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C^*-algebras and derivation ranges

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Dedicated to P. R. Halmos

1. Introduction and Summary

Let T be an element of the algebra $\mathcal{B}(\mathfrak{H})$ of bounded linear operators on a complex Hilbert space $\mathfrak{H}$ and let $\delta_T(X)=TX-XT$ be the corresponding inner derivation. There are two natural closed subalgebras of $\mathcal{B}(\mathfrak{H})$ associated with T, namely the *inclusion algebra* $\mathcal{I}(T)$ of operators A for which the range $\mathcal{R}(\delta_A)$ of δ_A is contained in the norm closure $\mathcal{R}(\delta_T)^-$, and the *multiplier algebra* $\mathcal{M}(T)=$ $=\{Z\in\mathcal{B}(\mathfrak{H}):Z\mathcal{R}(\delta_T)+\mathcal{R}(\delta_T)Z\subseteqq\mathcal{R}(\delta_T)^-\}$. Most of the recent results [1, 3, 16, 19, 20, 21, 22] about the range of a derivation can be interpreted as assertions about these algebras or the two algebras that are defined similarly by replacing $\mathcal{R}(\delta_T)^-$ by $\mathcal{R}(\delta_T)$. In the finite dimensional case, $\mathcal{M}(T)=\{T\}'$ and $\mathcal{I}(T)=\{T\}''$ are the commutant and bicommutant of T.

In this paper we study the situation in which either (and, therefore, both) of these is a C^*-subalgebra of $\mathcal{B}(\mathfrak{H})$. The corresponding operators T, those for which $\mathcal{R}(\delta_T)^-=\mathcal{R}(\delta_{T^*})^-$ is a self-adjoint subspace of $\mathcal{B}(\mathfrak{H})$, are called *d-symmetric operators*. Any isometry or normal operator is d-symmetric and so is the image of a d-symmetric operator under an irreducible representation of the C^*-algebra $C^*(T)$ generated by T and the identity operator. However, if N is normal then $\mathcal{R}(\delta_N)$ is itself self-adjoint only if the spectrum of N has empty interior [11, Theorem 4.1].

If T is d-symmetric then $\mathcal{R}(\delta_T)^-$ is determined by the *T-central states* on $\mathcal{B}(\mathfrak{H})$, that is, linear functionals f with $f(I)=1=\|f\|$ and $f(TX)=f(XT)$ for all

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$X \in \mathscr{B}(\mathfrak{H})$. In fact, $\mathscr{R}(\delta_T)^-$ is even determined by those pure states of $\mathscr{B}(\mathfrak{H})$ whose restrictions to $\mathscr{I}(T)$ are multiplicative. These satisfy $f(AX)=f(XA)=f(A)f(X)$ for $X \in \mathscr{B}(\mathfrak{H})$ and $A \in \mathscr{I}(T)$ so that in particular $C^*(T)$ must have a character.

The C^*-algebra $C(T)$ of operators C for which $C\mathscr{B}(\mathfrak{H})+\mathscr{B}(\mathfrak{H})C$ is contained in $\mathscr{R}(\delta_T)^-$ plays a fundamental role here. For example, T is d-symmetric if and only if $T^*T-TT^* \in \mathscr{C}(T)$ and a d-symmetric operator has the Fuglede property: $TX-XT \in \mathscr{C}(T)$ for some operator X on $\mathfrak{H}$ only if $T^*X-XT^* \in \mathscr{C}(T)$. For a d-symmetric operator $\mathscr{C}(T)$ coincides with the commutator ideal of $\mathscr{I}(T)$. It is non-separable in general.

The inclusion algebra $\mathscr{I}(T)$ of a d-symmetric operator T is identified in the two extreme cases in which T has no reducing eigenvalues (complex numbers λ for which $\ker(T-\lambda I)$ reduces T) and in which T has a spanning set of orthonormal eigenvectors (T is a *diagonal operator*). In the first case $\mathscr{I}(T)=C^*(T)+\mathscr{C}(T)$, while in the second $\mathscr{I}(T)$ is the C^*-algebra generated by T and those projections onto eigenspaces corresponding to eigenvalues of finite multiplicity that are limit points of the spectrum of T.

Various criteria for d-symmetry are given in § 2 and the ideal $\mathscr{C}(T)$ is studied in § 3. We study the T-central states in § 4, present examples, counterexamples and information about special cases in § 5, and mention several questions we have been unable to resolve in the final section of the paper.

2. Conditions for d-symmetry

The proof of our first result was inspired by ROSENBLUM's proof [14] of the Fuglede theorem.

Theorem 2.1. *For T in $\mathscr{B}(\mathfrak{H})$ the following are equivalent:*

(a) *T is d-symmetric,* (b) $T^*T-TT^* \in \mathscr{C}(T)$,

(c) $T^*\mathscr{R}(\delta_T)+\mathscr{R}(\delta_T)T^* \subseteqq \mathscr{R}(\delta_T)^-$.

Proof. The equivalence of (b) and (c) is a consequence of the identities:

$$(T^*T-TT^*)X = T^*\delta_T(X)-\delta_T(T^*X), \quad X(T^*T-TT^*) = \delta_T(X)T^*-\delta_T(XT^*).$$

Since $T^*\delta_{T^*}(X)=\delta_{T^*}(T^*X)$ and $\delta_{T^*}(X)T^*=\delta_{T^*}(XT^*)$, (a) implies (c).

Now assume (c) holds. To prove that T is d-symmetric it suffices to show that $f(\mathscr{R}(\delta_{T^*}))=0$ for all f in $\mathscr{B}(\mathfrak{H})^*$ satisfying $f(\mathscr{R}(\delta_T))=0$. If $X \in \mathscr{B}(\mathfrak{H})$ then $f(TX)=f(XT)$ and

$$f(T^{*n}TX) = f(T^{*n}(TX-XT))+f(T^{*n}XT) = 0+f(T^{*n}XT) = f(TT^{*n}X)$$

since $T^*\mathscr{R}(\delta_T) \subseteqq \mathscr{R}(\delta_T)^-$ by (c). By induction $f(T^{*n}T^mX)=f(T^mT^{*n}X)$ for all non-negative integers n and m. From this one obtains

$$f(\exp(\alpha T+\beta T^*)X)=f(\exp(\alpha T)\exp(\beta T^*)X)=f(\exp(\beta T^*)\exp(\alpha T)X)$$

for all complex numbers α and β by imitating the standard proof of the identity $\exp(A+B)=\exp(A)\exp(B)$ (for commuting A and B) as given in [12, p. 397] for example. A similar argument, using $\mathscr{R}(\delta_T)T^*\subseteq\mathscr{R}(\delta_T)^-$, gives

$$f(X\exp(\alpha T+\beta T^*))=f(X\exp(\alpha T)\exp(\beta T^*)=f(X\exp(\beta T^*)\exp(\alpha T)).$$

Since $f(TX)=f(XT)$, it follows by induction that $f(T^nX)=f(XT^n)$ for all n and hence $f(\exp(\alpha T)X)=f(X\exp(\alpha T))$ or $f(\exp(\alpha T)X\exp(-\alpha T))=f(X)$. These relations yield:

$$f(\exp(i\lambda T^*)X\exp(-i\lambda T))=f(\exp(i\bar{\lambda}T)\exp(i\lambda T^*)X\exp(-i\lambda T^*)\exp(-i\bar{\lambda}T))=$$
$$=f(\exp(i2\operatorname{Re}(\bar{\lambda}T))X\exp(-i2\operatorname{Re}(\bar{\lambda}T)))$$

for any complex λ. The right hand side of this equation is bounded, so by Liouville's theorem the entire function on the left hand side must be constant. In particular, the derivative vanishes at $\lambda=0$. This gives $f(T^*X-XT^*)=0$. □

Corollary 2.2. *Every normal operator is d-symmetric.*

Theorem 2.3. *Every isometry V is d-symmetric.*

Proof. If $Q=I-VV^*$ then $\delta_{V^*}(X)=\delta_V(-V^*XV^*)-QXV^*$ so it suffices to show that $QX\in\mathscr{R}(\delta_V)^-$ for all $X\in\mathscr{B}(\mathfrak{H})$. Let $T_n=\sum_{k=0}^{n-1}(k/n-1)V^kQXV^{*(k+1)}$ for $n=2,3,\ldots$. Then $\delta_V(T_n)-QX=-n^{-1}\sum_{k=1}^{n}V^kQXV^{*k}$. Since $(V^jQx, V^kQy)=0$ for $j\neq k$ and x,y in $\mathfrak{H}$,

$$\left\|\sum_{k=1}^{n}V^kQXV^{*k}x\right\|^2=\sum_{k=1}^{n}\|V^kQXV^{*k}x\|^2\leqq n\|QX\|^2\|x\|^2.$$

Hence $n^{-1}\|\sum_{k=1}^{n}V^kQXV^{*k}\|\leqq n^{-1/2}\|QX\|$ and $QX\in\mathscr{R}(\delta_V)^-$.

Remark. The proof of 2.3 shows that $Q\mathscr{B}(\mathfrak{H})\subseteq\mathscr{R}(\delta_V)^-$. The closure cannot be deleted here, however, as $\mathscr{R}(\delta_V)$ contains no non-zero right ideal of $\mathscr{B}(\mathfrak{H})$ [21]. But $\mathscr{R}(\delta_V)$ *does* contain the left ideal of $\mathscr{B}(\mathfrak{H})$ generated by Q [18].

Let $\mathscr{K}=\mathscr{K}(\mathfrak{H})$ denote the compact operators on $\mathfrak{H}$. An operator T is *essentially d-symmetric* if it is d-symmetric in the Calkin algebra $\mathscr{B}(\mathfrak{H})/\mathscr{K}$, that is, if $[\nu(T),\nu(\mathscr{B}(\mathfrak{H}))]^-$ is a self-adjoint subspace of the Calkin algebra. (Here ν denotes the canonical homomorphism of $\mathscr{B}(\mathfrak{H})$ onto $\mathscr{B}(\mathfrak{H})/\mathscr{K}$.) We now determine the relationship between d-symmetric and essentially d-symmetric operators.

A closed subspace of $\mathscr{B}(\mathfrak{H})$ is self-adjoint if and only if its annihilator $\mathscr{F}$ is self-adjoint in the sense that $f\in\mathscr{F}$ implies $f^*\in\mathscr{F}$, where $f^*(X)=f(X^*)^*$. Now each

$f \in \mathscr{B}(\mathfrak{H})^*$ has a unique representation $f = f_0 + f_J$ where f_0 is a bounded linear functional on $\mathscr{B}(\mathfrak{H})$ that vanishes on $\mathscr{K}$ and f_J is induced by an operator J in the trace class by the formula $f_J(X) = \text{trace}(XJ)$ for X in $\mathscr{B}(\mathfrak{H})$. (See [9, 2.11.7 and 4.1.2].) Moreover, $f = f_0 + f_J$ is T-central for an operator T if and only if both f_0 and f_J are T-central, and f_J is T-central if and only if $TJ = JT$ [20, Theorem 3]. These facts give

Proposition 2.4. *An operator T on $\mathfrak{H}$ is d-symmetric if and only if*

(a) *T is essentially d-symmetric, and*

(b) *$TJ = JT$ for an operator J in the trace class implies $TJ^* = J^*T$.*

Corollary 2.5. (a) *An essentially normal operator T is d-symmetric if and only if $TJ = JT$ for an operator J in the trace class implies $TJ^* = J^*T$.*

(b) *An operator in the trace class is d-symmetric if and only if it is normal.*

Proof. Since the proof of Theorem 2.1 is valid in any C^*-algebra, any essentially normal operator is essentially d-symmetric. □

Corollary 2.6. *The following are equivalent for a d-symmetric operator T:*

(a) $\mathscr{K} \subseteqq \delta_T(\mathscr{K})^-$.

(b) $\mathscr{K} \subseteqq \mathscr{R}(\delta_T)^-$.

(c) *T has no reducing eigenvalues.*

Proof. If T has a reducing eigenvalue, then $(Sx, x) = 0$ for all S in $\mathscr{R}(\delta_T)^-$ and some non-zero $x \in H$ and $\mathscr{K}$ non $\subseteqq \mathscr{R}(\delta_T)^-$. Thus, (b) implies (c). If $\delta_T(\mathscr{K})$ is not dense in $\mathscr{K}$, then since $\mathscr{K}^*$ is the trace class operators, there is a non-zero J in the trace class such that f_J vanishes on $\delta_T(\mathscr{K})$, that is, $TJ = JT$. Since T is d-symmetric $TJ^* = J^*T$ by 2.4 and T commutes with a non-zero self-adjoint trace class operator. Therefore T has a finite dimensional reducing subspace $\mathscr{M}$. Clearly, any direct summand of a d-symmetric operator is d-symmetric so $T|\mathscr{M}$ is normal by 2.5(b). Hence T has a reducing eigenvalue and (c) implies (a). The remaining implication is obvious. □

Remarks. (a) If S and T are d-symmetric operators with disjoint spectra, then an easy application of Rosenblum's theorem [13] shows that $S \oplus T$ is d-symmetric.

(b) If λ is an eigenvalue of T but $\bar{\lambda}$ is not an eigenvalue of T^*, then $T \oplus \lambda I$ is not d-symmetric, where I is the identity on any non-zero Hilbert space. In particular, if U_+ denotes the unilateral shift and $|\lambda| < 1$, $U_+ \oplus \lambda I$ is not d-symmetric. However, if $|\lambda| \geqq 1$ then 2.4(b) and a calculation show that $U_+ \oplus \lambda I$ is d-symmetric.

(c) STAMPFLI [16] constructed a compact weighted shift K that commutes with no non-zero trace class operator and therefore $\mathscr{R}(\delta_K)^- = \mathscr{K}$. This operator K is then d-symmetric and quasinilpotent. As $K \oplus n^{-1} I$ is d-symmetric by (a) above and $K \oplus 0$ is not d-symmetric by (b), it follows that *the set of d-symmetric operators is not norm closed.* Stampfli has independently pointed out this same fact to us.

The proof of our next theorem requires non-separable versions of two known results. We now present these (slight) generalizations. Let $\mathscr{A}$ denote a unital separable C^*-algebra of operators in $\mathscr{B}(\mathfrak{H})$, where $\mathfrak{H}$ is *separable*. In [17] Voiculescu showed that if π is a representation of $\mathscr{A}$ in $\mathscr{B}(\mathfrak{H}_\pi)$, where $\mathfrak{H}_\pi$ is separable and $\pi(\mathscr{A} \cap \mathscr{K}) = 0$, then there is a sequence of unitary transformations U_n of $\mathfrak{H} \oplus \mathfrak{H}_\pi$ onto $\mathfrak{H}$ such that $A - U_n(A \oplus \pi(A)) U_n^*$ is compact for all A in $\mathscr{A}$ and $\lim_n \|A - U_n(A \oplus \pi(A)) U_n^*\| = 0$ for all A in $\mathscr{A}$. (In symbols, $\mathrm{id} \underset{a}{\sim} \mathrm{id} \oplus \pi$). This fact was used in [2] to show that if f is a state on $\mathscr{A}$ that is zero on $\mathscr{A} \cap \mathscr{K}$ then f extends to a pure state on $\mathscr{B}(\mathfrak{H})$.

Proposition 2.7. *Let $\mathscr{A}$ denote a unital separable C^*-algebra of operators acting on a Hilbert space $\mathfrak{H}$ (of any dimension).*

(a) *If π is a representation of $\mathscr{A}$ in $\mathscr{B}(\mathfrak{H}_\pi)$, where $\mathfrak{H}_\pi$ is separable and $\pi(\mathscr{A} \cap \mathscr{K}) = 0$, then $\mathrm{id} \underset{a}{\sim} \mathrm{id} \oplus \pi$.*

(b) *If f is a state on $\mathscr{A}$ such that f vanishes on $\mathscr{A} \cap \mathscr{K}$, then f extends to a pure state on $\mathscr{B}(\mathfrak{H})$.*

Proof. Choose a dense sequence $\{A_n\}$ of operators in $\mathscr{A}$ and select unit vectors in $\mathfrak{H}$ as follows. For each n choose an infinite orthonormal sequence $\{e_{nk}\}$ such that $\|A_n\|_e = \lim_k \|A_n e_{nk}\|$ and choose a sequence $\{x_{nj}\}$ such that $\|A_n\| = \lim_j \|A_n x_{nj}\|$. (Here, $\|A_n\|_e$ denotes the norm of $A_n + \mathscr{K}$ in $\mathscr{B}(\mathfrak{H})/\mathscr{K}$.) Write $\mathfrak{M}$ for the subspace of $\mathfrak{H}$ generated by $\{\mathscr{A} e_{nk}\} \cup \{\mathscr{A} x_{nj}\}$. Then the restriction map Φ induced by the projection P of $\mathfrak{H}$ onto M is an isometric isomorphism. Furthermore, an operator A in $\mathscr{A}$ is compact if and only if $\Phi(A)$ is compact. Now suppose π is a representation of $\mathscr{A}$ as in (a) above. Then $\pi' = \pi \circ \Phi^{-1}$ is a representation of $\Phi(\mathscr{A})$ which satisfies the hypotheses of Voiculescu's theorem. (If $\Phi(A)$ is compact, then $\pi'(\Phi(A)) = \pi(A) = 0$ since A is compact.) Hence $\mathrm{id}_{\Phi(A)} \underset{a}{\sim} \mathrm{id}_{\Phi(A)} \oplus \pi'$. Let Ψ denote the restriction of $\mathscr{A}$ to $P^\perp \mathfrak{H}$. Then

$$\mathrm{id}_{\mathscr{A}} = \Psi \oplus \Phi \underset{a}{\sim} \Psi \oplus (\mathrm{id}_{\Phi(\mathscr{A})} \oplus \pi') = \mathrm{id}_{\mathscr{A}} \oplus \pi$$

and (a) is established.

Now suppose that f is a state on $\mathscr{A}$ that is zero on $\mathscr{A} \cap \mathscr{K}$. Then $f' = f \circ \Phi^{-1}$ is a state on $\Phi(\mathscr{A})$ that is zero on $\Phi(\mathscr{A}) \cap \mathscr{K}(\mathfrak{M})$ and so by [2] there is a pure state g' on $\mathscr{B}(\mathfrak{M})$ which extends f'. Define g'' on $\mathscr{B}(P^\perp \mathfrak{H}) \oplus \mathscr{B}(\mathfrak{M})$ by $g''(X \oplus Y) = g'(Y)$. Then g'' is a pure state on $\mathscr{B}(P^\perp \mathfrak{H}) \oplus \mathscr{B}(\mathfrak{M})$ and if $A \in \mathscr{A}$, $A = A_1 \oplus \Phi(A)$

and $g''(A)=g'(\Phi(A))=f'(\Phi(A))=f(A)$. Thus g'' is a pure state that extends f, so we may choose a pure state on $\mathscr{B}(\mathfrak{H})$ that extends g''. □

Recall that a representation $\pi:\mathscr{A}\to\mathscr{B}(\mathfrak{H}_\pi)$ of a C^*-algebra $\mathscr{A}$ into the operators on the Hilbert space $\mathfrak{H}_\pi$ is called *cyclic* if there is a vector x in $\mathfrak{H}_\pi$ such that $\pi(\mathscr{A})x$ is dense in $\mathfrak{H}_\pi$.

Theorem 2.8. *If T is a d-symmetric operator on a Hilbert space $\mathfrak{H}$ and $\pi:C^*(T)\to\mathscr{B}(\mathfrak{H}_\pi)$ is a cyclic representation such that either*
(a) $\pi(C^*(T)\cap\mathscr{K})=0$, *or* (b) $\pi(C^*(T))$ *is irreducible, then $\pi(T)$ is d-symmetric.*

Proof. Assume that $\pi(C^*(T)\cap\mathscr{K})=0$. Then by Proposition 2.7(a) there is a unitary transformation U mapping $\mathfrak{H}\oplus\mathfrak{H}_\pi$ onto $\mathfrak{H}$ such that $T-U(T\oplus\pi(T))U^*$ is compact. Since T is d-symmetric, $U(T\oplus\pi(T))U^*$ is essentially d-symmetric and so $T\oplus\pi(T)$ is essentially d-symmetric. Therefore $\pi(T)$ is essentially d-symmetric. Let f denote a $\pi(T)$-central bounded linear functional on $\mathscr{B}(\mathfrak{H}_\pi)$. We must show that f^* is $\pi(T)$-central. Write $f=f_0+f_w$ where f_0 vanishes on $\mathscr{K}(\mathfrak{H}_\pi)$ and f_w is ultraweakly continuous (that is, induced by a trace class operator.) Then f_0 and f_w are $\pi(T)$-central [20] and f_0^* is $\pi(T)$-central because $\pi(T)$ is essentially d-symmetric. We need only show that f_w^* is $\pi(T)$-central. Fix a cyclic vector x for $\pi(C^*(T))$ and define a state ω on $C^*(T)$ by $\omega(A)=(\pi(A)x, x)$. Since $C^*(T)$ is separable and ω vanishes on $\mathscr{A}\cap\mathscr{K}$, there is a pure state ϱ on $\mathscr{B}(\mathfrak{H})$ that extends ω by Proposition 2.7(b). It follows (as in [9, 2.10.2]) that there is a Hilbert space $\mathfrak{H}'$ containing $\mathfrak{H}_\pi$ and an irreducible representation π' of $\mathscr{B}(\mathfrak{H})$ in $\mathscr{B}(\mathfrak{H}')$ such that the projection P of $\mathfrak{H}'$ onto $\mathfrak{H}_\pi$ reduces $\pi'(C^*(T))$ and $P\pi'(A)|\mathfrak{H}_\pi=\pi(A)$ for all A in $C^*(T)$. Define a linear functional g on $\mathscr{B}(\mathfrak{H})$ by $g(X)=f_w(P\pi'(X)P|\mathfrak{H}_\pi)$.) Then

$$g(TX)=f_w(P\pi'(TX)P|\mathfrak{H}_\pi)=f_w(\pi(T)P\pi'(X)P|\mathfrak{H}_\pi)=$$
$$=f_w((P\pi'(X)P|\mathfrak{H}_\pi)\pi(T))=f_w(P\pi'(XT)P|\mathfrak{H}_\pi)=g(XT),$$

since f_w is $\pi(T)$-central. Thus g is T-central; so, since T is d-symmetric, g is T^*-central. Therefore, for all X in $\mathscr{B}(\mathfrak{H})$ we have

$$f_w((P\pi'(X)P|\mathfrak{H}_\pi)\pi(T)^*)=f_w(\pi(T)^*P\pi'(X)P|\mathfrak{H}_\pi).$$

Since π' is irreducible and f_w is ultraweakly continuous, f_w is $\pi(T)^*$-central.

Now suppose that π is irreducible. By the first part of the proof, we may assume that π is not zero on $C^*(T)\cap\mathscr{K}$. Then $\pi_0=\pi|C^*(T)\cap\mathscr{K}(\mathfrak{H})$ is irreducible [9, 2.11.3] and by [5, 1.4.4] there is a subspace $\mathfrak{M}$ of $\mathfrak{H}$ such that π_0 is unitarily equivalent to the restriction to $\mathfrak{M}$ of the identity representation of $C^*(T)\cap\mathscr{K}$. Since $C^*(T)\cap\mathscr{K}$ is irreducible on $\mathfrak{M}$, $\mathfrak{M}$ must reduce T. A similar argument shows that $\pi(C^*(T))$ is unitarily equivalent to $C^*(T)|\mathfrak{M}$. Thus $\pi(T)$ is unitarily equivalent to a direct summand of T and $\pi(T)$ is d-symmetric. □

Remarks. (a) The operator $T\oplus\pi(T)$ in the proof of Theorem 2.8 need not be d-symmetric. Indeed, let K denote the compact d-symmetric operator in Remark (c) following 2.6 and define π on $C^*(K)=\mathcal{K}(\mathfrak{H})+\mathbf{C}I$ by $\pi(K_1+\lambda I)=\lambda$. Then $K\oplus\pi(K)=K\oplus 0$ is not a d-symmetric operator.

(b) If T is d-symmetric and π is an irreducible representation of $C^*(T)$ then $\mathfrak{H}_\pi$ is either infinite dimensional or one dimensional. For $\pi(T)$ is d-symmetric by 2.8 and if $\mathfrak{H}_\pi$ has dimension $n<\infty$, then $\pi(T)$ is normal (2.5(b)), and irreducible, hence $n=1$. Thus, *if T is essentially n-normal and d-symmetric, then T is essentially normal.*

(c) We shall show (3.6) that if T is d-symmetric, then $C^*(T)$ has a character. Hence $C^*(\pi(T))$ has a character for every irreducible representation π of $C^*(T)$.

3. The inclusion and multiplier algebras

As noted prior to Proposition 2.4 d-symmetry of an operator is equivalent to the condition that the annihilator of its derivation range be a self-adjoint subspace of $\mathcal{B}(\mathfrak{H})^*$. We now show that the annihilator is actually determined by the states it contains.

Let $E(T)$ denote the set of all T-central states on $\mathcal{B}(\mathfrak{H})$; that is, the set of states f on $\mathcal{B}(\mathfrak{H})$ such that $f(TX)=f(XT)$ for all X in $\mathcal{B}(\mathfrak{H})$.

Theorem 3.1. *If T is a d-symmetric operator, then $\mathcal{R}(\delta_T)^-=\bigcap\{\ker(f):f\in E(T)\}$.*

Proof. Fix $f=f^*$ in the annihilator of $\mathcal{R}(\delta_T)$. Then there are unique positive linear functionals f^+ and f^- on $\mathcal{B}(\mathfrak{H})$ such that $f=f^+-f^-$ and $\|f\|=\|f^+\|+\|f^-\|$ [9, 12.3.4]. To prove the theorem, it suffices to show that f^+ and f^- are T-central. To do this we use an argument due to Effros and Hahn [10, p. 24].

Since f is self-adjoint, the set $\{A\in\mathcal{B}(\mathfrak{K}):f(AX)=f(XA)$ for all X in $\mathcal{B}(\mathfrak{H})\}$ is a C^*-algebra containing T. Fix a unitary owerator U in $C^*(T)$ and write $g_1(X)=f^+(U^*XU)$, $g_2(X)=f^-(U^*XU)$ for X in $\mathcal{B}(\mathfrak{H})$. Then g_1 and g_2 are positive linear functionals with $g_1(X)-g_2(X)=f(U^*XU)=f(X)$ and $\|g_1\|+\|g_2\|=$ $=g_1(I)+g_2(I)=\|f\|$. So, by the uniqueness of the decomposition of f, $f^+=g_1$ and $f^-=g_2$. Hence f^+ and f^- are U-central for every unitary U in $C^*(T)$. Since the unitaries in $C^*(T)$ span $C^*(T)$, f^+ and f^- are T-central. □

Remark. The proof of Theorem 3.1 shows that an operator T has a T-central state if and only if the commutator subspace $[C^*(T),\mathcal{B}(\mathfrak{H})]$ is not norm dense in $\mathcal{B}(\mathfrak{H})$. (See [6].) This is equivalent to the non-density of $\mathcal{R}(\delta_T)+\mathcal{R}(\delta_{T^*})$ or to the condition that 0 belong to the closure of the numerical range of every commutator $TX-XT$ [19]. The mere existence of a central state, however, does not imply d-symmetry as the example $T=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ shows.

Corollary 3.2. *If T is a d-symmetric operator, then:*
(a) $\mathcal{R}(\delta_T)^-$ *is an hereditary subspace of* $\mathcal{B}(\mathfrak{H})$: *that is, if* $0 \leqq X \leqq Y$ *and* $Y \in \mathcal{R}(\delta_T)^-$, *then* $X \in \mathcal{R}(\delta_T)^-$.
(b) $\mathcal{R}(\delta_A) \subseteqq \mathcal{R}(\delta_T)^-$ *for all A in $C^*(T)$.*
(c) $\mathcal{C}(T)$ *is the linear span of the positive elements in* $\mathcal{R}(\delta_T)^-$ *and* $\mathcal{C}(T)$ *is hereditary in* $\mathcal{B}(\mathfrak{H})$.
(d) $\mathcal{K}(\mathfrak{H}) \subseteqq \mathcal{C}(T)$ *if and only if T has no reducing eigenvalues.*

Proof. Parts (a) and (b) are clear from 3.1. We prove part (c). If C is a positive operator in $\mathcal{R}(\delta_T)^-$, then $f(C)=0$ for each T-central state f, and so, $|f(XC^{1/2})|^2 \leqq f(XX^*)\, f((C^{1/2})^2)=0$. Similarly, $f(C^{1/2}X)=0$. Hence, by Theorem 3.1, $C^{1/2} \in \mathcal{C}(T)$ and so $C=C^{1/2}C^{1/2} \in \mathcal{C}(T)$. On the other hand, if $C \in \mathcal{C}(T)$ is self-adjoint with spectral measure $E(\cdot)$, then $C=CE([0,\infty))+E((-\infty,0))C$ is a linear combination of positive operators in $\mathcal{R}(\delta_T)^-$. $\mathcal{C}(T)$ is a hereditary subspace of $\mathcal{B}(\mathfrak{H})$ by (a). Part (d) follows from (c) and 2.6. □

We now study the sets $\mathcal{C}(T)$, $\mathcal{I}(T)$, and $\mathcal{M}(T)$ in more detail.

Theorem 3.3: *If T is a d-symmetric operator, then:*
(a) $\mathcal{C}(T)$, $\mathcal{I}(T)$, *and* $\mathcal{M}(T)$ *are C^*-algebras.*
(b) $\mathcal{C}(T)$ *is a norm closed two-sided ideal in* $\mathcal{M}(T)$ *which is properly contained in* $\mathcal{I}(T)$. *Furthermore,* $\mathcal{I}(T) \subseteqq \mathcal{I}(T)+\{T\}' \subseteqq \mathcal{M}(T)$.
(c) $\mathcal{I}(T)/\mathcal{C}(T)$ *is contained in the center of* $\mathcal{M}(T)/\mathcal{C}(T)$.
(d) $\mathcal{M}(T)=\{Z \in \mathcal{B}(\mathfrak{H}):[Z,\mathcal{I}(T)] \subseteqq \mathcal{C}(T)\}=\{Z \in \mathcal{B}(\mathfrak{H}):[Z,T] \in \mathcal{C}(T)\}= =\{Z \in \mathcal{B}(\mathfrak{H}):[Z,T] \in \mathcal{I}(T)\}$.
(e) $\mathcal{C}(T)=\mathcal{I}(T) \cap \mathcal{R}(\delta_T)^- = \mathcal{M}(T) \cap \mathcal{R}(\delta_T)^-$.

Proof. As $\mathcal{R}(\delta_T)^-$ is self-adjoint it is clear that $\mathcal{M}(T)$ and $\mathcal{C}(T)$ are C^*-algebras. It is also clear that $\mathcal{C}(T) \subseteqq \mathcal{I}(T)$ and that $\{T\}' \subseteqq \mathcal{M}(T)$.

If $A \in \mathcal{I}(T)$ then $A\delta_T(X)=\delta_T(AX)+\delta_A(TX)-T\delta_A(X)$ is in $\mathcal{R}(\delta_T)^-$. Hence $A\mathcal{R}(\delta_T) \subseteqq \mathcal{R}(\delta_T)^-$. Similarly $\mathcal{R}(\delta_T)A \subseteqq \mathcal{R}(\delta_T)^-$ so that $\mathcal{I}(T) \subseteqq \mathcal{M}(T)$. Therefore if $A_1, A_2 \in \mathcal{I}(T)$ then $A_1A_2X-XA_1A_2=A_1(A_2X-XA_2)+(A_1X-XA_1)A_2 \in \in \mathcal{R}(\delta_T)^-$ and $A_1A_2 \in \mathcal{I}(T)$. Hence $\mathcal{I}(T)$ is a norm closed subalgebra of $\mathcal{B}(\mathfrak{H})$. Since $\mathcal{R}(\delta_T)^-$ is self-adjoint, $\mathcal{I}(T)$ is a C^*-algebra.

If $Z \in \mathcal{M}(T)$, $C \in \mathcal{C}(T)$, and X is any operator then

$$X(CZ)=(XC)Z \in \mathcal{R}(\delta_T)^-Z \subseteqq \mathcal{R}(\delta_T)^- \quad \text{and} \quad (CZ)X=C(ZX) \in \mathcal{R}(\delta_T)^-.$$

Hence $\mathcal{C}(T)$ is right ideal of $\mathcal{M}(T)$. Since $\mathcal{C}(T)$ is a C^*-algebra, it is a norm closed two sided ideal of $\mathcal{M}(T)$. Also, $I \notin \mathcal{C}(T)$ because $\mathcal{R}(\delta_T)^- \neq \mathcal{B}(\mathfrak{H})$ [16, Theorem 1], so $\mathcal{C}(T)$ is properly contained in $\mathcal{I}(T)$. This proves (a) and (b).

If $Z \in \mathcal{M}(T)$, $A \in \mathcal{I}(T)$, and X is any operator, then

$$\delta_Z(A)X=Z\delta_A(X)-\delta_A(ZX) \in \mathcal{R}(\delta_T)^- \quad \text{and} \quad X\delta_Z(A)=\delta_A(X)Z-\delta_A(XZ) \in \mathcal{R}(\delta_T)^-.$$

Hence $\delta_Z(A)\in\mathscr{C}(T)$. This proves (c). It also shows that

$$\mathscr{M}(T)\subseteqq\{Z\in\mathscr{B}(\mathfrak{H}):[Z,\mathscr{I}(T)]\subseteqq\mathscr{C}(T)\}\subseteqq\{Z\in\mathscr{B}(\mathfrak{H}):[Z,T]\in\mathscr{C}(T)\}\subseteqq$$
$$\subseteqq\{Z\in\mathscr{B}(\mathfrak{H}):[Z,T]\in\mathscr{I}(T)\}.$$

Before showing the reverse inclusions, we establish (e). Suppose that $A\in\mathscr{M}(T)\cap\cap\mathscr{R}(\delta_T)^-$. Then $AA^*+A^*A\in A\mathscr{R}(\delta_T)^-+\mathscr{R}(\delta_T)^-A\subseteqq\mathscr{R}(\delta_T)^-$. Hence both AA^* A^*A belong to $\mathscr{C}(T)$ as $\mathscr{C}(T)$ is hereditary. By considering the polar decompositions of A and A^* one gets $A\in\mathscr{C}(T)$. Thus $\mathscr{M}(T)\cap\mathscr{R}(\delta_T)^-\subseteqq\mathscr{C}(T)$ and the inclusions $\mathscr{C}(T)\subseteqq\mathscr{I}(T)\cap\mathscr{R}(\delta_T)^-\subseteqq\mathscr{M}(T)\cap\mathscr{R}(\delta_T)^-$ are trivial.

To finish the proof of (d), suppose $\delta_T(Z)\in\mathscr{I}(T)$ and X is an operator. Then $\delta_T(Z)\in\mathscr{C}(T)$ by (e) and so

$$Z\delta_T(X)=\delta_T(ZX)-\delta_T(Z)X\in\mathscr{R}(\delta_T)^- \quad\text{and}\quad \delta_T(X)Z=\delta_T(XZ)-X\delta_T(Z)\in\mathscr{R}(\delta_T)^-.$$

Hence $Z\in\mathscr{M}(T)$. □

The following is a version of the Fuglede theorem for d-symmetric operators.

Corollary 3.4. *Let T be d-symmetric and let $X\in\mathscr{B}(\mathfrak{H})$. If $TX-XT\in\mathscr{C}(T)$ then $TX^*-X^*T\in\mathscr{C}(T)$.*

Example. Let K denote an irreducible compact operator that does not commute with any trace class operator (as in remark (c) following 2.6, for example.) Then $\mathscr{C}(K)=\mathscr{R}(\delta_K)^-=\mathscr{K}$ so that $\mathscr{M}(K)=\mathscr{B}(\mathfrak{H})$ and $\mathscr{I}(K)=\mathscr{K}+\mathbf{C}I$ by [7, Theorem 2.9].

We now show that $\mathscr{C}(T)$ is the commutator ideal of $\mathscr{I}(T)$ if T is a d-symmetric operator.

Recall [4, § 3.3] that the *commutator ideal* Comm $(\mathscr{A})$ of a C^*-algebra $\mathscr{A}$ is the smallest closed two-sided ideal of $\mathscr{A}$ containing all of the commutators $A_1A_2-A_2A_1$ for A_1, A_2 in $\mathscr{A}$. Comm $(\mathscr{A})$ is also the smallest closed ideal $\mathscr{C}$ such that $\mathscr{A}/\mathscr{C}$ is commutative and, furthermore, $\mathrm{Comm}\,\mathscr{A}=\bigcap\ker(\varphi)$, where the intersection is taken over all the characters (non-zero complex homomorphisms) of $\mathscr{A}$. If T is an operator, then $\mathrm{Comm}\,(C^*(T))=\mathrm{Comm}\,C^*(T)$ is the ideal generated by T^*T-TT^*.

We make use of the fact that *if f is a state on a C^*-algebra $\mathscr{B}$ whose restriction φ to a C^*-subalgebra $\mathscr{A}$ is a character, then f is $\mathscr{A}$-multiplicative on $\mathscr{B}$* in the sense that $f(XA)=f(X)f(A)=f(AX)$ for all X in $\mathscr{B}$ and all A in $\mathscr{A}$. Indeed, $A-f(A)I$ belongs to the left kernel of f because $\varphi((A-\varphi(A))^*(A-\varphi(A)))=0$.

Theorem 3.5. *If T is a d-symmetric operator, then:*

(a) $\mathscr{C}(T)=\mathrm{Comm}\,(\mathscr{I}(T))$.

(b) $\mathrm{Comm}\,C^*(T)=C^*(T)\cap\mathscr{R}(\delta_T)^-=C^*(T)\cap\mathscr{C}(T)$.

(c) *The map α of $C^*(T)/\mathrm{Comm}\,C^*(T)$ into $\mathscr{I}(T)/\mathscr{C}(T)$ given by $\alpha(A+\mathrm{Comm}\,C^*(T))=A+\mathscr{C}(T)$ is an isomorphism.*

Proof. Let φ be a character on $\mathcal{I}(T)$ and let f be any extension of φ to a state on $\mathcal{B}(\mathfrak{H})$. Then f is T-central by the remark preceding the statement of the theorem, hence $\varphi(\mathcal{C}(T))=f(\mathcal{C}(T))=0$ as $\mathcal{C}(T)\subseteq\mathcal{R}(\delta_T)^-$. Thus $\mathcal{C}(T)\subseteq\mathrm{Comm}\,\mathcal{I}(T)$. The reverse inclusion is clear from Theorem 3.3(c).

The same remark shows that any character of $C^*(T)$ vanishes on $C^*(T)\cap\cap\mathcal{R}(\delta_T)^-$ so that $C^*(T)\cap\mathcal{R}(\delta_T)^-\subseteq\mathrm{Comm}\,C^*(T)\subseteq C^*(T)\cap\mathcal{C}(T)\subseteq C^*(T)\cap\cap\mathcal{R}(\delta_T)^-$ by the first part of the argument. This proves (b) and (c) is then clear. □

Corollary 3.6. *If T is a d-symmetric operator, then $C^*(T)$ has a character.*

Proof. Since $I\notin\mathcal{C}(T)$ (3.3(b)), $\mathrm{Comm}\,C^*(T)\neq C^*(T)$ by 3.5(b).

Remark. Note that $C^*(T)$ may have only one character, however. For example, this is the case for the compact operator considered in the example following 3.4.

We now derive additional results about the inclusion and multiplier algebras under the additional hypothesis that T has no reducing eigenvalues. Then $\mathcal{R}(\delta_T)^-\supseteq\mathcal{K}$ by 2.6 and so $\mathcal{C}(T)\supseteq\mathcal{K}$.

Theorem 3.7. *If T is a d-symmetric operator that has no reducing eigenvalues, then*

(a) $\mathcal{I}(T)=C^*(T)+\mathcal{C}(T)$, (b) *The center of* $\mathcal{M}(T)/\mathcal{C}(T)$ is $\mathcal{I}(T)/\mathcal{C}(T)$.

Proof. Suppose there is an operator S in $\mathcal{I}(T)$ such that $S\notin C^*(T)+\mathcal{C}(T)$. Then the commutative C^*-algebra $(C^*(S,T)+\mathcal{C}(T))/\mathcal{C}(T)$ properly contains $(C^*(T)+\mathcal{C}(T))/\mathcal{C}(T)$ and so by the Stone—Weierstrass theorem, there are distinct characters φ_1 and φ_2 on $C^*(S,T)+\mathcal{C}(T)$ that vanish on $\mathcal{C}(T)$ and agree on $C^*(T)+\mathcal{C}(T)$. Hence there are one-dimensional representations π_1 and π_2 of $C^*(S,T)+\mathcal{K}$ such that $\pi_1(S)\neq\pi_2(S)$, π_1 and π_2 agree on $C^*(T)+\mathcal{K}$, and π_1 and π_2 vanish on $\mathcal{K}$ (since $\mathcal{K}\subseteq\mathcal{C}(T)$). Let π denote the direct sum of $\aleph_0$ copies of π_1 and π_2. By Proposition 2.7(a) id is unitarily equivalent to $\mathrm{id}\oplus\pi$ modulo the compacts and it follows that there are infinite dimensional projections P_1 and P_2 on $\mathfrak{H}$ such that $P_iA-\varphi_i(A)P_i$ and $AP_i-\varphi_i(A)P_i$ are compact for $i=1,2$ and all A in $C^*(S,T)+\mathcal{K}$. Choose orthonormal bases $\{e_n\}$ and $\{f_n\}$ for $P_1\mathfrak{H}$ and $P_2\mathfrak{H}$, respectively and define W in $\mathcal{B}(\mathfrak{H})$ by $We_n=f_n$, $n=1,2,\ldots$, and $Wx=0$ for x in $(P_1\mathfrak{H})^\perp$. If $X\in\mathcal{B}(\mathfrak{H})$, then for $n=1,2,\ldots$, $((TX-XT)e_n,f_n)=((P_2TX-XTP_1)e_n,f_n)=(\varphi_2(T)-\varphi_1(T))(Xe_n,f_n)+(Ke_n,f_n)$, where K is a compact operator. Since $\varphi_1(T)=\varphi_2(T)$ and $\|Ke_n\|\to 0$, $\|W-(TX-XT)\|\geqq 1$ and $W\notin\mathcal{R}(\delta_T)^-$. On the other hand, $SW-WS=SP_2W-WP_1S=(\varphi_2(S)-\varphi_1(S))W+K$, where K is a compact operator. Since $\varphi_1(S)\neq\varphi_2(S)$ and $S\in\mathcal{I}(T)$, $W\in\mathcal{R}(\delta_S)+\mathcal{K}\subseteq\mathcal{R}(\delta_T)^-$, a contradiction. This proves part (a) of the theorem.

Now suppose that $Z+\mathcal{I}(T)$ is in the center of $\mathcal{M}(T)/\mathcal{C}(T)$ but $Z\notin\mathcal{I}(T)=C^*(T)+\mathcal{C}(T)$. Then by the argument given in the first part of the proof, there are characters φ_1 and φ_2 on $C^*(Z, T)+\mathcal{K}$ such that $\varphi_1(Z)\neq\varphi_2(Z)$, φ_1 and φ_2 vanish on $\mathcal{K}$, and φ_1 and φ_2 agree on $C^*(T)+\mathcal{K}$. Also, there are orthogonal infinite dimensional projections P_1 and P_2 on $\mathfrak{H}$ such that $P_iA-\varphi_i(A)P_i$ and $AP_i-\varphi_i(A)P_i$ are compact for all A in $C^*(Z, T)+\mathcal{K}$ and $i=1,2$. If $X\in\mathcal{B}(\mathfrak{H})$, then $(P_1+P_2)(TX-XT)(P_1+P_2)=(\varphi_1(T)-\varphi_2(T))(P_1XP_2-P_2XP_1)+K=K$, for some compact operator K, since $\varphi_1(T)=\varphi_2(T)$. Thus $(P_1+P_2)\mathcal{R}(\delta_T)^-(P_1+P_2)= =\mathcal{K}((P_1+P_2)\mathfrak{H})$. Let W denote a partial isometry of $P_1\mathfrak{H}$ onto $P_2\mathfrak{H}$ as in the first part of the proof. Then $W\delta_T(X)-\delta_T(WX)=-\delta_T(W)X=-(TP_2W-WP_1T)X= =(\varphi_1(T)-\varphi_2(T))WX+K=K$, for some compact operator K, since $\varphi_1(T)= =\varphi_2(T)$ and so $W\mathcal{R}(\delta_T)\subseteqq\mathcal{R}(\delta_T)+\mathcal{K}\subseteqq\mathcal{R}(\delta_T)^-$. A similar argument shows that $\mathcal{R}(\delta_T)W\subseteqq\mathcal{R}(\delta_T)^-$ so that $W\in\mathcal{M}(T)$. Hence, $ZW-WZ\in\mathcal{C}(T)\subseteqq\mathcal{R}(\delta_T)^-$. But $P_2(ZW-WZ)P_1=(\varphi_2(Z)-\varphi_1(Z))W+K$ for some compact operator K and since $\varphi_2(Z)\neq\varphi_1(Z)$, $P_2(ZW-WZ)P_1$ is not compact, a contradiction. □

Corollary 3.8. *If T is a d-symmetric operator that has no reducing eigenvalues, then* $C^*(T)/\text{Comm}\, C^*(T)\cong\mathcal{J}(T)/\mathcal{C}(T)$.

In the concluding result of this section we show that $\mathcal{C}(T)$ can be quite large.

Theorem 3.9. *Supposse $\mathfrak{H}$ is separable and that T is a d-symmetric operator with no reducing eigenvalues. If T is not essentially normal, or if T is essentially normal with uncountable spectrum, then $\mathcal{C}(T)$ contains a C^*-algebra that is spatially isomorphic to $\mathcal{B}(\mathfrak{H})\oplus\mathcal{K}(\mathfrak{H})$.*

Proof. It is enough to show that $\mathcal{R}(\delta_T)^-$ contains a projection P of infinite rank. For then, since P is positive, $P\in\mathcal{C}(T)$ by 3.2(c) and $P\mathcal{B}(\mathfrak{H})P+P^\perp\mathcal{K}(\mathfrak{H})P^\perp$ is the desired subalgebra of $\mathcal{C}(T)$.

If $\mathcal{R}(\delta_T)^-$ fails to contain a projection of infinite rank, then $\mathcal{C}(T)\subseteqq\mathcal{K}$ by 3.2(c) and spectral theory. Hence T is essentially normal. Since $\mathcal{K}\subseteqq\mathcal{R}(\delta_T)^-$, Remark 1 of [22] implies that the spectrum of T is countable. □

4. The T-central states

The set $E(T)$ of all T-central states on $\mathcal{B}(\mathfrak{H})$ is convex and weak*-compact. We begin this section by examining the extreme points of $E(T)$. Recall that a state f on a C^*-algebra $\mathcal{B}$ is $\mathcal{A}$-multiplicative if $f(AX)=f(A)f(X)=f(XA)$ for all X in $\mathcal{B}$ and all A in $\mathcal{A}$ and that the extreme points in the set of all states on $\mathcal{B}$ are also called pure states.

Theorem 4.1. *If T is a d-symmetric operator and f is an extreme point of $E(T)$, then f is $\mathcal{J}(T)$-multiplicative on $\mathcal{B}(\mathfrak{H})$ and f is a pure state on $\mathcal{B}(\mathfrak{H})$.*

Proof. Fix a self-adjoint element A in $\mathcal{I}(T)$ with $0<\varepsilon<A<I-\varepsilon$, for some $\varepsilon>0$. Define f_1 and f_2 on $\mathcal{B}(\mathfrak{H})$ by $f_1(X)=f(A)^{-1}f(XA)$ and $f_2(X)=$ $=f(I-A)^{-1}f(X(I-A))$. Then $f(XA)=f(XA^{1/2}A^{1/2})=f(A^{1/2}XA^{1/2})$ as $A^{1/2}\in$ $\in\mathcal{I}(T)$. Hence f_1 and (similarly) f_2 are states on $\mathcal{B}(\mathfrak{H})$. Since $TA-AT\in\mathcal{C}(T)$ by 3.3(c),

$$f(A)f_1(XT)=f(XTA)=f\big(X(TA-AT)\big)+f(XAT)=0+f(XAT)=$$
$$=f(TXA)=f(A)f_1(TX).$$

Thus, f_1 and (similarly) f_2 are T-central. Since $f=f(A)f_1+f(I-A)f_2$ is an extreme point of $E(T)$, $f=f_1$ and so f is A-multiplicative. Since $\mathcal{I}(T)$ is the linear span of operators of this form, the first assertion is proved.

Now suppose that there are states f_1 and f_2 on $\mathcal{B}(\mathfrak{H})$ and $0<\alpha<1$ such that $f=\alpha f_1+(1-\alpha)f_2$, where f is an extreme point of $E(T)$. Since f is multiplicative on $\mathcal{I}(T)$ by the first part of the proof, f is a pure state on $\mathcal{I}(T)$ and so f, f_1, and f_2 agree on $\mathcal{I}(T)$. In particular, each f_i is multiplicative on $C^*(T)$. It follows (see our remark preceding 3.5) that each f_i is T-central. Hence, $f=f_1=f_2$. □

Corollary 4.2. *If T is a d-symmetric operator, then each character on $C^*(T)$ extends to a character on $\mathcal{I}(T)$.*

Proof. Fix a character φ on $C^*(T)$ and let f be a pure state on $\mathcal{B}(\mathfrak{H})$ that extends φ. Then f is T-multiplicative since it extends φ and, therefore, f is an extreme point of $E(T)$. Hence f is multiplicative on $\mathcal{I}(T)$ by the theorem. □

Remark. It follows from 4.1 that if T is d-symmetric then $\mathcal{R}(\delta_T)^-$ is the intersection of the kernels of the T-multiplicative states on $\mathcal{B}(\mathfrak{H})$. Also, $\mathcal{I}(T)$ is the set of operators A such that every extreme point of $E(T)$ is A-multiplicative.

It is natural at this point to ask: Which states on $C^*(T)$ extend to T-central states on $\mathcal{B}(\mathfrak{H})$? The answer is what one might expect.

Theorem 4.3. *If T is a d-symmetric operator, then:*

(a) *A state f on $C^*(T)$ extends to a T-central state on $\mathcal{B}(\mathfrak{H})$ if and only if $f\big(\mathrm{Comm}\, C^*(T)\big)=0$.*

(b) *A state g on $\mathcal{I}(T)$ extends to a T-central state on $\mathcal{B}(\mathfrak{H})$ if and only if $g\big(\mathcal{C}(T)\big)=0$.*

Proof. Since $\mathcal{C}(T)\subseteqq\mathcal{R}(\delta_T)^-$, each T-central state on $\mathcal{B}(\mathfrak{H})$ vanishes on $\mathcal{C}(T)$ and so on $\mathrm{Comm}\, C^*(T)$ (by 3.5(b)) so that the conditions $f\big(\mathrm{Comm}\, C^*(T)\big)=$ $=g\big(\mathcal{C}(T)\big)=0$ are necessary. Now suppose g is a state on $\mathcal{I}(T)$ such that $g\big(\mathcal{C}(T)\big)=0$. Then g may be viewed as a state on the commutative C^*-algebra $\mathcal{I}(T)/\mathcal{C}(T)$. Hence, g is the weak*-limit of a net of convex combinations of characters on $\mathcal{I}(T)$. Each of the characters appearing in these convex combinations has an extension to a pure state on $\mathscr{B}(\mathfrak{H})$ which is T-multiplicative by 4.1. By taking the same convex combinations of the extended states, we obtain a net of T-central states that has a subnet that converges to a T-central extension of g. The proof of sufficiency in part (a) is the same. □

5. Examples

In this section we consider the C^*-algebras $\mathscr{C}(T)$, $\mathscr{I}(T)$, and $\mathscr{M}(T)$ for special d-symmetric operators.

I. *Normal operators without eigenvalues.* Let N denote a normal operator without eigenvalues. Then the spectrum σ of N is uncountable, N is d-symmetric, and $\mathscr{K}\subseteqq\mathscr{C}(N)$ (2.2 and 3.2(d)). Hence $\mathscr{C}(N)$ is nonseparable by 3.9. Also, $\mathscr{I}(N)=C^*(N)+\mathscr{C}(N)$ by 3.7(a) and if $C(\sigma)$ denotes the continuous functions on σ, then $C(\sigma)\cong\mathscr{I}(N)/\mathscr{C}(N)$ is the center of $\mathscr{M}(N)/\mathscr{C}(N)$ by 3.7(b). Further, $\mathscr{M}(N)$ contains the von Neumann algebra $\{N\}'$ by 3.3(b).

Recall [15, 4.4.19] that there is a norm one projection $\mathscr{P}$ of $\mathscr{B}(\mathfrak{H})$ onto $\{N\}'$ such that $\mathscr{P}(AXB)=A\mathscr{P}(X)B$ for A and B in $\{N\}'$ and X in $\mathscr{B}(\mathfrak{H})$. Thus, $\mathscr{P}(\mathscr{R}(\delta_N)^-)=0$ and so if $A\in\{N\}'$ and $X\in\mathscr{R}(\delta_N)^-$, $\|A\|=\|\mathscr{P}(A+X)\|\leqq\|A+X\|$ and $\{N\}'+\mathscr{C}(N)$ is an orthogonal direct sum in $\mathscr{M}(N)$. However, $\{N\}'+\mathscr{C}(N)\neq\mathscr{M}(N)$. Otherwise the center of $\mathscr{M}(N)/\mathscr{C}(N)$ would be isomorphic to $\{N\}'$. This is not the case. In fact, as noted above, the center of $\mathscr{M}(N)/\mathscr{C}(N)$ is isomorphic to $C^*(N)$.

II. *Diagonal operators.* In this example and the next all operators will be assumed to be acting on separable Hilbert space. An operator D is *diagonal* if there is a sequence $\{E_n\}$ of orthogonal projections such that $\Sigma E_n=I$ and a bounded sequence $\{d_n\}$ of distinct complex numbers such that $D=\Sigma d_n E_n$.

Proposition 5.1. *The following are equivalent for a d-symmetric operator T:*

(a) *T is a diagonal operator*, (b) *$\mathscr{I}(T)$ is commutative.* (c) $\mathscr{C}(T)=0$.

(d) $\mathscr{M}(T)=\{T\}'$.

Proof. Since $\mathscr{C}(T)=\operatorname{Comm}\mathscr{I}(T)$ by 3.5(a), (b) and (c) are equivalent. By 2.1(b) and 3.2(c), the condition $\mathscr{C}(T)=0$ is equivalent to the conditions that T be normal and $\mathscr{R}(\delta_T)^-$ contain no nonzero positive operator. Therefore (c) and (a) are equivalent by [22]. Finally (c) and 3.3(d) imply (d), and if (d) holds, then $\{T\}'$ is self-adjoint and T is normal. Hence $\mathscr{C}(T)=\mathscr{M}(T)\cap\mathscr{R}(\delta_T)^-=\{T\}'\cap\mathscr{R}(\delta_T)^-=0$. (This latter intersection is 0 for any normal operator T as shown in example 1. See also [1]).

Thus, for a diagonal operator D, $\mathscr{C}(D)$ and $\mathscr{M}(D)$ are easily described. The C^*-algebra $\mathscr{I}(D)$ is more complicated. Before describing it we need a preliminary result.

Lemma 5.2. *If* $D=\Sigma d_n E_n$ *is a diagonal operator, then* $C^*(D)\subseteq\mathscr{I}(D)\subseteq$ $\subseteq C^*(D, E_1, E_2, ...)$.

Proof. The first inclusion is trivial. Also $\mathscr{C}(D)=0$ as D is a diagonal operator, hence $\{D\}'=\mathscr{M}(D)=\mathscr{I}(D)'$ by 3.3(d). Therefore $\mathscr{I}(D)\subseteq(D)''$. To finish the proof, fix a diagonal operator $D'=\Sigma a_n E_n$ in $\{D\}''$ that is not in $C^*(D, E_1, E_2, ...)$. We must show that $D'\notin\mathscr{I}(D)$. Choose a sequence $\{e_n\}$ of unit vectors in $\mathfrak{H}$ such that $E_n e_n=e_n$ for each n, and let ω_n denote the vector state induced by e_n (so that $\omega_n(X)=(Xe_n, e_n)$). Then each ω_n is a character on $\mathscr{A}=C^*(D, D', E_1, ...)$ and if φ is a character on $\mathscr{A}$ then either $\varphi(E_n)=1$ for some unique integer n and $\varphi=\omega_n$, or else $\varphi(E_n)=0$ for all n and $\varphi=\lim_n \omega_{\sigma(n)}$ is the weak*-limit of a subsequence of the ω_n's induced by an injective map σ of the natural numbers $\mathbf{N}$ into $\mathbf{N}$. Since $C^*(D, E_1, ...)$ is a proper C^*-subalgebra of $\mathscr{A}$, there are distinct characters φ and ψ on $\mathscr{A}$ that agree on $C^*(D, E_1, E_2, ...)$ by the Stone—Weierstrass theorem. If $\varphi(E_n)=1$ for some n, then $\psi(E_n)=1$ and $\varphi=\psi=\omega_n$ because $E_n\in C^*(D, E_1, ...)$. Hence, $\varphi(E_n)=\psi(E_n)=0$ for all n and $\varphi=\lim_n \omega_{\sigma(n)}$, $\psi=\lim_n \omega_{\tau(n)}$ are weak*-limits of disjoint subsequences of $\{\omega_n\}$ induced by injective maps σ and τ of $\mathbf{N}$ into disjoint subsets of $\mathbf{N}$. Write

$$\alpha = \varphi(D') = \lim_n (D' e_{\sigma(n)}, e_{\sigma(n)}) = \lim_n a_{\sigma(n)},$$

$$\beta = \psi(D') = \lim_n (D' e_{\tau(n)}, e_{\tau(n)}) = \lim_n a_{\tau(n)}$$

and

$$\gamma = \varphi(D) = \psi(D) = \lim_n d_{\sigma(n)} = \lim_n d_{\tau(n)}$$

so that $\alpha\neq\beta$. Define an operator W by $We_{\sigma(n)}=e_{\tau(n)}$ for $n=1, 2, ...$ and $Wx=0$ if $x\in\{e_{\sigma(1)}, e_{\sigma(2)}, ...\}^\perp$. Then if X is any operator, $\|D', W]-[D, X]\|\geqq$ $\geqq \lim_n |([D', W]e_{\sigma(n)}, e_{\tau(n)})-([D, X]e_{\sigma(n)}, e_{\tau(n)})| = \lim_n |a_{\tau(n)}-a_{\sigma(n)})(We_{\sigma(n)}, e_{\tau(n)})-$ $-(d_{\tau(n)}-d_{\sigma(n)})(Xe_{\sigma(n)}, e_{\tau(n)})|=|\beta-\alpha|$. Thus $[D', W]\notin\mathscr{R}(\delta_D)^-$ and $D'\notin\mathscr{I}(D)$. □

Theorem 5.3. *Suppose* $D=\Sigma d_n E_n$ *is a diagonal operator and write*

$\Lambda=\{n\in\mathbf{N}: E_n$ has finite rank and d_n is not an isolated point of the spectrum of $D\}$.

Then $\mathscr{I}(D)=C^*(D, \{E_n\}_{n\in\Lambda})$.

Proof. First note that if d_n is an isolated point of the spectrum, then $E_n\in C^*(D)$ by the Gelfand theory. Now fix an eigenvalue d_n of D that is a limit point of the spectrum of D. By 5.2 it suffices to show that $E_n\in\mathscr{I}(D)$ if and only if E_n has finite rank. Suppose that E_n has infinite rank and choose an orthonormal basis

$\{e_1, e_2, \ldots\}$ for E_n. Since d_n is a limit point, there are projections E_{n_j} and unit vectors f_j such that $d_{n_j} \to d_n$ and $E_{n_j} f_j = f_j$. Define W in $\mathscr{B}(\mathfrak{H})$ by $We_j = f_j$ and $Wx = 0$ if $x \in (E_n \mathfrak{H})^\perp$. Then if X is any operator,

$$\|[E_n, W] - [D, X]\| \geqq \lim_j |([E_n, W]e_j, f_j) - ([D, X]e_j, f_j)| =$$
$$= \lim_j |(f_j, f_j) - 0 - (d_n - d_{n_j})(Xe_j, f_j)| = 1.$$

Thus, $[E_n, W] \notin \mathscr{R}(\delta_D)^-$ and $E_n \notin \mathscr{I}(D)$. Now suppose E_n has finite rank. Fix vectors x in $E_n \mathfrak{H}$ and y in $E_m \mathfrak{H}$, where $n \neq m$. Let $x \otimes y$ denote the rank one operator given by $x \otimes y(z) = (z, y)x$. Then $[D, x \otimes y] = Dx \otimes y - x \otimes D^* y = (d_n - d_m) x \otimes y$. Since $d_m \neq d_n$, $x \otimes y \in \mathscr{R}(\delta_D)^-$. If $z \in (E_n \mathfrak{H})^\perp$, then $z = \sum_{n \neq m} a_m y_m$, where $E_m y_m = y_m$, and $\Sigma |a_m|^2 = \|z\|^2$. Thus, $x \otimes z = \Sigma \bar{a}_m (x \otimes y_m)$ is in $\mathscr{R}(\delta_D)^-$ for all $x \in E_n \mathfrak{H}$ and $z \in (E_n \mathfrak{H})^\perp$. It follows that $E_n X E_n^\perp \in \mathscr{R}(\delta_D)^-$ for all X in $\mathscr{B}(\mathfrak{H})$ and so since $\mathscr{R}(\delta_D)^-$ is self-adjoint $\mathscr{R}(\delta_{E_n}) = \mathscr{R}(\delta_{E_n})^- = E_n \mathscr{B}(\mathfrak{H}) E_n^\perp + E_n^\perp \mathscr{B}(\mathfrak{H}) E_n \subseteqq \mathscr{R}(\delta_D)^-$. Thus, $E_n \in \mathscr{I}(D)$. □

Corollary 5.4. *If* $D = \Sigma d_n E_n$, *where each* E_n *has infinite rank, then* $\mathscr{I}(D) = C^*(D)$.

Corollary 5.5. *If* $D = \Sigma d_n E_n$, *where each* E_n *has rank one and* $\{d_n\}$ *is an enumeration of the rationals between* 0 *and* 1, *then:*

(a) $E_n \notin C^*(D)$, $n = 1, 2, \ldots$, (b) $\mathscr{I}(D) = C^*(D, E_1, E_2, \ldots)$.

Remarks. (a) Let D be the diagonal operator defined in 5.5. Then $\{D\}' = \{D\}'' = \mathscr{M}(D)$ (5.2(d)) so that $\mathscr{M}(D)$ is commutative, $\mathscr{C}(D) = 0$ by 5.2(c) and $C^*(D) = C^*(D) + \mathscr{C}(D) \neq \mathscr{I}(D)$ by 5.5. Thus, if the condition T has no reducing eigenvalues is omitted, Theorems 3.7 and 3.9 and Corollary 3.8 are no longer true.

(b) Let $\{E_n\}$ denote a sequence of orthogonal rank one projections with $\Sigma E_n = I$ and write

$$D = \sum_{n=1}^{\infty} n^{-1} E_n, \quad D' = \sum_{n=2}^{\infty} n^{-1} E_n.$$

Then $C^*(D) = C^*(E_1, E_2, \ldots) \neq C^*(D') = C^*(E_2, E_3, \ldots)$. However, by Theorem 5.1, $\mathscr{I}(D) = \mathscr{I}(D') = C^*(D)$ and by (the last part of) the proof of 5.4, $\mathscr{R}(\delta_D)^- = \mathscr{R}(\delta_{D'})^-$. Thus, in general, neither the includion algebra nor the derivation range determines $C^*(T)$.

(c) If T is an essentially normal d-symmetric operator with countable spectrum, then $\mathscr{C}(T) \subseteqq \mathscr{K}$; and $\mathscr{C}(T) = 0$ if and only if T is normal. Indeed, since the spectrum is countable, there is a non-zero representation π of $\mathscr{B}(\mathfrak{H})$ on a Hilbert space $\mathfrak{H}_\pi$ such that $\ker \pi = \mathscr{K}(\mathfrak{H})$ and $\pi(T)$ is a diagonal operator on $\mathfrak{H}_\pi$ [22].

Then $\mathscr{C}(\pi(T))=0$ by 5.2(c), and by 3.2(c) we have $\pi(\mathscr{C}(T))\subseteq\mathscr{C}(\pi(T))$. Hence $\mathscr{C}(T)\subseteq\ker(\pi)=\mathscr{K}$. The spectral theorem and 5.1(c) imply that $\mathscr{C}(T)=0$ if and only if T is normal.

III. *Pure isometries.* Let V denote a pure isometry (that is, an isometry with no unitary direct summand). Then V is d-symmetric by 2.3 and has no reducing eigenvalues; hence, $\mathscr{K}(\mathfrak{H})\subseteq\mathscr{C}(V)$ by 3.2(d) and so $\mathscr{I}(V)=C^*(V)+\mathscr{C}(V)$ and the center of $\mathscr{M}(V)/\mathscr{C}(V)$ is $\mathscr{I}(V)/\mathscr{C}(V)\cong C^*(V)/\text{Comm}\, C^*(V)\cong C$ (unit circle), the continuous functions on the unit circle by 3.5, 3.7 and [8, Theorem 3]. Also, by 3.9 $\mathscr{C}(V)$ contains a subalgebra that is spatially isomorphic to $\mathscr{B}(\mathfrak{H})\oplus\mathscr{K}(\mathfrak{H})$. We now show that $\mathscr{M}(V)/\mathscr{C}(V)$ is also large.

Proposition 5.6. *An operator Z is in $\mathscr{M}(V)$ if and only if $V^*ZV-Z\in\mathscr{C}(V)$. Hence $\mathscr{M}(V)\supseteq\mathscr{C}(V)\oplus\mathscr{T}_V$, where $\mathscr{T}_V=\{X\in\mathscr{B}(\mathfrak{H}): V^*XV=X\}$ is the set of Toeplitz operators associated with V. Thus $\mathscr{M}(V)/\mathscr{C}(V)$ is non-separable.*

Proof. If $Z\in\mathscr{M}(V)$ then $V^*ZV-Z=V^*(ZV-VZ)\in\mathscr{C}(V)$ by 3.3(d). Conversely, suppose that $V^*ZV-Z\in\mathscr{C}(V)$. Then $Z(VX-XV)=\delta_V(ZX)+$ $+(I-VV^*)ZVX+V(V^*ZV-Z)X$ belongs to $\mathscr{R}(\delta_V)^-$ for any operator X because $I-VV^*=[V^*V],\in\mathscr{C}(V)$. Also, $V^*Z^*V-Z^*\in\mathscr{C}(V)$ so that $Z^*\mathscr{R}(\delta_{V^*})^-=$ $=Z^*\mathscr{R}(\delta_V)^-\subseteq\mathscr{R}(\delta_V)^-$ and this implies $\mathscr{R}(\delta_V)Z\subseteq\mathscr{R}(\delta_V)^-$ on taking adjoints. Thus Z is a two-sided multiplier of $\mathscr{R}(\delta_V)^-$ and therefore $Z\in\mathscr{M}(V)$.

That the subspaces $\mathscr{C}(V)$ and $\mathscr{T}_V$ have trivial intersection and are in fact "orthogonal" follows from the existence of a norm one projection of $\mathscr{B}(\mathfrak{H})$ onto $\mathscr{T}_V$ that vanishes on $\mathscr{R}(\delta_V)$. (See [21]). □

6. Some open problems

(a) If T is a d-symmetric operator, must $C^*(T)$ be a postliminaire or GCR C^*-algebra [9, Paragraphe IV]? If T is d-symmetric and $C^*(T)$ is GCR, then by direct integral theory and Theorem 2.8 T would be a direct integral of irreducible d-symmetric operators. Which irreducible operators are d-symmetric? We do not know when a direct integral of d-symmetric operators is d-symmetric.

(b) The example in Remark (c) following Corollary 2.6 raises the question: Is the set $\{T+K\colon T\ d\text{-symmetric},\ K \text{ compact}\}$ norm-closed?

(c) It follows from Proposition 5.2 that there does not exist a normal operator N such that $\mathscr{R}(\delta_N)^-=\mathscr{R}(\delta_K)^-$, where K is the compact d-symmetric operator of Remark (c) following Corollary 2.6. If V is the simple unilateral shift does there exist a normal operator N such that $\mathscr{I}(N)=\mathscr{I}(V)$? If N is normal must $\mathscr{I}(N)=$ $=\mathscr{I}(A)$ for some self-adjoint operator A?

(d) Is there a property of δ_T as an element of the Banach algebra $\mathscr{B}(\mathscr{B}(\mathfrak{H}))$, which characterizes when T is d-symmetric?

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Sur certaines suites pseudo-aléatoires

JEAN COQUET

I. Introduction

I. 1. Notations et définitions. Comme d'habitude, on désigne par $\mathbf{N}$ l'ensemble des entiers naturels, $\mathbf{N}^*$ l'ensemble $\mathbf{N}-\{0\}$, $\mathbf{Z}$ l'ensemble des entiers relatifs, $\mathbf{Q}$ celui des nombres rationnels, $\mathbf{R}$ celui des nombres réels et $\mathbf{C}$ celui des nombres complexes.

Pour tout $x\in\mathbf{R}$, on pose $e(x)=e^{2i\pi x}$,

$$\|x\| = \operatorname{Min}\{|x-n| : n\in\mathbf{Z}\},\quad [x] = \operatorname{Max}\{n\in\mathbf{Z} : n \leqq x\},\quad \{x\} = x-[x].$$

Soit $g:\mathbf{N}\to\mathbf{C}$ une suite infinie. On dit que g est pseudo-aléatoire au sens de BERTRANDIAS [2] si les deux conditions suivantes (a) et (b) sont réalisées:

(a) $\gamma(t) = \lim\limits_{N\to\infty} \frac{1}{N}\sum\limits_{n=0}^{N-1} g(n+t)\overline{g(n)}$ existe pour tout $t\in\mathbf{N}$

(γ s'appelle corrélation de g),

(b) $\lim\limits_{N\to\infty} \frac{1}{N}\sum\limits_{t=0}^{N-1} |\gamma(t)|^2 = 0.$

Si, au lieu de (b), la condition suivante (b′), plus forte, est réalisée:

(b′) $\lim\limits_{t\to\infty}\gamma(t) = 0,$

g est dite pseudo-aléatoire au sens de BASS [1].

Dans cet article, $(a_k)_{k\in\mathbf{N}}$ est une suite de réels, $(s_k)_{k\in\mathbf{N}}$ désigne une suite strictement croissante de réels positifs tendant vers l'infini. On pose $g(n)=e\left(\sum\limits_{k=0}^{\infty} a_k\left[\frac{n}{s_k}\right]\right)$.

I. 2. Résultats. Dans un article écrit en collaboration avec M. MENDÈS-FRANCE [3], nous avions établi le résultat suivant:

THÉORÈME 1. *Soit q un entier $\geqq 2$. Posons $s_k=q^k$ pour tout $k\in\mathbf{N}$. Les trois assertions suivantes sont équivalentes:*

1) *g est pseudo-aléatorie au sens de Bertrandias,*

2) $\sum_{k=0}^{\infty} \|a_k\|^2 = \infty,$

3) *g est à spectre vide.*

Dans l'exposé [4] nous avons démontré que le théorème 1 se généralisait au cas où la suite (s_k) est une suite d'entiers naturels, telle que pour tout $k \in \mathbf{N}$, s_k divise s_{k+1}.

Nous nous proposons ici d'examiner le cas "diamétralement opposé" où les nombres s_k sont deux à deux premiers entre eux:

Théorème 2. *Soit $(s_k)_{k\in\mathbf{N}}$ une suite croissante de nombres entiers naturels, deux à deux premiers entre eux et telle que:*

$$\sum_{k=0}^{\infty} \frac{1}{s_k} < \infty.$$

g est pseudo-aléatoire au sens de Bertrandias si et seulement si $\sum_{k=0}^{\infty} \|a_k\|^2 = \infty$.

Nous démontrons également le

Théorème 3. *Soit $(s_k)_{k\in N}$ une suite de nombres irrationnels positifs telle que $\sum_{k=0}^{\infty} \frac{1}{s_k} < \infty$ et que les nombres $\frac{1}{s_k}$, $k \in \mathbf{N}$, soient* **Q***-linéairement indépendants. g est pseudo-aléatoire au sens de Bertrandias si et seulement si* $\sum_{k=0}^{\infty} \|a_k\|^2 = \infty$.

Le théorème 3 admet évidemment comme corollaire le théorème suivant, à comparer au théorème 1:

Théorème 4. *Soit τ un nombre réel transcendant >1. On pose $s_k = \tau^k$ pour tout $k \in \mathbf{N}$. g est pseudo-aléatoire au sens de Bertrandias si et seulement si*

$$\sum_{k=0}^{\infty} \|a_k\|^2 = \infty.$$

I. 3. Remarques. 1) Nous pouvons démontrer facilement à l'aide des relations de corrélation établies dans [3] que les suites envisagées dans le théorème 1 et, plus généralement, les suites q-multiplicatives de module 1, ne sont pas pseudo-aléatoires au sens de Bass. Par contre, lorsque $\sum_{k=0}^{\infty} \|a_k\|^2 = \infty$, les suites envisagées au théorème 2 peuvent être ou ne pas être pseudo-aléatoires au sens de Bass, comme nous le verrons dans le paragraphe III.

2) La méthode utilisée pour démontrer les théorèmes 2 et 3 diffère sensiblement de celle utilisée dans [3] pour prouver le théorème 1. Ici, nous déterminons effectivement la corrélation γ de g.

La démonstration de l'existence de γ est faite sous la seule hypothèse que $(s_k)_{k\in\mathbf{N}}$ est une suite de réels positifs telle que $\sum_{k=0}^{\infty}\frac{1}{s_k}<\infty$. Est-ce que, dans ce cas général, la condition $\sum_{k=0}^{\infty}\|a_k\|^2=\infty$ est encore nécessaire et suffisante pour que g soit pseudo-aléatoire au sens de Bertrandias?

II. Démonstration des théorèmes 2 et 3

II. 1. Existence de la corrélation de g. Considérons la fonction tronquée g_r définie par

$$g_r(n)=e\left(\sum_{k=0}^{r}a_k\left[\frac{n}{s_k}\right]\right).$$

Montrons d'abord que g_r a une corrélation γ_r. Soit $t\in\mathbf{N}^*$. On a

$$g_r(n+t)\overline{g_r(n)}=e\left(\sum_{k=0}^{r}a_k\left(\left[\frac{n+t}{s_k}\right]-\left[\frac{n}{s_k}\right]\right)\right)=$$

$$=e\left(\sum_{k=0}^{r}\frac{a_k t}{s_k}\right)\cdot\prod_{k=0}^{r}e\left(a_k\left\{\frac{n}{s_k}\right\}\right)\cdot e\left(-a_k\left\{\frac{n+t}{s_k}\right\}\right).$$

La suite $\left(g_r(n+t)\overline{g_r(n)}\right)$ est presque-périodique-B_1. Elle a donc une valeur moyenne $\gamma_r(t)$.

Montrons maintenant que g a une corrélation γ donnée par: $\gamma(t)=\lim_{r\to\infty}\gamma_r(t)$. On a

$$\text{Card }\{n:0\leqq n\leqq N-1 \text{ et } g(n+t)\overline{g(n)}\neq g_r(n+t)\overline{g_r(n)}\}$$

$$\leqq\sum_{k\geqq r+1}\text{Card}\left\{n:0\leqq n\leqq N-1 \text{ et }\left[\frac{n+t}{s_k}\right]\neq\left[\frac{n}{s_k}\right]\right\}\leqq\sum_{k\geqq r+1}\frac{tN}{s_k}.$$

L'hypothèse $\sum_{k=0}^{\infty}\frac{1}{s_k}<\infty$ implique

$$\frac{1}{N}\sum_{n=0}^{N-1}|g(n+t)\overline{g(n)}-g_r(n+t)\overline{g_r(n)}|\leqq 2t\sum_{k\geqq r+1}\frac{1}{s_k}\to 0(r\to\infty).$$

Un résultat classique sur l'interversion de deux passages à la limite montre que la suite $\left(g(n+t)\overline{g(n)}\right)$ a une valeur moyenne $\gamma(t)$.

De plus, pour tout $t\in\mathbf{N}^*$, $\gamma(t)=\lim_{r\to\infty}\gamma_r(t)$.

II. 2. Calcul de la corrélation de g. Nous allons montrer que, sous les hypothèses du théorème 2 ou celles du théorème 3, $\gamma_r(t)=\prod_{k=0}^{r}\psi_k(t)$ où

$$\psi_k(t)=e\left(a_k\left[\frac{t}{s_k}\right]\right)\cdot\left(1-\left\{\frac{t}{s_k}\right\}+\left\{\frac{t}{s_k}\right\}e(a_k)\right).$$

Il est évident que:

$$\left[\frac{n+t}{s_k}\right]-\left[\frac{n}{s_k}\right]=\begin{cases}\left[\frac{t}{s_k}\right] & \text{si } 0\leqq\left\{\frac{n}{s_k}\right\}<1-\left\{\frac{t}{s_k}\right\}\\ \left[\frac{t}{s_k}\right]+1 & \text{si } 1-\left\{\frac{t}{s_k}\right\}\leqq\left\{\frac{n}{s_k}\right\}<1.\end{cases}$$

La suite $\left(e\left(a_k\left[\frac{n+t}{s_k}\right]-a_k\left[\frac{n}{s_k}\right]\right)\right)_{n\in\mathbf{N}}$ a donc une valeur moyenne égale à $\psi_k(t)$.

Les hypothèses du théorème 2 ou du théorème 3 assurent l'indépendance statistique des ensembles $E_k=\left\{n\in\mathbf{N}:\left\{\frac{n}{s_k}\right\}\in A_k\right\}$, $0\leqq k\leqq r$, où les A_k sont des sous-intervalles arbitraires de $[0, 1[$.

Autrement dit, si $\delta(E)$ est la densité asymptotique de $E\subset\mathbf{N}$,

$$\delta(E_0\cap\ldots\cap E_r)=\prod_{k=0}^{r}\delta(E_k).$$

Cette indépendance statistique entraîne que $\gamma_r(t)=\prod_{k=0}^{r}\psi_k(t)$. Il en résulte que:

$$|\gamma_r(t)|^2=\prod_{k=0}^{r}|\psi_k(t)|^2=\prod_{k=0}^{r}\left(1-4\left\{\frac{t}{s_k}\right\}\left(1-\left\{\frac{t}{s_k}\right\}\right)\sin^2(\pi a_k)\right).$$

Comme $\gamma(t)=\lim_{r\to\infty}\gamma_r(t)$,

$$|\gamma(t)|^2=\prod_{k=0}^{\infty}\left(1-4\left\{\frac{t}{s_k}\right\}\left(1-\left\{\frac{t}{s_k}\right\}\right)\sin^2(\pi a_k)\right).$$

II. 3. Fin de la démonstration du théorème 3. s_k étant irrationnel, la valeur moyenne de $|\psi_k(t)|^2$ est égale à

$$1-4\sin^2(\pi a_k)\int_0^1 x(1-x)\,dx=1-\frac{2}{3}\sin^2(\pi a_k).$$

L'indépendance linéaire sur $\mathbf{Q}$ des nombres $\frac{1}{s_0},\ldots,\frac{1}{s_r}$ permet d'affirmer que $|\gamma_r(t)|^2$ a une valeur moyenne égale à

$$\mu_r=\prod_{k=0}^{r}\left(1-\frac{2}{3}\sin^2(\pi a_k)\right).$$

Cas $\sum_{k=0}^{\infty} \|a_k\|^2 = \infty$. Comme $|\gamma(t)| \leqq |\gamma_r(t)|$ pour tout $r \in \mathbf{N}$ et tout $t \in \mathbf{N}^*$, on a:

$$\overline{\lim_{N\to\infty}} \sum_{t=0}^{N-1} |\gamma(t)|^2 \leqq \mu_r \quad \text{pour tout} \quad r \in \mathbf{N}.$$

Puisque $\mu_r \to 0$ lorsque $r \to \infty$, g est pseudo-aléatoire au sens de Bertrandias.

Cas $\sum_{k=0}^{\infty} \|a_k\|^2 < \infty$. La suite $(|\gamma_r(t)|^2)_{r\in\mathbf{N}}$ converge, uniformément par rapport à t, vers $|\gamma(t)|^2$. En effet,

$$|\gamma_r(t)|^2 - |\gamma(t)|^2 \leqq 1 - \prod_{k \geqq r+1} \left(1 - 4\left\{\frac{t}{s_k}\right\}\left(1 - \left\{\frac{t}{s_k}\right\}\right) \sin^2(\pi a_k)\right) \leqq \pi^2 \sum_{k\geqq r+1} \|a_k\|^2.$$

On en déduit que $\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} |\gamma(t)|^2$ existe et est égale à $\lim_{r\to\infty} \mu_r > 0$. g n'est pas pseudo-aléatoire.

II. 4. Fin de la démonstration du théorème 2. Elle est semblable à celle du théorème 3. Dans ce cas, r étant fixé, la suite $(|\gamma_r(t)|^2)_{t\in\mathbf{N}}$ est périodique de période $s_0 \dots s_r$. Elle a une valeur moyenne

$$\mu_r = \prod_{k=0}^{r} \left(1 - \frac{2(s_k^2 - 1)}{3 s_k^2} \sin^2(\pi a_k)\right).$$

On conclut de la même façon.

III. Suites pseudo-aléatoires au sens de Bass

III. 1. Soit $\alpha \notin \mathbf{Q}$. Posons $a_k = \alpha$ pour tout $k \in \mathbf{N}$.

Supposons que la suite $(s_k)_{k\in\mathbf{N}}$ satisfasse aux conditions du théorème 2 et à la condition supplémentaire suivante

$$\text{Card}\,\{k\colon T \leqq s_k < 2T\} \to +\infty \quad (T \to \infty).$$

On a alors

$$|\gamma(t)|^2 \leqq \prod_{\frac{3t}{2} \leqq s_k < 3t} \left(1 - \frac{4t}{s_k}\left(1 - \frac{t}{s_k}\right) \sin^2 \pi\alpha\right) \leqq \prod_{\frac{3t}{2} \leqq s_k < 3t} \left(1 - \frac{8}{9} \sin^2 \pi\alpha\right) \to 0 \quad (t \to \infty).$$

La suite $\left(e\left(\alpha \sum_{k=0}^{\infty} \left[\frac{n}{s_k}\right]\right)\right)_{n\in\mathbf{N}}$ est donc pseudo-aléatoire au sens de Bass.

III. 2. Au contraire, sous les hypothèses du théorème 2, la condition $\sum_{k=0}^{\infty} \|a_k\|^2 = \infty$ peut être réalisée sans que g soit pseudo-aléatoire au sens de Bass.

Supposons cette fois que la suite (s_k) ait une croissance rapide dans le sens suivant:

$$\lim_{r\to\infty} \frac{s_0 \dots s_r}{s_{r+1}} = 0.$$

On a alors:

$$|\gamma_r(t)|^2 - |\gamma(t)|^2 \leqq \sum_{k\geqq r+1} 4\left(1-\left\{\frac{t}{s_k}\right\}\right)\left\{\frac{t}{s_k}\right\} \sin^2(\pi a_k) \leqq 4 \sum_{k\geqq r+1} \frac{t}{s_k}.$$

Donc $|\gamma_r(s_0\dots s_r)|^2 - |\gamma(s_0\dots s_r)|^2 \leqq 4 \sum_{k\geqq r+1} \frac{s_0 \dots s_r}{s_k}$. Comme $|\gamma_r(s_0\dots s_r)| = 1$, on a $|\gamma(s_0\dots s_r)| \to 1$ lorsque $r\to\infty$ la et suite g n'est pas pseudo-aléatoire au sens de Bass.

On peut montrer que g est pseudo-aléatoire au sens de Bertrandias si et seulement si son spectre est vide.

IV. Généralisation. Application

IV. 1. Généralisation. On peut généraliser les théorèmes 2 et 3 en considérant des suites du type:

$$g^*(n) = e\left(\sum_{k=0}^{\infty}\left(\sum_{i=1}^{m} a_{k,i}\left[\frac{n+q_i}{s_k}\right]\right)\right)$$

où les nombres q_i sont entiers naturels.

La suite g^* est pseudo-aléatoire si et seulement si la suite g de terme général $g(n) = e\left(\sum_{k=0}^{\infty}\left(\sum_{i=1}^{m} a_{k,i}\right)\left[\frac{n}{s_k}\right]\right)$ est pseudo-aléatoire.

IV. 2. Application. Les théorèmes 2 et 3 admettent des applications en théorie de l'équirépartition modulo 1. Nous donnerons sans démonstration une nouvelle caractérisation des nombres de Pisot ([3], [4], [5], [6]).

Théorème 5. *Soit $\theta > 1$ un nombre réel, soit $(s_k)_{k\in\mathbf{N}}$ une suite de nombres réels satisfaisant aux conditions du théorème 2 ou à celles du théorème 3. On pose*

$$g(n) = e\left(\sum_{k=0}^{\infty} \theta^k \left[\frac{n}{s_k}\right]\right) \quad et \quad h(n) = e\left(\sum_{k=0}^{\infty} \|\theta^k\| \left\{\frac{n}{s_k}\right\}\right).$$

Les assertions suivantes sont équivalentes.

1) *θ est un nombre de Pisot,*
2) *g n'est pas pseudo-aléatoire,*
3) *g est presque-périodique-B_1 à spectre non vide,*
4) *h n'est pas pseudo-aléatoire,*
5) *h est presque-périodique-B_1 à spectre non vide.*

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Near normality of a class of transforms

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1. Introduction. The class of bounded linear transformations $T: L^2(\mathbf{R}_+) \to L^2(\mathbf{R}_+)$ which satisfy the functional equation $Tt(a) = t(a)T$, all $a \in \mathbf{R}_+$, where $t(a)$ is the operator $\big(t(a)f\big)(x) = f(ax)$, is characterised by

$$T = M^{-1} m[K] M, \quad K \text{ some function in } L^\infty(\mathbf{R}),$$

where M denotes the Mellin transform operator and $m[K]$ denotes multiplication by the function K. Suitably choosing K, another equivalent characterisation, which is the familiar integral representation ([3], [6], [8]) for this class of mappings, is given by

$$\int_0^u Tf(t)\,dt = \int_0^\infty k(ux^{-1}) f(x)\,dx, \quad u \in \mathbf{R}_+.$$

Define a mapping S by $S = TR$, where R is the linear mapping $Rf(x) = x^{-1} f(x^{-1})$, $x \in \mathbf{R}_+$, and T is some member of the class of mappings defined above.

Mappings $S = TR$ have been called Watson transforms, and a study of these mappings has been carried out by a large number of authors (see [3], [6], [7], [8], [9], and [10], for further references). In this note we study Watson transforms from the point of view of bounded linear mappings acting on a functional Hilbert space, and show that although the class of Watson transforms is non-normal, it displays a large number of properties enjoyed by normal operators. In particular, we show that the class of Watson transforms consists of centered, normaloid operators for which the concepts of normal, quasi-normal, subnormal, hyponormal, quasi-hyponormal and paranormal coincide. It is shown that reducing subspaces for Watson transforms exist, and that the determination of the spectrum of a member of this class is very much linked with the determination of the spectrum of a normal transformation.

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2. Preliminaries. Let T, where T satisfies $Tt(a)=t(a)T$, be continuous on $L^2(\mathbf{R}_+)\to L^2(\mathbf{R}_+)$. Then the mapping (Banach space) conjugate to T is given by $T'=RTR$ (see [3]; Kober [3] calls T' the *concrete adjoint* of T). Let J denote the operation of complex conjugation, i.e. $Jg=\bar{g}$. We define the *Hilbert space adjoint,* henceforth called simply the *adjoint*, of T by $T^*=JT'J$. Noting that $P=JR=P^*=P^{-1}$, a simple argument shows that if $S=TR$, then $S^*=JSJ$. Let Q denote the mapping $Qf(t)=f^{\sim}(t)=f(-t)$, $t\in\mathbf{R}$. It is a simple matter to see that (use the following properties of the Mellin transforms: $MR=QM$ and $MJ=JQM$).

Theorem. (cf. [6, Theorem (2.3)]) *If* $S=M^{-1}m[K]MR$, *then S is normal* $\Leftrightarrow SS^*=JSS^*J\Leftrightarrow|K^{\sim}|=|K|$. *Furthermore, if K is even (i.e.* $K^{\sim}=K$), *then these conditions are equivalent to the implication that* $S=TR=RT$ *for some T satisfying the functional equation* $Tt(a)=t(a)T$.

In the sequel we will write S_K for S to denote its dependence on the function K.

3. Main result. Let A be a bounded linear mapping on a Hilbert space H to itself. The mapping A is said to be (i) *normal* if A and A^* commute; (ii) *quasi-normal* if A commutes with A^*A; (iii) *subnormal* if A has a normal extension; (iv) *hyponormal* if $\|A^*f\|\leqq\|Af\|$ for all $f\in H$; (v) *quasi-hyponormal* if $\|A^*Af\|\leqq\|AAf\|$ for all $f\in H$; (vi) *paranormal* if $\|Af\|^2\leqq\|A^2f\|$ for all unit vectors $f\in H$; (vii) *normaloid* if $w(A)=\|A\|$, where $w(A)$ denotes the numerical radius [2, p. 114] of A; (viii) *spectraloid if* $w(A)=r(A)$, where $r(A)$ denotes the spectral radius [2, p. 45] of A. We have the following inclusion relations for these classes of operators:

$$\text{(i)}\subseteqq\text{(ii)}\subseteqq\text{(iii)}\subseteqq\text{(iv)}\subseteqq\text{(v)}\subseteqq\text{(vi)}\subseteqq\text{(vii)}\subseteqq\text{(viii)}.$$

The reverse inclusions, in general, do not hold, and this remains true for Watson transforms. However, a partial result holds for Watson transforms, as we now show.

Theorem 1. *S_K is paranormal if and only if it is normal.*

Proof. Clearly, normality of S_K implies paranormality of S_K. We divide the proof of the reverse implication into three steps.

Step 1: S_K^* is hyponormal if and only if $|K^{\sim}|\leqq|K|$. Clearly, S_K^* is hyponormal if and only if $S_KS_K^*-S_K^*S_K\geqq 0$; since $S_K^*=M^{-1}m[\bar{K}^{\sim}]MR$, this holds if and only if

$$M^{-1}m[K]MRM^{-1}m[\bar{K}^{\sim}]MR-M^{-1}m[\bar{K}^{\sim}]MRM^{-1}m[K]MR\geqq 0,$$

or if and only if $M^{-1}m[|K|^2-|K^{\sim}|^2]M\geqq 0$, i.e. if and only if $|K|\geqq|K^{\sim}|$.

Step 2: S_K is paranormal only if $|K^{\sim}|\leqq|K|$. It is not very difficult to see (see [5], for example) that a mapping A on a Hilbert space H is paranormal if and only if $A^{*2}A^2+2\lambda A^*A+\lambda^2 I\geqq 0$ for all real λ. Substituting S_K for A, and noting that

$$(S_K^*)^2 = M^{-1}m[\bar{K}^{\sim}\bar{K}]M, \quad S_K^2 = M^{-1}m[KK^{\sim}]M \quad \text{and} \quad S_K^* S_K = M^{-1}m[|K^{\sim}|^2]M,$$

we now see that S_K is paranormal if and only if

$$M^{-1}m[|K|^2|K^{\sim}|^2]M+2\lambda M^{-1}m[|K^{\sim}|^2]M+\lambda^2 I \geqq 0$$

for all real λ, or what is the same (use the definition of a positive operator), $|K|^2|K^{\sim}|^2+2\lambda|K^{\sim}|^2+\lambda^2\geqq 0$ for all real λ. But the last inequality holds only if $|K^{\sim}|\leqq|K|$.

Step 3: S_K is paranormal only if $|K|=|K^{\sim}|$. From Steps 1 and 2 it now follows that if S_K is paranormal, then S_K^* is hyponormal, and hence that S_K^* is paranormal. Using once again the definition of paranormality we see that S_K^* is paranormal if and only if $|K|^2|K^{\sim}|^2+2\lambda|K|^2+\lambda^2\geqq 0$ for all real λ. This last inequality clearly implies that $|K|\leqq|K^{\sim}|$, and so we have that $|K|=|K^{\sim}|$.

The proof, once one takes into consideration the fact that S_K is normal if and only if $|K|=|K^{\sim}|$, is now complete.

Remark. The proof of Step 3 can also be deduced from the properties of the self-commutator [2, p. 132] of an operator. By Step 2, if S_K is paranormal then S_K^* is hyponormal, and hence $D=S_K S_K^*-S_K^* S_K=M^{-1}m[|K|^2-|K^{\sim}|^2]M\geqq 0$. Now if $|K|=|K^{\sim}|$, then there is nothing to prove; if, on the other hand, $|K|>|K^{\sim}|$, then the mapping D (clearly a multiplier transform) is invertible. But this is in contradiction with the fact that a positive self-commutator can not be invertible [2, Problem 188].

Since not all Watson transforms are normal (an example of this is provided by the Watson transform S_K for which the function K is given by $K(t)=2^{it}\Gamma(1/2+\nu/2+it/2)/\Gamma(1/2+\nu/2-it/2)$, $\operatorname{Re}\nu>-1$), and since concepts (i)—(vi) coincide for Watson transforms, the next best thing to happen to the class of Watson transforms (after it has failed to be normal) would be that the members of the class are normaloid. That this indeed is the case is shown by our Theorem 2. The following lemma will be useful (see also [9, p. 24], where the formulae of the lemma are used in the spectral resolution of Watson transforms).

Lemma.

$$(1)\qquad (S_K)^n(S_K^*)^m = \begin{cases} M^{-1}m[K^{n/2}\bar{K}^{m/2}(K^{\sim})^{n/2}(\bar{K}^{\sim})^{m/2}]M, & \text{if } n, m \text{ are positive even integers,} \\ M^{-1}m[K^{(n+1)/2}\bar{K}^{(m+1)/2}(K^{\sim})^{(n-1)/2}(\bar{K}^{\sim})^{(m-1)/2}]M, & \text{if } n, m \text{ are positive odd integers.} \end{cases}$$

Proof. A straightforward calculation, using again the identities $MR=QM$ and $MJ=JQM$, shows that

$$(2)\qquad (S_K)^n = \begin{cases} M^{-1}m[K^{(n+1)/2}(K^{\sim})^{(n-1)/2}]MR, & \text{if } n \text{ is odd,} \\ M^{-1}m[K^{n/2}(K^{\sim})^{n/2}]M, & \text{if } n \text{ is even;} \end{cases}$$

a similar calculation shows that

$$(3)\qquad (S_K^*)^m = \begin{cases} M^{-1}m[(\bar{K}^{\sim})^{(m+1)/2}\bar{K}^{(m-1)/2}]MR, & \text{if } m \text{ is odd,} \\ M^{-1}m[(\bar{K}^{\sim})^{m/2}\bar{K}^{m/2}]M, & \text{if } m \text{ is even.} \end{cases}$$

Substituting in $(S_K)^n(S_K^*)^m$, the lemma follows.

Theorem 2. *S_K is normaloid.*

Proof. To prove the theorem it is enough to show that $\|S_K\|^n=\|(S_K)^n\|$. It is easily seen that $\|S_K\|=\|K\|_\infty$ (cf. [6, Theorem (2.7)]). Since $S_K^*=JS_KJ=$ $=M^{-1}m[\bar{K}^{\sim}]M$, $\|S_K^*\|=\|S_K\|=\|K^{\sim}\|_\infty$. Clearly $\|S_K\|^n=\|K\|_\infty^n$. From (2) of the proof of the preceding lemma

$$\|(S_K)^n\| = \begin{cases} \|K^{(n+1)/2}\|_\infty\|(K^{\sim})^{(n-1)/2}\|_\infty, & \text{if } n \text{ is odd,} \\ \|K^{n/2}\|_\infty\|(K^{\sim})^{n/2}\|_\infty, & \text{if } n \text{ is even,} \end{cases}$$

i.e. $\|(S_K)^n\|=\|K\|_\infty^n$. This completes the proof.

The norm power and power norm equality is satisfied by each hyponormal operator. A further proof of the nice (near normal) behaviour of Watson transforms is provided by the following equality:

$$w(S_KS_G) = w(M^{-1}m[KG^{\sim}]M) = \|K\|_\infty\|G^{\sim}\|_\infty = w(S_K)w(S_G).$$

(In general, $w(AB)\neq w(A)w(B)$ even for normal A and B: the best one can have is $w(AB)\leqq w(A)w(B)$ for normal operators, and $w(AB)\leqq 4w(A)w(B)$ for operators in general [2, p. 116].)

Having seen earlier that S_K and S_K^* do not commute in general, let us see if any commutativity property is satisfied by $(S_K)^n(S_K^*)^n$ and $(S_K^*)^m(S_K)^m$, where n and m are positive integers. We say that a mapping A (on the Hilbert space H) is *binormal* if A^*A and AA^* commute; the mapping A is siad to be *centered* if the operators in the sequence $\ldots, A^2(A^*)^2, AA^*, A^*A, (A^*)^2A^2, \ldots$ are mutually commuting [4]. (Clearly, a centered operator is in particular binormal.) For Watson transforms we have

Theorem 3. *S_K is a centered operator.*

Proof. Letting $m=n$ in (1), we have that

$$(S_K)^n(S_K^*)^n = \begin{cases} M^{-1}m[|K|^n|K^{\sim}|^n]M, & \text{if } n \text{ is even,} \\ M^{-1}m[|K|^{n+1}|K^{\sim}|^{n-1}]M, & \text{if } n \text{ is odd.} \end{cases}$$

On the other hand, by (2) and (3) we have that

$$(S_K^*)^n(S_K)^n = \begin{cases} M^{-1}m[|K|^n|K^{\sim}|^n]M, & \text{if } n \text{ is even,} \\ M^{-1}m[|K|^{n-1}|K^{\sim}|^{n+1}]M, & \text{if } n \text{ is odd.} \end{cases}$$

The mutual commutativity is now obvious.

4. Spectra, reducing subspaces and unitary Watson transforms. The near normality of Watson transforms is manifest in many other properties that they have. Thus, just as for normal transformations, the residual spectrum $\sigma_r(S_K)$ is empty. If A is a normal transform on a functional Hilbert space H, then A can be represented (use the spectral theorem) as a multiplication, induced by a bounded measurable function φ (say), on some L^2 space, and so the spectrum of A $(=\sigma(A))$ is the essential range of φ $(=e_r(\varphi))$. The spectral resolution of the class of Watson transforms has been considered by AKUTOWICZ [1] and DE SNOO [9]. We have

$$\lambda \in \sigma(S_K) \quad \textit{if and only if} \quad \lambda^2 \in \sigma(S_K S_K).$$

This follows from Theorem (3.4) of [6]. Note that $(S_K)^2$ is normal, and that $\sigma(S_K S_K) = e_r(KK^{\sim})$. Another important property that normal transformations have is that there exist, at least one, non-trivial subspaces that reduce the operator. That the same holds for Watson transforms is shown by the following

Theorem 4. *There exists a subspace V of $L^2(\mathbf{R}_+)$ such that V reduces S_K.*

Proof. Let T_G be the mapping $T_G = M^{-1}m[G]M$. Then $T_G S_K = S_K T_G'$. It follows that if G is even, then the linear manifold $L_G = \{g \in L^2(\mathbf{R}_+): g = T_G f$ for some $f \in L^2(\mathbf{R}_+)\}$ is invariant for S_K (see DE SNOO [7, Corollaries (2.6) and (2.8)]). The validity of the theorem is now easily deduced upon suitably choosing G so that $V = L_G$ is closed (e.g., let G be the characteristic function of the interval $[-1, 1]$).

Turning now to unitary transforms, it is well known that a normal operator A is unitary if and only if $\sigma(A)$ lies on the unit circle. That a similar result is true for Watson transforms is contained in the following.

Theorem 5. *The following conditions on S_K are equivalent:* (a) $|K|=1$; (b) $S_K S_K^* = S_K^* S_K = I$; (c) $S_K = J S_{\bar{K}}^{-1} J$; (d) $\sigma(S_K)$ *lies on the unit circle.*

Furthermore, if the function k is defined by $xk(x) = M^{-1}(K(t)/(1/2-it))(x)$, then these conditions are equivalent to the implication that

$$\text{(e)} \qquad \int_0^\infty k(ax^{-1})\bar{k}(x^{-1})\,dx = \min(a, 1), \quad a \in \mathbf{R}_+.$$

(Condition (e) is of course the classical characterisation of unitary Watson transforms (see, for example, [10])).

Proof. That (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d) is not difficult to see. Suppose then that (d) is satisfied. Since the mappings S_K are normaloid, $r(S_K)=w(S_K)=\|S_K\|=1$, and so $|K|\leqq 1$. Similarly, $|K^{\sim}|\leqq 1$. Since $\lambda\in\sigma(S_K)$ if and only if $\lambda^2\in\sigma(S_K^2)$, the normal transformation S_K^2 has spectrum on the unit circle, and so is unitary. But S_K^2 is unitary if and only if $|KK^{\sim}|=1$. Hence, upon combining with the previous inequalities, $|K|=1$. Thus (d)$\Rightarrow$(a).

To complete the proof, suppose that there is a function k satisfying the hypotheses of the theorem. Then an argument following closely that in [8, p. 56] shows that

$$\int_0^\infty k(tx)\bar{k}(vx)\,dx/x^2 = \min(t, v), \quad t, v\in \mathbf{R}_+,$$

if and only if S_K is unitary. A suitable change of variable now gives (e).

We conclude with the remarks that (i) the condition that $\sigma(S_K)$ lies on the real axis is not, in general, enough to ensure that the Watson transform S_K be self-adjoint; (ii) Watson transforms are, in general, not convexoid. As an example, consider the Hankel transform of order ν, Re $\nu>-1$.

Finally, I would like to thank the referee for pointing out references [1] and [9], and for making a number of very helpful comments on the original draft of the paper.

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On products of integers. II

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1. Throughout this paper, $c_1, c_2, \ldots$ denote absolute constants; $k_0(\alpha, \beta, \ldots)$, $k_1(\alpha, \beta, \ldots), \ldots, x_0(\alpha, \beta, \ldots), \ldots$ denote constants depending only on the parameters $\alpha, \beta, \ldots$; $\nu(n)$ denotes the number of the prime factors of the positive integer n, counted according to their multiplicity. The number of the elements of a finite set S is denoted by $|S|$.

Let k, n be any positive integers, $A=\{a_1, a_2, \ldots, a_n\}$ any finite, strictly increasing sequence of positive integers satisfying

$$a_1 = 1,\ a_2 = 2, \ldots, a_k = k \tag{1}$$

(consequently, $|A|=n\geqq k$). Let us denote the number of integers which can be written in form

$$\prod_{i=1}^{n} a_i^{\varepsilon_i} \quad (\varepsilon_i = 0 \text{ or } 1) \tag{2}$$

or

$$a_i a_j \quad (1 \leqq i, j \leqq n),$$

respectively by $f(A, n, k)$ and $g(A, n, k)$. Let us write

$$F(n, k) = \min_A f(A, n, k) \quad \text{and} \quad G(n, k) = \min_A g(A, n, k)$$

where the minimums are extended over all sequences A satisfying (1) and $|A|=n$.

Starting out from a conjecture of G. Halász, the second author showed in the first part of this paper (see [4]) that

$$G(n, k) > n \cdot \exp\left(c_1 \frac{\log k}{\log\log k}\right).$$

Note that to get many distinct products of form $a_i a_j$, we need a condition of type (1); otherwise e.g. the sequence $A=\{1, 2, 2^2, \ldots, 2^{n-1}\}$ is a counterexample, namely for this sequence the number of the distinct products is $2n-1=O(n)$.

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Furthermore, $G(n, k)/n$ is not much greater for fixed k and large n than for $n=k$, i.e. for $A=B_k$ where

$$B_k = \{1, 2, \dots, k\}.$$

This can be shown by the following construction: let $A^*=\{a_1^*, a_2^*, \dots, a_n^*\}$ be the sequence of the integers of form $p^i j$ where p is a fixed prime number greater than k, $i=1, 2, \dots, m$, $j=1, 2, \dots, k$, and m is any positive integer. Clearly,

$$\frac{g(A^*, n, k)}{n} < 2\frac{g(B_k, k, k)}{k} = 2\frac{G(k, k)}{k}$$

thus

$$\frac{G(n, k)}{n} < 2\frac{G(k, k)}{k} \quad \text{for} \quad k/n,$$

hence

$$\frac{G(n, k)}{n} < 4\frac{G(k, k)}{k} \quad (= o(k)) \quad \text{for every } n.$$

The authors conjectured that

$$\frac{G(n, k)}{n} > c_2\frac{G(k, k)}{k} \tag{3}$$

for every $n \geqq k$, and furthermore, that for any $\omega>0$, $k>k_0(\omega)$ and $n\geqq k$, we have

$$F(n, k) > n^2 k^\omega$$

or perhaps

$$n^2 \exp\left(c_3 \frac{k}{\log k}\right) < F(n, k) < n^2 \exp\left(c_4 \frac{k}{\log k}\right) \tag{4}$$

for large k and $n \geqq k$. (See [4], also Problem 9 in [3].)

The aim of this paper is to disprove (3) (Theorem 1) and to prove a slightly weaker form of (4) (Theorem 2).

2. In this section, we will disprove (3).

P. Erdős showed in [1] (see Theorem 1) that for any $\varepsilon>0$ and $k>k_0(\varepsilon)$,

$$\frac{k^2}{(\log k^2)^{1+\varepsilon}}(e\log 2)^{\frac{\log\log k^2}{\log 2}} = g(B_k, k, k) = \sum_{\substack{m \leqq k^2 \\ m=xy \\ x\leqq k,\, y \leqq k}} 1 < \frac{k^2}{(\log k^2)^{1-\varepsilon}}(e\log 2)^{\frac{\log\log k^2}{\log 2}}.$$

This inequality can be written in the equivalent form

$$\frac{k^2}{(\log k)^{c_5+\varepsilon}} < G(k, k) < \frac{k^2}{(\log k)^{c_5-\varepsilon}}$$

where

$$c_5 = 1 - \frac{1+\log\log 2}{\log 2}.$$

An easy computation shows that

$$0,086 < c_5 < 0,087.$$

Hence, for large k,

(5) $$\frac{k}{(\log k)^{0,087}} < \frac{G(k, k)}{k} < \frac{k}{(\log k)^{0,086}}.$$

Thus to disprove (3), it is sufficient to show that for large k, there exist a positive integer n ($\geqq k$) and a sequence A such that $|A|=n$, (1) holds and

(6) $$\frac{g(A, n, k)}{n} < \frac{k}{(\log k)^{c_6}}$$

where

(7) $$c_6 > 0,087.$$

In fact, by (5) and the definition of the function $G(n, k)$, this would imply

(8) $$\frac{G(n, k)}{n} < \frac{k}{(\log k)^{c_6}} < \frac{1}{(\log k)^{c_7}} \cdot \frac{G(k, k)}{k}$$

where

$$c_7 = c_6 - 0,087 > 0$$

by (7).

Let us write $\varphi(x) = 1 + x \log x - x$ and let z denote the single real root of the equation

(9) $$\varphi(x) = \varphi(1+x).$$

A simple computation shows that

(10) $$0,54 < z < 0,55.$$

Theorem 1. *For any $\varepsilon > 0$ and $k > k_1(\varepsilon)$, there exist a positive integer $n(\geqq k)$ and a sequence A such that $|A|=n$, (1) holds and*

(11) $$\frac{g(A, n, k)}{n} < \frac{k}{(\log k)^{c_8 - \varepsilon}}$$

where

(12) $$c_8 = \varphi(z).$$

(The function $\varphi(x)$ is decreasing for $0 < x < 1$. Thus with respect to (10), we obtain by a simple computation that

$$c_8 = \varphi(z) > \varphi(0,55) > 0,121.$$

Hence, Theorem 1 yields that for large k, (6) holds with $c_6 = 0,121$ which satisfies (7). Thus in fact, (8) holds with $c_7 = 0,121 - 0,087 = 0,034$ which disproves (3).)

Proof. Let k be a positive integer which is sufficiently large (in terms of ε) and let m be any positive integer satisfying

$$m > k^2. \tag{13}$$

Let D_k denote the set of those integers d for which

$$1 \leqq d \leqq k \tag{14}$$

and

$$\nu(d) > \log\log k \tag{15}$$

hold. Let p be a prime number satisfying

$$p > k. \tag{16}$$

Let E_k denote the set of those integers e which can be written in form $p^\alpha d$ where

$$1 \leqq \alpha \leqq m \tag{17}$$

and

$$d \in D_k. \tag{18}$$

Finally, let

$$A = E_k \cup B_k.$$

We are going to show that for large enough k, this sequence A satisfies (11).

Obviously,

$$n = |A| = |E_k| + |B_k| \leqq mk + k < 2mk. \tag{19}$$

Furthermore, by a theorem of P. Erdős and M. Kac [2], we have

$$|D_k| > \frac{1}{3} k.$$

Thus (with respect to (16))

$$n = |A| > |E_k| = m \cdot |D_k| > \frac{1}{3} mk. \tag{20}$$

To estimate the number of the distinct products of form $a_i a_j$, we have to distinguish four cases.

Case 1. Assume at first that $a_i \in B_k$, $a_j \in B_k$. Since B_k consists of k elements, the pair a_i, a_j can be chosen in at most

$$k^2 < m < n$$

ways (with respect to (13) and (20)).

Case 2. Assume now that $a_i = p^\alpha d \in E_k$ (where (14), (15) and (16) hold),

$$a_j \in B_k \tag{21}$$

and

$$\nu(a_j) \leqq z \log\log k. \tag{22}$$

Then

(23) $$a_i a_j = p^\alpha d a_j.$$

Let $\pi_i(x)$ denote the number of those integers u for which $u \leqq x$ and $\nu(u)=i$ hold. By a theorem of Hardy and Ramanujan, for any $\omega>0$ there exists a constant $c_9=c_9(\omega)$ such that for large x and $1 \leqq i \leqq \omega \log x$, we have

(24) $$\pi_i(x) < c_9 \frac{x}{\log x} \frac{(\log\log x)^{i-1}}{(i-1)!}.$$

Choosing here $\omega=1$ and using Stirling's formula, we obtain that for $k>k_2(\omega)$, the number of the integers a_j satisfying (21) and (22) is at most

(25) $$\sum_{0 \leqq i \leqq z \log\log k} \pi_i(k) <$$

$$< 1 + \sum_{1 \leqq i \leqq z \log\log k} c_9 \frac{k}{\log k} \frac{(\log\log k)^{i-1}}{(i-1)!} <$$

$$< 1 + c_9 \frac{k}{\log k} \sum_{1 \leqq i \leqq z \log\log k} \frac{(\log\log k)^{[z \log\log k]-1}}{([z \log\log k]-1)!} \leqq$$

$$< 1 + c_9 \frac{k}{\log k} z \log\log k \frac{(\log\log k)^{[z \log\log k]-1}}{([z \log\log k]-1)!} <$$

$$< 1 + c_{10} \frac{k}{\log k} \frac{(\log\log k)^{[z \log\log k]}}{([z \log\log k]-1)^{[z \log\log k]-1/2} e^{-[z \log\log k]-1}} <$$

$$< 1 + c_{11} \frac{k}{\log k} \frac{(\log\log k)^{[z \log\log k]}}{(z \log\log k)^{[z \log\log k]-1/2} e^{-z \log\log k}} <$$

$$< c_{12} \frac{k}{\log k} \frac{1}{(\log k)^{z \log z} (\log\log k)^{-1/2} (\log k)^{-z}} < \frac{k}{(\log k)^{c_8 - \varepsilon/3}}$$

(where c_8 is defined by (12)) since $\dfrac{(\log\log k)^{i-1}}{i-1!}$ is increasing for $1 \leqq i \leqq \log\log k$.

By (14), (17) and (18), α and d can be chosen in at most m and k ways, respectively. Thus the number of the products of form (23) is less than

$$m \cdot k \cdot \frac{k}{(\log k)^{c_8 - \varepsilon/3}} < n \frac{k}{(\log k)^{c_8 - \varepsilon/2}}$$

(with respect to (20)).

Case 3. Assume that $a_i = p^\alpha d \in E_k$ (where (14), (15) and (16) hold),

(26) $$a_j \in B_k$$

and

(27) $$\nu(a_j) > z \log\log k.$$

Then

$$a_i a_j = (p^\alpha d) a_j = p^\alpha (d a_j). \tag{28}$$

By (14), (15), (18), (26) and (27),

$$d a_j \leqq k \cdot k = k^2$$

and

$$\nu(d a_j) = \nu(d) + \nu(a_j) > \log\log k + z \log\log k = (1+z)\log\log k.$$

Thus applying (24) with $\omega = 100$, we obtain that for any $0 < \delta < z/2$ and $k > k_3(\delta)$, and writing $r = [(1+z-\delta)\log\log k^2]$, the number of the distinct products of form $d a_j$ is at most

$$\sum_{(1+z)\log\log k < i} \pi_i(k^2) < \sum_{(1+z-\delta)\log\log k^2 < i} \pi_i(k^2) = \tag{29}$$

$$= \sum_{r < i \leqq 100 \log\log k^2} \pi_i(k^2) + \sum_{100 \log\log k^2 < i} \pi_i(k^2) <$$

$$< \sum_{r < i \leqq 100 \log\log k^2} c_9 \frac{k^2}{\log k^2} \frac{(\log\log k^2)^{i-1}}{(i-1)!} + R(k^2) <$$

$$< c_{13} \frac{k^2}{\log k} \frac{(\log\log k^2)^r}{r!} \sum_{j=0}^{+\infty} \left(\frac{\log\log k^2}{r} \right)^j + R(k^2) <$$

$$< c_{14} \frac{k^2}{\log k} \frac{(\log\log k^2)^r}{r!} \sum_{j=0}^{+\infty} \left(\frac{1}{1+z-\delta} \right)^j + R(k^2) <$$

$$< c_{15} \frac{k^2}{\log k} \frac{(\log\log k^2)^r}{r!} + R(k^2)$$

where

$$R(x) = \sum_{100 \log\log x < i} \pi_i(x).$$

Applying Stirling's formula, we obtain that for $k > k_4(\delta)$,

$$\frac{k^2}{\log k} \frac{(\log\log k^2)^r}{r!} < \tag{30}$$

$$< c_{16} \frac{k^2}{\log k} \frac{(\log\log k^2)^{[(1+z-\delta)\log\log k^2]}}{([(1+z-\delta)\log\log k^2])^{[(1+z-\delta)\log\log k^2]+1/2} e^{-[(1+z-\delta)\log\log k^2]}} <$$

$$< c_{17} \frac{k^2}{\log k} \frac{(\log\log k^2)^{[(1+z-\delta)\log\log k^2]}}{((1+z-\delta)\log\log k^2)^{[(1+z-\delta)\log\log k^2]+1/2} e^{-(1+z-\delta)\log\log k}} <$$

$$< c_{18} \frac{k^2}{\log k} \frac{1}{e^{(1+z-\delta)\log(1+z-\delta)\log\log k} (\log\log k)^{1/2} (\log k)^{-(1+z-\delta)}} <$$

$$< c_{18} \frac{k^2}{(\log k)^{\varphi(1+z-\delta)}}.$$

The function $\varphi(x)$ is continuous at $x=1+z$. Thus if δ is sufficiently small in terms of ε then for $k>k_5(\delta)=k_5(\delta(\varepsilon))=k_6(\varepsilon)$, we obtain from (30) that

(31) $$\frac{k^2}{\log k}\frac{(\log\log k^2)^r}{r!} < \frac{k^2}{(\log k)^{\varphi(1+z)-\varepsilon/3}} = \frac{k^2}{(\log k)^{c_8-\varepsilon/3}}$$

(since $\varphi(1+z)=\varphi(z)=c_8$ by the definition of z).

Furthermore, P. ERDŐS proved in [1] (see formulae (5) and (6)) that for large x,

(32) $$R(x) < 2\frac{x}{(\log x)^2}.$$

(29), (31) and (32) yield that the number of the distinct products of form da_j is at most

(33) $$\sum_{(1+z)\log\log k<i} \pi_i(k^2) < c_{15}\frac{k^2}{(\log k)^{c_8-\varepsilon/3}} + 2\frac{k^2}{(\log k^2)^2} < c_{19}\frac{k^2}{(\log k)^{c_8-\varepsilon/3}}.$$

Finally, by (17), α in (28) can be chosen in m ways. Thus with respect to (20), we obtain that the number of the distinct products of form (28) is less than

$$m\cdot c_{19}\frac{k^2}{(\log k)^{c_8-\varepsilon/3}} < n\frac{k}{(\log k)^{c_8-\varepsilon/2}}.$$

Case 4. Assume that $a_i=p^\alpha d_1\in E_k$, $a_j=p^\beta d_2\in E_k$ where

(34) $$1\leqq \alpha,\ \beta\leqq m$$

and

(35) $$d_1,\ d_2\in D_k.$$

Then the product a_ia_j can be written in form

(36) $$a_ia_j = (p^\alpha d_1)(p^\beta d_2) = p^{\alpha+\beta}d_1d_2 = p^\gamma d$$

where by (34) and (35),

(37) $$2\leqq\gamma\leqq 2m$$

and

(38) $$d = d_1d_2 \leqq k\cdot k = k^2,\quad \nu(d) = \nu(d_1)+\nu(d_2) > 2\log\log k.$$

By (37), γ can be chosen in at most $2m-1<2m$ ways, while in view of (33), at most

$$\sum_{2\log\log k<i}\pi_i(k^2) < \sum_{(1+z)\log\log k<i}\pi_i(k^2) < c_{19}\frac{k^2}{(\log k)^{c_8-\varepsilon/3}}$$

integers d satisfy (38). Thus the number of the distinct products a_ia_j of form (36) is less than

$$2m\cdot c_{19}\frac{k^2}{(\log k)^{c_8-\varepsilon/3}} < n\frac{k}{(\log k)^{c_8-\varepsilon/2}}.$$

Summarizing the results obtained above, we get that for $k > k_7(\varepsilon)$,

$$g(A, n, k) < n + 3 \cdot n \cdot \frac{k}{(\log k)^{c_8 - \varepsilon/2}} < n \cdot \frac{k}{(\log k)^{c_8 - \varepsilon}}$$

which completes the proof of Theorem 1.

3. In this section, we will estimate $F(n, k)$.

Theorem 2. *There exist absolute constants* c_{20}, c_{21} *such that for* $k > k_8$ *and* $n \geqq k$,

$$n^2 \exp\left(c_{20} \frac{k}{\log^2 k}\right) < F(n, k) < n^2 \exp\left(c_{21} \frac{k}{\log k}\right). \tag{39}$$

Proof. First we prove the upper estimate. We will show at first that

$$F(k, k) = f(B_k, k, k) < \exp\left(c_{22} \frac{k}{\log k}\right). \tag{40}$$

In case $A = B_k = \{1, 2, \ldots, k\}$ (and $n = k$), all the products of form (2) are divisors of $k!$. Thus applying Legendre's formula and the prime number theorem (or a more elementary theorem), we obtain that

$$F(k, k) \leqq d(k!) = \prod_{p \leqq k} \left(1 + \sum_{\alpha=1}^{+\infty} \left[\frac{k}{p^\alpha}\right]\right) \leqq$$

$$\leqq \prod_{p \leqq k} \left(2 \sum_{\alpha=1}^{+\infty} \left[\frac{k}{p^\alpha}\right]\right) < \prod_{p \leqq k} \left(\sum_{\alpha=1}^{+\infty} \frac{2k}{p^\alpha}\right) = \prod_{p \leqq k} \frac{2k}{p-1} \leqq \prod_{p \leqq k} \frac{4k}{p} =$$

$$= \prod_{j=1}^{\left[\frac{\log k}{\log 2}\right]} \prod_{\frac{k}{2^j} < p \leqq \frac{k}{2^{j-1}}} \frac{4k}{p} < \prod_{j=1}^{\left[\frac{\log k}{\log 2}\right]} \prod_{\frac{k}{2^j} < p \leqq \frac{k}{2^{j-1}}} 4k \cdot \frac{2^j}{k} \leqq$$

$$\leqq \prod_{j=1}^{\left[\frac{\log k}{\log 2}\right]} (4 \cdot 2^j)^{\pi\left(\frac{k}{2^{j-1}}\right)} < \exp\left\{c_{23} \left(\sum_{j=1}^{\left[\frac{\log k}{\log 2}\right]} \frac{k}{2^{j-1}} \cdot \frac{1}{\log \frac{k}{2^{j-1}}} \cdot \log 4 \cdot 2^j\right)\right\} <$$

$$< \exp\left\{c_{24} \left(\sum_{j=1}^{\left[\frac{1}{2} \cdot \frac{\log k}{\log 2}\right]} \frac{k}{2^j} \cdot \frac{1}{\log \sqrt{k}} \cdot j + \sum_{j=\left[\frac{1}{2} \cdot \frac{\log k}{\log 2}\right]+1}^{\left[\frac{\log k}{\log 2}\right]} \frac{k}{2^j} \cdot j\right)\right\} <$$

$$< \exp\left\{c_{25} \left(\frac{k}{\log k} + \sqrt{k}\right)\right\} < \exp\left(c_{26} \frac{k}{\log k}\right)$$

which proves (40).

Assume now that $n > k$. Let p denote a prime number satisfying $p > k$ and let

$$A = \{1, 2, \ldots, k, p, p^2, \ldots, p^{n-k}\}.$$

For this sequence A, $|A|=n$, and the products (2) can be written in form

$$\prod_{i=1}^{k} i^{\varepsilon_i} \prod_{j=1}^{n-k} p^{j\delta_j} = a \cdot p^{\beta} \tag{41}$$

where $\varepsilon_i=0$ or 1 and $\delta_j=0$ or 1. Here a may assume $F(k, k)$ different values, and obviously, β may assume any integer value (independently of α) from the interval

$$0 \leqq \alpha \leqq \sum_{j=1}^{n-k} 1 = \frac{(n-k)(n-k+1)}{2}$$

of length $\frac{(n-k)(n-k+1)}{2}$. Furthermore, the prime factors of a are less than p, thus for different pairs a, β, we obtain different products of form (41). Thus with respect to (40),

$$F(n, k) \leqq f(A, n, k) = F(k, k) \cdot \frac{(n-k)(n-k+1)}{2} <$$

$$< \exp\left(c_{22}\frac{k}{\log k}\right)\cdot\frac{n^2}{2} < n^2 \exp\left(c_{22}\frac{k}{\log k}\right)$$

which completes the proof of the second inequality in (39).

Now we are going to prove that the first inequality in (39) holds with $c_{20}=\frac{1}{92}$, in other words,

$$F(n, k) > n^2 \exp\left(\frac{1}{92}\frac{k}{\log^2 k}\right). \tag{42}$$

Let us assume at first that

$$n \leqq \exp\left(\frac{1}{3}\frac{k}{\log k}\right).$$

Then for large k, the right hand side of (42):

$$\begin{aligned} n^2 \exp\left(\frac{1}{92}\frac{k}{\log^2 k}\right) &\leqq \exp\left(\frac{2}{3}\frac{k}{\log k}+\frac{1}{92}\frac{k}{\log^2 k}\right) < \\ &< \exp\left(\frac{2}{3}\frac{k}{\log k}+\frac{1}{100}\frac{k}{\log k}\right) = \exp\left(\frac{68}{100}\frac{k}{\log k}\right). \end{aligned} \tag{43}$$

On the other hand, let A denote any sequence satisfying (1). Let us form all those products of form (2) for which

$$\varepsilon_i = \begin{cases} 0 & \text{or 1 if } a_i \text{ is a prime numbes and } a_i \leqq k, \\ 0 & \text{otherwise.} \end{cases}$$

By (1), A contains all the $\pi(k)$ prime numbers $p \leqq k$, thus the number of these products is $2^{\pi(k)}$. Hence, by the prime number theorem, we have

$$(F(n,k)\geqq)f(A,n,k)\geqq 2^{\pi(k)}=\exp(\log 2\pi(k))> \tag{44}$$

$$>\exp\left(\frac{69}{100}\pi(k)\right)>\exp\left(\frac{68}{100}\frac{k}{\log k}\right).$$

(43) and (44) yield (42) in this case.

Let us assume now that

$$n>\exp\left(\frac{1}{3}\frac{k}{\log k}\right). \tag{45}$$

Let

$$l=\left[\frac{1}{7}\frac{k}{\log^2 k}\right].$$

Denote the i^{th} prime number by p_i ($p_1=2, p_2=3, \ldots$) and let $q_i=p_{i+1}$ for $i=1,2,\ldots,l$, $Q=\{q_1,q_2,\ldots,q_l\}$, $R=\{q_1,2q_1,q_2,2q_2,\ldots,q_l,2q_l\}$. Obviously, (45) implies that $R\subset\{a_1,a_2,\ldots,a_{[n/2]}\}$. Let us define the sequence $E=\{e_1,e_2,\ldots,e_m\}$ by

$$\{a_1,a_2,\ldots,a_{[n/2]}\}=E\cup R,\quad E\cap R=\emptyset.$$

For $s=1,2,\ldots,\left[\frac{n}{4}\right]+1$, we denote the interval $[n-2[n/4]-1+2s, n]$ by I_s, and let F_s denote the set of those products of form (2) for which

$$\varepsilon_i=0 \quad \text{if} \quad a_i\in R,\quad \sum_{i:a_i\in B}\varepsilon_i=2,$$

$$\varepsilon_i=0 \quad \text{if} \quad \left[\frac{n}{2}\right]<i\leqq n-2[n/4]-2+2s,$$

and

$$\varepsilon_i=1 \quad \text{if} \quad i\in I_s \quad (\text{i.e. } n-2[n/4]-1+2s\leqq i\leqq n).$$

In other words, F_s denotes the set of those numbers which can be written in form

$$\Big(\prod_{\mu\in I_s}a_\mu\Big)\cdot e_ie_j$$

where $1\leqq i$, $j\leqq m$, $i\neq j$. Let F denote the set of those numbers which can be written in form

$$e_ie_j \quad \text{where} \quad 1\leqq i,\quad j\leqq m,\quad i\neq j.$$

Then obviously,

$$|F_s|=|F|, \tag{46}$$

independently of s.

Furthermore, for $s=1, 2, \ldots, \left[\frac{n}{4}\right]+1$, let G_s denote the set of those products of form (2) for which

$$\varepsilon_i = 0 \text{ or } 1 \quad \text{if} \quad a_i \in R, \quad \sum_{i: a_i \in B} \varepsilon_i = 1,$$

$$\varepsilon_i = 0 \quad \text{if} \quad \left[\frac{n}{2}\right] < i \leqq n-2[n/4]-2+2s$$

and

$$\varepsilon_i = 1 \quad \text{if} \quad i \in I_s \quad (\text{i.e. } n-2[n/4]-1+2s \leqq i \leqq n).$$

In other words, G_s denotes the set of those numbers which can be written in form

$$\left(\prod_{\mu \in I_s} a_\mu\right) \cdot e_i \prod_{j=1}^{l} q_j^{\varepsilon_j} \prod_{t=1}^{l} (2q_t)^{\varphi_t}$$

(where $\varepsilon_j = 0$ or 1, $\varphi_t = 0$ or 1). Then $|G_s|$ is equal to the number of the products of form

$$e_i \prod_{j=1}^{l} q_j^{\varepsilon_j} \prod_{t=1}^{l} (2q_t)^{\varphi_t} = 2^\alpha e_i \prod_{j=1}^{l} q_j^{\delta_j} \tag{47}$$

where

$$\delta_j = 0, 1 \quad \text{or} \quad 2 \tag{48}$$

and

$$0 \leqq \alpha \leqq l. \tag{49}$$

Let G denote the set of those numbers which can be written in form

$$e_i \prod_{j=1}^{l} q_j^{\delta_j}$$

where (48) holds. Obviously, for any product of this form, there exist exponents ε_j, φ_t and α, satisfying (47), (49), $\varepsilon_j = 0$ or 1 and $\varphi_t = 0$ or 1. A product of form (47) can be obtained from at most $l+1$ distinct elements of G; namely, by (49), α may assume only at most $l+1$ distinct values. Thus

$$|G_s| \geqq \frac{|G|}{l+1} \tag{50}$$

(again, independently of s).

We are going to show that for $s \neq t$,

$$(F_s \cup G_s) \cap (F_t \cup G_t) = \emptyset. \tag{51}$$

In fact, assume that $s > t$. Then for $y \in F_t \cup G_t$,

$$\begin{aligned} y \geqq \prod_{\mu \in I_t} a_\mu &= \prod_{n-2[n/4]-1+2t \leqq \mu < n-2[n/4]-1+2s} a_\mu \cdot \prod_{\mu \in I_s} a_\mu \geqq \\ &\geqq a_{n-2[n/4]-1+2t} a_{n-2[n/4]+2t} \cdot \prod_{\mu \in I_s} a_\mu > (a_{[n/2]})^2 \prod_{\mu \in I_s} a_\mu \quad (\text{for } y \in F_t \cup G_t). \end{aligned} \tag{52}$$

On the other hand, for $z\in F_s$,

$$z = e_i e_j \prod_{\mu\in I_s} a_\mu \leqq (a_{[n/2]})^2 \prod_{\mu\in I_s} a_\mu \quad (\text{for } z\in F_s). \tag{53}$$

Finally, if $v\in G_t$, then we have

$$v \leqq e_i \prod_{j=1}^{l} q_j \prod_{t=1}^{l} 2q_t \cdot \prod_{\mu\in I_s} a_\mu \leqq a_{[n/2]}\cdot 2^l \left(\prod_{j=1}^{l} q_j\right)^2 \cdot \prod_{\mu\in I_s} a_\mu . \tag{54}$$

By the prime number theorem,

$$\log\left(\prod_{i=1}^{x} p_i\right) \sim x\log x.$$

Thus if k (and consequently l) are sufficiently large then with respect to (45) we have

$$2^l\left(\prod_{j=1}^{l} q_j\right)^2 = 2^l\left(\prod_{i=2}^{l+1} p_i\right)^2 < 2^l\left(\exp\left\{\frac{35}{34}(l+1)\log(l+1)\right\}\right)^2 <$$

$$< \exp\left(\frac{1}{7}\frac{k}{\log^2 k}\cdot\log 2\right)\exp\left\{\frac{35}{17}\left(\frac{1}{7}\frac{k}{\log^2 k}+1\right)\log\left(\frac{1}{7}\frac{k}{\log^2 k}+1\right)\right\} <$$

$$< \exp\left(\frac{k}{\log^2 k}\right)\exp\left(\frac{5}{16}\frac{k}{\log^2 k}\log k\right) =$$

$$= \exp\left(\frac{k}{\log^2 k}+\frac{5}{16}\frac{k}{\log k}\right) < \frac{1}{3}\exp\left(\frac{5}{15}\frac{k}{\log k}\right) < \frac{1}{3}n < \left[\frac{n}{2}\right] \leqq a_{[n/2]}.$$

Putting this into (54), we obtain that

$$v \leqq (a_{[n/2]})^2 \prod_{\mu\in I_s} a_\mu \quad (\text{for } v\in G_s); \tag{55}$$

(52), (53) and (55) yield (51).

By (46), (50) and (51), we have

$$f(A, n, k) \geqq \left|\bigcup_{s=1}^{[n/4]+1} (F_s\cup G_s)\right| = \sum_{s=1}^{[n/4]+1} |F_s\cup G_s| \geqq \tag{56}$$

$$\geqq \sum_{s=1}^{[n/4]+1} \max\{|F_s|, |G_s|\} \geqq \sum_{s=1}^{[n/4]+1} \max\left\{|F|, \frac{|G|}{l+1}\right\} =$$

$$= ([n/4]+1)\max\left\{|F|, \frac{|G|}{l+1}\right\} > \frac{n}{4}\frac{1}{l+1}\max\{|F|, |G|\}.$$

Thus to complete the proof of Theorem 2, we need a lower estimate for $\max\{|F|, |G|\}$. In the next section, we will prove the following lemma (using the same method as in [4]):

Lemma 1. *Let* $Q=\{q_1, q_2, \dots, q_l\}$ *be any set consisting of* l *(distinct) prime numbers. Let* $E=\{e_1, e_2, \dots, e_m\}$ *(where* $e_1<e_2<\dots<e_m$*) be any sequence of positive*

integers. Let F and G denote the sets consisting of those integers which can be respectively written in form

$$e_i e_j \quad (1 \leqq i, j \leqq m,\ i \neq j) \quad \text{and} \quad e_i \prod_{j=1}^{l} q_j^{\delta_j} \quad (\delta_j = 0, 1 \text{ or } 2).$$

Then for

$$l > l_0, \tag{57}$$

we have

$$\max\{|F|, |G|\} > m \exp\left(\frac{2}{25} l\right). \tag{58}$$

Let us suppose now that Lemma 1 has been proved. Then the proof of Theorem 2 can be completed in the following way:

For large k, (57) holds by the definition of l. Thus we may apply Lemma 1. We obtain that (58) holds. Putting this into (56), we get that for large k and any sequence A (satisfying (1) and $|A|=n$),

$$f(A, n, k) > \frac{n}{4} \frac{1}{l+1} m \exp\left(\frac{2}{25} l\right). \tag{59}$$

With respect to (45),

$$m = |E| = [n/2] - |R| = [n/2] - 2l = \left[\frac{n}{2}\right] - 2\left[\frac{1}{7} \frac{k}{\log^2 k}\right] >$$

$$> \frac{n}{3} - \frac{2}{7} \frac{k}{\log^2 k} > \frac{n}{3} - \frac{1}{3} \frac{k}{\log k} > \frac{n}{3} - \log n > \frac{n}{4}.$$

Thus we obtain from (59) that for large k,

$$f(A, n, k) > \frac{n}{4} \frac{1}{l+1} \frac{n}{4} \exp\left(\frac{2}{25} l\right) > \frac{n^2}{16} \exp\left(\frac{2}{26} l\right) =$$

$$= \frac{n^2}{16} \exp\left\{\frac{1}{13}\left[\frac{1}{7} \frac{k}{\log^2 k}\right]\right\} > n^2 \exp\left(\frac{1}{92} \frac{k}{\log^2 k}\right)$$

which proves (42) and thus also Theorem 2.

4. To complete the proof of Theorem 2, we still have to give a

Proof of lemma 1. Let us write every $e \in E$ in form

$$e = (rs^2)(q_1^{\varepsilon_1} q_2^{\varepsilon_2} \dots q_l^{\varepsilon_l}) = bd \tag{60}$$

where r, s are positive integers, $\varepsilon_i = 0$ or 1 (for $i = 1, 2, \dots, l$), p/r implies that $p \notin Q$, p/s implies that $p \in Q$ (also $r=1$ and $s=1$ may occur) and $b = rs^2$, $d = q_1^{\varepsilon_1} q_2^{\varepsilon_2} \dots q_l^{\varepsilon_l}$. Let us denote the occuring values of b by $b_1, b_2, \dots, b_z$ ($b_i \neq b_j$

for $i \neq j$), let $B = \{b_1, b_2, \ldots, b_z\}$ and let us denote the set of those numbers $e \in E$ for which $b = b_i$ in (60) (for fixed i, $1 \leqq i \leqq z$), by $E(b_i)$. Then obviously,

$$E = \bigcup_{i=1}^{z} E(b_i) \quad \text{and} \quad E(b_i) \cap E(b_j) = \emptyset \quad \text{for} \quad i \neq j,$$

thus

$$m = |E| = \sum_{i=1}^{z} |E(b_i)|. \tag{61}$$

For $b \in B$, let $F(b)$ denote the set of those numbers which can be written in form

$$e_x e_y \quad \text{where} \quad e_x \in E(b), \quad e_y \in E(b), \quad e_x \neq e_y.$$

Furthermore, for fixed $b \in B$ and for each $e_x = b q_1^{\varepsilon_1} q_2^{\varepsilon_2} \ldots q_l^{\varepsilon_l}$, let us form all the products of form

$$e_x (q_1^{\gamma_1} q_2^{\gamma_2} \ldots q_l^{\gamma_l}) = (b q_1^{\varepsilon_1} q_2^{\varepsilon_2} \ldots q_l^{\varepsilon_l})(q_1^{\gamma_1} q_2^{\gamma_2} \ldots q_l^{\gamma_l}) \tag{62}$$

where

$$\gamma_i = \begin{cases} 0 \text{ or } 1 & \text{if } \varepsilon_i = 1 \\ 1 \text{ or } 2 & \text{if } \varepsilon_i = 0 \end{cases}$$

and let us denote the set of these products by $G(b)$.

Obviously,

$$F \supset \bigcup_{i=1}^{z} F(b_i) \tag{63}$$

and

$$G \supset \bigcup_{i=1}^{z} G(b_i). \tag{64}$$

We are going to show that

$$F(b_i) \cap F(b_j) = \emptyset \quad \text{for} \quad i \neq j \tag{65}$$

and

$$G(b_i) \cap G(b_j) = \emptyset \quad \text{for} \quad i \neq j. \tag{66}$$

In fact, let us assume that

$$b_i = r_i s_i^2 \neq b_j = r_j s_j^2, \tag{67}$$

$$e_x = b_i q_1^{\varepsilon_1} q_2^{\varepsilon_2} \ldots q_l^{\varepsilon_l} \in E(b_i), \quad e_y = b_i q_1^{\varphi_1} q_2^{\varphi_2} \ldots q_l^{\varphi_l} \in E(b_i),$$

$$e_u = b_j q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_l^{\alpha_l} \in E(b_j) \quad \text{and} \quad e_v = b_j q_1^{\beta_1} q_2^{\beta_2} \ldots q_l^{\beta_l} \in E(b_j).$$

Then

$$e_x e_y = r_i^2 s_i^4 q_1^{\varepsilon_1 + \varphi_1} q_2^{\varepsilon_2 + \varphi_2} \ldots q_l^{\varepsilon_l + \varphi_l} \quad (\in F(b_i)) \tag{68}$$

and

$$e_u e_v = r_j^2 s_j^4 q_1^{\alpha_1 + \beta_1} q_2^{\alpha_2 + \beta_2} \ldots q_l^{\alpha_l + \beta_l} \quad (\in F(b_j)). \tag{69}$$

If $r_i \neq r_j$ then there exists a prime power p^γ such that $p \notin Q$ and $p^\gamma / e_x e_y$ but $p^\gamma \nmid e_x e_y$, or conversely; this implies that $e_x e_y \neq e_u e_v$. If $r_i = r_j$ then by (67), $s_i \neq s_j$ must hold. Thus there exists a prime power q_t^μ such that $q_t \in Q$ and q_t^μ / s_i but $q_t^\mu \nmid s_j$ (or conversely). Then the exponent of q_t is at least $4\mu + \varepsilon_i + \varphi_i \geqq 4\mu$ in the canonical form of $e_x e_y$ and at most $4(\mu - 1) + \alpha_i + \beta_i \leqq 4\mu - 2$ in the canonical form of $e_u e_v$, thus $e_x e_y \neq e_u e_v$ holds also in this case, which proves (65).

In order to prove (66), note that we may write the product (62) in form

$$r(s^2 q_1 q_2 \dots q_l) q_1^{\alpha_1} q_2^{\alpha_2} \dots q_l^{\alpha_l} \quad \text{where} \quad \alpha_i = 0 \text{ or } 1 \quad \text{for} \quad i = 1, 2, \dots, l.$$

Obviously, a number of this form uniquely determines each of the factors $r, s, q_1^{\alpha_1}, \dots, q_l^{\alpha_l}$, which proves (66).

(63), (64), (65) and (66) imply that

$$\max\{|F|, |G|\} \geqq \max\left\{\left|\bigcup_{i=1}^{z} F(b_i)\right|, \left|\bigcup_{i=1}^{z} G(b_i)\right|\right\} = \tag{70}$$

$$= \max\left\{\sum_{i=1}^{z} |F(b_i)|; \sum_{i=1}^{z} |G(b_i)|\right\} \geqq \frac{1}{2}\left(\sum_{i=1}^{z} |F(b_i)| + \sum_{i=1}^{z} |G(b_i)|\right) =$$

$$= \frac{1}{2} \sum_{i=1}^{z} \left(|F(b_i)| + |G(b_i)|\right) \geqq \frac{1}{2} \sum_{i=1}^{z} \max\{|F(b_i)|, |G(b_i)|\}.$$

Thus in order to prove (58), it suffices to show that for $b \in B$, $\max\{|F(b)|, |G(b)|\}$ is large.

Let us assume that $b \in B$. We have to distinguish two cases.

Case 1:

$$(0<)\,|E(b)| \leqq 2^{\frac{7}{8}l-1}. \tag{71}$$

We are going to show that in this case $|G(b)|$ is large (in terms of $|E(b)|$). Let us fix an element e_x of $E(b)$ and for this e_x, form all the products of form (62). Obviously, the factor $q_1^{\gamma_1} q_2^{\gamma_2} \dots q_l^{\gamma_l}$ can be chosen in 2^l ways thus the number of these products is 2^l. Hence, with respect to (71),

$$|G(b)| \geqq 2^l = 2^{\frac{1}{8}l+1} \cdot 2^{\frac{7}{8}l-1} = 2^{\frac{1}{8}l+1} |E(b)|. \tag{72}$$

Case 2:

$$|E(b)| > 2^{\frac{7}{8}l-1}. \tag{73}$$

In this case, we shall need the following lemma:

Lemma 2. *Let ϱ be any real number, satisfying*

$$0 < \varrho < \frac{1}{2} \tag{74}$$

and

(75) $$f(\varrho) \stackrel{\text{def}}{=} -\varrho\log\varrho-(1-\varrho)\log(1-\varrho)-\left(1-\frac{\varrho}{2}\right)\log 2<0,$$

and let l be any integer, sufficiently large depending on ϱ:

(76) $$l>l_1(\varrho).$$

Put

$$\varphi(l)=2^{-\frac{\varrho}{2}l-1}.$$

Let S denote the set of the 2^l l-tuples $(\mu_1, \mu_2, \dots, \mu_l)$, satisfying $\mu_h=0$ or 1 for $h=1, 2, \dots, l$. Let R be any subset of S for which

(77) $$|R|>\varphi(l)2^l.$$

Then the number of the distinct sums of form

(78) $$(\mu_1+\nu_1, \dots, \mu_l+\nu_l)=(\mu_1, \dots, \mu_l)+(\nu_1, \dots, \nu_l),$$

where $(\mu_1, \dots, \mu_l)\in R$ and $(\nu_1, \dots, \nu_l)\in R$, is greater than $(\varphi(l))^{-1}|R|$.

This lemma is identical with Lemma 2 in [4].

Using Lemma 2, we are going to show that (73) implies that $|F(b)|$ is large. Let us choose $\varrho=\dfrac{1}{4}$ in Lemma 2. Then (74) holds trivially, and a simple computation shows that

$$f\left(\frac{1}{4}\right)=\frac{3}{8}(\log 8-\log 9)<0,$$

thus ϱ satisfies also (75). Furthermore, we choose R as the set of those l-tuples $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_l)$ (where $\varepsilon_i=0$ or 1) for which $bq_1^{\varepsilon_1}q_2^{\varepsilon_2}\dots q_l^{\varepsilon_l}\in E(b)$ holds. Then by (73), also (77) holds:

$$|R|=|E(b)|>2^{\frac{7}{8}l-1}=2^{-\frac{1}{8}l-1}\cdot 2^l=\varphi(l)2^l.$$

Thus we may apply Lemma 2. We obtain that the number of the distinct sums of form (78) $\big($where $(\mu_1, \dots, \mu_l)\in R$ and $(\nu_1, \dots, \nu_l)\in R\big)$ is greater than $(\varphi(l))^{-1}|R|$. But distinct sums of form (78) determine distinct products of form

$$e_x e_y=(bq_1^{\mu_1}\dots q_l^{\mu_l})(bq_1^{\nu_1}\dots q_l^{\nu_l})=b^2q_1^{\mu_1+\nu_1}\dots q_l^{\mu_l+\nu_l},$$

and with at most $|E(b)|$ exception, also $e_x\neq e_y$ holds. Thus

(79) $$|F(b)|>(\varphi(l))^{-1}|R|-|E(b)|=\left(2^{-\frac{1}{8}l-1}\right)^{-1}|E(b)|-|E(b)|=$$
$$=\left(2^{\frac{1}{8}l+1}-1\right)|E(b)|>2^{\frac{1}{8}l}|E(b)|.$$

(72) and (79) yield that for any $b\in B$,

$$\max\{|F(b)|, |G(b)|\}>2^{\frac{1}{8}l}|E(b)|.$$

Putting this into (70), we obtain (with respect to (61)) that

$$\max\{|F|,|G|\} \geqq \frac{1}{2}\sum_{i=1}^{z}\max\{|F(b_i)|,|G(b_i)|\} >$$

$$> \frac{1}{2}\sum_{i=1}^{z} 2^{\frac{1}{8}l}|E(b_i)| = 2^{\frac{1}{8}l-1}\sum_{i=1}^{z}|E(b_i)| = m2^{\frac{1}{8}l-1} =$$

$$= m\exp\left\{\log 2\left(\frac{1}{8}l-1\right)\right\} > m\exp\left\{\left(\frac{\log 2}{8}-\frac{1}{1000}\right)l\right\} > m\exp\left(\frac{2}{25}l\right)$$

which completes the proof of Lemma 1.

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A note on congruence extension property

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Congruence extension property (further on CEP) is very useful when trying to find identities for equational classes (see e.g. [3]). However, CEP is not an equational property. Therefore one has to prove it for all members of a given equational class. The only simplification is that subalgebra preserves CEP. The actual aim of this paper is to give an example to prove that homomorphic image does not preserve CEP. We shall put down some other ideas in this topic, too.

We shall use the terminology of [5]. We say that CEP holds for $(\mathbf{A}, \mathbf{B}, \theta)$ if $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, θ is a congruence relation on $\mathbf{B}$ and there exists a congruence relation θ' on $\mathbf{A}$ the restriction of which to $\mathbf{B}$ coincides with θ. The pair $(\mathbf{A}, \mathbf{B})$ satisfies CEP exactly if $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and for each congruence relation θ on $\mathbf{B}$ CEP holds for $(\mathbf{A}, \mathbf{B}, \theta)$. An algebra $\mathbf{A}$ satisfies CEP iff for all subalgebras $\mathbf{B}$ of $\mathbf{A}$ $(\mathbf{A}, \mathbf{B})$ satisfy CEP. CEP holds for a class of algebras iff each element of this class satisfies CEP.

We define *strong congruence extension property* (further on SCEP) as follows: A quadruple $(\mathbf{A}, \mathbf{B}, \theta, \Phi')$ satisfies SCEP iff $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, θ is a congruence relation on $\mathbf{B}$, Φ' is a congruence relation on $\mathbf{A}$ the restriction of which to $\mathbf{B}$ is contained in θ and there exists a congruence relation θ' on $\mathbf{A}$ containing Φ' the restriction of which to $\mathbf{B}$ coincides with θ. $(\mathbf{A}, \mathbf{B}, \theta)$ satisfies SCEP iff for each congruence relation Φ' on $\mathbf{A}$ the quadruple $(\mathbf{A}, \mathbf{B}, \theta, \Phi')$ satisfies SCEP, provided $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and the restriction of Φ' to $\mathbf{B}$ is contained in the congruence relation θ of $\mathbf{B}$. We define that a pair $(\mathbf{A}, \mathbf{B})$, an algebra $\mathbf{A}$ and a class of algebras satisfy SCEP as we have defined that for CEP substituting, everywhere, CEP by SCEP, respectively.

Proposition 1. *An algebra* $\mathbf{A}$ *has* SCEP *iff all homomorphic images of* $\mathbf{A}$ *have* CEP.

Proof. Let $\bar{\mathbf{A}}$ be the homomorphic image of $\mathbf{A}$ under the homomorphism φ, $\bar{\mathbf{B}}$ a subalgebra of $\bar{\mathbf{A}}$ and $\bar{\theta}$ a congruence relation on $\bar{\mathbf{B}}$. We define $\mathbf{B}=\varphi^{-1}(\bar{\mathbf{B}})$,

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$\theta=\varphi^{-1}(\bar{\theta})$ and $\Phi'=\mathrm{Ker}\,\varphi$. SCEP for $(\mathbf{A}, \mathbf{B}, \theta, \Phi')$ yields CEP for $(\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\theta})$. On the other hand, for given $\mathbf{A}, \mathbf{B}, \theta$ and Φ' CEP for $(\mathbf{A}/\Phi', \mathbf{B}/\Phi', \theta/\Phi')$ implies SCEP for $(\mathbf{A}, \mathbf{B}, \theta, \Phi')$.

Now, we are going to show that SCEP is "more" equational then CEP.

Proposition 2. *If a class K has* SCEP *so does* $HS(K)$.

Proof. We have to prove that if $\mathbf{A}$ satisfies SCEP so do all homomorphic images and subalgebras of $\mathbf{A}$. Proposition 1 takes care of homomorphic images. Now, let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$, $\mathbf{C}$ be a subalgebra of $\mathbf{B}$, Φ' a congruence relation on $\mathbf{B}$ with the restriction Φ to $\mathbf{C}$ and $\theta\geqq\Phi$ be a congruence relation on $\mathbf{C}$. Since CEP holds for $(\mathbf{A}, \mathbf{B}, \Phi')$ there exists a congruence relation Φ'' on $\mathbf{A}$ the restriction of which to $\mathbf{C}$ coincides with Φ. Then SCEP for $(\mathbf{A}, \mathbf{C}, \theta, \Phi'')$ gives us that there exists a congruence relation $\theta''\geqq\Phi''$ on $\mathbf{A}$ the restriction of which to $\mathbf{C}$ equals θ. Hence, for the restriction θ' of θ to $\mathbf{B}$ we have $\theta'\geqq\Phi'$ and the restriction of θ' to $\mathbf{C}$ equals θ proving that $(\mathbf{B}, \mathbf{C}, \theta, \Phi')$ satisfies SCEP.

The next proposition describes a typical situation when SCEP holds.

Proposition 3. *Let* $\mathbf{B}$ *be a subalgebra of* $\mathbf{A}$, Φ' *a congruence relation on* $\mathbf{A}$ *with the restriction* Φ *to* $\mathbf{B}$ *and* $\theta\geqq\Phi$ *a congruence relation on* $\mathbf{B}$. *If each congruence class of* Φ', *containes a (nonempty) class of* Φ, *then* CEP *for* $(\mathbf{A}, \mathbf{B}, \theta)$ *implies* SCEP *for* $(\mathbf{A}, \mathbf{B}, \theta, \Phi')$.

Proof. Actually, CEP for $(\mathbf{A}, \mathbf{B}, \theta)$ is not needed; the statement is an obvious consequence of the first isomorphism theorem.

Theorem. CEP *does not imply* SCEP.

We shall prove the statement by giving an example. The method used can give examples for $4n$-elements algebras with integers n greater than 1. The situation is somewhat more complicated if n is not a prime. The choice $n=2$ has the advantage that there exists a field with eight elements, thus, we can express the functions by the operations of the field, for finite fields are functionally complete.

Example. Let a be a root of the polynomial x^3+x+1 over the two elements field Q_2 and we denote the underlying set of $Q_2(a)$ by A. We define the following algebra:

$$\mathbf{A} = \langle A|0, f, g, p, F\rangle$$

where the operations are given as follows: 0 is a nullary operation assigning the zero element 0; f, g and p are unary operations defined by $f(x)=x+1$, $g(x)=x+a$, $p(x)=x^7$; F is a binary operation defined by $F(x,y)=(x^7+1)\left(a^2((ay)^4+(ay)^2+(ay))\right)$. The elements 0, $f(0)=1$, $g(0)=a$ and $f(g(0))=a+1$ are constants. We denote $B=\{0, 1, a, a+1\}$.

The underlying set of the only proper subalgebra **B** of **A** is B. The only non-trivial congruence θ of **B** consists of the cosets $\{0, 1\}$ and $\{a, a+1\}$. The congruence θ' of **A** consisting of the cosets $\{0, 1, a^2, a^2+1\}$ and $\{a, a+1, a^2+a, a^2+a+1\}$ is an extension of θ. Hence, CEP holds for **A**.

The cosets $U=\{a^2, a^2+a\}$, $V=\{a^2+1, a^2+a+1\}$ and all the singletons disjoint to both U and V form a congruence Φ' of **A** the restriction of which to **B** is ω. Since $\theta' \vee \Phi' = \iota$ we have $\theta' \not\equiv \Phi'$. Thus, $(\mathbf{A}, \mathbf{B}, \theta, \Phi')$, hence **A** do not satisfy SCEP. The details are left to the reader.

Remark. We are going to list some straightforward properties of SCEP to show how close it is to equational classes.

1) Proposition 1 shows that for an equational class CEP implies SCEP.

2) If an element of an equational class has SCEP so do its subdirect irreducible components.

3) If each direct product of subdirect irreducible elements of an equational class has SCEP so does the whole equational class.

4) If each direct product of some subdirect irreducible elements of a congruence distributive equational class has SCEP so does the equational class they generate (comp. [6]).

Problem. Let K be a class of algebras with SCEP. Prove or disprove:

a) K need not be closed under finite direct products.

b) K need not be closed under prime products.

c) Though the class K is closed both under finite direct products and prime products it need not be closed under direct products.

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On automorphism groups of subalgebras of a universal algebra

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Let A be a universal algebra and let Con (A), Sub (A), Aut (A) denote the lattice of all congruences of A, the lattice of all subalgebras of A, and the automorphism group of A, respectively. First in a series of so-called independence results is that of E. T. SCHMIDT [6] asserting that Aut (A) is independent of Sub (A). W. A. LAMPE [5] gave a construction representing any pair of nontrivial algebraic lattices and an arbitrary group as Sub (A), Con (A), and Aut (A) of a finitary algebra A.

Once these results are established, somewhat more detailed investigations of the structures associated with a universal algebra appear to be in order; we would like to formulate further possible questions in this field. For every finitary algebra A there are two obvious homomorphisms H_1: Aut $(A) \to$ Aut (Sub (A)) and H_2: Aut $(A) \to$ Aut (Con (A)) of the respective groups. Given a quintuple (G, L_1, H_1, L_2, H_2) in which G is a group, L_1 and L_2 are algebraic lattices, and H_i: $G \to$ Aut (L_i) are group homomorphisms, one may ask under what circumstances there is an algebra A with Aut $(A) \cong G$, $L_1 \cong$ Sub (A), $L_2 \cong$ Con (A), and H_1, H_2 the two natural homomorphisms as above. [1] states that an arbitrary triple (G, L_1, H_1) is representable in this way. There appears to be no corresponding result for the triple (G, L_2, H_2).

The aim of this note is to prove a partial result concerning the relationship of the subalgebra lattice and the automorphism groups of subalgebras of a finitary algebra. It is well known that automorphism groups of pairs algebra-subalgebra can be chosen arbitrarily, and similar claim is valid for endomorphism monoids as well ([3] and [4], see also [2]). The question we ask is this: what are the systems $(G_x$: $x \in L)$ of groups appearing as automorphism groups of subalgebras of a finitary algebra A whose subalgebra lattice is isomorphic to L?

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To be more precise, let A be a finitary algebra and let

$$H_A\colon \operatorname{Aut}(A) \to \operatorname{Aut}(\operatorname{Sub}(A)) \tag{1}$$

be defined by $(H_A(\alpha))(B)=\alpha^+(B)=\{\alpha(b)\colon b\in B\}$ for $\alpha\in\operatorname{Aut}(A)$ and $B\in\operatorname{Sub}(A)$. H_A is a group homomorphism; if $\operatorname{Aut}_B(A)$ denotes the subgroup of $\operatorname{Aut}(A)$ consisting of all those automorphisms α of A for which $\alpha^+(B)=B$, then

$$\operatorname{Ker}(H_A) \subseteqq \operatorname{Aut}_B(A) \quad \text{for every} \quad B\in\operatorname{Sub}(A). \tag{2}$$

The restriction $R_{AB}(\beta)$ of a $\beta\in\operatorname{Aut}_B(A)$ to B is an automorphism of B and the mapping

$$R_{AB}\colon \operatorname{Aut}_B(A) \to \operatorname{Aut}(B) \tag{3}$$

is a group homomorphism.

We will restrict our attention to the special case

$$\operatorname{Ker}(H_B) = \operatorname{Aut}(B) \quad \text{for all} \quad B\in\operatorname{Sub}(A), \tag{4}$$

that is, it will be assumed that, for every $\alpha\in\operatorname{Aut}(B)$, α^+ acts trivially on $\operatorname{Sub}(B)$ for each $B\in\operatorname{Sub}(A)$. It follows that $\operatorname{Aut}_C(B)=\operatorname{Aut}(B)$ and thus $\operatorname{Aut}(B)$ is the domain of R_{BC} for any pair $C\subseteqq B$ of subalgebras of A.

An algebraic lattice L is isomorphic to the lattice $I(C)$ of all ideals of the join semilattice C of all non-zero compact elements of L. If $J\in I(C)\cong L\cong\operatorname{Sub}(A)$, let A_J denote the subalgebra of A corresponding to the ideal J of C; for a principal ideal $J=(c]$ write A_c instead of A_J. Recall that J is principal if and only if A_J is finitely generated and that $A_J=\bigcup(A_c\colon c\in J)$ for every $J\in I(C)$. It is easy to see that an automorphism $\alpha\colon A_J\to A_J$ acts trivially on $\operatorname{Sub}(A_J)$ if and only if

$$\alpha^+(A_c) = A_c \quad \text{for all} \quad c\in J. \tag{5}$$

Thus the restriction (4) is equivalent to (5) being valid for all $J\in I(C)$. If $c\geqq d$ is a pair of elements of C, let $R_{cd}(\alpha)$ denote the restriction of $\alpha\in\operatorname{Aut}(A_c)$ to A_d. The system of homomorphisms

$$(R_{cd}\colon \operatorname{Aut}(A_c) \to \operatorname{Aut}(A_d),\ c\geqq d \text{ in } C) \tag{6}$$

satisfies

$$\begin{aligned} R_{de}\circ R_{cd} &= R_{ce} \quad \text{for all} \quad c\geqq d\geqq e \text{ in } C,\\ R_{ee} &= \mathrm{id}_{A_e} \quad \text{for all} \quad e\in C \end{aligned} \tag{7}$$

under the restriction (4).

If $d, e\in C$, $c=d\vee e$, then $R_{cd}(\alpha)\in\operatorname{Aut}(A_d)$ and $R_{ce}(\alpha)\in\operatorname{Aut}(A_e)$; if both $R_{cd}(\alpha)$ and $R_{ce}(\alpha)$ are identity automorphisms, then α is the identity automorphism of A_c since A_c is generated by $A_d\cup A_e$. Thus $\operatorname{Aut}(A_c)$ is a subgroup of $\operatorname{Aut}(A_d)\times\operatorname{Aut}(A_e)$; in other words, $\operatorname{Ker}(R_{cd})\cap\operatorname{Ker}(R_{ce})$ is trivial whenever $c=d\vee e$ in C.

If $J \in I(C)$ is non-principal, then $A_J = \bigcup (A_c : c \in J)$ and, because of (5), each $\alpha \in \operatorname{Aut}(A_J)$ determines a system $(\alpha_c \in \operatorname{Aut}(A_c) : c \in J)$ such that $R_{cd}(\alpha_c) = \alpha_d$ whenever $c \geqq d$ belong to J. Conversely, let $(\alpha_c : c \in J)$ be a system of automorphisms $\alpha_c \in \operatorname{Aut}(A_c)$ such that $R_{cd}(\alpha_c) = \alpha_d$ for all pairs $c \geqq d$ in J. If $d, e \in J$, then $d \vee e = f \in J$ and the equality $\alpha_d(x) = R_{fd}(\alpha_f)(x) = R_{fe}(\alpha_f)(x) = \alpha_e(x)$ holds for all $x \in A_d \cap A_e$. Thus we may define a mapping $\alpha : A_J \to A_J$ by $\alpha(x) = \alpha_c(x)$ for all $x \in A_c$; it is easy to see that α is an automorphism of A_J; α is the identity automorphism if and only if all α_c are identities. $\operatorname{Aut}(A_J)$ is therefore uniquely determined by the system

$$S = (R_{cd} : c \geqq d \text{ in } J)$$

of group homomorphisms. S is closed under composition; let $R_c : \operatorname{Aut}(A_J) \to \operatorname{Aut}(A_c)$ be the homomorphism that assigns to every $\alpha \in \operatorname{Aut}(A_J)$ its restriction $\alpha_c : A_c \to A_c$. A straightforward argument shows that $\operatorname{Aut}(A_J)$ is isomorphic to the inverse limit of the diagram S with the homomorphisms R_c playing the role of projections.

Now let $L \cong I(C)$ be an algebraic lattice, let G_x be a group for every $x \in L$, and let $r_{cd} : G_c \to G_d$ be a group homomorphism for every pair $c \geqq d$ of elements of C, let r_{cc} be the identity endomorphism of G_c. We say that a system

(8) $$\Sigma = (L, (G_x : x \in L), (r_{cd} : c \geqq d \text{ in } C))$$

is *representable* if there is a finitary algebra A such that

(9) $$\operatorname{Sub}(A) \cong L,$$

(10) $$\alpha^+(A_y) = A_y \quad \text{for all} \quad y \leqq x \quad \text{and all} \quad \alpha \in \operatorname{Aut}(A_x),$$

(11) $$\operatorname{Aut}(A_x) \cong G_x \quad \text{for every} \quad x \in L,$$

(12) each r_{cd} represents the restriction homomorphism $R_{cd} : \operatorname{Aut}(A_c) \to \operatorname{Aut}(A_d)$.

The statement below characterizes representability of Σ.

Theorem. *Σ is representable if and only if*

(a) $r_{de} \circ r_{cd} = r_{ce}$ *for all* $c \geqq d \geqq e$ *in* C,

(b) $\operatorname{Ker}(r_{cd}) \cap \operatorname{Ker}(r_{ce})$ *is trivial whenever* $d \vee e = c$,

(c) *if $x \in L$ is not compact, then G_x is the inverse limit of the diagram* $(r_{cd} : x > c \geqq d,\ c, d \in C)$.

Proof. We have already seen that (a), (b), (c) are consequences of representability of Σ. To prove the converse, define an algebra A as follows: its underlying set is the disjoint union of all groups G_c for $c \in C$ and its operations are defined by the formulae below.

(13) If $g \in G_c$, define a unary operation $\bar{g}$ by

$$\bar{g}(h) = hg \quad \text{if} \quad h \in G_c;$$
$$\bar{g}(h) = h \quad \text{if} \quad h \notin G_c,$$

(14) If $c>d$ are elements of C, F_{cd} is a unary operation defined as

$$F_{cd}(h) = r_{cd}(h) \quad \text{if} \quad h\in G_c;$$
$$F_{cd}(h) = h \qquad \text{if} \quad h\notin G_c.$$

(15) A single binary operation $*$:

$$g_1 * g_2 = g \quad \text{if} \quad g_1\in G_d, \quad g_2\in G_e, \quad g\in G_c, \quad c = d\vee e, \quad r_{cd}(g) = g_1, \quad r_{ce}(g) = g_2;$$
$$g_1 * g_2 = g_1 \quad \text{otherwise.}$$

Note that (b) implies that $*$ is well-defined.

First we will show that B is a subalgebra of A if and only if B is the (disjoint) union A_I of the groups $G_c (c\in I)$ for some ideal I of C (including $I=\emptyset$); this yields (9) immediately. It is easy to see that each A_I is a subalgebra of A; conversely, if $B\in \mathrm{Sub}(A)$, set

$$I = \{c\in C\colon B\cap G_c \neq \emptyset\}.$$

If $I=\emptyset$, then $B=\emptyset$ as well; let $c\in I$ and let $h\in B\cap G_c$. If $g\in G_c$, then $h^{-1}g=k$ belongs to G_c and $\bar{k}(h)=hh^{-1}g=g\in B$. Hence $I=\{c\in C\colon G_c\subseteq B\}$. If $d\in C$ and $d<c\in I$, then $F_{cd}(1_c)=r_{cd}(1_c)=1_d$ for the unit elements $1_c\in G_c$ and $1_d\in G_d$; thus $1_d\in B$, and $d\in I$ as well. $1_c * 1_d = 1_{c\vee d}\in B$ whenever $c, d\in I$; hence I is an ideal, and $B=A_I$. A nonempty A_I is finitely generated (one-generated, in fact) if and only if I is a principal ideal.

Let $I=(c]$. $A_I=\bigcup(G_d\colon d\leqq c)$ in this case; for every $g\in G_c$ define a mapping $g^*\colon A_I\to A_I$ by $g^*(h)=r_{cd}(g)\cdot h$ for $h\in G_d$, $d\leqq c$. Observe that $(g_1g_2)^*(h)=$ $=r_{cd}(g_1g_2)\cdot h=r_{cd}(g_1)\cdot r_{cd}(g_2)\cdot h=g_1^*(r_{cd}(g_2)\cdot h)=g_1^*(g_2^*(h))$, and that g^* is the identity mapping on A_I only if $1_c=g^*(1_c)=r_{cc}(g)\cdot 1_c=g$. Hence $g\to g^*$ is a one-to-one homomorphism of G_c into the symmetric group on A_I. To show that $g^*\in \mathrm{Aut}(A_I)$, choose a $k\in G_e (e\leqq c)$ first. If $h\in G_d\subseteq A_I$, then $\bar{k}(g^*(h))=\bar{k}(r_{cd}(g)\cdot h)=$ $=r_{cd}(g)\cdot h=g^*(h)=g^*(\bar{k}(h))$ if $d\neq e$, and $\bar{k}(g^*(h))=\bar{k}(r_{cd}(g)\cdot h)=r_{cd}(g)\cdot h\cdot k=$ $=g^*(h\cdot k)=g^*(\bar{k}(h))$ if $d=e$. Secondly, let $d>e$ in C. For any $h\in G_f$ with $f\neq d$ we have $g^*(F_{de}(h))=g^*(h)=F_{de}(g^*(h))$. If $f=d$; then $F_{de}(g^*(h))=F_{de}(r_{cd}(g)\cdot h)=$ $=r_{de}(r_{cd}(g)\cdot h)=r_{de}(r_{cd}(g))\cdot r_{de}(h)=r_{ce}(g)\cdot r_{de}(h)=r_{ce}(g)\cdot F_{de}(h)=g^*(F_{de}(h))$ since all r_{cc} are homomorphisms satisfying (a).

Now let $d, e\leqq c$, $f=d\vee e$ and let $h_1\in G_d$, $h_2\in G_e$ be such that there is an $h\in G_f$ with $r_{fd}(h)=h_1$ and $r_{fe}(h)=h_2$. Then $g^*(h_1 * h_2)=g^*(h)=r_{cf}(g)\cdot h$, and $g^*(h_1) * g^*(h_2)=(r_{cd}(g)\cdot h_1) * (r_{ce}(g)\cdot h_2)=(r_{cd}(g)\cdot r_{fd}(h)) * (r_{ce}(g)\cdot r_{fe}(h))=$ $=r_{fd}(r_{cf}(g)\cdot h) * r_{fe}(r_{cf}(g)\cdot h)=r_{cf}(g)\cdot h$ by (15). To deal with the second clause of (15), assume $g^*(h_1)=r_{fd}(k)$ and $g^*(h_2)=r_{fe}(k)$ for a $k=g^*(h_1) * g^*(h_2)$ in G_f. Then $r_{fd}(k)=r_{fd}(r_{cf}(g))\cdot h_1$ and $r_{fe}(k)=r_{fe}(r_{cf}(g))\cdot h_2$ imply that $h_1=$ $=r_{fd}(r_{cf}(g^{-1})\cdot k)$ and $h_2=r_{fe}(r_{cf}(g^{-1})\cdot k)$. Thus $h_1 * h_2=r_{cf}(g^{-1})\cdot k\in G_f$ and $g^*(h_1 * h_2)=r_{cf}(gg^{-1})\cdot k=k=g^*(h_1) * g^*(h_2)$ as required. This proves that $g\to g^*$ is an embedding of G_c into $\mathrm{Aut}(A_c)$.

Let $I\neq\emptyset$ be an ideal of C, let $c\in I$, and let $\alpha\in \operatorname{Aut}(A_I)$ be arbitrary. If $I=\{c\}$, then $\alpha\in \operatorname{Aut}(A_c)$ and, in particular, $\alpha(h)=\alpha(\bar{h}(1_c))=\bar{h}(\alpha(1_c))=\alpha(1_c)\cdot h$ for every $h\in G_c$. $g=\alpha(1_c)\in G_c$ and $\alpha=g^*$, that is, we know that $\operatorname{Aut}(A_I)\cong G_I$ in this case. If I is not a singleton, then for every $c\in I$ there is a $d\in I$ such that either $c>d$ or $c<d$. Assume that $c\in I$ is not a minimal element, let $d<c$. Note that $G_c=\{h\in A_I\colon F_{cd}(h)\neq h\}$; hence $F_{cd}(\alpha(1_c))=\alpha(F_{cd}(1_c))=\alpha(1_d)\neq\alpha(1_c)$ implies $\alpha(1_c)\in G_c$. If, on the other hand, c is minimal in I then there is a $d>c$ in I and $\alpha(1_c)=\alpha(F_{dc}(1_d))=F_{dc}(\alpha(1_d))$ belongs to G_c since $\alpha(1_d)\in G_d$ by the previous argument. $\alpha(g)=\alpha(\bar{g}(1_c))=\bar{g}(\alpha(1_c))=\alpha(1_c)\cdot g\in G_c$ for all $g\in G_c$, $c\in I$. Thus $\alpha^+(A_c)=A_c$ for all $c\in I$ and this implies (10). Denote $g_c=\alpha(1_c)$ for $c\in I$. If $d\leqq c$, then $\alpha(h)=\alpha(\bar{h}(1_d))=\bar{h}(\alpha(1_d))=\bar{h}(\alpha(F_{cd}(1_c)))=\bar{h}(F_{cd}(\alpha(1_c)))=\bar{h}(r_{cd}(g_c))=r_{cd}(g_c)\cdot h$ and $\alpha(h)=g_d\cdot h$ for all $h\in G_d$. Therefore $r_{cd}(g_c)=g_d$ for $c\geqq d$ in I. If $I=(c]$, then $\alpha(h)=g_c^*(h)$ for all $h\in A_c$ and, consequently, $\operatorname{Aut}(A_c)\cong G_c$. This proves (11) for non-zero compact elements of L. If I is not principal, then every $\alpha\in\operatorname{Aut}(A_I)$ determines a system

$$(g_c\in G_c\colon c\in I)$$

such that g_c^* is the restriction of α to A_c. As $r_{cd}(g_c)=g_d$ for all $c\geqq d$ in I, there is a unique $g\in G_I$ whose projection in G_c is g_c. It is now clear that $G_I\cong\operatorname{Aut}(A_I)$ for every ideal I of C.

Finally, let $c\geqq d\geqq e$ in $C, g\in G_c, k\in G_e$. Then $g^*(k)=r_{ce}(g)\cdot k=r_{de}(r_{cd}(g))\cdot k=(r_{cd}(g))^*(k)$ and (12) is satisfied as well. This finishes the proof.

Example 1. The set C of nonzero compact elements of an algebraic chain L consists of those $x\in L$ that cover some $y\in L$. If G_c is arbitrary for $c\in C$, $|G_x|=1$ for $x\notin C$, and if all r_{cd} are constant homomorphisms for $c>d$, then the system Σ is representable. This generalizes the independence of automorphism groups of pairs algebra-subalgebra.

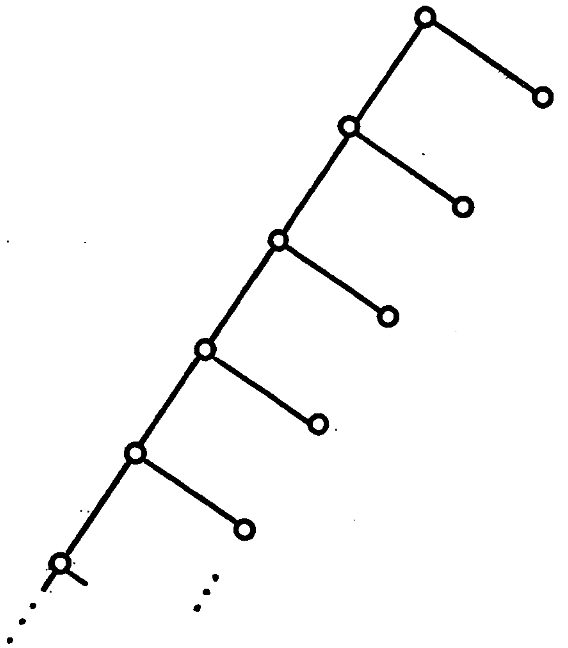
Example 2. Under the restriction (4) assumed throughout this note, the automorphism groups of subalgebras not finitely generated are uniquely determined by the automorphism groups of their finitely-generated subalgebras. A simple example shows that this is not generally the case.

Let L be the chain Z of integers extended by a largest element e and a smallest element z. L is an algebraic chain with $C=Z\cup\{z\}$. Let $G_c=\{1\}$ for $c\in C$ and let $(r_{cd}\colon c\geqq d)$ be the obvious homomorphisms. If $G_e=\{1\}$ as well, then the system Σ formed by these data is representable. On the other hand, if $f\colon Z\to Z$ is defined by $f(n)=n-1$, then the algebra $A=(Z,f)$ satisfies $\operatorname{Sub}(A)\cong L$, $|\operatorname{Aut}([n])|=1$ for all nonempty subalgebras $[n]=\{k\colon k\leqq n\}$, while $\operatorname{Aut}(A)$ is isomorphic to the additive group of integers.

Example 3. If $L\cong I(C)$ is an algebraic lattice and if all ideals of C are automorphism-free, then our special-case theorem describes the possible choices of

$(G_x : x \in L)$ completely. This is the case if, for instance, C is the join semilattice indicated by the Figure below.

Note that any non-empty ideal of C that is not a singleton is isomorphic to C; C is automorphism free as a semilattice — which implies (4) for any representable system with $L = I(C)$.



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Attractivity theorems for non-autonomous systems of differential equations

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Dedicated to Professor Béla Szőkefalvi-Nagy on his 65th birthday

1. Introduction

By the classical theorem proved by A. M. LJAPUNOV [1] in 1892 the zero solution of the system $\dot{x}=f(t,x)$ $(t\geqq 0,\ x\in R^n;\ f(t,0)\equiv 0)$ is asymptotically stable provided that there exists a positive definite scalar function $V(t,x)$ tending to zero uniformly in $t\in[0,\infty)$ as $x\to 0$ and having a negative definite derivative $\dot{V}(t,x)$ with respect to the system. Since the early days of stability theory numerous authors have dealt with weakening the conditions of this theorem. There are two main types of attempts.

In theorems belonging to the first type special assumptions are required of the vector field $f(t,x)$ independently of the Ljapunov function $V(t,x)$. The first theorem of this type is due to M. MARAČKOV [2], who assumed $f(t,x)$ to be bounded for all t when x belongs to an arbitrary compact set instead of the condition of $V(t,x)$ tending to 0, uniformly with respect to t, as $x\to 0$. Considering autonomous systems E. A. BARBAŠIN and N. N. KRASOVSKIĬ [3] generalized Ljapunov's theorem to the case when the function $\dot{V}(t,x)$ is not negative definite. By the method of several Ljapunov functions V. M. MATROSOV [4] extended this result to those non-autonomous systems whose right-hand side $f(t,x)$ is bounded for all t when x belongs to an arbitrary compact set. For the systems of the same kind T. YOSHIZAWA [5] and J. P. LASALLE [6] gave sufficient conditions for the attractivity of closed sets. A given set $H\subset R^n$ is called *attractive* if every solution starting from some neighbourhood of H tends to H as $t\to\infty$. In 1976 LaSalle extended his theorem by weakening the condition of boundedness of $f(t,x)$ [7, Th. 1].

Results of the second type are characterized by the fact that the direct conditions on the right-hand side $f(t,x)$ are omitted but certain relations between

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the function $\dot{V}(t, x)$ and the norm $\|f(t, x)\|$ of the right-hand side are required. The most important theorems of this type are due to T. A. BURTON [8] and J. R. HADDOCK [9].

The purpose of this paper is to improve some results of both types in the following two directions. On the one hand, we give the role of $f(t, x)$ to the derivative $\dot{W}(t, x)$ of a function $W: R^n \to R^k$ with respect to the system. On the other hand, in the theorems of the first type we refine the estimates on $\dot{V}(t, x)$ so that we should be able to take into account the finer structure of the "dangerous set" defined by $\dot{V}(t, x)=0$, which depends on the time t in the non-autonomous case. At the end of our paper we give examples to illustrate how our results relate to the above mentioned ones, and applications are given to the study of the asymptotic behaviour of solutions of non-linear second order differential equations.

2. Notations and definitions

The basic differential equation is

$$\dot{x} = f(t, x), \tag{2.1}$$

where $t \in R_+ = [0, \infty)$, and x belongs to the n-dimensional Euclidean space R^n. The function f is defined and continuous on the set $\Gamma^* = R_+ \times G^*$; G^* is an open set in R^n.

Denote by (x, y), $\|x\|$ and $d(x, y)$ the scalar product, norm and distance in R^n, respectively; namely $(x, y) = \sum_{i=1}^{n} x_i y_i$, $\|x\| = (x, x)^{1/2}$ and $d(x, y) = \|x-y\|$. Let R^n_∞ denote the one-point compactification of R^n and define $d(x, \infty) = 1/\|x\|$. For a set $H \subset R^n$ we denote the complement of H by H^c, the closure of H by $\bar{H}$, and the set $H \cup \{\infty\}$ in R^n_∞ by H_∞. For a set $K \subset R^n_\infty$, define $d(x, K) = \inf\{d(x, y): y \in K\}$. If $d(u(t), K) \to 0$ as $t \to \omega - 0$ for a continuous function $u: [0, \omega) \to R^n$, we shall say $u(t) \to K$ as $t \to \omega - 0$.

For $H \subset R^n$, $\varepsilon > 0$ the set $S(H, \varepsilon) = \{x \in R^n: d(x, H) < \varepsilon\}$ is called the ε-neighbourhood of H. We shall need another neighbourhood system. Let a set $G \subset R^n$ and a continuous function $W: G \to R^k$ be given. If $p \in G$ and $\varrho > 0$, we shall use the notation $S^*(p, \varrho) = W^{-1}[S(W(p), \varrho)]$, where $W^{-1}[H]$ denotes the inverse image of $H \subset R^k$ with respect to W.

Let $x(t)$ be a solution of (2.1) defined on a maximal right interval $[t_0, \omega)$ $(t_0 < \omega \leqq \infty)$. A point p is a *positive limit point* of $x(t)$ if there exists a sequence $\{t_m\}$ such that $t_m \to \omega - 0$ and $x(t_m) \to p$ as $m \to \infty$. The *positive limit set* Ω *of* $x(t)$ is the set of all its positive limit points. If $x(t)$ is bounded and $\Omega \subset G^*$, then $\omega = \infty$, Ω is nonempty, compact, connected and is the smallest closed set that $x(t)$ approaches as $t \to \infty$.

Denote by $C^1(D; R^k)$ the family of all functions $W: D(\subset R^m) \to R^k$ whose components have continuous first partial derivatives. For a function $u \in C^1(R_+ \times R^n; R)$ define the function

$$\dot{u}(t, x) = \sum_{i=1}^{n} \frac{\partial u(t, x)}{\partial x_i} f_i(t, x) + \frac{\partial u(t, x)}{\partial t}$$

which is said to be the *derivative of u with respect to equation* (2.1). The derivative of a vector-function $U \in C^1(R_+ \times R^n; R^k)$ with respect to (2.1) is the vector of the derivatives of the components of U with respect to (2.1).

System (2.1) is non-autonomous, so its solutions $x(t)$ can be represented by the graph $(t, x(t))$ in R^{n+1}. A solution $x(t)$ is said to be in $\Gamma \subset \Gamma^*$ if $(t, x(t)) \in \Gamma$ for all $t \in [t_0, \omega)$. For a given set $\Gamma \subset \Gamma^*$ we shall use the notations

$$G(t) = \{x: (t, x) \in \Gamma\}, \quad G = \bigcup_{t \geqq 0} G(t).$$

Denote by $[a]_+$ and $[a]_-$ the positive and negative part of the real number a, respectively.

Definition 2.1. Let Γ be a subset of Γ^*. We say that $V \in C^1(\Gamma; R)$ is a *Ljapunov function on* Γ if there exists a continuous function $\eta: R_+ \to R_+$ such that

$$\int_0^\infty \eta(t)\, dt < \infty, \quad [\dot{V}(t, x)]_+ \leqq \eta(t) \quad ((t, x) \in \Gamma).$$

Let A be a property concerning the functions V and $\dot{V}$. "Property A is satisfied on the set $t \geqq T, x \in H(\subset R^n)$" if it is satisfied on the subset of $[T, \infty) \times H$ where the Ljapunov function is defined, i.e. on the set $\{(t, x): t \geqq T, x \in H \cap G(t)\}$.

3. Theorems and proofs

In this section we study attractivity conditions of a given set with respect to system (2.1). Namely, we seek conditions assuring that the set contains the positive limit sets of solutions of (2.1).

Assume that we have a Ljapunov function V on Γ and an auxiliary function $W \in C^1(\bar{G} \cap G^*; R^k)$.

Lemma 3.1. *Let $x(t)$ be a solution with maximal right-interval of definition $[t_0, \omega)$, and let $M \subset G$ be a set such that $x(t) \in M$ for $t \in [t_0, \omega)$.*

Suppose that for a point $p \in \bar{G} \cap G^*$ there exist $\delta, \varrho > 0$ and T such that

(i) $V(t, x)$ is bounded from below and

(3.1) *(ii) $\dot{V}(t, x) \leqq -\delta \|\dot{W}(t, x)\| + \eta(t)$*

on the set $t \geqq T, x \in S^*(p, \varrho) \cap M$.

Then either a) $p \notin \Omega$ or b) $\omega = \infty$ and $\Omega \cap G^* \subset W^{-1}[W(p)]$.

Proof. Suppose the contrary of a), i.e. $p \in \Omega$. Since $p \in G^*$ and G^* is open, according to the theorem of continuation of solutions $\omega = \infty$ holds. Suppose b) is false, too. Then there exist $q \in \Omega \cap G^*$ and σ $(0 < \sigma < \varrho/\sqrt{k})$ such that $q \notin S^*(p, \sqrt{k}\sigma)$. Because of $p, q \in \Omega$ there exist a natural number l $(1 \leq l \leq k)$ and two sequences $\{t'_m\}$, $\{t''_m\}$ with the following properties:

$$T \leqq t'_1 < t''_1 < \ldots < t'_m < t''_m < \ldots; \quad \lim_{m \to \infty} t'_m = \infty; \tag{3.2}$$

$$\|W(p) - W(x(t))\| < \sigma\sqrt{k} \quad (t'_m \leqq t \leqq t''_m); \tag{3.3}$$

$$|W_l(x(t''_m)) - W_l(x(t'_m))| = \frac{\sigma}{2} \quad (m = 1, 2, \ldots). \tag{3.4}$$

By assumption (ii), for the function $v(t) = V(t, x(t))$ the estimation

$$v(t''_m) - v(t'_m) \leqq -\delta\frac{\sigma}{2} + \int_{t'_m}^{t''_m} \eta(t)\,dt \quad (m = 1, 2, \ldots) \tag{3.5}$$

is satisfied, from which it follows that

$$v(t''_m) \leqq v(t'_1) - m\delta\frac{\sigma}{2} + \int_T^{t''_m} \eta(t)\,dt \to -\infty \quad (m \to \infty),$$

and this contradicts assumption (i).

The lemma is proved.

Remark 3.1. If either the function W is scalar $(k=1)$ or assumptions (i)—(ii) are required on the set $t \geqq T$, $x \in M$, then assumption (ii) may be required of the function $[\dot{W}]_+$ instead of $\dot{W}$. In the first case the statement is unchanged; in the second case it can be stated that either a) $\Omega \cap G^*$ is empty or b) $\omega = \infty$ and there exists a $p \in \overline{M} \cap G^*$ such that $\Omega \cap G^* \subset W^{-1}[W(p)]$.

Indeed, if either the function W is scalar or property (3.3) is not required, then we may also assume property (3.4) is true without the absolute value sign. Then for deduction of inequality (3.5) it is sufficient to require assumption (ii) of the function $[\dot{W}]_+$ instead of $\dot{W}$.

Theorem 3.1. *Let the sets $H \subset R^n$, $M \subset G$ be given and suppose that for any $p \in H^c$ there exist $\varrho(p) > 0$, $\delta(p) > 0$ and $T(p)$ such that assumptions (i)—(ii) in Lemma 3.1 are satisfied on the set $t \geqq T(p)$, $x \in S^*(p, \varrho(p)) \cap M$.*

1) *If $x(t)$ is a solution and $x(t) \in M$ for $t \in [t_0, \omega)$, then either a) $\Omega \cap G^* \subset H$ or b) $\omega = \infty$ and there exists a $d \in R^k$ such that the set $W^{-1}[d] \cap H^c$ is non-empty and $\Omega \cap G^* \subset W^{-1}[d]$.*

2) *If also assumption $\bar{G} \subset G^*$ is satisfied, then either a) $x(t) \to H_\infty$ as $t \to \omega - 0$ or b) $\omega = \infty$ and there exists a $d \in R^k$ such that the set $W^{-1}[d] \cap H^c$ is non-empty and $x(t) \to W^{-1}[d]_\infty$ as $t \to \infty$.*

In the case of a scalar function W $(k=1)$ *the statements remain true after replacing function* $\dot{W}$ *with* $[\dot{W}]_+$ *in assumption* (3.1).

Proof. 1) If the set $\Omega\cap G^*$ is empty, then a) is true. Suppose that it is not empty, and there exists a $p\in\Omega\cap G^*$ such that $p\in H^c$. Then, by Lemma 3.1 (and Remark 3.1) b) is true, namely $d=W(p)$.

2) The statements follow from those under 1) and from the fact that $\Omega\subset G^*$.

Theorem 3.2. *Let the set* $M\subset G$ *be given, and suppose that there exist* $\delta>0$, $T\geqq 0$ *such that*

(i) $V(t, x)$ *is bounded from below and*

(ii) $\dot{V}(t, x)\leqq -\delta\|[\dot{W}(t, x)]_+\|+\eta(t)$

on the set $t\geqq T$, $x\in M$.

If $x(t)$ *is any solution and* $x(t)\in M$ *for* $t\in[t_0, \omega)$, *then either a) the set* $\Omega\cap G^*$ *is empty or b)* $\omega=\infty$, *and there exists a* $d\in R^k$ *such that* $\Omega\cap G^*\subset W^{-1}[d]$.

Proof. Applying Lemma 3.1 and Remark 3.1, the theorem can be proved in the same manner as Th. 3.1.

Our theorems can be used not only for studying stability properties of sets but also for establishing various kinds of asymptotic properties of solutions. For example, let us take $G^*=R^n$, $H=\{0\}$, $W(x)=(x, x)$; furthermore, let $V(t, x)$ be a Ljapunov function on the set $R_+\times R^n$ bounded from below for all $t\in R_+$ when x belongs to an arbitrary compact set. Suppose that for any point $p\neq 0$ there exist $\delta>0$, $\varrho>0$, T such that

$$\dot{V}(t, x)\leqq -\delta[(f(t, x), x)]_{+(-)}+\eta(t)$$

for $t\geqq T$, $\big|\|x\|-\|p\|\big|<\varrho$, where the symbol $[\cdot]_{+(-)}$ means that either the positive part $[\cdot]_+$ or the negative part $[\cdot]_-$ is considered for all (t, x). By Th. 3.1 these assumptions imply that for any solution $x(t)$ either a) the function $\|x(t)\|$ has a finite limit as $t\to\infty$ or b) $x(t)\to\infty$ as $t\to\omega-0$.

From Th. 3.1 by the choice of $W(x)=x$ an important result mentioned in the Introduction follows.

Corollary 3.1. (J. Haddock [9, Th. 3]). *Let* $G^*=R^n$, $H\subset R^n$ *be a closed set, and* $V(t, x)$ *be a Ljapunov function on* $R_+\times R^n$ *bounded from below for all* $t\in R_+$ *when* x *belongs to an arbitrary compact set. Suppose that for any* $\varepsilon>0$ *and any compact set* $K\subset R^n$ *there exist* $\delta(\varepsilon, K)>0$, $T(\varepsilon, K)$ *such that*

$$\dot{V}(t, x)\leqq -\delta\|f(t, x)\|+\eta(t) \tag{3.6}$$

on the set $t\geqq T$, $x\in K\cap S^c(H, \varepsilon)$.

If $x(t)$ *is any solution, then either a)* $x(t)\to H_\infty$ *as* $t\to\omega-0$ *or b)* $x(t)\to p$ *as* $t\to\infty$ *for some* $p\in H^c$.

In certain cases the fact that assumption (3.6) contains the non-monotonic function $\|\cdot\|$ can cause difficulties. This can be avoided by means of the last statement of Th. 3.1 in the following way: Suppose that $V(t, x)$ is a Ljapunov function on $R_+ \times R^n$, and for any $\varepsilon>0$ and any $C>0$ there exist $\delta(\varepsilon, C)>0$ and $T(\varepsilon, C)$ such that $V(t, x)$ is bounded from below and

$$\dot{V}(t, x) \leqq -\delta[f_i(t, x)]_{+(-)} + \eta(t) \quad (i = 1, 2, \ldots) \tag{3.7}$$

on the set $t \geqq T$, $x \in S^c(H, \varepsilon) \cap \{x \in R^n : |x_i| \leqq C\}$. Then the statement of Cor. 3.1 is true. If (3.7) is satisfied only for a fixed i, then instead of b) it can be stated only $x_i(t) \to p_i$ as $t \to \infty$ (see Th. 3.1, $W(x) = +(-)x_i$).

Having certain "a priori" (independent of the function $V(t, x)$) informations about the function $\dot{W}(t, x)$, we can replace assumption (ii) in Lemma 3.1 with another one to improve the previous theorems in some respects.

Definition 3.1. A measurable function $\varphi: R_+ \to R$ is said to be *integrally positive* (see [4], [11]) if $\int_I \varphi(t)\,dt = \infty$ holds on every set $I = \bigcup_{m=1}^{\infty} [\alpha_m, \beta_m]$ such that

$$\alpha_m < \beta_m < \alpha_{m+1}, \quad \beta_m - \alpha_m \geqq \delta > 0 \quad (m = 1, 2, \ldots).$$

A function $\varphi(t)$ is said to be *integrally negative* if $-\varphi(t)$ is integrally positive.

Lemma 3.2. *Let $x(t)$ be a solution and let $M \subset G$ be an arcwise connected set such that $x(t) \in M$ for all $t \in [t_0, \omega)$.*

Suppose that for a point $p \in \bar{G} \cap G^*$ there exist $\varrho > 0$ and T such that for any continuous function $u: [T, \infty) \to L = S^*(p, \varrho) \cap M$ the following conditions are satisfied:

(i) $\int_T^t \dot{W}(s, u(s))\,ds$ is uniformly continuous,

(ii) $\dot{V}(t, u(t))$ is integrally negative, and

(iii) $V(t, u(t))$ is bounded from below on the interval $[T, \infty)$.

Then $p \notin \Omega$.

Lemma 3.3. *The statement of Lemma 3.2 remains true if conditions* (i)—(ii) *are replaced with the following: for any continuous function $u: [T, \infty) \to L$*

(i′) $\left\| \int_T^{\infty} |\dot{W}(t, u(t))|\,dt \right\| < \infty,$

(ii′) $\int_T^{\infty} \dot{V}(t, u(t))\,dt = -\infty.$

Proof of Lemmas 3.2 and 3.3. Assume the contrary, i.e. $p\in\Omega$. Then $\omega=\infty$, and there exists a sequence $\{t_m\}$ such that $t_m\to\infty$ and $x(t_m)\to p$ as $m\to\infty$. On the other hand, however large the time T^* may be, the set $S^*(p,\varrho)$ must not contain the point $x(t)$ for *all* $t\geqq T^*$ because of assumptions (ii) ((ii′), respectively) and (iii). Consequently, in the same manner as in the proof of Lemma 3.1, there are $\sigma>0$, l $(1\leqq l\leqq k)$ and sequences $\{t'_m\}$, $\{t''_m\}$ with properties (3.2)—(3.4). Then we have

(3.8) $$\Big\|\int_{t'_m}^{t''_m}\dot W(t,x(t))\,dt\Big\|\geqq\frac{\sigma}{2}\quad(m=1,2,\ldots).$$

This contradicts (i′); therefore Lemma 3.3 is proved.

To prove Lemma 3.2 we show that (3.8) contradicts assumptions (i)—(iii), too. Indeed, (3.8) and (i) imply that $t''_m-t'_m\geqq\delta$ for all m with some $\delta>0$. The function $\dot V(t,x(t))$ is integrally negative, consequently

$$V(x(t''_m))\leqq\text{const.}+\sum_{i=1}^{m}\int_{t'_i}^{t''_i}\dot V(t,x(t))\,dt\to-\infty\quad(m\to\infty),$$

which contradicts the boundedness from below of the function $V(t,x(t))$.

The proof of both lemmas is complete.

Remark 3.2. If the function W is scalar $(k=1)$ then assumption (i) (assumption (i′), respectively) may be required of the function $[\dot W]_+$ instead of $\dot W$ ($|\dot W|$, respectively); the statements remain true.

Now suppose that for the derivative of the Ljapunov function V an inequality

(3.9) $$\dot V(t,x)\leqq\varphi(t)U(x)+\eta(t)\quad((t,x)\in\Gamma)$$

holds with continuous functions $\varphi\colon R_+\to R_+$, $U\colon\bar G\cap G^*\to R_-$, $\eta\colon R_+\to R_+$ (the function η is integrable on $[0,\infty)$ by Def. 2.1). Denote by F the so called "dangerous set":

$$F=\{x\in\bar G\cap G^*\colon U(x)=0\},$$

which is closed with respect to G^*.

Theorem 3.3. *Let $M\subset G$ be an arcwise connected set, and suppose that for any $p\in F^c$ there exist $\varrho(p)>0$, $T(p)$ such that:*

(i) $\sup\{U(x)\colon x\in L(p)=S^*(p,\varrho)\cap M\}<0$;

(ii) $\varphi(t)$ *is integrally positive;*

moreover, for any continuous function $u\colon[T(p),\infty)\to L(p)$

(iii) $\int_T^t\dot W(s,u(s))\,ds$ *is uniformly continuous, and*

(iv) $V(t, u(t))$ *is bounded from below on the interval* $[T(p), \infty)$.

If $x(t)$ *is any solution and* $x(t)\in M$ $(t_0\leqq t<\omega)$, *then* $\Omega\cap G^*\subset F$.

If also assumption $\bar{G}\subset G^*$ *is satisfied, then* $x(t)\to F_\infty$ *as* $t\to\omega-0$.

Theorem 3.4. *The statements of Th. 3.3 remain true if assumptions (ii)—(iii) are replaced with the following ones:*

(ii') $\int\limits_0^\infty \varphi(t)\,dt=\infty$;

(iii') $\left\|\int\limits_T^\infty |\dot{W}(t, u(t))|\,dt\right\|<\infty$.

Proof of Theorems 3.3 and 3.4. Suppose the contrary, i.e. the set $\Omega\cap G^*$ contains a point p not belonging to the set F. Then, by assumptions, there are $\varrho(p)>0$, $\delta(p)>0$, $T(p)$ such that inequality

$$\dot{V}(t, x)\leqq-\delta\varphi(t)+\eta(t)\quad (t\geqq T(p),\ x\in L(p))$$

holds. Hence, using Lemma 3.2 (and Lemma 3.3, respectively) we get $p\notin\Omega$, which contradicts our earlier assumption on p.

$\bar{G}\subset G^*$ implies inclusion $\Omega\subset G^*$. Consequently we have $\Omega\subset F$, so $x(t)\to F_\infty$ as $t\to\omega-0$.

The proof is complete.

Remark 3.3. In case of scalar function $W(k=1)$ the statements of Theorems 3.3 and 3.4 remain true after replacing function $\dot{W}$ (function $|\dot{W}|$, respectively) with $[\dot{W}]_+$ in assumption (iii) (assumption (iii'), respectively).

For example, by the choice $W(x)=(x, x)$, from Th. 3.3 we get the following: Suppose $V(t, x)$ is a Ljapunov function on $R_+\times R^n$ bounded from below for all $t\in R_+$ when x belongs to an arbitrary compact set. Further, suppose there exist continuous functions $\varphi\colon R_+\to R_+$, $a\colon R_+\to R_+$ such that $a(0)=0$, $a(r)>0$ for $r>0$; φ is integrally positive, and

$$\dot{V}(t, x)\leqq-\varphi(t)a(\|x\|)\quad ((t, x)\in\Gamma=R_+\times R^n).$$

If for any r_1, r_2 $(0<r_1<r_2)$ and for any continuous function $u\colon [T, \infty)\to\to\{x\in R^n\colon r_1\leqq\|x\|\leqq r_2\}$ the function $\int\limits_0^t [(f(s, u(s)), u(s))]_{+(-)}\,ds$ is uniformly continuous on $[0, \infty)$, then for any solution $x(t)$ either $x(t)\to 0$ or $x(t)\to\infty$ as $t\to\omega-0$.

Similarly to the previous ones, Th. 3.3 yields an important result when $W(x)=x$.

Corollary 3.2. *Suppose the estimation* (3.9) *is satisfied with an integrally positive function* φ. *Further suppose that for any compact set* $K\subset R^n$ *and any continuous function* $u:[T,\infty)\to K$ *the function* $\int_T^t f(s,u(s))\,ds$ *is uniformly continuous and* $V(t,u(t))$ *is bounded from below on the interval* $[T,\infty)$. *Then for any solution the inclusion* $\Omega\cap G^*\subset F$ *holds.*

From the point of view of applications the most important case is when the function V is differentiable and f is continuous, and this corollary is an improvement of the LaSalle theorem [7, Th. 1], which can be obtained from it by setting $\varphi(t)\equiv 1$. It will be shown by examples taken from the theory of nonlinear oscillations that even in simple cases it is necessary to introduce the function φ into estimation (3.9).

Remark 3.4. Associate with the functions V and W the set $\mathcal{F}\subset R^n$ defined as follows: $x\in\mathcal{F}$ iff there exists a sequence (t_m, x^m) such that $t_m\to\infty$, $W(x^m)\to W(x)$ and $\dot V(t_m, x^m)-\eta(t)\to 0$ as $m\to\infty$. Similarly as in Th. 3.3, from Lemma 3.2 we can derive a statement assuring the inclusion $\Omega\cap G^*\subset\mathcal{F}$ for any solution of (2.1). In this way it can be generalized a result given by N. Onuchic et al. [13, Th. 1], who introduced the set $\mathcal{F}$ in case of $W(x)=x$, $\eta(t)=0$. Even in this special case, obviously, there are functions f, V, φ and U such that $\mathcal{F}\supset F$, and what is more the set $\mathcal{F}$ is too large to obtain any information about the place of Ω in R^n by the inclusion $\Omega\cap G^*\subset\mathcal{F}$ (e.g. $\dot V(t,x)=\sin^2 t\cdot U(x)$). This fact motivates estimation (3.9). Moreover, if the functions φ and U are chosen in (3.9) "sufficiently well" and φ is bounded, then $\mathcal{F}\supset F$.

Remark 3.5. The key assumption in Th. 3.3 is (iii), which assures the point $x(t)$ not to go away in the same distance from the attractor F infinitely many times within a shorter and shorter time. Even in the special case of $W(x)=x$, the uniform continuity of the function $\int_T^t \sup\{\|f(s,x\|: x\in S(p,\varrho)\cap M\}\,ds$ on $[T,\infty)$ is often checked instead of assumption (iii). Before LaSalle's paper [7], in [12] the author used already an assumption equivalent to this one to assure the above mentioned property of the solutions.

4. Applications and examples

I. Let us consider the non-linear differential equation of second order

$$(p(t)\dot x)^{\cdot}+q(t)f(x)=0 \quad (x\in R), \tag{4.1}$$

where the functions $p,q:R_+\to R_+$ are continuously differentiable and $\dot p(t)\geqq 0$,

$\dot{q}(t) \leqq 0$ $(t \in R_+)$; the function $f: R \to R$ is continuous and $xf(x) \geqq 0$ $(x \in R)$; $F(x) = \int_0^x f(s)\,ds$.

Apply Th. 3.2 to the study of asymptotic behaviour of the coordinate x and the momentum $y = p(t)\dot{x}$. In terms of these Hamiltonian variables equation (4.1) has the form

$$\dot{x} = \big(1/p(t)\big)y, \quad \dot{y} = -q(t)f(x). \tag{4.1'}$$

Let us now consider functions

$$V(t, x, y) = \frac{1}{p(t)}\,y^2 + 2q(t)F(x), \quad W(x, y) = xy,$$

whose derivatives by virtue of (4.1′) are

$$\dot{V} = -\frac{\dot{p}(t)}{p^2(t)}\,y^2 + 2\dot{q}(t)F(x), \quad \dot{W} = \frac{1}{p(t)}\,y^2 - q(t)xf(x).$$

On the one hand, if there exist $\gamma_1 > 0$, $T_1 \in R_+$ such that

$$\dot{p}(t)/p(t) \geqq \gamma_1 > 0 \quad (t \geqq T_1), \tag{4.2}$$

then

$$\dot{V} \leqq -\frac{\dot{p}(t)}{p^2(t)}\,y^2 \leqq -\gamma_1 \frac{1}{p(t)}\,y^2 \leqq \gamma_1[\dot{W}]_+ \quad (t \geqq T_1;\ x, y \in R).$$

On the other hand, if there exist $\gamma_2, \gamma_3 > 0$ and $T_2 \in R$ such that

$$\frac{\dot{q}(t)}{q(t)} \leqq -\gamma_2 < 0 \quad (t \geqq T_2); \quad \gamma_3 F(x) > xf(x) \quad (x \in R), \tag{4.3}$$

then

$$\dot{V} \leqq 2\dot{q}(t)F(x) \leqq -\frac{2\gamma_2}{\gamma_3}\cdot\gamma_3 q(t)F(x) \leqq -\frac{2\gamma_2}{\gamma_3}\,[-W]_+$$

for $t \geqq T_2;\ x, y \in R$.

Applying Th. 3.2 with $M = G = R^2$ and with the functions V, W and V, $-W$, respectively, we get: for any solution of (4.1′) either $|x(t)| + |y(t)| \to \infty$, as $t \to \omega - 0$ or $\omega = \infty$ and $\lim_{t\to\infty}\big(x(t)y(t)\big)$ exists. By the first equation of system (4.1′) $(x^2)^{\cdot} = 2xy/p(t)$, hence, if

$$\int_0^\infty \big(1/p(t)\big)\,dt < \infty, \quad \int_0^\infty q(t)\,dt < \infty, \tag{4.4}$$

then $\lim_{t\to\infty} x(t)$, $\lim_{t\to\infty} y(t)$ exist. Thus, we have:

Suppose either (4.2) *or* (4.3), *and let* $x(t)$ *be any solution of* (4.1) *with the maximal right interval* $[t_0, \omega)$. *Then either a)* $|x(t)|+|p(t)\dot{x}(t)|\to\infty$ *as* $t\to\omega-0$ *or b)* $\omega=\infty$ *and* $\lim_{t\to\infty}(p(t)x(t)\dot{x}(t))$ *exists. If also condition* (4.4) *is satisfied, then in case b) we can state* $\lim_{t\to\infty}x(t)$, $\lim_{t\to\infty}(p(t)\dot{x}(t))$ *exist.*

II. Let us now consider the equation

$$\ddot{x}+a(t)\dot{x}+b(t)f(x)=0 \quad (x\in R), \tag{4.5}$$

where the functions $a: R_+\to R$, $f: R\to R$ are continuous, $b: R_+\to R$ is continuously differentiable. By the aid of Th. 3.3, we seek for conditions which assure that the derivative of any solution of (4.5) tends to 0 as $t\to\infty$.

Introducing the variable $y=\dot{x}$, we can transform equation (4.5) into the system

$$\dot{x}=y, \quad \dot{y}=-b(t)f(x)-a(t)y. \tag{4.5'}$$

Choose the Ljapunov functions

$$V_1(t,x,y)=\frac{y^2}{b(t)}+2F(x); \quad V_2(t,x,y)=\frac{y^2}{2}+b(t)F(x)$$

(see [11]), where $F(x)=\int_0^x f(s)\,ds$, and the auxiliary function $W(x)=(x^2/2, y^2/2)$. Their total derivatives by virtue of (4.5′) are

$$\dot{V}_1=-\varphi_1(t)y^2, \quad \varphi_1(t)=\frac{2a(t)}{b(t)}+\frac{\dot{b}(t)}{b^2(t)};$$

$$\dot{V}_2=-a(t)y^2+b(t)F(x), \quad \dot{W}=\big(xy, -b(t)yf(x)-a(t)y^2\big).$$

Applying Th. 3.3 with the functions V_1, W and V_2, W, respectively, we obtain the following results: *Suppose the function* $\int_0^t(|a(s)|+|b(s)|)\,ds$ *is uniformly continuous on* R_+.

1) *If either* $b(t)>0$ *or there exists a* $\gamma>0$ *such that* $b(t)\leqq-\gamma$ *for values of t large enough, and* $\varphi_1(t)$ *is integrally positive, then for any solution* $x(t)$ *of* (4.5) *either a)* $|x(t)|+|\dot{x}(t)|\to\infty$ *as* $t\to\omega-0$ *or b)* $\dot{x}(t)\to 0$ *as* $t\to\infty$.

2) *If* $F(x)\geqq 0$ $(x\in R)$, $b(t)$ *is bounded from below* $(t\in R_+)$, *and* $a(t)$ *is integrally positive, then for any solution* $x(t)$ *of* (4.5) *either a) or b) is satisfied.*

III. Finally, in order to compare our results with those of LaSalle and Haddock, we investigate attractivity properties of the solutions of the linear system

$$\begin{aligned}\dot{x}&=-r(t)x+q(t)y\\ \dot{y}&=-q(t)x-p(t)y\end{aligned} \quad (x, y\in R), \tag{4.6}$$

where $p, q, r: R_+\to R$ are continuous, and $p(t)\geqq 0$, $r(t)\geqq 0$ $(t\in R_+)$. Choose the

Ljapunov function $V(x, y)=(x^2+y^2)/2$. Its derivative by virtue of (4.6) $\dot{V}(t, x, y)=-r(t)x^2-p(t)y^2$ is non-positive, so any solution of (4.6) exists and is bounded on the whole R_+. The LaSalle—Yoshizawa theorem yields the following statement:

A) (J. P. LASALLE [6]: $r(t)\equiv 0$, $q(t)\equiv 1$). If $0<c<p(t)\leqq C$ ($t\in R_+$; $c, C=$ $=$const.), then for every solution of (4.6) $x(t)\to$const., $y(t)\to 0$ as $t\to\infty$.

Haddock deduced from his theorem (see Cor. 3.1 in this paper) the following result:

B) (J. HADDOCK [10]: $r(t)\equiv 0$). If there exists $\alpha>0$ such that

$$|q(t)|<\alpha p(t) \quad (t\in R_+), \tag{4.7}$$

and $\int_0^\infty p(t)\,dt=\infty$, then for any solution of (4.6) $y(t)\to 0$, $x(t)\to$const. as $t\to\infty$.

Let us now consider the auxiliary function $W(y)=y^2/2$, whose derivative is $\dot{W}=-q(t)xy-p(t)y^2$, and denote by H the set of the points of x-axis on the plane (x, y). We prove all the conditions of Th. 3.1 are satisfied, provided that (4.7) is true. For any solution $(x(t), y(t))$ of (4.6) there exists a C such that $(x(t), y(t))\in M=\{(x, y): x^2+y^2\leqq C\}$. It is sufficient to show that for every $\varepsilon>0$ there exists a $\delta>0$ such that $(x, y)\in M$, $|y|\geqq\varepsilon$ imply $\dot{V}(t, x)\leqq -\delta[\dot{W}(t, x)]_+$ for all $t\in R_+$. Let $\delta=2\varepsilon^2/(\alpha C)$. Then from (4.7) it follows that $-p(t)y^2\leqq$ $\leqq -\delta|q(t)|(x^2+y^2)/2$ ($t\in R_+$), which implies the desired inequality.

By Th. 3.1, using also the fact that $\lim_{t\to\infty} V(x(t), y(t))$ exists, we obtain the following result:

1) *Suppose* (4.7). *Then both of the components of any solution of* (4.6) *tend to a finite limit as* $t\to\infty$. *If* $\int_0^\infty p(t)\,dt=\infty$ *is also satisfied, then* $x(t)\to$const., $y(t)\to 0$ *as* $t\to\infty$.

It is worth noting that if we applied Haddock's theorem to this case, then in order to get the same result we would also have to require the condition analogous to (4.7) of the function $r(t)$. On the other hand, condition (4.7) is too strong, since it requires much of $q(t)$ *locally* at certain points (e.g. $p(t_0)=0$ implies $q(t_0)=0$), nevertheless the conclusion is only about the *limits* of the solutions. By the aid of Theorems 3.3 and 3.4 ($\varphi(t)=p(t)$, $U=-y^2$, $[\dot{W}]_+\leqq[q(t)xy]_+$) we do without assumption (4.7):

2) *If* $p(t)$ *is integrally positive and* $\int_0^t |q(s)|\,ds$ *is uniformly continuous on* R_+, *then for any solution of* (4.6) $x(t)\to$const, $y(t)\to 0$ *as* $t\to\infty$.

3) *If* $\int_0^\infty p(t)\,dt=\infty$ *and* $\int_0^\infty |q(t)|\,dt<\infty$, *then for any solution of* (4.6) $x(t)\to\text{const.}$, $y(t)\to 0$ *as* $t\to\infty$.

*

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Dependence of associativity conditions for ternary operations

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1. A ternary operation on a set S is a map on $S\times S\times S$ to S. For the sake of convenience we denote the ternary image of an ordered triple (a, b, c) on S by (a, b, c) itself. If

$$((a, b, c)d, e) = (a, (b, c, d), e) = (a, b, (c, d, e)) \quad \text{for} \quad a, b, c, d, e \in S,$$

we say that the associativity conditions for the sequence of elements a, b, c, d, e, corresponding to the ternary operation $(*, *, *)$ hold [1]. The associativity conditions for ternary operations in S are said to be independent if none of them is implied by the rest [3]. If the associativity conditions are not independent, they are said to be dependent. In other words, the associativity conditions are dependent if for every ternary operation some of the associativity conditions are implied by the rest [3].

2. In an earlier paper [2] we have shown that if a set S consists of more than five elements then the associativity conditions for ternary operations are necessarily independent. In this paper we complete the information by establishing the following theorem:

Theorem. *For a set S having five or less than five elements, the associativity conditions for ternary operations are dependent.*

For proving the theorem, we divide all the associativity conditions of S into two sets P and Q, P being the set of all associativity conditions in which at least two of the five elements are different, and Q the set of those in which all the five elements are the same. To establish our theorem we show that holding of all associativity conditions in P implies at least one of Q. We do this by showing that not holding of all the conditions in Q and holding of all in P lead to a contradiction.

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3. We use the following lemmas in the proof of the theorem. In all the lemmas it is assumed that the associativity conditions in P hold.

Lemma 1. *If* $(a, a, a)=b$ $(\neq a)$ *then* $(a, b, b)=(b, a, b)=(b, b, a)$.

Proof.

$$(a, b, b) = \big(a, (a, a, a), b\big) = \big((a, a, a), a, b\big) = (b, a, b),$$
$$(b, a, b) = \big(b, a, (a, a, a)\big) = \big(b, (a, a, a), a\big) = (b, b, a).$$

Lemma 2. *If* $(a, a, a)=b(\neq a)$ *and any one of the triples* (b, a, a), (a, b, a), (a, a, b) *is* a, *then all of them are* a.

Proof. If $(b, a, a)=a$ then

$$a = (b, a, a) = \big(b, (b, a, a), a\big) = \big(b, b, (a, a, a)\big) = (b, b, b);$$

if $(a, b, a)=a$ then

$$a = (a, b, a) = \big(a, b, (a, b, a)\big) = \big(a, (b, a, b), a\big) = \big(a, (a, b, b), a\big) =$$
$$= \big(a, a, (b, b, a)\big) = \big(a, a, (a, b, b)\big) = \big((a, a, a), b, b\big) = (b, b, b),$$

by Lemma 1; while if $(a, a, b)=a$ then

$$a = (a, a, b) = \big(a, (a, a, b), b\big) = \big((a, a, a), b, b\big) = (b, b, b).$$

In each case we have got $(b, b, b)=a(\neq b)$ which implies

$$(a, b, a) = (b, a, a) = (a, a, b) = a$$

by Lemma 1. This proves the lemma.

Lemma 3. *If* $(a, a, a)=b(\neq a)$ *and any one of the triples* (b, a, a), (a, b, a), (a, a, b) *is* b, *then all of them are* b.

Proof. If $(b, a, a)=b$ then

$$b = (b, a, a) = \big((b, a, a), a, a\big) = \big(b, (a, a, a), a\big) = (b, b, a) = (b, a, b) = (a, b, b)$$

by Lemma 1; if $(a, b, a)=b$ then

$$(b, a, a) = \big((a, b, a), a, a\big) = \big(a, b, (a, a, a)\big) = (a, b, b),$$
$$(a, a, b) = \big(a, a, (a, b, a)\big) = \big((a, a, a), b, a\big) = (b, b, a)$$

and, again by Lemma 1,

$$b = (a, b, a) = \big(a, (a, b, a), a\big) = \big((a, a, b), a, a\big) = \big((b, b, a), a, a\big) = \big((a, b, b), a, a\big) =$$
$$= \big((b, a, a), a, a\big) = \big(b, a, (a, a, a)\big) = (b, a, b) = (a, b, b) = (b, b, a);$$

if $(a, a, b)=b$ then

$$b = (a, a, b) = \big(a, a, (a, a, b)\big) = \big(a, (a, a, a), b\big) = (a, b, b) = (b, a, b) = (b, b, a).$$

Further, if $b=(a, b, b)=(b, a, b)=(b, b, a)$ then

$$(b, a, a) = \big((a, b, b), a, a\big) = \big(a, (b, b, a), a\big) = (a, b, a),$$

$$(a, b, a) = \big(a, (a, b, b), a\big) = \big(a, a, (b, b, a)\big) = (a, a, b).$$

Hence the lemma is proved.

Lemma 4. *Let* $(a, a, a)=b$ *and* $(b, a, a)=c$. *Then*
(i) $(b, b, b)=(a, b, c)=(b, a, c)=(b, c, a)=(c, a, b)=(c, b, a)$ *and* $(c, a, c)=(c, c, a)$. *Moreover,*
(ii) *if* $(a, a, b)=d$ *then* $(b, b, b)=(b, d, a)=(b, a, d)=(d, a, b)$ *and* $(d, a, d)=(d, d, a)$;
(iii) *if* $(a, b, a)=e$ *then* $(b, b, b)=(b, a, e)=(b, e, a)=(a, b, e)$ *and* $(a, e, e)=(b, b, e)$;
(iv) *if* $(a, a, b)=c$ *and* $(a, b, a)=d$ *then* $(b, b, b)=(b, a, d)=(b, d, a)=(a, b, d)$ *and* $(a, d, d)=(b, b, d)$.

Proof. Since $(a, a, a)=b$, by Lemma 1 we have $(a, b, b)=(b, a, b)=(b, b, a)$.

(i)
$$(a, b, c) = \big(a, b, (b, a, a)\big) = \big((a, b, b), a, a\big) = \big((b, b, a), a, a\big) = \big(b, b, (a, a, a)\big) = (b, b, b),$$
$$(b, a, c) = \big((a, a, a), a, c\big) = \big(a, (a, a, a), c\big) = (a, b, c) = (b, b, b),$$
$$(b, c, a) = \big(b, (b, a, a), a\big) = \big(b, b, (a, a, a)\big) = (b, b, b),$$
$$(c, a, b) = \big((b, a, a), a, b\big) = \big(b, (a, a, a), b\big) = (b, b, b),$$
$$(c, b, a) = \big(c, (a, a, a), a\big) = \big(c, a, (a, a, a)\big) = (c, a, b) = (b, b, b),$$
$$(c, a, c) = \big(c, a, (b, a, a)\big) = \big((c, a, b), a, a\big) = \big((c, b, a), a, a\big) = \big(c, (b, a, a), a\big) = (c, c, a).$$

(ii)
$$(b, d, a) = \big(b, (a, a, b), a\big) = \big((b, a, a), b, a\big) = (c, b, a) = (b, b, b),$$
$$(b, a, d) = \big(b, a, (a, a, b)\big) = \big(b, (a, a, a), b\big) = (b, b, b),$$
$$(d, a, b) = \big((a, a, b), a, b\big) = \big(a, a, (b, a, b)\big) = \big(a, a, (a, b, b)\big) = \big((a, a, a), b, b\big) = (b, b, b).$$
$$(d, a, d) = \big((a, a, b), a, d\big) = \big(a, a, (b, a, d)\big) = \big(a, a, (b, d, a)\big) = \big((a, a, b), d, a\big) = (d, d, a).$$

(iii)
$$(b, a, e) = \big(b, a, (a, b, a)\big) = \big((b, a, a), b, a\big) = (c, b, a) = (b, b, b),$$
$$(b, e, a) = \big(b, (a, b, a), a\big) = \big(b, a, (b, a, a)\big) = (b, a, c) = (b, b, b),$$
$$(a, b, e) = \big(a, b, (a, b, a)\big) = \big(a, (b, a, b), a\big) = \big(a, (b, b, a), a\big) = \big(a, b, (b, a, a)\big) = (a, b, c) = (b, b, b).$$

$$(a, e, e) = (a, (a, b, a), e) = (a, a, (b, a, e)) = (a, a, (a, b, e)) =$$
$$= ((a, a, a), b, e) = (b, b, e).$$

(iv) follows immediately from (iii) if e and d are replaced by d and c, respectively.

4. Proof of the theorem. We proceed step by step choosing first sets with one element only, then with two elements and so on, and finally with five elements. We note that our hypothesis, not holding of all the associativity conditions in Q, is the same as

$$\text{for all } x \in S, \text{ either } ((x, x, x), x, x) \neq (x, (x, x, x), x)$$
$$\text{or } (x, x, (x, x, x)) \neq (x, (x, x, x), x)$$
$$\text{or } ((x, x, x), x, x) \neq (x, x, (x, x, x)),$$

which implies in particular that

$$(x, x, x) \neq x. \tag{4.1}$$

Step I. Let S consist of one element only, say, a. Then clearly $(a, a, a)=a$, which contradicts (4.1).

Step II. Let S consist of two distinct elements a, b, say. Then, in view of (4.1), $(a, a, a)=b$ and $(b, b, b)=a$. Hence $((a, a, a), a, a)=(b, a, a)$, $(a, (a, a, a), a)= =(a, b, a)$, $(a, a, (a, a, a))=(a, a, b)$. On the other hand,

$$(b, a, a) = (b, (b, b, b), a) = ((b, b, b), b, a) = (a, b, a),$$
$$(a, b, a) = (a, b, (b, b, b)) = (a, (b, b, b), b) = (a, a, b),$$

which contradicts our hypothesis concerning Q.

Step III. Let S consist of three distinct elements a, b, c, say. Under the hypothesis $(a, a, a)\neq a$, let us denote $b=(a, a, a)$. Further, since all the triples (b, a, a), (a, b, a), (a, a, b) cannot be equal to c (see Step II), at least one of them must be either a or b. However, if any one of them equals a then all equal a by Lemma 2, and if any one of them equals b, all of them get equal to b by Lemma 3. Both cases are in contradiction with the hypothesis as laid down in (4.1).

Step IV. Let S consist of four distinct elements a, b, c, d, say. Then, as in Step III, we fix $b=(a, a, a)$. Further, as demonstrated in Step III, none of the triples (b, a, a), (a, b, a), (a, a, b) can be equal to a or b. Hence it will be sufficient for us to consider the case when two of these triples are equal either to c or d and the third one to d or c, respectively. Thus without loss of generality we can consider the cases

$$(b, a, a) = c, \quad (a, a, b) = d \quad \text{and} \quad (a, b, a) = c \quad \text{or} \quad d, \tag{1}$$

$$(b, a, a) = c = (a, a, b) \quad \text{and} \quad (a, b, a) = d. \tag{2}$$

As $(a, a, (b, a, a)) = ((a, a, b), a, a) = (a, (a, b, a), a)$, we have

(3) in case (1): $(a, a, c) = (d, a, a)\big(= (a, c, a)$ or $(a, d, a)\big)$

and

(4) in case (2): $(a, a, c) = (c, a, a) = (a, d, a)$.

We distinguish four subcases according to the value of (a, a, c).

Step IVA. If $(a, a, c) = a$ then by Lemma 4 (i)

$$b = (a, a, a) = \big(a, a\, (a, a, c)\big) = \big(a, (a, a, a), c\big) = (a, b, c) = (b, b, b)$$

which is in contradiction with (4.1).

Step IVB. Case (1). If $(a, a, c) = b$ then $(d, a, a) = b$ (by (3)). Furthermore,

$$c = (b, a, a) = \big((d, a, a), a, a\big) = \big(d, a, (a, a, a)\big) = (d, a, b) = (b, b, b)$$

by Lemma 4 (ii) and

$$d = (a, a, b) = \big(a, a, (a, a, c)\big) = \big(a, (a, a, a), c\big) = (a, b, c) = (b, b, b)$$

by Lemma 4 (i).

Case (2). If $(a, a, c) = b$ then

$$c = (a, a, b) = \big(a, a, (a, a, c)\big) = \big(a, (a, a, a), c\big) = (a, b, c) = (b, b, b)$$

and

$$d = (a, b, a) = \big(a, (a, a, c), a\big) = \big((a, a, a), c, a\big) = (b, c, a) = (b, b, b)$$

by Lemma 4 (i). Thus in both the cases $c = d$, which is a contradiction.

Step IVC. If $(a, a, c) = c$ then

$$c = (a, a, c) = \big(a, a, (a, a, c)\big) = \big((a, a, a), a, c\big) = (b, a, c) =$$
$$= \big(b, a, (a, a, c)\big) = \big((b, a, a), a, c\big) = (c, a, c) = (c, c, a)$$

by Lemma 4 (i) and hence

$$c = (c, a, c) = \big((c, c, a), a, c\big) = \big(c, c, (a, a, c)\big) = (c, c, c).$$

But $(c, c, c) = c$ is in contradiction with (4.1).

Step IVD. Case (1). If $(a, a, c) = d$ then $(d, a, a) = d$, too (by (3)), whence

$$d = (d, a, a) = \big((d, a, a), a, a\big) = \big(d, a, (a, a, a)\big) = (d, a, b) =$$
$$= \big((d, a, a), a, b\big) = \big(d, a, (a, a, b)\big) = (d, a, d) = (d, d, a)$$

by Lemma 4 (ii), so that

$$d = (d, a, b) = \big((d, d, a), a, b\big) = \big(d, d, (a, a, b)\big) = (d, d, d).$$

Case (2). If $(a, a, c)=d$ then $(a, d, a)=d$, too (by (4)). Furthermore,

$$d = (a, d, a) = \big(a, (a, a, c), a\big) = \big((a, a, a), c, a\big) = (b, c, a) = (b, b, b) = (b, d, a)$$

by Lemma 4 (i), (iv) and thus

$$d = (b, d, a) = \big(b, (b, d, a), a\big) = \big(b, (b, a, d), a\big) = \big(b, b, (a, d, a)\big) =$$
$$= (b, b, d) = (a, d, d) = \big(a, d, (a, d, d)\big) = \big((a, d, a), d, d\big) = (d, d, d)$$

by Lemma 4 (iv) again. But $(d, d, d)=d$ is again a contradiction.

Step V. Let S consist of five distinct elements a, b, c, d, e, say. We have to consider only the case when $(b, a, a)=c$, $(a, a, b)=d$ and $(a, b, a)=e$.

As $\big(a, a, (b, a, a)\big)=\big((a, a, b), a, a\big)=\big(a, (a, b, a), a\big)$, we have $(a, a, c)= =(d, a, a)=(a, e, a)$. If $(a, a, c)=a$ or b or c or d, a contradiction can be established as in the case (1) of Step IV. If $(a, a, c)=(d, a, a)=(a, e, a)=e$ then

$$e = (a, e, a) = \big(a, (a, a, c), a\big) = \big((a, a, a), c, a\big) = (b, c, a) = (b, d, a) = (b, e, a)$$

by Lemma 4 (i)—(iii), whence

$$e = (b, e, a) = \big(b, (b, d, a), a\big) = \big(b, b, (d, a, a)\big) = (b, b, e) = (a, e, e)$$

by Lemma 4 (iii), implying that

$$e = (a, e, e) = \big(a, e, (a, e, e)\big) = \big((a, e, a), e, e\big) = (e, e, e).$$

But $(e, e, e)=e$ is in contradiction with (4.1).

This completes the proof of the theorem.

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Automorphism groups of subalgebras; a concrete characterization

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FRIED and SICHLER [1] raise the following question: Let L be an algebraic lattice and let G_x be a group for every $x \in L$. Under what circumstances is there an algebra $\mathfrak{A}$ with $\mathrm{Sub}(\mathfrak{A}) \cong L$ such that if $\mathfrak{A}_x$ denotes the subalgebra corresponding to x, then $\mathrm{Aut}(\mathfrak{A}_x) \cong G_x$. In this note we solve the concrete version of the problem (Theorem 1).

The proof uses the techniques developed by B. JÓNSSON [3], M. G. STONE [4], J. JEŽEK [2], L. SZABÓ [5]. A direct application of our result to the original problem yields a new proof of Theorem 1 in [1] (Corollary 2).

In the second part of this note we obtain a partial result on the representability of a small concrete category as a category of universal algebras with prescribed subalgebras.

1. Let A be a set, L a family of subsets of A and let a permutation group G_R, acting on R, be assigned to each $R \in L$. The elements of G_R may then be regarded as partial mappings of A. Let φ and ψ be two partial mappings of A (into itself). Then we can define their product $\psi\varphi$, where $(\psi\varphi)a$ is defined iff $a \in \mathrm{Dom}\,\varphi$, $\varphi a \in \mathrm{Dom}\,\psi$ and in that case $(\psi\varphi)a = \psi(\varphi a)$. So, all the partial mappings of A, including the empty mapping $\emptyset \to \emptyset$ form a semigroup. Let $\bar{G}$ denote its subsemigroup, generated by $\{G_R : R \in L\}$.

We shall call G_R locally $\bar{G}$ closed if the following condition holds: For every permutation h of R, if for every finite subset X of R there exists a member of $\bar{G}$ that agrees with h on X, then $h \in G_R$.

For $\varphi : A \to A$ a partial injection (thus $x \neq y$ implies $\varphi x \neq \varphi y$ provided $x, y \in \mathrm{Dom}\,\varphi$), φ^{-1} can be defined in a natural way.

With the above notation, we have

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Theorem 1. *Let A be a set, L a family of subsets of A, and let a permutation group G_R, acting on R, be assigned to each $R\in L$. A universal algebra $\mathfrak{A}=\langle A, F\rangle$ with precisely the elements of L for its subuniverses, and with precisely the elements of G_R for the automorphism of $R\in L$, exists if and only if*

(i) *L is an algebraic closure system on A,*

(ii) *G_R is locally $\bar{G}$-closed for any $R\in L$,*

(iii) *$\sigma R\in L$ for any $\sigma\in\bar{G}$ and $R\in L$,*

(iv) *$\sigma|X=\tau|X$ implies $\sigma|\langle X\rangle=\tau|\langle X\rangle$ provided $\sigma,\tau\in\bar{G}$ and $X\subset A$ is finite.*

(Here $\langle X\rangle$ denotes the closure of X with respect to L.)

Proof. The conditions listed are necessary, since the elements of $\bar{G}$ are isomorphisms between certain subalgebras of A.

To prove sufficiency, let Y denote the set of the one-to-one sequences of finite length composed of the elements of A.

For any $y=(y_1,\dots,y_n)\in Y$, $a\in\langle y_1,\dots,y_n\rangle$ let us define an n-ary operation $f_{(y,a)}$ on A by

$$f_{(y,a)}(w_1,\dots,w_n)=\begin{cases}\sigma a & \text{if } \exists\sigma\in\bar{G}, w=\sigma y\\ w_1 & \text{otherwise.}\end{cases}$$

(Note, in particular, that $f_{(y,a)}(y)=a(y\in A^n, a\in A)$.)

The definition makes sense owing to condition (iv) ($\sigma_1 y=\sigma_2 y$ implies $\sigma_1 a=\sigma_2 a$.). In this way we have constructed a universal algebra $\mathfrak{A}=\langle A, f_{(y,a)}:y\in Y, a\in\langle y\rangle\rangle$. We assert that $\mathfrak{A}$ complies with the requirements.

Let $R\in L$; $w_1,\dots,w_n\in R$ and let us consider $f_{(y,a)}(w)$. If the second situation obtains, then $f_{(y,a)}(w)\in R$. Let therefore $w=\sigma y$. Then $\sigma^{-1}w=y$, whence $\langle w\rangle=\sigma\langle y\rangle$; but since $a\in\langle y\rangle$, we have $\sigma a\in\langle w\rangle\subset R$. Hence $R\in\mathrm{Sub}(\mathfrak{A})$.

Conversely let $Q\in\mathrm{Sub}(\mathfrak{A})$, and let $a\in\langle Q\rangle$. In that case there exist some $y_1,\dots,y_n\in Q$ such that $a\in\langle y_1,\dots,y_n\rangle$. But then $f_{(y,a)}(y)=a\in Q$. Consequently $Q=\langle Q\rangle$, implying that the subuniverses are precisely the elements of L.

Given $\tau\in\bar{G}$ and $w\in(\mathrm{dom}\,\tau)^n$, we assert that $f_{(y,a)}(\tau w)=\tau f_{(y,a)}(w)$. For, let $w=\sigma y$. Then $\tau w=\tau\sigma y$, whence $f_{(y,a)}(\tau w)=\tau\sigma a=\tau f_{(y,a)}(w)$. If $\tau w=\sigma y$, then $\tau w=\sigma y=\tau(\tau^{-1}\sigma)y$, whence $w=\tau^{-1}\sigma y$, implying $f_{(y,a)}(\tau w)=\tau(\tau^{-1}\sigma)a=\tau f_{(y,a)}(w)$. In particular, the elements of the G_R-s are automorphisms of $\mathfrak{A}|R$.

If $\varphi\notin G_R$, then by (ii) there exist $y_1,\dots,y_n\in R$ such that $\varphi y\neq\sigma y$ for any $\sigma\in\bar{G}$, and we may assume that $n\geqq 2$. But then $\varphi f_{(y,y_2)}(y)=\varphi y_2$; $f_{(y,y_2)}(\varphi y)=\varphi y_1$, since, however, $y_1\neq y_2$, also $\varphi y_2\neq\varphi y_1$, whence $\varphi\notin\mathrm{Aut}(\mathfrak{A}|R)$ completing the proof.

2. Now we derive some corollaries of Theorem 1. Let $\mathfrak{A}$ be a universal algebra, $L=\mathrm{Sub}(\mathfrak{A})$, $G_x=\mathrm{Aut}(x):x\in L$, $L_x=\{y\in L:y\subseteqq x\}$. Then there exist natural homomorphisms $\Phi_x:G_x\to\mathrm{Aut}(L_x)$.

Let $H_x=\mathrm{Ker}\,\Phi_x$. Let us first prove the following statement.

Corollary 1. *Given an algebra* $\langle A, F\rangle$ *there exists an algebra* $\langle A, G\rangle$ *such that* $\mathrm{Sub}\langle A, G\rangle=\mathrm{Sub}\langle A, F\rangle$ *and* $\mathrm{Aut}_G(x)=H_x$ *for any* $x\in\mathrm{Sub}\langle A, G\rangle$.

Proof. Of the conditions listed under Theorem 1, (i) clearly holds for the system $(L, H_x : x\in L)$; and so do (iii) and (iv), since $\bar{H}\subseteq\bar{G}$. If the permutation $\varphi : x\to x$ coincides on any finite set with a permutation in $\bar{H}$, then it coincides with a permutation in $\bar{G}$, whence $\varphi\in G_x$. On the other hand, the elements of $\bar{H}$ leave any subuniverse in place. We conclude that φ belongs to H_x: hence, (ii) also holds, which completes the proof.

If now $\varphi\in H_x$, and $y\subseteq x$. Then $\varphi|y\in H_y$, and we get a homomorphism $r_{xy}: H_x\to H_y$. Let $\Sigma=(L, (K_x : x\in L), (P_{xy} : x, y\in L, x\geqq y))$; where L is an algebraic lattice, K_x a group for each $x\in L$, and P_{xy} a homomorphism of K_x into K_y $(x, y\in L)$.

We say that Σ is representable if there exists an algebra $\mathfrak{A}$ such that $\mathrm{Sub}(\mathfrak{A})\cong L$, $K_x\cong H_x=G_x$ for each $x\in L$, each P_{xy} represents the restriction homomorphism r_{xy}.

Now we can prove the following

Corollary 2. (Fried—Sichler [1]) Σ *is representable iff*

(v) $P_{yx}\cdot P_{zy}=P_{zx}$ *for all* $x\leqq y\leqq z$,

(vi) $\mathrm{Ker}\,P_{xy}\cap\mathrm{Ker}\,P_{xz}$ *is trivial whenever* $x=y\cup z$,

(vii) *if* $x\in L$ *is not compact then* K_x *is the inverse limit of the diagram* $(P_{cd} : x>c\geqq d)$ *with the limit homomorphisms* P_{xc} $(c<x)$,

(viii) $K_0=1$.

Proof. The necessity of the conditions can be easily checked (cf. [1]). For the sufficiency we use the construction given in [1]. The correctness of the construction will easily follow from our Theorem 1.

Let $C=\{c\in L : c \text{ compact}\}$. $A=\{(c,\alpha) : c\in C,\ \alpha\in K_c\}$. $A_x=\{(c,\alpha) : c\leqq x,\ \alpha\in K_c\}$. $\bar{L}=\{A_x : x\in L\}$. Then $\bar{L}$ is an algebraic closure system, and regarded as a lattice it is isomorphic with L.

For any $\varphi\in K_x$ define $T_\varphi : A_x\to A_x$ by $T_\varphi(c,\alpha)=(c,(P_{xc}\varphi)\alpha)$. $G_x=\{T_\varphi : \varphi\in K_x\}$. Now $G_x\cong K_x$, and the elements of G_x leave the subuniverses in place, whence $G_x=H_x$.

Because of $G_x=H_x$ we have $\bar{G}=\bigcup G_x$, so (iii) holds. Now if X is a finite subset of A and $\sigma,\tau\in\bar{G}$ and $\sigma|X=\tau|X$ then for all $x\in X$ $\sigma|\langle x\rangle=\tau|\langle x\rangle$ by the definition of T_φ, hence by (vi) we also have $\sigma|\langle X\rangle=\tau|\langle X\rangle$. If now $A_x\in\bar{L}$, $\psi : A_x\to A_x$ and ψ coincides on any finite subset of A_x with an element of $\bar{G}$, then clearly we have $\psi(c,\alpha)=(c,\psi_c(\alpha))$ for some permutation ψ_c of K_c, since T_φ maps K_c into itself for all φ and c. Now considering our condition for the two element set $\{(c,1),(c,\alpha)\}$ we have $\psi(c,1)=T_\varphi(c,1)$, $\psi(c,\alpha)=T_\varphi(c,\alpha)$ for some T_φ and we have $\psi(c,\alpha)=(c,\psi_c(1)\alpha)$. If we consider $\{(c,1),(d,1)\}$ for any pair $c\leqq d$ then we have $\psi_c(1)=P_{dc}\psi_d(1)$, hence if $A_y \subset A_x$ and A_y is finitely generated then $\psi_y(1)$ determines uniquely $\psi|A_y$ hence $\psi|A_y=T_\varphi|A_y$ for some $T_\varphi \in G_y$. Applying (vii) on the system of the T_φ-s (as A_y runs over the set of finitely generated subalgebras of A_x) we get that $\psi=T_\sigma$ for some $\sigma \in G_x$. Hence Theorem 1 can be applied to prove corollary 2.

3. In [4] STONE considered the subalgebras and the automorphisms of subalgebras, in [5] SZABÓ (Theorem 1) the subalgebras and the isomorphisms between subalgebras, and in [2] JEŽEK (Theorem 2) considered a small category of algebras and the injective morphism. We are going to derive a common generalization of these results.

Let K be a small subcategory of SETS. The elements of Mor K can be considered as partial mapping of $\cup \mathrm{Ob}\, K$ (we consider this union to be disjoint). Let $\bar{S}$ denote the semigroup generated by the injective elements of Mor K and their inverses. With this notation we have

Theorem 2. *Let K be a small subcategory of* SETS *such that if $\alpha \in \mathrm{Mor}\, K$, then α is either injective or a mapping onto a single point. For each $A \in \mathrm{Ob}\, K$, let $L(A)$ be a family of subsets of A. There exists a set F of operation symbols and universal algebras $\langle A, F\rangle$ to each $A \in \mathrm{Ob}\, K$, such that* $\mathrm{Sub}\,\langle A, F\rangle = L(A)$ *and* $\mathrm{Hom}(\langle A, F\rangle, \langle B, F\rangle) = \mathrm{Mor}(A, B)$ *iff*

(ix) *$L(A)$ is an algebraic closure system for every $A \in \mathrm{Ob}\, K$,*

(x) *$R \in L(A)$, $\varphi: A \to B \in \bar{S}$, $R \subset \mathrm{Dom}\, \varphi$ implies $\varphi R \in L(B)$,*

(xi) *$\mathrm{Mor}(A, B)$ is locally $\bar{S}$ closed,*

(xii) *if $R \in L(A)$, $|R|=1$, $B \in \mathrm{Ob}\, K$ then $\varphi_{B,R}: B \to R \in \mathrm{Mor}(B, A)$,*

(xiii) *if $\sigma, \tau \in \bar{S}$, $X \subset A$ is finite then $\sigma|X=\tau|X$ implies $\sigma|\langle X\rangle=\tau|\langle X\rangle$ where $\langle X\rangle$ denotes the closure of X with respect to $L(A)$.*

Proof. Necessity of (ix), (x) and (xii) is obvious. The elements of $\bar{S}$ are compatible with the operations, therefore (xi) and (xiii) are also necessary.

In order to prove the sufficiency let $Y(A)$ denote the set of one-to-one sequences formed from the elements of A and $Y=\cup\{Y(A): A \in \mathrm{Ob}\, K\}$. For each $y=(y_1, \dots, y_n) \in Y$, $a \in \langle y_1, \dots, y_n\rangle$, we define an n-ary operation $f_{(y,a)}$: for $B \in \mathrm{Ob}\, K$, $w_1, \dots, w_n \in B$ set

$$f_{(y,a)}(w_1, \dots, w_n) = \begin{cases} \sigma a & \text{if there exists a } \sigma \in \bar{S} \text{ such that } w = \sigma y \\ w_1 & \text{otherwise.} \end{cases}$$

This definition makes sense owing to condition (xiii). Having endowed each $A \in \mathrm{Ob}\, K$ with this set $F=\{f_{(y,a)}\}$ of operations, we assert that the resulting set $\{\langle A, F\rangle: A \in \mathrm{Ob}\, K\}$ of universal algebras complies with our requirements.

As in Theorem 1, we can deduce that $\mathrm{Sub}\langle A, F\rangle = L(A)$, and $\mathrm{Mor}(A, B) \subseteq \mathrm{Hom}(\langle A, F\rangle, \langle B, F\rangle)$. Therefore it suffices to prove that $\mathrm{Hom}(\langle A, B\rangle, \langle B, F\rangle) \subseteq \mathrm{Mor}(A, B)$.

Let $\varphi \in \mathrm{Hom}(\langle A, F\rangle, \langle B, F\rangle)$. If $|\mathrm{Im}\,\varphi| = 1$ then $\varphi \in \mathrm{Mor}(A, B)$ because of (xii).

If φ is injective, and $y_1, \dots, y_n \in A$ $(n \geqq 2)$, then $\varphi f_{(y, y_2)}(y) = \varphi y_2$, hence $f_{(y, y_2)}(\varphi y) = \varphi y_2$, but $\varphi y_1 \neq \varphi y_2$ and therefore $\varphi y = \sigma y$ for some $\sigma \in \bar{S}$ and we conclude from (xi) that $\varphi \in \mathrm{Mor}(A, B)$.

If φ is neither injective nor a mapping onto a single point, then there exist $y_1, y_2, y_3 \in A$ such that $\varphi y_1 \neq \varphi y_2 = \varphi y_3$. But then $\varphi y \neq \sigma y$ any $\sigma \in \bar{S}$, for the elements of $\bar{S}$ are injective and hence $\varphi f_{(y, y_2)}(y) = \varphi y_2 \neq \varphi y_1 = f_{(y, y_2)}(\varphi y)$. Hence $\varphi \notin \mathrm{Hom}(\langle A, F\rangle, \langle B, F\rangle)$, completing the proof.

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Characterizations of completely regular elements in semigroups

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Let S be an arbitrary semigroup, $a \in S$. The element a of S is said to be *completely regular* (c.r.) if there exists an element x in S such that

$$(1) \qquad axa = a \quad \text{and} \quad ax = xa$$

holds. Evidently, every c.r. element of S is regular, but not conversely. A semigroup S is called c.r. if all its elements are c.r.

We shall make use of the following well-known result:

Lemma. (CROISOT [2]) *An element a of a semigroup S is completely regular if and only if $a \in a^2 S a^2$.*

Our first result is stated in the following

Theorem 1. *An element a of a semigroup S is c.r. if and only if there exists an idempotent element e in S such that*

$$(2) \qquad B(a) = B(e)$$

holds.[1])

Proof. Let a be a c.r. element of a semigroup S. Then S has an element x with property (1). Hence it follows easily that $B(a) = aSa$. Let us introduce the notation $e = ax$. Then it is easy to see that $e^2 = e$ and $B(e) = (ax)S(xa) \subseteq aSa$. On the other hand, we conclude that $B(a) = (axa)S(axa) = e(aSa)e \subseteq eSe$. Since $B(e) = eSe$, we obtain (2).

Conversely, suppose that for an element a of a semigroup S there exists an idempotent e such that condition (2) holds. This means that

$$(3) \qquad \{a\} \cup aS^1a = eSe.$$

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[1]) $B(a)$ denotes the principal bi-ideal of S generated by the element a of S. For other notations and terminology we refer to [1] and [5].

Hence it follows that $a=ese$, where $s\in S$. Thus we obtain the equations

(4) $$ea = a = ae.$$

Condition (3) implies also that $e=a$ or $e=ata$, where $t\in S^1$. In the latter case we get $ata^2=a=a^2ta$, whence $a=a^2(tat)a^2$, that is, the element a is c.r., indeed. This holds in the case $e=a$ too, whence Theorem 1 is proved.

Corollary 1. *Let $\mathcal{M}_n$ be the multiplicative semigroup of all $n\times n$ complex matrices. An element A of $\mathcal{M}_n$ has a group inverse (or Drazin* 1*-inverse) if and only if there is an idempotent matrix E in $\mathcal{M}_n$ such that $A\mathcal{M}_nA=E\mathcal{M}_nE$ holds.*

A matrix X is a *group inverse* of A if $AXA=A$, $XAX=X$, and $AX=XA$ (cf. Erdélyi [4]). This notion is a particular case of the Drazin pseudo-inverse (cf. Drazin [3]).

Corollary 1 follows at once from Theorem 1 because $\mathcal{M}_n$ is a regular semigroup.

Corollary 2. *A semigroup S is c.r. if and only if for every element a of S there exists an idempotent e_a in S such that $B(a)=B(e_a)$.*

It may be remarked that Theorem 1 and Corollary 2 remain true with quasi-ideal instead of bi-ideal.

Theorem 2. *An element a of a semigroup S is c.r. if and only if there exists an idempotent element e in S such that*

(5) $$Q(a) = Q(e)$$

holds.

Proof. Let a be a c.r. element of a semigroup S, i.e., there is an element x in S with property (1). Using the notation $e=ax$ we have $L(a)=Se$ and $R(a)=$ $=eS$. Thus it follows that $Q(a)=aS\cap Sa=L(a)\cap R(a)=eS\cap Se=Q(e)$.

Conversely, if to an element a of a semigroup S there is an idempotent e admitting the property (5), then

(6) $$aS^1\cap S^1a = eS\cap Se.$$

Hence $a=es=te$, where $s, t\in S$. Hence (4) follows. On the other hand, (6) implies $e\in aS^1\cap S^1a$, that is, $e=a$ or $e=au=va$ with $u, v\in S$. From the latter equations we get that $a^2u=a=va^2$, i.e., $a\in a^2Sa^2$. Therefore a is c.r. by the Lemma. Thus Theorem 2 is completely proved.

Corollary 1. *A semigroup S is c.r. if and only if for each element a of S there exists an idempotent e_a in S such that $Q(a)=Q(e_a)$.*

Corollary 2. *An element a of a semigroup S is c.r. if and only if the $\mathcal{H}$-class of a is a group.*

This follows from Theorem 2 and from a result by PONDĚLÍČEK [12].

Some further characterizations of c.r. elements and c.r. semigroups can be given, but the proofs are similar to the above proofs of Theorem 1 and Theorem 2, and we omit them.

Theorem 3. *For an element a of a semigroup S the following conditions are equivalent:*

(A) *a is completely regular.*
(B) $\exists e\in E(S)$ *such that* $B(a)=B(e)$.
(C) $\exists e\in E(S)$ *such that* $Q(a)=Q(e)$.
(D) $\exists e\in E(S)$ *such that* $B(a)=R(e)L(e)$.
(E) $\exists e\in E(S)$ *such that* $Q(a)=R(e)L(e)$.
(F) $\exists e\in E(S)$ *such that* $B(a)=L(e)\cap R(e)$.
(G) $\exists e\in E(S)$ *such that* $Q(a)=eS\cap Se$.
(H) $\exists e\in E(S)$ *such that* $B(a)\cap Q(a)=eSe$.
(I) $\exists e\in E(S)$ *such that* $B(a)\cap Q(a)=R(e)L(e)$.
(J) $\exists e\in E(S)$ *such that* $L(a)\cap R(a)=R(e)L(e)$.
(K) *The principal bi-ideal $B(a)$ is a monoid.*
(L) *The principal quasi-ideal $Q(a)$ is a monoid.*
(M) *The principal (m, n)-ideal generated by a is a monoid* $(m, n>0)$.
(N) $\exists e\in E(S)$ *such that* $\{a\}_{(m,n)}=\{e\}_{(m,n)}$, *where* $m, n>0$.

For the definition of (m, n)-ideals, see [6]. For earlier characterizations of completely regular semigroups as well as of completely regular elements, see [7], [8], [9], [10].

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Derivations and translations on l-semigroups

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Introduction. Functions of lattices into themselves have been studied in [4], [7], [9], [10]. With respect to pointwise intersection $\wedge$ and union $\vee$ and to composition of functions $\circ$ the set $(F(L), \wedge, \vee, \circ)$ of all transformations of a lattice $(L, \wedge, \vee)$ forms a "*right-lattice ordered semigroup*" (rl-semigroup; see [3], [4]). This means a set S with three binary operations $\wedge$, $\vee$ and $\cdot$, such that $(S, \cdot)$ is a semigroup, $(S, \wedge, \vee)$ is a lattice and

$$(x\vee y)z = (xz)\vee(yz), \quad (x\wedge y)z = (xz)\wedge(yz) \quad \text{for all} \quad x, y, z\in S.$$

Note that with respect to the order-relation induced by the lattice-operations multiplication satisfies: $x\leqq y\Rightarrow xz\leqq yz$ for each $z\in S$.

Recently SZÁSZ [9], [10] started the investigation of special functions on lattices $(L, \wedge, \vee)$, so-called "*derivations*", motivated by the formal rules of derivations in rings, i.e. transformations φ of L which satisfy

$$\varphi(x\vee y) = \varphi(x)\vee\varphi(y) \quad \text{and} \quad \varphi(x\wedge y) = [\varphi(x)\wedge y]\vee[x\wedge\varphi(y)] \quad \text{for all} \quad x, y\in L.$$

Since in $F(L)$ also the composition of functions is defined, it is natural to consider transformations of the rl-semigroup $(F(L), \wedge, \vee, \circ)$ which satisfy also a formal *chain rule:*

$$\varphi(f\circ g) = [\varphi(f)\circ g]\wedge\varphi(g) \quad \text{for all} \quad f, g\in F(L).$$

For rings with a third operation $\circ$, so-called *composition-rings,* such derivations with chain-rule have been studied — especially for polynomial-rings — in [6], [8].

In the following we suppose S to be a right-lattice ordered semigroup and investigate transformations φ of $(S, \wedge, \vee, \cdot)$ — so-called *C-derivations* — which have the following properties:

$$\left.\begin{array}{ll} \text{I.} & \varphi(x\vee y) = \varphi(x)\vee\varphi(y) \\ \text{II.} & \varphi(x\wedge y) = [\varphi(x)\wedge y]\vee[x\wedge\varphi(y)] \\ \text{III.} & \varphi(xy) = [\varphi(x)y]\wedge\varphi(y) \end{array}\right\} \quad \text{for all} \quad x, y\in S.$$

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Standard examples for S will be the rl-semigroups $(F(L), \wedge, \vee, \circ)$ of all transformations of a lattice L, $(L[x], \wedge, \vee, \circ)$ of all polynomials on L in the indeterminate x and $(P(L), \wedge, \vee, \circ)$ of all polynomial-functions on L (see [3]).

We shall use also the concept of *lattice ordered semigroup* (l-semigroup), which is defined as an rl-semigroup $(S, \wedge, \vee, \cdot)$ satisfying also the two left-distributive laws:

$$x(y\vee z) = (xy)\vee(xz) \quad \text{and} \quad x(y\wedge z) = (xy)\wedge(xz) \quad \text{for all} \quad x, y, z\in S.$$

Note that now multiplication also satisfies: $x\leqq y \Rightarrow zx\leqq zy$ for each $z\in S$; for the general theory see [2].

1. Reduction to translations

The main purpose of this section is to show, that every C-derivation φ on an rl-semigroup S is a special meet-translation [9], i.e. $\varphi(x)=x\wedge a$ for all $x\in S, a\in S$ fixed, if the lattice $(S, \wedge, \vee)$ has a greatest element or if the semigroup $(S, \cdot)$ has an identity.

Properties. Let φ be a C-derivation on S; then

0) $\varphi(x\wedge y)=\varphi(x)\wedge\varphi(y)$ for all $x, y\in S$; $\varphi(x)\leqq x$ for all $x\in S$ ([10]).

1) If S has a least element o, then $\varphi(o)=o$ (by II).

2) If c is a left-zero of $(S, \cdot)$ with least element o, then $\varphi(c)=o$ $(\varphi(c)=\varphi(co)= =\varphi(c)o\wedge\varphi(o)=o)$.

3) If S does not have a least element but admits a left-zero, then there is no C-derivation on S. $(\varphi(c)=\varphi(cx)=\varphi(c)x\wedge\varphi(x)\leqq\varphi(x)\leqq x$ for all $x\in S$ by 0): contradiction.)

4) If S has a right-identity e, then $\varphi(x)\leqq\varphi(e)$ for all $x\in S$. $(\varphi(x)=\varphi(xe)= =\varphi(x)e\wedge\varphi(e)\leqq\varphi(e)$ for all $x\in S$.)

5) If S admits a right-identity e and a least element o, then $\varphi(e)=\varphi(o)$ implies $\varphi(x)=o$ for all $x\in S$.

6) If $(S, \cdot)$ is 0-right-simple with o such that $ox=o$ for all $x\in S$, then $\varphi(a)=o$ for an element $a\neq o$ implies $\varphi(x)=o$ for all $x\in S$. (Since $aS=S$, hence for all $x\in S$ there exists a $y\in S$ with $ay=x$; thus $\varphi(x)=\varphi(ay)=\varphi(a)y\wedge\varphi(y)= =oy\wedge\varphi(y)=o$ for all $x\in S$.)

Examples. 1) If S admits a least element o, then $\varphi(x)=o$ for all $x\in S$ is a C-derivation, the *trivial* C-derivation.

2) The identity-function $\varphi(x)=x$ for all $x\in S$ is a C-derivation, iff $xy\leqq y$ for all $x, y\in S$.

3) The constant function $\varphi(x)=a$ for all $x\in S$, $a\in S$ fixed, is a C-derivation,

iff $a=o$ (if $o\in S$ exists) ($a=\varphi(x\wedge x)=\varphi(x)\wedge x\leqq x$ for all x).

4) Concerning meet-translations we know by Corollary 3 of [10]:

Lemma 1.1. *Let S be an rl-semigroup with greatest element i. Then every C-derivation φ on S has the form $\varphi(x)=x\wedge a$ for all $x\in S$ and a suitable element $a\in S$.*

In order to determine the suitable elements $a\in S$ we prove:

Lemma 1.2. *Let S be an rl-semigroup. The function $\varphi(x)=x\wedge a$ for all $x\in S$ is a C-derivation, iff* 1) *a is a neutral element of $(S, \wedge, \vee)$ and* 2) *$xy\wedge a\leqq ay\wedge y$ for all $x, y\in S$.*

Proof. If $a\in S$ satisfies 1), then $\varphi(x)=x\wedge a$ does I and II of the definition, too. If a also satisfies 2), then $\varphi(xy)=xy\wedge a=(xy\wedge a)\wedge(ay\wedge y)=(x\wedge a)y\wedge(a\wedge y)=$ $=\varphi(x)y\wedge\varphi(y)$ for all $x, y\in S$. The converse is clear.

Combining Lemmas 1.1 and 1.2 we get the following

Theorem 1.3. *Let S be an rl-semigroup with greatest element. Then the C-derivations on S are the functions φ of the form $\varphi(x)=x\wedge a$ with a fixed neutral element a of S such that $xy\wedge a\leqq ay\wedge y$ for each pair x, y of elements of S.*

Example. 5) Let S be an rl-semigroup with identity e admitting an invertible element $a\neq e$; then $\varphi(x)=x\wedge e$ is not a C-derivation. (If φ is a C-derivation, then by Lemma 1.2: $xy\wedge e\leqq y$ for all $x, y\in S$; but since $aa'=a'a=e$ for $a'\in S$, this implies in particular $e\leqq a$ and $e\leqq a'$; consequently we get $a'\leqq aa'=e$, thus $a'=a=e$: contradiction.)

Theorem 1.4. *Let S be an rl-semigroup with greatest element i, such that $ix=i$ for all $x\in S$. Then there is no C-derivation on S except the trivial one (if defined).*

Proof. By Theorem 1.3 every C-derivation on S has the form $\varphi(x)=x\wedge a$ such that $xy\wedge a\leqq ay\wedge y$ for all $x, y\in S$. For $x=i$ we get $a\leqq ay\wedge y\leqq y$ for all $y\in S$. If S has a least element o, then $a=o$ and φ is the trivial C-derivation; if not, then we have a contradiction.

The existence of $i\in S$ ensured that every C-derivation is a special meet-translation. The same is true if an identity exists:

Lemma 1.5. *Let S be an rl-semigroup with right-identity e. Then every C-derivation on S has the form $\varphi(x)=x\wedge a$ for all $x\in S$ with $a=\varphi(e)$.*

Proof. Since $x=x\vee(x\wedge e)$, hence $\varphi(x)\geqq x\wedge\varphi(e)$ for all $x\in S$. But $\varphi(x)\leqq$ $\leqq\varphi(e)$ by 4) and $\varphi(x)\leqq x$ for all $x\in S$ by 0); thus $\varphi(x)\leqq x\wedge\varphi(e)$ and equality follows.

Corollary. *If e is the identity of S and $\varphi(x)=x\wedge a$ is a C-derivation, then:* 1) $a\leqq e$; 2) $a^2=a$; 3) *$xy=y$ for all $y\leqq a\leqq x\leqq e$.*

Proof. Since $a=\varphi(e)=e\wedge a$, we have $a\leqq e$; thus $a^2\leqq a$, $ay\leqq y$ for all $y\in S$. By Lemma 1.2: $xy\wedge a\leqq ay\wedge y=ay$ for all $x, y\in S$; for $x=e$ we get $y\wedge a\leqq ay$. For $y=a$ we obtain $a\leqq a^2$, thus $a^2=a$; for $y\leqq a$ we conclude $y\leqq ay$, so that $ay=y$ for all $y\leqq a$. Now if $a\leqq x\leqq e$, then $y=ay\leqq xy\leqq y$ for all $y\leqq a$ and the assertion follows.

Remark. If S is an rl-semigroup with right-identity which is the least element of S, then there is only the trivial C-derivation on S. The same is true in the following case:

Lemma 1.6. *Let S be a left-simple rl-semigroup with right-identity. Then there is no C-derivation on S except the trivial one (if defined).*

Proof. Again for every C-derivation on S we have: $\varphi(x)=x\wedge a$ with $xy\wedge a\leqq y$ for all $x, y\in S$. Since $Sy=S$ for all $y\in S$, for each $y\in S$ there is an $x\in S$ with $xy=e$; thus by Corollary 1) of Lemma 1.5 we conclude $a=e\wedge a\leqq y$ for all $y\in S$ and $a=o$ (if $o\in S$ exists).

Corollary. *Let $S\neq\{e\}$ be an rl-group; then there is no C-derivation on S.*

Proof. Since a semigroup S is a group iff S is left- and right-simple (see [1]), there is at most the trivial C-derivation $\varphi(x)=o$ on S. But an rl-group cannot have a least element o: $o\leqq e$ implies $o^2=o$ and since the only idempotent in S is e, we get $o=e$; thus $e\leqq a$ for all $a\in S$ implies $a^{-1}\leqq e$, so that $a^{-1}=e$ and $a=e$ for all $a\in S$.

Example. 6) Concerning semigroup-left-translations (see [1]) we note the following: if S is a semigroup with left-identity e and φ a mapping of S into itself such that $\varphi(xy)=\varphi(x)y$ for all $x, y\in S$, then for $x=e$ one gets $\varphi(y)=\varphi(e)y$ for all $y\in S$ and $\varphi(x)=ax$ for all $x\in S$.

Lemma 1.7. *Let S be an rl-semigroup with right-identity e. Then the mapping $\varphi(x)=ax$ for all $x\in S$, $a\in S$ fixed, is a C-derivation iff* 1) *$a\in S$ is left-distributive with respect to $\vee$ and* 2) *$ab=a\wedge b$ for all $b\in S$.*

Proof. If $a\in S$ satisfies 1), then $\varphi(x\vee y)=a(x\vee y)=ax\vee ay=\varphi(x)\vee\varphi(y)$ for all $x, y\in S$. If it also satisfies 2), then $\varphi(x\wedge y)=a(x\wedge y)=a\wedge(x\wedge y)=[\varphi(x)\wedge y]\vee$ $\vee[x\wedge\varphi(y)]$ for all $x, y\in S$. Furthermore, since $ax=a\wedge x\leqq a$ implies $axy\leqq ay$ for all $x, y\in S$, it follows: $\varphi(xy)=axy=(ax)y\wedge ay=\varphi(x)y\wedge\varphi(y)$ for all $x, y\in S$. Conversely, let $\varphi(x)=ax$ be a C-derivation; then by I of the definition: $a(x\vee y)=$ $=ax\vee ay$ for all $x, y\in S$; by Lemma 1.5 we have $ax=\varphi(x)=x\wedge\varphi(e)=x\wedge a$, that is $ab=a\wedge b$ for all $b\in S$.

Combining Lemmas 1.5 and 1.7 we get similarly to Theorem 1.3:

Theorem 1.8. *Let S be an rl-semigroup with right-identity. Then the C-derivations on S are the functions φ of the form $\varphi(x)=ax$ with a fixed element $a\in S$ which is left-distributive with respect to $\vee$ such that $ab=a\wedge b$ for all $b\in S$.*

Remark. If S is an rl-semigroup with (right-identity e and) greatest element i, then $\varphi(x)=ax$ such that $ai=i$ is not a C-derivation except $\varphi(x)=x$ (if possible). In fact: if $\varphi(x)=ax=a\wedge x$ for all $x\in S$, then $i=ai=\varphi(i)=a\wedge i=a$ and $\varphi(x)=x$ for all $x\in S$.

For l-semigroups we have:

Theorem 1.9. *Let S be an l-semigroup with identity e, which is the greatest element of S. Then the C-derivations on S are exactly the left-translations $\varphi(x)=ax$ such that $ab=a\wedge b$ for all $b\in S$.*

Proof. On an l-semigroup every function $\varphi(x)=ax$ with $ab=a\wedge b$ for all $b\in S$ is a C-derivation by Lemma 1.7. Conversely, if φ is any C-derivation on S, then by Lemmas 1.2 and 1.5: $\varphi(x)=a\wedge x$ with $xy\wedge a\leqq ay$ for all $x, y\in S$. For $x=e$ we get $y\wedge a\leqq ay$; but $a, y\leqq e$ implies $ay\leqq a$ and $ay\leqq y$, thus $ay\leqq a\wedge y$ and $ay=a\wedge y$ for all $y\in S$. Consequently $\varphi(x)=ax$ for all $x\in S$ with $ab=a\wedge b$ for all $b\in S$.

Corollary. *Let S be a Boolean l-semigroup with identity e (resp. a uniquely complemented l-semigronp with e as greatest element); then the C-derivations on S are exactly the left-translations of S.*

Proof. By the Corollary (resp. Remark) in § 6 of [4] we have in both cases $e=i$ and $xy=x\wedge y$ for all $x, y\in S$.

Returning to general rl-semigroups with identity we show:

Lemma 1.10. *Let S be an rl-semigroup with right-identity e (resp. with greatest element i). Then the set of all C-derivations on S is a commutative, idempotent semigroup with respect to composition of functions: $(\varphi\circ\psi)(x)=\varphi[\psi(x)]$ for all $x\in S$.*

Proof. Let $\varphi(x)=a\wedge x$, $\psi(x)=b\wedge x$ with $a=\varphi(e)$, $b=\psi(e)$ be arbitrary C-derivations on S (see Theorems 1.3 resp. 1.8). Then $(\varphi\circ\psi)(x)=(a\wedge b)\wedge x=c\wedge x$ for all $x\in S$ with $(\varphi\circ\psi)(e)=c\wedge e=c$, since by Corollary 1) to Lemma 1.5: $a\leqq e$, $b\leqq e$, hence $c=a\wedge b\leqq e$. Furthermore, since a and b are neutral, $c=a\wedge b$ is neutral, too. Since $xy\wedge a\leqq ay\wedge y$ and $xy\wedge b\leqq by\wedge y$ for all $x, y\in S$, we get $xy\wedge(a\wedge b)\leqq$ $\leqq(a\wedge b)y\wedge y$ for all $x, y\in S$ and we can apply Lemma 1.2. Trivially we have $(\varphi\circ\psi)(x)=(\psi\circ\varphi)(x)$ and $(\varphi\circ\varphi)(x)=\varphi(x)$ for all $x\in S$.

The results deduced above show, that the class of rl-semigroups which admit non-trivial C-derivations is quite restricted. For concrete examples of rl-semigroups we can prove even more:

Theorem 1.11. *Let* $(L, \wedge, \vee)$ *be an arbitrary lattice. Then on the rl-semigroups* $(F(L), \wedge, \vee, \circ)$ *resp.* $(P(L), \wedge, \vee, \circ)$ *there is no C-derivation except the trivial one (if* $o \in L$ *exists).*

Proof. We give the proof for $F(L)$. If a least element does not exist, then there is no C-derivation by Property 3. If a least element exists, then for the constant functions $f_a(x)=a, f_o(x)=o$ for all $x \in L$ we have $f_a \circ f_o = f_a$ and $f_o \circ f_a = f_o$ for all $a \in L$. If φ is a C-derivation on $F(L)$, then $\varphi(f_a)=[\varphi(f_a)\circ f_o] \wedge \varphi(f_o) \leqq \varphi(f_o)$ and conversely $\varphi(f_o) \leqq \varphi(f_a)$; thus $\varphi(f_a)=\varphi(f_o)$ for all $a \in L$. Since $F(L)$ has an identity $\mathrm{id}(x)=x$ for all $x \in S$, with respect to $\circ$, we know by Lemma 1.5 that $\varphi(f)=f \wedge \varphi(\mathrm{id})$ for all $f \in F(L)$. Moreover, $\varphi(\mathrm{id}) \leqq \mathrm{id}$ by Property 0). Consequently: $[\varphi(\mathrm{id})](a)=a \wedge [\varphi(\mathrm{id})](a)=f_a(a) \wedge [\varphi(\mathrm{id})](a)=[\varphi(f_a)](a)=[\varphi(f_o)](a) \leqq f_o(a)=o$. Therefore $[\varphi(\mathrm{id})](a)=o$ for all $a \in L$. Thus $\varphi(\mathrm{id})=\theta$, the zero-function on L and $\varphi(f)=f \wedge \theta=\theta$ for all $f \in F(L)$.

The proof of this Theorem depends essentially on the constant functions on L, which are left-zeroes of the semigroup $(F(L), \circ)$. We can generalize it to left-zero l-semigroups with identity e, that means l-semigroups S, such that $xy=x$ for all $x \neq e, y \in S$ (see [1]) — for example the set of all constant functions on a lattice:

Lemma 1.12. *Let* S *be a left-zero l-semigroup with identity* e. *Then there are no C-derivations on* S *except* $\varphi(x)=o$ *and* $\varphi(x)=x$ *for all* $x \in S$ *(if defined).*

Proof. By Lemma 1.5, $\varphi(x)=x \wedge \varphi(e)$ for all $x \in S$. If there is no least element in S, then by Property 3) there is no C-derivation on S. If there is $o \in S$, then $\varphi(x)=o$ for all $x \neq e$ in S by Property 2) Thus we have to determine only $\varphi(e)$: if $\varphi(e) \neq e$, $\varphi(e)$ is a left-zero of S and $\varphi(e)=\varphi[\varphi(e)]=o$ by Lemma 1.10; if $\varphi(e)=e$, we have for any $x \neq e$: $o=\varphi(x)=x \wedge e$. If e is not the greatest element, then there is an $x > e$ and $o=x \wedge e=e=\varphi(e)$; if e is the greatest element, then $o=x \wedge e=x$ for all $x \neq e$ in S, $S=\{o, e\}$ and $\varphi(x)=x$ for all $x \in S$.

2. Derivations with dual chain-rule

As mentioned above, a large class of rl-semigroups admits only the trivial C-derivation (if defined). Even the standard examples of mappings resp. polynomial-functions on lattices belong to this class. Therefore the abstraction of derivation of functions, which formalizes the rules of differentiating a sum, a product and the composite of functions, turns out to be not very useful. Also if axiom III of a C-derivation is replaced by its dual:

III′. $\varphi(xy)=\varphi(x)y \vee \varphi(y)$ for all $x, y \in S$

we get nothing new. We can show even more:

Theorem 2.1. *Let S be an rl-semigroup with identity e resp. $o=ox$ for all $x\in S$ (if $o\in S$ exists). Then there is no derivation with dual chain-rule except the trivial one (if defined).*

Proof. If S admits no least element and if φ is any mapping satisfying I, II and III′, then $\varphi(x)=\varphi(xe)=\varphi(x)e\vee\varphi(e)\geqq\varphi(e)$ for all $x\in S$; but $\varphi(x)\leqq x$ for all $x\in S$ by Property 0) (valid also in this case) and thus $\varphi(e)$ is the least element of S: contradiction. If S admits o with $ox=o$ for all $x\in S$, then $\varphi(o)=\varphi(ox)=$ $=\varphi(o)x\vee\varphi(x)\geqq\varphi(x)$ for all $x\in S$; by Axiom I the mapping φ is order-preserving, hence $\varphi(o)\leqq\varphi(x)$ for all $x\in S$ and $\varphi(x)=\varphi(o)=a$ with some $a\in S$, for all $x\in S$; by Axiom II we have $a=\varphi(x)=\varphi(x\wedge x)=\varphi(x)\wedge x\leqq x$ for all $x\in S$ and $a=o$; thus $\varphi(x)=o$ for all $x\in S$.

Remark. Motivated by the properties of "derivations of formal languages", which are in short lattice-endomorphisms of the l-semigroup of all formal languages on an alphabet X satisfying the dual chain rule III′, the Axiom II of a derivation finally may be replaced by

II′. $\varphi(x\wedge y)=\varphi(x)\wedge\varphi(y)$ for all $x, y\in S$.

Such "derivations" are studied in [5].

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On Jordan models of C_0-contractions

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In [7] the following theorem was proved:

Let T be a contraction of class C_0 on a separable Hilbert space. Then there exists an (up to constant factors of modulus 1) unique sequence $\{m_i\}_1^\infty$ of inner scalar functions such that:

1) $m_{i+1}|m_i$, *i.e.* m_{i+1} *divides* m_i, *for each i,*
2) *T is quasisimilar to* $S(m_1)\oplus S(m_2)\oplus\ldots$ *(the "Jordan model" of T).*

In [1] and [2] it was proved that if T has finite defect indices $\delta_T=\delta_{T^*}=n$ then, for $i=1, 2, \ldots, n$, m_i is equal to the $(n-i+1)$-th invariant factor of the characteristic function of T.

At the end of [8] the problem was raised what is the relation of the functions m_i to the characteristic function of T in the general case. We are going to give an answer to this question.

The main result (Theorem 1) can be also deduced from [9], Corollary 3.4. The methods of the proof, however, are quite different.

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We shall use the notations introduced in [2], [4], [6] and [8]. By H^∞ we mean the Banach algebra of bounded holomorphic functions on the disc $|\lambda|<1$. If $u, v\in H^\infty$ then $u\wedge v$ means the largest common inner divisor of u and v. For $1\leqq n\leqq\infty$ we define, as in [8], $\mathcal{M}(n)$ as the set of $n\times n$ matrices $A=(a_{ij})$ over H^∞ for which

$$\sum_i \Big|\sum_j \xi_j a_{ij}(\lambda)\Big|^2 \leqq K^2 \sum_j |\xi_j|^2 \quad (K\geqq 0)$$

holds for $|\lambda|<1$ and for every square-summable sequence of complex numbers, i.e. whose values $A(\lambda)$ $(|\lambda|<1)$ are operators on the complex euclidean n-space E_n, bounded by the constant K.

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For $A\in\mathcal{M}(n)$ and a natural number r, $r\leqq n$, we define $\mathcal{D}_r(A)$ as the largest common inner divisor of all minors of A of order r. The invariant factors $\mathcal{E}_r(A)$ of A are then defined by

$$\mathcal{E}_1(A)=\mathcal{D}_1(A) \quad \text{and} \quad \mathcal{E}_r(A)=\mathcal{D}_r(A)/\mathcal{D}_{r-1}(A) \quad \text{for} \quad r\geqq 2$$

(we put $\mathcal{E}_r(A)=0$ if $\mathcal{D}_r(A)=0$). A matrix $A\in\mathcal{M}(n)$ is called inner (inner from both sides) if A is isometry (unitary) valued almost everywhere on the unit circle.

For $A\in\mathcal{M}(n)$ inner we define the operator $S(A)$ on the Hilbert space $\mathfrak{H}(A)=H^2(E_n)\ominus AH^2(E_n)$ by $S(A)u=P_{\mathfrak{H}(A)}(\lambda u)$. If T is an operator on $\mathfrak{H}$ and T' is an operator on $\mathfrak{H}'$ we write $T\overset{i}{\prec}T'$ if there exists an injective operator $X\colon\mathfrak{H}\to\mathfrak{H}'$ such that $XT=T'X$. If X can be chosen such that $\overline{X\mathfrak{H}}=\mathfrak{H}'$ we write $T\prec T'$. T and T' are called quasisimilar ($T\sim T'$) if $T\prec T'$ and $T'\prec T$. We write $T\overset{u}{\sim}T'$ if T and T' are unitarily equivalent.

Lemma 1. *Let the matrix $A\in\mathcal{M}(n)$ $(1\leqq n\leqq\infty)$ be inner from both sides and have a scalar multiple $\psi\in H^\infty$, ψ inner. Let $\Omega\in\mathcal{M}(n)$ be such that $\Omega A=A\Omega=\psi I_n$. Then $\mathcal{E}_k(\Omega)|\psi$ for $k=1,2,\ldots$. Let $\psi_k=\psi|\mathcal{E}_k(\Omega)$. Then for every natural number $r\leqq n$ there exist matrices Δ, $\Lambda\in\mathcal{M}(n)$ with a common scalar multiple $h\in H^\infty$, $h\wedge\psi=1$. and a matrix $B_r\in\mathcal{M}(n')$ $(n'+r=n)$ inner from both sides and having the scalar multiple $\psi_r\cdot h$ such that $\Delta A=B\Lambda$, where $B=\operatorname{diag}[\psi_1,\ldots,\psi_r,B_r]\in\mathcal{M}(n)$.*

Proof. According to Theorem 1 of [8] there exist matrices $M,N\in\mathcal{M}(n)$ with the respective scalar multiples φ_1 and φ_2, such that $\varphi_1\cdot\varphi_2\wedge\psi=1$, $M\Omega=\Omega' N$ where Ω' is a matrix of the form $\Omega'=\operatorname{diag}[\mathcal{E}_1(\Omega),\ldots,\mathcal{E}_r(\Omega),\Omega_r']$ with $\Omega_r'\in\mathcal{M}(n')$ $(n'+r=n)$ and $\mathcal{E}_1(\Omega)|\mathcal{E}_2(\Omega)|\ldots|\mathcal{E}_r(\Omega)|\Omega_r'$.

Because M and N also have the scalar multiple $\varphi=\varphi_1\cdot\varphi_2$ there exist matrices M^a, $N^a\in\mathcal{M}(n)$ such that $MM^a=M^aM=NN^a=N^aN=\varphi\cdot I_n$. Set $A'=NAM^a$. Then we have $\Omega'A'=\Omega'NAM^a=M\Omega AM^a=\varphi\psi\cdot I_n$ and $\varphi A'\Omega'=A'\Omega'NN^a=NAM^aM\Omega N^a=\varphi^2\psi\cdot I_n$, hence $A'\Omega'=\varphi\psi\cdot I_n$. A' is necessarily of the form $\operatorname{diag}[\varphi\psi_1,\ldots,\varphi\psi_r,A_r]$, where $A_r\in\mathcal{M}(n)$ and $A_r\dfrac{\Omega_r'}{E_r(\Omega)}=\dfrac{\Omega_r'}{E_r(\Omega)}A_r=\varphi\psi_r\cdot I_n$. Let $\varphi=\varphi_i\cdot\varphi_e$ and $A_r=A_{ri}A_{re}$ be the canonical inner-outer factorizations of φ and A_r, respectively. Set $B=\operatorname{diag}[\psi_1,\ldots,\psi_r,A_{ri}]$, $\Delta=\varphi N$, $\Lambda=\operatorname{diag}[\underbrace{\varphi,\ldots,\varphi}_{r\text{ times}},A_{re}]M$. Then we have $B\Lambda=\operatorname{diag}[\varphi\psi_1,\ldots,\varphi\psi_r,A_r]\cdot M=A'M=NAM^aM=\varphi NA=\Delta A$. By [4] (V. 6. 4) A_{re} has the scalar multiple φ_e so the matrices Δ and Λ have the scalar multiple $h=\varphi^2$. On the other hand, A_{ri} has the scalar multiple $\varphi_i\cdot\psi_r$. So A_{ri} is inner from both sides by [4] (V. 6.2), and B, Δ and Λ satisfy all the conditions required.

Lemma 2. *Let the matrices $A,B\in\mathcal{M}(n)$ $(1\leqq n\leqq\infty)$ be inner from both sides and have scalar multiples ψ and ψh, respectively, where $\psi\wedge h=1$ and ψ is* *inner. Let* Δ *and* $\Lambda\in\mathcal{M}(n)$ *be matrices with the scalar multiple* h, *i.e.* $\Delta\Delta^a=\Delta^a\Delta=\Lambda\Lambda^a=\Lambda^a\Lambda=hI_n$ *for some* Δ^a, $\Lambda^a\in\mathcal{M}(n)$. *Suppose* $\Delta A=B\Lambda$. *Then:*

1) *the operator* $X\colon\mathfrak{H}(A)\to\mathfrak{H}(B)$ *defined by* $X=P_{\mathfrak{H}(B)}\Delta|\mathfrak{H}(B)$ *is injective and satisfies* $XS(A)=S(B)X$;
2) *the operator* $Y\colon\mathfrak{H}(B)\to\mathfrak{H}(A)$ *defined by* $Y=P_{\mathfrak{H}(A)}\Delta^a|\mathfrak{H}(B)$ *satisfies* $YS(B)=S(A)Y$ *and we have* $h(S(B)|\mathfrak{N})=0$, *where* $\mathfrak{N}=\ker Y$.

Proof. The proof of (1) is the same as in [1] or [6], and we repeat it only to be seen how does case 2 differ from case 1.

We have $\Delta AH^2(E_n)=B\Lambda H^2(E_n)\subset BH^2(E_n)$ so $P_{\mathfrak{H}(B)}\Delta P_{\mathfrak{H}(A)}=P_{\mathfrak{H}(B)}\Delta$. Hence $XS(A)=P_{\mathfrak{H}(B)}\Delta P_{\mathfrak{H}(A)}U_+|\mathfrak{H}(A)=P_{\mathfrak{H}(B)}\Delta U_+|\mathfrak{H}(A)=P_{\mathfrak{H}(B)}U_+\Delta|\mathfrak{H}(A)=P_{\mathfrak{H}(B)}U_+P_{\mathfrak{H}(B)}\Delta|\mathfrak{H}(A)=S(B)X$ (where $U_+\colon H^2(E_n)\to H^2(E_n)$ is defined by $U_+u=\lambda u$). Let $u\in\mathfrak{H}(A)$ and $Xu=0$, i.e. $\Delta u\in BH^2(E_n)$. As A and B are analytic functions inner from both sides, the corresponding multiplication operators on $L^2(E_n)$ are unitary. Set $f=A^{-1}u$. Then $B\Lambda f=\Delta Af=\Delta u\in BH^2(E_n)$; hence $\Lambda f\in H^2(E_n)$. On the other hand, we have $Af=u\in H^2(E_n)$. Hence $hf=\Lambda^a\Lambda f\in\Lambda^aH^2(E_n)\subset H^2(E_n)$ and $\psi f=A^aAf=A^au\in A^aH^2(E_n)\subset H^2(E_n)$ (this is the place where the proof does not work in case 2 because we get only $\psi hf=B^aBf\in H^2(E_n)$ then). Since $h\wedge\psi=1$, $hf\in H^2(E_n)$ and $\psi f\in H^2(E_n)$ imply $f\in H^2(E_n)$ by the Lemma of [3]. So $u=Af\in AH^2(E_n)$. Since, on the other hand, $u\in H(A)=H^2(E_n)\ominus AH^2(E_n)$ we conclude that $u=0$. Thus X is an injective operator.

As for 2, note that $hA\Lambda^a=\Delta^a\Delta A\Lambda^a=\Delta^aB\Lambda\Lambda^a=h\Delta^aB$ and hence $A\Lambda^a=\Lambda^aB$. We prove as above that $YS(B)=S(A)Y$. From this it follows that the subspace $\mathfrak{N}=\ker Y$ is $S(B)$-invariant. Let $u\in N$, i.e. $\Delta^au\in AH^2(E_n)$. Then $hu=\Delta\Delta^au\in\Delta AH^2(E_n)=B\Lambda H^2(E_n)\subset BH^2(E_n)$. We have $h(S(B))=uP_{\mathfrak{H}(B)}(hu)\in P_{\mathfrak{H}(B)}BH^2(E_n)=0$. Hence $h(S(B)|\mathfrak{N})=0$.

Now we are able to prove our main theorem:

Theorem 1. *Let* T *be an operator of class* C_0 *acting on a separable Hilbert space. Let* Θ *be the characteristic function of* T *and let* Ω *be a contractive analytic function such that* $\Theta\Omega=\Omega\Theta=\psi I_n$, *where* $\psi\in H^\infty$ *is inner and* n *is the defect index of* T *(such an* Ω *exists by* [4], VI. 5.1). *Let* $S(m_1)\oplus S(m_2)\oplus\cdots$ *be the Jordan model of* T. *Then* $m_r=\psi/\mathcal{E}_r(\Omega)$ *for every natural number* $r\leqq n$ *(if* n *is finite then in this notation* $m_i=1$ *for* $i>n$*).*

Proof. Let r be an integer, $r\leqq n$. By Lemma 1 there exist matrices Δ, Λ, $\Theta'\in\mathcal{M}(n)$ such that $\Delta\Theta=\Theta'\Lambda$, Δ and Λ have a scalar multiple h, $h\wedge\psi=1$ and $\Theta'=\operatorname{diag}[\psi_1,\dots,\psi_r,\Theta_r']$, where $\psi_i=\psi/\mathcal{E}_i(\Omega)$ $(i=1,\dots,r)$, and Θ_r' is inner from both sides and has the scalar multiple $h\psi_r$.

I. We prove first $m_r|\psi_r$.

By Lemma 2 the operator $X=P_{\mathfrak{H}(\Theta')}\Delta\,|\,\mathfrak{H}(\Theta)$ is injective and $XS(\Theta)=S(\Theta')X$, i.e. $S(\Theta)\overset{i}{\prec}S(\Theta')$. In the same time $S(\Theta)\overset{u}{\sim}T\sim S(M)$, where $M=\mathrm{diag}\,[m_1, m_2, \ldots]$, $S(M)=S(m_1)\oplus S(m_2)\oplus\cdots$ and $\mathfrak{H}(M)=\mathfrak{H}(m_1)\oplus\mathfrak{H}(m_2)\oplus\cdots$. Hence $S(M)\overset{i}{\prec}S(\Theta')$. Let $Z\colon \mathfrak{H}(M)\to\mathfrak{H}(\Theta')$ be an injective operator such that $ZS(M)=S(\Theta')Z$. Put $\varphi=h\psi_r$, $\mathfrak{M}=\overline{\varphi(S(M))\mathfrak{H}(M)}$ and $\mathfrak{M}'=\overline{\varphi(S(\Theta'))\mathfrak{H}(\Theta')}$. We proceed as in [5]. We have $Z\mathfrak{M}\subset\overline{Z\varphi(S(M))\mathfrak{H}(M)}=\overline{\varphi(S(\Theta'))Z\mathfrak{H}(M)}\subset\subset\overline{\varphi(S(\Theta'))\mathfrak{H}(\Theta')}=\mathfrak{M}'$ and $\mathfrak{M}$ and $\mathfrak{M}'$ are obviously subspaces invariant to $S(M)$ and $S(\Theta')$, respectively. Hence $S(M)|\mathfrak{M}\overset{i}{\prec}S(\Theta')|\mathfrak{M}'$. But $S(\Theta')|\mathfrak{M}'$ is unitarily equivalent to the operator $S(\psi_1/(\psi_1\wedge\varphi))\oplus\ldots\oplus S(\psi_{r-1}/(\psi_{r-1}\wedge\varphi))=S(\psi_1/\psi_r)\oplus\ldots\oplus S(\psi_{r-1}/\psi_r)=S(\mathscr{E}_r(\Omega)/\mathscr{E}_1(\Omega))\oplus\ldots\oplus S(\mathscr{E}_r(\Omega)/\mathscr{E}_{r-1}(\Omega))$ (see [6]). In the same way $S(M)|\mathfrak{M}$ is unitarily equivalent to the operator

$$S(m_1/(m_1\wedge\varphi))\oplus S(m_2/(m_2\wedge\varphi))\oplus\cdots.$$

From Proposition 2 of [5] it follows that $S(M)|\mathfrak{M}\in C_0(r-1)$; hence $m_r|\varphi$. Since $m_r\wedge h=1$ it is necessarily $m_r|\psi_r$.

II. Next we prove $\psi_r|m_r$.

By Lemma 2 there exists an operator $Y\colon\mathfrak{H}(\Theta')\to\mathfrak{H}(\Theta)$ such that $YS(\Theta')=S(\Theta)Y$ and $h(S(\Theta')|\mathfrak{N})=0$, where $\mathfrak{N}=\ker Y$. Let Y_1 be the restriction of Y to the subspace $M=\mathfrak{H}(\psi_1)\oplus\ldots\oplus\mathfrak{H}(\psi_r)\oplus\{0\}$ of the Hilbert space $\mathfrak{H}(\Theta')$. Denote $M'=\mathrm{diag}\,[\psi_1, \ldots, \psi_r, I_{n'}]$ $(n'+r=n)$, $S(M')=S(\Theta')|\mathfrak{M}$ and $\mathfrak{N}_1=\mathfrak{N}\cap\mathfrak{M}$. Obviously, $Y_1S(M')=S(\Theta)Y_1$ and $h(S(M')|\mathfrak{N}_1)=0$. On the other hand, $\psi(S(M')|\mathfrak{N}_1)=\psi(S(M'))|\mathfrak{N}_1=0$. As $\psi\wedge h=1$, the minimal function of $S(M')|\mathfrak{N}_1$ is 1, i.e. $\mathfrak{N}_1=\{0\}$. So Y_1 is an injective operator and $S(M')\overset{i}{\prec}S(\Theta)$. Now we have $S(\psi_1)\oplus\ldots\oplus S(\psi_r)\overset{u}{\sim}S(M')\overset{i}{\prec}S(\Theta)\sim S(m_1)\oplus S(m_2)\oplus\ldots$. It follows as in I (or [6]) that $\psi_r|m_r$. Together with I this gives $m_r=\psi_r$, thus finishing the proof.

Remark 1. We return now to the case of n finite. Then we can take $\psi=\det\Theta_T$ (in an arbitrary choice of orthonormal bases of the defect spaces of T), $\Omega=\Theta_T^A$ (the adjoint matrix of Θ_T). The theorem above gives $T\sim S(\psi_1)\oplus\cdots\oplus S(\psi_n)$ with $\psi_{i+1}|\psi_i$ where $\psi_i=\det\Theta_T/\mathscr{E}_i(\Theta_T^A)$. In the same time by [2] it holds $T\sim S(\mathscr{E}_n)\oplus\cdots\oplus S(\mathscr{E}_1)$ with $\mathscr{E}_i|\mathscr{E}_{i+1}$ where $\mathscr{E}_i=\mathscr{E}_i(\Theta_T)$. From the unicity of the Jordan model of T it follows $\psi_i=\mathscr{E}_{n-i+1}$ $(i=1, \ldots, n)$, i.e.

$$\det\Theta_T=\mathscr{E}_i(\Theta_T^A)\cdot\mathscr{E}_{n-i+1}(\Theta_T),\quad i=1, \ldots, n.$$

We shall prove this relation directly, by using the following well-known fact (see e.g. [10]):

Proposition. *Let M be an $n\times n$ matrix (n finite) over the complex numbers, M^A its adjoint matrix. Let M_1 be an $r\times r$ submatrix of M formed by the rows $i_1, \ldots, i_r$ $(1\leq i_1<\ldots<i_r\leq n)$ and the columns $j_1, \ldots, j_r$ $(1\leq j_1<\ldots<j_r\leq n)$. Let M_2 be the $(n-r)\times(n-r)$ submatrix of M^A obtained by leaving out of M^A its $i_1, \ldots, i_r$-th rows and $j_1, \ldots, j_r$-th columns. Then*

$$\det M_2 = (\det M)^{n-r-1}\cdot\det M_1\cdot(-1)^c \quad \textit{where} \quad c=\sum_{k=1}^{r}(i_k+j_k).$$

Now let N be an $n\times n$ matrix over H^∞. From the Proposition easily follows $\mathscr{D}_{n-r}(N^A)=(\det N)^{n-r-1}\mathscr{D}_r(N)$ $(r=1, \ldots, n)$ and $\mathscr{E}_{n-r+1}(N^A)=\mathscr{D}_{n-r+1}(N^A)/\mathscr{D}_{n-r}(N^A)=$ $=\det N\cdot\mathscr{D}_{r-1}(N)/\mathscr{D}_r(N)=\det N/\mathscr{E}_r(N)$, $r=1, \ldots, n$; whence (1).

Remark 2. Let $A=\operatorname{diag}[\varphi, \psi, \varphi, \psi, \ldots]$, $B=\operatorname{diag}[\varphi\psi, \varphi\psi, \ldots]=\varphi\psi I_\infty$, where $\varphi, \psi\in H^\infty$ are inner, $\varphi, \psi\neq 1$, $\varphi\wedge\psi=1$. Then A and B are matrices inner from both sides with a scalar multiple $\varphi\psi$. Obviously, $A^a=\operatorname{diag}[\psi, \varphi, \psi, \varphi, \ldots]$, $B^a=\operatorname{diag}[1, 1, \ldots]=I_\infty$. According to Theorem 1 we have $S(A)\sim\bigoplus_1^\infty S(\varphi\psi)=S(B)$. An easy computation shows on the other hand that A and B are not quasiequivalent. This situation cannot happen in the case of finite matrices (see [1]).

In the same manner the matrix $A^a=\operatorname{diag}[\psi, \varphi, \psi, \varphi, \ldots]$ is not quasiequivalent to the diagonal matrix formed by its invariant factors $\operatorname{diag}[\mathscr{E}_1(A^a), \mathscr{E}_2(A^a), \ldots]=$ $=B^a=I_\infty$.

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A remark on Gehér's theorem

ADAM MYSIOR

By SMIRNOV's theorem [3] every metrizable space X can be homeomorphically embedded into a Hilbert space. In [2] GEHÉR proved that for every metric on X this embedding can be chosen to be uniformly continuous. The aim of this note is to give a short and simple proof of the Gehér's result.

Theorem. *Every metric space (X, d) can be embedded into a Hilbert space by a uniformly continuous homeomorphism.*

Proof. By the Bing Metrization Theorem [1] the space X has a σ-discrete base $\mathscr{B}$. Let $\mathscr{B}=\{U_{(s,n)}\}_{(s,n)\in S\times N}$ where $U_{(s,n)}\cap U_{(s',n)}=\emptyset$ for every $s, s'\in S$, $s\neq s'$ and $n\in N$ (natural numbers). We may assume that every element of $\mathscr{B}$ has a diameter less than 1.

Denote by H the Hilbert space with $S\times N$ as the index set.

We show that the function $f\colon X\to H$ (well)-defined by

$$f(x)=\{2^{-n/2}(d(x, X-U_{(s,n)}))\}_{(s,n)\in S\times N}$$

is the embedding we were to construct.

The function f is uniformly continuous — for every two points $x, y\in X$ we have

$$\|f(x)-f(y)\|^2=\sum_{(s,n)\in S\times N}\frac{1}{2^n}[d(x, X-U_{(s,n)})-d(y, X-U_{(s,n)})]^2\leqq$$

$$\leqq\sum_{n\in N}\frac{1}{2^n}[d(x,y)]^2=[d(x,y)]^2.$$

On the other hand, for every open set U in X and every point $x\in U$ there is a pair $(s,n)\in S\times N$ such that $x\in U_{(s,n)}\subset U$. Hence, if $y\in X-U_{(s,n)}$, then

$$\|f(x)-f(y)\|^2\geqq\frac{1}{2^n}\{d(x, X-U_{(s,n)}\}^2$$

which proves that f is one-to-one and f^{-1} is continuous.

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Necessary and sufficient conditions for imbedding of classes of functions

J. NÉMETH

Dedicated to Professor Béla Szőkefalvi-Nagy on his 65th birthday

1. Let $\Psi(p)$ $(p>1)$ denote the family of non-negative functions $\psi(u)$ on $(0,\infty)$ such that $\frac{\psi(u)}{u}$ is non-decreasing and $\frac{\psi(u)}{u^p}$ is non-increasing, and let $\Psi:=\bigcup_{p>1}\Psi(p)$. Let $\Psi_+(p)$ $(p>1)$ denote the family of non-negative functions $\psi_+(u)$ on $(0,\infty)$ such that $\frac{\psi_+(u)}{u^p}$ is non-decreasing, while for any $p'>p$, $\frac{\psi_+(u)}{u^{p'}}$ is non-increasing, and let $\Psi_+:=\bigcup_{p>1}\Psi_+(p)$; furthermore let $\Psi_-(p)$ $(0<p<1)$ denote the family of non-negative functions $\psi_-(u)$ on $(0,\infty)$ such that $\frac{\psi_-(u)}{u^p}$ is non-increasing, while for any p', $0<p'<p$, $\frac{\psi_-(u)}{u^{p'}}$ is non-decreasing, and let $\Psi_-:=\bigcup_{0<p<1}\Psi_-(p)$.

Let $P=P(C)$ $(C\geqq 1)$ denote the family of non-negative and continuous functions $\varrho(u)$ on $(0,\infty)$ which are non-decreasing and satisfy $\varrho(u^2)\leqq C\cdot\varrho(u)$ on $[1,\infty)$, while on $(0,1]$ are defined by $\varrho(u)=\varrho\left(\frac{1}{u}\right)$, and for 0 by $\varrho(0)=0$; and let $P:=\bigcup_{C\geqq 1}P(C)$.

Let $\Lambda(M)$ denote the family of non-negative monotonic sequences $\lambda=\{\lambda_K\}_1^\infty$ such that $\lambda_{k^2}\leqq M\lambda_k$, and let $\Lambda:=\bigcup_{M>0}\Lambda(M)$, and for $\lambda\in\Lambda$ let $\lambda(u)$ denote the function $\lambda(u)=\sum_{k=1}^{[u]}\frac{\lambda_k}{k}$ for $u\geqq 1$, $\lambda(u)=\lambda\left(\frac{1}{u}\right)$ for $0<u\leqq 1$ and $\lambda(0)=0$.

For $\varrho\in P$ let ϱ_1 and ϱ_2 denote the functions which are equal to $\varrho(u)$ on $1\leqq u<\infty$ and $0\leqq u<1$ respectively, and equal to 0 elsewhere on $[0,\infty)$.

For $\lambda\in\Lambda$ we define the functions λ_1 and λ_2 in an analogous way.

For a non-negative, piecewise continuous function σ on $(0,\infty)$ we denote by $\sigma(L[a,b])$ $(0\leqq a<b\leqq\infty)$ the set of measurable functions f on (a,b) for which

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$\int_a^b \sigma(|f(x)|)\,dx<\infty$. In the case $[a,b]=[0,1]$ we simply write $\sigma(L)$ instead of $\sigma(L[0,1])$.

If φ is a non-negative, continuous and strictly monotonic function on $(0,\infty)$ and if $f\in\varphi(L[a,b])$ then the modulus of continuity of f with respect to φ is defined by

$$\omega_\varphi(\delta;f)=\sup_{0\leqq h\leqq\delta}\bar\varphi\Big(\int_a^{b-h}\varphi\big(|f(x+h)-f(x)|\big)\,dx\Big)\quad(0<\delta\leqq b-a)$$

and if $f\in\varphi(L(0,\infty))$ we also define

$$\hat\omega_\varphi(\delta;f)=\bar\varphi\Big(\int_{1/\delta}^{\infty}\varphi(|f(x)|)\,dx\Big),\quad \tilde\omega_\varphi(\delta;f)=\omega_\varphi(\delta;f)+\hat\omega_\varphi(\delta;f),$$

where $\bar\varphi$ denotes the inverse of φ.

Let f^* denote a non-increasing function, equidistributed with $|f|$, that is, such that

$$\operatorname{mes}\{x\colon x\in[a,b],\ |f(x)|>y\}=\operatorname{mes}\{z\colon z\in[a,b],\ f^*(z)>y\}.$$

2. Recently many papers deal with imbedding problems. Among others UL'JANOV [10], [11], [12] gave conditions which assure that a function $f\in L^p$ $(p\geqq 1)$ should belong to another space L^ν $(\nu>p)$. LEINDLER [2] generalized this result and gave conditions assuring the transition from L^p to $L^p(\ln^+ L)^\beta$ and from L^p to $\psi(L)$ where $\psi\in\Psi$; and in [3] the latter results were further generalized. More precisely he proved:

Theorem A. (LEINDLER [3], Theorem 2) *Let φ, $\psi\in\Psi$, and let ϱ be a non-negative, non-decreasing, continuous function with*

$$\sum_{k=m}^{\infty}\frac{\varrho(k)}{k^2}\leqq K\frac{\varrho(m)}{m}.$$

Then $f\in\varphi(L)$ and

$$\sum_{n=1}^{\infty}\frac{\varrho(n)}{n^2}(\psi\circ\bar\varphi\circ n\varphi)\left[\omega_\varphi\left(\frac{1}{n};f\right)\right]<\infty$$

imply $f\in\psi(L)\varrho(L)$.

STOROŽENKO [9] gave necessary conditions, in terms of the modulus of continuity $\omega_p(\delta;f^*)$, that a function $f\in L^p$ should belong to the class $L^q\varrho_1(L)$ $(q>p)$, where $\varrho\in P$ is absolutely continuous on any interval $(0,A)$, $A\geqq 1$. She proved:

Theorem B. ([9], Theorem 1) *If $\varrho\in P$ and $f\in L^q\varrho_1(L)$, then*

$$\int_0^1 x^{-q/p}\omega_p^q(x;f^*)\varrho_1\left(\frac{1}{x}\right)dx<\infty\quad for\quad q>p\geqq 1 \tag{1}$$

and

$$\int_0^1 x^{-2}\omega_p^p(x;f^*)\varrho_1'\left(\frac{1}{x}\right)dx<\infty \quad \textit{for} \quad q=p\geqq 1. \tag{2}$$

Later LEINDLER [5] gave a generalization of (2) and certain converse of Theorem A which is similar to (1), that is, he proved:

Theorem C. ([5], Theorem 1) *If* $\varphi\in\Psi$, $\lambda\in\Lambda$ *and* $f\in\varphi(L)\lambda_1(L)$ *then*

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\varphi\left(\omega_\varphi\left(\frac{1}{n};f^*\right)\right)<\infty.$$

Answering a problem of Leindler we gave a necessary condition in terms of the modulus of continuity $\omega_\varphi(\delta;f^*)$ that f should belong to $\psi(L)\varrho_1(L)$ where $\psi\in\Psi$, $\varrho\in P$. Namely we proved:

Theorem D. ([8]) *Let* $\varphi,\psi\in\Psi$ *and* $\varrho\in P$. *Suppose that* $\psi\circ\bar\varphi$ *belongs to* Ψ_+. *If* $f\in\psi(L)\varrho_1(L)$ *then we have*

$$\sum_{n=1}^{\infty}\frac{\varrho(n)}{n^2}(\psi\circ\bar\varphi\circ n\varphi)\left(\omega_\varphi\left(\frac{1}{n};f^*\right)\right)<\infty.$$

We remark that all of the above mentioned results are valid on the interval [0, 1].

UL'JANOV [12], GAĬMNASAROV [1] and the present author [7] have gave sufficient conditions for imbedding of classes of functions on the interval $(0,\infty)$ which are similar to the results concerning the interval [0, 1].

In this paper we prove a theorem concerning the interval $(0,\infty)$ which is similar to above mentioned results and gives necessary and sufficient conditions for general imbedding problems.

Theorem. *Let* $\varphi\in\Psi$, $\varrho\in P$ *and* $f\in\varphi(L(0,\infty))$.
If $\psi\in\Psi$ *is such that* $\psi\circ\bar\varphi\in\Psi_+$ *then we have*

$$f\in\psi(L(0,\infty))\varrho(L(0,\infty)) \tag{3}$$

if and only if

$$\sum_{n=1}^{\infty}\frac{\varrho(n)}{n^2}(\psi\circ\bar\varphi\circ n\varphi)\left(\omega_\varphi\left(\frac{1}{n};f^*\right)\right)<\infty. \tag{4}$$

If $\psi\in\Psi$ *is such that* $\psi\circ\bar\varphi\in\Psi_+$ *then*

$$f\in\psi(L(0,\infty))\varrho(L(0,\infty)) \tag{5}$$

if and only if

$$\sum_{n=1}^{\infty}\varrho(n)\left(\psi\circ\bar\varphi\circ\frac{1}{n}\varphi\right)\left(\hat\omega_\varphi\left(\frac{1}{n};f^*\right)\right)<\infty. \tag{6}$$

If $\lambda \in \Lambda$ *then*

$$f \in \varphi(L(0, \infty))\lambda(L(0, \infty)) \tag{7}$$

if and only if

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n} \varphi\left(\tilde{\omega}_\varphi\left(\frac{1}{n}; f^*\right)\right) < \infty. \tag{8}$$

Corollary. *Let* $f \in L^p(0, \infty)$, $p \geqq 1$, $\varrho \in P$.
If $v > p$ *then* $f \in L^v(0, \infty)\varrho(L(0, \infty))$ *if and only if*

$$\sum_{n=1}^{\infty} \varrho(n) n^{v/p-2} \omega_p^v\left(\frac{1}{n}; f^*\right) < \infty.$$

If $1 \leqq v < p$ *then* $f \in L^v(0, \infty)\varrho(L(0, \infty))$ *if and only if*

$$\sum_{n=1}^{\infty} \varrho(n) n^{-v/p} \hat{\omega}_p^v\left(\frac{1}{n}; f^*\right) < \infty.$$

If $\lambda \in \Lambda$ *then* $f \in L^p \lambda(L)$ *if and only if*

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n} \tilde{\omega}_p^p\left(\frac{1}{n}; f^*\right) < \infty.$$

3. We require the following lemmas

Lemma 1. ([13] p. 29) *If* χ *is a non-negative non-decreasing function on* $(0, \infty)$ *then*

$$\int_0^1 (\chi \circ |f|)(x)\, dx = \int_0^1 (\chi \circ f^*)(x)\, dx. \tag{9}$$

Lemma 2. ([5], Lemma 2) *If* $\psi \in \Psi$ *and* $\varrho \in P$ *then*

$$\int_0^1 ((\psi \varrho_1) \circ f^*)(x)\, dx < \infty \tag{10}$$

implies

$$\int_0^1 (\psi \circ f^*)(x) \varrho_1\left(\frac{1}{x}\right) dx < \infty. \tag{11}$$

Lemma 3. ([8], Lemma 6) *If* $\psi_+ \in \Psi_+$ *and*

$\varrho \in P$, $f(x) \geqq 0$, $F(x) = \int_0^x f(t)\, dt$ *then*

$$\int_0^1 \varrho_1\left(\frac{1}{x}\right) \psi_+\left(\frac{F(x)}{x}\right) dx \leqq K(\psi_+) \int_0^1 \varrho_1\left(\frac{1}{x}\right) (\psi_+ \circ f)(x)\, dx. \tag{12}$$

Lemma 4. *If* $\psi\in\Psi$, $\varrho\in P$ *then*

$$\int_0^\infty ((\psi\varrho)\circ f^*)(x)\,dx<\infty \Leftrightarrow \int_0^\infty ((\psi\varrho)\circ|f|)(x)\,dx<\infty. \tag{13}$$

The proof of this lemma is by an easy application of (9), using the definition of f^* and the properties of ψ and ϱ.

Lemma 5. *If* $\psi\in\Psi$ *and* $f\in\psi(L(0,\infty))$ *then*

$$\psi\left[\omega_\psi\left(\frac{1}{n};f^*\right)\right]\leqq\int_0^{1/n}(\psi\circ f^*)(x)\,dx. \tag{14}$$

The proof is similar to that of Lemma 3 of LEINDLER [5].

Lemma 6. *If the non-negative sequence* $\{a_n\}$ *is quasi-decreasing* $(a_{n+j}\leqq K\cdot a_n$ *for any n and* $j\leqq n)$, *and if* $\{\lambda_k\}$ *is a non-negative sequence and* $\psi_-\in\Psi_-$ *then*

$$\sum_{n=1}^\infty\lambda_n\psi_-\left(\sum_{k=n}^\infty a_k\right)\leqq M\sum_{n=1}^\infty\frac{\psi_-(n\cdot a_n)}{n}\left(n\cdot\lambda_n+\sum_{k=1}^n\lambda_k\right). \tag{15}$$

This is a trivial generalization of the inequality (4) of LEINDLER [4].

Lemma 7. *Let* $\psi\in\Psi$ *and* $\varrho\in P$. *Then we have*

$$\int_0^\infty ((\psi\varrho_2)\circ f^*)(x)\,dx<\infty \tag{16}$$

if and only if

$$\int_0^\infty (\psi\circ f^*)(x)\,\varrho_2\left(\frac{1}{x}\right)dx<\infty. \tag{17}$$

Proof. Let $\psi\in\Psi(p)$, $p>1$. (16) implies $\sum_{n=1}^\infty(\psi\circ f^*)(n)<\infty$ and since $(\psi\circ f^*)(n)\downarrow$ we get $(n\psi\circ f^*)(n)=0(1)$, whence

$$\sum_{n=1}^\infty(\psi\circ f^*)(n)\,\varrho_2\left(\frac{1}{n}\right)\leqq K\sum_{n=1}^\infty((\psi(\varrho_2\circ\psi))\circ f^*)(n)=\sum\nolimits_1. \tag{18}$$

Applying the following properties of ϱ_2 and ψ

$$(\varrho_2\circ\psi)(u)\leqq K_1\varrho_2(u^p)\leqq K_1\varrho_2(u)$$

we obtain

$$\sum\nolimits_1\leqq K_3\sum_{n=1}^\infty((\psi\varrho_2)\circ f^*)(n),$$

which by (18) proves the statement (16)⇒(17).

To prove (17)⇒(16) we mention that from the properties of ϱ_2 it follows

$$\sqrt{t}\,\varrho_2(t)\to 0 \quad \text{if} \quad t\to 0_+$$

(see for example [11] p. 664), therefore we can write

$$\sum_{n=1}^{\infty}((\psi\varrho_2)\circ f^*)(n)\leqq K_1\sum_{(\psi\circ f^*)(n)<n^{-4}}\sqrt{(\psi\circ f^*)}(n)+K_2\sum_{(\psi\circ f^*)(n)\geqq n^{-4}}(\psi\circ f^*)(n)\,\varrho_2(n^{-4})\leqq$$

$$\leqq K_3+K_4\sum_{n=1}^{\infty}(\psi\circ f^*)(n)\,\varrho_2(n^{-1}),$$

which gives the statement.

Lemma 8. *If* $\psi_-\in\Psi_-$, $\varrho\in P$ *and* $a_K\geqq 0$ *then*

$$\sum_{n=1}^{\infty}\varrho(n)\psi_-\left(\frac{1}{n}\sum_{k=n}^{\infty}a_k\right)\geqq K\sum_{n=1}^{\infty}\varrho(n)\psi_-(a_n). \tag{19}$$

Proof. Let $\psi\in{}_-\Psi_-(p)$, $0<p<1$. Using

$$\psi_-(tx)=\frac{\psi_-(tx)}{(tx)^p}(tx)^p\geqq\frac{\psi_-(x)}{x^p}t^px^p=\psi_-(x)t^p \quad \text{if} \quad t\leqq 1,\ x>0; \tag{20}$$

and applying the inequality (see [6], (11))

$$K\cdot\sum_{n=1}^{\infty}\lambda_n\psi\left(\sum_{k=n}^{\infty}a_k\right)\geqq\sum_{n=1}^{\infty}\lambda_n\psi\left(\frac{a_n}{\lambda_n}\sum_{k=1}^{n}\lambda_k\right)$$

with $\lambda_n=\varrho(n)n^{-p}$ and $\psi=\psi_-$ we get for an arbitrary integer l that

$$\begin{aligned}\sum_{n=1}^{l}\varrho(n)\psi_-\left(\frac{1}{n}\sum_{k=n}^{\infty}a_k\right)&\geqq\sum_{n=1}^{l}\varrho(n)\frac{1}{n^p}\psi_-\left(\sum_{k=n}^{l}a_k\right)\geqq\\&\geqq K_1\sum_{n=1}^{l}\frac{\varrho(n)}{n^p}\psi_-\left(\frac{a_n}{\varrho(n)}n^p\sum_{k=1}^{n}\frac{\varrho(k)}{k^p}\right)=S_1.\end{aligned} \tag{21}$$

Since from the properties of ϱ it follows that

$$\sum_{k=1}^{n}\frac{\varrho(k)}{k^p}\geqq K_2\varrho(n)n^{1-p}$$

we can write

$$S_1\geqq K_3\sum_{n=1}^{l}\frac{\varrho(n)}{n^p}\psi_-(n\cdot a_n)=S_2. \tag{22}$$

By

$$\psi_-(tx)=\frac{\psi_-(tx)}{(tx)^{p'}}(tx)^{p'}\geqq\frac{\psi_-(x)}{x^{p'}}t^{p'}x^{p'}=\psi_-(x)t^{p'} \quad \text{if} \quad x>0,\ t\geqq 1,\ p'<p, \tag{23}$$

choosing p_l' such that $0<p_l'<p$ and $l^{p-p_l'}<2$, we have

$$S_2 \geqq K_4 \sum_{n=1}^{l} \varrho(n) \frac{n^{p_l'}}{n^p} \psi_-(a_n) \geqq K_5 \sum_{n=1}^{l} \varrho(n)\psi_-(a_n). \tag{24}$$

Collecting (21), (22) and (24) we get (19).

Lemma 9. *If $\psi_- \in \Psi_-$, $\varrho \in P$ and $\{a_k\}$ is a non-negative non-increasing sequence then*

$$\sum_{n=1}^{\infty} \varrho(n)\psi_-\left(\frac{1}{n}\sum_{k=n}^{\infty} a_k\right) \leqq K \sum_{n=1}^{\infty} \varrho(n)\psi_-(a_n). \tag{25}$$

Proof. Let $\psi_- \in \Psi_-(p)$, $0<p<1$, and let l be an arbitrary integer and $0<p_l'<p$ such that $l^{p-p_l'}<2$; then applying (15) with $\lambda_n=\varrho(n)n^{-p_l'}$ (20) and (23) we get

$$\sum_{n=1}^{l} \varrho(n)\psi_-\left(\frac{1}{n}\sum_{k=n}^{l} a_k\right) \leqq M \sum_{n=1}^{l} \varrho(n)\frac{1}{n^{p_l'}}\psi_-\left(\sum_{k=n}^{l} a_k\right) \leqq$$

$$\leqq \frac{M_1}{1-p^{l}} \sum_{n=1}^{l} \frac{\psi_-(n\cdot a_n)}{n^{p_l'}}\varrho(n) \leqq \frac{M_1}{1-p'} \sum_{n=1}^{l} \psi_-(a_n)\frac{n^p}{n^{p_l'}}\varrho(n) \leqq M_2 \sum_{n=1}^{l} \psi_-(a_n)\varrho(n).$$

If $l\to\infty$ we obtain (25).

Proof of Theorem.

Implication (4)⇒(3) follows from Theorem 2 of [7], applying Lemma 4.

To prove (3)⇒(4) we apply Lemma 3 and Lemma 5 so we get

$$\sum_{n=1}^{\infty} \frac{\varrho(n)}{n^2}(\psi\circ\bar\varphi\circ n\varphi)\left[\omega_\varphi\left(\frac{1}{n};f^*\right)\right] \leqq \sum_{n=1}^{\infty} \frac{\varrho(n)}{n^2}(\psi\circ\bar\varphi)\left(n\cdot\int_0^{1/n}(\varphi\circ f^*)(x)\,dx\right) \leqq$$

$$\leqq K_1 \int_0^1 \varrho\left(\frac{1}{x}\right)(\psi\circ\bar\varphi)\left(\frac{1}{x}\int_0^x(\varphi\circ f^*)(t)\,dt\right)dx \leqq K_2\int_0^1 \varrho\left(\frac{1}{x}\right)(\psi\circ f^*)(x)\,dx.$$

Hence, using Lemmas 2 and 4, the statement follows.

To prove (6)⇒(5) we apply that from the properties of $\psi\circ\bar\varphi$ and ϱ we have

$$\psi(u)\varrho(u) \leqq K\varphi(u) \tag{26}$$

if u is large enough, furthermore applying Lemma 8 we get

$$\int_0^\infty (\psi\circ f^*)(x)\varrho\left(\frac{1}{x}\right)dx = K_1+K_2\sum_{n=1}^\infty \varrho(n)(\psi\circ\bar\varphi\circ\varphi\circ f^*)(n) \leqq$$

$$\leqq K_1+K_3\sum_{n=2}^\infty \varrho(n)(\psi\circ\bar\varphi)\left(\frac{1}{n}\sum_{k=n}^\infty(\varphi\circ f^*)(k)\right) \leqq$$

$$\leqq K_1+K_4\sum_{n=1}^\infty \varrho(n)(\psi\circ\bar\varphi)\left(\frac{1}{n}\hat\omega_\varphi\left(\frac{1}{n}; f^*\right)\right).$$

Using Lemmas 4 and 7, and (26) we get that (6)⇒(5).

The proof of (5)⇒(6) runs similarly to that of (6)⇒(5) using Lemma 9 instead of Lemma 8.

To prove (8)⇒(7) we remark, first of all, that from [7] Theorem 1, and from Lemma 4 of the present paper we get

$$\sum_{n=1}^\infty \frac{\lambda_n}{n}\varphi\left(\omega_\varphi\left(\frac{1}{n}; f^*\right)\right) < \infty \Rightarrow f\in\varphi(L(0,\infty))\lambda_1(L(0,\infty)). \tag{27}$$

To prove

$$\sum_{n=1}^\infty \frac{\lambda_n}{n}\varphi\left(\hat\omega_\varphi\left(\frac{1}{n}; f^*\right)\right) < \infty \Rightarrow f\in\varphi(L(0,\infty))\lambda_2(L(0,\infty)) \tag{28}$$

let $[\alpha_n, \alpha_{n+1})$ denote the interval of values x for which $\frac{1}{(n+1)^2} < f^*(x) \leqq \frac{1}{n^2}$ ($\alpha_0=0$). If we apply the properties of λ and, furthermore, the property

$$\varphi(tx) \leqq t\varphi(x) \quad (x>0,\ 0<t\leqq 1) \tag{29}$$

of φ then we get

$$\int_0^\infty ((\varphi\lambda_2)\circ f^*)(x)\,dx \leqq K\sum_{n=0}^\infty \int_{\alpha_n}^{\alpha_{n+1}} (\psi\circ f^*)(x)\,dx \sum_{k=1}^n \frac{\lambda_k}{k} \leqq$$

$$\leqq K_1+K_2\sum_{k=1}^\infty \frac{\lambda_k}{k}\left\{\int_{\alpha_k}^k (\varphi\circ f)(x)\,dx + \int_k^\infty (\varphi\circ f^*)(x)\,dx\right\} \leqq$$

$$\leqq K_1+K_3\sum_{k=1}^\infty \frac{\lambda_k}{k}\cdot k\cdot\frac{1}{k^2} + K_4\sum_{k=1}^\infty \frac{\lambda_k}{k}\varphi\left(\hat\omega_\varphi\left(\frac{1}{k}; f^*\right)\right) \leqq$$

$$\leqq K_5+K_6\sum_{k=1}^\infty \frac{\lambda_k}{k}\varphi\left(\hat\omega_\varphi\left(\frac{1}{k}; f^*\right)\right).$$

Hence we get (28), and by (27) and (28), applying Lemma 4 we have (8)⇒(7).

To prove (7)⇒(8) we show that

(30) $$\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\varphi\left(\omega_\varphi\left(\frac{1}{n};f^*\right)\right) \leqq K\int_0^\infty (\varphi\circ f^*)(x)\lambda_1\left(\frac{1}{x}\right)dx$$

and

(31) $$\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\varphi\left(\hat\omega_\varphi\left(\frac{1}{n};f^*\right)\right) \leqq K\int_0^\infty (\varphi\circ f^*)(x)\lambda_2\left(\frac{1}{x}\right)dx.$$

Now, (30) follows by Lemma 5, namely

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\varphi\left(\omega_\varphi\left(\frac{1}{n};f^*\right)\right) \leqq \sum_{n=1}^{\infty}\frac{\lambda_n}{n}\sum_{k=n}^{\infty}\int_{\frac{1}{k+1}}^{\frac{1}{k}}(\varphi\circ f^*)(x)\,dx \leqq$$

$$\leqq 2\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\sum_{k=n}^{\infty}\frac{1}{k^2}(\varphi\circ f^*)\left(\frac{1}{k}\right) = 2\sum_{k=1}^{\infty}\frac{1}{k^2}(\varphi\circ f^*)\left(\frac{1}{k}\right)\lambda_1(k) \leqq$$

$$\leqq K\int_0^1 (\varphi\circ f^*)(x)\lambda_1\left(\frac{1}{x}\right)dx.$$

The proof of (31) runs as follows

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\varphi\left(\hat\omega_\varphi\left(\frac{1}{n};f^*\right)\right) = \sum_{n=1}^{\infty}\frac{\lambda_n}{n}\sum_{k=n}^{\infty}\int_k^{k+1}(\varphi\circ f^*)(x)\,dx \leqq$$

$$\leqq \sum_{n=1}^{\infty}\frac{\lambda_n}{n}\sum_{k=n}^{\infty}(\varphi\circ f^*)(k) = \sum_{k=1}^{\infty}(\varphi\circ f^*)(k)\lambda_2\left(\frac{1}{k}\right) \leqq$$

$$\leqq 2(\varphi\circ f^*)(x)\lambda_2\left(\frac{1}{x}\right)dx.$$

So from (30) and (31), and applying Lemmas 2, 4 and 7, we get (7)⇒(8).

Thus our Theorem is completely proved.

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Über Schreiersche Gruppenerweiterungen und ihre Kommutatorgruppen

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Gegenstand dieser Arbeit sind Untersuchungen zur Bestimmung der Kommutatorgruppe vorgegebener Erweiterungen G von M mit $\mathcal{G}$ und zur Konstruktion solcher Erweiterungen mit vorgegebener Gruppe M als Kommutatorgruppe. In diesem Zusammenhang spielen allgemeine Aussagen über Erweiterungen von M mit einem direkten Produkt $\mathcal{G}=\times\mathcal{G}_\lambda$ beliebiger bzw. zyklischer Gruppen $\mathcal{G}_\lambda$ eine Rolle, die den Sätzen II und III der grundlegenden Arbeit [6] von O. Schreier entsprechen. Wir zeigen, daß beim Auftreten unendlicher zyklischer Gruppen die in Satz III von [6] angegebenen Bedingungen nicht ausreichen und geben für beide Fälle kürzere Beweise.

§ 1. Einleitung

Eine Gruppe G heißt Erweiterung einer Gruppe M mit einer Gruppe $\mathcal{G}$, wenn es Homomorphismen μ und Γ gibt, so daß die Sequenz

$$E \to M \xrightarrow{\mu} G \xrightarrow{\Gamma} \mathcal{G} \to E$$

exakt ist. Ist G mit dem Normalteiler $M=\{\alpha, \beta, \ldots\}$ und der Faktorgruppe $G/M=\mathcal{G}=\{A, B, \ldots\}$ vorgegeben, so lassen sich nach Wahl eines Repräsentantensystems $\{r_A\}_{A\in\mathcal{G}}$ von G nach M die Elemente von G auf genau eine Weise in der Form $r_A\alpha$ mit $A\in\mathcal{G}$, $\alpha\in M$ schreiben. Für die Multiplikation gilt dann

$$(1.1)\qquad (r_A\alpha)\cdot(r_B\beta) = r_{AB}[A, B]\alpha^{\varphi(B)}\beta$$

mit dem Parametersystem

$$[A, B] = r_{AB}^{-1} r_A r_B \in M, \quad \alpha^{\varphi(B)} = r_B^{-1}\alpha r_B \in M.$$

Normieren wir noch das Repräsentantensystem bezüglich der Einselemente gemäß

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$r_E = \varepsilon$, so gilt bekanntlich (vgl. etwa [5]) für alle $A, B, C \in \mathcal{G}$; $\alpha, \beta \in M$

$$(1.2) \qquad [A, E] = [E, B] = \varepsilon; \quad \alpha^{\varphi(E)} = \alpha; \quad \varepsilon^{\varphi(B)} = \varepsilon,$$

$$(1.3) \qquad (\alpha\beta)^{\varphi(B)} = \alpha^{\varphi(B)}\beta^{\varphi(B)},$$

$$(1.4) \qquad \alpha^{\varphi(BC)}[B, C] = [B, C]\alpha^{\varphi(B)\varphi(C)},$$

$$(1.5) \qquad [A, BC][B, C] = [AB, C][A, B]^{\varphi(C)}.$$

Umgekehrt ergibt zu vorgegebenen Gruppen M und $\mathcal{G}$ jedes diesen Gleichungen genügende Parametersystem $[A, B]$: $\mathcal{G} \times \mathcal{G} \to M$, $\alpha^{\varphi(B)}$: $M \times \mathcal{G} \to M$ eine Erweiterung von M mit $\mathcal{G}$, nämlich $\mathcal{G} \circ M = \{(A, \alpha) | A \in \mathcal{G}, \alpha \in M\}$ mit der (1.1) entsprechenden Multiplikation und $\mu: M \to \mathcal{G} \circ M$ gemäß $\alpha \to (E, \alpha)$ sowie $\Gamma: \mathcal{G} \circ M \to \mathcal{G}$ gemäß $(A, \alpha) \to A$.

Wir bezeichnen stets mit $\mathbf{A}(M)$ bzw. $\mathbf{I}(M)$ die Gruppe aller bzw. der inneren Automorphismen einer beliebigen Gruppe M; für den durch $\beta \in M$ bestimmten inneren Automorphismus $\alpha \to \beta^{-1}\alpha\beta$ schreiben wir $\mathcal{I}(\beta)$.

In § 2 geben wir zunächst mit den Formeln (2.1) und (2.2) den Kommutator zweier Elemente $r_A\alpha$, $r_B\beta$ einer Erweiterung G von M mit $\mathcal{G}$ explizit mit Hilfe des zugehörigen Parametersystems an. Dann wenden wir uns dem Problem zu, die Kommutatorgruppe G' von G zu bestimmen, falls $\mathcal{G}$ abelsch ist. Nach Satz 2.1 gilt dann $M \supseteq G' \supseteq M'$, und G' kann über M' (sogar rein multiplikativ) durch die Elemente $[B, A]^{-1}[A, B]$ (2.3) und $\alpha^{-1}\alpha^{\varphi(B)}$ (2.4) erzeugt werden. Dabei genügt es sogar, A, B und α je ein multiplikatives Erzeugendensystem von $\mathcal{G}$ bzw. M durchlaufen zu lassen, wie wir in Satz 2.2 zeigen. Für den Fall, daß sowohl M als auch $\mathcal{G}$ endliche abelsche Gruppen sind, wird dieses Ergebnis von O. Schreier [7, Satz 2] unter Verwendung der in [6] entwickelten Theorie (vgl. hier § 3, § 4) und weiterer Hilfsmittel bewiesen.

Um zu untersuchen, welche Rolle die Elemente $\alpha^{-1}\alpha^{\varphi(B)}$ bei der Erzeugung von G' spielen, definieren wir in Anlehnung an L. Kaloujnine [4] eine von einer Automorphismenmenge $\mathfrak{A} \subseteq \mathbf{A}(M)$ abhängige Untergruppe $K(M, \mathfrak{A})$ von M, die wir in Hilfssatz 2.3 näher kennzeichnen. Für $\mathfrak{A} = \mathbf{I}(M) \cup \{\varphi(B) | B \in \mathcal{G}\}$ gilt $M \supseteq G' \supseteq$ $\supseteq K(M, \mathfrak{A}) \supseteq M'$, und $K(M, \mathfrak{A})$ ist genau der von den Elementen $\alpha^{-1}\alpha^{\varphi(B)}$ über M' erzeugte Bestandteil der Kommutatorgruppe G' einer Erweiterung G von M mit $\mathcal{G}$. Daraus gewinnen wir in Satz 2.4 bzw. Folg. 2.5 hinreichende Bedingungen dafür, daß eine vorgegebene Erweiterung $G' = M$ erfüllt bzw. zu einer Gruppe M Erweiterungen G mit $G' = M$ existieren. Diese Problemstellung tritt bei der Untersuchung von Ringen mit nichtkommutativer Addition auf (vgl. H. J. Weinert [8], [9]), worauf wir an anderer Stelle näher eingehen wollen. Als weitere Anwendung dieser Überlegungen kennzeichnen wir in Satz 2.6 alle Erweiterungen G einer beliebigen zyklischen Gruppe M mit einer beliebigen zyklischen Gruppe $\mathcal{G}$, für die $G' = M$ gilt.

Die folgenden beiden Paragraphen geben eine allgemeine Theorie von Gruppenerweiterungen G von M mit $\mathcal{G}$, wobei $\mathcal{G}$ das (diskrete) direkte Produkt beliebiger bzw. zyklischer Gruppen $\mathcal{G}_\lambda$ ist. Unsere zusammenfassenden Sätze 3.1 bzw. 4.2 entsprechen im wesentlichen den Sätzen II bzw. III von O. Schreier [6]. Allerdings reichen die dort in Satz III formulierten Bedingungen nicht aus, um die ausgesprochenen Behauptungen beim Auftreten unendlicher zyklischer Gruppen $\mathcal{G}_\lambda$ zu gewährleisten. Wir zeigen dies im Anschluß an Satz 4.2 und verweisen auch für nähere Einzelheiten auf den Text. Es sei noch erwähnt, daß nach Hilfssatz 4.1 in jeder Gruppe mit Hilfe des Kommutators $\langle y, x\rangle$ die Kommutatoren beliebiger Potenzen von y und x explizit angegeben werden können. Als Anwendung zeigen wir schließlich in Folg. 4.3, daß jede abelsche Gruppe M Kommutatorgruppe einer geeigneten Erweiterung G von M ist.

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§ 2. Die Kommutatorgruppe einer Gruppenerweiterung

In jeder Erweiterung G von M mit $\mathcal{G}$ gilt für den Kommutator zweier Elemente

(2.1) $$\langle r_A\alpha, r_B\beta\rangle = \alpha^{-1}(\beta^{\varphi(A)})^{-1}\langle r_A, r_B\rangle\alpha^{\varphi(B)}\beta$$

wie sofort aus

$$(r_A\alpha)^{-1}(r_B\beta)^{-1}r_A\alpha r_B\beta = \alpha^{-1}\cdot r_A^{-1}\beta^{-1}r_A\cdot r_A^{-1}r_B^{-1}r_Ar_B\cdot r_B^{-1}\alpha r_B\cdot\beta$$

folgt. Dabei liegt in (2.1) höchstens der Faktor $\langle r_A, r_B\rangle$ nicht in M, und wir erhalten aus $\langle r_A, r_B\rangle = r_A^{-1}r_B^{-1}r_{AB}r_{AB}^{-1}r_Ar_B$ und

$$r_{AB} = r_{BA\langle A,B\rangle} = r_Br_Ar_{\langle A,B\rangle}[A,\langle A,B\rangle]^{-1}[B,A\langle A,B\rangle]^{-1}$$

(2.2) $$\langle r_A, r_B\rangle = r_{\langle A,B\rangle}[A,\langle A,B\rangle]^{-1}[B,A\langle A,B\rangle]^{-1}[A,B].$$

Genau für abelsche Gruppen $\mathcal{G}$ geht (2.2) für alle $A, B\in\mathcal{G}$ über in

(2.3) $$\langle r_A, r_B\rangle = [B,A]^{-1}[A,B]\in M.$$

Andere spezielle Kommutatoren ergeben sich in der Form

(2.4) $$\langle\alpha, r_B\rangle = \alpha^{-1}r_B^{-1}\alpha r_B = \alpha^{-1}\alpha^{\varphi(B)} = \langle r_B,\alpha\rangle^{-1},$$

(2.5) $$\langle r_A,\beta\rangle = r_A^{-1}\beta^{-1}r_A\beta = (\beta^{\varphi(A)})^{-1}\beta = \langle\beta, r_A\rangle^{-1},$$

(2.6) $$\langle\alpha,\beta\rangle = \alpha^{-1}\beta^{-1}\alpha\beta.$$

Satz 2.1. *Es sei G eine Erweiterung von M mit einer abelschen Gruppe $\mathcal{G}$. Dann besteht die Kommutatorgruppe G' von G aus allen Produkten von Elementen* (2.3), (2.4) *und* (2.6), *d. h. es gilt* $M \supseteqq G' \supseteqq M'$ *und* G' *ist die kleinste* M' *und alle Kommutatoren* (2.3) *und* (2.4) *enthaltende Unterhalbgruppe von* M. *Die gleiche Aussage gilt mit* (2.5) *anstelle von* (2.4).

Beweis. Es genügt zu zeigen, daß jeder Kommutator (2.1) ein solches Produkt ist. Bis auf einen Faktor aus M' folgt aber aus (2.1)

$$\langle r_A\alpha, r_B\beta\rangle \equiv \langle\alpha, r_B\rangle\langle\beta^{-1}, r_A\rangle\langle r_A, r_B\rangle \quad (M')$$

bzw.

$$\langle r_A\alpha, r_B\beta\rangle \equiv \langle r_B, \alpha^{-1}\rangle\langle r_A, \beta\rangle\langle r_A, r_B\rangle \quad (M').$$

Für die bereits angekündigte Verschärfung dieses Satzes, die wir nur mit (2.3) und (2.4) formulieren, nennen wir eine Teilmenge $\mathfrak{M}(\mathscr{G})$ einer beliebigen Gruppe $\mathscr{G}$ ein multiplikatives (oder Halbgruppen-) Erzeugendensystem von $\mathscr{G}$, wenn jedes Element von $\mathscr{G}$ Produkt endlich vieler Elemente aus $\mathfrak{M}(\mathscr{G})$ ist.

Satz 2.2. *Es sei* G *eine Erweiterung von* M *mit einer abelschen Gruppe* $\mathscr{G}$ *und* $\mathfrak{M}(\mathscr{G})$ *bzw.* $\mathfrak{M}(M)$ *je ein multiplikatives Erzeugendensystem von* $\mathscr{G}$ *bzw.* M. *Dann besteht die Kommutatorgruppe* G' *von* G *aus allen Produkten folgender Elemente:*

(2.3)′ $\quad \langle r_X, r_Y\rangle$ *mit* $X, Y \in \mathfrak{M}(\mathscr{G})$,

(2.4)′ $\quad \langle\xi, r_X\rangle$ *mit* $\xi \in \mathfrak{M}(M)$, $X \in \mathfrak{M}(\mathscr{G})$,

(2.6) $\quad \langle\alpha, \beta\rangle$ *mit* $\alpha, \beta \in M$.

Beweis. Ein beliebiges Element aus G' sei nach Satz 2.1 Produkt von Elementen (2.3), (2.4) und (2.6). Wir zeigen als erstes, daß man alle Faktoren (2.3) durch Faktoren (2.3)′ ersetzen kann. Es gilt für beliebige $A, B, C, D \in \mathscr{G}$:

$$\begin{aligned}
\langle r_{AB}, r_{CD}\rangle &= r_{AB}^{-1} r_{CD}^{-1} r_{AB} r_{CD} = \\
&= [A,B] r_B^{-1} r_A^{-1} [C,D] r_D^{-1} r_C^{-1} r_A r_B [A,B]^{-1} r_C r_D [C,D]^{-1} = \\
&= [A,B][C,D]^{\varphi(A)\varphi(B)} r_B^{-1} r_A^{-1} r_D^{-1} r_C^{-1} r_A r_B r_C r_D ([A,B]^{-1})^{\varphi(C)\varphi(D)} [C,D]^{-1} = \\
&= [C,D]^{\varphi(AB)} [A,B] (r_B^{-1} r_A^{-1} r_D^{-1} r_A r_D r_B)(r_B^{-1} r_D^{-1} r_{BD})(r_{BD}^{-1} r_A^{-1} r_C^{-1} r_A r_C r_{BD}) \cdot \\
&\quad \cdot (r_{BD}^{-1} r_B r_D)(r_D^{-1} r_B^{-1} r_C^{-1} r_B r_C r_D)[C,D]^{-1}([A,B]^{-1})^{\varphi(CD)} = \\
&= [C,D]^{\varphi(AB)} [A,B] \langle r_A, r_D\rangle^{\varphi(B)} [D,B]^{-1} \langle r_A, r_C\rangle^{\varphi(BD)} \cdot \\
&\quad \cdot [B,D] \langle r_B, r_C\rangle^{\varphi(D)} [C,D]^{-1} ([A,B]^{-1})^{\varphi(CD)}.
\end{aligned}$$

Diese Faktoren aus M können nun modulo M' beliebig vertauscht werden, und man erhält damit

$$\begin{aligned}
\langle r_{AB}, r_{CD}\rangle \equiv\ & \langle r_A, r_C\rangle\langle r_A, r_D\rangle\langle r_B, r_C\rangle\langle r_B, r_D\rangle\langle r_A, r_C\rangle^{-1}\langle r_A, r_C\rangle^{\varphi(BD)} \cdot \\
& \cdot \langle r_A, r_D\rangle^{-1}\langle r_A, r_D\rangle^{\varphi(B)}\langle r_B, r_C\rangle^{-1}\langle r_B, r_C\rangle^{\varphi(D)} \cdot \\
& \cdot [C,D]^{-1}[C,D]^{\varphi(AB)}[A,B]([A,B]^{-1})^{\varphi(CD)} \quad (M').
\end{aligned}$$

Wir können also annehmen, daß unser Produkt nur noch Faktoren der Form (2.3)′, (2.4) und (2.6) enthält. Weiter sind $\alpha^{\varphi(AB)}$ und $\alpha^{\varphi(A)\varphi(B)}$ nach (1.4) kongruent modulo M'; damit gilt für $\alpha \in M$; $A, B \in \mathfrak{G}$

$$\langle \alpha, r_{AB} \rangle = \alpha^{-1}\alpha^{\varphi(AB)} \equiv \alpha^{-1}\alpha^{\varphi(A)}(\alpha^{\varphi(A)})^{-1}\alpha^{\varphi(A)\varphi(B)} \quad (M')$$

und wir können die Faktoren (2.4) zunächst durch Faktoren der Form $\langle \alpha, r_X \rangle$ mit $\alpha \in M$, $X \in \mathfrak{M}(\mathfrak{G})$ ersetzen. Die Reduktionsmöglichkeit auf Faktoren (2.4)′ folgt nun aus

$$(\alpha\beta)^{-1}(\alpha\beta)^{\varphi(X)} \equiv \alpha^{-1}\alpha^{\varphi(X)}\beta^{-1}\beta^{\varphi(X)} \quad (M').$$

Aus Satz 2.1 folgt zunächst unmittelbar, daß für eine Erweiterung G von M mit abelscher Faktorgruppe jedenfalls dann $G' = M'$ gilt, wenn alle Automorphismen $\varphi(B)$, $B \in \mathfrak{G}$ in $\mathbf{I}(M)$ liegen und das Faktorensystem gemäß $[A, B] = [B, A]$ symmetrisch gewählt werden kann. Insbesondere treten also vollständige Gruppen M mit $M \supset M'$ nie als Kommutatorgruppen auf.

Wir wollen nun allgemeiner untersuchen, welchen Beitrag die Kommutatoren (2.4) bzw. (2.5) zur Kommutatorgruppe G' einer Erweiterung G von M mit $\mathfrak{G}$ liefern. In Anlehnung an [4] definieren wir dazu für eine beliebige Gruppe M und eine (nichtleere) Automorphismenmenge $\mathfrak{A} \subseteq \mathbf{A}(M)$ als $K(M, \mathfrak{A})$ die Untergruppe von M, die von allen Elementen der Form

$$\alpha^{-1}\alpha^{\varphi} \quad \text{mit} \quad \alpha \in M,\ \varphi \in \mathfrak{A} \tag{2.7}$$

erzeugt wird. Ersichtlich gilt $K(M, \mathbf{I}(M)) = M'$, und für

(2.8) $\mathfrak{A} = \mathbf{I}(M) \cup \{\varphi(B) \mid B \in \mathfrak{G}\}$, G Erweiterung von M mit abelschem $\mathfrak{G}$

gilt $M \supseteq G' \supseteq K(M, \mathfrak{A}) \supseteq M'$, wobei $K(M, \mathfrak{A})$ gerade der von M' und den Kommutatoren (2.4) erzeugte Bestandteil von G' ist.

Hilfssatz 2.3. a) *$K(M, \mathfrak{A})$ ist stets Normalteiler von M und der Durchschnitt aller Untergruppen H von M mit der Eigenschaft*

$$(\alpha H)^{\varphi} = \alpha H \quad \text{für alle} \quad \alpha \in M,\ \varphi \in \mathfrak{A}. \tag{2.9}$$

M. a. W.: $H = K(M, \mathfrak{A})$ ist der kleinste Normalteiler von M, so daß alle $\varphi \in \mathfrak{A}$ auf der Faktorgruppe M/H den identischen Automorphismus induzieren.

b) *Aus $\mathfrak{A} \supseteq \mathbf{I}(M)$ folgt $K(M, \mathfrak{A}) = K(M, \langle \mathfrak{A} \rangle)$ für die von $\mathfrak{A}$ erzeugte Untergruppe $\langle \mathfrak{A} \rangle$ von $\mathbf{A}(M)$.*

c) *Für $\mathfrak{A} \supseteq \mathbf{I}(M)$ besteht $K(M, \mathfrak{A})$ aus allen Produkten mit Faktoren aus M' und der Form $\xi^{-1}\xi^{\varphi}$ mit $\varphi \in \mathfrak{A}$ und ξ aus einem beliebigen multiplikativen Erzeugendensystem $\mathfrak{M}(M)$ von M (vgl. (2.4)′).*

Beweis. a) Die Normalteilereigenschaft folgt mit $\alpha, \beta \in M$, $\varphi \in \mathfrak{A}$ sofort aus

$$\beta^{-1}(\alpha^{-1}\alpha^{\varphi})\beta = (\alpha\beta)^{-1}(\alpha\beta)^{\varphi} \cdot (\beta^{-1}\beta^{\varphi})^{-1} \in K(M, \mathfrak{A}).$$

Weiter erfüllt eine Untergruppe H von M genau dann (2.9), wenn zu beliebigen $\alpha \in M$, $\varphi \in \mathfrak{A}$ stets ein $\gamma \in H$ mit $\alpha^\varphi = \alpha\gamma \in \alpha H$ existiert. Für $H = K(M, \mathfrak{A})$ ist dies nach Definition erfüllt. Ist umgekehrt H eine Untergruppe mit (2.9), so folgt $\alpha^{-1}\alpha^\varphi = \alpha^{-1}\alpha\gamma = \gamma \in H$, also $K(M, \mathfrak{A}) \subseteq H$.

b) Für $\varphi, \varphi' \in \mathfrak{A}$ gilt $\alpha^{-1}\alpha^{\varphi\varphi'} = \alpha^{-1}\alpha^\varphi(\alpha^\varphi)^{-1}(\alpha^\varphi)^{\varphi'}$ und

$$\alpha^{-1}\alpha^{(\varphi^{-1})} \equiv \beta^{-1}\beta^\varphi \ (M') \quad \text{mit} \quad \beta^{-1} = \alpha^{(\varphi^{-1})},$$

wobei $\mathfrak{A} \supseteq \mathbf{I}(M)$, also $K(M, \mathfrak{A}) \supseteq M'$, nur bei dem φ^{-1} betreffenden Teil verwendet wurde.

c) Wegen $K(M, \mathfrak{A}) \supseteq M'$ lassen sich zunächst alle Inversen von Elementen der Form (2.7) gemäß $(\alpha^{-1}\alpha^\varphi)^{-1} \equiv \alpha(\alpha^{-1})^\varphi \ (M')$ als Produkte mit Faktoren aus M' und der Form (2.7) gewinnen, und $(\alpha\beta)^{-1}(\alpha\beta)^\varphi \equiv \alpha^{-1}\alpha^\varphi \cdot \beta^{-1}\beta^\varphi \ (M')$ gestattet die Zurückführung auf Faktoren der Form $\xi^{-1}\xi^\varphi$.

Für die folgenden Aussagen setzen wir $\mathfrak{A} \supseteq \mathbf{I}(M)$ voraus, lassen aber offen, ob $\mathfrak{A}$ gemäß (2.8) durch eine Erweiterung G von M mit $\mathcal{G}$ vorgegeben ist (ersichtlich hängt dann $\mathfrak{A}$ nur von der Automorphismenklassengruppe dieser Erweiterung ab) oder ob wir umgekehrt zu geeignetem $\mathfrak{A}$ eine Erweiterung G von M konstruieren wollen.

Satz 2.4. *Es sei M eine Gruppe und $\mathbf{I}(M) \subseteq \mathfrak{A} \subseteq \mathbf{A}(M)$.*

a) *Erfüllt der Index $[M: M'] = 2$, so gilt stets $K(M, \mathfrak{A}) = M'$.*

b) *Gilt $[M: M'] > 2$ und existiert zu jedem Paar $\mu, \varkappa \in M \setminus M'$ mit $\mu \not\equiv \varkappa \ (M')$ ein Automorphismus $\varphi \in \mathfrak{A}$ mit $\mu^\varphi \equiv \varkappa \ (M')$, m. a. W., wirkt $\mathfrak{A}$ transitiv auf $(M/M') \setminus M'$, dann gilt $K(M, \mathfrak{A}) = M$.*

c) *Gilt $[M: M'] < \infty$ und existiert ein $\varphi \in \mathfrak{A}$ mit $\mu^\varphi \not\equiv \mu \ (M')$ für alle $\mu \in M \setminus M'$, m. a. W., enthält $\mathfrak{A}$ einen für M/M' fixpunktfreien Automorphismus, dann gilt $K(M, \mathfrak{A}) = M$.*

Beweis. a) Ersichtlich gilt $M'^\varphi = M'$ und $(M \setminus M')^\varphi = M \setminus M'$ für alle $\varphi \in \mathbf{A}(M)$.

b) Wir setzen $K(M, \mathfrak{A}) = H$ und wenden den Hilfssatz an. Aus $M' = H$ folgt wegen $\mu H \neq \varkappa H$ sofort der Widerspruch $\mu H = (\mu H)^\varphi = \mu^\varphi H = \varkappa H$. Für $M' \subset H \subset M$ wählen wir $\mu \in H$, $\varkappa \in M \setminus H$; dann gilt $\mu \not\equiv \varkappa \ (M')$ und mit $\mu H = H \neq \varkappa H$ erhalten wir den gleichen Widerspruch. Also folgt $H = K(M, \mathfrak{A}) = M$.

c) Aus $\mu \not\equiv \varkappa \cdot (M')$ folgt $\mu^{-1}\mu^\varphi \not\equiv \varkappa^{-1}\varkappa^\varphi \ (M')$, da das Gegenteil $\varkappa\mu^{-1} \equiv (\varkappa\mu^{-1})^\varphi \ (M')$ ergeben würde, was nach der Voraussetzung über φ nur für $\varkappa\mu^{-1} \in M'$, also $\mu \equiv \varkappa \ (M')$ möglich wäre. Damit bildet aber $\mu \to \mu^{-1}\mu^\varphi$ Elemente verschiedener Nebenklassen modulo M' auf Elemente verschiedener Nebenklassen ab. Aus $[M: M'] < \infty$ folgt die Behauptung.

Für eine gegebene Erweiterung G von M mit abelschem $\mathcal{G}$ ergeben b) bzw. c) dieses Satzes für $\mathfrak{A}$ gemäß (2.8) sofort hinreichende Kriterien für $G'=M$. Wir formulieren nur folgende Umkehrung:

Folgerung 2.5. *Es sei M eine Gruppe mit abelscher Automorphismengruppe. Genügen M' und $\mathfrak{A}=\mathbf{A}(M)$ den bei* b) *bzw.* c) *von Satz* 2.4 *formulierten Voraussetzungen, so tritt M als Kommutatorgruppe wenigstens einer Erweiterung G von M auf.*

Beweis. Es genügt, G als die faktorenfreie Erweiterung von M mit $\mathcal{G}=\mathfrak{A}=\mathbf{A}(M)$ und $\varphi(A)=A$ zu wählen. Aus $M\supseteqq G'\supseteqq K(M,\mathfrak{A})\supseteqq M'$ und $M=K(M,\mathfrak{A})$ folgt $G'=M$.

Als Beispiel für spezielle Anwendungen untersuchen wir noch Erweiterungen G einer zyklischen Gruppe $M=\langle\xi\rangle\cong\mathbf{Z}_m$ der Charakteristik m mit einer zyklischen Gruppe $\mathcal{G}=\langle X\rangle\cong\mathbf{Z}_n$ der Charakteristik n. Für jede ganze Zahl a sei $\bar{a}\equiv a$ (n) und $\bar{a}\in\{0, 1, \dots, n-1\}$ für $n\neq 0$. Wie üblich wählen wir als Parametersystem

$$(2.10)\qquad [X^a, X^b]=\begin{cases}\varepsilon & \text{falls}\quad n=0 \quad\text{oder}\quad \bar{a}+\bar{b}<n\neq 0\\ \nu & \text{falls}\quad \bar{a}+\bar{b}\geqq n\neq 0,\end{cases}$$

$$\alpha^{\varphi(X^b)}=\alpha^{\varphi(X)^{\bar{b}}}=\alpha^{\mathcal{A}^{\bar{b}}},$$

wobei $\varphi(X)=\mathcal{A}\in\mathbf{A}(M)$ durch die Wahl von k in $\xi^{\mathcal{A}}=\xi^k$ festgelegt ist $\big(k=\pm 1$ für $m=0$, $(k,m)=1$ für $m\neq 0\big)$. Für $n\neq 0$ gilt dabei

$$(2.11)\qquad \nu^{\mathcal{A}}=\nu \quad\text{und}\quad \mathcal{A}^n=\mathcal{I}(\nu), \quad\text{also}\quad \mathcal{A}^n=\mathrm{id}_M.$$

Satz 2.6. *Es sei G Erweiterung einer zyklischen Gruppe $M=\langle\xi\rangle\cong\mathbf{Z}_m$ mit einer zyklischen Gruppe $\mathcal{G}=\langle X\rangle\cong\mathbf{Z}_n$. Für $m=0$ bzw. $2|m$ gilt stets $G'\subset M$, während für $2\nmid m$ genau folgende Erweiterungen G mit $G'=M$ existieren:*

a) *Für $n=0$ alle G mit einem $k\in\{1, \dots, m\}$, welches neben $(k,m)=1$ auch $(k-1, l-1, m)=1$ für $kl\equiv 1$ (m) erfüllt.*

b) *Für $n\neq 0$ alle G mit $\nu=\varepsilon$ und einem $k\in\{1, \dots, m\}$, welches $(k,m)=(k-1,m)=1$ und $k^n\equiv 1$ (m) erfüllt.*

Bemerkung. Im Falle $2\nmid m$ erfüllt $k=m-1$ die Bedingung a) stets und die Bedingung b) für $2|n$; dagegen braucht für $2\nmid n$ je nach Wahl von m und n kein k mit b) zu existieren.

Beweis. Da M abelsch und das Faktorensystem symmetrisch ist, wird G' nach Satz 2.2 nur von den Faktoren (2.4)′ erzeugt. Für $m=0$ kommen dafür höchstens $\xi^{-1}\xi^{\pm 1}$ und $\xi\xi^{\mp 1}$ in Frage, woraus schon $G'\subset M$ folgt. Für $m\neq 0$, $n=0$ handelt es sich bei (2.4)′ um die Faktoren $\xi^{-1}\xi^k$ und $\xi^{-1}\xi^l$, also gilt $G'=M$ genau für $(k-1, l-1, m)=1$, was wegen $(k,m)=(l,m)=1$ für $2|m$ nicht möglich ist. Für

$m \neq 0$, $n \neq 0$ tritt in (2.4)′ nur $\zeta^{-1}\zeta^k$ auf, d. h. $G' = M$ gilt genau für $(k-1, m) = 1$. Wegen $(k, m) = 1$ entfällt wieder $2|m$, und für b) haben wir alle Parametersysteme mit $(k, m) = (k-1, m) = 1$ zu bestimmen. Sei $\nu = \zeta^y$; aus (2.11) folgt dann $ky \equiv y\ (m)$, also $(k-1)y \equiv 0\ (m)$ und damit $\nu = \varepsilon$. Der zweite Teil von (2.11) ist mit $k^n \equiv 1\ (m)$ gleichwertig.

Insbesondere ist nach Satz 2.6 jede endliche zyklische Gruppe M ungerader Ordnung Kommutatorgruppe einer geeigneten Erweiterung; wir werden später (vgl. Folg. 4.3) sehen, daß diese Aussage für jede abelsche Gruppe M zutrifft.

Für Potenzen $m = p^x$ und $n = p^y$ einer ungeraden Primzahl p sind die Bedingungen $k^n \equiv 1\ (m)$ und $(k-1, m) = 1$ aus Satz 2.6b) unverträglich, denn aus

$$k^{p^y} \equiv 1\ (p^x), \quad \text{also} \quad k^{p^y} \equiv 1\ (p)$$

folgt $k \equiv 1\ (p)$ und damit $(k-1, p^x) \neq 1$. Dies zeigt (vgl. auch [7], Satz 6):

Folgerung 2.7. *Es sei G Erweiterung einer zyklischen Gruppe M der Ordnung $m = p^x$ mit einer zyklischen Gruppe $\mathcal{G}$ der Ordnung $n = p^y$ für eine beliebige Primzahl p. Dann gilt $G' \subset M$.*

§ 3. Gruppenerweiterungen mit einem direkten Produkt

Es sei G eine Erweiterung von M mit $\mathcal{G}$ und $\mathcal{G}$ das (diskrete) direkte Produkt $\mathcal{G} = \times \mathcal{G}_\lambda$ von Untergruppen $\mathcal{G}_\lambda$, $\lambda \in \Lambda$. Dann enthält G in natürlicher Weise Erweiterungen $G_\lambda = \Gamma^{-1}(\mathcal{G}_\lambda)$ von M mit $\mathcal{G}_\lambda$, und wir werden Parametersysteme von $M \to G \to \mathcal{G}$ soweit als möglich auf Parametersysteme der Erweiterungen $M \to G_\lambda \to \to \mathcal{G}_\lambda$ $(\lambda \in \Lambda)$ zurückführen. Wir betrachten zunächst den Fall $\mathcal{G} = \mathcal{G}_i \times \mathcal{G}_j$, wobei wir die Indexmenge $\Lambda = \{i, j\}$ gemäß $i < j$ ordnen. Wir schreiben $A_\lambda, B_\lambda, \ldots$ jeweils für Elemente aus $\mathcal{G}_\lambda$ und $A = A_i A_j = A_j A_i, \ldots$ für Elemente aus $\mathcal{G}$. Für beliebig gewählte Repräsentanten $r_{A_\lambda} \in G_\lambda \subseteq G$ $(r_E = \varepsilon)$ definieren wir Repräsentanten von $M \to G \to \mathcal{G}$ gemäß

$$(3.1) \qquad r_A = r_{A_i A_j} = r_{A_i} r_{A_j}, \quad i < j.$$

Das zugehörige Parametersystem

$$(3.2) \qquad [A, B], \quad \alpha^{\varphi(B)} \quad \text{von} \quad M \to G \to \mathcal{G}$$

enthält dann die (1.2) bis (1.5) erfüllenden Parametersysteme

$$(3.3) \qquad [A_\lambda, B_\lambda], \quad \alpha^{\varphi(B_\lambda)} \quad \text{von} \quad M \to G_\lambda \to \mathcal{G}_\lambda.$$

Weiter gilt wegen (3.1) für $i < j$ stets $[A_i, B_j] = \varepsilon$, während

$$(3.4) \qquad [A_j, B_i] = r_{A_j B_i}^{-1} r_{A_j} r_{B_i} = (r_{B_i} r_{A_j})^{-1} r_{A_j} r_{B_i} = \langle r_{A_j}, r_{B_i} \rangle, \quad i < j$$

gerade der Kommutator der Repräsentanten ist. Aus

$$r^{-1}_{A_iA_jB_iB_j} r_{A_iA_j} r_{B_iB_j} = r^{-1}_{A_jB_j} r^{-1}_{A_iB_i} r_{A_i} r_{A_j} r_{B_i} r_{B_j}$$

ergibt sich folgende Zurückführung des Faktorensystems (3.2)

$$(3.5) \qquad [A_iA_j, B_iB_j] = [A_i, B_i]^{\varphi(A_jB_j)}[A_j, B_j][A_j, B_i]^{\varphi(B_j)}, \quad i<j$$

und entsprechend des Automorphismensystems (3.2)

$$(3.6) \qquad \alpha^{\varphi(B_iB_j)} = \alpha^{\varphi(B_i)\varphi(B_j)}, \quad i<j$$

auf (3.3), wobei nur die Bestandteile (3.4) zusätzlich auftreten. Letztere sind Abbildungen der Produktmenge $\mathscr{G}_j \times \mathscr{G}_i$ in M und genügen (zusammen mit den Bestandteilen von (3.3) und natürlich für alle $\alpha \in M$; A_λ, B_λ, $C_\lambda \in \mathscr{G}_\lambda$) folgenden Beziehungen

$$\text{I} \qquad \alpha^{\varphi(B_i)\varphi(B_j)}[B_j, B_i] = [B_j, B_i]\alpha^{\varphi(B_j)\varphi(B_i)}, \quad i<j;$$

$$\text{II} \qquad [A_j, B_iC_i][B_i, C_i] = [B_i, C_i]^{\varphi(A_j)}[A_j, C_i][A_j, B_i]^{\varphi(C_i)}, \quad i<j;$$

$$\text{III} \qquad [A_jB_j, C_i][A_j, B_j]^{\varphi(C_i)} = [A_j, B_j][A_j, C_i]^{\varphi(B_j)}[B_j, C_i], \quad i<j.$$

Sie ergeben sich unmittelbar aus (1.4) und (3.6) bzw. (1.5) und (3.5) oder auch durch direktes Rechnen mit den Kommutatoren (3.4) in G; im Falle $|\Lambda|=2$ werden sie sich auch für die Umkehrung als hinreichend erweisen (vgl. Satz 3.1).

Im allgemeinen Falle definieren wir bezüglich einer willkürlich gewählten Ordnung $<$ der Indexmenge Λ analog wie oben

$$(3.1)' \qquad r_A \doteq r_{A_{\lambda1}A_{\lambda2}\dots A_{\lambda t}} = r_{A_{\lambda1}} r_{A_{\lambda2}} \dots r_{A_{\lambda t}}, \quad \lambda1 < \lambda2 < \dots < \lambda t.$$

Wegen $r_E = \varepsilon$ braucht dabei das A enthaltende Teilprodukt $\mathscr{G}_{\lambda1} \times \mathscr{G}_{\lambda2} \times \dots \times \mathscr{G}_\lambda$ von $\mathscr{G} = \times \mathscr{G}_\lambda$ $(\lambda \in \Lambda)$ nicht minimal gewählt zu werden, und wir können jeweils endlich viele $A, B, \dots \in \mathscr{G}$ als Elemente des gleichen Teilproduktes ansehen. Zur Vereinfachung schreiben wir im folgenden $1, 2, \dots, t$ für jeweils geeignet auszuwählende Indizes $\lambda1 < \lambda2 < \dots < \lambda t$ aus Λ. Damit erhält man für das durch (3.1)′ festgelegte Parametersystem (3.2) wieder eine Zurückführung auf (3.3) und (3.4): Für $\alpha^{\varphi(B)}$ gilt unmittelbar

$$(3.6)' \qquad \alpha^{\varphi(B_1B_2\dots B_t)} = \alpha^{\varphi(B_1)\varphi(B_2)\dots\varphi(B_t)}.$$

Für $[A, B]$ erhält man aus (3.5) mit $\mathscr{G}_i = \mathscr{G}_1 \times \dots \times \mathscr{G}_{t-1}$, $\mathscr{G}_j = \mathscr{G}_t$ zunächst

$$(3.7) \qquad [A, B] = [A_1 \dots A_{t-1}A_t, B_1 \dots B_{t-1}B_t] = $$
$$= [A_1 \dots A_{t-1}, B_1 \dots B_{t-1}]^{\varphi(A_tB_t)}[A_t, B_t][A_t, B_1 \dots B_{t-1}]^{\varphi(B_t)}.$$

Die bekannte Kommutatorbeziehung (etwa aus [3], III 1.2 durch Induktion)

$$(3.8) \qquad [A_t, B_1 \dots B_{t-1}] = [A_t, B_{t-1}][A_t, B_{t-2}]^{\varphi(B_{t-1})} \dots [A_t, B_1]^{\varphi(B_2)\dots\varphi(B_{t-1})}$$

liefert dann für $[A, B]$ die rekursive Zurückführung

$$(3.5)' \qquad [A, B] = [A_1 \dots A_{t-1}, B_1 \dots B_{t-1}]^{\varphi(A_t B_t)}[A_t, B_t] \cdot$$
$$\cdot [A_t, B_{t-1}]^{\varphi(B_t)}[A_t, B_{t-2}]^{\varphi(B_{t-1})\varphi(B_t)} \dots [A_t, B_1]^{\varphi(B_2)\dots\varphi(B_t)}$$

auf (3.3) und die Kommutatoren $[A_j, B_i]$ von (3.4) für alle $i, j \in \Lambda$ mit $i<j$. Ersichtlich gelten jetzt I, II und III für alle Indexpaare dieser Art; für $|\Lambda| \geqq 3$ kommt dazu noch für alle $i, j, k \in \Lambda$

$$\text{IV} \quad [A_k, B_j][A_k, C_i]^{\varphi(B_j)}[B_j, C_i] = [B_j, C_i]^{\varphi(A_k)}[A_k, C_i][A_k, B_j]^{\varphi(C_i)}, \quad i<j<k.$$

Beide Seiten von IV reduzieren sich nämlich sofort auf $r_{A_k}^{-1} r_{B_j}^{-1} r_{C_i}^{-1} r_{A_k} r_{B_j} r_{C_i}$. Wir erinnern daran, daß für unendliche Indexmengen Λ stets $1, 2, \dots, t$ stellvertretend für geordnete endliche Teilsysteme $\lambda 1 < \lambda 2 < \dots < \lambda t$ steht und formulieren (vgl. [6], Satz II):

Satz 3.1. *Es sei M eine beliebige Gruppe und $\mathscr{G} = \times \mathscr{G}_\lambda$ $(\lambda \in \Lambda)$ ein (diskretes) direktes Produkt. Dann hat jede Erweiterung G von M mit $\mathscr{G}$ ein Parametersystem $[A, B]$, $\alpha^{\varphi(B)}$, welches sich gemäß* (3.5)′ *und* (3.6)′ *zurückführen läßt auf Parametersysteme $[A_\lambda, B_\lambda]$, $\alpha^{\varphi(B_\lambda)}$ je einer Erweiterung G_λ von M mit $\mathscr{G}_\lambda$ $(\lambda \in \Lambda)$ und Abbildungen $[A_j, B_i]$ der Produktmenge $\mathscr{G}_j \times \mathscr{G}_i$ in M $(i, j \in \Lambda,\ i<j)$, welche den Bedingungen* I—IV *genügen. Umgekehrt entsteht auf diese Weise stets ein Parametersystem $[A, B]$, $\alpha^{\varphi(B)}$ einer Erweiterung G von M mit $\mathscr{G}$.*

Beweis. Wir haben nur noch die Umkehrung zu zeigen. Dazu ordnen wir jeder Gruppe $\mathscr{G}_\lambda = \{E_\lambda, A_\lambda, B_\lambda, \dots\}$ eine gleichmächtige Menge $\bar{\mathscr{G}}_\lambda = \{r_{E_\lambda}, r_{A_\lambda}, r_{B_\lambda}, \dots\}$ zu, so daß alle $\bar{\mathscr{G}}_\lambda$ $(\lambda \in \Lambda)$ und $M = \{\varepsilon, \alpha, \beta, \dots\}$ paarweise disjunkt sind. Wir erzeugen eine Halbgruppe G von der Vereinigungsmenge aller $\bar{\mathscr{G}}_\lambda$ und M mit den definierenden Relationen (jeweils für alle auftretenden Elemente):

(R_0) $r_{E_\lambda} = \varepsilon$, (R_{00}) $r_{A_\lambda}\varepsilon = r_{A_\lambda}$,

(R_1) $\alpha\beta = \gamma$ (Multiplikation in M),

(R_2) $\alpha r_{A_\lambda} = r_{A_\lambda}\alpha^{\varphi(A_\lambda)}$, (R_3) $r_{A_\lambda} r_{B_\lambda} = r_{A_\lambda B_\lambda}[A_\lambda, B_\lambda]$,

(R_4) $r_{A_j} r_{B_i} = r_{B_i} r_{A_j}[A_j, B_i]$, $i<j$.

Zur Lösung des Wortproblems (vgl. [1], Theorem 9.3) legen wir die direkten Schritte für alle Relationen von links nach rechts fest und zeigen:

a) *Kann ein Wort W der zugehörigen freien Halbgruppe durch je einen direkten Schritt S_1 bzw. S_2 in Worte W_1 bzw. W_2 übergeführt werden, so gibt es ein Wort W', in welches W_1 und W_2 durch jeweils endlich viele direkte Schritte übergeführt werden können* (Beweis s. u.).

Damit läßt sich jedes Element der Halbgruppe G auf genau eine Weise in der Form

(3.9) $$r_{A_{\lambda 1}} r_{A_{\lambda 2}} \dots r_{A_{\lambda t}} \alpha, \quad \lambda 1 < \lambda 2 < \dots < \lambda t$$

(Worte der Form $r_{A_{\lambda 1}} \dots r_{A_{\lambda t}}$ und α eingeschlossen) schreiben. Aufgrund unserer Relationen und wegen (1.2) ist ε Einselement von G und jedes erzeugende Element invertierbar, also G Gruppe. Wir lassen nun in der Schreibweise (3.9) wieder das Auftreten zusätzlicher Faktoren $\varepsilon = r_{E_\lambda}$ zu und erhalten mit der oben eingeführten Abkürzung für die auftretenden Indizes gemäß

(3.10) $$r_{A_1} r_{A_2} \dots r_{A_t} \alpha \to (A_1 A_2 \dots A_t, \alpha)$$

eine Bijektion von G auf die Produktmenge $(\times \mathscr{G}_\lambda) \times M$. Unseren Beweis beendet der Nachweis von

b) *Für die Multiplikation in G gilt*

(3.11) $$r_{A_1} r_{A_2} \dots r_{A_t} \alpha \cdot r_{B_1} r_{B_2} \dots r_{B_t} \beta = \\ = r_{A_1 B_1} r_{A_2 B_2} \dots r_{A_t B_t} [A_1 \dots A_t, B_1 \dots B_t] \alpha^{\varphi(B_1 \dots B_t)} \beta$$

mit den bei (3.5)′ *und* (3.6)′ *definierten* $[A, B]$ *und* $\alpha^{\varphi(B)}$.

Nachweis von a) Wir brauchen nur die Fälle zu diskutieren, wo die direkten Schritte S_1 und S_2 nicht in beiden Reihenfolgen angewendet werden können und so von W über W_1 bzw. W_2 zum gleichen Wort W' führen. Solche „kollidierenden" Fälle beziehen sich nach der Form unserer Relationen jeweils auf ein Teilwort w von W, welches aus 2 oder 3 erzeugenden Elementen besteht. Wir beginnen mit den schwierigeren Fällen: Für

$$w = r_{A_\lambda} r_{B_\lambda} r_{C_\lambda} = \begin{cases} r_{A_\lambda B_\lambda} [A_\lambda, B_\lambda] r_{C_\lambda} & \text{nach } (R_3) \text{ links} \\ r_{A_\lambda} r_{B_\lambda C_\lambda} [B_\lambda, C_\lambda] & \text{nach } (R_3) \text{ rechts} \end{cases}$$

führen Schritte mit (R_2) und (R_3) oben und (R_3) unten wegen (1.5) zu

$$r_{A_\lambda B_\lambda C_\lambda} [A_\lambda B_\lambda, C_\lambda][A_\lambda, B_\lambda]^{\varphi(C_\lambda)} = r_{A_\lambda B_\lambda C_\lambda} [A_\lambda, B_\lambda C_\lambda][B_\lambda, C_\lambda].$$

Beim Auftreten von r_{E_λ} kommen noch Schritte mit (R_0) in Frage, wobei (R_{00}), (R_1) und (R_2) wegen (1.2) leicht die erwünschten Gleichheiten liefern; entsprechende Überlegungen sind auch im folgenden beim Auftreten von r_{E_λ} anzustellen. Mit $i < j$ für

$$w = r_{A_j} r_{B_i} r_{C_i} = \begin{cases} r_{B_i} r_{A_j} [A_j, B_i] r_{C_i} & \text{nach } (R_4) \text{ links} \\ r_{A_j} r_{B_i C_i} [B_i, C_i] & \text{nach } (R_3) \text{ rechts} \end{cases}$$

führen Schritte mit (R_2), (R_4), (R_3) und (R_2) oben und (R_4) unten wegen II zu

$$r_{B_i C_i} r_{A_j} [B_i, C_i]^{\varphi(A_j)} [A_j, C_i][A_j, B_i]^{\varphi(C_i)} = r_{B_i C_i} r_{A_j} [A_j, B_i C_i][B_i, C_i].$$

Entsprechendes gilt für $w=r_{A_j}r_{B_j}r_{C_i}$ mit III. Mit $i<j<k$ für

$$w = r_{A_k}r_{B_j}r_{C_i} = \begin{cases} r_{B_j}r_{A_k}[A_k, B_j]r_{C_i} & \text{nach } (\mathrm{R}_4) \text{ links} \\ r_{A_k}r_{C_i}r_{B_j}[B_j, C_i] & \text{nach } (\mathrm{R}_4) \text{ rechts} \end{cases}$$

führen Schritte mit (R_2), (R_4), (R_4) und (R_2) oben sowie (R_4), (R_2), (R_4) unten wegen IV zu

$$r_{C_i}r_{B_j}r_{A_k}[B_j, C_i]^{\varphi(A_k)}[A_k, C_i][A_k, B_j]^{\varphi(C_i)} = r_{C_i}r_{B_j}r_{A_k}[A_k, B_j][A_k, C_i]^{\varphi(B_j)}[B_j, C_i].$$

Damit sind alle kollidierenden Fälle mit Teilworten der Form $w=r_{A_\lambda}r_{B_\varkappa}r_{C_\mu}$ erschöpft. Mit anderen Teilworten der gleichen Länge ergeben sich im wesentlichen nur noch vier Fälle, die wir tabellarisch zusammenfassen:

Teilwort w	Typ des Schrittes links	rechts	Gleichheit ergibt sich mit Schritten vom Typ	wegen
$\alpha r_{A_\lambda}r_{B_\lambda}$	(R_2)	(R_3)	(R_2), (R_3); (R_2)	(1.4)
$\alpha r_{A_j}r_{B_i}$	(R_2)	(R_4)	(R_2), (R_4); (R_2)	I
$\alpha\beta r_{A_\lambda}$	(R_1)	(R_2)	(R_2); (R_2)	(1.3)
$\alpha\beta\gamma$	(R_1)	(R_1)	Assoziativität in M	

Nur beim Auftreten von r_{E_λ} bzw. ε ergeben sich bei diesen und weiteren Teilworten aus 3 oder 2 erzeugenden Elementen noch kollidierende Fälle mit Schritten vom Typ (R_0) oder (R_{00}), die jedoch ersichtlich trivial sind.

Nachweis von b) Mit mehrfacher Anwendung von (R_2), (R_4) und (R_3) gilt

$$r_{A_1}\dots r_{A_{t-1}}r_{A_t}\alpha r_{B_1}r_{B_2}\dots r_{B_t}\beta =$$
$$= r_{A_1}\dots r_{A_{t-1}}r_{A_t}r_{B_1}r_{B_2}\dots r_{B_t}\alpha^{\varphi(B_1)\dots\varphi(B_t)}\beta =$$
$$= r_{A_1}\dots r_{A_{t-1}}r_{B_1}r_{A_t}r_{B_2}\dots r_{B_t}[A_t, B_1]^{\varphi(B_2)\dots\varphi(B_t)}\alpha^{\varphi(B_1)\dots\varphi(B_t)}\beta = \dots =$$
$$= r_{A_1}\dots r_{A_{t-1}}r_{B_1}\dots r_{B_{t-1}}r_{A_tB_t}[A_t, B_t][A_t, B_{t-1}]^{\varphi(B_t)}[A_t, B_{t-2}]^{\varphi(B_{t-1})\varphi(B_t)}\dots$$
$$\dots[A_t, B_1]^{\varphi(B_2)\dots\varphi(B_t)}\alpha^{\varphi(B_1)\dots\varphi(B_t)}\beta.$$

Der Fall $t=1$ liefert unmittelbar die in § 1 formulierte „Umkehrung" des grundlegenden Satzes über Erweiterungen $M\to G\to\mathcal{G}$ als Induktionsanfang. Die Gültigkeit von (3.11) für $t-1$ liefert mit $\alpha=\beta=\varepsilon$

$$r_{A_1}\dots r_{A_{t-1}}\cdot r_{B_1}\dots r_{B_{t-1}} = r_{A_1B_1}\dots r_{A_{t-1}B_{t-1}}[A_1\dots A_{t-1}, B_1\dots B_{t-1}].$$

Multiplizieren wir diese Gleichung mit $r_{A_tB_t}$ und wenden rechts noch einmal (R_2) an, ergibt dies oben eingesetzt gerade (3.11) mit dem bei (3.5)′ und (3.6)′ definierten Parametersystem. Damit ist Satz 3.1 bewiesen.

§ 4. Gruppenerweiterungen mit einem direkten Produkt zyklischer Gruppen

In Fortsetzung von § 3 nehmen wir nun an, daß in $\mathcal{G} = \times \mathcal{G}_\lambda$ $(\lambda \in \Lambda)$ jeder Faktor eine zyklische Gruppe $\mathcal{G}_\lambda = \langle X_\lambda \rangle$ der Charakteristik $n_\lambda \in \mathbf{N}_0$ ist. Sei G eine Erweiterung von M mit $\mathcal{G}$. Für beliebig gewählte Repräsentanten $r_{X_\lambda} = x_\lambda \in G_\lambda \subseteqq G$ definieren wir für $a \in \mathbf{Z}$

$$r_{X_\lambda^a} = x_\lambda^{\bar{a}} \quad \text{mit} \quad \bar{a} \equiv a \ (n_\lambda), \quad \bar{a} \in \{0, 1, \ldots, n_\lambda - 1\} \quad \text{für} \quad n_\lambda \neq 0.$$

Wir schreiben sogleich $1, 2, \ldots, t$ für $\lambda 1 < \lambda 2 < \ldots < \lambda t$ und erhalten aus (3.1)′

$$r_A = r_{X_1^{a_1} X_2^{a_2} \ldots X_t^{a_t}} = x_1^{\bar{a}_1} x_2^{\bar{a}_2} \ldots x_t^{\bar{a}_t}. \tag{4.1}$$

Die zugehörigen Parametersysteme (3.3) sind dann (vgl. (2.10)) gemäß

$$[X_\lambda^a, X_\lambda^b] = \begin{cases} \varepsilon & \text{falls} \quad n_\lambda = 0 \quad \text{oder} \quad \bar{a} + \bar{b} < n_\lambda \neq 0 \\ v_\lambda & \text{falls} \quad \bar{a} + \bar{b} \geqq n_\lambda \neq 0, \end{cases} \tag{4.2}$$

$$\alpha^{\varphi(X_\lambda^b)} = \alpha^{\varphi(X_\lambda)^{\bar{b}}} = \alpha^{\mathcal{A}_\lambda^{\bar{b}}} \tag{4.3}$$

durch je einen Automorphismus $\varphi(X_\lambda) = \mathcal{A}_\lambda$ von M und für $n_\lambda \neq 0$ durch Elemente $x_\lambda^{n_\lambda} = v_\lambda \in M$ bestimmt, wobei

$$v_\lambda^{\mathcal{A}_\lambda} = v_\lambda \quad \text{und} \quad \mathcal{A}_\lambda^{n_\lambda} = \mathcal{I}(v_\lambda) \quad \text{für} \quad n_\lambda \neq 0 \tag{$*$}$$

gilt. Auch die Kommutatoren (3.4) lassen sich auf jeweils einen, nämlich auf

$$[X_j, X_i] = \langle x_j, x_i \rangle = \gamma_{ji} \in M \quad \text{für alle} \quad i < j \tag{4.4}$$

zurückführen, wie sich aus folgender allgemeiner Aussage ergibt:

Hilfssatz 4.1. *Es seien x und y Elemente einer beliebigen Gruppe G. Die Formeln*

$$\langle y^{-1}, x \rangle = \langle x, y \rangle^{\mathcal{I}(y^{-1})}, \quad \langle y, x^{-1} \rangle = \langle x, y \rangle^{\mathcal{I}(x^{-1})}, \quad \langle y^{-1}, x^{-1} \rangle = \langle y, x \rangle^{\mathcal{I}(y^{-1})\mathcal{I}(x^{-1})}$$

lassen sich mit $\sigma, \tau \in \{1, -1\}$ und $\sigma^* = -1$ für $\sigma = -1$, sonst $\sigma^* = 0$ gemäß

$$\langle y^\tau, x^\sigma \rangle = (\langle y, x \rangle^{\tau\sigma})^{\mathcal{I}(y)^{\tau^*} \mathcal{I}(x)^{\sigma^*}} \tag{4.5}$$

zusammenfassen. Damit kann der Kommutator beliebiger Potenzen von x und y mit positiven ganzen Zahlen a und b angegeben werden gemäß

$$\langle y^{\tau a}, x^{\sigma b} \rangle = \langle y^\tau, x^\sigma \rangle^{\sum_{r=1}^{a} \left(\sum_{s=0}^{b-1} \mathcal{I}(x^\sigma)^s \mathcal{I}(y^\tau)^{a-r} \right)}. \tag{4.6}$$

Bemerkung. Das Auftreten von Exponenten wie in (4.6) wird gerechtfertigt, indem man auf der Menge $\mathbf{T}(G)$ aller Abbildungen $\psi, \chi, \ldots$ von G in G Addition und Multiplikation gemäß

$$g^{\psi + \chi} = g^\psi \cdot g^\chi, \quad g^{\psi \cdot \chi} = (g^\psi)^\chi \quad \text{für alle} \quad g \in G$$

definiert. Damit wird $(\mathbf{T}(G), +, \cdot)$ zu einem (linksdistributiven) Fastring mit ω $(g^{\omega}=e)$ als Nullelement, $\iota=\mathrm{id}_G$ als Einselement und $g^{-\psi}=(g^{\psi})^{-1}$; genau die Endomorphismen von G sind (auch von rechts) distributiv in $\mathbf{T}$ (vgl. [2], [10]). Für beliebige Elemente $\psi_s, \chi_r \in \mathbf{T}(G)$ gilt dann

$$\left(\sum_{s=1}^{b} \psi_s\right)\left(\sum_{r=1}^{a} \chi_r\right) = \sum_{r=1}^{a}\left(\sum_{s=1}^{b} \psi_s\right)\chi_r,$$

und falls alle χ_r distributiv in $\mathbf{T}(G)$ sind, weiter

$$= \sum_{r=1}^{a}\left(\sum_{s=1}^{b} \psi_s \chi_r\right).$$

Beweis. Wir zeigen (4.6) o. B. d. A. mit $\tau=\sigma=1$. Für $a=b=1$ reduziert sich der Exponent auf die identische Abbildung. Wir übergehen den Induktionsschluß nach b mit $a=1$ und geben nur die zweite Induktion nach a mit beliebigem b, indem wir von (4.6) ausgehend (unter Verwendung der Distributivität von $\mathscr{I}(y)$) berechnen:

$$\langle y^{a+1}, x^b\rangle = \langle y^a y, x^b\rangle = \langle y^a, x^b\rangle^{\mathscr{I}(y)}\langle y, x^b\rangle =$$

$$= \langle y, x\rangle^{\sum_{r=1}^{a}\left(\sum_{s=0}^{b-1} \mathscr{I}(x)^s \mathscr{I}(y)^{a-r}\right)\mathscr{I}(y)} \langle y, x\rangle^{\sum_{s=0}^{b-1} \mathscr{I}(x)^s} = \langle y, x\rangle^{\sum_{r=1}^{a+1}\left(\sum_{s=0}^{b-1} \mathscr{I}(x)^s \mathscr{I}(y)^{a+1-r}\right)}.$$

Da in der Erweiterung G von M mit $\mathscr{G}$ für $\mathscr{I}(x_\lambda)=\mathscr{I}(r_{X_\lambda})$ gerade $\mathscr{I}(x_\lambda)|M=\mathscr{A}_\lambda$ gilt, erhalten wir mit den oben eingeführten Kommutatoren (4.4) aus (4.5)

$$[X_j^\tau, X_i^\sigma] = \gamma_{ji}^{\tau\sigma\mathscr{A}_j^{\tau*}\mathscr{A}_i^{\sigma*}}, \quad i<j, \tag{4.7}$$

wobei wir grundsätzlich verabreden, daß Exponenten τ, σ, ϱ bei X_λ^τ beide Werte aus $\{1, -1\}$ annehmen, falls $n_\lambda=0$ gilt, und sonst nur 1 zugelassen ist. Weiter folgt aus (4.5) und (4.6)

$$[X_j^\tau, X_i][X_j^\tau, X_i^{n_i-1}]^{\mathscr{A}_i} = \gamma_{ji}^{\tau\mathscr{A}_j^{\tau*}\sum_{s=0}^{n_i-1}\mathscr{A}_i^s}, \quad i<j,\ n_i\neq 0. \tag{4.8}$$

Wir bemerken, daß in solchen Formeln ein Exponent $\tau=-1$ wegen $\gamma^{-1}=\gamma^{-\iota}$ stets als Faktor $(-\iota)$ des Fastringelementes aufzufassen ist; man beachte $(-\iota)(\psi+\chi)=(-\iota)\psi+(-\iota)\chi$, $-(\psi+\chi)=-\chi+(-\psi)$.

Unser Ziel ist ein zu Satz 3.1 analoger Satz 4.2, in den außer den Bestimmungsstücken $\mathscr{A}_\lambda \in \mathbf{A}(M)$ und $v_\lambda \in M$ (für $n_\lambda \neq 0$) von Erweiterungen $M \to G_\lambda \to \mathscr{G}_\lambda = \langle X_\lambda\rangle$ nur noch Elemente $\gamma_{ji} \in M$ entsprechend (4.4) eingehen. Dafür geeignete Beziehungen zwischen diesen Bestimmungsstücken erhalten wir aus I—IV durch spezielle Wahl der dort auftretenden Elemente. Dabei berücksichtigen wir für jedes $\mathscr{G}_\lambda=\langle X_\lambda\rangle$ mit $n_\lambda=0$ neben X_λ zunächst auch X_λ^{-1}; mit den so entstehenden Formeln I**—IV** werden wir auch beim Beweis der Umkehrung arbeiten. Die Beschränkung auf positive Exponenten ergibt jeweils die Formeln I*—IV*, die im wesentlichen den in [6], Satz III angegebenen Bedingungen entsprechen. *Wir stellen daher auch sogleich fest, daß* I*—III* *wiederum* I**—III** *implizieren; dagegen ist* IV* *schwächer als* IV** *und für den angestrebten Satz* 4.2 *nicht ausreichend,* wie sich aus einem diesbezüglichen Gegenbeispiel zu Satz III aus [6] ergeben wird.

Aus I folgt mit $B_i = X_i^\sigma$, $B_j = X_j^\tau$ wegen (4.7)

I** $$\mathscr{A}_j^\tau \mathscr{A}_i^\sigma = \mathscr{A}_i^\sigma \mathscr{A}_j^\tau \mathscr{I}(\gamma_{ji}^{\tau\sigma \mathscr{A}_j^{\tau*} \mathscr{A}_i^{\sigma*}}), \quad i < j.$$

Diese Aussage für alle Kombinationen der Exponenten ist mit

I* $$\mathscr{A}_j \mathscr{A}_i = \mathscr{A}_i \mathscr{A}_j \mathscr{I}(\gamma_{ji}), \quad i < j$$

gleichwertig, wie sich aus der Anwendung von (4.5) auf die Automorphismengruppe $\mathbf{A}(M)$ und $\mathscr{I}(\gamma)^{\mathscr{I}(\mathscr{A})} = \mathscr{A}^{-1}\mathscr{I}(\gamma)\mathscr{A} = \mathscr{I}(\gamma^{\mathscr{A}})$ ergibt.

Aus II folgt mit $A_j = X_j^\tau$, $B_i = X_i^{n_i - 1}$, $C_i = X_i$ wegen (4.8)

II** $$v_i = v_i^{\mathscr{A}_j^\tau} \gamma_{ji}^{\tau \mathscr{A}_j^{\tau*} \sum_{s=0}^{n_i - 1} \mathscr{A}_i^s}, \quad i < j, \; n_i \neq 0.$$

Wir werden zeigen, daß bei Gültigkeit von I* auch hier aus

II* $$v_i = v_i^{\mathscr{A}_j} \gamma_{ji}^{\sum_{s=0}^{n_i - 1} \mathscr{A}_i^s}, \quad i < j, \; n_i \neq 0$$

der noch in II** enthaltene Fall mit $\tau = -1$ folgt. Folgende Umformung

(II*) $$v_i^{\mathscr{A}_j} = v_i (\gamma_{ji}^{-1})^{\sum_{r=1}^{n_i} \mathscr{A}_i^{n_i - r}} = v_i \gamma_{ji}^{\sum_{s=n_i - 1}^{0} - \mathscr{A}_i^s}$$

von II* zeigt nach Anwendung von $\mathscr{A}_j^{-1}$, daß unsere Behauptung aus

(4.9) $$\gamma_{ji}^{(\sum_{s=k}^{0} - \mathscr{A}_i^s) \mathscr{A}_j^{-1}} = \gamma_{ji}^{-\mathscr{A}_j^{-1} (\sum_{s=0}^{k} \mathscr{A}_i^s)}$$

für $k = n_i - 1$ folgt. Wir weisen (4.9) induktiv für alle $k \in \mathbf{N}_0$ nach, wobei $k = 0$ trivial ist. Für $k+1$ lautet (4.9) leicht umgeformt

(4.10) $$\gamma_{ji}^{-\mathscr{A}_i^{k+1} \mathscr{A}_j^{-1}} \gamma_{ji}^{(\sum_{s=k}^{0} - \mathscr{A}_i^s) \mathscr{A}_j^{-1}} = \gamma_{ji}^{-\mathscr{A}_j^{-1} (\sum_{s=0}^{k} \mathscr{A}_i^s)} \gamma_{ji}^{-\mathscr{A}_j^{-1} \mathscr{A}_i^{k+1}}.$$

Wegen I* und (4.6), angewendet auf $\mathbf{A}(M)$, ist der von dem Element (4.9) bestimmte innere Automorphismus von M gerade der Kommutator $\langle \mathscr{A}_j^{-1}, \mathscr{A}_i^{k+1} \rangle$, woraus (4.10) folgt.

Analog ergibt sich aus III mit $A_j = X_j^{n_j - 1}$, $B_j = X_j$, $C_i = X_i^\sigma$

III** $$v_j^{\mathscr{A}_i^\sigma} = v_j \gamma_{ji}^{\sigma \mathscr{A}_i^{\sigma*} \sum_{r=1}^{n_j} \mathscr{A}_j^{n_j - r}}, \quad i < j, \; n_j \neq 0,$$

was bei Gültigkeit von I* bereits wieder aus

$$\text{III}^* \qquad v_j^{\mathscr{A}_i} = v_j \gamma_{ji}^{\sum_{r=1}^{n_j} \mathscr{A}_j^{n_j - r}}, \quad i < j, \ n_j \neq 0$$

folgt. Den Beweis erspart die Bemerkung, daß II* (vgl. (II*)) und III* und entsprechend die Fälle mit $\tau = -1$ bzw. $\sigma = -1$ durch Vertauschung der Indizes i und j auseinander hervorgehen, wobei $\gamma_{ij} = \gamma_{ji}^{-1}$ zu setzen ist, im Einklang mit $\langle x_i, x_j \rangle = \langle x_j, x_i \rangle^{-1}$. (Man beachte aber, daß für $i<j$ stets $\gamma_{ji} = \langle x_j, x_i \rangle = [X_j, X_i]$, im allgemeinen aber $\gamma_{ji}^{-1} = \gamma_{ij} = \langle x_i, x_j \rangle \neq [X_i, X_j]$ gilt.) Auf diese Weise steht bei [6], Satz III die Bedingung III* mit $i \neq j$ für III* und II* mit $i<j$.

Schließlich ergibt IV mit $A_k = X_k^{\varrho}$, $B_j = X_j^{\tau}$, $C_i = X_i^{\sigma}$ wegen (4.7)

$$\text{IV}^{**} \quad \gamma_{kj}^{\varrho\tau \mathscr{A}_k^{\varrho *} \mathscr{A}_j^{\tau *}} \gamma_{ki}^{\varrho\sigma \mathscr{A}_k^{\varrho *} \mathscr{A}_i^{\sigma *} \mathscr{A}_j^{\tau}} \gamma_{ji}^{\tau\sigma \mathscr{A}_j^{\tau *} \mathscr{A}_i^{\sigma *}} = \gamma_{ji}^{\tau\sigma \mathscr{A}_j^{\tau *} \mathscr{A}_i^{\sigma *} \mathscr{A}_k^{\varrho}} \gamma_{ki}^{\varrho\sigma \mathscr{A}_k^{\varrho *} \mathscr{A}_i^{\sigma *}} \gamma_{kj}^{\varrho\tau \mathscr{A}_k^{\varrho *} \mathscr{A}_j^{\tau *} \mathscr{A}_i^{\sigma}}, \quad i < j < k.$$

Die auch bei Gültigkeit von I*—III* schwächere Aussage IV* mit $\sigma = \tau = \varrho = 1$ diskutieren wir später.

Satz 4.2. *Es sei M eine beliebige Gruppe und $\mathscr{G} = \times \mathscr{G}_\lambda$ ($\lambda \in \Lambda$) ein (diskretes) direktes Produkt zyklischer Gruppen $\mathscr{G}_\lambda = \langle X_\lambda \rangle$ der Charakteristik n_λ. Dann hat jede Erweiterung G von M mit $\mathscr{G}$ ein Parametersystem, welches nach Satz* 3.1 *über* (4.2), (4.3) *und* (4.4) *durch Automorphismen $\mathscr{A}_\lambda \in \mathrm{A}(M)$, Elemente $v_\lambda \in M$ für $n_\lambda \neq 0$ und Elemente $\gamma_{ji} \in M$ ($\lambda, i, j \in \Lambda$; $i<j$) festgelegt ist, welche den Bedingungen* ($*$), I*—III* *und* IV** *genügen. Umgekehrt bestimmt jedes System dieser Art eine Erweiterung G von M mit $\mathscr{G}$, wobei G als Halbgruppe von M und Elementen x_λ^{σ} ($\lambda \in \Lambda$) mit $\sigma = \pm 1$ für $n_\lambda = 0$, $\sigma = 1$ für $n_\lambda \neq 0$ mit folgenden definierenden Relationen (jeweils für alle auftretenden Elemente) erzeugt wird:*

(R_0^*) $x_\lambda^{\sigma} x_\lambda^{\sigma'} = \varepsilon$ *für* $\sigma\sigma' = -1$, (R_{00}^*) $x_\lambda^{\sigma} \varepsilon = x_\lambda^{\sigma}$,

(R_1^*) $\alpha\beta = \gamma$ *(Multiplikation in M)*,

(R_2^*) $\alpha x_\lambda^{\sigma} = x_\lambda^{\sigma} \alpha^{\mathscr{A}_\lambda^{\sigma}}$, (R_3^*) $x_\lambda^{n_\lambda} = v_\lambda$ *nur für* $n_\lambda \neq 0$,

(R_4^*) $x_j^{\tau} x_i^{\sigma} = x_i^{\sigma} x_j^{\tau} \gamma_{ji}^{\tau\sigma \mathscr{A}_j^{\tau *} \mathscr{A}_i^{\sigma *}}$, $i < j$.

Beweis. Es bleibt nur die Umkehrung zu zeigen. Zur Lösung des Wortproblems legen wir wieder die direkten Schritte für alle Relationen von links nach rechts fest. Aus dem sich anschließenden Nachweis des bei Satz 3.1 formulierten Kriteriums a) folgt, daß G eine Gruppe ist und sich jedes Element von G eindeutig in der Form

$$x_1^{\bar{a}_1} x_2^{\bar{a}_2} \dots x_t^{\bar{a}_t} \alpha \quad \text{mit} \quad \begin{cases} \bar{a}_\lambda \in \mathbf{Z} & \text{für} \quad n_\lambda = 0 \\ \bar{a}_\lambda \in \{0, 1, \dots, n_\lambda - 1\} & \text{für} \quad n_\lambda \neq 0 \end{cases}$$

schreiben läßt, wobei wieder $1, 2, \dots, t$ für $\lambda 1 < \lambda 2 < \dots < \lambda t$ steht und wir bereits mehrfaches Auftreten des Einselementes ε zugelassen haben. Ersichtlich ist die Untergruppe M Normalteiler von G und $G/M \cong \mathscr{G} = \times \mathscr{G}_\lambda$.

Nachweis von a) Wir dürfen uns auf die „kollidierenden" Fälle beschränken, bei denen wenigstens zwei x_i, x_j mit verschiedenen Indizes $i<j$ beteiligt sind. Für

$$w = \alpha x_j^\tau x_i^\sigma = \begin{cases} x_j^\tau \alpha^{\mathscr{A}_j^\tau} x_i^\sigma & \text{nach } (\mathrm{R}_2^*) \text{ links} \\ \alpha x_i^\sigma x_j^\tau \gamma_{ji}^{\tau\sigma \mathscr{A}_j^{\tau*} \mathscr{A}_i^{\sigma*}} & \text{nach } (\mathrm{R}_4^*) \text{ rechts} \end{cases}$$

führen Schritte mit (R_2^*) und (R_4^*) oben und zweimal (R_2^*) unten zu

$$x_i^\sigma x_j^\tau \gamma_{ji}^{\tau\sigma \mathscr{A}_j^{\tau*} \mathscr{A}_i^{\sigma*}} \alpha^{\mathscr{A}_j^\tau \mathscr{A}_i^\sigma} = x_i^\sigma x_j^\tau \alpha^{\mathscr{A}_i^\sigma \mathscr{A}_j^\tau} \gamma_{ji}^{\tau\sigma \mathscr{A}_j^{\tau*} \mathscr{A}_i^{\sigma*}},$$

wobei die Gleichheit wegen I** gilt. Mit $n_i \neq 0$ gilt für

$$w = x_j^\tau x_i^{n_i} = \begin{cases} x_i x_j^\tau \gamma_{ji}^{\tau \mathscr{A}_j^{\tau*}} x_i^{n_i - 1} & \text{nach } (\mathrm{R}_4^*) \text{ links} \\ x_j^\tau v_i & \text{nach } (\mathrm{R}_3^*) \text{ rechts.} \end{cases}$$

Die mehrfache Anwendung von Schritten mit (R_2^*) und (R_4^*) und schließlich mit (R_3^*), (R_2^*) ergibt oben

$$x_i^{n_i} x_j^\tau \gamma_{ji}^{\tau \mathscr{A}_j^{\tau*} \sum_{s=0}^{n_i - 1} \mathscr{A}_i^s} = x_j^\tau v_i^{\mathscr{A}_j^\tau} \gamma_{ji}^{\tau \mathscr{A}_j^{\tau*} \sum_{s=0}^{n_i - 1} \mathscr{A}_i^s},$$

was mit $x_j^\tau v_i$ nach II** übereinstimmt. Entsprechendes gilt für $w = x_j^{n_j} x_i^\sigma$ mit $n_j \neq 0$ unter Verwendung von III**. Als nächstes betrachten wir für $n_i = 0$ mit $\sigma\sigma' = -1$

$$w = x_j^\tau x_i^\sigma x_i^{\sigma'} = \begin{cases} x_i^\sigma x_j^\tau \gamma_{ji}^{\tau\sigma \mathscr{A}_j^{\tau*} \mathscr{A}_i^{\sigma*}} x_i^{\sigma'} & \text{nach } (\mathrm{R}_4^*) \text{ links} \\ x_j^\tau \varepsilon & \text{nach } (\mathrm{R}_0^*) \text{ rechts.} \end{cases}$$

Hier erhalten wir oben nach Schritten mit (R_2^*), (R_4^*), (R_0^*), (R_2^*) und (R_1^*) wegen $\sigma^* + \sigma' = \sigma'^*$

$$x_i^\sigma x_j^\tau x_i^{\sigma'} \gamma_{ji}^{\tau\sigma \mathscr{A}_j^{\tau*} \mathscr{A}_i^{\sigma'*}} = x_j^\tau \gamma_{ji}^{\tau\sigma' \mathscr{A}_j^{\tau*} \mathscr{A}_i^{\sigma'*} + \tau\sigma \mathscr{A}_j^{\tau*} \mathscr{A}_i^{\sigma'*}} = x_j^\tau \varepsilon.$$

Die analoge Überlegung für $w = x_j^\tau x_j^{\tau'} x_i^\sigma$ mit $n_j = 0$ und $\tau\tau' = -1$ benötigt am Ende

$$\gamma_{ji}^{\tau\sigma \mathscr{A}_j^{\tau*} \mathscr{A}_i^{\sigma*} \mathscr{A}_j^{\tau'} + \tau'\sigma \mathscr{A}_j^{\tau'*} \mathscr{A}_i^{\sigma*}} = \varepsilon,$$

was leicht aus I** folgt. Schließlich führen mit $i<j<k$ für

$$w = x_k^\varrho x_j^\tau x_i^\sigma = \begin{cases} x_j^\tau x_k^\varrho \gamma_{kji}^{\varrho\tau \mathscr{A}_k^{\varrho*} \mathscr{A}_j^{\tau*}} x_i^\sigma & \text{nach } (\mathrm{R}_4^*) \text{ links} \\ x_k^\varrho x_i^\sigma x_j^\tau \gamma_{ji}^{\tau\sigma \mathscr{A}_j^{\tau*} \mathscr{A}_i^{\sigma*}} & \text{nach } (\mathrm{R}_4^*) \text{ rechts} \end{cases}$$

Schritte mit (R_2^*), (R_4^*), (R_4^*), (R_2^*) oben und mit (R_4^*), (R_2^*), (R_4^*) unten genau auf die mit $x_i^\sigma x_j^\tau x_k^\varrho$ multiplizierte Gleichung IV**. Damit sind (abgesehen von dem trivialen Fall $x_j^\tau x_i^\sigma \varepsilon$) alle zu betrachtenden Möglichkeiten erschöpft.

Bemerkung. In dem zu Satz 4.2 analogen (und nur für endliche zyklische Gruppen $\mathscr{G}_\lambda$ bewiesenen) Satz III von [6] treten sechs Bedingungen (a)—(f) auf. Die ersten drei entsprechen $\mathscr{A}_\lambda \in \mathrm{A}(M)$ und $(*)$. Bedingung (d) entspricht I* mit $i \neq j$ $(\gamma_{ij} = \gamma_{ji}^{-1})$, also I* in zwei ersichtlich gleichwertigen Versionen. Bedingung (e) entspricht III* mit $i \neq j$, also wie oben bereits festgestellt II* und III*. Bedingung (f) läuft auf

$$\text{IV*} \qquad \gamma_{kj}\gamma_{ki}^{\mathscr{A}_j}\gamma_{ji} = \gamma_{ji}^{\mathscr{A}_k}\gamma_{ki}\gamma_{kj}^{\mathscr{A}_i}, \quad i < j < k$$

hinaus, allerdings wird (f) in [6] für $i \neq j, j \neq k, k \neq i$ formuliert, enthält also sechs Bedingungen statt einer für jedes Tripel aus Λ. Wir werden zeigen:

i) *Jede dieser sechs Bedingungen impliziert jeweils die fünf anderen.*

ii) *Beim Auftreten unendlicher zyklischer Gruppen reichen die Bedingungen* $(*)$, I*—IV* *(also die Bedingungen* (a)—(f) *in* [6]*) nicht aus, um die Existenz einer Erweiterung* G *von* M *mit* $\mathscr{G}$ *zu gewährleisten, so daß* $\alpha x_\lambda = x_\lambda \alpha^{\mathscr{A}_\lambda}$ (R_2^*) *und* $x_j x_i = x_i x_j \gamma_{ji}$ (R_4^*) *gilt.*

Beweis von i) Multiplizieren wir IV* für ein beliebiges Tripel $i \neq j, j \neq k, k \neq i$ unter Beachtung von $\gamma_{ij} = \gamma_{ji}^{-1}$ usw. von links mit $\gamma_{ij}^{\mathscr{A}_k}$ bzw. γ_{jk} und von rechts mit γ_{ij} bzw. $\gamma_{jk}^{\mathscr{A}_i}$, ergibt sich

$$\gamma_{ij}^{\mathscr{A}_k}\gamma_{kj}\gamma_{ki}^{\mathscr{A}_j} = \gamma_{ki}\gamma_{kj}^{\mathscr{A}_i}\gamma_{ij} \quad \text{bzw.} \quad \gamma_{ki}^{\mathscr{A}_j}\gamma_{ji}\gamma_{jk}^{\mathscr{A}_i} = \gamma_{jk}\gamma_{ji}^{\mathscr{A}_k}\gamma_{ki}.$$

Dies sind gerade die aus IV* bei den Permutationen (ij) bzw. (jk) hervorgehenden Formeln, was schon alles zeigt.

Beispiel zu ii) Man wähle $\Lambda = \{1, 2, 3\}$, $\mathscr{G}_\lambda = \langle X_\lambda \rangle$ mit $n_\lambda = 0$, als Normalteiler $M = S_3$, alle Automorphismen $\mathscr{A}_\lambda$ identisch und

$$\gamma_{21} = (132), \quad \gamma_{31} = (123), \quad \gamma_{32} = (12).$$

Dann sind alle Forderungen $(*)$, I*, II*, III* trivialerweise erfüllt, und es gilt IV** für $\sigma = \tau = \varrho = 1$, also IV* und nach i) auch [6] (f) gemäß

$$\gamma_{32}\gamma_{31}\gamma_{21} = (12)(123)(132) = (132)(123)(12) = \gamma_{21}\gamma_{31}\gamma_{32}.$$

Dagegen ist IV** z. B. für $\sigma = \tau = 1$, $\varrho = -1$ wegen

$$(13) = (12)^{-1}(123)^{-1}(132) \neq (132)(123)^{-1}(12)^{-1} = (23)$$

nicht erfüllt. Wendet man Satz 4.2 bzw. Satz III von [6] trotzdem an, um auf die angegebene Weise aus den Elementen von M und Elementen x_1, x_2, x_3 eine Gruppe G als Erweiterung von M mit $\mathscr{G}_1 \times \mathscr{G}_2 \times \mathscr{G}_3$ zu erzeugen, ergibt sich aus folgendem Vergleich

$$x_3^{-1}x_2x_1 = x_3^{-1}x_1x_2\gamma_{21} = x_1x_3^{-1}\gamma_{31}^{-1}x_2\gamma_{21} = x_1x_2x_3^{-1}\gamma_{32}^{-1}\gamma_{31}^{-1}\gamma_{21} = x_1x_2x_3^{-1}(13),$$

$$x_3^{-1}x_2x_1 = x_2x_3^{-1}\gamma_{32}^{-1}x_1 = x_2x_1x_3^{-1}\gamma_{31}^{-1}\gamma_{32}^{-1} = x_1x_2x_3^{-1}\gamma_{21}\gamma_{31}^{-1}\gamma_{32}^{-1} = x_1x_2x_3^{-1}(23)$$

der Widerspruch (13)=(23); die in [6] angegebenen Bedingungen gewährleisten also nicht die Konstruktion einer Gruppe G mit $G \supseteqq M$.

Folgerung 4.3. *Jede abelsche Gruppe M ist Kommutatorgruppe einer geeigneten Erweiterung G von M.*

Beweis. Es sei $\mathfrak{M}(M)$ ein (multiplikatives) Erzeugendensystem von M. Wir wählen eine Indexmenge Λ so, daß jedes Element $\xi \in \mathfrak{M}(M)$ in der Form $\xi = \gamma_{ji}$ mit $i, j \in \Lambda$, $i < j$ geschrieben werden kann. Mit den so gewählten Elementen $\gamma_{ji} \in M$ bestimmen wir nach Satz 4.2 eine automorphismenfreie Erweiterung G von M mit dem direkten Produkt $\mathcal{G} = \times \mathcal{G}_\lambda$ $(\lambda \in \Lambda)$ unendlicher zyklischer Gruppen, wobei alle Bedingungen $(*)$ und I*—III* entfallen und IV** für paarweise kommutative Elemente und nur identischen Automorphismen trivial wird. Da die Elemente γ_{ji} gemäß (R_4^*) die Kommutatoren $\langle x_j, x_i \rangle$ erzeugender Elemente x_j, x_i von G sind, folgt $G' = M$ (vgl. auch Satz 2.2).

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The Taylor coefficients of certain infinite products

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In memory of Paul Turán

1. Let q be any positive fundamental discriminant; that is, squarefree and $\equiv 1 \pmod 4$ or $q=4d$ where d is squarefree and $\equiv 2$ or $3 \pmod 4$. Let $\chi(j)=\left(\frac{q}{j}\right)$ be the Kronecker symbol. Let us define C_m, $m=0, 1, 2, \ldots$ by

$$\sum_{m=0}^{\infty} C_m t^m = \prod_{n=0}^{\infty} \prod_{j=1}^{q-1} (1-t^{qn+j})^{-\zeta\chi(j)} = F(t), \tag{1.0}$$

where ζ is either 1 or -1. Note that when $q=5$ and $\zeta=1$, the infinite product is

$$\frac{(1-t^2)(1-t^7)\ldots(1-t^3)(1-t^8)\ldots}{(1-t)(1-t^6)\ldots(1-t^4)(1-t^9)\ldots} = 1+\frac{t}{1+}\,\frac{t^2}{1+\ldots};$$

that is, $F(t)$ is Ramanujan's continued fraction [6; p. 294]. If $q=8$ and $\zeta=1$, $F(t)$ has a similar continued fraction representation

$$\frac{(1-t^3)(1-t^{11})\ldots(1-t^5)(1-t^{13})\ldots}{(1-t)(1-t^9)\ldots(1-t^7)(1-t^{15})\ldots} = 1+t+\frac{t^2}{1+t^3+}\,\frac{t^4}{1+t^5+}\,\frac{t^6}{1+t^7+\ldots}.$$

This representation is due to BASIL GORDON [5], but there are indications, according to Gordon, that it might have been known to Ramanujan.

In this paper we shall determine the asymptotic behaviour of the coefficients C_m, first by the saddle point method using a transformation due to ISEKI [7], and then more precisely by the circle method of Hardy and Ramanujan as modified by Rademacher, to obtain a convergent series representation of the C_m. Throughout the paper we make extensive use of results of ISEKI [8]. The exact formula for C_m is given by equation (4.14) in Theorem 4.1. The asymptotic formula (3.9) shows the interesting fact that if the product is turned upside down; that is, if the sign of ζ is reversed, the coefficients have the same asymptotic behaviour in the sense that they

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oscillate with the same amplitude and a common period of oscillation. In the classical case of Ramanujan the amplitude is $(5m)^{-3/4}\exp\left(\frac{4\pi}{5\sqrt{5}}\sqrt{m}\right)$ and the oscillation is a pure cosine wave of the form $\cos\left(\frac{2\pi}{5}\left(m-\frac{2}{5}\right)\right)$. In the case of Gordon's continued fraction the oscillating part of the asymptotic term has the form $\cos\frac{(m-1)\pi}{4}$, hence vanishes for $m\equiv 3 \pmod 4$. This suggests that $C_{4k+3}=0$, for all $k\geqq 0$ and we shall be able to verify this by means of the exact series. If the product is turned upside down $(q=8,\ \zeta=-1)$ then we shall find similarly that $C_{4k+2}=0$ for $k\geqq 0$.

We require the following results concerning the Kronecker symbol:

$$\chi(q-j)=\chi(j) \tag{1.1}$$

$$\sum_{j=1}^{q}\chi(j)=\sum_{j=1}^{q}j\chi(j)=0 \tag{1.2}$$

$$\sum_{j=1}^{q}\chi(j)\exp\left(2\pi i\frac{n}{q}j\right)=\sqrt{q}\,\chi(n). \tag{1.3}$$

These results are found for example in LANDAU [11]. Equation (1.3) is Theorem 215 of [11]. We shall often use without mention that $\chi(j)=0$ if and only if $(q,j)>1$ and that $\chi(mn)=\chi(m)\chi(n)$.

We also note that

$$\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\cos 2n\pi\lambda=((\lambda))^2-\frac{1}{12} \tag{1.4}$$

where, for any real λ, $((\lambda))=\lambda-[\lambda]-\frac{1}{2}$ (see [9], formula 573)[1]). Hence by (1.2) and (1.3)

$$\begin{aligned}\frac{\sqrt{q}}{\pi^2}\sum_{n=0}^{\infty}\frac{\chi(n)}{n^2}&=\frac{1}{\pi^2}\sum_{n=1}^{\infty}\sum_{j=1}^{q-1}\frac{1}{n^2}\chi(j)\exp\left(2\pi i\frac{n}{q}j\right)=\\&=\sum_{j=1}^{q-1}\chi(j)\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\cos\left(2\pi\frac{n}{q}j\right)=Q/q^2\end{aligned} \tag{1.5}$$

where

$$Q=\sum_{j=1}^{q-1}j^2\chi(j). \tag{1.6}$$

[1]) Note that when λ is an integer then $((\lambda))=-1/2$ which is not the usual convention.

From (1.3) and (1.5) we obtain for any integer m

$$\sum_{j=1}^{q-1} \chi(j) \sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2} \cos\left(2\pi \frac{n}{q} mj\right) = \chi(m)Q/q^2. \tag{1.7}$$

Formula (1.5) shows incidentally that $Q>0$ since $\sum_{n=1}^{\infty} \frac{\chi(n)}{n^2} > 1 - \sum_{n=2}^{\infty} \frac{1}{n^2} > 0$.

2. In this section we derive a transformation equation for the generating function $F(t)$. The transformation is obtained from a formula of Iseki and we largely follow his notation in [7] and [8]. Let z be a complex number with $\mathcal{R}z>0$ and h, k be co-prime positive integers with $k>1$. Let D and $K=kq/D$ denote the g.c.d. and l.c.m. of k and q respectively. Put $k=k_1 D$, $q=q_1 D$ so that $(k_1, q_1)=1$ and $K=k_1 q=kq_1$. Choose integers γ, δ satisfying

$$\gamma k_1 - \delta q_1 = 1. \tag{2.0}$$

Let H be any solution of

$$hH \equiv \delta \pmod{k}. \tag{2.1}$$

Set

$$x = \exp\left(2\pi i \frac{h}{k} - 2\pi \frac{z}{k}\right), \quad \tilde{x} = \exp\left(2\pi i \frac{H}{k} - \frac{2\pi}{Kz}\right),$$

and for $1 \leqq a < q$,

$$F_a(\tilde{x}; b, D, \varrho) = \prod_{m=0}^{\infty} (1-\varrho \tilde{x}^{Dm+b})^{-\zeta\chi(a)} (1-\bar{\varrho}\tilde{x}^{Dm+d-b})^{-\zeta\chi(a)} \tag{2.2}$$

where

$$b = ha - D\left[\frac{ha}{D}\right] = \{ha\}_D = D\left(\left(\left(\frac{ha}{D}\right)\right) + \frac{1}{2}\right), \quad \varrho = \varrho_a = \exp\left(-2\pi i a \frac{\gamma}{q}\right). \tag{2.3}$$

The notation $\{x\}_r$ will be used to denote the reduced residue of the integer x modulo r, that is $0 \leqq \{x\}_r < r$, $x \equiv \{x\}_r \pmod{r}$.

Finally let

$$\sigma_a(h, k) = \sum_{\mu \equiv a \pmod{q}}^{(K)} \left(\left(\frac{\mu}{K}\right)\right) \left(\left(\frac{h\mu}{k}\right)\right) \tag{2.4}$$

where $\sum^{(K)}$ signifies that μ runs through a complete set of residues modulo $K=k_1 q$, subject to the condition $\mu \equiv a \pmod{q}$. In particular

$$\sigma_a(k, q) = \left(\left(\frac{a}{q}\right)\right) \left(\left(\frac{ah}{q}\right)\right). \tag{2.5}$$

We also note that for $1 \leqq a < q$, $(a, q)=1$, $D>1$

$$\sigma_{q-a}(h, k) = \sum_{\mu \equiv -a \pmod{q}}^{(K)} \left(\left(\frac{\mu}{K}\right)\right) \left(\left(\frac{h\mu}{k}\right)\right) = \sigma_a(h, k) \tag{2.6}$$

since under the conditions μ is not divisible by D, hence neither $\frac{\mu}{K}$ nor $\frac{h\mu}{k}$ are integers, and $((-x))=-((x))$ for non-integer x. If $D=1$, $(k,q)=1$ then there is a unique $\mu_a\equiv a \pmod q$ such that $\mu_a\equiv 0 \pmod k$ and we obtain, noting that for integer x, $((x))=-1/2$,

$$\sigma_{q-a}(h,k)=\sigma_a(h,k)+\frac{\mu_a}{K}. \tag{2.6'}$$

In the following $\sum_a'$, $\prod_a'$ denote sums and products over $a=1,2,\dots,\left[\frac{q}{2}\right]$.

Theorem 2.1. *Let* $\omega^*(h,k)=\exp\{2\pi i\zeta\sigma^*(h,k)\}$ *where*

$$\sigma^*(h,k)=\sum_a{}'\chi(a)\sigma_a(h,k). \tag{2.7}$$

Then

$$F(x)=\omega^*(h,k)\exp\left\{\frac{\zeta\pi Q}{2kq}\left(\frac{D^2}{q^2}\chi\left(\frac{qh}{D}\right)z^{-1}-z\right)\right\}\times\prod_a{}'F_a(\tilde{x};b,D,\varrho).$$

Proof. From Theorem 1 of [8] we obtain that

$$\prod_{m=0}^{\infty}(1-x^{qm+a})^{-\zeta\chi(a)}(1-x^{qm+q-a})^{-\zeta\chi(a)}= \tag{2.8}$$

$$=\omega_a(h,k)\exp\left\{\frac{\pi\zeta\chi(a)}{6qk}(Bz^{-1}-Az)\right\}\times F_a(\tilde{x};b,D,\varrho)$$

where $\omega_a(h,k)=\exp\{2\pi i\zeta\chi(a)\sigma_a(h,k)\}$ and

$$A=6a^2-6qa+q^2,\quad B=6b^2-6Db+D^2=6D^2\left(\left(\left(\frac{ha}{D}\right)\right)^2-\frac{1}{12}\right) \tag{2.9}$$

by (2.3). It follows at once that

$$\prod_a{}'\omega_a(h,k)=\omega^*(h,k) \tag{2.10}$$

Next we show that

$$\sum_a{}'\chi(a)A=3\sum_{j=1}^{q-1}j^2\chi(j)=3Q. \tag{2.11}$$

Using equation (1.1)

$$\sum_a{}'\chi(a)A=\sum_a{}'\chi(a)\{6a^2-6qa+q^2\}=\sum_a{}'\chi(a)\{6(q-a)^2-6q(q-a)+q^2\}=$$

$$=\frac{1}{2}\sum_{j=1}^{q-1}\chi(j)\{6j^2-6jq+q^2\}=3\sum_{j=1}^{q-1}j^2\chi(j)$$

by equation (1.2) which proves (2.11):

Finally we prove

$$\sum_a{}' \chi(a)B = 3\frac{QD^2}{q^2}\chi\left(\frac{hq}{D}\right). \tag{2.12}$$

By definition

$$\sum_a{}' \chi(a)B = \sum_a{}' \chi(a)\{6b^2-6Db+D^2\} =$$

$$= 6D^2\sum_a{}' \chi(a)\left\{\left(\left(\frac{ha}{D}\right)\right)^2-\frac{1}{12}\right\} = 3D^2\sum_{j=1}^{q-1}\chi(j)\left\{\left(\left(\frac{hj}{D}\right)\right)^2-\frac{1}{12}\right\}$$

since if $D=1$, the last two expressions are 0 by (1.1) and (1.2), and if $D>1$ then

$$\left(\left(\frac{h(q-a)}{D}\right)\right) = \left(\left(-\frac{ha}{D}\right)\right) = -\left(\left(\frac{ha}{D}\right)\right) \tag{2.13}$$

when $D \nmid a$ and $\chi(a)=0$ otherwise. Hence the value of B is unchanged by substituting $q-a$ for a. It follows by equations (1.4) and (1.7) that

$$\sum_a{}' \chi(a)B = 3D^2\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\sum_{j=1}^{q-1}\chi(j)\cos\left(2\pi\frac{nh}{D}j\right) = \frac{3QD^2}{q^2}\chi\left(\frac{hq}{D}\right).$$

Since the product of the left side of equation (2.8) taken over $a=1, 2, \ldots, \left[\frac{q}{2}\right]$ is $F(x)$, the theorem follows at once from (2.8), (2.10), (2.11), and (2.12).

Lemma 2.1. *If $D=q$ and $\chi(h)=\zeta$ then*

$$\prod_a{}' F_a(\tilde{x}; b, q, \varrho) = F(\tilde{x})^{\zeta}.$$

Proof. Because $D=q$, we can take $\gamma=0$, $\delta=-1$ in (2.0), hence $\varrho=1$ and

$$\prod_a{}' F_a(\tilde{x}; b, q, \varrho) = \prod_a{}' \prod_{m=0}^{\infty}[(1-\tilde{x}^{qm+b})(1-\tilde{x}^{qm+q-b})]^{-\zeta\chi(a)}. \tag{2.14}$$

Now $b=ha-q\left[\frac{ha}{q}\right]$, $\chi(b)=\chi(ha)=\zeta\chi(a)$, and $0\leqq b<q$. Hence the exponent in (2.12) is $-\zeta\chi(a)=-\chi(b)$. Since $1=(h,k)=(h,qk_1)$, we have $(h,q)=1$. Thus as a runs through a reduced residue system mod q, so does ah, hence b. The lemma now follows from (2.14) and the definition (1.0) of $F(t)$.

Next we derive an alternative expression for the generalized Dedekind sum $\sigma^*(h,k)$ and hence for $\omega^*(h,k)$. Following Iseki we define integers f, g as follows:

$$\begin{aligned} &f=12,\quad g=1 \quad\text{for}\quad (k,6)=1; \qquad f=3,\quad g=4 \quad\text{for}\quad (k,6)=2;\\ &f=4,\quad g=3 \quad\text{for}\quad (k,6)=3; \qquad f=1,\quad g=12 \quad\text{for}\quad (k,6)=6. \end{aligned} \tag{2.15}$$

In all cases $fg=12$ and $(f, k)=1$. Thus

(2.16) $$(h, k) = 1 \Leftrightarrow (h, gDk) = 1,$$

(2.17) $$(f, gDk) = 1.$$

We define integers φ and ψ to be any solution of

(2.18) $$f\varphi + gDk\psi = 1$$

and choose the solution H of (2.1) so that

(2.19) $$hH \equiv \delta \pmod{gDk}.$$

Set $K_1=\frac{1}{2}k_1(k_1-1)$, $K_2=\frac{1}{6}k_1(k_1-1)(2k_1-1)$,

$$U_a = gD\gamma(aK_1+qK_2)-\varphi\delta(2K^2+3K(2a-q)+A),$$

$$V_a = \varphi(k^2-B),$$

$$W_a = -\frac{\gamma k_1}{4D}(k+2b-2D)+\left(\frac{a\gamma}{2Dq}+\frac{3\varphi\delta q_1}{gD}\right)(2b-D)-g\psi\delta\left(\frac{1}{2}a+\frac{1}{2}q+\frac{1}{2}\left(b-\frac{1}{2}\right)q_1\right),$$

where A, B are defined in (2.9). It is shown in [8, p. 947] that

(2.20) $$\sigma_a(h, k) = \frac{1}{gDk}(U_a h+V_a H)+W_a \pmod 1.$$

Now it follows from (2.6), (2.6′) and (2.7) that

(2.21)
$$\sigma^*(h, k) = \sum_a{}' \chi(a)\sigma_a(h, k) = \frac{1}{2}\sum_{j=1}^{q-1}\chi(j)\sigma_j(h, k) \text{ if } D>1$$
$$= \frac{1}{2}\sum_{j=1}^{q-1}\chi(j)\sigma_j(h, k)-\frac{1}{2}\sum_a{}'\chi(a)\frac{\mu_a}{kq} \text{ if } D=1$$

where $\mu_a=a+v_a q\equiv 0 \pmod k$, $0\leqq v_a<k$. Writing $\mu_a=a+v_a q=r_a k$, $1\leqq r_a<q$ and noting that $r_a\equiv ak^{-1} \pmod q$, we can rewrite the expression $\sum_a{}'\chi(a)\frac{\mu_a}{kq}$ in the form $\frac{\chi(k)}{q}\sum_{\substack{r=1\\ 1\leqq\{kr\}\leqq q/2}}^{q-1} r\chi(r)$, and we obtain for $D=(k, q)=1$

(2.21′) $$2\sigma^*(h, k) = \sum_{j=1}^{q-1}\chi(j)\sigma_j(h, k)-\frac{1}{q}\chi(k)\sum_{\substack{r=1\\ 1\leqq\{kr\}_q<q/2}}^{q-1} r\chi(r).$$

Now substitute for $\sigma_j(h, k)$ from (2.20) into (2.21), (2.21′). Making use of (2.11), (2.12), (1.2) and

$$\sum_{j=1}^{q-1}\chi(j)b = \chi(h)\sum_{j=1}^{q-1}\chi(hj)\{hj\}_q = \chi(h)\sum_{b=1}^{q-1}b\chi(b) = 0,$$

we obtain

$$\sum_{j=1}^{q-1}\chi(j)U_j = -\varphi\delta\sum_{j=1}^{q-1}A\chi(j) = -6\varphi\delta Q,$$

$$\sum_{j=1}^{q-1}\chi(j)V_j = -6\varphi Q\frac{D^2}{q^2}\chi\left(\frac{hq}{D}\right),$$

$$\sum_{j=1}^{q-1}\chi(j)W_j = \frac{\gamma}{Dq}\sum_{j=1}^{q-1}\chi(j)j\{hj\}_q,$$

hence

$$\begin{aligned}2\sigma^*(h,k) \equiv -\frac{6\varphi\delta Q}{gDk}h-\frac{6\varphi Q}{gDk}\frac{D^2}{q^2}\chi\left(\frac{hq}{D}\right)H+\frac{\gamma}{Dq}\sum_{j=1}^{q-1}\chi(j)j\{hj\}_q- \\ -\frac{1}{q}\chi(k)\delta_{D,1}\sum_{\substack{r=1\\ 1\le\{kr\}_q<q/2}}^{q-1} r\chi(r) \pmod 1\end{aligned} \tag{2.22}$$

where $\delta_{D,1}=0$ if $D>1$, 1 if $D=1$.

We apply formula (2.22) to the case when $D=q$, $\chi(h)=\zeta$.

Lemma 2.2. *Let $\omega^*(h,k)$ be as in Theorem 2.1 and suppose that $D=q, k=qk_1$, $\chi(h)=\zeta$. Let h^* be any solution of*

$$hh^* \equiv 1 \pmod{qk}. \tag{2.23}$$

Then

$$\omega^*(h,k) = \mu(h,k)\exp\left\{\pi i(\zeta h+h^*)\frac{Q}{2qk}\right\} \tag{2.24}$$

where $\mu(h,k)=+1$ or -1. In particular if $q\equiv 1 \pmod 4$ and $k=q$, $k_1=1$ then

$$\begin{aligned}\mu(h,q) = \chi(h) &= \left(\frac{h}{q}\right) \text{ (the Legendre symbol) if } q \text{ is prime} \\ &= 1 \text{ if } q \text{ is composite.}\end{aligned} \tag{2.25}$$

Proof. We first note that the value of the expression in (2.24) is independent of the solution h^* in (2.23). We have to verify that

$$Q \equiv 0 \pmod 4) \tag{2.26}$$

Indeed $Q=\sum_{j=1}^{q-1} j^2\chi(j) \equiv \sum_{j\equiv 1 \pmod 2}\chi(j) \pmod 4$ and $\sum_{j\equiv 1 \pmod 2}\chi(j)=0$, trivially from (1.2) if q is even, and from $\sum_{j\equiv 0 \pmod 2}\chi(j)=\chi(2)\sum_{j=1}^{[q/2]}\chi(j)=0$ if $q\equiv 1 \pmod 4$.

Now if $D=q$, $q_1=1$ we can take $\gamma=0$, $\delta=-1$ in (2.0), $h^*=-H$ in (2.23), and (2.22) simplifies to

$$2\sigma^*(h,k_1q) \equiv \frac{6\varphi Q}{gq^2k_1}(h+\zeta h^*) = \frac{6f\varphi Q}{fgq^2k_1}(h+\zeta h^*) \equiv$$

$$\equiv \frac{Q}{2q^2k_1}(h+\zeta h^*) \pmod 1 \text{ by (2.18).}$$

From here and from $\omega^*(h, k_1q)=\exp\{2\pi i\zeta\sigma^*(h, k_1q)\}$ equation (2.24) follows at once.

The sign of $\mu(h, k)$ in (2.24) depends on whether

$$2\sigma^*(h, k_1q)-\frac{Q}{2q^2k_1}(h+\zeta h^*) \tag{2.27}$$

is an even or an odd integer. We want to show that if $q\equiv 1 \pmod 4$ then $\mu(h, q)= =\chi(h)=\left(\frac{q}{h}\right)=\left(\frac{h}{q}\right)$ if q is prime, 1 if q is composite.

From (2.4) and (2.21)

(2.28)
$$2\sigma^*(h, k_1q)=\frac{1}{k}\sum_{r=0}^{k_1-1}\sum_{j=1}^{q-1}\chi(j)\left(\frac{j+rq}{k}-\frac{1}{2}\right)\left(\{h(j+rq)\}_k-\frac{1}{2}k\right)=\frac{1}{k^2}\sum_{j=1}^{k-1}j\chi(j)\{hj\}_k$$

since

$$\sum_{j=1}^{k-1}\chi(j)\{hj\}_k=\chi(h)\sum_{j=1}^{k-1}\chi(hj)\{hj\}_k=\sum_{r=0}^{k_1-1}\sum_{j=1}^{q-1}\chi(j)(rq+j)=0.$$

Comparing (2.27) and (2.28) we find that

$$\frac{1}{k_1}\sum_{j=1}^{k-1}\chi(j)j\{hj\}_k=\frac{1}{2}Q(h+\zeta h^*)+M(h, k)qk \tag{2.29}$$

for some integer $M(h, k)$. Clearly

$$\mu(h, k)=(-1)^{M(h,k)} \tag{2.30}$$

in (2.24).

Now suppose that $k=q\equiv 1 \pmod 4$. Then by (2.26) $\frac{1}{2}Q(h+\zeta h^*)\equiv 0 \pmod 2$ and hence

$$M(h, q)\equiv\sum_{j=1}^{q-1}\chi(j)j\{hj\}_q\equiv\sum_{\substack{(j,q)=1\\ j\equiv 1 \pmod 2}}^{(q)}\{hj\}_q \pmod 2.$$

Equation (2.25) follows from here, (2.28) and from

Lemma 2.3. *Let q be odd and squarefree, $(h, q)=1$. Set $\nu=\nu(h, q)= =\sum_{\substack{(j,q)=1\\ j\equiv 1 \pmod 2}}^{(q)}\{hj\}_q$. Then, $(-1)^{\nu(h,q)}$ is equal to $\left(\frac{h}{q}\right)$ if q is prime, and to 1 if q is composite.*

Proof. The first half of the lemma is a trivial corollary of Gauss' lemma, (see e.g. Bachmann [3], p. 266) but we give a direct proof. Each hj, j odd, has a unique odd residue m_j in the interval $-q<m_j<q$, and $m_i=m_j$ if and only if $i=j$ since $ih\equiv -jh \pmod q$, $(h, q)=1$ implies $i+j\equiv 0 \pmod q$ which is impossible since $i+j$ is even and less than $2q$.

Let λ be the number of negative ones among the m_j so that $\nu(h,q)\equiv$ $\equiv\lambda \pmod 2$. Set

$$P = \Pi\{j | j = 1, \ldots, q-1, (j,q) = 1, j \equiv 1 \pmod 2\}.$$

Then

$$\prod_j m_j = (-1)^\lambda p \equiv h^{\frac{1}{2}\varphi(q)} p \pmod q$$

since exactly one of j, $q-j$ is odd hence exactly half of the relatively prime (to q) reduced residues modulo q are odd. Now if q is prime then $h^{1/2\varphi(q)}=h^{1/2(q-1)}\equiv$ $\equiv\left(\frac{h}{q}\right) \pmod q$, giving $(-1)^{\nu(h,q)}=(-1)^\lambda=\left(\frac{h}{q}\right)$. If q is composite and square-free, $q=q_1\ldots q_r$, $r>1$ then $\frac{1}{2}\varphi(q)$ is a multiple of $\varphi(q_i)$, $i=1,\ldots,r$, hence $h^{1/2\varphi(q)}\equiv 1 \pmod{q_i}$, $i=1,\ldots,r$, $h^{\frac{1}{2}\varphi(q)}\equiv 1 \pmod q$, giving $(-1)^{\nu(h,q)}=1$.

In the case of even q no simple interpretation of $\mu(h,q)$ has been found. As the case $q=8$, $Q=16$ is of special interest, we show:

Lemma 2.4. *Let* $q=8$, $k_1\geqq 1$, $(h, 2k_1)=1$, $\chi(h)=\zeta$. *Then*

$$\mu(h+2k_1, 8k_1) = \mu(h, 8k_1) \quad \textit{if} \quad k_1 \quad \textit{is odd and} \quad hk_1 \equiv 3 \pmod 4,$$

$$\mu(4k_1-h, 8k_1) = \mu(h, 8k_1) \quad \textit{if} \quad k_1 \quad \textit{is even.}$$

Proof. Throughout the proof $\chi(j)$ will denote the Kronecker character modulo 8 i.e. $\chi(j)=1$ for $j\equiv\pm 1 \pmod 8$, $\chi(j)=-1$ for $j\equiv\pm 3 \pmod 8$, $\chi(j)=0$ for j even. Equation (2.29) now has the form

$$\text{(2.31)} \qquad \frac{1}{k_1}\sum_{j=1}^{k-1}\chi(j)j\{hj\}_k = 8(h+\zeta h^*)+64k_1M(h,k), \quad k = 8k_1,$$

$hh^*\equiv 1 \pmod{64\, k_1}$.

Suppose first that k_1 is odd and $k_1h\equiv 3 \pmod 4$. Then $\chi(h+2k_1)=\chi(h)$ since $h+(h+2k_1)=2(h+k_1)\equiv 0 \pmod 8$, and

$$\text{(2.32)} \qquad (h+2k_1)^* \equiv h^*+2k_1 \pmod{16}$$

as seen from $(h+2k_1)(h^*+2k_1)\equiv hh^*+2k_1(h+h^*)+4\equiv 1 \pmod{16}$. Hence

$$\text{(2.33)} \qquad 8\big(h+2k_1+\zeta(h+2k_1)^*\big)-8(h+\zeta h^*) \equiv 16k_1(1+\zeta) \pmod{128}.$$

Furthermore, writing for the moment j' for $\{hj\}_k$, it is easily seen that

$$\text{(2.34)} \qquad \frac{1}{k_1}\sum_{j=1}^{k-1}\chi(j)j\big(\{h+2k_1)j\}_k-\{hj\}_k\big) = $$
$$= 2\sum_{\substack{j\equiv 1 \pmod 4\\ 0<j'<6k_1}} j\chi(j)-6\sum_{\substack{(j\equiv 1 \bmod 4)\\ 6k_1<j'<k_1}} j\chi(j)+6\sum_{\substack{j\equiv 3 \pmod 4\\ 0<j'<2k_1}} j\chi(j)-2\sum_{\substack{j\equiv 3 \pmod 4\\ 2k_1<j'<8k_1}} j\chi(j)$$

where all summations go from $j=1$ to $j=k-1$. For instance

$$\{(h+2k_1)j\}_k = j'+2k_1 \quad \text{if} \quad j \equiv 1 \pmod 4 \quad \text{and} \quad 0 < j'+2k_1 < 8k_1,$$
$$j'-6k_1 \quad \text{if} \quad j \equiv 1 \pmod 4 \quad \text{and} \quad j'+2k_1 > 8k_1, \quad \text{etc.}$$

Now $\sum_{j\equiv 1 \pmod 4} j\chi(j) = -4k_1$, $\sum_{j\equiv 3 \pmod 4} j\chi(j) = 4k_1$, hence the expression in (2.34) is

$$-16k_1 - 8\Big(\sum_{\substack{j\equiv 1 \pmod 4\\ 6k_1<j'<8k_1}} j\chi(j) - \sum_{\substack{j\equiv 3 \pmod 4\\ 0<j'<2k_1}} j\chi(j)\Big) =$$
$$= -16k_1 - 16 \sum_{\substack{j\equiv 1 \pmod 4\\ 6k_1<j'<8k_1}} j\chi(j) + 64k_1 \sum_{\substack{j\equiv 1 \pmod 4\\ 6k_1<j'<8k_1}} \chi(j)$$

and we have to show, by (2.31) and (2.33), if we denote by S_h the set of residues j for which $6k_1 < \{hj\}_k < 8k_1$, that

$$k_1(2+\zeta) + \sum_{\substack{j\equiv 1 \pmod 4\\ j\in S_h}} j\chi(j) \equiv 4 \sum_{\substack{j\equiv 1 \pmod 4\\ j\in S_h}} \chi(j) \pmod 8,$$

or

$$3 \sum_{\substack{j\equiv 1 \pmod 8\\ j\in S_h}} + \sum_{\substack{j\equiv 5 \pmod 8\\ j\in S_h}} \equiv k_1(2+\zeta) \pmod 8 \tag{2.35}$$

provided that $hk_1 \equiv 3 \pmod 4$.

Now if $h\equiv 1 \pmod 8$, $k_1 \equiv 3 \pmod 4$ and $(h, 2k_1)=1$, the elements $hj, j\equiv$ $\equiv 1 \pmod 8$, $j\in S_h$ are exactly the elements $\equiv 1 \pmod 8$ between $6k_1$ and $8k_1$, namely $6k_1+7, 6k_1+15, \dots, 8k_1-7$, hence their total number is $\frac{1}{4}(k_1-3)$. Similarly the elements hj, $j\equiv 5 \pmod 8$, $j\in S_h$ are $6k_1+3, 6k_1+11, \dots, 8k_1-3$, and their total number is $\frac{1}{4}(k_1+1)$. Hence the left hand side of (2.33) is $\frac{3}{4}(k_1-3)+$ $+\frac{1}{4}(k_1+1) = k_1-2$ which is $\equiv 3k_1 \pmod 8$ since $k_1\equiv 3 \pmod 4$.

If $h\equiv 3 \pmod 8$, $k_1 \equiv 1 \pmod 8$, the elements hj, $j\equiv 1 \pmod 8$, $j\in S_h$ are $6k_1+5, 6k_1+13, \dots, 8k_1-5$, and the elements hj with $j\equiv 5 \pmod 8$, $j\in S_h$ are $6k_1+1, \dots, 8k_1-1$. Hence we get, by counting their respective numbers, $\frac{3}{4}(k_1-1)+$ $+\frac{1}{4}(k_1+3) = k_1$ for both sides of (2.35).

A similar count for $h\equiv 5 \pmod 8$, $k_1\equiv 3 \pmod 4$ gives $\frac{3}{4}(k_1+1)+\frac{1}{4}(k_1-3)=k_1$ and for $h\equiv 7 \pmod 8$, $k_1 \equiv 1 \pmod 4$, $\frac{3}{4}(k_1+3)+\frac{1}{4}(k_1-1) = k_1+2$ for the left hand side of (2.35), which agrees with the right hand side in each case. Thus the first half of the Lemma is proved.

Suppose next that k_1 is even. Then

$$(4k_1-h)^* \equiv 12k_1 - h^* \pmod{16k_1} \tag{2.36}$$

as seen from $(4k_1-h)(12k_1-h^*) \equiv hh^* - 4k_1(3h+h^*) \equiv 1 \pmod{16k_1}$. Hence

$$8\big(4k_1-h+\zeta(4k_1-h)^*\big) \equiv -8(h+\zeta h^*) \pmod{128k_1} \quad \text{if} \quad \zeta = 1$$
$$\equiv -8(h+\zeta h^*) + 64k_1 \pmod{128k_1} \quad \text{if} \quad \zeta = -1.$$

Furthermore, if we denote by R_h the subset of odd residues $\{1, 3, \dots, k-1\}$ for which $\{hj\}_k > 4k_1$,

$$\frac{1}{k_1}\sum_{j=1}^{k-1}\chi(j)j(\{hj\}_k+\{(4k_1-h)j\}_k) = 8\sum_{j\in R_h} j\chi(j)$$

since $\{hj\}_k+\{(4k_1-h)j\}_k$ is equal to $4k_1$ if $\{hj\}_k<4k_1$, and to $12k_1$ if $4k_1<\{hj\}_k<$ $<8k_1=k$, and since $4\sum_{j=1}^{k-1} j\chi(j)=0$. To prove the second half of Lemma 2.4 we must therefore show that

$$\sum_{j\in R_h} j\chi(j) \equiv 0 \pmod{16k_1} \quad \text{if} \quad h\equiv \pm 1 \pmod 8$$
$$\equiv 8k_1 \pmod{16k_1} \quad \text{if} \quad h\equiv \pm 3 \pmod 8.$$

Now $0<j<4k_1$, $\{hj\}_k>4k_1 \Rightarrow \{h(4k_1-j)\}_k=12k_1-\{hj\}_k>4k_1$, hence both j and $4k_1-j$ are in R_h and $j\chi(j)+(4k_1-j)\chi(4k_1-j)=4k_1\chi(j)$. Similarly $4k_1<j<8k_1$ $\{hj\}_k>4k_1 \Rightarrow \{h(12k_1-j)\}_k=12k_1-\{hj\}_k>4k_1$, and $j\chi(j)+(12k_1-j)=\chi(12k_1-j)=$ $=12k_1\chi(j)$. Hence

$$\sum_{j\in R_h} j\chi(j) = 2k_1\sum_{\substack{j<4k_1\\ j\in R_h}}\chi(j)+6k_1\sum_{\substack{4k_1<j<8k_1\\ j\in R_h}}\chi(j).$$

But $\{h(k-j)\}_k=k-\{hj\}_k$ therefore exactly one of j, $k-j$ is in R_h and since $\chi(j)=$ $=\chi(k-j)$, we conclude that among the residues j in R_h exactly half have $\chi(j)=\pm 1$. Hence $\sum_{j\in R_h}\chi(j)=0$ and we are finished with the proof if we can show that

$$\sum_{\substack{j=1\\ j\in R_h}}^{k_1-1}\chi(j) \equiv 1-\chi(h) \pmod 4,$$

or, since for $0<j<2k_1$, $2k_1-j\in R_h \Leftrightarrow 2k_1+j\in R_h$ (the condition for both is $4k_1<$ $<\{hj\}_k<6k_1$),

$$\sum_{\substack{j=1\\ j\in R_h}}^{2k_1-1}\chi(j) \equiv \frac{1}{2}(1-\chi(h)) \pmod 2.$$

But $\chi(j)\equiv 1 \pmod 2$ hence the last condition is equivalent to

$$\sum_{\substack{j=1\\ j\in R_h}}^{2k_1-1} \equiv \frac{1}{2}(1-\chi(h)) \pmod 2$$

and this again is equivalent to

$$\sum_{\substack{j=1\\ j\in R_h}}^{4k_1-1} 1 \equiv 1-\chi(h) \pmod 4. \tag{2.37}$$

We formulate the statement in congruence (2.37) as a separate lemma as it has some interest of its own.

Lemma 2.5. *Let* $k=16k'$, $0<h<k$, $(h;k)=1$. *Consider the set* $T_h=\{jh|j=1,3,\ldots,8k'-1\}$ *of the first* $4k'$ *odd multiples of* h, *and denote by* N_h *the number of those members of* T_h *whose reduced residues modulo* k *are in the top half of the interval* $(0,k)$, *i.e.* $8k'<\{hj\}_k<16k'$. *Then*

$$N_h \equiv \begin{matrix} 0 \pmod 4 & \text{if} & h\equiv \pm 1 \pmod 8 \\ 2 \pmod 4 & \text{if} & h\equiv \pm 3 \pmod 8. \end{matrix}$$

Proof. The Lemma is not a direct corollary of Gauss' lemma and we give an independent proof. The number of odd multiples of h between $(16r-8)k'$ and $16rk'$, $r=1,2,\ldots,\frac{1}{2}(h-1)$ is $\left[\frac{16rk'-h}{2h}\right]-\left[\frac{(16r-8)k'-h}{2h}\right]$ and we must show that

$$\sum_{r=1}^{h-1}(-1)^r\left[\frac{8rk'-h}{2h}\right]\equiv \begin{matrix} 0 \pmod 4 & \text{if} & h\equiv \pm 1 \pmod 8 \\ 2 \pmod 4 & \text{if} & h\equiv \pm 3 \pmod 8. \end{matrix}$$

Set $k'=mh+k_0$, $0<k_0<h$, then

$$\left[\frac{8rk'-h}{2h}\right]=4mr+\left[\frac{8rk_0-h}{2h}\right]$$

and we have to show that for $0<k_0<h$, $(2k_0,h)=1$,

$$\sum_{r=1}^{h-1}(-1)^r\left[\frac{8rk_0-h}{2h}\right]\equiv 1-\chi(h) \pmod 4.$$

The left hand side here is

$$\sum_{r=1}^{h-1}(-1)^r\left\{\frac{4rk_0}{h}-1-\left(\left(\frac{8rk_0-h}{2h}\right)\right)\right\}=\frac{2k_0(h-1)}{2h}-\sum_{r=1}^{h-1}(-1)^r\left(\left(\frac{8rk_0-h}{2h}\right)\right).$$

As r runs through the non-zero residues modulo h, so does rk_0 and the congruence reduces to

$$h\sum_{\lambda=1}^{h-1}(-1)^\lambda\left(\left(\frac{8\lambda-h}{2h}\right)\right)\equiv h(\chi(h)-1)\equiv\chi(h)-1 \pmod 4. \tag{2.38}$$

Break up the summation in (2.38) into

$$\sum_{1\le\lambda\le\left[\frac{h-1}{8}\right]}+\sum_{\left[\frac{h-1}{8}\right]<\lambda\le\left[\frac{3h-1}{8}\right]}+\sum_{\left[\frac{3h-1}{8}\right]<\lambda\le\left[\frac{5h-1}{8}\right]}+\sum_{\left[\frac{5h-1}{8}\right]<\lambda\le\left[\frac{7h-1}{8}\right]}+\sum_{\left[\frac{7h-1}{8}\right]\le\lambda\le h-1}.$$

Then in the i-th sum, $i=1,2,3,4,5$, $\left(\left(\frac{8\lambda-h}{2h}\right)\right)=\frac{4\lambda}{h}-1-t_i$ where the value of t_i is $-1,0,1,2,3$ respectively, and we get for the left side of (2.38)

$$-\sum_1(-1)^\lambda+\sum_3(-1)^\lambda+2\sum_4(-1)^\lambda+3\sum_5(-1)^\lambda \pmod 4 \tag{2.39}$$

where $\sum_1, \ldots, \sum_5$ are summed for the respective ranges in the five sums above. But clearly $\sum_i (-1)^\lambda$ only depends on the residue class of h modulo 16 and therefore it is sufficient to calculate (2.39) for $h=1, 3, 5, 7, 9, 11, 13, 15$. The respective values are 0, 2, 2, 4, 4, 2, 2, 4 and in each case they are congruent to $1-\chi(h)$ modulo 4. This proves Lemma 2.5, and the proof of Lemma 2.4 is complete.

3. We shall first use a direct saddle point method to obtain the main asymptotic expression for C_m. From Cauchy's integral formula

$$C_m = \frac{1}{2\pi i} \int_\Gamma t^{-m-1} F(t)\, dt \tag{3.1}$$

where Γ is any circle of positive radius less than 1 centred at the origin. We set $t=\exp\{-2\pi(\beta-i\theta)\}$,

$$C_m = \int_{-1}^{1} F(e^{-2\pi(\beta-i\theta)}) e^{2\pi m(\beta-i\theta)}\, d\theta, \quad \text{where} \quad \beta = \frac{1}{2q}\sqrt{\frac{Q}{qm}}. \tag{3.2}$$

This in fact is the saddle point condition as one can show that the derivative of $F(t)t^{-m-1}$ is zero for $t=\exp\left\{-2\pi\left(\beta-i\frac{h}{q}\right)\right\}$, $\beta=\frac{1}{2q}\sqrt{\frac{Q}{qm}}+O(m^{-3/2})$ and for h satisfying $\chi(h)=\zeta$. We omit verification as it will not be needed explicitly.

We break the range of integration up into Farey intervals of order $N=[\beta^{-3/4}]$ For the relevant properties of Farey dissections see [6], Chapter III. Thus

$$C_m = \sum_{(h,k)=1} \int_{I_{h,k}} F(e^{-2\pi(\beta-i\theta)}) e^{2\pi m(\beta-i\theta)}\, d\theta \tag{3.3}$$

where $I_{h,k}$ is the Farey interval about h/k and the summation extends for $0\leqq h< <k\leqq N$. The Farey intervals with $k=q$, $\zeta\chi(h)=1$ give the dominant terms; however, we require a few lemmas to prove this.

First of all, from Theorem 2.1, letting $\theta=\frac{h}{k}+\varphi$, $z=k(\beta-i\varphi)$ it follows that

$$\begin{aligned} &F(e^{-2\pi(\beta-i\frac{h}{k}-i\varphi)}) = \\ &= \omega^*(h,k)\exp\left\{\frac{\zeta\pi Q D^2 \chi(hq/D)}{2k^2q^3}\,\frac{\beta+i\varphi}{\beta^2+\varphi^2}-\frac{\zeta\pi Q}{2q}(\beta-i\varphi)\right\}\prod_a{}' F_a(\tilde{x}; b, D, \varrho). \end{aligned} \tag{3.4}$$

Lemma 3.1. *There exists a constant $c>0$, independent of the Farey interval $I_{h,k}$ of order N such that*

$$\prod_a{}' F_a(\tilde{x}; b, D, \varrho) = O\{\exp(c\beta^{-1/2})\}$$

on $I_{h,k}$. Furthermore there exists another constant $c'>0$ such that on $I_{h,q}$

$$\prod_a{}' F_a(\tilde{x}; b, D, \varrho) = 1+O\{\exp(-c'\beta^{-1/2})\}.$$

Proof. It is easily seen that

$$|F_a(\tilde{x}; b, D, \varrho)| \leqq \sum_{n=0}^{\infty} p(n)|\tilde{x}|^n$$

where $p(n)$ is the unrestricted partition function of n. Now

$$|\tilde{x}| = \exp\left(-\frac{2\pi}{kK}\frac{\beta}{\beta^2+\varphi^2}\right).$$

Since $k \leqq N = [\beta^{-3/4}]$, $K \leqq qk$ and $|\varphi| \leqq \frac{1}{kN}$ (the length of the Farey interval $I_{h,k}$), we have

$$\frac{2\pi}{kK}\frac{\beta}{\beta^2+\varphi^2} > c_1\beta^{1/2}$$

for some $c_1 > 0$, hence $|\tilde{x}| \leqq \exp(-c_1\beta^{1/2}n)$. It is well known that $p(n) \leqq \exp(c_2 n^{1/2})$ for some $c_2 > 0$. Thus

$$\sum_{n=1}^{\infty} p(n)\exp(-c_1\beta^{1/2}n) = O\{\exp(c_3\beta^{-1/2})\}.$$

This proves the first half of the Lemma.

If $k=q=K$ then

$$\frac{2\pi}{kK}\frac{\beta}{\beta^2+\varphi^2} > 2\pi\beta N^2 > c_4\beta^{-1/2}$$

and $|\tilde{x}| < \exp(-c_4\beta^{-1/2})$, from which the second part of the lemma follows, by the definition (2.1) of F_a. In the following c will denote a suitable positive constant, not necessarily identical with the constant in Lemma 3.1.

Lemma 3.2. *Let $k \neq q$ or $k=q$ and $\chi(h) \neq \zeta$. Then*

$$F\left(e^{-2\pi(\beta - i\frac{h}{k} - i\varphi)}\right) = O\left(\exp\frac{Q\pi}{4q^3\beta}\right) \quad \textit{for} \quad \varphi \in I_{h,k}.$$

Proof. This follows at once from (3.4) and Lemma 3.1 since $\chi(q/D)=0$ if $D \neq q$, and if $k \neq q$ then the smallest multiple of q that k can be is $2q$. Hence the expression in (3.4) is $O\left\{\exp\left(\frac{\pi Q}{8q^3\beta} + c\beta^{-1/2}\right)\right\}$. If $k=q$ and $\zeta\chi(h) = -1$ then the expression is $O\{\exp(c\beta^{-1/2})\}$.

Lemma 3.2 shows that the total contribution in (3.3) of all the Farey arcs except those with $k=q$ and $\chi(h)=\zeta$ is $O\left(\exp\frac{Q\pi}{4q^3\beta}\right)$. We now evaluate the contribution from those arcs in (3.3) with $k=D=q$ and $\chi(h)=\zeta$. From equation (3.4) and Lemma 3.1 we obtain

$$C_m = \sum_{\chi(h)=\zeta} \exp\left(-2\pi i m \frac{h}{q}\right) \omega^*(h, q) \int_{I_{h,q}} \exp\left\{\frac{\pi Q}{2q^3} \frac{1}{\beta - i\varphi} + \left(2\pi m - \frac{\zeta \pi Q}{2q}\right)(\beta - i\varphi)\right\} d\varphi \tag{3.5}$$
$$+ O\left\{\exp\left(\frac{\pi Q}{2q^3 \beta} + 2\pi m \beta - c\beta^{-1/2}\right)\right\} + O\left\{\exp \frac{\pi Q}{4q^3 \beta}\right\}.$$

Here $\int\limits_{I_{h,q}}$ can be written as

$$\frac{1}{i} \int_{\beta - i/qN}^{\beta + i/qN} \exp(E_1/w + E_2 w)\, dw, \quad E_1 = \frac{\pi Q}{2q^3}, \quad E_2 = 2\pi m - \frac{\zeta \pi Q}{2q} \tag{3.6}$$

and this can be changed into a contour integral

$$\frac{1}{i} \int_{(0+)} \exp(E_1/w + E_2 w)\, dw$$

with an error $O\{\exp(c\beta N^2)\} = O\{\exp(c\beta^{-1/2})\}$ (see Ayoub [2, p. 185]). Now

$$\frac{1}{i} \int_{(0+)} \exp(E_1/w + E_2 w)\, dw = 2\pi \operatorname*{res}_{w=0} \{\exp(E_1/w + E_2 w)\}\, dw =$$

$$= 2\pi \sqrt{E_1/E_2}\, I_1(2\sqrt{E_1 E_2}) = \frac{2\pi}{q} \sqrt{\frac{Q}{4qm - \zeta Q}} \cdot I_1\left(\frac{\pi}{q^2} \sqrt{Q(4qm - \zeta Q)}\right)$$

where $I_1(t) = \frac{1}{i} J_1(it)$ is the modified Bessel function of order 1. Hence by (3.5)

$$C_m = \frac{2\pi}{q} \sqrt{Q/(4qm - \zeta Q)}\, I_1\left(\frac{\pi}{q^2} \sqrt{Q(4qm - \zeta Q)}\right) \times \sum_{\chi(h)=\zeta} \omega^*(h, q) \exp\left(-2\pi i m \frac{h}{q}\right) +$$
$$+ O\left\{\exp\left(\frac{\pi Q}{2q^3 \beta} + 2\pi m \beta - c\beta^{-1/2}\right)\right\}.$$

Using the expression (2.24) for $\omega^*(h, q)$ and the saddle point condition (3.2) we obtain

$$C_m = \frac{2\pi}{q} \sqrt{Q/(4qm - \zeta Q)}\, I_1\left(\frac{\pi}{q^2} \sqrt{Q(4qm - \zeta Q)}\right) \times \tag{3.7}$$
$$\times \sum_{\chi(h)=\zeta} \mu(h, q) \cos\left\{2\pi \left(m \frac{h}{q} - (\zeta h + h^*) \frac{Q}{4q^2}\right)\right\} + O\left\{\exp\left(\frac{2\pi}{q} \sqrt{\frac{mQ}{q}} - cm^{1/4}\right)\right\}$$

for some positive constant c, where $Q = \sum_{j=1}^{q-1} \chi(j) j^2$, $h^* h \equiv 1 \pmod{q^2}$ and $\mu(h, q)$ is given for odd q by equation (2.25), otherwise by (2.29), (2.30).

By the well known asymptotic formula

$$I_1(t) = \frac{1}{\sqrt{2\pi t}} e^t \left(1 + O\left(\frac{1}{t}\right)\right) \quad (t \to \infty)$$

(e.g. [1, formula 9.7.1]) (3.7) reduces to

$$C_m = \frac{Q^{1/4}}{(qm)^{3/4}} \exp\left(\frac{2\pi}{q}\sqrt{\frac{mQ}{q}}\right)\times \tag{3.8}$$
$$\times\left\{\sum_{\substack{h=1\\ \chi(h)=\zeta}}^{[q/2]} \mu(h,k)\cos\left[2\pi\left(m\frac{h}{q}-(\zeta h+h^*)\frac{Q}{4q^2}\right)\right]+O\left(\frac{1}{\sqrt{m}}\right)\right\}.$$

This shows in particular that the asymptotic expression (3.7) gives C_m with a relative accuracy of $\exp(-cm^{1/4})$, except possibly when

$$\sum_{\chi(h)=\zeta} \mu(h,q)\cos\left\{2\pi\left(m\frac{h}{q}-(\zeta h+h^*)\frac{Q}{4q^2}\right)\right\}=0.$$

Thus, for Ramanujan's continued fraction ($q=5, \zeta=1, Q=4$) we obtain

$$C_m = \frac{4\pi}{\sqrt{5m-1}} I_1\left(\frac{4\pi}{25}\sqrt{5m-1}\right)\times\left\{\cos\left(\frac{2\pi}{5}\left(m-\frac{2}{5}\right)\right)+O\left(\exp(-cm^{1/4})\right)\right\} = \tag{3.9}$$
$$= \frac{\sqrt{2}}{(5m)^{3/4}}\exp\left(\frac{4\pi}{25}\sqrt{5m}\right)\times\left\{\cos\left(\frac{2\pi}{5}\left(m-\frac{2}{5}\right)\right)+O(m^{-1/2})\right\}.$$

When $q=5, \zeta=-1$, we obtain

$$C_m = \frac{\sqrt{2}}{(5m)^{3/4}}\exp\left(\frac{4\pi}{25}\sqrt{5m}\right)\times\left\{\cos\left(\frac{4\pi}{5}\left(m+\frac{3}{20}\right)\right)+O(m^{-1/2})\right\}.$$

In the case of Gordon's continued fraction ($q=8, \zeta=1, Q=16$) we get $C_m=\frac{1}{2(2m^3)^{1/4}}\exp\left(\frac{\pi}{4}\sqrt{2m}\right)\times\left\{\cos\frac{(m-1)\pi}{4}+O(m^{-1/2})\right\}$ hence the asymptotic term is 0 for $m\equiv 3 \pmod 4$. Similarly if $q=8, \zeta=-1$ then the oscillating part is $\cos\frac{3m\pi}{4}$, hence 0 for $m\equiv 2 \pmod 4$.

4. Finally we consider the representation of C_m as a convergent series. Starting from the integral formula (3.1) we again break the range of integration up into Farey arcs of order N where N is some positive integer. The saddle point condition is now of no help and we take $\exp(-2\pi N^{-2})$ for the radius of the circle Γ. We write (3.1) in the form

$$C_m = \sum_{(h,k)=1} \exp\left(-2\pi i m\frac{h}{k}\right)\int_{I_{h,k}} F(e^{2\pi i\frac{h}{k}-2\pi w})e^{2\pi m w}\,d\varphi, \quad w = N^{-2}-i\varphi, \tag{4.1}$$

where $I_{h,k}$ is the Farey interval about h/k and the summation extends over $0\leqq h< <k\leqq N$.

To evaluate the integral (4.1) we again make use of Iseki's transformation. Let us define r_ν by

$$\prod_a{}' F_a(\tilde{x}; b, D, \varrho) = \sum_{\nu=0}^{\infty} r_\nu \tilde{x}^\nu$$

where $F_a(\tilde{x}; b, D, \varrho)$ is as in (2.1). Then applying Theorem 2.1 to (4.1) with $z=kw$,

$$C_m = \sum_{(h,k)=1} \omega^*(h,k) \exp\left(-2\pi i m \frac{h}{k}\right) \int_{I_{h,k}} \sum_{\nu=0}^{\infty} r_\nu \cdot \exp\left(\frac{2\pi i \nu H}{k}\right) \times$$
$$\times \exp\left\{\left(\frac{\zeta\pi Q D^2}{2k^2 q^3}\chi\left(\frac{hq}{D}\right) - \frac{2\pi\nu}{kK}\right) w^{-1} + \left(2\pi m - \frac{\zeta\pi Q}{2q}\right) w\right\} d\varphi. \tag{4.2}$$

We break the summation over ν into two parts: $\sum_\nu = \sum_{\nu=0}^{\bar{\nu}} + \sum_{\nu>\bar{\nu}}$ where $\bar{\nu}$ is the greatest integer such that

$$\bar{\nu} < \frac{\zeta K Q D^2}{4q^3 k}\chi\left(h\frac{q}{D}\right) = \frac{\zeta D Q}{4q^2}\chi\left(h\frac{q}{D}\right). \tag{4.3}$$

Thus the coefficient of w^{-1} is positive for $\nu=0, 1, \ldots, \bar{\nu}$, and zero or negative for $\nu>\bar{\nu}$. The sum $\sum_{\nu=0}^{\bar{\nu}}$ is of course empty if $\bar{\nu}<0$, in particular if $D\neq q$.

Next we split (4.2) in three sums as follows:

(4.4)
$$C_m = \sum_{\substack{k=1\\ D\neq q}}^{N} \sum_k{}' \omega^*(h,k) \exp\left(-2\pi i m\frac{h}{k}\right) \int_{I_{h,k}} \sum_{\nu=0}^{\infty} r_\nu \exp\left(2\pi i \nu\frac{H}{k}\right) \cdot \exp(-E_0/w + F_0 w)\, d\varphi +$$
$$+ \sum_{\substack{k=1\\ D=q}}^{N} \sum_h{}' \omega^*(h,k) \exp\left(-2\pi i m\frac{h}{k}\right) \int_{I_{h,k}} \sum_{\nu>\bar{\nu}} r_\nu \exp\left(2\pi i \nu\frac{H}{k}\right) \exp(-E_1/w + F_0 w)\, d\varphi +$$
$$+ \sum_{\substack{k=1\\ D=q}}^{N} \sum_{\substack{h\\ \chi(h)=\zeta}}{}' \omega^*(h,k) \exp\left(-2\pi i m\frac{h}{k}\right) \int_{I_{h,k}} \sum_{\nu=0}^{\bar{\nu}} r_\nu \exp\left(2\pi i \nu\frac{H}{k}\right) \exp(-E_2/w + F_0 w)\, d\varphi$$

where

$$E_0 = \frac{2\pi\nu}{kK}, \quad E_1 = \frac{2\pi\nu}{k^2} - \frac{\pi\zeta Q}{2qk^2}\chi(h), \quad E_2 = -\frac{2\pi\nu}{k^2} + \frac{\pi Q}{2qk^2},$$
$$F_0 = 2\pi m - \frac{\zeta\pi Q}{2q} \tag{4.5}$$

and $\sum_h{}'$ denotes summation over those h for which $(h,k)=1$, $0<h<k$.

The estimation of the three parts in (4.4) is based upon the following Lemma which will be proved at the end of this section.

Lemma 4.1. *Let m, ν be integers, H as defined in* § 2, *and* ε>0 *arbitrary. Then*

$$\sum_h^{(\eta)} \exp\left\{-2\pi i m \frac{h}{k} + 2\pi i \nu \frac{H}{k} + 2\pi i \sigma^*(h, k)\right\} = O\left\{k^{\frac{2}{3}+\varepsilon} m^{\frac{1}{3}}\right\}, \quad \eta = +1 \quad or \quad -1,$$

where $\sum\limits_h^{(\eta)}$ *denotes summation over those h for which* $\chi(h) = \eta\zeta$.

Consider now the first summation in (4.4). Since the length of the Farey interval $I_{h,k}$ is $\geqq \frac{1}{kN}$ and $k \leqq N$, we have by (4.1) and (4.5)

$$\mathscr{R}(E_0/w) = \mathscr{R}\left(\frac{2\pi\nu}{kKw}\right) \geqq \frac{2\pi\nu}{qk^2} \frac{N-2}{N^{-4}+(kN)^{-2}} \geqq \frac{2\pi\nu}{q(k^2N^{-2}+1)} \geqq \frac{\pi\nu}{q}.$$

Thus

$$\int\limits_{I_{h,k}} \exp(-E_0/w + F_0 w)\, d\varphi = O\left\{\frac{1}{kN} \exp\left(-\frac{\pi\nu}{q} + 2\pi \frac{m}{N^2}\right)\right\};$$

and upon interchanging the order of summation of h and ν and applying Lemma 4.1 we obtain

$$\sum_{\substack{k=1 \\ D\neq q}}^{N} \sum_h{}' \exp(\dots) \int\limits_{I_{h,k}} \dots = O\left\{N^{-1} \sum_{k=1}^{N} \sum_{\nu=0}^{\infty} |r_\nu| e^{-\frac{\pi\nu}{q}+2\pi m N^{-2}} k^{-\frac{1}{3}+\varepsilon} m^{\frac{1}{3}}\right\} =$$

$$= O\left\{e^{2\pi m N^{-2}} m^{\frac{1}{3}} N^{-1} \sum_{k=1}^{N} k^{-\frac{1}{3}+\varepsilon}\right\}$$

since the radius of convergence of the infinite series is 1. Thus for the first summation of (4.4)

$$\sum_{\substack{k=1 \\ D\neq q}}^{N} \sum_h{}' \dots = O\left\{e^{2\pi m N^{-2}} N^{-\frac{1}{3}+\varepsilon} m^{\frac{1}{3}}\right\}. \tag{4.6}$$

In a similar manner we obtain for the second summation of (4.4), by the remark after the definition (4.4) of $\bar{\nu}$,

$$\sum_{\substack{k=1 \\ D=2}}^{N} \sum_h{}' \dots \int\limits_{I_{h,k}} \sum_{\nu>\bar{\nu}} \dots = O\left\{e^{2\pi m N^{-2}} N^{-\frac{1}{3}+\varepsilon} m^{\frac{1}{3}}\right\}. \tag{4.7}$$

Let us now consider the third (principal) part of (4.4). Define C_m^+ by equation (1.0) with $\zeta = 1$. It follows from Lemma 2.1 that

$$r_\nu = c_\nu^+. \tag{4.8}$$

Transforming $\int\limits_{I_{h,k}}$ as in (3.6), we obtain by the method of Ayoub

$$\int\limits_{I_{h,k}} \exp\{E_2/w + F_0 w\}\, d\varphi = \frac{1}{i} \int\limits_{(0+)} \exp\{E_2/w + F_0 w\}\, dw + O\{e^{2\pi m N^{-2}} k^{-1} N^{-1}\}. \tag{4.9}$$

Upon interchanging the order of summation of h and v and employing Lemma 4.1 and (4.9), it follows that the third summation of (4.4) is

$$(4.10)\quad \sum_{\substack{k=1\\D=q}}^{N} \sum_{\substack{h\\ \chi(h)=\zeta}}' \cdots \int_{I_{h,k}} \sum_{v=0}^{\bar{v}} \cdots = \sum_{\substack{k=1\\D=q}}^{N} \sum_{v=0}^{\bar{v}} c_v^{+} A_k(m, v) L_k(m, v) + O\left\{e^{2\pi m N^{-2}} m^{\frac{1}{3}} N^{-\frac{1}{3}+\varepsilon}\right\}$$

where

$$(4.11)\quad A_k(m, v) = \sum_{\substack{h=1\\(h,kq)=1\\ \chi(h)=\zeta}}^{hq-1} \exp\left\{2\pi i\left(v\frac{H}{kq} - m\frac{h}{kq} + \sigma^*(h, kq)\right)\right\}, \quad hH \equiv -1 \pmod{kq}$$

and

$$L_k(m, v) = \frac{1}{i} \int_{(0+)} \exp\{E_2/w + F_0 w\}\, dw = 2\pi \sqrt{E_2/F_0}\, I_1\left(2\sqrt{F_0 E_2}\right)$$

provided that $F_0 > 0$, i.e. $m > \dfrac{Q}{4q}$. Hence for $m > \dfrac{Q}{4q}$

$$(4.12)\quad L_k(m, v) = \frac{2\pi}{kq}(Q-4vq)^{1/2}(4qm-\zeta Q)^{-1/2} I_1\left\{\frac{\pi}{kq^2}(Q-4vq)^{1/2}(4qm-\zeta Q)^{1/2}\right\}.$$

If we let $N \to \infty$, equations (4.4), (4.6), (4.7) and (4.10) yield with Lemma 2.2:

Theorem 4.1. *Let C_m be given by equation* (1.0) *and C_m^+ by* (1.0) *with* $\zeta = 1$. *Let* $Q^* = \dfrac{1}{4}\sum_{j=1}^{q-1} \chi(j) j^2$ *and h^* be a solution of* $hh^* \equiv 1 \pmod{q^2 k}$. *Then for* $m > Q^*/q$

$$(4.13)\quad C_m = \sum_{k=1}^{\infty} \sum_{0 \le v < Q^*/q} C_v^{+} L_k(m, v) A_k^{(\zeta)}(m, v)$$

where $L_k(m, v)$ is given by (4.12),

$$(4.14)\quad A_k^{(\zeta)}(m, v) = \sum_{\substack{h=1\\ \chi(h)=\zeta}}^{kq-1} \mu(h, kq) \cos\frac{2\pi}{kq}\left(mh + vh^* - \frac{Q^*}{q}(\zeta h + h^*)\right)$$

with $\mu(h, kq)$ given by (2.25) *and* (2.29), (2.30) *when* $(h, kq) = 1$, $\mu(h, kq) = 0$ *otherwise.*

The following are the first thirty values of Q^*/q:

q	5	8	12	13	17	21	24	28	29	33	37	40	41	44	53
Q^*/q	$\frac{1}{5}$	$\frac{1}{2}$	1	1	2	2	3	4	3	6	5	7	8	7	7
q	56	57	60	61	65	69	73	76	77	85	88	89	92	93	97
Q^*/q	10	14	12	11	16	12	22	19	12	18	23	26	20	18	34

To prove Lemma 4.1, we estimate

$$\sum_{\substack{h=1\\ \chi(h)=\eta\zeta}}^{k-1} \exp\left\{-2\pi i m\frac{h}{k}+2\pi i v\frac{H}{k}+2\pi i\sigma^*(h,k)\right\} \tag{4.15}$$

by means of Kloosterman sums. Define the trigonometric sum

$$S(u,v;\lambda,\Lambda;r) = \sum_{\substack{0<h<r\\ (h,r)=1\\ h\equiv\lambda \,(\mathrm{mod}\, \Lambda)}} \exp\left\{\frac{2\pi i}{r}(uh+vh^*)\right\} \tag{4.16}$$

for integers $u, v, \lambda, \Lambda, r>0$ where Λ is a positive divisor of r and $hh^*\equiv 1 \pmod r$. It was proven by KLOOSTERMAN [10] that there exists a $\beta>0$ such that with $\varepsilon>0$ arbitrary,

$$S(u,v;\lambda,\Lambda;r) = O\left(r^{1-\beta+\varepsilon}(u,r)^\beta\right). \tag{4.17}$$

According to SALIÉ [12] and DAVENPORT [4], β can be taken as $\beta=\frac{1}{3}$, and we assume this for convenience.

By making use of the expression (2.22) for $\sigma^*(h,k)$, the sum in (4.15) can be written as

$$\sum_{\substack{h=1\\ \chi(h)=\eta\zeta}}^{k-1} \lambda(h,k)\exp\left\{\frac{2\pi i}{gqk}\left[-(3\varphi\delta Q+gqm)h+\delta\left(gqv-3\varphi Q\chi(h)\right)h^*\right]\right\} \quad \text{if} \quad D=q,$$

$$\sum_{\substack{h=1\\ \chi(h)=\eta\zeta}}^{k-1} \lambda(h,k)\exp\left\{\frac{2\pi i}{gDk}\left[-(3\varphi\delta Q+gDm)h+v\delta gDh^*\right]\right\} \quad \text{if} \quad D\neq q$$

where we have taken $H=\delta h^*$, $hh^*\equiv 1 \pmod{gDk}$, by the definition (2.19) of H. The value of

$$\lambda(h,k) = \pm\exp\left\{\pi i\left[\frac{\gamma}{Dq}\sum_{j=1}^{q-1}\chi(j)j\{hj\}_q - \frac{1}{q}\chi(k)\delta_{D,1}\sum_{\substack{r=1\\ 1\leqq\{hr\}_q<q/2}}^{q-1} r\chi(r)\right]\right\}$$

only depends on the residue class to which h and k belong modulo q, provided that we select the solution γ of $\gamma k_1-\delta q_1=1$ in (2.0) always in the interval $0\leqq\gamma<q_1$.

Thus the sum (4.15) splits up in at most q^2 sums of the form $cS(u,v;\lambda,\Lambda;r)$ with $\Lambda=q$, $r=gDk$ and $u=-(gDM+3\varphi\delta Q)$. But $uq=-\left(gDqm+3\varphi Q(\gamma k-D)\right)$ by (2.0) and so $(uq,k)=(gDqm-3\varphi QD,k)\leqq gDqm+3|\varphi QD|$, $(u,r)=O(m)$, and (with $\beta=\frac{1}{3}$)

$$r^{1-\beta+\varepsilon}(u,r)^\beta = O\left(k^{\frac{2}{3}+\varepsilon}m^{\frac{1}{3}}\right).$$

Hence by (4.17) the expression (4.15) itself is $O(k^{2/3+\varepsilon}m^{1/3})$, which is precisely the statement of Lemma 4.1. The proof of Theorem 4.1 is now complete.

5. We apply Theorem 4.1 to the case when $q=8$, $Q=16$ and $\zeta=1, m\equiv 3 \pmod 4$ or $\zeta=-1$, $m\equiv 2 \pmod 4$. Suppose first that k is odd and that $hk\equiv 3 \pmod 4$. Then $\chi(h+2k)=\chi(h)$, as in the proof of Lemma 2.4, and we find, by the first case of the Lemma and (2.32)

(5.1)
$$\mu(h, 8k)\cos\frac{\pi}{4k}\left(mh-\frac{1}{2}(\zeta h+h^*)\right)+\mu(h+2k, 8k)\cos\frac{\pi}{4k}\left(m(2k+h)-\right.$$
$$\left.-\frac{1}{2}(\zeta(h+2k)+(h+2k)^*)\right) =$$
$$=\mu(h, 8k)\left\{\cos\frac{\pi}{4k}\left(mh-\frac{1}{2}(\zeta h+h^*)\right)+\cos\frac{\pi}{4k}\left(mh-\frac{1}{2}(\zeta h+h^*)+\right.\right.$$
$$\left.+k(2m-\zeta-1)\right) = 0$$

since $m\equiv 3 \pmod 4$ if $\zeta=1$ and $m\equiv 2 \pmod 4$ if $\zeta=-1$. Clearly the co-prime residues h modulo $8k$ with $\chi(h)=\zeta$ can be uniquely grouped in pairs $h, h+2k$ satisfying the condition $kh\equiv 3 \pmod 4$ and each pair of corresponding terms in (4.14) cancels, by (5.1). Therefore $A_k^{(\zeta)}(m, v)=0$ for odd k in (4.14).

Suppose next that k is even. Then $\chi(h)=\chi(4k-h)$ and we find, by the second case of Lemma 2.4 and (2.36)

$$\mu(h, 8k)\cos\frac{\pi}{4k}\left(mh-\frac{1}{2}(\zeta h+h^*)\right)+\mu(4k-h, 8k)\cos\frac{\pi}{4k}\left(m(4k-h)-\right.$$
$$\left.-\frac{1}{2}(\zeta(4k-h)+(4k-h)^*)\right) =$$
$$=\mu(h, 8k)\left\{\cos\frac{\pi}{4k}\left(mh-\frac{1}{2}(\zeta h+h^*)\right)+\cos\frac{\pi}{4k}\left(mh-\frac{1}{2}(\zeta h+h^*)+\right.\right.$$
$$\left.\left.+2k(3+\zeta-2m)\right)\right\} = 0$$

since m is odd when $\zeta=1$, even when $\zeta=-1$. Thus the terms in the sum (4.14) cancel in pairs and $A_k^{(\zeta)}(m, v)=0$ also for even k. Thus $A_k^{(\zeta)}(m, v)=0$ for all k, and we obtain the following corollary of Theorem 4.1:

Theorem 5.1. *If* $q=8$, $\zeta=1$ *(Gordon's continued fraction) then* $C_m=0$ *in* (1.0) *for all* $m\equiv 3 \pmod 4$. *If* $q=8$, $\zeta=-1$ *then* $C_m=0$ *for all* $m\equiv 2 \pmod 4$.

Another interesting case is $q=12$ when the principal asymptotic term of C_m vanishes for $m\equiv 5 \pmod 6$ if $\zeta=1$ and for $m\equiv 3 \pmod 6$ if $\zeta=-1$. It is quite likely that C_m is zero for these values of m but at present we do not have the appropriate modification of Lemma 2.4. It would be interesting to prove these results independently from the series representation (4.13).

It is easy to give an interpretation of Theorem 5.1 in terms of partitions. Take first Gordon's product

$$F(x) = \frac{(1-x^3)(1-x^{11})\dots(1-x^5)(1-x^{13})\dots}{(1-x)(1-x^9)\dots(1-x^7)(1-x^{15})\dots} = \sum C_m X^m.$$

Since $C_m=0$ for $m\equiv 3 \pmod 4$, we have $F(x)-F(-x)+i(F(ix)-F(-ix))=0$ and upon expressing this equation as a sum of four fractions of products and bringing the fractions to common denominator we obtain for the product

$$G(x) = \prod_{m=0}^{\infty} (1+x^{8m+1})(1-x^{8m+3})(1-x^{8m+5})(1-x^{8m+7})(1+x^{16m+2})(1+x^{16m+14}) =$$

$$= \sum d_n x^n,$$

$G(x)-G(-x)+i(G(ix)-G(-ix))=0$, that is $d_n=0$ for $n\equiv 3 \pmod 4$. Or, if we take the partitions of n into distinct positive integers of the form $8m+1$, $16m+2$, $8m+7$, $16m+14$, $8m+3$, $8m+5$, and if $n\equiv 3 \pmod 4$ then the number of such partitions in which parts $8m\pm 3$ appear an even number of times is the same as the number of those partitions in which parts $8m\pm 3$ appear an odd number of times. By reinterpreting parts $16m+2$ and $16m+14$ as $(8m+1)+(8m+1)$, $(8m+7)+$ $+(8m+7)$ respectively, we obtain

Theorem 5.2. *Denote by Π_n the set of those partitions of n into positive odd parts in which summands* $\equiv \pm 3 \pmod 8$ *appear at most once and summands* $\equiv \pm 1 \pmod 8$ *appear with multiplicity at most three. Then if* $n\equiv 3 \pmod 4$, *exactly half of the partitions belonging to Π_n contain an even (odd) number of summands* $\equiv \pm 3 \pmod 8$.

For instance Π_{19} contains the 14 partitions (1, 1, 1, 3, 13), (1, 1, 1, 5, 11), (1, 1, 1, 7, 9), (1, 1, 17), (1, 1, 3, 5, 9), (1, 5, 13), (1, 9, 9), (1, 1, 3, 7, 7), (1, 3, 15), (1, 7, 11), (3, 5, 11), (3, 7, 9), (5, 7, 7), (19). The first seven of these contain an even number of summands $\equiv \pm 3 \pmod 8$, and the last seven an odd number of such summands.

By turning Gordon's product upside down, a similar theorem is obtained for $n\equiv 2 \pmod 4$, with ± 1 and ± 3 interchanged. For instance in the ten partitions (1, 17), (5, 13), (1, 3, 5, 9), (3, 5, 5, 5), (1, 5, 5, 7), (1, 3, 3, 11), (3, 3, 3, 9), (3, 3, 5, 7), (3, 15), (7, 11) of 18, exactly five contain an even number of summands $\equiv \pm 1 \pmod 8$.

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On Möbius bounded operators

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An operator T (that is, a bounded linear transformation) on a Banach space X is said to be power bounded if $\|T^n\| \leqq M$ $(n=1, 2, \ldots)$. It is said to be Möbius bounded if $\|\varphi(T)\| \leqq c$ $(\varphi \in \mathcal{M})$. Here $\mathcal{M}$ denotes the Möbius group of analytic homeomorphisms of the unit disc in the complex plane onto itself. The elements of $\mathcal{M}$ have the form

(1) $$\varphi(z) = \alpha(z-a)(1-\bar{a}z)^{-1} \quad (|\alpha| = 1, \ |a| < 1).$$

We assume that the spectrum of T is contained in the closed unit disc, so that $\varphi(T)$ is defined. (It is known that Möbius boundedness is equivalent to a first order growth condition on the resolvent; see Proposition 3.)

In this note we present a simple example of an operator that is Möbius bounded but not power bounded (in fact, $\|T^n\| = n+1$); this answers a question of B. M. Schreiber. We first present two propositions indicating the relationship between the two concepts.

In preparing this note the author benefitted from discussions with C. Foiaş and J. G. Stampfli.

Proposition 1. *If T is power bounded then it is Möbius bounded.*

Proof. Let $\varphi(z) = \sum \hat{\varphi}(n) z^n$ where φ is given by (1). One verifies that $\hat{\varphi}(0) = -\alpha a$, and $\hat{\varphi}(n) = \alpha(1-|a|^2)(\bar{a})^{n-1}$ $(n>0)$. Hence $\sum |\hat{\varphi}(n)| = 1 + 2|a| < 3$, and so $\|\varphi(T)\| < 3M$.

Proposition 2. *Let T be Möbius bounded with constant c, then*

$$\|T^n\| \leqq \frac{ce}{2}(n+1) \quad (n > 0).$$

Proof. $\varphi(T) = \sum \hat{\varphi}(n) T^n$. Hence

$$\hat{\varphi}(n) T^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(e^{i\theta} T) e^{-in\theta}\, d\theta,$$

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and so

(2) $$|\hat{\varphi}(n)|\,\|T^n\| \leqq \frac{1}{2\pi}\int \|\varphi(e^{i\theta}T)\|\,d\theta \leqq c.$$

Thus
$$\|T^n\| \leqq \frac{c}{(1-|a|^2)|a|^{n-1}} \quad (n>0).$$

For fixed n we let $a^2=1-2/(n+1)$. Then $\|T^n\|\leqq(n+1)c/2w_n$, where $w_n=[1-2/(n+1)]^{(n-1)/2}$. Since $(1-1/x)^{x-1}$ decreases to $1/e$ we see that w_n decreases to $1/e$, which completes the proof.

Inequality (2) shows that the proposition remains true under the weaker hypothesis that $\int\|\varphi(e^{i\theta}T)\|\,d\theta$ is bounded $(\varphi\in\mathcal{M})$. Thus one might hope that a better result could be obtained from the hypothesis that T is Möbius bounded. The following example shows that this is not the case.

Theorem 1. *There exists a Banach space X and an operator T such that $\|\varphi(T)\|=\|T\|=2$ $(\varphi\in\mathcal{M})$, and $\|T^n\|=n+1$ $(n\geqq 0)$.*

Proof. The elements of X are all those functions $f(z)$, analytic in the open unit disc, for which $f'\in H^1$. Geometrically this is equivalent to saying that f maps the unit disc onto a Riemann surface with a perimeter of finite length (see Duren [2], Theorem 3.12, for the case when f is a conformal map onto a plane domain bounded by a Jordan curve). By an inequality of G. H. Hardy ([2], Corollary to Theorem 3.15) each such function has an absolutely convergent power series, and hence is continuous on the closed disc. We norm X by taking the sum of the supremum norm of f and the H^1 norm of f':

$$\|f\| = \|f\|_\infty+\|f'\|_1.$$

One verifies that f is a commutative Banach algebra with identity under ordinary multiplication: $\|fg\|\leqq\|f\|\,\|g\|$, $\|1\|=1$.

The elements of X may be viewed as operators on X, operating by multiplication; the operator norm coincides with the norm in X.

For our operator T we take the operator M_z of multiplication by z. By the remark above: $\|T^n\|=\|z^n\|=n+1$ $(n\geqq 0)$.

Finally, we show that T is Möbius bounded. Let $\varphi\in\mathcal{M}$. Then one verifies that $\varphi(T)=M_\varphi$, the operator of multiplication by φ. Hence $\|\varphi(T)\|=\|\varphi\|$. Also, if φ is given by (1), then

$$\|\varphi'\|_1=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|a|^2}{|1-\bar{a}e^{i\theta}|^2}\,d\Theta=1.$$

(The integrand is the Poisson kernel.) Hence $\|\varphi\|=2$, which completes the proof.

Incidentally, it can be shown that the Möbius group operates on X by composition as a group of isometries:

(3) $$\|f\circ\varphi\| = \|f\| \quad (f\in X,\ \varphi\in\mathcal{M}).$$

This is obvious from the geometric interpretation of the elements of X; and it can be shown analytically by a change of variables in calculating $\|(f\circ\varphi)'\|_1$.

Question. If T is a Möbius bounded operator on Hilbert space do we have $\|T^n\|\leqq c(n+1)^{1/2}$?

We can prove this if $\|T^n\|$ is increasing, and if there is a unit vector f such that $\|T^n f\|\geqq\|T^n\|/2$ $(n>0)$. We omit the details. C. A. McCarthy proves this under a stronger hypothesis than Möbius boundedness (see Remark 3 at the end of this paper).

It is known that the condition of Möbius boundedness is equivalent to first order growth of the resolvent. We include a proof for completeness.

Proposition 3. *Let T be an operator with spectrum in the closed unit disc. Then T is Möbius bounded if and only if there is a constant d such that*

$$\|(T-\lambda)^{-1}\|\leqq\frac{d}{|\lambda|-1}\quad(1<|\lambda|<\infty). \tag{4}$$

Proof. Let $\lambda=1/a$ $(|a|<1)$. One verifies that (4) is equivalent to

$$\|(1-aT)^{-1}\|\leqq\frac{d}{1-|a|}\quad(|a|<1). \tag{5}$$

If φ is given by (1) then $1+\overline{a}\alpha\varphi(z)=(1-|a|^2)(1-\bar{a}z)^{-1}$. Thus

$$\|1+\overline{a}\alpha\varphi(T)\|=(1-|a|^2)\|(1-\bar{a}T)^{-1}\|. \tag{6}$$

Finally, (4) is equivalent to the boundedness of the right side of (6), while Möbius boundedness is equivalent to the boundedness of the left side of (6) (in showing that T is Möbius bounded it is sufficient to restrict attention to the parameter range $\frac{1}{2}<|a|<1$). This completes the proof.

We make a few remarks concerning the special case when T is an operator on Hilbert space.

Remark 1. B. Sz.-Nagy and C. Foiaş ([9], Remark 3, p. 20) have shown that if T satisfies (4) merely for $1<|\lambda|<1+\varepsilon$ for some $\varepsilon>0$, but with $d=1$, then T is in some class C_ϱ and hence is power bounded.

Remark 2. In [3] (Satz 4.1, p. 164) H.-O. Kreiss showed that if an operator on a finite-dimensional space satisfies (4), then it is power bounded (the bound depends on the dimension of the space). A shorter proof was given by K. W. Morton [7]. (This result was needed in studying the stability of finite-difference approximations to partial differential equations.)

On an infinite-dimensional space even the stronger assumption that the spectrum of T is a subset of the unit circumference and that (4) holds for all $|\lambda|\neq 1$ does not imply power boundedness. An example is given, somewhat implicitly, by C. A. McCarthy and J. T. Schwartz ([6], p. 199) (they state the growth condition (4) only for $|\lambda|>1$).

Closely related to this is an example due to A. S. MARCUS ([4], p. 544) of an operator A with real spectrum, that is not similar to a self-adjoint operator, but for which $\|(A-\lambda)^{-1}\| \leqq c|\text{Im}\,\lambda|^{-1}$ ($\text{Im}\,\lambda \neq 0$). (Such an example can also be obtained from the McCarthy—Schwartz example.)

In the positive direction, if (4) holds for all $|\lambda| \neq 1$, with $d=1$, then T is a unitary operator (see W. F. DONOGHUE [1]; see J. G. STAMPFLI [8], Theorem 2, for a generalization characterizing normal operators with spectrum contained in a smooth curve).

Remark 3. C. A. MCCARTHY [5] has considered the strong resolvent condition

$$\|(T-\lambda)^{-k}\| \leqq \frac{d}{(|\lambda|-1)^k} \quad (k = 1, 2, \ldots). \tag{7}$$

He shows that if T is an operator on a Banach space and if T satisfies (7), then $\|T^n\| \leqq 4n^{1/2}$ ($n=1, 2, \ldots$). Also, given $\varepsilon > 0$ he produces an example of an operator on Hilbert space that satisfies (7) with $d=1+\varepsilon$, but is not power bounded (the powers grow like $(\log\log n)^{1/2}$). Finally, he gives a more complicated example of an operator T whose spectrum is the unit circumference, such that both T and T^{-1} satisfy (7) with $d=1+\varepsilon$, but neither T nor T^{-1} is power bounded (again the powers grow like $(\log\log n)^{1/2}$).

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Unary algebras with regular endomorphism monoids

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The pair (A,f) where A is a non-void set and f is a unary operation will be briefly called a *unar*. For simplicity we often write A instead of (A,f). Let f^0 be the identity transformation and $f^n=ff^{n-1}$ for every $n\geqq 1$. We define a relation $\sim$ on the unar A as follows:

$$a \sim b \underset{\text{def}}{\Leftrightarrow} f^m(a)=f^n(b) \quad \text{for some} \quad m, n \geqq 0.$$

This relation turns out to be an equivalence relation, the classes of which are called *components*. A unar consisting of a single component is termed *connected*. An element a of a unar is *cyclic* if $f^n(a)=a$ for some $n\geqq 1$. A unar is called a *cycle of length n* if it consists of the distinct elements $a, f(a), \dots, f^{n-1}(a)$ with $f^n(a)=a$. The term *loop* stands for a cycle of length 1. The set

$$a^{\triangle} \underset{\text{def}}{=} \{f^n(a) \mid n=0,1,2,\dots\}$$

is called the *upper cone* of the element a. If $f(x)=a$ then the element x is called a *parent* of a. A connected unar which is not a cycle but in which every element has a unique parent is said to be a *line*. A connected unar A is called a *cycle*, a *loop* or a *line with short tails* if A contains a cycle, resp. a loop or a line C such that $f(x)\in C$ for every $x\in A$. We agree on denoting the cardinality of a set A by $|A|$. If $X\subseteqq A$, set $f(X)\underset{\text{def}}{=}\{f(x)\mid x\in X\}$.

The mapping φ of the unar A into the unar B is called a *homomorphism* if $\varphi(f(x))=f(\varphi(x))$ for all $x\in A$. In particular, if $A=B$ then we obtain the definition of an *endomorphism* of A. The set of all endomorphism of A forms a monoid which is denoted by End A. The set of all *automorphisms* (i.e. bijective endomorphisms) of A forms a group denoted by Aut A.

If m and n are positive integers or ∞ then the symbol $m|n$ means that either $n=\infty$ or $m, n\neq\infty$ and m divides n.

In the present paper the following results are established:

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Theorem 1. *The endomorphism monoid of a unar is regular if and only if each component of the unar is either a cycle with short tails or a line with short tails and for any components K, L and M the following conditions are satisfied:*

(1) *if* $|f(L)|\,\big|\,|f(K)|$, $|f(M)|\,\big|\,|f(L)|$ *and* $L \neq M$ *then* $|f(K)| = |f(L)|$;

(2) *if* $|f(L)|\,\big|\,|f(K)|$, $K \neq f(K)$ *and* $L \neq f(L)$ *then* $|f(K)| = |f(L)|$;

(3) *if* $|f(L)|\,\big|\,|f(K)|$ *and* $|L \setminus f(L)| \geqq 2$ *then* $K = f(K)$ *or* $K = L$.

Theorem 2. *The endomorphism monoid of a unar is an inverse semigroup if and only if every element in the unar has at most two parents, each of its components is either a cycle with short tails or a line with short tails and beyond conditions* (1)—(3), *the following are also fulfilled for any components K, L and M:*

(4) *if* $|f(L)|\,\big|\,|f(K)|$ *and* $|f(M)|\,\big|\,|f(K)|$ *then* $K = L$ *or* $K = M$ *or* $L = M$;

(5) *if* $K \neq L$ *and* $|f(L)|\,\big|\,|f(K)|$ *then* $|f(L)| = 1$ *and* $|f(K)| > 1$, *and if, in addition,* $L \neq f(L)$ *then* $K = f(K)$.

Theorem 3. *The endomorphism monoid of a unar is a group if and only if each of its components is either a cycle or a line and for arbitrary components K and L the relation* $|L|\,\big|\,|K|$ *implies* $K = L$.

In the proof of these theorems we need some lemmas. The first one characterizes inverse semigroups, while the others concern the unar (A, f).

Lemma 1. (cf. [1] Theorem 1.17) *The following conditions on a semigroup S are equivalent:*

(i) *S is regular and any two idempotents of S commute with each other;*

(ii) *S is an inverse semigroup (i.e., every element of S has a unique inverse).*

Lemma 2. (cf. [2] Theorem 2.4) *In a connected unar A the following conditions are equivalent:*

(i) *A is either a cycle or a line;*

(ii) *f is bijective;*

(iii) *the endomorphisms of A are the elements of the set* $\{f^k\colon k = 0, \pm 1, \pm 2, \dots\}$.

Lemma 3. (cf. [2] Lemma 2.8) *If C is a cycle of length n in A and* $a \in C$ *then for every endomorphism* φ *the element* $a\varphi$ *is contained in a cycle of length p where p divides n.*

Lemma 4. (cf. [2] Lemma 2.11) *If* $a, b \in A$ *belong to the same component K,* $\varphi \in \operatorname{End} A$ *and* $a\varphi$ *belongs to the component L then* $b\varphi$ *also belongs to L.*

The following lemma is easily verified.

Lemma 5. *The set* $f(A)$ *is a subalgebra in A which is invariant with respect to every endomorphism in* End A.

Lemma 6. *If* End A *is regular then* $f \in \operatorname{Aut} f(A)$.

Proof. Since $f \in \operatorname{End} A$, we have $f\Phi f = f$ for some $\Phi \in \operatorname{End} A$. If $x \in f(A)$, i.e., $x = f(y)$ for some $y \in A$ then we have

$$f\Phi(x) = f\Phi f(y) = f(y) = x \quad \text{and} \quad \Phi f(x) = f\Phi(x) = x,$$

which completes the proof.

Lemma 7. *If K is a component in A then* End K *can be embedded in* End A. *If* End A *is a regular or an inverse semigroup or a group then* End K *has the same property.*

Proof. If $\varphi \in \operatorname{End} K$ then put

$$\Phi(\varphi)(x) = \begin{cases} \varphi(x) & \text{if } x \in K \\ x & \text{otherwise} \end{cases}$$

for every x in A. It is easy to see that Φ embeds End K in End A. If there exists $(\Phi(\varphi))^{-1} \in \operatorname{End} A$ then K is invariant with respect to $(\Phi(\varphi))^{-1}$ and, consequently, the restriction of $(\Phi(\varphi))^{-1}$ to K can be chosen as φ^{-1}. Hence, End K is a group provided End A is a group. Assume now that the monoid End A is regular. Then $\Phi(\varphi)\Psi\Phi(\varphi) = \Phi(\varphi)$ for some $\Psi \in \operatorname{End} A$. If $\Psi(K) \subseteq K$ then the regularity of End K follows. In the opposite case we have $\Psi(a) \notin K$ for some $a \in K$. Then Lemma 4 implies that

$$\Phi(\varphi)\Psi\Phi(\varphi)(a) = \Phi(\varphi)\Psi(\varphi(a)) = \Psi(\varphi(a)) \notin K,$$

in contrary to the fact that $\Phi(\varphi)(a) = \varphi(a) \in K$. Finally, it remains to note that the rest follows from Lemma 1 since $\varepsilon^2 = \varepsilon$ implies $(\Phi(\varepsilon))^2 = \Phi(\varepsilon)$.

Lemma 8. *Let K and L be cycles with short tails or lines with short tails such that $|f(L)| \,\big|\, |f(K)|$. Let $a \in K$ and $b \in L$. Then there exists a homomorphism $\varphi\colon K \to L$ such that $\varphi(a) = b$ and $\varphi(x) \in f(L)$ for every $x \neq a$.*

Proof. If $f^m(a) = f^n(a)$ and $m > n$ then $|f(K)| \,\big|\, (m-n)$ and, moreover, $n \geqq 1$ provided $a \notin f(K)$. Since $|f(L)| \,\big|\, (m-n)$, we have $f^m(b) = f^n(b)$. Thus there exists a homomorphism $\varphi\colon a^{\triangle} \to b^{\triangle}$. If $|f(K)| < \infty$ then $\varphi(f(K)) = f(L)$. If $|f(K)| = \infty$, i.e. $f(K)$ is a line, then φ can be naturally continued to a homomorphism $\varphi\colon (a \cup f(K)) \to L$ such that again $\varphi(f(K)) = f(L)$. If $x \in K \setminus f(K)$ then $\varphi(f(x))$ is defined and there exists a unique element $x' \in f(L)$ such that $f(x') = \varphi(f(x))$. Choosing $\varphi(x) = x'$ we obtain the required homomorphism.

The proof of Theorem 1. Let A be a unar and End A a regular monoid. If A is connected then, by Lemma 6, we have $f \in \operatorname{Aut} f(A)$. Then Lemma 2 implies $f(A)$ to be a cycle or a line. In view of Lemma 7 the components of A have the required structure. Now suppose the components K, L and M satisfy the assumptions of property (1). Owing to Lemma 8, there exist homomorphisms $\varphi\colon K \to L$

and ψ: $L \to M$. For every $x \in A$ put

$$\Phi(x) = \begin{cases} \varphi(x) & \text{if } x \in K \\ \psi(x) & \text{if } x \in L \\ x & \text{otherwise.} \end{cases}$$

Clearly, $\Phi \in \text{End } A$. Since End A is regular, we have $\Phi\Psi\Phi = \Phi$ for some $\Psi \in \text{End } A$. If $x \in K$ then $\Phi(x) \in L$. Since $\Phi\Psi\Phi(x) = \Phi(x)$, we conclude $\Psi\Phi(x) \in K$. Hence it follows by Lemma 4 that $\Psi(L) \subseteq K$. Consequently, $|f(K)| \,|\, |f(L)|$ by Lemma 3 and therefore $|f(K)| = |f(L)|$. Let us assume now that the components K and L fulfil the assumptions of property (2). Choose $a \in K \setminus f(K)$, $b \in L \setminus f(L)$ and, making use of Lemma 8, let φ: $K \to L$ and ψ: $L \to L$ be homomorphisms satisfying $\varphi(a) = b$ and $\psi(L) \subseteq f(L)$, respectively. We define $\Phi \in \text{End } A$ as above and select $\Psi \in \text{End } A$ such that $\Phi\Psi\Phi = \Phi$. If $|f(K)| \neq |f(L)|$ then $\Psi(L) \cap K = \emptyset$ by Lemmas 3 and 4. Consequently, $\Phi\Psi\Phi(a) \neq b = \Phi(a)$ which is a contradiction. Finally, let K and L satisfy the assumptions of property (3). Choose $b, c \in L \setminus f(L)$ such that $b \neq c$. Suppose there exists $a \in K \setminus f(K)$. By Lemma 8, we can find homomorphisms φ: $K \to L$, ψ: $L \to L$ such that $\varphi(a) = b$, $\psi(b) = c$ and $\varphi(x)$, $\psi(y) \in f(L)$ provided $x \neq a$ and $y \neq b$. Define Φ as above and choose Ψ such that $\Phi\Psi\Phi = \Phi$. If $K \neq L$ then, by Lemma 4, we have $\Psi(L) \cap K = \emptyset$ or $\Psi(L) \cap L = \emptyset$. In the first case we obtain that $\Phi\Psi\Phi(a) = \Phi\Psi(b) \neq b = \Phi(a)$ while in the second case we have $\Phi\Psi\Phi(b) = \Phi\Psi(c) \neq c = \Phi(b)$. But, of course, both cases are impossible. Thus the necessity of the conditions of Theorem 1 is proved.

Conversely, suppose now that the unar A satisfies these conditions and $\Phi \in \text{End } A$. For every component L consider the set of components

$$L^{\triangle} = \{K \mid \Phi(K) \subseteq L\}.$$

We establish that the following statement is valid:

If $L^{\triangle} \neq \emptyset$ then there exists a component L^0 and a homomorphism ψ_L: $L \to L^0$ such that $\Phi\psi_L(x) = x$ for every $x \in \text{Im } \Phi \cap L$.

In fact, taking into consideration Lemma 4, denote by M the component containing $\Phi(L)$. By the structure of the components of A we have $f(L) \subseteq \text{Im } \Phi$. Suppose first that $\text{Im } \Phi \cap L = f(L)$. If $M = L$ then choose an element $a \in f(L)$ and, putting $L^0 = L$, choose an element $b \in f(L^0)$ with $\Phi(b) = a$. Applying Lemma 8 we can find a homomorphism ψ_L: $L \to L^0$ with $\psi_L(a) = b$. If $x \in \text{Im } \Phi \cap L$ and $x = f^k(a)$ for some k then

$$\Phi\psi_L(x) = f^k \Phi\psi_L(a) = f^k \Phi(b) = f^k(a) = x.$$

If there exists no such k then $f(L)$ is a line. Therefore $f^k(x) = a$ for some k whence we have

$$f^k(\Phi\psi_L(x)) = \Phi\psi_L(a) = \Phi(b) = a = f^k(x).$$

Since $\Phi(L)=f(L)$, it follows that $\Phi\psi_L(x)=x$. If $M\neq L$ then, by Lemma 4, we can see that $\Phi(K)\subseteq L$ for a component $K\neq L$. Lemma 3 and property (1) imply that $|f(K)|=|f(L)|$. Then we can set $L^0=K$ and literally repeat the foregoing argument. Assume now that $\operatorname{Im}\Phi\cap L\neq f(L)$. If there exists a component K in $L^\triangle$ such that $K\neq L$ and $K\neq f(K)$ then, by property (3) and Lemma 3, we obtain that $L\setminus f(L)$ consists of a single element, say a. Then $a=\Phi(b)$ for some $b\in A$ and we can choose L^0 to be the component containing b. It is easy to see that $b\notin f(L^0)$. Due to property (2), $|f(L^0)|=|f(L)|$ which allows us to apply the above reasoning again. It remains to treat the case when $\operatorname{Im}\Phi\cap L\neq f(L)$ and $K=f(K)$ for each $K\in L^\triangle\setminus\{L\}$. Then $L\in L^\triangle$. There is no difficulty in verifying that Φ induces an automorphism, say φ, on $f(L)$. Let $\psi_L: f(L)\to f(L)$ be the inverse of this automorphism. For every $x\in(\operatorname{Im}\Phi\cap L)\setminus f(L)$, choose and fix an $x'\in L$ with $\Phi(x')=x$ and set $\psi_L(x)=x'$. Then ψ_L maps $\operatorname{Im}\Phi\cap L$ into L and

$$\psi_L(f(x))=\psi_L f\Phi(x')=\psi_L\Phi(f(x'))=\psi_L\varphi(f(x'))=f(x')=f\psi_L(x).$$

Just as above, we extend ψ_L to a homomorphism of L into L for which we will use the same notation ψ_L and set $L^0=L$.

Returning to the proof of the theorem, put

$$\Psi(x)=\begin{cases}\psi_L(x) & \text{if } x\in L \text{ with } L^\triangle\neq\emptyset\\ x & \text{otherwise.}\end{cases}$$

Obviously, $\Psi\in\operatorname{End}A$. Moreover, we have $L^\triangle\neq\emptyset$ provided L is a component containing $\Phi(x)$ for some $x\in A$. Hence, utilizing the property of the homomorphism ψ_L we conclude that $\Phi\Psi\Phi(x)=\Phi\psi_L\Phi(x)=\Phi(x)$ which proves the regularity of the monoid End A.

The proof of Theorem 2. Let A be a unar and End A an inverse monoid. Suppose a, b, c are distinct elements in A and $f(a)=f(b)=f(c)$. Denote by K the component containing these elements. By Theorem 1, $f(K)$ is a cycle or a line. Therefore, for example, $a, b\notin f(K)$. The transformations ε and δ defined by

$$\varepsilon(x)=\begin{cases}b & \text{if } x=a\\ x & \text{otherwise}\end{cases}\quad\text{and}\quad \delta(x)=\begin{cases}a & \text{if } x=b\\ x & \text{otherwise,}\end{cases}$$

respectively, turn out to be endomorphisms of K. Here $\varepsilon^2=\varepsilon$, $\delta^2=\delta$,

$$\varepsilon\delta(a)=\varepsilon(a)=b \quad\text{and}\quad \delta\varepsilon(a)=\delta(b)=a.$$

Since the idempotents in an inverse semigroup commute with each other by Lemma 1, this contradicts Lemma 7. Thus, every element of A has at most two parents. The validity of conditions (1)—(3) is implied by Theorem 1. Assume now that the distinct components K, L and M satisfy the assumptions of property (4). Owing to Lemma 8, there exist homomorphisms $\varphi\colon K\to L$ and $\psi\colon K\to M$. The transformations Φ and Ψ where

$$\Phi(x)=\begin{cases}\varphi(x) & \text{if } x\in K\\ x & \text{if } x\notin K\end{cases}\quad\text{and}\quad \Psi(x)=\begin{cases}\psi(x) & \text{if } x\in K\\ x & \text{if } x\notin K\end{cases}$$

are easily shown to be endomorphisms of A. Here $\Phi^2=\Phi$, $\Psi^2=\Psi$. Still, if $x\in K$, we have

$$\Phi\Psi(x)=\Phi(\psi(x))=\psi(x)\in M\quad\text{and}\quad \Psi\Phi(x)=\Psi(\varphi(x))=\varphi(x)\in L,$$

which, by Lemma 1, fails to hold in the inverse monoid End A. If K and L are distinct components with $|f(L)|\,\big|\,|f(K)|$ and $|f(L)|\geqq 2$ then select elements $a\in f(K)$ and $b, c\in f(L)$ such that $b\neq c$. Lemma 8 implies the existence of homomorphisms $\varphi\colon K\to L$ and $\psi\colon K\to L$ such that $\varphi(a)=b$ and $\psi(a)=c$. Furthermore, we define endomorphisms Φ and Ψ by setting

$$\Phi(x)=\begin{cases}\varphi(x) & \text{if } x\in K\\ x & \text{if } x\notin K\end{cases}\quad\text{and}\quad \Psi(x)=\begin{cases}\psi(x) & \text{if } x\in K\\ x & \text{if } x\notin K.\end{cases}$$

Then $\Phi^2=\Phi$, $\Psi^2=\Psi$ and $\Phi\Psi(a)=c\neq b=\Psi\Phi(a)$. If $f(K)=\{v\}$ and $f(L)=\{w\}$ then $\Phi^2=\Phi$, $\Psi^2=\Psi$ and $\Phi\Psi(v)=v\neq w=\Psi\Phi(v)$, where

$$\Phi(x)=\begin{cases}v & \text{if } x\in K\cup L\\ x & \text{if } x\notin K\cup L\end{cases}\quad\text{and}\quad \Psi(x)=\begin{cases}w & \text{if } x\in K\\ x & \text{if } x\notin K.\end{cases}$$

This contradicts Lemma 1 as above. If $|f(L)|=1$ and assume $L\neq f(L)$ and $K\neq f(K)$ then, by property (3), $L=\{b, w\}$ where $f(b)=f(w)=w$. Putting

$$\Phi(x)=\begin{cases}w & \text{if } x\in K\\ x & \text{if } x\notin K\end{cases}\quad\text{and}\quad \Psi(x)=\begin{cases}w & \text{if } x\in f(K)\\ b & \text{if } x\in K\setminus f(K)\\ x & \text{if } x\notin K,\end{cases}$$

we can see that $\Phi, \Psi\in\text{End } A$, $\Phi^2=\Phi$ and $\Psi^2=\Psi$. However, for every $x\in K\setminus f(K)$ we have

$$\Phi\Psi(x)=\Phi(b)=b\quad\text{and}\quad \Psi\Phi(x)=\Psi(w)=w,$$

which is impossible. Thus we have proved the necessity of the conditions of Theorem 2.

Assume now that these conditions are satisfied in the unar A. In consequence of Theorem 1, End A is a regular monoid. Let $\Phi, \Psi\in\text{End } A$ such that $\Phi^2=\Phi$ and $\Psi^2=\Psi$. By Lemma 1, we have only to show that $\Phi\Psi=\Psi\Phi$. Let x be an arbitrary element in A and K the component containing x. Denote by L and M the components containing $\Phi(x)$ and $\Psi(x)$, respectively. By Lemma 4, $\Phi(K)\subseteqq L$ and $\Psi(K)\subseteqq M$. By virtue of Lemma 3 and property (4) we have $K=L$, $K=M$

or $L=M$. If $K=L=M$ then both Φ and Ψ induce idempotent endomorphisms on $f(K)$. Thus Lemma 2 implies that $\Phi(z)=\Psi(z)=z$ for every $z\in f(K)$, i.e. $\Phi\Psi(x)==x=\Psi\Phi(x)$ provided $x\in f(K)$. Otherwise, if $x\notin f(K)$ then, since $f(x)$ has at most two parents, we obtain $\Phi(x)=x$ or $\Phi(x)=x'$ where $x'\in f(K)$ and $f(x')=f(x)$. A similar statement holds for Ψ, too. If $\Phi(x)=\Psi(x)=x$ then $\Phi\Psi(x)=x=\Psi\Phi(x)$. If $\Phi(x)=x'$ or $\Psi(x)=x'$ then we have $\Phi\Psi(x)=x'=\Psi\Phi(x)$. Suppose now $L=M$ but $K\neq L$. Then property (5) implies that $|f(L)|=1$ and either $K=f(K)$ or $L=f(L)$. Hence we have $\Phi(z)=\Psi(z)=w$ for every $z\in K$ where w denotes the single element in $f(L)$. Moreover, $\Phi(w)=\Phi^2(z)=\Phi(z)=w$. Analogously, $\Psi(w)=w$. Thus

$$\Phi\Psi(x) = \Phi(w) = w = \Psi(w) = \Psi\Phi(x).$$

Finally, consider the case when $K=L$ but $L\neq M$. Property (5) implies that $|f(M)|=1$ and either $K=f(K)$ or $M=f(M)$. Denoting by w the single element of $f(M)$, we conclude as above that $\Psi(z)=w$ for every $z\in K$. In addition, properties (4) and (5) imply $\Phi(M)\subseteq M$ by Lemma 3. Thus

$$\Phi\Psi(x) = \Phi(w) = w = \Psi\Phi(x).$$

The case when $K=M$ but $M\neq L$ is handled similarly. Therefore $\Phi\Psi=\Psi\Phi$ which completes the proof.

The proof of Theorem 3. Let A be a unar and End A a group. If $|A|=1$ then the conditions of Theorem 3 are trivially fulfilled. Let $|A|\neq 1$. Lemmas 2 and 7 imply each component to be a cycle or a line. If we have distinct components K and L with $|L|\,|\,|K|$ then, according to property (5) in Theorem 2, $|L|=1$. If w is the single element in L then, defining Φ by $\Phi(x)=w$ for every $x\in A$, we have $\Phi\in\text{End } A=\text{Aut } A$. Consequently, $A=L$, contradicting our assumption. The proof of the necessity of the conditions of Theorem 3 is complete. In the case when these conditions are satisfied it is not difficult to show by Lemma 3 that every endomorphism induces an endomorphism on each component. To complete the proof it remains only to make use of Lemma 2.

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The singular sequence problem

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Introduction

If A and B are bounded selfadjoint operators in a Hilbert space H, and $B-A$ is compact, then A and B have the same essential spectrum.

This well-known result of H. WEYL [7] (cf. [1], section 94, Satz 1, [6], Satz 9.9) is easily proved by using Weyl's characterization of the essential spectrum by singular sequences. At the same time this proof shows that more is valid, namely: A and B have the same singular sequences. (For definitions see the end of the introduction.)

In this note we treat the question if the converse of this statement is valid, i.e.: *Let A and B be bounded selfadjoint operators with the same singular sequences. Is it possible to conclude that $B-A$ is compact?* We remark that we do not know the complete answer to this question. The purpose of this note is to present this problem and to give a positive answer in a special case.

We remark that a kind of converse of the above theorem of Weyl was proved by VON NEUMANN [4] (cf. [1], section 94, Satz 3): If A and B are bounded selfadjoint operators in a separable Hilbert space, with the same essential spectrum, then there exists a unitary operator U such that $B-UAU^{-1}$ is compact. It is easy to see by examples that $B-A$ need not be compact under this assumption; also A and B need not have the same singular sequences.

In Section 1 we review some results for unbounded operators in order to motivate the form in which we finally state the "singular sequence problem" for unbounded operators. In Section 2 we give a positive solution for the case that $\sigma(A)$ (or, equivalently, $\sigma_e(A)$) is countable. Here we need only that every singular sequence for A and s is also a singular sequence for B and s. In section 3 we show by an example that in the general case this assumption alone is not sufficient.

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We conclude the introduction by some basic facts and some notations. Let A be a selfadjoint operator in a Hilbert space H. The *essential spectrum* $\sigma_e(A)$ of A is the set consisting of the limit points of the spectrum $\sigma(A)$ of A and the eigenvalues of infinite multiplicity; this is just the set of the points of $\sigma(A)$ which are not isolated eigenvalues of finite multiplicity ([1], section 93, [3], section 1, [6], section 7.4). A real number s is in $\sigma_e(A)$ if and only if there is a *singular sequence for A and s*, i.e., a sequence (f_n) in $D(A)$ (the domain of definition of A) such that $\liminf \|f_n\| > 0$, $f_n \rightharpoonup 0$, and $(A-s)f_n \to 0$ ([7]; cf. [3], Theorem 11, [6], Satz 7.24). Let B also be a selfadjoint operator in H. If $D(A) \subset D(B)$, and for any $s \in \sigma_e(A)$ and any singular sequence (f_n) for A and s, (f_n) is also a singular sequence for B and s, then we say that *$\sigma_e(A)$ is contained in $\sigma_e(B)$ in the sense of singular sequences*, abbreviated $\sigma_e(A) \overset{S}{\subset} \sigma_e(B)$. Obviously, $\sigma_e(A) \overset{S}{\subset} \sigma_e(B)$ implies $\sigma_e(A) \subset \sigma_e(B)$. If $\sigma_e(A) \overset{S}{\subset} \sigma_e(B)$ and $\sigma_e(B) \overset{S}{\subset} \sigma_e(A)$, we say that *A and B have the same singular sequences*, $\sigma_e(A) \overset{S}{=} \sigma_e(B)$.

The singular sequence problem was posed by K. Jörgens in connection with the work [3]. He gave this problem to W. Tafel as the topic for his diploma thesis [5].

1. Statement of the problem

In order to give the first formulation of the problem for unbounded operators let us recall the statement of Weyl's theorem for unbounded operators (cf. [6], Satz 9.9): *Let A be selfadjoint, V symmetric and A-compact. Then $B = A + V$ is selfadjoint and $\sigma_e(A) \overset{S}{=} \sigma_e(B)$.* The following example shows that the problem for unbounded operators cannot simply be the question if the converse of the foregoing statement is true.

1.1. Example. Let $\dim H = \infty$, A selfadjoint with $\sigma_e(A) = \emptyset$, $B = 2A$. Then obviously $\sigma_e(A) \overset{S}{=} \sigma_e(B)$, but $B - A = A$ is not A-compact.

We remark that from the assumption that V is A-compact one also concludes that V is A-bounded with A-bound zero ([2], Corollary V.3.8, [6], Satz 9.7). We include this in our first formulation of the problem for unbounded operators.

1.2. Problem. Let A and B be selfadjoint operators, $\sigma_e(A) \overset{S}{=} \sigma_e(B)$, $V = B - A$ A-bounded with A-bound zero. Is it then possible to conclude that V is A-compact?

Let us note that V is A-compact (A-bounded with A-bound zero) if and only if V is B-compact (B-bounded with B-bound zero); therefore Problem 1.2 is symmetric with respect to A and B.

In our second formulation of the problem for unbounded operators we do not want to assume the A-boundedness with A-bound zero. Instead of the A-compactness we want to conclude a "local" compactness of V ("local" with respect to the spectral measure of A).

1.3. Definition. Let A be a selfadjoint operator, with spectral measure E. An operator V is called *A-locally compact* if $R(E(J))\subset D(V)$ and $VE(J)$ is compact for all compact intervals J.

Let us recall some known facts.

1.4. Theorem (cf. [6], Satz 9.8, Satz 9.11 b, c). *Let A be a selfadjoint operator.*

a) *An operator V is A-compact if and only if V is A-locally compact and A-bounded with A-bound zero.*

b) *An A-bounded operator V is A-locally compact if and only if V is A^p-compact for some (and then for all) $p>1$.*

1.5. Theorem (cf. [6], Satz 9.13). *Let A and B be selfadjoint operators, $D(A)=D(B)$. Let $V=B-A$ be A-locally compact. Then $\sigma_e(A) \overset{s}{=} \sigma_e(B)$.*

We conjecture that the converse of Theorem 1.5 is true.

1.6. Problem. Let A and B be selfadjoint operators, $\sigma_e(A) \overset{s}{=} \sigma_e(B)$. Is it then possible to conclude that $V=B-A$ is A-locally compact?

If the answer to Problem 1.6 is yes, then Theorem 1.4 shows that the answer to Problem 1.2 is also yes.

Also we remaık that for bounded operators A and B Problems 1.2 and 1.6 are just the problem formulated in the introduction.

Finally let us note that both Theorem 1.5 and Problem 1.6 are symmetric with respect to A and B. To see this it is sufficient to show: If A, B are selfadjoint operators, $D(A)=D(B)$, then $V=B-A$ is A-locally compact if and only if V is B-locally compact. This statement follows from [6], Satz 9.11 b, c and Satz 9.12.

2. A special case

In this section let A and B be selfadjoint operators in a Hilbert space H, with $D(A)\subset D(B)$. Let E be the spectral measure of A.

2.1. Lemma. *Let $s\in\mathbf{R}$ and $\varepsilon>0$. Assume that every singular sequence for A and s is also a singular sequence for B and s. Then there exist $\delta>0$ and a finite dimensional subspace M of H such that for all $f\in R(E((s-\delta, s+\delta)))\cap M^{\perp}$[1] we have the inequality $\|(B-A)f\|\leqq\varepsilon\|f\|$.*

[1]) R denotes range.

Proof. We proceed by contradiction. So we can define inductively a sequence (f_n) in H, with the following properties:

$$f_n \in R\left[E\left(\left(s-\frac{1}{n},\ s+\frac{1}{n}\right)\right)\right] \cap \operatorname{span}\{f_1, \dots, f_{n-1}\}^{\perp}, \quad \|f_n\|=1, \quad \|(B-A)f_n\| > \varepsilon,$$

for all $n \in \mathbf{N}$. Obviously (f_n) is a singular sequence for A and s, and therefore by assumption also a singular sequence for B and s. This implies $\|(B-A)f_n\| \leqq \leqq \|(B-s)f_n\| + \|(A-s)f_n\| \to 0$ $(n \to \infty)$, in contradiction to $\|(B-A)f_n\| > \varepsilon$ $(n \in \mathbf{N})$. □

2.2. Theorem. *Assume that for some compact interval J the set $\sigma(A) \cap J$ is countable and that every singular sequence for A and $s \in J$ is also a singular sequence for B and s. Then $(B-A)E(J)$ is compact.*

Proof. Let (f_n) be a sequence in H with $f_n \rightharpoonup 0$ and $\|f_n\| \leqq 1$ $(n \in \mathbf{N})$; we have to show $(B-A)E(J)f_n \to 0$.

Let $\varepsilon > 0$. Let $\sigma(A) \cap J = \{s_1, s_2, \dots\}$. (We disregard the trivial case $\sigma(A) \cap \cap J = \emptyset$.) For s_j and $\varepsilon 2^{-j}$, $j \in \mathbf{N}$, we choose δ_j and M_j according to Lemma 2.1. Then $\sigma(A) \cap J \subset \bigcup_{j=1}^{\infty} J_j$, where $J_j := (s_j - \delta_j, s_j + \delta_j)$, and by the compactness of $\sigma(A) \cap J$ we find $m \in \mathbf{N}$ such that $\sigma(A) \cap J \subset \bigcup_{j=1}^{m} J_j$.

For $j = 1, \dots, m$ define $K_j := J_j \setminus \bigcup_{i=1}^{j-1} J_i$. Then $\sigma(A) \cap J \subset \bigcup_{j=1}^{m} K_j$, and $K_1, \dots$ $\dots, K_m$ are mutually disjoint. For $j = 1, \dots, m$, denote by P_j, P_j' the orthogonal projections onto $R(E(K_j))$, $R(E(K_j)) \cap M_j^{\perp}$, and define $P_j'' = P_j - P_j'$. P_j'' is finite dimensional because $R(P_j'') = \overline{P_j M_j}$, and M_j has finite dimension. Now we decompose

$$E(J) = \sum_{j=1}^{m} P_j E(J) = \sum_{j=1}^{m} P_j' E(J) + P,$$

where $P = \sum_{j=1}^{m} P_j'' E(J)$ is finite dimensional and therefore compact. Also the assumptions imply that $(B-A)E(J)$ is a bounded operator, and so $(B-A)P = (B-A)E(J)P$ is compact. This implies

$$\begin{aligned} &\limsup \|(B-A)E(J)f_n\| \\ &\leqq \sum_{j=1}^{m} \limsup \|(B-A)P_j' E(J) f_n\| + \limsup \|(B-A)P f_n\| \\ &\leqq \sum_{j=1}^{m} \limsup (\varepsilon 2^{-j}) \|P_j' E(J) f_n\| + 0 \leqq \varepsilon \sum_{j=1}^{m} 2^{-j} < \varepsilon. \end{aligned}$$

This shows $(B-A)E(J)f_n \to 0$ $(n \to \infty)$. □

2.3. Corollary. *Let $\sigma_e(A) \overset{s}{\subset} \sigma_e(B)$, and assume that $\sigma(A)$ is countable. Then $B-A$ is A-locally compact.*

Proof. By Theorem 2.2 $(B-A)E(J)$ is compact for each compact interval J. □

We note that Corollary 2.3 applies especially to the case that A has purely discrete spectrum, i.e., $\sigma_e(A)=\emptyset$.

3. An example

In this section we show by an example that in the general setting of Problem 1.6 the assumption $\sigma_e(A) \overset{s}{=} \sigma_e(B)$ cannot be replaced by $\sigma_e(A) \overset{s}{\subset} \sigma_e(B)$, as was done in the special case of Corollary 2.3.

3.1. Example. We are going to construct bounded selfadjoint operators A and V with the properties:

(i) V is not compact,

(ii) $\sigma_e(A) \overset{s}{\subset} \sigma_e(A+V)$,

(iii) $[0, 1]=\sigma_e(A)\neq\sigma_e(A+V)$.

Property (iii) shows that the example is not a counterexample to Problem 1.6.

We take the Hilbert space $H=L_2(0, 1)$, and as A we take the multiplication by the independent variable, $Af(x)=xf(x)$. The spectral measure of A is then given by $E(\Sigma)f=\chi_\Sigma\cdot f$ (Σ Borel set of $\mathbf{R}$). Also $\sigma_e(A)=\sigma(A)=[0, 1]$.

To construct V, we define the function ψ: $(0, \infty)\to\mathbf{R}$ by

$$\psi(x) = (-1)^m \quad \text{for} \quad m < x \leqq m+1, \quad m\in\mathbf{N}_0$$

($\mathbf{N}_0=\{0, 1, 2, \dots\}$), and we define $v_m\in L_2(0, 1)$ by $v_m(x):=\psi(2^m x)$ for $m\in\mathbf{N}_0$; clearly (v_m) is an orthonormal sequence. We define V to be the orthogonal projection onto the subspace spanned by $\{v_m;\ m\in\mathbf{N}_0\}$, i.e. $Vf=\sum_{m=0}^{\infty}\langle v_m, f\rangle v_m$. Now we show that (i), (ii), (iii) are valid.

(i) is obvious.

(ii) Let $s\in[0, 1]=\sigma_e(A)$, and let (f_n) be a singular sequence for A and s. We are done if we show $Vf_n\to 0$. Without restriction we may assume $\|f_n\|\leqq 1$. Let $\varepsilon>0$. There exist $m'\in\mathbf{N}_0, p, q\in\mathbf{Z}, p<q$, such that $s\in J:=(p/2^{m'}, q/2^{m'})$, $(q-p)/2^{m'}\leqq\leqq\varepsilon^2$. From $(A-s)f_n\to 0$ we obtain $(I-E(J))f_n\to 0$, and therefore $V(I-E(J))f_n\to 0$. Next, we define $v'_m:=E(J)v_m=\chi_J\cdot v_m$ ($m\in\mathbf{N}_0$). It is easy to see from the definition of the v_m that $(v'_m)_{m\geqq m'}$ is an orthogonal sequence in $L_2(0, 1)$ with $0<\|v'_m\|^2\leqq\varepsilon^2$. In

$$VE(J)f = \sum_{m=0}^{\infty}\langle v_m, E(J)f\rangle v_m = \sum_{m=0}^{m'-1}\langle v'_m, f\rangle v_m + \sum_{m=m'}^{\infty}\langle v'_m, f\rangle v_m$$

we estimate

$$\left\| \sum_{m=m'}^{\infty} \langle v'_m, f\rangle v_m \right\|^2 = \sum_{m=m'}^{\infty} |\langle v'_m, f\rangle|^2 = \sum_{m=m'}^{\infty} \|v'_m\|^2 |\langle v'_m/\|v'_m\|, f\rangle|^2 \leqq \varepsilon^2 \|f\|^2.$$

This estimate together with $\langle v'_m, f_n\rangle \to 0$ $(n\to\infty)$ for all $m\in \mathbf{N}_0$ implies

$$\limsup \|VE(J)f_n\| \leqq \varepsilon,$$
$$\limsup \|Vf_n\| \leqq \limsup \|VE(J)f_n\| + \limsup \left\|V(I-E(J))f_n\right\| \leqq \varepsilon+0.$$

This shows $Vf_n\to 0$.

(iii) Consider the sequence $(v_m)_{m\in \mathbf{N}_0}$. It is orthonormal, and therefore $v_m \rightharpoonup 0$. Also,

$$\langle v_m, (A+V)v_m\rangle = \langle v_m, Av_m\rangle + \langle v_m, Vv_m\rangle = \int_0^1 x\,dx + \|v_m\|^2 = 3/2.$$

Now the following lemma shows that there exists $s\in\sigma_e(A+V)$ with $s\geqq 3/2$.

3.2. Lemma. *Let A be a bounded selfadjoint operator, E its spectral measure. Let* $s\in \mathbf{R}$. *If there exists a sequence (f_n) in H with* $f_n\rightharpoonup 0$, $\|f_n\|=1$ $(n\in \mathbf{N})$, *such that* $\limsup \langle f_n, Af_n\rangle \geqq s$, *then* $\sigma_e(A)\cap[s,\infty)\neq\emptyset$.

Proof. If we assume $\sigma_e(A)\cap[s,\infty)=\emptyset$, then there exists $\varepsilon>0$ such that $E((s-\varepsilon,\infty))$ is a finite dimensional projection. This would imply

$$\begin{aligned}&\limsup \langle f_n, Af_n\rangle\\ &\leqq \limsup \langle E((-\infty, s-\varepsilon])f_n, Af_n\rangle + \limsup \langle E((s-\varepsilon,\infty))f_n, Af_n\rangle\\ &\leqq (s-\varepsilon)\limsup \left\|E((-\infty, s-\varepsilon])f_n\right\|^2 + 0 = s-\varepsilon,\end{aligned}$$

in contradiction with the assumption $\limsup \langle f_n, Af_n\rangle \geqq s$. □

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Factorisations régulières et sous-espaces hyperinvariants

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1. Soient $\mathfrak{E}$ et $\mathfrak{E}_*$ deux espaces de Hilbert complexes séparables, soit $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ une fonction analytique contractive pure[1]) et soit T l'opérateur qui lui est associé par

$$T^*(u \oplus v) = e^{-it}(u(e^{it}) - u(0)) \oplus e^{-it} v(t)$$

sur l'espace $H = K_+ \ominus G$ où $K_+ = H^2(\mathfrak{E}_*) \oplus \overline{\Delta L^2(\mathfrak{E})}$, $G = \{\Theta u \oplus \Delta u : u \in H^2(\mathfrak{E})\}$ et $\Delta(t) = [I - \Theta^*(e^{it})\Theta(e^{it})]^{1/2}$.

Envisageons une factorisation $\Theta = \Theta_2 \Theta_1$ de $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ où les facteurs $\{\mathfrak{E}, \mathfrak{F}, \Theta_1(\lambda)\}$ et $\{\mathfrak{F}, \mathfrak{E}_*, \Theta_2(\lambda)\}$ sont des fonctions analytiques contractives et soit Z le prolongement à $\overline{\Delta L^2(\mathfrak{E})}$ de l'opérateur isométrique $Z_0 \colon \Delta L^2(\mathfrak{E}) \to \overline{\Delta_2 L^2(\mathfrak{F})} \oplus \oplus \overline{\Delta_1 L^2(\mathfrak{E})}$ défini par $Z_0(\Delta v) = \Delta_2 \Theta_1 v \oplus \Delta_1 v$, $v \in L^2(\mathfrak{E})$.

Rappelons que la factorisation $\Theta = \Theta_2 \Theta_1$ est dite régulière (cf. [H] ch. VII) si Z est un opérateur unitaire. On connait (cf. [H] ch. VII) les suivants résultats:

(a) A chaque sous-espace $H_1 \subset H$ invariant pour l'opérateur T il correspond une factorisation régulière $\Theta = \Theta_2 \Theta_1$ telle que le sous-espace H_1 et son complément orthogonal $H_2 = H \ominus H_1$ ont les répresentations suivantes:

$$H_1 = \{\Theta_2 u \oplus Z^{-1}(\Delta_2 u \oplus v) : u \in H^2(\mathfrak{F}),\ v \in \overline{\Delta_1 L^2(\mathfrak{E})},\ \Theta_1^* u + \Delta_1 v \perp H^2(\mathfrak{E})\} \tag{1}$$

$$H_2 = \{u \oplus Z^{-1}(v \oplus 0) : u \in H^2(\mathfrak{E}_*),\ v \in \overline{\Delta_2 L^2(\mathfrak{F})},\ \Theta_2^* u + \Delta_2 v \perp H^2(\mathfrak{F})\}; \tag{1'}$$

(b) Pour toute factorisation régulière $\Theta = \Theta_2 \Theta_1$ le sous-espace H_1 donné par la formule (1) est un sous-espace invariant pour T.

Le but de cette Note est de caractériser les factorisations régulières telles que le sous-espace correspondant H_1 est même hyperinvariant pour T, c'est-à-dire invariant pour tout opérateur S qui permute à T. Plus précisément nous allons démontrer le suivant

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[1]) Pour toutes les notions qui ne sont pas explicitement définies ainsi que pour la notation utilisée cf. [H].

Théorème. *Soit* $\Theta=\Theta_2\Theta_1$ *une factorisation régulière de la fonction analytique contractive pure* $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ *et soit* H_1 *le sous-espace invariant correspondant à cette factorisation. Supposons de plus que* $\Theta_1(e^{it})$ *et* $\Theta_1^*(e^{it})$ *sont injectifs pour presque tout* $t\in[0, 2\pi]$. *Pour que le sous-espace* H_1 *soit hyperinvariant pour* T *il faut et il suffit que les conditions suivantes soient vérifiées:*

(i) $\Delta_1(t)=O$ *où* $\Delta_2(t)=O$ *pour presque tout* $t\in[0, 2\pi]$,

(ii) *pour chaque couple* (A, A_0) *de fonctions analytiques bornées* $\{\mathfrak{E}_*, \mathfrak{E}_*, A(\lambda)\}$, $\{\mathfrak{E}, \mathfrak{E}, A_0(\lambda)\}$ *telles que* $A\Theta=\Theta A_0$ *il existe une fonction analytique bornée* $\{\mathfrak{F}, \mathfrak{F}, \Phi(\lambda)\}$ *vérifiant*

$$A(\lambda)\Theta_2(\lambda)=\Theta_2(\lambda)\Phi(\lambda) \quad et \quad \Phi(\lambda)\Theta_1(\lambda)=\Theta_1(\lambda)A_0(\lambda).$$

2. Avant de démontrer la nécessité des conditions (i) et (ii) nous rappelons (cf. [1]) que si $S\in\mathscr{B}(H)$ est un élément du commutant $\{T\}'=\{S\in\mathscr{B}(H): ST=TS)$ alors il existe des fonctions analytiques bornées $\{\mathfrak{E}_*, \mathfrak{E}_*, A(\lambda)\}$, $\{\mathfrak{E}, \mathfrak{E}, A_0(\lambda)\}$, des fonctions mesurables bornées $B(\cdot): \mathfrak{E}_*\to\overline{\Delta\mathfrak{E}}$, $C(\cdot): \overline{\Delta\mathfrak{E}}\to\overline{\Delta\mathfrak{E}}$, liées par les équations

$$A\Theta=\Theta A_0 \quad \text{et} \quad B\Theta+C\Delta=\Delta A_0, \tag{2}$$

et telles qu'on ait $S=P_+Y|H$ où P_+ est la projection orthogonale $K_+\to H$, et où

$$Y=\begin{bmatrix} A & O \\ B & C \end{bmatrix}.$$

Nous allons faire usage dans la suite de la remarque de [1], Lemme 2.1′, selon laquelle les fonctions $B(\cdot)$ et $C(\cdot)$ peuvent être exprimées sous la forme

$$B=D\Delta_*+\Delta A_0\Theta^*, \quad C=[-D\Theta+\Delta A_0\Delta]|\overline{\Delta L^2(\mathfrak{E})} \tag{3}$$

où $D:\overline{\Delta_* L^2(\mathfrak{E}_*)}\to\overline{\Delta L^2(\mathfrak{E})}$ est une fonction opératorielle mesurable bornée. Notons aussi que la factorisation régulière $\Theta(\lambda)=\Theta_2(\lambda)\Theta_1(\lambda)$ étant fixée nous pouvons considérer le modèle fonctionnel unitairement équivalent donné par

$$\mathbf{T}^*(u\oplus v_2\oplus v_1)=e^{-it}(u(e^{it})-u(0))\oplus e^{-it}v_2(t)\oplus e^{-it}v_1(t)$$

sur l'espace $\mathbf{H}=\mathbf{K}_+\ominus\mathbf{G}$ où $\mathbf{K}_+=H^2(\mathfrak{E}_*)\oplus\overline{\Delta_2L^2(\mathfrak{F})}\oplus\overline{\Delta_1L^2(\mathfrak{E})}$ et $\mathbf{G}=\{\Theta_2\Theta_1u\oplus\Delta_2\Theta_1u\oplus\Delta_1u: u\in H^2(\mathfrak{E})\}$; le sous-espace $\mathbf{H}_1$ correspondant à H_1 est alors donné par

$$\mathbf{H}_1=\{\Theta_2u\oplus\Delta_2u\oplus v: u\in H^2(\mathfrak{F}), v\in\overline{\Delta_1L^2(\mathfrak{E})}, \Theta_1^*u+\Delta_1v\perp H^2(\mathfrak{E})\}.$$

Dans ce cas, si l'on note

$$ZB=\begin{bmatrix} B_2 \\ B_1 \end{bmatrix} \quad \text{et} \quad ZCZ^{-1}=\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

l'opérateur $\mathbf{S}\in\{\mathbf{T}\}'$ correspondant à S est donné par $\mathbf{S}=\mathbf{P}_+\mathbf{Y}|\mathbf{H}$ où $\mathbf{P}_+$ est la projection orthogonale $\mathbf{K}_+\to\mathbf{H}$ et $\mathbf{Y}$ a la forme

$$\mathbf{Y} = \begin{bmatrix} A & O & O \\ B_2 & C_{11} & C_{12} \\ B_1 & C_{21} & C_{22} \end{bmatrix} \tag{4}$$

où

$$\begin{aligned} A\Theta &= \Theta A_0, \\ B_2\Theta + C_{11}\Delta_2\Theta_1 + C_{12}\Delta_1 &= \Delta_2\Theta_1 A_0, \\ B_1\Theta + C_{21}\Delta_2\Theta_1 + C_{22}\Delta_1 &= \Delta_1 A_0, \end{aligned} \tag{2'}$$

En revenant aux propriétés (i) et (ii) nous allons d'abord démontrer le

Lemme 1. *Si le sous-espaces* $\mathbf{H}_1$ *est invariant pour* $\mathbf{S}\in\{\mathbf{T}\}'$ *alors* $C_{12}=O$.

Démonstration. Notons que $\mathbf{H}_1$ étant invariant pour $\mathbf{S}=\mathbf{P}_+\mathbf{Y}$, $\mathbf{L}=\{\Theta_2 u\oplus\Delta_2 u\oplus v : u\in H^2(\mathfrak{F}),\ v\in\overline{\Delta_1 L^2(\mathfrak{E})}\}$ est invariant pour $\mathbf{Y}$. Donc si $0\oplus 0\oplus v\in\mathbf{L}$, alors $\mathbf{Y}(0\oplus 0\oplus v)\in\mathbf{L}$. Or, comme on a

$$\mathbf{Y}(0\oplus 0\oplus v) = 0\oplus C_{12}v\oplus v', \quad v' = C_{22}v\in\overline{\Delta_1 L^2(\mathfrak{E})}$$

il existe $w\in H^2(\mathfrak{F})$ tel que

$$\Theta_2 w = 0 \quad \text{et} \quad \Delta_2 w = C_{12}v.$$

De la première relation il dérive $\Delta_2^2 w=w$, donc $\Delta_2 w=w$, d'où $C_{12}v=w\in H^2(\mathfrak{F})$. Donc C_{12} applique $\overline{\Delta_1 L^2(\mathfrak{E})}$ dans $H^2(\mathfrak{F})$ et permute à la multiplication par e^{it}. Il en résulte $C_{12}=O$. C.q.f.d.

Dans la suite nous calculons la matrice de l'opérateur $\mathbf{S}_0=\mathbf{P}_+\mathbf{Y}_0\in\{\mathbf{T}\}'$ correspondant à un couple (A, A_0) satisfaisant à la première relation (2) et à $D=O$.

En tenant compte de (3) il en résulte que

$$B_0 = \Delta A_0\Theta^* \quad \text{et} \quad C_0 = \Delta A_0\Delta|\overline{\Delta L^2(\mathfrak{E})}$$

d'où $(ZB_0)(u)=\Delta_2\Theta_1 A_0\Theta^* u\oplus\Delta_1 A_0\Theta^* u$ pour $u\in L^2(\mathfrak{E}_*)$. Soit maintenant $v_2\oplus v_1\in\overline{\Delta_2 L^2(\mathfrak{F})}\oplus\overline{\Delta_1 L^2(\mathfrak{E})}$; on a

$$v_2\oplus v_1 = \lim_{n\to\infty}(\Delta_2\Theta_1 u_n\oplus\Delta_1 u_n) \quad \text{où} \quad u_n\in L^2(\mathfrak{E}),$$

$$\begin{aligned} ZC_0Z^{-1}(v_2\oplus v_1) &= ZC_0Z^{-1}(\lim \Delta_2\Theta_1 u_n\oplus\Delta_1 u_n) = \lim(Z\Delta A_0\Delta^2 u_n) = \\ &= \lim(\Delta_2\Theta_1 A_0\Delta^2 u_n\oplus\Delta_1 A_0\Delta^2 u_n) = \\ &= \lim[\Delta_2\Theta_1 A_0(\Theta_1^*\Delta_2^2+\Delta_1^2)u_n\oplus\Delta_1 A_0(\Theta_1^*\Delta_2^2\Theta_1+\Delta_1^2)u_n] = \\ &= [(\Delta_2\Theta_1 A_0\Theta_1^*\Delta_2 v_2+\Delta_2\Theta_1 A_0\Delta_1 v_1)\oplus(\Delta_1 A_0\Theta_1^*\Delta_2 v_2\oplus\Delta_1 A_0\Delta_1 v_3)]. \end{aligned}$$

Donc la matrice de $\mathbf{Y}_0$ est donnée par

$$\text{(5)} \qquad \mathbf{Y}_0 = \begin{bmatrix} A & O & O \\ \Delta_2\Theta_1 A_0 \Theta^* & \Delta_2\Theta_1 A_0 \Theta_1^* \Delta_2 & \Delta_2\Theta_1 A_0 \Delta_1 \\ \Delta_1 A_0 \Theta^* & \Delta_2 A_0 \Theta_1^* \Delta_2 & \Delta_1 A_0 \Delta_1 \end{bmatrix}.$$

Conséquence 1. *Si le sous-espace* $\mathbf{H}_1$ *est invariant pour l'opérateur* $\mathbf{S}_0 = \mathbf{P}_+\mathbf{Y}_0$ *où* $\mathbf{Y}_0$ *est donnée par* (5), *alors* $\Delta_2\Theta_1 A_0 \Delta_1 = O$.

En effet si $\mathbf{H}_1$ est invariant pour $\mathbf{S}_0 = \mathbf{P}_+\mathbf{Y}_0$ où $\mathbf{Y}_0$ est donnée par (5); en appliquant le Lemme 1, nous obtenons $\Delta_2\Theta_1 A_0 \Delta_1 = O$.

Corollaire 1. *Si le sous-espace* $\mathbf{H}_1$ *est hyperinvariant pour* $\mathbf{T}$, *alors* $\Delta_2\Theta_1\Delta_1 = O$.

En effet si le sous-espace $\mathbf{H}_1$ est hyperinvariant pour $\mathbf{T}$ alors il est invariant pour l'opérateur qui est déterminé par le couple $(I_{\mathfrak{E}_*}, I_{\mathfrak{E}})$ et $D = O$, donc qui s'obtient de (5) en choisissant $A = I_{\mathfrak{E}_*}$ et $A_0 = I_{\mathfrak{E}}$; donc d'après la conséquence précédente on a $\Delta_2\Theta_1\Delta_1 = O$.

Proposition 1. *Si le sous-espace* $\mathbf{H}_1$ *est hyperinvariant pour* $\mathbf{T}$ *et si* $\Theta_1(e^{it})$ *est injectif pour presque tout* $t \in [0, 2\pi]$, *alors*

$$\Delta_1(t) = [I - \Theta_1^*(e^{it})\Theta_1(e^{it})]^{1/2} = O \quad ou \quad \Delta_2(t) = [I - \Theta_2^*(e^{it})\Theta_2(e^{it})]^{1/2} = O$$

pour presque tout $t \in [0, 2\pi]$.

Démonstration. Soit $X: \overline{\Delta_1 L^2(\mathfrak{E})} \to \overline{\Delta_2 L^2(\mathfrak{F})}$ un opérateur mesurable borné qui permute à la multiplication par e^{it}. Envisageons l'opérateur $\mathbf{S}_1 = \mathbf{P}_+\mathbf{Y}_1 \in \{\mathbf{T}\}'$ où $\mathbf{Y}_1$ est déterminé par le couple $(A(\lambda) \equiv O,\ A_0(\lambda) \equiv O)$ et où

$$D = -Z^{-1}\begin{bmatrix} O & X \\ O & O \end{bmatrix} Z\Theta^* | \overline{\Delta_* L^2(\mathfrak{E}_*)}.$$

Pour calculer la matrice de $\mathbf{Y}_1$ notons d'abord qu'on a

$$Z\Delta^2 Z^{-1}(v_2 \oplus v_1) = [(\Delta_2\Theta_1\Theta_1^*\Delta_2 v_2 + \Delta_2\Theta_1^{\Delta_1} v_1) \oplus (\Delta_1\Theta_1^*\Delta_2 v_2 + \Delta_1^2 v_1)].$$

En effet pour $v \in \overline{\Delta L^2(\mathfrak{E})}$ soit $v = \lim_{n\to\infty} \Delta u_n$ où $u_n \in L^2(\mathfrak{E})$; alors,

$$Zv = Z(\lim \Delta u_n) = \lim (\Delta_2\Theta_1 u_n \oplus \Delta_1 u_n) = v_2 \oplus v_1,$$

$$Z\Delta^2 Z^{-1}(v_2 \oplus v_1) = Z\Delta^2 v = \Delta_2\Theta_1\Delta v \oplus \Delta_1\Delta v =$$

$$= \lim (\Delta_2\Theta_1\Delta^2 u_n \oplus \Delta_1\Delta^2 u_n) = \lim [\Delta_2\Theta_1(\Theta_1^*\Delta_2^2\Theta_1 + \Delta_1^2)u_n \oplus \Delta_1(\Theta_1^*\Delta_2^2\Theta_1 + \Delta_1^2)u_n] =$$

$$= [(\Delta_2\Theta_1\Theta_1^*\Delta_2 v_2 + \Delta_2\Theta_1\Delta_1 v_1) \oplus (\Delta_1\Theta_1^*\Delta_2 v_2 + \Delta_1^2 v_1)];$$

d'où en tenant compte du corollaire 1 on déduit

$$Z\Delta^2Z^{-1} = \begin{bmatrix} \Delta_2\Theta_1\Theta_1^*\Delta_2 & O \\ O & \Delta_1^2 \end{bmatrix}.$$

Nous montrons que la matrice de $ZD\Theta Z^{-1}$ est donnée par

$$ZD\Theta Z^{-1} = \begin{bmatrix} O & -X\Theta_1^*\Theta_1 \\ O & O \end{bmatrix} \Big| \overline{\Delta_2 L^2(F)} \oplus \overline{\Delta_1 L^2(E)}.$$

En effet,

$$ZD\Theta Z^{-1} = -ZZ^{-1}\begin{bmatrix} O & X \\ O & O \end{bmatrix} Z\Theta^*\Theta Z^{-1} =$$

$$= -\begin{bmatrix} O & X \\ O & O \end{bmatrix} + \begin{bmatrix} O & X \\ O & O \end{bmatrix} Z\Delta^2 Z^{-1} = \begin{bmatrix} O & -X\Theta_1^*\Theta_1 \\ O & O \end{bmatrix} \Big| \overline{\Delta_2 L^2(\mathfrak{F})} \oplus \overline{\Delta_1 L^2(\mathfrak{E})}.$$

Donc la matrice de $\mathbf{Y}_1$ a la forme

$$\mathbf{Y}_1 = \begin{bmatrix} O & O & O \\ B_2 & O & X\Theta_1^*\Theta_1 \\ B_1 & O & O \end{bmatrix}.$$

En appliquant le lemme 1 il en dérive que $X\Theta_1^*\Theta_1 | \overline{\Delta_1 L^2(\mathfrak{E})} = O$ mais $\overline{\Delta_1 L^2(\mathfrak{E})} = \overline{\Theta_1^*\Theta_1\Delta_1 L^2(\mathfrak{E})} \oplus \ker \Theta_1^*\Theta_1$, et d'après l'hypothèse on a $\ker \Theta_1^*\Theta_1 = \ker \Theta_1 = \{0\}$ donc $X = O$. Donc le seul opérateur $X\colon \overline{\Delta_1 L^2(\mathfrak{E})} \to \overline{\Delta_2 L^2(\mathfrak{F})}$ mesurable borné qui permute à e^{it} est l'opérateur nul, donc $\Delta_1(t) = O$ ou $\Delta_2(t) = O$ pour presque tout $t \in [0, 2\pi]$.

Proposition 2. *Si le sous-espace $\mathbf{H}_1$ est hyperinvariant pour $\mathbf{T}$ alors pour tout couple (A, A_0) de fonctions vérifiant $A\Theta = \Theta A_0$ il existe une fonction analytique bornée $\{\mathfrak{F}, \mathfrak{F}, \Phi(\lambda)\}$ telle que*

(6) $$A\Theta_2 = \Theta_2\Phi \quad et \quad \Phi\Theta_1 = \Theta_1 A_0.$$

Démonstration. Soit $\mathbf{S}_0 = \mathbf{P}_+\mathbf{Y}_0 \in \{\mathbf{T}\}'$ où $\mathbf{Y}_0$ est donné par (5). De la condition que $\mathbf{L}$ est invariant pour $\mathbf{Y}_0$ on déduit que pour tout $u \in H^2(\mathfrak{F})$ on a

(7) $$Y_0(\Theta_2 u \oplus \Delta_2 u \oplus 0) = \Theta_2 w \oplus \Delta_2 w \oplus v'$$

où $w \in H^2(\mathfrak{F})$ et $v' \in \overline{\Delta_1 L^2(\mathfrak{E})}$. Remarquons d'abord que l'application $\Phi\colon u \to w$ est univoque. En effet si l'on a $\Theta_2 w \oplus \Delta_2 w \oplus v' = \Theta_2 w_1 \oplus \Delta_2 w_1 \oplus v_1'$ alors $0 = \|\Theta_2(w-w_1)\|^2 + \|\Delta_2(w-w_1)\|^2 + \|v'-v_1'\|^2 = \|w-w_1\|^2 + \|v'-v_1'\|^2$ d'où $w = w_1$ et $v' = v_1'$. On obtient ainsi une application $\Phi\colon H^2(\mathfrak{F}) \to H^2(\mathfrak{F})$ qui permute à la multiplication par e^{it} et dont on peut vérifier facilement qu'elle est fermée donc continue. Donc Φ est l'opérateur de multiplication par une fonction analytique bornée $\{\mathfrak{F}, \mathfrak{F}, \Phi(\lambda)\}$. Les deux premières composantes de l'égalité (7) nous donnent

$$A\Theta_2 = \Theta_2\Phi \quad \text{et} \quad \Delta_2\Theta_1 A_0\Theta^*\Theta_2 + \Delta_2\Theta_1 A_0\Theta_1^*\Delta_2^2 = \Delta_2\Phi.$$

Après des transformations évidentes la deuxième relation obtenue devient

(8) $$\Delta_2 \Theta_1 A_0 \Theta_1^* = \Delta_2 \Phi.$$

Notons que de la première relation (2) et de la relation $A\Theta_2 = \Theta_2 \Phi$ démontrée auparavant on obtient

(9) $$\Theta_2(\Phi\Theta_1 - \Theta_1 A_0) = O.$$

De la relation (8) on déduit, tenant compte de la conséquence 1,

$$\Delta_2 \Phi \Theta_1 = \Delta_2 \Theta_1 A_0 \Theta_1^* \Theta_1 = -\Delta_2 \Theta_1 A_0 \Delta_1^2 + \Delta_2 \Theta_1 A_0 = \Delta_2 \Theta_1 A_0$$

donc

(9′) $$\Delta_2(\Phi\Theta_1 - \Theta_1 A_0) = O$$

Les relations (9) et (9′) impliquent $\Phi\Theta_1 = \Theta_1 A_0$. C. q. f. d.

3. Nous allons démontrer que les conditions (i) et (ii) sont aussi suffisantes pour que le sous-espace $\mathbf{H}_1$ soit hyperinvariant pour $\mathbf{T}$ sous l'hypothèse que $\Theta_1^*(e^{it})$ est injectif pour presque tout $t \in [0, 2\pi]$. Soit pour cela $\mathbf{S} = \mathbf{P}_+ \mathbf{Y} \in \{\mathbf{T}\}'$; d'après (i) nous obtenons pour $\mathbf{Y}$ une matrice de la forme

$$\mathbf{Y} = \begin{bmatrix} A & O & O \\ B_2 & C_{11} & O \\ B_1 & O & C_{22} \end{bmatrix}.$$

Les relations (2′) deviennent

(2″) $$A\Theta = \Theta A_0, \qquad B_2\Theta + C_{11}\Delta_2\Theta_1 = \Delta_2\Theta_1 A_0, \quad B_1\Theta + C_{22}\Delta_1 = \Delta_1 A_0.$$

Mais d'après (ii) il existe une fonction analytique bornée $\{\mathfrak{F}, \mathfrak{F}, \Phi(\lambda)\}$ telle que $A\Theta_2 = \Theta_2 \Phi$ et $\Phi\Theta_1 = \Theta_1 A_0$. Eu égard aussi à la deuxième des relations (2″) on obteint $(B_2\Theta_2 + C_{11}\Delta_2)\Theta_1 = \Delta_2 \Phi \Theta_1$; comme $\Theta_1^*(e^{it})$ est injectif il en résulte

$$B_2\Theta_2 + C_{11}\Delta_2 = \Delta_2\Phi.$$

Les relations $A\Theta_2 = \Theta_2\Phi$ et $B_2\Theta_2 + C_{11}\Delta_2 = \Delta_2\Phi$ démontrées auparavant montrent que le sous-espace $\mathbf{L}$ est invariant pour $\mathbf{Y}$ donc $\mathbf{H}_1$ est invariant pour $\mathbf{S}$. Le théorème annoncé au début de cette Note est ainsi complètement démontré.

Remarque. La démonstration de la nécessité du théorème utilise l'hypothèse que $\Theta_1(e^{it})$ est injectif pour presque tout $t \in [0, 2\pi]$, tandis que la condition que $\Theta_2^*(e^{it})$ est injectif pour presque tout $t \in [0, 2\pi]$ est utilisée dans la démonstration de la suffisance. De plus la démonstration de la nécessité de la condition (ii) reste valable sans l'hypothèse supplémentaire que $\Theta_1(e^{it})$ soit injectif.

4. Dans ce dernier alinéa nous envisageons le cas particulier où la factorisation $\Theta = \Theta_2\Theta_1$ est celle canonique, en produit d'un facteur extérieur Θ_1 et d'un facteur

intérieur Θ_2, et en outre nous donnons une application concernant les factorisations des fonctions analytiques contractives $*$-extérieures. On a le suivant

Corollaire 2. *Soit* $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ *une fonction analytique contractive pure et soit* $\Theta = \Theta_i \Theta_e$ *sa factorisation canonique. Pour tout couple de fonctions analytiques bornées* (A, A_0) *telles que* $A\Theta = \Theta A_0$, *il existe une fonction analytique bornée* $(\mathfrak{F}, \mathfrak{F}, \Phi(\lambda))$ *qui jouit des propriétés*

$$A\Theta_i = \Theta_i \Phi \quad et \quad \Phi\Theta_e = \Theta_e A_0.$$

En effet nous n'avons qu'à remarquer que le sous-espace invariant correspondant à la factorisation canonique est hyperinvariant et que la démonstration de la proposition 2 n'utilise pas l'hypothèse que $\Theta_e(e^{it})$ est injectif.

En tenant compte de la remarque précédente on peut énoncer une propriété analogue pour la factorisation $*$-canonique.

Pour les fonctions analytiques contractives $*$-extérieures $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ on a associé, dans la monographie [H], à chaque ensemble borelien $\alpha \subset \mathbf{C} = \{\lambda : |\lambda| = 1\}$ une factorisation régulière $\Theta = \Theta_{2\alpha}\Theta_{1\alpha}$ telle que:

(1) $\Theta_{1\alpha}^*(e^{it})\Theta_{1\alpha}(e^{it}) = I$ pour presque tout $t \in \alpha$;

(2) $\Theta_{2\alpha}^*(e^{it})\Theta_{2\alpha}(e^{it}) = I$ pour presque tout $t \in \alpha' = \mathbf{C} \setminus \alpha$;

(3) le sous-espace H_1 correspondant à cette factorisation est hyperinvariant pour T et de plus $T|H_1 \in C_{11}$.

Dans la suite nous allons démontrer que de cette manière on obtient tous les sous-espaces H' hyperinvariants pour T tels que $T|H' \in C_{11}$. Plus exactement on démontrera:

Proposition 3. *Soit* $\Theta = \Theta_2\Theta_1$ *une factorisation régulière de la fonction analytique contractive pure* $*$*-extérieure* $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ *et soit* H_1 *le sous-espace invariant correspondant à cette factorisation. Pour que* H_1 *soit hyperinvariant pour* T *et tel que* $T|H_1 \in C_{11}$, *il faut et il suffit qu'il existe un ensemble borelien* $\alpha \subset \mathbf{C} = \{\lambda : |\lambda| = 1\}$ *tel que*

$\Theta_1^*(e^{it}) \cdot \Theta_1(e^{it}) = I$ *pour presque tout* $t \in \alpha$,

$\Theta_2^*(e^{it}) \cdot \Theta_2(e^{it}) = I$ *pour presque tout* $t \in \alpha' = \mathbf{C} \setminus \alpha$.

Démonstration. Vu que la suffisance est démontrée dans la monographie [H] ch. VII, th. 5.2, il nous reste seulement à vérifier la nécessité. Pour cela nous remarquons qu'on désignant $\alpha = \{t \in \mathbf{C} : \Delta_1(t) = O\}$ on a d'après (i) $\Delta_2(t) = O$ pour presque tout $t \in \mathbf{C} \setminus \alpha$. Donc pour presque tout $t \in \alpha$ on a $\Theta_1^*(e^{it})\Theta_1(e^{it}) = I$ et de même, pour presque tout $t \in \alpha' = \mathbf{C} \setminus \alpha$ on a $\Theta_2^*(e^{it})\Theta_2(e^{it}) = I$. C. q. f. d.

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Bibliographie

M. S. Bartlett, An Introduction to Stochastic Processes, with Special Reference to Methods and Applications, third edition, XVII + 388 pages, Cambridge University Press, Cambridge—London—New York—Melburne, 1978.

Professor Bartlett is a leading statistician of the last forty years, and his book is one of the first texts on stochastic processes. The first edition has appeared in 1955 [MR 16 (1955) p. 939] and it was four times reprinted. The second revised and enlarged edition appeared in 1966 [MR 35 (1968) # 3785]. The primary aim of the book is to acquaint (mainly) statisticians and other applied mathematicians with the techniques for studying stochastic processes, but it can also be interesting to the pure mathematician who wants to know the kinds of applications. Written in the best tradition of English scholarship, this twenty-three year old text is still fresh and elastic enough to incorporate some recent developments without breaking the unity. Seven new sections are included, and four sections are enlarged, and the opportunity has also been taken to make a number of corrections and small changes to the existing text. The condensation in places is natural since the book covers a great amount of matarial in a moderate number of pages.

Sándor Csörgő (Szeged)

J. Brey and R. B. Jones, Ed., **Critical Phenomena,** Sitges International School on Statistical Mechanics, June 1976, Sitges, Barcelona/Spain, Springer Lecture Notes in Physics, Vol. 54, 383 pp. 1976.

Statistical physics, both rigorous and non-rigorous, has developed very vividly in the last decade. The most suitable form for expounding and understanding new ideas seems to be something like lecture notes (in rigorous statistical physics, the exception was Ruelle's excellent 1969 book): collections of surveys, like this one, and those written by one author (or group of authors), like Simon's book on $P(\varphi)_2$ or Sinai's lecture notes on some mathematical problems of statistical physics (to appear in 1979).

This collection contains the lectures held at the International School on Statistical Mechanics, June 1976, at Sitges, Barcelona. The lectures are about both rigorous and non-rigorous results: those of Miracle-Solé, Lebowitz and Gallavotti belonging to the first direction and those of Wegner, Green, Ma, Enz, Szépfalusy, Kadanoff and Brout belonging to the second one. Haag uses this more free genre to speak to physicists about mathematical results and to include his "hopes for the future". The topics of the lectures are: phase transitions in classical equilibrium systems, quantum equilibrium states, renormalization group methods and scaling, critical dynamics and critical fluctuations, spontaneous broken symmetry, gauge theory.

For the reader with mathematical or physical intelligence and intuition, these lectures offer a quick way to understand the state of affairs in several branches of statistical physics and they can be recommended to scholars and graduate students in mathematics or physics.

D. Szász (Budapest)

A. Brown—C.Pearcy, Introduction to Operator Theory. I, Elements of Functional Analysis (Graduate Texts in Mathematics 55), XIV+474 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1977.

This book was written to serve as a textbook for a one- or two-semester introductory graduate course in functional analysis. Its companion volume "Operators on Hilbert Space" will be published soon and is planned to be a textbook for a subsequent course in operator theory. The only critical prerequisite for the volume under review is the ability to follow and construct $\varepsilon-\delta$ arguments. The reader of part II of this volume is supposed to be familiar with the equivalent of a one-semester course in each of the following areas: linear algebra, general topology, complex analysis, and measure theory. As most courses in these subjects fail to treat certain topics that are needed in the study of functional analysis and operator theory, in part I the authors compiled the material that a student must know in order to study functional analysis and operator theory. There are many examples and exercises. The exercises constitute an integral part of the text, many topics are first introduced in problems. Because of the abundance of examples and problems the authors think this textbook will be of use also to those who wish to study functional analysis individually. The following list of the chapter headings of part II may give more insight to the content of the book: Normed linear spaces; Bounded linear transformations; The open mapping theorem; The Hahn—Banach theorem; Local convexity and weak topologies; Duality; Banach spaces and integration theory; The spaces $\mathscr{C}(X)$; Vector sums and bases.

J. Szűcs (Szeged)

Tosio Kato, Perturbation theory for linear operators (Grundlehren der mathematischen Wissenschaften — A Series of Comprehensive Studies in Mathematics, 132), XXI+619 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1976.

Just 10 years after its first edition this excellent monograph already necessitated a new edition. This success is firstly motivated by the fundamental importance and continuous use of perturbation theoretical arguments and techniques in various areas of modern physics and by the intrinsic mathematical interest of the analytical and operator theoretical methods which are applied in, or even were invented for the needs of perturbation problems. Indeed this, nowadays very extended, area of research is a striking result of interplay of problems and methods of a great variety of physical and mathematical disciplines.

Professor T. Kato, one the foremost creative experts in all branches of Perturbation Theory, succeeded in this monograph to make this interplay clear and vividly felt throughout his work. Although the emphasis is on the purely and rigorously mathematical aspects of the theory, he never loses contact with the physical origins of the problems: another reason for the success of the book.

In view of recent developments of the theory, some supplementary notes and a 10 page supplementary bibliography were added in the new edition, and — besides several minor changes — three of the sections were completely rewritten.

Béla Sz.-Nagy (Szeged)

John Laperti, Stochastic Processes, A Survey of the Mathematical Theory (Applied Mathematical Sciences, Volume 23), XV+266 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1977.

This tiny book gives an excellent introduction to the theory of stochastic processes. It is at least as excellent as the author's previous book (Probability, A Survey of the Mathematical Theory, Benjamin, 1966) with the same intention. The author writes: "I did not discuss specific applications of the theory; I *did* strive for a spirit friendly to application by coming to grips as fast as I could with the major problems and techniques and by avoiding too high levels of abstraction and completeness. At the same time, I tried to make the proofs both vigorous and motivated and to show how certain results have *evolved* rather than just presenting them in polished final form." There are ten chapters (General introduction — Second-order random functions — Stationary second-order processes — Interpolation and prediction — Strictly-stationary processes and ergodic theory — Markov transition functions — The application of semigroup theory — Markov processes — Strong Markov processes — Martingale theory), and two appendices (Existence of random processes with given finite-dimensional distributions — Review of conditional probability). A carefully compiled index helps the orientation. There are (on the average) six-seven problems to solve in each chapter. The prerequisites (carefully listed after the preface) for reading the book with profit are an adequate knowledge of mathematical analysis (including measure theory, standard Hilbert and Banach space ideas and techniques, elementary differential equations, potential theory and harmonic functions, familiarity with Laplace transforms and topology), knowledge of basic probability mathematics, and "familiarity with examples and applications from elementary probability, preferably including finite Markov chains". If all these are previously given, then the quality of the book guarantees the fulfillment of the author's hope "that after finishing this book readers will be prepared either to go on to the frontiers of mathematical research through more specialised literature, or to turn toward applied problems with an ability to relate them to the general theory and to use its tools and ideas as far as may be possible". Besides the sympathetic modest style — which is so rare nowadays — of a formemost researcher, special attention must be paid to the words (in the Preface) of the responsible mathematician urging to organize the scientific community to struggle against militarism and oppression.

Sándor Csörgő (Szeged)

R. S. Liptser—A. N. Shiryayev, Statistics of Random Processes. I. General Theory, II. Applications (Applications of Mathematics, Volumes 5 and 6), X+394 and X+339 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1977/1978.

This two-volume book, being an extensively revised and expanded translation of the Russian original (Statistika sluchaĭnyh protsessov, Nauka, Moskva, 1974), is an outstanding contribution to mathematical stochastics. The authors, both among the foremost researchers in the field, take the problems of nonlinear filtering as their central theme in this study. But the prerequisites for doing this are such that they had to write seven chapters (296 pages) in the first volume on the general theory of martingales, stochastic differential equations, the absolute continuity of probability measures, Itô and various diffusion processes. This part of the book is probably the best condensed presentation of today's knowledge on these topics. Chapters 8—10 present the main filtration theorems, and these are used in volume II, which is mainly devoted to various aspects of applications. Chapters 18 and 19 were specifically written for the English edition. The material covered is so wide and deep that we must restrict ourselves to the below listing of the contents. Many new important results and many new proofs of known ones are first published here. The book is designed primarily for research workers, but the clear and detailed presentation makes it accessible to graduate students also. No doubt, this work will be a leading reference book in the field. In the Bibliography, Russian versions of non-Russian authors' names and article titles were "translated" back into English. This results in a great number of inaccuracies.

Volume I: 1. Essentials of probability theory and mathematical statistics; 2. Martingales and semimartingales: discrete time; 3. Martingales and semimartingales: continuous time; 4. The Wiener process, the stochastic integral over the Wiener process, and stochastic differential equations; 5. Square integrable martingales, and structure of the functionals on a Wiener process; 6. Nonnegative supermartingales and martingales, and the Girsanov theorem; 7. Absolute continuity of measures corresponding to the Ito processes and processes of the diffusion type; 8. General equations of optimal nonlinear filtering, interpolation and extrapolation of partially observable random processes; 9. Optimal filtering, interpolation and extrapolation of Markov processes with a countable number of states; 10. Optimal linear nonstationary filtering.

Volume II: *11. Conditionally Gaussian processes*; *12. Optimal nonlinear filtering: interpolation* and extrapolation of components of conditionally Gaussian processes; 13. Conditionally Gaussian sequences: filtering and related problems; 14. Application of filtering equations to problems of statistics of random sequences; 15. Linear estimation of random processes; 16. Application of optimal nonlinear filtering equations to some problems in control theory and information theory; 17. Parameter estimation and testing of statistical hypotheses for diffusion type processes; 18. Random point processes: Stieltjes stochastic integrals; 19. The structure of local martingales, absolute continuity of measures for point processes, and filtering.

Sándor Csörgő (Szeged)

Cristopher J. Preston, Gibbs states on countable sets (Cambridge Tracts in Mathematics 68) IX+128 pages, Cambridge University Press, Cambridge 1974.

This book considers a relatively new field in mathematics, the theory of phase transitions in mathematical physics. The aim of this purely mathematical theory is to explain some physical phenomena like the possibility of magnetization at a low temperature.

The author beings the book with the definition of the Gibbs states and Markov random fields. First he considers them on a finite and then on an infinite lattice. He proves the equivalence of these notions. It is shown that under mild conditions for any potential and parameter β (β means the inverse temperature) there exists a Gibbs state also in an infinite lattice. But the main problem of the theory is the unicity of the Gibbs states. In order to tackle with this problem the author presents some basic identities and inequalities (Holley inequality, Kirkwood-Salzburg equations, Griffith inequality, Lee-Yang circle theorem etc.). With their help it is shown that at small β there is a unique Gibbs distribution. Physically this means that at a high temperature there is no phase transition. In the last chapter one of the most important models in this theory the so-called Ising model is investigated in detail. It is shown that if the lattice is two or more dimensional, then at large β there are several Gibbs distributions with a fixed potential. This means that at a low temperature a phase transition may occur.

The book is clear and well presented. It is a good introduction to a theory which could be studied previously only from the original articles.

Péter Major (Budapest)

M. Loève, Probability Theory, I—II, 4th edition (Graduate Texts in Mathematics, Volumes 45 and 46), XVII+425/XVI+413 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1977/1978.

There is no need to advertise this book or to review it thoroughly, since every reader of this note has at least heard of it. After the 1955 [MR **16** p. 598], 1960 [MR **23** # A 670] and 1963 [MR **34** # 3596] editions a fourth one became necessary. The second and third editions improved the quality and the size was expanded moderately. Now twenty per cent of the text of the fourth edition is new and "the additions increased the book to an unwieldy size and it had to be split into two volumes". The main additions are the following: Section 12 (Convergence of probabilities on metric spaces), Section 25 (Regular variation and domains of attraction), Section 26 (Random walk) — the latter two constituting a new Chapter VII (Independent identically distributed summands) —, and the new Chapter XIII (Brownian motion and limit distributions). The "Complements and details" sections are also expanded according to these additions. Of these sections the new one following Chapter XIII deserves special mention which contains a note "An extension of Donsker's theorem" written by LeCam.

The general experience seems to be that the book can hardly be recommended as a textbook. It has been extremely successful as a reference book for research workers in the last twenty three years, and the fourth edition certainly will maintain this role for a long time to come.

Sándor Csörgő (Szeged)

C. R. Rao, Lineare statistische Methoden und ihre Anwendungen, (Mathematische Lehrbücher und Monographien), XIV+519 Seiten, Akademie-Verlag, Berlin, 1973.

Deutsche Übersetzung der originalen englischen Ausgabe (Linear Statistical Inference and Its Applications, Wiley, 1965). Die exakte Formulierung wird durch die mathematischen Methoden und wahrscheinlichkeitstheoretischen Begriffe, die in den ersten drei Kapiteln eingeführt werden, gesichert. Der weitere Teil des Buches beschäftigt sich mit der modernen Theorie und Technik statistischer Schlussweisen. Neben der mathematischen Theorie der Statistik werden auch die Anwendungen auf Probleme der Praxis behandelt. Das ermöglicht auch ein besseres Verständnis der hinter dieser Methode stehenden Theorie. Zusätzlich wird am Ende der einzelnen Kapitel eine grosse Anzahl von Aufgaben angegeben.

Die Titel der Kapitel sind: I. Vektoralgebra und Matrizenkalkül; II. Wahrscheinlichkeitstheorie, Hilfsmittel und Verfahren; III. Stetige Wahrscheinlichkeitsmodelle; IV. Die Theorie der kleinsten Quadate und die Varianzanalyse; V. Kriterien und Methoden der Schätzung; VI. Theorie und Methoden bei grossen Stichproben; VII. Theorie der statistische Schlussweisen; VIII. Mehrdimensionale Theorie.

Das Buch wird, wegen der ausführlichen mathematischen Formulierung und wegen der Vielzahl praktischer Beispiele sowohl den theoretischen als auch den angewandten Mathematikern empfohlen.

János Csirik (Szeged)

I. E. Segal—R. A. Kunze, Integrals and Operators, Second Revised and Enlarged Edition (Grundlehren der mathematischen Wissenschaften 228), XIV+371 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1978.

Since the publication of the first edition of this book several treatments of various advanced topics in analysis have appeared. As these treatises assume much prerequisite knowledge on the part of the reader, in this second edition the authors give an introduction to some of these topics which meshes with the material of the first edition. Consequently, four chapters have been added. They give brief introductions to semigroups and perturbation theory, operator rings and spectral multiplicity, C^*-algebras and their applications, and to the trace as a non-commutative integral. These topics are in connection, for example, with partial differential equations, harmonic analysis, quantum mechanics, group representations, and the analysis on manifolds. The authors have taken the opportunity to correct errors, terminological variations, and expository lapses of the first edition.

J. Szűcs (Szeged)

Larry Smith, Linear Algebra (Undergraduate Texts in Mathematics), VII+280 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1978.

This text is written for students, versed in one-variable calculus and having little contact with complex numbers and abstract algebra. It deals almost exclusively with real finitedimensional vector spaces in a setting and formulation that permits easy generalization to abstract vector spaces. The parallel complex theory is developed in exercises.

The first 7 chapters contain an elementary introduction. The notions of a vector space, subspace, linear independence, bases etc. are illuminated by a large number of examples and exercises.

The central topic of the book is the principal axis theorem for real symmetric linear transformations, in which a more or less direct path is followed. This is done in the subsequent chapters from 8 to 16, in which the notions of a linear transformation, matrices, eigenvalue and eigenvector, inner product space, quadratic form etc. and the more important properties, interrelations and applications (for example, to systems of linear equations) are developed.

The main value of the book is that the presentation is as concrete as possible, and it provides a wide selection of examples of vector spaces and linear transformations that may serve as a testing ground for the theory.

The book is a good introduction to linear algebra. Although there are many areas that are not included (and this is intentional on the part of the author), the theory developed contains the essentials of linear algebra. The book will be useful for both students and lecturers.

F. Móricz (Szeged)

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