilde{c} > c^*$ for which

$$\min_{y\in K_2}\max_{x\in K_1} f(x, y) \geqq \tilde{c} > c^*.$$

Therefore $\max_{x\in K_1} f(x, y) \geqq \tilde{c}$ for every $y\in K_2$, hence $\{x\colon f(x, y)\geqq\tilde{c}\}\neq\emptyset$ for every $y\in K_2$, but this contradicts the choice of c^*.

On the other hand, because of (1), we have

$$A \overset{\text{def}}{=} \bigcap_{y\in K_2} H_y^{(c^*)} \neq \emptyset.$$

Let $x^*\in A$. From the definition of H_y we obtain $f(x^*, y)\geqq c^*$ for every $y\in K_2$; thus

(10) $$\min_{y\in K_2} f(x^*, y)\geqq c^* \quad \text{and} \quad \max_{x\in K_1}\min_{y\in K_2} f(x, y) \geqq c^*.$$

From (9) and (10) we deduce

$$\min_{y\in K_2}\max_{x\in K_1} f(x, y) \leqq \max_{x\in K_1}\min_{y\in K_2} f(x, y).$$

Since

$$\min_{y\in K_2}\max_{x\in K_1} f(x, y) \geqq \max_{x\in K_1}\min_{y\in K_2} f(x, y)$$

is obvious, the theorem is proved.

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BOLYAI INTÉZET
UNIVERSITY OF SZEGED
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

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Remarks on a paper of L. Szabó and Á. Szendrei

H. K. KAISER and L. MÁRKI*

The aim of this note is to give an infinite version of the Theorem of L. SZABÓ and Á. SZENDREI [4]. We shall do this without using I. Rosenberg's Theorem [3] and those parts of [4] which make use of it. We adopt the terminology of [2] and [4].

Theorem. *An at least four element non-trivial algebra with triply transitive automorphism group either has the interpolation property or is equivalent to an affine space over* GF (2).

Most of the proof follows closely that of L. SZABÓ and Á. SZENDREI [4], we shall write out only those parts which are different. We do not need Proposition 1 of [4]. We formulate Proposition 2 in a slightly different way: we consider not necessarily finite algebras and local term functions instead of term functions. The proof is literally the same.

Lemma 1. *Let A be an algebra with at least four elements and with a triply transitive automorphism group. If A does not have the interpolation property but has a three-place non-trivial local term function f, then f is a minority function such that $f(a, b, c) \notin \{a, b, c\}$ whenever the elements $a, b, c \in A$ are all different.*

Proof. The proof that $f(a, b, c) \notin \{a, b, c\}$ if $|\{a, b, c\}|=3$ and that condition $(*)$ of [4] holds, is literally the same as in [4]. This is the beginning of their proof of Lemma 1; thereby we need the infinite version of B. Csákány's Theorem, which is an immediate consequence of the finite one. For, given an (infinite) algebra A with a pattern function $p(x_1, \ldots, x_k)$ which can be interpolated on every finite subset of A^k, and a partial function f on a finite subset $H \subset A^k$, let B denote the subset of A which consists of the elements occurring as coordinates in H or being values of f on H. Then take the polynomial function $\tilde{p}$ which interpolates p on B^k;

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$(B, \tilde{p})$ is, by Csákány's Theorem, functionally complete, and this gives a representation of f in terms of $\tilde{p}$, hence as a polynomial function on A.

Now it suffices to show that if the local term function f is not a minority function, then A has the interpolation property. For this end we show first that in this case A has the 2-interpolation property. Further, it suffices to consider functions in one variable only: if we take two distinct elements of A^k for some $k \in N$, they differ in at least one component i, and then we consider the i-th projection. Given arbitrary elements $x, y, u, v \in A$, $x \neq y$, we have to show the existence of a unary polynomial function g such that $g(x)=u$, $g(y)=v$. Supposing that A has at least five elements, it is sufficient to prove this if x, y, u, v are all distinct. (In fact, in the other case we can choose two elements e, f both distinct from x, y, u, v, and then send x, y first to e, f and then e, f to u, v.) Since f is not a minority function, at least one of the values $f(x, y, y)$, $f(y, x, y)$, $f(y, y, x)$ is equal to y. Suppose e.g. $f(y, y, x)=y$. By $(*)$ we have elements $c, d \in A$ such that $f(y, x, d)=v, f(x, v, c)=u$. Then we take g to be a (unary) polynomial function which interpolates $f(f(\xi, x, d), v, c)$ at $\xi=x, y$. (In case A has four elements, by somewhat more, but still elementary, computation one can construct this polynomial function g, thus avoiding the use of Rosenberg's Theorem.)

Now we use induction and prove that if A has the $(n-1)$-interpolation property $(n>2)$ then it has the n-interpolation property, too. Let $g\colon A^k \to A$ and $x_1, \dots, x_n \in A^k$ be different elements and put $a_i=g(x_i)$, $i=1, \dots, n$. Since g has the $(n-1)$-interpolation property, we have polynomial functions $f_1, \dots, f_5$ such that

$$f_1(x_i) = a_i, \quad i = 1, 2, 4, \dots, n; \quad f_2(x_i) = a_i, \quad i = 1, 3, 4, \dots, n;$$

$$f_3(x_i) = \begin{cases} a_i & i = 4, \dots, n, \\ f_1(x_3) & i = 3, \\ f_2(x_2) & i = 2, \end{cases}$$

and for arbitrary elements $d, u \in A$,

$$f_4(x_i) = \begin{cases} a_i & i = 2, 4, \dots, n, \\ d & i = 3; \end{cases}$$

$$f_5(x_i) = \begin{cases} a_i & i = 1, 4, \dots, n, \\ u & i = 3. \end{cases}$$

If $f_1(x_3)=a_3$, then we are done. Suppose therefore $f_1(x_3) \neq a_3$ and by using $(*)$ choose d, u so that $f(f_1(x_3), d, u)=a_3$. By assumption, f is not a minority function, hence we have, say, $f(y, y, x)=y$. If we have in addition $f(y, x, y)=f(x, y, y)=x$, then we take a polynomial function p which interpolates $f(f_1, f_2, f_3)$ on $\{x_1, \dots, x_n\}$. It is easy to see that $p(x_i)=a_i$; $i=1, \dots, n$. If $f(y, x, y)$ or $f(x, y, y)$, say $f(y, x, y)$, is also y, then we consider a polynomial function q which interpolates $f(f_1, f_4, f_5)$ on $\{x_1, \dots, x_n\}$ and again we obtain that $q(x_i)=a_i$, $i=1, \dots, n$.

Lemma 2. *Let A be an algebra with at least four elements and with a triply transitive automorphism group. Suppose that there exists an at least quaternary (say n-ary) non-trivial local term function f which turns into a projection whenever we identify any two of its variables. Then A has the interpolation property.*

Proof. Again we repeat the beginning of the proof in [4] and obtain property $(**)$. Along the same lines as in Lemma 1 we show first that A has the 2-interpolation property. Take again four different elements $x, y, a, b \in A$. By $(**)$ there exist elements $d_3, \ldots, d_n, d_3', \ldots, d_n'$ in A such that $f(x, y, d_3, \ldots, d_n) = a$ and $f(y, a, d_3', \ldots, d_n') = b$. Consider now a polynomial function g which interpolates $f(f(\xi, y, d_3, \ldots, d_n), a, d_3', \ldots, d_n')$ at $\xi = x, y$. This function does the job.

Suppose next that A has the $(m-1)$-interpolation property $(m > 2)$. We show that it has the m-interpolation property as well. Consider a function $h\colon A^k \to A$ and put $a_i = h(x_i)$, $i = 1, \ldots, m$. By assumption we have a polynomial function f_1 such that $f_1(x_i) = a_i$, $i = 2, 3, \ldots, m$. If $f_1(x_1) = a_1$ then we are done. Suppose $f_1(x_1) \neq a_1$, then choose an element $b \notin \{a_1, f_1(x_1)\}$, and consider a polynomial function f_2 such that:

$$f_2(x_i) = \begin{cases} b & i = 1 \\ a_i & i = 3, \ldots, m. \end{cases}$$

By $(**)$ there are $t_3, \ldots, t_n$ in A such that $f(f_1(x_1), b, t_3, \ldots, t_n) = a_1$. Next we choose a polynomial function f_3 such that:

$$f_3(x_i) = \begin{cases} t_3 & i = 1 \\ a_i & i = 2, 4, \ldots, m. \end{cases}$$

Finally, we take a polynomial function r which interpolates $f(f_1, f_2, f_3, t_4, \ldots, t_n)$ on $\{x_1, \ldots, x_m\}$, then we have $h(x_i) = r(x_i)$, $i = 1, \ldots, m$.

As a next step, we transfer Lemma 3 of [4], together with its proof, with the obvious modifications to the infinite case.

Lemma 4. *Let A be an algebra with at least four elements and with triply transitive automorphism group. If A does not have the interpolation property, then A admits no essentially quaternary local term function.*

Proof. Suppose h is an essentially quaternary local term function on A, then it has the properties (1)—(7) of Lemma 3. Since h depends on the first variable, one can find elements a, b, c, d in A such that $h(a, b, c, d) := s \neq h(b, b, c, d) = m(b, c, d) := t$, where m is the unique non-trivial ternary local term function on A. A short elementary computation shows that (at least) b, c, d, t must be all different. Let Θ be a congruence of A and $u \Theta v$ with $u \neq v$, and choose an arbitrary $z \notin \{u, v\}$. If $h(a, b, c, d) \neq a$, then just as it is done at the corresponding place in the proof of the Theorem in [4], we see that a, $m(b, c, d)$, $h(a, b, c, d)$ are all different. Now we can find a $\pi\in\operatorname{Aut} A$ such that $\pi(a)=v$, $\pi(h(a, b, c, d))=z$, $\pi(m(b, c, d))=u$, and we have $h(v, \pi b, \pi c, \pi d)=z$, $h(u, \pi b, \pi c, \pi d)=u$, which implies $z=h(v, \pi b, \pi c, \pi d)\Theta h(u, \pi b, \pi c, \pi d)=u$, hence $\Theta=A^2$. Suppose now $h(a, b, c, d)=a$, then again we follow the corresponding lines in the proof of the Theorem in [4] and obtain that $a, b, m(b, c, d)$ are all different. Further we choose a $\pi\in\operatorname{Aut} A$ with $\pi a=u$, $\pi b=v$, $\pi(m(b, c, d))=z$, and conclude that $u=h(u, v, \pi c, \pi d)\Theta h(v, v, \pi c, \pi d)=z$, whence $\Theta=A^2$. By this we have that A is simple, and by Lemma 3, A has a unique non-trivial ternary local term function m, which is a minority function. This implies that m remains unchanged if we permute its variables, furthermore $m(m(x, y, z), y, z)=x$ for all $x, y, z\in A$ (cf. (8) in [4]). In particular, since $m(b, c, d)=t$, we get $m(t, c, d)=b$.

On the other hand, A does not have the interpolation property, hence by M. Istinger, H. K. Kaiser and A. F. Pixley [1], Corollary 3.9, we know: If q is a binary local polynomial function and r an element of A such that $q(x, r)=q(r, x)=r$ (for all $x\in A$), then q is the constant function with value r. Consider $q(x, y)=h(a, m(x, y, t), x, y)$. Then we have $q(x, y)=t$ for all $x, y\in A$, which contradicts $q(c, d)=h(a, m(c, d, t), c, d)=h(a, m(t, c, d), c, d)=h(a, b, c, d)=s\neq t$. This completes the proof of Lemma 4.

Now we continue the proof of the Theorem exactly as it is done in [4].

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(H. K. K.)
INSTITUT FÜR ALGEBRA
TECHNISCHE UNIVERSITÄT WIEN
ARGENTINIERSTRASSE 8
1040 WIEN, AUSTRIA

(L.M.)
MATHEMATICAL INSTITUTE
HUNGARIAN ACADEMY OF SCIENCES
REÁLTANODA U. 13—15
1053 BUDAPEST, HUNGARY

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Kanonische Zahlensysteme in der Theorie der quadratischen algebraischen Zahlen

I. KÁTAI und B. KOVÁCS

1. Bekanntlich kann jede nichtnegative ganze Zahl in jeder der beiden Formen

$$N = a_0 + a_1 A + \ldots + a_n A^n \quad \text{und} \quad N = a_0 + a_1(-A) + \ldots + a_n(-A^n)$$

eindeutig aufgeschrieben werden, wobei $a_j \in \{0, 1, \ldots, A-1\}$ $(j=0, 1, \ldots, n)$ und $A \geqq 2$ ganze Zahlen sind.

I. KÁTAI und J. SZABÓ [1] untersuchten das folgende Problem: Es seien α eine ganze Gaußsche Zahl, $N(\alpha)$ ihre Norm und $\mathcal{N}_0 = \{0, 1, \ldots, |N(\alpha)|-1\}$. Unter welchen Bedingungen kann man die Gaußsche Zahl γ eindeutig in der Form

$$(1.1) \qquad \gamma = a_0 + a_1\alpha + \ldots + a_n\alpha^n \quad \text{mit} \quad a_j \in \mathcal{N}_0 \quad (j = 0, \ldots, n)$$

aufschreiben? Sie haben bewiesen, daß dies für α dann und nur dann gilt, wenn $\alpha = -A \pm i$ ist, wobei A eine positive ganze Zahl bedeutet. Sie haben noch gezeigt, daß in diesem Falle jede komplexe Zahl z in der Form

$$(1.2) \qquad z = \sum_{j=k}^{-\infty} a_j \alpha^j \quad \text{mit} \quad a_j \in \mathcal{N}_0$$

aufgeschrieben werden kann.

Es sei jetzt $N > 0$ eine quadratfreie rationale ganze Zahl. Es ist bekannt, daß jeder reelle quadratische algebraische Zahlenkörper die Form $R(\sqrt{N})$ hat. Im Weiteren bezeichnen wir die ganzen Zahlen von $R(\sqrt{N})$ mit $\alpha, \beta, \ldots$, die rationalen ganzen Zahlen mit $A, B, C, \ldots$. Das Paar $\{\alpha, \mathcal{N}_0\}$ wird ein *Zahlensystem* in $R(\sqrt{N})$ genannt, wenn jede algebraische ganze Zahl $\gamma \in R(\sqrt{N})$ eindeutig in der Form (1.1) aufgeschrieben werden kann.

Wir bewiesen die folgenden Sätze.

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Satz 1. *$\{\alpha, \mathcal{N}_0\}$ ist dann und nur dann ein Zahlensystem in $R(\sqrt{N})$, wenn*

1) $\alpha = A \pm \sqrt{N}$ *und* $0 < -2A \leqq A^2 - N \geqq 2$, *für* $N \not\equiv 1 \pmod 4$,

2) $\alpha = \frac{1}{2}(B \pm \sqrt{N})$ *und* $0 < -B \leqq \frac{1}{4}(B^2 - N) \geqq 2$, *für* $N \equiv 1 \pmod 4$,

wobei B eine ungerade ganze Zahl ist.

Satz 2. *Es sei $\{\alpha_0, \mathcal{N}_0\}$ ein Zahlensystem in irgendeinem reellen quadratischen Zahlenkörper. Dann kann jede reelle Zahl x auf mindestens eine Weise in der Form (1.2) aufgeschrieben werden.*

2. Einige Bemerkungen und Hilfssätze. Es ist bekannt, daß für den Diskriminanten D von $R(\sqrt{N})$ gilt:

1) $D = 4N$ falls $N \not\equiv 1 \pmod 4$,

2) $D = N$ falls $N \equiv 1 \pmod 4$.

Ist $\alpha = A + B\sqrt{N} \in R(\sqrt{N})$ eine quadratische algebraische ganze Zahl, dann folgt aus der Berechnung des Diskriminanten der Basis $\{1, \alpha\}$, daß $\{1, \alpha\}$ genau dann eine ganze Basis von $R(\sqrt{N})$ ist, wenn, im Falle $N \not\equiv 1 \pmod 4$ $\alpha = A \pm \sqrt{N}$ mit einer beliebigen rationalen ganzen Zahl A, und im Falle $N \equiv 1 \pmod 4$, $\alpha = 1/2(B \pm \sqrt{N})$ mit einer ungerader ganzer Zahl B ist. Diese Zahlen α sind die Wurzel den folgenden quadratischen Gleichungen mit rationalen ganzen Koeffizienten:

$$x^2 - 2Ax + (A^2 - N), \quad \text{bzw.} \quad x^2 - Bx + \frac{1}{4}(B^2 - N).$$

Ist $\{\alpha, \mathcal{N}_0\}$ ein Zahlensystem in $R(\sqrt{N})$, dann ist $|N(\alpha)| > 1$, und α eine quadratische ganze Zahl, weiterhin ist $\{1, \alpha\}$ eine Basis in $R(\sqrt{N})$.

Lemma 1. *Es sei $\{\alpha, \mathcal{N}_0\}$ ein Zahlensystem in $R(\sqrt{N})$. Dann ist $\{1, \alpha\}$ eine ganze Basis von $R(\sqrt{N})$.*

Beweis. Es genügt zu beweisen: Ist $\gamma \in R(\sqrt{N})$ eine ganze Zahl, dann gilt $\gamma = X_\gamma + Y_\gamma \alpha$, wobei X_γ, Y_γ rationale ganze Zahlen sind. Es sei α eine Wurzel der Gleichung mit rationalen ganzen Koeffizienten $x^2 + Cx + D = 0$. Dann gilt für jede natürliche Zahl $s \geqq 2$:

(2.1) $$\alpha^s = X_s + Y_s \alpha$$

mit einem rationalen ganzen Zahlenpaar X_s, Y_s. Da $\{\alpha, \mathcal{N}_0\}$ ein Zahlensystem in $R(\sqrt{N})$ ist, hat jede ganze Zahl $\gamma \in R(\sqrt{N})$ die form (1.1). Durch Einsetzen von (2.1) ergibt sich, daß $\gamma = X_\gamma + Y_\gamma \alpha$ ist, wobei X_γ, Y_γ rationale ganze Zahlen sind.

Lemma 2. *Ist α eine nichtnegative ganze Zahl in $R(\sqrt{N})$, dann ist $\{\alpha, \mathcal{N}_0\}$ kein Zahlensystem in $R(\sqrt{N})$.*

Beweis. Es sei γ eine negative ganze Zahl in $R(\sqrt{N})$. Wir nehmen an, daß $\{\alpha, \mathcal{N}_0\}$ ein Zahlensystem in $R(\sqrt{N})$ ist. Dann kommen wir wegen $a_i \in \mathcal{N}_0$ $(i=0, 1, ..., n)$ und $\alpha \geqq 0$ zu einem Widerspruch:

$$0 > \gamma = \sum_{i=1}^{\infty} a_i \alpha^i \leqq 0.$$

Lemma 3. *Ist $\alpha \in R(\sqrt{N})$ eine algebraische ganze Zahl mit $|\alpha| < 1$, dann ist $\{\alpha, \mathcal{N}_0\}$ kein Zahlensystem in $R(\sqrt{N})$.*

Beweis. Wir nehmen an, daß $\{\alpha, \mathcal{N}_0\}$ ein Zahlensystem in $R(\sqrt{N})$ ist. Wegen Lemma 2 genügt es die Behauptung nur im Falle $-1 < \alpha < 0$ zu beweisen. Es sei $\gamma \in R(\sqrt{N})$ eine algebraische ganze Zahl mit

$$\gamma \geqq \frac{|N(\alpha)|-1}{1-\alpha^2}. \tag{2.2}$$

Auf Grund von $\gamma = \sum_{i=0}^{2n} a_i \alpha^i$; $a_i \in \mathcal{N}_0$ ist aber

$$\begin{aligned} \gamma &= (a_0 + a_2\alpha^2 + \dots + a_{2n}\alpha^{2n}) + (a_1\alpha + a_3\alpha^3 + \dots + a_{2n-1}\alpha^{2n-1}) \leqq \\ &\leqq a_0 + a_2\alpha^2 + \dots + a_{2n}\alpha^{2n} \leqq (|N(\alpha)|-1)(1+\alpha^2+\dots+\alpha^{2n}) \leqq \\ &\leqq (|N(\alpha)|-1)/(1-\alpha^2), \end{aligned}$$

im Widerspruch mit (2.2).

Lemma 4. *Es sei $\alpha \in R(\sqrt{N})$ eine Wurzel der Gleichung $X^2 + Ux + V = 0$, wobei $U \geqq 0$ und $V \geqq 1$ rationale ganze Zahlen sind. Ist $\gamma = X + Y\alpha$ mit rationalen ganzen Zahlen X, Y, dann existieren solche nichtnegativen rationalen ganzen Zahlen C, D, E, F, mit welchen $\gamma = C + D\alpha + E\alpha^2 + F\alpha^3$ gilt.*

Beweis. Da $\alpha^2 + U\alpha + V = 0$ und $V \geqq 1$ ist, existieren rationale ganze Zahlen $L_0 \geqq 0$ und $L_1 \geqq 1$, für welche $L_0 V + X \geqq 0$ und $L_1 V + Y \geqq 0$ ist. Dann gilt

$$\begin{aligned} \gamma = X + Y\alpha &= L_0(\alpha^2 + U\alpha + V) + L_1(\alpha^3 + U\alpha^2 + V\alpha) + X + Y\alpha = \\ &= (X_0 + L_0 V) + (Y + L_1 V + L_0 U)\alpha + (L_1 U + L_0)\alpha^2 + L_1\alpha^3, \end{aligned}$$

wobei jeder Koeffizient eine nichtnegative rationale ganze Zahl ist.

Lemma 5. *Es sei $\alpha \in R(\sqrt{N})$ eine Wurzel der Gleichung $x^2 + Ux + V = 0$, wobei $0 < U \leqq V \geqq 2$, und U, V rationale ganze Zahlen sind, ferner sei $\{1, \alpha\}$ eine ganze Basis in $R(\sqrt{N})$. Dann ist $\{\alpha, \mathcal{N}_0\}$ ein Zahlensystem in $R(\sqrt{N})$.*

Beweis. Da $\{1, \alpha\}$ eine ganze Basis in $R(\sqrt{N})$ ist, kann jede algebraische ganze Zahl $\gamma \in R(\sqrt{N})$ mit rationalen ganzen Zahlen X, Y eindeutig in der Form $\gamma = X + Y\alpha$ aufgeschrieben werden. Wegen Lemma 4 gilt

(2.3) $$\gamma = d_0 + d_1\alpha + \ldots + d_k\alpha^k, \quad k \geqq 3, \quad d_j \geqq 0 \quad (j = 0, 1, \ldots, k),$$

wobei d_j rationale ganze Zahlen sind. Es sei $L(\gamma, d) = d_0 + d_1 + \ldots + d_k$, $L(\gamma, d)$ ist eine nichtnegative ganze Zahl. Wegen $V \geqq 2$ kann $d_0 = r_0 + tV$ geschrieben werden, wobei $t \geqq 0$ eine rationale ganze Zahl ist, $r_0 \in \mathcal{N}_0$, d. h., wegen $\alpha^2 + U\alpha + V = 0$,

$$d_0 = r_0 + tV = r_0 + t(-\alpha^2 U\alpha) = r_0 + t\{(V-U)\alpha + (U-1)\alpha^2 + \alpha^3\}.$$

Setzen wir das in (2.3) ein, so ergibt sich

$$\gamma = r_0 + \{d_1 + t(V-U)\}\alpha + \{d_2 + t(U-1)\}\alpha^2 + (d_3 + t)\alpha^3 + d_4\alpha^4 + \ldots + d_k\alpha^k =$$
$$= d_0^* + d_1^*\alpha + \ldots + d_k^*\alpha^k,$$

wobei $d_0^* \in \mathcal{N}_0$, d_i^* nichtnegative ganze Zahlen sind, ferner $L(\gamma, d^*) = L(\gamma, d)$ gilt. Es sei $\gamma_1 = d_1^* + d_2^*\alpha + \ldots + d_k^*\alpha^{k-1}$, dann ist $\gamma = r_0 + \alpha\gamma_1$, $L(\gamma_1, d^*) \leqq L(\gamma, d^*)$, und Gleichung besteht nur im Falle $r_0 = 0$. Setzen wir diesen Algorithmus fort, so bekommen wir

(2.4) $$\gamma = r_0 + \alpha\gamma_1, \gamma_1 = r_1 + \alpha\gamma_2, \ldots, \gamma_n = r_n + \alpha\gamma_{n+1}, \quad \text{mit} \quad r_i \in \mathcal{N}_0 \ (i = 0, 1, \ldots)$$

und

$$L(\gamma, d) \geqq L(\gamma_1, d) \geqq \ldots \geqq L(\gamma_n, d) \geqq \ldots;$$

Gleichungen bestehen nur im Falle $r_i = 0$.

Da $L(\gamma, d) \geqq 0$ und $L(\gamma_i, d) \geqq 0$ ganze Zahlen sind, ist notwendigerweise $r_k = 0$ $(k \geqq M)$. Dann gilt aber wegen (2.4) für jede natürliche Zahl $s \geqq 1$, daß $\alpha^s | \gamma_M$, was auf Grund von $|N(\alpha)| = V \geqq 2$ nur dann möglich ist, wenn $\gamma_M = 0$ ist. Aus (2.4) bekommen wir die behauptete Darstellung

$$\gamma = r_0 + r_1\alpha + \ldots + r_{M-1}\alpha^{M-1}, \quad r_i \in \mathcal{N}_0.$$

Es soll noch gezeigt werden, daß diese Darstellung eindeutig ist. Dies folgt daraus, daß wenn $0 = s_0 + s_1\alpha + \ldots + s_k\alpha^k$ $(s_j \in \mathcal{N}_0)$, dann $\alpha | s_0$ und deshalb $s_0 = 0$, und aus ähnlichem Grund $s_1 = 0, s_2 = 0, \ldots, s_k = 0$ sind.

Lemma 6. *$\{\alpha, \mathcal{N}_0\}$ ist dann und nur dann ein Zahlensystem in $R(\sqrt{N})$, wenn $\{\bar{\alpha}, \mathcal{N}_0\}$ auch ein Zahlensystem ist.*

Beweis. Klar, denn die Darstellungen

$$\bar{\gamma} = \sum_{j=0}^{m} b_j\alpha^j \quad \text{und} \quad \gamma = \sum_{j=0}^{m} b_j(\bar{\alpha})^j \quad (b_j \in \mathcal{N}_0)$$

für $\gamma \in R(\sqrt{N})$ sich gegenseitig implizieren.

3. Beweis des ersten Satzes. Ist $\{\alpha, \mathcal{N}_0\}$ ein Zahlensystem in $R(\sqrt{N})$, so ist $\{1, \alpha\}$, auf Grund von Lemma 1, eine ganze Basis in $R(\sqrt{N})$. Im Falle $N\equiv 1 \pmod 4$ gilt also $\alpha=\frac{1}{2}(B\pm\sqrt{N})$ und $\alpha^2-B\alpha+\frac{1}{4}(B^2-N)=0$, wobei B eine ungerade ganze Zahl ist. Im Falle $N\not\equiv 1 \pmod 4$ aber ist $\alpha=A\pm\sqrt{N}$ und $\alpha^2-2A\alpha+(A^2-N)=0$, wobei A eine beliebige ganze Zahl bedeutet. Auf Grund von Lemma 6 genügt es die algebraischen ganzen Zahlen $\alpha=\frac{1}{2}(B+\sqrt{N})$, bzw. $\alpha=A+\sqrt{N}$ nur im Falle $N\equiv 1 \pmod 4$, bzw., im Falle $N\not\equiv 1 \pmod 4$ zu untersuchen. Nach Lemma 2 ist $B<-\sqrt{N}$, $-B\geqq 3$ und $A<-\sqrt{N}$, $-A\geqq 2$.

Es sei zuerst $N\not\equiv 1 \pmod 4$. Da $\mathcal{N}_0$ mindestens zwei Elemente hat, ist

$$|N(\alpha)| = |A^2-N| = A^2-N \geqq 2. \tag{3.1}$$

Da $A<-\sqrt{N}$, ist (3.1) im Falle $A\leqq-\sqrt{N+2}$ erfüllt. Wenn außerdem noch $1\leqq -2A\leqq A^2-N$ gilt, so ist — auf Grund von Lemma 5 — $\{\alpha, \mathcal{N}_0\}$ ein Zahlensystem. Wegen $-A\geqq 2$ ist $-2A\leqq A^2-N$ im Falle $A>-\sqrt{N+1}-1$ nicht erfüllt. Wir brauchen darum nur die α zu untersuchen, für welche $-\sqrt{N+1}-1<A<$ $<-\sqrt{N+2}$ ist. Auf Grund von Lemma 3 im Falle $|A+\sqrt{N}|<1$ wegen $A<-\sqrt{N}$ ist $-A-\sqrt{N}<1$, d. h. $\{\alpha, \mathcal{N}_0\}$ ist kein Zahlensystem in $R(\sqrt{N})$. Da $-\sqrt{N}-1$ keine ganze Zahl ist, genügt es die ganzen Zahlen A zu untersuchen, die die folgende Bedingung erfüllen:

$$-\sqrt{N+1}-1 < A < -\sqrt{N}-1. \tag{3.2}$$

Der Bedingung (3.2) genügt aber keine ganze Zahl A, da aus (3.2) $N<(-A+1)^2<$ $<N+1$ folgt.

Zusammengefaßt: $\alpha=A\pm\sqrt{N}$ ist dann und nur dann ein Zahlensystem in $R(\sqrt{N})$ im Falle $N\not\equiv 1 \pmod 4$, wenn $A\leqq-\sqrt{N+1}-1$, oder — was damit äquivalent ist — wenn $1\leqq -2A\leqq A^2-N\geqq 2$ ist, wobei A eine rationale ganze Zahl ist.

Es sei jetzt $N\equiv 1 \pmod 4$. Da $\mathcal{N}_0$ mindestens zwei Elemente hat, so ist

$$|N(\alpha)| = \left|\frac{1}{4}(B^2-N)\right| = \frac{1}{4}(B^2-N) \geqq 2. \tag{3.3}$$

Da $B<-\sqrt{N}$ ist, wird (3.3) dann erfüllt, wenn $B\leqq-\sqrt{N+8}$ gilt. Wenn außerdem auch $1\leqq -B\leqq\frac{1}{4}(B^2-N)$ gilt, so ist $\{\alpha, \mathcal{N}_0\}$ nach Lemma 5 ein Zahlensystem. Da $-B\geqq 3$ ist, wird $-B\leqq\frac{1}{4}(B^2-N)$ dann nicht erfüllt, wenn $B>-\sqrt{N+4}-2$

ist. Deshalb genügt es nur die α zu untersuchen, für welche

$$-\sqrt{N+4}-2 < B \leqq -\sqrt{N+8}$$

gilt. Auf Grund von Lemma 3 im Falle $\left|\frac{1}{2}(B+\sqrt{N})\right|<1$ wegen $B<-\sqrt{N}$ gilt $-B-\sqrt{N}<2$, d. h. es ist $B>-\sqrt{N}-2$ und so ist $\{\alpha, \mathcal{N}_0\}$ kein Zahlensystem in $R(\sqrt{N})$. Da $-\sqrt{N}-2$ keine ganze Zahl sein kann, genügt es die ganzen Zahlen B zu untersuchen, die die Bedingung

$$-\sqrt{N+4}-2 < B < -\sqrt{N}-2 \tag{3.4}$$

erfüllen. Genügt B (3.4), so gilt $(-B-2)^2=N+1$, oder $(-B-2)^2=N+2$, oder $(-B-2)^2=N+3$. Da B und N ungerade ganze Zahlen sind, so gilt $(-B-2)^2 \neq N+1$, und $(-B-2)^2 \neq N+3$. Da $N\equiv 1 \pmod 4$ und B ungerade ist, deshalb gilt $N+2\equiv -1 \pmod 4$, und $(-B-2)^2\equiv 1 \pmod 4$, daraus folgt $(-B-2)^2 \neq N+2$.

Zusammengefaßt: $\alpha=\frac{1}{2}(B\pm\sqrt{N})$ ist dann und nur dann ein Zahlensystem in $R(\sqrt{N})$ im Falle $N\equiv 1 \pmod 4$, wenn $B\leqq -\sqrt{N+4}-2$, oder — was damit äquivalent ist — wenn $1\leqq -B\leqq \frac{1}{4}(B^2-N)\geqq 2$ ist, wobei B eine ungerade ganze Zahl bedeutet.

4. Beweis des zweiten Satzes. Zum Beweis des zweiten Satzes wird eine Ungleichung benutzt.

Lemma 7. *Es sei* $N\not\equiv 1 \pmod 4$, *und* $\{\alpha, \mathcal{N}_0\}$ *ein Zahlensystem in* $R(\sqrt{N})$. *Dann ist* $|A^2-N-1|\geqq|\alpha|$, *wobei* $\alpha=A\pm\sqrt{N}$.

Beweis. Wir nehmen an, daß $|A^2-N-1|<|\alpha|$ ist. Wegen $\alpha=A\pm\sqrt{N}$ gilt

$$-1 < \frac{(A+\sqrt{N})(A-\sqrt{N})}{A\pm\sqrt{N}} - \frac{1}{A\pm\sqrt{N}} < 1.$$

Im Fall $\alpha=A+\sqrt{N}$ müßte

$$-1 < A-\sqrt{N}-\frac{1}{A+\sqrt{N}} < 1$$

gelten. Das ist aber wegen den Bedingungen $N\geqq 2$, $A<-\sqrt{N}$ und $|A+\sqrt{N}|>1$ unmöglich

Wenn aber $\alpha=A-\sqrt{N}$ ist, dann müßte

$$-1 < A+\sqrt{N}-\frac{1}{A-\sqrt{N}} < 1 \tag{4.1}$$

gelten. Wir zeigen aber, daß (4.1) unmöglich ist. Bei festgelegtem $N \geqq 2$ ist die Funktion $y = x + \sqrt{N} - (x - \sqrt{N})^{-1}$ im Intervall $(-\infty, 0)$ monoton wachsend und in diesem Intervall ist $y = -1$ nur für

(4.2) $$x = \frac{1}{2}\left(-1 - \sqrt{1 + 4(N + \sqrt{N} + 1)}\right).$$

So ist (4.1) nicht erfüllt, wenn

(4.3) $$A \leqq \frac{1}{2}\left(-1 - \sqrt{1 + 4(N + \sqrt{N} + 1)}\right).$$

Da $\{\alpha, \mathcal{N}_0\}$ in $R(\sqrt{N})$ ein Zahlensystem ist, gilt $A \leqq -\sqrt{N+1} - 1$. Eine einfache Rechnung ergibt

$$-\sqrt{N+1} - 1 < \frac{1}{2}\left(-1 - \sqrt{1 + 4(N + \sqrt{N} + 1)}\right).$$

Das bedeutet — wegen (4.3), (4.2) und weil die Funktion $y = x + \sqrt{N} - (X - \sqrt{N})^{-1}$ im Intervall $(-\infty, 0)$ monoton wachsend ist —, da (4.1) nicht erfüllt ist. Damit ist unsere Behauptung bewiesen.

Bemerkung. Wegen $|\alpha| > 1$ folgt aus diesem Lemma, daß $A^2 - N \geqq 3$.

Lemma 8. *Es sei* $N \equiv 1 \pmod 4$, *und sei* $\{\alpha, \mathcal{N}_0\}$ *ein Zahlensystem in* $R(\sqrt{N})$. *Dann ist* $\left|\frac{1}{4}(B^2 - N) - 1\right| \geqq |\alpha|$, *wobei* $\alpha = \frac{1}{2}(B \pm \sqrt{N})$.

Beweis. Wir nehmen an, daß $\left|\frac{1}{4}(B^2 - N) - 1\right| < |\alpha|$ gilt. Wegen $\alpha = \frac{1}{2}(B \pm \sqrt{N})$ ist:

$$-1 < \frac{\frac{1}{2}(B + \sqrt{N}) \cdot \frac{1}{2}(B - \sqrt{N})}{\frac{1}{2}(B \pm \sqrt{N})} - \frac{1}{\frac{1}{2}(B \pm \sqrt{N})} < 1.$$

Ist $\alpha = \frac{1}{2}(B + \sqrt{N})$, so gilt $-1 < \frac{1}{2}(B - \sqrt{N}) - 2(B + \sqrt{N})^{-1} < 1$. Das ist aber wegen den Bedingungen $N \geqq 5$, $B < -\sqrt{N}$ und $|\alpha| > 1$ unmöglich. Ist aber $\alpha = \frac{1}{2}(B - \sqrt{N})$, so muß gelten:

(4.4) $$-1 < \frac{1}{2}(B + \sqrt{N}) - \frac{2}{B - \sqrt{N}} < 1.$$

Wir zeigen, daß auch (4.4) unmöglich ist. Bei festgelegtem $N \geqq 5$ ist die Funktion $y = x + \frac{1}{2}\sqrt{N} - \left(x - \frac{1}{2}\sqrt{N}\right)^{-1}$ im Intervall $(-\infty, 0)$ monoton wachsend und in diesem Intervall ist $y = -1$ nur für

$$x = \frac{1}{2}\left(-1 - \sqrt{5 + N + 2\sqrt{N}}\right). \tag{4.5}$$

Deshalb, im Falle

$$B \leqq -1 - \sqrt{5 + N + 2\sqrt{N}} \tag{4.6}$$

kann (4.4) nicht gelten. Da $\{\alpha, \mathcal{N}_0\}$ ein Zahlensystem in $R(\sqrt{N})$ ist, deshalb gilt $B \leqq -\sqrt{N+4} - 2$. Durch eine einfache Rechnung ergibt sich

$$-\sqrt{N+4} - 2 < -1 - \sqrt{5 + N + 2\sqrt{N}}.$$

Auf Grund von (4.6), (4.5) und da die Funktion $y = x + \frac{1}{2}\sqrt{N} - \left(x - \frac{1}{2}\sqrt{N}\right)^{-1}$ im Intervall $(-\infty, 0)$ monoton wachsend ist, kann (4.4) nicht erfüllt sein. Damit ist unsere Behauptung bewiesen.

Bemerkung. Wegen $|\alpha| > 1$, auf Grund von diesem Lemma gilt $\frac{1}{4}(B^2 - N) \geqq 3$.

Jetzt sind wir imstande Satz 2 zu beweisen.

Es sei $\{\alpha, \mathcal{N}_0\}$ ein Zahlensystem in $R(\sqrt{N})$ und bezeichne C das größte Element von $\mathcal{N}_0$. Nach Lemma 7 und 8 ist $|C| \geqq |\alpha|$ und $C \geqq 2$. Daraus folgt für eine beliebige natürliche Zahl $n \geqq 1$:

$$C\left|\frac{1}{\alpha^n}\right| \geqq \left|\frac{1}{\alpha^{n-1}}\right|. \tag{4.7}$$

Da N quadratfrei und A, B ganze Zahlen sind, sind $A \pm \sqrt{N}$ und $\frac{1}{2}(B \pm \sqrt{N})$ irrationale Zahlen. Deshalb bilden die ganzzahligen Vielfachen von α mod 1 eine überall dichte Menge in $[0, 1]$. Daraus folgt, daß die reellen Zahlen der Form $X + Y\alpha$ (X, Y sind ganze Zahlen) — die ganzen Zahlen von $R(\sqrt{N})$ — im Intervall $(-\infty, \infty)$ überall dicht sind.

Es sei X eine beliebige reelle Zahl. Wir zeigen, daß X in der Form (1.2) geschrieben werden kann. Es sei $\beta \in R(\sqrt{N})$ eine ganze Zahl, für welche $\beta \in (x, x+1)$. Es existiert wegen (4.7) ein $r_1 \in \mathcal{N}_0$ mit

$$\beta_1 = \beta + \frac{r_{-1}}{\alpha} < x \leqq \beta + \frac{r_{-1} - 1}{\alpha}$$

(weil $\alpha<0$ ist). Durch wiederholte Anwendung von (4.7) erhalten wir, daß es ein $r_{-2}\in\mathcal{N}_0$ existiert, mit

$$\beta_1+\frac{r_{-2}-1}{\alpha^2}\leqq x<\beta_1+\frac{r_{-2}}{\alpha^2}=\beta_2$$

(weil $\alpha^2>0$), ferner, wegen (4.7) existiert auch ein $r_{-3}\in\mathcal{N}_0$ mit

$$\beta_3=\beta_2+\frac{r_{-3}}{\alpha^3}<x\leqq\beta_2+\frac{r_{-2}-1}{\alpha^3}.$$

Setzen wir diesen Algorithmus fort, so erhalten wir eine Folge β_n $(n=1, 2, \ldots)$ für die man nach dem ersten Satz hat:

a) $r_{-n}\in\mathcal{N}_0$ $(n=1, 2, \ldots)$,
b) $\lim\beta_n=x$ (wegen $|\alpha|>1$ und $\beta_{2k+1}<x<\beta_{2k}$ $(k=0, 1, \ldots)$),
c) $\beta=r_l\alpha^l+\ldots+r_1\alpha+r_0$ mit $r_i\in\mathcal{N}_0$ $(i=0, 1, \ldots, l)$.

So ist $\beta_n=\sum_{i=l}^{-n} r_i\alpha^i$ und deshalb $x=\sum_{i=l}^{-\infty} r_i\alpha^i$ mit $r_i\in\mathcal{N}_0$. Offensichtlich ist die Wahl von β nicht eindeutig, da die ganzen Zahlen von $R(\sqrt{N})$ auf der Zahlengerade überall dicht liegen: so ist die Form (1.2) von x auch nicht eindeutig. Damit ist unsere Behauptung bewiesen.

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(I.K.)
EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF NUMERICAL ANALYSIS AND COMPUTER SCIENCE
MÚZEUM KRT. 6—8
1088 BUDAPEST, HUNGARY

(B.K.)
KOSSUTH LAJOS UNIVERSITY
DEPARTMENT OF MATHEMATICS
EGYETEM TÉR 1
4032 DEBRECEN, HUNGARY

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On C_0-operators with property (P)

L. KÉRCHY

1. H. BERCOVICI [1] has considered the class $\mathcal{P}$ of Hilbert space operators T of class C_0 having the following property:

(P) any injection $X \in \{T\}'$ is a quasi-affinity.

He has shown that $T \in \mathcal{P}$ if and only if $\bigwedge_{n=1}^{\infty} m_n[T]=1$, where $m_n[T]$ $(n=1, 2, \ldots)$ are the inner functions in the Jordan model of T. (Cf. Theorem 4.1 of [1].)

He has proved, furthermore, that every operator $T \in \mathcal{P}$ has the following stronger property also:

(P*) for any $X \in \{T\}'$ we have $\gamma_T(\ker X)=\gamma_T(\ker X^*)$.

(Cf. Theorem 7.9 of [1].) Here $\gamma_T(\ker X)$ and $\gamma_T(\ker X^*)$ are generalized inner functions; they play the roles of determinants of the operators $T|\ker X$ and $T_{\ker X^*}$. (Cf. sections 6 and 7 of [1].)

Let ϱ be the following relation on the class $\mathcal{P}$: $T_1 \varrho T_2$ if there exist $T \in \mathcal{P}$ and $X \in \{T\}'$ such that T_1 and T_2 are quasisimilar to $T|\ker X$ and $T_{\ker X^*}$, that is, $T_1 \sim T|\ker X$, and $T_2 \sim T_{\ker X^*}$. Then the previous statement can be written in the following form. If $T_1, T_2 \in \mathcal{P}$ and $T_1 \varrho T_2$, then $\gamma_{T_1}=\gamma_{T_2}$ (because γ_T is a quasi-similarity invariant).

Bercovici has also proved a partial converse of this statement. Namely, he has proved that if $T_1, T_2 \in \mathcal{P}$ are weak contractions and $\gamma_{T_1}=\gamma_{T_2}$, then $T_1 \varrho T_2$. On the other hand he has shown that if $T_1, T_2 \in \mathcal{P}$ are such that $\gamma_{T_1}=\gamma_{T_2}$, then there exists $S \in \mathcal{P}$ such that $T_1 \varrho S$ and $S \varrho T_2$. The main purpose of this note is to prove the complete converse of the statement mentioned above, namely,

Theorem. *If $T_1, T_2 \in \mathcal{P}$ are such that $\gamma_{T_1}=\gamma_{T_2}$, then $T_1 \varrho T_2$.*

Thus the operators of class $\mathcal{P}$ have, in general, no stronger property than (P*). In particular, in general it is not true that an operator $T \in \mathcal{P}$ has the property:

(Q) $T|\ker X$ and $T_{\ker X^*}$ are quasisimilar for any $X \in \{T\}'$.

(Cf. [2].)

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Furthermore, from the Theorem we can easily infer that ϱ is an equivalence relation on $\mathcal{P}$.

2. In the sections 6 and 7 of [1] BERCOVICI introduced the notions of "generalized inner function" and "C_0-dimension of a subspace" in the following way. Any inner function $m\in H_i^\infty$ has a factorization $m=cbs$, where c is a complex constant of modulus one, b is a Blaschke product and s is a singular inner function deriving from a finite Borel measure μ on $[0, 2\pi]$, singular with respect to Lebesgue measure. (Cf. [3], Ch. III.) Let us denote by $\sigma(z)$ the multiplicity of the zero z ($|z|<1$) in the Blaschke product b. Then $\gamma(m)$ will denote the pair $\gamma(m)=(\sigma, \mu)$. The class $\tilde{\Gamma}$ of "generalized inner functions" will be the set of pairs $\gamma=(\sigma, \mu)$, where σ is a natural number valued function defined on $D=\{z\colon |z|<1\}$ such that $\sum_{\sigma(z)\neq 0}(1-|z|)<\infty$, and μ is a (not necessarily finite) Borel measure on $[0, 2\pi]$, which is absolute continuous with respect to a finite Borel measure ν singular with respect to Lebesgue measure. We define addition and lattice operations in $\tilde{\Gamma}$ by components.

If $T\in\mathcal{P}$, then it can be proved that $\gamma_T:=\sum_{j=0}^{\infty}\gamma(m_j)\in\tilde{\Gamma}$, where the $m_j=m_j[T]$ are the inner functions in the Jordan model of T. (Cf. Theorem 4.1 and Proposition 6.6 of [1].) If T is an operator of class C_0 and $\mathfrak{M}\in\mathrm{Lat}_{\frac{1}{2}}(T)$ is such that $T_{\mathfrak{M}}\in\mathcal{P}$, then $\gamma_T(\mathfrak{M})$ is defined as $\gamma_T(\mathfrak{M})=\gamma_{T_{\mathfrak{M}}}$.

For two operators T and T' we denote by $\mathcal{I}(T', T)$ the set of intertwining operators $\mathcal{I}(T', T)=\{X|T'X=XT\}$. If $T'=T$, then $\mathcal{I}(T, T)=\{T\}'$ is the commutant of T.

The next Lemmas will be frequently used in the sequel.

Lemma 1. *Let $\{m_i\}_{i=0}^\infty$ be a sequence of pairwise relatively prime inner functions having a least common multiple m. Then the operator $T=\bigoplus_{i=0}^{\infty}S(m_i)$ is quasi-similar to $S(m)$.*

Proof. Cf. Theorem 2.7 of [4].

Lemma 2. *Let m_1, m_2 be inner functions.*

(i) *If m_2 divides m_1 ($m_1\geqq m_2$) and $Xu=P_{\mathfrak{H}(m_2)}u$ for all $u\in\mathfrak{H}(m_1)$, then $X\in\mathcal{I}(S(m_2), S(m_1))$ is surjective and $S(m_1)|\ker X$ is unitarily equivalent to $S\left(\frac{m_1}{m_2}\right)$ $\left(S(m_1)|\ker X\cong S\left(\frac{m_1}{m_2}\right)\right)$.*

(ii) *If $m_1\leqq m_2$ and $Xu=\frac{m_2}{m_1}u$ for all $u\in\mathfrak{H}(m_1)$, then $X\in\mathcal{I}(S(m_2), S(m_1))$ is injective and $S(m_2)_{\ker X^*}\cong S\left(\frac{m_2}{m_1}\right)$.*

Proof. We can easily verify this statement by a short computation.

Lemma 3. (Proposition 4.6 of [1]) *Let T be an operator of class C_0 acting on $\mathfrak{H}$ and let $\mathfrak{H}_j \in \mathrm{Lat}(T)$ be such that $\mathfrak{H}_j \subset \mathfrak{H}_{j+1}$ $(j=1, 2, \ldots)$, and $\mathfrak{H} = \bigvee_{j=1}^{\infty} \mathfrak{H}_j$. Then $T \in \mathcal{P}$ if and only if $T_{\mathfrak{R}_j} \in \mathcal{P}$, $\mathfrak{R}_j = \mathfrak{H}_{j+1} \ominus \mathfrak{H}_j$ $(j=0, 1, 2, \ldots;\ \mathfrak{H}_0 = \{0\})$ and* $\bigwedge_{j=1}^{\infty} m_0[T_{\mathfrak{H}_j^{\perp}}]=1$. (If S is an operator of class C_0, then $m_0[S]$ denotes its minimal function.)

3. Firstly we shall prove the statement of the Theorem in different special cases in the Propositions 1 and 2, from which the general situation can be derived. We remark that it can be always supposed that T_1 and T_2 are Jordan operators. In the proofs of Propositions 1 and 2 we shall need the next Lemma.

Let us denote by $\sum$ the set of injections $\sigma\colon N \to N \cup (-N) = \hat{N}$ satisfying the conditions:

(i) if $1 \leqq i < j$ and $\sigma(i)\sigma(j) \geqq 0$, then $|\sigma(i)| < |\sigma(j)|$;

(ii) if $r \in \sigma(N)$, then for all $s \in \hat{N}$ such that $s \cdot r \geqq 0$ and $|s| < |r|$ we have $s \in \sigma(N)$. (Here and in the sequel N is the set of natural numbers $1, 2, \ldots$.) Let $\mathcal{G}$ be the set of sequences: $a = \{a_n\}_{n=1}^{\infty}$ of real numbers such that $a_1 \geqq a_2 \geqq \ldots \geqq 0$ and $a_n \to 0$ as $n \to \infty$. If $a, b \in \mathcal{G}$, then let $F_{(a,b)}$ denote the mapping $\hat{N} \to R$ defined by

$$F_{(a,b)}(i) = \begin{cases} a_i, & \text{if } i \in N, \\ -b_i, & \text{if } i \in (-N). \end{cases}$$

Lemma 4. *Let $a, b \in \mathcal{G}$ satisfy the condition: if $b_n = 0$ for some $n \in N$, then there exists $m \in N$ such that $a_m = 0$. If $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$, then there exists a $\sigma \in \sum$ such that for all $n \in N$ we have*

$$0 \leqq \sum_{i=1}^{n} F_{(a,b)}(\sigma(i)) \leqq 2 \max(a_1, b_1)$$

furthermore $\sum_{i=1}^{n} F_{(a,b)}(\sigma(i))$ tends to 0, if n tends to ∞.

Proof. Let $\sigma(1)=1$. If we have already defined σ for $i=1, 2, \ldots, j$, and $\max\{\sigma(i) | i=1, \ldots, j\} = r_j$, $\min\{\{\sigma(i) | i=1, \ldots, j\} \cup \{0\}\} = -s_j$, then

$$\sigma(j+1) := \begin{cases} -(s_j+1) & \text{if } \sum_{i=1}^{j} F_{(a,b)}(\sigma(i)) \geqq b_{s_j+1}, \\ r_j + 1 & \text{otherwise.} \end{cases}$$

It can be easily seen that this $\sigma \in \sum$ will be suitable.

Proposition 1. *If $T_1, T_2 \in \mathcal{P}$ are such that $\gamma_{T_1} = \gamma_{T_2} = \gamma$ and γ has the form $\gamma = (\sigma, 0)$, then $T_1 \varrho T_2$.*

Proof. Let T_1 and T_2 be the Jordan operators $T_1=\bigoplus_{n=1}^{\infty} S(u_n)$ and $T_2=\bigoplus_{n=1}^{\infty} S(v_n)$. From the assumption it follows that u_1 and v_1 are Blaschke products having the same zeros (disregarding multiplicities): $\lambda_1, \lambda_2, \ldots$. For all n u_n and v_n have factorizations $u_n=\prod_{l=1}^{\infty} u_{n,l}$, $v_n=\prod_{l=1}^{\infty} v_{n,l}$, where $u_{n,l}$ and $v_{n,l}$ are Blaschke factors containing only λ_l as a zero. (If λ_l is not a zero of u_n (v_n), then $u_{n,l}:=1$ $(v_{n,l}:=1)$.)

Let us denote by $a_n^{(l)}$ and $b_n^{(l)}$ the multiplicities of λ_l as zero of $v_{n,l}$ and of $u_{n,l}$, respectively. Then $a_l=\{a_n^{(l)}\}_{n=1}^{\infty}$, $b_l=\{b_n^{(l)}\}_{n=1}^{\infty}\in\mathcal{G}$, and by virtue of $\gamma_{T_1}=\gamma_{T_2}$ we have $\sum_{n=1}^{\infty} a_n^{(l)}=\sum_{n=1}^{\infty} b_n^{(l)}$ for all $l\in N$.

By Lemma 4 there exists a $\sigma_l\in\sum$ such that

$$0\leqq \sum_{i=1}^{j} F_{(a_l,b_l)}(\sigma_l(i))\leqq 2\max(a_1^{(l)}, b_1^{(l)})$$

for all $j\in N$. Let $c_j^{(l)}$ be defined by $c_j^{(l)}=\sum_{i=1}^{j} F_{(a_l,b_l)}(\sigma_l(i))$, and let

$$w_j^{(l)}(z):=\begin{cases}\left(\dfrac{\bar\lambda_l}{|\lambda_l|}\dfrac{\lambda_l-z}{1-\bar\lambda_l z}\right)^{c_j^{(l)}} & \text{if} \quad \lambda_l\neq 0,\\ z^{c_j^{(l)}} \quad \text{if} \quad \lambda_l=0; \ j\in N,\ z\in D.\end{cases}$$

It is clear that $w_j^{(l)}=1$, if j is large enough. So the operator T_l defined by $T_l=\bigoplus_{j=1}^{\infty} S(w_j^{(l)})$ has finite multiplicity. On the other hand by the construction it follows that $m_0[T_l]\leqq (u_{1,l}\vee v_{1,l})^2$.

Let X_l be the contraction defined by $X_l(\bigoplus_{j=1}^{\infty} f_j)=\bigoplus_{j=1}^{\infty} g_j$, where $\bigoplus_{j=1}^{\infty} f_j$, $\bigoplus_{j=1}^{\infty} g_j\in\bigoplus_{j=1}^{\infty}\mathfrak{H}(w_j^{(l)})$ and $g_1=0$,

$$g_j=\begin{cases}P_{\mathfrak{H}(w_j^{(l)})}f_{j-1} & \text{if} \quad w_{j-1}^{(l)}\geqq w_j^{(l)},\\ \dfrac{w_j^{(l)}}{w_{j-1}^{(l)}}f_{j-1} & \text{if} \quad w_{j-1}^{(l)}\leqq w_j^{(l)} \quad \text{for} \quad j\geqq 2.\end{cases}$$

By Lemma 2 we infer that $X_l\in\{T_l\}'$ and $T_l|\ker X_l\cong\bigoplus_{n=1}^{\infty} S(u_{n,l})$, $(T_l)_{\ker X_l^*}\cong\bigoplus_{n=1}^{\infty} S(v_{n,l})$.

Since $\bigwedge_{j=1}^{\infty} m_0[\bigoplus_{l=j}^{\infty} T_l]\leqq\bigwedge_{j=1}^{\infty}(\prod_{l=j}^{\infty}(u_{1,l}\vee v_{1,l})^2)=1$, by Lemma 3 we see that $T=\bigoplus_{l=1}^{\infty} T_l\in\mathcal{P}$. Then $X=\bigoplus_{l=1}^{\infty} X_l\in\{T\}'$ and using Lemma 1 we get

$$T|\ker X=\bigoplus_{l=1}^{\infty} T_l|\ker X_l\cong\bigoplus_{l=1}^{\infty}\left(\bigoplus_{n=1}^{\infty} S(u_{n,l})\right)\cong\bigoplus_{n=1}^{\infty}\left(\bigoplus_{l=1}^{\infty} S(u_{n,l})\right)\sim\bigoplus_{n=1}^{\infty} S(u_n)=T_1$$

and similarly $T_{\ker X^*}\sim T_2$. Therefore, $T_1\varrho T_2$ and Proposition 1 is proved.

Proposition 2. *If $T_1, T_2 \in \mathcal{P}$ are such that $\gamma_{T_1}=\gamma_{T_2}=\gamma$ and γ has the form $\gamma=(0, \mu)$, then $T_1 \varrho T_2$.*

Proof.

(i) Let T_1 and T_2 be the Jordan operators $T_1=\bigoplus_{n=1}^{\infty} S(u_n)$ and $T_2=\bigoplus_{n=1}^{\infty} S(v_n)$. From the assumption it follows that there exist a finite Borel measure ν in $[0, 2\pi]$, singular with respect to Lebesgue measure, and non-increasing sequences $\{f_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$ of non-negative Borel functions from $L^1(\nu)$ which are tending to 0 and such that

$$\operatorname{Exp}[f_n] = u_n \quad \text{and} \quad \operatorname{Exp}[g_n] = v_n \quad \text{for all } n.$$

Here and in the sequel we use the notations

$$\operatorname{Exp}[f, E](z) = \exp\left[-\int_E \frac{e^{it}+z}{e^{it}-z} f(t)\, d\nu(t)\right] \quad (z\in D), \text{ and } \operatorname{Exp}[f] = \operatorname{Exp}[f, [0, 2\pi]],$$

for any non-negative Borel function $f\in L^1(\nu)$, and measurable set $E\subset[0, 2\pi]$.

Therefore we see that $f(t)=\{f_n(t)\}_{n=1}^{\infty}$, $g(t)=\{g_n(t)\}_{n=1}^{\infty}\in\mathcal{G}$ for all t in $[0, 2\pi]$. Furthermore we can assume that

$$\sum_{n=1}^{\infty} f_n(t) = \sum_{n=1}^{\infty} g_n(t) \quad \text{for all } t \text{ in } [0, 2\pi].$$

(ii) Let E be the measurable set of points t in $[0, 2\pi]$ such that $a=g(t)$ and $b=f(t)$ satisfy the assumptions of Lemma 4. If $t\in E$ let $\sigma_t\in\sum$ be the function constructed in the proof of Lemma 4 taking $a=g(t)$ and $b=f(t)$. For all $j\in N$ let $h_j\in L^1(\nu)$ be the measurable function defined by

$$h_j(t) = \begin{cases} \sum_{i=1}^{j} F_{(g(t), f(t))}(\sigma_t(i)) & \text{if} \quad t\in E, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 4 we infer that

$$0 \leqq h_j(t) \leqq 2\max(f_1(t), g_1(t))$$

for all $j\in N$, $t\in[0, 2\pi]$, and

$$\lim_{j\to\infty} h_j(t) = 0 \quad \text{for all} \quad t\in[0, 2\pi].$$

Introducing the inner functions $\{w_j\}_{j=1}^{\infty}$ by $w_j=\operatorname{Exp}[h_j]$, we consider the operator $\bigoplus_{j=1}^{\infty} S(w_j)$.

(iii) We shall show that $\bigoplus_{j=1}^{\infty} S(w_j)\in\mathcal{P}$. By Lemma 3 it is enough to prove that $m=\bigwedge_{k=1}^{\infty} m_0\left[\bigoplus_{j=k}^{\infty} S(w_j)\right]=1$.

Let ε be an arbitrary positive number. There exists a positive δ such that if H is a Borel set and $\nu(H)<\delta$, then $\int_H 2\max\{f_1(t), g_1(t)\}d\nu(t)<\varepsilon$. By Egorov's theorem we infer that there exists a Borel set H_ε such that $\nu(H_\varepsilon)<\delta$ and the sequence $\{h_j\}_{j=1}^\infty$ converges uniformly to zero on the complement $CH_\varepsilon=[0, 2\pi]\setminus H_\varepsilon$. So there exists a k_0 such that for all $j>k_0$ and $t\in CH_\varepsilon$ we have $h_j(t)<\varepsilon$. Therefore if $j>k_0$, then for all $t\in[0, 2\pi]$ we have $h_j(t)\leqq\tilde{h}_\varepsilon(t)$, where $\tilde{h}_\varepsilon$ is the function defined by

$$\tilde{h}_\varepsilon(t) = \begin{cases} \varepsilon \quad \text{if} \quad t\in CH_\varepsilon, \\ 2\max\{f_1(t), g_1(t)\} \quad \text{if} \quad t\in H_\varepsilon. \end{cases}$$

We infer that the inner function m satisfies the inequality

$$m \leqq \operatorname{Exp}[\tilde{h}_\varepsilon].$$

Therefore we have

$$|m(0)| \geqq |\operatorname{Exp}[\tilde{h}_\varepsilon](0)| = \exp\Big[-\int_0^{2\pi} \tilde{h}_\varepsilon(t)\,d\nu(t)\Big] =$$

$$= \exp\Big[-\int_{H_\varepsilon} \tilde{h}_\varepsilon(t)\,d\nu(t) - \int_{CH_\varepsilon} \tilde{h}_\varepsilon(t)\,d\nu(t)\Big] \geqq \exp\big[-\varepsilon-\varepsilon\cdot\nu([0, 2\pi])\big].$$

Since ε can be chosen arbitrary small, so $|m(0)|=1$. That is, $m=1$.

(iv) Let $E_{j,i}$ denote the measurable subset of E defined by

$$E_{j,i} = \{t\in E\colon \sigma_t(j+1) = i\}$$

for all $j\in N$ and $i\in\hat{N}$. Then $\{E_{j,i}\}_{j\in N, i\in\hat{N}}$ will be a system of subsets of E such that the systems $\{E_{j,i}\}_{i\in\hat{N}}$ and $\{E_{j,i}\}_{j\in N}$ consist of pairwise disjoint sets for all fixed $j\in N$ and $i\in\hat{N}$, respectively; furthermore $\bigcup_{i\in\hat{N}} E_{j,i}=E$ for all $j\in N$, $(\bigcup_{j\in N} E_{j,i})\supset \{t\in E\,|\,g_i(t)>0\}$ if $i\in N$ and $(\bigcup_{j\in N} E_{j,i})\supset\{t\in E\,|\,f_i(t)>0\}$ if $i\in(-N)$.

For all $j\in N$ let S_j be the operator defined by $S_j=S_{j,1}\oplus S_{j,2}$, where $S_{j,1}= =\bigoplus_{i\in\hat{N}} S(\operatorname{Exp}[h_j, E_{j,i}])$ and $S_{j,2}=\bigoplus_{i\in\hat{N}} S(\operatorname{Exp}[h_{j+1}, E_{j,i}])$.

By Lemma 1 we infer that $S_{j,1}$ and $S_{j,2}$ are quasisimilar to $S(w_j)$ and $S(w_{j+1})$, respectively, for all $j\in N$. Therefore the operator $S=\bigoplus_{j=1}^\infty S_j$ is quasisimilar to the operator $(\bigoplus_{j=1}^\infty S(w_j))\oplus(\bigoplus_{j=2}^\infty S(w_j))$, which belongs to $\mathcal{P}$ by section (iii) and Proposition 4.4 of [1]. By Corollary 4.3 of [1] we see that $S\in\mathcal{P}$.

Since $S_{j,2}$ is quasisimilar to $S_{j+1,1}$, there exists a quasiaffinity $Y_j\in\mathcal{I}(S_{j+1,1}, S_{j,2})$ $(j\in N)$. We may assume that Y_j is a contraction.

For all $j\in N$, $i\in\hat{N}$ let

$$Z_{j,i}\in\mathcal{I}(S(\operatorname{Exp}[h_{j+1}, E_{j,i}]), S(\operatorname{Exp}[h_j, E_{j,i}]))$$

be the operator defined by

$$Z_{j,i}m=\begin{cases}\dfrac{\operatorname{Exp}[h_{j+1}, E_{j,i}]}{\operatorname{Exp}[h_j, E_{j,i}]}m & \text{if } i\in N,\\ P_{\mathfrak{H}(\operatorname{Exp}[h_{j+1}, E_{j,i}])}m & \text{if } i\in(-N),\end{cases}$$

where $m\in\mathfrak{H}(\operatorname{Exp}[h_j, E_{j,i}])$.

Then for all $j\in N$ we infer that $Z_j=\bigoplus_{i\in\hat{N}} Z_{j,i}\in\mathcal{I}(S_{j,2}, S_{j,1})$.

Let $X\in\{S\}'$ be the operator defined by

$$X|\bigoplus_{i\in\hat{N}}\mathfrak{H}(\operatorname{Exp}[h_j, E_{j,i}])=Z_j \quad\text{and}\quad X|\bigoplus_{i\in\hat{N}}\mathfrak{H}(\operatorname{Exp}[h_{j+1}, E_{j,i}])=Y_j$$

for all $j\in N$.

Then by Lemmas 1 and 2 we infer

$$S|\ker X\cong\bigoplus_{j=1}^{\infty}\left(\bigoplus_{i=1}^{\infty}S(\operatorname{Exp}[f_i, E_{j,i}])\right)\cong\bigoplus_{i=1}^{\infty}\left(\bigoplus_{j=1}^{\infty}S(\operatorname{Exp}[f_i, E_{j,i}])\right)\sim$$

$$\sim\bigoplus_{i=1}^{\infty}S(\operatorname{Exp}[f_i, E]),\quad\text{and similarly,}$$

$$S_{\ker X^*}\sim\bigoplus_{i=1}^{\infty}S(\operatorname{Exp}[g_i, E]).$$

(v) It is clear that for all $t\in CE=[0, 2\pi]\setminus E$ we have that $a=f(t)$ and $b=g(t)$ satisfy the assumptions of Lemma 4. Replacing E, $f_n(t)$, $g_n(t)$, $dv(t)$ by $(CE)^{\sim}=\{t\in[0, 2\pi]|2\pi-t\in CE\}$, $g_n(2\pi-t)$, $f_n(2\pi-t)$ and $dv(2\pi-t)$, respectively, we repeat the reasoning of the sections (ii), (iii) and (iv). Also taking adjoints we get that there exist operators $R\in\mathcal{P}$ and $Y\in\{R\}'$ such that

$$R|\ker Y\sim\bigoplus_{i=1}^{\infty}S(\operatorname{Exp}[f_i, CE]) \quad\text{and}\quad R_{\ker Y^*}\sim\bigoplus_{i=1}^{\infty}S(\operatorname{Exp}[g_i, CE]).$$

Therefore, the operator $T=S\oplus R$ will belong to $\mathcal{P}$, $Z=X\oplus Y\in\{T\}'$, and by Lemma 1

$$T|\ker Z\sim\bigoplus_{i=1}^{\infty}S(\operatorname{Exp}[f_i])=T_1,\quad T_{\ker Z^*}\sim\bigoplus_{i=1}^{\infty}S(\operatorname{Exp}[g_i])=T_2.$$

That is, $T_1\varrho T_2$ and the Proposition 2 is proved.

Proof of the Theorem. Let T_1 and T_2 be the Jordan operators $T_1=\bigoplus_{n=1}^{\infty} S(u_n)$ and $T_2=\bigoplus_{n=1}^{\infty} S(v_n)$. The inner functions u_n, v_n have canonical factorizations $u_n=u_{n,1}\cdot u_{n,2}$, $v_n=v_{n,1}\cdot v_{n,2}$, where $u_{n,1}$, $v_{n,1}$ are Blaschke products, $u_{n,2}$, $v_{n,2}$ are singular inner functions for all $n\in N$. Introducing the operators $T_{1,i}=\bigoplus_{n=1}^{\infty} S(u_{n,i})$ and $T_{2,i}=\bigoplus_{n=1}^{\infty} S(v_{n,i})$ $(i=1, 2)$ we infer by Propositions 1 and 2 that $T_{1,1}\varrho T_{2,1}$ and $T_{1,2}\varrho T_{2,2}$. Taking direct sums and using Lemma 1 we see that $T_1\varrho T_2$. The proof is done.

4. By this Theorem and Theorem 7.9 of [1] we infer:

Corollary 1. *For $T_1, T_2\in\mathscr{P}$ we have $T_1\varrho T_2$ if and only if $\gamma_{T_1}=\gamma_{T_2}$.*

We list some immediate consequences of this Corollary.

Corollary 2. *ϱ is an equivalence relation on $\mathscr{P}$.*

Corollary 3. *Let us suppose that $T_i\in\mathscr{P}$, $\mathfrak{H}_i\in\operatorname{Lat}(T_i)$ and $\gamma_{T_i}(\mathfrak{H}_i)=(\sigma_i, \mu_i)$, where μ_i is σ-finite $(i=1, 2)$. If $T_1\varrho T_2$ and $(T_1|\mathfrak{H}_1)\varrho(T_2|\mathfrak{H}_2)$, then $(T_1)_{\mathfrak{H}_1^\perp}\varrho(T_2)_{\mathfrak{H}_2^\perp}$.*

Proof. This follows from Corollary 7.10 and Lemma 6.5 of [1], and from the above Corollary 1.

Corollary 4. *Let T, S be operators of class $\mathscr{P}$ acting on the spaces $\mathfrak{H}$ and $\mathfrak{K}$, respectively, and let $\mathfrak{H}_j\in\operatorname{Lat}(T)$, $\mathfrak{K}_j\in\operatorname{Lat}(S)$ be such that $\mathfrak{H}_j\subset\mathfrak{H}_{j+1}$, $\mathfrak{K}_j\subset\mathfrak{K}_{j+1}$ $(j=1, 2, \ldots)$ and $\bigvee_{j=1}^{\infty}\mathfrak{H}_j=\mathfrak{H}$, $\bigvee_{j=1}^{\infty}\mathfrak{K}_j=\mathfrak{K}$. If $(T|\mathfrak{H}_j)\varrho(S|\mathfrak{K}_j)$ for all $j=1, 2, \ldots$, then $T\varrho S$.*

Proof. This follows from Corollary 1 and Lemma 7.4 of [1].

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BOLYAI INSTITUTE
UNIVERSITY SZEGED
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

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Contributions to the ideal theory of semigroups

S. LAJOS

Let S be a semigroup. A subsemigroup A of S is said to be an *(m, n)-ideal* of S if the inclusion $A^m S A^n \subseteqq A$ holds, where m, n are non-negative integers, A^0 is the empty symbol. The author [4] proved that the product of two (1, 1)-ideals of S is again a (1, 1)-ideal. Thus the collection of all (1, 1)-ideals of a semigroup S is a semigroup with respect to the ordinary set product. This semigroup will be denoted by $\mathbf{B}(S)$. Also, the collection of all left [right] ideals of S is a multiplicative semigroup. This semigroup will be denoted by $\mathbf{L}(S)[\mathbf{R}(S)]$. It is easy to see, that $\mathbf{L}(S)$ is a right ideal and $\mathbf{R}(S)$ is a left ideal of $\mathbf{B}(S)$. Their intersection, the multiplicative semigroup of all two-sided ideals of S is a quasi-ideal of $\mathbf{B}(S)$.

In this short note certain classes of semigroups will be characterized by properties of the semigroups $\mathbf{B}(S)$ and $\mathbf{L}(S)$. For the undefined notions and notations we refer to [1], [2], and [12].

We begin with two lemmas.

Lemma 1. *A semigroup S is regular if and only if $BSB=B$ holds for every bi-ideal B of S.*

This is an easy consequence of a result by J. Luh [10].

Lemma 2. *A semigroup S is a semilattice of groups if and only if the intersection of any two* (1, 1)*-ideals of S is equal to their product.*

For this criterion, see the author [5] or [6].

Our first main result is contained in the following

Theorem 1. *For a semigroup S the following conditions are equivalent:*

(1) *S is a semilattice of groups.*

(2) $\mathbf{B}(S)$ *is a distributive lattice with respect to the set product and the set-theoretical union.*

(3) $\mathbf{B}(S)$ *is a regular monoid with respect to the set product.*

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Proof. (1)⇒(2): If S is a semigroup which is a semilattice of groups, then every bi-ideal of S is a two-sided ideal of S. Hence this implication is straightforward by Lemma 2.

(2)⇒(1): by Theorem 1 of [6].

(1)⇒(3) is obvious.

(3)⇒(1): Suppose that S is a semigroup whose bi-ideal semigroup $\mathbf{B}(S)$ is a regular monoid with respect to set product. If A is the identity element of $\mathbf{B}(S)$, we have $S=ASA\subseteqq A$. Hence $A=S$. Therefore $BS=SB=B$ holds for any bi-ideal B of S, whence B is a two-sided ideal of S. On the other hand, the regularity of $\mathbf{B}(S)$ together with Lemma 1 implies that S is regular. Thus S is a regular duo semigroup which is a semilattice of groups.

Corollary 1. *If S is a semilattice, then $\mathbf{B}(S)$ is a distributive lattice. In particular, if S is a diagonal semilattice (i.e., every non-zero element of S is an atom), then $\mathbf{B}(S)$ is a Boolean algebra.*

Corollary 2. *The bi-ideal semigroup $\mathbf{B}(S)$ of a semigroup S is a Boolean algebra if and only if S is a diagonal semilattice of groups.*

The following criterion is due to the author [7].

Lemma 3. *A semigroup S is a semilattice of left groups if and only if $B\cap L=BL$ holds for every bi-ideal B and every left ideal L of S.*

By making use of Lemma 3, further characterizations can be given for semigroups that are semilattices of left groups in term of the bi-ideal semigroup $\mathbf{B}(S)$.

Theorem 2. *For a semigroup S the following conditions are equivalent:*

(1) S is a semilattice of left groups.

(2) $\mathbf{B}(S)$ is a band and S is a right identity of it.

(3) $\mathbf{B}(S)$ is a regular semigroup and S is a right identity of it.

Proof. (1)⇒(2): If S is a semigroup which is a semilattice of left groups, then, by Lemma 3, the relation $L\cap R=RL$ holds for every left ideal L and every right ideal R of S, thus S is regular. Moreover Lemma 3 implies $BS=B$ for every bi-ideal B of S, whence S is a right identity of $\mathbf{B}(S)$. Then Lemma 1 implies that every bi-ideal of S is globally idempotent, i.e., $\mathbf{B}(S)$ is a band.

(2)⇒(3) is clear.

(3)⇒(1): If (3) holds, then it follows that S is a regular left duo semigroup which is a semilattice of left groups.

T. Saitô [11] has proved the following criterion.

Lemma 4. *A semigroup S is a semilattice of left simple semigroups if and only if the intersection of any two left ideals of S is equal to their product.*

Now we are ready to prove the following result.

Theorem 3. *For a semigroup S the following conditions are equivalent:*

(1) *S is a semilattice of left simple semigroups.*

(2) $\mathbf{L}(S)$ *is a distributive lattice with respect to the set product and set-theoretical union.*

(3) $\mathbf{L}(S)$ *is a multiplicative semilattice.*

Proof. (1)⇒(2): If S is a semilattice of left simple semigroups, then, by Lemma 4, every left ideal of S is a two-sided ideal. Hence the implication follows by Lemma 4.

(2)⇒(3) is obvious. Finally, (3)⇒(1) by [11].

Next an ideal-theoretical characterization will be given for homogroups. A semigroup S is called a *homogroup* if it has a subgroup which is at the same time a two-sided ideal of S (for an equivalent definition see [13]). For instance, a semigroup with zero element is a homogroup.

Theorem 4. *A semigroup S is a homogroup if and only if the bi-ideal semigroup* $\mathbf{B}(S)$ *has a zero element.*

Proof. Let S be a homogroup with the group-ideal G. Let B be a bi-ideal of S. Then the product BG is a right ideal of S, and $BG\subseteqq G$. Hence it follows that $BG=G$, because a group has no proper right ideals. Similarly, we get $GB=G$ and G is the zero element of $\mathbf{B}(S)$.

Conversely, if S is a semigroup whose bi-ideal semigroup has a zero element Z, then we have $SZ=ZS=Z$. Hence Z is a two-sided ideal of S. For any element z of Z the product Zz is a left ideal of S. Thus we have $Z=Z(Zz)=Zz$, since the set product is associative for non-empty subsets of S. Similarly we get $zZ=Z$ for any element z of Z. Therefore Z is a subgroup of S, and S is a homogroup, indeed.

Remark. It is easy to see that Theorem 4 remains true with $\mathbf{P}(S)$ instead of $\mathbf{B}(S)$, where $\mathbf{P}(S)$ is the power semigroup of S, i.e., the multiplicative semigroup of all non-empty subsets of S.

Finally, we are interested in semigroups whose bi-ideal semigroup is a monoid.

Theorem 5. *For a semigroup S the bi-ideal semigroup* $\mathbf{B}(S)$ *is a monoid if and only if* (i) *every bi-ideal of S is a two-sided ideal of S, and* (ii) *every two-sided ideal of S is complete (i.e. $IS=SI=I$).*

Proof. First, let S be a semigroup having properties (i), (ii). Then the bi-ideal semigroup $\mathbf{B}(S)$ is the multiplicative monoid of all two-sided ideals of S with the identity S.

Secondly, if the (1, 1)-ideals of a semigroup S form a monoid with identity A, then we have $S=ASA\subseteqq A$, whence it follows that $A=S$. Thus $BS=SB=B$ for every bi-ideal B of S, that is, every (1, 1)-ideal B is a complete (two-sided) ideal of S. Theorem 5 is completely proved.

For the characterizations of completely regular semigroups in terms of (m, n)-ideals, see the author [8] and [9].

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KARL MARX UNIVERSITY FOR ECONOMICS
INSTITUTE FOR MATHEMATICS AND COMPUTER SCIENCE
1828 BUDAPEST, HUNGARY

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Selecting independent lines from a family of lines in a space

L. LOVÁSZ

0. Introduction. A family of flats in a projective space is called *independent*, if no member of the family intersects the flat spanned by the other members. It is an interesting combinatorial problem to select a maximum number of independent flats from a given family of flats. In the special case when all the flats are faces of a simplex, this question is equivalent to the so-called *matching problem* for hypergraphs: given a collection of sets, find the maximum number of disjoint ones among them. This problem is known to belong to the class of (in a sense) hardest combinatorial problems, the so-called NP-complete problems (see [6]). Hence there is no hope to solve it in a satisfactory way.

However, the special case of the matching problem when all the given sets are pairs, is well-solved [2, 4]. This suggests that probably the problem of selecting a maximum set of independent lines from a family of lines is solvable.

"Solution" here may mean two different things:

(a) find a minimax formula for the number in question;

(b) find an algorithm to determine this number such that the running time of the algorithm is polynomial in the number of data.

We shall present a solution in the sense of (a) (Theorem 2). It remains open if these methods can be extended (or other methods found) to yield a solution in the sense (b), but we hope the answer is affirmative. The problem we discuss can be considered as the so-called "matroid parity problem" for representable matroids (see LAWLER [3], Ch. 9). We shall discuss the difficulties of generalizing our methods, along with other connections to matroid theory in section 5.

1. Some special cases and equivalents. The famous *f-factor problem*, solved by TUTTE [5], is the following. Let G be a graph and f an integer-valued function on its vertex set $V(G)$. Does there exist a subgraph G' such that the degree of x in G' is $f(x)$, for every $x \in V(G)$?

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This problem can be reduced to the line-selection problem as follows. Let, for each $x \in V(G)$, A_x be a flat of rank* $f(x)$ in a projective space, such that the flats $\{A_x: x \in V(G)\}$ are independent. For each edge $e=(x, y)$, select two points $p_{e,x} \in A_x$ and $p_{e,y} \in A_y$ such that the points $\{p_{e,x}$: e is a line adjacent to $x\}$ are in general position on A_x (i.e. no $f(x)$ of them are contained in a proper subflat of A_x). Let $\hat{e}$ denote the line connecting $p_{e,x}$ to $p_{e,y}$. Then it is easy to verify that

a subgraph H has degree $\leqq f(x)$ at each point x iff the lines $\{\hat{e}: e \in E(H)\}$ are independent.

So G has an f-factor iff the family $\{\hat{e}: e \in E(G)\}$ contains $\frac{1}{2} \sum_x f(x)$ independent lines. Our results therefore yield a necessary and sufficient condition for the existence of an f-factor in a graph. Although our condition has features similar to Tutte's, to derive Tutte's theorem from it is somewhat lengthy.

We may place the points $p_{e,x}$ on A_x in such a way that they form an arbitrary matroid embeddable in the projective space we consider. This yields then a solution to the "matchoid problem" of Edmonds in the special case when the matroids prescribed at the vertices are representable.

Finally, we mention an equivalent version of our problem. Let $\mathcal{H}$ be a collection of lines which spans a rank r projective space P. Let $\nu(\mathcal{H})$ be the maximum number of independent lines in $\mathcal{H}$ and $\mu(\mathcal{H})$ the minimum number of lines in $\mathcal{H}$ which still span P. Then $\nu(\mathcal{H})+\mu(\mathcal{H})=r$. This identity is a generalization of Gallai's identity in graph theory, and can be proved along the same lines. So we also have a minimax formula for the minimum number of lines in a family which span the same flat as the whole family. The transformation of Theorem 2 to this version is left to the reader.

2. Preliminaries. Let P be a projective geometry over a (possibly skew) field. We shall denote by $\overline{X}$ the *span* of the set $X \subseteq P$, i.e. the smallest flat (subspace) containing X. Each flat A in P has a rank $r(A)$, which is one larger than its dimension. So $\emptyset$ has rank 0, points have rank 1, lines have rank 2. We extend the notation of rank over arbitrary subsets of P by $r(X)=r(\overline{X})$ and even over a collection $\mathcal{H}$ of subsets of P by $r(\mathcal{H})=r(\cup \mathcal{H})$. Similarly, if $\mathcal{H}$ is a collection of subsets of P we set $\overline{\mathcal{H}} = \overline{\cup \mathcal{H}}$.

The rank satisfies the important identity

$$r(A \cup B)+r(A \cap B) = r(A)+r(B),$$

where A, B are flats.

* The rank $r(A)$ of a projective space A is its dimension plus 1.

We shall make use of the following very simple lemma:

Lemma 1. *Let* $A_1, \dots, A_k$, D *be flats in a projective geometry and* $A_i \subseteqq D$. *Assume that*

$$\sum_{i=1}^{k} \{r(D) - r(A_i)\} < r(D).$$

Then $\bigcap_{i=1}^{k} A_i \neq \emptyset$.

Proof. We show by induction on k that

$$r(A_1 \cap \dots \cap A_k) \geqq r(D) - \sum_{i=1}^{k} \{r(D) - r(A_i)\}.$$

This is trivially true if $k=1$. Let $k \geqq 2$. Then

$$r(A_1 \cap \dots \cap A_k) = r(A_1 \cap \dots \cap A_{k-1}) + r(A_k) - r((A_1 \cap \dots \cap A_{k-1}) \cup A_k) \geqq$$
$$\geqq r(A_1 \cap \dots \cap A_{k-1}) + r(A_k) - r(D).$$

Applying the induction hypothesis the assertion follows.

Q.E.D.

Recall that a set of lines in a projective geometry is called *independent* if no member of the set meets the span of the rest. It is immediately seen that each subset of an independent set is independent.

Lemma 2. *Let* $\mathscr{F}$ *be a set of lines in a projective space. Then*

$$r(\mathscr{F}) \leqq 2|\mathscr{F}|. \tag{1}$$

Equality holds iff $\mathscr{F}$ *is independent.*

Proof. Let $e \in \mathscr{F}$. Then

$$r(\mathscr{F}) = r(e) + r(\mathscr{F}-e) - r(e \cap \overline{\mathscr{F}-e}) = r(\mathscr{F}-e) + 2 - r(e \cap \overline{\mathscr{F}-e}).$$

Hence the inequality (1) follows by induction. If $\mathscr{F}$ is independent, then clearly so is $\mathscr{F}-e$ and then equality in (1) follows by induction. On the other hand, if equality holds in (1) then the computation above implies that $r(e \cap \overline{\mathscr{F}-e}) = 0$, i.e. $e \cap \overline{\mathscr{F}-e} = \emptyset$. Since this holds for every $e \in \mathscr{F}$, it follows that $\mathscr{F}$ is independent.

Q.E.D.

A set $\mathscr{C}$ of lines is called a *circuit*, if $r(\mathscr{C}) = 2|\mathscr{C}| - 1$ but every proper subset of $\mathscr{C}$ is independent. Thus a circuit is a minimal dependent set of lines; but not every minimal dependent set of lines is a circuit, as shown by 3 lines in general position in the space.

Note that if $\mathscr{C}$ is a circuit and $e\in\mathscr{C}$ then

$$2\,|\mathscr{C}|-1=r(\mathscr{C})=r(\mathscr{C}-e)+2-r(e\cap\overline{\mathscr{C}-e})=2|\mathscr{C}|-r(e\cap\overline{\mathscr{C}-e}),$$

whence it is seen that e meets $\overline{\mathscr{C}-e}$ in exactly one point.

Lemma 3. *Let $\mathscr{K}$ be a set of lines such that $r(\mathscr{K})=2|\mathscr{K}|-1$. Then $\mathscr{K}$ contains exactly one circuit.*

Proof. Let $\mathscr{C}$ be a minimal subset of $\mathscr{K}$ with $r(\mathscr{C})=2|\mathscr{C}|-1$. We claim that all proper subsets of $\mathscr{C}$ are independent. In fact,

$$r(\mathscr{C}-e)=r(\mathscr{C})-2+r(e\cap\overline{\mathscr{C}-e})=2|\mathscr{C}|-3+r(e\cap\overline{\mathscr{C}-e})\geqq 2|\mathscr{C}-e|-1.$$

Equality here would contradict the minimality property of $\mathscr{C}$. Hence $\mathscr{C}-e$ is independent for every e. This implies that $\mathscr{C}$ is a circuit.

Assume now indirectly that there is another circuit $\mathscr{C}'$. Let e.g. $f\in\mathscr{C}-\mathscr{C}'$. We have

$$r(\mathscr{K}-f)=r(\mathscr{K})-2+r(f\cap\overline{\mathscr{K}-f})=2|\mathscr{K}|-3+r(f\cap\overline{\mathscr{K}-f}).$$

But $f\cap\overline{\mathscr{C}-f}\neq\emptyset$ and so $f\cap\overline{\mathscr{K}-f}\neq\emptyset$. Hence $\mathscr{K}-f$ is independent and so it cannot contain any circuit.

Q.E.D.

Let $\mathscr{H}$ be an arbitrary set of lines in a projective geometry. Let $\nu(\mathscr{H})$ denote the maximum number of independent lines in $\mathscr{H}$. A set of $\nu(\mathscr{H})$ independent lines will be called a *basis* of $\mathscr{H}$.

Let $\mathscr{B}$ be a basis of $\mathscr{H}$ and e a line not contained in $\overline{\mathscr{B}}$. Obviously, e must intersect $\overline{\mathscr{B}}$. Lemma 3 implies that $\mathscr{B}+e$ contains a unique circuit, which will be called the *fundamental circuit* of e relative to $\mathscr{B}$. Trivially, the fundamental circuit of e contains e. If e intersects a line $f\in\mathscr{B}$ then the fundamental circuit of e, relative to $\mathscr{B}$, is the set $\{e,f\}$.

If $\mathscr{B}$ is a basis, e a line not contained in $\overline{\mathscr{B}}$, and f a line of the fundamental circuit of e relative to $\mathscr{B}$, then $\mathscr{B}-f+e$ is another basis. We say that $\mathscr{B}-f+e$ arises from $\mathscr{B}$ by *elementary augmentation.* Trivially, the inverse of an elementary augmentation is an elementary augmentation as well.

3. Primitive sets of lines. In this section we discuss a special type of arrangement of lines. These sets will be the most difficult cases in the proof of the main result. A set $\mathscr{H}$ of lines in a projective space is called *primitive*, if the intersection of spans of all bases is void.

Lemma 4. *Let $\mathscr{H}$ be a primitive set of lines and $\mathscr{B}_1$, $\mathscr{B}_2$ two bases of $\mathscr{H}$. Then $\mathscr{B}_1$ can be transformed into $\mathscr{B}_2$ by elementary augmentations.*

Proof. Let $\mathscr{B}_1'$, $\mathscr{B}_2'$ be two bases such that $\mathscr{B}_i'$ arises from $\mathscr{B}_i$ by elementary augmentations and $|\mathscr{B}_1'\cap\mathscr{B}_2'|$ is maximal. If $\mathscr{B}_1'=\mathscr{B}_2'$ we are done by the remark after the definition of elementary augmentation.

We claim that $\overline{\mathscr{B}_1'}=\overline{\mathscr{B}_2'}$. In fact, if $\overline{\mathscr{B}_1'}\neq\overline{\mathscr{B}_2'}$ then there exists a line $e\in\mathscr{B}_1'$ such that $e\not\subseteq\overline{\mathscr{B}_2'}$. Let $\mathscr{C}$ be the fundamental circuit of e relative to $\mathscr{B}_2'$. Since $\mathscr{B}_1'$ is independent, there exists a line $f\in\mathscr{C}-\mathscr{B}_1'$. $\mathscr{B}_2''=\mathscr{B}_2'+e-f$ is a basis which arises from $\mathscr{B}_2$ by elementary augmentations and which has $|\mathscr{B}_1'\cap\mathscr{B}_2''|>|\mathscr{B}_1'\cap\mathscr{B}_2'|$, a contradiction.

So we know that $\overline{\mathscr{B}_1'}=\overline{\mathscr{B}_2'}$. We want to show that $\mathscr{B}_1'=\mathscr{B}_2'$. Assume indirectly that there exists a line $e\in\mathscr{B}_1'-\mathscr{B}_2'$. Consider a basis $\mathscr{B}_0$ which does not span e. Such a basis exists since $\mathscr{H}$ is primitive. Choose $\mathscr{B}_1'$, $\mathscr{B}_2'$ and $\mathscr{B}_0$ so that $|\mathscr{B}_1'\cap\mathscr{B}_0|$ is maximal. Obviously, $\overline{\mathscr{B}_0}\neq\overline{\mathscr{B}_1'}$, and hence, there exists a line $g\in\mathscr{B}_0$ such that $g\not\subseteq\overline{\mathscr{B}_1'}$. Let $\mathscr{C}_1$, $\mathscr{C}_2$ denote the fundamental circuits of g relative to $\mathscr{B}_1'$ and $\mathscr{B}_2'$. We distinguish two cases.

Case 1. $\mathscr{C}_1\neq\mathscr{C}_2$. Then $\mathscr{C}_1\not\subseteq\mathscr{B}_2'+g$, since otherwise, $\mathscr{B}_2'+g$ would contain two distinct circuits, contradicting Lemma 3. Similarly $\mathscr{C}_2\not\subseteq\mathscr{B}_1'+g$. So we can select lines $f_1\in\mathscr{C}_1-\mathscr{B}_2'-g$ and $f_2\in\mathscr{C}_2-\mathscr{B}_1'-g$. Now $\mathscr{B}_i''=\mathscr{B}_i'-f_i+g$ is a basis arising from $\mathscr{B}_i$ by elementary augmentations, and $|\mathscr{B}_1''\cap\mathscr{B}_2''|>|\mathscr{B}_1'\cap\mathscr{B}_2'|$, a contradiction.

Case 2. $\mathscr{C}_1=\mathscr{C}_2$. Then $e\notin\mathscr{C}_1=\mathscr{C}_2$. Let $f\in\mathscr{C}_1-\mathscr{B}_0$, and put $\mathscr{B}_i''=\mathscr{B}_i'+g-f$. Now $|\mathscr{B}_1''\cap\mathscr{B}_2''|=|\mathscr{B}_1'\cap\mathscr{B}_2'|$, $|\mathscr{B}_1''\cap\mathscr{B}_0|>|\mathscr{B}_1'\cap\mathscr{B}_0|$, and since $e\in\mathscr{B}_1''$, $e\notin\mathscr{B}_2''$, this is a contradiction.

Lemma 5. *Let $\mathscr{H}$ be a primitive set of lines and $\mathscr{K}\subseteqq\mathscr{H}$ such that $r(\mathscr{K})\geqq 2v+2$. Then the flats spanned by the circuits in $\mathscr{K}$ have no element in common.*

Proof. Suppose indirectly that a point p is contained in the span of each circuit in $\mathscr{K}$. Since $\mathscr{H}$ is primitive, there exists a basis $\mathscr{B}\subseteqq\mathscr{H}$ such that the span of $\mathscr{B}$ does not contain p. Choose such a $\mathscr{B}$ with $|\mathscr{B}\cap\mathscr{K}|$ maximal. Since $r(\mathscr{B}+p)=2v+1<r(\mathscr{K})$, there is a line $e\in\mathscr{K}$ such that $e\not\subseteq\overline{\mathscr{B}+p}$. Then $p\notin\overline{\mathscr{B}+e}$. Let $\mathscr{C}$ be the fundamental circuit of e relative to $\mathscr{B}$. Then $p\notin\overline{\mathscr{B}}$ and so by the definition of p, $\mathscr{C}\not\subseteq\mathscr{K}$. Let $f\in\mathscr{C}-\mathscr{K}$, then $\mathscr{B}'=\mathscr{B}-f+e$ is a basis such that $p\notin\overline{\mathscr{B}'}$ and $|\mathscr{B}'\cap\mathscr{K}|>|\mathscr{B}\cap\mathscr{K}|$, which is a contradiction.

Lemma 6. *Let $\mathscr{H}$ be a primitive set of lines, $\mathscr{B}$ a basis, $e,f\in\mathscr{H}$ such that $r(\mathscr{B}+e+f)=2v+2$. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be the fundamental circuits of e and f, respectively, relative to $\mathscr{B}$. Then $\mathscr{C}_1\cap\mathscr{C}_2=\emptyset$.*

Proof. Suppose indirectly that $\mathscr{C}_1\cap\mathscr{C}_2\neq\emptyset$. Let $\mathscr{C}_1, \ldots, \mathscr{C}_t$ denote the circuits in $\mathscr{K}=\mathscr{B}+e+f$ and let D_i be the flat spanned by $\mathscr{C}_i$. For each $u\in\mathscr{K}$, $r(\mathscr{K}-u)\leqq$

$\leqq 2v+1$, since otherwise $\mathcal{K}-u$ would be an independent set of $v+1$ lines. Let

$$\mathcal{K}_0 = \{u \in \mathcal{K}: r(\mathcal{K}-u) = 2v\}.$$

If $u \in \mathcal{K}-\mathcal{K}_0$ then $\mathcal{K}-u$ contains a unique circuit by Lemma 3. Let

$$\mathcal{K}_i = \{u \in \mathcal{K}-\mathcal{K}_0: \mathcal{C}_i \subseteqq \mathcal{K}-u\}.$$

Thus $\{\mathcal{K}_0, \mathcal{K}_1, \dots, \mathcal{K}_t\}$ is a partition of $\mathcal{K}$, and

$$\mathcal{C}_i = \mathcal{K}-\mathcal{K}_0-\mathcal{K}_i.$$

$\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$ implies now by Lemma 5 that $t \geqq 3$. Furthermore, we have

$$r(D_i) = 2|\mathcal{K}-\mathcal{K}_0-\mathcal{K}_i|-1, \quad r\left(\bigcup_{i=1}^{t} D_i\right) = 2|\mathcal{K}-\mathcal{K}_0|-2.$$

(Here $r(\bigcup_{i=1}^{t} D_i) \geqq 2|\mathcal{K}-\mathcal{K}_0|-2$ is trivial; in the case of strict inequality Lemma 3 would imply that $\bigcup D_i$ contains only one circuit, which is not the case.) Now apply Lemma 1:

$$\sum_{i=1}^{t}\left\{r\left(\bigcup_{j=1}^{t} D_j\right)-r(D_i)\right\} = \sum_{i=1}^{t}\{2|\mathcal{K}_i|-1\} = 2|\mathcal{K}-\mathcal{K}_0|-t < r\left(\bigcup_{i=1}^{t} D_i\right),$$

and so $\bigcap_{i=1}^{t} D_i \neq \emptyset$. But this contradicts Lemma 5.

Call two lines e, f of a primitive set $\mathcal{H}$ *coherent*, if $r(\mathcal{B}+e+f) \leqq 2v+1$ for every basis $\mathcal{B}$.

Lemma 7. *Coherence of lines is an equivalence relation.*

Proof. Symmetry and reflexivity of coherence are evident. Suppose e and f, moreover f and g, are coherent. We show that e and g are coherent. Suppose indirectly that there exists a basis $\mathcal{B}_1$ such that $r(\mathcal{B}_1+e+g)=2v+2$. Since $\mathcal{H}$ is primitive, there exists another basis $\mathcal{B}_2$ such that $f \not\subseteqq \overline{\mathcal{B}_2}$, and so $r(\mathcal{B}_2+f)=2v+1$. Choose $\mathcal{B}_2$ such that $|\mathcal{B}_1 \cap \mathcal{B}_2|$ is maximum. Since $r(\mathcal{B}_1+e+g) > r(\mathcal{B}_2+f)$, there exists a line $h \in \mathcal{B}_1+e+g$ such that $h \not\subseteqq \overline{\mathcal{B}_2+f}$, i.e. such that

$$r(\mathcal{B}_2+f+h) = 2v+2. \tag{3}$$

So h is not coherent with f and so $h \neq e, g$. Thus $h \in \mathcal{B}_1$. Let $\mathcal{C}$ denote the fundamental circuit of h relative to $\mathcal{B}_2$. Since $\mathcal{C} \not\subseteqq \mathcal{B}_1$, we may choose a line $l \in \mathcal{C}-\mathcal{B}_1$. Then $\mathcal{B}_2' = \mathcal{B}_2+h-l$ is a basis such that $f \not\subseteqq \overline{\mathcal{B}_2'}$ (since $f \not\subseteqq \overline{\mathcal{B}_2+h}$ by (3) and $\mathcal{B}_2' \subset \mathcal{B}_2+h$), and moreover, $|\mathcal{B}_2' \cap \mathcal{B}_1| > |\mathcal{B}_2 \cap \mathcal{B}_1|$. This contradicts the choice of $\mathcal{B}_2$.

Q.E.D.

Lemma 8. *Let $\mathcal{B}$ be a basis and e a line not in the span of $\mathcal{B}$. Let $\mathcal{C}$ denote the fundamental circuit of e relative to $\mathcal{B}$. Then all lines in $\mathcal{C}$ are coherent.*

Proof. Suppose indirectly that there is a line $f \in \mathscr{C}$ such that e, f are not coherent. Let $\mathscr{B}'$ be a basis such that $r(\mathscr{B}'+e+f)=2v+2$. From among all counter-examples choose one in which $|\mathscr{B}' \cap \mathscr{B}|$ is maximal. Since $r(\mathscr{B}'+e+f) > r(\mathscr{B}+f)$ there is a line $g \in \mathscr{B}'+e+f$ not contained in the span of $\mathscr{B}+f$. Obviously, $g \neq e, f$, so $g \in \mathscr{B}'$. Let $\mathscr{C}_2$ denote the fundamental circuit of g relative to $\mathscr{B}$. Since $r(\mathscr{B}+f+g)= =2v+2$, if follows by Lemma 6 that $\mathscr{C} \cap \mathscr{C}_2 = \emptyset$. Hence if we replace any element of $\mathscr{C}_2 \cap \mathscr{B}$ by g in $\mathscr{B}$, we obtain another basis $\mathscr{B}^*$ which has the property that the fundamental circuit of f relative to $\mathscr{B}^*$ is $\mathscr{C}$, but $|\mathscr{B}^* \cap \mathscr{B}'| > |\mathscr{B} \cap \mathscr{B}'|$, a contradiction.

Q.E.D.

Lemma 9. *If $\mathscr{H}$ is primitive, $e, f \in \mathscr{H}$ and e and f intersect, then e and f are coherent.*

Proof. Suppose not, then there exists a basis $\mathscr{B}$ such that $r(\mathscr{B}+e+f)=2v+2$. By an elementary augmentation we get a basis $\mathscr{B}'$ such that $e \in \mathscr{B}'$ but $f \not\subseteq \mathscr{B}'$. But by $e \cap f \neq \emptyset$ the fundamental circuit of f relative to $\mathscr{B}'$ is $\{e, f\}$, which contradicts Lemma 8.

Q.E.D.

Lemma 10. *Let $\mathscr{H}$ be a primitive set of lines, $\mathscr{B}$ a basis of $\mathscr{H}$, e a line not in the span of $\mathscr{B}$ and $\mathscr{A}$ the set of lines in $\mathscr{B}$ coherent to e. Then every line coherent to e is contained in $\overline{\mathscr{A}+e}$.*

Proof. Let f be a line coherent to e. Let $p \in e - \overline{\mathscr{B}}$ and $q \in f$, and denote by g the line pq. Set $\mathscr{H}' = \mathscr{H}+g$.

Claim 1. $v(\mathscr{H}')=v$. For suppose indirectly that $\mathscr{H}'$ contains an independent set $\mathscr{F}$ of $v+1$ lines. Obviously, $g \in \mathscr{F}$ and $\mathscr{F}-g$ is a basis of $\mathscr{H}$. But

$$r(\mathscr{F}-g+e+f) \geqq r(\mathscr{F}) = 2v+2,$$

which contradicts the assumption that e and f are coherent.

Claim 2. $\mathscr{H}'$ is primitive. This follows immediately from the fact that all bases of $\mathscr{H}$ are bases of $\mathscr{H}'$.

Claim 3. If two lines of $\mathscr{H}$ are coherent in $\mathscr{H}'$ then they are coherent in $\mathscr{H}$, for the same reason.

Claim 4. e, f and g are coherent in $\mathscr{H}'$. This follows by Lemma 9.

Now by Lemma 8, all lines in the fundamental circuit of g relative to $\mathscr{B}$ are coherent to g in $\mathscr{H}'$. By Claim 4, they are coherent to e in $\mathscr{H}'$ and so by Claim 3, they are coherent to e in $\mathscr{H}$. Thus $g \cap \overline{\mathscr{A}} \neq \emptyset$. Since $p \notin \overline{\mathscr{A}}$ but $p \in e$, it follows that g has at least two points in $\overline{\mathscr{A}+e}$. But then $g\subseteq\overline{\mathscr{A}+e}$, and consequently $q\in\overline{\mathscr{A}+e}$. q being an arbitrary point of f, it follows that $f\subseteq\overline{\mathscr{A}+e}$.

Q.E.D.

Let $\mathscr{H}_1, \ldots, \mathscr{H}_k$ denote the equivalence classes of the relation of coherence. Consider a basis $\mathscr{B}$ and set $v_i=|\mathscr{B}\cap\mathscr{H}_i|$. Observe that the numbers v_i are independent of the choice of $\mathscr{B}$: in fact, they remain the same when an elementary augmentation is carried out by Lemma 8, and every other basis can be obtained from $\mathscr{B}$ by elementary augmentations by Lemma 4.

Our result on primitive set of lines can be summarized as follows:

Theorem 1. *Let $\mathscr{H}$ be a primitive set of lines. Then there exist flats $A_1, \ldots, A_k$ with the following properties:*

(i) *$A_1, \ldots, A_k$ are disjoint.*

(ii) *Every line in $\mathscr{H}$ is contained in exactly one of $A_1, \ldots, A_k$.*

(iii) $r(A_i)=2v_i+1$.

(iv) *Every basis contains precisely v_i lines in A_i.*

(v) $v(\mathscr{H})=\sum_{i=1}^{k} v_i$.

Proof. Denote by A_i the flat spanned by $\mathscr{H}_i$. First we show that $r(A_i)=2v_i+1$. Let $e\in\mathscr{H}_i$ and $\mathscr{B}$ any basis not spanning e. Let $\mathscr{A}_i=\mathscr{B}\cap\mathscr{H}_i$. By Lemma 8, the fundamental circuit of e relative to $\mathscr{B}$ is contained in $\mathscr{A}_i+e$. Hence $r(\mathscr{A}_i+e)=2v_i+1$. On the other hand, Lemma 10 implies that all lines of $\mathscr{H}_i$ are contained in $\overline{\mathscr{A}_i+e}$. Hence $A_i=\overline{\mathscr{A}_i+e}$ and $r(A_i)=2v_i+1$.

Thus (iii) and (iv) are proved. (v) follows immediately. If we show (i) then (ii) will be trivially true.

So let $1\leqq i<j\leqq k$; we show that $A_i\cap A_j=\emptyset$. Let $e_i\in\mathscr{H}_i$, $e_j\in\mathscr{H}_j$, and let $\mathscr{B}$ be a basis such that $r(\mathscr{B}+e_i+e_j)=2v+2$ (such a basis exists by the definition of the sets $\mathscr{H}_i$). Let $\mathscr{A}_t=\mathscr{B}\cap\mathscr{H}_t$. By the argument above, e_t meets $\overline{\mathscr{A}_t}$ $(t=i,j)$ and $A_t=\overline{\mathscr{A}_t+e_t}$. But

$$r(A_i\cup A_j) = r(\mathscr{A}_i\cup\mathscr{A}_j\cup\{e_i, e_j\}) \geqq r(\mathscr{B}+e_i+e_j)-2|\mathscr{B}-\mathscr{A}_i-\mathscr{A}_j| =$$

$$= 2|\mathscr{B}|+2-2|\mathscr{B}-\mathscr{A}_i-\mathscr{A}_j| = |\mathscr{A}_i|+|\mathscr{A}_j|+2 = r(A_i)+r(A_j).$$

Hence A_i and A_j are disjoint.

Q.E.D.

4. The main result.

Theorem 2. *Let $\mathscr{H}$ be a set of lines in a projective geometry. Then the maximum number $v(\mathscr{H})$ of independent lines in $\mathscr{H}$ is the minimum of the expression*

$$r(A)+\sum_{i=1}^{k}\left[\frac{r(A_i)-r(A)}{2}\right],$$

where $A, A_1, \ldots, A_k$ are flats such that $A\subseteqq A_i$ $(i=1, \ldots, k)$ and for every $e\in\mathscr{H}$ either $e\cap A\neq\emptyset$ or there is an i such that $e\subseteqq A_i$.

Proof. I. First we show that if $\mathscr{F}$ is a set of independent lines, $A, A_1, \ldots, A_k$ are subspaces such that $A\subseteqq A_i$ and each line of $\mathscr{F}$ either meets A or is contained in one of the A_i's then

$$|\mathscr{F}|\leqq r(A)+\sum_{i=1}^{k}\left[\frac{r(A_i)-r(A)}{2}\right].$$

Let $\mathscr{F}_i$ and $\mathscr{F}_0$ denote the set of lines of $\mathscr{F}$ contained in A_i and meeting A, respectively. Let A_i' be the subspace spanned by $\mathscr{F}_i'=\mathscr{F}_i-\mathscr{F}_{i-1}-\ldots-\mathscr{F}_0$. Then $r(A_i')=2|\mathscr{F}_i'|$. Moreover, the subspaces A_i' are clearly independent and, therefore, so are the subspaces $A_i'\cap A$, $i=0, \ldots, k$. Hence

$$r(A)\geqq\sum_{i=0}^{k}r(A_i'\cap A).$$

Here $r(A_i'\cap A)=r(A_i')+r(A)-r(A_i'\cup A)\geqq r(A_i')+r(A)-r(A_i)$, whence

$$|\mathscr{F}_i'|=\frac{1}{2}r(A_i')\leqq\frac{r(A_i)-r(A)}{2}+\frac{r(A_i'\cap A)}{2}\leqq\frac{r(A_i)-r(A)}{2}+r(A_i'\cap A),$$

and using integrality,

$$|\mathscr{F}_i'|\leqq\left[\frac{r(A_i)-r(A)}{2}\right]+r(A_i'\cap A).$$

Moreover, obviously $|\mathscr{F}_0'|\leqq r(A\cap A_0')$. Hence

$$|\mathscr{F}|=\sum_{i=0}^{k}|\mathscr{F}_i'|\leqq r(A_0'\cap A)+\sum_{i=1}^{k}\left\{\left[\frac{r(A_i)-r(A)}{2}\right]+r(A_i'\cap A)\right\}\leqq$$

$$\leqq r(A)+\sum_{i=1}^{k}\left[\frac{r(A_i)-r(A)}{2}\right].$$

II. We want to construct subspaces $A, A_1, \ldots, A_k$ satisfying the conditions in the theorem. We use induction on $v(\mathscr{H})=v$. If $\mathscr{H}$ is primitive then the result is immediate by Theorem 1. So we may suppose that $\mathscr{H}$ is not primitive, i.e. there exists a point p contained in the span of each basis. Delete the lines containing p from $\mathscr{H}$ to obtain the system $\mathscr{H}_1$. Project the lines of $\mathscr{H}_1$ from p onto a hyperplane T. Let e' be the projection of e on T, and $\mathscr{H}_1'=\{e'\colon e\in\mathscr{H}_1\}$.

The system $\mathscr{H}_1'$ contains no v independent lines. In fact, if v lines from $\mathscr{H}_1$ are not independent, then neither are the corresponding lines in $\mathscr{H}_1'$; if v lines from $\mathscr{H}_1$ form a basis then p is contained in their span and hence the rank of their span decreases by the projection from p.

So by the induction hypothesis, there exist flats $D, D_1, \ldots, D_k$ in T such that $D\subseteqq D_i$ $(i=1, \ldots, k)$; for each $e\in\mathscr{H}_1'$ the line e either meets D or is contained in some D_i; and

$$r(D)+\sum_{i=1}^{k}\left[\frac{r(D_i)-r(D)}{2}\right]=v-1.$$

Consider now the subspaces $A=D+p$, $A_i=D_i+p$. Obviously, $A\subseteqq A_i$. Furthermore, the lines in $\mathscr{H}-\mathscr{H}_1$ meet A, and so do all lines e for which e' meets D. If $e'\subseteqq D_i$ then $e\subseteqq A_i$. Finally, $r(A)=r(D)+1$, $r(A_i)=r(D_i)+1$ and hence

$$r(A)+\sum_{i=1}^{k}\left[\frac{r(A_i)-r(A)}{2}\right]=r(D)+1+\sum_{i=1}^{k}\left[\frac{r(D_i)-r(D)}{2}\right]=v.$$

Q.E.D.

5. Connections with matroid theory. The first question which comes up is whether or not Theorem 2 remains valid in an arbitrary matroid. First of all, the definition of independence of lines has to be done more carefully; let us accept the natural solution that a set $\mathscr{F}$ of lines is independent if $r(\mathscr{F})=2|\mathscr{F}|$. In this case the problem is equivalent to the so-called matroid parity problem (see LAWLER [3], Chapter 9).

A counterexample to the analogue of Theorem 2 is any affine space, where $\mathscr{H}$ consists of all lines parallel to a given one. Of course, if we extend our affine space to a projective space then we could choose $k=0$, A the common ideal point of our lines. But in general, there seems to be no hope to extend the original matroid so as to achieve the validity of Theorem 2. The possiblity of "simulating" the flat A inside the matroid seems to be a difficult, and probably not only technical, question.

It is clear that independence of lines does not define, in general, a matroid. See e.g. two disjoint lines and a third one meeting both. There is a class of systems of lines, however, for which the situation is different. Let us call a set $\mathscr{H}$ of lines *flexible*, if $r(e\cap\overline{\mathscr{H}-e})\leqq 1$ for each line $e\in\mathscr{H}$. For each $e\in\mathscr{H}$, let $p(e)$ be the intersection of e with $\overline{\mathscr{H}-e}$, if this exists, and an arbitrary point of e otherwise.

The next proposition shows that independence of lines in a flexible set defines a matroid:

Proposition 1. *Let $\mathscr{H}$ be a flexible set of lines. Then $\mathscr{F} \subseteq \mathscr{H}$ is independent iff the set $\mathscr{F}' = \{p(e): e \in \mathscr{F}\}$ of points is independent.*

Proof. It is trivial that if $\mathscr{F}$ is independent then so is $\mathscr{F}'$. Assume now that $\mathscr{F}'$ is independent. Then we prove by induction on $|\mathscr{G}|$ that if $\mathscr{G} \subseteq \mathscr{F}$ then

$$r(\mathscr{F}' \cup \mathscr{G}) = |\mathscr{F}'| + |\mathscr{G}|. \tag{4}$$

For $\mathscr{G} = \mathscr{F}$ this will mean that $\mathscr{F}$ is independent.

(4) is trivially true for $\mathscr{G} = \emptyset$. Let $\mathscr{G} \neq \emptyset$ and $e \in \mathscr{G}$. Then

$$r(\mathscr{F}' \cup \mathscr{G}) = r\big(\mathscr{F}' \cup (\mathscr{G} - e)\big) + 1,$$

since $\mathscr{H}$ being flexible, e intersects $\overline{\mathscr{F}' \cup (\mathscr{G} - e)}$ in precisely one point. This proves (4) by induction.

Q.E.D.

Finally, let us point out one more matroid which is induced by a set of lines. This is a certain analogue of the *matching matroid* of graphs by EDMONDS and FULKERSON [1]. Let $\mathscr{H}$ be a set of lines. Call a subset $\mathscr{G} \subseteq \mathscr{H}$ *dispersive*, if there exists a basis $\mathscr{B}$ of $\mathscr{H}$ such that $r(\mathscr{B} \cup \mathscr{G}) = 2v(\mathscr{H}) + |\mathscr{G}|$.

Proposition 2. *Dispersive sets form the independent sets of a matroid.*

This proposition generalizes Lemma 8, and can be proved along the same lines. Details are omitted.

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BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

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On the divergence of multiple orthogonal series

F. MÓRICZ and K. TANDORI

1. Preliminaries. Let $I^d=\times_{j=1}^{d}[0,1]$ be the unit cube in the d-dimensional Euclidean space, where $d\geqq 1$ is a fixed integer. The points $(x_1, \ldots, x_d), (y_1, \ldots, y_d), \ldots$ of I^d are denoted by the corresponding bold letters $\mathbf{x}, \mathbf{y}, \ldots$. Let Z_+^d be the set of d-tuples $\mathbf{k}=(k_1, \ldots, k_d)$ with positive integral coordinates, the tuple $(1, \ldots, 1)$ is denoted by $\mathbf{1}$. Z_+^d is partially ordered by agreeing that $\mathbf{k}\leqq\mathbf{m}$ iff $k_j\leqq m_j$ for each j. Finally, we write

$$\mathbf{k}_* = \min_{1\leqq j\leqq d} k_j \quad \text{and} \quad \mathbf{k}^* = \max_{1\leqq j\leqq d} k_j.$$

Let $\{\varphi_{\mathbf{k}}(\mathbf{x})\colon \mathbf{k}\in Z_+^d\}$ be a *d-dimensional orthonormal system* on I^d, i.e. for every $\mathbf{k}$ and $\mathbf{m}$ in Z_+^d let

$$\int_{I^d} \varphi_{\mathbf{k}}(\mathbf{x})\varphi_{\mathbf{m}}(\mathbf{x})\,d\mathbf{x} = \delta_{\mathbf{km}} \qquad (d\mathbf{x} = dx_1 \ldots dx_d).$$

In particular, if for each $j=1, 2, \ldots, d$ the system $\{\varphi_k^{(j)}(x)\}_{k=1}^{\infty}$ is orthonormal on $I=[0, 1]$, then the functions

$$\varphi_{k_1,\ldots,k_d}(x_1, \ldots, x_d) = \prod_{j=1}^{d} \varphi_{k_j}^{(j)}(x_j)$$

are orthonormal on I^d.

We shall consider the d-multiple orthogonal series

$$\sum_{\mathbf{k}\geqq\mathbf{1}} a_{\mathbf{k}}\varphi_{\mathbf{k}}(\mathbf{x}) = \sum_{k_1=1}^{\infty}\cdots\sum_{k_d=1}^{\infty} a_{k_1,\ldots,k_d}\varphi_{k_1,\ldots,k_d}(x_1, \ldots, x_d), \tag{1}$$

where $\{a_{\mathbf{k}}\colon \mathbf{k}\in Z_+^d\}$ is a given system of numbers (coefficients). For any $\mathbf{m}\in Z_+^d$ set

$$S_{\mathbf{m}}(\mathbf{x}) = \sum_{\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{m}} a_{\mathbf{k}}\varphi_{\mathbf{k}}(\mathbf{x}) =$$

$$= \sum_{k_1=1}^{m_1}\cdots\sum_{k_d=1}^{m_d} a_{k_1,\ldots,k_d}\varphi_{k_1,\ldots,k_d}(x_1, \ldots, x_d),$$

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which is a *rectangular partial sum* of (1). In case $m_1 = \ldots = m_d$, $S_{\mathbf{m}}(\mathbf{x})$ is called a *square partial sum* of (1). The *spherical partial sums* of (1) are defined as

$$S_r(\mathbf{x}) = \sum_{k_1^2 + \ldots + k_d^2 \leqq r} a_{\mathbf{k}} \varphi_{\mathbf{k}}(\mathbf{x}) \qquad (r = d, d+1, \ldots).$$

The following Theorem A has been published by a few authors, while Theorems B and C were proved by the first author in [3] and [4].

Theorem A. *If*

$$\sum_{\mathbf{k} \geqq 1} a_{\mathbf{k}}^2 \prod_{j=1}^{d} (\log 2k_j)^2 < \infty,$$

then the rectangular partial sums $S_{\mathbf{m}}(\mathbf{x})$ *of* (1) *converge a.e. on* I^d *as* $\mathbf{m}_* \to \infty$.

Here and in the sequel log is of base 2.

Theorem B. *If*

$$\sum_{\mathbf{k} \geqq 1} a_{\mathbf{k}}^2 (\log \mathbf{k}^*)^2 < \infty,$$

then both the square partial sums $S_{n,\ldots,n}(\mathbf{x})$ *and the spherical partial sums* $S_n(\mathbf{x})$ *of* (1) *converge a.e. on* I^d *as* $n \to \infty$.

The part concerning the spherical partial sums was stated in [3] in a slightly different form, but the two statements are equivalent, because

$$(\mathbf{k}^*)^2 \leqq k_1^2 + \ldots + k_d^2 \leqq d(\mathbf{k}^*)^2.$$

Theorem C. *If*

$$\sum_{\mathbf{k} \geqq 1} a_{\mathbf{k}}^2 < \infty,$$

then

$$S_{\mathbf{m}}(\mathbf{x}) = o_{\mathbf{x}}\left(\prod_{j=1}^{d} \log 2m_j\right) \quad \textit{a.e. on } I^d \textit{ as } \quad \mathbf{m}^* \to \infty.$$

2. Results. In this paper we are going to show that these theorems are exact in the sense that $\log n$ cannot be replaced by any sequence $\varrho(n)$ tending to ∞ slower than $\log n$ as $n \to \infty$. More precisely, let $\{\varrho(n)\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive numbers for which

$$\varrho(n) = o(\log n) \quad (n \to \infty). \tag{2}$$

Theorem 1. *For every* $d \geqq 1$ *and* $\{\varrho(n)\}$ *satisfying* (2), *there exist an orthonormal system* $\{\varphi_{\mathbf{k}}(\mathbf{x})\colon \mathbf{k} \in Z_+^d\}$ *and a system* $\{a_{\mathbf{k}}\colon \mathbf{k} \in Z_+^d\}$ *of coefficients such that*

$$\sum_{\mathbf{k} \geqq 1} a_{\mathbf{k}}^2 (\log \mathbf{k}^*)^{2d-2} \varrho^2(\mathbf{k}^*) < \infty \tag{3}$$

and

(4) $$\limsup_{\mathbf{m}_*\to\infty} |S_{\mathbf{m}}(\mathbf{x})| = \infty \quad a.e. \text{ on } I^d.$$

By virtue of Theorem B in case $d \geqq 2$ both the square partial sums and the spherical partial sums of the series $\sum_{\mathbf{k}\geqq 1} a_{\mathbf{k}}\varphi_{\mathbf{k}}(\mathbf{x})$ occurring in Theorem 1 converge a.e.

Theorem 2. *For every* $d \geqq 1$ *and* $\{\varrho(n)\}$ *satisfying* (2), *there exist an orthonormal system* $\{\varphi_{\mathbf{k}}(\mathbf{x})\colon \mathbf{k}\in Z_+^d\}$ *and a system* $\{b_{\mathbf{k}}\colon \mathbf{k}\in Z_+^d\}$ *of coefficients such that*

$$\sum_{\mathbf{k}\geqq 1} b_{\mathbf{k}}^2 < \infty \quad \textit{and} \quad \limsup_{\mathbf{m}_*\to\infty} \frac{|S_{\mathbf{m}}(\mathbf{x})|}{(\log \mathbf{m}^*)^{d-1}\varrho(\mathbf{m}^*)} = \infty \quad a.e. \text{ on } I^d.$$

Theorems 1 and 2 for $d=1$ are well-known (see, e.g. [1, pp. 99—100]).

Theorem 3. *For every* $d \geqq 2$ *and* $\{\varrho(n)\}$ *satisfying* (2), *there exist an orthonormal system* $\{\varphi_{\mathbf{k}}(\mathbf{x})\colon \mathbf{k}\in Z_+^d\}$ *and a system* $\{c_{\mathbf{k}}\colon \mathbf{k}\in Z_+^d\}$ *of coefficients such that*

$$\sum_{\mathbf{k}\geqq 1} c_{\mathbf{k}}^2 (\log \mathbf{k}^*)^2 < \infty$$

and

$$\limsup_{\mathbf{m}_*\to\infty} \frac{|S_{\mathbf{m}}(\mathbf{x})|}{(\log \mathbf{m}^*)^{d-2}\varrho(\mathbf{m}^*)} = \infty \quad a.e. \text{ on } I^d.$$

Again by Theorem B, both the square partial sums and the spherical partial sums of the series $\sum_{\mathbf{k}\geqq 1} c_{\mathbf{k}}\varphi_{\mathbf{k}}(\mathbf{x})$ converge a.e.

Our last theorem states the a.e. divergence of the rectangular partial sums of series (1) for a whole class of coefficient systems. A system $\{a_{\mathbf{k}}\colon \mathbf{k}\in Z_+^d\}$ of coefficients is said to be *non-increasing in absolute value* if for every $\mathbf{k}, \mathbf{m}\in Z_+^d$,

$$\mathbf{k} \leqq \mathbf{m} \Rightarrow |a_{\mathbf{k}}| \geqq |a_{\mathbf{m}}|.$$

It is clear that this is equivalent to the following: for every $\mathbf{k}\in Z_+^d$ and j, $1\leqq j\leqq d$, we have

$$|a_{k_1,\dots,k_{j-1},k_j,k_{j+1},\dots,k_d}| \geqq |a_{k_1,\dots,k_{j-1},k_j+1,k_{j+1},\dots,k_d}|.$$

Theorem 4. *For every system* $\{a_{\mathbf{k}}\colon \mathbf{k}\in Z_+^d\}$ *of coefficients, which is non-increasing in absolute value and satisfies the relation*

(5) $$\sum_{\mathbf{k}\geqq 1} a_{\mathbf{k}}^2 \prod_{j=1}^{d} (\log 2k_j)^2 = \infty,$$

there exists an orthonormal system $\{\varphi_{\mathbf{k}}(\mathbf{x})\colon \mathbf{k}\in Z_+^d\}$ *such that*

(6) $$\limsup_{\mathbf{m}^*\to\infty} |S_{\mathbf{m}}(\mathbf{x})| = \infty \quad a.e. \text{ on } I^d.$$

If, in addition, for every $\mathbf{m}\in Z_+^d$ *we have*

$$\sum_{\mathbf{k}\geqq\mathbf{m}} a_{\mathbf{k}}^2 \prod_{j=1}^{d} (\log 2k_j)^2 = \infty, \tag{7}$$

then $\mathbf{m}^*\to\infty$ *can be replaced by* $\mathbf{m}_*\to\infty$ *in* (6).

The two parts of Theorem 4 coincide for $d=1$. In this case Theorem 4 was proved by the second author in [5].

3. Notations and an auxiliary result. For the sake of simplicity in notations, we present the proofs only for the case $d=2$. We write (x, y) instead of $\mathbf{x}=(x_1, x_2)$ and (k, l) instead of $\mathbf{k}=(k_1, k_2)$.

We agree that $\langle a, b\rangle$ means either the open interval (a, b), or one of the half-closed intervals $[a, b)$ and $(a, b]$, or the closed interval $[a, b]$. Given a function $f(x, y)$ defined on I^2 and a rectangle $R=\langle a, b\rangle\times\langle c, d\rangle\subseteqq I^2$, let us put

$$f(R; x, y) = \begin{cases} f\left(\dfrac{x-a}{b-a}, \dfrac{y-c}{d-c}\right) & \text{if} \quad (x, y)\in R, \\ 0 & \text{otherwise.} \end{cases}$$

Given a set $H\subseteqq I^2$, let $H(R)$ denote the set into which H is carried over by the linear transformation $\bar{x}=(b-a)x+a$ and $\bar{y}=(d-c)y+c$.

A set $H\subseteqq I$ (or $\subseteqq I^2$) is said to be *simple* if H can be represented as the union of finitely many disjoint intervals (or rectangles).

The proofs of all our theorems are based on the following basic result of Menšov [2].

Lemma. *For every positive integer n there exist a system* $\{\psi_k^{(n)}(x)\}_{k=1}^n$ *of step functions, orthonormal on the interval* $I=[0, 1]$, *and a simple set* $E^{(n)}$ *of I such that*

$$\operatorname{mes} E^{(n)} \geqq C_1, \tag{8}$$

and for every $x\in E^{(n)}$ *there exists an integer* $\varkappa(x)$, $1\leqq\varkappa(x)\leqq n$, *such that* $\psi_1^{(n)}(x)\geqq\geqq 0, \ldots, \psi_{\varkappa(x)}^{(n)}(x)\geqq 0$ *and*

$$\sum_{k=1}^{\varkappa(x)} \psi_k^{(n)}(x) \geqq C_2\sqrt{n}\log 2n. \tag{9}$$

Here C_1 and C_2 denote positive constants. Further, if $E\subseteqq I$ (or $\subseteqq I^2$), then $\operatorname{mes} E$ denotes the Lebesgue measure of the set E on the line (or on the plane).

4. Proof of Theorem 4. *Part* 1. By (5) and the non-increasing property of $\{a_{kl}^2\}_{k,l=1}^\infty$ we have

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} 2^{p+q}(p+1)^2(q+1)^2 a_{2^{p+1}-1,2^{q+1}-1}^2 = \infty.$$

With the notation

$$A_r = \sum_{\substack{0 \leqq p, q \leqq r \\ \max(p,q)=r}} 2^{p+q}(p+1)^2(q+1)^2 a^2_{2^{p+1}-1, 2^{q+1}-1} \quad (r = 0, 1, \ldots)$$

this can be rewritten into the form $\sum_{r=0}^{\infty} A_r = \infty$. We can find a sequence $\{s_r\}_{r=0}^{\infty}$ of positive numbers with the following properties:

$$\lim_{r\to\infty} s_r = 0, \quad s_r^2 A_r \leqq 1 \quad (r = 0, 1, \ldots)$$

and

$$\sum_{r=0}^{\infty} s_r^2 A_r = \infty. \tag{10}$$

Without loss of generality we may assume that $a_{kl} \geqq 0$ for every $k, l = 1, 2, \ldots$.

Our goal is to construct a system $\{\varphi_{kl}(x, y)\}_{k,l=1}^{\infty}$ of step functions and a system $\{H_{pq}\}_{p,q=0}^{\infty}$ of simple sets of I^2 such that these functions be orthonormal on I^2, these sets be stochastically independent with

$$\text{mes}\, H_{pq} \geqq C_1^2 2^{p+q}(p+1)^2(q+1)^2 s_r^2 a^2_{2^{p+1}-1, 2^{q+1}-1} \qquad (p, q = 0, 1, \ldots), \tag{11}$$

and for every $(x, y) \in H_{pq}$

$$\max_{2^p \leqq m < 2^{p+1}} \max_{2^q \leqq n < 2^{q+1}} \left| \sum_{k=2^p}^{m} \sum_{l=2^q}^{n} a_{kl}\varphi_{kl}(x, y) \right| \geqq \frac{C_2^2}{s_r}, \tag{12}$$

where $r = \max(p, q)$.

The construction will be done by induction on r. If $r=0$, then let $\varphi_{11}(x, y) = 1/s_0 a_{11}$ on a rectangle H_{00} of area $s_0^2 a_{11}^2$ and let $\varphi_{11}(x, y) = 0$ otherwise. Then (11) and (12) are satisfied for $p=q=0$ provided $C_1, C_2 \leqq 1$, which is the case.

Now let r_0 be a positive integer and assume that the step functions $\varphi_{kl}(x, y)$ are defined for $k, l = 1, 2, \ldots, 2^{r_0}-1$ and the simple sets H_{pq} are defined for $p, q = 0, 1, \ldots, r_0-1$ such that these functions are orthonormal on I^2, these sets are stochastically independent, and the relations (11) and (12) are satisfied for $p, q = 0, 1, \ldots, r_0-1$. We are going to define the step functions $\varphi_{kl}(x, y)$ of the r_0th block successively in the following arrangement: for

$k = 2^{r_0}, 2^{r_0}+1, \ldots, 2^{r_0+1}-1$ and $l = 1$;

$k = 2^{r_0}, 2^{r_0}+1, \ldots, 2^{r_0+1}-1$ and $l = 2, 3$;

....

$k = 2^{r_0}, 2^{r_0}+1, \ldots, 2^{r_0+1}-1$ and $l = 2^{r_0}, 2^{r_0}+1, \ldots, 2^{r_0+1}-1$;

$k = 2^{r_0-1}, 2^{r_0-1}+1, \ldots, 2^{r_0}-1$ and $l = 2^{r_0}, 2^{r_0}+1, \ldots, 2^{r_0+1}-1$;

....

$k = 2, 3$ and $l = 2^{r_0}, 2^{r_0}+1, \ldots, 2^{r_0+1}-1$;

$k = 1$ and $l = 2^{r_0}, 2^{r_0}+1, \ldots, 2^{r_0+1}-1$;

and the simple sets $H_{r_0,0}, H_{r_0,1}, \ldots, H_{r_0,r_0}, H_{r_0-1,r_0}, \ldots, H_{1,r_0}, H_{0,r_0}$ in such a way that the functions $\varphi_{kl}(x, y)$ $(k, l=1, 2, \ldots, 2^{r_0+1}-1)$ be orthonormal on I^2, the sets H_{pq} $(p, q=0, 1, \ldots, r_0)$ be stochastically independent, and the relations (11) and (12) be satisfied for $p, q=0, 1, \ldots, r_0$.

For the sake of definiteness, let us assume that the sets $H_{r_0,0}, H_{r_0,1}, \ldots, H_{r_0,q_0-1}$ $(1\leqq q_0\leqq r_0)$ and the functions $\varphi_{kl}(x, y)$ for $k=2^{r_0}, 2^{r_0}+1, \ldots, 2^{r_0+1}-1$ and $l=1, 2, \ldots$ $\ldots, 2^{q_0}-1$ have been appropriately defined. Let us apply Menšov's lemma firstly with $n=2^{r_0}$, secondly with $n=2^{q_0}$, and set

$$\bar{\varphi}_{2^{r_0}+k-1,2^{q_0}+l-1}(x, y) = \psi_k^{(2^{r_0})}(x)\psi_l^{(2^{q_0})}(y) \quad (k = 1, 2, \ldots, 2^{r_0};\ l = 1, 2, \ldots, 2^{q_0}).$$

Then by (9) for every $(x, y)\in F=E^{(2^{r_0})}\times E^{(2^{q_0})}$ we have

$$\max_{1\leqq m\leqq 2^{r_0}}\max_{1\leqq n\leqq 2^{q_0}}\left|\sum_{k=1}^{m}\sum_{l=1}^{n} a_{2^{r_0}+k-1,2^{q_0}+l-1}\bar{\varphi}_{2^{r_0}+k-1,2^{q_0}+l-1}(x, y)\right| \geqq$$

$$\geqq C_2^2\sqrt{2^{r_0+q_0}}(r_0+1)(q_0+1)a_{2^{r_0+1}-1,2^{q_0+1}-1}.$$

Let Q be an arbitrary rectangle in I^2 with

$$\operatorname{mes} Q = 2^{r_0+q_0}(r_0+1)^2(q_0+1)^2 s_{r_0}^2 a_{2^{r_0+1}-1,2^{q_0+1}-1}^2$$

(the quantity on the right is not greater than 1 because of the choice of s_{r_0}), and let us "contract" the functions $\bar{\varphi}$ from I^2 to Q:

$$\bar{\bar{\varphi}}_{2^{r_0}+k-1,2^{q_0}+l-1}(x, y) = \frac{\bar{\varphi}_{2^{r_0}+k-1,2^{q_0}+l-1}(Q;\ x, y)}{\sqrt{2^{r_0+q_0}}(r_0+1)(q_0+1)s_{r_0}a_{2^{r_0+1}-1,2^{q_0+1}-1}}$$

$$(k=1, 2, \ldots, 2^{r_0};\ l = 1, 2, \ldots, 2^{q_0}).$$

It is not hard to check that these step functions are also orthonormal on I^2, by (8)

$$\operatorname{mes} F(Q) = \operatorname{mes} F \operatorname{mes} Q = (\operatorname{mes} E^{(2^{r_0})})^2 \operatorname{mes} Q \geqq \tag{13}$$

$$\geqq C_1^2 2^{r_0+q_0}(r_0+1)^2(q_0+1)^2 s_{r_0}^2 a_{2^{r_0+1}-1,2^{q_0+1}-1}^2,$$

and for every $(x, y)\in F(Q)$

$$\max_{1\leqq m\leqq 2^{r_0}}\max_{1\leqq n\leqq 2^{q_0}}\left|\sum_{k=1}^{m}\sum_{l=1}^{n} a_{2^{r_0}+k-1,2^{q_0}+l-1}\bar{\bar{\varphi}}_{2^{r_0}+k-1,2^{q_0}+l-1}(x, y)\right| \geqq \frac{C_2^2}{s_{r_0}}. \tag{14}$$

Since the functions $\varphi_{kl}(x, y)$ and the sets H_{pq} defined so far are step functions and simple sets, respectively, we can divide I^2 into a finite number of disjoint rectangles $R_1, R_2, \ldots, R_\sigma$ such that each function $\varphi_{kl}(x, y)$ $(k, l=1, 2, \ldots, 2^{r_0}-1;$ $k=2^{r_0}, 2^{r_0}+1, \ldots, 2^{r_0+1}-1$ and $l=1, 2, \ldots, 2^{q_0}-1)$ is constant on each R_s

$(s=1, 2, \ldots, \sigma)$ and each set H_{pq} $(p, q=0, 1, \ldots, r_0-1; p=r_0$ and $q=0, 1, \ldots, q_0-1)$ is the union of certain R_s. Let R_s' and R_s'' denote the two halves of R_s, i.e., if $R_s= =\langle a, b\rangle\times\langle c, d\rangle$, then let $R_s'=\langle a, (a+b)/2]\times\langle c, d\rangle$ and $R_s''=((a+b)/2, b\rangle\times\langle c, d\rangle$. We set

$$\varphi_{2^{r_0}+k-1, 2^{q_0}+l-1}(x, y) = \sum_{s=1}^{\sigma} \bar{\bar{\varphi}}_{2^{r_0}+k-1, 2^{q_0}+l-1}(R_s'; x, y) -$$

$$- \sum_{s=1}^{\sigma} \bar{\bar{\varphi}}_{2^{r_0}+k-1, 2^{q_0}+l-1}(R_s''; x, y) \quad (k = 1, 2, \ldots, 2^{r_0};\ l = 1, 2, \ldots, 2^{q_0})$$

and

$$H_{r_0, q_0} = \left(\bigcup_{s=1}^{\sigma} G(R_s')\right) \cup \left(\bigcup_{s=1}^{\sigma} G(R_s'')\right),$$

where $G=F(Q)$.

It is easy to verify that the step functions $\varphi_{kl}(x, y)$ $(k, l = 1, 2, \ldots, 2^{r_0}-1;$ $k = 2^{r_0}, 2^{r_0}+1, \ldots, 2^{r_0+1}-1$ and $l = 1, 2, \ldots, 2^{q_0+1}-1)$ form an orthonormal system on I^2, the simple sets H_{pq} $(p, q=0, 1, \ldots, r_0-1; p=r_0$ and $q=0, 1, \ldots, q_0)$ are stochastically independent, by (13)

$$\operatorname{mes} H_{r_0, q_0} = \operatorname{mes} F(Q) \geqq C_1^2 2^{r_0+q_0}(r_0+1)^2(q_0+1)^2 s_{r_0}^2\, a^2_{2^{r_0+1}-1, 2^{q_0+1}-1},$$

and by (14) for every $(x, y)\in H_{r_0, q_0}$

$$\max_{1\leqq m\leqq 2^{r_0}} \max_{1\leqq n\leqq 2^{q_0}} \left| \sum_{k=1}^{m} \sum_{l=1}^{n} a_{2^{r_0}+k-1, 2^{q_0}+l-1}\varphi_{2^{r_0}+k-1, 2^{q_0}+l-1}(x, y)\right| \geqq \frac{C_2^2}{s_{r_0}}.$$

The above induction scheme shows that the orthonormal system $\{\varphi_{kl}(x, y)\}_{k, l=1}^{\infty}$ and the system $\{H_{pq}\}_{p, q=1}^{\infty}$ of stochastically independent sets can be defined so that the conditions (11) and (12) hold true.

Putting (10) and (11) together we see that

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \operatorname{mes} H_{pq} = \infty. \tag{15}$$

Thus the second Borel—Cantelli lemma implies that almost every $(x, y)\in I^2$ belongs to an infinite number of sets H_{pq}. Taking into account (12) this means that for almost every (x, y) there exist four sequences $\{m_i\}$, $\{M_i\}$, $\{n_i\}$ and $\{N_i\}$ of positive integers such that $m_i\leqq M_i$ and $n_i\leqq N_i$ $(i=1, 2, \ldots)$, $\max(m_i, n_i)\to\infty$ as $i\to\infty$, and

$$\lim_{i\to\infty} \left| \sum_{k=m_i}^{M_i} \sum_{l=n_i}^{N_i} a_{kl}\varphi_{kl}(x, y)\right| = \infty.$$

Since the double sum in absolute value is equal to

$$S_{M_i, N_i}(x, y) - S_{M_i, n_i-1}(x, y) - S_{m_i-1, N_i}(x, y) + S_{m_i-1, n_i-1}(x, y),$$

(6) follows.

Part 2. Now suppose that (7) is also satisfied, i.e. for every m and n we have

$$\sum_{k=m}^{\infty}\sum_{l=n}^{\infty} a_{kl}^2 (\log 2k)^2 (\log 2l)^2 = \infty.$$

Then, using the non-increasing property of $\{a_{kl}^2\}$, for every r we have

$$\sum_{p=r}^{\infty}\sum_{q=r}^{\infty} 2^{p+q}(p+1)^2(q+1)^2 a_{2^{p+1}-1,2^{q+1}-1}^2 = \infty.$$

This makes it possible to define a sequence $0=r_0<r_1<r_2<\dots$ of integers such that

$$A_i' = \sum_{p=r_i+1}^{r_{i+1}}\sum_{q=r_i+1}^{r_{i+1}} 2^{p+q}(p+1)^2(q+1)^2 a_{2^{p+1}-1,2^{q+1}-1}^2 \geqq 1 \quad (i=0,1,\dots).$$

Finally, let $\{s_i'\}_{i=0}^{\infty}$ be a sequence of positive numbers with the properties

$$\lim_{i\to\infty} s_i' = 0, \quad (s_i')^2 A_i' \leqq 1 \quad (i=0,1,\dots)$$

and

$$\sum_{i=0}^{\infty} (s_i')^2 A_i' = \infty. \tag{16}$$

After this modification we have only to repeat the construction of Part 1. Relations (11) and (16) imply that

$$\sum_{i=0}^{\infty}\sum_{p=r_i+1}^{r_{i+1}}\sum_{q=r_i+1}^{r_{i+1}} \operatorname{mes} H_{pq} = \infty,$$

which is stronger than (15). The second Borel—Cantelli lemma yields that almost every $(x,y)\in I^2$ now belongs to an infinite number of sets H_{pq} with $r_i<p, q\leqq r_{i+1}$, $i=0,1,\dots$. This already ensures that $\mathbf{m}^*\to\infty$ can be replaced by $\mathbf{m}_*\to\infty$ in (6).

The proof of Theorem 4 is complete.

5. Proofs of Theorems 1—3 run along the same lines as that of Theorem 4 with the exception that now there is no need of a "contraction" of the functions $\bar\varphi$. In particular, at present

$$\varphi_{2^r+k-1,2^r+l-1}(x,y) = \sum_{s=1}^{\sigma} \bar\varphi_{2^r+k-1,2^r+l-1}(R_s'; x,y) - \sum_{s=1}^{\sigma} \bar\varphi_{2^r+k-1,2^r+l-1}(R_s''; x,y),$$

where

$$\bar\varphi_{2^r+k-1,2^r+l-1}(x,y) = \psi_k^{(2^r)}(x)\psi_l^{(2^r)}(y) \quad (k,l=1,2,\dots,2^r;\ r=0,1,\dots),$$

while the other $\varphi_{kl}(x,y)$ are indifferent from our point of view (of course, they have to be normal and orthogonal to each other). Further,

$$H_{rr} = \left(\bigcup_{s=1}^{\sigma} F(R_s')\right) \cup \left(\bigcup_{s=1}^{\sigma} F(R_s'')\right),$$

where $F=E^{(2^r)}\times E^{(2^r)}$. By (8)

$$\text{mes}\, H_{rr} = \text{mes}\, F = (\text{mes}\, E^{(2^r)})^2 \geqq C_1^2. \tag{17}$$

Let $\bar\varrho(n)=(\varrho(n)\log n)^{1/2}$. Then by (2)

$$\varrho(n) = o(\bar\varrho(n)) \quad \text{and} \quad \bar\varrho(n) = o(\log n) \quad (n\to\infty).$$

Hence there exists a sequence $\{n_j=2^{r_j}\}_{j=1}^{\infty}$ of integers such that $n_j\geqq 2n_{j-1}$ $(n_0=1)$,

$$\frac{\varrho(n)}{\bar\varrho(n)} \leqq \frac{1}{j} \quad \text{and} \quad \frac{\bar\varrho(n)}{\log n} \leqq \frac{1}{j} \quad \text{if} \quad n\geqq n_j \quad (j=1,2,\ldots). \tag{18}$$

The definition of the coefficients is the following: set for $k, l=1, 2, \ldots, n_j$; $j=1, 2, \ldots$

$$a_{n_j+k-1,n_j+l-1} = \frac{1}{n_j\bar\varrho(2n_j)\log 2n_j} \quad \text{(in Theorem 1)},$$

$$b_{n_j+k-1,n_j+l-1} = \frac{\bar\varrho(2n_j)}{n_j\log 2n_j} \quad \text{(in Theorem 2)},$$

$$c_{n_j+k-1,n_j+l-1} = \frac{\bar\varrho(2n_j)}{n_j(\log 2n_j)^2} \quad \text{(in Theorem 3)};$$

and $a_{kl}=b_{kl}=c_{kl}=0$ for $k, l=1, 2, \ldots, n_1-1$;

$$k = n_j, \ldots, 2n_j-1 \quad \text{and} \quad l = 1, 2, \ldots, n_j-1;$$

$$k = 2n_j, \ldots, n_{j+1}-1 \quad \text{and} \quad l = 1, 2, \ldots, n_{j+1}-1;$$

$$k = 1, 2, \ldots, n_j-1 \quad \text{and} \quad l = n_j, \ldots, 2n_j-1;$$

$$k = 1, 2, \ldots, 2n_j-1 \quad \text{and} \quad l = 2n_j, \ldots, n_{j+1}-1; \quad j = 1, 2, \ldots.$$

After this preparation it is quite easy to conclude the proofs. For example, let us carry out the proof of Theorem 1.

If $(x, y)\in H_{r_j,r_j}$ (recall $n_j=2^{r_j}$), then by (9) and (18)

$$\max_{n_j\leqq m,n<n_{j+1}} \left|\sum_{k=n_j+1}^{m}\sum_{l=n_j+1}^{n} a_{kl}\varphi_{kl}(x,y)\right| \geqq \tag{19}$$

$$\geqq \frac{C_2^2 n_j(\log 2n_j)^2}{n_j\bar\varrho(2n_j)\log 2n_j} \geqq C_2^2 j \quad (j=1,2,\ldots).$$

By virtue of the second Borel—Cantelli lemma (17) implies that

$$\text{mes}\,(\limsup_{j\to\infty} H_{r_j,r_j}) = 1.$$

Thus (19) provides (4).

Besides, by (18)

$$\sum_{n_j \leqq k, l < 2n_j} a_{kl}^2 (\log \max(k, l))^2 \varrho^2(\max(k, l)) \leqq$$

$$\leqq n_j^2 \frac{(\log 2n_j)^2 \varrho^2(2n_j)}{n_j^2 \bar{\varrho}^2(2n_j)(\log 2n_j)^2} \leqq \frac{1}{j^2} \quad (j = 1, 2, \ldots).$$

Since the remaining $a_{kl}=0$, hence (3) follows immediately. This finishes the proof of Theorem 1.

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BOLYAI INSTITUTE

UNIVERSITY OF SZEGED
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

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Characterization of Lebesgue-type decomposition of positive operators

K. NISHIO

1. Introduction

Our concerns in this paper are (bounded linear) *positive*, i.e. non-negative definite, operators on a Hilbert space. Order relation among operators always refers to this notion of positivity; that is, $B \geqq A$ means $B-A$ is positive. For convenience, a positive operator B is said to *dominate* another A if $\alpha B \geqq A$ for some $\alpha \geqq 0$.

Given a positive operator A, we say a positive operator C to be *A-absolutely continuous* if there exists a sequence $\{C_n\}$ of positive operators such that $C_n \uparrow C$ and $C_n \leqq \alpha_n A$ for some $\alpha_n \geqq 0$ $(n=1, 2, \ldots)$. Here $C_n \uparrow C$ means that $C_1 \leqq C_2 \leqq \ldots$ and C_n converges strongly to C. In [2] ANDO showed that for any positive operator B there is the maximum of all A-absolutely continuous positive operators C such that $C \leqq B$, and established an algorithm for obtaining the maximum, denoted by $[A]B$, in terms of parallel addition;

$$[A]B = \lim_{n\to\infty} (nA):B.$$

Here the *parallel sum* $A:B$ of two positive operators A, B was introduced by ANDERSON and TRAPP [1] in study of electrical network connection; $A:B$ is defined, in operator matrix notation, as the maximum of all positive operators C such that

$$\begin{pmatrix} A & A \\ A & A+B \end{pmatrix} \geqq \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}.$$

Meanwhile, PEKAREV and SMULJAN [6] introduced the notion of *complement* of a positive operator B with respect to a positive operator A. When B dominates A^2 the complement, denoted by B_A, exists and is defined as the minimum of all positive operators C such that

$$\begin{pmatrix} B & A \\ A & C \end{pmatrix} \geqq 0.$$

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They developed detailed analysis of the map $B \mapsto B_A$ as well as the map $B \mapsto (B_A)_A$ in connection with the reverse operation of parallel addition, that is, parallel subtraction.

Our first aim in this paper is to present algorithms for obtaining $[A]B$ in terms of complement operation.

There is still an important binary operation for positive operators A, B. It is the *geometric mean* $A \# B$ introduced by PUSZ and WORONOWICZ [7]; $A \# B$ is defined as the maximum of all positive operators C such that

$$\begin{pmatrix} A & C \\ C & B \end{pmatrix} \geqq 0.$$

Our second aim is to show that $[A]B$ coincides with each of $[A:B]B$, $[(A:B)^2]B$, $[A \# B]B$ and $[(A \# B)^2]B$. Coincidence of $[A:B]B$ and $[(A:B)^2]B$ as well as that of $[A \# B]B$ and $[(A \# B)^2]B$ is not a trivial thing. As a consequence the identities $A \# B = A \# [A]B$ and $A:B = A:[A]B$ will be established.

In the next section fundamental properties and lemmas of parallel sum, complement and geometric mean are established in the form convenient for our aim. The main results will be presented in the final section.

2. Parallel sum, complement and geometric mean

In this section all operators are positive unless otherwise mentioned.

The *parallel sum* $A:B$ of two operators A, B is defined as the maximum of all operators C satisfying

(1) $$\begin{pmatrix} A & A \\ A & A+B \end{pmatrix} \geqq \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}.$$

Explicit representation for $A:B$ is given by

$$((A:B)f, f) = \inf\{(Ah, h) + (Bg, g);\ f = h + g\}.$$

If A and B are invertible

(2) $$A:B = (A^{-1} + B^{-1})^{-1}.$$

The following properties of parallel addition can be derived easily from definition (e.g. [1], [5]).

(3) $$A:B = B:A \quad \text{and} \quad (A:B):C = A:(B:C).$$

(4) $$(CAC):(CBC) \geqq C(A:B)C.$$

A consequence of (4) is

(5) $$(CAC):(CBC) = C(A:B)C \quad \text{for invertible } C.$$

(6) $$A_n \downarrow A \quad \text{and} \quad B_n \downarrow B \quad \text{implies} \quad A_n:B_n \downarrow A:B.$$

As mentioned in the Introduction, given positive operators A, B the maximum of all positive A-absolutely continuous C with $C \leqq B$ exists and is determined by

(7) $$[A]B = \lim_{n\to\infty} (nA):B.$$

Moreover, it is known (see [2]) that $[A]B=B$ is equivalent to the condition

(8) the linear manifold $\{h;\ B^{\frac{1}{2}}h \in \operatorname{ran}(A^{\frac{1}{2}})\}$ is dense in the whole space.

The following properties of the operation $[A]$ can be derived easily from definition (e.g. [2]).

(9) $$[A]B \geqq [C]B \quad \text{if } A \text{ dominates } C.$$

(10) $$[A](B+C) \geqq [A]B+[A]C.$$

A consequence of (10) is

(11) $$[A]B \geqq [A]C \quad \text{if} \quad B \geqq C.$$

More delicate properties are summarized in the following lemma.

Lemma 1. *For any positive operator A the operation $[A]$ has the following properties;*

(i) $[A](B+\alpha A)=[A]B+\alpha A$ *for* $\alpha>0$ *and* $B\geqq 0$.

(ii) *If positive operators B and A^p dominate each other for some $p>0$, then* $[A]B=B$.

Proof. (i) follows from the identity

$$(\beta A):(B+\alpha A) = \left(\frac{\beta}{\alpha+\beta}\right)^2 (((\alpha+\beta)A):B)+\frac{\alpha\beta}{\alpha+\beta}A \quad \text{for} \quad \alpha, \beta > 0,$$

which is easily checked for invertible A, B and then for general A, B by (6) through approximation of A by $A+\varepsilon I$ and B by $B+\varepsilon I$.

(ii) Suppose B and A^p dominate each other. If $p\geqq 1$, A dominates A^p, hence B. This implies $[A]B=B$. Suppose $0<p<1$. Then there is X such that $B=A^{p/2}XA^{p/2}$, $\ker(X)=\ker(A^p)$ (see [4]) and the restriction $X|_{\operatorname{ran}(A)^-}$ is an invertible operator on $\operatorname{ran}(A)^-$, the closure of the range of A. Now by (7) and (4)

$$B \geqq [A]B \geqq nA:B \geqq A^{p/2}(nA^{1-p}:X)A^{p/2}.$$

Since $X|_{\mathrm{ran}(A)^-}$ is invertible and $\ker(X)=\ker(A)=\ker(A^{1-p})$, by virtue of condition (8) $[A^{1-p}]X$ must coincide with X itself. Therefore by (7)

$$B \geqq [A]B \geqq A^{p/2}([A^{1-p}]X)A^{p/2} = A^{p/2}XA^{p/2} = B.$$

This completes the proof.

Let A, B be positive operators. It is known (e.g. [4], [7]) that there is a positive operator C for which

$$\begin{pmatrix} B & A \\ A & C \end{pmatrix} \geqq 0$$

if and only if B dominates A^2. In this case, there is the minimum of all such C. According to Pekarev—Smuljan [6] this minimum is called the *complement* of B with respect to A and is denoted by B_A. More explicit representation of B_A is given by

(12) $$B_A = Z^*Z,$$

where Z is the uniquely determined (bounded linear) operator such that $A=B^{\frac{1}{2}}Z$ and $\ker(Z^*)\supset\ker(B)$ (e.g. [4], [7]). If B and A^2 dominate each other, the restriction $Z|_{\mathrm{ran}(A)^-}$ is an invertible operator on $\mathrm{ran}(A)^-$. In particular,

(13) $$B_A = AB^{-1}A \quad \text{for invertible } B.$$

The following properties of complement can be derived easily from definition (e.g. [7]).

(14) $$B_A \leqq C_A \quad \text{if} \quad B \geqq C \quad \text{and } C \text{ dominates } A^2.$$

(15) $$B_n \downarrow B \quad \text{implies} \quad (B_n)_A \uparrow B_A \quad \text{if } B \text{ dominates } A^2.$$

As a consequence of (15), B dominates A^2 if and only if $A(B+\varepsilon I)^{-1}A$ is bounded above for $\varepsilon>0$. In this case

(16) $$B_A = \lim_{\varepsilon\downarrow 0} A(B+\varepsilon I)^{-1}A.$$

A little more effort will show

(17) $$B_A = \text{weak-}\lim_{\varepsilon\downarrow 0}(A+\varepsilon I)(B+\varepsilon I)^{-1}(A+\varepsilon I).$$

The following property and calculation rules of complement can be derived easily by (16), (2) and (4) through approximation.

(18) $$[A^2](B_A) = B_A \quad \text{if } B \text{ dominates } A^2.$$

(19) $$(A:B)_C = A_C + B_C \quad \text{if both } A \text{ and } B \text{ dominate } C^2.$$

(20) $$(A+B)_C \leqq A_C : B_C \quad \text{if both } A \text{ and } B \text{ dominate } C^2.$$

Lemma 2. (Pekarev and Smuljan [6]) *Let A, B be positive operators. Then B dominates A if and only if B does $A+B$. In this case, the following identity holds*

$$B_{A+B} = B_A + 2A + B.$$

Proof. The first assertion is immediate from definition. The expected identity is true when B is invertible. In fact, by (13)

$$B_{A+B}=(A+B)B^{-1}(A+B)=AB^{-1}A+2A+B=B_A+2A+B.$$

The general case results by (17) through approximation. This completes the proof.

Lemma 3. *The following three conditions for positive operators A, B and C are mutually equivalent,*

(i) $A:B\geqq C$,

(ii) $A-C\geqq (B-C)_C$,

(iii) $A-C\geqq (A+B)_A$.

If equality holds anywhere in (i) *or* (ii) *or* (iii) *then equality holds everywhere. In particular, the following identity holds*

$$A:B=A-(A+B)_A.$$

Proof. By definition of parallel addition, (i) is equivalent to

$$\begin{pmatrix} A-C & A \\ A & A+B \end{pmatrix}\geqq 0,$$

which is equivalent to (iii) by definition of complement. On the other hand, the identity

$$\begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}\begin{pmatrix} A-C & A \\ A & A+B \end{pmatrix}\begin{pmatrix} I & -I \\ 0 & I \end{pmatrix}=\begin{pmatrix} A-C & C \\ C & B-C \end{pmatrix}$$

implies the equivalence of (ii) and (iii). This completes the proof.

Given positive operators A, B there is the maximum of all positive operator C such that

$$\begin{pmatrix} A & C \\ C & B \end{pmatrix}\geqq 0.$$

This maximum is called the *geometric mean* of A and B, and is denoted by $A\# B$. The following properties of geometric mean can be derived from definition (e.g. [3], [7]).

(21) $$A\# B=A^{\frac12}(A^{-\frac12}BA^{-\frac12})^{\frac12}A^{\frac12}\quad\text{for invertible } A.$$

For general A, the geometric mean $A\# B$ can be computed by approximation.

(22) $$A_n\downarrow A\quad\text{and}\quad B_n\downarrow B\quad\text{imply}\quad A_n\# B_n\downarrow A\# B.$$

(23) $$(CAC)\#(CBC)\geqq C(A\# B)C.$$

A consequence of (23) is

(24) $$(CAC)\#(CBC)=C(A\# B)C\quad\text{for invertible } C.$$

Lemma 4. *For any positive operators A, B the following indentity holds*

$$A_{(A\#B)}\#B_{(A\#B)}=A\#B.$$

Proof. Let $C=A\#B$. Clearly (see the second sentence after the proof of Lemma 1) A, B and A_C dominate C^2 and moreover $A\geqq B_C$, $B\geqq A_C$ hold. This implies $A\#B\geqq A_C\#B_C$. Reverse inequality follows immediately from the inequality

$$A_C\#B_C\geqq(A\#B)_C,$$

because $(A\#B)_{A\#B}=A\#B$. This inequality is surely guaranteed whenever both A and B dominate C^2. In fact (13), (14) and (23) will yield

$$\begin{aligned}A_C\#B_C&\geqq C(A+\varepsilon I)^{-1}C\#C(B+\varepsilon I)^{-1}C\\&\geqq C\big((A+\varepsilon I)^{-1}\#(B+\varepsilon I)^{-1}\big)C\\&=C\big((A+\varepsilon I)\#(B+\varepsilon I)\big)^{-1}C\quad\text{for}\quad\varepsilon>0,\end{aligned}$$

the last equality resulting from (21), in which by (22) and (15) the last term $C\big((A+\varepsilon I)\#(B+\varepsilon I)\big)^{-1}C$ converges increasingly to $(A\#B)_C$ on taking limit $\varepsilon\to0$. This completes the proof.

Relations between parallel sum and geometric mean are gathered in the following lemma.

Lemma 5. *The following relations hold for parallel sum and geometric mean.*

(i) $2^{-1}(A\#B)\geqq A:B\geqq\|A+B\|^{-1}(A\#B)^2$.

(ii) $(A+A\#B):(B+A\#B)=A\#B$.

Proof. By using approximation, A can be assumed to be invertible. Let $C=A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. Then by (5) and (24) the first inequality of (i) is equivalent to

$$2^{-1}(I\#C)\geqq I:C,$$

which is, in turn, equivalent to

$$2^{-1}C^{\frac{1}{2}}\geqq C(I+C)^{-1}$$

by (21) and Lemma 3. But the last inequality is surely guaranteed by arithmetic-geometric mean inequality for a positive operator. In the same way the second inequality of (i) is equivalent to

$$C(I+C)^{-1}\geqq\|A+B\|^{-1}C^{\frac{1}{2}}AC^{\frac{1}{2}},$$

hence to

$$(I+C)^{-1}\geqq\|A+B\|^{-1}A.$$

Since the inverse map converts order-relation, this last inequality is equivalent to

$$\|A+B\|\, I \geqq A^{\frac{1}{2}}(I+C)A^{\frac{1}{2}},$$

which is surely guaranteed. (ii) results from the identity

$$(I+C^{\frac{1}{2}}):(C+C^{\frac{1}{2}}) = C^{\frac{1}{2}},$$

which is guaranteed by simple computation. This completes the proof.

3. Theorems

Theorem 1. *For any positive operators A, B*

$$[A]B = \lim_{\varepsilon \downarrow 0} \big((B+\varepsilon A)_{A^{\frac{1}{2}}}\big)_{A^{\frac{1}{2}}}.$$

If B dominates A, then

$$[A]B = (B_{A^{\frac{1}{2}}})_{A^{\frac{1}{2}}}.$$

Proof. Suppose first that A and B dominate each other. Then by (12)

$$B_{A^{\frac{1}{2}}} = Z^*Z,$$

where Z is an operator such that $\ker(Z)=\ker(A^{\frac{1}{2}})$, $A^{\frac{1}{2}}=B^{\frac{1}{2}}Z$ and $Z|_{\operatorname{ran}(A)^-}$ is an invertible operator on $\operatorname{ran}(A)^-$. This implies that

$$V(B_{A^{\frac{1}{2}}})^{\frac{1}{2}} = Z,$$

where V is a unitary operator on $\operatorname{ran}(A)^-$, and hence

$$(B_{A^{\frac{1}{2}}})^{\frac{1}{2}} V^* B^{\frac{1}{2}} = Z^* B^{\frac{1}{2}} = A^{\frac{1}{2}}.$$

Again by (12) the last identity leads to

$$(B_{A^{\frac{1}{2}}})_{A^{\frac{1}{2}}} = B^{\frac{1}{2}} V V^* B^{\frac{1}{2}} = B = [A]B,$$

the last equality resulting from domination by A. Thus the assertion is true in case A and B dominate each other.

Suppose next that B dominates A. Then for each n the operator $(nA):B$ and A dominate each other, hence

$$(nA):B = [A]\big((nA):B\big) = \big((nA):B\big)_{A^{\frac{1}{2}}}\big)_{A^{\frac{1}{2}}} \leqq (B_{A^{\frac{1}{2}}})_{A^{\frac{1}{2}}},$$

which implies, by definition of $[A]B$,

$$[A]B \leqq (B_{A^{\frac{1}{2}}})_{A^{\frac{1}{2}}}.$$

Since $(B_{A^{\frac{1}{2}}})_{A^{\frac{1}{2}}}$ is A-absolute continuous by (18), and $[A]B$ is the maximum of all A-absolutely continuous C with $C \leqq B$, the reverse inequality holds too, proving the second assertion of the theorem.

To prove the first assertion, remark that for any positive B and $\varepsilon>0$, $B+\varepsilon A$ dominates A. Therefore by Lemma 1 (i) and the second assertion already proved

$$[A]B+\varepsilon A=[A](B+\varepsilon A)=((B+\varepsilon A)_{A^{\frac{1}{2}}})_{A^{\frac{1}{2}}},$$

which leads to the first assertion on taking limit $\varepsilon\to 0$. This completes the proof.

Theorem 2. *For any positive operators A, B*

$$[A]B=[A:B]B=[(A:B)^2]B=(A-A:B)_{A:B}+A:B.$$

Proof. Let $C=A:B$. Since $A, B\geqq C$ and C dominates C^2, by (9)

$$[A]B\geqq[C]B\geqq[C^2]B.$$

Further (10) (concavity) implies

$$[C^2]B\geqq[C^2](B-C)+[C^2]C=[C^2](B-C)+C,$$

the last equality resulting from Lemma 1 (ii). On the other hand, by Lemma 3

$$B-C\geqq(A-C)_C,$$

which together with (11) and C^2-absolute continuity of $(A-C)_C$ implies

$$[C^2](B-C)\geqq(A-C)_C.$$

Now it remains to prove the relation

$$(A-C)_C+C=[A]B.$$

To this end, for each n let $B_n=(nC):B$ and $C_n=A:B_n$. Since A and $A+B_n$ dominate each other, Lemma 1 (ii) with $p=\frac{1}{2}$ and with A^2 instead of A shows

$$[A^2](A+B_n)=A+B_n,$$

and hence by Theorem 1

$$((A+B_n)_A)_A=A+B_n.$$

On the other hand, since the relation by Lemma 3

$$C_n=A:B_n=A-(A+B_n)_A$$

yields

$$((A+B_n)_A)_A=(A-C_n)_A,$$

combination of these two relations leads to

$$A+B_n=(A-C_n)_A.$$

Further Lemma 2, with $A-C_n$ and C_n instead of B and A respectively, shows

$$(A-C_n)_A=(A-C_n)_{C_n}+C_n+A.$$

Therefore the following relation has been proved

$$B_n = (A - C_n)_{C_n} + C_n.$$

Since (3) (associativity and commutativity) implies

$$B_n = \frac{n}{n+1}\left(((n+1)A):B\right) \quad \text{and} \quad C_n = \frac{n}{n+1}C,$$

the established relation becomes

$$((n+1)A):B = \frac{n}{n+1}\left(A - \frac{n}{n+1}C\right)_C + C,$$

which leads to

$$[A]B = (A-C)_C + C$$

by (15) because $A - \frac{n}{n+1}C$ converges decreasingly to $A-C$. This completes the proof.

Corollary 3. *For any positive operators A, B*

$$A:B = A:[A]B.$$

Proof. Let $C=A:B$. Then definitely both A and B dominate C^2. By Theorem 2 it suffices to show $A:[C^2]B=C$. Calculation rules (19), (20) and Theorem 1 will yield

$$C = [C^2]C = ((A:B)_C)_C = (A_C+B_C)_C \leqq (A_C)_C:(B_C)_C \leqq A:[C^2]B \leqq A:B = C.$$

This completes the proof.

Theorem 4. *For any positive operators A, B*

$$[A]B = [A \# B]B = [(A \# B)^2]B = A_{A \# B}.$$

Proof. With $C=A \# B$ Lemma 5 (ii) shows

$$(A+C):(B+C) = C.$$

Then, in the proof of Theorem 2, replacement of A and B by $A+C$ and $B+C$, respectively yields

$$[C](B+C) \geqq [C^2]B + C \geqq A_C + C = [C](B+C),$$

which implies by Lemma 1 (i)

$$[C]B = [C^2]B = A_C.$$

On the other hand, Lemma 5 (i) together with (9) implies

$$[C]B \geqq [A:B]B \geqq [C^2]B,$$

which completes the proof, because $[A:B]B=[A]B$ by Theorem 2.

Corollary 5. *For any positive operators A, B*

$$A \# B = A \# [A]B.$$

Proof. Let $C = A \# B$. By Theorem 4 it suffices to prove $A \# [C^2]B = C$. Twice applications of Lemma 4 will show

$$C = A_C \# B_C = (A_C)_C \# (B_C)_C,$$

hence by Theorem 1

$$C = [C^2]A \# [C^2]B \leqq A \# [C^2]B \leqq A \# B = C.$$

Therefore $A \# [C^2]B = C$. This completes the proof.

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DEPARTMENT OF INFORMATION SCIENCE
FACULTY OF ENGINEERING
IBARAKI UNIVERSITY
HITACHI, IBARAKI 316
JAPAN

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Integrability of Rees—Stanojević sums

BABU RAM

1. A sequence $\langle a_n \rangle$ of positive numbers is called quasi-monotone if $n^{-\beta} a_n \downarrow 0$ for some β, or equivalently if $a_{n+1} \leqq a_n(1+\alpha/n)$.

We say that a sequence $\langle a_k \rangle$ of numbers satisfies

Condition S^* if $a_k \to 0$ as $k \to \infty$ and there exists a sequence $\langle A_k \rangle$ such that $\langle A_k \rangle$ is quasi-monotone, $\sum_{k=0}^{\infty} A_k < \infty$, and $|\Delta a_k| \leqq A_k$ for all k.

Condition S^* is weaker than Condition S of Sidon introduced in [4].

Recently, REES and STANOJEVIĆ [3] (see also GARRETT and STANOJEVIĆ [2]) introduced the modified cosine sums

$$g_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx \tag{1}$$

and obtained a necessary and sufficient condition for the integrability of the limit of these sums.

The object of this paper is to show that Condition S^* is sufficient for integrability of the limit of (1).

2. We require the following lemmas for the proofs of our results:

Lemma 1. (FOMIN [1]) *If* $|c_k| \leqq 1$, *then*

$$\int_0^{\pi} \left| \sum_{k=0}^{n} c_k \frac{\sin(k+1/2)x}{2 \sin x/2} \right| dx \leqq C(n+1),$$

where C is a positive absolute constant.

Lemma 2. (SZÁSZ [5]) *If $\langle a_n \rangle$ is quasi-monotone with $\sum a_n < \infty$, then $na_n \to 0$ as $n \to \infty$.*

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3. We prove

Theorem. *Let the sequence $\langle a_k \rangle$ satisfy Condition S^*. Then*

$$g(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{1}{2}\Delta a_k + \sum_{j=k}^{n} \Delta a_j \cos kx\right]$$

exists for $x \in (0, \pi]$ *and* $g(x) \in L(0, \pi)$.

Proof. We have

$$g_n(x) = \sum_{k=1}^{n} \left[\frac{1}{2}\Delta a_k + \sum_{j=k}^{n} \Delta a_j \cos kx\right] =$$

$$= \sum_{k=1}^{n} \frac{1}{2}\Delta a_k + \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x) + \frac{1}{2} a_{n+1}.$$

Making use of Abel's transformation, we obtain

(2) $$g_n(x) =$$

$$= \sum_{k=1}^{n} \frac{1}{2}\Delta a_k + \sum_{k=1}^{n-1} \Delta a_k \left(D_k(x) + \frac{1}{2}\right) + a_n \left(D_n(x) + \frac{1}{2}\right) - a_{n+1} D_n(x) - a_1 + \frac{1}{2} a_{n+1} =$$

$$= \sum_{k=1}^{n-1} \Delta a_k D_k(x) + a_n D_n(x) - a_{n+1} D_n(x).$$

The last two terms tend to zero as $n \to \infty$ for $x \neq 0$ and since

$$|D_k(x)| = O(1/x) \quad \text{if} \quad x \neq 0 \quad \text{and} \quad \sum_{k=0}^{\infty} |\Delta a_k| < \infty,$$

the series $\sum_{k=1}^{\infty} \Delta a_k D_k(x)$ converges. Hence $\lim_{n\to\infty} g_n(x)$ exists for $x \neq 0$. Now applications of Abel's transformation and Lemma 1 yield

(3) $$\int_0^{\pi} |g(x)|\, dx = \int_0^{\pi} \left|\sum_{k=1}^{\infty} \Delta a_k D_k(x)\right| dx =$$

$$= \int_0^{\pi} \left|\sum_{k=1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x)\right| dx \leqq \sum_{k=1}^{\infty} |\Delta A_k| \int_0^{\pi} \left|\sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j(x)\right| dx \leqq$$

$$\leqq C \sum_{k=1}^{\infty} (k+1) |\Delta A_k| =$$

$$= C\left[\sum_{k=1}^{\infty} (k+1) \left|A_k\left(1 + \frac{\alpha}{k}\right) - \frac{\alpha A_k}{k} - A_{k+1}\right|\right] \leqq$$

$$\leqq C \sum_{k=1}^{\infty} (k+1) \left|A_k\left(1 + \frac{\alpha}{k}\right) - A_{k+1}\right| + C\alpha \sum_{k=1}^{\infty} \frac{k+1}{k} A_k =$$

$$= C \sum_{k=1}^{\infty} (k+1) \Delta A_k + 2C\alpha \sum_{k=1}^{\infty} \frac{k+1}{k} A_k,$$

the last step being the consequence of $A_k(1+\alpha/k)\geqq A_{k+1}$. But

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n-1} (k+1)\Delta A_k + (n+1)A_n - A_1.$$

Applications of $\sum_{0}^{\infty} A_k < \infty$ and Lemma 2 yield

$$\sum_{k=1}^{\infty}(k+1)\Delta A_k = \sum_{k=1}^{\infty} A_k + A_1 < \infty; \tag{4}$$

(3) and (4) now imply the conclusion of the Theorem.

Corollary. *Let $\langle a_k\rangle$ be a sequence satisfying the condition S^*. Then*

$$\frac{1}{x}\sum_{k=1}^{\infty} \Delta a_k \sin(k+1/2)x = \frac{h(x)}{x}$$

converges for $x\in(0,\pi]$ *and* $\frac{h(x)}{x}\in L(0,\pi)$.

Proof. This follows immediately, namely by (2), $2\sin\frac{x}{2}\,g(x)=h(x)$.

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DEPARTMENT OF MATHEMATICS
MAHARSHI DAYANAND UNIVERSITY
ROHTAK-124001, INDIA

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Minimax theorems beyond topological vector spaces

L. L. STACHÓ

1. Introduction

The numerous applications and generalizations of von Neumann's classical minimax theorem constitute an important chapter of modern convex analysis. However, all proofs make essential use of some variant of Brouwer's fixed point theorem, a result that has seemingly nothing to do with convexity but closely connected with the vector space structure of $\mathbf{R}^n$.

In his recent paper [3], I. Joó submitted a completely new and elementary proof of Ky Fan's minimax principle, based on a simple fixed point theorem that can be easily proved by the usual methods of convex analysis. Now the converse question arises: Is it possible to give an extension of the concept of convexity that allows us to find a proof of Brouwer's fixed point theorem proceeding along the lines of the fixed point theorem in [3].

Unfortunately, we cannot furnish yet a definitive answer to this problem. However, by an examination of the proofs in [1] and [3] we can find a deep argument that may provide some hope in an affirmative answer. Namely, these proofs do not touch the algebraic structure of the underlying vector spaces and the only property arising from convexity which is actually used is the trivial topological fact that the interval [0, 1] is connected.

The main purpose of the present article is to show how these remarks yield new generalizations of the Ky Fan and Brézis—Nirenberg—Stampacchia minimax principles, respectively, for topological spaces that are richer but axiomatically simpler than the familiar topological vector spaces.

Our goals will be the following three observations:

a) The most suitable concept in describing the topological situation that occurs in the minimax principles is perhaps the interval space defined (in Section 2) as a topological space equipped with a system of connected subsets that play the role

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of closed segments in vector spaces. In such spaces the convexity of sets and quasiconvexity of functions have a natural interpretation and Joó's method (even with some simplifications) can be applied to establish an extension of Ky Fan's minimax theorem.

b) On the other hand, by shifting the emphasis from the topology on the order structure of one of the underlying spaces, a little change in the crucial steps of [1] (summarized there in formulae (3), (4), (5)) leads to a new elementary proof and generalization for certain interval spaces of the Brézis—Nirenberg—Stampacchia minimax theorem [4, p. 289] that provides a deeper explanation of the asymmetry noted in [4, Remark p. 290].

c) We can answer by a counterexample a question of L. Nirenberg [5, p. 144] concerning the conjectured most general form of minimax theorems in topological vector spaces.

I am indebted to I. Joó for the stimulating discussions and for having called my attention to Nirenberg's question.

2. A Joó type minimax theorem in interval spaces

Definition. By an *interval space* we mean a topological space X endowed with a mapping $[.,.]$: $X\times X\to\{$connected subsets of $X\}$ such that $x_1, x_2\in[x_1, x_2]=[x_2, x_1]$ for all $x_1, x_2\in X$.

In interval spaces it makes sense to speak of convexity in a natural way:

Definition. A subset K of an interval space X is *convex* if for every $x_1, x_2\in K$ we have $[x_1, x_2]\subset K$. Obviously, this concept preserves the following fundamental properties of convexity in vector spaces:

Proposition 1. *In any interval space X, convex sets are connected or empty. The intersection of any family of convex sets is convex. The union of any increasing (with respect to inclusion) net is convex.*

For our purposes it is of more importance that, although convex functions cannot be defined on interval spaces in a reasonable manner, the concept of quasiconvexity of functions can be extended to interval spaces.

Definition. A function f mapping an interval space X into $\mathbf{R}$ is *quasiconvex* or *quasiconcave* if $f(z)\leqq\max\{f(x_1), f(x_2)\}$ or $f(z)\geqq\min\{f(x_1), f(x_2)\}$ whenever $x_1, x_2\in X$ and $z\subset[x_1, x_2]$. Thus f is quasiconvex [quasiconcave] iff the sets $\{x: f(x)\leqq\gamma\}$ $[\{x: f(x)\geqq\gamma\}]$ are convex for all $\gamma\in\mathbf{R}$.

To extend the proof in [1] for interval spaces, we need the following generalization of the fixed point theorem in [3]:

Proposition 2. *Let Y be an interval space, X a topological space and $K(\cdot)$ a mapping of Y into the family of compact subsets of X, such that*

(1) $K(y)\neq\emptyset$ *for all* $y\in Y$,

(2) $K(z)\subset K(y_1)\cup K(y_2)$ *whenever* $z\in[y_1, y_2]$ *and* $y_1, y_2\in Y$,

(3) $\bigcap_{i=1}^{n} K(y_i)$ *is connected or empty for every* $y_1, \dots, y_n\in Y$ $(n=1, 2, \dots)$,

(4) $x\in K(y)$ *whenever* $y=\lim_{i\in\mathscr{I}} y_i$, $x=\lim_{i\in\mathscr{I}} x_i$ *and* $x_i\in K(y_i)$ *for all* $i\in\mathscr{I}$. *Then we have* $\bigcap_{y\in Y} K(y)\neq\emptyset$.

Proof. We must show that the family $K(Y)$ has the finite intersection property, i.e.

$$(3')\qquad \bigcap_{i=1}^{n} K(y_i)\neq\emptyset \quad \text{for every} \quad y_1, \dots, y_n\in Y$$

for all $n\in\mathbf{N}$. We prove (3′) by induction on n. For $n=1$, (3′) follows from (1). Suppose that (3′) holds for $n=1, \dots, k$ but there are $y_1^*, \dots, y_{k+1}^*$ such that $\bigcap_{i=1}^{k+1} K(y_i^*)=\emptyset$. Consider now the mapping $y\mapsto K^*(y)\equiv K(y)\cap\bigcap_{i=3}^{k+1} K(y_i^*)$. It readily follows from our induction hypothesis that $K^*(y)\neq\emptyset$ for all $y\in Y$. Moreover, (2) and (3) ensure that

(5) $K^*(z)$ is a connected subset of $K^*(y_1^*)\cup K^*(y_2^*)$ for any $z\in[y_1^*, y_2^*]$.

By definition, $K^*(y_1^*)\cap K^*(y_2^*)=\emptyset$. (5) implies that for every $z\in[y_1^*, y_2^*]$, the connected set $K^*(z)$ is the disjoint union of the compact sets $K^*(z)\cap K^*(y_j^*)$ $(j=1, 2)$. Hence

(5′) either $K^*(z)\subset K^*(y_1^*)$ or $K^*(z)\subset K^*(y_2^*)$ for any $z\in[y_1^*, y_2^*]$.

Thus the sets $S_j\equiv\{z\in[y_1^*, y_2^*]:\ K^*(z)\subset K^*(y_j^*)\}$ $(j=1, 2)$ are disjoint non-empty and $S_1\cup S_2=[y_1^*, y_2^*]$. But from (4) we see that both S_1 and S_2 must be closed in $[y_1^*, y_2^*]$. (In fact, let $j=1$ or 2 be fixed and let $(y_i\colon i\in\mathscr{I})$ be a net in S_j with $y_i\to y\in[y_1^*, y_2^*]$. For any index $i\in\mathscr{I}$, pick a point $x_i\in K^*(y_i)$ arbitrarily. Since by the definition of S_j, the sets $K^*(y_i)$ are contained in the compact $K^*(y_j^*)$, for a suitable subnet $(x_{i_m}\colon m\in\mathscr{M})$ we have $x_{i_m}\to x$ for some $x\in K^*(y_j^*)$. Now (4) ensures that $x\in K^*(y)$ whence $K^*(y)\subset K^*(y_j^*)$.) However this contradicts our axiomatic assumption that intervals are connected.

Theorem 1. *Let X, Y be compact interval spaces and let $f\colon X\times Y\to\mathbf{R}$ be a continuous function such that*

(6^x) *the subfunctions* $x\mapsto f(x, y)$ *are quasiconcave for any fixed* $y\in Y$,

(6^y) *the subfunctions* $y\mapsto f(x, y)$ *are quasiconvex for any fixed* $x\in X$.

Then $\gamma_*\equiv \max_x \min_y f(x, y)= \min_y \max_x f(x, y)\equiv\gamma^*$.

Proof. A standard compactness argument establishes that both γ_* and γ^* are attained (thus the statement of Theorem 1 makes sense). Then obviously we have $\gamma_* \leqq \gamma^*$. The converse inequality $\gamma_* = \max_x \min_y f(x, y) \geqq \gamma^*$ is equivalent to the existence of some $x_0 \in X$ such that for all $y \in Y$ we have $f(x_0, y) \geqq \gamma^*$.

For each $y \in Y$, let $K(y)$ be defined by $K(y) \equiv \{x: f(x, y) \geqq \gamma^*\}$. Thus to $\gamma_* \geqq \gamma^*$ we have to show $\bigcap_{y \in Y} K(y) \neq \emptyset$.

From the definition of γ^* we see that $K(y) \neq \emptyset$ for any $y \in Y$. The continuity of f implies that $K(y)$ is compact and from (6^x) we obtain that $K(y)$ is convex for all $y \in Y$. From (6^y) it follows $K(z) = \{x: f(x, z) \geqq \gamma^*\} \subset \{x: \max\{f(x, y_j): j = 1, 2\} \geqq \gamma^*\} = \bigcup_{j=1}^{2} \{x: f(x, y_j) \geqq \gamma^*\} = \bigcup_{j=1}^{2} K(y_j)$ whenever $z \in [y_1, y_2]$. Finally, also from the continuity of f we deduce (4). Since convex sets are connected or empty, Proposition 2 can be applied, whose conclusion is $\bigcap_{y \in Y} K(y) \neq \emptyset$.

We close this section with the following question:

Question. Is there a choice of X, Y and K in Proposition 2 such that the conclusion $\bigcap_{y \in Y} K(y) \neq \emptyset$ be a known equivalent of Brouwer's fixed point theorem?

3. A generalization of the Brézis—Nirenberg—Stampacchia minimax theorem

Definition. We shall say that an interval space Y is *Dedekind complete* if for every pair of points $y_1, y_2 \in Y$ and convex subsets $H_1, H_2 \subset Y$ with $y_j \in H_j$ $(j = 1, 2)$ and $[y_1, y_2] \subset H_1 \cup H_2$ there exists $z \in H_1$ such that $[y_2, z] \setminus \{z\} \subset H_2$ or there exists $z \in H_2$ such that $[y_1, z] \setminus \{z\} \subset H_2$.

Lemma 1. *Let Y be a convex subset of some real Hausdorff topological vector space with its natural interval structure $[y_1, y_2] \equiv \{(1-\lambda)y_1 + \lambda y_2 : \lambda \in [0, 1]\}$ (for each $y_1, y_2 \in Y$). Then Y is a Dedekind complete interval space.*

Proof. Given y_1, y_2 and H_1, H_2 as above, set $z \equiv (1-\lambda^*)y_1 + \lambda^* y_2$ where $\lambda^* \equiv \sup\{\lambda \in [0, 1]: (1-\lambda)y_1 + \lambda y_2 \in H_1\}$. Then $z \in [y_1, y_2]$ and $[y_j, z] \setminus \{z\} \subset H_j$ $(j = 1, 2)$.

Proposition 3. *Let X be an interval space, Y a Dedekind complete Hausdorff interval space and $f: X \times Y \to \mathbf{R}$ a function such that*

(7^x) *the subfunctions $x \mapsto f(x, y)$ are quasiconcave on X and upper semicontinuous on any interval of X (for all fixed $y \in Y$).*

(7^y) *the subfunctions $y \mapsto f(x, y)$ are quasiconvex on Y and lower semicontinuous on any interval of Y (for all fixed $x \in X$). Then the family $\mathcal{F}$ of X-subsets defined by*

(8) $$\mathcal{F} \equiv \{\{x: f(x, y) \geqq \gamma\}: y \in Y, \gamma < \gamma^*\}, \quad \textit{where} \quad \gamma^* \equiv \inf_y \sup_x f(x, y),$$

has the finite intersection property whenever $\gamma^* > -\infty$.

Proof. The definition of γ^* ensures that $F\neq\emptyset$ for any $F\in\mathcal{F}$ (and $\mathcal{F}\neq\emptyset$ if $\gamma^*>-\infty$). Assume now that we have

(9) $$\bigcap_{i=1}^{n} F_i\neq\emptyset \quad \text{for every choice of} \quad F_1, \ldots, F_n\in\mathcal{F},$$

but $\bigcap_{i=1}^{n+1} F_i^*=\emptyset$ where $F_1^*, \ldots, F_{n+1}^*$ are some given elements of $\mathcal{F}$. To complete the proof, we show that this is impossible.

By (8) we may suppose that $F_i^*=\{x: f(x, y_i^*)\geqq\gamma_i^*\}$ $(i=1, \ldots, n+1)$ with $y_1^*, \ldots, y_{n+1}^*\in Y$ and $\gamma^*>\gamma_1^*\geqq\ldots\geqq\gamma_{n+1}^*$. Set

(10) $$G\equiv\bigcap_{i=3}^{n+1}\{x: f(x, y_i^*)>\gamma_1^*\} \quad \text{and} \quad K(y)\equiv\{x\in G: f(x, y)>\gamma_1^*\} \quad \text{for all} \quad y\in Y.$$

Now (7^x) implies that each set $K(y)$ is convex in X and from (10) and (9) we see that

$$K(y)\supset\left\{x: f(x, y)\geqq\frac{\gamma_1^*+\gamma^*}{2}\right\}\cap\bigcap_{i=3}^{n+1}\left\{x: f(x, y_i^*)\geqq\frac{\gamma_1^*+\gamma^*}{2}\right\}\neq\emptyset \quad (\text{for all} \quad y\in Y).$$

Also in this proof, the key property of the mapping $y\mapsto K(y)$ is that

(2^*) $K(z)\subset K(y_1)\cup K(y_2)$ whenever $z\in[y_1, y_2]$ (for all $y_1, y_2\in Y$) which can be deduced from (10) and (7) as follows: $K(z)=\{x\in G: f(x, z)>\gamma_1^*\}\subset\subset\{x\in G: \{\max f(x, y_j): j=1, 2\}>\gamma_1^*\}=\bigcup_{j=1}^{2}\{x\in G: f(x, y_j)>\gamma_1^*\}=K(y_1)\cup K(y_2)$.
Hence it follows that

(5^*) either $K(z)\subset K(y_1^*)$ or $K(z)\subset K(y_2^*)$ for any $z\in[y_1^*, y_2^*]$.

Indeed, $x_1\in K(z)\cap K(y_1^*)$ and $x_2\in K(z)\cap K(y_2^*)$ implies that for the sets $T_j\equiv\equiv[x_1, x_2]\cap F_j^*\cap\bigcap_{i=3}^{n+1} F_i^*$ $(j=1, 2)$ we have $T_1\cap T_2\subset\bigcap_{i=1}^{n+1} F_i^*=\emptyset$ and $[x_1, x_2]\supset T_1\cup T_2\supset\supset[x_1, x_2]\cap(F_1^*\cup F_2^*)\cap G\supset[x_1, x_2]\cap\bigcup_{j=1}^{2} K(y_j^*)\supset$ by (2) $\supset[x_1, x_2]\cap K(z)=[x_1, x_2]$. By ($7^x$) the sets F_i^* are closed in X $(i=1, \ldots, n+1)$ whence T_1 and T_2 are closed in $[x_1, x_2]$. But this contradicts the connectedness of $[x_1, x_2]$. Thus (5^*) holds.

(2^*) and (5^*) show that the sets

(11) $$H_j^*\equiv\{z: K(z)\subset K(y_j^*)\} \quad (j=1, 2)$$

are convex in Y, $H_1^*\cup H_2^*\supset[y_1, y_2]$ and $y_j^*\in H_j^*$ $(j=1, 2)$. Since the interval space Y was assumed to be Dedekind complete, there exist $j\in\{1, 2\}$ and $z^*\in H_j^*$ such that

(12) $$[y_k^*, z^*]\setminus\{z^*\}\subset H_k^* \quad \text{where} \quad k\in\{1, 2\}\setminus\{j\}.$$

From (10) and (11) we have

(13) $$f(x^*, z^*)>\gamma_1^* \quad \text{for all} \quad x^*\in K(z^*).$$

On the other hand, if $x^*\in K(z^*)$ then $x^*\notin K(y_k^*)$. From (12) and (11) it follows $K(y_k^*)\supset K(z)$ for all $z\in[y_k^*, z^*]\setminus\{z^*\}$ whence we obtain by (10) that

$$(13') \qquad f(x^*, z) \leqq \gamma_1^* \quad \text{for all} \quad z\in[y_k^*, z^*]\setminus\{z^*\} \quad \text{and} \quad x^*\in K(z^*).$$

Since the topology of Y was supposed to be Hausdorff and since the interval $[y_k^*, z^*]$ is connected, the point z^* belongs to the closure of $[y_k^*, z^*]\setminus\{z^*\}$. But then ($7^y$) and (13′) imply $f(x^*, z^*)\leqq\gamma_1^*$ for all $x^*\in K(z^*)$ $(\neq\emptyset)$ which contradicts (13).

Theorem 2. *Suppose that X is an interval space, Y is a Dedekind complete Hausdorff interval space and that the function $f\colon X\times Y\to\mathbf{R}$ has the properties* (7^y),

(7_*^x) *the subfunctions $x\mapsto f(x, y)$ are upper semicontinuous and quasiconcave on the whole X (for all fixed $y\in Y$),*

(7^c) *for some $\gamma<\inf_y\sup_x f(x, y)$ and $y\in Y$, the set $\{x\colon f(x, y)\geqq\gamma\}$ is compact.*

Then we have $\max_x\inf_y f(x, y)=\inf_y\sup_x f(x, y)$.

Proof. From the definition of the operations inf and sup it follows immediately that $\sup_x\inf_y f(x, y)\leqq\inf_y\sup_x f(x, y)$. Therefore again it suffices to prove that $\inf_y f(x_0, y)\geqq\gamma^*(\equiv\inf_y\sup_x f(x, y))$ for some $x_0\in X$, or equivalently that the family $\mathcal{F}$ defined by (8) admits a common point.

Now (7^c) ensures that $\gamma^*>-\infty$ and that some member of $\mathcal{F}$ is a non-empty compact set. By (7_*^x), each member of $\mathcal{F}$ is a closed subset of X. Hence $\bigcap\mathcal{F}\neq\emptyset$ if and only if $\mathcal{F}$ has the finite intersection property. But this is a directe consequence of Proposition 3.

Corollary. (Brézis—Nirenberg—Stampacchia) *If X is a convex subset of a real Hausdorff topological vector space, Y is a convex subset in a real vector space and $f\colon X\times Y\to\mathbf{R}$ is a function satisfying* (7_*^x), (7^y) *and* (7^c) *then we have* $\max_x\inf_y f(x, y)=\inf_y\sup_x f(x, y)$.

Proof. Let us endove Y with any locally convex Hausdorff vector space topology. (It is always possible e.g. by taking the convex core topology on the supporting vector space of Y, cf. [6, p. 110, (2.10)].) Then by Lemma 1 we can apply Theorem 2.

4. A counterexample concerning the extendibility of Theorem 2

In the light of the proof of Proposition 3, we can answer (negatively) the question raised by L. Nirenberg [5, p. 144] whether condition (7_*^x) can be replaced by the weaker condition (7^x) in the Brézis—Nirenberg—Stampacchia minimax theorem.

Theorem 3. *There exist locally convex Hausdorff topological vector spaces F, G and compact convex subsets $X\subset F$ and $Y\subset G$, further a function $f\colon X\times Y\to\{0,1\}$ satisfying* (7^x), (7^y), *and such that* $0=\max_x \min_y f(x,y)$ *and* $1=\max_y \min_x f(x,y)$.

Remark. It is well-known from elementary convex analysis that a convex subset K of a finite dimensional real Hausdorff topological vector space V is closed if and only if it is algebraically closed (i.e. if the sets $\{\lambda\in\mathbf{R}\colon u+\lambda\cdot v\in K\}$ are closed for all $u,v\in V$) [6, p. 59, p. 9]. Hence (7^x) [respectively (7^y)] implies that the subfunctions $x\mapsto f(x,y)$ [$y\mapsto f(x,y)$] restricted to the intersection of $X[Y]$ with any finite dimensional linear submanifold of F [G] are all upper [lower] semicontinuous.

Proof. Let G be the space of the functions mapping $\mathbf{N}(\equiv\{1,2,\ldots\})$ into $\mathbf{R}$ endowed with the pointwise convergence topology and let $Y\equiv\{y\in G\colon \text{range}\,(y)\subset[0,1]\}$. Thus Y is homeomorphic to the compact product space $[0,1]^{\mathbf{N}}$. For $i=1,2,\ldots$ let e_i denote the function $e_i\colon n\mapsto\delta_{in}(=1$ if $i=n$, 0 if $i\neq n)$. Set $H_n\equiv\text{co}\,\{e_i\colon i>n\}$ (the symbol *co* standing for the algebraic convex hull operation; $n=1,2,\ldots$). Clearly, the sets H_n are algebraically closed (because the vectors $e_1,e_2,\ldots$ are linearly independent). Further we have $\bigcap_{n=1}^{\infty} H_n=\emptyset$. Therefore the function

$$m(y)\equiv\min\,\{n\in\mathbf{N}\colon y\notin H_n\}$$

is well-defined for all $y\in G$. Now we define the space F as the set of the functions mapping Y into $\mathbf{R}$, also with the pointwise convergence topology, and we set $X\equiv\{x\in F\colon \text{range}\,(x)\subset[0,1]\}$. Again, X is homeomorphic to the compact product $[0,1]^Y$. To define the function f, first we introduce the following X-subset valued function $K(\cdot)$ on Y:

$$K(y)\equiv\text{co}\,\{1_{H_n}\colon n\geqq m(y)\}\quad(\text{for all } y\in Y)$$

where 1_{H_n} denotes the characteristic function of the set H_n (i.e. $1_{H_n}(y)=1$ if $y\in H_n$ and 0 else). Since the functions 1_{H_n} $(n\in\mathbf{N})$ are linearly independent, the sets $K(y)$ are algebraically closed (for all $y\in Y$). Then let

$$f(x,y)\equiv 1_{K(y)}(y)\quad\bigl(=1 \text{ if } x\in K(y),\ 0 \text{ if } x\notin K(y)\bigr)\quad\text{for all}\quad x\in X,\ y\in Y.$$

To show (7^x), we have to check that for all $\gamma\in\mathbf{R}$, the sets $\{x\colon f(x,y)\geqq\gamma\}$ are algebraically closed for any $y\in Y$. But $\{x\colon f(x,y)\geqq\gamma\}=X$ if $\gamma\leqq0$, $K(y)$ if $0<\gamma\leqq1$, $\emptyset$ if $\gamma>1$.

In particular, $\{x\colon f(x,y)\geqq1\}=K(y)\neq\emptyset$ for each $y\in Y$, whence $1=\max_{x,y} f(x,y)= = \min_y \max_x f(x,y)$.

For (7^y) we must show that $\{y\colon f(x,y)\leqq\gamma\}$ is algebraically closed for all $\gamma\in\mathbf{R}$ and $x\in X$. Now we have $\{y\colon f(x,y)\leqq\gamma\}=\emptyset$ if $\gamma<0$, Y if $\gamma\geqq1$, and if $0\leqq\gamma<1$ then

$\{y: f(x,y)\leqq\gamma\}=\{y: f(x,y)=0\}=\{y: x\notin K(y)\}=\{y: x_{H_n}\text{co}\,\{1_{H_n}: n\geqq m(y)\}\}$. In case of $x\notin\text{co}\,\{1_{H_n}: n\in\mathbf{N}\}$ we obviously have $\{y: x\notin\text{co}\,\{1_{H_n}: n\geqq m(y)\}\}=Y$. If $x\in\text{co}\,\{1_{H_n}: n\in\mathbf{N}\}$ then there exist finite sets $\mathscr{I}_x\subset\mathbf{N}$ and $\{\lambda_i^x: i\in\mathscr{I}_x\}\subset(0,\infty)$ such that $\sum_{i\in\mathscr{I}_x}\lambda_i^x=1$ and $x=\sum_{i\in\mathscr{I}_x}\lambda_i^x\cdot 1_{H_i}$, thus in this case we have $\{y: x\notin\text{co}\,\{1_{H_n}: n\geqq m(y)\}\}=$
$=\{y: \min\mathscr{I}_x<m(y)\}=\{y: \min\mathscr{I}_x<\min\{n: y\notin H_n\}\}=\{y: \exists n\leqq\min\mathscr{I}_x\ y\notin H_n\}=$
$=\{y: \forall n\leqq\min\mathscr{I}_x\ y\in H_n\}=\bigcap_{n=1}^{\min\mathscr{I}} H_n=H_{\min\mathscr{I}_x}$ which is also convex and algebraically closed.

Since for any $x\in X$ we have seen that $\{y: f(x,y)=0\}=Y$ or H_n for some $n\in\mathbf{N}$, i.e. $\{y: f(x,y)=0\}\neq\emptyset$, we can conclude $0=\min_{x,y}f(x,y)=\max_x\min_y f(x,y)$.

Question. Does $\sup_x\inf_y f(x,y)=\inf_y\sup_x f(x,y)$ hold if the function $f\colon X\times Y\to\mathbf{R}$ is such that X and Y are convex compact subsets of some locally convex Hausdorff topological vector spaces and every restriction to any straight line segment contained in X [in Y] of the subfunctions $x\mapsto f(x,y)$ [$y\mapsto f(x,y)$] is continuous and *concave [convex]*?

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BOLYAI INSTITUTE
JÓZSEF ATTILA UNIVERSITY
ARADI VÉRTANÚK TERE 1.
6720 SZEGED, HUNGARY

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Almost periodic functions and functional equations

L. SZÉKELYHIDI

1. Introduction. In this paper we deal with bounded solutions of a class of functional equations defined on topological groups. Our results are based on the fact that all characters of a group are almost periodic functions (see e.g. MAAK [5], [6]). This can be restated by saying that all bounded solutions of the functional equation $f(xy)=f(x)f(y)$ are almost periodic functions. In this work this result is generalized for the functional equation (1) which has been dealt by many authors (see [1], [7], [8], [9], [11], [12]) but has not been completely solved. Using our results we give all bounded solutions of (1) on commutative groups. Our other main result is the proof of the fact that all bounded solutions of (2), and in particular of (3), are almost periodic functions. Concerning these equations see [1].

We note that some of our results remain valid on topological semigroups as well. On the other hand the method used in Section 3 to solve equation (1) can be used successfully to solve other similar equations ([10]).

2. Preliminary facts and results. Let G be a group and X a uniform space. A function $f\colon G\to X$ is said to be *almost periodic* if for every X-entourage R there exists a finite covering $A_1, \ldots, A_n$ of G such that $\big(f(xz), f(yz)\big)\in R$ whenever $z\in G$, $x, y\in A_i$ $(i=1, \ldots, n)$.

Let H be a set and X a uniform space. A function $f\colon H\to X$ is said to be *totally bounded* if for every X-entourage R there exists a finite covering $B_1, \ldots, B_n$ of $\operatorname{ran} f$, the range of f, such that $(x, y)\in R$ whenever $x, y\in B_i$ $(i=1, \ldots, n)$.

If G is a group, X is a uniform space and $f\colon G\to X$ is an almost periodic function, then f is totally bounded. Indeed, if R is any X-entourage and $A_1, \ldots, A_n$ is a finite covering of G for which $\big(f(xz), f(yz)\big)\in R$ holds whenever $z\in G$, $x, y\in A_i$ $(i=1, \ldots, n)$ then $B_i=f(A_i)$ $(i=1, \ldots, n)$ yields an appropriate covering of $\operatorname{ran} f$.

If G is a topological group, X is a Banach space, then the continuous function $f\colon G\to X$ is almost periodic if and only if the orbit of f is relatively compact in the Banach space of all continuous, bounded X-valued functions on G. (The

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orbit of f is the set of all right translates of f. It can be proved that this is equivalent to the relative compactness of the set of all left translates of f.)

For more about almost periodic functions see e.g. [2], [3], [4], [5], [6].

3. Bounded solutions of functional equations.

Theorem 3.1. *Let G be a topological group, n a positive integer, and a_k, b_k: $G\to\mathbf{C}$ bounded functions, where the a_k's are continuous $(k=1, \dots, n)$. If f: $G\to\mathbf{C}$ is a function for which*

$$(1) \qquad f(xy) = \sum_{k=1}^{n} a_k(x)\, b_k(y)$$

holds whenever $x, y\in G$, then f is a continuous almost periodic function. If the a_k's are linearly independent, then the b_k's are also continuous almost periodic functions.

Proof. Let $B(G)$ denote the set of all complex valued continuous bounded functions on G equipped with the pointwise operations and sup-norm. $B(G)$ is a Banach space. Let

$$A_k = \{b_k(y)a_k\colon y\in G\} \quad (k = 1, \dots, n).$$

As a_k is continuous bounded and b_k is bounded, hence A_k is relatively compact in $B(G)$ $(k=1, \dots, n)$. Let F denote the orbit of f in $B(G)$, then by (1) we see that F is a subset of the set $A_1+\dots+A_n$, which is a continuous image of the relatively compact set $A_1\times\dots\times A_n$. Hence f is almost periodic. The continuity of f follows directly from (1) by the substitution $y=e$ (the unit element).

If the a_k's are linearly independent then there are elements $x_1, \dots, x_n$ of G for which the matrix $(a_i(x_j))$ is regular (see e.g. [11]). Substituting the x_i's into (1) in place of x, for fixed y we get that the numbers $b_k(y)$ satisfy a system of linear equations, the matrix of which is regular. Hence the functions b_k can be represented as a linear combination of some translates of f and thus they are continuous almost periodic functions.

Theorem 3.1 can be generalized as follows:

Theorem 3.2. *Let G be a topological group, L, M, N normed spaces, g: $G\to L$ a totally bounded continuous function, h: $G\to M$ a bounded function and F: $L\times M\to N$ a bounded bilinear operator. If f: $G\to N$ is a function for which*

$$(2) \qquad f(xy) = F\big(g(x), h(y)\big)$$

holds whenever $x, y\in G$, then f is a continuous almost periodic function.

Proof. The continuity of f follows by substituting $y=e$. Let $\varepsilon>0$ be arbitrary and let K be a bound for h. As g is totally bounded, there exists a finite covering $L_1, \dots, L_n$ of $\operatorname{ran} g$ such that $\|u-v\|<\varepsilon$ whenever $u, v\in L_i$ $(i=1, \dots, n)$. Let $A_i= =g^{-1}(L_i)$ $(i=1, \dots, n)$ then $A_1, \dots, A_n$ is a finite covering of G. If $x, y\in A_i$

$(i=1, \dots, n)$ then $g(x), g(y) \in L_i$ hence $\|g(x)-g(y)\|<\varepsilon$ which implies for every $z\in G$

$$\|f(xz)-f(yz)\| = \|F(g(x), h(z))-F(g(y), h(z))\| =$$
$$= \|F(g(x)-g(y), h(z))\| \leqq C\|g(x)-g(y)\|\cdot K \leqq C\cdot\varepsilon\cdot K$$

that is f is almost periodic.

The linearity of F in (2) can be replaced by uniform continuity. Namely, we have

Theorem 3.3. *Let G be a topological group, L, M, N uniform spaces, $g\colon G\to L$ a totally bounded continuous function, $h\colon G\to M$ a bounded function and $F\colon L\times M\to N$ a uniformly continuous function. If $f\colon G\to N$ is a function for which* (2) *holds whenever $x, y\in G$ then f is a continuous almost periodic function.*

Proof. The continuity of f follows by substituting $y=e$. Let R be an arbitrary X-entourage. By the uniform continuity of F there exists an $L\times M$-entourage S, for which $((u, v), (u', v'))\in S$ implies $(F(u, v), F(u', v'))\in R$. Further there exists an L-entourage T such that $(u, u')\in T$ and $v\in M$ implies $((u, v), (u', v'))\in S$. By the totally boundedness of g there exists a finite covering $L_1, \dots, L_n$ of ran g such that $u, u'\in L_i$ implies $(u, u')\in T$ $(i=1, \dots, n)$.

Let $A_i=g^{-1}(L_i)$ $(i=1, \dots, n)$, then $A_1, \dots, A_n$ is a finite covering of G, and for $x, y\in A_i$ $(i=1, \dots, n)$ we have $g(x), g(y)\in L_i$, hence for $z\in G$

$$(f(xz), f(yz)) = (F(g(x), h(z)), F(g(y), h(z)))\in R,$$

that is f is almost periodic.

Remark 3.4. The conditions of Theorem 3.3 are satisfied for instance if g, h are bounded functions with values in finite dimensional vector spaces (or, more generally, in Montel spaces), L, M denote the closures of their ranges respectively, and F is continuous on $L\times M$. Hence we have the corollaries:

Corollary 3.5. *Let G be a topological group, let $g, h\colon G\to \mathbf{C}$ (the set of complex numbers) be bounded functions, and let g be continuous. Let $F\colon$ $(\operatorname{ran} g\times \operatorname{ran} h)^-\to \mathbf{C}$ be a continuous function. If $f\colon G\to\mathbf{C}$ is a function for which* (2) *holds whenever $x, y\in G$, then f is a continuous almost periodic function.*

Corollary 3.6. *Let G be a topological group, $f\colon G\to\mathbf{C}$ be a continuous bounded function. Let $F\colon$ $(\operatorname{ran} f\times \operatorname{ran} f)^-\to\mathbf{C}$ be a continuous function. If the equality*

$$f(xy) = F(f(x), f(y)) \tag{3}$$

holds whenever $x, y\in G$, then f is almost periodic.

4. Bounded solutions of equation (1). In this section we exhibit all bounded solutions of equation (1) on commutative groups. More exactly, we show that f is a trigonometric polynomial and so are the functions a_k, b_k whenever the a_k's and also the b_k's are linearly independent. By trigonometric polynomial we mean a linear combination of continuous characters. Here the number of different characters is called the degree of the trigonometric polynomial.

In what follows we assume that G is a commutative topological group with sufficiently many continuous characters, that is any two elements of G can be separated by continuous character. For instance all locally compact Hausdorff groups posses this property and so does the additive group of any locally convex topological vectorspace. Then the Fourier transform of almost periodic functions can be defined as an injective mapping by the formula

$$\hat{f}(\gamma) = \int f\bar{\gamma}$$

(where $\int$ denotes the invariant mean on almost periodic functions) whenever γ is a continuous character of G (see [5], [6]).

Theorem 4.1. *Let G be a commutative group with sufficiently many continuous characters, n a positive integer and a_k, b_k, f: $G\to\mathbf{C}$ $(k=1, \dots, n)$ functions. If f is a continuous bounded function, then it is a trigonometric polynomial of degree at most n.*

Proof. First we assume that the a_k's and also the b_k's are linearly independent. Then there are elements $x_1, \dots, x_n$ of G for which the matrix $(a_i(x_j))$ is regular. As in Theorem 3.1 we obtain that the b_k's are continuous bounded functions. Similarly, we get the same for the a_k's.

By Theorem 3.1, f, a_k, b_k are almost periodic functions. On the other hand, the linear independence of the a_k's implies the same for their Fourier transforms.

Now let y be fixed and compute the Fourier transforms of both sides of (1) as functions of x. We obtain

$$\hat{f}(\gamma)\gamma(y) = \sum_{k=1}^{n} \hat{a}_k(\gamma)b_k(y) \tag{4}$$

where $y\in G$ and γ is a character of G. Now compute the Fourier transforms of both sides of (4) as functions of y. We obtain

$$\hat{f}(\gamma)\hat{\gamma}(\tau) = \sum_{k=1}^{n} \hat{a}_k(\gamma)\hat{b}_k(\tau) \tag{5}$$

where γ, τ are characters of G. Let $\gamma_1, \dots, \gamma_n$ be characters of G for which the matrix $(\hat{a}_k(\gamma_j))$ is regular. Substituting in (5) γ_j for γ we get that the numbers $\hat{b}_k(\tau)$ for $\tau\neq\gamma_j$ $(j=1, \dots, n)$ satisfy a homogeneous linear system of equations, the matrix of which is regular, hence $\hat{b}_k(\tau)=0$ for $\tau\neq\gamma_j$ $(j=1, \dots, n,\ k=1, \dots, n)$. Thus the Fourier transform of $b_k-\sum_{j=1}^{n}\hat{b}_k(\gamma_j)\gamma_j$ vanishes, and hence b_k is a trigonometric polynomial of degree at most n $(k=1, \dots, n)$. Similarly we get the statement for a_k, f.

In the general case, when the a_k's or the b_k's are linearly dependent, then by the successive decreasing of n we can make the a_k's and the b_k's simultaneously linearly independent and hence the statement remains valid for f.

Corollary 4.2. *Let G be a locally compact topological group. Then any finite dimensional translation invariant subspace of the Banach space of all continuous bounded complex valued functions on G consists of almost periodic functions. If G is commutative then this subspace consists of trigonometric polynomials.*

Proof. Let M be the subspace in question and let $a_1, \dots, a_n$ be a basis of M. Then for every $f \in M$ we have

$$f(xy) = \sum_{k=1}^{n} a_k(x)\, b_k(y) \tag{6}$$

whenever $x, y \in G$. Since the a_k's are linearly independent, hence the b_k's are continuous bounded functions. This implies that f is almost periodic. If G is commutative then, as in the proof of Theorem 4.1, we obtain that the b_k's are trigonometric polynomials and hence substituting $x=e$ in (6) we see that f is a trigonometric polynomial. In particular, the a_k's are trigonometric polynomials.

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DEPARTMENT OF MATHEMATICS
KOSSUTH LAJOS UNIVERSITY
4010 DEBRECEN, HUNGARY

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Bemerkung zu einem Satz von S. Kaczmarz

KÁROLY TANDORI

Für ein orthonormiertes System $\varphi=\{\varphi_k(x)\}_1^\infty$ im Intervall (0, 1) bilden wir die Lebesgueschen Funktionen

$$L_n(\varphi, x)=\int_0^1\left|\sum_{k=1}^n \varphi_k(x)\varphi_k(t)\right|dt \quad (n=1,2,\ldots).$$

S. KACZMARZ [2] hat bewiesen, daß im Falle

$$L_n(\varphi; x)=O(1) \quad (x\in(0,1);\ n=1,2,\ldots) \tag{1}$$

und für $a=\{a_k\}_1^\infty\in l^2$ die Reihe

$$\sum_{k=1}^\infty a_k\varphi_k(x) \tag{2}$$

fast überall in (0, 1) konvergiert.

Weiterhin haben wir in [3] Folgendes gezeigt:

Ist $a\notin l^2$, dann gibt es ein orthonormiertes System φ in (0, 1) derart, daß (1) erfüllt ist, und die Reihe (2) in (0, 1) fast überall divergiert.

In dieser Note werden wir für diese Behauptung einen einfacheren Beweis geben, der sogar noch etwas mehr ergibt.

Für ein orthonormiertes System φ in (0, 1) bilden wir

$$L_n^*(\varphi; x)=\int_0^1 \max_{1\le i\le n}\left|\sum_{k=1}^n \varphi_k(x)\varphi_k(t)\right|dt.$$

Offenbar gilt

$$L_n(\varphi; x)\le L_n^*(\varphi; x).$$

Wir beweisen.

Satz. *Ist $a\notin l^2$, so gibt es ein orthonormiertes System φ in* (0, 1) *mit*

$$L_n^*(\varphi; x)=O(1) \quad (x\in(0,1);\ n=1,2,\ldots) \tag{3}$$

derart, daß die Reihe (2) *in* (0, 1) *überall divergiert.*

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Bemerkung. Dieser Satz ist eine Verschärfung eines vorigen Resultates von Verf. [4]. Nach einem Satz von L. Csernyák [1] gilt im Falle (3) und $a\in l^2$

$$\sup_n \left|\sum_{k=1}^{n} a_k \varphi_k(x)\right| \in L^2(0, 1).$$

Beweis des Satzes. Ohne Beschränkung der Allgemeinheit können wir $0\leqq a_k\leqq 1$ voraussetzen. Es sei $0=n(1)<\ldots<n(l)<\ldots$ eine Indexfolge mit der Eigenschaft

$$A_l^2 = \sum_{k=n(l-1)+1}^{n(l)} a_k^2 \geqq 4^l \quad (l = 2, 3, \ldots). \tag{4}$$

Mit $k_1<\ldots<k_i<\ldots$ bezeichnen wir die Indizes k, für die $a_k=0$ ist. Es sei $Z(l)$ die Menge der Indizes k mit $n(l-1)<k\leqq n(l)$ und $a_k\neq 0$ $(l=2, 3, \ldots)$.

Es seien weiterhin $I_k(l)$, $J_k(l)$ $\big(k\in Z(l);\ l=2, 3, \ldots\big)$, J_i $(i=1, 2, \ldots)$ Teilintervalle von (0, 1) mit den Eigenschaften (für $l, l_1, l_2=2, 3, \ldots$)

$$I_{k_1}(l)\cap I_{k_2}(l) = \emptyset \quad \big(k_1, k_2\in Z(l), k_1 \neq k_2\big) \quad \bigcup_{k\in Z(l)} I_k(l) = (0, 1),$$

$$\operatorname{mes} I_k(l) = a_k^2/A_l^2 \quad \big(k\in Z(l)\big),\ I_k(l)\cap J_k(l) = \emptyset \quad \big(k\in Z(l)\big),$$

$$J_{k_1}(l_1)\cap J_{k_2}(l_2) = \emptyset \quad \big(k_1\in Z(l_1)\big),\ k_2\in Z(l_2),\ (k_1-k_2)^2+(l_1-l_2)^2 \neq 0\big),$$

$$\operatorname{mes} J_k(l) = \operatorname{mes} I_k(l)/l^2 \quad \big(k\in Z(l)\big), \quad J_{i_1}\cap J_{i_2} = \emptyset \quad (i_1, i_2 = 1, 2, \ldots;\ i_1 \neq i_2).$$

Unter den obigen Bedingungen kann man solche Intervalle leicht angeben.

Es sei $\varphi=\{\varphi_k(x)\}_1^\infty$ ein orthonormiertes System von Treppenfunktionen in (0, 1) mit den Eigenschaften

$$|\varphi_k(x)| = \begin{cases} A_l/a_k\cdot l, & x\in I_k(l) \\ (1-1/l^2)/\sqrt{\operatorname{mes} J_k(l)}, & x\in J_k(l) \\ 0, & \text{sonst} \end{cases} \quad \big(k\in Z(l)\big),$$

$$|\varphi_{k_i}(x)| = \begin{cases} 1/\sqrt{\operatorname{mes} J_i}, & x\in J_i \\ 0, & \text{sonst} \end{cases} \quad (i = 1, 2, \ldots).$$

Ein solches System kann leicht angegeben werden; man hat die Gruppe der Funktionen $\varphi_{n(l-1)+1}(x), \ldots, \varphi_{n(l)}(x)$ durch Rekursion zu definieren.

Es sei $x\in(0, 1)$. Auf Grund der Definition der Intervalle $J_k(l)$, J_i und der Funktionen $\varphi_k(x)$ gibt es einen Index l_0 derart, daß

$$x\notin\left(\bigcup_{l=l_0}^{\infty} \bigcup_{k\in Z(l)} J_k(l)\right)\cup\Big(\bigcup_{\substack{i\\ k_i>n(l_0-1)}} J_i\Big). \tag{5}$$

Ist $l\geqq l_0$, dann gibt es auf Grund von (5) und der Definition von $\varphi_k(x)$ einen Index $k(x, l)\in Z(l)$ mit

$$\left|\sum_{k=\nu(l-1)+1}^{\nu(l)} a_k \varphi_k(x)\right| = |a_{k(x,l)}\varphi_{k(x,l)}(x)| = A_l/l \geqq 2^l/l.$$

Daraus folgt, daß die Reihe (2) im Punkt x divergiert.

Es sei $x\in(0,1)$. Auf Grund der Definition der Intervalle $I_k(l)$, $J_k(l)$, J_i und der Funktionen $\varphi_k(x)$ gibt es für jedes l einen Index $k(x,l)\in Z(l)$ mit $x\in I_{k(x,l)}$; weiterhin existieren Indizes l_0, $k_0(x,l_0)(\in Z(l_0))$ und i_0 mit $x\in J_{k_0(x,l_0)}$, $x\in J_{i_0}$. Dann gilt für jedes n

$$\max_{1\le s\le n}\left|\sum_{k=1}^{s}\varphi_k(x)\varphi_k(t)\right| \le \sum_{l=2}^{\infty}|\varphi_{k(x,l)}(x)\varphi_{k(x,l)}(t)|+|\varphi_{k_0(x,l_0)}(x)\varphi_{k_0(x,l_0)}(t)|+|\varphi_{k_i}(x)\varphi_{k_i}(t)|.$$

Auf Grund der Definition der Funktion $\varphi_k(x)$ ergibt sich dann

$$\int_0^1 \max_{1\le s\le n}\left|\sum_{k=1}^{s}\varphi_k(x)\varphi_k(t)\right| dt \le$$

$$\le \sum_{l=2}^{\infty}\Big(|\varphi_{k(x,l)}(x)|\int_{I_{k(x,l)}}|\varphi_{k(x,l)}(t)|\,dt+|\varphi_{k(x,l)}(x)|\int_{J_{k(x,l)}}|\varphi_{k(x,l)}(t)|\,dt\Big)+$$

$$+|\varphi_{k_0(x,l_0)}(x)|\int_{I_{k_0(x,l_0)}}|\varphi_{k_0(x,l_0)}(t)|\,dt+|\varphi_{k_0(x,l_0)}(x)|\int_{J_{k_0(x,l_0)}}|\varphi_{k_0(x,l_0)}(t)|\,dt+$$

$$+|\varphi_{k_{i_0}}(x)|\int_{J_{i_0}}|\varphi_{k_{i_0}}(t)|\,dt=$$

$$=\sum_{l=2}^{\infty}\left(\frac{A_l^2}{a_{k(x,l)}^2\,l^2}\operatorname{mes} I_{k(x,l)}(l)+\frac{A_l}{a_{k(x,l)}\,l}(1-1/l^2)^{1/2}\frac{1}{\sqrt{\operatorname{mes} J_{k(x,l)}(l)}}\operatorname{mes} J_{k(x,l)}(l)\right)+$$

$$+(1-1/l_0^2)^{1/2}\frac{1}{\sqrt{\operatorname{mes} J_{k_0(x,l_0)}(l_0)}}\,\frac{A_l}{a_{k_0(x,l_0)}\,l_0}\operatorname{mes} I_{k_0(x,l_0)}(l)+$$

$$+(1-1/l_0^2)\frac{1}{\operatorname{mes} J_{k_0(x,l_0)}(l_0)}\operatorname{mes} J_{k_0(x,l_0)}(l_0)+\frac{1}{\operatorname{mes} J_{i_0}}\operatorname{mes} J_{i_0}=$$

$$=\sum_{l=2}^{\infty}\left(\frac{1}{l^2}+(1-1/l^2)^{1/2}\frac{1}{l^2}\right)+\left(1-\frac{1}{l_0^2}\right)^{1/2}+(1-1/l_0^2)+1\le 3\left(1+\sum_{l=2}^{\infty}\frac{1}{l^2}\right)<\infty.$$

Damit haben wir bewiesen, daß (3) für das System φ erfüllt ist.

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BOLYAI INSTITUT
UNIVERSITÄT SZEGED
ARADI VÉRTANÚK TERE 1
6720 SZEGED, UNGARN

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Über einen Satz von Alexits und Sharma

KÁROLY TANDORI

1. Es sei $(X, \mathscr{A}, \mu)$ ein Maßraum mit $\mu(X)<\infty$. Für ein System $\varphi=\{\varphi_k(x)\}_1^\infty$ von Funktionen in $L(X)$ betrachten wir die Lebesgueschen Funktionen

$$L_n(\varphi; x) = \int_X \left|\sum_{k=1}^n \varphi_k(x)\varphi_k(t)\right| d\mu(t) \quad (x\in X;\ n = 1, 2, \ldots).$$

Es sei weiterhin $\lambda=\{\lambda_k\}_1^\infty$ eine monoton nichtabnehmende Folge von positiven Zahlen; im folgenden werden wir auch $\lambda_k\to\infty$ $(k\to\infty)$ voraussetzen.

G. ALEXITS und A. SHARMA [1] haben im Fall

$$L_n(\varphi; x) = O(\lambda_n) \quad (x\in X;\ n = 1, 2, \ldots) \tag{1}$$

den Folgenden Satz bewiesen: Genügt eine Folge $\{a_k\}_1^\infty$ von reellen Zahlen der Bedingung

$$\sum_{k=1}^\infty a_k^2\lambda_k < \infty, \tag{2}$$

weiterhin besteht

$$\int_X \left|\sum_{k=1}^n a_k b_k \varphi_k(x)\right| d\mu(x) = O(1) \quad (n = 1, 2, \ldots),$$

für jede Folge $\{b_k\}_1^\infty$ mit $\sum_{k=1}^\infty a_k^2 b_k^2<\infty$, dann konvergiert die Reihe

$$\sum_{k=1}^\infty a_k\varphi_k(x) \tag{3}$$

in X fast überall.

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Man kann zeigen (s. z. B. [2]), daß im Falle (1) die Bedingung (2) *allein* für die Konvergenz fast überall der Reihe (3) nicht hinreichend ist. Es ist natürlich zu befragen, welche Bedingung für a im Falle (1) die Konvergenz fast überall der Reihe (3) sichert. In dieser Note werden wir auf diese Frage eine genaue Antwort geben.

2. Ohne Beschränkung der Allgemeinheit können wir $\lambda_1 \geqq 1$ voraussetzen. Für jede positive ganze Zahl l bezeichne $Z(l)$ die Menge der positiven ganzen Zahlen k, mit $2^l < \lambda_k \leqq 2^{l+1}$. Es seien $l_1 < \ldots < l_i < \ldots$ diejenigen Indizes, für die $Z(l_i) \neq \emptyset$ ist; die Elemente von $Z(l)$ seien in der natürlichen Anordnung $\nu(i)+1, \ldots, \nu(i+1)$. Für eine Folge a setzen wir

$$A_i^2 = \sum_{k=\nu(i)+1}^{\nu(i+1)} a_k^2 \lambda_k \quad (i = 1, 2, \ldots).$$

Satz I. *Ist* (1) *erfüllt, und gilt für die Folge a*

$$\sum_{i=1}^{\infty} A_i < \infty,$$

so konvergiert die Reihe (3) *fast überall in X.*

Beweis. Wir wenden die Methode von Alexits und Sharma an. Die n-te Partialsumme der Reihe (3) bezeichnen wir mit $s_n(x)$. Für eine positive ganze Zahl i setzen wir

$$\delta_i(x) = \max_{\nu(i) < n \leqq \nu(i+1)} |s_n(x) - s_{\nu(i)}(x)|,$$

$$E_i^+ = \left\{x \in X\colon \max_{\nu(i) < n \leqq \nu(i+1)} \left(s_n(x) - s_{\nu(i)}(x)\right)\right\},$$

$$E_i^- = \left\{x \in X\colon \max_{\nu(i) < n \leqq \nu(i+1)} \left(-(s_n(x) - s_{\nu(i)}(x))\right)\right\};$$

$n(x)$ bezeichne die kleinste positive ganze Zahl $\left(\nu(i) < n(x) \leqq \nu(i+1)\right)$, für die

$$s_{n(x)}x - s_{\nu(i)}(x) = \max_{\nu(i) < n \leqq \nu(i+1)} \left(s_n(x) - s_{\nu(i)}(x)\right) \quad (x \in E_i^+)$$

ist. Dann gibt mit dem Rademacherschen System $\{r_k(t)\}_1^\infty$

$$\int_{E_i^+} \max_{\nu(i) < n \leqq \nu(i+1)} \left(s_n(x) - s_{\nu(i)}(x)\right) d\mu(x) = \int_0^1 \left(\sum_{k=\nu(i)+1}^{\nu(i+1)} a_k \sqrt{\lambda_k}\, r_k(t)\right) \left(\sum_{p=\nu(i)+1}^{m(x)} \frac{r_p(t)\varphi_p(x)}{\sqrt{\lambda_p}}\right) dt$$
$$(x \in E_i^+).$$

Durch Anwendung der Bunjakowski—Schwarzschen Ungleichung und des Fubinischen Satzes ergibt sich:

$$\int\limits_{E_i^+} \big(\max_{\nu(i)<n\leq\nu(i+1)} (s_n(x)-s_{\nu(i)}(x)\big)\, d\mu(x) = \tag{5}$$

$$= \int\limits_0^1 \Bigg\{\Big(\sum_{k=\nu(i)+1}^{\nu(i+1)} a_k\sqrt{\lambda_k}\, r_k(t)\Big) \int\limits_{E_i^+}\Big(\sum_{p=\nu(i)+1}^{n(x)} \frac{r_p(t)\varphi_p(x)}{\sqrt{\lambda_p}}\Big)\, d\mu(x)\Bigg\}\, dt \leq$$

$$\leq A_i\Bigg\{\int\limits_0^1 \int\limits_{E_i^+}\int\limits_{E_i^+}\Big(\sum_{p=\nu(i)+1}^{n(x)} \frac{r_p(t)\varphi_p(x)}{\sqrt{\lambda_p}}\Big)\Big(\sum_{q=\nu(i)+1}^{n(y)} \frac{r_q(t)\varphi_q(y)}{\sqrt{\lambda_q}}\Big)\, d\mu(x)\, d\mu(y)\, dt\Bigg\}^{1/2} \leq$$

$$\leq A_i\Bigg\{\int\limits_{E_i^+}\int\limits_{E_i^+}\Bigg|\sum_{p=\nu(i)+1}^{\min(n(x),\, n(y))} \frac{\varphi_p(x)\varphi_p(y)}{\lambda_p}\Bigg|\, d\mu(x)\, d\mu(y)\Bigg\}^{1/2} \leq$$

$$\leq \sqrt{2}A_i\Bigg\{\int\limits_X\Bigg(\int\limits_X\Bigg|\sum_{k=\nu(i)+1}^{n(y)} \frac{\varphi_k(x)\varphi_k(y)}{\lambda_k}\Bigg|\, d\mu(x)\Bigg)\, d\mu(y)\Bigg\}^{1/2}.$$

Auf Grund der Voraussetzung (1) gibt es eine positive Konstante K, für die $L_n(\varphi;\, x)\leq$ $\leq K\lambda_n$ $(x\in X;\ n=1, 2, \ldots)$ erfüllt ist. Durch eine Abelsche Umformung bekommen wir

$$\sum_{k=\nu(i)+1}^{n(y)} \frac{\varphi_k(x)\varphi_k(y)}{\lambda_k} = \sum_{k=\nu(i)+1}^{n(y)-1}\Big(\frac{1}{\lambda_k}-\frac{1}{\lambda_{k+1}}\Big)\sum_{s=\nu(i)+1}^{k}\varphi_s(x)\varphi_s(y) + \frac{1}{\lambda_{n(y)}}\sum_{s=\nu(i)+1}^{n(y)}\varphi_s(x)\varphi_s(y),$$

woraus folgt

$$\int\limits_X \Bigg|\sum_{k=\nu(i)+1}^{n(y)} \frac{\varphi_k(x)\varphi_k(y)}{\lambda_k}\Bigg|\, d\mu(x) \leq$$

$$\leq \sum_{k=\nu(i)+1}^{n(y)-1}\Big(\frac{1}{\lambda_k}-\frac{1}{\lambda_{k+1}}\Big)\int\limits_X\Bigg|\sum_{s=\nu(i)+1}^{k}\varphi_s(x)\varphi_s(y)\Bigg|\, d\mu(x) +$$

$$+\frac{1}{\lambda_{n(y)}}\int\limits_X\Bigg|\sum_{s=\nu(i)+1}^{n(y)}\varphi_s(x)\varphi_s(y)\Bigg|\, d\mu(x) \leq$$

$$\leq \sum_{k=\nu(i)+1}^{n(y)-1}\Big(\frac{1}{\lambda_k}-\frac{1}{\lambda_{k+1}}\Big)\big(L_k(\varphi;\, y)+L_{\nu(i)}(\varphi;\, y)\big) + \frac{1}{\lambda_{n(y)}}\big(L_{n(y)}(\varphi;\, y)+L_{\nu(i)}(\varphi;\, y)\big) \leq$$

$$\leq 4K\lambda_{\nu(i+1)}/\lambda_{\nu(i)+1} \leq 8K$$

für jedes $x\in X$, auf Grund der Definition der Folge $\{\nu(i)\}_1^\infty$. Daraus und aus (5) erhalten wir

$$\int\limits_{E_i^+} \max_{\nu(i)<n\leq\nu(i+1)} (s_n(x)-s_{\nu(i)}(x))\, d\mu(x) \leq \sqrt{2}\,8K\mu(X)A_i. \tag{6}$$

Durch Anwendung dieser Ungleichung auf das System $\{-\varphi_k(x)\}_1^\infty$ ergibt sich

$$\int_{E_i^-} \max_{\nu(i)<n\leqq\nu(i+1)} (-(s_n(x)-s_{\nu(i)}(x)))\, d\mu(x) \leqq \sqrt{2}\, 8K\mu(X)A_i.$$

Daraus und aus (6) folgt

$$\int_X \delta_i(x)\, d\mu(x) \leqq \sqrt{2}\, 16K\mu(X)A_i \quad (i=1,2,\dots). \tag{7}$$

Endlich aus (4) bekommen wir, daß

$$\sum_{i=1}^\infty \delta_i(x) < \infty$$

in X fast überall besteht. Da auf Grund der Definition von $\delta_i(x)$ die Ungleichung $|s_{\nu(i+1)}(x)-s_{\nu(i)}(x)|\leqq\delta_i(x)$ $(x\in X;\ i=1,2,\dots)$ gilt, ergibt sich, daß

$$\sum_{i=1}^\infty |s_{\nu(i+1)}(x)-s_{\nu(i)}(x)| < \infty$$

in X fast überall besteht und so $\lim_{i\to\infty} s_{\nu(i)}(x)$ fast überall in X existiert. Im Falle $\nu(i)<n\leqq\nu(i+1)$ gilt weiterhin $|s_n(x)-s_{\nu(i)}(x)|\leqq\delta_i(x)\to 0$ $(i\to\infty)$ in X fast überall, und so konvergiert die Reihe (3) in X fast überall.

3. Wir zeigen, daß die Bedingung (4) genau ist.

Satz II. *Gilt*

$$\sum_{i=1}^\infty A_i = \infty, \tag{8}$$

so gibt es ein System $\Phi=\{\Phi_k(x)\}_1^\infty$ *von reellen Funktionen in* $L(0,1)$ *derart, daß*

$$L_n(\Phi;\, x) = \int_0^1 \left|\sum_{k=1}^\infty \Phi_k(x)\Phi_k(t)\right| dt \leqq 16\lambda_n \quad (x\in(0,1);\ n=1,2,\dots)$$

besteht und die Reihe

$$\sum_{k=1}^\infty a_k \Phi_k(x) \tag{9}$$

in (0, 1) *überall divergiert.*

Beweis. Für jede positive ganze Zahl i seien $I_s(i)$ $(s=\nu(i)+1,\dots,\nu(i+1))$ disjunkte Intervalle mit

$$\bigcup_{s=\nu(i)+1}^{\nu(i+1)} I_s(i) = (0,1), \mathrm{mes}\, I_s(i) = a_s^2 \Big/ \sum_{k=\nu(i)+1}^{\nu(i+1)} a_k^2 \quad \text{und} \quad I_s(i)=\emptyset, \quad \text{wenn} \quad a_s=0.$$

Für einen Index s mit $\nu(i)<s\leqq\nu(i+1)$ und $a_s\neq0$ setzen wir

$$\Phi_s(x)=\begin{cases}A_i/a_s, & x\in I_s(i),\\ 0, & \text{sonst};\end{cases}$$

im Falle $a_s=0$ sei $\Phi_s(x)\equiv0$.

Sei i_0 eine positive ganze Zahl und sei $x\in(0,1)$. Dann gibt es für jede positive ganze Zahl i $(1\leqq i\leqq i_0)$ einen Index $s(x;i)$ $\big(\nu(i)<s(x;i)\leqq\nu(i+1)\big)$ mit $x\in I_{s(x;i)}(i)$. Man hat dann

$$\sum_{k=1}^{\nu(i_0+1)} a_k\Phi_k(x)=\sum_{i=1}^{i_0} a_{s(x;i)}\Phi_{s(x;i)}(x).$$

Daraus, auf Grund der Definition der Funktionen $\Phi_k(x)$, folgt

$$\sum_{k=1}^{\nu(i_0)+1} a_k\Phi_k(x)=\sum_{i=1}^{i_0} A_i \quad (x\in(0,1);\ i_0=1,2,\ldots). \tag{10}$$

Aus (8) ergibt sich, daß die Reihe (9) in (0, 1) überall divergiert.

Es sei i eine positive ganze Zahl, $\nu(i)<n\leqq\nu(i+1)$ und $x\in(0,1)$. Dann gibt es einen Index $s(x;i)$ $\big(\nu(i)<s(x;i)\leqq\nu(i+1)\big)$ mit $x\in I_{s(x;i)}(i)$, und so gilt

$$\int_0^1\left|\sum_{k=\nu(i)+1}^{n}\Phi_k(x)\Phi_k(t)\right|dt\leqq\int_{I_{s(x;i)}(i)}|\Phi_{s(x;i)}(x)\Phi_{s(x;i)}(t)|\,dt=\frac{A_i^2}{a_{s(x;i)}^2}\operatorname{mes} I_{s(x;i)}(i).$$

Daraus folgt, auf Grund der Definition von A_i und $\nu(i)$,

$$\int_0^1\left|\sum_{k=\nu(i)+1}^{n}\Phi_k(x)\Phi_k(t)\right|dt\leqq 2\lambda_n \quad (x\in(0,1;\ \nu(i)<n\leqq\nu(i+1);\ i=1,2,\ldots). \tag{11}$$

Es sei n eine beliebige positive ganze Zahl. Dann gibt es einen Index i_0 mit $\nu(i_0)<$ $<n\leqq\nu(i_0+1)$, und gilt

$$L_n(\Phi;x)\leqq\sum_{i=1}^{i_0-1}\int_0^1\left|\sum_{k=\nu(i)+1}^{\nu(i+1)}\Phi_k(x)\Phi_k(t)\right|dt+\int_0^1\left|\sum_{k=\nu(i_0)+1}^{n}\Phi_k(x)\Phi_k(t)\right|dt\leqq$$
$$\leqq 2(\lambda_{\nu(2)}+\ldots+\lambda_{\nu(i_0+1)}+\lambda_n)\leqq 4(2^2+\ldots+2^{i_0+1})\leqq 16\cdot2^{i_0}\leqq 16\lambda_{\nu(i_0)+1}\leqq 16\lambda_n$$

für jedes $x\in(0,1)$.

Damit haben wir Satz II bewiesen.

4. Für eine positive Konstante K bezeichne $\Omega(\lambda,K)$ die Klasse der Systeme $\varphi=\{\varphi_k(x)\}_1^\infty$ von reellen Funktionen in $L(0,1)$ für die

$$L_n(\varphi;x)=\int_0^1\left|\sum_{k=1}^{n}\varphi_k(x)\varphi_k(t)\right|dt\leqq K\lambda_n \quad (x\in(0,1);\ n=1,2,\ldots)$$

gilt, und sei $\Omega(\lambda)$ die Klasse der Systeme φ mit

$$L_n(\varphi;x)=O(\lambda_n) \quad (x\in(0,1);\ n=1,2,\ldots).$$

$M(\lambda)$ bezeichne die Klasse der Folgen a, für die die Reihe (3) bei jedem System $\varphi \in \Omega(\lambda)$ in (0, 1) fast überall konvergiert. Endlich wird für eine Folge a

$$\|a; \lambda\| = \sup_{\varphi \in \Omega(\lambda; 1)} \int_0^1 \sup_n |s_n(x)| \, dx$$

gesetzt. In [3] haben wir bewiesen:

$a \in M(\lambda)$ gilt dann und nur dann, wenn $\|a; \lambda\| < \infty$.

Nach den vorigen Resultaten kann man $\|a; \lambda\|$ auswerten.

Satz III. *Für jede Folge a gilt*

$$C_1 \sum_{i=1}^{\infty} A_i \leqq \|a; \lambda\| \leqq C_2 \sum_{i=1}^{\infty} A_i$$

mit positiven Konstanten C_1, C_2.

Beweis. Da

$$\sup_n |s_n(x)| \leqq \sum_{i=1}^{\infty} \delta_i(x)$$

ist, erhalten wir

$$\int_0^1 \sup_n |s_n(x)| \, dx \leqq \sqrt{2}\, 16 \sum_{i=1}^{\infty} A_i$$

auf Grund von (7) für jedes System $\varphi \in \Omega(\lambda; 1)$; woraus die zweite Ungleichung mit $C_2 = \sqrt{2}\, 16$ folgt.

Weiterhin sei $\varphi_k(x) = \Phi_k(x)/4$ $(k = 1, 2, \ldots)$ mit den in § 3 definierten Funktionen $\Phi_k(x)$. Dann gilt $\varphi \in \Omega(\lambda; 1)$ nach dem Satz II. Weiterhin bekommen wir aus (10)

$$\int_0^1 \sup_n |s_n(x)| \, dx \geqq \frac{1}{4} \sum_{i=1}^{\infty} A_i,$$

also besteht die erste Ungleichung mit $C_1 = 1/4$.

5. Bemerkungen. 1) G. Alexits und A. Sharma [1] haben Systeme $\varphi = \{\varphi_n(x)\}_1^{\infty}$ von Funktionen in $L(X)$ betrachtet, für die

$$L_n(\varphi; x) = O(1) \quad (x \in X;\ n = 1, 2, \ldots)$$

gilt, und haben Folgendes bewiesen: Ist die Summe

$$\sum_{k=1}^{\infty} a_k^2 \tag{12}$$

endlich, so konvergiert die Reihe (3) in X fast überall.

Weiterhin haben wir in [4] Folgendes bewiesen: Ist die Summe (12) unendlich, so gibt es ein orthonormiertes System $\Phi=\{\Phi_k(x)\}_1^\infty$ im Grundintervall (0,1) mit

$$L_n(\Phi; x) = O(1) \quad (x\in(0,1);\ n=1,2,\ldots)$$

derart, daß die Reihe (9) in (0, 1) fast überall divergiert.

Diese Sätze sind in den Sätzen I—II enthalten. Im Falle $\lambda_k=4$ $(k=1,2,\ldots)$ ist nämlich die Konvergenz der Reihe (4) mit der Konvergenz der Reihe (12) äquivalent.

2) Im Falle $\lambda_k\to\infty$ $(k\to\infty)$ kann in Satz II das System Φ im allgemeinen nicht normiert gewählt werden. Ist nämlich $\sum_{k=1}^{\infty}|a_k|<\infty$, und gilt

$$\int_X \Phi_k^2(x)\,d\mu(x) = 1 \quad (k=1,2,\ldots),$$

so konvergiert die Reihe (9) in X fast überall.

3) Es gilt auch der folgende Satz.

Satz IV. *Es sei* $\varphi=\{\varphi_k(x)\}_1^\infty$ *ein System von Funktionen in* $L(X)$ *mit* $L_n(\varphi; x)\leqq K\lambda_n (x\in X;\ n=1,2,\ldots)$. *Dann gilt*

$$\int_X |\varphi_k(x)|\,d\mu(x) \leqq 2\sqrt{K\mu(X)\lambda_n} \quad (n=1,2,\ldots). \tag{13}$$

Beweis. Für eine positive ganze Zahl n seien

$$E_n^+ = \{x\in X\colon\ \varphi_n(x)>0\},\quad E_n^- = \{x\in X\colon\ \varphi_n(x)<0\}.$$

Da

$$\varphi_n(x) = \int_0^1 r_n(t)\Big(\sum_{k=1}^{n} r_k(t)\varphi_k(x)\Big)\,dt$$

gilt, hat man

$$\int_{E_n^+}\varphi_n(x)\,d\mu(x) = \int_0^1 r_n(t)\Big(\int_{E_n^+}\Big(\sum_{k=1}^{n} r_k(t)\varphi_k(x)\Big)d\mu(x)\Big)dt \leqq$$

$$\leqq \Big\{\int_0^1\int_{E_n^+}\int_{E_n^+}\Big(\sum_{p=1}^{n} r_p(t)\varphi_p(x)\Big)\Big(\sum_{q=1}^{n} r_q(t)\varphi_q(y)\Big)d\mu(x)\,d\mu(y)dt\Big\}^{1/2} \leqq$$

$$\leqq \Big\{\int_X\Big(\int_X\Big|\sum_{k=1}^{n}\varphi_k(x)\varphi_k(y)\Big|\,d\mu(x)\Big)d\mu(y)\Big\}^{1/2} \leqq \Big\{\int_X L_n(\varphi;\ y)\,d\mu(y)\Big\}^{1/2} \leqq$$

$$\leqq \sqrt{K\mu(X)\lambda_n}.$$

Durch Anwendung dieser Ungleichung auf das System $\{-\varphi_k(x)\}_1^\infty$ ergibt sich

$$\int_{E_n^-}(-\varphi_n(x))\,d\mu(x) \leqq \sqrt{K\mu(X)\lambda_n}.$$

Diese zwei Ungleichungen ergeben die Behauptung (13).

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BOLYAI INSTITUT
UNIVERSITÄT SZEGED
ARADI VÉRTANÚK TERE 1
6720 SZEGED, UNGARN

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The maximal function of a contraction

ILIE VALUŞESCU

1. Let $\mathfrak{E}$ be a separable Hilbert space. An $\mathcal{L}(\mathfrak{E})$-valued *semi-spectral measure* F on the unit circle $\mathbf{T}$ is a map from the family of the Borel sets $\mathcal{B}(\mathbf{T})$ of the unit circle into $\mathcal{L}(\mathfrak{E})$, such that for any $a\in\mathfrak{E}$, $\sigma\to(F(\sigma)a, a)$ is a positive Borel measure. A semi-spectral measure E is *spectral* if for any σ_1, σ_2 in $\mathcal{B}(\mathbf{T})$ we have $E(\sigma_1\cap\sigma_2)=$ $=E(\sigma_1)E(\sigma_2)$ and $E(\mathbf{T})=I_{\mathfrak{E}}$.

By the Naimark dilation theorem, for any $\mathcal{L}(\mathfrak{E})$-valued semi-spectral measure F there exists a spectral dilation $[\mathfrak{K}, V, E]$, i.e., a Hilbert space $\mathfrak{K}$, a bounded operator V from $\mathfrak{E}$ into $\mathfrak{K}$ and an $\mathcal{L}(\mathfrak{K})$-valued spectral measure E on $\mathbf{T}$ such that for any $\sigma\in\mathcal{B}(\mathbf{T})$

$$F(\sigma) = V^*E(\sigma)V. \tag{1.1}$$

For a Hilbert space $\mathfrak{F}$, we denote by $E_{\mathfrak{F}}^{\times}$ the spectral measure corresponding to the multiplication by e^{it} on $L^2(\mathfrak{F})$.

An $\mathcal{L}(\mathfrak{E})$-valued semi-spectral measure F is of *analytic type* if it admits a spectral dilation of the form $[L^2(\mathfrak{F}), V, E_{\mathfrak{F}}^{\times}]$ such that $V\mathfrak{E}\subset L_+^2(\mathfrak{F})$. The name is justified by the fact that there exists an analytic function $\{\mathfrak{E}, \mathfrak{F}, \Theta(\lambda)\}$ (see [4], [5]) such that for each $a\in\mathfrak{E}$

$$\Theta(\lambda)a = (Va)(\lambda) \quad (\lambda\in\mathbf{D}). \tag{1.2}$$

Moreover, $\{\mathfrak{E}, \mathfrak{F}, \Theta(\lambda)\}$ is an L^2*-bounded analytic function*, i.e., there exists $M>0$ such that

$$\sup_{0\leq r<1}\frac{1}{2\pi}\int_0^{2\pi}\|\Theta(re^{it})a\|^2\,dt \leq M^2\|a\|^2 \quad (a\in\mathfrak{E}). \tag{1.3}$$

Conversely, to any L^2-bounded analytic function $\{\mathfrak{E}, \mathfrak{F}, \Theta(\lambda)\}$ it corresponds an analytic type semi-spectral measure F_Θ, with a dilation as $\{L^2(\mathfrak{F}), V_\Theta, E_{\mathfrak{F}}^{\times}\}$, such that (1.2) is verified.

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An L^2-bounded analytic function $\{\mathfrak{E}, \mathfrak{F}, \Theta(\lambda)\}$ is called *outer* if

$$\bigvee_0^\infty e^{int} V_\Theta \mathfrak{E} = L_+^2(\mathfrak{F}). \tag{1.4}$$

To an arbitrary $\mathscr{L}(\mathfrak{E})$-valued semi-spectral measure F on $\mathbf{T}$ a unique outer L^2-bounded analytic function $\{\mathfrak{E}, \mathfrak{F}_1, \Theta_1(\lambda)\}$ was attached in [4] such that the corresponding semi-spectral measure F_{Θ_1} is maximal among the $\mathscr{L}(\mathfrak{E})$-valued semi-spectral measures of analytic type dominated by F. This unique outer L^2-bounded analytic function is called the *maximal function* of the semi-spectral measure F. In the present note, for the semi-spectral measure F corresponding to a contraction T, some specific properties of the maximal function, in connection with the Sz.-Nagy—Foiaş model for T, are obtained.

2. Let T be a contraction on a Hilbert space $\mathfrak{H}$, and let U be its minimal unitary dilation acting on $\mathfrak{K}$. If E is the spectral measure of U, then the *semi-spectral measure of the contraction* T is the $\mathscr{L}(\mathfrak{H})$-valued semi-spectral measure obtained by the compression of E to $\mathfrak{H}$, i.e.

$$F_T(\sigma) = P_{\mathfrak{H}} E(\sigma)|\mathfrak{H} \quad (\sigma \in \mathscr{B}(\mathbf{T})). \tag{2.1}$$

Now, let us sketch the way to obtain the maximal function of F_T. If we put

$$\mathfrak{K}_+ = \bigvee_0^\infty U^n \mathfrak{H}, \tag{2.2}$$

then $U_+ = U|\mathfrak{K}_+$ is an isometry on $\mathfrak{K}_+$. Taking the Wold decomposition of U_+ on $\mathfrak{K}_+$ it follows that

$$\mathfrak{K}_+ = M_+(\mathfrak{L}_*) \oplus \mathfrak{R}, \tag{2.3}$$

where $\mathfrak{L}_* = \mathfrak{K}_+ \ominus U_+ \mathfrak{K}_+$, $M_+(\mathfrak{L}_*) = \bigoplus_0^\infty U_+^n \mathfrak{L}_*$ and $\mathfrak{R} = \bigcap_{n=0}^\infty U_+^n \mathfrak{K}_+$. Let $P^{\mathfrak{L}_*}$ be the orthogonal projection of $\mathfrak{K}_+$ onto $M_+(\mathfrak{L}_*)$, $\Phi^{\mathfrak{L}_*}$ the Fourier representation of $M_+(\mathfrak{L}_*)$ onto $L_+^2(\mathfrak{L}_*)$, and V_1 the bounded linear operator from $\mathfrak{H}$ into $L_+^2(\mathfrak{L}_*)$ defined by

$$V_1 = \Phi^{\mathfrak{L}_*} P^{\mathfrak{L}_*} | \mathfrak{H}. \tag{2.4}$$

Then the $\mathscr{L}(\mathfrak{H})$-valued semi-spectral measure defined by

$$F_1(\sigma) = V_1^* E_{\mathfrak{L}_*}^\times(\sigma) V_1 \quad (\sigma \in \mathscr{B}(\mathbf{T})) \tag{2.5}$$

is of analytic type. The L^2-bounded analytic function $\{\mathfrak{H}, \mathfrak{L}_*, \Theta_1(\lambda)\}$ attached to F_1, as in section 1, is the maximal function of F_T and is called the *maximal function of the contraction T*.

In the next Proposition (suggested by C. Foiaş) an explicit form of the maximal function of T is given.

Proposition 1. *The maximal function* $\{\mathfrak{H}, \mathfrak{L}_*, \Theta_1(\lambda)\}$ *of the contraction* T *on* $\mathfrak{H}$ *coincides with* $\{\mathfrak{H}, \mathfrak{D}_{T*}, \Theta(\lambda)\}$ *where*

$$\Theta(\lambda) = D_{T^*}(I-\lambda T^*)^{-1} \quad (\lambda \in \mathbf{D}). \tag{2.6}$$

Proof. We shall show that for any $\lambda \in \mathbf{D}$ and $h \in \mathfrak{H}$

$$\Theta(\lambda)h = \omega_* \Theta_1(\lambda)h \quad (\lambda \in \mathbf{D}), \tag{2.7}$$

where ω_* is the unitary operator from $\mathfrak{L}_*$ into $\mathfrak{D}_{T*}$ defined by

$$\omega_*(I_{\mathfrak{R}} - UT^*)h = D_{T^*}h. \tag{2.8}$$

If $\Theta_n\colon \mathfrak{H} \to \mathfrak{L}_*$ are the Taylor coefficients of the maximal function $\{\mathfrak{H}, \mathfrak{L}_*, \Theta_1(\lambda)\}$, then for any $h \in \mathfrak{H}$ and $l_* \in \mathfrak{L}_*$ we have

$$(\Theta_n h, l_*)_{\mathfrak{L}_*} = \frac{1}{2\pi}\int_0^{2\pi} ((V_1 h)(e^{it}), e^{int} l_*)_{\mathfrak{L}_*}\, dt = (V_1 h, e^{int}\Phi^{\mathfrak{L}_*} l_*)_{L^2(\mathfrak{L}_*)} =$$
$$= (\Phi^{\mathfrak{L}_*} P^{\mathfrak{L}_*} h, \Phi^{\mathfrak{L}_*} U^n l_*)_{L^2(\mathfrak{L}_*)} = (P^{\mathfrak{L}_*} h, U^n l_*)_{\mathfrak{R}} = (U^{*n} P^{\mathfrak{L}_*} h, l_*)_{\mathfrak{R}} = (P^{\mathfrak{L}_*} U^{*n} h, l_*)_{\mathfrak{L}_*}.$$

Hence, the coefficients of $\Theta_1(\lambda)$ are of the form

$$\Theta_n = P^{\mathfrak{L}_*} U^{*n} | \mathfrak{H}. \tag{2.9}$$

In order to prove (2.7) it is enough to show that

$$\Theta_n h = (I-UT^*)T^{*n}h \quad \text{for} \quad n \geqq 0,$$

or, by (2.9), that

$$U^{*n}h - (I-UT^*)T^{*n}h \perp \mathfrak{L}_*. \tag{2.10}$$

But, for any $h, h' \in \mathfrak{H}$ we have

$$(U^{*n}h - (I-UT^*)T^{*n}h, (I-UT^*)h') =$$
$$= (U^{*n}h - T^{*n}h + UT^{*n+1}h, h') - (U^{*n+1}h - U^*T^{*n}h + T^{*n+1}h, T^*h') =$$
$$= (T^{*n}h - T^{*n}h + TT^{*n+1}h, h') - (T^{*n+1}h - T^{*n+1}h + T^{*n+1}h, T^*h') =$$
$$= (TT^{*n+1}h, h') - (T^{*n+1}h, T^*h') = 0.$$

Hence (2.10) is valid and then so is (2.7). Thus the proof of Proposition 1 is done.

Remark that the maximal function of T is not zero unless T is a coisometric operator.

From the fact that the characteristic function of T

$\Theta_T(\lambda) = [-T + \lambda D_{T^*}(I-\lambda T^*)^{-1} D_T] | \mathfrak{D}_T$ verifies $\Theta_T(\lambda) D_T = D_{T^*}(I-\lambda T^*)^{-1}(\lambda I - T)$

(see [5]) it results the following relation between the maximal function and the characteristic function of the contraction T:

$$\Theta_T(\lambda) D_T = \Theta(\lambda)(\lambda I - T) \quad (\lambda \in \mathbf{D}). \tag{2.11}$$

If the contraction T belongs to the class $C_{\cdot 0}$ (i.e. $T^{*n}\to 0$ as $n\to\infty$) then $\mathfrak{K}=M(\mathfrak{L}_*)$, and thus by (2.3)—(2.5) it results that the semi-spectral measure of T is of analytic type if and only if $T\in C_{\cdot 0}$. In this case the contraction T is uniquely determined (up to unitary equivalence) by its maximal function. Moreover, in the $C_{\cdot 0}$ case, the maximal function gives an explicit form of the imbedding of $\mathfrak{H}$ into the space $\mathbf{H}=H^2(\mathfrak{D}_{T*})\ominus\Theta_T H^2(\mathfrak{D}_T)$ of the Sz.-Nagy—Foiaş functional model for T.

Proposition 2. *Let T be a contraction of the class $C_{\cdot 0}$ on the Hilbert space $\mathfrak{H}$ and let $\{\mathfrak{H}, \mathfrak{D}_{T*}, \Theta(\lambda)\}$ be its maximal function. The image of an element $h\in\mathfrak{H}$ in the space of the functional model* $\mathbf{H}$ *is the function $u\in H^2(\mathfrak{D}_{T*})$ defined by*

$$u(\lambda)=\Theta(\lambda)h \quad (\lambda\in\mathbf{D}). \tag{2.12}$$

Proof. The functional model (see [5], Ch. VI) is obtained by a unitary imbedding Φ of the dilation space $\mathfrak{K}$ of T into a functional space.

In the $C_{\cdot 0}$ case $\mathfrak{K}=M(\mathfrak{L}_*)$, $\Phi=\Phi^{\mathfrak{D}_{T^*}}$ and it follows that

$$\mathbf{H}=\Phi\mathfrak{H}=\Phi^{\mathfrak{D}_{T^*}}\mathfrak{H}.$$

From (2.4) and (2.7) it results that $\mathbf{H}=V_\Theta\mathfrak{H}$ and, consequently, $u\in\mathbf{H}$ is given by

$$u(\lambda)=(V_\Theta h)(\lambda)=\Theta(\lambda)h \quad (\lambda\in\mathbf{D}).$$

The proof is finished.

3. In general, the maximal function of a contraction is not bounded. If $\{\mathfrak{H}, \mathfrak{L}_*, \Theta_1(\lambda)\}$ is bounded, then there exists $\Theta_1(e^{it})$ a.e. as non-tangential strong limit of $\Theta_1(\lambda)$ and

$$dF_{\Theta_1}=\frac{1}{2\pi}\Theta_1(e^{it})^*\Theta_1(e^{it})\,dt \quad \text{a.e.} \tag{3.1}$$

Concerning the boundedness of $\Theta_1(\lambda)$, we have the following

Proposition 3. *The L^2-bounded analytic function $\{\mathfrak{E}, \mathfrak{F}, \Theta(\lambda)\}$ is bounded if and only if the corresponding semi-spectral measure F_Θ is boundedly dominated by the Lebesgue measure dt on* $\mathbf{T}$.

Proof. If $\{\mathfrak{E}, \mathfrak{F}, \Theta(\lambda)\}$ is bounded, then for any analytic polynomial p and for $a\in\mathfrak{E}$

$$\int_0^{2\pi}|p|^2 d(F_\Theta(t)a, a)=\frac{1}{2\pi}\int_0^{2\pi}|p|^2(\Theta(e^{it})^*\Theta(e^{it})a, a)\,dt=$$

$$=\frac{1}{2\pi}\int_0^{2\pi}|p|^2\|\Theta(e^{it})a\|^2dt\leqq M^2\frac{1}{2\pi}\int_0^{2\pi}|p|^2\|a\|^2dt.$$

It results that $dF_\Theta \leqq M^2 \frac{1}{2\pi}\, dt$. Conversely, if $dF_\Theta \leqq M^2 \frac{1}{2\pi}\, dt$, then

$$\int_0^{2\pi} |p|^2 \|(V_\Theta a)(t)\|^2\, dt = \|pV_\Theta a\|^2_{L^2(\mathfrak{F})} = \int_0^{2\pi} |p|^2 d(E_{\mathfrak{F}}^{\times} V_\Theta a, V_\Theta a) =$$

$$= \int_0^{2\pi} |p|^2 d\big(F_\Theta(t)a, a\big) \leqq M^2 \frac{1}{2\pi} \int_0^{2\pi} |p|^2 \|a\|^2\, dt.$$

It follows that

$$\|(V_\Theta a)(t)\| \leqq M\|a\| \quad \text{a.e.} \tag{3.2}$$

Using the Poisson integral of $\Theta(\lambda)$ and (3.2), it results that

$$\|\Theta(\lambda)a\| = \left\| \frac{1}{2\pi} \int_0^{2\pi} P_\lambda(t)(V_\Theta a)(t)\, dt \right\| \leqq$$

$$\leqq \frac{1}{2\pi} \int_0^{2\pi} P_\lambda(t) \|(V_\Theta a)(t)\|\, dt \leqq M\|a\| \frac{1}{2\pi} \int_0^{2\pi} P_\lambda(t)\, dt = M\|a\|$$

and the proof is finished.

It is known [2] that the contraction T with the spectral radius $\varrho(T)<1$ is characterized by the fact that the semi-spectral measure F_T has bounded derivative. Therefore the following holds.

Corollary. *If the spectrum of the contraction T is in the open unit disc, then the maximal function $\{\mathfrak{H}, \mathfrak{L}_*, \Theta_1(\lambda)\}$ is bounded.*

Moreover, the above quoted result of Schreiber can be completed in the following manner.

Proposition 4. *A contraction T on a Hilbert space $\mathfrak{H}$ has the semi-spectral measure F_T of the form $dF_T = \Theta(e^{it})^* \Theta(e^{it})dt$, with $\{\mathfrak{H}, \mathfrak{F}, \Theta(\lambda)\}$ a bounded analytic function, if and only if $T \in C_{\cdot 0}$ and $\varrho(T)<1$. Moreover, the bounded analytic function $\{\mathfrak{H}, \mathfrak{F}, \Theta(\lambda)\}$ has a bounded inverse if and only if T is a strict contraction.*

Proof. By the form of dF_T it follows that F_T is of analytic type i.e. $T \in C_{\cdot 0}$. The boundedness of $\Theta(\lambda)$ implies that F_T has bounded derivative and from [2] it results that $\varrho(T)<1$.

Conversely, if $T \in C_{\cdot 0}$ and $\varrho(T)<1$, then $F_T = F_\Theta$, and using the above Corollary the function $\Theta(\lambda)$ is bounded and $dF_T = \Theta(e^{it})^* \Theta(e^{it})dt$.

If, moreover, $dF_T = \Theta(e^{it})^* \Theta(e^{it})dt$ and $\{\mathfrak{H}, \mathfrak{F}, \Theta(\lambda)\}$ has a bounded inverse, then the associated operator Θ from $L^2_+(\mathfrak{H})$ into $L^2_+(\mathfrak{L}_*)$ defined by

$$(\Theta u)(e^{it}) = \Theta(e^{it})u(t) \quad \big(u \in L^2_+(\mathfrak{H})\big) \tag{3.3}$$

is boundedly invertible, and for any trigonometric polynomial p and for $h\in\mathfrak{H}$ we have

$$\int_0^{2\pi} |p(e^{it})|^2 d\big(F_T(t)h, h\big) = \int_0^{2\pi} |p(e^{it})|^2\big(\Theta(e^{it})^*\Theta(e^{it})h, h\big)\,dt =$$

$$= \int_0^{2\pi} \|\Theta(e^{it})p(e^{it})h\|^2 dt \leqq \|\Theta\|^2 \int_0^{2\pi} |p(e^{it})|^2\|h\|^2 dt.$$

Also, we have

$$\int_0^{2\pi} \|\Theta(e^{it})p(e^{it})h\|^2 dt \geqq \|\Theta^{-1}\|^{-2} \int_0^{2\pi} \|\Theta(e^{it})^{-1}\Theta(e^{it})p(e^{it})h\|^2 dt =$$

$$= \|\Theta^{-1}\|^{-2} \int_0^{2\pi} |p(e^{it})|^2\|h\|^2 dt.$$

For any positive continuous function φ on $\mathbf{T}$ it follows that

$$\|\Theta^{-1}\|^{-2} \int_0^{2\pi} \varphi\, dt \leqq \int_0^{2\pi} \varphi\, dF_T(t) \leqq \|\Theta\|^2 \int_0^{2\pi} \varphi\, dt.$$

Therefore, there exists a positive constant c such that

$$c\,dt \leqq dF_T \leqq c^{-1}dt. \tag{3.4}$$

But (3.4) holds (see [1], [3]) if and only if T is a strict contraction.

Now, let us suppose that T is a strict contraction. Then F_T is of analytic type, $F_T = F_\Theta$ where $\Theta(\lambda)$ is the maximal function of T, and (3.4) implies that the bounded operator Θ defined by (3.3) has a bounded inverse. By the fact that Θ intertwines the shift operators in $L^2_+(\mathfrak{H})$ and $L^2_+(\mathfrak{L}_*)$, using Lemma 3.2 from [5] it follows that $\Theta(\lambda)$ has a bounded inverse.

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NATIONAL INSTITUTE FOR SCIENTIFIC AND
TECHNICAL CREATION
DEPARTMENT OF MATHEMATICS
BD. PĂCII 220
77538 — BUCHAREST, ROMANIA

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A problem of Sz.-Nagy

JAN A. VAN CASTEREN

1. Introduction

Let $\mathfrak{H}$ be a complex Hilbert space. Relatively simple proofs of the following results are given.

(a) A power bounded operator T on $\mathfrak{H}$ is similar to a unitary operator if and only if T is surjective and if there exists a constant M such that

$$(1-|\lambda|)\|x\| \leqq M\|Tx-\lambda x\|, \quad |\lambda|<1, \quad x\in\mathfrak{H}.$$

(b) Let iA be the generator of a strongly continuous group $\{P_t\colon t\in\mathbf{R}\}$ in $\mathfrak{H}$. Suppose that $\sup\{\|P_{-t}\|\colon t\geqq 0\}$ is finite. Then A is similar to a selfadjoint operator if and only if there is a constant M such that

$$\operatorname{Re}\lambda\|x\| \leqq M\|\lambda x-iAx\|, \quad \operatorname{Re}\lambda>0, \quad x\in D(A).$$

By spectral theory the "only if" parts are obvious. For a contraction T, statement (a) is due to GOHBERG and KREIN [3], who deduced it from a theorem of SZ.-NAGY and FOIAŞ [10]. In the latter theorem the authors provide a sufficient condition for an invertible contraction T to be similar to a unitary operator, in terms of the characteristic operator function $\Theta_T(\lambda)$ of T. This condition is that a constant N exists for which

$$\|x\| \leqq N\|\Theta_T(\lambda)x\|, \quad |\lambda|<1, \quad x\in\mathfrak{H}.$$

For the concept of characteristic operator function and its connection with the theory of unitary dilations we refer to SZ.-NAGY and FOIAŞ [11, Chapitre VI, pp. 228—230, and Chapitre IX, p. 334].

The problem of finding a simpler proof of statement (a), avoiding characteristic functions and dilation theory, was pointed out by SZ.-NAGY in [2]. In the present paper we shall give a solution. We shall even do it for non-contractive, but power

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bounded operators. Indeed, the proof of (a) shall be reduced to the comparatively simpler theorem of SZ.-NAGY [9] which asserts that an invertible operator S is similar to a unitary operator if (and only if) $\sup\{\|S^n\|: n\in\mathbf{Z}\}$ is finite.

Statement (b), the continuous counterpart of (a), is entirely new.

2. Main results

We shall need a few definitions. A linear operator T on $\mathfrak{H}$ is said to be *power bounded* if $\sup\{\|T^n\|: n\in\mathbf{N}\}$ is finite. Let A and B be linear operators with domain and range in $\mathfrak{H}$. Then A is said to be *similar* to B if there exists a bounded linear operator V with bounded everywhere defined inverse such that $AV=VB$.

Theorem 1. *A power bounded operator T on $\mathfrak{H}$ is similar to a unitary operator if and only if it satisfies one of the following conditions (in* (ii)′ *T is supposed to be a contraction):*

(i) *T has power bounded inverse S.*

(ii) *The operators $(T-\lambda I)^{-1}$, $|\lambda|<1$, exist and*

$$\sup\{(1-|\lambda|)\|(T-\lambda I)^{-1}\|: |\lambda|<1\}<\infty.$$

(ii)′ *The operators $\Theta_T(\lambda)^{-1}$, $|\lambda|<1$, exist and*

$$\sup\{\|\Theta_T(\lambda)^{-1}\|: |\lambda|<1\}<\infty.$$

(iii) *T has an inverse S for which the operators $(I-\lambda S)^{-1}$, $|\lambda|<1$, exist and for which*

$$\liminf_{r\uparrow 1}\sup\{(1-r^2)\|(I-\lambda S)^{-1}\|: |\lambda|=r\}<\infty.$$

(iv) *T is surjective and there is a constant M such that*

$$(1-|\lambda|)\|x\|\leqq M\|Tx-\lambda x\|,\quad |\lambda|<1,\quad x\in\mathfrak{H}.$$

Proof. SZ.-NAGY [9] proves the sufficiency of (i) by means of an invariant mean on $\mathbf{Z}$. The necessity of (i) is trivial. The implications (i)⇒(ii), (ii)⇒(iii) and (iii)⇒(iv) are more or less trivial. The implication (iv)⇒(ii) follows from the fact that boundary points of the spectrum of a closed linear operator are approximate eigenvalues; e.g. HALMOS [4, Problem 63, p. 39]. In [10] SZ.-NAGY and FOIAŞ use unitary dilation theory to prove the sufficiency of (ii)′. By establishing certain mutual inequalities between $\|\Theta_T(\lambda)^{-1}\|$ and $\|(T-\lambda I)^{-1}\|$, $|\lambda|<1$, GOHBERG and KREIN [3] prove the equivalency of (ii) and (ii)′. See also KREIN [5, 6] and SZ.-NAGY and FOIAŞ [11, Chapitre IX, p. 334].

A simple proof of the implication (iii)⇒(i) runs as follows. Since it neither uses unitary dilation theory nor characteristic functions it solves a problem posed by SZ.-NAGY in [2, p. 585].

Fix x in $\mathfrak{H}$ and r in $[0, 1)$. Denote

$$M(r) = \sup\{(1-r^2)\|(I-\lambda S)^{-1}\|: |\lambda| = r\}$$

for $0<r<1$ and put $M_0=\sup\{\|T^n\|: n\in \mathbf{N}\}$. From (iii) it follows that the spectral radius $\varrho(S)$ of S satisfies $\varrho(S)\leqq 1$. Since $\|T^n\|\leqq M_0$, $n\in\mathbf{N}$, it also follows that $\varrho(T)\leqq 1$. Hence, for $|\lambda|<1$, we have norm convergence in both expansions

$$(I-\lambda S)^{-1} = \sum_{n=0}^{\infty} \lambda^n S^n, \quad (I-\bar{\lambda} T)^{-1} = \sum_{n=0}^{\infty} \bar{\lambda}^n T^n.$$

So, since $ST=I$, we have with $\lambda=re^{it}$, $0\leqq r<1$,

$$\sum_{n=-\infty}^{\infty} r^{|n|} e^{int} S^n = \sum_{n=0}^{\infty} r^n e^{int} S^n + \sum_{n=1}^{\infty} r^n e^{-int} T^n =$$

$$= (I-re^{it}S)^{-1} + re^{-it}T(I-re^{-it}T)^{-1} = (1-r^2)(I-re^{it}S)^{-1}(I-re^{-it}T)^{-1}.$$

Thus, by (iii), it follows that

$$\sum_{n=-\infty}^{\infty} r^{2|n|}\|S^n x\|^2 = \frac{1}{2\pi}\int_{-\pi}^{+\pi}\Big\|\sum_{n=-\infty}^{\infty} r^{|n|} e^{int} S^n x\Big\|^2 dt =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \|(1-r^2)(I-re^{it}S)^{-1}(I-re^{-it}T)^{-1}x\|^2 dt \leqq$$

$$\leqq M(r)^2\cdot\frac{1}{2\pi}\int_{-\pi}^{+\pi}\|(I-re^{-it}T)^{-1}x\|^2 dt =$$

$$= M(r)^2\cdot\frac{1}{2\pi}\int_{-\pi}^{+\pi}\Big\|\sum_{n=0}^{\infty} r^n e^{-int} T^n x\Big\|^2 dt = M(r)^2 \sum_{n=0}^{\infty} r^{2n}\|T^n x\|^2.$$

Consequently,

$$\sum_{n=1}^{\infty} r^{2n}\|S^n x\|^2 \leqq \big(M(r)^2-1\big)\sum_{n=0}^{\infty} r^{2n}\|T^n x\|^2.$$

Next, fix m in $\mathbf{N}$, $m\geqq 1$. Then,

$$r^{2m}\|S^m x\|^2 = (1-r^2)\sum_{n=m}^{\infty} r^{2n}\|T^{n-m}S^n x\|^2 \leqq$$

$$\leqq (1-r^2) - M_0^2 \sum_{n=m}^{\infty} r^{2n}\|S^n x\|^2 \leqq (1-r^2) M_0^2 \sum_{n=1}^{\infty} r^{2n}\|S^n x\|^2$$

and, by what is proved above,

$$r^{2m}\|S^m x\|^2 \leqq (1-r^2)M_0^2\big(M(r)^2-1\big)\sum_{n=0}^{\infty} r^{2n}\|T^n x\|^2 \leqq$$

$$\leqq (1-r^2)M_0^2\big(M(r)^2-1\big)M_0^2(1-r^2)^{-1}\|x\|^2 = M_0^4\big(M(r)^2-1\big)\|x\|^2.$$

Since $0<r<1$ is arbitrary, we conclude that

$$\|S^m\| \leqq \liminf_{r\uparrow 1} M_0^2(M(r)^2-1)^{1/2}, \quad m \geqq 1.$$

Hence (i) follows.

Remark 1. The operator $(1-r^2)(I-rS)^{-1}(I-rS^{-1})^{-1}$ can be considered as kind of an operator valued Poisson kernel.

Remark 2. In [7] SHIELDS discusses a number of boundedness properties of powers of an operator in relation to the boundedness properties of its resolvent family. See also VAN CASTEREN [12] where similar questions are considered.

*

Next we describe the continuous analogue of Theorem 1. For a proof the reader will need Stone's theorem and some other standard facts on strongly continuous semigroups. For all this we refer to YOSIDA [13].

Theorem 2. *Let iA be the generator of a strongly continuous group $\{P_t\colon t\in\mathbf{R}\}$. Assume that* $\sup\{\|P_{-t}\|\colon t\geqq 0\}$ *is finite. Then A is similar to a selfadjoint operator if and only if it satisfies one of the following conditions:*

(i) $\sup\{\|P_s\|\colon s\geqq 0\}<\infty$.

(ii) *The inverses* $(\lambda I-iA)^{-1}$, $\operatorname{Re}\lambda>0$, *exist and*

$$\sup\{\operatorname{Re}\lambda\|(\lambda I-iA)^{-1}\|\colon \operatorname{Re}\lambda>0\}<\infty.$$

(iii) *The inverses* $(\lambda I-iA)^{-1}$, $\operatorname{Re}\lambda>0$, *exist and*

$$\liminf_{\omega\downarrow 0}\sup\{\omega\|\lambda I-iA)^{-1}\|\colon \operatorname{Re}\lambda=\omega\}<\infty.$$

(iv) *There is a constant M such that*

$$\operatorname{Re}\lambda\|x\| \leqq M\|\lambda x-iAx\|, \quad \operatorname{Re}\lambda>0, \quad x\in D(A).$$

Proof. We only prove the implication (iii)$\Rightarrow$(i). Here we use Plancherel's theorem in $L^2(\mathrm{R},\mathfrak{H})$; e.g. see EDWARDS and GAUDRY [1, § 3.4, p. 53] or STEIN [8, Chapter II, § 5, pp. 45—47].

Fix x in $\mathfrak{H}$ and $\omega>0$. Put

$$M(\omega)=\sup\{2\omega\|(\lambda I-iA)^{-1}\|\colon \operatorname{Re}\lambda=\omega\}.$$

From standard semigroup considerations it follows by (iii) that the integral

$$\int_{-\infty}^{\infty} e^{-\omega|s|-i\xi s}P_s x\,ds$$

exists and that

$$\int_{-\infty}^{\infty} e^{-\omega|s|-i\xi s}P_s x\,ds = \int_0^{\infty} e^{-\omega s-i\xi s}P_s x\,ds + \int_0^{\infty} e^{-\omega s+i\xi s}P_{-s}x\,ds =$$

$$=\big((\omega-i\xi)I-iA\big)^{-1}x+\big((\omega-i\xi)I+iA\big)^{-1}x = 2\omega\big((\omega+i\xi)I-iA\big)^{-1}\big((\omega-i\xi)I+iA\big)^{-1}x.$$

So by Plancherel's theorem it follows that

$$\int_{-\infty}^{\infty} e^{-2\omega|s|}\|P_s x\|^2\,ds = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\|\int_{-\infty}^{\infty} e^{-\omega|s|-i\xi s}P_s x\,ds\Big\|^2 d\xi =$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\big\|2\omega\big((\omega+i\xi)I-iA\big)^{-1}\big((\omega-i\xi)I+iA\big)^{-1}x\big\|^2 d\xi \leqq$$

$$\leqq M(\omega)^2\frac{1}{2\pi}\int_{-\infty}^{\infty}\big\|\big((\omega-i\xi)I+iA\big)^{-1}x\big\|^2 d\xi =$$

$$=M(\omega)^2\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\|\int_0^{\infty} e^{-\omega s+i\xi s}P_{-s}x\,ds\Big\|^2 d\xi = M(\omega)^2\cdot\int_0^{\infty} e^{-2\omega s}\|P_{-s}x\|^2\,ds.$$

Put $M_0=\sup\{\|P_{-t}\|:\ t\geqq 0\}$ and fix $S>0$. Then

$$e^{-2\omega S}\|P_S x\|^2 = 2\omega\int_S^{\infty} e^{-2\omega s}\|P_{-(s-S)}P_s x\|^2\,ds \leqq$$

$$\leqq 2\omega M_0^2\int_S^{\infty} e^{-2\omega s}\|P_s x\|^2\,ds \leqq 2\omega M_0^2\int_0^{\infty} e^{-2\omega s}\|P_s x\|^2\,ds$$

and by what is proved above,

$$e^{-2\omega S}\|P_S x\|^2 \leqq 2\omega M_0^2\big(M(\omega)^2-1\big)\int_0^{\infty} e^{-2\omega s}\|P_{-s}x\|^2\,ds \leqq$$

$$\leqq 2\omega M_0^4\big(M(\omega)^2-1\big)\int_0^{\infty} e^{-2\omega s}\,ds\cdot\|x\|^2 = M_0^4\big(M(\omega)^2-1\big)\|x\|^2.$$

Consequently, we conclude that

$$\|P_s\| \leqq M_0^2\liminf_{\omega\downarrow 0}\big(M(\omega)^2-1\big)^{1/2},\quad s\geqq 0.$$

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DEPARTMENT OF MATHEMATICS
UNIVERSITAIRE INSTELLING ANTWERPEN
UNIVERSITEITSPLEIN 1
2610 WILRIJK, BELGIUM

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A note on quasitriangularity and trace-class selfcommutators

DAN VOICULESCU

In [3] C. A. BERGER and B. I. SHAW proved that for a hyponormal operator T the following inequality holds:

$$\operatorname{Tr}[T^*, T] \leqq \frac{1}{\pi} m(T)\omega(\sigma(T))$$

where $\sigma(T)$ is the spectrum of T, ω is planar Lebesgue measure and $m(T) \in \mathbf{N} \cup \{\infty\}$ denotes the multicyclicity of T. The aim of the present note is to give a new proof and an extension of the result of Berger and Shaw by connecting it with quasitriangularity relative to the Hilbert—Schmidt class. Thus, we obtain that the hyponormality condition can be replaced by the condition that the negative part $([T^*, T])_-$ of $[T^*, T]$ be trace class (the author has learned that this result has been obtained about a year ago by C. A. Berger using different methods). But even more, for such T we prove that

$$\operatorname{Tr}[T^*, T] \leqq \frac{1}{\pi} m(T+X)\,\omega(\sigma(T+X))$$

where X is any Hilbert—Schmidt operator. In particular if

$$\operatorname{Tr}[T^*, T] > \frac{1}{\pi}\,\omega(\sigma(T))$$

then every Hilbert—Schmidt perturbation of T has a non-trivial invariant subspace.

Quasitriangular operators were introduced by P. R. HALMOS [6] and it was shown by APOSTOL, FOIAŞ and VOICULESCU [2] that there is a spectral characterization of these operators. A refinement of the notion of quasitriangular operator relative to a norm-ideal was considered in [11].

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Throughout, H will denote a complex separable Hilbert space of infinite dimension. By $\mathcal{L}(H)$ we denote the bounded operators on H and by $\mathcal{P}(H)$ the set of finite-rank orthogonal projections on H with its natural order. Then the analogue of Apostol's modulus of quasitriangularity relative to a Schatten—von Neumann class is:

$$q_p(T) = \liminf_{P\in\mathcal{P}(H)} |(I-P)TP|_p$$

where $T\in\mathcal{L}(H)$ and $|X|_p=\operatorname{Tr}((X^*X)^{p/2})$ $(1\le p<\infty)$.

Then if $P_n\in\mathcal{P}(H)$ and $P_n\xrightarrow{w} I$ we have

$$\liminf_{n\to\infty} |(I-P_n)TP_n|_p \geqq q_p(T).$$

Moreover, one can find $P_n\in\mathcal{P}(H)$ such that $P_n\uparrow I$ and

$$\lim_{n\to\infty} |(I-P_n)TP_n|_p = q_p(T).$$

For $T\in\mathcal{L}(H)$ we shall denote by $\operatorname{Rat}(T)$ the algebra of operators of the form $f(T)$ where f is a rational function with poles off the spectrum $\sigma(T)$ of T. The multicyclicity $m(T)\in\mathbf{N}\cup\{\infty\}$ is the least cardinal of a set $\Xi\subset H$ such that the closed linear span of $\operatorname{Rat}(T)\Xi$ is H.

Proposition 1. *For $T\in\mathcal{L}(H)$ and $1\le p<\infty$ we have*

$$q_p(T) \le (m(T))^{1/p}\|T\|.$$

Proof. If $m(T)=\infty$ there is nothing to prove. So assume $m(T)=n<\infty$ and consider a multicyclic set $\{\xi_1, \dots, \xi_n\}$ for T_j. Consider

$$H_j = \bigvee_{k=1}^{j} \operatorname{Rat}(T)\xi_k, \quad H_0=0, \quad K_j = H_j\ominus H_{j-1}, \quad T_j = P_{K_j}T|K_j, \quad \eta_j = P_{K_j}\xi_j.$$

Then, using Proposition 2.1 of [11] we have $q_p(T)\le\left(\sum_{k=1}^{n}(q_p(T_k))^p\right)^{1/p}$. Now, it is easily seen that $\sigma(T_k)\subset\sigma(T)$ and η_k is a multicyclic vector for T_k. This reduces the proof of the proposition to the case $n=1$.

Consider a sequence $\{\lambda_j\}_{j=1}^{\infty}$ of points contained and dense in the union of the bounded components of $\mathbf{C}\setminus\sigma(T)$. Since ξ_1 is multicyclic for T, it is easily seen that denoting by P_m the projection onto the finite-dimensional subspace of H spanned by the vectors $T^k(T-\lambda_1)^{-1}\dots(T-\lambda_m)^{-1}\xi_1$ where $0\le k\le 2m$, we have $P_m\le P_{m+1}$, $P_m\uparrow I$ and $\operatorname{rank}((I-P_m)TP_m)=1$. It follows that $|(I-P_m)TP_m|_p\le\|T\|$ and hence $q_p(T)\le\|T\|$. Q.E.D.

For a hermitian operator $A\in\mathcal{L}(H)$ such that the negative part A_- of A is trace-class, we shall denote by $\operatorname{Tr} A$ the trace of A, in case A is trace-class and ∞ in case A is not trace-class.

Proposition 2. *Let $T\in\mathcal{L}(H)$ be an operator such that the negative part $([T^*, T])_-$ of $[T^*, T]$ is trace-class. Then we have* $\operatorname{Tr}[T^*, T]\leqq(q_2(T))^2$.

Proof. Let $P_m\in\mathcal{P}(H)$, $P_m\uparrow I$ be such that $\lim\limits_{m\to\infty} |(I-P_m)TP_m|_2=q_2(T)$.
We have

$$(q_2(T))^2 = \lim_{m\to\infty} |(I-P_m)TP_m|_2^2 = \lim_{m\to\infty} \operatorname{Tr}(P_mT^*TP_m-P_mT^*P_mTP_m) =$$

$$= \lim_{m\to\infty} \operatorname{Tr}\big(P_m[T^*, T]P_m+P_mT(I-P_m)T^*P_m\big) \geqq$$

$$\geqq \limsup_{m\to\infty} \operatorname{Tr}(P_m[T^*, T]P_m) = Tr[T^*, T].$$

Q.E.D.

Proposition 3. *Let $T\in\mathcal{L}(H)$ be an operator such that the negative part $([T^*, T])_-$ of $[T^*, T]$ be trace-class and let $X\in\mathcal{L}(H)$ be a Hilbert—Schmidt operator. Then we have*

$$\operatorname{Tr}[T^*,T]\leqq\frac{1}{\pi}\, m(T+X)\,\omega\big(\sigma(T+X)\big)$$

where ω denotes planar Lebesgue-measure.

Proof. It is clearly sufficient to consider the case when $m(T+X)=n<\infty$. Given $\varepsilon>0$ and denoting by Ω the open set

$$\Omega = \{z\in\mathbf{C}\colon\ |z| \leqq \|T+X\|+\varepsilon\}\setminus\sigma(T+X)$$

we can find a hyponormal operator D such that:

$$\sigma(D)\subset\Omega,\quad m(D)=n,\quad \|D\|\leqq\|T+X\|+\varepsilon,\quad [D^*, D]\geqq 0,$$

$$\operatorname{Tr}[D^*, D]\geqq\frac{n}{\pi}\big(\omega(\Omega)-\varepsilon\big).$$

Such a D is easily constructed by using the "computational lemma" of the paper of Berger and Shaw [3], or more elementarily by considering an appropriate direct sum of operators of the form $\lambda I+\mu S$ where $\lambda, \mu\in\mathbf{C}$ and S is the unilateral shift.

Using Proposition 1 we have

$$\operatorname{Tr}[(T\oplus D)^*, (T\oplus D)]\leqq\big(q_2(T\oplus D)\big)^2,$$

and hence

$$\mathrm{Tr}[T^*, T] + \frac{n}{\pi}(\omega(\Omega) - \varepsilon) \leqq (q_2(T \oplus D))^2.$$

But $q_2(T \oplus D) = q_2((T+X) \oplus D)$ since X is Hilbert—Schmidt.

Moreover, $m((T+X) \oplus D) = n = m(T+X)$ and hence, using Proposition 2, we have

$$(q_2((T+X) \oplus D))^2 \leqq (m(T+X))(\|T+X\| + \varepsilon)^2 = \frac{1}{\pi} m(T+X)(\omega(\Omega) + \omega(\sigma(T+X))).$$

It follows that $\mathrm{Tr}\,[T^*, T] \leqq \frac{1}{\pi} m(T+X)(\omega(\sigma(T+X)) - \varepsilon)$. Since $\varepsilon > 0$ is arbitrary, we have

$$\mathrm{Tr}\,[T^*, T] \leqq \frac{1}{\pi} m(T+X)\,\omega(\sigma(T+X))$$

which is the desired result. Q.E.D.

Consider $\sigma_{le}(T)$, $\sigma_{re}(T)$ the left-essential and the right-essential spectra of T and remark that if $\sigma(T+X)$ in the proposition above is bigger than $\sigma_{le}(T) \cap \sigma_{re}(T)$ then $T+X$ has a non-trivial invariant subspace. This together with Proposition 3 gives the following:

Corollary 1. *If T is an operator with $([T^*, T])_-$ trace class and*

$$\mathrm{Tr}[T^*, T] > \frac{1}{\pi}(\sigma_{le}(T) \cap \sigma_{re}(T))$$

then every operator $T+X$ with X Hilbert—Schmidt has a non-trivial invariant subspace.

Consider also $E(\sigma(T))$ the polynomially convex hull of $\sigma(T)$, i.e., the complement of the unbounded component of $\mathbf{C} \setminus \sigma(T)$ and remark that for X a compact operator $\sigma(T+X) \cap (\mathbf{C} \setminus E(\sigma(T)))$ is an at most countable set and hence $\omega(\sigma(T+X)) \leqq \omega(E(\sigma(T)))$. This together with Proposition 3 gives:

Corollary 2. *If T is an operator with $([T^*, T])_-$ trace-class and if*

$$\mathrm{Tr}[T^*, T] > \frac{1}{\pi}\omega(E(\sigma(T)))$$

then $m(T+X) > 1$ for every Hilbert—Schmidt operator X.

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INCREST BD. PĂCII 220
DEPARTMENT OF MATH.
77538 BUCHAREST, ROMANIA

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The functional model of a contraction and the space L^1

CIPRIAN FOIAŞ, CARL PEARCY and BÉLA SZ.-NAGY

The present Note is a straightforward continuation of the recent paper [I]. Indeed, we have noticed subsequently that, under slightly changed assumptions, the results of that paper can be extended from the factor space L^1/H_0^1 to the space L^1 itself, and "localized" on parts of the unit circle C.

The ingredients of these extensions are mostly taken over, with some changes, from the paper [I], and so are the notations and the terminology. When referring to a specified lemma or formula of [I] we indicate it by the subscript I. Applications to the invariant subspace problem are to be given later.

1. Let us begin with some lemmas requiring little changes with respect to [I].

Lemma 1. *If $\{a_n\}$ converges weakly to 0 in $\mathfrak{E}_*$ then for any $\varphi \in H^2$ and $h \in \mathfrak{H}$ we have*

$$\|(\varphi \circ a_n) \cdot h^*\|_{L^1} \to 0 \quad and \quad \|h \cdot (\varphi \circ a_n)^*\|_{L^1} \to 0 \quad as \quad n \to \infty.$$

(This is a strengthening of Lemma 3_{I}, where only convergence in the factor space L^1/H_0^1 was established.)

Proof. For any $h, k \in \mathfrak{H}$ the function $k \cdot h^*$ is the complex conjugate of $h \cdot k^*$ so they have the same norm in L^1. Therefore it suffices to prove the first convergence. Now, by $(4.3)_{\text{I}}$ and $(4.7)_{\text{I}}$ we have

$$\|(\varphi \circ a_n) \cdot h^*\|_{L^1} \leqq \|\varphi(a_n, h)_{\mathfrak{E}_*}\|_{L^1} + \|([\Theta^* \varphi a_n]_+, h_2)_{\mathfrak{E}}\|_{L^1} \to 0 \quad \text{as} \quad n \to \infty.$$

Lemma 2. *For any $\varphi, \psi \in H^2$ and $a \in \mathfrak{E}_*$ we have*

$$\big\|(\psi \circ a) \cdot (\varphi \circ a)^* - \psi \bar{\varphi} \|a\|^2_{\mathfrak{E}_*}\big\|_{L^1} \leqq \|\psi a\|_{H^2(\mathfrak{E}_*)} \|[\Theta^* \varphi a]_+\|_{H^2(\mathfrak{E})} + \|[\Theta^* \psi a]_+\|_{H^2(\mathfrak{E})} \|\varphi a\|_{H^2(\mathfrak{E}_*)}.$$

(This takes over the role of Lemma 4_{I}, with the unpleasant difference that here we have to increase the right hand side of the inequality by a second term.)

Proof. It readily follows from $(4.2)_{\text{I}}$ and $(4.3)_{\text{I}}$ that

$$(\psi \circ a) \cdot (\varphi \circ a)^* = \psi \bar{\varphi} \|a\|^2_{\mathfrak{E}_*} - (\psi a, \Theta [\Theta^* \varphi a]_+)_{\mathfrak{E}_*} - (\Theta [\Theta^* \psi a]_+, \varphi a)_{\mathfrak{E}_*} +$$
$$+ ([\Theta^* \psi a]_+, [\Theta^* \varphi a]_+)_{\mathfrak{E}} = \psi \bar{\varphi} \|a\|^2_{\mathfrak{E}_*} - ([\Theta^* \psi a]_-, [\Theta^* \varphi a]_+)_{\mathfrak{E}} - ([\Theta^* \psi a]_+, \Theta^* \varphi a)_{\mathfrak{E}},$$

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where $[\cdot]_- = [\cdot] - [\cdot]_+$; hence,

$$\|(\psi\circ a)\cdot(\varphi\circ a)^* - \psi\bar\varphi\|a\|^2_{\mathfrak{E}_*}\|_{L^1} \leqq$$
$$\leqq \|[\Theta^*\psi a]_-\|_{L^2(\mathfrak{E})}\|[\Theta^*\varphi a]_+\|_{L^2(\mathfrak{E})} + \|[\Theta^*\psi a]_+\|_{L^2(\mathfrak{E})}\|\Theta^*\varphi a\|_{L^2(\mathfrak{E})}.$$

Since $\|[\cdot]_-\|_{L^2(\mathfrak{E})} \leqq \|[\cdot]\|_{L^2(\mathfrak{E})}$ and since Θ^* is also contractive, the proof is done.

Lemma 3. *Suppose $\mathfrak{E}_*$ is (countably) infinite dimensional, and let $h, k \in \mathfrak{H}$; $\varphi_1, \ldots, \varphi_r, \psi_1, \ldots, \psi_r \in H^2$ and $\varepsilon > 0$ be given. Then there exist $h', k' \in \mathfrak{H}$ such that*

$$\left\|(h+h')\cdot(k+k')^* - h\cdot k^* - \sum_1^r \psi_j\bar\varphi_j\right\|_{L^1} \leqq \sum_1^r \|\psi_j\|_{H^2}\|\varphi_j\|_{H^2}\big(\eta_\Theta(\psi_j) + \eta_\Theta(\varphi_j)\big) + \varepsilon,$$
$$\|h'\|^2 \leqq \sum_1^r \|\psi_j\|^2_{H^2}, \quad \|k'\|^2 \leqq \sum_1^r \|\varphi_j\|^2_{H^2}.$$

Remark. *One can choose h', k' even to run over sequences $h^{(n)}, k^{(n)}$ $(n = 1, 2, \ldots)$ such that, for every $l \in \mathfrak{H}$, $h^{(n)}\cdot l^*$ and $k^{(n)}\cdot l^*$ tend to 0 in L^1 as $n \to \infty$.*

Proofs. Almost identical with those of Lemma 5_{I} and $\mathrm{Remark}_{\mathrm{I}}$, by using Lemmas 1 and 2 in place of Lemmas 3_{I} and 4_{I}, and applying inequality $(5.3)_{\mathrm{I}}$ both to φ_j and ψ_j.

2. More essential change is needed with Lemma 2_{I}. Its role will be taken by

Lemma 4. *Given a subset S of the open unit disc $D = \{\lambda: |\lambda| < 1\}$ let s be the set of non-tangential limit points of S on the unit circle C.* [1]) *Then for any $f \in L^1(s)$ and $\varepsilon > 0$ there exist $\mu_1, \ldots, \mu_n \in S$ and $c_1, \ldots, c_n \in \mathbf{C}$ such that*

(1) $$\left\|f - \sum_1^n c_j P_{\mu_j}\right\|_{L^1(s)} < \varepsilon \quad \textit{and} \quad \sum_1^n |c_j| \leqq \|f\|_{L^1(s)},$$

where P_μ is the Poisson kernel function on C corresponding to the point $\mu (\in D)$, i.e.

(2) $$P_\mu(e^{it}) = \frac{1-|\mu|^2}{|1-\bar\mu e^{it}|^2}.$$

Proof. Suppose there exist $f_0 \in L^1(s)$ and $\varepsilon_0 > 0$ for which the assertion does not hold, i.e. such that the open ball G in $L^1(s)$ with centre f_0 and radius ε_0 is disjoint from the set X of all finite linear combinations $\sum c_j P_{\mu_j}$ with $\mu_j \in S$, $c_j \in \mathbf{C}$, and $\sum |c_j| \leqq \|f_0\|_{L^1(s)}$. Since both G and X are convex, and G is open, there exist, by the Hahn—Banach separation theorem, a function $g_0 \in L^\infty(s)$ (the Banach dual of $L^1(s)$) and a real number α such that

(3) $$\mathrm{Re} \int_s h g_0\, dm \leqq \alpha < \mathrm{Re} \int_s f g_0\, dm$$

[1]) For any $S \subset D$, the corresponding set $s \subset C$ is a Borel set, indeed an $F_{\sigma\delta\sigma}$.

for all $h \in X$ and $f \in G$ (in particular for $f = f_0$); m denotes normalized Lebesgue measure on C.

Thus if we set

$$\tilde{g}_0(\mu) = \int_s g_0(e^{it}) P_\mu(e^{it})\, dm \quad (\mu \in D)$$

and observe that

$$\|P_\mu\|_{L^1(s)} \leqq \|P_\mu\|_{L^1} = 1, \quad \text{and hence,} \quad \|f_0\|_{L^1(s)} P_\mu \in X,$$

the first inequality in (3) shows that

$$\|f_0\|_{L^1(s)} |\tilde{g}_0(\mu)| \leqq \alpha \quad \text{for all} \quad \mu \in S. \tag{4}$$

Since $\tilde{g}_0$ is a bounded harmonic function on D, by the Fatou theorem we infer from (4) that

$$\|f_0\|_{L^1(s)} |g_0(e^{it})| \leqq \alpha \quad \text{almost everywhere on} \quad s,$$

so that

$$\mathrm{Re} \int_s f_0 g_0\, dm \leqq \|f_0\|_{L^1(s)} \|g_0\|_{L^\infty(s)} \leqq \alpha.$$

This contradicts the second inequality (3). The proof of Lemma 4 is complete.

3. In the sequel the functional $\eta_\Theta(\varphi)$ defined in [I] will again play a basic part. Let us recall, in particular, that for $\varphi = p_\mu$, where

$$p_\mu(e^{it}) = (1 - \bar{\mu} e^{it})^{-1} \quad (\mu \in D),$$

we have

$$\eta_\Theta(p_\mu) = \inf_{\mathfrak{A}} \|\Theta(\mu)^* | \mathfrak{A}\|,$$

where $\mathfrak{A}$ runs through the family of subspaces of $\mathfrak{E}_*$ of finite codimension; cf. $(2.6)_{\mathrm{I}}$.

For any number ϑ, $0 \leqq \vartheta < 1$, consider the subset

$$S_\vartheta = \{\mu \in D: \quad \eta_\Theta(p_\mu) \leqq \vartheta\} \tag{5}$$

of D, and the corresponding set s_ϑ of non-tangential limit points of S_ϑ on C.

We are going to prove the following substitute for Lemma 5_{I}:

Lemma 5. *Suppose $\mathfrak{E}_*$ is (countably) infinite dimensional and suppose $f \in L^1(s_\vartheta)$ and $h, k \in \mathfrak{H}$, and also $\varepsilon > 0$ are given. Then there exist $h', k' \in \mathfrak{H}$ such that*

$$\|(h+h') \cdot (k+k')^* - h \cdot k^* - f\|_{L^1(s_\vartheta)} \leqq 2\vartheta \|f\|_{L^1(s_\vartheta)} + 2\varepsilon,$$
$$\|h'\|, \|k'\| \leqq \|f\|_{L^1(s_\vartheta)}.$$

Proof. By Lemma 4 there exist $\mu_1, \ldots, \mu_n \in S$ and $c_1, \ldots, c_n \in \mathbf{C}$ satisfying (1) (with $s = s_\vartheta$). One can obviously assume that $c_j \neq 0$ for all j, so we can set

$$\varphi_j = |c_j|^{1/2} (1 - |\mu_j|^2)^{1/2} p_{\mu_j}, \quad \psi_j = (\operatorname{sgn} c_j) \cdot \varphi_j$$

$(j = 1, 2, \ldots, n)$. Then we have

$$\psi_j \bar{\varphi}_j = c_j P_{\mu_j}, \quad \|\psi_j\|_{H^2}^2 = \|\varphi_j\|_{H^2}^2 = |c_j|$$

so that by Lemma 3 we obtain $h', k' \in \mathfrak{H}$ such that

$$\left\| (h+h')\cdot(k+k')^* - h\cdot k^* - \sum_1^n c_j P_{\mu_j} \right\|_{L^1(s_\vartheta)} \leqq 2\vartheta \sum_1^m |c_j| + \varepsilon$$

and

$$\|h'\|^2, \|k'\|^2 \leqq \sum_1^n |c_j|.$$

Taking also account of (1) we conclude the proof.

4. Now we are ready to state the following:

Theorem. *Suppose* $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ *is a contractive analytic function, with separable* $\mathfrak{E}, \mathfrak{E}_*$ *and with* $\dim \mathfrak{E}_* = \infty$. *Suppose that for some* ϑ, $0 \leqq \vartheta < \frac{1}{2}$, *the set* s_ϑ *of non-tangential limit points of the set* S_ϑ (defined by (5)) *on C has positive Lebesgue measure. Then for every* $f \in L^1(s_\vartheta)$ *there exist* $h, k \in \mathfrak{H}$ *such that*

$$f = h\cdot k^* \quad \textit{almost everywhere on } s_\vartheta. \tag{6}$$

Proof. As in the proof of Theorem$_{\mathrm{I}}$ we choose a number ω such that $2\vartheta < \omega < 1$ and consider an $f \in L^1(s_\vartheta)$ with $\|f\|_{L^1(s_\vartheta)} \leqq 1$. Setting $h_{-1} = h_0 = k_{-1} = k_0 = 0$ (in $\mathfrak{H}$) we show by induction that there exist $h_n, k_n \in \mathfrak{H}$ $(n = 1, 2, \ldots)$ such that

$$\|f - h_n\cdot k_n^*\|_{L(s_\vartheta)} \leqq \omega^n \quad \text{and} \quad \|h_n - h_{n-1}\|^2, \ \|k_n - k_{n-1}\|^2 \leqq \omega^{n-1} \quad (n = 0, 1, \ldots). \tag{7}$$

This being obvious for $n=0$ we assume h_n, k_n to be already found for $n = 0, \ldots, q$. Setting $f_q = f - h_q\cdot k_q^*$ and $\varepsilon_q = (\omega - 2\vartheta)\omega^q/2$, by Lemma 5 we infer that there exist $h_{q+1}, k_{q+1} \in \mathfrak{H}$ such that

$$\|h_{q+1}\cdot k_{q+1}^* - h_q\cdot k_q^* - f_q\|_{L^1(s_\vartheta)} \leqq 2\vartheta\cdot\|f_q\|_{L^1(s_\vartheta)} + 2\varepsilon_q$$

and

$$\|h_{q+1} - h_q\|^2, \quad \|k_{q+1} - k_q\|^2 \leqq \|f_q\|_{L^1(s_\vartheta)} \leqq \omega^q.$$

Then we have

$$\|f - h_{q+1}\cdot k_{q+1}^*\|_{L^1(s_\vartheta)} = \|(f_q + h_q\cdot k_q^*) - h_{q+1}\cdot k_{q+1}^*\|_{L^1(s_\vartheta)} \leqq 2\vartheta\cdot\omega^q + (\omega - 2\vartheta)\omega^q = \omega^{q+1},$$

and the proof of (7) by induction is done.

By account of (7), the sequences $\{h_n\}$, $\{k_n\}$ are strongly convergent (in $\mathfrak{H}$) and their respective limits h, k satisfy (6). Theorem is proved.

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INDIANA UNIVERSITY
BLOOMINGTON, INDIANA

MICHIGAN UNIVERSITY
ANN ARBOR, MICHIGAN

UNIV. OF SZEGED
SZEGED, HUNGARY

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On contractions of class $C_1.$

PEI YUAN WU

It has been known that C_{11} contractions are quasi-similar to unitary operators. One may come to wonder what the corresponding result for the larger class of $C_{1\cdot}$ contractions is. Along this line SZ.-NAGY and FOIAŞ ([3], pp. 71—72) showed that an arbitrary $C_{1\cdot}$ contraction is a quasi-affine transform of an isometry. This result was also proved by DOUGLAS ([2], Lemma 4.5) using a different method. In the present paper we will refine the Sz.-Nagy and Foiaş technique more deeply to derive a "canonical" isometry for a completely non-unitary (c.n.u.) $C_{1\cdot}$ contraction whose defect indices are finite.

After we fix the notation and terminology in Section 1, we prove our main result in Section 2 in a series of lemmas. The notion of "multiplicity-free" $C_{1\cdot}$ contractions will be taken up in Section 3. We show that a c.n.u. multiplicity-free $C_{1\cdot}$ contraction with finite defect indices must be either of class C_{11} or of class C_{10}.

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1. Preliminaries. A contraction T ($\|T\|\leqq 1$) is *completely non-unitary (c.n.u.)* if there exists no reducing subspace on which T is unitary. The *defect indices* of T are, by definition, $d_T=\operatorname{rank}(I-T^*T)^{1/2}$ and $d_{T*}=\operatorname{rank}(I-TT^*)^{1/2}$. $T\in C_{1\cdot}$ (resp. $C_{\cdot 1}$) if $T^n x\not\to 0$ (resp. $T^{*n}x\not\to 0$) for all $x\neq 0$; $C_{11}=C_{1\cdot}\cap C_{\cdot 1}$. For every $C_{1\cdot}$ contraction T we have $d_T\leqq d_{T*}$. $T\in C_{0\cdot}$ (resp. $C_{\cdot 0}$) if $T^n x\to 0$ (resp. $T^{*n}x\to 0$) for all x; $C_{10}=C_{1\cdot}\cap C_{\cdot 0}$.

Let $\mathbf{C}$ be the complex plane. For a positive integer n, let L_n^2 and H_n^2 denote the standard Lebesgue and Hardy spaces of $\mathbf{C}^n$-valued functions defined on the unit circle C. We will use "t" to denote the argument of a function defined on C and for

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analytic functions, we will freely identify $h(t)$ on the circle with its extension to the open unit disk $h(\lambda)$. If T is a c.n.u. contraction with defect indices $d_T=m$ and $d_{T*}=n$, in the discussion of the following we shall consider its *functional model*, that is, we consider T being defined on $\mathfrak{H}\equiv[H_n^2\oplus\overline{\Delta L_m^2}]\ominus\{\Theta_T w\oplus\Delta w\colon w\in H_m^2\}$ by $T(f\oplus g)=P(e^{it}f\oplus e^{it}g)$ for $f\oplus g\in\mathfrak{H}$, where Θ_T denotes the characteristic function of T, $\Delta=(I-\Theta_T^*\Theta_T)^{1/2}$ and P denotes the (orthogonal) projection onto $\mathfrak{H}$. If Θ_T is the characteristic function of T, then the characteristic function of T^* is $\Theta_T^{\sim}$, where $\Theta_T^{\sim}(\lambda)=\Theta_T(\bar\lambda)^*$. For the details, the readers are referred to [3].

For arbitrary operators T_1, T_2 on $\mathfrak{H}_1$, $\mathfrak{H}_2$, respectively, $T_1\overset{i}{\prec}T_2$ denotes that T_1 is *injected into* T_2, that is, there exists an injection $X\colon \mathfrak{H}_1\to\mathfrak{H}_2$ such that $T_2X=XT_1$. If X also has dense range, then we say that X is a *quasi-affinity* and T_1 is a *quasi-affine transform* of T_2 (denoted by $T_1\prec T_2$). T_1, T_2 are *quasi-similar* $(T_1\sim T_2)$ if $T_1\prec T_2$ and $T_2\prec T_1$. For an arbitrary operator T on $\mathfrak{H}$, let μ_T denote the *multiplicity* of T, that is, the least cardinal number of a subset $\mathfrak{K}$ of elements in $\mathfrak{H}$ for which $\mathfrak{H}=\bigvee_{n=0}^{\infty}T^n\mathfrak{K}$. Note that if $T_1\overset{\sim}{\prec}T_2$ then $\mu_{T_2}\leqq\mu_{T_1}$.

2. $C_{1\cdot}$ contractions in general. Our purpose in this section is to prove the following main result.

Theorem 2.1. *Let T be a completely non-unitary $C_{1\cdot}$ contraction with defect indices $d_T=n\leqq d_{T*}=m<\infty$. Then $T\prec S_{m-n}\oplus U$, where S_{m-n} denotes the unilateral shift on H_{m-n}^2 and U denotes the operator of multiplication by e^{it} on $\overline{\Delta L_n^2}$.*

If T is a $C_{1\cdot}$ contraction as above, then T^* is of class $C_{\cdot 1}$ and we may consider T^* being defined on $\mathfrak{H}\equiv[H_n^2\oplus\overline{\Delta^{\sim}L_m^2}]\ominus\{\Theta_T^{\sim}w\oplus\Delta^{\sim}w\colon w\in H_m^2\}$ by $T^*(f\oplus g)=P^{\sim}(e^{it}f\oplus e^{it}g)$ for $f\oplus g\in\mathfrak{H}$, where $\Delta^{\sim}=(I-\Theta_T^{\sim *}\Theta_T^{\sim})^{1/2}$ and $P^{\sim}$ denotes the (orthogonal) projection onto $\mathfrak{H}$. Let $P_1\colon\mathfrak{H}\to\overline{\Delta^{\sim}L_m^2}$ be the operator $P_1(f\oplus g)=g$ and let V be the operator of multiplication by e^{it} on $\overline{\Delta^{\sim}L_m^2}$. Then it is easily seen that $V^*P_1=P_1T$ and P_1 is injective (cf. [3], pp. 71—72). Thus $T\prec V^*|\overline{P_1\mathfrak{H}}$. What Lemmas 2.2, 2.3 and 2.4 below show is that $V^*|\overline{P_1\mathfrak{H}}$ is unitarily equivalent to $S_{m-n}\oplus U$.

Lemma 2.2. $\overline{P_1\mathfrak{H}}=\overline{\Delta^{\sim}L_m^2}\ominus\overline{\Delta^{\sim}\mathfrak{L}}$, *where* $\mathfrak{L}=\{f\in H_m^2\colon \Theta_T^{\sim}f=0\}$.

Proof. Let k be an element of $L_n^2\oplus\overline{\Delta^{\sim}L_m^2}$. We first show that $k\in\overline{\Delta^{\sim}L_m^2}\ominus\overline{P_1\mathfrak{H}}$ if and only if $k\perp L_n^2$ and $k\perp\mathfrak{H}$. Indeed, any $h\in\mathfrak{H}$ can be written in the form $h=f+g$, where $f\perp\overline{\Delta^{\sim}L_m^2}$ and $g\in\overline{P_1\mathfrak{H}}$. If k is orthogonal to any two of the elements h, f and g, then it is also orthogonal to the third element. Our assertion follows immediately. Since $L_n^2\oplus\overline{\Delta^{\sim}L_m^2}=(L_n^2\ominus H_n^2)\oplus\mathfrak{H}\oplus\{\Theta_T^{\sim}w\oplus\Delta^{\sim}w\colon w\in H_m^2\}$, the following three conditions are equivalent:

$$k \in \overline{\Delta^{\sim} L_m^2} \ominus \overline{P_1 \mathfrak{H}},$$

$$k \in \overline{\Delta^{\sim} L_m^2} \quad \text{and} \quad 0 \oplus k = \Theta_T^{\sim} w \oplus \Delta^{\sim} w \quad \text{for some} \quad w \in H_m^2,$$

$$k \in \overline{\Delta^{\sim} L_m^2} \quad \text{and} \quad k = \Delta^{\sim} w \quad \text{for some} \quad w \in \mathfrak{L},$$

which shows that $\overline{P_1 \mathfrak{H}} = \overline{\Delta^{\sim} L_m^2} \ominus \overline{\Delta^{\sim} \mathfrak{L}}$.

Lemma 2.3. *Let S_m and S_{m-n} denote the unilateral shifts on H_m^2 and H_{m-n}^2, respectively, and let $\mathfrak{L} = \{f \in H_m^2 \colon \Theta_T^{\sim} f = 0\}$. Then $S_m | \mathfrak{L} \cong S_{m-n}$.*

Proof. Since $\mathfrak{L}$ is an invariant subspace for S_m, $\mathfrak{L} = \Phi H_q^2$ for some inner function $\{\mathbf{C}^q, \mathbf{C}^m, \Phi\}$ where $q \leqq m$. Θ_T is $*$-outer implies that $\Theta_T^{\sim}$ is outer. Hence $\ker \Theta_T^{\sim}(t)$ has dimension $m-n$ for almost all t (cf. [3], p. 191), and it follows that $q \leqq m-n$.

On the other hand, considering the quotient field derived from the algebra H^∞, we see that the equation $\Theta_T^{\sim} f = 0$ has $m-n$ linearly independent solutions: $\psi_1, \ldots, \psi_{m-n}$. That is, $\psi_1, \ldots, \psi_{m-n} \in \mathfrak{L}$ and $\psi_1(t), \ldots, \psi_{m-n}(t)$ is a linearly independent system for almost all t (cf. [4], the proof of Theorem 5). Therefore, $m-n \leqq q$. Thus $q = m-n$ and the assertion follows.

Lemma 2.4. *$V^* | \overline{P_1 \mathfrak{H}}$ is unitarily equivalent to $S_{m-n} \oplus U$.*

Proof. Let $\mathfrak{L} = \Phi H_{m-n}^2$ be as in Lemma 2.3 and let $\psi_j = \Phi \eta_j$ for $j = 1, \ldots, m-n$, where η_j denotes the column vector with $m-n$ components whose j-th component is 1 and other components are 0. It is easily seen that for almost all t, $\psi_1(t), \ldots, \psi_{m-n}(t)$ are orthonormal eigenvectors of $\Delta^{\sim}(t)$ whose corresponding eigenvalues $\delta_1(t), \ldots, \delta_{m-n}(t)$ all constantly equal to 1. Since for almost all t, $\Delta^{\sim}(t)$ is a self-adjoint operator on $\mathbf{C}^m$ bounded by 0 and 1, we can extend $\{\psi_j(t)\}_1^{m-n}$ to an orthonormal base $\{\psi_j(t)\}_1^m$ of $\mathbf{C}^m$ consisting of eigenvectors of $\Delta^{\sim}(t)$, that is, such that $\Delta^{\sim}(t) \psi_j(t) = \delta_j(t) \psi_j(t)$, $j = 1, \ldots, m$, where the eigenvalues $\delta_j(t)$ are arranged in decreasing order:

$$1 = \delta_1(t) = \ldots = \delta_{m-n}(t) \geqq \delta_{m-n+1}(t) \geqq \ldots \geqq \delta_m(t) \geqq 0 \quad \text{a.e.}$$

Let $E_j = \{t \colon \operatorname{rank} \Delta^{\sim}(t) \geqq j\}$, $j = 1, \ldots, m$. Define $X \colon \overline{\Delta^{\sim} L_m^2} \to L^2(E_1) \oplus \ldots \oplus L^2(E_m)$ by $X(\Delta^{\sim} v) = x_1 \delta_1 \oplus \ldots \oplus x_m \delta_m$, where for any $v \in L_m^2$, $x_j(t) = (v(t), \psi_j(t))_{\mathbf{C}^m}$, $j = 1, \ldots, m$, and $(\ ,\ )_{\mathbf{C}^m}$ denotes the Euclidean inner product in $\mathbf{C}^m$. It was shown on pp. 272—273 of [3] that X can be extended to a unitary transformation from $\overline{\Delta^{\sim} L_m^2}$ onto $L^2(E_1) \oplus \ldots \oplus L^2(E_m)$ such that $XV = V'X$, where V' is the operator of multiplication by e^{it} on $L^2(E_1) \oplus \ldots \oplus L^2(E_m)$.

We complete the proof of this lemma in several steps. In each step the first statement is proved.

(i) $X\overline{\Delta^{\sim}\mathfrak{L}}=H^2_{m-n}\oplus\underbrace{0\oplus\ldots\oplus 0}_{n}$. Let S_{m-n} and S_m denote the unilateral shifts on H^2_{m-n} and H^2_m, respectively. We have $\Phi S_{m-n}=S_m\Phi$. So $\bigvee_{i=0}^{\infty}\{S_m^i\psi_j, j=1,\ldots,m-n\}=\mathfrak{L}$ and $\overline{\Delta^{\sim}\mathfrak{L}}=\bigvee_{i=0}^{\infty}\{\Delta^{\sim}S_m^i\psi_j,\ j=1,\ldots,m-n\}=\bigvee_{i=0}^{\infty}\{V^i\Delta^{\sim}\psi_j,\ j=1,\ldots,m-n\}$. Since X is a unitary operator for which $XV=V'X$, $X\overline{\Delta^{\sim}\mathfrak{L}}=\bigvee_{i=0}^{\infty}\{XV^i\Delta^{\sim}\psi_j,\ j=1,\ldots,m-n\}=$ $=\bigvee_{i=0}^{\infty}\{V'^iX\Delta^{\sim}\psi_j,\ j=1,\ldots,m-n\}=H^2_{m-n}\oplus 0\oplus\ldots\oplus 0$, where in the last equation we used the relation $X\Delta^{\sim}\psi_j=\eta_j$ for $j=1,\ldots,m-n$.

(ii) $V'^*|\underbrace{0\oplus\ldots\oplus 0}_{m-n}\oplus L^2(E_{m-n+1})\oplus\ldots\oplus L^2(E_m)$ *is unitarily equivalent to* U *on* $\overline{\Delta L^2_n}$. Let U be unitarily equivalent to the operator U' of multiplication by e^{it} on $L^2(F_1)\oplus\ldots\oplus L^2(F_n)$, where $F_j=\{t:\ \operatorname{rank}\Delta(t)\geqq j\}$, $j=1,\ldots,n$, are Borel subsets of C satisfying $F_1\supseteq F_2\supseteq\ldots\supseteq F_n$ (cf. [3], pp. 272—273). An elementary argument shows that $m+\operatorname{rank}\Delta(t)=n+\operatorname{rank}\Delta_*(t)=n+\operatorname{rank}\Delta^{\sim}(-t)$ a.e., where $\Delta_*=$ $=(I-\Theta_T\Theta_T^*)^{1/2}$. Hence $\operatorname{rank}\Delta(t)\geqq j$ if and only if $\operatorname{rank}\Delta^{\sim}(-t)\geqq m-n+j$. It follows that $F_j=E^{\sim}_{m-n+j}\equiv\{t\in C:\ -t\in E_{m-n+j}\}$, for $j=1,\ldots,n$. We infer that U', hence U, is unitarily equivalent to $V'^*|0\oplus\ldots\oplus 0\oplus L^2(E_{m-n+1})\oplus\ldots\oplus L^2(E_m)$.

(iii) $V^*|\overline{P_1\mathfrak{H}}$ *is unitarily equivalent to* $S_{m-n}\oplus U$. By (i) and Lemma 2.2 we have $X^*[(L^2_{m-n}\ominus H^2_{m-n})\oplus L^2(E_{m-n+1})\oplus\ldots\oplus L^2(E_m)]=\overline{\Delta^{\sim}L^2_m}\ominus\overline{\Delta^{\sim}\mathfrak{L}}=\overline{P_1\mathfrak{H}}$. Hence $V^*|\overline{P_1\mathfrak{H}}$ is unitarily equivalent to $V'^*|(L^2_{m-n}\ominus H^2_{m-n})\oplus L^2(E_{m-n+1})\oplus\ldots\oplus L^2(E_m)$, which is, in term, unitarily equivalent to $S_{m-n}\oplus U$ by (ii).

This completes the proof.

We remark that from the proof above we can easily deduce that if T is a c.n.u. $C_{1\cdot}$ contraction with defect indices $d_T=n\leqq d_{T*}=m<\infty$, and U, V and W denote the operators of multiplication by e^{it} on $\overline{\Delta L^2_n}$, $\overline{\Delta_* L^2_m}$ and L^2_{m-n}, respectively, then $V\cong W\oplus U$.

Note that the isometry of which T is a quasi-affine transform is, in general, not unique as is evident from the following lemma.

Lemma 2.5. *Let* S *and* U *be the unilateral and bilateral shifts on* H^2 *and* L^2, *respectively. Then* $S\prec U$.

Proof. Let g be an essentially bounded function in L^2 which is cyclic for U, that is, $L^2=\bigvee_{n=0}^{\infty}U^ng$ (cf. [5], proof of Lemma 4). Define X: $H^2\to L^2$ by $Xf=gf$ for $f\in H^2$. It is easily verified that X is a quasi-affinity intertwining S and U.

Corollary 2.6. *Let T be a C_{10} contraction with defect indices* $d_T=n\leqq d_{T*}=m<\infty$. *Then* $T\prec S_{m-n}$.

Proof. For a C_{10} contraction T we have $\Delta=(I-\Theta_T^*\Theta_T)^{1/2}=0$. The assertion follows immediately from Theorem 2.1.

Actually, in the preceding situation Sz.-Nagy and Foiaş showed that T is completely injection-similar to the uniquely determined S_{m-n} (cf. [4]).

3. Multiplicity-free $C_1.$ contractions. A $C_1.$ contraction T is said to be *multiplicity-free* if it admits a cyclic vector, that is, $\mu_T=1$. The following theorem gives equivalent conditions for multiplicity-free C_{10} contractions, which generalizes Proposition 2 of [4].

Theorem 3.1. *Let T be a C_{10} contraction with defect indices* $d_T=n\leqq d_{T*}=m<\infty$ *and let S denote the simple unilateral shift. Then the following are equivalent:*

(1) *T is multiplicity-free;*

(2) $S\prec T$;

(3) $S\sim T$;

(4) $m-n=1$ *and there exists an* $m\times 1$ *matrix Δ over H^∞ such that* $[\Delta, \Theta_T]$ *is outer;*

(5) $m-n=1$ *and there exist elements* $x_1, \dots, x_m$ *in* H^∞ *such that*

$$x_1\theta_1-x_2\theta_2+\dots+(-1)^{m+1}x_m\theta_m$$

is outer, where θ_j denotes the determinant of the $n\times n$ matrix obtained by deleting the j-th row from the matrix of Θ_T, $j=1, \dots, m$.

The proof essentially follows the same line of arguments as given by Sz.-Nagy and Foiaş [4] for the case $m=2$, $n=1$. We leave the verification to the readers.

Theorem 3.2. *Let T be a c.n.u. $C_1.$ contraction with defect indices* $d_T=n\leqq d_{T*}=m<\infty$. *Then the following are equivalent:*

(1) *T is multiplicity-free;*

(2) *either T is of class C_{10} and $T\sim S$ or T is of class C_{11} and $T\sim M_E$, where S denotes the simple unilateral shift and M_E denotes the operator of multiplication by e^{it} on $L^2(E)$ for some Borel subset $E\subseteqq C$.*

Proof. (2)⇒(1). This is trivial since $\mu_T=\mu_S=\mu_{M_E}=1$.

(1)⇒(2). By Theorem 2.1, $T\prec J\equiv S_{m-n}\oplus U$, where S_{m-n} denotes the unilateral shift on H^2_{m-n} and U denotes the operator of multiplication by e^{it} on $\overline{\Delta L^2_n}$. Thus (1) implies that $\mu_J\leqq\mu_T=1$. It is an easy matter to check that either $J=S$

and T is of class C_{10} or $J=M_E$ for some Borel subset $E\subseteqq C$ and T is of class C_{11}. In the former case, $T\sim S$ follows from Theorem 3.1; in the latter, $T\sim M_E$ follows from Lemma 4.1 of [1], since T is itself quasi-similar to a unitary operator. This completes the proof.

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DEPARTMENT OF APPLIED MATHEMATICS
NATIONAL CHIAO TUNG UNIVERSITY
HSINCHU, TAIWAN, CHINA

Current address:
DEPARTMENT OF MATHEMATICS
INDIANA UNIVERSITY
BLOOMINGTON, INDIANA 47401, U.S.A.

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On a partial solution of the transitive algebra problem

B. S. YADAV and S. CHATTERJEE

Let $B(H)$ denote the Banach algebra of all bounded linear operators on an infinite-dimensional separable complex Hilbert space H. A subalgebra $\mathcal{A}$ of $B(H)$ is called transitive if it is weakly closed, contains the identity operator and its only invariant subspaces are $\{0\}$ and H. $B(H)$ is obviously transitive. Whether there exists any other transitive algebra is a well known open problem, the so-called 'transitive algebra problem'. The problem was first raised by KADISON [5] and it continues to be still unsolved. However, partial solutions of the problem have been obtained by many mathematicians; see, for example, ARVESON [1], BARNES [2], DOUGLAS and PEARCY [3], NORDGREN [8], NORDGREN, RADJAVI and ROSENTHAL [9], and RADJAVI and ROSENTHAL [10], [11]. The first such solution was given by ARVESON [1] who proved that if a transitive algebra $\mathcal{A}$ contains a maximal abelian self-adjoint algebra, then $\mathcal{A}=B(H)$. In the same paper, he also proved that $B(H)$ is the only transitive algebra containing a simple unilateral shift. By using Arveson's techniques, NORDGREN, RADJAVI and ROSENTHAL [9] have shown that a transitive algebra containing a Donoghue operator (backward weighted shift with a monotone decreasing and square-summable weight sequence) equals $B(H)$. The purpose of this note is to go a step further in this direction and show that every transitive algebra containing a certain type of weighted shift, more general than a Donoghue operator, coincides with $B(H)$. Our result assumes significance in the light of the conjecture that every transitive algebra containing a weighted shift is equal to $B(H)$.

We shall denote by $H^{(n)}$ the direct sum of n copies of H, and by $A^{(n)}$ the operator on $H^{(n)}$ which is the direct sum of n copies of A.

Let $\{w_k\}_{k=1}^{\infty}$ be a bounded sequence of non-zero complex numbers and let $\{e_k\}_{k=0}^{\infty}$ be an orthonormal basis of H. The operator T on H defined by the requirement

$$Te_0=0 \quad \text{and} \quad Te_k=w_k e_{k-1} \quad (k=1, 2, \ldots)$$

is called a *weighted unilateral* (backward) *shift* with the weight sequence $\{w_k\}_{k=1}^{\infty}$.

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We may and shall assume, without any loss of generality, that the weights w_k are positive real numbers [4]. In this case, $\{w_k\}_{k=1}^{\infty}$ is said to be of bounded pth-power variation if

$$\sum_{k=1}^{\infty} |w_k - w_{k+1}|^p < \infty.$$

(For $p=1$, we simply say "bounded variation".)

The following theorem is an important tool to obtain our results:

Theorem A. [9, Corollary 1] *If a transitive algebra $\mathscr{A}$ contains an operator A such that*

(i) *every eigenspace of A is one-dimensional, and*
(ii) *for every n, each non-trivial invariant subspace of $A^{(n)}$ contains an eigenvector of $A^{(n)}$,*

then $\mathscr{A}=B(H)$.

In the rest of this paper, $\mathscr{A}$ will denote a transitive algebra containing a weighted unilateral shift T with the weight sequence $\{w_k\}_{k=1}^{\infty}$. Our first result is

Theorem 1. *If $\{w_k\}_{k=1}^{\infty}$ is of bounded variation and*

$$\delta = \delta(n) = \sum_{k=0}^{\infty} \left(\frac{w_{k+2} \dots w_{k+n}}{w_2 \dots w_n}\right)^2 < \infty \tag{1}$$

for all $n \geqq 2$, then $\mathscr{A}=B(H)$.

Proof. We know that there is a disc of eigenvalues for a backward weighted shift, but they are all of multiplicity one. Thus T satisfies condition (i) of Theorem A. Next, let $(x_1, x_2, \dots, x_n)$ be a non-zero element of a non-zero invariant subspace M of $T^{(n)}$ and let

$$x_j = \sum_{i=0}^{\infty} x_{ij} e_i, \quad 1 \leqq j \leqq n.$$

If, for each j, the sequence $\{x_{ij}\}_{i=0}^{\infty}$ has only finitely many non-zero terms, then the invariant subspace of $T^{(n)}$ generated by $(x_1, x_2, \dots, x_n)$ is finite-dimensional and thus contains an eigenvector. We therefore assume, without loss of generality, that for every $m \geqq 0$, there is a number $r=r(m) \geqq m$ and a number $s=s(m)$, $1 \leqq s(m) \leqq n$, such that

$$|x_{r,s}| = \max_{i \geqq m;\, 1 \leqq j \leqq n} \{|x_{ij}|\} > 0. \tag{2}$$

Now, for a given integer m, we have

$$\frac{(T^{(n)})^r (x_1, x_2, \dots, x_n)}{x_{r,s} w_r \dots w_1} = \left(\frac{x_{r,1}}{x_{r,s}} e_0, \frac{x_{r,2}}{x_{r,s}} e_0, \dots, \frac{x_{r,n}}{x_{r,s}} e_0\right) + (y_{r,1}, y_{r,2}, \dots, y_{r,n}),$$

where

$$y_{r,j} = \sum_{k=r+1}^{\infty} \frac{x_{k,j} w_k \dots w_{k-r+1}}{x_{r,s} w_r \dots w_1} e_{k-r}.$$

Now

$$\|y_{r,j}\|^2 = \sum_{k=r+1}^{\infty} \left(\frac{w_k \dots w_{k-r+1}}{w_r \dots w_1}\right)^2 \left|\frac{x_{k,j}}{x_{r,s}}\right|^2 = \sum_{k=0}^{\infty} \left(\frac{w_{k+2} \dots w_{k+r+1}}{w_1 \dots w_r}\right)^2 \left|\frac{x_{k+r+1,j}}{x_{r,s}}\right|^2 \geqq$$

$$\leqq \sum_{k=0}^{\infty} \left(\frac{w_{k+2} \dots w_{k+r+1}}{w_1 \dots w_r}\right)^2, \quad \text{by (2)},$$

$$= \frac{1}{w_1^2} \sum_{k=0}^{\infty} \left(\frac{w_{k+2} \dots w_{k+r}}{w_2 \dots w_r}\right)^2 w_{k+r+1}^2 = \frac{1}{w_1^2} \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{w_{i+2} \dots w_{i+r}}{w_2 \dots w_r}\Bigg)^2 (w_{k+r+1}^2 - w_{k+r+2}^2)$$

(by Abel's transformation [12])

$$\leqq \frac{\delta}{w_1^2} \sum_{k=0}^{\infty} |w_{k+r+1}^2 - w_{k+r+2}^2|, \quad \text{by (1)},$$

$$= \frac{\delta}{w_1^2} \sum_{k=0}^{\infty} |w_{k+r+1} - w_{k+r+2}| (w_{k+r+1} + w_{k+r+2}) \leqq$$

$$\leqq \frac{2\delta\mu}{w_1^2} \sum_{k>r} |w_k - w_{k+1}|, \quad \text{where} \quad \mu = \sup_k \{w_k\},$$

and hence $y_{rj} \to 0$ as $m \to \infty$.

Also, for each j $(1 \leqq j \leqq n)$, the sequence $\left\{\frac{x_{r,j}}{x_{r,s}}\right\}_{m=1}^{\infty}$ is contained in the unit disc, and hence admits a convergent subsequence converging to a number, say z_j. A routine check reveals that a number j_0 lying between 1 and n will occur infinitely often as a value $s = s(m)$ and corresponding to this j_0, we have $z_{j_0} = 1$. The upshot of the above deliberation is that M contains an eigenvector of $T^{(n)}$, viz. $(z_1 e_0, z_2 e_0, \dots, z_n e_0)$. Thus, T also satisfies condition (ii) of Theorem A and we are done.

Theorem 2. *If $\{w_k\}_{k=1}^{\infty}$ is of bounded pth-power variation and*

$$\delta = \delta(n) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \frac{w_{j+2} \dots w_{j+n}}{w_2 \dots w_n}\right)^q < \infty \tag{3}$$

for all $n \geqq 2$, where $1 < p < \infty$ and q is the Hölder conjugate of p, then $\mathcal{A} = B(H)$.

Proof. Proceeding as in the proof of Theorem 1, we have

$$\|y_{r,j}\| = \left(\sum_{k=r+1}^{\infty}\left(\frac{w_k\dots w_{k-r+1}}{w_r\dots w_1}\right)^2\left|\frac{x_{k,j}}{x_{r,s}}\right|^2\right)^{1/2} \leqq \sum_{k=r+1}^{\infty}\left(\frac{w_k\dots w_{k-r+1}}{w_r\dots w_1}\right)\left|\frac{x_{k,j}}{x_{r,s}}\right| \leqq$$

$$\leqq \sum_{k=r+1}^{\infty}\frac{w_k\dots w_{k-r+1}}{w_r\dots w_1}, \quad \text{by (2)}$$

$$= \frac{1}{w_1}\sum_{k=0}^{\infty}\frac{w_{k+2}\dots w_{k+r}}{w_2\dots w_r}w_{k+r+1} \leqq \frac{1}{w_1}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{w_{j+2}\dots w_{j+r}}{w_2\dots w_r}\right)|w_{k+r+1}-w_{k+r+2}|$$

(by Abel's transformation [12])

$$= \frac{1}{w_1}\left(\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{w_{j+2}\dots w_{j+r}}{w_2\dots w_r}\right)^q\right)^{1/q}\left(\sum_{k>r}|w_k-w_{k+1}|^p\right)^{1/p} \quad \text{(by Hölder's inequality)}$$

$$= \frac{\delta^{1/q}}{w_1}\left(\sum_{k>r}|w_k-w_{k+1}|^p\right)^{1/p}, \quad \text{by (3);}$$

and hence, $y_{r,j}\to 0$ as $m\to\infty$.

The rest of the proof follows as that for Theorem 1.

Let l^p, $1<p<\infty$, be the Banach space of all complex pth-power summable sequences $x=\{x_0, x_1, x_2, \dots\}$ with the norm

$$\|x\| = \left(\sum_{k=0}^{\infty}|x_k|^p\right)^{1/p}.$$

Then a weighted unilateral (backward) shift T on l^p appears as

$$T\{x_0, x_1, x_2, \dots\} = \{w_1x_1, w_2x_2, \dots\}.$$

We denote by $\mathscr{L}$ a strongly closed subalgebra of $B(l^p)$ containing the identity operator and with no non-trivial invariant subspaces. We have the following analogue of Theorem 1 for l^p spaces, which we state without proof:

Theorem 3. *If $\mathscr{L}$ contains T with*

$$\sum_{k=0}^{\infty}\left(\frac{w_{k+2}\dots w_{k+n}}{w_2\dots w_n}\right)^p<\infty \quad \textit{for all} \quad n\geqq 2,$$

then $\mathscr{L}=B(l^p)$.

Remark. A subalgebra $\mathcal{L}$ of $B(H)$ is called *strictly cyclic* if there exists a vector $x_0 \in H$ such that $\{Ax_0: A\in\mathcal{L}\}=H$, and an operator $A\in B(H)$ is strictly cyclic if the algebra generated by A is strictly cyclic. LAMBERT [7] has shown that every transitive algebra which contains a strictly cyclic algebra equals $B(H)$. It follows, in particular, that every transitive algebra containing a strictly cyclic operator is equal to $B(H)$. Every Donoghue operator is strictly cyclic [6]. Whether the weighted shifts T in our Theorems 1 and 2 are also strictly cyclic, is not known. In case they are, these theorems will follow as corollaries to LAMBERT's theorem [7, Theorem 4.5]. In fact, we strongly feel that the following is true:

Conjecture. Every weighted unilateral shift whose weight sequence is of bounded variation and square-summable is strictly cyclic.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF DELHI
DELHI-110007, INDIA

Bibliographie

D. W. Barnes and J. M. Mack, An Algebraic Introduction to Mathematical Logic, V+121 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1975.

According to the authors' intention declared in the Preface, "this book is intended to make mathematical logic available to mathematicians working in other branches of mathematics". Despite of the title the presentation uses very few algebraic means.

Chapter I accumulates some simple notions such as the free algebra, relatively free algebra and variety of universal algebras. Chapters II and III deal with Propositional calculus. Chapter IV develops both syntax and semantics of Predicate calculus and proves Gödel's Completeness Theorem by the method of Henkin. In Chapter V mathematical theories based on the first order predicate calculus are investigated. In particular, the Löwenheim-Skolem Theorem and the elimination of quantifiers are studied. Chapter VI lists the axioms of the Zermelo—Frankel set theory. Chapter VII introduces the notions of ultraproduct, ultrapower and direct limit and there is a nice proof of the theorem that every field has an algebraic closure. Non-standard models are discussed in Chapter VIII with applications to elementary non-standard analysis. In Chapter IX Turing machines and Gödel numbers are introduced to explain the notion of calculability and solvability. In particular, Church's theorem on undecidability of the predicate calculus is included. Finally, Hilbert's Tenth Problem and a brief outline of its solution by Matiyaševič are presented in Chapter X.

The book is very clearly written, supplied with excercises at the end of sections (some of them need far more knowledge than provided by the text).

P. E.-Tóth (Szeged)

B. Bollobás, Graph theory: An Introductory Course (Graduate Texts in Mathematics, Vol. 63), X+180 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1979.

The 8 chapters of the book (Fundamentals; Electrical Networks; Flows, Connectivity and Matching; Extremal Problems; Colouring; Ramsey Theory; Random Graphs; Graphs and Groups) contain gradually more and more involved results, with several relations to other branches of mathematics.

The reviewer was pleasantly surprised to find a full chapter on electrical networks, and feels some competence to criticize this chapter in a more detailed way. The order of the presentation is quite unusual. In other texts, Theorem 1 is presented usually much later than Theorem 7. (However, the other texts are written mainly to students in engineering, while the order in this book seems to be more adequate for mathematicians.) On the other hand, one sees no reason why should the material of §2 separate those in §§1 and 3. The references at the end of the chapter refer to §2 only and the exercises, related to §2 are also more adequate than the rest. Probably a few remarks on electric network duality and its relation to planar graphs could also be in order.

The author successfully meets two contradicting requirements: most of the important branches of the theory are presented; still, several deep results are included. The presentation is clear, there is a strong attempt to present typical methods and ways of reasoning in addition to the results.

A great advantage of the book is the good selection of exercises, containing quite a few unusual and deep results.

The book is a valuable addition to the literature and is highly suggested for students and teachers of graph theory.

A. Recski (Budapest)

Carl de Boor, A Practical Guide to Splines (Applied Mathematical Sciences, 27) XXIV+392 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1978.

The textbook grew out of lectures on splines delivered by the author at Redstone Arsenal in 1976 and at White Sands Missile Range in 1977. It stresses the representation of splines as linear combinations of B-splines, provides proofs only for some of the results stated but offers many Fortran programs. The reader is requested to consult a few books listed in the bibliography if he wishes to develop a more complete picture of spline theory. As the author says in the Preface, his book presents only those parts of spline theory which he found useful in calculations. Indeed, this book is an excellent one for everyone who deals with applied mathematical problems involving polynomial splines.

The following outline may provide an idea of the content. Chapters I and II recapitulate material needed later from the ancient theory of polynomial interpolation. The next four chapters follow somewhat the historical development, with piecewise linear, piecewise cubic, and piecewise parabolic approximation discussed. The computational handling of piecewise polynomial functions is the subject of Chapters VII and VIII. B-splines are introduced in Ch. IX, while Chs. X and XI are intended to familiarize the reader with them. The remaining chapters contain various applications, all involving B-splines: the smoothing spline and least-squares spline approximation for noisy data, the use of splines in solving differential equations, approximation of curves etc. The final chapter treat with the simplest generalization of splines to several variables.

Each chapter ends with some problems to test the reader's understanding of the material, to bring in additional material and to urge numerical experimentation with the programs provided. The Bibliography does not claim completeness, it contains only items referred to in the text. For the reader's convenience a Postscript on Things not Covered, a List of Fortran Programs, and a Subject Index complete the book.

F. Móricz (Szeged)

Yuan Shih Chow and Henry Teicher, Probability Theory (Independence, Interchangeability, Martingales), XV+455 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1978.

The concern of this book is the measure theoretical foundations of probability theory and the major theorems of the subject. The main topics treated are independence, interchangeability and martingales as indicated in the title. Thus, such important concepts as Markov and stationary processes are not even defined, although the special cases of sums of independent random variables and interchangeable random variables are dealt with extensively. Likewise, continuous parameter stochastic processes, although alluded to, are not discussed.

The book is intended to serve as a graduate text in probability theory. No knowledge of measure or probability is presupposed. A novel feature is the attempt to intertwine measure and probability rather than, as customary, to set up between them a sharp demarkation. Particular emphasis is placed upon stopping times, on the one hand, as tools in proving theorems, and on the other, as objects of interest in themselves. For example, optimal stopping problem, limit destributions of sequences of stopping rules (i.e. finite stopping times), randomly stopped sums are of special interest. Many of the proofs given and a few of the results are new. Occasionally, a classical notion is looked at through new lenses (e.g. reformulation of the Lindeberg condition).

Chapter 1—3 contain the elements of measure theory, binomial random variables and independence involving the Borel—Cantelli theorem and Kolmogorov zero-one law. It is surprising how much probability can be developed without even a mention of integration. A number of topics treated later in generality are foreshadowed in the very tractable binomial case. Ch. 4 is devoted to integration in a probability space, while Ch. 6 to measure extensions, Lebesgue—Stieltjes measure and the Kolmogorov consistency theorem.

Readers familiar with measure theory can plunge into Ch. 5 after reading Section 3.2. A one-year course presupposing measure theory can be built around Chapters 5, 7, 8, 9, 10, 11 and 12. In more detail, Ch. 5 treats the sums of independent random variables, Ch. 7 introduces the notions of conditional expectation, conditional independence, and martingales. Ch. 8 deals with distribution functions and characteristic functions, involving the Fréchet—Shohat, Glivenko—Cantelli and Cramér—Lévy theorems. The central limit theorems are studied for the independent case, interchangeable case and martingale case (Ch. 9), while the laws of large numbers, the law of the iterated logarithm for independent case (Ch. 10), Martingales are introduced in Section 7.4, where the upward case is treated, and then developed more generally in Ch. 11. The final Ch. 12 contains material concerning infinite divisible laws.

The book is complemented by a List of Abbreviations, a List of Symbols and Conventions, and an (author and subject) Index. Each section ends with exercises, and each chapter with references. The exercises are used to extend theory, to illustrate a theorem, or to obtain a classical result from one recently proven.

The presentation is self-contained and unified. It is highly recommended for every graduate student or mathematician who wishes to begin studies in Probability Theory.

F. Móricz (Szeged)

Combinatorial Mathematics. VI, Proceedings of the Sixth Australian Conference on Combinatorial Mathematics, Armidale, August 1978. Edited by A. F. Horadam and W. D. Wallis (Lecture Notes in Mathematics, Vol. 748), IX+206 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1979.

The volume contains texts of three of the invited addresses (R. B. Eggleton and D. A. Holton on graphic sequences, S. O. Macdonald on the interaction between combinatorics and graph theory, B. D. McKay and R. G. Stanton on generalized Moore-graphs) and 15 contributed papers (about 40—40% of which refer to designs and graphs, respectively).

A. Recski (Budapest)

George Grätzer, Universal Algebra, 2nd edition, XVIII+581 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1979.

The first edition of Grätzer's Universal Algebra came out in 1968 and instantly became *the* reference book of its topic. The very successful choice of the material is testified by the fact that, after eleven years and about a thousand new articles in the area, a second edition containing the unchanged text of the first one has been justified and necessitated.

Clearly, in order to remain the reference book also in the future, this second edition has had to mirror the rapid development of universal algebra in the seventies. For this aim, it contains seven appendices, partly written by invited experts, and an abundant additional bibliography which includes even several important articles not in print yet.

The first appendix (Shortly: A1) is a survey of recent research not covered in the further appendices. A2 is a review of the solved problems, posed in the first edition. A3 (by B. Jónsson) introduces into Mal'cev conditions and congruence varieties, A4 (by W. Taylor) surveys equational theories, A5 (by R. N. Quackenbush) gives a picture on primal algebras and other generalizations of Boolean algebras, and A6 (by G. H. Wenzel) deals with equational compactness. Finally, A7 contains the proof of a deep new result of the author and W. A. Lampe, namely, that the congruence lattice, the subalgebra lattice and the automophism group are independent for infinitary algebras.

The time-tested basic text with these nicely written mini-monographs added will serve, no doubt, as the standard universal algebra book, in the eighties, too.

B. Csákány (Szeged)

S. W. Hawking and W. Israel, editors, General Relativity. An Einstein Centenary Survey, XV + 920 pages, Cambridge University Press, Cambridge—London—New York—Melbourne, 1979.

The Einstein centenary arose widerspread new interest for general relativity all over the world. The present book is a most appropriate commemoration on Einstein's hundredth birthday. Twenty-one of the world's leading relativists collaborated on this survey and gave a render of the current state of research.

The book starts with an introductory survey of S. W. Hawking and W. Israel. The papers of C. M. Will, D. H. Douglass and V. B. Braginsky deal with the confrontation between gravitation theory and experiments. The work of A. E. Fischer and J. E. Marsden discusses the initial value problem and the Cauchy problem for relativity and they give the dynamical formulation of general relativity too.

The discovery of exotic astronomical objects (quasars, pulsars, and X-ray sources) necessitated the development of theories which can explain the complex behaviour of these objects theoretically. Such are the theory of cosmology, black hole physics, theory of singularity, the early history of the universe, e.t.c. A lot of papers discuss these fields by the authorities who, strictly speaking, created these theories. We can mention here, e.g., the following names: R. Gerock and G. T. Horowith (Global structure of spacetimes), B. Carter (The general theory of the mechanical, electromagnetic and thermodynamic properties of black holes), S. Chandrasekhar (An introduction to the theory of the Kerr metric and its perturbations), R. D. Blanford and K. S. Thorne (Black hole astrophysics), R. H. Dicke and P. J. E. Peebles (The big bang cosmology—enigmas and nostrums), Ya. B. Zel'dovich (Cosmology and the early universe), M. A. H. MacCallum (Anisotropic and inhomogeneous relativistic cosmologies), R. Penrose (Singularities and time-asymmetry).

One of the most exciting problem of physics is the unification of general relativity with quantization and with other laws of physics. The book treats the present status of this field with an abundant and profound material. C. W. Gibbons surveys the present quantum field theory in curved spacetime. B. S. DeWitt gives a new synthesis of quantum gravity. The article of S. W. Hawking shows how the path integral approach can be applied to the quantization of gravity and how it leads to the concepts of black hole temperature and intrinsic quantum mechanical entropy. In the last article of the book S. Weinberg deals, in connection with ultraviolet divergences in quantum gravity, with the future of quantum gravity and gives several conjectures concerning the evolution of the quantization in relativity.

We want to emphasize also the highly intelligent editorial work and the very nice appearance of the book. The editors should be congratulated for presenting us a work which will remain the "Bible" of relativity for many decades to come.

Z. I. Szabó (Szeged)

H Hermes, Introduction to Mathematical Logic, XI+242 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1973.

This volume is a valuable introductory text in the classical two-valued predicate logic.

After three editions in German, the original text was translated into English by D. Schmidt. Concerning the material covered is no difference between the English and the third German edition.

Both syntactical and semantical approaches are developed with a little more emphasis on the latter. After an excellent introduction the language and calculus of the first-order predicate logic are given in Chs. II—IV. The treatment leads to the Gödel's Completeness Theorem in Ch. V. In Ch. VI, the axiomatic number theory and the second order predicate logic are introduced, on making the notion of completeness clearer. Ch. VIII includes pure model-theoretic proofs of some basic results in definition theory (such as theorems of Robinson, Craig, Beth, etc).

In the remaining chapters (VII and IX) useful techniques are presented to derive some well-known logical connectives and normal forms. A systematic treatment of substitution is also included here.

P. E.-Tóth (Szeged)

Joram Lindenstrauss and Lior Tzafriri, Classical Banach Spaces. II (Function Spaces) (Ergebnisse der Mathematik und ihrer Grenzgebiete, 97), X+243 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1979.

The second volume on classical Banach spaces by the same authors [Volume I: *Classical Banach Spaces.* I *(Squence spaces),* Springer-Verlag, Berlin—Heidelberg—New York, 1977] is devoted to the study of Banach lattices.

A partially ordered Banach space X over the reals is called a *Banach lattice* if the following conditions are satisfied:

(i) $x \leqq y$ implies $x+z \leqq y+z$, for every $x, y, z \in X$;

(ii) $ax \geqq 0$, for every $x \geqq 0$ in X and every non-negative real a;

(iii) for all $x, y \in X$ there exists a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$;

(iv) there exists a constant M such that $\|x\| \leqq M \|y\|$ whenever $|x| \leqq |y|$, where the absolute value $|x|$ of $x \in X$ is defined by $|x| = x \vee (-x)$.

The structure of Banach lattices is much simpler than that of general Banach spaces and their theory is therefore more complete and satisfactory. Many of the results concerning Banach lattices are not valid and sometimes even do not make sense for general Banach spaces. The theory of Banach lattices has many tools which are specific to this theory, in particular, the notions of p-convexity and p-concavity seem to be especially useful. These notions play a central role in the present volume and presumably will continue to dominate the theory of Banach lattices.

The book consists of two chapters, both subdividing into seven sections. The table of contents is quite detailed and gives a clear idea of the material discussed in each section. The basic standard theory of Banach lattices is contained is Sections 1 a)—c). The theory of p-convexity and p-concavity is presented in Sections 1 d)—f).

Chapter 2 is devoted to a detailed study of the structure of rearrangement invariant function spaces (r.i.f.s.) on [0, 1] and [0, ∞): a) Basic definitions, examples and results; b) The Boyd indices; c) The Haar and the trigonometric systems; d) Some results on complemented subspaces; e) Isomorphisms between r.i.f.s. and uniqueness of the r.i. structure; f) Applications of the Poisson process to r.i.f.s.

Three of the sections are concerned with the general theory of Banach spaces rather than with Banach lattices. Section 1 e) deals with the theory of uniform convexity, 1 g) with the approximation property, and 2 g) with geometric aspects of interpolation theory in general Banach spaces.

The prerequisities include, besides standard material from functional analysis and measure theory, only a superficial knowledge of the material presented in Volume I of this book. For the convenience of the reader the authors tried to discuss briefly in the appropiiate places the notions and results from probability theory which they apply.

The overlap between this volume and existing books on lattice theory is small and consists mostly of the standard material presented in Sections 1 a)—b). The books of W. A. J. Luxemburg and A. C. Zaanen [*Riesz spaces* I, North—Holland, Amsterdam, 1971] and H. H. Schaefer [*Banach lattices and positive operators*, Springer-Verlag, Berlin—Heidelberg—New York, 1974] contain much additional material rather on vector lattices. The volume under review comprises the substantial progress made in the seventies.

To sum up, the present book is a rich and up-to-date account on this fast-growing and important subject. It is warmly recommended to everyone who wants to learn, or do research in, the theory of Banach spaces.

F. Móricz (Szeged)

M. Schreiber, Differential forms (A heuristic introduction), X+150 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1977.

The theory of differential forms is one of the most frequently applied branches of mathematics not only in several fields of mathematics but also in theoretical physics. But the systematic treatment of differential forms requires an apparatus of topology and algebra which can be difficult for mathematicians and physicists working in other fields of research. The present book treats the theory of differential forms with minimal apparatus and very few prerequisites. The exposition is heuristic and concrete. A differential form is considered as a multi-dimensional integrand given on surfaces in Euclidean space, and the various operations (such as exterior derivation) are treated on an elementary level, from the geometrical point of view. Several formulas, such as Stokes'formula, are proved on such an elementary level as possible. The book contains a short introduction to integral geometry also.

It is addressed to mathematicians, physicists and students who are interested in a quick acquirement of differential forms techniques.

Z. I. Szabó (Szeged)

G. Takeuti and W. M. Zaring, Axiomatic Set Theory, V+238 pages, Springer-Verlag (Berlin—Heidelberg—New York, 1973).

This almost completely self-contained volume is a continuation of a previous one by the same authors ("Introduction to Axiomatic Set Theory", Springer-Verlag, 1971). The present book deals with three well-known methods for constructing models of the Zermelo—Fraenkel set theory: relative constructibility, Cohen's forcing and Boolean valued models. After developing Lévy-Schoenfield's theory of relative constructibility (Sections 7, 8, 9) a relationship is established between Cohen's technique of forcing (Sec. 10) and Scott-Solovay's theory of Boolean valued models (Sec. 13). In the first six sections some facts of Boolean algebras, Boolean σ-algebras, partial ordered structures, and topologies needed later on are collected. The remaining sections are devoted to a deeper investigation of the concepts introduced in the earlier sections.

The text is recommended for graduate students.

P. E.-Tóth (Szeged)

I. M. Yaglom, A simple non-euclidean geometry and its physical basis, XVIII+308 pages, Springer-Verlag (New York—Heidelberg—Berlin, 1979).

It is a hard problem of geometrical education to give a simple, relatively quick but deep synthetical treatment of classical non-euclidean geometries. The book of I. M. Yaglom proves that this program is realizable very elegantly from the mathematical point of view and the deep connections between these geometries and physics can also be illuminated on this level. This physical motivation of the classical geometries is the most important intrinsic value of the book.

Chapter I and II are simple but non-trivial introductions to plane Galilean geometry and to Galilean inversive geometry with plane and inversive Euclidean geometry. The next chapters treat the physical basis of Galilean geometry, the relativistic kinematic and relativistic Minkowskian geometry. At the end of the book the reader finds three supplements in which the author gives a systematical treatment of the nine plane geometries with their axiomatic characterization and analytic models.

The subject is accessible to anyone versed in elementary mathematics. The book is addressed mainly to students of mathematics, physics, and mathematical education.

Z. I. Szabó (Szeged)

Livres reçus par la rédaction

M. Aigner, Combinatorial theory (Grundlehren der mathematischen Wissenschaften — A Series of Comprehensive Studies in Mathematics, 234), VIII+483 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1979. — DM 79,50.

Algebraic Topology, Proceedings of a Conference sponsored by the Canadian Mathematical Society, NSERC (Canada), and the University of Waterloo, June 1978. Edited by P. Hoffman and V. Snaith (Lecture Notes in Mathematics, Vol. 741), XI+655 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1979. — DM 56,—.

Algebraic Topology, Proceedings of a Symposium held at Aarhus, Denmark, August 7—12, 1978. Edited by J. L. Dupont and I. H. Madsen (Lecture Notes in Mathematics, Vol. 763), VI+695 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1979. — DM 60,—.

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