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On a set-mapping problem of Hajnal and Máté

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In the course of a wide-ranging survey of combinatorial set theory, A. Hajnal and A. Máté prove by a forcing argument the consistency of the following combinatorial principle with the Generalized Continuum Hypothesis GCH, and ask whether if follows from the Axiom of Constructibility $V=L$ (see [4], Thm. 5.4 and Problem 8).

(HM) There is a function f: $\{(\alpha, \beta, \gamma): \alpha<\beta<\gamma<\omega_2\} \to \omega_2$ such that for any uncountable $A \subseteq \omega_2$ there exist $\alpha<\beta<\gamma$ in A with $f(\alpha, \beta, \gamma) \in A$.

(We are using the same standard set-theoretic notation as [4], except that we use ω_α rather than $\aleph_\alpha$ for the αth transfinite cardinal.) We present here a proof that $V=L$ implies HM by a metamathematical method which we feel has interest beyond this particular problem.

1. Jensen's Absoluteness Principle. The language $L\,[Q_1, Q_2]$ is just like ordinary first order logic, except for the presence of two generalized quantifiers:

$Q_1 x\varphi(x)$ meaning: There exist uncountably many x such that $\varphi(x)$.

$Q_2 x\varphi(x)$ meaning: There exist at least ω_2 many x such that $\varphi(x)$.

As is explained in some detail in the final paragraphs of [3], R. B. JENSEN's work on model theory establishes the following principle:

(∗) Let φ be a sentence of $L[Q_1, Q_2]$. Suppose there is a Boolean-valued extension $V^{\mathscr{B}}$ of the universe of set theory in which GCH holds, such that in $V^{\mathscr{B}}$ it is true that φ has a model. Then already in the constructible universe L it is true that φ has a model.

This principle provides a method for turning a consistency proof for a combinatorial principle ψ into a derivation of ψ from $V=L$. Namely, it suffices to find a sentence φ of $L[Q_1, Q_2]$ for which we can prove, using GCH if needs be, that φ has a model if and only if ψ holds. Unfortunately this method does not seem to

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apply directly to the principle HM. What we will show here is that it applies to a certain principle which implies HM.

2. Quagmires. The principle we have in mind is just a bit complicated. A *tree* is a partial order $\mathscr{T}=(T,<)$ in which the predecessors of any element are well ordered. The order type of the predecessors of $t\in T$ is called the *rank* $|t|$ of t. The αth *level* T_α of the tree is the set of t with $|t|=\alpha$, and its *height* the least α with $T_\alpha=\emptyset$. For present purposes a *Kurepa tree* may be defined as a tree of height ω_1+1 in which T_{ω_1} has cardinality ω_2, distinct elements of T_{ω_1} have distinct sets of predecessors, and T_α is countable for $\alpha<\omega_1$.

A *quagmire* $(T,<,\lhd,Q)$ is a Kurepa tree $(T,<)$ equipped with a binary relation $\lhd$ and a trinary function Q such that:

(1) $\lhd$ holds only between elements of equal rank, and linearly orders each level T_α of the tree.

(2) Q is defined on those triples (y',x',x) with $y'\lhd x'<x$, and for any such, $y'<Q(y',x',x)\lhd x$.

(3) *(Commutativity)* If $y''\lhd x''<x'<x$, then $Q(Q(y'',x'',x'),x',x)=$ $=Q(y'',x'',x)$.

(4) *(Coherence)* If $z'\lhd y'\lhd x'<x$, then $Q(z',y',Q(y',x',x))=Q(z',x',x)$.

(5) *(Completeness)* If $y\lhd x\in T_{\omega_1}$, then for some $\alpha<\omega_1$, $Q(P_\alpha(y),P_\alpha(x),x)=y$. Here P_α is the *projection* function which assigns to any t with $|t|\geqq\alpha$ the unique $u<t$ with $|u|=\alpha$. Note that the condition $Q(P_\alpha(y),P_\alpha(x),x)=y$ implies $P_\alpha(y)\lhd$ $\lhd P_\alpha(x)$, else Q would not be defined on this triple.

What we are going to show, assuming GCH, is that:

(A) The existence of a quagmire implies HM.

(B) There is a sentence of $L[Q_1,Q_2]$ which has a model if and only if there exists a quagmire.

(C) There is a Boolean-valued extension $V^{\mathscr{B}}$ of the universe of set theory in which GCH holds and there exists a quagmire.

3. Proof of (A). We will show, assuming CH, that *if* there exists a quagmire $(T,<,\lhd,Q)$, then HM holds. We begin by deriving from these assumptions the following combinatorial principle, due to Silver. (For its consequences, cf. [5].)

(W) There exists a Kurepa tree $(T,<)$ equipped with a function W defined on ω_1, such that:

For $\alpha<\omega_1$, $W(\alpha)$ is a countable family of subsets of the level T_α. For any countable $S\subseteqq T_{\omega_1}$ there exists $\alpha<\omega_1$ such that for any $\alpha\leqq\beta<\omega_1$, $\{P_\beta(x)\colon x\in S\}\in W(\beta)$.

Indeed, to derive W given CH and a quagmire, note that for each $\alpha<\omega_1$, the αth level T_α of the quagmire is countable, so its power set can be enumerated in an ω_1-sequence $X_{\alpha,\beta}$ for $\beta<\omega_1$. For $x\in T$ and $\alpha,\beta<|x|$ let $S(\alpha,\beta,x)$ be the image $\{Q(y', P_\alpha(x), x)\colon y' \lhd P_\alpha(x) \;\&\; y' \in X_{\alpha,\beta}\}$ of the βth subset of T_α under the map $Q(\cdot, P_\alpha(x), x)$. For $\gamma<\omega_1$, let $W(\gamma)=\{S(\alpha, \beta, x)\colon \alpha, \beta<\gamma \;\&\; x\in T_\gamma\}$, a countable family of subsets of T_γ.

Now it follows by the Completeness condition in the definition of quagmire that any $x\in T_{\omega_1}$ has at most ω_1 $\lhd$-predecessors. Hence given a countable $S\subseteqq T_{\omega_1}$, there must exist an x with $y\lhd x$ for all $y\in S$. Again by Completeness, for each $y\in S$ there is then an $\alpha(y)<\omega_1$ with $y=Q(P_{\alpha(y)}(y), P_{\alpha(y)}(x), x)$. Let $\alpha=\sup\{\alpha(y)\colon y\in S\}$. By Commutativity, for any $\alpha\leqq\delta<\omega_1$ and $y\in S$, the element $y'=Q(P_{\alpha(y)}(y), P_{\alpha(y)}(x), P_\delta(x))$ satisfies $P_{\alpha(y)}(y)<y'\lhd P_\delta(x)$ and $y'<Q(y', P_\delta(x), x)=Q(P_{\alpha(y)}(y), P_{\alpha(y)}(x), x)=y$. Hence $y'=P_\delta(y)$ and $Q(P_\delta(y), P_\delta(x), x)=y$.

If now we fix a β such that $\{P_\alpha(y)\colon y\in S\}=X_{\alpha,\beta}$ and let γ be $>\alpha$ and β, then for any $\gamma\leqq\delta<\omega_1$ it is readily verified that $\{P_\delta(y)\colon y\in S\}=S(\alpha, \beta, P_\delta(x))\in W(\delta)$, which suffices to prove Silver's principle W above. This established, we go on, still assuming CH and the existence of a quagmire, to derive the following combinatorial principle, due to Hajnal and Máté:

(HM′) There exists a sequence of functions $H_\alpha\colon \omega_2\to\omega_2$, for $\alpha<\omega_1$, such that for any infinite $S\subseteqq\omega_2$ there exists a $\gamma<\omega_1$ such that for any $\gamma\leqq\delta<\omega_1$ there exists an $x\in S$ with $H_\delta(x)\in S$.

Towards proving this, we first note that we may assume without loss of generality that in our quagmire no level T_α has a $\lhd$-least element. (Otherwise we can construct a new quagmire with this property by taking:

$$T' = \omega\times T,$$

$$(m, x) <' (n, y) \leftrightarrow m=n \;\&\; x<y,$$

$$(m, x) \lhd' (n, y) \leftrightarrow x\lhd y \text{ or } (x=y \;\&\; m>n),$$

$$Q'((m, y'), (n, x'), (n, x)) = (m, Q(y', x', x)),$$

i.e. by replacing each element x of the original quagmire by a sequence $\ldots(2, x), (1, x), (0, x)$.)

This settled, we go on to construct for each $\alpha<\omega_1$ a map $h_\alpha\colon T_\alpha\to T_\alpha$ such that $h_\alpha(x)\lhd x$ for each $x\in T_\alpha$, and for any infinite $S\in W(\alpha)$ there exists a $y\in S$ with $h_\alpha(y)\in S$. Since $W(\alpha)$ is countable, this can be accomplished by a simple diagonal construction in ω stages, whose details are left to the reader. Having the h_α, we define maps $H_\alpha\colon T_{\omega_1}\to T_{\omega_1}$ by $H_\alpha(x)=Q(h_\alpha(P_\alpha(x)), P_\alpha(x), x)$.

Now for any denumerably infinite $S\subseteqq T_{\omega_1}$, our arguments above establish two things. First, there is an $x\in T_{\omega_1}$ and an $\alpha<\omega_1$ such that for all $y\in S$ and $\alpha\leqq\delta<\omega_1$, $Q(P_\delta(y), P_\delta(x), x)=y\lhd x$. Second, there is a $\beta<\omega_1$ such that for all $\beta\leqq\delta<\omega_1$, $\{P_\delta(y)\colon y\in S\}\in W(\delta)$. If $\gamma=\max(\alpha, \beta)$, then for any $\gamma\leqq\delta<\omega_1$, by construction there exist $y, z\in S$ with $h_\delta(P_\delta(y))=P_\delta(z)$. Now by Coherence $H_\delta(y)=$

$=Q(h_\delta(P_\delta(y)), P_\delta(y), y)=Q(P_\delta(z), P_\delta(y), y)=Q(P_\delta(z), P_\delta(x), x)=z$, i.e. there is a $y\in S$ with $H_\delta(y)\in S$.

If we assume, as we may without loss of generality, that T_{ω_1} consists precisely of the ordinals $<\omega_2$, then this is precisely what is required to establish the principle HM′ above. Now as HAJNAL and MÁTÉ [4] show that HM′ and the existence of a Kurepa tree imply HM, our proof that CH and the existence of a quagmire imply HM is complete.

4. Proof of (B). VAUGHT [6] long ago proved that the existence of a Kurepa tree is equivalent to the existence of a model for a certain sentence φ of $L[Q_1, Q_2]$. For completeness we recall his argument here: φ will involve two singulary predicates T, O, plus two binary predicates $<_T$, $<_O$, plus a singulary function symbol r, plus a constant w. φ is the conjunction of the sentences (whose precise formalization we leave to the reader) expressing:

(1) $<_T$ partially orders T in such a way that the predecessors of any element are linearly ordered.

(2) $<_O$ linearly orders O, with last element w.

(3) r maps T onto O in such a way that for any $t\in T$ and $u\in O$, $u<_O r(t)$ if and only if there exists $t'<_T t$ with $u=r(t')$.

(4) $Q_1 uO(u)$ & $\forall u(u<_O w\to \neg Q_1 u'(u'<_O u))$

(5) $Q_2 t(T(t)$ & $r(t)=w)$ & distinct t with $r(t)=w$ have distinct sets of $<_T$-predecessors.

(6) $\forall u(u<_O w\to\neg Q_1 t(T(t)$ & $r(t)=u))$

If $(T, <_T)$ is a Kurepa tree, we get a model of this sentence φ by interpreting O as the set of ordinals $\leqq\omega_1$, $<_O$ as the usual order on this set, w as ω_1, and r as the rank function. Conversely, if $(T, <_T, O, <_O, w, r)$ is a model of φ, then using (4) above one easily sees that there is a $<_O$-cofinal subset Z of $\{u\in O\colon u<_O w\}$ which is well ordered by $<_O$ in order type ω_1. Then restricting $<_T$ to $\{t\in T\colon r(t)=w$ or $r(t)\in Z\}$ we get a Kurepa tree.

To get a formula φ' which has a model if and only if there exists a quagmire, simply take new symbols $\lhd$ and Q and conjoin the above φ with the sentences expressing conditions (1)—(5) in the definition of quagmire in § 2 above.

5. Proof of (C). It remains only to prove, assuming GCH, that some suitable set of forcing conditions gives rise to a Boolean-valued extension of the universe of set theory in which GCH holds and there exists a quagmire. The proof is so similar to the proof of the consistency of HM′ in [4] and the proof of the consistency of Silver's W in [2], that we leave most details to the reader.

As our forcing conditions we take the set $\mathscr{P}$ of all sixtuples $p=(\alpha_p, T_p, <_p, \lhd_p, Q_p, \Lambda_p)$ such that:

(0) T_p is a countable subset of ω_1.

(1) $(T_p, <_p)$ is a tree of height $\alpha_p+1<\omega_1$.

(2)—(5) in the definition of quagmire in § 2 above hold for $\lhd_p$ and Q_p.

(6) Λ_p maps a subset of ω_2 onto the α_pth level of the tree $(T_p, <_p)$, and is order preserving in the sense that for $\xi<\eta$ in dom Λ_p, we have $\Lambda_p(\xi)\lhd_p\Lambda_p(\eta)$. Note that the requirement that Λ_p be order preserving means that Λ_p is completely determined by its domain.

We partially order $\mathcal{P}$ by setting $p<q$ if and only if:

(7) $\alpha_p>\alpha_q$ and $T_p\supseteqq T_q$ and $<_p, \lhd_p, Q_p$ extend $<_q, \lhd_q, Q_q$ respectively and dom $\Lambda_p\supseteqq$ dom Λ_q.

(8) For all $\xi\in$ dom Λ_q, $\Lambda_q(\xi)<_p\Lambda_p(\xi)$; and for $\xi<\eta$ in dom Λ_q, $Q_p(\Lambda_q(\xi), \Lambda_q(\eta), \Lambda_p(\eta))=\Lambda_p(\xi)$.

In order to show that $\mathcal{P}$ does what it should, we need the following:

Lemma. (a) *$\mathcal{P}$ is σ-closed; i.e. whenever $p_n\in\mathcal{P}$ for $n\in\omega$ and $p_{n+1}<p_n$ for all n, then there exists $p\in\mathcal{P}$ with $p<p_n$ for all n.*

(b) *$\mathcal{P}$ has the ω_2-chain condition; i.e. no set of pairwise incompatible elements of $\mathcal{P}$ has cardinality ω_2.*

(c) *For each $\alpha<\omega_1$ and $\xi<\omega_2$, $\{p\colon \alpha_p>\alpha$ & sup dom $\Lambda_p>\xi\}$ is dense in $\mathcal{P}$.*

The proof of the easy parts (a) and (c) will be left to the reader. As for part (b), let $A\subseteqq\mathcal{P}$ have cardinality ω_2. Assuming CH, there must exist an $A'\subseteqq A$ of cardinality ω_2 and fixed α, T, $<$, $\lhd$, and Q such that for all $p\in A'$, $\alpha_p=\alpha$, $T_p=T$, $<_p=<$, $\lhd_p=\lhd$, $Q_p=Q$. For assuming CH there are only ω_1 possibilities for these items.

{dom Λ_p: $p\in A'$} forms a set of ω_2 countable subsets of ω_2. By a well-known result of Erdős and Rado (cf. Thm. 2.3 of [4] or Lemma 3.6 of [2]) there exists a sequence p_ν, $\nu<\omega_2$ of elements of A' and a fixed $X\subseteqq\omega_2$ such that for any $\mu<\nu<\omega_2$, dom $\Lambda_{p_\mu}\cap$dom $\Lambda_{p_\nu}=X$ and sup dom $\Lambda_{p_\mu}<$inf (dom $\Lambda_{p_\nu}-X$).

Let $p=p_0$, $q=q_0$, $Y=$dom Λ_p, $Z=$dom Λ_q. Note $\Lambda_p|X=\Lambda_q|X$. To establish part (b) of the Lemma it will suffice to construct an $r\in\mathcal{P}$ with $r<p$ and $r<q$. This may be accomplished by taking:

$\alpha_r=\alpha+1$,

$T_r=T\cup\{t_\xi\colon \xi\in Y\cup Z\}$ where the t_ξ are distinct elements of ω_1-T,

$<_r=$the extension of $<$ defined so that $\Lambda_p(\eta)<_r t_\eta$ for $\eta\in Y$ and $\Lambda_q(\zeta)<_r t_\zeta$ for $\zeta\in Z$,

$\lhd_r=$the extension of $\lhd$ defined so that $t_\eta\lhd_r t_\zeta$ for $\eta<\zeta$ in $Y\cup Z$,

$Q_r=$the extension of Q defined so that $Q_r(\Lambda_p(\xi), \Lambda_p(\eta), t_\eta)=t_\xi$ for $\xi<\eta$ in Y, and $Q_r(\Lambda_q(\xi), \Lambda_q(\zeta), t_\zeta)=t_\xi$ for $\xi<\zeta$ in Z,

$\Lambda_r=$the function $\Lambda_r(\xi)=t_\xi$ for $\xi\in Y\cup Z$.

Details are left to the reader.

With the Lemma established, we let $\mathscr{B}$=the complete Boolean algebra of regular open subsets of $\mathscr{P}$. Parts (a) and (b) of the above Lemma and standard forcing lemmas (for which see e.g. [2]) imply that, assuming GCH, in the Boolean-valued extension $V^{\mathscr{B}}$ all cardinals are preserved and GCH holds.

Moreover if $G\in V^{\mathscr{B}}$ is a generic subset of $\mathscr{P}$, then the $p\in G$ can be fitted together to produce a quagmire. Again details are left to the reader. This completes the proof that $V=L$ implies HM.

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Scalar central elements in an algebra over a principal ideal domain

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1. Introduction. Let A be an algebra (not necessarily associative) over a commutative ring R. A is called scalar commutative if, for each $x, y \in A$, there exists $\alpha \in R$ depending on x, y such that $xy = \alpha yx$. RICH [3] proves that if A is scalar commutative and if R is a field then A is either commutative or anticommutative. KOH, LUH, and PUTCHA [1] prove that if A is scalar commutative with 1 and if R is a principal ideal domain then A is commutative. Recently, LUH and PUTCHA [2] generalized these results by proving that if A is an algebra with 1 over a principal ideal domain R such that for each $x, y \in A$ there exist $\alpha, \beta \in R$ such that $(\alpha, \beta) = 1$ and $\alpha xy = \beta yx$, then A is commutative.

In this paper a "local" scalar commutativity will be studied. We shall call an element $x \in A$ *scalar central* if for each $y \in A$, there exist $\alpha, \beta \in R$ depending on y such that $(\alpha, \beta) = 1$ and $\alpha xy = \beta yx$. We shall prove that if A is an associative algebra over a principal ideal domain R and if $x \in A$ is scalar central then there exists a positive integer n such that $x^n y = x^{n-1} yx = x^{n-2} yx^2 = \ldots = yx^n$ for *all* $y \in A$. If, in addition, A has 1 then $x^2 y = xyx = yx^2$. Therefore the results of Rich, Koh, Luh and Putcha for associative algebras immediately follow.

Throughout this paper A will denote an associative algebra over a principal ideal domain R, C will denote the center of A, $\mathbf{Z}^+$ the set of all positive integers and $\mathbf{N}$ the set of natural numbers. If $a, b \in A$ then $[a, b] = ab - ba$. If $\alpha, \beta \in R$ then (α, β) denotes the greatest common divisor of α and β. If $a \in A$ then the order of a, denoted by $o(a)$, is the generator of the ideal $I = \{\alpha | \alpha \in R, \alpha a = 0\}$ of R. $o(a)$ is unique up to associates.

2. Main results. Throughout this section x will denote a scalar central element in A. Let y be an arbitrary element in A. We assume $\alpha, \beta, \alpha_1, \beta_1 \in R$ to be such that

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$(\alpha, \beta)=(\alpha_1, \beta_1)=1$ and

(1) $$\alpha xy = \beta yx,$$

(2) $$\alpha_1 x(x+y) = \beta_1(x+y)x.$$

From (1) and (2), we obtain

(3) $$(\alpha_1\beta-\alpha\beta_1)xy = \beta(\beta_1-\alpha_1)x^2,$$

(4) $$(\alpha_1\beta-\alpha\beta_1)yx = \alpha(\beta_1-\alpha_1)x^2.$$

We begin with

Lemma 2.1. *If* $(\alpha_1-\beta_1)qx^k=0$, *where* $k\in\mathbf{Z}^+$, $k\geqq 2$ *and* $q\in R$, *then* $q[x^iy, x^{k-i}]=0$ *for* $i=0, 1, 2, \dots, k-1$.

Proof. By (2), $\alpha_1 qx^{i+1}(x+y)x^{k-i-1}=\beta_1 qx^i(x+y)x^{k-i}$ which is reduced to

(5) $$\alpha_1 qx^{i+1}yx^{k-i+1} = \beta_1 qx^iyx^{k-i}.$$

In particular, $\beta_1^k qx^ky=\alpha_1^k qx^ky=\beta_1^k qyx^k=\alpha_1^k qyx^k$. Since $(\alpha_1, \beta_1)=1$, $qx^ky=qyx^k$. Thus, by (5), $\alpha_1^i qx^iyx^{k-i}=\beta_1^i qyx^k=\beta_1^i qx^ky=\alpha_1^i qx^ky$, and $\beta_1^i qx^iyx^{k-i}=\alpha_1^i qx^ky=$ $=\beta_1^i qx^ky$. Consequently, $\alpha_1^i q(x^iyx^{k-i}-x^ky)=\beta_1^i q(x^iyx^{k-i}-x^ky)=0$. Since $(\alpha_1^i, \beta_1^i)=1$, $q(x^iyx^{k-i}-x^ky)=0$. That is, $q[x^iy, x^{k-i}]=0$ as required.

It is clear that there exists an integer $n\geqq 3$ such that $o(x^n)=o(x^{n+1})$.

Lemma 2.2. *Suppose* $o(x^n)=p^m$, *where p is a prime element in R and* $m\in\mathbf{Z}^+$. *If* $p^lx^ny=0$ *for some* $l\in\mathbf{N}$, $l<m$, *then* $[x^iy, x^{n-i}]=0$ *for* $i=0, 1, 2, \dots, n-1$.

Proof. We proceed by induction on l. Suppose $l=0$. Then $x^ny=0$. By (3) and (4), we get $0=(\alpha_1\beta-\alpha\beta_1)x^ny=\beta(\beta_1-\alpha_1)x^{n+1}=(\alpha_1\beta-\alpha\beta_1)xyx^{n-1}$, and $0=(\alpha_1\beta-\alpha\beta_1)xyx^{n-1}=\alpha(\beta_1-\alpha_1)x^{n+1}$. Since $(\alpha, \beta)=1$, $(\beta_1-\alpha_1)x^{n+1}=0$ and $p^m|(\beta_1-\alpha_1)$. So $(\beta_1-\alpha_1)x^n=0$. Thus, by Lemma 2.1, $[x^iy, x^{n-i}]=0$ for $i=0, 1, 2, \dots, n-1$.

Now we assume $l>0$ and $\alpha_1\beta-\beta\alpha_1=p^t\delta$, where $(p, \delta)=1$, $t\in\mathbf{N}$.

Suppose $t\geqq l$. Then, by (3), $0=p^t\delta x^ny=\beta(\beta_1-\alpha_1)x^{n+1}=p^t\delta xyx^n$, and hence by (4), $0=p^t\delta xyx^n=\alpha(\beta_1-\alpha_1)x^{n+1}$. Since $(\alpha, \beta)=1$, $(\beta_1-\alpha_1)x^{n+1}=0$. Again by Lemma 2.1, $[x^iy, x^{n-i}]=0$ for $i=0, 1, 2, \dots, n-1$.

Suppose $t<l$. Then, by (3), $0=p^l\delta x^ny=p^{l-t}\beta(\beta_1-\alpha_1)x^{n+1}$. So $p^m|p^{l-t}\beta(\beta_1-\alpha_1)$. By (3), $p^{l-t}p^t\delta xyx^{n-1}=0$ and, by (4), $p^{l-t}\alpha(\beta_1-\alpha_1)x^{n+1}=p^l\delta xyx^{n-1}=0$. Hence, we have $p^{l-t}(\beta_1-\alpha_1)x^{n+1}=0$ and $p^m|p^{l-t}(\beta_1-\alpha_1)$. Since $l<m$, $p^t|(\beta_1-\alpha_1)$ and $\beta_1-\alpha_1=p^t\gamma$, where $\gamma\in R$. Thus, by (3), $p^t\delta x^ny=\beta p^t\gamma x^{n+1}$, i.e. $p^tx^n(\delta y-\beta\gamma x)=0$. Since $t<l$, $[x^i(\delta y-\beta\gamma x), x^{n-i}]=0$ for $i=0, 1, 2, \dots, n-1$, by the induction hypothesis. This implies that $\delta[x^iy, x^{n-i}]=0$. On the other hand, since $(\alpha_1-\beta_1)p^mx^{n+1}=0$, $p^m[x^iy, x^{n-i}]=0$ for $i=0, 1, 2, \dots, n-1$, by Lemma 2.1. Since $(p^m, \delta)=1$, we obtain $[x^iy, x^{n-i}]=0$ for $i=0, 1, 2, \dots, n-1$. This completes the proof.

Lemma 2.3. *Suppose* $o(x^n)=p^m$, *where* p *is a prime element in* R *and* $m\in\mathbf{N}$. *Then* $[x^i y, x^{n-i}]=0$ *for* $i=0, 1, 2, \ldots, n-1$.

Proof. Again we let $\alpha_1\beta-\alpha\beta_1=p^t\delta$. Suppose $t\geqq m$. Then, by (3) and (4) respectively, we have

$$0=p^t\delta x^n y=\beta(\beta_1-\alpha_1)x^{n+1} \quad \text{and} \quad 0=p^t\delta y x^n=\alpha(\beta_1-\alpha_1)x^{n+1}.$$

Since $(\alpha, \beta)=1$, $(\beta_1-\alpha_1)x^{n+1}=0$. By Lemma 2.1, $[x^i y, x^{n-i}]=0$ for $i=0, 1, 2, \ldots, n-1$.

Now suppose $t<m$. Then by (3), $0=p^m\delta x^n y=p^{m-t}\beta(\beta_1-\alpha_1)x^{n+1}$. So $p^t|\beta(\beta_1-\alpha_1)$. Let $\beta(\beta_1-\alpha_1)=p^t\gamma$, where $\gamma\in R$. Then, by (3), $p^t x^n(\delta y-\gamma x)=0$. By Lemma 2.2, $[x^i(\delta y-\gamma x), x^{n-i}]=0$. So $\delta[x^i y, x^{n-i}]=0$. On the other hand, since $(\alpha_1-\beta_1)p^m x^{n+1}=0$, $p^m[x^i y, x^{n-i}]=0$ by Lemma 2.1. Thus, $[x^i y, x^{n-i}]=0$ since $(p^m, \delta)=1$.

Lemma 2.4. *Suppose* $o(x^n)=p_1^{m_1}p_2^{m_2}\ldots p_s^{m_s}$, *where* $p_1, p_2, \ldots, p_s$ *are non-associate primes in* R, *and* $m_1, m_2, \ldots, m_s\in\mathbf{Z}^+$. *Then* $[x^i y, x^{n-i}]=0$ *for* $i=0, 1, 2, \ldots, n-1$.

Proof. Let $q_j=p_1^{m_1}\ldots p_{j-1}^{m_{j-1}}p_{j+1}^{m_{j+1}}\ldots p_s^{m_s}$, $j=1, 2, \ldots, s$. Then $q_j x$ is scalar central, $o((q_j x)^n)=o((q_j x)^{n+1})$, and hence, by Lemma 2.3, $q_j^n[x^i y, x^{n-i}]=[(q_j x)^i y, (q_j x)^{n-i}]=0$ for $j=1, 2, \ldots, s$; $i=0, 1, 2, \ldots, n-1$. Since the q_j's are relatively prime, we obtain $[x^i y, x^{n-i}]=0$ for $i=0, 1, 2, \ldots, n-1$.

Theorem 2.1. *Suppose* $x\in A$ *is scalar central and* $o(x^n)=o(x^{n+1})=0$, *where* $n\geqq 3$. *Then* $x\in C$.

Proof. Clearly $o(x^3)=0$. By (3), and (4) respectively, we obtain

$$(\alpha_1\beta-\alpha\beta_1)xyx=\beta(\beta_1-\alpha_1)x^3 \quad \text{and} \quad (\alpha_1\beta-\alpha\beta_1)xyx=\alpha(\beta_1-\alpha_1)x^3.$$

Hence $(\beta-\alpha)(\beta_1-\alpha_1)x^3=0$. This implies that $\beta=\alpha$ or $\beta_1=\alpha_1$. In either case, we have $xy=yx$. Since y is an arbitrary element in A, $x\in C$.

Theorem 2.2. *If* $x\in A$ *is scalar central then there exists* $n\in\mathbf{Z}^+$ *such that*

$$x^n y=x^{n-1}yx=x^{n-2}yx^2=\ldots=yx^n \quad \textit{for all} \quad y\in A.$$

Proof. This is an immediate consequence of Lemma 2.4 and Theorem 2.1.

3. Algebras with unity elements. We assume throughout this section that A is an algebra with 1 over a principal ideal domain R, and x is a scalar central element in A. Let y be an arbitrary element in A and $\alpha, \beta, \alpha_2, \beta_2\in R$ be such that $(\alpha, \beta)=(\alpha_2, \beta_2)=1$,

$$\alpha xy=\beta yx, \tag{1'}$$

$$\alpha_2 x(1+y)=\beta_2(1+y)x. \tag{2'}$$

Then

$$(3') \qquad (\alpha_2\beta-\alpha\beta_2)xy = \beta(\beta_2-\alpha_2)x,$$

$$(4') \qquad (\alpha_2\beta-\alpha\beta_2)yx = \alpha(\beta_2-\alpha_2)x.$$

Lemma 3.1. *If* $(\alpha_2-\beta_2)qx=0$, *where* $q\in R$, *then* $qxy=qyx$.

Proof. By (3′) and (4′), $(\alpha_2\beta-\alpha\beta_2)qxy=(\alpha_2\beta-\alpha\beta_2)qyx=0$. By (1′), $\alpha_2(\beta-\alpha)qxy=\beta_2(\beta-\alpha)qxy=0$. Since $(\alpha_2, \beta_2)=1$, $(\beta-\alpha)qxy=0$. So $\beta qxy= =\alpha qxy=\beta qyx$. It follows that $\beta(qxy-qyx)=0$. Similarly, $\alpha(qxy-qyx)=0$. Thus, $qxy=qyx$.

Similarly to the arguments in Section 2 but using identities (1′), (2′), (3′), (4′) instead of (1), (2), (3), (4), we can readily prove the following

Lemma 3.2. *Suppose* $o(x^2)=p^m$, *where* p *is a prime element in* R *and* $m\in\mathbf{Z}^+$. *If* $p^l x^2y=0$ *for some* $l\in\mathrm{N}, l<m$, *then* $x^2y=xyx=yx^2$.

Lemma 3.3. *Suppose* $o(x^2)=p^m$, *where* p *is a prime element in* R *and* $m\in\mathbf{Z}^+$. *Then* $x^2y=xyx=yx^2$.

Lemma 3.4. *Suppose* $o(x^2)=p_1^{m_1}p_2^{m_2}\dots p_s^{m_s}$, *where* $p_1, p_2, \dots, p_s$ *are non-associate prime elements in* A *and* $m_1, m_2, \dots, m_s\in\mathbf{Z}^+$. *Then* $x^2y=xyx=yx^2$.

Theorem 3.1. *If* $x\in A$ *is scalar central and if* $o(x^2)=0$, *then* $x\in C$.

Theorem 3.2. *If* $x\in A$ *is scalar central then* $x^2y=xyx=yx^2$ *for all* $y\in A$.

We should note that under the hypothesis of Theorem 3.2, one could not expect $x\in C$.

Example. Let $A=\left\{\begin{bmatrix} a & b \\ o & c\end{bmatrix}\middle| a, b, c\in\mathbf{Z}_2\right\}$ be the algebra of all upper triangular matrices over the ring $\mathbf{Z}_2$ of integers modulo 2. Let $x=\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}$. Then A has a unity element, x is scalar central, but $x\notin C$.

4. Some special cases. We noted in passing that in an algebra over a principal ideal domain, scalar central elements need not lie in the centre of the algebra. However, we have the following

Theorem 4.1. *Suppose A is a semi-prime algebra (with or without 1) over a principal ideal domain R. Then all scalar central elements in A are in the centre C of A.*

Proof. Let x be a scalar central element. By Theorem 2.2, there is a *least* positive integer n such that $x^ny=x^{n-1}yx=x^{n-2}yx^2=\dots=yx^n$ for all $y\in A$.

Suppose $n>1$. For $y\in A$, let $\alpha, \beta\in R$ be such that $(\alpha, \beta)=1$ and $\alpha xy=\beta yx$. Noting that $\alpha x^{2n-2}y=\beta x^{2n-2}y$ and $\alpha yx^{2n-2}=\beta yx^{2n-2}$, we have for any $z\in A$ and $i=0, 1, 2, \dots, n-2$,

$$\alpha^i(x^{n-1}y-x^iyx^{n-i-1})z\alpha^i(x^{n-1}y-x^iyx^{n-i-1})=$$
$$=\alpha^{2i}(x^{n-1}yzx^{n-1}y-x^iyx^{n-i-1}zx^{n-1}y-x^{n-1}yzx^iyx^{n-i-1}+x^iyx^{n-i-1}zx^iyx^{n-i-1})=$$
$$=\alpha^{2i}x^{2n-2}yzy-\alpha^i\beta^iyx^{n-1}zx^{n-1}y-\alpha^i\beta^ix^{n-1}yzyx^{n-1}+\beta^{2i}yx^{n-1}zyx^{n-1}=$$
$$=\alpha^{2i}x^{2n-2}yzy-\alpha^i\beta^iyx^{2n-2}zy-\alpha^i\beta^ix^{2n-2}yzy+\beta^{2i}yx^{2n-2}zy=0.$$

Thus, by the semiprimeness of A, $\alpha^i(x^{n-1}y-x^iyx^{n-i-1})=0$. Likewise, $\beta^{n-i-1}(x^{n-1}y-x^iyx^{n-i-1})=0$. Since $(\alpha^i, \beta^{n-i-1})=1$, $x^{n-1}y-x^iyx^{n-i-1}=0$ for $i=0, 1, 2, \dots, n-2$. So $x^{n-1}y=x^{n-2}yx=x^{n-3}yx^2=\dots=yx^{n-1}$ for all $y\in A$. This contradicts the minimality of n. Hence $n=1$ and $xy=yx$ for all $y\in A$.

Theorem 4.2. *Let A be an algebra with* 1 *over a principal ideal domain R. If x and $1+x$ are both scalar central then $x\in C$.*

Proof. By Theorem 3.2, for any $y\in A$, $xyx=x^2y$ and $(1+x)y(1+x)=(1+x)^2y$ which imply that $xy=yx$.

As a corollary we have the following result due to Luh and Putcha [2].

Corollary 4.1. *Let A be an algebra with* 1 *over a principal ideal domain R. If every element in R is scalar central then A is commutative.*

Remark. To generalize the concept of scalar central element one may call an element $x\in A$ scalar power central if for each $y\in A$ there exist $\alpha, \beta\in R$ and $n\in\mathbf{Z}^+$, depending on y, such that $\alpha x^ny=\beta yx^n$ and $(\alpha, \beta)=1$. It would be interesting to know whether analogous results remain true.

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On the concentration of distribution of additive functions

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1. We say that $g(n)$ is *additive* if $g(mn)=g(m)+g(n)$ holds for every coprime pairs m, n of positive integers. If, moreover, $g(p^\alpha)=g(p)^\alpha$ for every prime power p^α, then $g(n)$ is called *strongly additive*. By $p, p_1, p_2, \ldots, q, q_1, q_2, \ldots$ we denote prime numbers, $c, c_1, c_2, \ldots$ are suitable positive constants. $P(n)$ and $\varkappa(n)$ denote the largest and the smallest prime factor of n. The symbol $\ll$ is used instead of O; $\#\{\ \}$ is the counting function of the set indicated in brackets $\{.\}$. For a distribution function $H(x)$ let $\varphi_H(\tau)$ denote its characteristic function. Let

$$Q(h) = Q_H(h) = \sup_x \left(H(x+h)-H(x)\right)$$

be the continuity module — concentration — of H. We say that H satisfies a Lipschitz condition if $Q(h)\ll h$ as $h\to 0$.

We assume that $g(n)$ is strongly additive and that

$$\sum_p \frac{g^2(p)}{p} < \infty. \tag{1.1}$$

The theorem of Erdős—Wintner [1] guarantees that the function $g(n)-A_n$, where

$$A_n = \sum_{p<n} \frac{g(p)}{p}, \tag{1.3}$$

has a limit distribution, i.e. the relation

$$\frac{1}{N}\#\{n \leqq N \mid g(n)-A_n < x\} \to F(x) \tag{1.4}$$

holds at every continuity point of $F(x)$, where $F(x)$ is a distribution function. If,

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moreover, $\Sigma g(p)/p$ converges, then the values $g(n)$ have a limit distribution too, i.e.

$$\frac{1}{N}\#\{n \leqq N|g(n) < x\} \to G(x), \tag{1.5}$$

at every continuity point of the distribution function $G(x)$.

We have the relations

$$\varphi_F(\tau) = \prod_p \left(\left(1-\frac{1}{p}\right)e^{-i\tau\frac{g(p)}{p}} + \frac{1}{p}e^{i\tau\left(1-\frac{1}{p}\right)g(p)}\right), \tag{1.6}$$

$$\varphi_G(\tau) = \prod_p \left(1-\frac{1}{p}+\frac{e^{i\tau g(p)}}{p}\right). \tag{1.7}$$

From these forms we can see that both F and G can be represented as the distribution of the sum of infinitely many mutually independent random variables having purely discrete distributions. By the well-known theorem of P. Lévy [2] G and F are continuous if

$$\sum_{p \in Z_g} 1/p = \infty, \quad \text{where } Z_g = \{p|g(p) \neq 0\}. \tag{1.8}$$

Furthermore, assuming the validity of (1.3) we have that F and G are of pure type, either absolutely continuous or singular (see E. Lukács [3]). To decide the question if a distribution function were absolutely continuous or singular seems to be quite difficult. The first result upon this has been achieved by P. Erdős [4]; namely it was proved that if $g(p)=O(p^{-\delta})$, δ being any positive constant, then $G(x)$ is singular. Recently Jogesh Babu [5] has proved that $G(x)$ is absolutely continuous if $g(n)$ is generated by $g(p)=(\log p)^{-a}$ $(0<a<2)$. The main idea of the proof is that $\varphi_G(\tau)$ is square-integrable in $(-\infty, \infty)$, and so by using Plancherel's theory of Fourier integrals it must have an inverse in $L^2(-\infty, \infty)$ that is the density function of $G(x)$.

It is known that a distribution function H satisfies Lipschitz condition if $|\varphi_H(\tau)|$ is integrable in $(-\infty, \infty)$, and so it is absolutely continuous. The method of Jogesh Babu gives that G satisfies Lipschitz condition if $g(p)=(\log p)^{-a}$ $(0<a<1)$.

The aim of this paper is to investigate the singularity or absolute continuouity of distribution functions for some classes of additive functions.

We shall prove the following theorems.

Theorem 1. *Let $g(n)$ be a strongly additive function,*

$$D(y) = \sum_{p>y} \frac{|g(p)|}{p}, \tag{1.9}$$

and suppose that the inequalities

$$D(t^A) < 1/t, \tag{1.10}$$

$$|g(p_1)-g(p_2)| > 1/t \quad \textit{if} \quad p_1 \neq p_2 < t^\delta \tag{1.11}$$

hold, with suitable positive constants A and δ, for every large t. Then

$$(\log t)^{-1} \ll Q_G(1/t) \ll (\log t)^{-1} \quad (t \to \infty), \tag{1.12}$$

where the constants involved by $\ll$ may depend on g.

This result was achieved by TJAN [7] and P. ERDŐS [8] for $\log \frac{\varphi(n)}{n}$, and for $\log \frac{\sigma(n)}{n}$, resp.

Theorem 2. *Let $g(n)$ be strongly additive satisfying* (1.1). *Then for the concentration $Q(h)$ of $F(x)$ or $G(x)$ (if it exists) we have*

$$Q(4D_R) \geqq \frac{c}{\log R} \quad (R \geqq 2), \tag{1.13}$$

c being an absolute positive constant, and

$$D_R = \left(\sum_{p>R} \frac{g^2(p)}{p} \right)^{1/2}. \tag{1.14}$$

Remarks.

1) This assertion is non-trivial only if $D_R \log R \to 0$ $(R \to \infty)$, since $Q_H(1/t) \gg 1/t$ $(t \to \infty)$ for every $H(x)$.
2) If $g(p) = (\log p)^{-\gamma}$ ($\gamma \geqq 1$ constant), then $D_R = (1+o(1)) \frac{(\log R)^{-\gamma}}{\sqrt{2\gamma}}$ and so $Q_G(1/t) \gg \frac{1}{t^{1/\gamma}}$.

Theorem 3. *If the strongly additive $g(n)$ is generated by $g(p) = (\log p)^{-\gamma}$, then*

$$\frac{1}{t^{1/\gamma}} \ll Q_G(1/t) \ll \frac{(\log\log t)^2}{t^{1/\gamma}} \tag{1.15}$$

if $\gamma > 1$, while for $\gamma = 1$

$$\frac{1}{t} \ll Q_G(1/t) \ll \frac{(\log\log t)^2 \log t}{t}. \tag{1.16}$$

Remarks.

1) We guess that $Q_G(1/t) \ll \frac{1}{t^{1/\gamma}}$ for $\gamma > 1$ but we are unable to prove it.
2) We also guess that $G(x)$ is singular if $0 \leqq g(p) \leqq (\log p)^{-\gamma}$, $\gamma > 2$. This seems not to be known even if $g(p) = (\log p)^{-\gamma}$.
3) By our method we could estimate the concentration for other functions if $g(p)$ is monotonic. The following assertion holds. Let $t(u) > 0$ to monotonically decreasing in $(1, \infty)$, $g(p) = t(p)$ for primes p. Let $y(\tau), z(\tau)$ be defined by the

relations $t(y(\tau))=\frac{y(\tau)^{1/4}}{\tau}$; $t(z(\tau))=1/\tau$. Suppose that for large τ, $y(\tau)<\tau^c$, $z(\tau)>e^{\tau^{1+\varepsilon}}$ ($\varepsilon>0$ constant), and that the integral

$$\int_{y(\tau)}^{z(\tau)} \frac{\cos \tau t(u)}{u \log u}\, du$$

is bounded as $\tau\to\infty$. Then $Q_F(h)\ll 1/h$. These conditions hold if $g(p)$ decreases regularly and

$$\sum \frac{g^2(p)}{p}<\infty; \quad \sum \frac{g(p)}{p}=\infty.$$

Theorem 4. *There exists a monotonically decreasing function $t(u)$ satisfying the conditions*

$$\sum \frac{t(p)}{p}=\infty, \quad \sum \frac{t^2(p)}{p}<\infty,$$

for which the distribution function $F(x)$ of the strongly additive $g(n)$ defined by $g(p)=t(p)$ is singular.

2. Proof of Theorems 2 and 4. We shall prove Theorem 2 for $F(x)$ only. The proof is almost the same for $G(x)$.

$F(x)$ can be represented as the distribution function of θ_1; $\theta_R=\sum_{p>R}\xi_p$, where ξ_p are mutually independent random variables with the distribution

$$P\left(\xi_p=g(p)\left(1-\frac{1}{p}\right)\right)=\frac{1}{p}, \quad P(\xi_p=-g(p)/p)=1-\frac{1}{p},$$

for the mean value $M\theta_R$ and variance $D\theta_R$ we have $M\theta_R=0$, $D\theta_R=D_R$. Consequently, by the Chebyshev inequality,

$$P(|\theta_R|<\Lambda D_R)\geqq 1-\frac{1}{\Lambda^2}.$$

So by

$$d=\sum_{p\leqq R}\frac{g(p)}{p}$$

we have

$$F(-d+\Lambda D_R)-F(-d-\Lambda D_R)\geqq P\left(\xi_p=\frac{-g(p)}{p}\,(\forall p\leqq R)\Lambda|\theta_R|\leqq \Lambda D_R)\right)$$

$$\geqq\left(1-\frac{1}{\Lambda^2}\right)\prod_{p\leqq R}(1-1/p)\gg(1-1/\Lambda^2)\cdot\frac{1}{\log R} \quad (R\geqq 2).$$

By putting $\Lambda=2$ our assertion follows immediately.

To prove Theorem 4 we define our $g(p)$ as follows. Let $R_1=1$, R_{l+1} be defined by $R_l=\log\log\log\log\log R_{l+1}$, $\lambda_l=\exp(\exp(\exp R_l))$, $g(p)=\frac{1}{\lambda_l}$ if $p\in[R_l, R_{l+1})$. Then

$$\sum_{p>R_l}\frac{g(p)}{p}=\infty,\quad \sum_{p>R_l}\frac{g^2(p)}{p}\ll\frac{1}{\lambda_l^2}\log\log R_{l+1}\ll\frac{1}{\lambda_l}.$$

Let m run over the square-free integers all prime factor of which is less than R. By Theorem 2, for fixed m. the number of integers n with

$$n=mv\leqq N,\ \varkappa(m)\geqq R_l,\ g(v)-(A_N-A_{R_l})\in\left[-\frac{c}{\lambda_l},\frac{c}{\lambda_l}\right]$$

is greater than a constant time of

$$\frac{N}{m}\prod_{p<R_l}\left(1-\frac{1}{p}\right).$$

Summing up for m we have

$$\#\left\{n=mv\leqq N\,|\,g(n)\in\bigcup_m\left[g(m)-A_{R_l}-\frac{c}{\lambda_l},\ g(m)-A_{R_l}+\frac{c}{\lambda_l}\right]\right\}$$

$$\gg N\prod_{p<R_l}(1-1/p)\sum_{P(m)\leqq R_l}\frac{1}{m}\gg N.$$

So the intervals

$$\bigcup_m\left[g(m)-A_{R_l}-\frac{c}{\lambda_l},\ g(m)-A_{R_l}+\frac{c}{\lambda_l}\right]$$

cover a positive percentage of integers. The whole length of these intervals is less than $c2^{\pi(R_l)}/\lambda_l$. This quantity tends to zero as $l\to\infty$. By this the theorem is proved.

3. Lemmas. Let $\mathcal{S}(A)$ be an arbitrary set of distinct square free integers m having the following properties:

(1) $A\leqq\varkappa(m)$,

(2) if $p_1|m_1$, $p_2|m_2$, $m_1\neq m_2\in\mathcal{S}(A)$, then $\frac{m_1}{p_1}\neq\frac{m_2}{p_2}$.

Let $\varrho(n)$ be a multiplicative function such that $0\leqq\varrho(p)\leqq 1+O(1/p^\delta)$ ($\delta>0$ constant). Moreover, let

$$T(A)=\sum_{m\in\mathcal{S}(A)}\frac{\varrho(m)}{m}. \tag{3.1}$$

Lemma 1. *For* $2\leqq A$ *we have*

$$T(A)\leqq\frac{c_1}{A\log A}, \tag{3.2}$$

c_1 *being an absolute constant.*

Proof. We split the elements of $\mathcal{S}(A)$ according to $P(m)\in[A^{2^h}, A^{2^{h+1}})$. Let $T_h(A)$ denote the part of the sum (3.1) corresponding to this interval. From (2) we have

$$T_h(A) \leqq \frac{1}{A^{2^h}} \sum \frac{\varrho(n)}{n}$$

where the sum extends over the square free n with $A \leqq \varkappa(n) < P(n) \leqq A^{2^{h+1}}$. So

$$\sum \frac{\varrho(m)}{m} \ll \prod_{A<p\leqq A^{2^{h+1}}} \left(1+\frac{\varrho(p)}{p}\right) \ll \frac{\log A^{2^{h+1}}}{\log A}.$$

Using this inequality for every $h \geqq 0$ we have (3.2).

Remark. Since $T(1) \leqq 1+T(2)$, therefore by Lemma 1, $T(1)$ is bounded.

We shall use the following Esseen type inequality due to A. S. Fainleib [6] which we quote as

Lemma 2. *For an arbitrary distribution function $H(x)$ we have*

$$Q_H(h) \leqq C \sup_{t\geqq 1/h} \frac{1}{t} \int_0^t |\varphi_H(\tau)|\, d\tau. \tag{3.3}$$

Lemma 3. *Let $\gamma>0$ be fixed,*

$$S = \sum_{\tau^{10}<p<e^{\tau^{1/\gamma}}} \frac{\cos \tau (\log p)^{-\gamma}}{p}. \tag{3.4}$$

Then S is bounded as $\tau\to\infty$.

Proof. First of all we shall prove that

$$E = \sum_{\tau^{10}\leqq n\leqq e^{\tau^{1/\gamma}}} \frac{\cos \tau (\log n)^{-\gamma}}{n \log n}$$

is bounded as $\tau\to\infty$. Indeed,

$$\left| E - \int_{\tau^{10}}^{e^{\tau^{1/\gamma}}} \frac{\cos \tau (\log u)^{-\gamma}}{u \log u}\, du \right| \ll \sum_{\tau^{10}\leqq n} \frac{1}{n \log n} \left(\frac{\tau}{(\log n)^{\gamma}} - \frac{\tau}{(\log (n+1))^{\gamma}} \right)$$

$$\ll \frac{\tau}{\tau^{10} (\log \tau)^{1+\gamma}}.$$

To estimate the integral we substitute $y=\tau/(\log u)^{\gamma}$, and we get immediately that

$$\int_{\tau^{10}}^{e^{\tau^{1/\gamma}}} \frac{\cos \tau (\log u)^{-\gamma}}{u \log u}\, du = \frac{1}{\gamma} \int_1^{\tau/(10 \log \tau)^{\gamma}} \frac{\cos y}{y^{1/\gamma}}\, dy = O(1).$$

So it is enough to prove that $S-E=O(1)$ as $\tau\to\infty$.

Let $\tau^{10} \leqq M \leqq e^{\tau^{1/\gamma}}$; $N_1 = M + jN^{3/4}$ $(j=0, 1, \ldots, [M^{1/4}])$, $N = M^{3/4}$, $N_2 = N_1 + N$, and consider the quantity

$$S(N_1, N_2) = \sum_{N_1 \leqq p < N_2} \frac{\cos\tau(\log p)^{-\gamma}}{p} - \sum_{N_1 \leqq n < N_2} \frac{\cos\tau(\log n)^{-\gamma}}{n \log n}.$$

To estimate it we use the prime number theorem for short intervals in the form

$$\Delta_{N_1}(u) = \sum_{n=N_1}^{u} \big(\Lambda(n) - 1\big) \ll \frac{N}{(\log N_1)^{10}} \quad (N_1 \leqq u \leqq N_2). \tag{3.5}$$

Since

$$\frac{1}{N_1 \log N_1} - \frac{1}{n \log n} = -\int_{N_1}^{n} \frac{\log x + 1}{x^2 (\log x)^2} dx \leqq \frac{2(n - N_1)}{N_1^2 \log N_1}$$

for $N_1 \leqq n \leqq N_2$, therefore

$$S(N_1, N_2) \ll \sum_{N_1 \leqq p < N_2} 1/p^2 + \frac{N^2}{N_1^2} + \frac{|L(N_1, N_2)|}{N_1 \log N_1}, \tag{3.6}$$

where

$$L(N_1, N_2) = \sum_{n=N_1}^{N_2} \big(\Lambda(n) - 1\big) \cos\tau(\log n)^{-\gamma}.$$

By using partial summation,

$$L(N_1, N_2) = \Delta_{N_1}(N_2) \cos\tau(\log N_2)^{-\gamma} + \sum_{n=N_1}^{N_2 - 1} \Delta_{N_1}(n) \left(\cos\frac{\tau}{(\log n)^\gamma} - \cos\frac{\tau}{(\log(n+1))^\gamma}\right).$$

Hence, by (3.6) we get

$$L(N_1, N_2) \ll \frac{N}{(\log N_1)^{10}} \left(1 + \sum_{n=N_1}^{N_2 - 1} \left|\cos\frac{\tau}{(\log n)^\gamma} - \cos\frac{\tau}{(\log(n+1))^\gamma}\right|\right).$$

Since $\tau/(\log n)^\gamma$ is monotonic and cosine satisfies Lipschitz condition, the last sum is majorated by

$$\frac{\tau}{\log N_1} - \frac{\tau}{\log N_2}.$$

Consequently,

$$\sum_{0 \leqq j \leqq M^{1/4}} S(N_1, N_2) \ll \sum_{M \leqq p \leqq 2M} \frac{1}{p^2} + \frac{M^{3/2} M^{1/4}}{M^2} + \frac{1}{(\log M)^{11}} +$$

$$+ \frac{M^{-1/4}}{(\log M)^{10}} \left(\frac{\tau}{\log M} - \frac{\tau}{\log 2M}\right).$$

By putting $M = 2^h \tau^{10}$, $h = 0, 1, 2, \ldots,$ up to $M \leqq e^{\tau^{1/\gamma}}$ we have $S - E = O(1)$.

By this Lemma 3 has been proved.

4. Proof of Theorem 3. Let

$$\varphi(\tau) = \prod_p \left(1 + \frac{e^{i\tau(\log p)^{-\gamma}} - 1}{p}\right)$$

be the characteristic function of the limit distribution of $g(n)$ defined by $g(p)=(\log p)^{-\gamma}$. First we observe that

$$\log|\varphi(\tau)| \leqq \operatorname{Re} \sum_{p \leqq e^{\tau^{1/\gamma}}} \frac{e^{i\tau(\log p)^{-\gamma}}-1}{p} + O(1). \tag{4.1}$$

Lemma 3 and the relation

$$\sum_{p \leqq y} \frac{1}{p} = \log\log y + O(1)$$

gives that

$$\log|\varphi(\tau)| \leqq -\frac{1}{\gamma}\log\tau + O(1) + \operatorname{Re} \sum_{p \leqq \tau^{10}} \frac{e^{i\tau(\log p)^{-\gamma}}}{p}. \tag{4.2}$$

Consequently, $\int\limits_0^\infty |\varphi(\tau)| < \infty$ for $\gamma<1$. Let $\gamma \geqq 1$. From (4.2) we have

$$|\varphi(\tau)| \ll \tau^{-1/\gamma}|\psi(\tau)|, \tag{4.3}$$

where

$$\psi(\tau) = \prod_{p \leqq R^{1/2}} \left(1+\frac{e^{i\tau(\log p)^{-\gamma}}}{p}\right), \quad R \leqq \tau \leqq 2R. \tag{4.4}$$

Let $\psi(\tau)=\psi_1(\tau)\cdot\psi_2(\tau)$, where

$$\psi_1 = \prod_{p \leqq (\log R)^4}, \quad \psi_2 = \prod_{(\log R)^4 < p \leqq R^{1/2}}.$$

So we have

$$\int\limits_R^{2R} |\varphi(\tau)|\,d\tau \ll \frac{1}{R^{1/\gamma}}\left(B_1(R)+B_2(R)\right), \tag{4.5}$$

where

$$B_j(R) = \int\limits_R^{2R} |\psi_j(\tau)|^2\,d\tau \quad (j=1,2). \tag{4.6}$$

First we estimate $B_2(R)$. We have

$$\psi_2(\tau) = 1+\sum \frac{e^{i\tau g(m)}}{m},$$

where the summation is extended for the square-free m's satisfying $(\log R)^4 \leqq \varkappa(m) \leqq P(m) \leqq R^{1/4}$. We have

$$B_2(R) \ll R + \sum \frac{1}{m}\min\left(R, \frac{1}{|g(m)|}\right) + \sum_{m,n} \frac{1}{mn}\min\left(R, \frac{1}{|g(m)-g(n)|}\right),$$

n runs over the same set as m.

Let

$$K(1/R) = \sup_x \sum_{g(m)\in[x,x+1/R)} 1/m. \tag{4.7}$$

Let x be fixed. We observe that the set of m's standing in the right hand side satisfies the conditions of Lemma 1 with $A=(\log R)^4$, $\varrho\equiv 1$. Indeed, if $|g(m_1)-g(m_2)|\leqq 1/R$, $p_1/m_1, p_2/m_2$, then

$$\left|g\left(\frac{m_1}{p_1}\right)-g\left(\frac{m_2}{p_2}\right)\right| \geqq |g(p_1)-g(p_2)|-|g(m_1)-g(m_2)| \geqq$$
$$\geqq \left|\frac{1}{(\log p_1)^\gamma}-\frac{1}{(\log p_2)^\gamma}\right|-1/R>0,$$

and so $\frac{m_1}{p_1}\neq\frac{m_2}{p_2}$. So we have

$$K(1/R)\ll\frac{\log\log R}{(\log R)^4}.$$

Furthermore, the contribution of the pairs m, n for which $|g(m)-g(n)|\geqq R^2$ is majorated by

$$\frac{1}{R^2}\prod_{p\leqq R^{1/2}}(1+1/p)^2\ll\frac{(\log R)^2}{R^2}.$$

Consequently

$$B_2(R)\ll R+\sum_n\frac{1}{n}\left(\sum_{0\leqq j\leqq R^3}\frac{R}{j+1}\left\{\sum_{|g(m)-g(n)|\in\left[j/R,\frac{j+1}{R}\right]}1/m\right\}\right)+ \tag{4.8}$$
$$+\sum_{0\leqq j\leqq R^3}\frac{R}{j+1}\left(\sum_{|g(m)|\in\left[j/R,\frac{j+1}{R}\right]}1/m\right)\ll R.$$

Since $|\psi_1(\tau)|\leqq\prod_{p\leqq(\log R)^4}(1+1/p)\ll\log\log R$, therefore $B_1(R)\ll(\log\log R)^2R$. So we have

$$\int_R^{2R}|\varphi(\tau)|\,d\tau\ll R^{1-1/\gamma}(\log\log R)^2.$$

Applying this inequality for $R=T/2^h$ $(h=1, 2, \ldots)$ we get

$$\frac{1}{T}\int_1^T|\varphi(\tau)|\,d\tau\ll\begin{cases}\dfrac{(\log\log T)^2}{T^{1/\gamma}}, & \text{if}\quad \gamma>1,\\[2ex] \dfrac{(\log\log T)^2\log T}{T}, & \text{if}\quad \gamma=1.\end{cases}$$

From Lemma 2 our theorem immediately follows.

5. Proof of Theorem 1. First we prove the second inequality in (1.12). Let

$$g(n;\ y)=\sum_{p|n,\ p\geqq y}g(p).$$

Since from (1.10)

$$\sum_{n\leqq N}|g(n;\ t^{2A})|\leqq ND(t^{2A})\leqq\frac{N}{t^2},$$

we have

$$\#\{n \leqq N;\ |g(n;\ t^{2A})| \geqq 1/t\} \leqq \frac{N}{t}. \tag{5.1}$$

For a natural number n let $e(n)$ denote the product of those prime factors of n that are less than t^{2A}; let $f(n)=n/e(n)$. From (5.1) we get that with the exception of at most N/t integers if $n \leqq N$ and $g(n)\in[x, x+1/t]$, then $g(e(n))\in[x-1/t, x+1/t]$. Let x and t be fixed, and $a_1<a_2<\ldots<a_R$ be the sequence of those square-free integers all prime divisors of which is less than t^{2A} and $g(a_j)\in[x-1/t, x+1/t]$, Let $E(a_j)$ be the number of those $n \leqq N$ for which $a_j|e(n)$ and $(a_j, e(n))=a_j$ holds. By using the Eratosthenian sieve we have

$$E(a_j) \leqq 1+O(1)\frac{N\varrho(a_j)}{a_j}\prod_{p<t^{2A}}\left(1-\frac{1}{p}\right) \quad (N\to\infty), \tag{5.2}$$

where $\varrho(m)=\prod_{p|m}\frac{1}{1-1/p}$. Since $\prod_{p<t^A}(1-1/p) \ll (\log t)^{-1}$,

we have

$$Q_G(1/t) \ll \frac{1}{t}+\frac{1}{\log t}\sup_x \sum_{g(a_j)\in[x-1/t,\, x+1/t]}\frac{\varrho(a_j)}{a_j}. \tag{5.3}$$

It has only remained to prove that

$$U_{x,t} = \sum_{g(a_j)\in[x,\, x+1/t]}\frac{\varrho(a_j)}{a_j} \ll 1 \tag{5.4}$$

uniformly for $x\in(-\infty, \infty)$ as $t\to\infty$.

We write every a_j as mv where $P(m)<t^\delta$, $\varkappa(v)\geqq t^\delta$, or $v=1$. So

$$U_{x,t} = \sum_v \frac{\varrho(v)}{v}\left\{\sum_{g(m)\in[x-g(v),\, x+1/t-g(v)]}\frac{\varrho(m)}{m}\right\}.$$

The set of m's satisfies the conditions of Lemma 1 (see (1.11)) so the inner sum is bounded, and we have

$$U_{x,t} \ll \prod_{t^\delta\leqq p\leqq t^{2A}}\left(1+\frac{\varrho(p)}{p}\right) \ll 1.$$

We shall prove that

$$G(1/t)-G(-1/t) \geqq \frac{c}{\log t} \quad (t\to\infty),$$

and by this the proof will be finished.

Let $P=\prod_{p<t^{c_1}} p$. It is obvious that

$$\sum_{n\leqq N,(n,P)=1} 1=((1+o(1))N\prod_{p<t^{c_1}}(1-1/p)\geqq\frac{c_2N}{c_1\log t}\quad (N\to\infty), \tag{5.5}$$

c_2 is an absolute constant. Furthermore,

$$\sum_{n\leqq N,(n,P)=1}|g(n)|\leqq\sum_{q>t^{c_1}}|g(q)|\sum_{qm\leqq N,(m,P)=1}1\leqq c_3N\prod_{p\leqq t^{c_1}}\left(1-\frac{1}{p}\right)\sum_{q>t^{c_1}}\frac{|g(q)|}{q}.$$

By choosing $c_1=2A$, from (1.9) we have

$$\sum_{\substack{n\leqq N,(n,P)=1\\ |g(n)|\leqq 1/t}}1\leqq t\sum_{n\leqq N,(n,P)=1}|g(n)|\leqq tN\prod_{p|P}\left(1-\frac{1}{p}\right)\sum_{p>t^{2A}}\frac{|g(p)|}{p}\leqq\frac{c_4N}{t\log t}.$$

This and (5.5) gives that

$$F(1/t)-F(-1/t)\geqq\frac{c_2}{2A\log t}-\frac{c_4}{t\log t}\geqq\frac{c_5}{\log t}.$$

By this the proof of our theorem is finished.

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On the essential maximal numerical range

C. K. FONG

1. Introduction

In [4], STAMPFLI introduced the concept of maximal numerical range and used it to derive an identity for the norm of a derivation on $\mathscr{B}(\mathfrak{H})$. If T is a bounded operator on a Hilbert space $\mathfrak{H}$, then the *maximal numerical range* of T, denoted by $W_0(T)$, is defined to be the set

$$\{\lambda\colon (Tx_n, x_n) \to \lambda \text{ where } \|x_n\| = 1 \text{ and } \|Tx_n\| \to \|T\|\}.$$

For an operator T on $\mathfrak{H}$ the inner derivation δ_T is a map on $\mathscr{B}(\mathfrak{H})$ defined by $\delta_T(X)=TX-XT$ $(X\in\mathscr{B}(\mathfrak{H}))$. Stampfli showed that $\|\delta_T\|=2\inf\{\|T-\lambda\|\colon \lambda\in\mathbf{C}\}$, and $\|\delta_T\|=2\|T\|$ if and only if $0\in W_0(T)$.

In the present paper, we consider the analogous concept called *essential maximal numerical range* to derive the norm of an inner derivation on the Calkin algebra. Let $T\in\mathscr{B}(\mathfrak{H})$ and t be the image of T in the Calkin algebra $\mathscr{B}(\mathfrak{H})/\mathscr{K}(\mathfrak{H})$. The inner derivation d_t on $\mathscr{B}(\mathfrak{H})/\mathscr{K}(\mathfrak{H})$ is defined by $d_t(x)=tx-xt$. The essential maximal numerical range of T, denoted by ess $W_0(T)$, is defined to be the set

$$\{\lambda\colon (Tx_n, x_n) \to \lambda \text{ where } \|x_n\| = 1,\ x_n \to 0 \text{ weakly and } \|Tx_n\| \to \|t\|\}.$$

We shall see that $\|d_t\|=2\inf\{\|t-\lambda\|\colon \lambda\in\mathbf{C}\}$ and $\|d_t\|=2\|t\|$ if and only if $0\in\text{ess } W_0(T)$. Also, we shall show that $W_0(T)=\text{ess } W_0(T)$ under the following mild condition: $\|Tx\|\neq\|T\|$ for every unit vector x. In the final section we consider the maximal numerical range $V_0(T)$ for an element T in a general C^*-algebra and we show that $V_0(T)=W_0(T)$ if $T\in\mathscr{B}(\mathfrak{H})$ and $V_0(t)=\text{ess } W_0(T)$ where t is the image of $T\in\mathscr{B}(\mathfrak{H})$ in the Calkin algebra.

To close this introduction, we state and prove two technical but simple lemmas which will be used several times in the following sections. Recall that the essential

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norm of $T\in\mathscr{B}(\mathfrak{H})$, denoted by $\|T\|_e$, is $\inf\{\|T+K\|: K \text{ is compact}\}$. Note that $\|T\|_e=\|t\|$ where t is the image of T in the Calkin algebra.

Lemma 1.1. *If* $\|x_n\|=1$ *and* $x_n\to 0$ *weakly, then* $\limsup \|Tx_n\|\leqq\|T\|_e$.

Proof. For every compact operator K, $\|Tx_n\|\leqq\|T+K\|+\|Kx_n\|$. Since $\|Kx_n\|\to 0$, we have $\limsup\|Tx_n\|\leqq\|T+K\|$. Therefore the lemma follows.

Lemma 1.2. *If* $T\in\mathscr{B}(\mathfrak{H})$, *then there exists an orthonormal sequence* $\{x_n\}$ *such that* $\|Tx_n\|\to\|T\|_e$. *Furthermore, if P is an infinite rank prejection and* $TP=T$, *then we can choose* $\{x_n\}$ *so that the additional condition* $Px_n=x_n$ *for all n is satisfied.*

Proof. Suppose $x_1, x_2, \ldots, x_{k-1}$ have been constructed so that $Px_n=x_n$ and $\|Tx_n\|\geqq\|T\|_e-n^{-1}$ for $n=1, \ldots, k-1$. Let E be the projection onto the linear span of $x_1, \ldots, x_{k-1}$. Then $\|T(I-E)P\|=\|T(I-E)\|\geqq\|T(I-E)\|_e=\|T\|_e$. Hence there exists a unit vector x_k such that $(I-E)Px_k=x_k$ and $\|Tx_k\|\geqq\|T\|_e-k^{-1}$. The sequence $\{x_n\}$ constructed as above is the required one.

2. Essential maximal numerical ranges

The following proposition is similar to Theorem 5.1 in [2].

Proposition 2.1. *Let* $T\in\mathscr{B}(\mathfrak{H})$ *and* $\lambda\in\mathbf{C}$. *Then the following conditions are equivalent:*

(1) *There exists an orthonormal sequence* $\{x_n\}$ *in* $\mathfrak{H}$ *such that* $\|Tx_n\|\to\|T\|_e$ *and* $(Tx_n, x_n)\to\lambda$.

(2) *There exists a sequence* $\{x_n\}$ *of unit vectors such that* $x_n\to 0$ *weakly,* $\|Tx_n\|\to\|T\|_e$ *and* $(Tx_n, x_n)\to\lambda$.

(3) *There is a projection P of infinite rank such that* $PTP-\lambda P$ *is compact and* $\|TP\|_e=\|T\|_e$.

Proof. That (1) implies (2) is obvious.

(2)$\Rightarrow$(1): Suppose that $\{y_n\}$ is a sequence of unit vectors such that $y_n\to 0$ weakly, $\|Ty_n\|\to\|T\|_e$ and $(Ty_n, y_n)\to\lambda$. We construct an orthonormal sequence $\{x_n\}$ such that $\|Tx_n\|\geqq\|T\|_e-n^{-1}$ and $|(Tx_n, x_n)-\lambda|<n^{-1}$ as follows. Assume that $x_1, \ldots, x_{k-1}$ have been constructed. Let E be the projection onto the subspace spanned by $x_1, \ldots, x_{k-1}$. Then $\|Ey_n\|\to 0$ as $n\to\infty$. Let $z_n=\|(I-E)y_n\|^{-1}(I-E)y_n$. (Note that $(I-E)y_n\neq 0$ and hence z_n is well defined when n is large enough.) We have $\|z_n-y_n\|\to 0$. Hence $\|Tz_n\|\geqq\|T\|_e-k^{-1}$ and $|(Tz_n, z_n)-\lambda|<k^{-1}$ for some large n. Let x_k be such a z_n.

(1)⇒(3): Assume that (1) holds. By the proof of Theorem 5.1 in [2], we can choose a subsequence $\{y_n\}$ of $\{x_n\}$ such that

$$\sum_{m,n} |((T-\lambda)y_n, y_m)|^2 < \infty.$$

Let P be the projection onto the subspace spanned by $\{y_n\}$. Then $PTP-\lambda P$ is a Hilbert—Schmidt operator and hence compact. Since $\{y_n\}$ is orthonormal, by Lemma 1.1, $\|TP\|_e \geqq \limsup \|(TP)y_n\|$. Hence $\|TP\|_e = \|T\|_e$.

(3)⇒(1): Assume that (3) holds. By Lemma 1.2, there exists an orthonormal sequence $\{x_n\}$ such that $Px_n = x_n$ for all n and $\|Tx_n\| \to \|TP\|_e = \|T\|_e$. Since $PTP= =\lambda P+K$ where K is compact, we have $(Tx_n, x_n) = \lambda + (Kx_n, x_n) \to \lambda$ as $n\to\infty$. (Note that, since $x_n \to 0$ weakly and K is compact, we have $\|Kx_n\| \to 0$.) Hence (1) holds.

The proof is complete.

Definition. Let $T\in\mathcal{B}(\mathfrak{H})$. The *essential maximal numerical range* of T, denoted by ess $W_0(T)$, is defined to be the set of all those $\lambda\in\mathbf{C}$ satisfying one of the conditions in Proposition 2.1.

Remark. By Lemma 1.2, we see that ess $W_0(T)$ is always non-empty. Obviously, ess $W_0(T)=$ess $W_0(T+K)$ if K is a compact operator.

By condition (2) in Proposition 2.1, we can follow the argument of Lemma 2 in [4] to prove the convexity of ess $W_0(T)$. Thus we obtain:

Proposition 2.2. *The set* ess $W_0(T)$ *is non empty, compact, convex and contained in the essential numerical range of T.*

The following proposition is simple but useful.

Proposition 2.3. *Suppose that $T\in\mathcal{B}(\mathfrak{H})$ and U is a neighborhood of* ess $W_0(T)$. *Then there exists $\delta>0$ and a subspace $\mathfrak{M}$ of $\mathfrak{H}$ of finite codimension such that $x\in\mathfrak{M}$, $\|x\|=1$ and $\|Tx\|\geqq\|T\|_e-\delta$ imply $(Tx, x)\in U$.*

Proof. We may assume that U is open. Suppose that no such $\mathfrak{M}$ and δ exist. Then we can construct an orthonormal sequence $\{x_n\}$ such that $\|Tx\|_n \to \|T\|_e$ and $(Tx_n, x_n)\notin U$. Let $\{y_n\}$ be a subsequence of $\{x_n\}$ such that the limit $\lambda = \lim_{n\to\infty} (Ty_n, y_n)$ exists. Then $\lambda\notin U$. This is impossible because by definition we have $\lambda\in$ess $W_0(T)$.

A consequence of the above proposition is the upper semicontinuity of the map $T\mapsto$ess $W_0(T)$. (This result resembles Theorem 6 in [4].)

Corollary 2.4. *Let $A\in\mathcal{B}(\mathfrak{H})$ and let U be a neighborhood of* ess $W_0(A)$. *Then there exists $\delta>0$ such that $T\in\mathcal{B}(\mathfrak{H})$ and $\|T-A\|_e<\delta$ imply* ess $W_0(T)\subseteqq U$.

Proof. We may choose a neighborhood V of ess $W_0(T)$ such that $V+\{\lambda\in\mathbf{C}\colon |\lambda|\leqq\varepsilon\}\subseteqq U$ for some positive number ε. By Proposition 2.3, there is a subspace $\mathfrak{M}$ of finite codimension and $\delta>0$ such that $x\in\mathfrak{M}$, $\|x\|=1$ and $\|Ax\|\geqq\geqq\|A\|_e-4\delta$ imply $(Ax, x)\in V$. We may assume that $2\delta<\varepsilon$. Suppose that $\|T-A\|_e<\delta$ and $\lambda\in\operatorname{ess} W_0(T)$. Then there exists a sequence $\{x_n\}$ of unit vectors such that $x_n\to 0$ weakly, $\|Tx_n\|\to\|T\|_e$ and $(Tx_n, x_n)\to\lambda$. When n is sufficiently large, we have $\|Tx_n\|>\|T\|_e-\delta$, $\|(T-A)x_n\|\leqq\|T-A\|_e+\delta\leqq 2\delta$ (by Lemma 1.1) and hence $\|Ax_n\|\geqq\geqq\|Tx_n\|-\|(T-A)x_n\|>\|T\|_e-3\delta>\|A\|_e-4\delta$. Let P be the projection onto $\mathfrak{M}$. Since $x_n\to 0$ weakly and $I-P$ is a finite rank projection, we have $\|Px_n-x_n\|\to 0$. Let $y_n=\|Px_n\|^{-1}Px_n$. Then $y_n\in\mathfrak{M}$ and $\|y_n-x_n\|\to 0$. Therefore $\|Ay_n\|\geqq\|A\|_e-4\delta$ and hence $(Ay_n, y_n)\in V$ when n is sufficiently large. By Lemma 1.1, when n is large enough, $\|(T-A)y_n\|\leqq\|T-A\|_e+\delta\leqq 2\delta$ and hence $(Ty_n, y_n)\in U$. With no loss of generality, we may assume that U is closed at the very beginning. Therefore $\lambda=\lim (Tx_n, x_n)=\lim (Ty_n, y_n)\in U$. The proof is complete.

3. The norm of an inner derivation on the Calkin algebra

Let $T\in\mathscr{B}(\mathfrak{H})$ and t be the image of T in the Calkin algebra $\mathscr{B}(\mathfrak{H})/\mathscr{K}(\mathfrak{H})$. Recall that d_t is the derivation defined on $\mathscr{B}(\mathfrak{H})/\mathscr{K}(\mathfrak{H})$ given by $d_t(x)=tx-xt$. The main result of the present section is the following identity:

$$\|d_t\| = 2\inf\{\|T-\lambda\|_e\colon \lambda\in\mathbf{C}\}.$$

Proposition 3.1. *If $\lambda\in\operatorname{ess} W_0(T)$, then $\|d_t\|\geqq 2(\|T\|_e-|\lambda|)$.*

Proof. By Proposition 2.1, there exists a projection P of infinite rank such that $ptp=\lambda p$ and $\|tp\|=\|t\|$. (Again, p is the image of P in the Calkin algebra.) Hence

$$\|d_t\| \geqq \|d_t(2p-1)\| = \|t(2p-1)-(2p-1)t\| =$$
$$= 2\|tp-pt\| \geqq 2\|tp-ptp\| \geqq 2(\|tp\|-\|ptp\|) = 2(\|t\|-|\lambda|).$$

The proof is complete.

Proposition 3.2. *We have $0\in\operatorname{ess} W_0(T)$ if and only if $\|T\|_e\leqq\|T-\lambda\|_e$ for all $\lambda\in\mathbf{C}$.*

Proof. If $0\in\operatorname{ess} W_0(T)$, then, by Proposition 3.1, we have $2\|T\|_e\leqq\|d_t\|\leqq\leqq 2\|T-\lambda\|_e$ for all $\lambda\in\mathbf{C}$. Conversely, suppose that $0\notin\operatorname{ess} W_0(T)$. Then, by a suitable scalar multiple of T, we may assume that $\operatorname{Re}\lambda\geqq 2\varepsilon$ $(\lambda\in\operatorname{ess} W_0(T))$ for some $\varepsilon>0$. By Proposition 2.3, there exist $\delta>0$ and a subspace $\mathfrak{M}$ of finite codimension such that $x\in\mathfrak{M}$, $\|x\|=1$ and $\|Tx\|\geqq\|T\|_e-3\delta$ imply $\operatorname{Re}(Tx, x)\geqq\varepsilon$. We may assume that $\delta\leqq\varepsilon$. Let $\{x_n\}$ be an orthonormal sequence in $\mathfrak{M}$ such that $\|(T-\delta)x_n\|\to\to\|T-\delta\|_e$. (The existence of such a sequence follows from Lemma 1.2.) For sufficiently large n, we have $\|(T-\delta)x_n\| \geqq \|T-\delta\|_e - \delta$ and hence $\|Tx_n\| \geqq \|T\|_e - 3\delta$. Therefore, when n is large enough, we have $\operatorname{Re}(Tx_n, x_n) \geqq \delta$ and hence

$$\|(T-\delta)x_n\|^2 = \|Tx_n\|^2 - 2\delta \operatorname{Re}(Tx_n, x_n) + \delta^2 \leqq \|Tx_n\|^2 - 2\delta^2 + \delta^2 = \|Tx_n\|^2 - \delta^2.$$

Let $n \to \infty$. Then we get $\|T-\delta\|_e^2 \leqq \|T\|_e^2 - \delta^2$. Thus $\|T-\delta\|_e < \|T\|_e$. Therefore, if $\|T\|_e \leqq \|T-\lambda\|_e$ for all $\lambda \in \mathbf{C}$, then we have $0 \in \operatorname{ess} W_0(T)$.

Theorem 3.3. *Suppose that $T \in \mathscr{B}(\mathfrak{H})$ and t is the image of T in the Calkin algebra. Then* $\|d_t\| = 2 \inf \{\|T-\lambda\|_e \colon \lambda \in \mathbf{C}\}$.

Proof. It is easy to see that there exists some $\lambda_0 \in \mathbf{C}$ such that

$$\|T-\lambda_0\|_e = \inf \{\|T-\lambda\|_e \colon \lambda \in \mathbf{C}\}.$$

By Proposition 3.2, we have $0 \in \operatorname{ess} W_0(T-\lambda_0)$. Hence, by Proposition 3.1, $\|d_t\| = \|d_{t-\lambda_0}\| \geqq 2\|T-\lambda_0\|_e$. Therefore the theorem is valid.

Corollary 3.4. $\|d_t\| = 2\|t\|$ *if and only if* $0 \in \operatorname{ess} W_0(T)$.

4. Relation between $W_0(T)$ and ess $W_0(T)$.

Let $T \in \mathscr{B}(\mathfrak{H})$. Then the following proposition follows from the definitions of $W_0(T)$ and ess $W_0(T)$.

Proposition 4.1. *If* $\|T\| = \|T\|_e$, *then* $\operatorname{ess} W_0(T) \subseteqq W_0(T)$.

In case $\|T\| > \|T\|_e$, nothing much can be said about the relation between $W_0(T)$ and ess $W_0(T)$. However, in that case, $W_0(T)$ is the "numerical range over the maximal vectors":

Proposition 4.2. *If* $\|T\| > \|T\|_e$, *then*

$$W_0(T) = \{(Tx, x) \colon \|x\| = 1 \text{ and } \|Tx\| = \|T\|\}.$$

Proof. Since $\|T^*T\|_e = \|T\|_e^2 < \|T^*T\|$, there is a finite rank projection P commuting with T^*T such that $\|T^*T(I-P)\| < \|T^*T\|$. Hence $\|T(I-P)\| < \|T\|$. Now the proposition follows from the following lemma.

Lemma 4.3. *If P is a projection commuting with T^*T such that $\|T(I-P)\| < \|T\|$, then $W_0(T) = W_0(TP)$.*

Proof. Note that $\|TP\| = \|T\|$. Suppose that $\lambda \in W_0(T)$. Then there exists a sequence $\{x_n\}$ of unit vectors such that $\|Tx_n\| \to \|T\|$ and $(Tx_n, x_n) \to \lambda$. Since $\|T\|^2 \geqq \|T^*Tx_n\| \geqq (T^*Tx_n, x_n) = \|Tx_n\|^2 \to \|T\|^2$, we have $\|T^*Tx_n\| \to \|T\|^2$. Write

$x_n=\alpha_n y_n+\beta_n z_n$ with $\|y_n\|=\|z_n\|=1$, $|\alpha_n|^2+|\beta_n|^2=1$, $Py_n=y_n$ and $Pz_n=0$. Now, since P commutes with T^*T,

$$\begin{aligned}\|T\|^2 &\geqq |\alpha_n|^2\|(T^*T)^{1/2}y_n\|^2+|\beta_n|^2\|(T^*T)^{1/2}z_n\|^2 = \\ &= \|(T^*T)^{1/2}(\alpha_n y_n)+(T^*T)^{1/2}(\beta_n z_n)\|^2 = \\ &= \|(T^*T)^{1/2}x_n\|^2 = \|Tx_n\|^2 \to \|T\|^2.\end{aligned}$$

Since $\|(T^*T)^{1/2}z_n\|^2=\|Tz_n\|^2=\|T(I-P)z_n\|^2\leqq\|T(I-P)\|^2<\|T\|^2$, $\lim\beta_n=0$. Hence $\|Ty_n\|\to\|T\|$ and $(Ty_n, y_n)\to\lambda$. Now it is easy to see that $\lambda\in W_0(TP)$. The proof of $W_0(TP)\subseteqq W_0(T)$ is straightforward and hence omitted.

Next we show that $W_0(T)=\text{ess } W_0(T)$ under a rather mild condition.

Proposition 4.4. *If $T\in\mathscr{B}(\mathfrak{H})$ fails to attain its norm (in the sense that $\|Tx\|\neq\|T\|\,\|x\|$ unless $x=0$), then $W_0(T)=\text{ess } W_0(T)$.*

Proof. From the proof of Proposition 4.2 and the given condition we see that $\|T\|=\|T\|_e$. Now the proposition follows from the following lemma.

Lemma 4.5. (Holmes and Kripke [3; Lemma 2]) *If $T\in\mathscr{B}(\mathfrak{H})$ fails to attain its norm and if $\{x_n\}$ is a sequence of unit vectors in $\mathfrak{H}$ such that $\|Tx_n\|\to\|T\|$, then $x_n\to 0$ weakly.*

Corollary 4.7. *If $\mathfrak{H}$ is a separable Hilbert space and $T\in\mathscr{B}(\mathfrak{H})$, then there is a compact operator K such that $W_0(T+K)=\text{ess } W_0(T)$.*

Proof. It suffices to show that there exists a compact operator K such that $T+K$ fails to attain its norm. Let the polar decomposition of T be $T=VP$ where $P=(T^*T)^{1/2}$ and V is a partial isometry. By considering eigenvalues of P, we can show that there exists a hermitian compact operator J such that $P+J$ has the following three properties: first, $P+J$ remains to be positive; second, the range of $P+J$ is in the initial space of V; third, $P+J$ has no eigenvalue greater than or equal to $\|P\|_e$. By the third property, it is easy to see that $P+J$ does not attain its norm. Let $K=VJ$. Then $T+K=V(P+J)$. If $\|(T+K)x\|=\|T+K\|\,\|x\|$, then by the second property of $P+J$, we have

$$\begin{aligned}\|(P+J)x\| &= \|V(P+J)x\| = \|V(P+J)\|\,\|x\| \\ &\geqq \|V^*V(P+J)\|\,\|x\| = \|P+J\|\,\|x\|\end{aligned}$$

and hence $x=0$. Therefore $T+K$ does not attain its norm. The proof is complete.

5. Maximal numerical range of an element in C^*-algebra

Let $\mathscr{A}$ be a C^*-algebra with identity I and let T be an element in $\mathscr{A}$. Recall that a linear functional f on $\mathscr{A}$ is called a *state* if $f(I)=\|f\|=1$. We call a state f is *maximal* for T if $f(T^*T)=\|T\|^2$. We shall denote by $S_0(T, \mathscr{A})$ the set of all maximal states of T. It is easy to show that $S_0(T, \mathscr{A})$ is non-empty.

Definition. The *(algebraic) maximal numerical range* of an element T in a C^*-algebra $\mathscr{A}$, denoted by $V_0(T, \mathscr{A})$, is defined to be the set $\{f(T): f\in S_0(T, \mathscr{A})\}$.

Note that $V_0(T, \mathscr{A})$ is a non-empty convex compact subset of $V(T, \mathscr{A})$, the (algebraic) numerical range of T. Because of the following proposition, $V_0(T, \mathscr{A})$ can be abbreviated as $V_0(T)$.

Proposition 5.1. *If $\mathscr{A}$ is a sub-C^*-algebra of $\mathscr{B}$ containing I and T, then $V_0(T, \mathscr{A})=V_0(T, \mathscr{B})$.*

The proof of the above proposition follows from a standard argument of Hahn—Banach type and hence is omitted.

Remark. It is easy to check that $S_0(T, \mathscr{A})$ is a *face* of the state space, that is, if f and g are two states such that $\lambda f+(1-\lambda)g$ is in $S_0(T, \mathscr{A})$ for some λ with $0<\lambda<1$, then $f, g\in S_0(T, \mathscr{A})$. However, $V_0(T)$, the image of $S_0(T, \mathscr{A})$ under the evaluation map $f\to f(T)$, is in general not a face of $V(T, \mathscr{A})$. For example, if $\mathfrak{H}=\mathbf{C}^2$ and $T\in\mathscr{B}(\mathfrak{H})$ is given by the matrix $\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$, then $V_0(T)$ is $\{0\}$ while $V(T, \mathscr{B}(\mathfrak{H}))$ is a disk centred at 0.

Proposition 5.2. *For an element T in the C^*-algebra $\mathscr{A}$, we have $V_0(T^*)=$ $=V_0(T)^*$. (For a set S in $\mathbf{C}$, we write S^* for $\{\lambda\in\mathbf{C}: \bar{\lambda}\in S\}$.)*

Proof. It suffices to show that $S_0(T, \mathscr{A})=S_0(T^*, \mathscr{A})$. Let f be in $S_0(T, \mathscr{A})$. By Schwarz's inequality, we have $f(T^*T)^2\leqq f((T^*T)^2)$. Hence $\|T\|^4=f(T^*T)^2\leqq$ $\leqq f((T^*T)^2)=f(T^*(TT^*)T)\leqq\|T\|^2 f(TT^*)$. Hence $f(TT^*)\geqq\|T\|^2$. Therefore $f\in S_0(T^*, \mathscr{A})$. Thus we have shown that $S_0(T^*, \mathscr{A})=S_0(T, \mathscr{A})$ and hence the proposition follows.

Now we are going to show the main result of the present section: the algebraic maximal numerical range of an operator on Hilbert space is the same as the usual one. First we need a lemma similar to Proposition 2.3.

Lemma 5.2. *If U is a neighbourhood of $W_0(T)$, then there is a positive number δ such that $(Tx, x)\in U$ for the unit vectors x satisfying $\|Tx\|\geqq\|T\|-\delta$.*

The proof is the same as that of Proposition 2.3 and hence omitted.

Remark. By using this lemma, we can show that the map $T\mapsto W_0(T)$ is upper semi-continuous.

Theorem 5.4. *If T is an operator on a Hilbert space $\mathfrak{H}$, then $W_0(T)=V_0(T,(\mathscr{B}(\mathfrak{H}))$.*

Proof. Suppose that $\lambda\in W_0(T)$. Then there exists a sequence $\{x_n\}$ of unit vectors such that $\|Tx_n\|\to\|T\|$ and $(Tx_n, x_n)\to\lambda$. For each n, define a state f_n on $\mathscr{B}(\mathfrak{H})$ by $f_n(A)=(Ax_n, x_n)$. By the compactness of the state space, $\{f_n\}$ has a subsequence converging in the weak*-topology to some state, say f. Then $\|T\|^2=\lim\|Tx_n\|^2=\lim(T^*Tx_n, x_n)=f(T^*T)$ and $\lambda=\lim(Tx_n, x_n)=\lim f_n(T)=f(T)$. Therefore $\lambda\in V_0(T,\mathscr{B}(\mathfrak{H}))$.

Conversely, suppose that $\lambda\in V_0(T)$. We assume on the contrary that $\lambda\notin W_0(T)$. Since, by Lemma 2 in [4], $W_0(T)$ is compact and convex, there is an open half-space H containing $W_0(T)$ such that $\lambda\notin H^-$, the closure of H. By Lemma 5.3, there exists a positive number δ such that $(Tx, x)\in H$ for all x with $\|x\|=1$ and $\|Tx\|^2\geqq\|T\|^2-\delta$. We can choose δ small enough so that $3\delta\|T\|<\operatorname{dist}(\lambda, H)$, the distance from λ to H. It is well-known that convex combinations of vector states are dense in the state space in the weak*-topology. Hence there exists a linear functional f of the form $f(A)=\sum_n \mu_n(Ax_n, x_n)$ with $\mu_n\geqq 0$, $\sum\mu_n=1$ and $\|x_n\|=1$ such that $f(T^*T)\leqq\|T\|^2-\delta^2$ and $|f(T)-\lambda|<\delta\|T\|$. Let

$$\mathscr{S}=\{n:\ \|Tx_n\|^2\geqq\|T\|^2-\delta\}.$$

Then

$$\|T\|^2-\delta^2\leqq f(T^*T)=\sum\mu_n\|Tx_n\|^2\leqq(\|T\|^2-\delta)\Big(\sum_{n\notin\mathscr{S}}\mu_n\Big)+\|T\|^2\Big(\sum_{n\in\mathscr{S}}\mu_n\Big)=$$
$$=\|T\|^2-\delta\Big(\sum_{n\notin\mathscr{S}}\mu_n\Big).$$

Hence $\sum_{n\notin\mathscr{S}}\mu_n\leqq\delta$. Therefore,

$$\Big|f(T)-\sum_{n\in\mathscr{S}}\mu_n(Tx_n, x_n)\Big|=\Big|\sum_{n\notin\mathscr{S}}\mu_n(Tx_n, x_n)\Big|\leqq\delta\|T\|.$$

Let $\lambda_n=\big(\sum_{n\in\mathscr{S}}\mu_n\big)^{-1}\mu_n$. Then we have $\sum_{n\in\mathscr{S}}\lambda_n(Tx_n, x_n)\in H$ and

$$\Big|\sum_{n\in\mathscr{S}}\lambda_n(Tx_n, x_n)-\sum_{n\in\mathscr{S}}\mu_n(Tx_n, x_n)\Big|=\Big|\Big(1-\sum_{n\in\mathscr{S}}\mu_n\Big)\Big(\sum_{n\in\mathscr{S}}\lambda_n(Tx_n, x_n)\Big)\Big|\leqq\delta\|T\|.$$

Hence $\operatorname{dist}(f(T), H)\leqq 2\delta\|T\|$. From this and $\|f(T)-\lambda|<\delta\|T\|$ we see that $\operatorname{dis}(\lambda, H)<3\delta\|T\|$. This contradicts our choice of δ which satisfies the inequality $3\delta\|T\|<\operatorname{dist}(\lambda, H)$.

Next we prove a theorem similar to Theorem 5.4 for an element in the Calkin algebra. First we need a simple lemma.

Lemma 5.5. *Let $T\in\mathscr{B}(\mathfrak{H})$ and t be its image in the Calkin algebra. If $\|T\|=\|t\|$, then $V_0(t)\subseteqq V_0(T)$.*

Proof. Let $\lambda\in V_0(t)$. Then there is a state g on the Calkin algebra $\mathscr{B}(\mathfrak{H})/\mathscr{K}(\mathfrak{H})$ such that $g(t^*t)=\|t\|^2$ and $g(t)=\lambda$. Let p be the canonical projection from $\mathscr{B}(\mathfrak{H})$ to the Calkin algebra. Then $f=g\circ p$ is a state on $\mathscr{B}(\mathfrak{H})$ satisfying $f(T^*T)=\|T\|^2$ and $f(T)=\lambda$. Hence $\lambda\in V_0(T)$.

Theorem 5.6. *If $\mathfrak{H}$ is a separable Hilbert space, T is an operator on $\mathfrak{H}$ and t is its image in the Calkin algebra, then* $\operatorname{ess} W_0(T)=V_0(t, \mathscr{B}(\mathfrak{H})/\mathscr{K}(\mathfrak{H}))$.

Proof. By Corollary 4.6, there is a compact operator K such that $\|T+K\|=\|t\|$ and $\operatorname{ess} W_0(T)=W_0(T+K)$. By Theorem 5.4, $W_0(T+K)=V_0(T+K)$. By Lemma 5.5, we have $V_0(T+K)\supseteqq V_0(t)$. Hence $\operatorname{ess} W_0(T)\supseteqq V_0(t)$.

On the other hand, suppose $\lambda\in\operatorname{ess} W_0(T)$. Then, by Proposition 2.1, there exists a projection P in $\mathscr{B}(\mathfrak{H})$ such that its image p in the Calkin algebra satisfies $ptp=\lambda p$ and $\|tp\|=\|t\|$. Let $\mathscr{C}$ be the commutative algebra generated by $1, p$ and pt^*tp. Then it is easy to see that there is a multiplicative linear functional g on $\mathscr{C}$ such that $g(p)=g(1)=1$ and $g(pt^*tp)=\|pt^*tp\|=|tp\|^2=\|t\|^2$. Let h be a state on the Calkin algebra which extends g and let f be the functional given by $f(x)=h(pxp)$. Then it is easy to check that f is a state on the Calkin algebra, $f(t^*t)=\|t\|^2$ and $f(t)=\lambda$. Therefore $\lambda\in V_0(t)$.

Remarks 1. The above theorem can be proved in the same way as Theorem 5.4, by using Proposition 2.3 instead of Lemma 5.5. This alternative proof does not require the underlying Hilbert space $\mathfrak{H}$ to be separable. 2. Because of Theorem 5.4, many results concerning maximal numerical ranges of operators can be extended to corresponding results for maximal numerical ranges of elements in C^*-algebras. For instance, Pythagorean relation for operators in [4] becomes: if T is an element in a C^*-algebra, then there exists a unique $z_0\in\mathbf{C}$ such that $\|T-z_0\|^2+|\lambda|^2\leqq\|(T-z_0)+\lambda\|^2$ for all λ in $\mathbf{C}$; moreover, $0\in V_0(T-\lambda)$ if and only if $\lambda=z_0$.

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On Fong and Sucheston's mixing property of operators in a Hilbert space

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1. Introduction

Let T be a bounded linear operator on a (real or complex) Hilbert space $\mathfrak{H}$. A matrix (a_{ni}) $(n, i=0, 1, 2, \ldots)$ is said to be *uniformly regular* (U. R.) if

$$\sup_n \sum_i |a_{ni}| = a < \infty, \quad \sup_i |a_{ni}| = b_n \to 0 \quad (n \to \infty), \quad \text{and} \quad \lim_n \sum_i a_{ni} = 1.$$

In this article we consider the problem of the equivalence of assertions (a) and (b) below. h is an element of $\mathfrak{H}$.

(a) $T^n h$ converges weakly.

(b) For every U. R. matrix (a_{ni}), $\sum_i a_{ni} T^i h$ converges strongly (to the weak limit in (a)).

In the more general context of a Banach space $\mathfrak{B}$, (b)$\Rightarrow$(a) is always true ([8]), but (a)$\Rightarrow$(b) may fail even if T is a contraction and (a) holds for every $h \in \mathfrak{B}$ ([3]). This equivalence (in a weaker form) was first proved for the special case where $\mathfrak{H}=L_2$ of a probability space and $Th=h\circ\overline{T}$ for an invertible, measure preserving transformation $\overline{T}$ on that space ([4]). This was recently generalized to an arbitrary contraction on $\mathfrak{H}$ in [1], [13] and, in the form as stated above, FONG and SUCHESTON [8]. In this article we shall prove in Theorem 1 the equivalence for a much wider class of operators. This class contains all operators similar to contractions, and we shall give some sufficient conditions for such similarity to hold. By an application of the uniform boundedness principle, it is easy to show that conditions (2.0—1) in Theorem 1 imply that T is power-bounded, i.e. $\sup \|T^n\| < \infty$, provided that the operator B is bounded. Whether the equivalence is true for a general power-

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bounded operator is an open question. Note that if (a) holds, then $\{T^i h\colon i\geqq 0\}$ is bounded and the expression in (b) is meaningful. For (b)⇒(a) with T not power-bounded, we require such expressions to be finite sums. With this modification, (b) implies that $\{T^i h\colon i\geqq 0\}$ is bounded (see [1, p. 237]), and hence (a) ([8]).

2. Main Theorem

Theorem 1. *Let T be an operator on a Hilbert space $\mathfrak{H}$. Assume that there exist Hilbert spaces $\mathfrak{G}$, $\mathfrak{K}$, a contraction C on $\mathfrak{K}$, and operators $A\colon \mathfrak{K}\to\mathfrak{G}$, $R\colon \mathfrak{G}\to\mathfrak{H}$ and $B, S\colon \mathfrak{H}\to\mathfrak{G}$ which are bounded except possibly B such that*

$$(2.0)\qquad RS = \textit{identity operator on } \mathfrak{H}$$

and

$$(2.1)\qquad \lim \|AC^n A^* Bh - ST^n h\| = 0 \quad \textit{for all} \quad h\in\mathfrak{H}.$$

Then for any fixed $h\in\mathfrak{H}$, the following conditions (a) *and* (b) *are equivalent:*

(a) *$T^n h$ converges weakly.*

(b) *For every U. R. matrix (a_{ni}), $\sum_i a_{ni} T^i h$ converges strongly (to the weak limit in* (a)*).*

Proof. We only need to prove (a)⇒(b). In (2.1), C can be assumed to be an isometry. In fact, there exists an isometry U on a Hilbert space $\mathfrak{L}\supset\mathfrak{K}$ satisfying $C^n = PU^n|\mathfrak{K}$, $n\geqq 0$, where P is the orthoprojector from $\mathfrak{L}$ onto $\mathfrak{K}$ (see e.g. [18], p. 11), thus implying $AC^n A^* = (AP)U^n(AP)^*$. Henceforth we shall replace C by an isometry U.

Suppose (a) holds. Since the limit is a fixed point of T, we can and do assume that it is 0. Given $\varepsilon>0$, there exists an integer N such that for all $m\geqq N$, $\|AU^m A^* Bh - ST^m h\|\leqq\varepsilon$. Hence

$$(2.2)\qquad \Big\|\sum_i a_{ni} T^i h\Big\| \leqq b_n \sum_{i=0}^{N-1} \|T^i h\| + \|R\| \Big\|\sum_{i=N}^{\infty} a_{ni} ST^i h\Big\|,$$

$$(2.3)\quad \Big\|\sum_{i=N}^{\infty} a_{ni} ST^i h\Big\| \leqq a\varepsilon + \Big\|\sum_{i=N}^{\infty} a_{ni} AU^i A^* Bh\Big\| \leqq a\varepsilon + \|A\| \Big\|\sum_{i=N}^{\infty} a_{ni} U^i A^* Bh\Big\|.$$

By the assumption, there exists a positive integer $M\geqq N$ such that for all $m\geqq M$,

$$(2.4)\qquad |\langle ST^m h, Bh\rangle| = |\langle Bh, ST^m h\rangle| = |\langle S^* Bh, T^m h\rangle| \leqq \varepsilon.$$

Hence for all $m \geqq M$ and $i, j \geqq 0$, we have

$$(2.5)\qquad |\langle U^i A^* Bh, U^{i+m} A^* Bh\rangle| = |\langle Bh, AU^m A^* Bh\rangle| \leqq |\langle Bh, ST^m h\rangle| + \\ + |\langle Bh, AU^m A^* Bh - ST^m h\rangle| \leqq \varepsilon + \varepsilon \|Bh\|;$$

and similarly,

$$(2.6)\qquad |\langle U^{j+m} A^* Bh, U^j A^* Bh\rangle| \leqq \varepsilon + \varepsilon \|Bh\|.$$

Hence

$$(2.7)\qquad \left\| \sum_{i=N}^{\infty} a_{ni} U^i A^* Bh \right\|^2 = \sum_{i=N}^{\infty} \sum_{j=N}^{\infty} a_{ni} \bar{a}_{nj} \langle U^i A^* Bh, U^j A^* Bh\rangle \leqq \\ \leqq (2M+1) a b_n \|A^* Bh\|^2 + a^2 \varepsilon (1 + \|Bh\|),$$

as can be seen by dividing the double sum into parts where $|i-j| \leqq M$ and $|i-j| > M$, respectively, and using (2.5) and (2.6). Finally (2.2), (2.3) and (2.7) imply $\lim \|\sum_i a_{ni} T^i h\| = 0$.

Remarks. (1) The proof actually shows that (c) $\langle T^n h, S^* Bh\rangle \to 0$ implies (b). This together with (b)$\Rightarrow$(a) shows that (c) is equivalent to $T^n h \to 0$ weakly. In fact, we have for all k, $z \in \mathfrak{K}$, $\limsup |\langle C^n k, z\rangle| \leqq \|z\| \cdot \limsup |\langle C^n k, k\rangle|^{1/2}$ ([6], Lemma 2.1). Applying this to $k = A^* Bh$, $z = A^* R^* y$ for any $y \in \mathfrak{H}$ and utilizing (2.0) and (2.1), it is not hard to show that $\limsup |\langle T^n h, y\rangle| \leqq \|A^* R^* y\| \cdot \limsup |\langle T^n h, S^* Bh\rangle|^{1/2}$. In the case of T being a contraction, (c)$\Rightarrow$(b) was implicitly proved in [8] by a somewhat different method. We can also prove the general case from this by observing that (c) implies, by (2.1), $\langle C^n A^* Bh, A^* Bh\rangle \to 0$, and hence, applying the contraction case and using (2.1) again and (2.0), (b).

(2) An operator T on $\mathfrak{H}$ is said to be *similar* to a contraction C on $\mathfrak{K}$ if there exists a (boundedly) invertible operator $A: \mathfrak{K} \to \mathfrak{H}$ such that $T = ACA^{-1}$. Then $T^n = AC^n A^{-1} = AC^n A^* (A^{*-1} A^{-1})$, $n \geqq 0$, and the condition (2.1) is satisfied.

(3) Theorem 1 applies to operators of the C_A classes of H. Langer (see [18], p. 55) and the now classical $C_\varrho = C_{\varrho I}$ classes, $\varrho > 0$, of Sz.-Nagy and Foiaş ([17]). They are those operators T on $\mathfrak{H}$ satisfying $T^n = A^{1/2} P_{\mathfrak{H}} U^n A^{1/2}$, $n \geqq 1$, for a positive and (boundedly) invertible operator A on $\mathfrak{H}$ and a unitary operator U on a Hilbert space $\mathfrak{K} \supset \mathfrak{H}$. Note also that $C_A \subset C_{\|A\|}$ ([12]) and that the union of all C_ϱ classes is dense (in the norm topology) in the set of power-bounded operators ([10]). (b) is valid for all $h \in \mathfrak{H}$ in case of operators with their spectra lying inside the open unit disc. This follows from the fact that $\lim \|T^n\|^{1/n} < 1$ implies $\lim \|T^n h\| = 0$ for all $h \in \mathfrak{H}$. We should also mention that the operators considered here are all similar to contractions (see a general theorem in [11]), and that some power-bounded operators are not similar to any contraction ([7]).

3. Similarity to contractions

We shall give three sufficient conditions for T on $\mathfrak{H}$ to be similar to a contraction. The Corollary below generalizes a result of Sz.-Nagy [16]. The special case where T is power-bounded and $\limsup \|T^n h\| \geqq m\|h\|$, $h \in \mathfrak{H}$, is tacitly contained in [1, p. 238]. In [1] and [16] Banach limits are used as the main tool. Our proof is of a more constructive nature. Theorem 2 will also be used in the proof of Theorem 3.

Theorem 2. *Let T be an operator on $\mathfrak{H}$ satisfying, for a positive number M,*

$$n^{-1}\sum_{i=0}^{n-1}\|T^i h\|^2 \leqq M^2\|h\|^2 \quad (n \geqq 1,\ h \in \mathfrak{H}). \tag{3.1}$$

Then there exists a positive operator R on $\mathfrak{H}$ such that

$$T^*RT = R \quad \text{and} \quad R \leqq M^2 I. \tag{3.2}$$

If, in addition, there exists a positive number m such that

$$m^2\|h\|^2 \leqq n^{-1}\sum_{i=0}^{n-1}\|T^i h\|^2 \quad (n \geqq 1,\ h \in \mathfrak{H}) \tag{3.3}$$

then R and its positive square root P are invertible and

$$PTP^{-1} \text{ is an isometry and } mI \leqq P \leqq MI. \tag{3.4}$$

Corollary. *If T is an operator on $\mathfrak{H}$ and there exist positive numbers m, M, p such that*

$$m^p\|h\|^p \leqq \limsup n^{-1}\sum_{i=0}^{n-1}\|T^i h\|^p \leqq M^p\|h\|^p \quad (h \in \mathfrak{H}), \tag{3.5}$$

then

$$(m/M)\|h\| \leqq \|T^n h\| \leqq (M/m)\|h\| \quad (n \geqq 0,\ h \in \mathfrak{H}), \tag{3.6}$$

and T is similar to a contraction.

The same conclusion holds if we replace in the middle term of (3.5) lim sup *by* lim inf *and even if we replace this middle term by* $\limsup \|T^n h\|^p$ *or by* $\liminf \|T^n h\|^p$.

Proof of Corollary. The middle term in (3.5) is unchanged if we change h to $T^j h$, for any $j \geqq 0$. Hence $m^p\|T^i h\|^p \leqq M^p\|T^j h\|^p$, for any $i, j \geqq 0$. The first conclusion then follows. Theorem 2 applies now to give the second conclusion.

Proof of Theorem 2. Consider first the separable case. So assume that there is a countable dense subset $\{h_1, h_2, \dots\}$ of $\mathfrak{H}$. Let $R_n = n^{-1}\sum_{i=0}^{n-1} T^{*i}T^i$, $n \geq 1$. R_n is positive and (3.1) implies $R_n \leq M^2 I$, $n \geq 1$. Hence for each $j \geq 1$, $\{R_n h_j : n \geq 1\}$ is bounded and so weakly sequentially compact ([5, II.3.28]). Using the diagonal process, we can extract a subsequence $\{R'_n\}$ such that $R'_n h_j$ converges weakly for each j. It follows that R'_n converges in the weak operator topology to a positive operator $R \leq M^2 I$. $T^* R'_n T$ converges to $T^* R T$. On the other hand, $T^* R_n T - R_n = n^{-1}(T^{*n} T^n - I)$, $n \geq 1$. We claim that $n^{-1} T^{*n} T^n$ converges weakly to 0. This then implies that $T^* R'_n T$ has to converge to R, and thus $T^* R T = R$. For the claim, observe that for each $h \in \mathfrak{H}$ and each positive integer n,

$$n^{-1}\|T^n h\|^2 \sum_{j=1}^{n} j^{-1} = n^{-1} \sum_{j=1}^{n} (j^{-1}\|T^{j-1} T^{n-j+1} h\|^2) \leq$$

$$\leq n^{-1} \sum_{j=1}^{n} (M^2 \|T^{n-j+1} h\|^2) = M^2 n^{-1} \sum_{i=0}^{n-1} \|T^i T h\|^2 \leq M^4 \|Th\|^2,$$

by applying (3.1) twice. But $\sum_{j=1}^{n} j^{-1}$ diverges, and hence $n^{-1}\|T^n h\|^2 \to 0$. Now for any $h, k \in \mathfrak{H}$, $|n^{-1}\langle T^{*n} T^n h, k\rangle| \leq (n^{-1}\|T^n h\|^2)^{1/2}(n^{-1}\|T^n k\|^2)^{1/2} \to 0$, proving the claim. Thus (3.2) is proved.

If in addition (3.3) is assumed, then $m^2 I \leq R_n$, $n \geq 1$. In particular $m^2 I \leq R'_n$, $n \geq 1$, whence $m^2 I \leq R$. Thus $m^2 I \leq R \leq M^2 I$ and so $mI \leq P \leq MI$, and R and P are invertible. From $T^* P^2 T = P^2$, we get $(PTP^{-1})^*(PTP^{-1}) = I$, showing that PTP^{-1} is an isometry.

When $\mathfrak{H}$ is not separable, we proceed as follows. Given any $h \in \mathfrak{H}$, the closed subspace generated by $\{h\} \cup \{S_1 \dots S_n h : n \geq 1,\ S_i = T \text{ or } T^*,\ 1 \leq i \leq n\}$ is separable, contains h, and reduces T. Utilising this construction and employing transfinite induction, $\mathfrak{H}$ can be decomposed into a direct sum of a family of mutually orthogonal, separable, closed subspaces, each reducing T. The construction for the separable case applies to each of these subspaces, and we get a positive operator R on $\mathfrak{H}$ satisfying (3.2) and, if (3.3) is assumed, $m^2 I \leq R$. The rest of the proof is as before.

Theorem 3 below generalizes the result of G.-C. Rota [15, Th. 2] that every operator T with spectral radius $r < 1$ is similar to a proper contraction (one of norm < 1). This is because $r = \lim \|T^n\|^{1/n}$ and so by the root test for series, $\sum_{n=0}^{\infty} \|T^n\|^2 < \infty$, implying the case $s = 0$ of Theorem 3. Another case, $s = 1$, was treated by Holbrook ([11]) under the assumption that T is power-bounded.

Theorem 3. *Let T be an operator on $\mathfrak{H}$, $0 \leqq s \leqq 1$ a fixed number, and $Q = Q(T, s) = |I - sT^*T|^{1/2}$ (by symbolic calculus). Assume that there exist positive numbers M, N such that* (3.1) *is satisfied and*

$$\sum_{n=0}^{\infty} \|QT^n h\|^2 \leqq N^2 \|h\|^2 \quad (h \in \mathfrak{H}). \tag{3.7}$$

Then there exists a positive operator P on $\mathfrak{H}$ satisfying

$$I \leqq P \leqq (N^2 + sM^2)^{1/2} I \tag{3.8}$$

such that PTP^{-1} is a contraction, and a proper one in the case $s=0$.

Condition (3.1) is redundant in the case $s=0$, i.e., $Q=I$.

Proof. Condition (3.7) implies that the increasing sequence of positive operators $S_n = \sum_{i=0}^{n-1} T^{*i} Q^2 T^i$, $n \geqq 1$, converges in the weak operator topology to a positive operator $S \leqq N^2 I$. In fact for any $h, k \in \mathfrak{H}$, and any $n > m \geqq 0$,

$$|\langle (S_n - S_m) h, k \rangle| = \left| \sum_{i=m}^{n-1} \langle QT^i h, QT^i k \rangle \right| \leqq$$

$$\leqq \sum_{i=m}^{n-1} \|QT^i h\| \cdot \|QT^i k\| \leqq \left(\sum_{i=m}^{n-1} \|QT^i h\|^2 \right)^{1/2} \left(\sum_{i=m}^{n-1} \|QT^i k\|^2 \right)^{1/2},$$

whence the assertion follows. From the identities $Q^2 + T^* S_n T = S_{n+1}$, $n \geqq 1$, we get $Q^2 + T^* S T = S$.

For each positive integer n,

$$S + sR_n \geqq \sum_{i=0}^{n-2} s^i T^{*i} Q^2 T^i + n^{-1} \sum_{j=1}^{n-1} s^j T^{*j} T^j \geqq n^{-1} \sum_{j=1}^{n-1} \left(\sum_{i=0}^{j-1} s^i T^{*i} Q^2 T^i + s^j T^{*j} T^j \right).$$

Since $Q^2 + sT^*T = |I - sT^*T| + sT^*T \geqq I$, it follows by easy induction that the terms in the first summation form an increasing sequence of positive operators, each $\geqq I$. Hence $S + sR_n \geqq (1 - 1/n) I$. By Theorem 2, there exists a positive operator $R \leqq M^2 I$ on $\mathfrak{H}$ with $T^* R T = R$, and by the above inequalities, and considerations as in the proof of Theorem 2, $S + sR \geqq I$. Summing up the results in this and the last paragraphs, we get $Q^2 + T^* P^2 T = P^2$ and $I \leqq P^2 \leqq (N^2 + sM^2) I$, where P is the positive square root of $S + sR$. Hence (3.8) follows and P is invertible. With $C = PTP^{-1}$,

$$(QP^{-1})^* (QP^{-1}) + C^* C = P^{-1} (Q^2 + T^* P^2 T) P^{-1} = P^{-1} P^2 P^{-1} = I.$$

This shows that $C^* C \leqq I$ and C is a contraction. In the case $s=0$, we have $Q=I$, and the above equality becomes $P^{-2}+C^*C=I$. Hence for each $h\in\mathfrak{H}$,

$$\|Ch\|^2=\langle h, C^*Ch\rangle=\langle h,h\rangle-\langle h, P^{-2}h\rangle=\|h\|^2-\|P^{-1}h\|^2\leqq\|h\|^2(1-\|P\|^{-2}).$$

Thus C is a proper contraction.

We now present a similarity theorem in a measure-theoretic setting. Let $(X,\mathfrak{F},\mu)=(X,\mu)$ be a σ-finite measure space, and $L_p=L_p(X,\mathfrak{F},\mu)$, $1\leqq p<\infty$, the usual Banach spaces of functions. Let $\overline{M}^+$ be the set of extended-valued nonnegative measurable functions (modulo μ-null functions) on (X,μ). A linear operator τ on $\overline{M}^+$ is *monotone* if f_n, $f\in\overline{M}^+$, $f_n\uparrow f$ a.e. implies $\tau f_n\uparrow\tau f$ a.e. (cf. [2], p. 389). For such a τ, its adjoint is uniquely defined as a (linear) operator τ^* on $\overline{M}^+$ satisfying $\int f\cdot\tau^*g\,d\mu=\int g\cdot\tau f\,d\mu$, for all $f,g\in\overline{M}^+$. It is easy to show that τ^* is also monotone and that $\tau^{**}=\tau$. If for a fixed $1\leqq p<\infty$, T is a positive (in the sense that $TL_p^+\subset L_p^+$), bounded linear operator on L_p, then it extends uniquely to a monotone operator τ on $\overline{M}^+$, according to the definition: $\tau f=\lim Tf_n$ a.e., where $f\in\overline{M}^+$, $f_n\in L_p^+$, and $f_n\uparrow f$. For each $f\in\overline{M}^+$, such a sequence f_n always exists and the definition of τf is unambiguous. We shall simply write T for the extended τ.

Theorem 4. *Let τ be a monotone operator on $\overline{M}^+$ and $1\leqq p<\infty$ a fixed number. Assume that, for $p=1$, $\tau^*k\leqq k$; and for $p>1$,*

$$\tau^*\big(k(\tau h)^{p-1}\big)\leqq kh^{p-1}\ \text{for some functions}\ 0<h,k<\infty. \tag{3.9}$$

Then σ, defined on $\overline{M}^+$ as $\sigma f=k^{1/p}\tau(fk^{-1/p})$, is a positive L_p contraction. Further, (3.9) *is equivalent to*

$$\tau\big(k_1(\tau^*h_1)^{p'-1}\big)\leqq k_1h_1^{p'-1}\ \text{for some functions}\ 0<h_1,k_1<\infty,\ \text{with} \tag{3.10}$$

$$k_1=k^{-p'+1}\ \text{and}\ 1/p+1/p'=1.$$

Corollary. *Suppose T is a positive (in the sense that $TL_2^+\subset L_2^+$), bounded operator on L_2, and $T^*(kTh)\leqq kh$ or $T(kT^*h)\leqq kh$ for some functions $0<h<\infty$ $m\leqq k\leqq M$, where m, M are positive constants. Then T is similar to a positive contraction on L_2.*

Proof of Theorem 4. The case $p=1$ is easy. Consider the case $p>1$. First we show (3.9)$\Rightarrow$(3.10). Suppose (3.9) holds. Then $\tau h<\infty$. For if $\tau h=\infty$ on a set E of positive measure, then for all positive numbers N,

$$N\tau^*(k1_E)=\tau^*(Nk1_E)\leqq\tau^*\big(k(\tau h)^{p-1}\big)\leqq kh^{p-1}<\infty,$$

implying $\tau^*(k1_E)=0$. So $0=\int h\cdot\tau^*(k1_E)d\mu=\int_E k\cdot\tau h\,d\mu=\infty$, a contradiction. Let

$F=\{\tau h=0\}$. Then $\int h\cdot\tau^* 1_F d\mu=\int_F \tau h d\mu=0$, and hence $\tau^* 1_F=0$. Define $h_1=$ $=k(\tau h)^{p-1}+1_F$. Then $0<h_1<\infty$, and (3.10) can be verified as follows: $\tau^* h_1=$ $=\tau^*(k(\tau h)^{p-1})+0\leqq kh^{p-1}$ by (3.9); consequently, $(\tau^* h_1)^{p'-1}=(\tau^* h_1)^{1/(p-1)}\leqq$ $\leqq(kh^{p-1})^{1/(p-1)}=k^{1/(p-1)}h=k_1^{-1}h$, and hence, $\tau(k_1(\tau^* h_1)^{p'-1})\leqq\tau h\leqq(k^{-1}h_1)^{1/(p-1)}=$ $=k_1 h_1^{1/(p-1)}=k_1 h_1^{p'-1}$, which is (3.10). Implication (3.10)⇒(3.9) can be proved similarly, by replacing (τ, h, k, p) by (τ^*, h_1, k_1, p'). From the definition of σ we can show that $\sigma^* f=k^{-1/p}\tau^*(k^{1/p}f), f\in\overline{M}^+$. Hence (3.9) transforms into $\sigma^*(\sigma u)^{p-1}\leqq$ $\leqq u^{p-1}$, where $u=hk^{1/p}$. This implies that σ is a contraction on L_p. In case of a Borel space, this implication follows from a dilation theorem in [2]. The general case is proved here by adapting the proof in [9] for the case $\sigma 1\leqq 1$, $\sigma^* 1\leqq 1$. In fact, we have $\sigma u<\infty$, just as $\tau h<\infty$. For $f\in\overline{M}^+$ and any $\lambda>0$,

$$\int 1_{\{\sigma f\geqq\lambda\sigma u>0\}}(\sigma f-\lambda\sigma u)\cdot(\sigma u)^{p-1}d\mu\leqq$$

$$\leqq\int\sigma(f-\lambda u)^+\cdot(\sigma u)^{p-1}d\mu=\int(f-\lambda u)^+\cdot\sigma^*(\sigma u)^{p-1}d\mu\leqq\int 1_{\{f\geqq\lambda u\}}(f-\lambda u)u^p d^{-1}\mu.$$

Multiplying both sides by λ^{p-2}, and integrating with respect to λ from 0 to ∞, we obtain, by the Fubini—Tonelli Theorem,

$$\left(\frac{1}{p-1}-\frac{1}{p}\right)\int(\sigma f)^p d\mu\leqq\left(\frac{1}{p-1}-\frac{1}{p}\right)\int f^p d\mu,$$

showing that σ is an L_p contraction.

Remarks (4). If $\sigma\colon\overline{M}^+(X,\mu)\to\overline{M}^+(Y,\nu)$ is monotone, $1\leqq p\leqq q<\infty$, $0<u\in L_q(X,\mu)$, and $\sigma^*(\sigma u)^{p-1}\leqq u^{q-1}$, then σ extends to a bounded, positive linear operator from $L_q(X,\mu)$ to $L_p(Y,\nu)$ with norm $\leqq\|u\|_q^{(q/p)-1}$. Indeed, by the method of the proof of Theorem 4, we have for all $f\in L_q^+(X,\mu)$, $\int(\sigma f)^p d\nu\leqq\int f^p u^{q-p}d\mu$. (This is trivial when $p=1$, for which case the condition on σ reads $\sigma^* 1\leqq u^{q-1}$.) By the Schwarz inequality, the last integral is $\leqq\left(\int f^q d\mu\right)^{p/q}\left(\int u^q d\mu\right)^{(q-p)/q}$. The conclusion follows. This generalizes a result in [14] for non-negative infinite matrices, as it can be easily shown that non-negative matrices are monotone. Analogous to Theorem 4, the inequality for σ is equivalent to $\sigma(\sigma^* v)^{q'-1}\leqq v^{p'-1}$ for some $0<v\in L_{p'}(Y,\nu)$ when $1<p$, where $1/p+1/p'=1$ and $1/q+1/q'=1$.

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Mean ergodicity in G-semifinite von Neumann algebras

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Introduction. Let A be a von Neumann algebra in a complex Hilbert space H, and let G be a semigroup of normal endomorphisms of A. Denote by A^G the set of all elements of A which are invariant with respect to each element of G. If the identity I belongs to A^G, then A^G is a von Neumann algebra too, but if this isn't so, then A^G is 'only' an ultraweakly closed involutive subalgebra of A, and hence there exists a largest projection $P \neq I$ in A such that for every element T of A one has $PT=TP=T$ ([7], Chap. I. § 3, Théorème 2.).

Let Q denote the set of positive, normal, linear mappings of A into itself obtained from the elements of G by forming convex combinations. The operators in A of the form $V(T)$, where $V \in Q$ and $T \in A$ are called the means of the operator T. For any $T \in A$ let $K_0(T, G)$ denote the set of all means of T. The investigation of the 'behaviour' of the means is one of the subjects of mean ergodic theory ([9], Kap. 1, § 2.). Concerning von Neumann algebras we refer only to the classical results of J. DIXMIER ([6]) and the paper of I. KOVÁCS and J. SZŰCS ([10]).

The purpose of this paper is to investigate a special class of von Neumann algebras.

§ 1 contains preliminary results without their proofs.

In § 2 we define the notion of 'weak ergodicity in means' to express a 'good behaviour' of the means of an operator. This section is devoted to establishing the simplest consequences of this definition.

Let $K(T, G)$ be the weak closure of $K_0(T, G)$. In § 3 we shall give sufficient conditions for T in order that $K(T, G) \cap A^G$ be nonempty (Theorem 3.1.), and that $K(T, G) \cap A^G$ consist of exactly one operator.

1. Definitions and preliminaries. Let us consider a pair (A, G) of a von Neumann algebra A and a semigroup G of normal endomorphisms of A. We shall denote by A^+ the positive portion of A.

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A non-negative, finite or infinite valued function φ defined on A^+ is called a weight on A^+, if it has the following properties:

(i) $$\varphi(T+S) = \varphi(T)+\varphi(S) \quad \text{for every} \quad T, S \in A^+; \quad \text{and}$$

(ii) $$\varphi(cT) = c\varphi(T) \quad \text{for every} \quad c \geqq 0 \quad \text{and} \quad T \in A^+,$$

(with the convention that $0 \cdot \infty = 0$).

We call φ G-invariant if for every $T \in A^+$ and $g \in G$ we have $\varphi(T) = \varphi(g(T))$.

The notion of a G-invariant weight is a very natural generalization of that of a trace.

A weight φ on A^+ is said to be faithful if the conditions $T \in A^+$ and $\varphi(T)=0$ imply $T=0$; normal if, for every increasing directed set $\mathscr{F} \subset A^+$ with $\sup_{S \in \mathscr{F}} S = T \in A^+$, we have $\varphi(T) = \sup_{S \in \mathscr{F}} \varphi(S)$; semi-finite if, for every $T \in A^+$, $T \neq 0$ there exists $S \in A^+$, $S \neq 0$ such that $S \leqq T$ and $\varphi(S) < \infty$.

A weight φ on A^+ is said to be non-infinite if there exists $S \in A^+$, $S \neq 0$ such that $\varphi(S) < \infty$.

For later purposes we state an important fact concerning weights.

Proposition 1.1. ([8], Lemma 1.5) *For any weight φ on A^+ the following conditions are equivalent:*

(i) *φ is normal,*

(ii) *φ is ultraweakly lower semicontinuous,*

(iii) *there exists a family of vectors $\{x_\iota\}$ in H such that*

$$\varphi(T) = \sum_\iota (Tx_\iota, x_\iota) \quad \textit{for every} \quad T \in A^+.$$

Now we shall define special subspaces of A. Denote by Γ the set of normal faithful G-invariant non-infinite and non-zero weights defined on A^+.

Definition 1.1. A *projection* $E \in A$ is called *finite*, if there is a $\varphi \in \Gamma$ such that $\varphi(E) < \infty$. An *operator* in A is called *simple*, if it is a linear combination of finite projections. Denote the set of simple operators by M_0.

Let $\varphi \in \Gamma$ and let $M_\varphi^+ = \{T \in A^+ | \varphi(T) < \infty\}$. Denote by M the smallest norm closed subspace of A that contains M_φ^+ for every $\varphi \in \Gamma$. Since φ defines a linear form $\dot{\varphi}$ on the linear span of M_φ^+, it is not hard to see that the norm closure of M_0 is identical with M.

Let $N_\varphi = \{T \in A | \varphi(T^*T) < \infty\}$. N_φ is a left ideal in A. Denote by N the norm closed linear hull of all N_φ. It is obvious that $M_0 \subseteqq N$ and hence $M \subseteqq N$.

Definition 1.2. A pair (A, G) is said to have *property Π* if for every proper projection $P \in A$ such that $g(P) \leqq P$ for every $g \in G$, we have that $P \in A^G$.

We classify the pairs (A, G) by their weights.

Definition 1.3. A *pair* (A, G) is called *finite* (resp. *semifinite*) if for every $T \in A^+$, $T \neq 0$ we can find a normal G-invariant finite (resp. semifinite) weight φ such that $\varphi(T) \neq 0$.

To facilitate the statement of the next proposition it will be convenient to introduce the following notations.

Definition 1.4. Let E be a projection in A^G. Let us consider the restricted von Neumann algebra A_E. Since $E \in A^G$, every element g of G induces a normal endomorphism g_E on A_E. These restricted endomorphisms form a semigroup. Let us denote this semigroup by G_E. The pair (A_E, G_E) is called a *restriction* of (A, G).

Proposition 1.2. ([5], Theorem 1) *If a pair (A, G) has property Π, then there exists a maximal projection E in A^G such that the restricted pair (A_E, G_E) is finite.*

For finite pairs the following theorem will play an important role in proving Theorem 3.3.

Theorem. (I. Kovács—J. Szűcs ([10])) *Let the pair (A, G) be finite. For every $T \in A$ the convex set $K(T, G) \cap A^G$ contains exactly one element.*

In the following paragraphs we shall deal with pairs (A, G) for which the set Γ is non-empty. This requirement is fulfilled for example in the classical case, when the group $\natural$ of inner automorphisms of A plays the role of G, and A is semifinite. We do not know if this is the case in general for semifinite pairs, but we can state the following:

Proposition 1.3. *If a semifinite pair (A, G) has property Π and $\natural \subset G$, then there exists a normal faithful G-invariant and semifinite weight on A^+.*

Property Π ensures that the support of any G-invariant weight defined on A^+ does belong to A^G. It follows from the condition $\natural \subset G$ that A^G is part of the center of A and hence Dixmier's reasoning ([7], Chap. 1, § 6, Proposition 9.) can be repeated essentially word by word.

The terms and symbols introduced here will be used in what follows without further reference.

2. Let $\mathscr{F}$ be an ultrafilter in Q. Denote by $\mathscr{F}(T)$ the image of $\mathscr{F}$ which is ultrafilter, too. Since the unit ball of A is weakly compact, $K(T, G)$ is weakly compact, too, for every $T \in A$, and so the ultrafilter $\mathscr{F}(T)$ of the means of T converges weakly to an element S of $K(T, G)$. Let this fact be expressed by the symbol $\lim\limits_{\mathscr{F}} V(T) = S$.

Now we define two notions to express 'good behaviour' of the means of an operator.

Definition 2.1. Let the operator $T\in A$ be called *weakly quasi-ergodic* if it has the following properties:

(Li) $K(T, G)\cap A^G$ is non-empty

(Lii) for each $R\in K(T, G)$ the set $K(R, G)\cap A^G$ is non-empty.

Denote by L the subset of weakly quasi-ergodic elements of A.

Definition 2.2. Let the operator $T\in A$ be called *weakly ergodic* if it has the following properties:

(Ei) $K(T, G)\cap A^G$ consists of exactly one element,

(Eii) for each $R\in K(T, G)$ the set $K(R, G)\cap A^G$ consists of exactly one element.

Denote by E the subset of weakly ergodic elements of A. It is obvious that $A^G\subset \subset E\subset L$.

Proposition 2.1. *L is a norm closed, G-invariant subspace of A.*

Proof. The G-invariance and the homogeneity of L are rather obvious. First we prove the additivity of L. Let T_1 and T_2 be arbitrary elements of L. We shall show that the operator $T=T_1+T_2$ belongs to L. By assumption there is an operator S_1 such that $S_1\in K(T_1, G)\cap A^G$. Let $\mathscr{F}_1$ be an ultrafilter in Q such that $\lim\limits_{\mathscr{F}_1} V(T_1)= =S_1$. The limits $\lim\limits_{\mathscr{F}_1} V(T)=S_0$ and $\lim\limits_{\mathscr{F}_1} V(T_2)=R_2$ exist, $S_0\in K(T, G)$ and $R_2\in K(T_2, G)$. By condition (Lii) there exists an ultrafilter $\mathscr{F}_2$ in Q such that $\lim\limits_{\mathscr{F}_2} V(R_2)=R\in K(R_2, G)\cap A^G$. It follows taking account of the facts that $S_0= =S_1+R_2$ and $K(S_0, G)\subset K(T, G)$ that $S=\lim\limits_{\mathscr{F}_2} V(S_0)=S_1+R\in K(T, G)\cap A^G$.

Now let us consider an arbitrary element Y of $K(T, G)$. Then we can find an ultrafilter $\mathscr{F}$ in Q such that $Y=\lim\limits_{\mathscr{F}} V(T)$. The limits $\lim\limits_{\mathscr{F}} V(T_1)=Y_1$ and $\lim\limits_{\mathscr{F}} V(T_2)= =Y_2$ exist, and both belong to L. Since $Y=Y_1+Y_2$, then using the previous result it is obvious that $K(Y, G)\cap A^G$ is non-empty, so we have finished proving that $T\in L$.

Now we are going to show that L is norm closed. Let the sequence $\{T_n\}$ of operators converge to the operator T uniformly. Let us suppose that for each n, $T_n\in L$. Passing, if necessary, to a subsequence, we can assume without loss of generality that $\|T_{n+1}-T_n\| < 1/2^{n+1}$ for each n.

Using the technique of the previous part of the present proof we can construct a sequence $\{S_n\}$ recursively in the following way:

$$S_n\in K(T_n, G)\cap A^G \quad \text{and} \quad S_{n+1}-S_n\in K(T_{n+1}-T_n, G)$$

for each n. It is an obvious consequence of these facts that the sequence $\{S_n\}$ converges in norm, and the limit S of it belongs to A^G.

Now we prove that for any $\varepsilon > 0$ and for any finite system of vectors $x_1, x_2, \ldots, x_k; y_1, y_2, \ldots, y_k$ of H we can find an operator $R \in K_0(T, G)$ such that

$$(*) \qquad |((S-R)x_i, y_i)| < \varepsilon \quad \text{for each} \quad i = 1, 2, \ldots, k.$$

Let us choose a sufficiently large index p, for which $\|S-S_p\|$ and $\|T-T_p\|$ are both sufficiently small. Since $S_p \in K(T_p, G)$, there exists a $V_0 \in Q$ such that $|((S_p - V_0(T_p))x_i, y_i)|$ is sufficiently small for each $i=1, 2, \ldots, k$. Let $R = V_0(T)$. This operator satisfies $(*)$, and this means that $S \in K(T, G) \cap A^G$.

Now let us consider an arbitrary element Y of $K(T, G)$. We can find an ultrafilter $\mathscr{F}$ in Q such that $Y = \lim\limits_{\mathscr{F}} V(T)$. Let us set $Y_n = \lim\limits_{\mathscr{F}} V(T_n)$. It is clear that $Y_n \in L$ for every n, and that the sequence $\{Y_n\}$ converges in norm to Y. Applying the preceding part to the sequence $\{Y_n\}$, we get that $K(Y, G) \cap A^G$ is non-empty.

The next proposition might bear the name 'The Theorem of Linear Choice'.

Proposition 2.2. *For every $T_0 \in L$ and $S_0 \in K(T_0, G) \cap A^G$ we can find a positive linear mapping τ of L onto A^G which possesses the following properties:*

(i) *$\tau(T) \in K(T, G)$ for each $T \in L$,*

(ii) *$\tau(TS) = \tau(T)S$ and $\tau(ST) = S\tau(T)$ for every $T \in L$ and $S \in A^G$,*

(iii) *$\tau(T_0) = S_0$.*

We omit the proof. It can be done by J. T. Schwartz's method developed in ([11], Lemma 5).

Proposition 2.3. *The weakly ergodic elements of A form a norm closed, G-invariant subspace E of A. Denote by $\tau_0(T)$ the single element of $K(T, G) \cap A^G$ for every $T \in E$. The mapping τ_0 is positive linear and has the property that*

$$\tau_0(TS) = \tau_0(T)S \quad \text{and} \quad \tau_0(ST) = S\tau_0(T) \quad \textit{for every } T \in E \textit{ and } S \in A^G.$$

Proof. The G-invariance of E is based upon the fact that for every $T \in A$ the elements of G map $K(T, G)$ into itself.

Denote by Λ the family of those linear mappings τ of L onto A^G which have properties (i) and (ii) of Proposition 2.2. Let τ and ψ be two arbitrary elements of Λ. Let us define the following subset

$$L_{\tau,\psi} = \{T \in L \mid \tau(T) = \psi(T)\}.$$

Taking into account the fact that every element of Λ is norm-continuous and linear it follows that $L_{\tau,\psi}$ is a norm closed subspace of A. Denote by L_0 the intersection of all such $L_{\tau,\psi}$ subspaces. It is obvious that L_0 is a norm closed subspace of A and by Proposition 2.2 it is identical with E.

If we restrict any τ ocĉuring in Proposition 2.2 to E, then we get the mapping τ_0 with the desired properties.

3. In this section we shall investigate pairs (A, G) for which Γ is non-empty and hence the subspaces M and N defined in Definition 1.1. are different from the trivial subspace $\{0\}$.

Theorem 3.1. *If for a pair (A, G) the set Γ is non-empty then all elements of the subspace N are weakly quasi-ergodic.*

Proof. By virtue of Proposition 2.1 it is enough to prove that for every $\varphi\in\Gamma$ $N_\varphi\subset L$. Proving this we follow S. M. ABDALLA ([1], Chap. 3, Theorem 3.4). For our purposes it is sufficient to show that for every $T\in N_\varphi$

(i) $K(T, G)\subset N_\varphi$ and (ii) $K(T, G)\cap A^G$ is non-empty.

Let $T\in N_\varphi$ and $R\in K(T, G)$. We can find a filter $\mathcal{F}$ in Q such that $\lim\limits_{\mathcal{F}} V(T)=R$ in the strong operator topology. As $K(T, G)$ is bounded, we have $\lim\limits_{\mathcal{F}}(V(T)^*V(T))=$ $=R^*R$ in the weak operator topology. On the other hand, if $V\in Q$ and $V=\sum\limits_{i=1}^{n}\alpha_i g_i$ $(\alpha_i>0, \sum\limits_{i=1}^{n}\alpha_i=1, g_i\in G)$, then we have by Schwarz's inequality

$$\varphi(V(T)^*V(T))=\varphi\left[\left(\sum_{i=1}^{n}\alpha_i g_i(T)^*\right)\left(\sum_{j=1}^{n}\alpha_j g_j(T)\right)\right]=\sum_{i,j}\alpha_i\alpha_j\varphi\big(g_i(T)^*g_j(T)\big)\leqq$$

$$\leqq\sum_{i,j}\alpha_i\alpha_j\varphi\big(g_i(T^*)g_i(T)\big)^{\frac{1}{2}}\cdot\varphi\big(g_j(T^*)g_j(T)\big)^{\frac{1}{2}}=\sum_{i,j}\alpha_i\alpha_j\varphi(T^*T)=\varphi(T^*T).$$

Since φ is normal, it is ultraweakly lower semicontinuous and so it is weakly lower semicontinuous on any bounded part of A^+, thus $\varphi(R^*R)\leqq\varphi(T^*T)$. This proves (i).

Since φ is normal it can be represented in the following form: $\varphi(T)=\sum\limits_{\iota}(Tx_\iota, x_\iota)$ for every $T\in A^+$, where the x_ι's are suitable vectors from H. It follows that the function $S\to\varphi(S^*S)$ is weakly lower semicontinuous on any bounded part of A and thus it attains its minimum on the weakly compact bounded set $K(T, G)$. Taking into account the fact that φ is faithful it follows that the function $S\to(\varphi(S^*S))^{1/2}=$ $=\|S\|_2$ is a pre-Hilbert norm on N_φ, therefore the minimum is attained only at one point. Denote by T_0 this element. It is not hard to see that for every element g of G $g(T_0)\in K(T, G)$. On the other hand, it is evident that $\varphi(T_0^*T_0)=\varphi\big(g(T_0)^*g(T_0)\big)$ and this implies that $g(T_0)=T_0$. This means that $T_0\in A^G$ and proves (ii).

The next theorem is a generalisation of J. B. CONWAY's result ([4], Lemma 6).

Theorem 3.2. *If for a pair (A, G) the set Γ is non-empty and A^G does not contain any finite projection except 0, then for every $T\in M$, $K(T, G)\cap A^G=\{0\}$.*

Proof. Let P be a finite projection in A. Then we can find a $\varphi\in\Gamma$ such that $\varphi(P)<\infty$. By Theorem 3.1 it follows that $K(P, G)\cap A^G$ is non-empty. Denote by S an arbitrary element of this set. Since φ is weakly lower semicontinuous on $K(P, G)$ and finite constant on $K_0(P, G)$, the values of φ are finite on $K(P, G)$, thus $\varphi(S)<\infty$. On the other hand, $P\in A^+$, hence $S\in A^+$.

Let $S=\int \lambda dE_\lambda$ be the spectral decomposition of S, where E_λ is right-continuous. Let $\mu>\lambda$ be arbitrary positive reals. It is clear that $E_\mu-E_\lambda$ belongs to A^G and that $\lambda(E_\mu-E_\lambda)\leqq S$. It follows that $\lambda\cdot\varphi(E_\mu-E_\lambda)\leqq\varphi(S)$ so the projection $E_\mu-E_\lambda$ can't be infinite, and therefore $E_\mu=E_\lambda$. This proves that $S=0$.

Now let $T\in M$ be arbitrary. For any $\varepsilon>0$ we can find finite projections $P_1, P_2, \dots, P_n$ and complex numbers $c_1, c_2, \dots, c_n$ such that $\left\|T-\sum_{i=1}^{n} c_iP_i\right\|<\varepsilon$. By Theorem 3.1 it follows that $K(T, G)\cap A^G$ is non-empty. Denote by S an arbitrary element of this set. By Proposition 2.2 there exists a positive linear mapping τ of L onto A^G such that for every $R\in L$, $\tau(R)\in K(R, G)\cap A^G$ and $\tau(T)=S$. Since $\|\tau\|\leqq 1$, we have $\left\|\tau(T)-\sum_{i=1}^{n} c_i\tau(P_i)\right\|<\varepsilon$. By the preceding part of the present proof we have $\tau(P_i)=0$ for all indices i, hence $\|\tau(T)\|<\varepsilon$. This proves that $\tau(T)=S=0$.

Theorem 3.3. *Let the pair (A, G) possess property Π. Let us suppose that Γ is non-empty and that $\natural\subset G$. In this case for every $T\in M$, $K(T, G)\cap A^G$ consists of a single element. In other words, $M\subset E$.*

Proof. Denote the largest projection of A^G by P. If $P=0$ then the statement of the theorem is trivial. If $P\neq 0$, then necessarily $P=I$. Indeed, if we set $R=I-P$ then we have $g(R)g(P)=0$ and $g(P)=P$ for every $g\in G$ and thus $g(R)\leqq R$ for every $g\in G$. It follows from property Π that $R\in A^G$, and, consequently, $I=P+$ $+R\in A^G$.

In virtue of Proposition 2.3 and Theorem 3.1 it is sufficient to show that for every $\varphi\in\Gamma$ and $T\in M_\varphi^+$ the set $K(T, G)\cap A^G$ contains exactly one element.

Denote by Y the maximal projection of A^G for which the restriction (A_Y, G_Y) of (A, G) is finite (Proposition 1.2.). Let $Z=I-Y$. Taking into account that $\natural\subset G$ the projections Y and Z belong to the center of A. It follows immediately from this that for every $S\in A$ the operator S is uniquely determined by its 'parts' S_Y and S_Z.

By Theorem 3.1 $K(T, G)\cap A^G$ is non-empty. Denote by R and S two elements of it. Using the facts that

$$(K(T, G)\cap A^G)_Y \subseteqq K(T_Y, G_Y)\cap A_Y^{G_Y} \quad \text{and} \tag{1}$$

$$(K(T, G)\cap A^G)_Z \subseteqq K(T_Z, G_Z)\cap A_Z^{G_Z} \tag{2}$$

the restricted operators R_Y and S_Y belong to the set (1) and the restricted operators R_Z and S_Z belong to the set (2). By the theorem of I. Kovács—J. Szűcs the set (1) consists of a single element, so $R_Y = S_Y$. By Theorem 3.2 it follows that $R_Z = S_Z = 0$. This means that $R = S$, and thus the set $K(T, G) \cap A^G$ has only one element.

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Note on the convergence of Fourier series in the spaces Λ_ω^p

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In this note we consider the basis problem for the trigonometric system in the spaces Λ_ω^p defined as follows: Let ω be a modulus of continuity and let $1 \leqq p < \infty$ be a real number. The class Λ_ω^p consists of all functions $f \in L^p$ for which the norm

$$\|f\|_{p,\omega} = \|f\|_p + \|f\|_{p,\omega}^*$$

is finite, where

$$\|f\|_p = \left\{ \int_{-\pi}^{\pi} |f(x)|^p \, dx \right\}^{\frac{1}{p}}, \quad \|f\|_{p,\omega}^* = \sup_{0<\delta\leqq\pi} \frac{\omega_p(\delta, f)}{\omega(\delta)}.$$

(We refer to [1] for $\omega_p(f)$ and ω.) With respect to this norm Λ_ω^p is a nonseparable Banach space.

A sequence $\{f_n\}$ of elements in the Banach space B, which is a basis for its closed span $E(\{f_n\}, B) = E(B)$ is called a basic sequence.

Theorem 1. *For any ω and $1 < p < \infty$ the trigonometric system is a basic sequence in the space Λ_ω^p.*

If T_n is a trigonometric polynomial of degree n, then the inequality

$$\|T_n'\|_p \leqq n\omega_p\left(\frac{\pi}{n}, T_n\right) \tag{1}$$

holds [5].

For any $f \in L^p$ and $n \geqq 0$ the inequality[1])

$$\|S_n f\|_p \leqq C_p \|f\|_p \tag{2}$$

is true [4], where $S_n f$ denotes the n-th partial sum of the Fourier series of f.

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[1]) C_p will always denote positive constants depending only on p, not necessarily the same at each occurrence.

Proof of Theorem 1. Since for any absolutely continuous function f with $f' \in L^p$ the inequality

$$\omega_p(\delta, f) \leqq \delta \|f'\|_p \quad (0 < \delta \leqq \pi)$$

holds, by (1) we obtain

$$\omega_p(\delta, T_n) \leqq n\delta\omega_p\left(\frac{\pi}{n}, T_n\right) \tag{3}$$

for any trigonometric polynomial T_n. Furthermore, from (2) and a theorem of Jackson type in the space L^p (see [6]) for any $f \in L^p$ and $n \geqq 1$ the inequality

$$\|f - S_n f\|_p \leqq C_p \omega_p\left(\frac{\pi}{n}, f\right) \tag{4}$$

follows.

Using the inequality (3) we have

$$\omega_p(\delta, S_n f) \leqq n\delta\omega_p\left(\frac{\pi}{n}, S_n f\right) \leqq n\delta\left[\omega_p\left(\frac{\pi}{n}, f\right) + \omega_p\left(\frac{\pi}{n}, f - S_n f\right)\right] \leqq$$

$$\leqq n\delta\left[\omega_p\left(\frac{\pi}{n}, f\right) + 2\|f - S_n f\|_p\right]$$

so, by inequality (4),

$$\omega_p(\delta, S_n f) \leqq C_p n\delta\omega_p\left(\frac{\pi}{n}, f\right) \tag{5}$$

holds. From (5) and by a familiar inequality (see e.g. [8] p. 111)

$$\omega(\delta) \leqq 2\delta\eta^{-1}\omega(\eta) \quad (0 < \eta \leqq \delta \leqq \pi)$$

it follows that

$$\omega_p(\delta, S_n f) \leqq C_p \omega(\delta) \|f\|^*_{p,\omega} \quad \left(0 < \delta \leqq \frac{\pi}{n}\right).$$

If $\frac{\pi}{n} \leqq \delta \leqq \pi$, then by (4) we have

$$\omega_p(\delta, S_n f) \leqq \omega_p(\delta, f) + \omega_p(\delta, f - S_n f) \leqq$$

$$\leqq \omega_p(\delta, f) + 2\|f - S_n f\|_p \leqq C_p \omega_p(\delta, f).$$

From the last two inequalities we obtain

$$\|S_n f\|^*_{p,\omega} \leqq C_p \|f\|^*_{p,\omega}$$

and by (2)

$$\|S_n f\|_{p,\omega} \leqq C_p \|f\|_{p,\omega}.$$

Now our statement follows from a known theorem (see e.g. [7], p. 58). The proof is complete.

In order to describe the subspaces $E(\Lambda^p_\omega)$ we consider the classes

$$\lambda^p_\omega = \left\{ f \in \Lambda^p_\omega : \lim_{\delta\to 0} \frac{\omega_p(\delta, f)}{\omega(\delta)} = 0 \right\}$$

which are closed subspaces in Λ^p_ω.

We show that *if the condition*

(6) $$\lim_{\delta\to 0} \frac{\delta}{\omega(\delta)} = 0$$

is fulfilled, then

(7) $$E(\Lambda^p_\omega) = \lambda^p_\omega.$$

In fact, if the function $f \in \lambda^p_\omega$, then

$$\omega_p(\delta, f) \leqq \varepsilon_p(\delta, f)\, \omega(\delta),$$

where $\varepsilon_p(\delta, f) \downarrow 0$ as $\delta \downarrow 0$. We can take for example

$$\varepsilon_p(\delta, f) = \omega_\infty \left(\delta, \frac{\omega_p(f)}{\omega} \right).$$

For $\frac{\pi}{n} < \delta \leqq \pi$, by (4), we have

$$\frac{\omega_p(\delta, f - S_n f)}{\omega(\delta)} \leqq \frac{\omega_p(1, f - S_n f)}{\omega\left(\frac{\pi}{n}\right)} \leqq C_p \varepsilon_p\left(\frac{\pi}{n}, f\right)$$

and for $0 < \delta \leqq \frac{\pi}{n}$ from (5) the inequality

$$\frac{\omega_p(\delta, f - S_n f)}{\omega(\delta)} \leqq \frac{\omega_p(\delta, f)}{\omega(\delta)} + C_p \frac{n\delta}{\omega(\delta)} \omega_p\left(\frac{\pi}{n}, f\right)$$

follows. As in the proof of Theorem 1, we obtain hence

$$\|f - S_n f\|_p \leqq C_{p,} \left[\omega_p\left(\frac{\pi}{n}, f\right) + (1 + \|f\|^*_{p,\omega}) \varepsilon_p\left(\frac{\pi}{n}, f\right) \right]$$

and thus $\lambda^p_\omega \subset E(\Lambda^p_\omega)$.

Since by condition (6) $\sin nx, \cos nx \in \lambda^p_\omega$ $(n \geqq 0)$ and λ^p_ω is a closed subspace of Λ^p_ω, thus $E(\Lambda^p_\omega) \subset \lambda^p_\omega$ and (7) is proved.

If the condition (6) is not fulfilled, then Λ^p_ω contains only the functions which are equivalent to constants. Consequently, by Theorem 1 we have

Theorem 2. *The trigonometric system forms a basis in the space* λ^p_ω, $1 < p < \infty$ *if and only if condition* (6) *holds.*

A system, which is a basic sequence for every permutation of its terms, is called an unconditional basic sequence.

For any $f \in L^2$ we have by the Parseval formula

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x+h)-f(x-h)|^2 \, dx = 4 \sum_{n=1}^{\infty} (a_n^2+b_n^2) \sin^2 nh \quad (h>0),$$

where a_n and b_n are Fourier coefficients of n. Hence it is easy to obtain that *the trigonometric system is an unconditional basic sequence in* Λ_ω^p *for every* ω.

On the other hand, *if* $p \neq 2$, *then the trigonometric system does not form an unconditional basic sequence in the space* Λ_ω^p, *where* $\omega(\delta)=\delta^\alpha$ $(0<\alpha\leqq 1)$. This statement follows from Konjushkov [2], Theorems 8 and 10.

In [3] we have given necessary and sufficient conditions that the Haar system should be a basic or unconditional basic sequence in the spaces Λ_ω^p, $1 \leqq p < \infty$.

Finally we remark that Theorem 1 is also true for other spaces. So we can consider the spaces defined by the modulus of smoothness of order k of functions; or, for example, we can take the spaces

$$W^r\Lambda_\omega^p = \{f\colon f^{(r-1)} \in AC, f^{(r)} \in \Lambda_\omega^p\}.$$

Since the proofs are the same as before, we omit them.

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Multiparameter strong laws of large numbers. II (Higher order moment restrictions)

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§ 1. Introduction

We use the notations introduced in [5] with the exceptions that at present

(i) it is more convenient to write $\zeta_{\mathbf{k}}$ into the form $\zeta_{\mathbf{k}}=a_{\mathbf{k}}\varphi_{\mathbf{k}}(x)$, where $\{a_{\mathbf{k}}\}=\{a_{\mathbf{k}}\colon \mathbf{k}\in Z_+^d\}$ is a set of numbers (coefficients) and $\{\varphi_{\mathbf{k}}(x)=\{\varphi_{\mathbf{k}}(x)\colon \mathbf{k}\in Z_+^d\}$ is a set of measurable functions defined on a positive measure space (X, A, μ);

(ii) by $\mathbf{m}=(m_1, \ldots, m_d)\to\infty$ we always mean that only $\max(m_1, \ldots, m_d)\to\infty$ (and $\min(m_1, \ldots, m_d)\not\to\infty$ may also occur).

We consider the d-multiple series

$$\sum_{\mathbf{k}\geqq 1} a_{\mathbf{k}}\varphi_{\mathbf{k}}(x) = \sum_{j=1}^{d}\sum_{k_j=1}^{\infty} a_{k_1,\ldots,k_d}\varphi_{k_1,\ldots,k_d}(x), \tag{1.1}$$

where the multiindex $\mathbf{k}=(k_1, \ldots, k_d)$ belongs to Z_+^d, the partially ordered set of the d-tuples of positive integers, d being a fixed positive integer. The set of d-tuples of non-negative integers is denoted by Z^d. For $\mathbf{b}\in Z^d$ and $\mathbf{m}\in Z_+^d$ write

$$S(\mathbf{b}, \mathbf{m};\, x) = \sum_{\mathbf{b}+1\leqq\mathbf{k}\leqq\mathbf{b}+\mathbf{m}} a_{\mathbf{k}}\varphi_{\mathbf{k}}(x) = \sum_{j=1}^{d}\sum_{k_j=b_j+1}^{b_j+m_j} a_{k_1,\ldots,k_d}(x)\;\varphi_{k_1,\ldots,k_d}(x)$$

and

$$M(\mathbf{b}, \mathbf{m};\, x) = \max_{1\leqq\mathbf{k}\leqq\mathbf{m}} |S(\mathbf{b}, \mathbf{k};\, x)| = \max_{1\leqq j\leqq d}\;\max_{1\leqq k_j\leqq m_j} |S(b_1, \ldots, b_d;\, k_1, \ldots, k_d;\, x)|.$$

In case $\mathbf{b}=\mathbf{0}$ write $S(\mathbf{0}, \mathbf{m}; x)=S(\mathbf{m}; x)$ (rectangular partial sums of (1.1)) and $M(\mathbf{0}, \mathbf{m}; x)=M(\mathbf{m}; x)$.

Throughout the paper we assume that there exist a number $r>2$ and a constant C such that the inequality

$$\int |S(\mathbf{b}, \mathbf{m};\, x)|^r\, d\mu(x) \leqq C\Big(\sum_{\mathbf{b}+1\leqq\mathbf{k}\leqq\mathbf{b}+\mathbf{m}} a_{\mathbf{k}}^2\Big)^{r/2} \tag{1.2}$$

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holds for all $\mathbf{b}\in Z^d$ and $\mathbf{m}\in Z_+^d$, and either for all sets $\{a_\mathbf{k}\}$ (in §§ 1—2) or for only the single set $\{a_\mathbf{k}\equiv 1\}$ of coefficients (in §§ 3—4).

Here and in the sequel the integrals, unless stated otherwise, are taken over X; $C, C_1, C_2, \ldots$ denote positive constants, not necessarily the same at different occurrences.

Example 1. Let r be an integer, $r\geqq 2$. The set $\{\varphi_\mathbf{k}(x)\}$ is said to be *multiplicative of order r* if for all systems of pairwise distinct $\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_r$ from Z_+^d we have

$$\int\Big(\prod_{p=1}^{r}\varphi_{\mathbf{k}_p}(x)\Big)\,d\mu(x)=0. \tag{1.3}$$

This definition for $d=1$ is due to Alexits [1, p. 186].

The arguments of Gapoškin [2], Komlós and Révész [3] in the case $d=1$ obviously apply to the case $d\geqq 2$ and lead to the following result: *Let r be an even integer, $r\geqq 4$. If $\{\varphi_\mathbf{k}(x)\}$ is multiplicative of order r and*

$$\int\varphi_\mathbf{k}^r(x)\,d\mu(x)\leqq C \tag{1.4}$$

for all $\mathbf{k}\in Z_+^d$, then we have (1.2) *for all $\{a_\mathbf{k}\}$.*

Example 2. The vanishing of the integrals in (1.3) is of no relevance, only their "smallness" in a certain sense is needed.

In case $d=1$, according to Gapoškin [2], a sequence $\{\varphi_i(x)\}_{i=1}^\infty$ is said to be *weakly multiplicative of order r*, where r is an even positive integer, if there exists a non-negative function $h(l)$ such that for every $1\leqq i_1<i_2<\ldots<i_r$ we have

$$\left|\int\Big(\prod_{p=1}^{r}\varphi_{i_p}(x)\Big)\,d\mu(x)\right|\leqq h(l)$$

with $l=\min(i_2-i_1, i_4-i_3, \ldots, i_r-i_{r-1})$ and

$$\sum_{l=1}^{\infty} l^{(r-2)/2}h(l)<\infty.$$

Now it is proved in [2] (and announced in [3]) that *if $r\geqq 4$, $\{\varphi_i(x)\}_{i=1}^\infty$ is a weakly multiplicative sequence of order r, which satisfies* (1.4), *then we have* (1.2) *for all $\{a_i\}_{i=1}^\infty$.*

In case $d\geqq 2$, let (X_j, A_j, μ_j) be a positive measure space, $\{\varphi_i^{(j)}(x_j)\}_{i=1}^\infty$ a sequence of measurable functions on X_j for each $j=1, 2, \ldots, d$. Let

$$(X, A, \mu)=\mathop{\times}_{j=1}^{d}(X_j, A_j, \mu_j)$$

be the product measure space and let

$$\varphi_{\mathbf{k}}(x)=\prod_{j=1}^{d}\varphi_{k_j}^{(j)}(x_j),\quad \text{where}\quad \mathbf{k}=(k_1,\ldots,k_d)\quad\text{and}\quad x=(x_1,\ldots,x_d).$$

The following statement holds: *If for some* $r\geqq 2$ *each sequence* $\{\varphi_i^{(j)}(x_j)\}_{i=1}^{\infty}$ $(j=1,2,\ldots,d)$ *satisfies the inequality*

$$\int_{X_j}\left|\sum_{i=b+1}^{b+m}a_i\varphi_i^{(j)}(x_j)\right|^r d\mu_j(x_j)\leqq C_j\left(\sum_{i=b+1}^{b+m}a_i^2\right)^{r/2} \tag{1.5}$$

for all $\{a_i\}_{i=1}^{\infty}$, $b\geqq 0$ *and* $m\geqq 1$, *then* $\{\varphi_{\mathbf{k}}(x)\colon \mathbf{k}\in Z_+^d\}$ *satisfies inequality* (1.2) *for all* $\{a_{\mathbf{k}}\colon \mathbf{k}\in Z_+^d\}$, $\mathbf{b}\in Z^d$ *and* $\mathbf{m}\in Z_+^d$ *with* $C=\prod_{j=1}^{d}C_j$.

For simplicity, assume that $d=2$. Then by (1.5), Fubini's theorem, and Minkowski's inequality we get that

$$\int_{X_1}\int_{X_2}\left|\sum_{i=b+1}^{b+m}\sum_{k=c+1}^{c+n}a_{ik}\varphi_i^{(1)}(x_1)\varphi_k^{(2)}(x_2)\right|^r d\mu_1(x_1)\,d\mu_2(x_2)=$$

$$=\int_{X_2}\left\{\int_{X_1}\left|\sum_{i=b+1}^{b+m}\left(\sum_{k=c+1}^{c+n}a_{ik}\varphi_k^{(2)}(x_2)\right)\varphi_i^{(1)}(x_1)\right|^r d\mu_1(x_1)\right\}d\mu_2(x_2)\leqq$$

$$\leqq C_1\int_{X_2}\left\{\sum_{i=b+1}^{b+m}\left(\sum_{k=c+1}^{c+n}a_{ik}\varphi_k^{(2)}(x_2)\right)^2\right\}^{r/2}d\mu_2(x_2)\leqq$$

$$\leqq C_1\left\{\sum_{i=b+1}^{b+m}\left(\int_{X_2}\left|\sum_{k=c+1}^{c+n}a_{ik}\varphi_k^{(2)}(x_2)\right|^r d\mu_2(x_2)\right)^{2/r}\right\}^{r/2}\leqq$$

$$\leqq C_1C_2\left(\sum_{i=b+1}^{b+m}\sum_{k=c+1}^{c+n}a_{ik}^2\right)^{r/2}.$$

This is the wanted inequality (1.2).

The results below will be obtained by adaptation of more or less standard arguments well-known in probability theory concerning limit theorems, and by making use of a recent maximal inequality of the author [4, Theorem 7]. It is worth stating the special case of this inequality for $\alpha=r/2$, $\gamma=r$ and $f(\mathbf{b},\mathbf{m})=\sum_{\mathbf{b}+\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{b}+\mathbf{m}}a_{\mathbf{k}}^2$ in the form of a separate lemma.

Lemma 1. *Let* $r>2$ *and* $\{a_{\mathbf{k}}\}$ *be given. If inequality* (1.2) *holds for all* $\mathbf{b}\in Z^d$ *and* $\mathbf{m}\in Z_+^d$, *then*

$$\int M^r(\mathbf{b},\mathbf{m};\,x)\,d\mu(x)\leqq C_1\Big(\sum_{\mathbf{b}+\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{b}+\mathbf{m}}a_{\mathbf{k}}^2\Big)^{r/2} \tag{1.6}$$

also holds for all $\mathbf{b}\in Z^d$ *and* $\mathbf{m}\in Z_+^d$.

2§. A.e. convergence of series (1.1)

Theorem 1. *Let $r>2$ and let $\{a_{\mathbf{k}}\}$ be such that*

$$\sum_{\mathbf{k}\geqq 1} a_{\mathbf{k}}^2 < \infty. \tag{2.1}$$

If inequality (1.2) *holds for all* $\mathbf{b}\in Z^d$ *and* $\mathbf{m}\in Z_+^d$, *then*

$$S(\mathbf{b},\mathbf{m};\, x) \to 0 \quad \text{a.e. as} \quad \mathbf{b}\to\infty \quad \text{and} \quad \mathbf{m}\in Z_+^d; \tag{2.2}$$

furthermore,

$$\int \big(\sup_{\mathbf{b}\geqq 0}\sup_{\mathbf{m}\geqq 1} |S(\mathbf{b},\mathbf{m};\, x)|\big)^r \, d\mu(x) \leqq C_2 \big(\sum_{\mathbf{k}\geqq 1} a_{\mathbf{k}}^2\big)^{r/2}. \tag{2.3}$$

In particular, from (2.2) it follows that the d-multiple series (1.1) converges a.e. in the sense that its rectangular partial sums $S(\mathbf{m};x)$ converge a.e. as $\min(m_1, \dots, m_d)\to\infty$. (See more detailed in [6].)

Lemma 2 ([6, Lemma 1]). *For all* $\mathbf{b}\in Z^d$ *and* $\mathbf{m}\in Z_+^d$

$$\max_{1\leqq \mathbf{p}\leqq \mathbf{q}\leqq \mathbf{m}} \Big| \sum_{\mathbf{b}+\mathbf{p}\leqq \mathbf{k}\leqq \mathbf{b}+\mathbf{q}} a_{\mathbf{k}}\varphi_{\mathbf{k}}(x)\Big| \leqq 2^d M(\mathbf{b},\mathbf{m};x).$$

Proof of Theorem 1. Condition (2.1) implies the existence of a sequence $\{\mathbf{m}_l=(m_{1l}, \dots, m_{dl})\}_{l=1}^{\infty}$ in Z_+^d for which

(i) $1 \leqq m_{j1} < m_{j2} < \dots$ for each $j=1, 2, \dots, d$;

(ii) $\Big\{\sum_{\mathbf{k}\geqq 1} - \sum_{1\leqq \mathbf{k}\leqq \mathbf{m}_l}\Big\} a_{\mathbf{k}}^2 \leqq (l+1)^{-2(r+2)/r} \sum_{\mathbf{k}\geqq 1} a_{\mathbf{k}}^2 \quad (l=1, 2, \dots)$.

It follows from (i) that $\min(m_{1l}, \dots, m_{dl})\to\infty$ as $l\to\infty$, and from (ii) that

$$\sum_{l=0}^{\infty} (l+1)^r \Big\{\sum_{\mathbf{k}\geqq 1} - \sum_{1\leqq \mathbf{k}\leqq \mathbf{m}_l}\Big\} a_{\mathbf{k}}^2 \leqq 2\big(\sum_{\mathbf{k}\geqq 1} a_{\mathbf{k}}^2\big)^{r/2} \quad (\mathbf{m}_0=\mathbf{0}). \tag{2.4}$$

Motivating by the representation

$$S(\mathbf{m}_{l+1};\, x) - S(\mathbf{m}_l;\, x) = \sum_{\boldsymbol{\varepsilon}} S(\boldsymbol{\varepsilon}\mathbf{m}_l, \boldsymbol{\varepsilon}(\mathbf{m}_{l+1}-\mathbf{m}_l)+(\mathbf{1}-\boldsymbol{\varepsilon})\mathbf{m}_l;\, x),$$

where the summation $\sum_{\boldsymbol{\varepsilon}}$ is extended over all 2^d-1 choices of $\boldsymbol{\varepsilon}=(\varepsilon_1, \dots, \varepsilon_d)$ with coordinates $\varepsilon_j=0$ or 1, the case $\varepsilon_1=\dots=\varepsilon_d=0$ excluded, we introduce the following maxima:

$$M_{t,l}(x) = M(\boldsymbol{\varepsilon}\mathbf{m}_l, \boldsymbol{\varepsilon}(\mathbf{m}_{l+1}-\mathbf{m}_l)+(\mathbf{1}-\boldsymbol{\varepsilon})\mathbf{m}_l;\, x),$$

where $t=\varepsilon_1+2\varepsilon_2+\dots+2^{d-1}\varepsilon_d$. It is clear that $1\leqq t\leqq 2^d-1$.

We are going to show that

$$\sum_{l=0}^{\infty} (l+1)^r \Big(\sum_{t=1}^{2^d-1} M_{t,l}^r(x)\Big) < \infty \quad \text{a.e.} \tag{2.5}$$

Inequality (1.2), via Lemma 1, yields

$$\sum_{l=0}^{\infty}(l+1)^r\Big(\sum_{t=1}^{2^d-1}\int M_{t,l}^r(x)\,d\mu(x)\Big) \leqq \tag{2.6}$$

$$\leqq C_1\sum_{l=0}^{\infty}(l+1)^r\{(\sum_{1\leqq \mathbf{k}\leqq \mathbf{m}_{l+1}}-\sum_{1\leqq \mathbf{k}\leqq \mathbf{m}_l})a_{\mathbf{k}}^2\}^{r/2}\leqq 2C_1(\sum_{\mathbf{k}\geqq 1}a_{\mathbf{k}}^2)^{r/2},$$

the last inequality is owing to (2.4). Hence B. Levi's theorem implies (2.5).

Let us now estimate $S(\mathbf{b},\mathbf{m};x)$ with arbitrary $\mathbf{b}\in Z^d$ and $\mathbf{m}\in Z_+^d$. Recall that $\mathbf{b}\nleqq\mathbf{m}_l$ iff $b_j>m_{jl}$ for at least one j $(1\leqq j\leqq d)$. In the special case when there exists a non-negative integer l such that $\mathbf{b}\nleqq\mathbf{m}_l$ and $\mathbf{b}+\mathbf{m}\leqq\mathbf{m}_{l+1}$, by Lemma 2 we obviously have

$$|S(\mathbf{b},\mathbf{m};\,x)|\leqq 2^d\sum_{t=1}^{2^d-1}M_{t,l}(x).$$

In the general case let us determine non-negative integers u and v such that

$$\mathbf{b}\nleqq\mathbf{m}_u \quad\text{and}\quad \mathbf{b}\leqq\mathbf{m}_{u+1}\ ;\quad \mathbf{b}+\mathbf{m}\nleqq\mathbf{m}_v \quad\text{and}\quad \mathbf{b}+\mathbf{m}\leqq\mathbf{m}_{v+1}.$$

It is clear that such u and v (uniquely) exist, and $0\leqq u\leqq v$. Again by virtue of Lemma 2 we have

$$|S(\mathbf{b},\mathbf{m};\,x)|\leqq 2^d\sum_{l=u}^{v}\Big(\sum_{t=1}^{2^d-1}M_{t,l}(x)\Big),$$

whence, using Hölder's inequality,

$$|S(\mathbf{b},\mathbf{m};\,x)|\leqq 2^d\Big\{\sum_{l=u}^{v}(l+1)^r\Big(\sum_{t=1}^{2^d-1}M_{t,l}^r(x)\Big)\Big\}^{1/r}\times \tag{2.7}$$

$$\times\Big\{\sum_{l=u}^{v}\frac{2^d-1}{(l+1)^{r'}}\Big\}^{1/r'}\quad\text{with}\quad r'=\frac{r}{r-1}>1.$$

By (2.5) we can conclude that $|S(\mathbf{b},\mathbf{m};x)|$ is a.e. as small as required if $\max(b_1,\dots,b_d)$, and consequently u is large enough. This proves (2.2).

From (2.7) we obtain that

$$(\sup_{\mathbf{b}\geqq 0}\sup_{\mathbf{m}\geqq 1}|S(\mathbf{b},\mathbf{m};\,x)|)^r\leqq C_3\sum_{l=0}^{\infty}(l+1)^r\Big(\sum_{t=1}^{2^d-1}M_{t,l}^r(x)\Big).$$

Integrating both sides over X, the wanted inequality (2.3) comes from (2.6). This completes the proof of Theorem 1.

§ 3. Multiparameter SLLN

In the sequel we assume that all $a_{\mathbf{k}}=1$ in (1.1), i.e. from now on

$$S(\mathbf{b}, \mathbf{m};\ x) = \sum_{\mathbf{b}+\mathbf{1} \leqq \mathbf{k} \leqq \mathbf{b}+\mathbf{m}} \varphi_{\mathbf{k}}(x)$$

and

$$M(\mathbf{b}, \mathbf{m};\ x) = \max_{\mathbf{1} \leqq \mathbf{k} \leqq \mathbf{m}} \Big| \sum_{\mathbf{b}+\mathbf{1} \leqq \mathbf{l} \leqq \mathbf{b}+\mathbf{k}} \varphi_{\mathbf{l}}(x) \Big| \quad (\mathbf{b} \in Z^d \text{ and } \mathbf{m} \in Z_+^d),$$

although our results remain valid in the more general setting when $\sum_{\mathbf{k} \geqq \mathbf{1}} a_{\mathbf{k}}^2 = \infty$ and $\{a_{\mathbf{k}}\}$ behaves sufficiently "regularly".

Our permanent assumption is now that inequality (1.2) holds true only in this special $a_{\mathbf{k}} \equiv 1$ case, i.e. there exists a number $r > 2$ such that

$$\int |S(\mathbf{b}, \mathbf{m};\ x)|^r \, d\mu(x) \leqq C |\mathbf{m}|^{r/2} \tag{3.1}$$

holds for all $\mathbf{b} \in Z^d$ and $\mathbf{m} \in Z_+^d$, where $|\mathbf{m}|$ stands for $\prod_{j=1}^{d} m_j$. Hence Lemma 1 implies

$$\int M^r(\mathbf{b}, \mathbf{m};\ x) \, d\mu(x) \leqq C_1 |\mathbf{m}|^{r/2}. \tag{3.2}$$

Theorem 2. *If inequality* (3.1) *holds for all* $\mathbf{b} \in Z^d$ *and* $\mathbf{m} \in Z_+^d$ *with an* $r > 2$, *then for any* $\delta > 0$ *we have*

$$\lim_{\mathbf{m} \to \infty} \frac{S(\mathbf{m};\ x)}{|\mathbf{m}|^{1/2} \left(\sum_{j=1}^{d} \log 2m_j \right)^{1/r} (\log\log 4|\mathbf{m}|)^{(d+\delta)/r}} = 0 \quad a.e. \tag{3.3}$$

and

$$\lim_{\mathbf{m} \to \infty} \frac{S(\mathbf{m};\ x)}{|\mathbf{m}|^{1/2} (\log 2|\mathbf{m}|)^{d/r} (\log\log 4|\mathbf{m}|)^{(1+\delta)/r}} = 0 \quad a.e. \tag{3.4}$$

Here and in the sequel all logarithms are of base 2. Further, $S(\mathbf{m};\ x) = S(\mathbf{0}, \mathbf{m};\ x)$.

This result for $d=1$ (in a slightly weaker form) was proved by Serfling [7, Theorem 3.1].

Remark 1. For $d=1$ relations (3.3) and (3.4) coincide. For $d \geqq 2$, if $\mathbf{m} \to \infty$ is such a way that $m_1 = m_2 = \ldots = m_d$ then (3.3) is stronger than (3.4), while if $\mathbf{m} \to \infty$ in such a way that e.g. $m_2 = \ldots = m_d = 1$ then the situation is converse: (3.4) is stronger than (3.3).

Both (3.3) and (3.4) improve as r increases. By letting $r\to\infty$ we find, for any $\delta>0$,

$$\lim_{\mathbf{m}\to\infty}\frac{S(\mathbf{m};\, x)}{|\mathbf{m}|^{1/2}(\log 2|\mathbf{m}|)^{\delta}}=0 \quad \text{a.e.}$$

This result is not far from the "$\leqq$" part of the law of the iterated logarithm.

Lemma 3. *For any* $\delta>0$, *we have*

$$\sum_{\mathbf{k}\geqq\mathbf{0}}|\mathbf{1}+\mathbf{k}|^{-1}\left\{\log\left(2+\sum_{j=1}^{d}k_j\right)\right\}^{-d-\delta}<\infty$$

and

$$\sum_{\mathbf{k}\geqq\mathbf{0}}\left(1+\sum_{j=1}^{d}k_j\right)^{-d}\left\{\log\left(2+\sum_{j=1}^{d}k_j\right)\right\}^{-1-\delta}<\infty.$$

Proof of Lemma 3. For simplicity, we only prove in the case $d=2$. Then the first series can be rewritten and estimated as follows

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{ik(\log(i+k))^{2+\delta}}\leqq\sum_{i=1}^{\infty}\frac{1}{i}\left\{\sum_{k=1}^{i}\frac{1}{k(\log(1+i))^{2+\delta}}+\right.$$

$$\left.+\sum_{k=i+1}^{\infty}\frac{1}{k(\log(1+k))^{2+\delta}}\right\}\leqq C_4\sum_{i=1}^{\infty}\frac{1}{i(\log(1+i))^{1+\delta}}<\infty.$$

The convergence of the second series can be verified similarly.

Proof of Theorem 2. Lemma 1 constitutes the basis of the proof. Applying Chebyshev's inequality to (3.2) we obtain that

$$\mu\{M(\mathbf{b},\mathbf{m};\, x)\geqq\lambda\}\leqq\frac{C_1|\mathbf{m}|^{r/2}}{\lambda^r}\quad(\mathbf{b}\in Z^d,\mathbf{m}\in Z_+^d \text{ and } \lambda>0). \tag{3.5}$$

Substituting here

$$\lambda(\mathbf{m})=|\mathbf{m}|^{1/2}\left(\prod_{j=1}^{d}\log 2m_j\right)^{1/r}(\log\log 4|\mathbf{m}|)^{(d+\delta)/r}\quad\text{or}$$

$$|\mathbf{m}|^{1/2}(\log 2|\mathbf{m}|)^{d/r}(\log\log 4|\mathbf{m}|)^{(1+\delta)/r}$$

for λ, we get that

$$\mu\{M(\mathbf{m};\, x)\geqq\lambda(\mathbf{m})\}\leqq C_1\left(\prod_{j=1}^{d}\log 2m_j\right)^{-1}(\log\log 4|\mathbf{m}|)^{-d-\delta}\quad\text{or}$$

$$C_1(\log 2|\mathbf{m}|)^{-d}(\log\log 4|\mathbf{m}|)^{-1-\delta}.$$

Let $\mathbf{m}=2^{\mathbf{k}}$ where $\mathbf{k}$ runs over Z^d. Then, by Lemma 3,

$$\sum_{\mathbf{k}\geqq\mathbf{0}} \mu\{M(2^{\mathbf{k}};\, x) \geqq \lambda(2^{\mathbf{k}})\} < \infty.$$

Hence, via the Borel—Cantelli lemma, we have

$$M(2^{\mathbf{k}};\, x) < \lambda(2^{\mathbf{k}}) \quad \text{a.e.},$$

with the exception of a finite number (depending on x) of $\mathbf{k}$.

It is obvious that if $2^{\mathbf{k}} \leqq \mathbf{m} \leqq 2^{\mathbf{k}+1}$ with some $\mathbf{k}\geqq\mathbf{0}$, then we have

$$\lambda(\mathbf{m}) \geqq \lambda(2^{\mathbf{k}}) \quad \text{and} \quad |S(\mathbf{m};\, x)| \leqq M(2^{\mathbf{k}+1};\, x).$$

Consequently,

$$\frac{|S(\mathbf{m};\, x)|}{\lambda(\mathbf{m})} \leqq \frac{M(2^{\mathbf{k}+1};\, x)}{\lambda(2^{\mathbf{k}})} < \frac{\lambda(2^{\mathbf{k}+1})}{\lambda(2^{\mathbf{k}})} \quad \text{a.e.}, \tag{3.6}$$

provided $\max(k_1, \ldots, k_d)$ is large enough. Since the right-most member in (3.6) is bounded as $\mathbf{k}\to\infty$, it follows that

$$S(\mathbf{m};\, x) = O\{\lambda(\mathbf{m})\} \quad \text{a.e.} \tag{3.7}$$

Taking into consideration that δ may be chosen arbitrarily small (but positive), we can change "O" to "o" in (3.7), as a result of which we get the wanted (3.3) and (3.4).

§ 4. Rates of convergence

Turning to the rate of convergence in (3.3) and (3.4), we can state

Theorem 3. *If inequality* (3.1) *holds for all* $\mathbf{b}\in Z^d$ *and* $\mathbf{m}\in Z_+^d$ *with an* $r>2$, *then for any choices of* α *and* β *satisfying*

$$0 \leqq \beta < \alpha r - 1 \tag{4.1}$$

and for any $\varepsilon>0$ *we have*

$$\sum_{\mathbf{m}\geqq\mathbf{1}} \frac{1}{|\mathbf{m}|\,(\log 2|\mathbf{m}|)^{d-\beta}}\, \mu\left\{ \sup_{\substack{k_j \geqq m_j \\ \text{for at least one } j, \\ 1\leqq j\leqq d}} \frac{|S(\mathbf{k};\, x)|}{|\mathbf{k}|^{1/2}\left(\prod_{j=1}^{d} \log 2k_j\right)^{\alpha}} \geqq \varepsilon \right\} < \infty \tag{4.2}$$

and

$$\sum_{\mathbf{m}\geqq\mathbf{1}} \frac{1}{|\mathbf{m}|\left(\prod_{j=1}^{d} \log 2m_j\right)^{1-\beta}}\, \mu\left\{ \sup_{\mathbf{k}\geqq\mathbf{m}} \frac{|S(\mathbf{k};\, x)|}{|\mathbf{k}|^{1/2}(\log 2|\mathbf{k}|)^{\alpha d}} \geqq \varepsilon \right\} < \infty. \tag{4.3}$$

This result for $d=1$ was also established by SERFLING [7, Theorem 5.3].

Remark 2. Observe that the more restrictive " $\sup\limits_{\substack{k_j \geq m_j \\ \text{for at least one } j, \\ 1 \leq j \leq d}}$ " in (4.2) is weakened into " $\sup\limits_{\substack{k_j \geq m_j \\ \text{for every } j, \\ 1 \leq j \leq d}}$ " in (4.3).

If inequality (3.1) is satisfied for all $\mathbf{b} \in Z^d$ and $\mathbf{m} \in Z^d_+$ with arbitrarily large exponents r, then (4.2) and (4.3) hold for each choice of $\alpha > 0$ and $\beta > 0$.

The proof of Theorem 3 is based on (3.5) and on the following auxiliary result, which for the sake of brevity is stated only for $d=2$.

Lemma 4. *If* (4.1) *holds, then*

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{ik(\log 2ik)^{2-\beta}(\log 2i)^{\alpha r-1}} < \infty.$$

Proof of Lemma 4. An easy computation shows that the series in question can be estimated from above as follows

$$\sum_{k=1}^{\infty} \frac{1}{k} \sum_{l=0}^{\infty} \sum_{i=2^l}^{2^{l+1}-1} \frac{1}{i(\log 2ik)^{2-\beta}(\log 2i)^{\alpha r-1}} \leq \sum_{k=1}^{\infty} \frac{1}{k} \sum_{l=0}^{\infty} \frac{1}{(l+\log 2k)^{2-\beta}(l+1)^{\alpha r-1}}.$$

Now let us deal with the inner series:

$$\left\{\sum_{l=0}^{[\log 2k]} + \sum_{l=[\log 2k]+1}^{\infty}\right\} \frac{1}{(l+\log 2k)^{2-\beta}(l+1)^{\alpha r-1}} \leq \frac{1}{(\log 2k)^{2-\beta}} \sum_{l=0}^{[\log 2k]} \frac{1}{(l+1)^{\alpha r-1}} +$$

$$+ \sum_{l=[\log 2k]+1}^{\infty} \frac{1}{(l+1)^{\alpha r-\beta+1}} \leq \frac{C_5}{(\log 2k)^{\alpha r-\beta}},$$

where $[.]$ denotes integral part. Taking into account that by (4.1) we have $\alpha r - \beta > 1$, the proof is ready.

Proof of Theorem 3. We prove for $d=2$ only. The general case $d>2$ can be handled in the same way, merely the technical details become more complicated.

In virtue of Lemma 4, for (4.2) it is enough to demonstrate that

$$\mu(m,n) = \mu\left\{\sup_{i \geq m \text{ or } k \geq n} \frac{|S(i,k;x)|}{(ik)^{1/2}(\log 2i \log 2k)^{\alpha}} \geq \varepsilon\right\} \leq \tag{4.4}$$

$$\leq C_6\left(\frac{1}{(\log 2i)^{\alpha r-1}} + \frac{1}{(\log 2k)^{\alpha r-1}}\right).$$

To this end, let the non-negative integers p and q be defined by

$$2^p \leqq m < 2^{p+1} \quad \text{and} \quad 2^q \leqq n < 2^{q+1}.$$

It is obvious that

$$\mu(m, n) \leqq \left\{ \sum_{u=p}^{\infty} \sum_{v=q}^{\infty} + \sum_{u=p}^{\infty} \sum_{v=0}^{q-1} + \sum_{u=0}^{p-1} \sum_{v=q}^{\infty} \right\} \nu(u, v) \tag{4.5}$$

with

$$\nu(u, v) = \mu \left\{ \max_{2^u \leqq i < 2^{u+1}} \max_{2^v \leqq k < 2^{v+1}} \frac{|S(i, k; x)|}{(ik)^{1/2} (\log 2i \log 2k)^\alpha} \geqq \varepsilon \right\}.$$

By (3.5) it is not hard to check that

$$\nu(u, v) \leqq \mu\{M(2^{u+1}, 2^{v+1}; x) \geqq \varepsilon(u+1)^\alpha (v+1)^\alpha 2^{(u+v)/2}\} \leqq$$

$$\leqq C_1 2^r \varepsilon^{-r} \big((u+1)(v+1)\big)^{-\alpha r}.$$

Since

$$\sum_{u=p}^{\infty} \sum_{v=q}^{\infty} \nu(u, v) \leqq C_7 \varepsilon^{-r} \big((p+1)(q+1)\big)^{-\alpha r+1} \leqq C_8 \varepsilon^{-r} (\log 2m \log 2n)^{-\alpha r+1},$$

$$\sum_{u=p}^{\infty} \sum_{v=0}^{q-1} \nu(u, v) \leqq C_7 \varepsilon^{-r} (p+1)^{-\alpha r+1} \leqq C_8 \varepsilon^{-r} (\log 2m)^{-\alpha r+1},$$

and similarly,

$$\sum_{u=0}^{p-1} \sum_{v=q}^{\infty} \nu(u, v) \leqq C_8 \varepsilon^{-r} (\log 2m)^{-\alpha r+1},$$

from (4.5) we obtain the desired (4.4). This proves (4.2).

The proof of (4.3) can be executed in a similar manner as that of (4.2). The proof of Theorem 3 is complete.

It is clear that under the conditions of Theorem 2 we have $S(\mathbf{m}; x)/|\mathbf{m}| \to 0$ a.e. as $\mathbf{m} \to \infty$. For this SLLN we can prove essentially better convergence rates, however, now only with the weaker "$\sup\limits_{\substack{k_j \geqq m_j \\ \text{for every } j, \\ 1 \leqq j \leqq d}}$" instead of "$\sup\limits_{\substack{k_j \geqq m_j \\ \text{for at least one } j, \\ 1 \leqq j \leqq d}}$".

Theorem 4. *If inequality* (3.1) *holds for all* $\mathbf{b} \in Z^d$ *and* $\mathbf{m} \in Z_+^d$ *with an* $r > 2$, *then for any* $\delta > 0$ *and* $\varepsilon > 0$ *we have*

$$\sum_{\mathbf{m} \geqq 1} \frac{|\mathbf{m}|^{(r-2)/2}}{\left(\prod_{j=1}^{d} \log 2m_j \right) (\log \log 4|\mathbf{m}|)^{d+\delta}} \mu \left\{ \sup_{\mathbf{k} \geqq \mathbf{m}} \frac{|S(\mathbf{k}; x)|}{|\mathbf{k}|} \geqq \varepsilon \right\} < \infty \tag{4.6}$$

and

$$\sum_{\mathbf{m}\geqq 1} \frac{|\mathbf{m}|^{(r-2)/2}}{(\log 2|\mathbf{m}|)^d (\log\log 4|\mathbf{m}|)^{1+\delta}} \mu\left\{\sup_{\mathbf{k}\geqq \mathbf{m}} \frac{|S(\mathbf{k};\, x)|}{|\mathbf{k}|} \geqq \varepsilon\right\} < \infty. \tag{4.7}$$

Remark 3. Both convergence rates improve as r increases. Letting $r\to\infty$ results, for any $\alpha>0$ and $\varepsilon>0$,

$$\sum_{\mathbf{m}\geqq 1} |\mathbf{m}|^{\alpha} \mu\left\{\sup_{\mathbf{k}\geqq \mathbf{m}} |S(\mathbf{k};\, x)|/|\mathbf{k}| \geqq \varepsilon\right\} < \infty.$$

The proof of Theorem 4 runs along the same lines as that of Theorem 3. First we infer that

$$\mu\left\{\sup_{\mathbf{k}\geqq \mathbf{m}} |S(\mathbf{k};\, x)|/|\mathbf{k}| \geqq \varepsilon\right\} \leqq C_9 \varepsilon^{-r} |\mathbf{m}|^{-r/2}$$

(for $d=1$ see also in [7, Theorem 5.1]), then (4.6) and (4.7) follow from the fact that, for any $\delta>0$,

$$\sum_{\mathbf{m}\geqq 1} |\mathbf{m}|^{-1} \left\{\prod_{j=1}^{d} \log 2m_j\right\}^{-1} (\log\log 4|\mathbf{m}|)^{-d-\delta} < \infty$$

and

$$\sum_{\mathbf{m}\geqq 1} |\mathbf{m}|^{-1} (\log 2|\mathbf{m}|)^{-d} (\log\log 4|\mathbf{m}|)^{-1-\delta} < \infty.$$

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On the indicatrix of orbits of 1-parameter subgroups in a homogeneous space

P. T. NAGY

§ 1. Preliminaries

In the following $H, K \subset G$ denote Lie groups, $\mathfrak{g}, \mathfrak{h}, \mathfrak{k}$ the corresponding Lie algebras, which can be identified with the tangent spaces T_eG, T_eH, T_eK at the unity $e \in G, H, K$, respectively.

Let be $L(M)$ the bundle of linear frames on the manifold M and $p\colon L(M) \to M$ the natural projection in this bundle.

The isotropy group H of the homogeneous space $M = G/H$ leaves the origin $o \in M$ of the space $M = G/H$ fixed. Hence the differential z_{*o} of the map $z\colon M \to M$ $(z \in H)$ is a linear transformation on the tangent space T_oM. This representation $z \mapsto z_{*o}$ $(z \in H)$ of the isotropy group on the tangent space T_oM is called the linear isotropy group. The action $\alpha\colon G \times M \to M$ of the group G on M induces an action $\tilde{\alpha}\colon G \times L(M) \to L(M)$ of the group G on the linear frame bundle $L(M)$. It is clear that the action $\tilde{\alpha}$ is effective if and only if the linear representation of the isotropy group is faithful, i.e. the map $z \mapsto z_{*o}$ $(z \in H)$ is one-to-one.

It is well-known that the faithfulness of the linear representation of the isotropy group is a necessary condition for the existence of invariant connections in a homogeneous space. Therefore in the following this condition will be supposed.

Let be given a frame $u_0 \in L_oM$ at the point $o \in M$. The action $\tilde{\alpha}$ of G on $L(M)$ yields an embedding of G in $L(M)$ so that to the unity $e \in G$ corresponds the frame u_0. In the following we use this embedding and we will regard the principal bundle $\{G, \pi, G/H\}$ as a subbundle of $\{L(M), p, M\}$.

We recall Wang's theorem on invariant connections, cf. [2], 186—190.

Let be $M = G/H$ a homogeneous space. There exists a one-to-one correspondence between the set of G-invariant connections in $L(M)$ and the set of linear maps $\Lambda\colon \mathfrak{g} \to \mathfrak{gl}(n)$ satisfying the conditions

(i) $\Lambda(X) = \lambda(X)$ if $X \in \mathfrak{h}$,

(ii) $\Lambda([Z, X]) = [\lambda(Z), \Lambda(X)]$ if $Z \in \mathfrak{h}, X \in \mathfrak{g}$,

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where λ denotes the homomorphism of the Lie algebras $\mathfrak{h}\to\mathfrak{gl}(n)$ induced by the linear representation of the isotropy group.

Let φ denote a G-invariant connection form on $L(M)$, than the corresponding linear map $\Lambda\colon \mathfrak{g}\to\mathfrak{gl}(n)$ satisfies

$$\Lambda(X)=\varphi(\hat{X}) \quad \text{if} \quad X\in\mathfrak{g},$$

where $\hat{X}$ denotes the vector field on $L(M)$, defined by the tangent vectors of orbits in $L(M)$ of the one-parameter subgroup $\exp tX\subset G$.

Let $\mathfrak{m}$ denote a complementary subspace to the subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ that is

$$\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}.$$

Let be given a leftinvariant coframe $\{\omega^1, \dots, \omega^n, \omega^{n+1}, \dots, \omega^{n+k}\}$ on the group G such that the equations $\omega^1=\dots=\omega^n=0$ define the subalgebra $\mathfrak{h}$ and the equations $\omega^{n+1}=\dots=\omega^{n+k}=0$ define the subspace $\mathfrak{m}$. In the following the indices have the values: $a, b, c=1, \dots, n$; $\alpha, \beta, \gamma=n+1, \dots, n+k$, where $n=\dim M$ and $n+k=\dim G$. The structure equations of the group G have the form

$$d\omega^a=-\sum_{\beta,c} c^a_{\beta c}\omega^\beta\wedge\omega^c-\frac{1}{2}\sum_{b,c} c^a_{bc}\omega^b\wedge\omega^c,$$

$$d\omega^\alpha=-\frac{1}{2}\sum_{\beta,\gamma} c^\alpha_{\beta\gamma}\omega^\beta\wedge\omega^\gamma-\sum_{\beta,c} c^\alpha_{\beta c}\omega^\beta\wedge\omega^c-\frac{1}{2}\sum_{b,c} c^\alpha_{bc}\omega^b\wedge\omega^c.$$

The connection form φ can be expressed by

$$\varphi(\hat{X})=\sum_{a,c}\varphi^a_c(\hat{X})E^c_a=\sum_{a,c}\Big(\sum_\beta c^a_{\beta c}\omega^\beta(X)+\frac{1}{2}\sum_b c^a_{bc}\omega^b(X)+\frac{1}{2}\sum_b l^a_{bc}\omega^b(X)\Big)E^c_a,$$

where l^a_{bc} are constant and $\{E^c_a\}$ denotes the canonical basis of the linear Lie algebra $\mathfrak{gl}(n)$.

§ 2. The indicatrix of orbits of 1-parameter subgroups

Let be M a differentiable manifold and suppose that there is linear connection on M. Let $y(t)$ be given a differentiable curve in M. The operator of the parallel translation along the curve $y(t)$ will be denoted by $\tau_{t,t_0}\colon T_{y(t)}M\to T_{y(t_0)}M$.

The indicatrix of the curve $y(t)$ at the point $y(t_0)$ is the curve $Y(t)$ in the tangent space $T_{y(t_0)}M$, defined by the parallel translation of the tangent vector $\dot{y}(t)$ of the curve to the point $y(t_0)$:

$$Y(t)=\tau_{t,t_0}\dot{y}(t).$$

Theorem 1. *Let $M=G/H$ be a homogeneous space, and let a G-invariant connection on M be given by a map $\Lambda\colon \mathfrak{g}\to\mathfrak{gl}(n)$, according to Wang's theorem. The indicatrix of the orbit $y(t)=\alpha(\exp tX, o)$ at the origin $o\in M$ $(X\in\mathfrak{g})$ is the curve*

$Y(t)=\varkappa^{-1}(\exp t\Lambda(X))\varkappa Y_0$, *where* $\varkappa\colon T_0M\to\mathbf{R}^n$ *is the coordinate map* *defined by the frame u and* $Y_0=\pi_*(X)\in T_oM$ *is the tangent vector to the curve* $y(t)$ *at the initial point o.*

Proof. Since we regard the group G as a submanifold of $L(M)$, the 1-parameter subgroup $x(t)=\exp tX$ $(X\in \mathfrak{g})$ is a curve in $L(M)$ with tangent vectors $\hat{X}(t)\in T_{x(t)}L(M)$. The equations of $x(t)$ in $G\subset L(M)$ are

$$\frac{d}{dt}\omega^a(\hat{X}(t))=0 \ \ (a=1, \dots, n), \ \ \frac{d}{dt}\omega^\alpha(\hat{X}(t))=0 \ \ (\alpha=n+1, \dots, n+k),$$

with respect to the given G-left invariant coframe $\{\omega^1, \dots, \omega^{n+k}\}$. Hence the equations of the orbit $y(t)=\alpha(\exp tX, o)=p\cdot x(t)$ are

$$\frac{d}{dt}\omega^a(\hat{X}(t))=0 \quad (a=1, \dots, n).$$

On the other hand, using the following lemma, the components of the covariant derivative $\nabla_t\dot{y}=\nabla_{\frac{\partial}{\partial t}}\dot{y}$ of the tangent vector $\dot{y}(t)$ of the orbit $y(t)$ can be expressed as

$$\omega^a(\nabla_t\dot{y})=\frac{d}{dt}\omega^a(\hat{X})+\sum_c \varphi^a_c(\hat{X})\omega^c(X).$$

Lemma. *Let M be a manifold equipped with a connection form* φ *on* $L(M)$. *Let* $y(t)$ *be a curve in M,* $X(t)$ *a vector field along* $y(t)$. *The components* $\omega^1, \dots, \omega^n$ *of the* R^n*-valued canonical form* ω *on the covariant derivative vector* $\nabla_tX=\nabla_{\frac{\partial}{\partial t}}X$ *along the curve* $y(t)$ *satisfy*

$$\omega^a(\nabla_tX)=\frac{d}{dt}\omega^a(\hat{X})+\sum_c \varphi^a_c(\hat{y})\omega^c(\hat{X})$$

where $\hat{y}$ *and* $\hat{X}$ *denote the horizontal lifts of the vectors* $\dot{y}$ *and X, and* φ^a_c *are the components of connection form* φ.

This lemma is a version of Theorem 11 in § 6.4 [1]. $\Lambda(X)=\varphi(\hat{X})$ and $\frac{d}{dt}\omega^a(X)=0$, we get $\nabla_t\dot{y}=\varkappa^{-1}\Lambda(X)\varkappa\dot{y}$, where $\varkappa\colon T_0M\to\mathbf{R}^n$ is the coordinate map defined by the chosen frame field, or equivalently, we get the equation of the indicatrix $Y(t)$ of $y(t)$ in the form

$$\frac{d}{dt}Y(t)=\varkappa^{-1}\Lambda(X)\varkappa Y(t).$$

It is well-known that the solution of this ordinary differential equation with constant coefficients is

$$Y(t)=\varkappa^{-1}(\exp t\Lambda(X))\varkappa Y_0,$$

where $Y_0=Y(0)=\pi_*X$. The theorem is proved.

Corollary. *The k-th covariant derivative* $\nabla^{(k)}_t\dot{y}$ *of tangents of the orbit* $y(t)=\alpha(\exp tX, o)$ *at the initial point* $o\in M$ *is* $(\Lambda(X))^kY_0$, *where* $Y_0=\pi_*Y$.

§ 3. The indicatrix of orbits in a reductive space

If there is given a reductive complement $\mathfrak{m}\subset\mathfrak{g}$ to the subalgebra $\mathfrak{h}$ in the Lie algebra $\mathfrak{g}$, characterized by

$$\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m} \quad\text{and}\quad [\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m},$$

then it is clear that the map $\Lambda\colon\ \mathfrak{g}\to\mathfrak{gl}(n)$ defined by

$$\Lambda(X)=\lambda(X) \quad\text{if}\quad X\in\mathfrak{h},\quad \Lambda(X)=0 \quad\text{if}\quad X\in\mathfrak{m}$$

satisfies the assumptions of Wang's theorem. The corresponding G-invariant connection is called the canonical connection of the reductive space $\{M=G/H,\mathfrak{m}\}$. From Theorem 1 it follows immediately:

Theorem 2. *Let $\{M=G/H,\mathfrak{m}\}$ be a reductive homogeneous space. The curve $y(t)$ in M is the orbit of a* 1*-parameter subgroup of G if and only if its indicatrix with respect to the canonical connection is an orbit of a* 1*-parameter subgroup of linear isotropy group. In detail, the indicatrix of the orbit $\alpha(\exp tX, o)$ at the origin $o\in M$ is the curve $Y(t)=(\exp t\,\mathrm{ad}\,Z)Y_0$, where $Z=X_{\mathfrak{h}}$ and $Y_0=X_{\mathfrak{m}}$ are the components of the vector X in the subspaces $\mathfrak{h}$ and $\mathfrak{m}$, respectively, and the tangent space T_oM is identified with the reductive complement* $\mathfrak{m}$.

Proof. From the property $[\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m}$ of the reductive complement $\mathfrak{m}$ follows that the homomorphism $\lambda\colon\mathfrak{h}\to\mathfrak{gl}(n)$ induced by the linear representation of isotropy group has the form: $\lambda(Z)=\mathrm{ad}\,Z\colon\ \mathfrak{m}\to\mathfrak{m}$ $(Z\in\mathfrak{h})$. The theorem is proved.

Corollary. *The k-th covariant derivative $\nabla_t^{(k)}\dot{y}$ of the tangents of the orbit $y(t)=\alpha(\exp tX, o)$ at the initial point $o\in M$ is* $(\mathrm{ad}\,Z)^k Y_0$.

§ 4. Geodesics in a fibering of reductive space

Let $\{M=G/H,\mathfrak{m}\}$ be a reductive homogeneous space. Let be given a subgroup $K\subset H$ and a reductive complementum $\mathfrak{f}$ on the homogeneous space $F=H/K$. The homogeneous space $N=G/K$ has a structure of a fibre bundle $\{N,\pi,M,F\}$, where N, M and F are the total, basic and the fiber type manifolds, respectively. We have the decompositions of Lie algebras

$$\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m},\quad \mathfrak{h}=\mathfrak{k}\oplus\mathfrak{f},\quad \mathfrak{g}=\mathfrak{k}\oplus\mathfrak{f}\oplus\mathfrak{m}$$

satisfying

$$[\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m},\quad [\mathfrak{k},\mathfrak{f}]\subset\mathfrak{f},\quad [\mathfrak{k},\mathfrak{f}\oplus\mathfrak{w}]\subset\mathfrak{f}\oplus\mathfrak{m}.$$

It is clear that $\mathfrak{f}\oplus\mathfrak{m}$ is a reductive complement on the homogeneous space $N=G/K$.

We investigate the projection to M of the geodesics in the homogeneous space $N=G/K$ with respect to the canonical connection corresponding to the reductive complement $\mathfrak{f}\oplus\mathfrak{m}$.

Theorem 3. *The curve $y(t)$ in $M=G/H$ through the origin $o\in M$ is a projection of a geodesic in $N=G/K$ $(K\subset H)$ with respect to the canonical connection if and only if its indicatrix at the origin $o\in M$ is an orbit of a* 1*-parameter subgroup* $\exp t \operatorname{ad} Z$ *of the linear isotropy group, where* $Z\in\mathfrak{k}$.

(Here and in the following $\operatorname{ad} Z$: $\mathfrak{g}\to\mathfrak{g}$ denotes the operator $X\to[Z, X]$ on $\mathfrak{g}$. Since $[\mathfrak{h}, \mathfrak{m}]\subset\mathfrak{m}$, this operator can be restricted to the subspace $\mathfrak{m}\subset\mathfrak{g}$; this restriction is denoted by the same way.)

Proof. Since $N=G/K$ is a reductive homogeneous space equipped with canonical connection, the geodesics in N are the orbits of 1-parameter subgroups $\exp tX$ of the group G, where $X\in\mathfrak{k}\oplus\mathfrak{m}$. From Theorem 2, it follows that the indicatrix of the orbit of subgroup $\exp tX$ at the point $o\in M$ is the curve $Y(t)=(\exp t \operatorname{ad} Z)Y$, where $Z=X_{\mathfrak{h}}$ and $Y=X_{\mathfrak{m}}$. From $X\in\mathfrak{k}\oplus\mathfrak{m}$ follows that $Z=X_{\mathfrak{h}}\in\mathfrak{k}$.

On the other hand, if $Y\in\mathfrak{m}(=T_oM)$, $Z\in\mathfrak{k}$, then it is clear that $Y(t)=(\exp t \operatorname{ad} Z)Y$ is the indicatrix of the orbit of the subgroup $\exp t(Y+Z)$. But we know that the orbit of a 1-parameter subgroup $\exp t(Y+Z)$ in the space $N=G/K$ is geodesic. The theorem is proved.

§ 5. Geodesics in the tangent sphere bundle of a 2-transitive Riemannian homogeneous space

We apply our results to the characterization of the projections of geodesics of the tangent sphere bundle of a 2-transitive Riemannian homogeneous space with respect to the Sasaki metric. We get a generalization of a result ([5], [4], [3]) asserting that the projection of a geodesic of the tangent sphere bundle of a space of constant curvature is a helix.

Let be $M=G/H$ a 2-transitive Riemannian homogeneous space, that is the group G is supposed to act transitively on the tangent sphere bundle N of the manifold M. It is well-known that from the 2-transitivity of the isometry group G of M follows that M is symmetric space (cf. [6], 289). On a Riemannian symmetric space $M=G/H$ there is a natural reductive complement $\mathfrak{m}\subset\mathfrak{g}$ whose canonical connection has the same geodesics as the Riemannian connection of the symmetric space M [6].

From the 2-transitivity of G on $M=G/H$ it follows that there exists a subgroup $K\subset H$ such that the tangent sphere bundle N can be written in the form $N=G/K$. The isotropy group H is isomorphic to a subgroup of the orthogonal group $O(n)$, and hence we have an invariant metric on H. This metric induces on the homogeneous space $F=H/K$ a naturally reductive Riemannian metric, which defines on F the geometry of n-sphere. Let $\mathfrak{m}$ and $\mathfrak{k}$ denote the reductive complements on M

and F, respectively, i.e. we have $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$, $\mathfrak{h}=\mathfrak{k}\oplus\mathfrak{f}$. Now we can apply Theorem 3 to this case.

Theorem 4. *Let $M=G/H$ be a 2-transitive Riemannian homogeneous space. The curve $y(t)$ in M is a projection of a geodesic in the tangent sphere bundle if and only if $y(t)$ is a 3-dimensional helix (i.e. the first two curvatures $\varkappa_1$, $\varkappa_2$ are arbitrary constants, and the others zero: $\varkappa_3=\ldots=\varkappa_{n-1}=0$).*

Proof. From Theorem 3 we know that $y(t)$ is a projection of a geodesic in N if and only if its indicatrix has the form $\exp(t \operatorname{ad} Z)Y$, where $Y\in\mathfrak{m}$, $Z\in\mathfrak{f}\subset\mathfrak{h}$.

After identifying an orthogonal frame at $o\in M$ with the identity of H the adjoint representation maps the group H isomorphically on a subgroup of the orthogonal group $O(n)$ acting on the unit $(n-1)$-sphere of the tangent space T_oM $(=\mathfrak{m})$. In the following we identify the group H with the subgroup of $O(n)$ by this isomorphism. The reductive complement $\mathfrak{f}$ of the subalgebra $\mathfrak{k}$ in $\mathfrak{h}$ corresponds to the tangent space at the initial point of the $(n-1)$-sphere $F=H/K$. Since the reductive complement $\mathfrak{f}$ on $F=H/K$ is identified with the reductive complement on the $(n-1)$-sphere $S^{n-1}=O(n)/O(n-1)$, the 1-parameter subgroup $\exp(t \operatorname{ad} Z)$ $(Z\in\mathfrak{f})$ of $O(n)$ is a 1-parameter rotation group around the $(n-2)$-plane in T_oM, orthogonal to the 2-plane of the geodesic great circle which is the orbit of $\exp(t \operatorname{ad} Z)$ in $S^{n-1}=F$ through the initial point. It follows that the curve $Y(t)=\exp(t \operatorname{ad} Z)Y$ $(Y\in\mathfrak{m}, Z\in\mathfrak{f})$ is a circle. The indicatrix of a curve $y(t)$ is a circle if and only if it is a 3-dimensional helix. Theorem 4 is proved.

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Some equivalent formulations of the intersection problem of finitely generated classes of graphs

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Introduction. An ordering $\prec$ on the class of finite undirected graphs without loops is defined by $G \prec F$ iff there exists a (partial) subgraph G of the graph F which is a subdivision of the graph G. A class L of graphs is called *closed* if $G \in L$, $G \prec F \Rightarrow F \in L$. By $L(G_1, \ldots, G_n)$ we denote the smallest class of graphs which is closed and contains the graphs $G_1, \ldots, G_n$. The graphs $G_1, \ldots, G_n$ are called *generators* of the class $L(G_1, \ldots, G_n)$. A class L is called *finitely generated* if it is closed and there are graphs $G_1, \ldots, G_n$ such that $L = L(G_1, \ldots, G_n)$. If L is a closed class we denote by $B(L)$ the set of all minimal members of L in $\prec$. The set $B(L)$ is called the *base* of L. Evidently, L is finitely generated iff its base $B(L)$ is finite.

The following problem was posed by L. LOVÁSZ [1] and by P. UNGAR [3]: Is the class $L \cap L'$ finitely generated for every pair L, L' of finitely generated classes of graphs? It is not difficult to see that the essence of the problem lies in the investigation of "braids" of subdivisions of pairs of graphs. The problem is equivalent to the question whether the number of "critical braids" is finite or infinite.

Our method shows that it is sufficient to investigate such "braids" of subdivisions G', H' of graphs G, H that G' does not contain vertices of H and H' does not contain vertices of G. Every edge of a graph determines a path in its subdivision. If we decompose the graphs G, H into single edges, it is sufficient to investigate the "braids" of corresponding paths. This "braid" of paths will be called a *crossing system* (see the definition below). We hope that the investigation of "braids" of paths is easier than the investigation of "braids" of general graphs and could lead to a solution of the problem. It also follows that the problem does not depend on concrete graphs. We prove that it is sufficient to solve it for special pairs $L(G)$, $L(H)$ where G is a disjoint union of complete graphs K_6 and H is a disjoint union of complete bipartite graphs $K_{2,6}$.

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Notions and results. A graph $c=(\{v_0, \ldots, v_t\}, \{e_1, \ldots, e_t\})$ is called a *path* if e_i is edge adjacent to vertices v_{i-1}, v_i, $1 \leqq i \leqq t$. Denote by $V(c)$ the set $\{v_0, \ldots, v_t\}$ of vertices of the path c, and by $K(c)=\{v_0, v_t\}$ the set of endvertices of the path c. A set of paths $C=\{c_1, \ldots, c_m\}$ is called a *disjoint system of paths* if every two paths of C are vertex disjoint. Put $V(C)=\bigcup_{i=1}^{m} V(c_i)$, $K(C)=\bigcup_{i=1}^{m} K(c_i)$. $K(C)$ is called the set of endvertices of C.

Let $C=(c_1, \ldots, c_m)$, $D=(d_1, \ldots, d_n)$ be two disjoint systems of paths which satisfy $K(C)\cap V(D)=V(C)\cap K(D)=\emptyset$. In this case the couple (C, D) is called an *(m, n)-crossing system*. By $\mathrm{gr}\,(C, D)$ we denote the graph on the set of vertices $V(C)\cup V(D)$ which is the union of all paths of C, D. A vertex $v\in V(C)\cap V(D)$ is called a *crossing* of (C, D) if $N(v, C)\neq N(v, D)$ where $N(v, C)$, resp. $N(v, D)$, is the set of all neighbours of the vertex v in the graph C, resp. D.

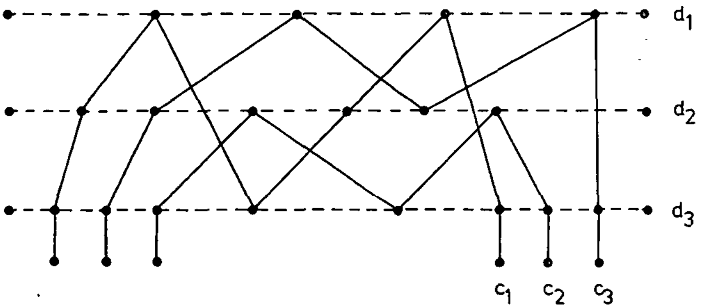

Fig. 1a

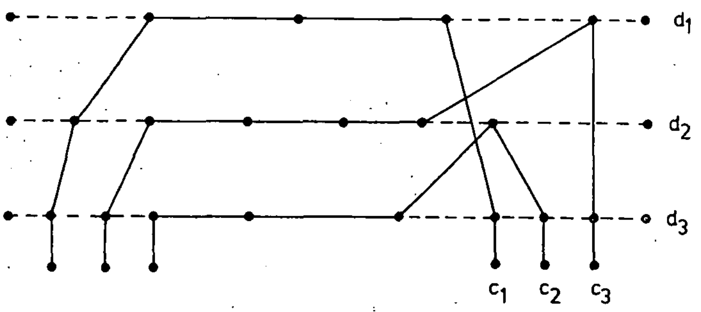

Fig. 1b

Fig. 1a, 1b are examples of (3, 3)-crossing systems. The crossing system 1a is reducible. An example of its reduction is the crossing system in Fig. 1b.

There is a crossing at every vertex in the crossing system in Fig. 1a but not in Fig. 1b.

We say that an (m, n)-crossing system (C, D) is *reducible* if in $\mathrm{gr}\,(C, D)$ there exist two disjoint systems of paths C', D' such that

1) $C' = (c'_1, \dots, c'_m)$, $D' = (d'_1, \dots, d'_n)$;

2) $K(c_i) = K(c'_i)$, $K(d_j) = K(d'_j)$ for every i, j, $i = 1, \dots, m$, $j = 1, \dots, n$;

3) the crossing system (C', D') has strictly fewer crossings than (C, D). (See Fig. 1.)

Denote by $G+H$ the disjoint sum of G and H.

Theorem. *The following conjectures are equivalent:*

1) *$L_1 \cap L_2$ is a finitely generated class for every pair L_1, L_2 of finitely generated classes of graphs.*

2) *The class $L(G) \cap L(H)$ is finitely generated for every two graphs G, H.*

3) *The class $L(K_6 + \dots + K_6) \cap L(K_{2,6} + \dots + K_{2,6})$ (the graphs in brackets are disjoint unions of n copies of K_6, resp. $K_{2,6}$) is finitely generated for every natural number n.*

4) *For every m and n there exists a k such that every (m, n)-crossing system with more than k crossings is reducible.*

Proofs. Evidently 1)$\Leftrightarrow$2) and 2)$\Rightarrow$3). We prove 3)$\Rightarrow$4) and 4)$\Rightarrow$2). The crossing system (C, D) is called *minimal* if (C, D) is not reducible and every vertex of $\mathrm{gr}\,(C, D)$ is crossing. The implication 4)$\Rightarrow$2) immediately follows from the following lemma.

Lemma 1. *Let a graph B belong to the base of the class $L(G) \cap L(H)$ and let m and n denote the numbers of the edges of graphs G and H, resp. Then there exists a minimal (m, n)-crossing system with at least $|B|$ vertices.*

Proof. Let a graph R contain subdivisions G', H' of the graphs G, H. We may suppose G, H have no isolated vertices. In general these subdivisions can be placed differently in the graph R. Therefore we introduce the following notation. We denote by $\varphi_G \colon G \to R$ the morphism which maps the graph G on its subdivision $G' = \varphi G$ in the graph R: the morphism φ_G maps the vertices of G on distinct vertices of the graph R and the edges of the graph G on openly disjoint paths. The location of the subdivision of the graph H we denote similarly by $\varphi_H \colon H \to R$. Put $\varphi = (\varphi_G, \varphi_H)$. In the sequel a *morphism* will always mean such a pair $\varphi = (\varphi_G, \varphi_H)$. Every morphism $\varphi = (\varphi_G, \varphi_H)$ induces a *vertex-mapping* $f_\varphi \colon V(G+H) \to V(R)$ which is the restriction of the morphism φ to the set of vertices of $G+H$. Clearly a vertex-mapping $f \colon V(G+H) \to V(R)$ can be induced by various morphisms. If $e = (a_1, a_2)$ is an edge of G then the image of the edge e is a path $\varphi(e) = \varphi(a_1, a_2) = (f(a_1) = x_0, x_1, \dots$ $\dots, x_{k-1}, x_k = f(a_2))$, $x_i \in V(\varphi G)$, $k \geqq 1$. A vertex $a \in V(G)$ is called a *tied vertex* in R with respect to φ if $f(a)\in\varphi H$. Likewise $b\in V(H)$ is a *tied vertex* in R with respect to φ if $f(b)\in\varphi G$. The set of tied vertices of the graphs G, H is denoted by $W\varphi=W$. (So $W\subseteqq V(G)\cup V(H)$.)

We shall study quadruples (R, φ, f, W) where R is a graph, φ is a morphism, f is a vertex-mapping and W is a set of tied vertices. The quadruple (R, φ, f, W) is *admissible* if $\varphi=(\varphi_G, \varphi_H)\colon G+H\to R$ and $f=f_\varphi$ is the vertex mapping induced by φ and $W=W\varphi$ is the set of tied vertices with respect to φ. The admissible quadruple (R, φ, f, W) is called *critical* if:

1) after removing any edge e of R, there is no φ' and no $W'\subseteqq W$ such that $(R-e, \varphi', f, W')$ is an admissible quadruple;

2) there is no couple φ'', $W''\subsetneqq W$ such that (R, φ'', f, W'') is an admissible quadruple;

3) if $x\in V(R)$ has degree 2 then $x\in f(G+H)$.

Put $L=L(G)\cap L(H)$. Evidently, for every graph $B\in B(L)$ there exist φ, f, W such that the admissible quadruple (B, φ, f, W) is critical.

The following lemma will finish the proof of Lemma 1.

Induction Lemma. *For every critical quadruple $Q=(R, \varphi, f, W)$, $W\neq\emptyset$, there exists a critical quadruple $Q'=(R', \varphi', f', W')$ such that $|R'|\geqq|R|$ and $W'\subsetneqq W$.*

Using the Induction Lemma, Lemma 1 may be proved as follows. For every $B\in B(L)$ there exists a critical quadruple $Q=(R, \varphi, f, W)$ such that $|R|\geqq|B|$ and $W=\emptyset$. We construct an (m, n)-crossing system from the quadruple Q by splitting every vertex $f(x)$, $x\in V(G+H)$ into $d(x)$ vertices of degree 1 where $d(x)$ is the degree of the vertex $f(x)$ in R. Since Q is critical, this (m, n)-crossing system is minimal.

Proof of the Induction Lemma. Let $Q=(R, \varphi, f, W)$ be a critical quadruple, $W\neq\emptyset$. Take a point $u\in W$. We will construct a quadruple $Q'=(R', \varphi', f', W')$ such that $W'\subseteqq W-\{u\}$ and $|R'|\geqq|R|$. Put $w=f(u)$. There are three possibilities:

a) $w\in f(G)\cap f(H)$,

b) $w\in f(G)\cap(\varphi H-f(H))$,

c) $w\in f(H)\cap(\varphi G-f(G))$.

Cases b) and c) are symmetric, consequently it suffices to treat a) and b) only. Denote by $N(x, \varphi G)$, resp. $N(x, \varphi H)$, the neighbourhood of the vertex $x\in V(R)$ in φG, resp. φH.

Case a). Let $w=f(a)=f(b)$ where $a\in V(G)$, $b\in V(H)$. Clearly, $|N(w, \varphi G)|=$ $=d_G(a)$, $|N(w, \varphi H)|=d_H(b)$, and from condition 1) in the definition of critical quadruples, $d_R(w)=|N(w, \varphi G)\cup N(w, \varphi H)|$. Next we define the admissible quadruple Q'. Let $V(R')=(V(R)-\{w\})\cup\{a', b'\}$ and defines the edges of R' by

$$e\in E(R') \quad \text{for} \quad w\notin e\in E(R),$$

$$(x, a')\in E(R') \quad \text{for} \quad x\in N(w, \varphi G),$$

$$(x, b')\in E(R') \quad \text{for} \quad x\in N(w, \varphi H).$$

The vertex mapping f' is defined by $f'(x)=f(x)$ for $x\neq a, b, f'(a)=a', f'(b)=b'$. Now we define the morphism φ'. If an edge e is not adjacent to a or b in $G+H$, put $\varphi'(e)=\varphi(e)$. If $e=(a, v)$, resp. $e=(b, v)$, $\varphi(e)=(w, x_1, \ldots, x_k, f(v))$, put $\varphi'(e)=$ $=(a', x_1, \ldots, x_k, f(v))$, resp. $\varphi'(e)=(b', x_1, \ldots, x_k, f(v))$.

Evidently, $W\varphi'=W\varphi-\{a, b\}$. We verify that the quadruple Q' satisfies condition 1) in the definition of critical quadruples. By way of contradiction let us suppose that there is an edge $e_0\in E(R')$ and a morphism ψ' such that $(R'-e_0, \psi', f', W\psi')$ is an admissible quadruple with $W\psi'\subseteq W\varphi'$. Since $d_{R'}(a')=d_G(a)$ and $d_{R'}(b')=$ $=d_G(b)$, neither a' nor b' is adjacent to e_0. If e is an edge of G, resp. H, which is not adjacent to the vertex a, resp. b, then $b'\notin\psi'(e)$, resp. $a'\notin\psi'(e)$, because $a, b\notin W\psi'$. Thus, we can define an admissible quadruple $(R-e_0, \psi, f, W\psi)$ where the morphism ψ is defined from ψ' by the reverse procedure to the one we used to obtain φ' from φ. Clearly, $W\psi=W\psi'\cup\{a, b\}\subseteq W\varphi$. This contradicts the fact that Q is critical. Condition 3) is obvious. If Q does not satisfy condition 2), it is sufficient to replace φ' by a suitable φ'' and W' by a smaller set W''.

Case b) will be divided into three subcases b_0), b_1), b_2) where b_i) means $|N(w, \varphi G)\cap N(w, \varphi H)|=i$. Let $w=f(a)$, $a\in V(G)$.

Cases b_0) and b_1) will be considered together. Denote by x_0 an arbitrary element of $N(w, \varphi G)$ in the case b_0) and the only element of $N(w, \varphi G)\cap N(w, \varphi H)$ in the case b_1).

1. Put $V(R')=V(R)\cup\{a'\}$. Define the edges of R' by

$$e\in E(R') \quad \text{for} \quad w\notin e\in E(R),$$

$$(x, a')\in E(R') \quad \text{for} \quad x\in N(w, \varphi G)\cup\{w\}, \quad x\neq x_0,$$

$$(x, w)\in E(R') \quad \text{for} \quad x\in N(w, \varphi H)\cup\{x_0\}.$$

2. The mapping f' is defined by $f'(a)=a'$ and $f'(v)=f(v)$ for $v\neq a$.

3. The morphism φ' is defined by

$$\varphi'(e) = \varphi(e) \quad \text{for} \quad a \notin e \in E(G+H),$$

$$\varphi'(e) = (a', x_1, \ldots, x_k) \quad \text{for} \quad a \in e, \quad \varphi(e) = (w, x_1, \ldots, x_k), \quad x_1 \neq x_0,$$

$$\varphi'(e) = (a', w, x_0, x_1, \ldots, x_k) \quad \text{for the only } e \text{ with } \quad \varphi(e) = (w, x_0, x_1, \ldots, x_k).$$

4. $W' = W - \{a\}$.

Case b_2). Denote by x_1, x_2 the elements of $N(w, \varphi G) \cap N(w, \varphi H)$.

1. Put $V(R') = (V(R) \cup \{a'\}) - \{w\}$. Define the edges of R' by

$$e \in E(R') \quad \text{for} \quad w \notin e \in E(R),$$

$$(x, a') \in E(R') \quad \text{for} \quad x \in N(w, \varphi G),$$

$$(x_1, x_2) \in E(R').$$

2. Put $f'(a) = a'$ and $f'(v) = f(v)$ for $v \neq a$.

3. Put $\varphi'(e) = \varphi(e)$ for $\varphi(e)$ not containing w,

$$\varphi'(e) = (a', y_1, \ldots, y_k) \quad \text{for} \quad \varphi(e) = (w, y_1, \ldots, y_k),$$

$$\varphi'(e) = (\ldots, x_1, x_2, \ldots) \text{ for the only } e \in E(H) \text{ for which } \varphi(e) = (\ldots, x_1, w, x_2, \ldots) \text{ contains } w.$$

4. $W' = W - \{a\}$.

The proof of case b) goes like the proof of case a) (but in case b_2) it is also necessary to use condition 2) in the definition of critical quadruples), and we omit it. (See Fig. 2.)

Lemma 2 below implies immediately the proof of the implication 3)$\Rightarrow$4) and completes the proof of the Theorem.

Lemma 2. *Let S be a minimal (m, m)-crossing system. Then there exists a graph B from the base of the class $L(K_6^1 + \ldots + K_6^m) \cap L(K_{2,6}^1 + \ldots + K_{2,6}^m)$ such that the number of vertices of B is greater than the number of vertices of S. (K_6^i, $K_{2,6}^i$ denote the i-th copy of K_6, $K_{2,6}$, respectively.)*

Proof. Let S be a minimal (m, m)-crossing system formed by two disjoint systems of paths C, D where $C = (c_1, \ldots, c_m)$, $D = (d_1, \ldots, d_m)$. First, take the disjoint union of m copies of the graph K_6 (denote the *i-th* copy by K_6^i). The vertices of

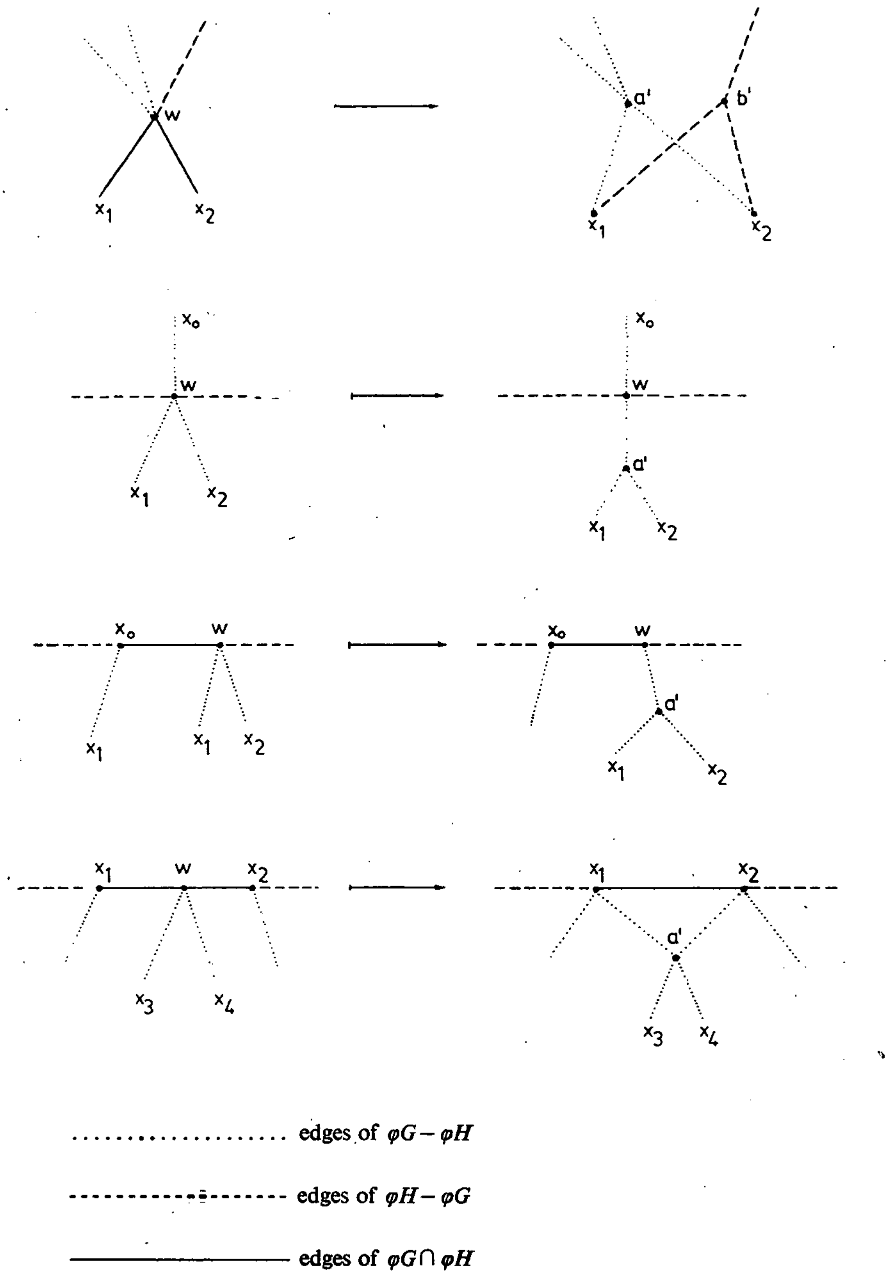
w
x1
x2
a'
b'
x0
x3
x4
edges of $\varphi G - \varphi H$
edges of $\varphi H - \varphi G$
edges of $\varphi G \cap \varphi H$

Fig. 2

K_6^i are denoted by $c_i^1, c_i^2, d_i^1, d_i^2, u_i^1, u_i^2$. We shall construct the graph B from $\sum K_6$ in two steps:

a) we construct a subdivision of $\sum K_6$,

b) we add further edges.

Put a new vertex u_i^3 on every edge (c_i^1, c_i^2). Identify the vertices c_i^1, c_i^2 with the endpoints of the path c_i. Subdivide the edge (d_i^1, d_i^2) by the number of vertices of the path d_i and identify this subdivision with the path d_i. Now, all vertices of the crossing system S are identified with some vertices of the graph B. Hence we may assume $V(S) \subseteq V(B)$. Add to B all edges $(x, y) \in E(\mathrm{gr}(C, D))$. This completes the construction of B.

The graph B evidently contains the subdivision of the graph $G = \sum K_6$. We shall show that it contains the subdivision of $H = \sum K_{2,6}$, too. The graph $K_{2,6}$ is formed by six paths of length 2 which have common endpoints. The subdivision of $K_{2,6}^i$ is in the graph B formed by the paths c_i^1, u_i^j, c_i^2, $j=1, 2, 3$, c_i^1, d_i^j, c_i^2, $j=1, 2$ and c_i^1, c_i, c_i^2 (c_i is the path of the system C).

We shall prove that the graph B does not contain other subdivisions of $\sum K_6$ and $\sum K_{2,6}$ than those described above. The vertices $u_i^1, u_i^2, c_i^1, c_i^2, d_i^1, d_i^2$, $i=1, \ldots, m$ are the only vertices of B of degree $\geqq 5$. Hence, the only subdivision of $\sum K_6$ in B, possibly with the exception of edges (d_i^1, d_i^2), is that described above. Since the vertices c_i^1, c_i^2, $i=1, \ldots, m$, are the only ones of degree 6 in B, the vertices of degree 6 in $K_{2,6}^i$ must be put on them. Further, 5 vertices of $K_{2,6}^i$ must be put on vertices $d_i^1, d_i^2, u_i^1, u_i^2, u_i^3$. Thus subdivisions of edges (d_i^1, d_i^2) and the remaining paths between c_i^1, c_i^2 in $K_{2,6}^i$ correspond to the paths in the crossing system S. The minimality of B follows from the minimality of S.

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A Jordan form for certain infinite-dimensional operators

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We derive several theorems about invariant subspaces of operator algebras of finite strict multiplicity. We generalize a result of EMBRY [3] to show that such algebras have maximal invariant subspaces (Theorem 2), and we prove some related theorems. The main results are Theorems 5 and 6, which give a "Jordan form" for operators which inherit finite strict multiplicity.

The first theorem is a slight sharpening of its corollary, which is due to HERRERO [7]. Herrero's result generalizes LAMBERT's result for strictly cyclic algebras [11] to algebras of finite strict multiplicity. Our proof combines ideas from HERRERO [7] and from RADJAVI and ROSENTHAL [18].

We will use the following notation throughout this article. We use $\mathfrak{H}$ to denote a separable Hilbert space, and $\mathscr{B}(\mathfrak{H})$ is the algebra of all bounded, linear operators on $\mathfrak{H}$. If $\mathscr{A}$ is a subalgebra of $\mathscr{B}(\mathfrak{H})$ or if T is an operator in $\mathscr{B}(\mathfrak{H})$, Lat $\mathscr{A}$ or Lat T denotes the lattice of invariant subspaces of $\mathscr{A}$ or of T. $\mathscr{A}(T)$ will be used for the subalgebra of $\mathscr{B}(\mathfrak{H})$ generated by T and the identity. Finally, if $\mathfrak{M}$ and $\mathfrak{N}$ are subsets of $\mathfrak{H}$, $\mathfrak{M}\vee\mathfrak{N}$ is the closed linear span of $\mathfrak{M}$ and $\mathfrak{N}$.

Recall that a subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathfrak{H})$ has *finite strict multiplicity* if there is a finite collection of vectors $\{x_1, x_2, \ldots, x_n\}$ such that

$$\{A_1x_1+A_2x_2+\ldots+A_nx_n\colon A_i\in\mathscr{A}\} = \mathfrak{H}.$$

In that case, $\{x_1, x_2, \ldots, x_n\}$ is called an *FSM set* for $\mathscr{A}$, and the minimal cardinality of all such sets of vectors is called the *strict multiplicity* of $\mathscr{A}$. If $\mathscr{A}$ has strict multiplicity 1, $\mathscr{A}$ is said to be *strictly cyclic*. The operator T has *finite strict multiplicity* if $\mathscr{A}(T)$ does, and T is *strictly cyclic* if $\mathscr{A}(T)$ is. $\mathscr{A}$ is said to *inherit finite strict multiplicity* if the uniform closure of its restriction to every invariant subspace has finite strict multiplicity, and $\mathscr{A}$ is said to be *hereditarily strictly cyclic* if the uniform closure of its restriction to every invariant subspace is strictly cyclic. We will reserve the terms strictly cyclic and finite strict multiplicity for infinite-dimensional operators

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(unless we are talking about the restriction of an infinite-dimensional operator to a finite-dimensional invariant subspace).

Theorem 1. *Let $\mathscr{A}$ be a uniformly closed subalgebra of $\mathscr{B}(\mathfrak{H})$ containing the identity, and let $\mathfrak{M}\in \operatorname{Lat}\mathscr{A}$ be such that $\mathscr{A}|\mathfrak{M}$ has finite strict multiplicity; assume that $\{x_1, x_2, \ldots, x_n\}$ is an FSM set for $\mathscr{A}|\mathfrak{M}$. Then every invariant linear manifold of $\mathscr{A}|\mathfrak{M}$ whose closure contains the vector $x_1+x_2+\ldots+x_n$ is closed.*

Proof. Let $\tilde{\mathscr{A}}$ be the (uniformly closed) algebra of all $n\times n$ matrices with entries from $\mathscr{A}$, and define $\varphi\colon \tilde{\mathscr{A}}\to\mathfrak{M}^{(n)}$ by

$$\varphi(A_{ij}) = (A_{ij})\bar{x}$$

where $\bar{x}=(x_1, x_2, \ldots, x_n)$. Then φ is obviously bounded, and φ is onto since $\mathscr{A}$ has finite strict multiplicity on $\mathfrak{M}$.

Let $\mathfrak{N}$ be an invariant linear manifold of $\mathscr{A}|\mathfrak{M}$ whose closure contains x_1+ $+x_2+\ldots+x_n$, let $\tilde{\mathscr{B}}=\varphi^{-1}(\overline{\mathfrak{N}}^{(n)})$, and let $\tilde{\mathscr{N}}=\varphi^{-1}(\mathfrak{N}^{(n)})$. If $\tilde{\mathscr{N}}=\tilde{\mathscr{B}}$, then $\mathfrak{N}=\overline{\mathfrak{N}}$ since $\varphi(\tilde{\mathscr{N}})=\varphi(\tilde{\mathscr{B}})=\overline{\mathfrak{N}}^{(n)}$. We show that $\tilde{\mathscr{N}}=\tilde{\mathscr{B}}$ by assuming that $\tilde{\mathscr{N}}\neq\tilde{\mathscr{B}}$ and finding a contradiction.

Since we are assuming $\tilde{\mathscr{N}}\neq\tilde{\mathscr{B}}$, the invariance of $\mathfrak{N}$ implies that $\tilde{\mathscr{N}}$ is a proper left ideal in $\tilde{\mathscr{B}}$. Also, $1\in\tilde{\mathscr{B}}$ since $\sum x_i\in\overline{\mathfrak{N}}$; i.e.,

$$1\bar{x} = \left(\sum x_i, \sum x_i, \sum x_i, \ldots, \sum x_i\right),$$

so $1\bar{x}\in\overline{\mathfrak{N}}^{(n)}$. Since $\tilde{\mathscr{N}}$ is a proper ideal, 1 is not in the closure of $\tilde{\mathscr{N}}$, and so $\tilde{\mathscr{N}}$ is not dense in $\tilde{\mathscr{B}}$. Now, let $\mathscr{U}\subset\tilde{\mathscr{B}}$ be an open set such that $\mathscr{U}\cap\tilde{\mathscr{B}}=\emptyset$. Then $\varphi(\mathscr{U})$ is open by the open mapping theorem, and $\varphi(\mathscr{U})\cap\mathfrak{N}^{(n)}=\emptyset$. But then $\mathfrak{N}^{(n)}$ is not dense in $\overline{\mathfrak{N}}^{(n)}$. But of course $\overline{\mathfrak{N}^{(n)}}=\overline{\mathfrak{N}}^{(n)}$, giving a contradiction.

The following special case of this theorem is the basis for many important known results, some of which are listed below.

Corollary 1. (Herrero [7]) *A uniformly closed algebra of finite strict multiplicity has no dense invariant linear manifolds other than $\mathfrak{H}$.*

Corollary 2. (Embry [3]) *If $\mathscr{A}$ is a uniformly closed algebra of finite strict multiplicity, and if x_0 is a cyclic vector for $\mathscr{A}$, then x_0 is a strictly cyclic vector.*

Proof. $\{Ax_0\colon A\in\mathscr{A}\}$ is dense since x_0 is a cyclic vector, and hence is all of $\mathfrak{H}$ by Corollary 1.

One of the best known unsolved problems in operator theory is the transitive algebra problem. Recall that an operator algebra is *transitive* if its only invariant subspaces are $\{0\}$ and $\mathfrak{H}$. The problem is whether $\mathscr{B}(\mathfrak{H})$ is the only (weakly closed) transitive algebra? Many partial results have been obtained, beginning with Arve-

SON's work [1]. An affirmative answer would imply that every operator has a nontrivial invariant subspace — see [18, Chapter 8]. Finding results such as the following appears to have been the main goal of LAMBERT [11], [13] and HERRERO [7], [8] in studying algebras of finite strict multiplicity.

Corollary 3. (HERRERO [7]) *The only weakly closed transitive algebra of finite strict multiplicity is* $\mathscr{B}(\mathfrak{H})$.

Proof. Let $\mathscr{A}$ be a transitive, weakly closed algebra of finite strict multiplicity. Since $\mathscr{A}$ is transitive, every invariant linear manifold of $\mathscr{A}$ (other than $\{0\}$) must be dense in $\mathfrak{H}$. Hence, by Corollary 1, $\{0\}$ and $\mathfrak{H}$ are the only invariant linear manifolds of $\mathscr{A}$. The Rickart—Yood theorem (cf. [18, Corollary 8.5]) then implies that $\mathscr{A}=\mathscr{B}(\mathfrak{H})$.

Lemma 1. *If $\mathscr{A}$ is an algebra of strict multiplicity n, and if $\mathfrak{M}\in\operatorname{Lat}\mathscr{A}$, then the compression of $\mathscr{A}$ to $\mathfrak{M}^{\perp}$ has strict multiplicity at most n.*

Proof. Let P be the projection onto $\mathfrak{M}^{\perp}$, and let $e_1, e_2, e_3, \ldots, e_n$ be vectors such that

$$\{A_1e_1+A_2e_2+\ldots+A_ne_n\colon A_i\in\mathscr{A}\}=\mathfrak{H}.$$

Let $\tilde{\mathscr{A}}=\{PA\colon A\in\mathscr{A}\}$, and let $f_i=Pe_i$, $m_i=e_i-f_i$. It suffices to show that

$$\{\tilde{A}_if_1+\tilde{A}_2f_2+\ldots+\tilde{A}_nf_n\colon \tilde{A}_i\in\tilde{\mathscr{A}})=\mathfrak{M}^{\perp}.$$

Note that $PAm_i=0$ for every $A\in\mathscr{A}$ since $\mathfrak{M}\in\operatorname{Lat}\mathscr{A}$. Given $x\in\mathfrak{M}^{\perp}$, choose $\{A_1, A_2, \ldots, A_n\}\subset\mathscr{A}$ such that $x=A_1e_1+A_2e_2+\ldots+A_ne_n$. Then

$$PA_1f_1+\ldots+PA_nf_n=PA_1(f_1+m_1)+\ldots+PA_n(f_n+m_n)=Px=x.$$

In the above proof we found n vectors to prove that $\tilde{\mathscr{A}}$ had finite strict multiplicity. Some of these vectors might be 0. In the strictly cyclic case, since the strict multiplicity does not increase, the compression algebra will also be strictly cyclic. This proves

Corollary. *If $\mathscr{A}$ is a strictly cyclic algebra, and if $\mathfrak{M}\in\operatorname{Lat}\mathscr{A}$, the compression of $\mathscr{A}$ to $\mathfrak{M}^{\perp}$ is strictly cyclic.*

EMBRY [3, Theorem 2] proves that every intransitive strictly cyclic algebra has a maximal invariant subspace. The next theorem is a generalization of Embry's theorem.

Theorem 2. *If $\mathscr{A}$ is an algebra of finite strict multiplicity, then every (proper) invariant subspace of $\mathscr{A}$ is contained in a (proper) maximal invariant subspace.*

Proof. Since the uniform closure of $\mathscr{A}$ has the same invariant subspace lattice as $\mathscr{A}$, we can assume that $\mathscr{A}$ is closed. Let $\mathfrak{M}\in\operatorname{Lat}\mathscr{A}$ with $\mathfrak{M}\neq\mathfrak{H}$. By the Hausdorff Maximality Principle there exists a maximal chain $\{\mathfrak{M}_\alpha\}$ of proper invariant subspaces containing $\mathfrak{M}$. Choose a countable dense subset $\{x_i\colon i=1, 2, \dots, \infty\}$ of $\bigcup_\alpha \mathfrak{M}_\alpha$, and choose $\mathfrak{M}_{\alpha_i}$ so that $x_i\in\mathfrak{M}_{\alpha_i}$. Then

$$\overline{\cup\mathfrak{M}_{\alpha_i}} = \overline{\{x_i\}} = \overline{\cup\mathfrak{M}_\alpha}.$$

If $\overline{\cup\mathfrak{M}_{\alpha_i}}=\mathfrak{H}$, $\cup\mathfrak{M}_{\alpha_i}$ is dense in $\mathfrak{H}$. By Corollary 1 to Theorem 1, $\cup\mathfrak{M}_{\alpha_i}=\mathfrak{H}$. By the Baire Category Theorem, some $\mathfrak{M}_{\alpha_i}=\mathfrak{H}$, which is impossible. Thus, $\overline{\cup\mathfrak{M}_{\alpha_i}}\neq\mathfrak{H}$, and so $\overline{\cup\mathfrak{M}_\alpha}$ is a maximal invariant subspace containing $\mathfrak{M}$.

For a large class of algebras of finite strict multiplicity, maximal invariant subspaces have co-dimension 1.

Theorem 3. *If $\mathscr{A}$ is an algebra of finite strict multiplicity such that for every $\mathfrak{M}\in\operatorname{Lat}\mathscr{A}$, the compression of $\mathscr{A}$ to $\mathfrak{M}^\perp$ is not strongly dense in $\mathscr{B}(\mathfrak{M}^\perp)$, then every maximal invariant subspace of $\mathscr{A}$ has co-dimension* 1.

Proof. Let $\mathfrak{M}$ be a maximal invariant subspace. If the co-dimension of $\mathfrak{M}$ is greater than 1, the compression of $\mathscr{A}$ to $\mathfrak{M}^\perp$ has a non-trivial invariant subspace $\mathfrak{N}$. This follows from Corollary 3 to Theorem 1 since the compression is not $\mathscr{B}(\mathfrak{M}^\perp)$ by hypothesis. If we show that $\mathfrak{N}\oplus\mathfrak{M}\in\operatorname{Lat}\mathscr{A}$, we will be done since this will contradict the maximality of $\mathfrak{M}$.

Since $\mathfrak{M}\in\operatorname{Lat}\mathscr{A}$, it is enough to show that if $y\in\mathfrak{N}$, then $Ay\in\mathfrak{N}\oplus\mathfrak{M}$ for every $A\in\mathscr{A}$. So let $P_{\mathfrak{M}}$ and $P_{\mathfrak{M}^\perp}$ be the projections onto $\mathfrak{M}$ and $\mathfrak{M}^\perp$, respectively. Then

$$Ay = (P_{\mathfrak{M}^\perp}+P_{\mathfrak{M}})Ay = P_{\mathfrak{M}^\perp}Ay+P_{\mathfrak{M}}Ay.$$

Note that $P_{\mathfrak{M}^\perp}\mathscr{A}$ is in the compression algebra, so $P_{\mathfrak{M}^\perp}Ay\in\mathfrak{N}$. And of course $P_{\mathfrak{M}}Ay\in\mathfrak{M}$.

Corollary. *If $\mathscr{A}$ is an Abelian algebra of finite strict multiplicity, then every invariant subspace of $\mathscr{A}$ is contained in an invariant subspace of co-dimension* 1.

Proof. This follows immediately from the previous two theorems since $\mathscr{A}$ being Abelian guarantees that the compression of $\mathscr{A}$ to $\mathfrak{M}^\perp$ is also Abelian and hence not strongly dense in $\mathscr{B}(\mathfrak{M}^\perp)$.

We can even say more about algebras generated by certain strictly cyclic operators.

Lemma 2. *If T is a strictly cyclic operator, and if $\sigma(T)$ is a singleton, then T has a unique maximal invariant subspace.*

Proof. The Corollary to Theorem 3 implies that T has a maximal invariant subspace. Suppose that T has two distinct maximal invariant subspaces $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Let $\mathfrak{M}=\mathfrak{M}_1\cap\mathfrak{M}_2$, and choose unit vectors $e_1\in\mathfrak{M}_1^\perp\cap\mathfrak{M}_2$ and $e_2\in\mathfrak{M}_1\cap\mathfrak{M}_2^\perp$. Since $\mathfrak{M}_1$ and $\mathfrak{M}_2$ have co-dimension 1 by the last corollary, $\mathfrak{M}^\perp=\vee\{e_1, e_2\}$. Let

$$T=\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

be the decomposition of T with respect to $\mathfrak{M}\oplus\mathfrak{M}^\perp$.

Now, T_3 has one-point spectrum since T does, and T_3 is strictly cyclic by Corollary 1 to Theorem 1. Since a strictly cyclic operator with one-point spectrum on a finite-dimensional space is similar to a unilateral shift, T_3 has a one-dimensional eigenspace. Thus e_1 or e_2 is not an eigenvector; suppose e_1 is not. Since $e_1\in\mathfrak{M}_2$, $Te_1\in\mathfrak{M}_2$; i.e.,

$$\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}\begin{pmatrix} 0 \\ e_1 \end{pmatrix} = (T_2e_1, T_3e_1)\in\mathfrak{M}_2.$$

So $T_3e_1\in\mathfrak{M}_2$, and $T_3e_1\in\mathfrak{M}$; i.e., $T_3e_1\in\mathfrak{M}_2\cap\mathfrak{M}^\perp$. But $\mathfrak{M}_2\cap\mathfrak{M}^\perp=\vee\{e_1\}$, which shows that e_1 is an eigenvector of T_3. This contradiction completes the proof.

Corollary. *If T is a strictly cyclic operator whose spectrum is a singleton, if $\mathfrak{M}\in\operatorname{Lat} T$ and is the unique maximal invariant subspace of T, and if $e\in\mathfrak{M}^\perp$, $e\neq 0$, then e is a strictly cyclic vector for $\mathscr{A}(T)$.*

Proof. Since e is not contained in any proper invariant subspace, it is a cyclic vector. By Corollary 2 to Theorem 1, e is a strictly cyclic vector.

Theorem 4. *Let T be a strictly cyclic operator with one-point spectrum. Let $\mathfrak{M}$ be its maximal invariant subspace, let $e\in\mathfrak{M}^\perp$, $e\neq 0$, and let $B\in\mathscr{A}(T)$. Then*

(i) *$Be\in\mathfrak{M}$ if and only if $\mathfrak{R}(B)\subset\mathfrak{M}$.**)

(ii) *x is a strictly cyclic vector if and only if $(x, e)\neq 0$.*

(iii) *B is invertible if and only if $(Be, e)\neq 0$.*

(iv) *Every operator in $\mathscr{A}(T)$ has one-point spectrum.*

Proof. By the Corollary to Lemma 2, e is a strictly cyclic vector. Let $\mathscr{A}=\mathscr{A}(T)$.

(i) If $\mathfrak{R}(B)\subset\mathfrak{M}$, trivially $Be\in\mathfrak{M}$. If $Be=x\in\mathfrak{M}$, since $\mathscr{A}e=\mathfrak{H}$, we have $\mathfrak{R}(B)=B\mathscr{A}e=\mathscr{A}Be=\mathscr{A}x\subset\mathfrak{M}$.

(ii) If x is a strictly cyclic vector, $x\notin\mathfrak{M}$, so $(x, e)\neq 0$. If $(x, e)\neq 0$, $\mathscr{A}x\not\subset\mathfrak{M}$, so $\mathscr{A}x=\mathfrak{H}$ (since every element of Lat $\mathscr{A}$ is contained in $\mathfrak{M}$).

(iii) If B is invertible, $\mathfrak{R}(B)\not\subset\mathfrak{M}$, so $(Be, e)\neq 0$ by (i).

*) $\mathfrak{R}$ denotes "range".

If $(Be, e) \neq 0$, $\mathfrak{R}(B) \not\subset \mathfrak{M}$. Thus $\mathfrak{R}(B)$ is dense in $\mathfrak{H}$, and $\mathfrak{R}(B)$ is invariant under $\mathfrak{A}$. Hence $\mathfrak{R}(B) = \mathfrak{H}$. By a theorem of LAMBERT [11, Lemma 3.1] every point in the spectrum of B is compression spectrum. Thus B must be invertible.

(iv) Let $Be = \alpha e + m$, where $m \in \mathfrak{M}$. If $\lambda \neq \alpha$, then $((B-\lambda)e, e) \neq 0$. Thus, $B-\lambda$ is invertible, so $\lambda \notin \sigma(B)$; i.e., $\sigma(B) = \{\alpha\}$.

LAMBERT [11] proved that a unilateral shift whose weights are p-summable and decrease monotonically to 0 is strictly cyclic. Since such an operator has Donoghue lattice, and since that property is trivially inherited, such an operator is hereditarily strictly cyclic. The class of hereditarily strictly cyclic operators is much wider than this. For example, if S is any quasinilpotent hereditarily strictly cyclic shift with Donoghue lattice, let $T = S \oplus (S+1)$. Since the full spectra of S and $S+1$ are disjoint, $\mathscr{A}(T) = \mathscr{A}(S) \oplus \mathscr{A}(S+1)$. (Consider the Riesz decomposition.) Thus if e is a strictly cyclic vector for $\mathscr{A}(S)$, $e \oplus e$ is obviously a strictly cyclic vector for $\mathscr{A}(T)$.

An example of HEDLUND [6] shows that even for $\mathscr{A} = \mathscr{A}(T)$ where T is a unilateral weighted shift, $\mathscr{A}$ being strictly cyclic does not in general imply that $\mathscr{A}$ is hereditarily strictly cyclic. Thus, in the theorems that follow, we cannot remove the "hereditary" part of the hypothesis.

In the case of an hereditarily strictly cyclic operator with one-point spectrum, we can describe its invariant subspaces in some detail. This is done in the following theorem, which generalizes the well-known fact that an operator on a finite-dimensional space is unicellular if and only if it is cyclic and has one-point spectrum (see [18, Theorem 4.7]). There are operators which are unicellular but have spectra containing more than one point — see [4].

Theorem 5. *Let T be strictly cyclic. If T is unicellular, then $\sigma(T)$ is one point. Conversely, if T is hereditarily strictly cyclic and if $\sigma(T)$ is one point, then T has Donoghue lattice* (i.e. there is an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ such that the non-trivial invariant subspaces of T are the subspaces $\mathfrak{M}_k = \bigvee_{j=k}^{\infty} e_j$ for positive integers k).

Proof. Every point in $\sigma(T^*)$ is an eigenvalue of T^* since every point in $\sigma(T)$ is compression spectrum. If T^* had two linearly independent eigenvectors, T would have two non-comparable invariant subspaces. Thus T unicellular implies that $\sigma(T)$ is a singleton.

Conversely, suppose that T is hereditarily strictly cyclic, and that $\sigma(T) = \{\lambda_0\}$. Let $S = T - \lambda_0$. Since $\operatorname{Lat} S = \operatorname{Lat} T$, it suffices to show that S has Donoghue lattice.

Let $\mathfrak{M}_0 = \mathfrak{H}$. If $\mathfrak{M}_k$ has been defined, let $\mathfrak{M}_{k+1}$ be the unique maximal invariant subspace of $S|\mathfrak{M}_k$. Since 0 is compression spectrum for $S|\mathfrak{M}_k$ for each k, $S(\mathfrak{M}_k) \subset$

$\subset\mathfrak{M}_{k+1}$. Choose a unit vector $e_0\in\mathfrak{M}_1^{\perp}$. Then e_0 is a strictly cyclic vector, and $S^n e_0\in\mathfrak{M}_n$.

Let $S^n e_0=e_n+m_{n+1}$ where $e_n\in\mathfrak{M}_{n+1}^{\perp}$ and $m_{n+1}\in\mathfrak{M}_{n+1}$. Since S is strictly cyclic, $\{S^n e_0: n=0, 1, 2, \ldots\}$ spans $\mathfrak{H}$. Hence, $\mathfrak{E}=\{e_n: n=0, 1, 2, \ldots\}$ is an orthogonal spanning set. Each $e_i\notin\mathfrak{M}_j$ if $j>i$, so $\mathfrak{E}\perp(\bigcap\mathfrak{M}_n)$; i.e., $\bigcap\mathfrak{M}_n=\{0\}$. This shows that Lat T contains a Donoghue lattice. To complete the proof, we must show that there are no other invariant subspaces.

Let $\mathfrak{M}$ be any invariant subspace of T, and let n be the largest index such that $\mathfrak{M}\subset\mathfrak{M}_n$. (Since $\mathfrak{M}\subset\mathfrak{M}_0$ and since $\bigcap\mathfrak{M}_i=\{0\}$, such an n exists). If $\mathfrak{M}\neq\mathfrak{M}_n$, $\mathfrak{M}_{n+1}$ is the unique maximal invariant subspace of $T|\mathfrak{M}_n$, and so $\mathfrak{M}\subset\mathfrak{M}_{n+1}$, a contradiction. So $\mathfrak{M}=\mathfrak{M}_n$, and we are done.

Shields' article [20] contains many of the known results about weighted shifts. It includes some discussion of strictly cyclic shifts. The invariant subspace lattice of every weighted shift obviously contains the Donoghue subspaces, but there may be other invariant subspaces. Shields defines a shift to be *strongly strictly cyclic* if its restriction to each of its Donoghue subspaces is strictly cyclic. This definition is somewhat weaker than hereditarily strictly cyclic. Shields proves [20, Prop. 38] that a quasinilpotent strongly strictly cyclic shift is unicellular. Although the statement of Theorem 5 does not include Shields' theorem, its proof obviously yields his result too. (The proof in [20] depends on calculations with the weights.)

Theorem 5 suggests that hereditarily strictly cyclic operators might serve as "generalized Jordan blocks", and that operators which inherit finite strict multiplicity may have a "Jordan form" in some sense. This is the case, and it is proven in Theorem 6 below. We require two lemmas.

Lemma 3. *Let T be a strictly cyclic operator whose spectrum is a singleton, and suppose that T inherits finite strict multiplicity. Then T is hereditarily strictly cyclic.*

Proof. Let $\sigma(t)=\{\lambda\}$ and let $S=T-\lambda$. It suffices to prove the theorem for S. By Lemma 2, S has a unique maximal invariant subspace $\mathfrak{M}_1$. As in the proof of Theorem 5, let e_0 be a unit vector orthogonal to $\mathfrak{M}_1$. Then e_0 is a strictly cyclic vector by Theorem 4, and $e_1=Se_0$ is in $\mathfrak{M}_1$ since 0 is compression spectrum for S. Since S is strictly cyclic, $\{S^n e_0: n=0, 1, 2, \ldots\}$ spans $\mathfrak{H}$, and so $\{S^n e_0: n=1, 2, 3, \ldots\}=\{S^n e_1: n=0, 1, 2, \ldots\}$ spans $\mathfrak{M}_1$; i.e. e_1 is a cyclic vector for $S|\mathfrak{M}_1$. Hence, by Corollary 2 to Theorem 1, e_1 is a strictly cyclic vector since $S|\mathfrak{M}_1$ has finite strict multiplicity.

We now proceed as we did in the proof of Theorem 5 to construct a sequence of invariant subspaces $\mathfrak{M}_n$ such that $S|\mathfrak{M}_n$ is strictly cyclic. At each step, $S|\mathfrak{M}_n$ strictly cyclic implies the existence of a maximal invariant subspace $\mathfrak{M}_{n+1}$, and then $S|\mathfrak{M}_{n+1}$ will be strictly cyclic. To complete the proof, we must show that $\bigcap\mathfrak{M}_n=$

$=\{0\}$, and that S has no other invariant subspaces. But this follows exactly as in the proof of Theorem 5.

Lemma 4. *Let $\{x, y\}$ be an FSM set for the operator T, and suppose that T inherits finite strict multiplicity, and that $\sigma(T)=\{0\}$. Let $\mathfrak{M}=\overline{\mathscr{A}(T)x}$ and $\mathfrak{N}=\overline{\mathscr{A}(T)y}$, and assume that $\mathfrak{M}$ is infinite-dimensional. Then either $\mathfrak{M}\cap\mathfrak{N}=\{0\}$ or there exists a finite-dimensional invariant subspace $\mathfrak{K}$ of T complementary to $\mathfrak{M}$ such that $\mathfrak{M}\vee\mathfrak{K}=\mathfrak{M}\vee\mathfrak{N}=\mathfrak{H}$.*

Remark. The assumption that $\mathfrak{M}$ is infinite-dimensional is for convenience. If both $\mathfrak{M}$ and $\mathfrak{N}$ are finite-dimensional, since $T|\mathfrak{M}$ will then be cyclic and nilpotent, the lemma reduces to a well-known finite-dimensional theorem (see [5, Theorem 1, 57]).

Proof. Note first that $\mathfrak{M}\vee\mathfrak{N}=\mathfrak{H}$ since every vector in $\mathfrak{H}$ has the form $Ax+By$ where $A, B\in\mathscr{A}(T)$. By Corollary 2 to Theorem 1 and by Lemma 3, $T|\mathfrak{M}$ is hereditarily strictly cyclic, and we may apply Theorem 5. So let $\mathfrak{M}_0, \mathfrak{M}_1, \mathfrak{M}_2, \ldots$ be the non-zero invariant subspaces of $T|\mathfrak{M}$ in decreasing order ($\mathfrak{M}_0=\mathfrak{M}$). If $\mathfrak{N}$ were finite-dimensional, then $\mathfrak{M}\cap\mathfrak{N}=\{0\}$ since $T|\mathfrak{M}$ has no finite-dimensional invariant subspaces. So assume that $\mathfrak{N}$ is infinite-dimensional and that the non-zero invariant subspaces of $T|\mathfrak{N}$ are $\mathfrak{N}_0, \mathfrak{N}_1, \mathfrak{N}_2, \ldots$ in decreasing order.

Now, if $\mathfrak{M}\cap\mathfrak{N}=\{0\}$, we are done. If not, $\mathfrak{M}\cap\mathfrak{N}=\mathfrak{M}_k=\mathfrak{N}_m$ for some k and m. Thus, $\mathfrak{M}\cap\mathfrak{N}$ has finite co-dimension α in $\mathfrak{H}$. We proceed by induction on α. If $\alpha=0$, then $\mathfrak{M}=\mathfrak{N}=\mathfrak{H}$, and $\mathfrak{K}=\{0\}$ does the trick. So assume true for $\alpha=n-1$, and consider $\alpha=n$.

Since $\mathfrak{M}\cap\mathfrak{N}=\mathfrak{N}_m$, if $\mathfrak{M}\cap\mathfrak{N}\not\subset\mathfrak{N}_1$, then $m=0$ and $\mathfrak{N}\subset\mathfrak{M}$, and again $\mathfrak{K}=\{0\}$ suffices. So assume $\mathfrak{M}\cap\mathfrak{N}\subset\mathfrak{N}_1$, and let y_1 be a unit vector in $\mathfrak{N}_1\ominus\mathfrak{N}_2$. Then y_1 is a cyclic vector for $T|\mathfrak{N}_1$. (In fact, y_1 is a strictly cyclic vector.) Thus, $\{x, y_1\}$ is an FSM set for $\mathfrak{M}\vee\mathfrak{N}_1$, and $\mathfrak{M}\cap\mathfrak{N}=\mathfrak{M}\cap\mathfrak{N}_1\subset\mathfrak{M}\vee\mathfrak{N}_1$. Moreover, the co-dimension of $\mathfrak{M}\cap\mathfrak{N}$ in $\mathfrak{M}\vee\mathfrak{N}_1$ is exactly $n-1$, and the inductive hypothesis applies. So choose a finite-dimensional invariant subspace $\mathfrak{K}_0$ of T complementary to such that $\mathfrak{M}\vee\mathfrak{K}_0=\mathfrak{M}\vee\mathfrak{N}_1$, and let $\mathfrak{K}=\{z\in\mathfrak{H}\colon Tz\in\mathfrak{K}_0\}$. We will show that $\mathfrak{K}$ has the desired properties.

First, if $z\in\mathfrak{M}$ and $z\neq0$, then $Tz\notin\mathfrak{K}_0$ since $T|\mathfrak{M}$ has no finite-dimensional invariant subspaces. Hence, since the co-dimension of $\mathfrak{M}$ in $\mathfrak{H}$ is finite, $\mathfrak{K}$ is finite-dimensional, and since 0 is the only point in the spectrum of T, $T|\mathfrak{K}$ is nilpotent. Thus, the dimension of $\mathfrak{K}$ is greater than the dimension of $\mathfrak{K}_0$ since $T\mathfrak{K}\subset\mathfrak{K}_0$. Hence, the co-dimension of $\mathfrak{M}\vee\mathfrak{K}$ is less than the co-dimension of $\mathfrak{M}\vee\mathfrak{K}_0$, i.e. $\mathfrak{M}\vee\mathfrak{K}$ must be all of $\mathfrak{H}$. Finally, $T\mathfrak{K}\subset\mathfrak{K}_0\subset\mathfrak{K}$, so $\mathfrak{K}$ is an invariant subspace of T. The proof is complete.

The purpose of Lemma 4 is of course Theorem 6 below. Lemma 4 yields Theorem 6 fairly easily. But first, define the operator T to be a *Jordan operator* if T has n complementary invariant subspaces $\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_n$ such that $\mathfrak{H}$ is the (not necessarily orthogonal) direct sum of the $\mathfrak{M}_i$'s, and such that either the matrix of $T|\mathfrak{M}_i$ is a (finite-dimensional) Jordan block, or $T|\mathfrak{M}_i$ has Donoghue lattice.

Theorem 6. *Let T be an operator on $\mathfrak{H}$ which inherits finite strict multiplicity, and whose spectrum is finite. Then T is a Jordan operator.*

Proof. Let $\sigma(T)=\{\lambda_1, \lambda_2, \dots, \lambda_k\}$. Then T has k complementary invariant subspaces $\mathfrak{H}_1, \mathfrak{H}_2, \dots, \mathfrak{H}_k$ whose span is all of $\mathfrak{H}$ such that $\sigma(T|\mathfrak{H}_i)=\{\lambda_i\}$. It suffices to prove the theorem for each $T|\mathfrak{H}_i$, so assume that $\mathfrak{H}_1=\mathfrak{H}$ and $\lambda_1=0$ (otherwise, consider $T-\lambda_1$).

We now proceed by induction on n the strict multiplicity of T. For $n=1$, Theorem 5 applies. So assume true for strict multiplicity $n-1$, and let $\{x_1, x_2, \dots, x_n\}$ be an FSM set for T. Let $\mathfrak{N}_i=\overline{\mathscr{A}(T)x_i}$. By the inductive hypothesis, the theorem holds on $\bigvee_{i=1}^{n-1} \mathfrak{N}_i$. Let

$$\bigvee_{i=1}^{n-1} \mathfrak{N}_i = \bigvee_{j=1}^{m} \mathfrak{M}_j,$$

where the $\mathfrak{M}_j$'s are mutually complementary invariant subspaces of T, where $T|\mathfrak{M}_j$ has Donoghue lattice for $j<m$, and where $\mathfrak{M}_m$ is finite-dimensional. (We are thus throwing all of the finite-dimensional invariant subspaces of T into $\mathfrak{M}_m$. By the Jordan Canonical Form Theorem, this is equivalent to the above definition of Jordan operator.)

Now, if $\mathfrak{N}_n$ is finite-dimensional, since $T|\mathfrak{M}_j$ has Donoghue lattice for $j<m$, $\mathfrak{N}_n$ is necessarily complementary to $\mathfrak{M}_j$ for $j<m$. In that case, replacing $\mathfrak{M}_m$ by $\mathfrak{M}_m \vee \mathfrak{N}_n$ does the trick. If $\mathfrak{N}_n$ is infinite-dimensional, Lemma 4 applies to $\mathfrak{N}_n \vee \mathfrak{M}_1$. (If $m=1$, $\mathfrak{N}_n$ is complementary to $\mathfrak{M}_1$, and we are done.) So let $\mathfrak{N}_n \vee \mathfrak{M}_1=\mathfrak{K} \vee \mathfrak{M}_1$, where $\mathfrak{K}$ and $\mathfrak{M}$ are complementary, and where $\mathfrak{K}$ is a finite-dimensional invariant subspace of T. Then $\mathfrak{K}$ must be complementary to $\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_{m-1}$, and replacing $\mathfrak{M}_m$ by $\mathfrak{M}_m \vee \mathfrak{K}$ comples the proof.

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A note on integral operators

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Let (X, m) be a separable σ-finite measure space which is not purely atomic (it may include some atoms). A bounded linear operator T on $L^2(X)$ is called an integral operator if there exists a measurable function k on $X\times X$ such that $(Tf)(x)=\int k(x,y)f(y)m(dy)$ almost everywhere. It is known ([7], p. 35) that every Hilbert—Schmidt operator is an integral operator. It is also known that there are integral operators which are not Hilbert—Schmidt or even compact. For example, if k is the characteristic function of the set $\bigcup_{n=0}^{\infty}([n, n+1]\times[n, n+1])$, the operator induced by k on $L^2(0,\infty)$ is a projection of infinite rank. (This example is in HALMOS [3].) However, KOROTKOV [6] proved that every operator unitarily equivalent to T is an integral operator if and only if T is a Hilbert—Schmidt operator. The purpose of this note is to give a proof of Korotkov's theorem which seems to be conceptually simpler than the original. Unlike the proof in [6], we do not use any results about Fourier series. Our techniques are more operator-theoretic.

We start by establishing notation. Let $\mathfrak{H}$ be a separable infinite-dimensional Hilbert space, and let $\mathscr{B}(\mathfrak{H})$ be the algebra of bounded operators on $\mathfrak{H}$. If $T\in\mathscr{B}(\mathfrak{H})$, and if $\mathfrak{M}$ is a (closed) subspace of $\mathfrak{H}$, then the *compression* of T to $\mathfrak{M}$ is the operator $PTP|\mathfrak{M}$, where P is the projection onto $\mathfrak{M}$. We will always assume that $\mathfrak{M}$ is a "half" of $\mathfrak{H}$, that is, both $\mathfrak{M}$ and $\mathfrak{M}^\perp$ are infinite-dimensional.

If K is a compact operator, then the sequence of *s-numbers* of K is the sequence $s_1\geqq s_2\geqq\ldots$ of nonzero eigenvalues of the compact positive operator $(K^*K)^{1/2}$, each repeated according to its multiplicity. A compact operator is called *Hilbert—Schmidt* if its sequence of s-numbers is square summable. For a detailed discussion of ideals, s-numbers and related concepts see pp. 25—27 of [7]. Here we need only the following fact: If two compact operators have the same sequence of s-numbers, then they must belong to the same two-sided ideals ([7, p. 26]).

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Lemma 1. *If K is a compact operator, then* $K=\begin{bmatrix} K_1 & S_2 \\ S_3 & S_4 \end{bmatrix}$, *where every* S_i *is a Hilbert—Schmidt operator. Consequently, there is a Hilbert—Schmidt operator* S *such that* $\begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}$ *is unitarily equivalent to* $K+S$.

Proof. Let $\{e_n\}$ be an orthonormal basis for the underlying Hilbert space, then $\|Ke_n\|\to 0$ and $\|K^*e_n\|\to 0$. Choose a subsequence $\{f_n\}$ of $\{e_n\}$ such that $\sum \|Kf_n\|^2<\infty$ and $\sum \|K^*f_n\|^2<\infty$. Let $\mathfrak{M}$ be the orthogonal complement of the span of $\{f_n\}$. (By passing to a subsequence of $\{f_n\}$, if necessary, we can assume that $\mathfrak{M}$ is infinite-dimensional.) The matrix of K relative to the decomposition $\mathfrak{M}\oplus\mathfrak{M}^\perp$ has the required form. The second assertion of the Lemma follows easily from the first.

Lemma 2. *Let $\mathfrak{A}$ be a linear space of compact operators, and assume that $\mathfrak{A}$ is closed under unitary equivalence and under compression and that it contains every Hilbert—Schmidt operator. Then $\mathfrak{A}$ is a two-sided ideal in $\mathcal{B}(\mathfrak{H})$.*

Proof. Assume that $K\in\mathfrak{A}$, and apply Lemma 1 to conclude that the operator $T=\begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}$ belongs to $\mathfrak{A}$. Each of the following operators is unitarily equivalent to T and hence belongs to $\mathfrak{A}$:

$$T_1=\begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix},\quad T_2=\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}=\frac{1}{2}\begin{pmatrix} K & K \\ K & K \end{pmatrix},$$

$$T_3=\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} K & K \\ K & K \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}=\frac{1}{2}\begin{pmatrix} K & -iK \\ iK & K \end{pmatrix}.$$

By taking an appropriate linear combination, we see that $\begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}$ is also in $\mathfrak{A}$, and so is the operator

$$\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}\begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}\begin{pmatrix} U^* & 0 \\ 0 & V^* \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ VKU^* & 0 \end{pmatrix}$$

for any unitary operators U and V. Since every operator can be written as a linear combination of four unitary operators ([1, p. 4]), the operator $\begin{pmatrix} 0 & 0 \\ AKB & 0 \end{pmatrix}$ belongs to $\mathfrak{A}$ for any operators A and B. By unitary equivalence, the following operator also belongs to $\mathfrak{A}$

$$\frac{1}{2}\begin{pmatrix} AKB & AKB \\ -AKB & -AKB \end{pmatrix}.$$

Consequently AKB belongs to $\mathfrak{A}$. Therefore $\mathfrak{A}$ is a two-sided ideal.

Lemma 3. *Let (X, m) be a separable σ-finite measure space and Y a Borel subset of X such that $L^2(Y)$ and $L^2(X\setminus Y)$ are both infinite dimensional, and let $\mathfrak{H}$ be a separable Hilbert space, $\mathfrak{M}$ a subspace of $\mathfrak{H}$ with* $\dim\mathfrak{M}=\dim\mathfrak{M}^\perp=\infty$, *$T$ an* *operator on $\mathfrak{H}$ and A the compression of T to $\mathfrak{M}$. If every operator on $L^2(X)$ which is unitarily equivalent to T is an integral operator, then every operator on $L^2(Y)$ which is unitarily equivalent to A is an integral operator.*

Proof. Let $V: \mathfrak{M} \to L^2(Y)$ be a unitary operator and let W be any unitary mapping $\mathfrak{M}^\perp$ onto $L^2(X \setminus Y)$, and let $U = V \oplus W$. Thus UTU^* is an integral operator on $L^2(X)$. If k is the kernel of the latter, then VAV^* is an integral operator whose kernel is the restriction of k to $Y \times Y$.

Lemma 4. *Let T be a bounded operator on $\mathfrak{H}$ such that UTU^* is an integral operator for every unitary operator U mapping $\mathfrak{H}$ onto $L^2(X)$. Then T is compact.*

Proof. First we show that every non-compact operator has a compression (to an infinite dimensional subspace) which equals the sum of a non-zero scalar and a Hilbert—Schmidt operator. Let T be a non-compact operator and let $T = T_1 + iT_2$ where T_1 and T_2 are self-adjoint. One of the operators T_1 and T_2 (say T_1) is not compact. Let E be the spectral measure of T_1. Then there is a real number $\lambda \neq 0$ such that $\dim (E(\Delta)\mathfrak{H}) = \infty$ for every open set Δ containing λ. Consequently, there is a compression $PT_1P|P\mathfrak{H}$ of T_1 (to an infinite dimensional subspace) which is equal to $\lambda +$ a Hilbert—Schmidt operator. Since PT_2P is self-adjoint, the same argument shows that there is an infinite dimensional projection $Q \leqq P$ such that $QT_2Q|Q\mathfrak{H}$ is a scalar+a Hilbert—Schmidt operator (this scalar may be zero). Thus

$$QTQ|Q\mathfrak{H} = \mu + S$$

where $\mu \neq 0$ and S is a Hilbert—Schmidt operator. (This proof is due to the referee.)

Let Y be a non-atomic "half" of X. If T is non-compact and is always integral on X, then by Lemma 3, the compression $\mu + S$ is always integral on Y. It follows that the identity on $L^2(Y)$ is an integral operator, which is impossible (see [5, problem 134]). So T must be compact.

For clarity of exposition, we will prove the main result first when $X = [0, 1]$.

Theorem 1. *Let T be a bounded operator on $\mathfrak{H}$. Then UTU^* is an integral operator for every unitary operator U mapping $\mathfrak{H}$ onto $L^2(0, 1)$ if and only if T is a Hilbert—Schmidt operator.*

Proof. The "if" part is easy. To prove the converse, let $\mathcal{J}$ be the set of all operators T on $\mathfrak{H}$ with the property that UTU^* is an integral operator for every unitary U: $\mathfrak{H} \to L^2(0, 1)$. It is easy to see that $\mathcal{J}$ is a linear space and is closed under unitary equivalence. It is also closed under compression since if $T \in \mathcal{J}$ and A is a compression of T, then A is always integral on $\left(0, \frac{1}{2}\right)$ and hence is always integral on $(0, 1)$. By Lemma 2, $\mathcal{J}$ is a two-sided ideal. Let $K \in \mathcal{J}$ and let $\{\lambda_n\}$ be the sequence of s-numbers of K. Since $\mathcal{J}$ is an ideal, every operator on $L^2(0, 1)$ with the same sequence of s-numbers is an integral operator. We will now construct an operator on $L^2(0, 1)$ with s-numbers $\{\lambda_n\}$.

Let $\{e_n\}$ be an orthonormal basis of $L^2(0, 1)$ consisting of unimodular function, that is $|e_n(x)|=1$. (For example, the usual exponentials $\exp(2\pi ikx)$, arranged in a sequence.) Let $\{\alpha_n\}$ be a sequence of positive numbers such that $\sum \alpha_n^2=1$, and let $\{E_n\}$ be a sequence of disjoint measurable subsets of (0, 1) whose union is (0, 1) and such that $m(E_n)=\alpha_n^2$. Let $\varphi_n=\alpha_n^{-1}\chi_n$, where χ_n is the characteristic function of E_n. Therefore $\{\varphi_n\}$ is an orthonormal set in $L^2(0, 1)$. Define an operator C on $L^2(0, 1)$ by the equations

$$C\varphi_n = \lambda_n e_n, \quad \text{and} \quad Cf = 0 \quad \text{if} \quad f\in\{\varphi_n\}^{\perp}.$$

It is easy to see that $C^*e_n=\lambda_n\varphi_n$ and $CC^*e_n=\lambda_n^2 e_n$, and so C is a compact operator whose sequence of s-numbers is $\{\lambda_n\}$.

By the foregoing, the operator C must be an integral operator. Let k be the kernel of C, so

$$(Cf)(x) = \int k(x, y)f(y)m(dy) \quad \text{a.e.}$$

By considering only functions in $L^2(E_n)$ for a fixed n, we see that

$$Cf = (f, \varphi_n)\lambda_n e_n \quad \text{for} \quad f\in L^2(E_n),$$

so

$$(Cf)(x) = \int \alpha_n^{-1}\lambda_n e_n(x)f(y)m(dy) \quad \text{for} \quad f\in L^2(E_n).$$

By the uniqueness of the kernel, we must have $k(x, y)=\alpha_n^{-1}\lambda_n e_n(x)$ when $y\in E_n$. For every $f\in L^2(0, 1)$, the function $|k(x, \cdot)f(\cdot)|$ must be integrable for almost every x. By taking $f=1$, we have $\int |k(x, y)|m(dy)<\infty$ for almost every x, so $\sum \alpha_n\lambda_n<\infty$. Since this is true for any (normalized) square-summable sequence $\{\alpha_n\}$, we must have $\{\lambda_n\}$ square-summable, and so K is a Hilbert—Schmidt operator.

Theorem 2. *Let (X, m) be a separable σ-finite measure space with no atoms, and let T be a bounded operator on $\mathfrak{H}$. Then UTU^* is an integral operator for every unitary operator U mapping $\mathfrak{H}$ onto $L^2(X)$ if and only if T is a Hilbert—Schmidt operator.*

Proof. This theorem is proved by slight modifications of the proof of Theorem 1. We only indicate the necessary changes.

As before, let $\mathscr{I}$ be the set of all operators T on $\mathfrak{H}$ with the property that UTU^* is an integral operator for every unitary U: $\mathfrak{H}\to L^2(X)$. Unlike the case $X=[0, 1]$, it is not immediately obvious that $\mathscr{I}$ is closed under compression. (If $T\in\mathscr{I}$ and A is a compression of T, then we only know that A is always integral on every half of X.) So we introduce the class $\mathscr{J}$ of all compressions of operators in $\mathscr{I}$, that is $A\in\mathscr{J}$ if and only if there exist operators B, C, D such that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\mathscr{I}$. It is obvious that $\mathscr{J}$ is a linear space, and is closed under compression and under unitary equivalence, and so it is a two-sided ideal by Lemma 2. In view of Lemma 1, we need only show that every operator in $\mathcal{J}$ is Hilbert—Schmidt. Let Y be a "half" of X which has finite measure and we may assume that $m(Y)=1$. If $K\in\mathcal{J}$, then as before, every operator on $L^2(Y)$ with the same s-numbers as K must be an integral operator. An examination of the remainder of the proof of Theorem 1 shows that it depends on the two properties of Y which we now list and prove.

(i) There exists an orthonormal basis of $L^2(Y)$ consisting of unimodular functions.

Proof. There is an isomorphism of the measure algebra of (Y, m) onto the measure algebra of the unit interval [4, p. 173]. This isomorphism induces a linear map V of the linear space of equivalence classes of measurable functions on $[0, 1]$ onto the space of equivalence classes of measurable functions on Y (see [2, pp. 252—254] for details). This map can be seen to carry the exponentials $(\exp 2\pi inx)$ into a basis of $L^2(Y)$ consisting of unimodular functions.

(ii) If $\{\alpha_n\}$ is a sequence of positive numbers such that $\sum\alpha_n=1$, then there is a sequence of disjoint measurable subsets of Y whose union is Y and such that $m(E_n)=\alpha_n$.

Proof. Again this follows immediately from the isomorphism of the measure algebras.

This ends the proof of Theorem 2.

Corollary. *The conclusion of Theorem 2 is valid if X contains atoms but is not purely atomic.*

Proof. Let Y be a half of X which contains no atoms, and let T be an operator which is always integral on X. Every compression of T is always integral on Y, hence is Hilbert—Schmidt by Theorem 2. Therefore T must be Hilbert—Schmidt by Lemma 1.

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A short proof of the fact that biholomorphic automorphisms of the unit ball in certain L^p spaces are linear

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1. As a consequence of his investigations on the Carathéodory and Kobayashi distances on domains in locally convex vector spaces, E. VESENTINI [1] proved that biholomorphic automorphisms of the unit ball*) of $L^1(\Omega, \mu)$ are all linear, whenever the underlying measure space (Ω, μ) is not a unique atom. In this paper we shall provide a quite different approach to the problem which applies to $L^p(\Omega, \mu)$ as well, for every $p \in [1, \infty)$.

Theorem. *Let (Ω, μ) be a measure space having two disjoint subsets Ω', Ω'' such that $0 < \mu(\Omega')$, $\mu(\Omega'') < \infty$. Then for any $p \in [1, \infty) \setminus \{2\}$, all biholomorphic automorphisms of the unit ball of $L^p(\Omega, \mu)$ are linear.*

Our method is based on a result of W. KAUP and H. UPMEIER [2] concerning $\operatorname{Aut} B(E)$ for general Banach spaces E. Here we present a direct proof of the theorem, which may have interest because of its extreme brevity. However, we remark that one can also determine the general algebraic form of an element from $\operatorname{Aut} B(L^2(\Omega, \mu))$ in a similar way.

2. First we prove a lemma. To this end, let E denote an arbitrarily fixed Banach space with norm $\|.\|$, E^* the dual of E endowed with the norm $\|.\|_*$.

Lemma. $\operatorname{Aut} B(E)$ *contains only linear mappings if and only if the relation*

(1) $\langle q(x, x), \varphi \rangle = -\overline{\langle c, \varphi \rangle}$ *for all* $x \in E, \varphi \in E^*$ *with* $\|x\| = \|\varphi\|_* = 1 = \langle x, \varphi \rangle$

entails $c = 0$ whenever $c \in E$ and q is a bilinear form from $E \times E$ into E.

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*) In general, if $B(E)$ denotes the open unit ball of a Banach space E then the biholomorphic automorphisms of $B(E)$ are defined as those one-to-one mappings of $B(E)$ onto itself whose Fréchet derivative exists at every point $x \in B(E)$ as an invertible operator. We shall denote the group formed by the biholomorphic automorphisms of $B(E)$ by $\operatorname{Aut} B(E)$.

Proof. According to [2, p. 131], there can be found a subspace V in E and a conjugate-linear mapping $v \mapsto q_v$ from V into the space of the (continuous) E-bilinear forms such that Aut (D) is generated by the group G_0 of the surjective linear isometries of E onto itself any by the images under the exponential map of the vector fields $\left(v+q_v(z, z)\right)\dfrac{\partial}{\partial z}$ $(v \in V)$. Thus, for Aut $B(E)=G_0$ it is necessary and sufficient that there exist a $c \in E \setminus \{0\}$ and a bilinear form $q\colon E \times E \to E$ such that the vector field $\left(c+q(z, z)\right)\dfrac{\partial}{\partial z}$ be tangent to $\partial B(E)$ (the boundary of $B(E)$), i.e.

$$\text{(2)} \qquad \operatorname{Re}\langle c+q(z, z), \psi\rangle = 0 \quad \text{whenever} \quad \|z\| = \|\psi\|_* = 1 = \langle z, \psi\rangle.$$

Suppose now that the vectors $c, x \in E$, $\varphi \in E^*$ and the E-bilinear form q satisfy $\|x\|=\|\varphi\|_*=1=\langle x, \varphi\rangle$ and (2). Then for all $\lambda \in \mathbf{C}$ with $|\lambda|=1$ we have $\|\lambda x\| = \|\bar{\lambda}\varphi\|_* = 1 = \langle \lambda x, \bar{\lambda}\varphi\rangle$ whence $0=\operatorname{Re}\langle c+q(\lambda x, \lambda x), \bar{\lambda}\varphi\rangle = \operatorname{Re}[\lambda(\overline{\langle c, \varphi\rangle}+\langle q(x, x), \varphi\rangle)]$. Therefore $\overline{\langle c, \varphi\rangle}+\langle q(x, x), \varphi\rangle=0$ which completes the proof of the Lemma.

3. Now we shall proceed to the proof of the Theorem. Henceforth let $p \in [1, \infty)$ be arbitrarily fixed and set $E \equiv L^p(\Omega, \mu)$. As usual we shall identity E^* with $L^{p/(p-1)}(\Omega, \mu)$ and the pairing operation with $\langle x, \varphi\rangle \equiv \int_\Omega x(\xi) \cdot \varphi(\xi)\, d\mu(\xi)$ (for all $x \in E$ and $\varphi \in E^*$), respectively.

For any $x \in E$, let x denote the function $\xi \mapsto x(\xi) \cdot |x(\xi)|^{p-2}$ (with the convention $0 \cdot 0^{p-2} \equiv 0$). Observe that here

$$\text{(3)} \qquad x^* \in E^*, \quad \|x^*\|_* = \|x\|^{p-1}, \quad \langle x, x^*\rangle = \|x\|^p \quad \text{for all} \quad x \in E.$$

Then assume that the function $x \in E$ and the E-bilinear form q satisfy (1). Applying (3) we see that

$$\langle q(x/\|x\|, x/\|x\|), (x/\|x\|)^*\rangle = -\overline{\langle c, (x/\|x\|)^*\rangle} \quad \text{for all} \quad x \in E \setminus \{0\},$$

that is

$$\text{(1')} \qquad \langle q(x, x), x^*\rangle = -\|x\|^2 \overline{\langle c, x^*\rangle} \quad \text{for all} \quad x \in E.$$

In particular, if F and G are any two disjoint subsets of Ω such that $0<\mu(F)$, $\mu(G)<\infty$ then

$$\int_\Omega q(1_F+\lambda \cdot 1_G, 1_F+\lambda \cdot 1_G)(1_F+\bar{\lambda}|\lambda|^{p-2} 1_G)\, d\mu =$$
$$= -\left(\mu(F)+|\lambda|^p \cdot \mu(G)\right)^{2/p} \int_\Omega \bar{c}(1_F+\lambda \cdot |\lambda|^{p-2} 1_G)\, d\mu$$

for all $\lambda \in \mathbf{C}$. (For any μ-measurable subset $H \subset \Omega$ of finite μ-measure, 1_H denotes the characteristic function of H, considered as an element in E.)

Thus, by setting

$$\alpha_0 \equiv \int_F q(1_F, 1_F)\, d\mu, \quad \alpha_1 \equiv \int_F [q(1_F, 1_G) + q(1_G, 1_F)]\, d\mu, \quad \alpha_2 \equiv \int_F q(1_G, 1_G)\, d\mu,$$

$$\beta_0 \equiv \int_G q(1_F, 1_F)\, d\mu, \quad \beta_1 \equiv \int_G [q(1_F, 1_G) + q(1_G, 1_F)]\, d\mu, \quad \beta_2 \equiv \int_G q(1_G, 1_G)\, d\mu,$$

$$\mu_1 \equiv \mu(F), \quad \mu_2 \equiv \mu(G), \quad \gamma_1 \equiv \int_F \bar{c}\, d\mu, \quad \gamma_2 \equiv \int_G \bar{c}\, d\mu$$

we obtain

$$\sum_{k=0}^{2} \alpha_k \lambda^k + \bar{\lambda} |\lambda|^{p-2} \cdot \sum_{k=0}^{2} \beta_k \lambda^k = -(\mu_1 + |\lambda|^p \mu_2)^{2/p} (\gamma_1 + \lambda \cdot |\lambda|^{p-2} \gamma_2)$$

for all $\lambda \in \mathbf{C}$. Therefore for any $\varrho > 0$ and $\vartheta \in \mathbf{C}$ with $|\vartheta| = 1$,

$$(\beta_0 \cdot \varrho^{p-1}) \vartheta^{-1} + (\alpha_0 + \beta_1 \cdot \varrho^p) + (\alpha_1 \cdot \varrho + \beta_2 \cdot \varrho^{p+1}) \vartheta + (\alpha_2 \cdot \varrho^2) \vartheta^2 =$$
$$= -(\mu_1 + \mu_2 \cdot \varrho^p)^{2/p} [\gamma_1 + (\gamma_2 \cdot \varrho^{p-1}) \vartheta].$$

In particular, we have

$$\alpha_0 + \beta_1 \cdot \varrho^p = -(\mu_1 + \mu_2 \cdot \varrho^p)^{2/p} \gamma_1 \quad \text{for all} \quad \varrho > 0.$$

Hence $-\mu_2^{2/p} \cdot \gamma_1 = \lim_{\varrho \uparrow \infty} [-(\mu_1 + \mu_2 \cdot \varrho^p)^{2/p} \varrho^{-2}] = \lim_{\varrho \uparrow \infty} (\alpha_0 + \beta_1 \cdot \varrho^p) \cdot \varrho^{-2}$. This is possible only if $p = 2$ or $\gamma_1 = 0$. Thus if $p \neq 2$ then by definition of γ_1 we have

$$(4) \qquad \int_F \bar{c}\, d\mu = 0 \quad \text{whenever} \quad 0 < \mu(G) < \infty \quad \text{for some} \quad G \subset \Omega \setminus F.$$

But (4) immediately implies $c = 0$ because of our assumption on the measure space (Ω, μ). Thus, by the Lemma, $B(E)$ admits in case $p \neq 2$ only linear biholomorphic automorphisms. Q.E.D.

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On the tensor product of weights on W^*-algebras

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1. Let φ and ψ be normal semifinite weights on the W^*-algebras $\mathcal{M}$ and $\mathcal{N}$, respectively. Using the Tomita—Takesaki theory ([13]) and the Pedersen—Takesaki theorem on the equality of weights ([10]), CONNES ([3], 1.1.3) (see also [9]) proved that there exists a unique normal semifinite weight $\varphi\bar{\otimes}\psi$ on $\mathcal{M}\bar{\otimes}\mathcal{N}$ such that

(1) $\quad a\in\mathfrak{M}_\varphi^+,\ b\in\mathfrak{M}_\psi^+ \Rightarrow a\otimes b\in\mathfrak{M}_{\varphi\bar{\otimes}\psi}^+$ and $(\varphi\bar{\otimes}\psi)(a\otimes b)=\varphi(a)\psi(b)$,

(2) $$\mathrm{s}(\varphi\bar{\otimes}\psi)=\mathrm{s}(\varphi)\bar{\otimes}\mathrm{s}(\psi),$$

(3) $\quad \sigma_t^{\varphi\bar{\otimes}\psi}(x\bar{\otimes}y)=\sigma_t^{\varphi}(x)\bar{\otimes}\sigma_t^{\psi}(y)$ for $t\in\mathbf{R}$, $x\in\mathrm{s}(\varphi)\mathcal{M}\mathrm{s}(\varphi)$, $y\in\mathrm{s}(\psi)\mathcal{N}\mathrm{s}(\psi)$.

Here and in the sequel we use the standard notations in the Tomita—Takesaki theory ([12], [13]). In particular, $\mathrm{s}(\varphi)$ is the support projection of φ and $\mathfrak{M}_\varphi^+=\{x\in\mathcal{M}^+;\ \varphi(x)<+\infty\}$. If φ is not faithful, then $\{\sigma_t^\varphi\}_{t\in\mathbf{R}}$ means, of course, the modular automorphism group associated with the restriction of φ to $\mathrm{s}(\varphi)\mathcal{M}\mathrm{s}(\varphi)$.

If φ and ψ are normal positive functionals, then condition (1) alone is sufficient to insure the uniqueness in the definition of $\varphi\bar{\otimes}\psi$. However, in the general case it is often difficult to check condition (3) above for some candidates for $\varphi\bar{\otimes}\psi$.

The aim of this Note is to offer alternative equivalent definitions for $\varphi\bar{\otimes}\psi$ and to prove some very natural properties of the tensor product of weights.

2. From the works of COMBES ([1]), HAAGERUP ([7]) and PEDERSEN and TAKESAKI ([10]) (see also [6]) we know that for every normal weight φ on $\mathcal{M}$ there exists a family $\{\varphi_i\}_{i\in I}$ of normal positive functionals on $\mathcal{M}$ such that $\varphi=\sum_{i\in I}\varphi_i$, i.e.;

(4) $$\varphi(x)=\sum_{i\in I}\varphi_i(x)\quad \textit{for all}\quad x\in\mathcal{M}^+.$$

In particular, there is an increasing net $\{\varphi_i\}_{i\in I}$ of normal positive functionals on $\mathcal{M}$ such that $\varphi_i\uparrow\varphi$, i.e.:

(5) $$\varphi(x)=\sup_i\varphi_i(x)=\lim_i\varphi_i(x)\quad \textit{for all}\quad x\in\mathcal{M}^+.$$

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On the other hand, and this is the main technical tool we shall use, from the recent work of Connes ([4]) it follows that

(6) *if φ is a normal semifinite weight on $\mathcal{M}$ and $\{\varphi_i\}_{i\in I}$ is an increasing net of normal weights on $\mathcal{M}$ such that $\varphi_i\uparrow\varphi$, then*

$$\sigma_t^{\varphi_i}(x)\xrightarrow{s}\sigma_t^{\varphi}(x)\quad (t\in\mathbf{R})$$

for every $x\in\bigcup_{i\in I} s(\varphi_i)\mathcal{M}s(\varphi_i)$.

Here $\xrightarrow{s}$ means convergence in the ultra-strong topology on $\mathcal{M}$ of some section $\{i\in I\colon i\geqq i_0\}$ of the net involved.

Finally, from the proof of ([10], Lemma 5.2) it is easy to infer the following improvement of ([10], Lemma 5.2):

(7) *if φ_1, φ_2 are normal semifinite weights on $\mathcal{M}$ such that $s(\varphi_1)\leqq s(\varphi_2)$ and there exists an s-dense σ^{φ_2}-invariant $*$-subalgebra $\mathcal{A}$ of $\mathfrak{M}_{\varphi_2}$ such that*

$$\varphi_1(a^*a)\leqq\varphi_2(a^*a)\quad \text{for all}\quad a\in\mathcal{A},$$

then $\varphi_1\leqq\varphi_2$, i.e. $\varphi_1(x)\leqq\varphi_2(x)$ for all $x\in\mathcal{M}^+$.

In all this paper $\mathcal{M}$ and $\mathcal{N}$ will denote two W^*-algebras.

3. Lemma. *Let φ_1, φ_2 be normal semifinite weights on $\mathcal{M}$ and ψ a normal semifinite weight on $\mathcal{N}$. If $\varphi_1\leqq\varphi_2$, then $\varphi_1\bar{\otimes}\psi\leqq\varphi_2\bar{\otimes}\psi$.*

Proof. If $\varphi_1\leqq\varphi_2$, then $s(\varphi_1)\leqq s(\varphi_2)$, whence, by (2), $s(\varphi_1\bar{\otimes}\psi)=s(\varphi_1)\bar{\otimes}s(\psi)\leqq$ $\leqq s(\varphi_2)\bar{\otimes}s(\psi)=s(\varphi_2\bar{\otimes}\psi)$. Moreover, by (1) and (3), the algebraic tensor product $\mathcal{A}=\mathfrak{M}_{\varphi_2}\otimes\mathfrak{M}_\psi$ is an s-dense $\sigma^{\varphi_2\bar{\otimes}\psi}$-invariant $*$-subalgebra of $\mathfrak{M}_{\varphi_2\bar{\otimes}\psi}$. Since $\varphi_1\leqq\varphi_2$ are positive linear functionals on the $*$-algebra $\mathfrak{M}_{\varphi_2}$ and $\psi\geqq 0$ on the $*$-algebra $\mathfrak{M}_\psi$, it follows that $\varphi_1\otimes\psi\leqq\varphi_2\otimes\psi$ on the $*$-algebra $\mathcal{A}$. Thus $\varphi_1\bar{\otimes}\psi\leqq\varphi_2\bar{\otimes}\psi$, by (7).

4. Theorem. *Let φ, ψ be normal semifinite weights and $\{\varphi_i\}_{i\in I}$, $\{\psi_j\}_{j\in J}$ be increasing nets of normal weights on $\mathcal{M}$, $\mathcal{N}$, respectively. If $\varphi_i\uparrow\varphi$ and $\psi_j\uparrow\psi$, then $\varphi_i\bar{\otimes}\psi_j\uparrow\varphi\bar{\otimes}\psi$.*

Proof. By Lemma 3, $\{\varphi_i\bar{\otimes}\psi_j\}_{i\in I, j\in J}$ is an increasing net of normal weights on $\mathcal{M}\bar{\otimes}\mathcal{N}$ and $\varphi_i\bar{\otimes}\psi_j\leqq\varphi\bar{\otimes}\psi$ for all $i\in I$, $j\in J$. Consequently, the formula

$$\omega(z)=\sup_{ij}(\varphi_i\bar{\otimes}\psi_j)(z)=\lim_{ij}(\varphi_i\bar{\otimes}\psi_j)(z),\quad (z\in(\mathcal{M}\bar{\otimes}\mathcal{N})^+)$$

defines a normal semifinite weight ω on $\mathcal{M}\bar{\otimes}\mathcal{N}$. For $a\in\mathfrak{M}_\varphi^+$, $b\in\mathfrak{M}_\psi^+$, we have

$$\omega(a\bar{\otimes}b)=\sup_{ij}\varphi_i(a)\psi_j(b)=\sup_i\varphi_i(a)\sup_j\psi_j(b)=\varphi(a)\psi(b).$$

On the other hand, it is easy to see that $\mathbf{s}(\varphi_i)\uparrow\mathbf{s}(\varphi)$, $\mathbf{s}(\psi_j)\uparrow\mathbf{s}(\psi)$ and $\mathbf{s}(\varphi_i\bar{\otimes}\psi_j)\uparrow\mathbf{s}(\omega)$, hence

$$\mathbf{s}(\omega) = \mathbf{s}(\varphi)\bar{\otimes}\mathbf{s}(\psi).$$

Finally, by the result (6), for $t\in\mathbf{R}$, $x\in\bigcup_{i\in I}\mathbf{s}(\varphi_i)\mathcal{M}\mathbf{s}(\varphi_i)$, $y\in\bigcup_{j\in J}\mathbf{s}(\psi_j)\mathcal{N}\mathbf{s}(\psi_j)$, we have

$$\sigma_t^{\varphi_i}(x)\xrightarrow{s}\sigma_t^{\varphi}(x),\quad \sigma_t^{\psi_j}(y)\xrightarrow{s}\sigma_t^{\psi}(y)$$

and

$$\sigma_t^{\varphi_i\bar{\otimes}\psi_j}(x\bar{\otimes}y)\xrightarrow{s}\sigma_t^{\omega}(x\bar{\otimes}y).$$

Hence,

$$\sigma_t^{\omega}(x\bar{\otimes}y) = \sigma_t^{\varphi}(x)\bar{\otimes}\sigma_t^{\psi}(y).$$

Since $\mathbf{s}(\varphi_i)\uparrow\mathbf{s}(\varphi)$, $\mathbf{s}(\psi_j)\uparrow\mathbf{s}(\psi)$, the above equality still holds for $x\in\mathbf{s}(\varphi)\mathcal{M}\mathbf{s}(\varphi)$, $y\in\mathbf{s}(\psi)\mathcal{N}\mathbf{s}(\psi)$.

Thus, ω satisfies all conditions (1), (2), (3) which define $\varphi\bar{\otimes}\psi$. Consequently $\omega=\varphi\bar{\otimes}\psi$, i.e. $\varphi_i\bar{\otimes}\psi_j\uparrow\varphi\bar{\otimes}\psi$.

5. In particular if the φ_i's and the ψ_j's are normal positive functionals such that $\varphi_i\uparrow\varphi$, $\psi_j\uparrow\psi$, then

$$(8)\qquad (\varphi\bar{\otimes}\psi)(z) = \sup_{ij}(\varphi_i\bar{\otimes}\psi_j)(z)\quad \left(z\in(\mathcal{M}\bar{\otimes}\mathcal{N})^+\right)$$

is an alternative equivalent definition of the weight $\varphi\bar{\otimes}\psi$, independent of the choice of the families $\{\varphi_i\}$, $\{\psi_j\}$, whose existence is guaranteed by (5).

6. As a first application we obtain the distributivity of the tensor product with respect to addition:

Corollary. *Let φ_1, φ_2 be normal semifinite weights on $\mathcal{M}$ such that $\varphi_1+\varphi_2$ is semifinite and ψ is a normal semifinite weight on $\mathcal{N}$. Then*

$$(\varphi_1+\varphi_2)\bar{\otimes}\psi = \varphi_1\bar{\otimes}\psi+\varphi_2\bar{\otimes}\psi.$$

Proof. Let $\{\psi_j\}$ be an increasing net of normal positive functionals on $\mathcal{N}$ such that $\psi_j\uparrow\psi$.

Assume that φ_1, φ_2 are normal positive functionals. Since the distributivity property is obvious for normal positive functionals, by Theorem 4 we obtain

$$(\varphi_1+\varphi_2)\bar{\otimes}\psi = \sup_j(\varphi_1+\varphi_2)\bar{\otimes}\psi_j = \sup_j\varphi_1\bar{\otimes}\psi_j+\sup_j\varphi_2\bar{\otimes}\psi_j = \varphi_1\bar{\otimes}\psi+\varphi_2\bar{\otimes}\psi.$$

Now, in the general case, let $\{\varphi_{1i}\}$, $\{\varphi_{2k}\}$ be increasing nets of normal positive functionals on $\mathcal{M}$ such that $\varphi_{1i}\uparrow\varphi_1$, $\varphi_{2k}\uparrow\varphi_2$. It is then obvious that $\varphi_{1i}+\varphi_{2k}\uparrow\varphi_1+\varphi_2$. Using Theorem 4 and the first part of the proof, we obtain

$$(\varphi_1+\varphi_2)\bar{\otimes}\psi = \sup_{ik}(\varphi_{1i}+\varphi_{2k})\bar{\otimes}\psi = \sup_{ik}(\varphi_{1i}\bar{\otimes}\psi+\varphi_{2k}\bar{\otimes}\psi) =$$
$$= \sup_i\varphi_{1i}\bar{\otimes}\psi+\sup_k\varphi_{2k}\bar{\otimes}\psi = \varphi_1\bar{\otimes}\psi+\varphi_2\bar{\otimes}\psi.$$

7. If $\varphi=\sum_i \varphi_i$ and $\psi=\sum_j \psi_j$, then from Corollary 6 and Theorem 4 it follows that

(9) $$(\varphi\bar{\otimes}\psi)(z)=\sum_{ij}(\varphi_i\bar{\otimes}\psi_j)(z)\quad (z\in(\mathscr{M}\bar{\otimes}\mathscr{N})^+).$$

In particular if the φ_i's and the ψ_j's are normal positive functionals, then the above relation gives another alternative equivalent definition of $\varphi\bar{\otimes}\psi$, independent of the choice of the families $\{\varphi_i\}$ and $\{\psi_j\}$, whose existence is guaranteed by (4).

The weight φ is called strictly semifinite ([2]) if there exists a family $\{\varphi_i\}$ of normal positive functionals with mutually orthogonal supports such that $\varphi=\sum_i \varphi_i$. If both φ and ψ are strictly semifinite, then, by (9), $\varphi\bar{\otimes}\psi$ is again strictly semifinite. This result is originally due to Combes ([2]).

Other particular cases of (8) and (9) are mentioned in ([11], 0.1.2).

8. Another application concerns the relation between the tensor product and the balanced weight. Let us recall ([3], 1.2.2) that if φ_1, φ_2 are normal semifinite weights on $\mathscr{M}$, then the balanced weight $\theta(\varphi_1,\varphi_2)$ on the W^*-algebra $\mathrm{Mat}_2(\mathscr{M})\cong\mathscr{M}\otimes\mathrm{Mat}_2(\mathbf{C})$ of 2 by 2 matrices over $\mathscr{M}$ is defined by

$$\theta(\varphi_1,\varphi_2)\begin{pmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}=\varphi_1(x_{11})+\varphi_2(x_{22}).$$

Now let ψ be a normal semifinite weight on $\mathscr{N}$. Then $\theta(\varphi_1,\varphi_2)\bar{\otimes}\psi$ and $\theta(\varphi_1\bar{\otimes}\psi,\varphi_2\bar{\otimes}\psi)$ are both normal semifinite weights on the W^*-algebra

$$\mathrm{Mat}_2(\mathscr{M})\bar{\otimes}\mathscr{N}\cong\mathscr{M}\bar{\otimes}\mathscr{N}\otimes\mathrm{Mat}_2(\mathbf{C})\cong\mathrm{Mat}_2(\mathscr{M}\bar{\otimes}\mathscr{N})$$

and we have the following

Corollary. $\theta(\varphi_1\bar{\otimes}\psi,\varphi_2\bar{\otimes}\psi)=\theta(\varphi_1,\varphi_2)\bar{\otimes}\psi$.

Proof. It is obvious that if $\varphi_{1i}\uparrow\varphi_1$ and $\varphi_{2k}\uparrow\varphi_2$, then $\theta(\varphi_{1i},\varphi_{2k})\uparrow\theta(\varphi_1,\varphi_2)$. Also, the stated equality is obvious for normal positive functionals. Thus the corollary follows using (5) and Theorem 4.

9. Consider again the balanced weight $\theta(\varphi_1,\varphi_2)$ and assume that $\mathrm{s}(\varphi_2)\leqq\mathrm{s}(\varphi_1)$. Then the Connes cocycle ([3], 1.2.2) $u_t=[D\varphi_2:D\varphi_1]_t$, $(t\in\mathbf{R})$, is defined by the equality

$$\sigma_t^{\theta(\varphi_1,\varphi_2)}\begin{pmatrix}0 & 0\\ \mathrm{s}(\varphi_2) & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\ u_t & 0\end{pmatrix}\quad (t\in\mathbf{R}).$$

Thus, using Corollary 8, for $v_t=[D(\varphi_2\bar{\otimes}\psi):D(\varphi_1\bar{\otimes}\psi)]_t$ we get

$$\begin{pmatrix}0 & 0\\ v_t & 0\end{pmatrix}=\sigma_t^{\theta(\varphi_1\bar{\otimes}\psi,\varphi_2\bar{\otimes}\psi)}\begin{pmatrix}0 & 0\\ \mathrm{s}(\varphi_2\bar{\otimes}\psi) & 0\end{pmatrix}=\sigma_t^{\theta(\varphi_1,\varphi_2)\bar{\otimes}\psi}\begin{pmatrix}0 & 0\\ \mathrm{s}(\varphi_2)\otimes\mathrm{s}(\psi) & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\ u_t\otimes\mathrm{s}(\psi) & 0\end{pmatrix}.$$

Consequently,

$$[D(\varphi_2\bar{\otimes}\psi):D(\varphi_1\bar{\otimes}\psi)]_t=[D\varphi_2:D\varphi_1]_t\bar{\otimes}\mathrm{s}(\psi).$$

Using this equality and the chain rule for the Connes cocycle ([3], 1.2.3), we obtain the following

Corollary. *Let φ_1, φ_2 be normal semifinite weights on $\mathcal{M}$ with* $\mathbf{s}(\varphi_2) \leqq \mathbf{s}(\varphi_1)$ *and let ψ_1, ψ_2 be normal semifinite weights on $\mathcal{N}$ with* $\mathbf{s}(\psi_2) \leqq \mathbf{s}(\psi_1)$. *Then*

$$[D(\varphi_2 \bar{\otimes} \psi_2) \colon D(\varphi_1 \bar{\otimes} \psi_1)]_t = [D\varphi_2 \colon D\varphi_1]_t \bar{\otimes} [D\psi_2 \colon D\psi_1]_t \quad (t \in \mathbf{R}).$$

This result is stated by Digernes ([5], 2.4), where the proposed proof consists of checking the *KMS* conditions insuring the uniqueness of the Connes' cocycle ([5], 2.2), but only for decomposable elements

$$z_1 \in (\mathfrak{N}^*_{\varphi_2} \cap \mathfrak{N}_{\varphi_1}) \otimes (\mathfrak{N}^*_{\psi_2} \cap \mathfrak{N}_{\psi_1}), \quad z_2 \in (\mathfrak{N}^*_{\varphi_1} \cap \mathfrak{N}_{\varphi_2}) \otimes (\mathfrak{N}^*_{\psi_1} \cap \mathfrak{N}_{\psi_2}).$$

However, it is not obvious *à priori* that this entails the *KMS* condition for all

$$z_1 \in \mathfrak{N}^*_{\varphi_2 \bar{\otimes} \psi_2} \cap \mathfrak{N}_{\varphi_1 \bar{\otimes} \psi_1}, \quad z_2 \in \mathfrak{N}^*_{\varphi_1 \bar{\otimes} \psi_1} \cap \mathfrak{N}_{\varphi_2 \bar{\otimes} \psi_2},$$

which is the real requirement for the uniqueness.

On the other hand, if $\psi_1 = \psi_2 = \psi$, then *using* Corollary 8 it is easy to show that for the S-operators ([5], (2.6)) we have

$$S_{\varphi_2 \bar{\otimes} \psi, \varphi_1 \bar{\otimes} \psi} = S_{\varphi_2, \varphi_1} \bar{\otimes} S_\psi .$$

Once this equality is obtained, the proof in ([5], 2.4) holds indeed.

10. For every normal semifinite weight φ on $\mathcal{M}$ and every positive self-adjoint operator A affiliated with the centralizer $\mathcal{M}_\varphi$ of φ there exists a unique normal semifinite weight φ_A on $\mathcal{M}$ such that $[D\varphi_A \colon D\varphi]_t = A^{it}$, $t \in \mathbf{R}$, ([10]). From Corollary 9 we infer the following result, originally obtained by Katayama ([9]):

Corollary. *Let φ, ψ be normal semifinite weights on $\mathcal{M}$, $\mathcal{N}$, respectively, and let A, B be positive self-adjoint operators affiliated to $\mathcal{M}_\varphi$, $\mathcal{N}_\psi$, respectively. Then $A \bar{\otimes} B$ is a positive self-adjoint operator affiliated to $(\mathcal{M} \bar{\otimes} \mathcal{N})_{\varphi \bar{\otimes} \psi}$ and*

$$(\varphi \bar{\otimes} \psi)_{A \bar{\otimes} B} = \varphi_A \bar{\otimes} \psi_B .$$

11. Arguing as in the proof of Corollaries 6 and 8, with the help of (5) and Theorem 4 we obtain:

Corollary. *Let φ, ψ be normal semifinite weights on M, N, respectively, and let $\pi \colon \mathcal{M}_1 \to \mathcal{M}$; $\varrho \colon \mathcal{N}_1 \to \mathcal{N}$ be normal completely positive linear maps. If the weights $\varphi \circ \pi$, $\psi \circ \varrho$ are semifinite, then*

$$(\varphi \bar{\otimes} \psi) \circ (\pi \bar{\otimes} \varrho) = (\varphi \circ \pi) \bar{\otimes} (\psi \circ \varrho).$$

12. A final application concerns some operator valued weights ([8]) called Fubini mappings ([14]). For every normal semifinite weight ψ on $\mathcal{N}$ there is a unique normal semifinite operator valued weight $E^\psi_{\mathcal{M}}$ defined on $(\mathcal{M} \bar{\otimes} \mathcal{N})^+$ with values in the extended positive part ([8]) $\bar{\mathcal{M}}^+$ of $\mathcal{M}$, such that

$$(10) \qquad \varphi(E^{\psi}_{\mathcal{M}}(z)) = (\varphi \bar{\otimes} \psi)(z) \quad (z \in (\mathcal{M} \bar{\otimes} \mathcal{N})^+)$$

for every normal positive functional φ on $\mathcal{M}$ (cf. also [11], 0.1.6). From Theorem 4 it follows that:

Corollary. *If ψ, ψ_j are normal semifinite weights on $\mathcal{N}$ and $\psi_j \uparrow \psi$, then*

$$E^{\psi}_{\mathcal{M}}(z) = \sup_j E^{\psi_j}_{\mathcal{M}}(z) \quad (z \in (\mathcal{M} \bar{\otimes} \mathcal{N})^+).$$

Also, the equality (10) extends to any normal semifinite weight φ on $\mathcal{M}$.

Actually, the operator valued weight $E^{\psi}_{\mathcal{M}}$ is nothing but the tensor product operator valued weight $\iota_{\mathcal{M}} \bar{\otimes} \psi$ ([8]), where $\iota_{\mathcal{M}}$ stands for the identity mapping on $\mathcal{M}$. We remark that Corollary 12 can be extended to an arbitrary normal semifinite operator valued weight instead of $\iota_{\mathcal{M}}$. Moreover, Theorem 4 can be extended to operator valued weights.

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Almost all algebras with triply transitive automorphism groups are functionally complete

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1. Introduction

The present work is a continuation of a series of results on the functional completeness of algebras with high symmetry. It is also a contribution to the solution of Problem 20 in GRÄTZER [4]. WERNER [14] proved that every finite algebra $\langle A; t\rangle$ where t is Pixley's ternary discriminator function on A is functionally complete. Recently, FRIED and PIXLEY [2] showed that for $3\leqq|A|<\aleph_0$, the algebra $\langle A; d\rangle$ with d the dual discriminator function on A is also functionally complete. A considerable generalization of these results was found by CSÁKÁNY [1] who proved that, up to equivalence, except for six algebras every non-trivial finite algebra whose automorphism group is the full symmetric group is functionally complete. Our contribution to this topic is the following theorem: an at least four element non-trivial finite algebra whose automorphism group is triply transitive is either functionally complete or equivalent to an affine space over the two element field. In the proof our main tool is Rosenberg's completeness criterium which provides a powerful method for checking functional completeness.

There is an interesting phenomenon which is worth being referred to in connection with our result. This is the connection of our theorem to the Słupecki type criteria for completeness due to SALOMAA [10] and SCHOFIELD [11], saying that any set F of functions over a finite set A ($|A|\geqq 4$) which contains a function satisfying the Słupecki condition and a triply transitive group of permutations of A, generates the set of all functions on A, except for the case when all functions in F are linear in each variable, relative to some representation of A as a vector space over the two element field. Making use of Rosenberg's completeness criterium, this theorem can be further improved to doubly transitive permutation groups and then the excep-

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tions are exactly those sets of functions that are linear with respect to a vector space over an arbitrary prime field (see ROSENBERG [8], also KNOEBEL [5]). It would be worthwhile to find out whether our theorem could be generalized for finite algebras with doubly transitive automorphism groups.

2. Preliminaries

Let A be a non-empty set. By an operation we always mean a finitary operation. The set of n-ary operations on A will be denoted by $\mathbf{O}_A^{(n)}$ $(n \geqq 1)$. Furthermore, we set $\mathbf{O}_A = \bigcup_{n=1}^{\infty} \mathbf{O}_A^{(n)}$. An operation $f \in \mathbf{O}_A^{(n)}$ is said to depend on its i'th variable $(1 \leqq i \leqq n)$ if there exist elements $a_1, \ldots, a_n, a_i' (\neq a_i)$ in A such that

$$f(a_1, \ldots, a_n) \neq f(a_1, \ldots, a_{i-1}, a_i', a_{i+1}, \ldots, a_n).$$

f is called *essentially k-ary*, if it depends on exactly k of its variables. f is termed *idempotent*, if for every $a \in A$, we have $f(a, \ldots, a) = a$. f is called *non-trivial* if it is not a projection.

We adopt the terminology of [4] except that polynomials and algebraic functions are called *term functions* and *polynomial functions*, respectively. Accordingly, the set of polynomial functions and the set of term functions of an algebra $\mathfrak{A}$ are denoted by $\mathbf{P}(\mathfrak{A})$ and $\mathbf{T}(\mathfrak{A})$, respectively. Two algebras (with a common base set) are said to be *equivalent* if they have the same term functions. By a *clone* we mean a subset $\mathbf{C}$ of $\mathbf{O}_A$ for some set $A (\neq \emptyset)$, which contains the projections and is closed with respect to superposition. In particular, both $\mathbf{P}(\mathfrak{A})$ and $\mathbf{T}(\mathfrak{A})$ are clones for any algebra $\mathfrak{A}$. An algebra $\mathfrak{A} = \langle A; F \rangle$ is called *functionally complete* if $\mathbf{P}(\mathfrak{A}) = \mathbf{O}_A$ and *trivial* if $\mathbf{T}(\mathfrak{A})$ contains projections only. An algebra is said to be *idempotent* if its fundamental operations (and hence all term functions) are idempotent. For a field K, an *affine space over* K is defined to be an algebra $\langle A; I \rangle$ where I is the set of all idempotent term functions of a vector space over K with base set A.

The automorphism group of an algebra $\mathfrak{A}$ is denoted by Aut $\mathfrak{A}$. If Aut $\mathfrak{A}$ is the full symmetric group then $\mathfrak{A}$ is called *homogeneous*.

Now we are going to formulate Rosenberg's Theorem [6, 7] which is our main tool in proving our theorem. First, however, we need some further definitions.

Let $A (\neq \emptyset)$ be a finite set, $k, n \geqq 1$, $f \in \mathbf{O}_A^{(n)}$ and $\varrho \subseteq A^k$ an arbitrary k-ary relation. f is said to *preserve* ϱ if ϱ is a subalgebra of the k'th direct power of the algebra $\langle A; f \rangle$; in other words, f preserves ϱ if for any $n \times k$ matrix with entries in A, whose rows belong to ϱ, the row of column values of f also belongs to ϱ. It is easy to verify that the set of operations preserving a relation ϱ forms a clone, which will be denoted by Pol ϱ.

A k-ary relation ϱ on A is called *central* if $\varrho \neq A^k$ and there exists a non-void proper subset C of A such that

(a) $\langle a_1, \ldots, a_k \rangle \in \varrho$ whenever at least one $a_j \in C$ $(1 \leqq j \leqq k)$;

(b) $\langle a_1, \ldots, a_k \rangle \in \varrho$ implies $\langle a_{1\pi}, \ldots, a_{k\pi} \rangle \in \varrho$ for every permutation π of the indices $1, \ldots, k$;

(c) $\langle a_1, \ldots, a_k \rangle \in \varrho$ if $a_i = a_j$ for some $i \neq j$ $(1 \leqq i, j \leqq k)$.

Let $2 < k \leqq |A|$ and $m \geqq 1$. A family $T = \{\Theta_1, \ldots, \Theta_m\}$ of equivalence relations on A is termed *k-regular* if

(d) each Θ_j has k equivalence classes $(j = 1, \ldots, m)$;

(e) the intersection $\bigcap_{i=1}^{m} \varepsilon_i$ of arbitrary equivalence classes ε_i of Θ_i $(i = 1, \ldots, m)$ is non-empty.

The relation ϱ *determined by* T consists of all $\langle a_1, \ldots, a_k \rangle \in A^k$ having the property that for each j $(j = 1, \ldots, m)$ at least two elements among $a_1, \ldots, a_k$ are equivalent modulo Θ_j. Notice that ϱ has properties (b) and (c).

We shall use the following version of Rosenberg's Theorem (see [9]):

Theorem. (ROSENBERG [6, 7]) *For a non-empty finite set A,* Pol ϱ *is a maximal subclone of $\mathbf{O}_A$, provided ϱ is one of the following relations on A:*

(α) *a bounded partial order;*

(β) *a binary relation $\{\langle a, a\pi \rangle | a \in A\}$ where π is a permutation of A with $|A|/p$ cycles of the same prime length p;*

(γ) *a quaternary relation $\{\langle a_1, a_2, a_3, a_4 \rangle \in A^4 | a_1 + a_2 = a_3 + a_4\}$ where $\langle A; + \rangle$ is an elementary abelian p-group (p is a prime number);*

(δ) *a non-trivial equivalence relation;*

(ε) *a central relation;*

(ζ) *a relation determined by a k-regular family of equivalence relations on A $(k \geqq 3)$.*

Moreover, every proper subclone of $\mathbf{O}_A$ is contained in at least one of the clones listed above.

In the proof of our theorem we need two other results.

Lemma. (ŚWIERCZKOWSKI [12]; see also [1; Lemma 4]). *If an at least quaternary operation turns into projection whenever we identify any two of its variables, then it always turns into the same projection.*

Theorem. (URBANIK [13; Lemma 9]) *Let $\mathfrak{A} = \langle A; F \rangle$ $(|A| \geqq 2)$ be an idempotent algebra which has essentially ternary term functions but has neither essentially binary nor essentially quaternary term functions. Then $\mathfrak{A}$ is equivalent to an algebra $\langle A; I \cup G \rangle$ where*

(i) *$\langle A; I \rangle$ is an affine space over the two element field* GF (2);

(ii) *either* $G=\emptyset$ *or there exists an integer* $r \geqq 5$ *such that* G *contains an* r*-ary operation depending on every variable, furthermore, every* $g \in G$ *depends on at least* r *variables and satisfies the equation* $g(x_1, \ldots, x_n)=x_1$ *whenever the elements* $x_1, \ldots, x_n$ *belong to a subalgebra of* $\langle A; I\rangle$ *generated by less than* r *elements.*

3. Results

Our main theorem was inspired by the following

Theorem. (Csákány [1]) *A non-trivial finite homogeneous algebra is functionally complete unless it is equivalent to one of the following algebras:*

$\langle \mathbf{2}; n\rangle$ *with* $n(x) \equiv x+1 \pmod 2$;

$\langle \mathbf{2}; s\rangle$ *with* $s(x, y, z) \equiv x+y+z \pmod 2$ (i.e. *the two element affine space over* GF (2));

$\langle \mathbf{2}; \bar{s}\rangle$ *with* $\bar{s}(x, y, z) \equiv x+y+z+1 \pmod 2$;

$\langle \mathbf{2}; d\rangle$ *with* $d(x, y, z) \equiv xy+yz+xz \pmod 2$;

$\langle \mathbf{3}; \circ\rangle$ *with* $x \circ y \equiv 2x+2y \pmod 3$;

the four element affine space over GF (2).

The proof of this result in [1] depends upon the Słupecki criterium. Trying to prove it by means of Rosenberg's Theorem, the first author noticed that it suffices to require Aut $\mathfrak{A}$ to be quadruply transitive. Moreover, the major part of his proof used 3-fold transitivity only. This observation led us to the following

Theorem. *An at least four element non-trivial finite algebra with triply transitive automorphism group is either functionally complete or equivalent to an affine space over* GF (2).

Remark. Examining the proof presented in the next section one can observe that the hypotheses of this theorem can be slightly weakened so that the conclusion still remain valid. Namely, it suffices to assume that the endomorphism monoid be *weakly triply transitive* in the sense that any three distinct elements of the algebra can be sent into any other tree distinct elements by an endomorphism.

It is easy to check that a more than four element affine space over GF (2) has a triply but not quadruply transitive automorphism group. Hence we get

Corollary 1. *An at least four element non-trivial finite algebra with quadruply transitive automorphism group is functionally complete unless it is equivalent to the four element affine space over* GF (2).

Corollary 2. *An at least four element non-trivial finite algebra with triply transitive automorphism group is simple or equivalent to an affine space over* GF (2).

This is a sharpening of a result of GANTER, PŁONKA and WERNER [3] on the simplicity of finite homogeneous algebras.

Corollary 3. *An at least four element finite simple algebra with triply transitive automorphism group is functionally complete.*

4. Proof of the main theorem

We start with two simple observations.

Proposition 1. *A finite algebra* $\mathfrak{A}=\langle A; F\rangle$ *is either functionally complete or* $\mathbf{P}(\mathfrak{A})\subseteq \operatorname{Pol}\varrho$ *for a relation* ϱ *(on* A*) of type* (α), (γ), (δ), (ζ) *or*

(ε') *an at least binary central relation.*

Proof. Notice that if ϱ is a unary central relation or a relation of type (β) then $\operatorname{Pol}\varrho$ fails to contain all constant functions on A. Thus the statement follows from Rosenberg's Theorem.

Proposition 2. *Let* $\mathfrak{A}$ *be an at least four element finite algebra whose automorphism group is triply transitive. Then any non-trivial term function of* $\mathfrak{A}$ *is at least ternary. In particular,* $\mathfrak{A}$ *is idempotent.*

Proof. Let $f\in\mathbf{T}(\mathfrak{A})$, f binary, and $a\neq b$ arbitrary elements in the base set A of $\mathfrak{A}$. Then $f(a,b)\in\{a,b\}$, else there would exist $\pi\in\operatorname{Aut}\mathfrak{A}$ with $a\pi=a$, $b\pi=b$ and $f(a,b)\pi=c\notin\{a,b,f(a,b)\}$, implying that $f(a,b)=f(a\pi,b\pi)=c$ which contradicts the choice of c. Similarly, $g(x)\in\{x\}$ for any unary $g\in\mathbf{T}(\mathfrak{A})$ and $x\in A$. Thus $f(x,x)=x$ for any $x\in A$. Furthermore, if, say, $f(a,b)=a$ then by the 2-fold transitivity of $\operatorname{Aut}\mathfrak{A}$, $f(x,y)=x$ for any distinct $x,y\in A$. Hence f is a projection, what was to be proved.

Lemma 1. *Let* A *be a finite set,* $|A|\geqq 4$, *and* f *a non-trivial ternary operation on* A *such that the algebra* $\langle A;f\rangle$ *is functionally incomplete and has a triply transitive automorphism group. Then*

(i) f *is a minority function, i.e.* $f(x,y,y)=f(y,x,y)=f(y,y,x)=x$ *for all* $x,y\in A$, *and for any distinct elements* $a,b,c\in A$, $f(a,b,c)\notin\{a,b,c\}$;

(ii) $\mathbf{P}(\langle A;f\rangle)\nsubseteq\operatorname{Pol}\varrho$ *if* ϱ *is a relation of type* (α), (γ) *with* $p>2$, (ε') *or* (ζ).

Proof. Recall that f turns into projection if we identify any two of its variables. Suppose that there exist distinct elements $a,b,c\in A$ such that $f(a,b,c)\in\{a,b,c\}$, say, $f(a,b,c)=a$. Then the 3-fold transitivity of $\operatorname{Aut}\langle A;f\rangle$ implies $f(x,y,z)=x$ for any distinct $x,y,z\in A$. Hence the algebra $\langle A;f\rangle$ is homogeneous, so that by Csákány's Theorem $\langle A;f\rangle$ must be equivalent to the four element affine space over

GF (2), else it would be functionally complete. However, then f is necessarily the "parallelogram operation" $x+y+z$, which does not satisfy our assumption on f. This contradiction shows that $a, b, c, f(a, b, c)$ are pairwise different provided the first three of them are such. Let $a, b, c \in A$ be pairwise different. Then there exists an automorphism π of $\langle A; f\rangle$ which sends a, b and $f(a, b, c)$ into a, b and c, respectively. Hence $f(a, b, c\pi)=c$. Consequently,

($*$) for any distinct elements $a, b, c \in A$ there exists $d \in A$ such that $f(a, b, d)=c$.

Now we are going to show that $\mathbf{P}(\langle A; f\rangle) \subsetneqq \operatorname{Pol} \varrho$ if ϱ is a relation of type (α), (ε') or (ζ). We do this by constructing matrices with entries in A such that each row belongs to ϱ but the row of column values of f fails to belong to ϱ. We have to construct various matrices according to the various possibilities for the behaviour of f when identifying two of its variables. Consider first a partial order $\leqq$ with lower bound 0 and upper bound 1, further, let $0<a<1$ $(a \in A)$. Owing to ($*$), we can choose $d \in A$ such that $f(0, a, d)=1$. As regards the behaviour of f when identifying two of its variables, by symmetry, it suffices to deal with the following two cases: $f(x, y, y)=x$ for all $x, y \in A$ or f is a majority function (i.e. $f(x, y, y)=$ $=f(y, x, y)=f(y, y, x)=y$ for all $x, y \in A$). Accordingly, the two matrices disproving $\mathbf{P}(\langle A; f\rangle) \subseteqq \operatorname{Pol} \leqq$ are

$$\begin{array}{cc} 0 & 0 \\ a & 1 \\ d & 1 \\ \hline 1 & 0 \end{array} \quad \text{and} \quad \begin{array}{cc} 0 & a \\ a & a \\ d & d \\ \hline 1 & a \end{array}$$

Let ϱ be a k-ary central relation $(k \geqq 2)$ and select $\langle a_1, \ldots, a_k\rangle \in A^k-\varrho$. Furthermore, let $c \in C$, the centre of ϱ. By the definition of a central relation, a_1, a_2, c are pairwise different, so that, by ($*$), there exists $d \in A$ such that $f(c, a_2, d)=a_1$. Now the matrices

$$\begin{array}{cccc} c & a_2 & a_3 \ldots & a_k \\ a_2 & a_2 & a_3 \ldots & a_k \\ d & d & d \ldots & d \\ \hline a_1 & a_2 & a_3 \ldots & a_k \end{array} \quad \text{and} \quad \begin{array}{cccc} a_1 & c & a_3 \ldots & a_k \\ c & c & a_3 \ldots & a_k \\ c & a_2 & a_3 \ldots & a_k \\ \hline a_1 & a_2 & a_3 \ldots & a_k \end{array}$$

show that $\mathbf{P}(\langle A; f\rangle) \subsetneqq \operatorname{Pol} \varrho$ whether $f(x, x, y)=x$ for all $x, y \in A$ or f is a minority function. By symmetry, all other cases can be reduced to one of these. Similarly, if ϱ is a k-ary relation of type (ζ) $(k \geqq 3)$ and $\langle a_1, \ldots, a_k\rangle \in A^k-\varrho$ then, a_1, a_2, a_3 being pairwise different (by property (c)), there is a $d \in A$ with $f(a_2, a_3, d)=a_1$. Hence the two matrices

$$\begin{array}{ccccc} a_2 & a_2 & a_3 & a_4 \ldots & a_k \\ a_3 & a_2 & a_3 & a_4 \ldots & a_k \\ d & d & d & d \ldots & d \\ \hline a_1 & a_2 & a_3 & a_4 \ldots & a_k \end{array} \quad \text{and} \quad \begin{array}{ccccc} a_1 & a_2 & a_2 & a_4 \ldots & a_k \\ a_2 & a_2 & a_2 & a_4 \ldots & a_k \\ a_2 & a_2 & a_3 & a_4 \ldots & a_k \\ \hline a_1 & a_2 & a_3 & a_4 \ldots & a_k \end{array}$$

meet our requirements if $f(x, x, y)=x$ for all $x, y \in A$ or f is a minority function, respectively.

Suppose f is not a minority function, say $f(x, x, y)=x$ for all $x, y \in A$. Then $\mathbf{P}(\langle A; f\rangle) \nsubseteq \mathrm{Pol}\, \varrho$ for any relation of type (γ) or (δ). Indeed, assume first ϱ is a non-trivial equivalence relation, $a\varrho b$, $a \neq b$ and $a\bar{\varrho}c$ (i.e. $\langle a, c\rangle \in A^2-\varrho$), $a, b, c \in A$. Then, by ($*$), there exists $d \in A$ such that $f(a, b, d)=c$, hence the matrix

$$\begin{array}{cc} a & a \\ a & b \\ d & d \\ \hline a & c \end{array}$$

proves $\mathbf{P}(\langle A; f\rangle) \nsubseteq \mathrm{Pol}\, \varrho$. If in turn ϱ is a relation of type (γ), take into consideration that f is essentially ternary, hence in particular, f depends on the third variable, i.e. there exist elements $a, b, c, d \in A$, $c \neq d$ such that $f(a, b, c) \neq f(a, b, d)$. Then the matrix

$$\begin{array}{cccc} a & a & a & a \\ b & a & b & a \\ c & a & d & c-d+a \\ \hline f(a, b, c) & a & f(a, b, d) & a \end{array}$$

shows that $\mathbf{P}(\langle A; f\rangle) \nsubseteq \mathrm{Pol}\, \varrho$, what was to be proved. By Proposition 1, this contradicts the functional incompleteness of $\langle A; f\rangle$. Thus f is a minority function.

It remains to verify that if f is a minority function and $\mathbf{P}(\langle A; f\rangle) \subseteq \mathrm{Pol}\, \varrho$ for a relation ϱ of type (γ) then $p=2$. This is done by the following matrix:

$$\begin{array}{cccc} a & 0 & a & 0 \\ a & 0 & 0 & a \\ a & a & a & a \\ \hline a & a & 0 & 0 \end{array}$$

where 0 is the zero element of the abelian group $\langle A; +\rangle$, $a \in A$ is arbitrary and, by definition, $\langle a, a, 0, 0\rangle \in \varrho$ iff $a+a=0$.

Lemma 2. *Consider a finite set A, $|A| \geqq 4$, and an at least quaternary non-trivial operation f on A such that the algebra $\langle A; f\rangle$ has a triply transitive automorphism group and f turns into projection whenever we identify any two of its variables. Then $\langle A; f\rangle$ is functionally complete.*

Proof. Suppose f is n-ary, $n \geqq 4$. By Świerczkowski's Lemma f always turns into the same, say the first, projection if we identify any two of its variables. Since f itself is not the first projection, there exist (necessarily distinct) elements e_i $(1 \leqq i \leqq n)$

such that $f(e_1, \ldots, e_n) \neq e_1$. Then $f(e_1, \ldots, e_n) \neq e_2$ or e_3. We can assume without loss of generality that $f(e_1, \ldots, e_n) \neq e_2$, i.e. e_1, e_2 and $f(e_1, \ldots, e_n)$ are pairwise different. Hence, 3-fold transitivity of Aut $\langle A; f\rangle$ implies

(∗ ∗) for any distinct elements $a, b, c \in A$ there exist elements $d_3, \ldots, d_n$ such that $f(a, b, d_3, \ldots, d_n) = c$.

By Proposition 1, we are done if we show that $\mathbf{P}(\langle A; f\rangle)$ is not contained in Pol ϱ for any relation ϱ of type (α), (γ), (δ), (ε') or (ζ). To this end we have to construct matrices with entries in A whose rows belong to ϱ but the row of column values of f fails to belong to ϱ. The five matrix schemes corresponding to the five types are the following:

$$(\alpha)\ \begin{array}{cc} 0 & 0 \\ b & 1 \\ d_3 & 1 \\ \vdots & \vdots \\ d_n & 1 \\ \hline 1 & 0 \end{array} \qquad (\gamma)\ \begin{array}{cccc} a & a & a & a \\ b & a & b & a \\ d_3 & a & a & d_3 \\ \vdots & \vdots & \vdots & \vdots \\ d_n & a & a & d_n \\ \hline c & a & a & a \end{array} \qquad (\delta)\ \begin{array}{cc} a & a \\ b & a \\ d_3 & d_3 \\ \vdots & \vdots \\ d_n & d_n \\ \hline c & a \end{array}$$

$$(\varepsilon')\ \begin{array}{cccc} c & a_2 & a_3 \ldots & a_k \\ a_2 & a_2 & a_3 \ldots & a_k \\ d_3 & d_3 & d_3 \ldots & d_3 \\ \vdots & \vdots & \vdots & \vdots \\ d_n & d_n & d_n \ldots & d_n \\ \hline a_1 & a_2 & a_3 \ldots & a_k \end{array} \ (k \geqq 2) \qquad (\zeta)\ \begin{array}{ccccc} a_2 & a_2 & a_3 & a_4 \ldots & a_k \\ a_3 & a_2 & a_3 & a_4 \ldots & a_k \\ d_3 & d_3 & d_3 & d_3 \ldots & d_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d_n & d_n & d_n & d_n \ldots & d_n \\ \hline a_1 & a_2 & a_3 & a_4 \ldots & a_k \end{array} \ (k \geqq 3).$$

If ϱ is a partial order, 0 and 1 denote the lower and upper bounds, respectively, b is another element, $b \neq 0, 1$. The existence of the elements $d_3, \ldots, d_n$ is ensured by (∗ ∗). Similar argument can be applied in the other cases, too. In case (γ) a, b, c are arbitrary distinct elements of A while in case (δ) a, b and c are selected such that $a \neq b$, $a\varrho b$ and $a\bar{\varrho}c$ (i.e. $\langle a, c\rangle \in A^2 - \varrho$); $d_3, \ldots, d_n$ is chosen according to (∗ ∗). Finally, if ϱ is of type (ε') or (ζ) then we fix a k-tuple $\langle a_1, \ldots, a_k\rangle \in A^k - \varrho$. By definition, its components are necessarily pairwise different, moreover, if ϱ is a central relation, none of them belong to the centre C. Thus $c(\in C)$, a_2, a_1, resp. a_2, a_3, a_1 are pairwise different, hence (∗ ∗) implies the existence of the elements $d_3, \ldots, d_n \in A$ completing the first columns of the corresponding matrices.

Lemma 3. *Let* $\mathfrak{A} = \langle A; F\rangle$ $(|A| \geqq 4)$ *be a functionally incomplete non-trivial finite algebra with a triply transitive automorphism group. Then*

(i) $\mathfrak{A}$ *has a unique non-trivial ternary term function* m. *It is a minority function and has the property that for any distinct elements* $a, b, c \in A$, $m(a, b, c) \notin \{a, b, c\}$;

(ii) *any non-trivial quaternary term function h of $\mathfrak{A}$ satisfies the identities*

$$h(x, x, y, z) = m(x, y, z), \tag{1}$$

$$h(x, y, x, z) = m(x, y, z), \tag{2}$$

$$h(x, y, z, x) = m(x, y, z), \tag{3}$$

$$h(x, y, y, z) = z, \tag{4}$$

$$h(x, y, z, z) = y, \tag{5}$$

$$h(x, y, z, y) = z, \tag{6}$$

$$h\big(m(x, y, z), x, y, z\big) = m(x, y, z), \tag{7}$$

or arises from such a term function by interchanging its variables.

Proof. Let n denote the minimum of the arities of non-trivial term functions of $\mathfrak{A}$. By Proposition 2, $n \geqq 3$. If $n \geqq 4$, arbitrary non-trivial n-ary term function f turns into projection when we identify any two of its variables. Hence, by Lemma 2, $\langle A; f\rangle$ (consequently, also $\mathfrak{A}$) is functionally complete, contradicting our hypothesis. Thus $n=3$, i.e. $\mathfrak{A}$ has a non-trivial ternary term function. By Lemma 1, every such term function enjoys property (i).

In order to prove uniqueness we first show that for any non-trivial ternary term functions $f, g \in \mathbf{T}(\mathfrak{A})$, the following identity holds:

$$f\big(g(x, y, z), y, z\big) = x. \tag{8}$$

Indeed, $f\big(g(x, y, z), y, z\big)$, being a ternary term function of $\mathfrak{A}$, must be a minority function or a projection. Since by the identification $x=y$ we get x, the former case is excluded. Thus $f\big(g(x, y, z), y, z\big)=x$ or y. On the other hand, by the identification $x=z$ we also get x, so the proof of (8) is concluded. Taking into consideration that (8) holds for any $f, g \in \mathbf{T}(\mathfrak{A})$, in particular for $g=f$, too, we get the identity

$$g(x, y, z) = f\big(f(g(x, y, z), y, z), y, z\big) = f(x, y, z).$$

This completes the proof of (i).

Let h be a non-trivial quaternary term function of $\mathfrak{A}$. If we identify any two of its variables, we either get a projection or the (unique) non-trivial ternary term function m. The latter must occur at least once, otherwise, by Lemma 2, the algebra $\langle A; h\rangle$ (hence also $\mathfrak{A}$) would be functionally complete. Suppose e.g. that $h(x, x, y, z)=m(x, y, z)$. Thus $h(x, x, z, z)=x$, so that $h(x, y, z, z)=x$ or y (since neither $h(x, y, z, z)=z$ nor $h(x, y, z, z)=m(x, y, z)$ can hold). We can assume without loss of generality that $h(x, y, z, z)=y$. So far, we have (1) and (5). They imply

$$\text{(I)}\quad h(x, x, x, z) = z, \qquad \text{(II)}\quad h(x, x, y, x) = y,$$

$$\text{(III)}\quad h(x, y, y, y) = y, \qquad \text{(IV)}\quad h(x, y, x, x) = y.$$

By (II) and (III) $h(x, y, z, y) \neq x, y, m(x, y, z)$, which proves (6). Similarly, (I) and (IV) exclude all possibilities for $h(x, y, x, z)$ but (2). (4) follows from (I) and (III), while (3) from (II) and (IV). In order to verify (7) one has to check that $h(m(x, y, z), x, y, z)$ is a minority function, which is straightforward by the preceding identities. The proof is complete.

Now we are ready to prove our main result formulated in Section 3.

Proof of the Theorem. Let $\mathfrak{A} = \langle A; F \rangle$ $(|A| \geqq 4)$ be a non-trivial finite algebra which is functionally incomplete and has a triply transitive automorphism group. By Proposition 2, $\mathfrak{A}$ is idempotent and has no essentially binary term function. On the other hand, by Lemma 3, $\mathfrak{A}$ has an essentially ternary term function. We are going to prove that $\mathfrak{A}$ has no essentially quaternary term function. Suppose the contrary and choose an essentially quaternary $h \in \mathbf{T}(\mathfrak{A})$ such that it satisfy identities (1)—(7) in Lemma 3. Since h depends on its first variable, there exist elements $a, b, c, d \in A$ such that $h(a, b, c, d) \neq h(b, b, c, d) = m(b, c, d)$. Then the matrix

$$\begin{array}{cccc} a & b & a & b \\ b & b & b & b \\ c & c & b & b \\ d & d & b & b \\ \hline h(a, b, c, d) & h(b, b, c, d) & b & b \end{array}$$

shows that $\mathbf{P}(\mathfrak{A}) \nsubseteq \operatorname{Pol} \varrho$ if ϱ is a relation of type (γ) with $p=2$. By Lemma 1, $\mathbf{P}(\mathfrak{A}) \nsubseteq \operatorname{Pol} \varrho$ if ϱ is a relation of type (α), (γ) with $p>2$, (ε') or (ζ). Thus $\mathbf{P}(\mathfrak{A}) \subseteq \operatorname{Pol} \varrho$ where ϱ is a non-trivial equivalence relation. Select distinct elements $a', b', c' \in A$ such that $a' \varrho b'$ but $a' \bar{\varrho} c'$ (i.e. $\langle a', c' \rangle \in A^2 - \varrho$). Assume first $h(a, b, c, d) \neq a$. Then, by (7)

(9) $$h(a, b, c, d) \neq h(m(b, c, d), b, c, d) = m(b, c, d)$$

where a, $m(b, c, d)$ and $h(a, b, c, d)$ are pairwise different $(a = m(b, c, d)$ would imply equality in (9), contradicting the choice of $a, b, c, d)$. Hence, by the 3-fold transitivity of $\operatorname{Aut} \mathfrak{A}$ there exists $\pi \in \operatorname{Aut} \mathfrak{A}$ which sends $m(b, c, d)$, a, $h(a, b, c, d)$ into a', b' and c', respectively. Thus we have the matrix

$$\begin{array}{cc} a' & b' \\ b\pi & b\pi \\ c\pi & c\pi \\ d\pi & d\pi \\ \hline a' & c' \end{array}$$

with its rows belonging to ϱ but $a' \bar{\varrho} c'$, contradicting the inclusion $\mathbf{P}(\mathfrak{A}) \subseteq \operatorname{Pol} \varrho$.

Assume now that $h(a, b, c, d)=a$. Then, by (1) and (7)

$$a = h(a, b, c, d) \neq h(b, b, c, d) = m(b, c, d) = h\big(m(b, c, d), b, c, d\big). \tag{10}$$

Thus $a, b, m(b, c, d)$ are pairwise different. (10) implies immediately that $a \neq b$, $m(b, c, d)$. If $b=m(b, c, d)$ then by Lemma 3(i), b, c, d are not distinct, so that, since m is a minority function, we have $c=d$. However, then by (5), $h(a, b, c, d)=$ $=b=h(b, b, c, d)$, which is impossible by (10). By the 3-fold transitivity of Aut $\mathfrak{A}$ there exists an automorphism π sending $a, b, m(b, c, d)$ into a', b', c', respectively. Hence we get the matrix

$$\begin{array}{cc} a' & b' \\ b' & b' \\ c\pi & c\pi \\ d\pi & d\pi \\ \hline a' & c' \end{array},$$

again contradicting the inclusion $\mathbf{P}(\mathfrak{A}) \subseteq \mathrm{Pol}\ \varrho$.

It follows from the foregoing argument that $\mathfrak{A}$ has no essentially quaternary term function. Thus $\mathfrak{A}$ satisfies the hypotheses of Urbanik's Theorem, so that $\mathfrak{A}$ is equivalent to an affine space over GF(2) or arises from such a space by adding new at least $r(\geqq 5)$-ary fundamental operations among which there is an essentially r-ary operation which turns into projection if we identify any two of its variables. However, by Lemma 2, the existence of such an operation would imply functional completeness. Hence $\mathfrak{A}$ is equivalent to an affine space over GF(2), what was to be proved.

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The function model of a contraction and the space L^1/H_0^1

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Recently, new techniques were invented for obtaining invariant subspaces for rather general classes of operators on Hilbert space, see [2]—[5]. The present note constitutes a first step to exploit similar techniques in the understanding of the fine structure of the functional model, in the sense of [1], of completely non-unitary contractions.

1. Recalling the canonical model of a completely non-unitary contraction on a separable Hilbert space we consider a contractive analytic function $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ on the unit disc $D=\{\lambda: |\lambda|<1\}$; $\mathfrak{E}$ and $\mathfrak{E}_*$ being separable Hilbert spaces. Setting $\Delta=\Delta(e^{it})=(I-\Theta(e^{it})^*\Theta(e^{it}))^{1/2}$ we define the Hilbert function spaces

$$\mathfrak{K}_+ = H^2(\mathfrak{E}_*)\oplus\overline{\Delta L^2(\mathfrak{E})}, \quad \mathfrak{H} = \mathfrak{K}_+\ominus\{\Theta w\oplus\Delta w:\ w\in H^2(\mathfrak{E})\} \tag{1.1}$$

(see [1], Chapter VI). $P_{\mathfrak{H}}$ will denote orthogonal projection of $\mathfrak{K}_+$ onto $\mathfrak{H}$.

We shall also have to do with spaces L^1, H^1, H_0^1, H^∞, all with respect to normalized Lebesgue measure $dm=dt/(2\pi)$ on the unit circle $\{e^{it}: 0\leq t<2\pi\}$. Recall that H^∞ is the Banach dual of the factor space L^1/H_0^1, through the bilinear form

$$\langle f^{\cdot}, u\rangle = \int fu\, dm \quad (f\in L^1, u\in H^\infty),$$

$f\mapsto f^{\cdot}$ denoting the natural map of L^1 onto L^1/H_0^1 (see e.g. [6]).

With any (ordered) pair $\{h, k\}$ of elements of H we associate the element hk^* of L^1 defined by

$$hk^*(e^{it}) = (h(e^{it}), k(e^{it}))_{\mathfrak{E}_*\oplus\mathfrak{E}} \quad (0\leq t<2\pi). \tag{1.2}$$

For sake of simplicity we shall also write, for any $f\in L^1$,

$$\|f\|_{L^1/H_0^1} \quad \text{instead of} \quad \|f^{\cdot}\|_{L^1/H_0^1},$$

and scalar product and norm of vectors without subscript will always mean those in the space $\mathfrak{H}$.

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2. With the operator valued function $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ we associate the multiplication operator

$$(2.1)\qquad \Theta_\times: H^2(\mathfrak{E}) \to H^2(\mathfrak{E}_*) \text{ defined by } (\Theta_\times u)(e^{it}) = \Theta(e^{it})u(e^{it}) \quad (u \in H^2(\mathfrak{E}))$$

and its adjoint $\Theta_\times^*$ (i.e. the coanalytic Toeplitz operator denoted in [6] by $T(\Theta^\sim)$); we have

$$(2.2)\qquad (\Theta_\times^* u)(e^{it}) = [\Theta(e^{it})^* u(e^{it})]_+ \quad (u \in H^2(\mathfrak{E})),$$

where $[\cdot]_+$ denotes the natural orthogonal projection of any (scalar or vector valued function space) L^2 onto its subspace H^2.

Observe that for any fixed $\mu \in D$ the function

$$(2.3)\qquad p_\mu(\lambda) = (1-\bar{\mu}\lambda)^{-1}$$

belongs to H^2, and has norm

$$\|p_\mu\|_{H^2} = (1-|\mu|^2)^{-1/2}.$$

It is easy to deduce from (2.2) that

$$(2.4)\qquad \Theta_\times^*(p_\mu a) = p_\mu \Theta(\mu)^* a \quad \text{for any} \quad a \in \mathfrak{E}_*.$$

The following functional η_Θ on H^2 will play an important part:

$$(2.5)\qquad \eta_\Theta(\varphi) = \inf_{\mathfrak{A}\in\Phi} \sup_{a\in\mathfrak{A}} s(\varphi, a), \quad \text{where} \quad s(\varphi, a) = \frac{\|\Theta_\times^* \varphi a\|_{H^2(\mathfrak{E})}}{\|\varphi a\|_{H^2(\mathfrak{E}_*)}} \; (=0 \text{ if } \varphi a = 0)$$

and Φ denotes the family of subspaces of $\mathfrak{E}_*$ with *finite codimension.*

Obviously, $\eta_\Theta(c\varphi) = \eta_\Theta(\varphi)$ for any complex number $c \neq 0$. By virtue of (2.4) we have, in particular,

$$(2.6)\qquad \eta_\Theta(p_\mu) = \inf_{\mathfrak{A}\in\Phi} \sup_{a\in\mathfrak{A}} \frac{\|\Theta(\mu)^* a\|_{\mathfrak{E}}}{\|a\|_{\mathfrak{E}_*}}$$

In what follows we shall assume that $\mathfrak{E}_*$ is infinite dimensional.

Lemma 1. *Given any sequence $\{\varphi_j\}_1^\infty$ of elements of H^2 there exists an orthonormal sequence $\{a_n\}_1^\infty$ in $\mathfrak{E}_*$ such that,*

$$(2.7)\qquad s(\varphi_j, a_n) \leqq \eta_\Theta(\varphi_j) + \frac{1}{n} \quad \text{for} \quad j = 1, 2, \ldots, n; \; n = 1, 2, \ldots.$$

Proof. By virtue of the definition (2.5) there exist $\mathfrak{A}_{j,n} \in \Phi$ $(j=1, 2, \ldots; n=1, 2, \ldots)$ such that

$$\sup_{a\in\mathfrak{A}_{j,n}} s(\varphi_j, a) \leqq \eta_\Theta(\varphi_j) + \frac{1}{n}.$$

Set

$$\mathfrak{A}_n = \Big(\bigvee_{\substack{1\leqq j\leqq n\\ 1\leqq m\leqq n}} \mathfrak{A}_{j,m}^{\perp}\Big)^{\perp} \quad (n = 1, 2, \ldots).$$

Clearly, $\mathfrak{A}_n \in \Phi$, $\mathfrak{A}_n \subset \mathfrak{A}_{n-1}$ and $\mathfrak{A}_n \subset \mathfrak{A}_{j,m}$ $(1\leqq j\leqq n,\ 1\leqq m\leqq n)$. From the last inclusion we infer

$$\sup_{a\in\mathfrak{A}_n} s(\varphi_j, a) \leqq \sup_{a\in\mathfrak{A}_{j,n}} s(\varphi_j, a) \leqq \eta_\Theta(\varphi_j) + \frac{1}{n}.$$

Choose inductively a sequence $\{a_n\}_1^\infty$ of unit vectors in $\mathfrak{E}_*$ such that $a_n \in \mathfrak{A}_n$ and that a_n be orthogonal to $a_1, \ldots, a_{n-1}$ $(n=2, 3, \ldots)$. Then we shall have (2.7.)

Notice, for further use, that each infinite orthonormal sequence weakly converges to 0.

3. A subset $\mathcal{S}$ of the (open) unit ball $\mathcal{D}$ of H^2 will be called *dominant* if

$$\sup_{\varphi\in\mathcal{S}} \|[\bar{u}\varphi]_+\|_{H^2} = \|u\|_{H^\infty} \quad \text{for every} \quad u\in H^\infty. \tag{3.1}$$

This is an obvious analogue of that a subset S of the unit disc D be dominant in the sense of [8], namely that

$$\sup_{\mu\in S} |u(\mu)| = \|u\|_{H^\infty} \quad \text{holds for every} \quad u\in H^\infty. \tag{3.2}$$

Moreover, if S is dominant in D in the sense (3.2) then

$$\mathcal{S}_S = \{1-|\mu|^2)^{1/2} p_\mu :\ \mu\in S\} \tag{3.3}$$

is dominant in $\mathcal{D}$ in the sense (3.1). Indeed, $\mathcal{S}_S \subset \mathcal{D}$ is obvious and in analogy with (2.4) we easily obtain

$$[\bar{u}p_\mu]_+ = p_\mu \overline{u(\mu)} \quad \text{for} \quad u\in H^\infty \quad \text{and} \quad \mu\in D.$$

Hence,

$$\|[\bar{u}(1-|\mu|^2)^{1/2} p_\mu]_+\|_{H^2} = |u(\mu)| \tag{3.4}$$

so validity of (3.2) for S implies that of (3.1) for $\mathcal{S}_S$.

Lemma 2. *If $\mathcal{S}$ is dominant in the unit ball $\mathcal{D}$ of H^2 then the convex hull of the set*

$$\{(\psi\bar{\varphi})^\cdot :\ \varphi\in\mathcal{S},\ \psi\in\mathcal{D}\} \tag{3.5}$$

is dense in the unit ball of L^1/H_0^1.

Proof. If not, there exist in the Banach dual H^∞ of L^1/H_0^1 an element u and a unit vector f in L^1/H_0^1 such that

$$\text{(3.6)} \qquad \operatorname{Re}\langle f^\cdot, u\rangle > \sup_{\substack{\varphi\in\mathscr{S}\\ \psi\in\mathscr{D}}} \operatorname{Re}\langle(\psi\bar\varphi)^\cdot, u\rangle = \sup_{\varphi\in\mathscr{S}}\sup_{\psi\in\mathscr{D}}\left|\int \overline{\bar u\varphi}\,\psi\, dm\right| =$$

$$= \sup_{\varphi\in\mathscr{S}}\sup_{\psi\in\mathscr{D}}\left|\int\overline{[\bar u\varphi]_+}\,\psi\, dm\right| = \sup_{\varphi\in\mathscr{S}} \|[\bar u\varphi]_+\|_{H^2}.$$

$\mathscr{S}$ being dominant in $\mathscr{D}$ the last member equals $\|u\|_{H^\infty}$, and hence is $\geqq \operatorname{Re}\langle f^\cdot, u\rangle$, in contradiction with the strict inequality in (3.6).

4. For fixed $\varphi\in H^2$ and $a\in\mathfrak{E}_*$ we denote

$$\text{(4.1)} \qquad \varphi\circ a = P_{\mathfrak{H}}(\varphi a\oplus 0).$$

It is easy to show that

$$\text{(4.2)} \qquad \varphi\circ a = (\varphi a - \Theta[\Theta^*\varphi a]_+)\oplus(-\Delta[\Theta^*\varphi a]_+).$$

For any $h=h_0\oplus h_1\in\mathfrak{H}$ $\left(h_0\in H^2(\mathfrak{E}_*),\ h_1\in\overline{\Delta L^2(\mathfrak{E})}\right)$ we have therefore

$$\text{(4.3)} \qquad (\varphi\circ a)h^* = \varphi(a, h_0)_{\mathfrak{E}_*} - (\Theta[\Theta^*\varphi a]_+, h_0)_{\mathfrak{E}_*} - (\Delta[\Theta^*\varphi a]_+, h_1)_{\mathfrak{E}} =$$

$$= \varphi(a, h)_{\mathfrak{E}_*} - ([\Theta^*\varphi a]_+, \Theta^* h_0 + \Delta h_1)_{\mathfrak{E}},$$

where the last term belongs to H_0^1 since

$$\text{(4.4)} \qquad h_2 \overset{\text{def}}{=} \Theta^* h_0 + \Delta h_1 \in L^2(\mathfrak{E})\ominus H^2(\mathfrak{E})$$

because of the definition (1.1) of $\mathfrak{H}$.

Therefore,

$$\text{(4.5)} \qquad (\varphi\circ a)h^* \equiv \varphi(a, h_0)_{\mathfrak{E}_*} \bmod H_0^1.$$

It also follows from (4.3) and (4.4) that

$$\text{(4.6)} \qquad h(\varphi\circ a)^* = \overline{(\varphi\circ a)h^*} = \bar\varphi(h_0, a)_{\mathfrak{E}_*} - (h_2, [\Theta^*\varphi a]_+)_{\mathfrak{E}}.$$

Suppose $\{a_n\}$ is a sequence of vectors in $\mathfrak{E}_*$, tending weakly to 0. Then by (4.5) and by the Lebesgue dominated convergence theorem,

$$\|(\varphi\circ a_n)h^*\|_{L^1/H_0^1} \leqq \|\varphi(a_n, h_0)_{\mathfrak{E}_*}\|_{L^1} \leqq \|\varphi\|_{H^2}\left[\int|(a_n, h_0(e^{it}))|^2\, dm\right]^{1/2} \to 0 \quad (n\to 0).$$

We shall also show that $\|h(\varphi\circ a_n^*)\|_{L^1/H_0^1}\to 0$. Since $\|\bar\varphi(h_0, a_n)_{\mathfrak{E}_*}\|_{L^1}\to 0$ by part of the preceding argument, by (4.6) it suffices to prove that

$$\text{(4.7)} \qquad \|(h_2, [\Theta^*\varphi a_n]_+)_{\mathfrak{E}}\|_{L^1} \to 0 \quad \text{as} \quad n\to 0.$$

It even suffices to prove (4.7) for $\varphi = e^{irt}$ $(r=0, 1, \ldots)$. Indeed, (4.7) then holds for all partial sums $\varphi_N(e^{it})$ of the L^2-expansion $\varphi(e^{it})=\sum_0^\infty c_r e^{irt}$, and since $\{a_n\}$ is bounded, say $\|a_n\|_{\mathfrak{E}_*}\leqq A$, we have, setting $\psi_N=\varphi-\varphi_N$,

$$(4.8)\qquad \|(h_2, [\Theta^*\psi_N a_n]_+)_{\mathfrak{E}}\|_{L_1} = \int \|h_2\|_{\mathfrak{E}}\|[\Theta^*\psi_N a_n]_+\|_{\mathfrak{E}}\,dm =$$
$$\leqq \|h_2\|_{L^2(\mathfrak{E})}\|[\Theta^*\psi_N a_n]_+\|_{L^2(\mathfrak{E})} \leqq \|h_2\|_{L^2(\mathfrak{E})}\|\psi_N\|_{H^2}A,$$

and this bound is independent of n and as small as we wish upon choosing N large enough.

Now to prove (4.7) for $\varphi=e^{irt}$ $(r\geqq 0)$ observe that if $\Theta(\lambda)=\Theta_0+\lambda\Theta_1+\lambda^2\Theta_2+\dots$ then we have

$$[\Theta(e^{it})^* e^{irt}a_n]_+ = \sum_{j=0}^{r}\Theta_{r-j}e^{ijt}a_n,$$

and hence,

$$\|(h_2, [\Theta^* e^{irt}a_n]_+)_{\mathfrak{E}}\|_{L^1} = \int\left|\sum_{j=0}^{r}(e^{-ijt}\Theta^*_{r-j}h_2, a_n)_{\mathfrak{E}_*}\right|dm,$$

which tends to 0 as $n\to 0$, again by the weak convergence of $\{a_n\}$ to 0 and by the Lebesgue dominated convergence theorem.

So we have proved, in particular,

Lemma 3. *If $\{a_n\}$ converges to 0 weakly in $\mathfrak{E}_*$ then for any $\varphi\in H^2$ and $h\in\mathfrak{H}$ we have*

$$\|(\varphi\circ a_n)h^*\|_{L^1/H_0^1}\to 0,\quad \|h(\varphi\circ a_n)^*\|_{L^1/H_0^1}\to 0\quad \textit{as}\quad n\to\infty.$$

We shall also need the following

Lemma 4. *For all φ, $\psi\in H^2$ and $a\in\mathfrak{E}_*$ we have*

$$(4.9)\qquad \|(\psi\circ a)(\varphi\circ a)^*-\psi\bar\varphi\,\|a\|^2_{\mathfrak{E}_*}\|_{L^1/H_0^1}\leqq \|\psi\|_{H^2}\|\Theta^*_\times\varphi a\|_{H^2(\mathfrak{E})}\|a\|_{\mathfrak{E}_*}.$$

Proof. By virtue of (4.5) and (4.2) we have

$$(\psi\circ\varphi)(\varphi\circ a)^* = \varphi(a, \varphi a-\Theta[\Theta^*\varphi a]_+)_{\mathfrak{E}_*}\bmod H_0^1.$$

Because

$$\|\psi(a, \Theta[\Theta^*\varphi a]_+)_{\mathfrak{E}_*}\|_{L^1} = \|\psi\|_{H^2}\|a\|_{E_*}\|\Theta^*_\times\varphi a\|_{L^2(\mathfrak{E})}$$

(in analogy to (4.8)) we conclude to (4.9).

5. Next we prove the following

Lemma 5. *Suppose $\mathfrak{E}_*$ is (countably) infinite dimensional and suppose $h, k\in\mathfrak{H}$; $\varphi_1, \dots, \varphi_r, \psi_1, \dots, \psi_r\in H^2$, and $\varepsilon>0$ are given. Then there exist $h', k'\in\mathfrak{H}$ such that*

$$(5.1)\qquad \left\|(h+h')(k+k')^*-hk^*-\sum_1^r\psi_j\bar\varphi_j\right\|_{L^1/H_0^1}\leqq\sum_1^r\|\psi_j\|_{H^2}\|\varphi_j\|_{H^2}\eta_\Theta(\varphi_j)+\varepsilon,$$

$$(5.2)\qquad \|h'\|^2\leqq\sum_1^r\|\psi_j\|^2_{H^2},\quad \|k'\|^2\leqq\sum_1^r\|\varphi_j\|^2_{H^2}.$$

Proof. Let $\delta>0$ be fixed and choose by virtue of Lemma 1 an orthonormal sequence $\{a_n\}$ in $\mathfrak{E}_*$ such that

$$\|\Theta_\times^* \varphi_j a_n\|_{H^2(\mathfrak{E})} \leqq \left(\eta_\Theta(\varphi_j)+\frac{1}{n}\right)\|\varphi_j\|_{H^2} \quad \text{for} \quad j=1,\dots,r \quad \text{and} \quad n \geqq r.$$

Hence, and from Lemma 3 we deduce that for n large enough, say for $n\geqq n_0$, and for $j=1,\dots,r$ we have

$$\|\Theta_\times^* \varphi_j a_n\|_{H^2(\mathfrak{E})} \leqq \left(\eta_\Theta(\varphi_j)+\delta\right)\|\varphi_j\|_{H^2} \tag{5.3}$$

and

$$\|h(\varphi_j \circ a_n)^*\|_{L^1/H_0^1} \leqq \delta, \quad \|(\psi_j \circ a_n)k^*\|_{L^1/H_0^1} \leqq \delta. \tag{5.4}$$

Again by virtue of Lemma 3 we can choose, step by step, the integers $(n_0\leqq)n_1< <n_2<\dots<n_r$ such that

$$\|(\psi_i \circ a_{n_i})(\varphi_j \circ a_n)^*\|_{L^1/H_0^1} \leqq \delta, \quad \|(\psi_j \circ a_n)(\varphi_i \circ a_{n_i})^*\|_{L^1/H_0^1} \leqq \delta \tag{5.5}$$

$$(j=1,\dots,r;\ i=1,\dots,j-1;\ n \geqq n_j).$$

Rename a_{n_j} by b_j $(j=1,\dots,r)$ and set

$$h' = \sum_1^r (\psi_j \circ b_j)', \quad k' = \sum_1^r (\varphi_j \circ b_j). \tag{5.6}$$

Then we have

$$(h+h')(k+k')^* - hk^* - \sum_1^r \psi_j\overline{\varphi_j} = hk'^* + h'k^* + h'k'^* - \sum_1^r \psi_j\overline{\varphi_j} =$$

$$= \sum_1^r h(\varphi_j \circ b_j)^* + \sum_1^r (\psi_j \circ b_j)k^* + \sum_1^r \left[(\psi_j \circ b_j)(\varphi_j \circ b_j) - \psi_j\overline{\varphi_j}\right] +$$

$$+ \sum_{j=1}^r \sum_{i=1}^{j-1} \left[(\psi_i \circ b_i)(\varphi_j \circ b_j)^* + (\psi_j \circ b_j)(\varphi_i \circ b_i)^*\right] = \Omega.$$

Taking account of (5.3), (5.4), (5.5), and Lemma 4 we deduce that

$$\|\Omega\|_{L^1/H_0^1} \leqq r\delta + r\delta + \sum_1^r \|\psi_j\|_{H^2}\left(\eta_\Theta(\varphi_j)+\delta\right)\|\varphi_j\|_{H^2} + \frac{r(r-1)}{2}\delta + \frac{r(r-1)}{2}\delta$$

so we arrive at the conclusion (5.3) by choosing δ small enough, namely such that

$$\left[(r(r+1) + \sum_1^r \|\psi_j\|_{H^2}\|\varphi_j\|_{H^2}\right]\delta \leqq \varepsilon.$$

Finally, (5.2) follows at once from (5.6) and (4.1); e.g.,

$$\|h'\|^2 = \left\|P_{\mathfrak{H}}\left(\sum_1^r \psi_j b_j \oplus 0\right)\right\|^2 \leqq \left\|\sum_1^r \psi_j b_j\right\|^2_{H^2(\mathfrak{E}_*)} = \sum_{j,i=1}^r (\psi_j, \psi_i)_{H^2}(b_j, b_i)_{\mathfrak{E}_*} = \sum_1^r \|\psi_j\|^2$$

because of orthonormality of $\{b_j\}_1^r$.

Remark. The pair h', k' can obviously be replaced by any of the pairs $h^{(n)}, k^{(n)}$ $(n=1, 2, \ldots)$ defined by

$$h^{(n)} = \sum_{j=1}^{r} (\psi_j \circ b_{j+n}), \quad k^{(n)} = \sum_{j=1}^{r} (\varphi_j \circ b_{j+n}). \tag{5.7}$$

Then, for every $l \in \mathfrak{H}$,

$$l \circ h^{(n)*},\ h^{(n)} l^*,\ lk^{(n)*},\ k^{(n)} l^*$$

tend to 0 in L^1/H_0^1 as $n \to \infty$.

6. Now we are going to establish the main result of this paper.

Theorem. *Suppose $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ is a contractive analytic function, with separable $\mathfrak{E}, \mathfrak{E}_*$, and $\dim \mathfrak{E}_* = \infty$, and suppose that for some ϑ, $0<\vartheta<1$, the set*

$$\mathscr{S} = \{\varphi \in H^2 \colon \|\varphi\|_{H^2} = 1, \eta_\Theta(\varphi) \leqq \vartheta\}$$

is dominant in the unit ball $\mathscr{D}$ of H^2. Then

$$\{(hk^*)^\cdot : h, k \in \mathfrak{H}\} = L^1/H_0^1.$$

i.e. every $f \in L^1$ has a representation

$$f \equiv hk^* \bmod H_0^1 \quad \textit{with} \quad h, k \in \mathfrak{H}.$$

Proof. Consider an $f \in L^1$ with $\|f\|_{L^1/H_0^1} \leqq v_0$; it does not restrict generality to assume $v_0 = 1$. Choose a number ω such that $\vartheta < \omega < 1$ and set $v_s = \omega^s$, $\varepsilon_s = \dfrac{\omega - \vartheta}{2}\omega^s$; then

$$v_{s+1} = \vartheta v_s + 2\varepsilon_s. \tag{6.1}$$

Setting $h_0, h_{-1}, k_0, k_{-1} = 0$ (in $\mathfrak{H}$) we are going to prove that there exist $h_s, k_s \in \mathfrak{H}$ $(s=1, 2, \ldots)$ such that

$$\begin{gathered} \|f - h_s k_s^*\|_{L^1/H_0^1} \leqq v_s \\ \|h_s - h_{s-1}\|^2 \leqq v_{s-1}, \quad \|k_s - k_{s-1}\|^2 \leqq v_{s-1} \end{gathered} \qquad (s = 1, 2, \ldots). \tag{6.2}$$

This being obvious for $s=0$ we shall proceed by induction. Suppose h_s, k_s have been already found for $s=0, \ldots, q$, satisfying (6.2), and perform the step $q \to q+1$ as follows. Set

$$f' = f - h_q k_q^*; \tag{6.3}$$

then $\|f'\|_{L^1/H_0^1} \leqq v_q$ by (6.2) for $s=q$. It now follows from Lemma 2 that there exist $\varphi_j \in \mathscr{S}$, $\psi_j \in \mathscr{D}$, $c_j \geqq 0$ $(j=1, \ldots, r)$, with $\sum_1^r c_j = 1$ and

$$\Big\| f' - \sum_{j=1}^{r} c_j v_q \psi_j \overline{\varphi_j} \Big\|_{L^1/H_0^1} \leqq \varepsilon_q. \tag{6.4}$$

On the other hand, from Lemma 5 it follows that there exist

$h_{q+1}=h_q+h'$, $k_{q+1}=k_q+k'\in\mathfrak{H}$ such that

$$\left\|h_{q+1}k_{q+1}^*-h_qk_q^*-\sum_{j=1}^r c_j v_q\psi_j\overline{\varphi_j}\right\|_{L^1/H_0^1}\leqq \tag{6.5}$$

$$\leqq\sum_{j=1}^r\left\|\sqrt{c_jv_q}\psi_j\right\|_{H^2}\left\|\sqrt{c_jv_q}\varphi_j\right\|_{H^2}\eta_\Theta(\varphi_j)+\varepsilon_q\leqq\sum_{j=1}^r c_jv_q\vartheta+\varepsilon_q=v_q\vartheta+\varepsilon_q$$

and

$$\|h_{q+1}-h_q\|^2\leqq\sum_{j=1}^r\left\|\sqrt{c_jv_q}\psi_j\right\|_{H^2}^2\leqq v_q,\quad \|k_{q+1}-k_q\|^2\leqq\sum_{j=1}^r\left\|\sqrt{c_jv_q}\varphi_j\right\|_{H^2}^2\leqq v_q. \tag{6.6}$$

Because of the relation

$$f-h_{q+1}k_{q+1}^*=\left(f'-\sum_{j=1}^r c_jv_q\psi_j\overline{\varphi_j}\right)-\left(h_{q+1}k_{q+1}^*-h_qk_q^*-\sum_{j=1}^r c_jv_q\psi_j\overline{\varphi_j}\right),$$

from (6.4), (6.5) and (6.1) we deduce

$$\|f-h_{q+1}k_{q+1}^*\|_{L^1/H_0^1}\leqq\vartheta v_q+2\varepsilon_q=v_{q+1}; \tag{6.7}$$

and (6.6), (6.7) yield (6.2) for the h_{q+1}, k_{q+1} just defined. The construction by induction is thus established for all s.

From (6.2) now follows that h_s, k_s converge (strongly in $\mathfrak{H}$) to some limits h, k, and that $h_sk_s^*$ converges in L^1/H_0^1 to $f^{\cdot}$. Since $h_s\to h$, $k_s\to k$ obviously also imply $\|h_sk_s^*-hk^*\|_{L^1}\to 0$ we conclude that $\|f-hk^*\|_{L^1/H_0^1}=0$; thus completing the proof of the theorem.

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A note on the Radon—Nikodym theorem of Pedersen and Takesaki

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0. Introduction. The Radon—Nikodym theorem of PEDERSEN and TAKESAKI [4] shows the existence and uniqueness of a density of certain semi-finite weights ψ with respect to a given normal, faithful and semi-finite weight φ, the density being a self-adjoint, positive operator. Here, it is shown that — with a suitable extension of the definition of a density — this theorem remains true without the assumption of semifiniteness of the weight ψ. Paragraph 2 sums up some facts about projections which are used in the sequel.

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1. Basic notations and definitions. Let $\mathfrak{A}$ be a von Neumann algebra. A weight φ on $\mathfrak{A}$ is a map defined on $\mathfrak{A}^+$ with values in $\overline{\mathbf{R}}^+ := \mathbf{R}^+ \cup \{\infty\}$ which is additive and positive homogeneous $(0 \cdot \infty := 0)$.

A weight φ on $\mathfrak{A}$ defines the left ideals

$$\mathfrak{n}_\varphi := \{A \in \mathfrak{A} \mid \varphi(A^*A) < \infty\} \quad \text{and} \quad N_\varphi := \{A \in \mathfrak{A} \mid \varphi(A^*A) = 0\}$$

and the convex cone

$$\mathfrak{m}_\varphi^+ := \{A \in \mathfrak{A}^+ \mid \varphi(A) < \infty\}.$$

The weight φ is called faithful if it is strictly positive, semi-finite if the identity of $\mathfrak{A}$ is the ultraweak limit of elements of $\mathfrak{m}_\varphi^+$, and normal if $\varphi(\sup A_i) = \sup \varphi(A_i)$ for every increasing bounded net in $\mathfrak{A}^+$.

If φ is semi-finite, normal and faithful, then on $\mathfrak{n}_\varphi$ an inner product is defined by $(A, B) := \dot{\varphi}(B^*A)$ ($\dot{\varphi}$ the canonical extension of φ to $\mathfrak{m}_\varphi := \mathfrak{m}_\varphi^+ - \mathfrak{m}_\varphi^+ = \mathfrak{n}_\varphi^* \mathfrak{n}_\varphi$). The usual Gelfand—Naimark—Segal construction gives a faithful, normal represen-

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tation π_φ of $\mathfrak{A}$ on H_φ, the completion of $\mathfrak{n}_\varphi$ with respect to the inner product $(.,.)$. The involution * of $\mathfrak{A}$ extends from $\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ to a closed conjugate linear operator S on H_φ. If $S = J\Delta^{1/2}$ (the polar decomposition of S) then J is a conjugate linear isometry and Δ is a self-adjoint, positive, non-degenerate operator. Every $t \in \mathbf{R}$ defines a unitary operator Δ^{it} on H_φ, $A \to \Delta^{-it} A \Delta^{it}$ leaves $\pi_\varphi(\mathfrak{A})$ invariant and so gives rise to a *-automorphism σ_t of $\mathfrak{A}$. The strongly continuous one parameter group $\Sigma_\varphi := \{\sigma_t | t \in \mathbf{R}\}$ is called the modular automorphism group of φ. The weight φ fulfils the Kubo—Martin—Schwinger (KMS) condition with respect to Σ_φ, i.e. for all $A, B \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ there is a continuous, bounded function f on $\{z \in \mathbf{C} | 0 \leqq \operatorname{Im} z \leqq 1\}$, holomorphic in the interior and such that for all $t \in \mathbf{R}$

$$f(t) = \varphi(\sigma_t(A)B),\ f(t+i) = \varphi(B\sigma_t(A)).$$

If Σ' is a strongly continuous one parameter group on $\mathfrak{A}$ and φ is KMS with respect to Σ', then $\Sigma' = \Sigma_\varphi$.

A semi-finite, faithful, normal weight φ is a trace iff Σ_φ is trivial.

2. Semi-finite projections. Let φ be a fixed normal weight on $\mathfrak{A}$. If $A \in N_\varphi \cap \mathfrak{A}^+$ and if E is the spectral measure of A, then $\operatorname{supp} A = \sup E(]1/n, \infty[)$. Now, $0 \leqq 1/nE(]1/n, \infty[) \leqq A$, so $E(]1/n, \infty[)$ is in N_φ (since φ is additive), and we have that $\operatorname{supp} A$ is in N_φ (since φ is normal).

It follows that given two projections $P, Q \in N_\varphi$, their supremum (in the set of all projections of $\mathfrak{A}$) $P \vee Q = \operatorname{supp}(P+Q)$ is again in N_φ. So the set of all projections in N_φ is an increasing family with supremum P_φ. Since φ is normal, P_φ is again in N_φ, hence (Combes [1], p. 75): The set of all projections of N_φ has a largest element P_φ.

Remarks.

a) If $A \in N_\varphi$, then $\operatorname{supp} A^*A = \operatorname{supp} A$ is in N_φ, so $\operatorname{supp} A \leqq P_\varphi$. Thus, $A = A \operatorname{supp} A = AP_\varphi$ and $N_\varphi \subset \mathfrak{A}P_\varphi$. Since N_φ is a left ideal and $P_\varphi \in N_\varphi$, it follows that $N_\varphi = \mathfrak{A}P_\varphi$.

b) If g is a *-automorphism of $\mathfrak{A}$ and if φ is g-invariant, then $\varphi(g(P_\varphi)) = \varphi(P_\varphi) = \varphi(g^{-1}(P_\varphi)) = 0$, so $g(P_\varphi) \leqq P_\varphi \leqq g^{-1}(P_\varphi)$ and P_φ is g-invariant.

The following example shows that the set of projections of $\mathfrak{m}_\varphi^+$ is not an increasing family.

Example. Let H be an infinite-dimensional Hilbert space with an orthonormal basis $(e_n)_{n \in \mathbf{N}}$. Define $\mathfrak{A} := L(H)$, for $n \in \mathbf{N}$ define $f_n \in H$ by

$$f_n := (1 - 1/2^n)^{1/2} e_{2n-1} + (1/2^n)^{1/2} e_{2n},$$

and define projections $P_1, P_2 \in \mathfrak{A}$ by

$$P_1 := \sum_{n=1}^{\infty} e_{2n-1} \otimes e_{2n-1}, \quad P_2 := \sum_{n=1}^{\infty} f_n \otimes f_n.$$

Define the weight φ on $\mathfrak{A}$ by

$$\varphi := \sum_{n=1}^{\infty} \omega_{e_{2n}}.$$

Then φ is normal, $\varphi(P_1)=0$, $\varphi(\mathrm{Id})=\infty$ and $\varphi(e_{2n}\otimes e_{2n})=1$ for all $n\in\mathbf{N}$. Since $\mathrm{Id}=P_1+\sum_{n=1}^{\infty} e_{2n}\otimes e_{2n}$, this shows that φ is semi-finite.

$$\varphi(P_2) = \sum_{n=1}^{\infty}(P_2e_{2n}, e_{2n}) = \sum_{n=1}^{\infty}((e_{2n}, f_n)f_n, e_{2n}) = \sum_{n=1}^{\infty}|(e_{2n}, f_n)|^2 = \sum_{n=1}^{\infty} 1/2^n = 1.$$

Now, it is easy to see that $P_1 \vee P_2=\mathrm{Id}$. Thus we have: P_1 and P_2 are in $\mathfrak{m}_\varphi^+$ and $P_1 \vee P_2$ is not, i.e. the set of all projections in $\mathfrak{m}_\varphi^+$ is not an increasing family.

Definition. Let P be a projection in $\mathfrak{A}$.

a) P is called *semi-finite* (with respect to φ) if the restriction of φ to $P\mathfrak{A}^+P$ is semi-finite;

b) P is called *σ-finite* (with respect to φ) if there is a sequence $(P_n)_{n\in\mathbf{N}}$ of mutually orthogonal projections of $\mathfrak{m}_\varphi^+$ with $P=\sum P_n$.

Clearly, every σ-finite projection is semi-finite.

2.1 Lemma. *A projection P of $\mathfrak{A}$ is σ-finite iff there is an $A\in\mathfrak{m}_\varphi^+$ with $P=\operatorname{supp} A$.*

Proof. Let P be σ-finite, $P=\sum_{n=1}^{\infty} P_n$ and $P_n\in\mathfrak{m}_\varphi^+$. One can assume that $\varphi(P_n)\neq 0$ for all n. Define $A:=\sum_{n=1}^{\infty}\left(1/(2^n \max(\varphi(P_n), 1))\right)P_n$. Then $\varphi(A)\leqq 1$, so $A\in\mathfrak{m}_\varphi^+$. On the other hand, $\operatorname{supp} A=P$. To prove the other direction, let A be in $\mathfrak{m}_\varphi^+$ and $P:=\operatorname{supp} A$. Let E be the spectral measure of A. Define $E_1:=E(]1, \infty[)$, $E_n:=E(]1/n, 1/(n-1)[)$ for $n\geqq 2$. Then $1/nE_n\leqq A$, so $E_n\in\mathfrak{m}_\varphi^+$ and $P=\sum_{n=1}^{\infty} E_n$.

2.2 Corollary. *With P_1, P_2 σ-finite projections, $P_1\vee P_2$ is σ-finite.*

Proof. Let A_1, A_2 be in $\mathfrak{m}_\varphi^+$ with $\operatorname{supp} A_i=P_i$. Then $\operatorname{supp}(A_1+A_2)=P_1\vee P_2$.

2.3 Proposition (Characterization of semi-finite projections). *Let P be a projection in $\mathfrak{A}$. Then the following are equivalent:*

a) P is semi-finite;

b) $P=\vee P_i$, where $(P_i)_{i\in I}$ is a family of σ-finite projections;

c) $P=\vee P_i$, where $(P_i)_{i\in I}$ is a family of projections in $\mathfrak{m}_\varphi^+$;

d) $P=\sup P_i$, where $(P_i)_{i\in I}$ is an increasing family of σ-finite projections.

Proof. a)$\Rightarrow$b): Assume b) false and a) true.

Define $\mathfrak{S}:=\{S\in\mathfrak{A}|\ S\ \sigma\text{-finite projection},\ S\leqq P\}$, $Q:=P-\bigvee\mathfrak{S}$.

By assumption $Q\neq 0$, so there is an ultraweakly continuous state f on $\mathfrak{A}$ with $E:=\operatorname{supp} f\leqq Q$. Now, if $(A_i)_{i\in I}$ is a net in $P\mathfrak{A}P\cap\mathfrak{m}_\varphi^+$ converging ultraweakly to P, then for all $i\in I$, $\operatorname{supp} A_i$ is σ-finite and $\operatorname{supp} A_i\leqq P$; thus, $\operatorname{supp} A_i$ is in $\mathfrak{S}$, i.e. $A_iQ=0$. Now, $|f(P-A_i)|=|f(E(P-A_i)E)|=|f(EPE)|=1$, which is a contradiction to (A_i) converging ultraweakly to P. The proofs of the other implications are easy consequences of the definition of σ-finite.

2.4 Corollary. *For every family (P_i) of semi-finite projections, $\bigvee P_i$ is semi-finite.*

2.5 Corollary (cf. Pedersen—Takesaki [4]). *The set of all semi-finite projections (with respect to φ) contains a largest element denoted by Q_φ.*

2.6 Corollary. *If g is a *-automorphism of $\mathfrak{A}$ and φ is g-invariant, then Q_φ is g-invariant.*

Proof. The Proposition shows that

$$Q_\varphi=\bigvee\{P|\ P\ \text{projection},\ P\in\mathfrak{m}_\varphi^+\}.$$

This set is g-invariant by assumption and g is a *-automorphism, thus

$$g(Q_\varphi)=\bigvee\{g(P)|\ P\ \text{projection},\ P\in\mathfrak{m}_\varphi^+\}=Q_\varphi.$$

If H is an infinite-dimensional Hilbert space with an orthonormal basis $(e_n)_{n\in\mathbf{N}}$, then $\varphi:=\sum n\cdot\omega_{e_n}$ defines a normal, semi-finite weight on $\mathfrak{A}:=L(H)$.

If $x:=\sum_{n\in\mathbf{N}} e_n/n$ and P_x is the projection on $\langle x\rangle$, then $\varphi(P_x)=\infty$. Every $A\in P_x\mathfrak{A}P_x$ is a multiple of P_x, so P_x is not semi-finite. This shows that if P is a semi-finite projection and Q is a projection with $Q\leqq P$, then Q is not necessarily semi-finite. However, if φ is semi-finite, normal and faithful and if P is Σ_φ-invariant, then P is semi-finite (cf. Combes—Delaroche [2]). For then by [4], thm 3.6, $P\mathfrak{m}_\varphi\subset\mathfrak{m}_\varphi$ and $\mathfrak{m}_\varphi P\subset\mathfrak{m}_\varphi$; so, if (A_i) is a net in $\mathfrak{m}_\varphi^+$ which converges ultraweakly to the identity, (PA_iP) is a net in $P\mathfrak{A}P\cap\mathfrak{m}_\varphi^+$ which converges ultraweakly to P.

3. The Radon—Nikodym theorem. For the rest of this paragraph let $\mathfrak{A}$ be a von Neumann algebra and φ a semi-finite, normal, faithful weight on $\mathfrak{A}$. The von Neumann algebra of all invariant elements of $\mathfrak{A}$ with respect to the modular automorphism group Σ_φ will be denoted by $\mathfrak{A}^\varphi$.

For the convenience of the reader some of the notations and results of [4] will be given.

Let H be a self-adjoint, positive operator in $\mathfrak{A}^\varphi$. Then, the map $A \to \varphi(H^{1/2}AH^{1/2})$ is a normal, semi-finite, Σ_φ-invariant weight on $\mathfrak{A}$, denoted by φ_H. If H is a self-adjoint, positive operator affiliated to $\mathfrak{A}^\varphi$, then for every $\varepsilon > 0$ the operator H_ε is defined by $H_\varepsilon := H(1+\varepsilon H)^{-1}$. Then, $H_\varepsilon \in \mathfrak{A}^{\varphi^+}$ and the map $A \to \sup_{\varepsilon>0} \varphi_{H_\varepsilon}(A)$ is a normal, semi-finite, Σ_φ-invariant weight on $\mathfrak{A}$, again denoted by φ_H.

Then, the main result of [4] is the following:

Theorem (Radon—Nikodym theorem of Pedersen and Takesaki). *Let ψ be a semi-finite, normal Σ_φ-invariant weight on $\mathfrak{A}$. Then, there is a unique self-adjoint operator H affiliated with $\mathfrak{A}^\varphi$ such that $\psi = \varphi_H$.*

There is a commutative analogue of this theorem, cf. [3], p. 245, lemme 1: If $\mathfrak{A}$ is commutative, i.e. isomorphic to an $L^\infty(Z, \mu)$ with locally compact Z and positive Radon measure μ, denote by $\widehat{\mathfrak{Z}}^+$ the set of all positive, measurable functions on Z with values in $\overline{\mathbf{R}}^+$ modulo locally null-functions. The weight φ is then a semi-finite, normal, faithful trace on $\mathfrak{A}$. Then, for every normal weight ψ ($=$normal trace) on $\mathfrak{A}$ there is a unique $H \in \widehat{\mathfrak{Z}}^+$ such that

$$\hat{\psi}(A) = \hat{\varphi}(HA) \quad \text{for all} \quad A \in \widehat{\mathfrak{Z}}^+,$$

where $\hat{\psi}$, $\hat{\varphi}$ denote the canonical extensions of ψ and φ to $\widehat{\mathfrak{Z}}^+$.

Here, ψ need not be semi-finite. In the following it is shown that the same is true for the theorem of Pedersen and Takesaki with a suitable definition of the density H.

Definition. Let $\mathfrak{B}$ be a von Neumann algebra. A spectral measure on the Borel sets $B(\overline{\mathbf{R}}^+)$ of the extended positive real line with values in the set of self-adjoint projections of $\mathfrak{B}$ is called a ($\mathfrak{B}$*-valued*) *extended spectral measure.* The set of all $\mathfrak{B}$-valued extended spectral measures is denoted by $\widehat{\mathfrak{B}}^+$.

3.1 Lemma. *If $E \in \widehat{\mathfrak{A}^{\varphi^+}}$ and $A \in \mathfrak{A}^+$, then the map $m_{\varphi,A}$ on $B(\overline{\mathbf{R}}^+)$ with values in $\overline{\mathbf{R}}^+$, defined by*

$$m_{\varphi,A}(\Delta) := \varphi\big(E(\Delta)AE(\Delta)\big) \quad (\Delta \in B(\overline{\mathbf{R}}^+))$$

is a measure on $\overline{\mathbf{R}}^+$.

Proof. Since $E(\Delta) \in \mathfrak{A}^\varphi$, prop. 4.1 of [4] shows that $m_{\varphi,A}$ is additive. The spectral measure E is σ-additive and, by prop. 4.2 of [4], the map $E(\Delta) \to \varphi\big(E(\Delta)AE(\Delta)\big)$ is normal.

Definition. Let E be in $\widehat{\mathfrak{A}^{\varphi^+}}$. For $A \in \mathfrak{A}^+$ define $m_{\varphi,A}$ as in Lemma 3.1. Define φ_E: $\mathfrak{A}^+ \to \overline{\mathbf{R}}^+$ by

$$\varphi_E(A) := \int_{\overline{\mathbf{R}}^+} \lambda \, dm_{\varphi,A}(\lambda) \Big(= \int_{\overline{\mathbf{R}}^+} \lambda \, d\varphi(E_\lambda A E_\lambda)\Big) \quad (A \in \mathfrak{A}^+).$$

3.2 Lemma. *If* $E \in \widehat{\mathfrak{A}^{\varphi +}}$, *then* φ_E *is a normal and* Σ_φ*-invariant weight on* $\mathfrak{A}$.

Proof. By [4], prop. 4.1 the map $A \to \varphi(E(\Delta)AE(\Delta))$ is a weight on $\mathfrak{A}$ for every $\Delta \in B(\overline{\mathbf{R}}^+)$. Thus, the map φ_E is additive and positive homogeneous.

Next, take a $g \in \Sigma_\varphi$. Then, for all $A \in \mathfrak{A}^+$ and for all $\Delta \in B(\overline{\mathbf{R}}^+)$ we have

$$\varphi(E(\Delta)g(A)E(\Delta)) = \varphi(E(\Delta)AE(\Delta)),$$

from which it follows that φ_E is Σ_φ-invariant.

Now, we show that φ_E is normal. Take an increasing family (A_i) in $\mathfrak{A}^+$ with $\sup A_i = A$. Then by the positivity of all occurring values the following holds:

$$\begin{aligned}\varphi_E(A) &= \int_{\overline{\mathbf{R}}^+} \lambda \, d\varphi(E_\lambda A E_\lambda) = \\ &= \sup\left\{\sum \lambda_j \varphi(E(\Delta_j)AE(\Delta_j)) \,\middle|\, \sum \Delta_j = \overline{\mathbf{R}}^+, \lambda_j = \inf \Delta_j\right\} = \\ &= \sup\left\{\sum \lambda_j \varphi(E(\Delta_j)(\sup_i A_i)E(\Delta_j))\right\} = \\ &= \sup_i \sup\left\{\sum \lambda_j \varphi(E(\Delta_j)A_iE(\Delta_j))\right\} = \\ &= \sup_i \varphi_E(A_i).\end{aligned}$$

The following lemma shows that the definition of φ_E is indeed an extension of the definition of φ_H by Pedersen and Takesaki.

3.3 Lemma. *Let H be a self-adjoint, positive operator affiliated to* $\mathfrak{A}^\varphi$. *If E is the canonical spectral measure on* $B(\overline{\mathbf{R}}^+)$ *defined by H, then* $\varphi_H = \varphi_E$.

Proof. Take $A \in \mathfrak{A}^+$. First, if f is a simple real-valued function on $\overline{\mathbf{R}}^+$, $f = \sum_{i=1}^n \alpha_i 1_{\Delta_i}$, we have

$$\varphi_{f(H)}(A) = \sum_{i=1}^n \alpha_i \varphi(1_{\Delta_i}(H)A1_{\Delta_i}(H)) = \int_{\overline{\mathbf{R}}^+} f(\lambda)\, d\varphi(E_\lambda A E_\lambda).$$

Next, take an increasing sequence (f_n) of simple functions which converges to $\lambda(1+\varepsilon\lambda)^{-1}$. Then $H_\varepsilon = \sup_j f_n(H)$, so

$$\begin{aligned}\varphi_{H_\varepsilon}(A) = \sup_n \varphi_{f_n(H)}(A) &= \sup_n \int_{\overline{\mathbf{R}}^+} f_n(\lambda)\, d\varphi(E_\lambda A E_\lambda) = \\ &= \int_{\overline{\mathbf{R}}^+} \lambda(1+\varepsilon\lambda)^{-1}\, d\varphi(E_\lambda A E_\lambda).\end{aligned}$$

Finally, by definition we have

$$\begin{aligned}\varphi_H(A) = \sup_\varepsilon \varphi_{H_\varepsilon}(A) &= \sup_\varepsilon \int_{\overline{\mathbf{R}}^+} \lambda(1+\varepsilon\lambda)^{-1}\, d\varphi(E_\lambda A E_\lambda) = \\ &= \int_{\overline{\mathbf{R}}^+} \lambda\, d\varphi(E_\lambda A E_\lambda) = \varphi_E(A).\end{aligned}$$

3.4 Theorem (Radon—Nikodym theorem of Pedersen and Takesaki — generalized version). *Let ψ be a normal Σ_φ-invariant weight on $\mathfrak{A}$. Then, there is a unique $E\in\widehat{\mathfrak{A}^{\varphi+}}$ such that $\psi=\varphi_E$.*

Proof. *Existence:* Define $\tilde{\mathfrak{A}}:=Q_\psi\mathfrak{A}Q_\psi$. For a map f on $\mathfrak{A}^+$ define $\tilde{f}$ to be the restriction of f to $\tilde{\mathfrak{A}}^+$.

Since ψ is Σ_φ-invariant, so is Q_ψ (2.6), i.e. $Q_\psi\in\mathfrak{A}^\varphi$ and so Q_ψ is semi-finite with respect to φ (see the end of § 2). Thus, $\tilde{\varphi}$ is semi-finite, normal and faithful and $\tilde{\psi}$ is semi-finite and normal.

We show that $\tilde{\psi}$ is $\Sigma_{\tilde{\varphi}}$-invariant: Since Q_ψ is Σ_φ-invariant, Σ_φ leaves $\tilde{\mathfrak{A}}$ invariant; now, $\tilde{\varphi}$ fulfils the KMS condition with respect to the restriction of Σ_φ to $\tilde{\mathfrak{A}}$, so $\tilde{\Sigma}_\varphi$ and $\Sigma_{\tilde{\varphi}}$ coincide. Thus, for all $\hat{g}\in\Sigma_{\tilde{\varphi}}$ there is a $g\in\Sigma_\varphi$ such that $\hat{g}=\tilde{g}$; hence, $\tilde{\psi}(\hat{g}(A))=$ $=\tilde{\psi}(\tilde{g}(A))=\psi(g(A))=\psi(A)=\tilde{\psi}(A)$ $(A\in\tilde{\mathfrak{A}}^+)$.

So, the Radon—Nikodym theorem of Pedersen and Takesaki gives a unique self-adjoint and positive operator H affiliated to $\tilde{\mathfrak{A}}^{\tilde{\varphi}}$ such that $\tilde{\psi}=\tilde{\varphi}_H$. If E_H is the spectral measure of H, define $E\in\widehat{\mathfrak{A}^{\varphi+}}$ by $\mathrm{Rest}_{B(\mathbf{R}^+)}E:=E_H$ and $E(\{\infty\}):=Q_\psi^\perp$.

Next we show that $\psi=\varphi_E$:

Let A be in $\mathfrak{A}^+$.

Case 1: $Q_\psi^\perp AQ_\psi^\perp\neq0$. Since φ is faithful, $\varphi(Q_\psi^\perp AQ_\psi^\perp)\neq0$ and it follows that $\varphi_E(A)\geqq\infty\cdot\varphi(Q_\psi^\perp AQ_\psi^\perp)=\infty$. Assume that $\psi(A)$ is finite. Then, by Lemma 2.1, supp A is σ-finite, and it follows that supp $A\leqq Q_\psi$ and $Q_\psi^\perp AQ_\psi^\perp=0$ which is a contradiction. Thus $\psi(A)=\infty$ and so $\psi(A)=\varphi_E(A)$.

Case 2: $Q_\psi^\perp AQ_\psi^\perp=0$. Now, since A is positive, we have that $Q_\psi AQ_\psi=A$ (i.e. $A\in\tilde{\mathfrak{A}}^+$) and $\psi(A)=\tilde{\psi}(A)=\tilde{\varphi}_H(A)=\tilde{\varphi}_{E_H}(A)=\varphi_E(A)$.

Uniqueness. Suppose $F\in\widehat{\mathfrak{A}^{\varphi+}}$ with $\psi=\varphi_F$. Then $Q:=F(\overline{\mathbf{R}}^+)\in\mathfrak{A}^\varphi$, so Q is semi-finite with respect to φ. If f is a map on $\mathfrak{A}^+$, denote the restriction of f to $(Q\mathfrak{A}Q)^+$ by $\hat{f}$. Then, $\hat{\varphi}$ is semi-finite, normal and faithful and $\widehat{\varphi_F}(.)=\int_{\overline{\mathbf{R}}^+}\lambda d\varphi(F.F)$.

If K is the (canonical) self-adjoint operator with spectral measure $\mathrm{Rest}_{B(\mathbf{R}^+)}F$, then by 3.3, $\widehat{\varphi_F}=\hat{\varphi}_{\mathrm{Rest}_{B(\mathbf{R}^+)}F}=\hat{\varphi}_K$, so $\widehat{\varphi_F}$ is semi-finite by [4], prop. 4.2, from which it follows that Q is semi-finite with respect to $\varphi_F=\psi$. Thus $Q\leqq Q_\psi=Q_{\varphi_F}$. On the other hand, if $P\in\mathfrak{A}$ is a projection with $\varphi_F(P)<\infty$, then by the faithfulness of φ, $F(\{\infty\})PF(\{\infty\})=0$, so $P\leqq Q$. These facts together give $Q=Q_\psi$ (see § 2).

In particular, the argument applies to the spectral measure E (where E is as in the proof of existence), so one has $E(\mathbf{R}^+)=F(\mathbf{R}^+)$ (i.e. $\tilde{\psi}=\hat{\psi}=\hat{\varphi}_K$), and by the uniqueness of K it follows that $E=F$.

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On the very strong and mixed approximations

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1. Let f be a continuous and 2π-periodic function. Denote by $E_n(f)$, $\omega(f;\delta)$ and $s_k(x)=s_k(f;x)$ its best uniform approximation by trigonometric polynomials of degree at most n, its modulus of continuity, and the k-th partial sum of its Fourier series, respectively.

If ω is a modulus of continuity and $r\geqq 0$ is an integer we define W^rH^ω to be the class of those functions f for which $\omega(f^{(r)};\delta)\leqq K_f\omega(\delta)$ $(\delta\in[0,2\pi])$ holds with some constant K_f.

In [3], following works of ALEXITS, KRÁLIK and LEINDLER, we proved

Theorem A. *If $p,\beta,\gamma>0$ and $f\in W^rH^\omega$ then we have*

$$h_n(f,p,\beta;x)=\left\{\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_k(x)-f(x)|^p\right\}^{\frac{1}{p}}\leqq KH^{p,\beta,n}_{r,\omega}\ ^{*)}$$

and

$$\sigma_n^\gamma|f,p;x|=\left\{\frac{1}{A_n^\gamma}\sum_{k=0}^{n}A_{n-k}^{\gamma-1}|s_k(x)-f(x)|^p\right\}^{\frac{1}{p}}\leqq KH^{p,1,n}_{r,\omega}\quad\left(A_n^\gamma=\binom{n+\gamma}{n}\right),$$

where

$$H^{p,\beta,n}_{r,\omega}=\left\{\frac{1}{(n+1)^\beta}\sum_{k=1}^{n}(k+1)^{\beta-1}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p\right\}^{\frac{1}{p}}.$$

Moreover, there are functions $f\in W^rH^\omega$ for which

$$h_n(f,p,\beta;0)\geqq cH^{p,\beta,n}_{r,\omega}\quad\text{and}\quad\sigma_n^\gamma|f,p;0|\geqq cH^{p,1,n}_{r,\omega}\quad(n=1,2,\ldots)$$

for some $c>0$.

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*) K, c with or without subscripts denote constants not necessarily the same at each occurrence.

LEINDLER [2] raised the question: What can we say about the order of the strong approximation if we replace the sequence of the partial sums by a subsequence (very strong approximation) or by a permutation of such a subsequence (mixed approximation). In this paper we shall deal with these questions.

Our main result is

Theorem 1. *Let* $E_n(f)\leqq K\varrho_n$ $(n=1, 2, \ldots)$, *where the sequence* $\{\varrho_n\}$ *satisfies the condition*

(1.1) $$i\varrho_{2^i n} \leqq K\varrho_n \quad (i, n = 1, 2, \ldots).$$

There exists a constant K_p, *independent of* n *and of the sequence* $v=\{v_k\}_{k=0}^{\infty}$ *for which*

$$\left\{\frac{1}{n}\sum_{k=n+1}^{2n}|s_{v_k}(x)-f(x)|^p\right\}^{\frac{1}{p}} \leqq K_p\varrho_n \quad (p>0).$$

We shall use Theorem 1 to prove

Theorem 2. *Let us suppose that* $f\in W^r H^{\omega}$ *where either* $r\geqq 1$ *or* $r=0$, *and* ω *satisfies the condition*

(1.2) $$i\omega\left(\frac{1}{2^i n}\right) \leqq K\omega\left(\frac{1}{n}\right) \quad (i, n_i = 1, 2, \ldots).$$

We have for any $\gamma, \beta, p>0$ *and for an arbitrary sequence* $v=\{v_k\}$

(1.3) $$\left\{\frac{1}{(n+1)^{\beta}}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_{v_k}(x)-f(x)|^p\right\}^{\frac{1}{p}} \leqq KH_{r,\omega}^{p,\beta,n}$$

and

(1.4) $$\left\{\frac{1}{A_n^{\gamma}}\sum_{k=0}^{n}A_{n-k}^{\gamma-1}|s_{v_k}(x)-f(x)|^p\right\}^{\frac{1}{p}} \leqq KH_{r,\omega}^{p,1,n}$$

where K *is independent of* n *and* v.

If, moreover, for every function $f\in W^r H^{\omega}$ *and for every sequence* $\{v_k\}$ *we have*

(1.5) $$\left\{\frac{1}{n}\sum_{k=n+1}^{2n}|s_{v_k}(x)-f(x)|^p\right\}^{\frac{1}{p}} = O\left(\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\right)$$

then either $r\geqq 1$ *or* $r=0$, *and* (1.2) *is true.*

If $\omega(\delta)=\delta^{\alpha}$ $(0<\alpha\leqq 1)$ then (1.2) is satisfied and Theorem 2 shows that there is no difference with respect to the approximation order between the strong and the very strong approximation of functions in the classes W^r Lip α $(r=0, 1, \ldots; 0<\alpha\leqq 1)$. This is an answer to one of Leindler's problems (see the last two question of [2]).

We mention that the assumption "either $r \geqq 1$ or $r=0$ and (1.2)" is also necessary that (1.3) and (1.4) should be satisfied, namely for $\beta > (r+1)p$ we obtain by Corollary of [3, Theorem 1]

$$\left\{\frac{1}{n}\sum_{k=n+1}^{2n}|s_{\nu_k}(x)-f(x)|^p\right\}^{\frac{1}{p}} = O\left(\left\{\frac{1}{(2n+1)^\beta}\sum_{k=0}^{2n}(k+1)^{\beta-1}|s_{\nu_k}(x)-f(x)|^p\right\}^{\frac{1}{p}}\right) =$$

$$= O\left(\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\right)$$

so the second part of Theorem 2 is applicable.

Finally we turn to the mixed approximation. Let N be the collection of the natural numbers.

Theorem 3. *Let $\pi: N \to N$ be an injection, $p>0$ and $f \in W^r H^\omega$, where either $r \geqq 1$ or $r=0$, and* (1.2) *is true for ω.*

(i) *If $0<\beta\leqq 1$ then*

$$h_n(f, p, \beta, \pi; x) = \left\{\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_{\pi(k)}(x)-f(x)|^p\right\}^{\frac{1}{p}} \leqq KH_{r,\omega}^{p,\beta,n}.$$

(ii) *If $\beta>1$ and $\sum_{k=1}^{\infty}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p=\infty$ then $h_n(f, p, \beta, \pi; x) \leqq KH_{r,\omega}^{p,1,n}$.*

(iii) *If $\beta>1$ and $\sum_{k=1}^{\infty}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p<\infty$ then $h_n(f, p, \beta, \pi; x) = o(H_{r,\omega}^{p,1,n})$,*

uniformly in x.

(iv) *If $0<p<1$ and $\sum_{k=1}^{\infty}(k+1)^{\gamma-1}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p=\infty$ then*

$$\sigma_n^\gamma|f, p, \pi; x| = \left\{\frac{1}{A_n^\gamma}\sum_{k=0}^{n}A_{n-k}^{\gamma-1}|s_{\pi(k)}(x)-f(x)|^p\right\}^{\frac{1}{p}} \leqq KH_{r,\omega}^{p,\gamma,n}.$$

(v) *If $0<p<1$ and $\sum_{k=1}^{\infty}(k+1)^{\gamma-1}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p<\infty$ then*

$$\sigma_n^\gamma|f, p, \pi; x| = o(H_{r,\omega}^{p,\gamma,n}),$$

uniformly in x.

(vi) *If $\gamma\geqq 1$ then $\sigma_n^\gamma|f, p, \pi; x| \leqq KH_{r,\omega}^{p,1,n}$.*

The above constant K is independent of π, n and x.

These estimations are best possible, namely if $\varrho_n\to 0$ arbitrarily, then there exist $f\in W^r H^\omega$ and $c>0$ such that, according to the cases (i)—(vi) *separately, there* *are permutations π of N for which*

$$(1.8)\qquad h_n(f, p, \beta, \pi;\ x) = \begin{cases} cH_{r,\omega}^{p,\beta,n} & \text{(i)} \\ cH_{r,\omega}^{p,1,n} & \text{(ii)} \\ c\varrho_n H_{r,\omega}^{p,1,n} & \text{(iii)} \end{cases}$$

and

$$(1.9)\qquad \sigma_n^{\gamma}|f, p, \pi;\ 0| \geqq \begin{cases} cH_{r,\omega}^{p,\gamma,n} & \text{(iv)} \\ c\varrho_n H_{r,\omega}^{p,\gamma,n} & \text{(v)} \\ cH_{r,\omega}^{p,1,n} & \text{(vi)} \end{cases}$$

are satisfied for infinitely many n.

Corollary. *Under the assumptions of Theorem 3, for*

$$h_n(p, \beta) = \sup_{f:\ \omega(f;\ \delta)\leqq\omega(\delta)}\ \sup_{\pi;\ x} h_n(f, p, \beta, \pi;\ x)$$

and

$$\sigma_n(p, \beta) = \sup_{f:\ \omega(f;\ \delta)\leqq\omega(\delta)}\ \sup_{\pi;\ x} \sigma_n^{\beta}|f, p, \pi;\ x|$$

we have

$$c_1 H_{r,\omega}^{p,\beta^*,n} \leqq h_n(p, \beta),\ \sigma_n(p, \beta) \leqq c_2 H_{r,\omega}^{p,\beta^*,n} \quad (c_1 > 0,\ n = 1, 2, \ldots)$$

where $\beta^* = \min(1, \beta)$.

2. To prove our theorems we require the following two lemmas.

Lemma 1. [3, Theorem 4] *There exists a K_p depending only on $p(>0)$ for which*

$$\left\{\frac{1}{r}\sum_{i=1}^{r}|s_{k_i} - f|^p\right\}^{\frac{1}{p}} \leqq K_p E_{k_1}(f)\log\frac{2n}{r},$$

whenever $1 \leqq k_1 < k_2 < \ldots < k_r \leqq n$.

Lemma 2. [3, Lemma 5] *Let ω be an arbitrary modulus of continuity. Then there are functions $f \in W^0 H^\omega$ such that*

$$(2.1)\qquad |s_{n\pm\lambda}(f;\ 0) - f(0)| > 10^{-2}\omega\left(\frac{1}{n}\right)\log\frac{n}{\lambda} \quad (\lambda \leqq e^{-100}n)$$

is true for infinitely many n.

We can also require that (2.1) *be true for infinitely many n belonging to a given sequence.*

3. Proof of Theorem 1. Let k_i be the number of those ν_t for which

$$2^i n < \nu_t \leqq 2^{i+1} n \quad (n < t \leqq 2n,\ i = 0, 1, \ldots).$$

By Lemma 1 we have

$$\sum_{2^i n < \nu_t \leqq 2^{i+1} n} |s_{\nu_t}(x) - f(x)|^p \leqq Kk_i\left(E_{2^i n}(f)\right)^p \left(\log\frac{2^{i+1}n}{k_i}\right)^p \leqq Kk_i\varrho_{2^i n}^p \left(\log\frac{2^{i+1}n}{k_i}\right)^p,$$

and thus it is enough to show that

$$S=\frac{1}{n}\sum_{k_i>0}k_i\left(\frac{\varrho_{2^i n}}{\varrho_n}\right)^p\left(\log\frac{2^{i+1}n}{k_i}\right)^p\leqq K, \tag{3.1}$$

where K is independent from ν and n.

Now,

$$S\leqq K_p\left(\frac{1}{n}\sum_{k_i>0}k_i\left(\frac{\varrho_{2^i n}(i+1)}{\varrho_n}\right)^p+\frac{1}{n}\sum_{k_i>0}\left(\frac{\varrho_{2^i n}}{\varrho_n}\right)^p k_i\left(\log\frac{n}{k_i}\right)^p\right)=S_1+S_2.$$

(1.1) gives

$$S_1\leqq K\frac{1}{n}\sum_{k_i>0}k_i\leqq K,$$

$$S_2\leqq K\frac{1}{n}\sum_{k_i>0,\, i>0}\left(\frac{1}{i}\right)^p k_i\left(\log\frac{n}{k_i}\right)^p+\frac{k_0}{n}\left(\log\frac{n}{k_0}\right)^p=S_{21}+S_{22}+\frac{k_0}{n}\left(\log\frac{n}{k_0}\right)^p$$

(if $k_0=0$ then the last member is missing), where the summation in S_{21} is extended to the i's satisfying the condition $\frac{1}{i}\log\frac{n}{k_i}\leqq p$. We obtain

$$S_{21}\leqq K\sum_{k_i>0}\frac{k_i}{n}\leqq K.$$

In S_{22} we have $\log\frac{n}{k_i}>pi$ i.e. $\frac{n}{k_i}>e^{pi}$, and so $\left(\log\frac{n}{k_i}\right)^p\Big/\frac{n}{k_i}\leqq(pi)^p/e^{pi}$; hence,

$$S_{22}\leqq K\sum_{i=1}^{\infty}\frac{1}{i^p}\frac{(pi)^p}{e^{pi}}\leqq K.$$

Finally, $(\log x)^p/x\leqq K_p$ $(x\geqq 1)$ and so $\frac{k_0}{n}\left(\log\frac{n}{k_0}\right)^p\leqq K_p$.

Collecting the above estimations we obtain (3.1), and the proof is completed.

Proof of Theorem 2. $f\in W^r H^\omega$ implies by the well-known result of Jackson that $E_n(f)\leqq K\frac{1}{n^r}\omega\left(\frac{1}{n}\right)$; thus we can apply Theorem 1 with $\varrho_n=\frac{1}{n^r}\omega\left(\frac{1}{n}\right)$ and obtain

$$\left\{\frac{1}{n}\sum_{k=n=1}^{2n}|s_{\nu_k}(x)-f(x)|^p\right\}^{\frac{1}{p}}\leqq K\frac{1}{n^r}\omega\left(\frac{1}{n}\right).$$

Using this, we get for $2^{m_0-1}<n\leqq 2^{m_0}$

$$\left\{\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_{\nu_k}x)-f(x)|^p\right\}^{\frac{1}{p}}\leqq K\left\{\frac{1}{(n+1)^\beta}\sum_{k=0}^{1}|s_{\nu_k}(x)-f(x)|^p+\right.$$

$$\left.+\frac{1}{(n+1)^\beta}\sum_{m=0}^{m_0-1}(2^m)^{\beta-1}\sum_{k=2^m+1}^{2^m}|s_{\nu_k}(x)-f(x)|^p\right\}^{\frac{1}{p}}\leqq$$

$$\leqq K\left\{\frac{1}{(n+1)^\beta}\sum_{m=0}^{m_0-1}(2^m)^{\beta-1}2^m\left(\frac{1}{2^{mr}}\,\omega\left(\frac{1}{2^m}\right)\right)^p\right\}^{\frac{1}{p}}\leqq KH_{r,\omega}^{p,\beta,n}$$

which is (1.3).

(1.4) results by a similar argument using also the Hölder inequality (see e.g. the proof of [1, Theorem 3]).

Next we prove the last statement of Theorem 2. We have to show that if (1.2) is not satisfied then (1.5) does not hold for some f and ν.

Thus let us suppose that (1.2) is not true. Then for every n there are m_n and i_n such that

$$i_n\omega\left(\frac{1}{2^{i_n}m_n}\right)>n\omega\left(\frac{1}{m_n}\right).$$

Since $\omega\left(\frac{1}{2m}\right)\geqq\frac{1}{2}\,\omega\left(\frac{1}{m}\right)$, we may suppose that the sequence $\{i_n\}_{n=1}^{\infty}$ is increasing and that $m_{n+1}>2m_n$ $(n=1,2,\ldots)$.

Taking into account that surely $i_n\to\infty$ if $n\to\infty$, we have $2^{i_n}>e^{100}$ for all sufficiently large n. Now Lemma 2 gives a function $f\in W^0H^\omega$ such that

$$|s_{2^{i_n}m_n+\lambda}(f;\,0)-f(0)|>10^{-2}(\log 2^{i_n})\,\omega\left(\frac{1}{2^{i_n}m_n}\right)\quad(0<\lambda\leqq m_n)\tag{3.2}$$

holds for infinitely many n. Hence, if we construct a sequence $\{\nu_k\}$ for which

$$\nu_{m_n+1}=2^{i_n}m_n+1,\ \nu_{m_n+2}=2^{i_n}m_n+2,\ \ldots,\ \nu_{2m_n}=2^{i_n}m_n+m_n$$

for all n (this is clearly possible) then we get for infinitely many n

$$\left\{\frac{1}{m_n}\sum_{k=m_n+1}^{2m_n}|s_{\nu_k}(f;\,0)-f(0)|^p\right\}^{\frac{1}{p}}>10^{-2}(\log 2^{i_n})\,\omega\left(\frac{1}{2^{i_n}m_n}\right)\geqq$$

$$\geqq\frac{1}{2}\,10^{-2}\,i_n\omega\left(\frac{1}{2^{i_n}m_n}\right)>\frac{1}{2}\,10^{-2}n\omega\left(\frac{1}{m_n}\right)$$

i.e. f and $\{\nu_k\}$ do not satisfy (1.5).

We have completed our proof.

Proof of Theorem 3. First we prove (i) for $\beta=1$:

$$h_n(f, p, 1, \pi;\ x) \leqq K\left(\left\{\frac{1}{n+1}\sum_{k=0}^{n}|s_k(x)-f(x)|^p\right\}^{\frac{1}{p}}+\left\{\frac{1}{n+1}\sum_{\substack{0\leqq k\leqq m\\ \pi(k)>n}}|s_{\pi(k)}(x)-f(x)|^p\right\}^{\frac{1}{p}}\right) \leqq$$

$$\leqq KH_{r,\omega}^{p,1,n}+K\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right) \leqq KH_{r,\omega}^{p,1,n},$$

where we use Theorem A and Theorem 1.

This gives

$$\sum_{k=1}^{n}|s_{\pi(k)}(x)-f(x)|^p \leqq K\sum_{k=1}^{n}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p, \tag{3.3}$$

by which we have for $2^{m_0-1}<n\leqq 2^{m_0}$ and for $\beta<1$

$$\sum_{k=1}^{n}(k+1)^{\beta-1}|s_{\pi(k)}(x)-f(x)|^p \leqq K\sum_{m=0}^{m_0-1}(2^m)^{\beta-1}\sum_{k=2^m}^{2^{m+1}}|s_{\pi(k)}(x)-f(x)|^p \leqq$$

$$\leqq K\sum_{m=0}^{m_0-1}(2^m)^{\beta-1}\sum_{k=1}^{2^{m+1}}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p \leqq K\sum_{k=1}^{2^{m_0}}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p\sum_{m=\overset{2}{\log}k-1}^{m_0-1}(2^m)^{\beta-1} \leqq$$

$$\leqq K\sum_{k=1}^{2^{m_0}}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p(2^{\overset{2}{\log}k})^{\beta-1} \leqq K\sum_{k=1}^{n}(k+1)^{\beta-1}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p$$

and this is exactly (i).

(ii) follows from (3.3) since

$$h_n(f, p, \beta, \pi;\ x) \leqq \left\{\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(n+1)^{\beta-1}|s_{\pi(k)}(x)-f(x)|^p\right\}^{\frac{1}{p}} \leqq KH_{r,\omega}^{p,1,n}.$$

Now let us suppose that $\sum_{k=1}^{\infty}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p<\infty$. Lemma 1 gives that

$$\sum_{k=n}^{2n}|s_k(x)-f(x)|^p \leqq K\sum_{k=n}^{2n}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p$$

by which

$$\sum_{k=M}^{\infty}{}'|s_k(x)-f(x)|^p \leqq K\sum_{k=M}^{\infty}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p = o(1)\quad (M\to\infty).$$

Let $\varepsilon>0$ be arbitrary and let us choose M so that

$$\sum_{k=M}^{\infty}|s_k(x)-f(x)|^p < \varepsilon$$

be satisfied for all x. If $N \geqq \max_{0 \leqq i \leqq M} \pi^{-1}(i)$ then

$$\frac{1}{(n+1)^\beta} \sum_{k=0}^{n} (k+1)^{\beta-1} |s_{\pi(k)}(x)-f(x)|^p \leqq$$

$$\leqq \frac{(N+1)^{\beta-1}}{(n+1)^\beta} \sum_{k=0}^{M} |s_k(x)-f(x)|^p + \frac{1}{(n+1)^\beta} \sum_{\substack{0 \leqq k \leqq n \\ \pi(k) \geqq M}} (n+1)^{\beta-1} |s_{\pi(k)}(x)-f(x)|^p =$$

$$= o(n^{-1}) + \frac{1}{n+1} \sum_{k=M}^{\infty} |s_k(x)-f(x)|^p \leqq \frac{2\varepsilon}{n+1}$$

for all n large enough. Thus we have proved (iii), too.

(iv) follows from (i):

$$\sigma_n^\gamma |f, p, \pi;\, x| \leqq K \left\{ \frac{1}{(n+1)^\gamma} \sum_{k=0}^{n} (n+1-k)^{\gamma-1} |s_{\pi(k)}(x)-f(x)|^p \right\}^{\frac{1}{p}} =$$

$$= K \left\{ \frac{1}{(n+1)^\gamma} \sum_{k=0}^{n} (k+1)^{\gamma-1} |s_{\pi(n-k)}(x)-f(x)|^p \right\}^{\frac{1}{p}} \leqq K H_{r,\omega}^{p,\gamma n},$$

where we used the inequalities

$$c_1(\alpha) k^\alpha \leqq A_k^\alpha \leqq c_2(\alpha) k^\alpha \quad (\alpha > -1,\ c_1(\alpha) > 0,\ k = 1, 2, \ldots).$$

(vi) could be proved similarly with the aid of (ii) and (iii).

Finally let us suppose that $\gamma < 1$ and $\sum_{k=1}^{\infty} (k+1)^{\gamma-1} \left(\frac{1}{k^r} \omega\left(\frac{1}{k}\right) \right)^p < \infty$. It is known that the last condition implies

$$\sum_{k=1}^{n} \left(\frac{1}{k^r} \omega\left(\frac{1}{k}\right) \right)^p = o((n+1)^{1-\gamma}).$$

Thus to every $\varepsilon > 0$ there exists an $M = M(\varepsilon)$ for which

$$\sum_{k=2^M+1}^{\infty} (k+1)^{\gamma-1} \left(\frac{1}{k^r} \omega\left(\frac{1}{k}\right) \right)^p < \varepsilon^p \quad \text{and} \quad \sum_{k=1}^{2^M+1} \left(\frac{1}{k^r} \omega\left(\frac{1}{k}\right) \right)^p < \varepsilon^p (2^M)^{1-\gamma}$$

are satisfied. It is easy to see that (1.2) implies $\omega(\delta) \log \delta = o(1)$ $(\delta \to 0)$. Now the Dini—Lipschitz test gives that $s_k(x) - f(x) = o(1)$ uniformly in x, and so

$$|s_{\pi(n-k)}(x) - f(x)| < \frac{\varepsilon^p}{2^M+1} \quad (k = 0, 1, \ldots, 2^M)$$

for $n \geqq n_\varepsilon$.

Using the previous estimations and (3.3) we have for $n \geqq n_\varepsilon$ and $2^{m_0-1} < n \leqq 2^{m_0}$

$$\sigma_n^\gamma |f, p, \pi; x| \leqq K\left(\left\{\frac{1}{A_n^\gamma}\sum_{k=0}^{2^M} A_k^{\gamma-1}|s_{\pi(n-k)}(x)-f(x)|^p\right\}^{\frac{1}{p}} + \right.$$

$$+\left\{\frac{1}{A_n^\gamma}\sum_{m=M}^{m_0-1}\sum_{k=2^m+1}^{2^{m+1}} A_k^{\gamma-1}|s_{\pi(n-k)}(x)-f(x)|^p\right\}^{\frac{1}{p}} \leqq K\left(\left\{\frac{1}{A_n^\gamma}\sum_{k=0}^{2^M}|s_{\pi(n-k)}(x)-f(x)|^p\right\}^{\frac{1}{p}} + \right.$$

$$+\left\{\frac{1}{A_n^\gamma}\sum_{m=M}^{m_0-1}(2^m)^{\gamma-1}\sum_{k=2^m+1}^{2^{m+1}}|s_{\pi(n-k)}(x)-f(x)|^p\right\}^{\frac{1}{p}}\Bigg) \leqq K\left(\left\{\frac{1}{A_n^\gamma}(2^M+1)\frac{\varepsilon^p}{2^M+1}\right\}^{\frac{1}{p}} + \right.$$

$$+\left\{\frac{1}{A_n^\gamma}\sum_{m=M}^{m_0-1}(2^m)^{\gamma-1}\sum_{k=1}^{2^{M+1}}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p\right\}^{\frac{1}{p}}\Bigg) \leqq K\left(\frac{\varepsilon}{(A_n^\gamma)^{\frac{1}{p}}} + \right.$$

$$+\left\{\frac{1}{A_n^\gamma}\sum_{k=1}^{2^{M+1}}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p\sum_{m=M}^{m_0-1}(2^m)^{\gamma-1}\right\}^{\frac{1}{p}} + \left\{\frac{1}{A_n^\gamma}\sum_{k=2^{M+1}+1}^{2^{m_0}}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p\cdot\right.$$

$$\cdot\left.\sum_{m=\overset{2}{\log}k-1}^{m_0-1}(2^m)^{\gamma-1}\right\}^{\frac{1}{p}}\Bigg) \leqq K\left(\frac{\varepsilon}{(A_n^\gamma)^{\frac{1}{p}}} + \left\{\frac{1}{A_n^\gamma}\sum_{k=1}^{2^{M+1}}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p(2^M)^{\gamma-1}\right\}^{\frac{1}{p}} + \right.$$

$$+\left.\left\{\frac{1}{A_n^\gamma}\sum_{k=2^M+1}^{2^{m_0}}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p(k+1)^{\gamma-1}\right\}^{\frac{1}{p}}\right) \leqq K\frac{\varepsilon}{(A_n^\gamma)^{\frac{1}{p}}} \leqq K\varepsilon H_{r,\omega}^{p,\gamma,n}$$

which was to be proved.

So far we have proved (i)—(vi). It remains to show that these estimations are best possible.

Let f be the function given in Theorem A.

The first row of (1.8) immediately follows from Theorem A.

Let us suppose that $\sum_{k=1}^{\infty}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p = \infty$ and that $\beta > 1$. We shall define a π permutation of N as follows: If $\pi(0), \ldots, \pi(n_{m-1})$ are already known and $\pi(i) \leqq M_m$ $(i=0, 1, \ldots, n_{m-1})$, let

$$\pi(n_m+1) = 2M_m+1,\ \pi(n_m+2) = 2M_m+2, \ldots, \pi(n_m+n_m) = 2M_m+n_m,$$

where n_m will be chosen later.

However should π be defined between n_{m-1} and n_m we have in any case

$$\left\{\frac{1}{(2n_m+1)^\beta}\sum_{k=0}^{2n_m}(k+1)^{\beta-1}|s_{\pi(k)}(x)-f(x)|^p\right\}^{\frac{1}{p}} \geqq$$

$$\geqq \left\{\frac{1}{(2n_m+1)^\beta}(n_m)^{\beta-1}\sum_{k=n_m+1}^{2n_m}|s_{\pi(k)}(x)-f(x)|^p\right\}^{\frac{1}{p}} \geqq \left\{\frac{C}{n_m+1}\sum_{k=2M_m+1}^{n_m}|s_k(x)-f(x)|^p\right\}^{\frac{1}{p}}$$

and therefore

(3.4)

$$h_{2n_m}(f, p, \beta, \pi; 0) \geqq c \left\{ \frac{1}{n_m+1} \sum_{k=0}^{n_m} |s_k(0)-f(0)|^p - \frac{\sum_{k=0}^{2M_m} |s_k(0)-f(0)|^p}{n_m+1} \right\}^{\frac{1}{p}} \geqq \frac{c}{2} H_{r,\omega}^{p,1,n_m}$$

if n_m is large enough in comparison with M_m, because $\sum_{k=1}^{\infty} \left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p = \infty$ is equivalent with $(H_{r,\omega}^{p,1,n})^p \neq O\left(\frac{1}{n}\right)$.

Let us choose n_m so large that the above estimation should be satisfied, and then continue the procedure.

It is clear that the above, partly defined π could be extended to a permutation of N, and so (3.4) shows that (ii) cannot be improved.

Finally, if $\varrho_n \to 0$ arbitrarily, we follow the above construction and get

$$h_{2n_m}(f, p, \beta, \pi; 0) \geqq c \left\{ \frac{1}{n_m+1} \sum_{k=2M_m+1}^{n_m} |s_k(0)-f(0)|^p \right\}^{\frac{1}{p}} \geqq$$

$$\geqq c \left\{ \frac{1}{n_m} \sum_{k=2M_m+1}^{n_m} \left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p \right\}^{\frac{1}{p}} \geqq c\varrho_{n_m}\left(\frac{1}{n_m}\right)^{\frac{1}{p}} \geqq c\varrho_{n_m} H_{r,\omega}^{p,1,n}$$

(at the second inequality we used that for f we have $|s_k(0)-f(0)| \geqq c \frac{1}{k^r}\omega\left(\frac{1}{k}\right)$ $(5\cdot 2^\nu - 2^{\nu-1} \leqq k \leqq 5\cdot 2^\nu + 2^{\nu-1})$ (see the proof of [3, Theorem 1]) if n_m is large enough.

Thus the proof of (1.8) is completed.

The proof of (1.9) is similar, we omit the details.

The proof of the Corollary on the basis of the above arguments is easy. The right-hand estimations follow from the proof of (i)—(vi), while the left-hand sides are easy consequences of Theorem A.

The proof of Theorem 3 is thus completed.

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Quasi-similarity of restricted C_0 contractions

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1. A bounded linear operator X from a separable Hilbert space $\mathfrak{H}$ to a separable Hilbert space $\mathfrak{H}'$ is called a *quasi-affinity* if $K(X)=0$ and $K(X^*)=0$, where $K(X)$ denotes the kernel of X. The bounded operators T on $\mathfrak{H}$ and T' on $\mathfrak{H}'$ are called *quasi-similar* and denoted by $T \sim T'$ if there are quasi-affinities X and Y such that $XT=T'X$ and $TY=YT'$.

In this note we say thet T has *property* (Q) if $T|K(A)$ and $((T^*|K(A^*)^*)$ are quasi-similar for every A in $(T)'$. Not every bounded operator has property (Q); it is easy to contstruct even a self adjoint operator which has not property (Q).

2. Lemma 1. *If T on $\mathfrak{H}$ and S on $\mathfrak{H}'$ are similar, then T has property* (Q) *if and only if so is S.*

Proof. Let T have property (Q) and suppose $XT=SX$ for some invertible X. Set $B=X^{-1}AX$ for A commuting with S. Then it is clear that B commutes with T and that $T|K(B)$ and $T^*|K(B^*)$ are similar to $S|K(A)$ and $S^*|K(A^*)$, respectively. Therefore $S|K(A) \sim (S^*|K(A^*))^*$.

Lemma 2. *If both T on $\mathfrak{H}$ and S on $\mathfrak{H}'$ have property* (Q) *and $\sigma(T) \cap \sigma(S)=\emptyset$, then the direct sum $T \oplus S$ on $\mathfrak{H} \oplus \mathfrak{H}'$ has property* (Q) *also.*

Proof. From Rosenblum's corollary, $(T \oplus S)'=(T)' \oplus (S)'$ [2]. The rest is omitted.

Proposition 1. *If $\mathfrak{H}$ is finite dimensional, then every normal operator on $\mathfrak{H}$ has property* (Q).

Proof. From Lemma 1 and Lemma 2, we may assume that $T=\alpha I$ for some scalar α. The rest is obvious.

We will use the above results in the last example.

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3. SZ.-NAGY and C. FOIAŞ [7] conjectured that all C_0 contractions with finite multiplicity have property (Q). In this section we present a counter example. About the terminology and the notations see [4] and [1].

Example 1. Let ψ_1 and ψ_2 be relatively prime scalar inner functions defined on the unit circle. And define the 2×2 diagonal matrix valued inner function M by

$$M = \psi_1^2\psi_2 \oplus \psi_1^3\psi_2^2.$$

Then the class $C_0(2)$ contraction $S(M)$ on $\mathfrak{H}(M)$ defined by

$$\mathfrak{H}(M) = H_2^2 \ominus MH_2^2, \quad S(M)h = P(zh),$$

where H_2^2 denotes the 2-dimensional vector valued Hardy class and P is the projection from H_2^2 onto $\mathfrak{H}(M)$, does not have property (Q).

Proof. Setting

$$\Delta = \begin{bmatrix} \psi_1^2 & \psi_1^3 \\ \psi_1^2\psi_2^2 & 0 \end{bmatrix},$$

$A = P\Delta|\mathfrak{H}(M)$ commutes with $S(M)$, because $\Delta MH_2^2 \subset MH_2^2$. First we show that

$$K(A) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \psi_1 \\ \psi_2 & -1 \end{bmatrix}\left\{H_2^2 \ominus \frac{1}{\sqrt{2}}\begin{bmatrix} \psi_1 & \psi_1^3\psi_2 \\ \psi_1\psi_2 & 0 \end{bmatrix} H_2^2\right\}$$

and hence

$$S(M)|K(A) \sim S\left(\frac{1}{\sqrt{2}}\begin{bmatrix} \psi_1 & \psi_1^3\psi_2 \\ \psi_1\psi_2 & 0 \end{bmatrix}\right).$$

For this, it is sufficient to show that

$$\{h_1 \oplus h_2 \colon h_i \in H_2^2, \ \Delta(h_1 \oplus h_2) \in MH_2^2\} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \psi_1 \\ \psi_2 & -1 \end{bmatrix} H_2^2.$$

It is clear that the right hand side set is included to the left hand side set. Suppose that an element $h_1 \oplus h_2$ in the left hand side set is orthogonal to the right hand set. Then there are f_1 and f_2 in H_2^2 such that

$$h_1 + \psi_1 h_2 = \psi_2 f_1, \quad h_1 = \psi_1 f_2, \quad \text{and, therefore,} \quad \psi_1(f_2 + h_2) = \psi_2 f_1.$$

Since ψ_1 and ψ_2 are relatively prime, there exists f in H_2^2 such that $f_1 = \psi_1 f$ so $f_2 + h_2 = \psi_2 f$. On the other hand, for every g_1 and g_2 in H_2^2 it follows that

$$(h_1, \psi_1 g_2) + (h_2, \psi_2 g_1 - g_2) = 0.$$

Thus we have $f_2 = h_2$ and $(h_2, \psi_2 g_1) = 0$, which imply $f = 0$ and hence $h_1 = h_2 = 0$.

Next we show that

$$\text{closure of range } A = (\psi_1^2 \oplus \psi_1^2\psi_2^2) H_2^2 \ominus MH_2^2$$

and hence $(S(M)^*|K(\Delta^*))^* \sim S(\psi_1^2 \oplus \psi_1^2\psi_2^2)$. For this it suffices to show that

$$\Delta H_2^2 \vee MH_2^2 = (\psi_1^2 \oplus \psi_1^2\psi_2^2) H_2^2.$$

Since

$$\Delta = \begin{bmatrix} \psi_1^2 & 0 \\ 0 & \psi_1^2\psi_2^2 \end{bmatrix} \begin{bmatrix} 1 & \psi_1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad M = (\psi_1^2 \oplus \psi_1^2\psi_2^2)(\psi_2 \oplus \psi_1),$$

$\Delta H_2^2 \vee MH_2^2 \subset (\psi_1^2 \oplus \psi_1^2\psi_2^2) H_2^2$. Suppose that $\psi_1^2 h_1 \oplus \psi_1^2\psi_2^2 h_2$ is orthogonal to $\Delta H_2^2 \vee MH_2^2$. Then $h_1 \oplus h_2$ is orthogonal to

$$\begin{bmatrix} 1 & \psi_1 \\ 1 & 0 \end{bmatrix} H_2^2 \vee (\psi_2 \oplus \psi_1) H_2^2.$$

From this it follows that $h_1 + h_2 = 0$, and that h_1 and h_2 are orthogonal to $\psi_2 H^2$ and $\psi_1 H^2$, respectively. Since ψ_1 and ψ_2 are relatively prime, we have $h_1 = h_2 = 0$.

Last we must show that $S(M)|K(\Delta)$ and $(S(M)^*|K(\Delta^*))^*$ are not quasi-similar. But this is clear, because the minimal functions of these operators are $\psi_1^3\psi_2^2$ and $\psi_1^2\psi_2^2$, respectively.

4. We denote the lattice of invariant subspaces for T and the lattice of hyper-invariant subspaces for T by Lat T and Hyplat T, respectively.

Let θ and θ' be $n \times n$ matrix valued inner functions. Suppose $S(\theta)$ on $\mathfrak{H}(\theta)$ and $S(\theta')$ on $\mathfrak{H}(\theta')$ defined as Example 1 are quasi-similar. Then there are $n \times n$ matrices Γ and Λ over H^∞ such that

$$\Gamma\theta = \theta'\Lambda \quad \text{and} \quad (\det \Gamma)(\det \Lambda) \wedge (\det \theta)(\det \theta') = 1 \quad [1].$$

Moreover, it follows that

$$(\det \Lambda)\Gamma^a\theta' = \theta(\det \Gamma)\Lambda^a,$$

where Γ^a denotes the classical adjoint of Γ [6]. In this case, setting $X = P'\Gamma|\mathfrak{H}(\theta)$ and $Y = P(\det \Lambda)\Gamma^a|\mathfrak{H}(\theta')$, where P' and P are the projections from H_n onto $\mathfrak{H}(\theta')$ and $\mathfrak{H}(\theta)$, respectively, X and Y are quasi-affinities satisfying $XS(\theta) = S(\theta')X$ and $YS(\theta') = S(\theta)Y$ [1]; moreover, $XY = \varphi(S(\theta'))$ and $YX = \varphi(S(\theta))$, where $\varphi = (\det \Gamma)(\det \Lambda)$.

Proposition 2. *The mapping τ from* Lat $S(\theta)$ *to* Lat $S(\theta')$ *defined by $\tau\mathfrak{L} = \overline{X\mathfrak{L}}$ is a lattice isomorphism, and its inverse is given by $\tau^{-1}\mathfrak{L} = \overline{Y\mathfrak{L}}$.* Hyplat $S(\theta)$ *and* Hyplat $S(\theta')$ *are isomorphic. Similarly, the mapping τ' from* Lat $S(\theta)^*$ *to* Lat $S(\theta')^*$ *defined by $\tau'\mathfrak{L} = \overline{Y^*\mathfrak{L}}$ is a lattice isomorphism, and its inverse is given by $\tau'^{-1}\mathfrak{L} = \overline{X^*\mathfrak{L}}$.* Hyplat $S(\theta)^*$ *and* Hyplat $S(\theta')^*$ *are isomorphic.*

Proof. Let $\mathfrak{L} \neq 0$ belong to Lat $S(\theta)$. Then $\overline{X\mathfrak{L}} \neq 0$ belongs to Lat $S(\theta')$. Since $(X|\mathfrak{L})(S(\theta)|\mathfrak{L}) = (S(\theta')|\overline{X\mathfrak{L}})(X|\mathfrak{L})$, we have $S(\theta)|\mathfrak{L} \sim S(\theta')|\overline{X\mathfrak{L}}$ [1]. Similarly,

$S(\theta')|\overline{X\mathfrak{L}} \sim S(\theta)|\overline{YX\mathfrak{L}}$. Since $\overline{YX\mathfrak{L}} = \overline{\varphi(S(\theta))\mathfrak{L}} \subset \mathfrak{L}$, we have $\overline{YX\mathfrak{L}} = \mathfrak{L}$ (see [5] or [7]). Therefore, τ is one to one. Surjectivity is similarly shown. That τ preserve the lattice structure is obvious. That Hyplat $S(\theta)$ and Hyplat $S(\theta')$ are isomorphic was shown in [8]. Since

$$X^*Y^* = \tilde{\varphi}(S(\theta)^*) \quad \text{and } Y^*X^* = \tilde{\varphi}(S(\theta')^*)$$

we can show the rest similarly.

Proposition 3. *If $S(\theta)$ and $S(\theta')$ are quasi-similar, then $S(\theta)$ has property* (Q) *if and only if so is $S(\theta')$.*

Proof. Assume that $S(\theta')$ has property (Q). For each A commuting with $S(\theta)$ set $B = XAY$. Then B commutes with $S(\theta')$ and $Y\,K(B) \subset K(A)$. Since

$$BX = XAYX = XA\varphi(S(\theta)) = X\varphi(S(\theta))A$$

we have $XK(A) \subset K(B)$. Thus, by Proposition 2, it follows that

$$K(A) \supset \overline{YK(B)} \supset \overline{YXK(A)} = K(A).$$

Therefore, we have $K(A) = \overline{YK(B)}$ and $\overline{XK(A)} = \overline{XYK(B)} = K(B)$. Thus

$$S(\theta)|K(A) = S(\theta)|\overline{YK(B)} \sim S(\theta')|K(B).$$

Similarly, we have

$$S(\theta)^*|K(A^*) = S(\theta)^*|\overline{X^*K(B^*)} \sim S(\theta')^*|K(B^*).$$

Since $S(\theta')|K(B) \sim (S(\theta')^*|K(B^*))^*$, it follows that

$$S(\theta)|K(A) \sim (S(\theta)|K(A^*))^*,$$

concluding the proof.

Proposition 4. *If A belongs to $(S(\theta))''$, then*

$$S(\theta)|K(A) \sim (S(\theta)^*|K(A^*))^*.$$

Proof. Let $\theta' = \psi_1 \oplus \ldots \oplus \psi_n$ be the normal form of θ. Then $B = XAY$ belongs to $(S(\theta'))''$ so $B = \eta(S(\theta'))$ for some η in H^∞ [9]. Setting $\psi_i' = \psi_i/(\eta \wedge \psi_i)$ we have

$$K(B) = (\psi_1' \oplus \ldots \oplus \psi_n')H_n^2 \ominus (\psi_1 \oplus \ldots \oplus \psi_n)H_n^2.$$

Thus $S(\theta')|K(B) \sim S(\eta \wedge \psi_1 \oplus \ldots \oplus \eta \wedge \psi_n)$. On the other hand,

$$\eta H_n^2 \vee \theta' H_n^2 = (\eta \wedge \psi_1 \oplus \ldots \oplus \eta \wedge \psi_n)H_n^2$$

implies that

$$(S(\theta')^*|K(B^*))^* \sim S(\eta \wedge \psi_1 \oplus \ldots \oplus \eta \wedge \psi_n).$$

Since, by the proof of Proposition 3,

$$S(\theta)|K(A) \sim S(\theta')|K(B) \quad \text{and} \quad S(\theta)^*|K(A^*) \sim S(\theta')^*|K(B^*),$$

we have $S(\theta)|K(A) \sim (S(\theta)^*|K(A^*))^*$.

Corollary. *If $S(\theta)$ has a cyclic vector, then $S(\theta)$ has Property* (Q).

Proof. Since $(S(\theta))'=(S(\theta))''$ (see [3] and [4]), it is obvious.

To conclude we present a counterexample to the converse assertion of Corollary.

Example 2. Set $\psi_1(z)=\frac{z-\alpha}{1-\bar{\alpha}z}$ for $|\alpha|<1$ and $\psi_2(z)=\exp\left(\frac{z+1}{z-1}\right)$. Then $\theta=(\psi_1\oplus\psi_1\psi_2)$ is a 2×2 matrix valued inner function, and $S(\theta)$ has no cyclic vector [4]. But it follows that

$$S(\theta)=S(1\oplus\psi_1\oplus\psi_1\psi_2)\sim S(\psi_1\oplus\psi_1\oplus\psi_2)=S(\psi_1\oplus\psi_1)\oplus S(\psi_2).$$

Since $S(\psi_1\oplus\psi_1)$ is a 2×2 diagonal matrix, by Proposition 1, $S(\psi_1\oplus\psi_1)$ has property (Q). Since $S(\psi_2)$ has a cyclic vector, by Proposition 4, $S(\psi_2)$ has property (Q). Lemma 2 and relation

$$\sigma(S(\psi_1\oplus\psi_1))\cap\sigma(S(\psi_2))=\emptyset \quad \text{(cf. [4])},$$

imply that $S(\psi_1\oplus\psi_1)\oplus S(\psi_2)$ has property (Q). Thus, by Proposition 3, $S(\theta)$ also has property (Q).

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On the structure of standard regular semigroups

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We give a structure theorem for a class of regular semigroups and determine the smallest inverse semigroup congruence for this class of semigroups. Let S be a regular semigroup, let T denote the union of the maximal subgroups of S, and let $E(T)$ denote the set of idempotents of T. Assume T is a semigroup (equivalently T is a semilattice Y of completely simple semigroups $(T_y\colon y\in Y)$). If Y has a greatest element and $e, f, g\in E(T)$, $e\geqq f$, and $e\geqq g$ imply $fg=gf$, we term S a *standard regular semigroup*. The structure of S is given modulo standard inverse semigroups and standard completely regular semigroups by means of an explicit multiplication. In the case $|Y|=1$, our structure theorem reduces to the Rees theorem for completely simple semigroups. A structure theorem for standard completely regular semigroups is also given. The minimum inverse semigroup congruence on a standard regular semigroup is described.

Let us first state our structure theorem for standard regular semigroups. Let $(V, \circ)$ be a standard inverse semigroup with semilattice of idempotents Y, and let $(T, *)$ be a standard semilattice Y of completely simple semigroups $(T_y\colon y\in Y)$ with $y=y*y\in T_y$. Suppose $T_y\cap V=H_y$ for $y\in Y$ and $(H_y, \circ)$ $[(H_y, *)]$ is the maximal subgroup of $(V, \circ)$ $[(T, *)]$ containing y and assume $a*b=a\circ b$ for $a, b\in \bigcup(H_y\colon y\in Y)$. Let I_y $[J_y]$ denote the maximal left zero [right zero] subsemigroup of T_y containing y. Let (Y, T, V) denote $\{(i, b, j)\colon b\in V,\ i\in I_{b\circ b^{-1}},\ j\in J_{b^{-1}\circ b}\}$ under the multiplication $(i, b, j)(r, c, s)=(i*u,\ b\circ(j*r)\circ c,\ v*s)$ where $u\in I_{(b\circ c)\circ(b\circ c)^{-1}}$ and $v\in J_{(b\circ c)^{-1}\circ(b\circ c)}$. We show (Theorem 1.9) that (Y, T, V) is a standard regular semigroup and, conversely, every standard regular semigroup is isomorphic to some (Y, T, V).

In [4, Theorem 3.14], we gave a different structure theorem for standard regular semigroups.

The structure of standard inverse semigroups is clarified by [4, Theorem 5.5].

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In Section 1, we prove our structure theorem for standard regular semigroups (Theorem 1.9) and give some specializations of this theorem (Remarks 1.21 and 1.22). In Section 2, we describe standard completely regular semigroups in terms of groups by means of a "Rees type" multiplication (Theorem 2.1). In Section 3, we give the following description of the minimum inverse semigroup congruence on a standard regular semigroup $S=(Y, T, V)$. Let N denote the collection of all finite products of elements of the form $a^{-1}\circ s\circ a$ where $a\in V$ and s or $s^{-1}\in(\bigcup(J_y\colon y\in Y))*(\bigcup(I_y\colon y\in Y))$. Let $N_y=N\cap H_y$ for $y\in Y$. Let

$$\delta_N=\{((i,a,j),(p,b,q))\in S\times S\colon N_y\circ a=N_y\circ b \quad\text{where}\quad y=a\circ a^{-1}=b\circ b^{-1}\}.$$

Then, δ_N is the minimum inverse semigroup congruence on S.

We will use the definitions and notation of Clifford and Preston [1, 2] unless otherwise specified. The terms mainly used are: Green's relations ($\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, and $\mathscr{D}$), $\mathscr{R}$-class, regular semigroup, bisimple semigroup, inverses, inverse semigroup, left (right) zero semigroup, right group, idempotent, natural partial order of idempotents, semilattice, completely simple semigroup, semilattice of completely simple semigroups [groups, left (right) zero semigroups], maximal subgroup, congruence, and kernel of a homomorphism.

A semigroup is termed *completely regular* if it is a union of its subgroups. If X is a semigroup, $E(X)$ will denote the set of idempotents of X. A regular semigroup X is termed *locally inverse* if $e,f,g\in E(X)$, $e\geqq f$ and $e\geqq g$ imply $fg=gf$. (See [4] for an explanation of terminology.) A congruence ϱ on a semigroup X such that X/ϱ is an inverse semigroup is termed an inverse semigroup congruence on X. "Structure homomorphisms" are defined and discussed in [4, Section 1].

1. Standard regular semigroups. In this section, we establish our new structure theorem for standard regular semigroups (Theorem 1.9).

Let S be a standard regular semigroup and let T denote the union of the maximal subgroups of S. Hence, T is a semilattice Y' of completely simple semigroups $(T_y\colon y\in Y')$ [1, Theorem 4.6] where Y' has a greatest element y_0. Let $\{\zeta_{y,z}\colon y,z\in Y\}$ denote the set of structure homomorphisms of T [4, Section 1]. Let $E_y=E(T_y)$. Select and fix $e_{y_0}\in E(T_{y_0})$. For each $y\in Y'$, define $e_y=e_{y_0}\zeta_{y_0,y}$. Let $S_0=e_{y_0}Se_{y_0}$. Let I_{e_y} $[J_{e_y}]$ denote the set of idempotents of the $\mathscr{L}$-class [$\mathscr{R}$-class] of T_y containing e_y. Let H_{e_y} denote the $\mathscr{H}$-class of S containing e_y.

Lemma 1.1. ([4, Lemma 2.2]) *$y\to e_y$ defines an isomorphism of Y' onto $E(S_0)$.*

Lemma 1.2. *$H_{e_y}=T_y\cap S_0$ for $y\in Y'$.*

Proof. Utilize [4, Theorem 2.3].

Let $E(S_0)=Y$ and let $\mathscr{I}(a)$ denote the collection of inverses of a.

Lemma 1.3. (a) *T is a semilattice Y of completely simple semigroups* $(T_y\colon y\in Y)$ *where* $y^2=y\in T_y$. (b) $I=\bigcup(I_y\colon y\in Y)$ $[J=\bigcup(J_y\colon y\in Y)]$ *is the semilattice Y of left zero semigroups [right zero semigroups]* $(I_y\colon y\in Y)$ $[(J_y\colon y\in Y)]$.

Proof. (a) Let $T_{e_y}=T_y(y\in Y')$. Then, using Lemma 1.1, $T_{e_y}T_{e_z}=T_yT_z\subseteq T_{yz}=$ $=T_{e_{yz}}=T_{e_ye_z}$. (b) Utilize the proof of [4, Lemma 2.4] and its dual.

Lemma 1.4. *Every element of S may be uniquely expressed in the form* $x=gbh$ *where* $b\in S_0$, $g\in I_{bb^{-1}}$, *and* $h\in J_{b^{-1}b}$.

Proof. Let $a\in S$. Hence, $a\in R_e\cap L_f$ for some $e,f\in E(S)$. Suppose $e\in T_y$ and $f\in T_z$ $(y,z\in Y')$. Let r_e $[l_f]$ denote the $\mathscr{R}$-class [$\mathscr{L}$-class] of T_y $[T_z]$ containing e $[f]$. Using [1, Theorem 2.51], $r_e\cap I_{e_y}\neq\square$ and $l_f\cap J_{e_z}\neq\square$. Let $g\in r_e\cap I_{e_y}$ and $h\in l_f\cap J_{e_z}$. Hence, $g(e_{y_0}ae_{y_0})h=(ge_{y_0})a(e_{y_0}h)=(ge_ye_{y_0})a(e_{y_0}e_zh)=gah=a$. By the proof of [1, Theorem 2.18], since $a\in R_g\cap L_h$, there exists a unique $a^{-1}\in R_h\cap L_g\cap\mathscr{I}(a)$ such that $aa^{-1}=g$ and $a^{-1}a=h$. Thus, $(e_{y_0}ae_{y_0})(e_{y_0}a^{-1}e_{y_0})(e_{y_0}ae_{y_0})=e_{y_0}ae_{y_0}ha^{-1}ge_{y_0}ae_{y_0}=$ $=e_{y_0}ae_{y_0}$, and similarly, $(e_{y_0}a^{-1}e_{y_0})(e_{y_0}ae_{y_0})(e_{y_0}a^{-1}e_{y_0})=e_{y_0}a^{-1}e_{y_0}$. Thus, if $b=e_{y_0}ae_{y_0}$, $b^{-1}=e_{y_0}a^{-1}e_{y_0}$. Hence, as above, $bb^{-1}=e_y$ and $b^{-1}b=e_z$. Hence, every element of S may be expressed in the form gbh where $b\in S_0$, $g\in I_{bb^{-1}}$, and $h\in J_{b^{-1}b}$. We next show $gbh\in R_g\cap L_h$. Since $gbhb^{-1}bb^{-1}=g$, $g\in gbhS$. Thus, since $gbh\in gS$, $gbh\in R_g$. Similarly, $gbh\in L_h$. We are now in a position to establish uniqueness. Let $x=gbh=wcz$ where $c\in S_0$, $w\in I_{cc^{-1}}$, and $z\in J_{c^{-1}c}$. Hence, $g\mathscr{R}x\mathscr{R}w$ and, similarly, $h\mathscr{L}z$. Since $gw=w$, $wg=g$, and S_0 is an inverse semigroup, using [1, Theorem 1.17], $cc^{-1}=bb^{-1}cc^{-1}=cc^{-1}bb^{-1}=bb^{-1}$. Thus, $g=w$. Similarly $b^{-1}b=c^{-1}c$ and $h=z$. Hence, $b=bb^{-1}bb^{-1}b=bb^{-1}gbhb^{-1}b=cc^{-1}wczc^{-1}c=$ $=cc^{-1}cc^{-1}c=c$. Q.E.D.

Using Lemma 1.2, H_{e_y} is the $\mathscr{H}$-class of S_0 $[T_{e_y}]$ containing e_y.

Lemma 1.5. *If* $i\in I_{e_y}$ *and* $j\in J_{e_z}$, $ji\in H_{e_ye_z}$.

Proof. Apply the proof of [4, Lemma 2.11].

Lemma 1.6. *Let* $H=\bigcup(H_{e_y}\colon y\in Y')$. *Then H is the semilattice Y of groups* $(H_y\colon y\in Y)$. *Hence, $E(H)$ is contained in the center of H (*i.e. $eh=he$ *for all* $e\in E(H)$ *and* $h\in H$*)*.

Proof. Utilize [4, Proposition 1.9], Lemma 1.2, and [1, Lemma 4.8].

Lemma 1.7. *Let* $b,c\in S_0$, $j\in J_{b^{-1}b}$, *and* $p\in I_{cc^{-1}}$. *Then* $(b(jp)c)(b(jp)c)^{-1}=$ $=(bc)(bc)^{-1}$ *and* $(b(jp)c)^{-1}b(jp)c=(bc)^{-1}bc$.

Proof. Using Lemmas 1.5 and 1.6, $(b(jp)c)(b(jp)c)^{-1}=b(jp)cc^{-1}(jp)^{-1}b^{-1}=$ $=bcc^{-1}(jp)(jp)^{-1}b^{-1}=bcc^{-1}b^{-1}=(bc)(bc)^{-1}$ and, similarly, $(b(jp)c)^{-1}(b(jp)c)=$ $=(bc)^{-1}bc$.

For $a, b \in S_0$, define $a \circ b = ab$. For $a, b \in T$, define $a * b = ab$.

Lemma 1.8. *Let* $b, c \in S_0$, $i \in I_{b \circ b^{-1}}$, $j \in J_{b^{-1} \circ b}$, $p \in I_{c \circ c^{-1}}$, *and* $q \in J_{c^{-1} \circ c}$. *Then* $(ibj)(pcq) = (i * x)(b \circ (j * p) \circ c)(y * q)$ *where* $x \in I_{(b \circ c) \circ (b \circ c)^{-1}}$ *and* $y \in J_{(b \circ c)^{-1} \circ (b \circ c)}$. *Hence,* $S \cong \{(i, b, j): b \in S_0,\ i \in I_{b \circ b^{-1}},\ j \in J_{b^{-1} \circ b}\}$ *under the multiplication* $(i, b, j)(p, c, q) = (i * x,\ b \circ (j * p) \circ c,\ y * q)$.

Proof. Utilizing Lemma 1.7, $(ibj)(pcq) = i(b \circ (j * p) \circ c)q = (i * ((b \circ c) \circ (b \circ c)^{-1}))(b \circ (j * p) \circ c)(((b \circ c)^{-1} \circ (b \circ c)) * q)$. Let $b \circ b^{-1} = e_r$ and $(b \circ c) \circ (b \circ c)^{-1} = e_w$. Thus, $i * ((b \circ c) \circ (b \circ c)^{-1}) = i\zeta_{r,w} = i * x$ and, similarly, $((b \circ c)^{-1} \circ (b \circ c)) * q = y * q$. Hence, using Lemmas 1.4, 1.3, 1.5, 1.2, and 1.7 the last sentence of the lemma is established.

Theorem 1.9. *(Y, T, V) is a standard regular semigroup, and, conversely, every standard regular semigroup is isomorphic to some (Y, T, V).*

Proof. The converse is a consequence of Lemmas 1.1, 1.6, 1.3, 1.2, and 1.8. We next establish the direct part of Theorem 1.9. Let $S = (Y, V, T)$.

Lemma 1.10. *S is a groupoid.*

Proof. Let $(i, b, j), (r, c, s) \in S$. Let $\{\zeta_{y,z}: y, z \in Y\}$ denote the set of structure homomorphisms of $(T, *)$. Suppose $y \geqq z$. Hence, $z = y * z = y\zeta_{y,z} * z = z * y = z * y\zeta_{y,z}$ or $z \leqq y\zeta_{y,z}$. Thus, $y\zeta_{y,z} = z$. Hence, $i\zeta_{b \circ b^{-1}, (b \circ c) \circ (b \circ c)^{-1}} \mathscr{L} (b \circ c) \circ (b \circ c)^{-1}$, since $i \mathscr{L} b \circ b^{-1}$. Thus $i\zeta_{b \circ b^{-1}, (b \circ c) \circ (b \circ c)^{-1}} \in I_{(b \circ c) \circ (b \circ c)^{-1}}$. Hence, $i * x = i\zeta_{b \circ b^{-1}, (b \circ c) \circ (b \circ c)^{-1}}$ for $x \in I_{(b \circ c) \circ (b \circ c)^{-1}}$ and, similarly, $s\zeta_{c^{-1} \circ c, (b \circ c)^{-1} \circ b \circ c} \in J_{(b \circ c)^{-1} \circ b \circ c}$ and $y * s = s\zeta_{c^{-1} \circ c, (b \circ c)^{-1} \circ (b \circ c)}$ for $y \in J_{(b \circ c)^{-1} \circ (b \circ c)}$. Thus, $(i, b, j)(r, c, s)$ is independent of the choice of u and v. Furthermore, as in the proof of [2, Theorem 2.11], $j \in J_z$ and $i \in I_y$ implies $j * i \in H_{yz}$. Let $H = \bigcup (H_y: y \in Y)$. Then, Lemma 1.6 is valid for H. Thus, as in the proof of Lemma 1.7, $(b \circ (j * r) \circ c) \circ (b \circ (j * r) \circ c)^{-1} = (b \circ c) \circ (b \circ c)^{-1}$, and, similarly, $(b \circ (j * r) \circ c)^{-1} \circ (b \circ (j * r) \circ c) = (b \circ c)^{-1} \circ (b \circ c)$.

Lemma 1.11. *S obeys the associative law.*

Proof. Let $\alpha = (i, b, j)$, $\beta = (r, c, s)$, $\gamma = (w, d, z)$ be elements of S. Let $\alpha_1 = i$, $\alpha_2 = b$, and $\alpha_3 = j$. Then, $((\alpha\beta)\gamma)_1 = i\zeta_{b \circ b^{-1}, (b \circ c \circ d) \circ (b \circ c \circ d)^{-1}} = (\alpha(\beta\gamma))_1$, and, similarly, $((\alpha\beta)\gamma)_3 = (\alpha(\beta\gamma))_3$. Furthermore, $((\alpha\beta)\gamma)_2 = b \circ (j * r) \circ c \circ ((v * s) * w) \circ d$ where $v \in J_{(b \circ c)^{-1}(b \circ c)}$. However, $(v * s) * w = (((b \circ c)^{-1} \circ (b \circ c)) * s) * w = (c^{-1} \circ b^{-1} \circ b \circ c) \circ (s * w)$. Hence, $((\alpha\beta)\gamma)_2 = b \circ (j * r) \circ c \circ c^{-1} \circ b^{-1} \circ b \circ c \circ (s * w) \circ d = b \circ (j * r) \circ c \circ (s * w) \circ d$. Similarly, $(\alpha(\beta\gamma))_2 = b \circ (j * r) \circ c \circ (s * w) \circ d$. Q.E.D.

Lemma 1.12. $(b^{-1} \circ b, b^{-1}, b \circ b^{-1}) \in \mathscr{I}((i, b, j))$. *Hence, S is a regular semigroup.*

Proof. This lemma follows from a straightforward calculation.

Lemma 1.13. (a) $(i, b, j)\mathcal{R}(p, z, q)$ *if and only if* $i=p$. (b) $(i, b, j)\mathcal{L}(p, z, q)$ *if and only if* $j=q$. (c) $(i, b, j)\mathcal{H}(p, z, q)$ *if and only if* $i=p$ *and* $j=q$.

Proof. (a) First assume $i=p$ (hence, $b\circ b^{-1}=z\circ z^{-1}$). Thus, $(i, b, j)(b^{-1}\circ b, b^{-1}\circ z, q)=(i, z, q)$ and $(i, z, q)(z^{-1}\circ z, z^{-1}\circ b, j)=(i, b, j)$. If $(i, b, j)\mathcal{R}(p, z, q)$, there exist $x, y\in I$ such that $i*x=p$ and $p*y=i$. Thus, $i*p=p$ and $p*i=i$. Hence, $b\circ b^{-1}=z\circ z^{-1}$ and $i=p$.

Lemma 1.14. $(i, b, j)\mathcal{D}(p, z, w)$ *if and only if* $b\mathcal{D}z(\in V)$. *Hence, S is bisimple if and only if V is bisimple.*

Proof. Suppose $b\mathcal{D}z$ (in V). Hence, there exists $x\in V$ such that $b\circ b^{-1}=x\circ x^{-1}$ and $x^{-1}\circ x=z^{-1}\circ z$. Thus, $(i, b, j)\mathcal{R}(i, x, w)\mathcal{L}(p, z, w)$. Conversely, suppose $(i, b, j)\mathcal{D}(p, z, w)$. Hence, $(i, b, j)\mathcal{R}(u, x, v)\mathcal{L}(p, z, q)$, say. Thus, $b\circ b^{-1}=x\circ x^{-1}$ and $x^{-1}\circ x=z^{-1}\circ z$ or $b\mathcal{D}z$.

Lemma 1.15. $E(S)=\{(i, b, j)\colon j*i=b^{-1},\ i\in I_y,\ j\in J_y,\ y\in Y\}$.

Proof. Suppose $(i, b, j)(i, b, j)=(i, b, j)$. Hence, $b\circ(j*i)\circ b=b$. Thus, $(b^{-1}\circ b)\circ(j*i)\circ(b\circ b^{-1})=b^{-1}$. Hence, $b^{-1}\in H$ and $b\circ b^{-1}=b^{-1}\circ b$. Hence, $j*i\in H_{b\circ b^{-1}}$ and $j*i=b^{-1}$. Conversely, $(i, (j*i)^{-1}, j)(i, (j*i)^{-1}, j)=(i*((j*i)^{-1}\circ (j*i)), (j*i)^{-1}, ((j*i)\circ(j*i)^{-1})*j)=(i, (j*i)^{-1}, j)$.

Lemma 1.16. $T'=\{(i, b, j)\colon b\in H_y, i\in I_y, j\in J_y, y\in Y\}$ *is the union of the maximal subgroups of S.*

Proof. Let T' denote the union of the maximal subgroups of S. Hence, $(i, b, j)\in T'$ if and only if $(i, b, j)\mathcal{H}(p, c, q)\in E(S)$. Suppose $(i, b, j)\mathcal{H}(p, c, q)\in E(S)$. Using Lemmas 1.13 and 1.15, $c=(q*p)^{-1}\in H_y$, say, $i\in I_y$, $j\in J_y$, and $b\in H_y$. Suppose $i\in I_y$, $b\in H_y$, and $j\in J_y$. Hence, $(i, b, j)\mathcal{H}(i, (j*i)^{-1}, j)\in E(S)$. Q.E.D.

Lemma 1.17. *Let* $T'_y=\{(i, g, j)\colon g\in H_y, i\in I_y, j\in J_y\}$. *Then* T' *is the semilattice* Y *of completely simple semigroups* $(T'_y\colon y\in Y)$.

Proof. Let $(i, g, j), (p, h, q)\in T'_y$. Hence, $(i, g, j)(p, h, q)=(i, g\circ(j*p)\circ h, q)$. Hence, T'_y is completely simple. Let $(i, g, j)\in T'_y$ and $(p, h, q)\in T'_z$. Hence, $(i, g, j)(p, h, q)=(i*(y\circ z), g\circ(j*p)\circ h, (y\circ z)*q)\in T'_{yz}$.

Lemma 1.18. *Every element of* T_y *may be uniquely expressed in the form* $x=i*g*j$ *where* $i\in I_y$, $g\in H_y$, *and* $j\in J_y$.

Proof. Suppose $T_y=\mathcal{M}(G; M, K; P)$ (notation of [1]). Let $e_y=(p_{11}^{-1})_{11}$. Hence, $I_y=\{(p_{1i}^{-1})_{i1}\colon i\in M\}$, $J_y=\{(p_{j1}^{-1})_{1j}\colon j\in K\}$, and $H_y=\{(g)_{11}\colon g\in G\}$. Hence, $(g)_{ij}=(p_{1i}^{-1})_{i1}(x)_{11}(p_{j1}^{-1})_{1j}$ where $x=p_{11}^{-1}p_{1i}gp_{j1}p_{11}^{-1}$.

Lemma 1.19. *Let* $(i*g*j)\theta=(i,g,j)$ $(i\in I_y,\ g\in H_y,\ j\in J_y)$. *Then* θ *defines an isomorphism of* T *onto* T'. *Hence,* T' *is locally inverse.*

Proof. If $j\in H_y$ and $p\in I_z$, $j*p\in H_{y\circ z}$. Let $i*g*j\in T_y$ and $p*h*q\in T_z$ $(i\in I_y,\ g\in H_y,\ j\in J_y,\ p\in I_z,\ h\in H_z,\ q\in J_z)$. Hence, $((i*g*j)*(p*h*q))\theta=$ $=(i*g*(j*p)*h*q)\theta=(i*(y\circ z)*g*(j*p)*h*(y\circ z)*q)\theta=(i*(y\circ z), g\circ(j*p)\circ h, (y\circ z)*q)=(i,g,j)(p,h,q)=(i*g*j)\theta(p*h*q)\theta$.

Remark 1.20. The isomorphism $g\to(g\circ g^{-1}, g, g^{-1}\circ g)$ embeds $(V,\circ)$ into (Y,T,V). In fact, $\{(g\circ g^{-1}, g, g^{-1}\circ g)\colon g\in V\}=(y_0,y_0,y_0)(Y,T,V)(y_0,y_0,y_0)$ where y_0 is the greatest of Y.

The terms standard regular semigroup of type ωY, ωY inverse semigroup, locally inverse semigroup, rectangular group, orthodox semigroup, standard orthodox semigroup and standard $\mathcal{L}$-unipotent semigroup are defined in [4, pp. 540—542].

Remark 1.21. Using Lemmas 1.14—1.17, (Y,T,V) is a standard regular semigroup of type ωY if and only if V is an ωY inverse semigroup.

Remark 1.22. Let $(Y,T,V)_0$ denote (Y,T,V) with "completely simple semigroups" replaced by "rectangular groups" and "$b\circ(j*r)\circ c$" replaced by "$b\circ c$". Let $(Y,T,V)_{\mathcal{L}}$ denote $(Y,T,V)_0$ with "rectangular groups" replaced by "right groups". Then, $(Y,T,V)_0$ $[(Y,T,V)_{\mathcal{L}}]$ is a standard orthodox [standard $\mathcal{L}$-unipotent] semigroup, and conversely every standard orthodox [standard $\mathcal{L}$-unipotent] semigroup is isomorphic to some $(Y,T,V)_0$ $[(Y,T,V)_{\mathcal{L}}]$ (cf. [4, Theorems 5.1 and 5.3 and Remark 5.6]).

Remark 1.23. If we specialize Theorem 1.9 to orthodox semigroups, we obtain the specialization of Yamada's structure theorem for generalized inverse semigroups [6] to standard regular semigroups.

2. Standard completely regular semigroups. In this section, we give a structure theorem for standard completely regular semigroups (Theorem 2.1).

Let Y be a semilattice with greatest element. Let I $[J]$ be a locally inverse semilattice Y of left zero [right zero] semigroups $(I_\alpha\colon \alpha\in Y)$ $[(J_\alpha\colon \alpha\in Y)]$ with structure homomorphisms $(\xi_{\alpha,\beta})$ $[(\zeta_{\alpha,\beta})]$. Let G be a semilattice Y of groups $(G_\alpha\colon \alpha\in Y)$ with structure homomorphisms $\{\varphi_{\alpha,\beta}\}$. Let $(j,i)\to p_{j,i}$ be a function of $J\times I$ into G such that

(1) if $j\in J_\alpha$ and $i\in I_\alpha$, $p_{j,i}\in G_\alpha$;

(2) if $j\in J_\alpha$ and $i\in I_\beta$, $p_{j,i}=p_{j\zeta_{\alpha,\alpha\beta},\, i\xi_{\beta,\alpha\beta}}$;

(3) if $j\in J_\alpha$ and $i\in I_\alpha$ and $\alpha\geqq\beta$, $p_{j,i}\varphi_{\alpha,\beta}=p_{j\zeta_{\alpha,\beta},\, i\xi_{\alpha,\beta}}$.

Let $(Y, I, J, G, \zeta, \xi, \varphi)$ denote $\{(i, g, j)\colon i \in I_\alpha, g \in G_\alpha, j \in J_\alpha \text{ and } \alpha \in Y\}$ under the multiplication

(4) $(i, g, j)(w, h, z) = (iw, gp_{j,w}h, jz)$.

Theorem 2.1. *$(Y, I, J, G, \zeta, \xi, \varphi)$ is a standard completely regular semigroup, and, conversely, every such semigroup is isomorphic to some $(Y, I, J, G, \zeta, \xi, \varphi)$.*

Proof. Let S be a standard completely regular semigroup. Hence, S is a semilattice Y of completely simple semigroups $(S_\alpha\colon \alpha \in Y)$. Let α_0 denote the greatest element of Y. Let $\{\delta_{\alpha,\beta}\}$ denote the set of structure homomorphisms of S. Let $e_{\alpha_0} \in E(S_{\alpha_0})$ and define $e_\alpha = e_{\alpha_0}\delta_{\alpha_0,\alpha}$. Hence, $e_\alpha e_\beta = e_{\alpha\beta}$. Let I_α $[J_\alpha]$ denote the set of idempotents of the $\mathscr{L}$-class [$\mathscr{R}$-class] of S_α containing e_α. Hence, I_α $[J_\alpha]$ is a left zero [right zero] semigroup. As in the proof of Lemma 1.3, $I = \bigcup(I_\alpha\colon \alpha \in Y)$ $[J = \bigcup(J_\alpha\colon \alpha \in Y)]$ is a semilattice Y of left zero [right zero] semigroups $(I_\alpha\colon \alpha \in Y)$ $[(J_\alpha\colon \alpha \in Y)]$. Let $\zeta_{\alpha,\beta} = \delta_{\alpha,\beta}|J$ and $\xi_{\alpha,\beta} = \delta_{\alpha,\beta}|I$. Thus I and J are locally inverse by [4, Theorem 1.6]. Let G_α denote the $\mathscr{H}$-class of S_α containing e_α. Hence, using [4, Proposition 1.9], $G = \bigcup(G_\alpha\colon \alpha \in Y)$ is the semilattice Y of groups $(G_\alpha\colon \alpha \in Y)$ with structure homomorphisms $\varphi_{\alpha,\beta} = \delta_{\alpha,\beta}|G$. As in the proof of Lemma 1.18, every element of S may be uniquely expressed in the form $x = igj$ where $i \in I_\alpha$, $g \in G_\alpha$, and $j \in J_\alpha$. Let $j \in J_\alpha$ and $i \in I_\beta$. Hence, $ji = j\zeta_{\alpha,\alpha\beta}i\xi_{\beta,\alpha\beta} \in G_{\alpha\beta}$. For $j \in J_\alpha$ and $i \in I_\beta$, define $p_{j,i} = ji$. Hence, $(j, i) \to p_{j,i}$ defines a function of $J \times I$ into G satisfying (1) and (2). (3) is verified by a straightforward calculation. Let $x = igj \in S_\alpha$ and $y = whz \in S_\beta$. Hence $xy = (igj)(whz) = i(gp_{j,w}h)z = (i\zeta_{\alpha,\alpha\beta})(gp_{j,w}h)(z\xi_{\beta,\alpha\beta}) = (iw)(gp_{j,w}h)(jz)$. Thus, $igj \to (i, g, j)$ defines an isomorphism of S onto $X = (Y, I, J, G, \zeta, \xi, \varphi)$ under (4).

Next, we show $X = (Y, I, J, G, \zeta, \xi, \varphi)$ is a standard completely regular semigroup. Closure is a consequence of (1) and (2). For $\alpha \in Y$, let $T_\alpha = \{(i, g, j)\colon i \in I_\alpha, g \in G_\alpha, j \in J_\alpha\}$. Let $x = (i, g, j) \in T_\alpha$, $y = (m, h, n) \in T_\beta$, and $w = (c, z, d) \in T_\gamma$. Using (2) and (3), $p_{j,m}\varphi_{\alpha\beta,\alpha\beta\gamma} = p_{j\zeta_{\alpha,\alpha\beta},m\xi_{\beta,\alpha\beta}}\varphi_{\alpha\beta,\alpha\beta\gamma} = p_{j\zeta_{\alpha,\alpha\beta\gamma},m\xi_{\beta,\alpha\beta\gamma}} = p_{j\zeta_{\alpha,\alpha\beta\gamma},m\xi_{\beta,\alpha\beta\gamma}c\xi_{\gamma,\alpha\beta\gamma}} = p_{j\zeta_{\alpha,\alpha\beta\gamma},(mc)\xi_{\beta\gamma,\alpha\beta\gamma}} = p_{j,mc}$. Similarly, $p_{n,c}\varphi_{\beta\gamma,\alpha\beta\gamma} = p_{jn,c}$. Thus, $(xy)w = (imc, g\varphi_{\alpha,\alpha\beta\gamma}p_{j,m}\varphi_{\alpha\beta,\alpha\beta\gamma}h\varphi_{\beta,\alpha\beta\gamma}p_{jn,c}z\varphi_{\gamma,\alpha\beta\gamma}, jnd) = x(yw)$. Using (4), the Rees theorem [1, Theorem 3.5], (1) and (2), X is the semilattice Y of completely simple semigroups $(T_\alpha\colon \alpha \in Y)$. Hence, X is completely regular by [1, Theorem 4.6]. We next show X is locally inverse. Using (4), $E(X) = \{(i, p_{j,i}^{-1}, j)\colon i \in I_\alpha, j \in J_\alpha, \alpha \in Y\}$. Let $(i, p_{j,i}^{-1}, j) \in T_\alpha$ and $(a, p_{b,a}^{-1}, b) \in T_\beta$. Then, using (4), (3) and (2), $(i, p_{j,i}^{-1}, j) \geqq (a, p_{b,a}^{-1}, b)$ if and only if $\alpha \geqq \beta$, $i\xi_{\alpha,\beta} = a$, and $j\zeta_{\alpha,\beta} = b$. Thus, using (4), (3) and (2), S is locally inverse.

Remark 2.2. The structure of I, J, and G are given in terms of their respective structure homomorphisms (see [4, Section 1, especially Remark 1.7], [5, Theorem 1] and [1, Theorem 4.11]).

Remark 2.3. In [4], we used the term Cliffordian semigroup to describe a union of groups. In order not to conflict with the terminology of [3], we adopted our present terminology which appears to be the prevalent terminology.

3. The minimum inverse semigroup congruence. In this section, we describe the minimum inverse semigroup congruence on a standard regular semigroup $S=(Y, V, T)$. If φ is a homomorphism, $\ker\varphi$ will denote the kernel of φ.

Proposition 3.1. *Let ω be a homomorphism of V onto an inverse semigroup V^* such that $J*I\subseteqq \bigcup \ker\omega$. Then, $(i,b,j)\theta=b\omega$ defines a homomorphism of S onto V^*. Conversely, if θ is a homomorphism of S onto an inverse semigroup V^*, then $(i,b,j)\theta=b\omega$ where ω is a homomorphism of V onto V^* with $\bigcup\ker\omega\subseteqq J*I$.*

Proof. We first establish the direct part. Let (i,b,j), $(r,c,s)\in S$. Hence,

$$((i,b,j)(r,c,s))\theta=(b\circ(j*r)\circ c)\omega=b\omega\circ((b^{-1}\circ b)\circ(c\circ c^{-1}))\omega\circ c\omega=$$

$$=b\omega\circ c\omega=(i,b,j)\theta(r,c,s)\theta.$$

Conversely, let θ be a homomorphism of S onto V^*. For $b\in V$, define $b\omega=$ $=(b\circ b^{-1}, b, b^{-1}\circ b)\theta$. Thus, $b\omega c\omega=((b\circ b^{-1}, b, b^{-1}\circ b)(c\circ c^{-1}, c, c^{-1}\circ c))\theta=$ $=((b\circ c)\circ(b\circ c)^{-1}, b\circ c, (b\circ c)^{-1}\circ(b\circ c))\theta=(b\circ c)\omega$. Hence, ω is a homomorphism of V into V^*. Let $(i,b,j)\in S$. Then, $(i,b,j)=(i,b\circ b^{-1}, b\circ b^{-1})(b\circ b^{-1}, b, b^{-1}\circ b)(b^{-1}\circ b, b^{-1}\circ b, j)$. Using Lemma 1.13 (b), $(i, b\circ b^{-1}, b\circ b^{-1})\mathscr{L}(b\circ b^{-1}, b\circ b^{-1}, b\circ b^{-1})$. Hence, using Lemma 1.15, $(i, b\circ b^{-1}, b\circ b^{-1})\theta=(b\circ b^{-1}, b\circ b^{-1}, b\circ b^{-1})\theta$. Similarly, $(b^{-1}\circ b, b^{-1}\circ b, j)\theta=(b^{-1}\circ b, b^{-1}\circ b, b^{-1}\circ b)\theta$. Thus, using Lemmas 1.15 and 1.13, $(i,b,j)\theta=((b\circ b^{-1}, b\circ b^{-1}, b\circ b^{-1})(b\circ b^{-1}, b, b^{-1}\circ b)(b^{-1}\circ b, b^{-1}\circ b, b^{-1}\circ b))\theta=(b\circ b^{-1},$ $b, b^{-1}\circ b)\theta=b\omega$. Let $c\in V^*$. Hence, $c=(i,d,j)\theta=d\omega$ for some $(i,d,j)\in S$. Thus, ω is a homomorphism of V onto V^*. Let $j\in J_y$ and $i\in I_z$. Since $(y,y,j)\theta(i,z,z)\theta=$ $=(y*z, j*i, y*z)\theta=(j*i)\omega=y\omega z\omega=(yz)\omega$, $j*i\in\bigcup\ker\omega$. Q.E.D.

Let N denote the collection of all finite products of elements of the form $a^{-1}\circ s\circ a$ where $a\in V$ and s or $s^{-1}\in J*I$. Since $\mathscr{H}$ is a congruence relation on V by [4, Lemma 2.13] and $J*I\subseteqq H=\bigcup(H_y\colon y\in Y)$, $a^{-1}\circ s\circ a\in H$. Thus, N is an inverse subsemigroup of V and H. Since $E(V)$ is contained in the center of H, it follows that $x^{-1}\circ N\circ x\subseteqq N$ for all $x\in V$. Let $N_y=H_y\cap N$. Then N is the semilattice Y of groups $(N_y\colon y\in Y)$. Let $\varrho_N=\{(a,b)\in V\times V\colon a\circ a^{-1}, b\circ b^{-1}, a\circ b^{-1}\in N_y$ for some $y\in Y\}$. Then, using [2, Theorem 7.54 and Lemma 7.48], ϱ_N is a congruence relation on V with kernel $\{N_y\colon y\in Y\}$.

Proposition 3.2. *V/ϱ_N is the maximal inverse semigroup homomorphic image of S under the homomorphism $(i,b,j)\theta_N=b\varrho_N^{\#}$ where $\varrho_N^{\#}$ is the natural homomorphism of V onto V/ϱ_N.*

Proof. Using Proposition 3.1, θ_N is a homomorphism of S onto V/ϱ_N. Let θ be a homomorphism of S onto an inverse semigroup V^*. Define $(x\theta_N)\gamma=x\theta$ for $x\in S$. We will show that γ is a homomorphism of V/ϱ_N onto V^*. Suppose that $(i, b, j)\theta_N=(p, c, q)\theta_N$. Hence, $b\varrho_N^{\#}=c\varrho_N^{\#}$ and $(b, c)\in\varrho_N$. Thus, using [2, Theorem 7.55], $b=nc$ for some $n\in N_{c\circ c^{-1}}$. By Proposition 3.1, $(i, b, j)\theta=b\omega$ for some homomorphism ω of V onto V^* with $\bigcup\ker\omega\subseteqq J*I$. We note that $n=(a_1^{-1}s_1a_1)\dots(a_n^{-1}s_na_n)$ where $a_i\in V$ and s_i or $s_i^{-1}\in J*I$. Thus, $s_i\omega\in E(V^*)$ and, hence, $n\omega\in E(V^*)$. Thus, since $n\mathscr{H}c\circ c^{-1}$, $n\omega=(c\omega)(c\omega)^{-1}$. Hence $b\omega=n\omega c\omega=$ $=c\omega(c\omega)^{-1}c\omega=c\omega$. Thus, $(i, b, j)\theta=(p, c, q)\theta$. Q.E.D.

Theorem 3.3. *Let $S=(Y, V, T)$ be a standard regular semigroup. Let N denote the collection of all finite products of elements of the form $a^{-1}\circ s\circ a$ where $a\in V$ and s or $s^{-1}\in J*I$. Let $N_y=N\cap H_y$ for $y\in Y$. Let $\delta_N=\{((i, a, j), (p, b, q))\in S\times S\colon N_y\circ a=$ $=N_y\circ b$ where $y=a\circ a^{-1}=b\circ b^{-1}\}$. Then, δ_N is the minimum inverse semigroup congruence on S.*

Proof. Utilize Proposition 3.2 and its proof.

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A concept of characteristic for semigroups and semirings

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§ 1. Introduction

The characteristic $\gamma(R)$ of a Ring $R=(R, +, \cdot)$ corresponds to a congruence on the ring $\mathbf{Z}$ of integers via the ideal $\mathfrak{n}(R)=(\gamma(R))$, the annihilator $\mathfrak{n}(R)$ of $(R, +)$ regarded as a $\mathbf{Z}$-module in the natural way. Likewise, the *characteristic* $\gamma(a)$ *of an element* $a\in R$ is defined by the annihilator $\mathfrak{n}(a)=(\gamma(a))$, and it determines the structure of the submodule

$$\langle a\rangle = \mathbf{Z}a \cong \mathbf{Z}/(\gamma(a)).$$

Moreover, the characteristic $\gamma(R)$ of R is the least common multiple of all $\gamma(a)$, corresponding to the intersection $(\gamma(R))=\bigcap\{(\gamma(a))|a\in R\}$ of ideals or congruences. Clearly, $\gamma(a)=o(a)$ if the (additive) order $o(\mathrm{a})=|\langle a\rangle|$ of $a\in R$ is finite, and $\gamma(a)=0$ if $o(a)=\infty$. In particular, these considerations do not depend on the multiplication of R, and may be used to define the *characteristic* $\gamma(R)$ *of any (not necessarily commutative) group* $(R, +)$.

In a similar way we shall introduce the characteristic of a *semiring* $(S, +, \cdot)$, defined to be an algebra such that $(S, +)$ and $(S, \cdot)$ are arbitrary semigroups connected by ring-like distributivity, dealing basically with the characteristic of a semigroup $(S, +)$. For the latter, the additive notation does not mean any restriction, and may be changed if one is interested in semigroups only.

Let $(S, +)$ be a semigroup. For each $a\in S$, the cyclic subsemigroup

$$\langle a\rangle = \mathbf{N}a \cong \mathbf{N}/\varkappa(a)$$

is determined by a congruence $\varkappa(a)$ on the semigroup $(\mathbf{N}, +)$ of positive integers. Let $K=K(\mathbf{N})$ be the complete lattice of all congruences on $(\mathbf{N}, +)$. Then, analogously to the above procedure concerning rings or groups, the intersection $\bigcap\{\varkappa(a)|a\in S\}\in K(\mathbf{N})$ will be a first candidate for the characteristic of

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$(S, +)$ we want to define. However, the characteristics $\gamma(a)$ and $\gamma(R)$ above are integers, corresponding to congruences on $\mathbf{Z}$ by a lattice isomorphism of $(K(\mathbf{Z}), \supseteq)$ onto $(\mathbf{N}_0, |)$ with the divisiblity relation $|$, and this arithmetical aspect is important. As a substitute for the latter, we define a lattice monomorphism χ of $(K(\mathbf{N}), \supseteq)$ into a complete lattice $(L, \supseteq)$, the dual of the direct product $(\mathbf{N}_0^\infty, \leqq)\times(\mathbf{N}_0, |)$, determining each congruence $\varkappa$ on $(\mathbf{N}, +)$ by a pair $\chi(\varkappa)=(v, g)\in L$. In particular, the pair $\chi(\varkappa(a))=(v(a), g(a))$ corresponding to the congruence $\varkappa(a)$, will be called the characteristic of the element $a\in S$.

After these preparations in § 2, we define in § 3 the characteristic of a semigroup $(S, +)$ as the intersection $\chi(S)=(v(S), g(S))$ of all $\chi(\varkappa(a))=(v(a), g(a))$, the characteristics of the elements $a\in S$. In fact there are two ways to do this, depending on whether one takes this intersection in L or in $\chi(K)\subset L$. Both resulting concepts, clearly not very different, will prove fruitful and well-behaved e.g. with respect to subsemigroups and epimorphic images. In particular, if S is a ring or a group, the second component $g(S)$ of $\chi(S)$ will coincide in both cases with the usual characteristic $\gamma(S)$ discussed above.

In fact we deal in this paper (§ 3) with the more general concept of characteristic, taking the intersection in L, since it contains the first one by simplification, and provides more information in some cases. More details as well as some remarks concerning another concept of characteristic introduced in [5] and [6], are given in the text. Of our results, some of them being independent on any concept of "characteristic" used to prove them, we mention here the following ones on semirings: All elements of a semiring S which are multiplicatively (left, right or weakly) cancellable in S, have the same characteristic $\chi(s)$, coinciding with the characteristic $\chi(S)$ of S. Let S be a semiring which consists only of those elements (and, possibly, of a zero); then $\chi(S)$ is either $(0, 1)$, or $(0, p)$ for some prime p, or $(\infty, 0)$. If such a semiring S contains at most one idempotent and no element of infinite order, both with respect to $(S, +)$, then it is a ring (cf. Prop. 7, Thm. 8).

In § 4, we deal with semirings S embeddable into one with right identity or even with identity. Let T_r be a semiring containing S and a right identity e_r. Then $\chi(S)\supseteq\chi(T_r)=\chi(e_r)=\lambda\supseteq(\infty, 0)$ holds, and for each $\lambda\in\chi(K)$ contained in this interval there exists a semiring T_r' with a right identity e_r' such that $\chi(T_r')=\lambda$. The corresponding statements hold for a semiring S embeddable into one with identity (cf. Thm. 9). Further, using concepts due to [2] and given in the text, the universal identity extension U of such a semiring S has characteristic $\chi(U)=(\infty, 0)$, and, if $\chi(S)\in\chi(K)$, at least one strict identity extension T_0 of S has characteristic $\chi(T_0)=\chi(S)$. Moreover, for each $\lambda\in\chi(K)$ such that $\chi(S)\supseteq\lambda\supseteq(\infty, 0)$, there exists an identity extension U_λ of S with characteristic $\chi(U_\lambda)=\lambda$ which is universal with respect to all identity extensions T of S with characteristic $\chi(T)\supseteq\lambda$: each such T is an epimorphic image of U_λ (cf. Thm. 11).

§ 2. Basic concepts

The semiring $(\mathbf{N}, +, \cdot)$ of positive integers operates in a natural way on each semigroup $(S, +)$ or on each semiring $(S, +, \cdot)$ by

$$na = an = \sum_{i=1}^{n} a \quad \text{for all} \quad n \in \mathbf{N}, a \in S, \tag{1}$$

satisfying $(n+m)a = na+ma$, $(nm)a = n(ma)$ and, in case of semirings,

$$n(a \cdot b) = (na) \cdot b = a \cdot (nb) \quad \text{for all} \quad n \in \mathbf{N}, \quad a, b \in S. \tag{1'}$$

Sometimes we shall assume, for all $a, b \in S$ or some particular ones, that

$$n(a+b) = na+nb \quad \text{for all} \quad n \in \mathbf{N} \tag{1''}$$

holds, clearly not true in general and weaker than $a+b=b+a$. (For example, a semiring S embeddable into one with identity satisfies (1″) for all $a, b \in S$, and there are such semirings with non commutative addition, cf. [2].) Moreover, for each $a \in S$,

$$\varphi_a\colon \mathbf{N} \to \langle a \rangle = \mathbf{N}a \quad \text{defined by} \quad n \to n^{\varphi_a} = na \tag{2}$$

is an epimorphism of $(\mathbf{N}, +)$ onto the cyclic subsemigroup $\langle a \rangle$ of $(S, +)$, and the corresponding congruence on $(\mathbf{N}, +)$ will be denoted by $\varkappa(a)$.

According to the introduction, *we want to define the characteristic of* $a \in S$ *by* $\chi(a) = \chi(\varkappa(a))$ *with a suitable monomorphism* χ *of the lattice* K *of all congruences on* $(\mathbf{N}, +)$ *into a lattice* L, which extends the arithmetical structure of $(\mathbf{N}_0, |)$. We do this step by step in the following way.

Each congruence $\varkappa \in K$ on $(\mathbf{N}, +)$ is either the equality $\iota_{\mathbf{N}}$ or uniquely determined by two integers $v \in \mathbf{N}_0$ and $g \in \mathbf{N}$ (the minimal ones such that $v+1 \equiv v+1+g(\varkappa)$ holds, cf. [5], § 20) according to

$$(n, m) \in \varkappa \Leftrightarrow n \equiv m(\varkappa) \Leftrightarrow \begin{cases} n = m & \text{or} \\ n \equiv m & (g) \quad \text{for} \quad n, m > v, \end{cases} \tag{3}$$

where $n \equiv m(g)$ means the usual congruence of integers modulo g. Conversely, each pair $(v, g) \in \mathbf{N}_0 \times \mathbf{N}$ defines by (3) a congruence $\varkappa \neq \iota_{\mathbf{N}}$ on $(\mathbf{N}, +)$. Thus we can define the bijection

$$\chi\colon K \setminus \{\iota_{\mathbf{N}}\} \to L' = \mathbf{N}_0 \times \mathbf{N} \quad \text{by} \quad \varkappa \to \chi(\varkappa) = (v, g). \tag{4}$$

Applying this via (2) to an element $a \in (S, +)$, we obtain: If $\varkappa(a) = \iota_{\mathbf{N}}$, all elements $a, 2a, \ldots$ of $\langle a \rangle$ are mutually distinct, and $o(a) = |\langle a \rangle| = \infty$. If $\varkappa(a) \neq \iota_{\mathbf{N}}$, we use (4) to define the *characteristic of such an element* $a \in S$ by

$$\chi(a) = \chi(\varkappa(a)) = (v(a), g(a)) = (v, g).$$

It determines the mutually distinct elements of $\langle a \rangle$,

$$a, \ldots, va; \quad (v+1)a, \ldots, (v+g)a \tag{5}$$

such that $(v+g+1)a = (v+1)a$ is the first repetition, $v = v(a)$ the length of the *aperiodic part* $V(a)=\{a, \ldots, va\}$ of $\langle a\rangle$, and $g=g(a)$ the length or order of the *periodic part* $G(a)=\{(v+1)a, \ldots, (v+g)a\}$ of $\langle a\rangle$, known to be a group (which follows immediately by Lemma 3). Clearly we have $v(a)+g(a)=o(a)$.

Lemma 1. *Let* $(K, \subseteqq)$ *be the lattice of all congruences on* $(\mathbf{N}, +)$, *regarded as partially ordered set with respect to the inclusion relation* $\varkappa_2 \subseteqq \varkappa_1$. *Define, further, a relation on* $L'=\mathbf{N}_0 \times \mathbf{N}$ *by*

$$(v_1, g_1) \supseteqq (v_2, g_2) \Leftrightarrow v_1 \leqq v_2 \text{ and } g_1 | g_2. \tag{6}$$

Then $(L', \subseteqq)$ *is dual-isomorphic to the direct product of* $(\mathbf{N}_0, \leqq)$ *and* $(\mathbf{N}, |)$, *hence is likewise a lattice, and* (4) *becomes an isomorphism of the lattice* $(K \setminus \{\iota_{\mathbf{N}}\}, \subseteqq)$ *onto* $(L', \subseteqq)$ *according to*

$$\varkappa_1 \supseteqq \varkappa_2 \Leftrightarrow \chi(\varkappa_1) \supseteqq \chi(\varkappa_2) \Leftrightarrow v_1 \leqq v_2 \text{ and } g_1 | g_2. \tag{7}$$

Proof. It is easily seen by (3) that (7) holds for all $\varkappa_1, \varkappa_2 \in K \setminus \{\iota_{\mathbf{N}}\}$. Hence (4) becomes an isomorphism of the partially ordered sets $(K \setminus \{\iota_{\mathbf{N}}\}, \subseteqq)$ and $(L', \subseteqq)$, due to (6); since $(L', \subseteqq)$ is a lattice, so is $(K \setminus \{\iota_{\mathbf{N}}\}, \subseteqq)$.

In order to include $\iota_{\mathbf{N}} \in K$ in this context, we also want to define $\chi(\iota_{\mathbf{N}})$ as a pair (v, g) such that (3) remains meaningful. This could be done by choosing $g=0$ (for each v) or $v=\infty$ (for each g), adjoining a greatest element ∞ to $(\mathbf{N}_0, \leqq)$. With respect to the structure of L', we do both and define

$$\chi(\iota_{\mathbf{N}}) = (\infty, 0) \in L = \mathbf{N}_0^\infty \times \mathbf{N}_0, \tag{4'}$$

hence $\chi(a)=\chi(\varkappa(a))=(\infty, 0)$ for the *characteristic of an element* $a \in (S, +)$ *of infinite order*.

Lemma 2. *We use* (6) *to define a relation on* $L=\mathbf{N}_0^\infty \times \mathbf{N}_0$. *Then* $(L, \subseteqq)$ *is dual-isomorphic to the direct product of* $(\mathbf{N}_0^\infty, \leqq)$ *and* $(\mathbf{N}_0, |)$, *hence is likewise a complete lattice. Moreover, the bijection* $\chi\colon K \to \chi(K) = L' \cup \{(\infty, 0)\}$ *defined by* (4) *and* (4') *is a lattice monomorphism of* $(K, \subseteqq)$ *into* $(L, \subseteqq)$. *Hence* $(\chi(K), \subseteqq)$ *is a sublattice of* $(L, \subseteqq)$ *as well as a complete lattice, but* $\chi(K)$ *is not closed with respect to infinite intersections in* $(L, \subseteqq)$.

The proof is immediate using Lemma 1 and assertions like $\bigcap \{(v, 1) | v \in \mathbf{N}_0\} = (\infty, 1)$ or $\bigcap \{(0, g) | g \in \mathbf{N}\} = (0, 0)$. We conclude these preliminary considerations with the following statements, denoting by [ ] the greatest integer, by (,)* the greatest common divisor, and by [,]* the least common multiple.

Lemma 3. *Let* $\chi(a)=(v(a), g(a))$ *be the characteristic of an element* $a \in (S, +)$, $o(a) < \infty$, *and consider an element* $ha \in \langle a\rangle$, $1 \leqq h \leqq o(a)$. *Then*

$$v(ha) = \left[\frac{v(a)}{h}\right], \quad g(ha) = \frac{g(a)}{(g(a), h)^*} \tag{8}$$

holds, implying $\chi(ha)=(v(ha), g(ha)) \supseteqq \chi(a)$.

Proof. Suppose $\chi(ha)=(v', g')$. Then, by (3) or (5), $v'\in\mathbf{N}_0$ has to be maximal such that $v'ha$ is contained in the aperiodic part of $\langle a\rangle$, hence v' is given by the first formula (8). Similarly, $g'\in\mathbf{N}$ has to be minimal such that $(v'+1+g')ha=(v'+1)ha$ holds, which is equivalent to $(v'+1+g')h\equiv(v'+1)\cdot h$ modulo $g(a)$; the smallest solution $g'\in\mathbf{N}$ of this congruence is given by the second formula of (8).

As a consequence of (8), the periodic part $G(a)=\{ha|v(a)<h\leqq o(a)\}$ of $\langle a\rangle$ contains a unique idempotent h_0a, i.e. an element with characteristic (0, 1), determined by $(g(a), h_0)^*=g(a)$, or $g(a)|h_0$. This yields $\chi((h_0+1)a)=(0, g(a))$, hence $(h_0+1)a\in G(a)$ generates — like each element with a characteristic of this type — a cyclic group of order $g(a)$, proving that $G(a)$ is such a group with h_0a as its neutral element.

For formulas being equivalent to (8) cf. [6], § 2. Note that (8) as well as $v(a)+$ $+g(a)=\infty$ also hold in case $o(a)=\infty$, dealing with ∞ in a suitable way. (One can look at $(\mathbf{N}_0^\infty +, \cdot, \leqq)$ as an ordered semiring.) The proof of the next lemma, similar to that above, will be omitted.

Lemma 4. a) *Let $(S, +)$ be a semigroup and a, b elements of S satisfying* (1″). *Then we have*

$$v(a+b) \leqq \max\{v(a), v(b)\}, \quad g(a+b)|[g(a), g(b)]^*, \tag{9}$$

i.e. $\chi(a+b)\supseteq\chi(a)\cap\chi(b)$ *and likewise* $\chi(b+a)\supseteq\chi(a)\cap\chi(b)$.

b) *Let $(S, +, \cdot)$ be a semiring and $a, b\in S$. Then we have*

$$v(ab) \leqq \min\{v(a), v(b)\}, \quad g(ab)|(g(a), g(b))^*, \tag{9'}$$

i.e. $\chi(ab)\supseteq\chi(a)\cup\chi(b)$ *and likewise* $\chi(ba)\supseteq\chi(a)\cup\chi(b)$.

§ 3. The characteristic of semigroups and semirings

Definition. The *characteristic* $\chi(S)$ *of a semigroup* $(S, +)$ is defined by

$$\chi(S) = (v(S), g(S)) = (\sup\{v(a)|a\in S\}, \operatorname{lcm}\{g(a)|a\in S\}), \tag{10}$$

i.e. by the intersection of all characteristics $\chi(a)=\chi(\varkappa(a))=(v(a), g(a))$, $a\in S$, taken in the lattice $(L, \subseteq)$ introduced in Lemma 2. The *characteristic of a semiring* $(S, +, \cdot)$ is defined to be that of $(S, +)$.

If $(S, +)$ contains an element of infinite order, then $\chi(S)=(\infty, 0)$ by (4′) [1]. Hence, suppose $o(a)<\infty$, i.e. $\chi(a)=(v(a), g(a))\in L'$ for all $a\in S$. Then we have $v(S)<\infty$ or $v(S)=\infty$ depending on whether $\{v(a)|a\in S\}\subseteq\mathbf{N}_0$ has a maximum,

[1]) A possibility to distinguish this case from the following one with $\chi(S)=(\infty, 0)$ is to replace L by $\mathbf{N}_0^\infty\times\mathbf{N}_0^\infty$ and to define $\chi(\iota_\mathbf{N})=(\infty, \infty)$ instead of (4′).

and likewise $g(S) \neq 0$ or $g(S)=0$ with $\{g(a)|a \in S\} \subseteq \mathbf{N}$, and clearly there are semigroups and semirings which correspond to each of the resulting four cases.

In particular, if $v(S)<\infty$ and $g(S) \neq 0$, i.e. $\chi(S) \in L'$ (obviously satisfied if $|S|<\infty$), then each $\langle a \rangle \subseteq (S, +)$ is an epimorphic image of a single finite cyclic semigroup $(C, +)$, determined by $\chi(C)=\chi(S)$. Moreover, since a congruence on $(\mathbf{N}, +)$ is also one on $(\mathbf{N}, +, \cdot)$, the semiring $(\mathbf{N}, +, \cdot)$ operating on $(S, +)$ by (1) can be replaced by the semiring $(\mathbf{N}/\varkappa, +, \cdot)$ with $\varkappa=\chi^{-1}(\chi(S))$ if $\chi(S) \in \chi(K)$, and $\varkappa$ is the maximal congruence on $(\mathbf{N}, +)$ of this kind.

Note further that only in the mixed cases $v(S)<\infty$, $g(S)=0$ and $v(S)=\infty$, $g(S) \neq 0$, the characteristic $\chi(S)$ defined above is not contained in $\chi(K)=L' \cup \{(\infty, 0)\}$. Leaving certain information out of consideration, one could decide to define a characteristic $\bar{\chi}(S)$ such that $\bar{\chi}(S)=(\infty, 0)$ also holds in these two cases, or equivalently, to define $\bar{\chi}(S)$ by the intersection $\bigcap \{\chi(a)|a \in S\}$ taken in the sublattice $\chi(K)$ of L. But this would not simplify things considerably, and so we stay here [2]) with the more general concept defined above. Clearly, corresponding results for the other one may be obtained by identification of all $(v, g) \in L \setminus \chi(K)$ with $(\infty, 0)$.

Proposition 5. *Let E be a set of generators of the semigroup $S=(S, +)$ or of the semiring $S=(S, +, \cdot)$, of course using both operations in the latter case, and assume that* (1″) *holds for all $a, b \in S$. Then one can replace* (10) *by*

$$(10') \qquad \chi(S) = (\sup\{v(a)|a \in E\},\ \operatorname{lcm}\{g(a)|a \in E\}) = \bigcap \{\chi(a)|a \in E\}.$$

In particular, $|E|<\infty$ implies $\chi(S) \in \chi(K)$ if (1″) *holds for all $a, b \in S$.*

The proof is immediate by Lemma 4. The next statements on semigroups clearly apply to semirings, too.

Proposition 6. a) *Let $(H, +)$ be a subsemigroup or an epimorphic image of a semigroup $(S, +)$. Then we have $\chi(H) \supseteq \chi(S)$.*

b) *If $(S, +)$ is cancellative, we have $v(a) \in \{0, \infty\}$ for all $a \in S$, hence $v(S) \in \{0, \infty\}$. If $(S, +)$ is a group, $g(S)$ is the usual characteristic of S as defined in* § 1.

The proof is straightforward. Note that even a commutative semigroup $(S, +)$ with $\varkappa(S)=(0, p)$, p a prime, need not be cancellative. Any direct sum in the semigroup-theoretical sence (cf. [1], II, § 9.4) of two or more cyclic groups of order p provides a counter example.

An element s of a semigroup $(S, \cdot)$ is called *weakly cancellable* in $(S, \cdot)$ if $sx=sy$ and $xs=ys$ for $x, y \in S$ imply $x=y$ (cf. [3], I. 2).

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[2]) Some announcements given in [2], concerning the characteristic of semirings S, refer directly to the intersection $\varkappa(S)=\bigcap\{\varkappa(a)|a \in S\} \in K$, hence to a concept which clearly corresponds to $\bar{\chi}(S)$ above.

Proposition 7. *Let* $(S, +, \cdot)$ *be a semiring.*

a) *If* $s \in S$ *is weakly cancellable in* $(S, \cdot)$, *then* $\chi(s) \subseteqq \chi(a)$ *holds for all* $a \in S$. *Hence all elements* $s, s' \in S$ *which are weakly cancellable in* $(S, \cdot)$ *have the same characteristic, and* $\chi(s) = \chi(S) \in \chi(K)$ *holds if such an element* s *exists.*

b) *If* $(S, +)$ *has a neutral element, called the zero* o *of* $(S, +, \cdot)$, *and if* o *is weakly cancellable in* $(S, \cdot)$, *then* $\chi(S) = (0, 1)$ *or, equivalently,* $(S, +)$ *is idempotent.*

Proof. By assumption on s, $n(as) = m(as)$ and $n(sa) = m(sa)$ together imply $na = ma$, from which $\chi(as) \cap \chi(sa) \subseteqq \chi(a)$ follows. Using (9′), we obtain $\chi(s) \subseteqq \subseteqq \chi(as) \cap \chi(sa) \subseteqq \chi(a)$ for all $a \in S$, i.e. $\chi(s) \subseteqq \chi(a)$. This implies the other statements in a) and also b), since (0, 1) is the characteristic of idempotent elements in $(S, +)$, and the greatest element of L.

An example of a semiring S such that S has a zero o which is even cancellable in $(S, \cdot)$, is given in [8]. Moreover, a semiring S is called *multiplicatively cancellative*, briefly *mult. can.*, if each $a \neq o$ of S (meaning each $a \in S$ if there is no zero o of S) is cancellable in $(S, \cdot)$. Note that if S has a zero o and $|S| \geqq 2$, then either o is also cancellable in $(S, \cdot)$, or o is annihilating (i.e. $ao = oa = o$ for all $a \in S$) and $(S \setminus \{o\}, \cdot)$ is a cancellative subsemigroup of $(S, \cdot)$. A mult. can. semiring S does not have (proper) zero divisors, whereas the converse does not hold in general (cf. [8], [10]).

We introduce a wider class of semirings: A semiring S, containing a zero o or not, is called *weakly mult. can.*, if each $a \neq o$ of S is weakly cancellable in $(S, \cdot)$.

Theorem 8. a) *Let* S *be a weakly mult. can. semiring. Then all elements* $s \neq o$ *of* S *have the same characteristic* $\chi(s)$, *coinciding with* $\chi(S)$, *which is either* (0, 1), *or* $(0, p)$ *for some prime* p, *or* $(\infty, 0)$.

b) *Let* S *be a weakly mult. can. semiring with zero* o. *Then either* $(S, +)$ *is idempotent, or* o *is the only idempotent of* $(S, +)$ *and annihilating. Clearly, the first case corresponds to* $\chi(S) = (0, 1)$, *the second one to* $(0, p)$ *or* $(\infty, 0)$.

c) *Let* S *be a weakly mult. can. semiring,* $|S| \geqq 2$, *which contains at most one idempotent and no element of infinite order of* $(S, +)$. *Then* S *is a ring, whose additive group* $(S, +)$ *is the direct sum of cyclic groups of order* p.

Proof. a) By Prop. 7, a) we obtain $\chi(S) = \chi(s)$ for all $s \neq o$ of S. If $\chi(S) \neq \neq (\infty, 0)$, we have $v(s) = v(S) < \infty$ and $g(s) = g(S) \neq 0$ for each $s \neq o$; hence for each $h \in \mathbf{N}$, $1 \leqq h \leqq v(s) + g(s)$, either $hs = o$ holds or (8) implies

$$v(hs) = \left[\frac{v(s)}{h}\right] = v(s) \quad \text{and} \quad g(hs) = \frac{g(s)}{(g(s), h)^*} = g(s).$$

This proves $v(s) = 0$ and either $g(s) = 1$ or $g(s)$ prime.

b) If $(S, +)$ is not idempotent, we have $(0, 1)=\chi(o)\neq\chi(s)=\chi(S)$ for all $s\neq o$ of S by a). Hence o is the only idempotent of $(S, +)$, and $\chi(ao)\supseteqq\chi(o)$, $\chi(oa)\supseteqq\chi(o)$ by (9′) implies $ao=oa=o$ for all $a\in S$.

c) Using a) and the assumptions, we have $\chi(S)=(0, p)$. Hence each $s\neq o$ generates a group $\langle s\rangle$ of order p, whose zero o_s has characteristic (0, 1). Again by a), there is at most one element in S whose characteristic differs from $(0, p)$, hence all o_s collapse to the zero o of $(S, +)$, and $(S, +)$ is a group with $\varkappa(S)=(0, p)$. By well known facts on groups or modules, $(S, +)$ is the direct sum of cyclic groups of order p, if it is commutative. Suppose the contrary, then $(S, +, \cdot)$ would be a ring with non commutative addition, which always contains a two-sided annihilator ideal different from $\{o\}$ (cf. [9]), contradicting that S is weakly mult. can. .

An example of a weakly mult. can. semiring S such that each element of $(S, \cdot)$ is neither left nor right cancellable is given by the tables

$+$	a_1	a_2	b_1	b_2
a_1	a_1	a_2	b_1	b_2
a_2	a_2	a_2	b_2	b_2
b_1	b_1	b_2	b_1	b_2
b_2	b_2	b_2	b_2	b_2

$\cdot$	a_1	a_2	b_1	b_2
a_1	a_1	a_2	a_1	a_2
a_2	a_1	a_2	a_1	a_2
b_1	b_1	b_2	b_1	b_2
b_2	b_1	b_2	b_1	b_2.

Note that S is the direct composition (cf. the definition given in the proof of Prop.10) of two semirings $\{a, b\}$ and $\{1, 2\}$ with operations obvious from these tables, and that the zero a_1 of S is weakly cancellable in $(S, \cdot)$, too. On the other hand, one easily proves that a ring S is weakly mult. can. iff it is mult. can.. Further, using the other parts of Thm. 8, part c) may be reformulated as follows: A weakly mult. can. semiring S such that $\chi(S)=(0, p)$ is a ring. The corresponding statements in the other cases, $\chi(S)=(0, 1)$ or $\chi(S)=(\infty, 0)$, are far away from being true, even for semirings S which are mult. can.. In fact, such a semiring need not have a zero (e.g. $S=(\mathbf{N}, +, \cdot)$ for $\chi(S)=(\infty, 0)$, for $\chi(S)=(0, 1)$ see [8]).

Concluding this section, we mention a concept of characteristic introduced in [5], § 23 for semigroups, and similarly in [6], § 3 for semirings. Working only with the order of elements of $(S, +)$, the characteristic of S is defined to be

0 iff $o(a)=\infty$ for all $a\neq o$ of S,

n iff there is a (minimal) $n\in\mathbf{N}$ such that $o(a)|n$ for all $a\in S$, and

∞ for all other cases.

Applied to a weakly mult. can. semiring S, by Thm. 8a) S has either characteristic $n=1$, or $n=p$, or 0 in this sense, a result stated in [6], Satz 1 for mult. left (or right) can. semirings.

§ 4. Semirings embeddable into semirings with (one sided) identity

A semiring S need neither be embeddable into a semiring with right identity nor into one with left identity. There are also semirings S for which only one kind of these extensions exists, and semirings S which have both, extensions with right and no left identity as well as extensions with left and no right identity. The latter case is equivalent to S being embeddable into a semiring with identity. For corresponding examples as well as for necessary and sufficient conditions we refer to [2]. The following statements deal with the characteristic of a semiring S and of its extensions with a one-sided (say, right) or two-sided identity; assertions or concepts needed from [2] will be given.

Theorem 9. a) *Let S be a semiring embeddable into one with right identity. Then for each semiring T_r with a right identity e_r, containing S as a subsemiring, the characteristic $\lambda=\chi(T_r)$ satisfies $\chi(S)\supseteqq\chi(T_r)$ and $\lambda=\chi(T_r)=\chi(e_r)\in\chi(K)$, hence*

$$\chi(S) \supseteqq \lambda \supseteqq (\infty, 0), \quad \lambda\in\chi(K). \tag{11}$$

In particular, if $\chi(S)\notin L'=\mathbf{N}_0\times\mathbf{N}$, the characteristic of T_r is uniquely determined by $\lambda=\chi(T_r)=(\infty, 0)$.

Conversely, let S be a semiring as above and let $\lambda\in L$ satisfy (11). *Then there exists at least one such extension T_r of S satisfying $\chi(T_r)=\lambda$.*

b) *Let S be a semiring embeddable into one with identity. Then the same statements hold for the characteristic $\lambda=\chi(T)$ of each semiring T with identity which contains S, and for each $\lambda\in L$ satisfying* (11).

Remark. By the converse statements, *for a semiring S embeddable into one with (right) identity, there exists such an extension T_r or T of the same characteristic $\chi(T_r)=\chi(S)$ or $\chi(T)=\chi(S)$ if and only if $\chi(S)\subseteqq\chi(K)$ holds. This is always the case if S is finitely generated* (by Prop. 5, since (1″) holds for all $a, b\in S$ if S is embeddable as assumed above), *and also if $(S, \cdot)$ contains a weakly cancellable element* (by Prop. 7. a). But there are semirings S, embeddable as above, such that $\chi(S)\notin\chi(K)$, hence $\chi(S)\supset\chi(T_r)$ or $\chi(S)\supset\chi(T)$ for all extensions under discussion. For an example, let S be the zero ring on the Prüfer group $(S, +)$ (cf. [5], § 23); then S is even embeddable into a ring with identity, but $\chi(S)=(0, 0)\notin\chi(K)$.

Proof of Thm. 9. The first part of a) follows directly by Prop. 6a) and by Prop. 7a), and likewise the corresponding part of b). Moreover, both converse statements of a) and b) become trivial if $\chi(S)\notin L'$, since then only $\lambda=(\infty, 0)$ satisfies (11). Thus it remains to prove these statements with the assumption $\chi(S)\in L'$.

If S is embeddable into a semiring with right identity, in the proof of Thm. 1, [2] we have constructed a semiring T_r with the following properties: S is a subsemiring, T_r contains the identity mapping $\iota=\iota_S$ of S, $\iota+\iota$ is defined by $a(\iota+\iota)=a+a$ for all $a\in S$, and $e_r=\iota_S$ is the right identity of T_r. Hence for each $(n, m)\in \mathbf{N}\times\mathbf{N}$ we have

$$na = ma \text{ for all } a\in S \Leftrightarrow ne_r = me_r. \tag{12}$$

Since $\chi(S)=\bigcap\chi(a)\in L'$ by assumption, (12) yields $\chi(S)=\chi(e_r)=\chi(T_r)$. In the two-sided case, the semiring T constructed in the proof of Thm. 2, [2] in a similar way, satisfies (12) with respect to the identity e of T, hence $\chi(S)=\chi(e)=\chi(T)$. Thus in both cases there are extensions T_r resp. T of S whose characteristic equals $\chi(S)$, and the proof will be complete applying the following

Proposition 10. a) *Let T_r be a semiring with a right identity e_r. Then for each $\lambda\in\chi(K)$ such that $\chi(T_r)\supseteqq\lambda\supseteqq(\infty, 0)$, there exists an extension T_r' of T_r with a right identity e_r' of characteristic $\chi(T_r')=\lambda$.*

b) *The corresponding statement holds in the case of two-sided identities.*

Proof. It will be enough to deal with a). Since $\lambda=(v, g)$ corresponds to $\varkappa=\chi^{-1}(\lambda)\in K$, which is also a congruence on the semiring $(\mathbf{N}, +, \cdot)$, the semiring $\mathbf{N}'=\mathbf{N}/\varkappa=\{1', 2', \dots\}$ satisfies $\chi(\mathbf{N}')=\chi(1')=\lambda$ and $v=v(1')$, $g=g(1')$. If $\lambda=(0, g)$, $\mathbf{N}'\cong\mathbf{Z}/(g)$ has $g'=o'$ as annihilating zero, and we write $\mathbf{N}'=\mathbf{N}_0'$. In each other case (including $\mathbf{N}'=\mathbf{N}$ for $\lambda=(\infty, 0)$) we adjoin an annihilating zero o' to $\mathbf{N}'$ and obtain a semiring $\mathbf{N}_0'=\{o', 1', 2', \dots\}$, sharing with $\mathbf{N}'$ all properties stated above.

Now we use the semiring T_r, and define operations on the set $T_r'=T_r\times\mathbf{N}_0'$ by

$$(t_1, n_1')+(t_2, n_2') = (t_1+t_2, n_1'+n_2'). \tag{13}$$

This construction, called the *direct composition of T_r and $\mathbf{N}_0'$*, may clearly be applied to any two (or more) semirings, yielding a semiring again. In our case, T_r' contains an isomorphic copy of T_r by $t\to(t, o')$; hence we can consider T_r' as an extension of T_r. Moreover, $e_r'=(e_r, 1')$ is a right identity of T_r'. Since $\chi(T_r)=\chi(e_r)\supseteqq\lambda=$ $=\chi(1')$, we obtain $\chi(e_r')=\chi(1')$, again by (13), i.e. $\chi(T_r')=\lambda$ as we were to prove.

Now let S be a subsemiring of a semiring T with identity e_T. We call T an *identity extension of S*, and write $T=[S, e_T]$, if T is generated by S and e_T. In this case, each element $t\in T$ equals a sum

$$t = \sum_{i=1}^{n} t_i \quad \text{with} \quad t_i\in S\cup\langle e_T\rangle, \quad n\in\mathbf{N}.$$

Note that the addition in T is not assumed to be commutative. Clearly, each extension T' of S with an identity e' contains the identity extension $[S, e']\subseteqq T'$. Moreover, by Thm. 4 of [2], there exists a *universal identity extension $U=[S, e_U]$ of S*, defined by the property that for each $T=[S, e_T]$ there is an epimorphism

$$\psi\colon U\to T \quad \text{such that} \quad a\to a \quad \text{for all} \quad a\in S, \quad \text{and} \quad e_U\to e_T. \tag{14}$$

By (14), U is unique up to isomorphisms (relative w.r.t. S). Conversely, each congruence τ on $(U, +, \cdot)$ such that $\tau|S\times S$ is the equality on S, corresponds to an identity extension $T\cong U/\tau$ of S. Applying Zorn's Lemma to the set Θ of all these congruences (in fact a complete lattice), there is at least one maximal $\tau_0\in\Theta$, hence an identity extension $T_0\cong U/\tau_0$ of S with the property: Each epimorphism (relative w.r.t. S) of T_0 onto an identity extension is an isomorphism. We call such a T_0 (being "minimal with respect to epimorphisms") a *strict identity extension* of S. Note that T_0 also has no proper subsemirings containing S and any identity, and that there are semirings S with non isomorphic strict identity extensions (cf. [2]).

Theorem 11. *Let S be a semiring embeddable into one with identity.*

a) *The universal identity extension $U=[S, e_U]$ has characteristic $\chi(U)=(\infty, 0)$, regardless of the characteristic $\chi(S)$ of S.*

b) *For each $\lambda\in\chi(K)$ such that $\chi(S)\supseteqq\lambda\supseteqq(\infty, 0)$ (i.e. (11)), there exists an identity extension U_λ of S of characteristic $\chi(U_\lambda)=\lambda$, which is universal for all identity extensions T' of S of characteristic $\chi(T')$ satisfying $\chi(S)\supseteqq\chi(T')\supseteqq\lambda$. Clearly, U_λ is uniquely determined by S and λ, up to isomorphisms relative w.r.t. S.*

c) *If $\chi(S)\in\chi(K)$, then there exists at least one strict identity extension T_0 of S of characteristic $\chi(T_0)=\chi(S)$.*

Proof. a) By Thm. 9, there is a semiring T containing S and an identity e_T such that $\chi(T)=(\infty, 0)$. The identity extension $[S, e_T]\subseteqq T$ is an epimorphic image of U, $\psi\colon U\to[S, e_T]$, hence $(\infty, 0)=\chi([S, e_T])\supseteqq\chi(U)$ by Prop. 6a), proving $\chi(U)=(\infty, 0)$.

b) As just stated, the subsemiring $\mathbf{N}e_U=\langle e_U\rangle$ of $U=[S, e_U]$ is isomorphic to $(\mathbf{N}, +, \cdot)$, and may be identified with the latter. As a consequence of Thm. 9, for each $\lambda\in\chi(K)$ satisfying (11), there is an identity extension T of S with $\chi(T)=\lambda$. Hence for the congruence $\tau\in\Theta$ on $(U, +, \cdot)$ such that $T\cong U/\tau$, the restriction $\tau|\mathbf{N}\times\mathbf{N}$ coincides with $\chi^{-1}(\lambda)=\varkappa\in K$. Considering $\varkappa$ as a relation on U, let $\Theta_\varkappa$ be the set of all $\tau'\in\Theta$ such that $\tau'\supseteqq\varkappa$. Since $\tau\in\Theta_\varkappa$, the intersection $\tau_\varkappa=\bigcap\{\tau'\in\Theta_\varkappa\}\in\Theta_\varkappa$ satisfies $\tau_\varkappa|\mathbf{N}\times\mathbf{N}=\varkappa$, like τ. Hence $\tau_\varkappa$ provides an identity extension $U_\lambda\cong U/\tau_\varkappa$ of S which has characteristic $\chi(U_\lambda)=\chi(\varkappa)=\lambda$. Moreover, for each identity extension T' of S with $\chi(T')=\lambda'\supseteqq\lambda$, the corresponding congruence $\tau'\in\Theta$ such that $T'\cong U/\tau'$ satisfies $\tau'\in\Theta_\varkappa$, since $\tau'|\mathbf{N}\times\mathbf{N}=\chi^{-1}(\lambda')\supseteqq\varkappa$. Thus by $\tau_\varkappa\subseteqq\tau'$ there is an epimorphism of U_λ onto T', relative w.r.t. S, hence respecting identities.

c) If $\chi(S)\in\chi(K)$, then there is an identity extension T of S with $\chi(T)=$ $=\chi(S)$ by Thm. 9 (or, likewise, $U_{\chi(S)}$ of b) just proved). If T is not strict, there exists an epimorphism of T onto a strict identity extension T_0 of S. Using (11) and Prop. 6a) we obtain $\chi(S)\supseteqq\chi(T_0)\supseteqq\chi(T)=\chi(S)$, as we have to show.

Each semiring S has a unique maximal epimorphic image S^* of S which is embeddable into a semiring with identity, and $\varphi^*\colon S\to S^*$ is universal with respect to this property (cf. [2]). Hence the universal identity extension U^* of S^*, together with $\varphi^*\colon S \to U^*$, satisfies that each homomorphism $\varphi\colon S \to T''$ of S into any semiring T'' with identity has a (unique) decomposition

$$\varphi\colon S \xrightarrow{\varphi^*} U^* \xrightarrow{\psi} T'',$$

i.e. U^* is the *universal semiring with identity of* S as introduced first in [4]. Hence the results of this section, applied to S^* instead of S, provide also information on an arbitrary semiring S. For instance: *The universal semiring with identity* U^* *of* S *has characteristic* $\chi(U^*)=(\infty, 0)$.

Finally we note: *A mult. can. semiring* S *is always embeddable into one with identity, and has a unique strict identity extension* T_0. *It is the only identity extension of* S *being again mult. can.* ([2], Thm. 12; for semirings with commutative addition cf. [6], Satz 2). Moreover, by Thm. 11c), *the characteristic* $\chi(T_0)$ *of* T_0 *coincides with* $\chi(S)$. For similar results on identity extensions of rings compare [7].

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On intertwining dilations. V
(Letter to the Editor)

ZOIA CEAUŞESCU and CIPRIAN FOIAŞ

1. In the paper [3] the last two theorems (Lemma 5.1 and Proposition 5.1) are incorrect. Namely, the mapping constructed in the proof of Lemma 5.1 (yielded by the formula (5.6)) is not injective (as asserted at the end of the proof of Lemma 5.1). The error consists in the assumption which is implicitly made in § 5, that for any Ando dilation $\{U_1, U_2\}$ on $\mathfrak{K}$, U_2 is the minimal isometric dilation of

$$\hat{A} = P_K U_2|K \quad \text{where} \quad K = \bigvee_{n=0}^{\infty} U_1^n \mathfrak{H}$$

(the terminology and the notation are that of [3]). Here is a counterexample:

Set

$$\mathfrak{H} = \mathbf{C}\oplus\{0\}, \quad K = H^2\oplus\{0\}, \quad \mathfrak{K} = H^2\oplus S_+H^2$$

$$T_1 = T_2 = 0_{\mathfrak{H}}, \quad U_1 = S_+\oplus S_+, \quad U_2 = \begin{pmatrix} 0 & S_+ \\ S_+ & 0 \end{pmatrix}$$

where S_+ denotes the canonical multiplication unilateral shift on the classical Hardy space H^2. Then $\{U_1, U_2\}$ is an Ando dilation of $\{T_1, T_2\}$ but U_2 is not the minimal isometric dilation of $\hat{A}=P_K U_2|K=O_K$. Moreover, changing the role of U_1 and U_2 does not improve the situation since, if we set

$$K' = \bigvee_{n=0}^{\infty} U_2^n \mathfrak{H}, \quad \hat{A}' = P_{K'} U_1|K',$$

then analogously U_1 is not the minimal isometric dilation of $\hat{A}'=O_{K'}$.

2. Therefore one cannot range the present Ando dilation $\{U_1, U_2\}$ of $\{O_{\mathfrak{H}}, O_{\mathfrak{H}}\}$ in the frame considered in [3], § 5. Consequently, we must withdraw Lemma 5.1 and Proposition 5.1 from our paper [3]. However we take this opportunity to state

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that one can give a similar, but more complicated labeling of all Ando dilations by referring besides the paper [3] also to our next paper [4]. Since this correct form of Lemma 5.1 and Proposition 5.1 of [3] needs a much longer discussion, it will be given in a subsequent paper.

3. We should like to indicate a simple case in which Lemma 5.1 of [3] conserves its validity, namely *if the factorization* $T_1 \cdot T_2$ *is regular* (in the sense of [5], Ch. VII). Indeed the only fact we have to prove is that for any Ando dilation $\{U_1, U_2\}$, U_2 is the minimal isometric dilation of $\hat{A}$. In the present case this is equivalent to the relation

$$U_2(\mathfrak{K} \ominus K) \subset \mathfrak{H} \ominus K. \tag{1}$$

In proving (1) we firstly notice that for any $l \in \mathfrak{L} = ((U_1 - T_1)\mathfrak{H})^-$ there exists (because of the regularity of $T_1 \cdot T_2$) a sequence $\{h_j\}_{j=1}^{\infty} \subset \mathfrak{H}$ such that

$$D_{T_2} h_j \to 0 \quad \text{and} \quad (U_1 - T_1) T_2 h_j \to l. \tag{2}$$

From the first relation (2) we obtain

$$\|(U_2 - T_2) h_j\|^2 = \|D_{T_2} h_j\|^2 \to 0 \tag{3}$$

so that the second relation (2) becomes $(I-P) U_1 U_2 h_j \to l$. Therefore, setting $\mathfrak{H}_0 = \mathfrak{H}$ and $\mathfrak{H}_n = \mathfrak{H} \vee U_1 \mathfrak{H} \vee \ldots \vee U_1^n \mathfrak{H}$ $(n=1, 2, \ldots)$ as in [3], § 1 we have

$$(U_2 \mathfrak{H}_1 + \mathfrak{H})^- \ni U_2 U_1 h_j - T_1 T_2 h_j \to l,$$

whence $(U_2 \mathfrak{H}_1 + \mathfrak{H})^- \supset \mathfrak{L}$, and consequently,

$$(U_2 \mathfrak{H}_1 + \mathfrak{H})^- \supset \mathfrak{H}_1. \tag{4$_1$}$$

We can apply $(4)_1$ to the compressions $(T_1)_n$ and $\hat{A}_n$ to $\mathfrak{H}_n$ of U_1 and U_2, respectively (since by [1], $(T_1)_n \cdot \hat{A}_n$ is also regular) and obtain

$$(U_2 \mathfrak{H}_n + \mathfrak{H}_{n-1})^- \supset \mathfrak{H}_n \tag{4$_n$}$$

for all $n \geqq 1$. By iterating $(4)_n$ we finally obtain

$$(U_2 K + \mathfrak{H})^- \supset K. \tag{5}$$

Now

$$U_2(\mathfrak{K} \ominus K) \subset U_2(\mathfrak{K} \ominus \mathfrak{H}) \subset \mathfrak{K} \ominus \mathfrak{H} \perp \mathfrak{H} \tag{6}$$

(because of formula (5.8) of [3]) and

$$U_2(\mathfrak{K} \ominus K) \perp U_2 K. \tag{7}$$

Relation (1) follows now directly from Relations (5), (6), and (7).

The validity of Lemma 5.1 of [3] under the supplementary condition of regularity completes also the proof of Theorem 6.1 (3) and Corollary 6.1 of [2].

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Bibliographie

A. C. Bajpai, I. M. Calus, and J. A. Fairley, Numerical methods for engineers and scientists (A students' course book), XII+380 pages, Taylor & Francis Ltd, London, 1975.

The book comprises three 'Units': 1. Equations and Matrices, 2. Finite Differences and their Applications, 3. Differential Equations. The emphasis is on the practical side of the subject and the more theoretical aspects are omitted. The reader should be familiar with the items listed under the heading of Prerequisits at the beginning of each Unit. There are several references to the suitability of methods presented for programming on a computer. As different programming languages are in use, the various techniques discussed are not, with one exception, translated into computer programs, but a large number of flow diagrams are incorporated in the text.

The programmed method of presentation requires the active participation of the reader in many places where he is asked to answer a question or to solve, either partially or completely, a problem. The answers to these are always given so that the reader can check his attempt and thus obtain a continuous assessment of his understanding of the subject.

The book will certainly be useful as a textbook for both science and engineering students.

F. Móricz (Szeged)

H. Bühlmann—L. Loeffel—E. Neivergelt, Entscheidungs- und Spieltheorie. Ein Lehrbuch für Wirtschaftswissenschaftler (Hochschultext), XIII+311 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1975.

In everyday life, and especially in management praxis one often has to make decisions sequentially in a process in which some external effect modifies the evolution between two consecutive steps. The decision-maker wants, of course, to choose those decisions which ensure the most favourable evolution of the process, in other words, he wants to maximize his reward.

Decision and game theory deals with the mathematical analysis of such, so-called sequential, decision processes. If the influencing external effect is another decision-maker acting according to his own preference (reward) structure, then the corresponding process is called a game. If the distrubing effect is simply the chance, or, in other words, a non-interested decision-maker, then one faces a simple sequential decision problem. Risk theory, Wald's statistical decision theory and decision making under uncertainty are the most important sub-fields of decision theory.

The present book is a first introduction to decision and game theory. It was written for students in management science, and requires a mathematical education on secondary school level only.

The first part of the book deals with decision theory including utility theory. The second part is devoted to game theory, while the third one to statistical decision theory. Two mathematical appendices contain the more elaborate proofs, and a bibliography and an index close the volume. 121 figures and many explicitly solved examples help to understand the text.

D. Vermes (Szeged)

Surveys in Combinatorics, Proceedings of the 7^{th} British Combinatorial Conference, ed. B. Bollobás, VII+261 pages, Cambridge University Press, Cambridge—New York—London—Melbourne, 1979.

These excellent surveys cover many basic areas in combinatorics and give a good picture of recent developments of the field. The papers are the following: N. L. Biggs: Resonance and reconstruction; A. Gardiner: Symmetry conditions in graphs; D. J. Kleitman: Extremal hypergraph problems; W. Mader: Connectivity and edge-connectivity in finite graphs; J. Nešetřil and V. Rödl: Partition theory and its applications; J. J. Seidel: Strongly regular graphs; J. A. Thas: Geometries in finite projective and affine spaces; C. Thomessen: Long cycles in digraphs with constraints on the degrees; D. Welsh: Colouring problems and matroids.

L. Lovász (Szeged)

Siegfried Brehmer, Hilbert-Räume und Spektralmaße, 224 Seiten, Akademie-Verlag, Berlin, 1979.

Der Hauptteil dieses Bändchens in der Reihe „Wissenschaftliche Taschenbücher" ist der Theorie der beschränkten linearen Operatoren gewidmet. Im Mittelpunkt steht die Spektralzerlegung beschränkter selbstadjungierter Operatoren, die dann auch auf den Fall unbeschränkter selbstadjungierter Operatoren ausgedehnt wird. Der Rest bringt eine relativ elementare, aber gründlich ausgearbeitete Einführung in die Theorie der Spektralmaße und Spektralintegrale und gipfelt in der Bereitstellung der Funktionalkalküls für meßbare Funktionen von (nicht notwendig beschränkten) normalen Operatoren. Der Verf. stützt sich natürlich auf Standardwerken, macht aber gelegentlich auch Vereinfachungen und Erneuerungen, die teilweise seine Kollegen und Studenten gefunden haben.

Béla Sz.-Nagy (Szeged)

Shiing-shen Chern, Complex manifolds without potential theory (with an Appendix on the geometry of characteristic classes), V+152 pages. Second Edition, Springer Verlag, Berlin—Heidelberg—New York, 1979.

The new methods of complex manifold theory are very useful tools for investigations in algebraic geometry, complex function theory, differential operators and so on. The differential geometrical methods of this theory were developed essentially under the influence of Professor S.-S. Chern's works. The present book is a second edition; it was originally published by Van Nostrand in 1968. It can serve as an introduction to, and a survey of, this theory and is based on the author's lectures held at the University of California and at a summer seminar of the Canadian Mathematical Congress.

The methods of complex manifold theory have grown parallel to the Hodge—De Rham theory of harmonic integrals, which is an analogue of classical potential theory. The treatment of this book leaves out of consideration these analytical aspects of the theory; the title hints at this circumstance.

The text is illustrated by many examples. The reader in supposed to be acquainted with some differential geometry, fibre bundle and sheave theory. The book is warmly recommended to everyone interested in complex differential geometry.

P. T. Nagy (Szeged)

Shiing-shen Chern, Selected Papers, XXXII+476 pages. Springer Verlag, New York—Heidelberg—Berlin, 1978.

This book is a presentation of a fascinating personal Oeuvre and at the same time of the many-sided progress in differential geometry in the last 45 years. The volume contains approximately one third of Professor Chern's works, among them also some less known fundamental papers published in inaccessible journals.

The selection is introduced by three papers presenting Chern's mathematical and personal oeuvre written by André Weil, Phillip A. Griffiths and S.-S. Chern himself with the titles: "S.-S. Chern as Geometer and Friend", "Some Reflection on the Mathematical Contributions of S.-S. Chern" and "A Summary of My Scientific Life and Works", respectively.

Chern's investigations can be put into the following domains of differential geometry according to his own classification: projective differential geometry, euclidean differential geometry, geometric structures and their intrinsic connections, integral geometry, characteristic classes, holomorphic mappings, minimal submanifolds, webs. His results give programs for future research, and at the same time they pursue the geometric view of his masters: Wilhelm Blaschke and Elie Cartan.

This excellent book should not be missing in any mathematical library.

P. T. Nagy (Szeged)

P. Gänssler and W. Stute, Wahrscheinlichkeitstheorie (Hochschultext/Universitext), XII+418 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1977.

The book is intended to serve as a graduate text in probability theory. No knowledge of measure or probability theory is pressupposed, only a few notions and results from analysis, linear algebra and set theory are required. These prerequisities are collected in Ch. 0.

The text comprises the major theorems of probability theory and the measure theoretical foundations of the subject. The material of Chapters 1—6 may be considered as an introductory course in probability theory: 1. Measure theoretical tools and basic notions of probability theory, 2. Laws of large numbers, 3. Empirical distributions, 4. The central limit theorem, 5. Conditional expectations and distributions, 6. Martingales. The material of Chapters 7—10 may form the basis of an advanced course: 7. Stochastic processes, 8. Random elements in metric spaces, 9. Central limit theorems for martingale difference schemes, 10. Invariance principles.

There are exercises and remarks at the end of each chapter. The book is supplemented with a bibliography consisting of 154 items, a list of symbols and conventions, an author and subject index.

The textbook is written in a concise but always clear and well-readable way. We warmly recommend it to both students and lecturers at universities and technical colleges.

F. Móricz (Szeged)

I. I. Gihman—A. V. Skorohod, The theory of stochastic processes I, II, III (Grundlehren der mathematischen Wissenschaften 210, 218, 232), VIII+570, VII+441, VII+387 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1974, 1975, 1979.

Very few scientists show up in our age of specialization who would try to make an effort to penetrate in almost every important part of the whole branch of a mathematical field. This is what Professors Gihman and Skorohod do with the theory of probability and stochastic processes. That this is indeed so is recognized if the three-volume treatise under review is looked at as a part of a larger series of books. This series consists of three more books by the same authors, another joint book by Professor Skorohod and N. P. Slobodenyuk, and five books by Skorohod alone.

In these three volumes the authors endeavoured to present an exposition of the basic results, methods and applications of the theory of random processes. The various branches of the theory cannot, however, be treated in equal detail. A knowledge of basic probability and measure theory, as well as real and complex variable function theory and functional analysis (especially Hilbert space theory) is required from the reader. Therefore, these volumes are intended for professional mathematicians and graduate students rather than for undergraduates. A substantial number of the results are appearing in non-periodical literature for the first time, and there are results which have not been published even in periodicals. A great number of proofs of known results are also new. Since the authors are able to review the material in a long perspective, there is no doubt that this three-volume monograph will be one of the main references and sources of inspiration for research for a long time to come.

In what follows we can only try to indicate the contents by giving some key words.

Volume I. Chapter I (Basic notions of probability): axioms, independence, conditional expectation, random functions and mappings, Kolmogorov's fundamental theorem. *Chapter II* (Random sequences): martingales, semi-martingales, Markov chains, lattice random walk with vector jumps, stationary sequences, Birkhoff—Hinchin theorem. *Chapter III* (Random functions): Gaussian, Markov, independent increment processes, Doob's theorem on separable and measurable equivalents, criteria for the absence of second kind discontinuities, Kolmogorov's criterion for continuity. *Chapter IV* (Linear theory of random processes): second order random functions in a linear space, spectral decomposition of correlation functions of processes and fields, L^2-continuity, -differentiability, -integrability, and -decomposability into orthogonal series. Stochastic measures and integrals, integral and spectral representations of second-order processes and fields. Linear transformations, admissible and physically realizable filters, filtering of stationary processes with minimal mean square error, forecasting. *Chapter V* (Probability measures on fuction spaces): Conditions for realizability of measures on function spaces endowed with metric or vector structure, positive definite functionals and measures on a Hilbert space X, characteristic, linear and quadratic functionals and Gaussian measures on X. *Chapter VI* (Limit theorems for random processes): weak compactness and convergence of probability measures in metric and Hilbert spaces, limit theorems for sums of independent variables in a Hilbert space, convergence of continuous processes and processes with no second kind discontinuities. *Chapter VII* (Absolute continuity of measures associated with random processes): densities of measures, admissible shifts of measures on a Hilbert space, absolute continuity under mappings, applications for Gaussian and Markov processes. *Chapter VIII.* (Measurable functions on Hilbert spaces): conditions for continuous approximation (in measure) of linear functionals, operators and mappings, orthogonal polynomials for Gaussian measures.

Volume II. Chapter I (Basic definitions and properties of Markov processes), *Chapter II* (Homogeneous Markov processes): semigroup theory, strong Markov property, local behaviour of sample paths, Feller processes, processes in locally compact spaces, cut-off and non-cut-off processes, multiplicative and additive functionals, excessive functions. *Chapter III* (Jump processes): structure of sample paths, homogeneous Markov processes with a countable set of states, semi-Markov jump processes, Markov processes with a discrete component. *Chapter IV* (Processes with independent increments): decomposition into discrete and stochastically continuous processes, conditions for the latter to be Poisson, Lévy representation for the characteristic function of the increments, distribution of functionals concerning fluctuations (supremum, arrival time, size of jumps), local behaviour, growth at infinity, vector-valued jump processes. *Chapter V* (Branching processes): branching Markov processes with a finite number of particles, infinitesimal characteristics of branching processes with a continuum of states, general Markov processes with branching.

Volume III. Chapter I (Martingales and stochastic integrals): quasi-martingales, stopping and random time substitution, decomposition of supermartingales, (local) square integrable martingales, continuous characteristics. Stochastic integrals over locally square-integrable martingales and martingale measures. Itô's formula, stochastic differentials, bounds on moments, representation of martingales by integrals over a Wiener measure, decomposition of locally square integrable martingales. *Chapter II* (Stochastic differential equations): the stochastic line integral, existence and uniqueness, finite-difference approximations, solutions of stochastic differential equations without an after-effect as a Markov process, differentiability with respect to initial data of solutions, limit theorems of stochastic differential equations. *Chapter III* (Stochastic differential equations for continuous processes and continuous Markov processes in R^m): Itô processes, processes of diffusion type, existence and uniqueness, diffusion processes in R^m, homogeneous processes with integrable kernel of a potential, local structure of continuous homogeneous Markov processes in R^m, M-functionals; the rank of a process, continuous processes in R^1.

Apart from the bibliography, each volume ends with a section of historical and bibliographical remarks and a (not too rich) subject index. Also, an Appendix is included in Vol. III, correcting some errors in the first two volumes. All three volumes were translated by Samuel Kotz who has done a superb job.

Sándor Csörgő (Szeged)

S. A. Greibach, Theory of program structures: Schemes, Semantics, Verification (Lecture Notes in Computer Science, 36), 364 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1975.

Investigations concerning semantics play a fundamental rôle in computer science. This book contains the material of a first course on schematology, dealing with one approach to formalizing the elusive notion of the "semantics of programming languages". It is a nice introduction intending to make the reader familiar with the theory of program schemes and related topics.

In accordance with the introductory feature of the book, numerous examples are included to illustrate each new construction and many of the proofs, while in some cases the formal proofs are given in outline only. The book concludes with a large number of exercises. All these greatly help the reader to understand the main ideas.

As familiarity with formal languages and finite state machines makes the understanding of some chapters easier, we recommend this book first of all to students with this background.

G. Maróti (Szeged)

Maurice Holt, Numerical Methods in Fluid Dynamics (Springer Series in Computational Physics), VIII+253 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1977.

At the present time the majority of unsolved problems in fluid dynamics are governed by non-linear partial differential equations and can only be treated by a numerical approach. The development of large-scale computers have formed a basis for algoritmic constructions and extensive mathematical experiments in this area, too, as a result of which a lot of principal advances have been recently made in numerical methods.

The first part of this monograph describes two recent finite difference methods, both developed in the USSR. The first is due to Godunov (Ch. 2) originally presented in 1960 and revised in 1970. The second method was developed principally by Rusanov in 1964 in collaboration with Babenko, Voskresenskiĭ and Liubimov, and is familiarly known as the BVLR method (Ch. 3). Both the Godunov and BVLR methods have their origins in the method of characteristics (in two dimensions). Ch. 4 contains the method of characteristics for three-dimensional problems.

The second part treats the methods of integral relations (Ch. 5) introduced by Dorodnitsyn in 1950 and extended in 1960, the method of lines and Telenin's method (Ch. 6) developed from 1964 onwards. The objective of all these methods is to eliminate finite difference calculations in one or more coordinate directions by using interpolation formulae, especially polynomials or trigonometric functions, to represent the unknowns in selected directions.

The presentation is made for graduate students in engineering or applied mathematics with basic knowledge of fluid mechanics, partial differential equations and numerical analysis. Many applications and samples of numerical solutions of model problems are presented.

The book is warmly recommended to everyone practicing numerical analysis in industry or teaching at universities and technical colleges. It will certainly stimulate some of the readers to look for further effective numerical methods to attack the rather difficult problems of fluid dynamics.

F. Móricz (Szeged)

E. H. Lockwood and R. H. Macmillan, Geometric Symmetry, X+228 pages, Cambridge University Press, Cambridge—London—New York—Melbourne, 1978.

This large-scale summarizing work retrieves a long-time missing unified basic collection of discrete symmetry groups and present them not only for the specialists of this discipline, but also for artists and for the interested general public.

The book is divided into a "Descriptive" part and a part on "The mathematical structure". (Both parts discuss discrete symmetries of spaces of dimension not higher than 3 and this splitting of themes is no benefit for the user who wishes to find all information about say, the frieze-groups or the plane-groups in the same place.)

The book consists of the following chapters: Part I. 1. Reflexions and rotations, 2. Finite patterns in the plane, 3. Frieze patterns, 4. Wallpaper patterns, 5. Finite objects in three dimensions, 6. Rod patterns, 7. Layer patterns, 8. Space patterns, 9. Patterns allowing continuous movement, 10. Dilation symmetry, 11. Colour symmetry, 12. Classifying and identifying plane patterns, 13. Making patterns; — Part II. 14. Movements in the plane, 15. Symmetry groups. Point groups, 16. Line groups in two dimensions, 17. Nets, 18. Plane groups in two dimensions, 19. Movements in three dimensions, 20. Point groups in three dimensions, 21. Line groups in three dimensions, 22. Plane groups in three dimensions, 23. Lattices, 24. Space groups I, 25. Space groups II, 26. Limiting groups, 27. Colour symmetry.

It must be noted that the references are incomplete. For instance there is no reference to the works of Coxeter or Fejes Tóth in discrete geometry. Perhaps this is the cause of some mistake in the historical introduction: it were not Pólya or Niggli who first enumerated the 17 wallpaper groups in 1924, but Fedorov in 1891 and later, independently of him, Fricke and Klein in 1897.

A particular virtue of the book is the Notation and Axes supplementary chapter which symbolizes the beneficent endeavour of the authors to unify the notation system of the groups studied. Perhaps because the main user of these symmetry groups is crystallography, the authors' effors aim to generalize the crystallographic notation, although that is not ideal owing to its redundancy. As far as we know this book is a pionerring work not only in summarizing geometric symmetry but in the unification of its notation, too.

Its clear structure, neat way of exposition and abundant illustrations in color make this excellent book an attractive reading, a valuable and useful help for teachers on all levels (in secondary or high schools, or at universities), and even for artists, textile designers, architects, etc.

Dénes Nagy and *Szaniszló Bérczi* (Budapest)

László Lovász, Combinatorial Problems and Exercises, 551 pages, Akadémiai Kiadó and North Holland Publishing Company, Budapest, 1979.

Though the roots of combinatorics go back to the 18^{th} and 19^{th} centuries, it has become a coherent discipline in the last twenty years only. Mostly isolated theorems were known beside the earlier developed enumeration techniques. The recent extremely rapid development of combinatorics was influenced by the occurrence of combinatorial problems connected with computer science, operation research, statistics, coding theory etc. The enormous quantitative increase has been accompanied by the appearance of several new methods, techniques and theories. This development is manifested also by the increase in the number of books from no more than a dozen in the middle of this century to several hundreds today. Lovász' book is a masterpiece among them.

Inspite of the modest title it is not just a collection of problems but it builds up more than a dozen "theories" and techniques in combinatorics, some of them presented here for the first time as a coherent topic.

It is a three-level version of the classical book of Pólya—Szegő: *Aufgaben und Lehrsätze aus der Analysis,* containing parts as Problems, Hints and Solutions. These cover classical theorems and the latest results as well. A large part of the text has appeared previously in research papers only. In many cases the proofs are much simpler than the original ones.

The first four chapters are devoted to enumeration; generating-function techniques (the first developed techniques in Combinatorics), the famous Pólya method (used for some classical problems on partitions), sieve methods, a large part of the latter in probabilistic setting such as M. Hall's and Rényi's method for coding permutations, enumeration of trees and one-factors.

§ 5 is on duality and parity. Here the nature of the solutions unifies the material more than the problems themselves. § 6—§ 7 deal with connectivity, Menger—König—Hall—Tutte—Edmonds type factorization theorems, the 'max-flow — min cut' theorem, a subject which is strongly connected with linear programming.

Chromatic number is a concept whose origin goes back to the last century (Four Colour Conjecture). It is now of completely independent interest; e.g. chromatic polynomials and the problem of characterization of critical graphs concerning the chromatic number are considered in § 8.

§ 9 deals with independent sets, characterization of critical graphs concerning maximal independent sets and their applications; e.g. game-theory.

§ 10 contains extremal problems characterized by Turán's theorem, and several problems on Hamiltonian lines.

§ 11 and § 12 deal with algebraic graph theory, spectra of graphs and automorphims of graphs.

§ 13 contains hypergraph theory, including intersection theorems like the Sperner and the Erdős—Ko—Rado theorems, fractional and integer matching and covering, and it ends with Lovász' perfect graph theorem. Various proof techniques are demonstrated, some of them developed, partly or fully, by the author himself.

§ 14 contains the Ramsey theory. This includes Ramsey-type theorems for systems of finite sets and also other structures (as integers, vector spaces, arithmetic progressions). The last chapter is devoted to reconstruction problems.

Throughout the book there is a strong emphasis on "good characterization", on algorithmic aspects, on the connection of combinatorics with integral linear programming, on the use of linear algebra, and on probabilistic setting.

The exposition is extremely clear and elegant. The author seems always to find the simplest way to prove the deepest theorems.

The book is highly recommended not only to young researchers but also to the specialists in combinatorics and the mathematical public in general.

It will undoubtedly not only "help in learning existing techniques in combinatorics" but will also stimulate new ideas.

"Some fields have had to be completely omitted: random structures, integer programming, matroids, block designs, lattice geometry, etc. I hope eventually to write a sequel to this volume covering some of these latter topics." Having an outstanding book like this we are looking forward to the next volume.

Vera T. Sós (Budapest)

J. D. Monk, Mathematical Logic (Graduate Texts in Mathematics, 37), 531 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1976.

This book is based on the author's lectures given at various universities.

After a survey of recursive function theory and the elements of logic, the reader is made familiar with the concept of first order languages and the basic facts concerning them. This part of the book serves as a preparation for the following chapters, dealing with decidable and undecidable theories and other topics in model theory. The book concludes with touching upon several other kinds of logics, e.g., many-sorted logic, second-order logics, etc.

At the end of each chapter the reader finds references and a rich variety of interesting exercises.

We recommend this excellent work to everyone interested in, or dealing with, mathematical logic. First or second year graduate students can study sentential logic and its relationship to Boolean algebras by reading chapters 8 and 9 only. Because of the very abstract nature of the subject we suggest reading the whole book first of all to postgraduate students, as well as young logicians, who thereby can be helped efficiently in preparing the material of their lectures on mathematical logic

G. Maróti (Szeged)

H. Rademacher, Lectures on Elementary Number Theory, IX+146 pages, Robert E. Krieger Publishing Co., Huntington, New York, 1977.

Number theory is full of problems and results that most mathematicians know, but the general feeling about their proof is that it is very difficult and technical. We all know about the prime number theorem, quadratic reciprocity, Dirichlet's theorem on primes in arithmetic progressions, Brun's theorem on twin primes, just to mention some of the most classical examples. But very few of us have had the possibility of learning the proofs of these facts, although these proofs are not as inaccessible as believed. This, of course, is a pity, because a major contribution of number theory to mathematics is in the powerful methods which emerge from the solutions of its simple yet very difficult, challenging problems. Some of the "elementary" proofs in number theory may contain the kernel of other more general mathematical theories. This is why it is great to be able to read accounts of some of these classical difficult problems in number theory in a form accessible to non-specialists, in particular students.

This classic book, whose second printing is reviewed here, discusses some of these well-known but not generally well-understood problems in "elementary" number theory ("elementary" only means that no complex function theory is used: real calculus is used and the field is full with complicated, ingenious arguments). It is not a textbook but it does start with the basics: unique factorization, Farey fractions, linear Diophantine equations, congruences. It gets to quadratic reciprocity through an interesting detour to constructing regular heptadecagons, Lagrange resolvents and Gaussian sums. After discussing lattice point techniques and some results on prime distribution like Chebyshev's theorem, the book gives the proof of Dirichlet's theorem on primes in arithmetical progressions, and of Brun's theorem on the convergence of the sum of reciprocals of twin primes.

This book is indeed recommended to everyone, in particular to students: its material belongs to what may be regarded as "basic mathematical intelligence", its presentation is easy to follow and yet it leads the reader to the deepest "elementary" results in number theory.

L. Lovász (Szeged)

R. D. Richtmyer, Principles of advanced mathematical physics. I (Texts and Monographs in Physics), XV+422 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1978. — DM 44,—.

As the author points out in the preface, nowadays physics cannot apply intuitive methods as earlier, it needs a high level adequate mathematics of a wide range. However, branches of mathematics are used from a specific physical point of view, i.e., some of the mathematical theories are irrevelant to physics while some results marginal to the mathematical theories have great importance in physics. The aim of the book is to collect mathematics from this special physical point of view. The title is somewhat misleading because the book does not concern any principles; it concentrates on Hilbert and Banach spaces and distributions, linear operators and their spectra, with special attention to operators that emerge from differential equations in physics.

There is a great demand for such books which can serve as basic ones for students. That is why it is a pity that measure theory is not treated thoroughly, hence probability theory cannot be set forth in its natural way and the spectral theories of self-adjoint and unitary operators are formulated by spectral families (resolutions of the identity) instead of projection valued measures, involving thus more complicated tools.

The treated material is essential for general understanding of physics (except perhaps the last chapter: non-linear problems; fluid dynamics); the presentation of the subject is clear and suitable for the purpose of the author. The book will certainly prove very useful for students in physics.

T. Matolcsi (Budapest)

A. N. Shiryayev, Optimal Stopping Rules (Applications of Mathematics, Vol. 8), X+217 pages, New York—Heidelberg—Berlin, Springer-Verlag, 1978.

What is the secret of a successful life? Perhaps nothing but the ability to stop each activity at the right moment. The vital applicability of the theory of optimal stopping is consequently beyond doubt. But there was one more reason why to publish the present volume in the series "Applications of Mathematics". Namely the theory of optimal stopping itself serves as an interesting field of application for other deep mathematical disciplines.

The present book is a well-written, concise presentation of the beautiful round theory of optimal stopping of Markov processes. Although it is shown that all interesting non-Markovian stopping problems can be reduced to equivalent Markovian ones, the decision of the author to restrict himself to Markov processes was a step off the applications in favour of the methodological closedness of the theory. The emphasis of the volume lies on the demonstration how potential- and martingale-theoretical results can be applied to solve the mathematical problem of optimal stopping. Possible applications of optimal stopping theory are only outlined. But this incompleteness from the side of applications does not lessen the value of the book. On the contrary, it has the effect of forcing the reader to think it over and fill up the gaps by himself. This way the passive reader is converted into an active partner in research. Besides conciseness and theoretical clarity, this is the very property which makes the book extremely fitting to serve as a basis for a half-year course for advanced students in probability. The presented material can also be regarded as a first non-trivial introduction to the theory of filtration and control of stochastic processes.

The Russian original was substantially improved and enlarged before translation. The chapter-headings are 1. Random Processes: Markov Times; 2. Optimal Stopping of Markov Sequences; 3. Optimal Stopping of Markov Processes; 4. Some Applications to Problems of Mathematical Statistics. A detailed bibliography and an index close the volume.

D. Vermes (Szeged)

Dietrich Stoyan, Qualitative Eigenschaften und Abschätzungen stochastischer Modelle, X+198 pages, Akademie-Verlag, Berlin, 1977.

The theories of queues, inventories, dams, risks and reliability belong to the oldest spheres of applied probability, and even non-specialists know that they are merely different interpretations of the same mathematical discipline. It is not the lack of a common language that gives rise to the very non-homogeneous outlook of these theories, but rather the dissimilarity of the applied techniques.

The situation very much resembles to the early decades of the theory of differential equations, when the explicit form of the exact solutions was of primary interest. At that time the necessary approaches varied from equation to equation. Only Liapunov's direct method and the monotonicity methods (differential inequalities and fixed-point theorems) have opened the fundamentally new prospects of the uniform, so called qualitative, theory of differential equations.

The stochastic theory of queues, inventories, etc. now stands at the beginning of a similar vigorous development. The aim of the present booklet is to awake interest in this new field. Most of the book deals with monotonicity methods, based mainly on the author's own results. Although these techniques are far not as powerful at the present stage as their deterministic analogues (they are used only for obtaining some estimates), they appear to be a first step towards a uniform qualitative theory of queues, reliability etc. The last chapter of the book gives a short glimpse into the modern but already well-developed stability theory of stochastic models.

The purpose of the author is to give a first introduction and therefore the more laborious proofs are only sketched. The reader is supposed to have some pre-knowledge in the theory of queues, inventories, etc. The language of the book is clear, but, due to some long definitions and complicated formulations, it is not very easy-flowing. Some open problems, an extensive bibliography and an index supplement the volume.

D. Vermes (Szeged)

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