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A convolution theorem and a remark on uniformly closed Fourier algebras

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Let G be a locally compact group. Recently J. T. BURNHAM and R. R. GOLDBERG [3] gave a new and elementary proof to the following theorem of DIEUDONNÉ [4]: If G is abelian and not discrete then $f*L_p(G)$ is a proper subset of $L_p(G)$ for all f in $L_1(G)$ and all $p\geqq 1$. Their proof, while simpler than Dieudonné's, relies on the structure of $L_1(G)$ as a commutative Banach algebra and therefore does not seem to extend to nonabelian groups. In the first part of this paper we prove this result for nonabelian groups. Our proof depends only on the structure of $L_p(G)$ as a Banach space and as a Banach $L_1(G)$ module. A corollary of this result is that $L_1(G)$ is not countably generated, algebraically, as a right ideal.

In part two we use a characterization of multipliers on $L_\infty(G)$ to give a new proof to the following result due to M. RAJAGAPOLAN [10] and to L. T. GARDNER [7]: If $L_1(G)$ is equivalent to a C^*-algebra then G is finite.

I. The Convolution Theorem

Let left Haar measure be denoted by μ and let Δ denote the modular function of G. If f is a function on G we denote by $\tilde{f}$ the function defined by $\tilde{f}(x)=f(x^{-1})$ for all x in G. Let $C_{00}(G)$ denote the set of continuous functions on G with compact support and $C_0(G)$ denote the set of continuous functions on G that vanish at infinity.

Lemma 1.1. *Let h belong to $C_{00}(G)$ and g belong to $L_p(G)$, $p\geqq 1$. Then $h*g$ is an everywhere defined continuous function on G.*

Proof. Let $q=p(p-1)^{-1}$ if $p\neq 1$ and let $q=\infty$ if $p=1$. Then for x in G we have

$$|h*g| = \left|\int h(y)g(y^{-1}x)\,d\mu(y)\right| = \left|\int \Delta(y^{-1})h(y^{-1})g(yx)\,d\mu(y)\right| =$$
$$= \left|\int (\Delta h)^{\sim}(y)g(yx)\,d\mu(yx)\right| \leqq \Delta(x^{-1})\|(\Delta h)^{\sim}\|_q\|g\|_p.$$

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This shows that $h*g$ is everywhere defined. A similar computation shows that for x_1 and x_2 in G we have

$$|h*g(x_1)-h*g(x_2)| \leqq |\Delta(x_1^{-1})-\Delta(x_2^{-1})| \, \|(\Delta h)^{\sim}\|_q \|g\|_p =$$

$$= |\Delta(x_1)-\Delta(x_2)| \Delta(x_1x_2)^{-1} \|(\Delta h)^{\sim}\|_q \|g\|_p.$$

The continuity of $h*g$ now follows from the continuity of Δ.

Theorem 1.2. *Let G be a locally compact, non-discrete group. Then $f*L_p(G)$ is a proper subset of $L_p(G)$ for all f in $L_1(G)$ and all $p\geqq 1$.*

Proof. Suppose $f*L_p(G)=L_p(G)$ for some f in $L_1(G)$ and some $p\geqq 1$. Then the map T_f: $L_p(G)\to L_p(G)$ defined by $T_f(g)=f*g$ is continuous and surjective. By the open mapping theorem [8, E. 2 (iii)] there exists a constant $M>0$ such that given any g in $L_p(G)$ with $\|g\|_p\leqq 1$ there exists an h in $L_p(G)$ with $\|h\|_p\leqq M$ such that $T_f(h)=g$. Choose f_0 in $C_{00}(G)$ such that $\|f-f_0\|_1\leqq(2M)^{-1}$. Consider the map T_{f_0}: $L_p(G)\to L_p(G)$ defined as above. Given any g in $L_p(G)$ with $\|g\|_p\leqq 1$, we can choose h in $L_p(G)$ with $\|h\|_p\leqq M$ and $T_f(h)=g$. Then

$$\|T_{f_0}(h)-g\|_p = \|T_{f_0}(h)-T_f(h)\|_p = \|f_0*h-f*h\|_p \leqq \|f_0-f\|_1\|h\|_p < 2^{-1}.$$

By the theorem of W. Bade and P. C. Curtis [2, Thm. 1.2] this implies that T_{f_0} maps $L_p(G)$ onto $L_p(G)$. Therefore $f_0*L_p(G)=L_p(G)$. It follows from Lemma 1.1 that every function in $L_p(G)$ is equal to a continuous function locally almost everywhere. In particular if h belongs to $L_p(G)$ and K is a compact subset of G then there exists a constant $M\geqq 0$ such that the set $\{x\in K: |h(x)|\geqq M\}$ has measure zero. Since G is not discrete we can choose a decreasing sequence $\{U_n\}_{n=1}^{\infty}$ of compact neighborhoods of the identity such that $\mu(U_n)<n^{-p-1}$ for $n=1, 2, \ldots$. Let $h(x)=\sum_{n=1}^{\infty} n\xi_{U_n}(x)$ where $\xi_{U_n}(x)$ is the characteristic function of the set U_n. Then h belongs to $L_p(G)$. But the sets $\{x\in U_1: |h(x)|\geqq n\}$ have positive measure for all $n=1, 2, \ldots$. This contradiction proves the theorem.

Corollary 1.3. *Let G be a locally compact non-discrete group and M a countable subset of $L_1(G)$. Then* $\operatorname{span}(M*L_p(G))$ *is a proper subset of $L_p(G)$ for all $p\geqq 1$.*

Proof. By [8, 32.50 (c)] there exists an f in $L_1(G)$ such that M is contained in $f*L_1(G)$. Then we have that $\operatorname{span}(M*L_p(G))\subset f*L_1(G)*L_p(G)\subset f*L_p(G)$. The corollary now follows from Theorem 1.2.

Corollary 1.3 shows, in particular, that $L_1(G)$ is not countably generated, algebraically, as a right ideal.

II. Fourier Algebras

Let $B(G)$ denote the Fourier—Steltjes algebra of G, $A(G)$ the Fourier algebra of G, $C^*(G)$ the C^*-enveloping algebra of G and $W^*(G)$ the W^*-enveloping algebra of G. For the definitions of these objects the reader is referred to [6]. It is shown in [6] that $B(G)$ is the predual of $W^*(G)$. Now $A(G)$ is a closed, translation invariant subspace of $B(G)$ and so, as noted in [12, p. 23], there exists a central projection z in $W^*(G)$ such that $A(G)$ can be identified with those f in $B(G)$ such that $f(za)=f(a)$ for all a in $W^*(G)$. We write ${}_zf$ for the functional $a \to f(za)$ on $W^*(G)$ and $A(G)={}_zB(G)$.

A multiplier on $L_\infty(G)$ is a linear operator on $L_\infty(G)$ that commutes with left translation by elements G.

Lemma 2. 1. *Suppose $L_1(G)$ is equivalent to a C^*-algebra. Then $L_\infty(G)=B(G)$ and there exists a norm continuous multiplier P from $L_\infty(G)$ onto $C_0(G)$ such that $P^2=P$.*

Proof. Let $M(G)$ denote the set of finite, regular, Borel measures on G and ω the natural embedding of $M(G)$ into $W^*(G)$, see [6 and 12]. Since ω is a $*$-isomorphism and $L_1(G)$ is equivalent to a C^*-algebra we have by [11, Cor. 4.8.6] that $\omega|L_1(G)$ is bicontinuous. But $\omega(L_1(G))$ is dense in $C^*(G)$ and so it follows that $\omega(L_1(G))=C^*(G)$. By taking the adjoint of $\omega|L_1(G)$ we get $B(G)$, as the dual space of $C^*(G)$, bicontinuously isomorphic to $L_\infty(G)$, as the dual space of $L_1(G)$. But then by [6, p. 193] the image of $B(G)$ under the adjoint of $\omega|L_1(G)$ is $B(G)$ as a subspace of $L_\infty(G)$. Therefore $L_\infty(G)=B(G)$.

The subspace $A(G)$ is closed in $B(G)$ and dense in $C_0(G)$ and so $A(G)=C_0(G)$. We had $A(G)={}_zB(G)$ where z was a central projection in $W^*(G)$. Let x be in G and f in $B(G)$, then ${}_z({}_{\omega(x)}f)={}_{\omega(x)}({}_zf)$. Therefore the map $f \to {}_zf$ induces a projection P with the desired properties.

Theorem 2. 2. *If $L_1(G)$ is equivalent to a C^*-algebra, then G is finite.*

Proof. Let P be the projection in Lemma 2.1. Then by [9, Thm. 2.9] there exists an additive set function φ on G such that $P(f)=\varphi * f$ for all f in $L_\infty(G)$. By Lemma 2.1, $L_\infty(G)$ consists entirely of continuous functions and so φ is regular, in the sense of [5, III. 5. 11]. Let V be a relatively compact open subset of G. Then ξ_V belongs to $C_0(G)$. So $\varphi(V)=\int \xi_V(y)\,d\varphi(y)=\check{\varphi} * \xi_V(e)=P(\xi_V)(e)=\xi_V(e)$. Therefore by the regularity of φ we have that $\varphi(\{e\})=1$ and $\varphi(\{x\})=0$ if $x \neq e$. By [9, Prop. 2.4 (ii)] the function $x \to \varphi(\{x\})$ belongs to $L_\infty(G)$ and so in our case it is continuous. This implies that $\{e\}$ is open in G and so G is discrete. But then since P is a projection, $c_0(G)$ is complemented in $l_\infty(G)$. This is impossible by [1, Cor. 2.2] unless G is finite.

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Conditions involving universally quantified function variables

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In his paper [1], W. TAYLOR gave a characterization for properties of varieties which are expressible by means of Mal'cev conditions. The goal of this note is to show that many natural and usual properties of varieties may be expressed by means of a sort of conditions, similar to the Mal'cev type ones.

To emphasize the analogy, we shall use the language of *heterogeneous clones* (i.e., of heterogeneous algebras of type τ_0) due to Taylor. A complete introduction to this subject may be found in [1], § 2, the knowledge of which will be supposed in the sequel. Especially, our notations are adopted from there. Note that letters x, y with exponent n stand for variables of type n ($n=1, 2, \ldots$; see p. 360 in [1]). As in [1], no use of nullary operations will be made.

We shall say that two (or more) sentences (in prenex normal form) are uniformly quantified if, neglecting variables as well as repetitions, they have the same quantifier symbol sequences.

The definition of Mal'cev condition may be formulated as follows (see [1], 2.16): Let $\mathcal{L}$ be a class of varieties. Suppose that there exists a countable sequence $\langle f_1, f_2, \ldots \rangle$ such that

(α^*) each f_i is the *existential* quantification of a (finite) conjunction of equations in the language of heterogeneous clones,

(β^*) for each $n, f_n \rightarrow f_{n+1}$ is true,

(γ^*) $\mathcal{V} \in \mathcal{L}$ if and only if for some n, $\mathfrak{A}(\mathcal{V}) \models f_n$, (i.e., f_n is true in the heterogeneous clone of polynomials of the free $\mathcal{V}$-algebra on a countable generating set). Then we say that $\mathcal{L}$ is defined by a Mal'cev condition.

We shall describe some classes of varieties which may be characterized by means of conditions containing *universal* quantifiers too. Several such ones were touched in [1], 6.7; here we give, essentially, a more complete list and a classification for them, emphasizing the request in [1], 6.8., to have characterizations for classes defined by conditions involving universal quantifiers in the language of heterogeneous clones.

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Theorem. *Let $\mathcal{K}_i$ be the class of all varieties $\mathcal{V}$ having the property* (i), *where*

(1) *in the algebras of $\mathcal{V}$ all congruence classes are subalgebras,*

(2) *in the algebras of $\mathcal{V}$ any operation* *) *applied to endomorphisms furnishes an endomorphism,*

(3) *every direct product $\mathbf{B}\times\mathbf{C}\in\mathcal{V}$ can be decomposed into a direct sum of its subalgebras $\mathbf{B}_1$ and $\mathbf{C}_1$ with $\mathbf{B}_1\cong\mathbf{B}$ and $\mathbf{C}_1\cong\mathbf{C}$,*

(4) *$\mathcal{V}$ is a variety of groups with multiple operators in the Higgins' sense,*

(5) *in the algebras of $\mathcal{V}$ among the classes of every congruence there is exactly one subalgebra,*

(6) *in the algebras of $\mathcal{V}$ all subalgebras are congruence classes,*

(7) *in the algebras of $\mathcal{V}$ any operation applied to subalgebras gives subalgebras,*

(8) *in the algebras of $\mathcal{V}$ any operation applied to classes of a congruence furnishes a (full) class of the same congruence,*

(9) *in the algebras of $\mathcal{V}$ every subalgebra is a class of a unique congruence and* (5) *holds,*

(10) *$\mathcal{V}$ satisfies* (5) *[whence it has an essentially nullary operation 0] and every direct product $\mathbf{B}\times\mathbf{C}\in\mathcal{V}$ is a free product in $\mathcal{V}$ of its subalgebras $\mathbf{B}\times\mathbf{0}$ and $\mathbf{0}\times\mathbf{C}$,*

(11) *in the algebras of $\mathcal{V}$ every subalgebra is a class of a unique congruence and* (1) *holds.*

Let $\mathcal{M}$ denote any fixed one of the classes $\mathcal{K}_i$ $(i=1, \ldots, 11)$. Then there exists a countable sequence $\langle f_1, f_2, \ldots\rangle$ such that

(α) *each f_i is a sentence in the language of heterogeneous clones whose matrix is a (finite) conjunction of equations, and all f_i are uniformly quantified,*

(β) *for each n, $f_{n+1}\to f_n$ is true,*

(γ) *$\mathcal{V}\in\mathcal{M}$ if and only if $\mathfrak{A}(\mathcal{V})\models f_n$ for all n.*

Proof. (1) means that in algebras of $\mathcal{V}$ each operation is idempotent, i.e., for any natural n, every n-ary operation is idempotent. This may be reformulated as $\mathfrak{A}(\mathcal{V})\models f_n$, where

$$f_n \equiv (\forall x^n)\,(C_n^n(x^n, e_1^n, \ldots, e_1^n) = e_1^n).$$

Clearly, the sentences f_i satisfy (α). Finally, suppose that $C_{n+1}^{n+1}(x^{n+1}, e_1^{n+1}, \ldots, e_1^{n+1}) = e_1^{n+1}$ holds identically in $\mathfrak{A}(\mathcal{V})$. Then, substituting $x^{n+1}=C_{n+1}^n(x^n, e_1^{n+1}, \ldots, e_n^{n+1})$ and using the identities of heterogeneous clones, we get $C_{n+1}^n(x^n, e_1^{n+1}, \ldots, e_1^{n+1})=e_1^{n+1}$ for any $x^n\in\mathfrak{A}(\mathcal{V})$, whence also f_n holds there, and thus (β) is fulfilled, too.

(2) means that in algebras of $\mathcal{V}$ any two operations commute (see, e.g., [7] and [2], p. 127). Clearly, this is equivalent to the requirement that, for any natural n,

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*) Here and in what follows under operation we mean not only basic operations, but also polynomials of the considered algebra.

any two n-ary operations commute, i.e., $\mathfrak{A}(\mathscr{V})\models f_n$, where

$$f_n \equiv (\forall x^n)(\forall y^n)\big(C^n_{n^2}(x^n, C^n_{n^2}(y^n, e^{n^2}_1, \ldots, e^{n^2}_n), C^n_{n^2}(y^n, e^{n^2}_{n+1}, \ldots, e^{n^2}_{2n}), \ldots$$

$$\ldots, C^n_{n^2}(y^n, e^{n^2}_{n^2-n+1}, \ldots, e^{n^2}_{n^2})) =$$

$$= C^n_{n^2}(y^n, C^n_{n^2}(x^n, e^{n^2}_1, e^{n^2}_{n+1}, \ldots, e^{n^2}_{n^2-n+1}), \ldots, C^n_{n^2}(x^n, e^{n^2}_n, e^{n^2}_{2n}, \ldots, e^{n^2}_{n^2}))\big).$$

The remainder discussion may be performed as in the case of (1).

As follows from a result of J. Łoś ([3]; see also [1], 6.7), property (3) means that for any natural n, $\mathfrak{A}(\mathscr{V})\models f_n$, where

$$f_n \equiv (\exists x^1)(\exists x^2)(\forall y^n)\big(C^1_2(x^1, e^2_1) = C^1_2(x^1, e^2_2) \wedge C^2_1(x^2, e^1_1, C^1_1(x^1, e^1_1)) =$$

$$= C^2_1(x^2, C^1_1(x^1, e^1_1), e^1_1) \wedge C^n_1(y^n, C^1_1(x^1, e^1_1), \ldots, C^1_1(x^1, e^1_1)) = C^1_1(x^1, e^1_1)\big).$$

What concerns (4) the sentences defining groups with multiple operators can be written down in a similar fashion. Furthermore, (5) means the existence of an essentially nullary operation in $\mathscr{V}$, the result of which forms a subalgebra in every algebra of $\mathscr{V}$ (see [7], § 4). Equivalently, $\mathfrak{A}(\mathscr{V})\models f_n$ for any natural n, where

$$f_n \equiv (\exists x^1)(\forall y^n)\big(C^1_2(x^1, e^2_1) = C^1_2(x^1, e^2_2)\big) \wedge C^n_1(y^n, C^1_1(x^1, e^1_1), \ldots, C^1_1(x^1, e^1_1)) =$$

$$= C^1_1(x_1, e^1_1).$$

(6) is called the Hamiltonian property and it is fulfilled if and only if for any n-ary $(n=1, 2, \ldots)$ operation g there exists a ternary operation h_g such that $g(x_1, \ldots, x_n) = h_g(x_0, x_1, g(x_0, x_2, \ldots, x_n))$ holds identically in $\mathscr{V}$ (see [4]). The assumption (7) means that for any n-ary $(n=1, 2, \ldots)$ operations g and h there exist n-ary operations $g_1, \ldots, g_n$ such that $g(h(x_{11}, \ldots, x_{1n}), \ldots, h(x_{n1}, \ldots, x_{nn})) = h(g_1(x_{11}, \ldots, x_{n1}), \ldots, g_n(x_{1n}, \ldots, x_{nn}))$ holds identically in $\mathscr{V}$ (cf. [7], p. 205). Furthermore, (8) is valid if and only if for any n-ary operation g there exist $n+1$-ary operations $g_1, \ldots g_n$ such that $g_i(x_1, \ldots, x_n, g(x_1, \ldots, x_n)) = x_i$ $(i=1, \ldots, n)$ and $g(g_1(x_1, \ldots, x_n, x), \ldots, g_n(x_1, \ldots, x_n, x)) = x$ hold identically in $\mathscr{V}$ (see [5], p. 242). We shall consider only the property (8); (6) and (7) can be treated analogously.

From the above form of condition (8), $\mathscr{V}$ has property (8) exactly then if for any natural n, $\mathfrak{A}(\mathscr{V})\models f_n$, where

$$f_n \equiv (\forall x^n)(\exists y^{n+1}_1)\ldots(\exists y^{n+1}_n)\Big(\bigwedge_{i=1}^{n} (C^{n+1}_n(y^{n+1}_i, e^n_1, \ldots, e^n_n, x^n) = e^n_i) \wedge$$

$$\wedge C^n_{n+1}(x^n, y^{n+1}_1, \ldots, y^{n+1}_n) = e^{n+1}_{n+1}\Big).$$

We have to prove (β). Choose an arbitrary $x^n \in \mathfrak{A}(\mathscr{V})$. As f_{n+1} is valid, for $x^{n+1} = C^n_{n+1}(x^n, e^{n+1}_1, \ldots, e^{n+1}_n)$ there exist $y^{n+2}_1, \ldots, y^{n+2}_{n+1} \in \mathfrak{A}(\mathscr{V})$, satisfying the matrix of

f_{n+1}. Then a routine computation shows that x^n and

$$y_i^{n+1} = \left(C_{n+1}^{n+2}(y_i^{n+2}, e_1^{n+1}, \ldots, e_n^{n+1}, e_n^{n+1}, e_{n+1}^{n+1})\right)$$

satisfy the matrix of f_n, proving (β).

(9) means that $\mathcal{V}$ is equivalent to the variety of all unital modules over a ring with unit element (see [7], § 4). Furthermore, $\mathcal{V}$ is such a variety if and only if it has operations "addition" and "forming of inverse element" satisfying the axioms of Abelian groups, any operation in $\mathcal{V}$ commutes with the addition, and for any n-ary operation g there exist unary operations $g_1, \ldots, g_n$ such that $g(x_1, \ldots, x_n) = g_1(x_1) + \ldots + g_n(x_n)$ holds identically in $\mathcal{V}$. (The easy verification of this fact may be omitted.) Obviously, these conditions can be rewritten in the form of a countable sequence of $\exists \forall \exists$ sentences f_n, satisfying $(\alpha)-(\gamma)$.

(10) means that $\mathcal{V}$ is equivalent to the variety of all unital modules over a semiring with unit element (see [6], Theorem 2). Now we can proceed as in the case (9), observing that varieties of unital modules over semirings may be characterized by the following properties: they have operations "addition" and "forming of neutral element" (a unary operation!) satisfying the axioms of Abelian monoids, any operation in $\mathcal{V}$ commutes with the addition, for any n-ary operation g there exist unary operations $g_1, \ldots, g_n$ with the identity $g(x_1, \ldots, x_n) = g_1(x_1) + \ldots + g_n(x_n)$ in $\mathcal{V}$, and any unary operation is annihilated by the formation of neutral element.

As it was shown in [9], property (11) means that $\mathcal{V}$ is equivalent to the variety of all affine modules over a ring with unit element. Then a desired reformulation into the language of heterogeneous clones can be achieved using the following fact:

A variety $\mathcal{V}$ is equivalent to the variety of all affine modules over some ring **R** if and only if

(a) $\mathcal{V}$ has a ternary operation p commuting with itself and with every binary operation such that $p(x, y, x) = p(x, x, y) = y$ holds identically in $\mathcal{V}$,

(b) all binary operations are idempotent in $\mathcal{V}$,

(c) if $n \geqq 3$, for each n-ary operation g there exist binary operations $g_2, \ldots, g_n$ such that

$$g(x_1, \ldots, x_n) = p(x_1, \ldots, p(x_1, p(x_1, g_2(x_1, x_2), g_3(x_1, x_3)), g_4(x_1, x_4)) \ldots, g_n(x_1, x_n))$$

holds identically in $\mathcal{V}$.

To prove the necessity of (a)—(c), let us consider a ring **R** with unit element 1. In any affine **R**-module, take $p(x, y, z) = -x + y + z$ and for $g(x_1, \ldots, x_n) = \gamma_1 x_1 + \ldots + \gamma_n x_n$ $(\gamma_i \in \mathbf{R})$ let $g_k(x, y) = (1 - \gamma_k)x + \gamma_k y$ $(k = 2, \ldots, n)$. Then (a)—(c) can be verified immediately.

Now assume the validity of (a)—(c). We have to prove that $\mathcal{V}$ is idempotent, regular and Hamiltonian (cf. Theorem 2 in [9]). The idempotency of operations in $\mathcal{V}$ follows from (a)—(c) directly. To prove the regularity it is enough to find a ternary operation r with the identity $r(x, x, z)=z$ and identical implication $r(x, y, z)=z \to x=y$ (see [8]). We show that $r=p$ is adequate. Indeed, $p(x, x, z)=z$ was assumed; let, on the other hand, $p(x, y, z)=z$. Then (a) implies

$$x = p(z, z, x) = p(z, p(x, y, z), x) = p(p(x, x, z), p(x, y, z)p(x, x, x)) =$$
$$= p(p(x, x, x), p(x, y, x), p(z, z, x)) = p(x, y, x) = y.$$

To prove that $\mathscr{V}$ is Hamiltonian, for any n-ary g a ternary h_g is needed with the identity $g(x_1, \dots, x_n)=h_g(x_0, x_1, g(x_0, x_2, \dots, x_n))$ in $\mathscr{V}$. We assert that for any at least binary g the operation $h_g(x_1, x_2, x_3)=p(x_1, g(x_2, x_1, \dots, x_1), x_3)$ is good. From (a) and (c) it follows that p commutes with each operation; hence we get

$$h_g(x_0, x_1, g(x_0, x_2, \dots, x_n)) = p(g(x_0, \dots, x_0), g(x_1, x_0, \dots, x_0), g(x_0, x_2, \dots, x_n)) =$$
$$= g(p(x_0, x_1, x_0), p(x_0, x_0, x_2), \dots, p(x_0, x_0, x_n)) = g(x_1, x_2, \dots, x_n).$$

This completes the discussion of the case (11) and also the proof of our Proposition.

Remark that properties (1) and (2) are defined with the aid of $\forall$ sentences, (3)—(5) involve $\exists\forall$ sentences, (6)—(8) require $\forall\exists$ sentences and (9)—(11) may be expressed by $\exists\forall\exists$ sentences.

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Modification sets and transforms of discrete measures

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In this paper $\mathbf{T}$ is the circle group and $\mathbf{Z}$ the ring of integers. Let $M(\mathbf{T})$ denote the usual Banach convolution algebra of bounded Borel measures on $\mathbf{T}$; $M_a(\mathbf{T})$ those $\mu \in M(\mathbf{T})$ which are absolutely continuous with respect to Lebesgue measure on $\mathbf{T}$; $M_s(\mathbf{T})$ the set of $\mu \in M(\mathbf{T})$ which are concentrated on sets of Lebesgue measure zero and $M_d(\mathbf{T})$ those $\mu \in M_s(\mathbf{T})$ which are discrete.

The Fourier—Stieltjes coefficients $\hat{\mu}(n)$ of the measure $\mu \in M(\mathbf{T})$ are defined by

$$\hat{\mu}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta}\, d\mu(\theta) \qquad (n \in \mathbf{Z}).$$

A subset E of $\mathbf{Z}$ is called a *modification set* if

$$M_a(\mathbf{T})\hat{}\,|_{E^c} \subset M_s(\mathbf{T})\hat{}\,|_{E^c}. \tag{1}$$

If $S \subset \mathbf{Z}$, then let $\#(S, n)$ be the number of members of S which do not exceed n in modulus. If $\lim_{n\to\infty} \frac{\#(S,n)}{2n}$ exists then we call this limit the *natural density* of S and denote it by $d(S)$.

W. RUDIN in [5] proved the existence of sets $E \subset \mathbf{Z}$ satisfying (1) with arbitrarily small natural density. In [6] RUDIN showed the existence of modification sets with natural density zero.

Using a result of Pigno and Saeki (stated below) we show the existence of arithmetically interesting sets $E \subset \mathbf{Z}$ with arbitrarily small natural density satisfying

$$M_a(\mathbf{T})\hat{}\,|_{E^c} \subset M_d(\mathbf{T})\hat{}\,|_{E^c}. \tag{2}$$

Futhermore, in contrast to Rudin's result we prove that there are no sets E of natural density zero that satisfy (2).

Let $\overline{\mathbf{Z}}$ denote the Bohr compactification of $\mathbf{Z}$ and let E^a denote the set of accumulation points of E which are in $\mathbf{Z}$ (the topology is with respect to $\overline{\mathbf{Z}}$).

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Theorem 1. (PIGNO and SAEKI [4]) *The set $A \subset \mathbf{Z}$ satisfies*

$$M_a(\mathbf{T})\hat{}|_A \subset M_d(\mathbf{T})\hat{}|_A \tag{3}$$

if and only if $A^a \cap A = \emptyset$ and there exists a $\mu \in M(\mathbf{T})$ such that $\hat{\mu}(A)=1$ and $\hat{\mu}(A^a)=0$.

Theorem 2. *Given $\varepsilon > 0$ there is a set $E \subset \mathbf{Z}$ such that E has natural density less than ε and E^c satisfies (2). This result is best possible.*

To prove the first part of Theorem 2, we will need two lemmas.

Lemma 1. *Let $E \subset \mathbf{Z}^+$ be such that for infinitely many positive integers n, there exists a positive integer l_n, and a finite set E_n such that*

$$n - l_n \to \infty \tag{4}$$

and

$$E \subset E_n \cup \bigcup_{j=0}^{\infty} \big([jn, jn+l_n] \cap [(j+1)n - l_n, (j+1)n]\big) \text{ for each } n. \tag{5}$$

Then E has no limit point in $\mathbf{Z}$ with the relative Bohr topology.

Proof. The proof follows the lines of the proof of Proposition 5 of [3]. For $a<0$, the cited proof shows that a is not a limit point of E. For $a \geqq 0$, we can find an n with $a < n - l_n$ and the proof continues in the same way.

Lemma 2. *Let $b \geqq 2$ be a fixed positive integer. Let $\{n_s\}_{s=1}^{\infty}$ be any increasing sequence of positive integers and let $\{k_s\}_{s=1}^{\infty}$ be any sequence of positive integers with $n_s > k_s$ for all s, and $n_s - k_s \to \infty$. Let E be the set of positive integers t with the property:*

If $t = d_r d_{r-1} \dots d_0$ is the representation of t in the base b and $n_{s_t} \leqq r < n_{s_t+1}$, then for each $s \leqq s_t$ at least one of the digits $d_{ns-1}, d_{ns-2}, \dots, d_{ns-ks}$ is non-zero and at least one of these digits is not $b-1$.

Then E has no limit point in $\mathbf{Z}$ with the relative Bohr topology.

Proof. Given n_s and any $t \in E$, there is some $j \geqq 0$ with $t \in [jb^{n_s}, (j+1)b^{n_s}]$. For this j we have

$$jb^{n_s} + b^{n_s - k_s} \leqq t \leqq (j+1)b^{n_s} - b^{n_s - k_s}.$$

We may now apply Lemma 1 with the b^{n_s}'s playing the role of the n's and with $b^{n_s}(1-b^{-k_s})$ playing the role of l_n.

Proof of Theorem 2. For the first part, we will, given $\varepsilon > 0$, find a subset E of $\mathbf{Z}^+$ such that $d(E) > (1-\varepsilon)/2$ and E has no limit point in $\mathbf{Z}$ with the relative Bohr topology. To do this, fix any base $b \geqq 2$ and apply Lemma 2 with $\{n_s\}_{s=1}^{\infty}$, a sufficiently rapidly increasing sequence and $\{k_s\}_{s=1}^{\infty}$, also rapidly increasing with $n_s > k_s$ and

$n_s - k_s \to \infty$. For example, if we take

$$\prod_{s=1}^{\infty} (1-2/b^{k_s})/(1+2/b^{k_s}) > 1-\varepsilon/2$$

and then choose the sequence $\{n_s\}_{s=1}^{\infty}$ to be sufficiently rapidly increasing in terms of our already chosen sequence $\{k_s\}_{s=1}^{\infty}$, then E will have the desired properties. The set $E \cup -E$ satisfies the conclusion of the first part of the present theorem.

For the second part, we begin by observing that any set S for which $\overline{\lim}_{n\to\infty} \frac{\#(S,n)}{2n} = 1$ must contain arbitrarily long blocks of consecutive integers. To conclude our proof we establish the following lemma.

Lemma 3. *Any subset S of $\mathbf{Z}$ which contains arbitrarily long blocks of consecutive integers is dense in $\overline{\mathbf{Z}}$. In fact, $S^a = \overline{\mathbf{Z}}$.*

Proof. First, if U is any neighborhood of 0 in $\overline{\mathbf{Z}}$, then finitely many integer translates of U cover $\overline{\mathbf{Z}}$. If these translates are $x_1+U, x_2+U, \dots, x_n+U$, then set $x = \max(|x_1|, |x_2|, \dots, |x_n|)$. It is now clear that any block of $2x+1$ consecutive integers contains a member of U and we are done.

Note. The sequences of Theorem 2 have a nice arithmetical structure. If one is only interested in density properties, then the derivation of Theorem 2 can be simplified as follows:

Since arithmetic progressions are open sets in $\mathbf{Z}$ with the relative Bohr topology we begin by choosing a thin arithmetic progression containing 0 and, except for 0, delete all members of this arithmetic progression from $\mathbf{Z}$. We now go to the first negative member of this new set and place it in a thin arithmetic progression and again delete all other members of this arithmetic progression from the set just constructed. We next go to the first positive member of the set we now have and continue. If all arithmetic progressions are chosen sufficiently thin, then after the n^{th} step all deleted members of the arithmetic progressions chosen will have absolute value greater than n and it is immediate that our set is well defined. In addition, the sufficient thinness of the arithmetic progressions guarantees that our constructed set has the desired density property. Finally, its lack of a limit point is clear from the construction.

We conclude with the following two results:

Theorem 3. *Let $\mathcal{P} = \{p^k : p$ a prime, $k \in \mathbf{Z}^+\}$ be the set of prime powers. Then $\mathcal{P}$ satisfies*

$$M_a(\mathbf{T})\hat{\ }|_{\mathcal{P}} \subset M_d(\mathbf{T})\hat{\ }|_{\mathcal{P}}.$$

Proof. We show that $\mathcal{P}^a \subset \{-1,1\}$. If $n \neq 0, \pm 1$, consider the arithmetic progression $\{2n^2k+n : k \in \mathbf{Z}\}$. Since $(n, 2nk+1) = 1$, it follows that $n(2nk+1) = 2n^2k+n \in \mathcal{P}$

is impossible unless $|2nk+1|=1$. But if $|2nk+1|=1$, we have $k=0$ and $2n^2k+n=n$. Thus, the arithmetic progression $\{2n^2k+n\colon k\in\mathbf{Z}\}$ separates n from $\mathscr{P}\setminus\{n\}$.

Finally, 0 is separated from $\mathscr{P}$ by the arithmetic progression $\{6k\colon k\in\mathbf{Z}\}$. This shows $\mathscr{P}^a\subset\{-1,1\}$. Thus $\mathscr{P}\cap\mathscr{P}^a=\emptyset$ and our result now follows from Theorem 1.

Let $r\in\mathbf{Z}^+$ with $r\geqq 2$. Set $\mathscr{E}=\{r^k\colon k\in\mathbf{Z}^+\}$ and put $\mathscr{F}=2\mathscr{E}=\{r^n+r^m\colon n, m\in\mathbf{Z}^+\}$. Recall that $\mathscr{E}$ is an I_0 set; see for example [1, p.85].

Theorem 4. *The set* $\mathscr{F}$ *satisfies*

$$M_a(\mathbf{T})\hat{}|_{\mathscr{F}}\subset M_d(\mathbf{T})\hat{}|_{\mathscr{F}}.$$

Proof. We show that $\mathscr{F}^a\subset\{0\}$. For definiteness we shall take $\mathscr{E}=\{2^k\colon k\in\mathbf{Z}^+\}$. Consider the one point compactification of $\mathscr{E}$ which we realize in the following manner: Put

$$\mathbf{D}=\{e^{2\pi i m/2^n}\colon m\in\mathbf{Z} \text{ and } n\in\mathbf{Z}^+\}$$

and consider $\mathbf{D}$ as a discrete subgroup of $\mathbf{T}$. We then identify $\mathscr{E}$ with its image in the compact group $\hat{\mathbf{D}}$ (dual to $\mathbf{D}$) in the usual way; see [2, p. 107] and [2, p. 403]. The closure of $\mathscr{E}$ in $\hat{\mathbf{D}}$ is simply $\mathscr{E}\cup\{0\}$. The set of limit points of $\mathscr{F}$ in $\hat{\mathbf{D}}$ is $\{0\}\cup\mathscr{E}$. Since $\hat{\mathbf{D}}$ is a factor group of $\bar{\mathbf{Z}}$ and $\mathbf{D}$ is dense in $\mathbf{T}$ it follows that $\mathscr{F}^a\subset\{0\}\cup\mathscr{E}$.

Fix any 2^k and look at the arithmetic progression $\{3s2^{k+1}+2^k\colon s\in\mathbf{Z}\}$. Suppose we have $3s2^{k+1}+2^k=2^m+2^n$ $(m\geqq n)$.

Case 1. If $m>n$, then $2^k\|3s2^{k+1}+2^k$ and $2^n\|2^m+2^n$ and so $k=n$ and $3s2^{k+1}=2^m$, which is impossible.

Case 2. If $m=n$, then $3s2^{k+1}+2^k=2^{m+1}$ and since $2^k\|3s2^{k+1}+2^k$ we see that $k=m+1$. Thus, $s=0$ whence $3s2^{k+1}+2^k=2^m+2^n=2^k$.

It now follows from cases 1 and 2 that $\{3s2^{k+1}+2^k\colon s\in\mathbf{Z}\}$ separates 2^k from $\mathscr{F}\setminus\{2^k\}$. Thus $\mathscr{F}^a\cap\mathscr{F}=\emptyset$ and our result again follows from Theorem 1.

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Extensions of partial multiplications and polynomial identities on Abelian groups

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(i) In this paper G will denote an abelian group, and A will denote a subgroup of G. A multiplication on A is meant to be a homomorphism $\mu: A\times A\to A$, and a partial multiplication on A is meant to be a homomorphism $\mu: A\times A\to G$ [1, vol. II, pp. 281—284]. A multiplication φ on G is called an extension of a partial multiplication μ on A if the restriction of φ to A, $\varphi|_A=\mu$. In (ii) conditions are given for which every partial multiplication on A extends to a multiplication on G.

$P(X_1, \dots, X_n)$ will denote a polynomial in non-commuting variables over the ring of integers. A partial multiplication μ on A is said to satisfy a polynomial identity $P(X_1, \dots, X_n)$ if the elements of (A, μ) satisfy $P(X_1, \dots X_n)=0$. In (iii) conditions are given for which a multiplication on G extending a partial multiplication μ on A satisfies polynomial indentities statisfied by μ. Polynomial identities which a multiplication on a torsion free group can satisfy are examined in (iv).

(ii) Theorem 1. *Let A be a torsion free subgroup of G. Every partial multiplication on A can be extended to a multiplication on G under each of the following conditions:*

1. *G is divisible,*
2. *$(G\otimes G)/(A\otimes A)$ is free,*
3. *$(G\otimes G)/(A\otimes A)$ is a torsion group, and G is p-divisible for every prime p for which $(G\otimes G)/(A\otimes A)$ has a non-trivial p-component.*

Proof. The sequence

$$0 \to A\otimes A \to G\otimes G \to (G\otimes G)/(A\otimes A) \to 0$$

is exact [3, Theorem 2.8]. Therefore, the sequence

$$\mathrm{Hom}\,(G\otimes G, G) \to \mathrm{Hom}\,(A\otimes A, G) \xrightarrow{\varphi} \mathrm{Ext}\,\big((G\otimes G)/(A\otimes A), G\big)$$

is exact. Each of the conditions 1—3 assures that $\mathrm{Ext}\,\big((G\otimes G)/(A\otimes A), G\big)=0$, so that φ is an epimorphism.

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(iii) Theorem 2. *Let G be torsion free, and let A be an essential subgroup of G (i.e. G/A is a torsion group). Let μ be a partial multiplication on A satisfying a homogeneous polynomial identity $P(X_1, \dots, X_n)$. If $\bar{\mu}$ is a multiplication on G which extends μ, then $\bar{\mu}$ satisfies $P(X_1, \dots, X_n)$.*

Proof. Let $m = \deg P(X_1, \dots, X_n)$, and let $g_1, \dots, g_n \in G$. There exist positive integers l_i such that $l_i g_i \in A$, $1 \leqq i \leqq n$. Let $l = \prod_{i=1}^{n} l_i$. Then $l g_i \in A$, $1 \leqq i \leqq n$. Therefore $0 = P(lg_1, \dots, lg_n) = l^m P(g_1, \dots, g_n)$. G is torsion free, so that $P(g_1, \dots, g_n) = 0$.

Corollary 1. *Let G be a torsion free group, and let B be an A-high subgroup of G. Let μ be a partial multiplication on $A \otimes B$, and let μ_A and μ_B respectively be the restrictions of μ to A and to B. Let μ_A and μ_B satisfy a homogeneous polynomial $P(X_1, \dots, X_n)$. Then* 1. *μ satisfies $P(X_1, \dots, X_n)$, and* 2. *every multiplication $\bar{\mu}$ on G which extends μ satisfies $P(X_1, \dots, X_n)$.*

Proof. The homogeneity of $P(X_1, \dots, X_n)$ clearly implies 1. Let $\bar{\mu}$ be a multiplication on G which extends μ. $G/(A \otimes B)$ is a torsion group [1, vol I, p. 50 ex. 6]. By Theorem 2, $\bar{\mu}$ satisfies $P(X_1, \dots, X_n)$.

Corollary 2. *For every positive integer $n \geqq 2$ there exists a nilpotent ring R with degree of nilpotency n such that the additive group G of R satisfies:*

1. *G is divisible and torsion free.*
2. *G is the divisible hull of a group A whose nilstufe* [4] *is $n-1$.*

Proof. SZELE [4, Theorem 2] has shown that there exists a torsion free group A with nilstufe $n-1$. Let μ be a multiplication on A for which $A^{n-1} \neq 0$. μ satisfies $P(X_1, \dots, X_n) = X_1 X_2, \dots, X_n$. Let G be the divisible hull of A. By Theorem 1, μ can be extended to a multiplication on G, and by Theorem 2, $\bar{\mu}$ satisfies $P(X_1, \dots, X_n)$.

Theorem 3. *Let A be an essential subgroup of G, and let μ be a partial multiplication on A such that (A, μ) does not possess any nonzero left zero divisors. Then for any multiplication $\bar{\mu}$ on G extending μ, the nonzero elements of A are not left zero divisors in $(G, \bar{\mu})$.*

Proof. Let $0 \neq a \in A$. Define $\varphi_a \colon A \to G$, $\varphi_a(A') = \mu(a, a')$ for all $a' \in A$. φ_a is a homomorphism on A. Since a is not a left zero divisor in (A, μ), φ_a is a monomorphism. Let $\bar{\mu}$ be a multiplication on G extending μ. Define $\bar{\varphi}_a \colon G \to G$, $\bar{\varphi}_a(g) = \mu(a, g)$ for all $g \in G$. $\bar{\varphi}_a$ is an endomorphism of G, with the restriction of $\bar{\varphi}_a$ to A, $\bar{\varphi}_a|_A = \varphi_a$. By [1, Lemma 24.2] $\bar{\varphi}_a$ is a monomorphism. Hence a is not a left zero divisor in $(G, \bar{\mu})$.

(iv) Theorem 4. *Let G be a torsion free group, and let μ be a multiplication on G satisfying a homogeneous polynomial $P(X_1, \dots, X_n)$ of degree r. Let C be the sum of the coefficients of $P(X_1, \dots, X_n)$. Then either μ satisfies X^r, or $C = 0$.*

Proof. Let $0\neq g\in G$. Clearly, $0=P(g, \ldots, g)=Cg^r$. G is torsion free. Therefore, if $C\neq 0$, then $g^r=0$.

Theorem 5. *Let R be a ring satisfying the polynomial identity*

$$P(X_1, X_2) = aX_1^2+bX_2^2+CX_1X_2+dX_2X_1+eX_1+fX_2.$$

Then R satisfies $b(XY+YX)$.

Proof. If R satisfies $P(X_1, X_2)$, then R satisfies

$$P_1(X_1, X_2, X_3) = P(X_1+X_3, X_2)-P(X_1, X_2)-P(X_3, X_2) = $$
$$= a(X_1X_3+X_3X_1)-bX_2^2-fX_2.$$

R also satisfies

$$P_2(X_1, X_2, X_3, X_4) = P_1(X_1+X_4, X_2, X_3)-P(X_1, X_2, X_3)-P_1(X_4, X_2, X_3) =$$
$$= bX_2^2+fX_2,$$

or $P_2(X)=bX^2+fX$. This implies that R satisfies

$$P_3(X, Y) = P_2(X+Y)-P_2(X)-P_2(Y) = b(XY+YX).$$

The following are direct consequences of Theorem 5 or its proof:

Corollary 1. *Let G be a torsion free group, and let μ be a multiplication on G satisfying $P(X_1, X_2)$ of theorem 5 with $b\neq 0$. Then μ satisfies $XY+YX$. If μ is commutative, then μ satisfies XY.*

Corollary 2. *Let R be a commutative ring satisfying $P(X_1, X_2)$ of Theorem 5. Let π be the set of prime divisors of b and let π' be the set of primes p for which the additive group of R has a nonzero p-primary component. If $\pi\cap\pi'=\emptyset$, then R satisfies XY.*

Corollary 3. *Let R be a ring satisfying $P(X_1, X_2)$ of Theorem 5 with $b\neq 0$. Then for every $a\in R$, $\{a, a^2\}$ is a dependent set* [1 vol. I, p. 83].

Corollary 4. *Let R be a ring satisfying $P(X_1, X_2)$ of Theorem 5, with $b=0$, $f\neq 0$. Then the additive group of R is bounded.*

Theorem 6. *Let G be a torsion free group of finite rank n such that for every $0\neq g\in G$, the type of g, $T(g)$, is not idempotent. Then every multiplication on G satisfies X^{2^n}.*

Proof. Koehler [2, Theorem 1.6] has shown that every ascending chain of types realizable in G, $t_1<t_2, <\ldots<t_r$, with $t_r\neq(\infty, \ldots, \infty, \ldots)$ is of length less than or equal to n. Let $0\neq g\in G$. For every multiplication on G

$$(*) \qquad T(g) \leqq T(g^2) \leqq T(g^4) \leqq \ldots \leqq T(g^{2^n}).$$

Suppose that $T(g^{2^{k+1}})=T(g^{2^k})$, $T(g^{2^{k+1}})\geqq 2T(g^{2^k})$ for some $0\leqq k\leqq n$ so that $T(g^{2^k})$ is idempotent, and hence $g^{2^k}=0$. If $T(g^{2^k})<T(g^{2^{k+1}})$ for all k, $0\leqq k<n$, then $(*)$ is a chain of length $n+1$, and hence $T(g^{2^n})=(\infty, \ldots, \infty, \ldots)$ which implies that $g^{2^n}=0$.

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On products of abstract automata

F. GÉCSEG

Frequently two automata behave exactly in the same way as far as the transitions induced by their inputs are concerned, but none of them can be represented homomorphically by a (general) power of the other one; although the existence of homomorphisms between automata does not imply that they have common input sets. This situation can be avoided by allowing input words as input signals of the component automata. This modification leads to the concept of a generalized product introduced in this paper. Furthermore, we allow input words as counter images of input signals under homomorphic representations. The resulting representations will be called simulations.

The purpose of this paper is to study the generalized products and simulations from the point of view of isomorphic and homomorphic completeness. It will turn out that in most cases the generalized products and simulations are more effective than the classical products and representations. Furthermore, the results concerning generalized products and simulations will be interpreted in terms of classical products, representations and temporal products of automata.

By an *automaton* we mean a triplet $\mathbf{A}=(X, A, \delta)$, where X and A are nonvoid finite sets called the *input set* and *state set*, respectively. Moreover, $\delta\colon A\times X\to A$ denotes the *transition function* of $\mathbf{A}$.

Take an arbitrary finite group G, and form the automaton $\mathbf{G}=(G, G, \delta_{\mathbf{G}})$ with $\delta_{\mathbf{G}}(g_1, g_2)=g_1g_2$ for all $g_1, g_2\in G$, where g_1g_2 means that g_1 is multiplied by g_2 in G. $\mathbf{G}$ is a *grouplike automaton*.

For any nonvoid set X, let us denote by $F(X)$ the free monoid generated by X. If X is an input set of an automaton $\mathbf{A}=(X, A, \delta)$ then the elements $p\in F(X)$ are called *input words* of A. The transition function δ can be extended to $A\times F(X)\to A$ in a natural way: for any $p=p'x\in F(X)$ and $a\in A$, $\delta(a, p)=\delta(\delta(a, p'), x)$. Further on we shall use the more convenient notation $ap_{\mathbf{A}}$ for $\delta(a, p)$. If there is no danger of confusion then we omit the index $\mathbf{A}$.

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Let $\mathbf{A}=(X, A, \delta)$ be an automaton. Define a binary relation $\varrho_{\mathbf{A}}$ on $F(X)$ in the following manner: for two input words $p, q \in F(X), p \equiv q(\varrho_{\mathbf{A}})$ if and only if $ap_{\mathbf{A}}=aq_{\mathbf{A}}$ for all $a \in A$. The quotient semigroup $F(X)/\varrho_{\mathbf{A}}$ is called the *characteristic semigroup* of $\mathbf{A}$, and it will be denoted by $S(\mathbf{A})$. We use the notation $[p]_{\mathbf{A}}$ for the element of $S(\mathbf{A})$, containing $p \in F(X)$. Thus, $[p]_{\mathbf{A}}=[q]_{\mathbf{A}}$ $(p, q \in F(X))$ if and only if p and q induce the same transition in A. Again, if there is no danger of confusion, we omit the index $\mathbf{A}$ in $[p]_{\mathbf{A}}$.

Take an automaton $\mathbf{A}=(X, A, \delta)$, and let π be a partition of A. It is said that π has the *substitution property* (shortly, SP) if $a \equiv b(\pi)$ implies $\delta(a, x) \equiv \delta(b, x)(\pi)$ for all $a, b \in A$ and $x \in X$. (Let us note that we use the same symbol π for a partition and for the equivalence relation inducing it.) The quotient automaton induced by π will be denoted by $\mathbf{A}/\pi$.

Let $\mathbf{A}_i=(X_i, A_i, \delta_i)$ $(i=1, \ldots, n)$ be a system of automata. Moreover, let X be a finite nonvoid set, and φ a mapping of $A_1 \times \ldots \times A_n \times X$ into $F(X_1) \times \ldots \times F(X_n)$. We say that the automaton $\mathbf{A}=(X, A, \delta)$ with $A=A_1 \times \ldots \times A_n$ and

$$\delta((a_1, \ldots, a_n), x) = (a_1 p_1, \ldots, a_n p_n),$$

where $(p_1, \ldots, p_n)=\varphi(a_1, \ldots, a_n, x)$, is the *generalized product* of $\mathbf{A}_i$ $(i=1, \ldots, n)$ with respect to X and φ. For this product we use the shorter notation $\mathbf{A}=\prod_{i=1}^{n} \mathbf{A}_i[X, \varphi]$.

A generalized product $\mathbf{A}=\prod_{i=1}^{n} \mathbf{A}_i[X, \varphi]$ is a *generalized α_i-product* $(i=0, 1, \ldots)$ if φ can be given in the form

$$\varphi(a_1, \ldots, a_n, x) = (\varphi_1(a_1, \ldots, a_n, x), \ldots, \varphi_n(a_1, \ldots, a_n, x))$$

such that each φ_j $(1 \leqq j \leqq n)$ is independent of states having indices greater than or equal to $j+i$.

If in a generalized product [generalized α_i-product] φ is of the form $\varphi\colon A_1 \times \times \ldots \times A_n \times X \to X_1 \times \ldots \times X_n$ then we get the concept of a *product* [α_i-product] (see [3]). Moreover, if in a generalized product [product] $\mathbf{A}$, $\mathbf{A}_i=\mathbf{B}$ for all $i(=1, \ldots, n)$ then $\mathbf{A}$ is called a *generalized power [power]* of $\mathbf{B}$.

The concept of the generalized α_i-product (α_i-product) can be interpreted in the following way. For a given generalized product (product) take a well ordering on the set of its components. Assume that $\mathbf{A}_i$ is the i-th automaton under this well ordering. If for two j and i with $i \leqq j$ there is a feed-back from $\mathbf{A}_j$ to A_i then we say that the length of this feed-back is $j-i+1$. Now for any $i(=0, 1, \ldots)$, in the generalized α_i-products (α_i-products) the lengths of such feed-backs does not exceed i under the usual well ordering of natural numbers.

We say that an automaton $\mathbf{A}=(X, A, \delta)$ *homomorphically simulates* $\mathbf{B}=(X', B, \delta')$ if there exist a one-to-one mapping τ_1 of X' into $F(X)$ and a mapping τ_2 of a subset A' of A onto B such that $\tau_2(a\tau_1(x'))=\delta'(\tau_2(a), x')$ for any $a \in A'$ and $x' \in X'$. If τ_2 is one-to-one as well then we speak of an *isomorphic simulation*. Furthermore, if τ_1 is of the form $\tau_1: X' \to X$, then we speak of *homomorphic* and *isomorphic representations*.

The following result is trivial.

Lemma 1. *If* **A** *homomorphically simulates* **B** *and* **B** *homomorphically simulates* **C**, *then* **C** *can be simulated homomorphically by* **A**. *Similar statement is valid for isomorphic simulations.*

A system $\sum$ of automata is called *homomorphically S-complete* with respect to the generalized product [generalized α_i-product] if any automaton can be simulated homomorphically by a generalized product [generalized α_i-product] of automata from $\sum$. The concept of *isomorphic S-completeness* is defined similarly.

Take a system $\sum$ of automata. For any $\mathbf{A}=(X, A, \delta)\in\sum$ denote by $\mathbf{A}^*=(X^*, A, \delta^*)$ the automaton whose input set X^* is $S(\mathbf{A})$ and $\delta^*(a, [p])=ap_{\mathbf{A}}$ $([p]\in S(\mathbf{A}))$.

The following statement is obvious.

Lemma 2. *For every generalized product (generalized α_i-product)* $\mathbf{B}=\prod_{i=1}^{n}\mathbf{B}_i[X, \varphi]$ *there is a product (α_i-product)* $\mathbf{B}'=\prod_{i=1}^{n}\mathbf{B}_i^*[X, \varphi^*]$ *such that* **B** *is isomorphic to* $\mathbf{B}'$, *and conversely.*

Now we are ready for studying isomorphic and homomorphic S-completeness with respect to different types of generalized products.

1. α_0-products

For any natural number n, denote by $\mathbf{T}_n=(T_n, N, \delta_N)$ the automaton for which $N=\{1, \dots, n\}$, T_n is the set of all transformations t of N, and $\delta_N(j, t)=t(j)$ for all $j\in N$ and $t\in T_n$.

Theorem 1. *A system $\sum$ of automata is isomorphically S-complete with respect to the generalized α_0-product if and only if for any natural number n, there exists an automaton* $\mathbf{B}\in\sum$ *such that* **B** *isomorphically simulates* $\mathbf{T}_n$.

Proof. In order to prove the sufficiency of these conditions take an automaton $\mathbf{A}=(X, A, \delta)$ with n states. Let τ_2 be an arbitrary 1—1 mapping of A onto $N=\{1, \dots, n\}$. Form the α_0-product $\mathbf{T}_n'=(\mathbf{T}_n)[X, \varphi]$, where $\varphi(x)=t$ $(x\in X, t\in T_n)$ such that $\tau_2(\delta(a, x))=t(\tau_2(a))$ for any $a\in A$. Let τ_1 denote the identity mapping on X. Then (τ_1, τ_2^{-1}) gives an isomorphic simulation of **A** by an α_0-product of $\mathbf{T}_n$. Moreover, by our assumption, there exists an automaton **B** in $\sum$ which isomorphically simulates $\mathbf{T}_n$. Therefore, by Lemma 1, **A** can be simulated isomorphically by a generalized α_0-power of **B**.

Conversely, let $n>1$ be a natural number, and take $\mathbf{T}_n$. Assume that a generalized α_0-product $\mathbf{B}=(X, B, \delta')=\prod_{i=1}^{k} \mathbf{B}_i[X, \varphi]$ of automata from $\sum$ isomorphically simulates $\mathbf{T}_n$. Then, by Lemma 2, $\mathbf{T}_n$ can be simulated isomorphically by an α_0-product $\mathbf{B}'=(X, B, \delta'')=\prod_{i=1}^{k} \mathbf{B}_i^*[X, \varphi^*]$, under two mappings τ_1: $T_n\to F(X)$ and τ_2: $B'\to N$ $(B'\subseteqq B)$.

The elements b of B can be written in the vectorial form $b=(b_1, \ldots, b_k)$ ($b_j\in B_j$ and B_j is the state set of $\mathbf{B}_j^*$). Define partitions π_j' $(j=1, \ldots, k)$ on B in the following way:

$$b(=(b_1, \ldots, b_k)) \equiv ((b_1', \ldots, b_k')=)b'(\pi_j') \quad (b, b'\in B)$$

if and only if $b_1=b_1', \ldots, b_j=b_j'$. Now let π_j $(j=1, \ldots, k)$ be partitions on N given as follows: for any $b, b'\in B'$ we have $\tau_2(b)\equiv\tau_2(b')$ (π_j) if and only if $b\equiv b'$ (π_j'). It is easy to prove that the partitions π_j have SP.

On the other hand, on $\mathbf{T}_n$ only the two trivial partitions have SP. Thus, we get that each π_j has one-element blocks only, or it has one block only. Among these partitions there should be at least one which has more than one block, since $n>1$. Let l be the least index for which π_l has at least two blocks. Then the blocks of π_l consist of single elements. Therefore, the number of all blocks of π_l is n. We show that $\mathbf{B}_l^*$ isomorphically simulates $\mathbf{T}_n$.

By our assumption and the definition of π_j, all elements of B' coincide in their first $l-1$ components; let us denote them by $b_1', \ldots, b_{l-1}'$. Moreover, denote by B_l' the set of all l-th components of elements from B', and let X_l^* be the input set of $\mathbf{B}_l^*$. Define two mappings τ_1': $T_n\to F(X_l^*)$ and τ_2': $B_l'\to A$ in the following way: if $\tau_1(t)=x^{(1)}\ldots x^{(u)}$ then let

$$\tau_1'(t) = \varphi_l^*(b_1', \ldots, b_{l-1}', b_l, \ldots, b_k, x^{(1)}) \ldots$$
$$\ldots \varphi_l^*((b_1', \ldots, b_{l-1}', b_l, \ldots, b_k)(x^{(1)} \ldots x^{(u-1)})_{\mathbf{B}'}), (x^{(u)}),$$

and if $\tau_2(b)=a$ $(b\in B', a\in N)$ and b_l is the l-th component of b then let $\tau_2'(b_l)=a$. (Note that, by the definition of the α_0-product, φ_l^* is independent of states having indices greater than or equal to l.) It is obvious that τ_2' is a one-to-one mapping of B_l' onto N. Let us take a $b_l'\in B_l'$ and a $t\in T_n$. Then there exits a $b\in B'$ with $b=(b_1', \ldots, b_{l-1}', b_l', b_{l+1}, \ldots, b_k)$ such that $\tau_2(b)=\tau_2'(b_l')=a$. Therefore, if $\tau_1(t)=$ $=x^{(1)}\ldots x^{(u)}$ then

$$b\tau_1(t) = (b_1', \ldots, b_{l-1}', b_l'\,\varphi_l^*(b_1', \ldots, b_{l-1}', b_l', b_{l+1}, \ldots, b_k, x^{(1)}) \ldots$$
$$\ldots \varphi_l^*((b_1', \ldots, b_{l-1}', b_l', b_{l+1}, \ldots, b_k)(x^{(1)} \ldots x^{(u-1)})_{\mathbf{B}'}, x^{(u)}), \ldots),$$

since

$$b_v'\varphi_v^*(b_1', \ldots, b_l', b_{l+1}, \ldots, b_k, x^{(1)}) \ldots$$
$$\ldots \varphi_v^*((b_1', \ldots, b_l', b_{l+1}, \ldots, b_k)(x^{(1)} \ldots x^{(u-1)})_{\mathbf{B}'}, x^{(u)}) = b_v'$$

for any $v<l$. From this we get that the l-th component of $b\tau_1(t)$ is $b'_l\tau'_1(t)$, showing that $\tau'_2(b'_l\tau'_1(t))=\delta_N(\tau'_2(b'_l), t)$. Since τ'_2 is 1—1, thus $\mathbf{B}_l\in\sum$ isomorphically simulates $\mathbf{T}_n$.

The case $n=1$ can be proved by a similar argument.

From Theorem 1 we get the following

Corollary. *There exists no system of automata which is isomorphically S-complete with respect to the generalized α_0-product and minimal.*

Proof. Take a system $\sum$ of automata which is isomorphically S-complete with respect to the generalized α_0-product. Moreover, let $\mathbf{A}\in\sum$ be an automaton with n states, and take a natural number $m>n$. It is obvious that $\mathbf{A}$ is isomorphic to a subautomaton of an α_0-product of $\mathbf{T}_m$ (having one factor only). Furthermore, by Theorem 1, there exists a $\mathbf{B}\in\sum$ which isomorphically simulates $\mathbf{T}_m$. Therefore, $\mathbf{A}$ can be simulated isomorphically by a generalized α_0-power of $\mathbf{B}$. Thus, $\sum$-$\{\mathbf{A}\}$ is isomorphically S-complete with respect to the generalized α_0-product, showing that $\sum$ is not minimal.

Take the automaton $\mathbf{A}=(X, A, \delta)$ with $X=\{x, y, z\}$, $A=\{a_1, a_2\}$ and $\delta(a_1, x)=\delta(a_2, x)=\delta(a_2, z)=a_2$ and $\delta(a_2, y)=\delta(a_1, y)=\delta(a_1, z)=a_1$. This $\mathbf{A}$ is called a *two-state reset automaton.* Let us denote by H_2 the characteristic semigroup of $\mathbf{A}$.

For homomorphic simulations we have

Theorem 2. *A system $\sum$ of automata is homomorphically S-complete with respect to the generalized α_0-product if and only if the following conditions are satisfied:*

(i) *For any simple group G there exists a $\mathbf{B}\in\sum$ such that G is a homomorphic image of a subgroup of $S(\mathbf{B})$;*

(ii) *There exists $\mathbf{C}\in\sum$ such that H_2 is a homomorphic image of a subsemigroup of $S(\mathbf{C})$.*

Proof. The necessity of these conditions follows from the well known theorem of Krohn and Rhodes. (For a nice presentation of the Krohn—Rhodes theory, see [6].)

To prove the sufficiency of (i) and (ii), again, by the Krohn—Rhodes theorem, it is enough to show that: Every grouplike automaton $\mathbf{G}=(G, G, \delta_G)$ with a simple group G ($|G|>1$) and a two-state reset automaton can be given as a homomorphic image of a subautomaton of an α_0-product $\prod_{i=1}^{k} \mathbf{B}_i^*[X, \varphi^*]$, where $\mathbf{B}_i\in\sum$.

Take a grouplike automaton $\mathbf{G}=(G, G, \delta_G)$, where G ($|G|>1$) is a simple group. By condition (i), there exists a $\mathbf{B}\in\sum$ such that G is a homomorphic image of a subgroup G' of $S(\mathbf{B})$, under a homomorphism $\tau: G'\to G$. Let $\mathbf{B}$ be given in the form $\mathbf{B}=(X, B, \delta)$. Now define an α_0-product $\mathbf{B}'=(\mathbf{B}^*)[G, \varphi^*]$, where φ^* is an isomorphism of $F(G)$ into $F(G')$ such that $\tau(\varphi^*(g))=g$ for any $g\in G$. Take an arbitrary identity $up=vq$ over G, where u, v are variables and $p, q\in F(G)$. Assume that this identity holds on $\mathbf{B}'$. Since $S(\mathbf{B}')$ is a group (isomorphic to a subgroup of G'), thus there exists a subset B' of B such that each element of G induces a permutation of B' (in $\mathbf{B}'$), and distinct elements of G induce distinct permutations. It is obvious that $|B'|>1$. The identity $up=vq$ implies $up=vp$. But p induces a permutation of B'. Therefore, for any two elements a and b of B', we have $ap\neq bp$ if $a\neq b$. Thus, all identities holding on $\mathbf{B}'$ should have the form $up=uq$, i.e., $[\varphi^*(p)]=[\varphi^*(q)]$ in $S(\mathbf{B})$ whenever $up=uq$ holds in $\mathbf{B}'$. Now, by the choice of φ^*, $p=\tau(\varphi^*(p))=\tau(\varphi^*(q))=q$, i.e., $up=uq$ holds in $\mathbf{G}$. Therefore, we got that $\mathbf{G}$ is contained in the equational class generated by $\mathbf{B}'$. Thus, by the Theorem in [2], $\mathbf{G}$ is a homomorphic image of a subautomaton of a finite direct power of $\mathbf{B}'$. Since the direct product is a special case of the α_0-product, thus $\mathbf{G}$ is a homomorphic image of a subautomaton of an α_0-power of $\mathbf{B}'$. Consequently, by Lemma 2, $\mathbf{G}$ can be simulated homomorphically by a generalized α_0-power of $\mathbf{B}$.

Finally, if (ii) holds, then $\mathbf{C}^*$ has a subautomaton which is a two-state reset automaton (see [6], p. 148). This completes the proof of Theorem 2.

Since for any simple group G with n elements there exists a simple group G' with $|G'|>n$ such that G is isomorphic to a subgroup of G', thus from Theorem 2 we get

Corollary 1. *There exists no system of automata which is homomorphically S-complete with respect to the generalized α_0-product and minimal.*

Moreover, Theorems 1 and 2 imply

Corollary 2. *There exists a system $\sum$ of automata such that $\sum$ is homomorphically S-complete with respect to the generalized α_0-product and $\sum$ is not isomorphically S-complete with respect to the generalized α_0-product.*

2. α_1-products

We start with the study of homomorphic S-completeness with respect to the generalized α_1-products.

Theorem 3. *A system $\sum$ of automata is homomorphically S-complete with respect to the generalized α_1-product if and only if for any natural number n, there exist an automaton $\mathbf{A}=(X, A, \delta)$ in $\sum$, states $a_1, \dots, a_n\in A$ and input words $p_{jl}\in F(X)$ $(1\leq j, l\leq n)$ such that $a_j p_{jl}=a_l$.*

Proof. Let $\sum$ be a system of automata which is homomorphically S-complete with respect to the generalized α_1-product. Let n be a natural number, and take a prime $r>n$. Define an automaton $\mathbf{A}_r=(X', A_r, \delta_r)$ in the following way: $X'=\{x\}$,

$A_r=\{a_0, \ldots, a_{r-1}\}$ and

$$\delta_r(a_i, x) = \begin{cases} a_{i+1} & \text{if} \quad i < r-1, \\ a_0 & \text{if} \quad i = r-1. \end{cases}$$

Assume that $\mathbf{A}_r$ can be simulated homomorphically by a generalized α_1-product $\mathbf{B}=\prod_{i=1}^{k} \mathbf{B}_i[\bar{X}, \varphi]$ of automata from $\sum$. Thus, by Lemma 2, there exists an α_1-product $\mathbf{B}'=(\bar{X}, B, \delta')=\prod_{i=1}^{k} \mathbf{B}_i^*[\bar{X}, \varphi^*]$ which homomorphically simulates $\mathbf{A}_r$ under a set $B' \subseteq B$ and mappings $\tau_1(x)=p \in F(\bar{X})$ and $\tau_2 : B' \to A_r$.

Let us represent the elements of B in the vectorial form $b=(b_1, \ldots, b_k)$. Define the partitions π_j' $(j=1, \ldots, k)$ on B in the same way as in the proof of Theorem 1. It can be shown by a short computation that these partitions π_j' have SP.

By the choice of $\mathbf{A}_r$, there exists a subset $B''=\{b_0', \ldots, b_{u-1}'\}$ of B' such that $r|u$,

$$b_l' \, p_{\mathbf{B}'} = \begin{cases} b_{l+1}' & \text{if} \quad l < u-1, \\ b_0' & \text{if} \quad l = u-1. \end{cases}$$

and $\tau_2(b_l')=a_{l(\text{mod}\, r)}$, where $l(\text{mod}\, r)$ denotes the least nonnegative residue of l modulo r. Let π_j be the restriction of π_j' to B''. It can be proved that for any j, the blocks of π_j have the same cardinality. Donete by f_1 the number of blocks of π_1. Moreover, it is easy to show that $\pi_1 \geqq \pi_2 \geqq \ldots \geqq \pi_k$, and each block of π_j contains the same number f_{j+1} of blocks of π_{j+1} $(j=1, \ldots, k-1)$. Therefore, $u=f_1 f_2 \ldots f_k$. But $r|u$ and r is a prime. Thus, there exists an $l (1 \leqq l \leqq k)$ such that $r|f_l$. This means, by the definition of the partitions π_j, that the number of states of $\mathbf{B}_l^*$ occuring as l-th components in the elements of B'' is at least $f_j \geqq r$. Let us denote them by $c_1, \ldots, c_s$. Since for any two elements b' and b'' of B'' there exists an input word $q=p \ldots p$ such that $b' q_{\mathbf{B}'}=b''$, thus for any c_t, c_h $(1 \leqq t, h \leqq s)$ there is an input signal x_{th} of $\mathbf{B}_l^*$ with $c_t x_{th}=c_h$ in $\mathbf{B}_l^*$. Consequently, by the definition of $\mathbf{B}_l^*$, $\mathbf{B}_l \in \sum$ satisfies the conditions of Theorem 3.

Conversely, assume that the conditions of Theorem 3 are satisfied. Take an arbitrary automaton $\mathbf{B}=(X, B, \delta_{\mathbf{B}})$ with $B=\{b_1, \ldots, b_n\}$. Then there exist an automaton $\mathbf{A}=(\bar{X}, A, \delta_{\mathbf{A}}) \in \sum$, states $a_1, \ldots, a_n \in A$ and input signals x_{ij} $(1 \leqq i, j \leqq n)$ of $\mathbf{A}^*$ such that $\delta_{\mathbf{A}}^*(a_i, x_{ij})=a_j$. Now take the α_1-product $\mathbf{C}=(X, C, \delta_{\mathbf{C}})=(\mathbf{A}^*)[X, \varphi^*]$, where for any $x \in X$, $\varphi^*(a_i, x)=x_{ij}$ if $\delta_{\mathbf{B}}(b_i, x)=b_j$ $(i, j=1, \ldots, n)$, and in all other cases $\varphi^*(a, x) (a \in A)$ is defined arbitrarily. It is obvoius that $\mathbf{C}$ *isomorphically* simulates $\mathbf{B}$.

From the above proof we get

Corollary 1. *A system of automata is homomorphically S-complete with respect to the generalized α_1-product if and only if it is isomorphically S-complete with respect to the generalized α_1-product.*

Corollary 2. *There exists no system of automata which is homomorphically (or isomorphically) S-complete with respect to the generalized α_1-product and minimal.*

The following result shows that the homomorphic and isomorphic simulations with respect to the generalized α_1-product do not coincide if they are considered over an arbitrary system of automata.

Theorem 4. *There exist a system $\sum$ of automata and an automaton* **A** *such that* **A** *can be simulated homomorphically by a generalized α_1-product of automata from $\sum$ and* **A** *cannot be simulated isomorphically by any generalized α_1-product of automata from $\sum$.*

Proof. Take the following automaton $\mathbf{A}=(X, A, \delta)$, where $X=\{x, y\}$, $A=\{a, b, c\}$, $\delta(a, x)=\delta(c, y)=b$, $\delta(b, x)=\delta(c, x)=c$ and $\delta(b, y)=\delta(a, y)=a$. Moreover, let $\sum$ consist of all two-state automata. If **A** can be simulated isomorphically by a generalized α_1-product of automata from $\sum$, then, by the proof of Theorem 3, there exists a nontrivial partition of A having SP. But a short computation shows that only the two trivial partitions of A have SP.

Now define an automaton $\mathbf{B}=(X, B, \delta')$ such that $X=\{x, y\}$, $B=\{a, b, b', c\}$, $\delta'(a, x)=b$, $\delta'(b, x)=\delta'(b', x)=\delta'(c, x)=c$, $\delta'(a, y)=\delta'(b, y)=\delta'(b', y)=a$ and $\delta'(c, y)=b'$. It is obvious that the mapping τ of B onto A with $\tau(a)=a$, $\tau(b)=\tau(b')=b$ and $\tau(c)=c$ is a homomorphism of **B** onto **A**. Moreover, the partition π with two blocks $\{a, b'\}$ and $\{b, c\}$ has SP. Therefore, **B** is isomorphic to an α_0-product of two two-state automata (cf. [1], p. 184). This ends the proof of Theorem 4.

3. General products and α_i-products with $i>1$

Take a set A and a system $\pi_0, \ldots, \pi_n$ of partitions on A. We say that this system of partitions is *regular* if the following conditions are satisfied:

(i) π_0 has one block only,

(ii) π_n has one-element blocks only,

(iii) $\pi_0 \geqq \pi_1 \geqq \ldots \geqq \pi_n$.

Let π be a partition of A. For any $a \in A$, denote by $\pi(a)$ the block of π containing a. Moreover, set $M_{i,a}=\{\pi_{i+1}(b): b \in A$ and $b \equiv a(\pi_i)\}$, where $a \in A$ and $i=0, \ldots, n-1$. Finally, let $\pi_i/\pi_{i+1}=\max\{|M_{i,a}|: a \in A\}$.

Consider an automaton $\mathbf{A}=(X, A, \delta)$. Then $(X^*)_{g(\mathbf{A})}$ always denotes a generating set of $S(\mathbf{A})$.

Now we prove.

Theorem 5. *Let $l>2$ be a natural number and $i>1$. For an automaton* $\mathbf{A}=(X, A, \delta)$, $\mathbf{A}^*$ *is isomorphic to some* $\mathbf{B}^*$, *where* **B** *is a subautomaton of a generalized*

α_i-product of automata having fewer states than l, if and only if for some $(X^*)_{g(\mathbf{A})}$ there exists a regular system $\pi_0, \ldots, \pi_n$ of partitions of A such that

(I) $\pi_j/\pi_{j+1} \leqq l$ *for all* $j=0, \ldots, n-1$,

(II) $a \equiv b(\pi_j)$ *implies* $\delta^*(a, x^*) \equiv \delta^*(b, x^*)$ (π_{j-i+1}) *for all* $i-1 \leqq j \leqq n$, $x^* \in (X^*)_{g(\mathbf{A})}$ *and* $a, b \in A$.

Proof. Assume that for $\mathbf{A}=(X, A, \delta)$, $\mathbf{A}^*$ is isomorphic to $\mathbf{B}^*$, where $\mathbf{B}$ is a subautomaton of a generalized α_i-product $\prod_{j=1}^{n} \mathbf{A}_j[X', \varphi]$ of automata with $|A_j| \leqq l$, $l>2$ and $i>1$. By Lemma 2, $\mathbf{B}$ is isomorphic to a subautomaton of the α_i-product $\mathbf{A}'= =(X', \bar{A}, \bar{\delta})= \prod_{j=1}^{n} \mathbf{A}_j^*[X', \varphi^*]$. We may assume that $\mathbf{B}^*$ is a subautomaton of $\mathbf{A}'^*$. Moreover, let σ: $S(\mathbf{A}) \to S(\mathbf{B})$, η: $A \to B$ be an isomorphism of $\mathbf{A}^*$ onto $\mathbf{B}^*$. Define partitions π_j $(j=1, \ldots, n)$ on A in the following way: $a \equiv a'(\pi_j)$ if and only if $\eta(a)= =(a_1, \ldots, a_n)$, $\eta(a')=(a_1', \ldots, a_n')$ and $a_1=a_1', \ldots, a_j=a_j'$. It is obvious that $\pi_0, \pi_1, \ldots$ $\ldots, \pi_n$ is a regular system of partitions. Moreover, condition (I) is satisfied by this system. Indeed, if $\eta(a)=(a_1, \ldots, a_n)$ and $\eta(a')=(a_1', \ldots, a_n')$ then $\pi_{j+1}(a') \in M_{j,a}$ if and only if $a_1'=a_1, \ldots, a_j'=a_j$. Therefore, $M_{j,a}$ contains at most $|A_{j+1}| (\leqq l)$ blocks of π_{j+1}.

In order to prove the necessity of these conditions it remains to show that the system $\pi_0, \pi_1, \ldots, \pi_n$ satisfies (II) as well. Denote by $(X^*)_{g(\mathbf{A})}$ the subset of $S(\mathbf{A})$ consisting of all $[p]$ $(p \in F(X))$ for which $\sigma([p])$ contains an $x' \in X'$. Since the set $\{\sigma([p]) : [p] \in (X^*)_{g(\mathbf{A})}\}$ obviously generates $S(\mathbf{B})$ thus $(X^*)_{g(\mathbf{A})}$ is a generating system of $S(\mathbf{A})$.

Take a j with $i-1 \leqq j \leqq n$, and two elements $a, a' \in A$ such that $a \equiv a'(\pi_j)$. Assume that $\eta(a)=(a_1, \ldots, a_n)$ and $\eta(a')=(a_1', \ldots, a_n')$. Then, by the definition of π_j, we have $a_1=a_1', \ldots, a_j=a_j'$. Now choose an arbitrary $x^* \in (X^*)_{g(\mathbf{A})}$, and let $x' \in X'$ such that $x' \in \sigma(x^*)$. Moreover, let $\varphi^*(\eta(a), x')=(x_1^*, \ldots, x_n^*)$ and $\varphi^*(\eta(a'), x')=(\bar{x}_1^*, \ldots, \bar{x}_n^*)$. Thus, by the definition of the α_i-product, $x_1^*=\bar{x}_1^*, \ldots, x_{j-i+1}^*=\bar{x}_{j-i+1}^*$ since $a_1= =a_1', \ldots, a_j=a_j'$. Therefore, for $\bar{\delta}(\eta(a), x')=(b_1, \ldots, b_n)$ and $\bar{\delta}(\eta(a'), x')=(b_1', \ldots, b_n')$ we have $b_1=b_1', \ldots, b_{j-i+1}=b_{j-i+1}'$, showing that

$$\delta^*(a, x^*) \equiv \delta^*(a', x^*)\,(\pi_{j-i+1}).$$

Conversely, assume that for an $\mathbf{A}=(X, A, \delta)$ and $(X^*)_{g(\mathbf{A})}$ there exists a regular system $\pi_0, \pi_1, \ldots, \pi_n$ of partitions satisfying conditions (I) and (II). We construct automata $\mathbf{A}_j=(X_j, A_j, \delta_j)$ $(j=1, \ldots, n)$ with $|A_j|=\pi_{j-1}/\pi_j (\leqq l)$ such that for a subautomaton $\mathbf{B}$ of an α_i-product of the $\mathbf{A}_j$ we have $\mathbf{A}^* \cong \mathbf{B}^*$.

Let A_j be arbitrary abstract sets with $|A_j|=\pi_{j-1}/\pi_j$. Moreover, $X_j=A_1 \times \ldots$ $\ldots \times A_{j+i-1} \times (X^*)_{g(\mathbf{A})}$ if $j+i-1 \leqq n$, and $X_j=A_1 \times \ldots \times A_n \times (X^*)_{g(\mathbf{A})}$ otherwise.

Now let $\varkappa_j$ be a mapping of $M_j=\{\pi_j(a)\colon a \in A\}$ onto A_j such that the restriction of $\varkappa_j$ to any $M_{j-1,a}$ is one-to-one. Define the transition function δ_j by the following rules:

(1) $j \leqq n-i+1$. Then $\delta_j(a_j,(b_1, \ldots, b_{j+i-1}, x^*))=\varkappa_j(\pi_j(\delta^*(a, x^*)))$ $(a_j \in A_j;$ $(b_1, \ldots, b_{j+i-1}) \in A_1 \times \ldots \times A_{j+i-1}$ and $x^* \in (X^*)_{g(\mathbf{A})})$ if $a_j=b_j$ and there exists an $a \in A$ such that $\varkappa_t(\pi_t(a))=b_t$ for all $t=1, \ldots, j+i-1$.

(2) $j>n-i+1$. Then $\delta_j(a_j,(b_1, \ldots, b_n, x^*))=\varkappa_j(\pi_j(\delta^*(a, x^*)))$ if $a_j=b_j$ and there exists an $a \in A$ with $\varkappa_t(\pi_t(a))=b_t$ $(t=1, \ldots, n)$.

(3) In all other cases δ_j is defined arbitrarily.

First we prove that δ_j is well defined. Assume that in case (1) there exists a $b \in A$ with $\varkappa_t(\pi_t(b))=b_t$ $(t=1, \ldots, j+i-1)$. It is enough to show that $b \equiv a$ (π_{j+i-1}) (since this, by condition (II), implies that $\delta^*(b, x^*) \equiv \delta^*(a, x^*)$ (π_j) for any $x^* \in (X^*)_{g(\mathbf{A})}$). We proceed by induction on t. $b \equiv a(\pi_1)$ obviously holds since $\varkappa_1$ is a 1—1 mapping of M_1 onto A_1. Assume that our statement has been proved for $t-1$ $(1 \leqq t-1<j+i-1)$, i.e., $b \equiv a(\pi_{t-1})$. Therefore, since $\varkappa_t$ is 1—1 on $M_{t-1,a}$ and $\varkappa_t(\pi_t(b))=\varkappa_t(\pi_t(a))$ thus $\pi_t(b)=\pi_t(a)$.

Case (2) can be proved by a similar argument. Note that π_n is induced by the equality relation on A. Therefore, in case (2) we get $a=b$.

Now let us form the following α_i-product $\mathbf{C}=((X^*)_{g(\mathbf{A})}, C, \delta_{\mathbf{C}})=\prod_{j=1}^{n} \mathbf{A}_j[(X^*)_{g(\mathbf{A})}, \varphi]$, where $\varphi=(\varphi_1, \ldots, \varphi_n)$ and for any $j=1, \ldots, n$, $(a_1, \ldots, a_n) \in A_1 \times \ldots \times A_n$ and $x^* \in (X^*)_{g(\mathbf{A})}$,

$$\varphi_j(a_1, \ldots, a_n, x^*) = \begin{cases} (a_1, \ldots, a_{j+i-1}, x^*) & \text{if } j \leqq n-i+1, \\ (a_1, \ldots, a_n, x^*) & \text{otherwise.} \end{cases}$$

It is clear that $\mathbf{C}$ is an α_i-product.

Define a mapping $\tau\colon A \to C$ in the following way:

$$\tau(a) = (\varkappa_1(\pi_1(a)), \ldots, \varkappa_n(\pi_n(a)))$$

for any $a \in A$. We prove that τ is an isomorphism of the automaton $((X^*)_{g(\mathbf{A})}, A, \delta^*)$ into $\mathbf{C}$. First we show, by induction, that τ is 1—1. Assume that $a \neq a'$ $(a, a' \in A)$. Let t be the greatest index for which $\pi_t(a)=\pi_t(a')$. $t<n$, since otherwise $a=a'$, contradicting our assumption. Then $\pi_{t+1}(a) \neq \pi_{t+1}(a')$. Therefore, $\varkappa_{t+1}(a) \neq \varkappa_{t+1}(a')$, since $\varkappa_{t+1}$ is one-to-one on $M_{t,a}$.

Now take an arbitrary input signal $x^* \in (X^*)_{g(\mathbf{A})}$. Then

$$\delta_{\mathbf{C}}(\tau(a), x^*) = (\delta_1(\varkappa_1(\pi_1(a)), (\varkappa_1(\pi_1(a)), \ldots, \varkappa_i(\pi_i(a)), x^*)), \ldots$$

$$\ldots, \delta_n(\varkappa_n(\pi_n(a)), (\varkappa_1(\pi_1(a)), \ldots, \varkappa_n(\pi_n(a)), x^*))) =$$

$$= (\varkappa_1(\pi_1(\delta^*(a, x^*))), \ldots, \varkappa_n(\pi_n(\delta^*(a, x^*)))) = \tau(\delta^*(a, x^*)),$$

showing that τ is an isomorphism of $((X^*)_{g(\mathbf{A})}, A, \delta^*)$ onto the subautomaton $\mathbf{B}=((X^*)_{g(\mathbf{A})}, B, \delta^*)$ of $\mathbf{C}$, where $B=\{\tau(a) | a \in A\}$. This obviously implies that τ defines an isomorphism of $\mathbf{A}^*$ onto $\mathbf{B}^*$, which completes the proof of Theorem 5.

Let us denote by $\mathbf{A}^{(2)}=(X^{(2)}, A^{(2)}, \delta^{(2)})$ the automaton for which $X^{(2)}= =\{x^{(1)}, x^{(2)}\}$, $A^{(2)}=\{a^{(1)}, a^{(2)}\}$, $\delta^{(2)}(a^{(1)}, x^{(1)})=\delta^{(2)}(a^{(2)}, x^{(2)})=a^{(2)}$ and $\delta^{(2)}(a^{(2)}, x^{(1)})= =\delta^{(2)}(a^{(1)}, x^{(2)})=a^{(1)}$.

Theorem 6. *Every automaton can be simulated isomorphically by a generalized α_2-power of $\mathbf{A}^{(2)}$.*

Proof. Let $\mathbf{A}=(X, A, \delta)$ be an arbitrary automaton. It is obvious that $\mathbf{T}_n= =(T_n, N, \delta_n)$ with $n\geqq\max\{3, |A|\}$ isomorphically simulates $\mathbf{A}$. Therefore, in order to prove Theorem 6, by Lemma 1, it is enough to show that $\mathbf{T}_n$ can be simulated isomorphically by an α_2-power of $\mathbf{A}^{(2)}$.

Take the following elements t_1, t_2 and t_3 of T_n

$t_1(i)=i+1$ if $i<n$, and $t_1(n)=1$;

$t_2(1)=2$, $t_2(2)=1$, and $t_2(i)=i$ if $i>2$;

$t_3(1)=t_3(2)=1$, and $t_3(i)=i$ if $i>2$.

It can be proved (cf. [7]) that $\{[t_1], [t_2], [t_3]\}=(T_n^*)_{g(\mathbf{T}_n)}$ generates $S(\mathbf{T}_n)$.

First we prove that $\mathbf{T}_n$ can be simulated isomorphically by a generalized α_2-product of two-state automata. By Theorem 5, it is enough to show that there exists a regular system $\pi_0, \pi_1, \ldots, \pi_k$ of partitions of N such that

(i) $\pi_j/\pi_{j+1}\leqq 2$ for all $j=0, \ldots, k-1$;

(ii) $b\equiv c(\pi_j)$ implies that $\delta_n^*(b, t^*)\equiv\delta_n^*(c, t_n^*)(\pi_{j-1})$ for all $b, c\in N$, $t^*\in\{[t_1], [t_2], [t_3]\}$ and $1\leqq j\leqq k$.

Let π_1 consist of the following two blocks: $\{1, \ldots, k\}$ and $\{k+1, \ldots, n\}$, where $k=u$ if $n=2u$, and $k=u+1$ if $n=2u+1$. Let us assume that the partitions π_t have been defined for all $t\leqq m\leqq k$, and that π_m has the following blocks: $\{1, \ldots, k-m+1\}$, $\{k-m+2\}, \ldots, \{k\}$, $\{k+1, \ldots, k+n-m+1\}$, $\{k+n-m+2\}, \ldots, \{n\}$. Then π_{m+1} is defined to be the partition having the blocks:

$$\{1, \ldots, k-m\}, \{k-m+1\}, \ldots, \{k\}, \{k+1, \ldots, k+n-m\}, \{k+n-m+1\}, \ldots, \{n\}.$$

It is obvious that the resulting system of partitions $\pi_0, \pi_1, \ldots, \pi_k$ is regular and satisfies (i). Moreover, (ii) obviously holds for π_1 and π_k. Now take an arbitrary m with $1\leqq m<k-1$, and let $b, c\in N$ such that $b\equiv c(\pi_{m+1})$. We may assume that $b\neq c$. Then either $1\leqq b, c\leqq k-m$ or $k+1\leqq b, c\leqq k+n-m$. In the first case for any $t^*\in\{[t_1], [t_2], [t_3]\}$, $1\leqq\delta_n^*(b, t^*)$, $\delta_n^*(c, t^*)\leqq k-m+1$, and in the second case $k+1\leqq \leqq\delta_n^*(b, t^*)$, $\delta_n^*(c, t^*)\leqq k+n-m+1$, showing that (ii) holds for any π_j $(1\leqq j\leqq k)$. Thus we have proved that $\mathbf{A}$ can be simulated isomorphically by a generalized α_2-product of two-state automata.

One can easily prove that every two-state automaton is isomorphic to an α_1-power of $\mathbf{A}^{(2)}$, having one factor only. Since an α_2-product of α_1-products with single factors is an α_2-product, thus $\mathbf{A}$ can be simulated isomorphically by a generalized α_2-power of $\mathbf{A}^{(2)}$.

Theorem 7. *A system* $\sum$ *of automata is homomorphically S-complete with respect to the generalized product if and only if there exist an* $\mathbf{A}=(X, A, \delta)\in\sum$, $a\in A$ *and* $p_1, p_2, q_1, q_2\in F(X)$ *such that* $ap_1\neq ap_2$ and $a=ap_1q_1=ap_2q_2$.

Proof. The necessity of these conditions can be proved in the same way as that of the corresponding statement for products in [9].

Conversely, assume that the conditions of Theorem 7 are satisfied by $\sum$. Set $a_1=ap_1$ and $a_2=ap_2$. Now form the following generalized α_1-product $\mathbf{B}=(X^{(2)}, A, \delta')=(\mathbf{A})[X^{(2)}, \varphi]$, where $\varphi(a_1, x^{(1)})=q_1p_2$, $\varphi(a_1, x^{(2)})=q_1p_1$, $\varphi(a_2, x^{(1)})=q_2p_1$ and $\varphi(a_2, x^{(2)})=q_2p_2$; moreover, $\varphi(a, x)$ is defined arbitrarily if $a\neq a_1, a_2$ $(a\in A, x\in X^{(2)})$. It is obvious that the mapping $\eta\colon a^{(j)}\to a_j$ $(j=1, 2)$ is an isomorphism of $\mathbf{A}^{(2)}$ into $\mathbf{B}$. Thus, by Theorem 6, we get that $\sum$ is *isomorphically* S-*complete* with respect to the generalized α_2-product. This ends the proof of Theorem 7.

The proof of the sufficiency of Theorem 7 yields the following

Corollary. *A system* $\sum$ *of automata is homomorphically S-complete with respect to the generalized product if and only if for any* $i=2, 3, \ldots$, $\sum$ *is isomorphically S-complete with respect to the generalized* α_i*-product.*

Now we are going to prove a stronger result. First we introduce the following notation, and prove a lemma.

Let us denote by $\mathbf{E}_{(2)}=(X^{(2)}, E_2, \delta^{(2)})$ the automaton for which $X^{(2)}=\{x, x_e\}$, $E_2=\{e_1, e_2\}$, $\delta^{(2)}(e_1, x_e)=e_1$, $\delta^{(2)}(e_2, x_e)=e_2$, and $\delta^{(2)}(e_i, x)=e_2$ for $i=1, 2$.

Lemma 3. *Let* $\mathbf{B}=(Y, B, \delta)$ *be an automaton such that there exists a well ordering* $\leqq$ *on* B *with the property that* $b\leqq bp$ *for any* $b\in B$ *and* $p\in F(Y)$. *Then* B *is isomorphic to a subautomaton of an* α_0*-power of* $\mathbf{E}_{(2)}$.

Proof. Assume that the conditions of Lemma 3 are satisfied. Moreover, let $B=\{b_1, \ldots, b_n\}$, and $b_i<b_j$ if $i<j$. Now define partitions π_t $(t=1, \ldots, n-1)$ on B in the following way: $b_u\equiv b_v(\pi_t)$ implies $b_u=b_v$ if $u\leqq t$ or $v\leqq t$, and $b_u\equiv b_v$ (π_t) for all $u, v>t$. It is obvious that all π_t have SP, $\pi_1>\pi_2>\ldots>\pi_{n-1}$ and $\pi_t/\pi_{t+1}=2$.

For any $t(=1, \ldots, n-1)$ take an abstract set $A_t=\{a_t^{(1)}, a_t^{(2)}\}$. Furthermore, define mappings $\varkappa_t$ of $M_t=\{\pi_t(b)|b\in B\}$ onto A_t such that $\varkappa_t(\{b_j\})=a_t^{(1)}$ if $j\leqq t$ and $\varkappa_t(\{b_{t+1}, \ldots, b_n\})=a_t^{(2)}$. Obviously, $\varkappa_t$ is 1—1 on $M_{t-1,b}$ for any $b\in B$. (π_0 is the trivial partition of B having one block only.)

Now let us define the automata $\mathbf{A}_t=(X_t, A_t, \delta_t)$ in the following way: $X_1=Y$, and $X_t=A_1\times\ldots\times A_{t-1}\times Y$ if $1<t<n$. Moreover, $\delta_1(a_1, y)=\varkappa_1(\pi_1(\delta(b, y)))$ $(a_1\in A_1, y\in Y)$, where $b\in\varkappa^{-1}(a_1)$, and

(i) $\delta_t(a_t, (a_1, \ldots, a_{t-1}, y))=\varkappa_t(\pi_t(\delta(b, y)))$ if there exists a $b\in B$ such that $\varkappa_j(\pi_j(b))=a_j$ $(j=1, \ldots, t)$;

(ii) $\delta_t(a_t, (a_1, \ldots, a_{t-1}, y))=a_t$ otherwise,

where $y\in Y$ and $(a_1, \ldots, a_t)\in A_1\times\ldots\times A_t$.

Now form the α_0-product $\mathbf{C}=(Y, C, \delta_{\mathbf{C}})=\prod_{t=1}^{n-1} \mathbf{A}_t[Y, \varphi]$ for which $\varphi_1(a_1, \ldots, a_{n-1}, y)=y$, and $\varphi_t(a_1, \ldots, a_{n-1}, y)=(a_1, \ldots, a_{t-1}, y)$ if $t>1$ $(y\in Y, a_j\in A_j, j=1, \ldots, n-1)$. One can prove in a way similar to that in the proof of the sufficiency of Theorem 5, that the mapping $\tau: b\to(\varkappa_1(\pi_1(b)),\ldots, \varkappa_{n-1}(\pi_{n-1}(b)))$ is an isomorphism of $\mathbf{B}$ into $\mathbf{C}$.

Now let us order the elements of A_t by $a_t^{(1)}<a_t^{(2)}$. We prove that for any $x_t\in X_t$, $\delta_t(a_t^{(i)}, x_t)=a_t^{(j)}$ $(1\leqq i, j\leqq 2)$ implies $a_t^{(i)}\leqq a_t^{(j)}$. Take an arbitrary $x_t=(a_1, \ldots, a_{t-1}, y)\in X_t$. If there exists no $b\in B$ with $\varkappa_s(\pi_s(b))=a_s$ $(s=1, \ldots, t-1)$ and $\varkappa_t(\pi_t(b))=a_t^{(i)}$ then, by (ii) in Lemma 3, $\delta_t(a_t^{(i)}, x_t)=a_t^{(i)}$. Now assume that for a $b_u\in B$, $\varkappa_s(\pi_s(b_u))=a_s$ $(s=1, \ldots, t;\ a_t=a_t^{(i)})$ and $\delta(b_u, y)=b_v$. Then $b_u\leqq b_v$. Therefore, by the definition of $\varkappa_t$ and the ordering on A_t, $\varkappa_t(\pi_t(b_u))=a_t^{(i)}\leqq a_t^{(j)}=\varkappa_t(\pi_t(b_v))$.

Finally, we show that $\mathbf{A}_t$ can be represented isomorphically by an α_0-power of $\mathbf{E}_{(2)}$ (having a single factor). Take the α_0-power $\mathbf{D}_t=(X_t, E_2, \delta_{\mathbf{D}})=(\mathbf{E}_{(2)})[X_t, \psi]$, where for any $e_i\in E_2$ and $x_t\in X_t$,

$$\psi(e_i, x_t) = \begin{cases} x & \text{if } \ \delta_t(a_t^{(1)}, x_t) = a_t^{(2)}, \\ x_e & \text{if } \ \delta_t(a_t^{(1)}, x_t) = a_t^{(1)}. \end{cases}$$

It can be shown, by a short computation, that the mapping $\eta: a_t^{(i)}\to e_i$ $(i=1, 2)$ is an isomorphism of $\mathbf{A}_t$ onto $\mathbf{D}_t$.

Since the formation of the α_0-product is associative, thus we proved that $\mathbf{B}$ can be represented isomorphically by an α_0-power of $\mathbf{E}_{(2)}$.

Now we prove

Theorem 8. *Let $\sum$ be an arbitrary set of automata. An automaton $\mathbf{B}$ can be simulated homomorphically by a generalized product of automata from $\sum$ if and only if $\mathbf{B}$ can be simulated isomorphically by a generalized α_2-product of automata from $\sum$.*

Proof. If there is an $\mathbf{A}\in\sum$ satisfying the conditions of Theorem 7 then, by the Corollary to Theorem 7, $\sum$ is isomorphically S-complete with respect to the generalized α_2-product. Therefore, in the sequel we may assume that none of the automata in $\sum$ satisfies the conditions of Theorem 7.

Let $\mathbf{B}=(Y, B, \delta)$ be an automaton which can be simulated homomorphically by a generalized product of automata from $\sum$. It can be shown that $\mathbf{B}$ does not satisfy the conditions of Theorem 7. Consequently, one can define a well ordering $\leqq$ on B such that for any $b, c\in B$ and $p\in F(Y)$, $bp=c$ implies $b\leqq c$. Now assume that there exist $b, c\in B$ and $p\in F(Y)$ with $bp=c$ and $b\neq c$. It is easy to prove that in this case there exist an $\mathbf{A}=(X, A, \delta')$ in $\sum$, $a_1, a_2\in A$, $p_1, p_2\in F(Y)$ such that $a_1p_1=a_2p_1=a_2p_2=a_2$, $a_1p_2=a_1$ and $a_1\neq a_2$.

By Lemma 3, $\mathbf{B}$ can be represented isomorphically by an α_0-power of $\mathbf{E}_{(2)}$. Since the formation of the generalized α_0-product is associative, thus it is enough to show that $\mathbf{E}_{(2)}$ can be represented isomorphically by a generalized α_0-power of $\mathbf{A}$. Take the α_0-power $\mathbf{D}=(X^{(2)}, A, \delta_{\mathbf{D}})=(A^*)[X^{(2)}, \psi]$, where for any $a\in A$, $\psi(a, x)=[p_1]$ and $\psi(a, x_e)=[p_2]$. Then $\tau: e_i\to a_i$ $(i=1, 2)$ defines an isomorphism of $\mathbf{E}_{(2)}$ into $\mathbf{D}$.

Now if for any $b\in B$ and $y\in Y$, $\delta(b, y)=b$ and B has at least two elements then there exists an $\mathbf{A}\in\sum$ such that $\mathbf{A}$ has at least two states. Then $\mathbf{B}$ can be represented isomorphically by a generalized α_0-power of $\mathbf{A}$. Finally, if $|B|=1$ then $\mathbf{B}$ can be represented isomorphically by a generalized α_0-power of any automata from $\sum$. This ends the proof of Theorem 8.

4. T-products and (T, α_i)-products $(i=0,1,\ldots)$

In [8] G. I. Ivanov introduced the concept of the temporal composition as an abstract equivalent of the single-channel representation of multichannel finite state machines (see [5]). Now we restrict the definition of the temporal composition to automata.

Let $\mathbf{A}_i=(X_i, A, \delta_i)$ $(i=1,2)$ be arbitrary automata having a common state set A. Take a set X with $|X|=|X_1\times X_2|$ and a 1—1 mapping γ of X onto $X_1\times X_2$. Then the automaton $\mathbf{A}=(X, A, \delta)$ is the *temporal product* of $\mathbf{A}_1$ by $\mathbf{A}_2$ with respect to X and γ if for any $a\in A$ and $x\in X$, $\delta(a, x)=\delta_2(\delta_1(a, x_1), x_2)$, where $(x_1, x_2)=\gamma(x)$.

The concept of the temporal product can be generalized in a natural way for arbitrary finite family of automata. It should be noted that the formation of the temporal product is associative.

We say that an automaton $\mathbf{A}$ is a *(T, α_i)-product* $(i=0, 1, \ldots)$ [*T-product*] of automata from $\sum$ if there exists a sequence of classes of automata, $\sum=\sum_0, \sum_1, \sum_2, \sum_3$ such that the automata in $\sum_1$ and $\sum_3$ can be given as temporal products of automata in $\sum_0$ and $\sum_2$, respectively, the automata in $\sum_2$ are isomorphic copies of subautomata of α_i-products [products] of automata from $\sum_1$, and $A\in\sum_3$.

Let us note that in the definition of $\sum_2$ it would be enough to confine ourselves to isomorphic copies of α_i-products [products] of automata in $\sum_1$. However, it would make our computations more difficult, without yielding any further results.

In the sequel we assume that if $\sum$ is a system of automata then for any $\mathbf{A}=(X, A, \delta)\in\sum$ there exists an $x\in X$ inducing the identity mapping of A, i.e., $\delta(a, x)=a$ for all $a\in A$.

We say that an automaton $\mathbf{A}$ *can be represented homomorphically by a T-product* [*(T, α_i)-product*] of automata from $\sum$ if $\mathbf{A}$ is a homomorphic image of a subautomaton of a T-product [(T, α_i)-product] of automata in $\sum$. The concept of the *isomorphic representation* is defined similarly. Moreover, $\sum$ is *homomorphically complete* with respect to the T-product [(T, α_i)-product] if every automaton can be represented homomorphically by a T-product [(T, α_i)-product] of automata from $\sum$. A natural modification of this definition leads to the concept of the *isomorphic completeness* with respect to the T-product [(T, α_i)-product].

The following results show the relation between simulations by generalized products and representations by T-products and (T, α_i)-products of automata. One can easily prove that if $\sum$ is a system of automata and $\mathbf{A} \in \sum$ then $\mathbf{A}^*$ can be represented isomorphically by a temporal power of $\mathbf{A}$. Thus we have

Theorem 9. *If $\sum$ is isomorphically (homomorphically) S-complete with respect to the generalized α_0-product then $\sum$ is isomorphically (homomorphically) complete with respect to the (T, α_0)-product.*

The converse of Theorem 9 fails to hold which will follow from Theorems 1 and 11.

Theorem 10. *Assume that a set $\sum$ of automata is homomorphically complete with respect to the (T, α_0)-product. Then there exist an $\mathbf{A}=(X, A, \delta) \in \sum$, $a, b \in A$ and a word $p \in F(X)$ such that $a \neq b$ and $ap=bp=b$.*

Proof. Let $\sum_1$, $\sum_2$ and $\sum_3$ denote the same classes of automata as in the definition of the (T, α_0)-product.

Assume that $\sum$ is homomorphically complete with respect to the (T, α_0)-product. Then there exists a $\mathbf{B}=(X, B, \delta)$ in $\sum_3$ such that $\mathbf{E}_{(2)}$ is a homomorphic image of a subautomaton of $\mathbf{B}$. (For the definition of $\mathbf{E}_{(2)}$, see p. 32.) One can prove that there exist $a, b \in B$, $x \in X$ and a positive integer k such that $a \neq b$ and $ap=bp=b$, where $p=x^k$.

Suppose that $\mathbf{B}$ is a temporal product of $\mathbf{B}_1, \ldots, \mathbf{B}_l$ with respect to X and γ such that $\mathbf{B}_i=(X_i, B, \delta_i)$ $(i=1, \ldots, l)$, $\mathbf{B}_i \in \sum_2$ and $\gamma(x)=(x_1, \ldots, x_l)(\in X_1 \times \ldots \times X_l)$. For any $t(=0, 1, \ldots)$ and $1 \leqq i < l$, let $a_{t \cdot l+i}$ and $b_{t \cdot l+i}$ denote the elements $a(x^t)_{\mathbf{B}}(x_1)_{\mathbf{B}_1} \cdots (x_i)_{\mathbf{B}_i}$ and $b(x^t)_{\mathbf{B}}(x_1)_{\mathbf{B}_1} \cdots (x_i)_{\mathbf{B}_i}$, respectively. Thus, $a=a_0$, $b=b_0=a_{k \cdot l}=b_{k \cdot l}$. Now assume that $u < k \cdot l$ is the greatest nonnegative integer for which $a_u \neq b_u$. There exists such a u, since $a_0 \neq b_0$. Let u be given in the form $u=m \cdot l+v$, where m and v are nonnegative integers and $v < l$. Therefore, $\delta_{v+1}(a_u, x_{v+1})=\delta_{v+1}(b_u, x_{v+1})$. This means that there are $c, d \in B$ and a positive integer n such that $c \neq d$ and $c(x_{v+1}^n)_{\mathbf{B}_{v+1}}=d(x_{v+1}^n)_{\mathbf{B}_{v+1}}=d$.

Thus we have got that there exist a $\mathbf{C}=(Y, C, \delta_C)$ in $\sum_2$, $c, d \in C$, $y \in Y$ and a positive integer k such that $c \neq d$ and $cy^k=dy^k=d$. Assume that $\mathbf{C}$ can be given by an α_0-product $\mathbf{C}=(\mathbf{C}_1 \times \mathbf{C}_2)[Y, \varphi]$, where $\mathbf{C}_i=(Y_i, C_i, \delta_i')$ $(i=1, 2)$. Let $c=(c_1, c_2)$ and $d=(d_1, d_2)$. For a $p=y_1 \ldots y_n \in F(Y)$ and $c' \in C_1$ let $p(\mathbf{C}_1)=\varphi_1(y_1) \ldots \varphi_1(y_n)$ and $p(\mathbf{C}_2, c')=y_1' \ldots y_n'$, where $y_1'=\varphi_2(c', y_1), \ldots, y_n'=\varphi_2(c'(y_1 \ldots y_{n-1})(\mathbf{C}_1), y_n)$. Then, for $q=y^k$, we obviously have $c_1 q(\mathbf{C}_1)=d_1$, $d_1 q(\mathbf{C}_1)=d_1$ and $c_2 q(\mathbf{C}_2, c_1)=d_2$, $d_2 q(\mathbf{C}_2, d_1)=d_2$. Now if $c_1 \neq d_1$ then there exists a word $q'=q(\mathbf{C}_1) \in F(Y_1)$ such that $c_1 q'=d_1 q'=d_1$.

Let us assume that $c_1=d_1$. Then $q(\mathbf{C}_2, c_1)=q(\mathbf{C}_2, d_1)$, and $c_2\neq d_2$ since $c\neq d$. Therefore, in this case for $q''=q(\mathbf{C}_2, c_1)\in F(Y_2)$ we have $c_2q''=d_2q''=d_2$.

Since $\mathbf{C}\in\sum_2$ and the formation of the α_0-product is associative, thus we have got that there exist an automaton $\mathbf{D}=(Z, D, \delta_{\mathbf{D}})$ in $\sum_1$, two states $d, d'\in D$ and a word $p\in F(Z)$ such that $d\neq d'$ and $dp=d'p=d'$. Assume that $p=z_1...z_n$ $(z_i\in Z)$. Let us denote by d_i and d_i' the states dp_i and $d'p_i$, respectively, where p_i is the prefix of p of length i, for all $0\leqq i<n$. Suppose that $j<n$ is the greatest nonnegative integer with $d_j\neq d_j'$. Since $d_0\neq d_0'$ thus there exists such a j. Therefore, $\delta_{\mathbf{D}}(d_j, z_{j+1})=$ $=\delta_{\mathbf{D}}(d_j', z_{j+1})$. Thus, there are states $a', b'\in D$ and a positive integer t such that $a'\neq b'$ and $a'z_{j+1}^t=b'z_{j+1}^t=b'$. Now, since $\mathbf{D}$ is a temporal product of automata from $\sum$ thus there exist an $\mathbf{A}=(X, A, \delta)\in\sum$, $a, b\in A$ and a word $p\in F(X)$ such that $a\neq b$ and $ap=bp=b$. (See the proof of the similar statement concerning $\mathbf{B}$.) This ends the proof of Theorem 10.

Take an automaton $\mathbf{A}=(X, A, \delta)$, a state $a\in A$ and an input signal $x\in X$. Then the *cycle* generated by (a, x) in $\mathbf{A}$ means the set of elements $ax^0, ax, ..., ax^k, ...$. For this cycle we use the short notation (a, x). If $ax^0, ..., ax^u$ are pairwise different and u is the least exponent for which there exists a $w>u$ such that $ax^w=ax^u$ then $ax^0, ...$ $..., ax^{u-1}$ is the *preperiod* of (a, x) and u is the *length of* this *preperiod.* (When the preperiod is empty its length equals 0.) Furthermore, if $u+v$ is the smallest positive integer for which $ax^u=ax^{u+v}$ holds then $ax^u, ..., ax^{u+v-1}$ is the *period* of the cycle under question, and v is the *length of* this *period.* In this case we say that (a, x) is a cycle of *type* (u, v).

An automaton $\mathbf{A}=(X, A, \delta)$ is called *x-cyclic* $(x\in X)$ of *type* (k, l) if for some $a\in A$, the set A coincides with the cycle (a, x) in $\mathbf{A}$, and this cycle is of type (k, l), while the input signals different from x induce the identity mapping of A. $\mathbf{A}$ is said to be a *prime-power automaton* with respect to x if it is x-cyclic of type $(0, r^n)$, where r is a prime and n is a natural number. If $n=1$ then $\mathbf{A}$ is a *prime automaton.* Moreover, $\mathbf{A}$ is an *elevator* regarding x if it is x-cyclic of type $(k, 1)$ with $k\geqq 1$.

For any natural number r, let $\mathbf{C}_{(r)}=(X, C_r, \delta_r)$ denote the following automaton: $X=\{x, x_e\}$, $C_r=\{c_0^{(r)}, ..., c_{r-1}^{(r)}\}$, $\delta_r(c_j^{(r)}, x_e)=c_j^{(r)}$ $(0\leqq j<r)$ and $\delta_r(c_j^{(r)}, x)=$ $=c_{(j+1)(\mathrm{mod}\, r)}^{(r)}$. Moreover, let $\mathbf{E}_{(t)}=(X, E_t, \delta^{(t)})$ be the elevator of type $(t, 1)$, where $X=\{x, x_e\}$, $E_t=\{e_1, ..., e_t\}$, $\delta^{(t)}(e_j, x_e)=e_j$ $(j=1, ..., t)$, $\delta^{(t)}(e_j, x)=e_{j+1}$ if $j<t$, and $\delta^{(t)}(e_t, x)=e_t$. Finally, let $\sum_P$ denote the system consisting of $\mathbf{E}_{(2)}$ and of $\mathbf{C}_{(r)}$ for all prime number r.

Now we prove

Lemma 4. *Let $\mathbf{A}=(X, A, \delta)$ be an automaton with two input signals such that one of them induces the identity mapping of A. Then $\mathbf{A}$ can be represented isomorphically by an α_0-product of automata from $\sum_P$.*

Proof. Let $\mathbf{A}=(X, A, \delta)$ be an arbitrary automaton with $X=\{x, x_e\}$ such that x_e induces the identity mapping of $\mathbf{A}$. Then $\mathbf{A}$ can be given as a union of pairwise disjoint subsets $A_1, \ldots, A_k$ such that $\mathbf{A}_i=(X, A_i, \delta_i)$ $(i=\mathrm{i}, \ldots, k)$ are connected subautomata of $\mathbf{A}$, where δ_i denotes the restriction of δ to A_i.

For an $a \in A$ we say that it is *initial* if (a, x) is of type (s, r) with $s>0$ and there exists no $b \in A$ and $p \in F(X)$ such that $b \neq a$ and $bp=a$. Assume that $\{a_{i1}, \ldots, a_{il_i}\}$ is the set of all the initial elements of A_i $(i=1, \ldots, k)$. For any a_{ij} take the cycle (a_{ij}, x) in $\mathbf{A}_i$. It is obvious that these cycles (a_{ij}, x) $(j=1, \ldots, l_i)$ have the same period, say of type $(0, t_i)$. Define a partition π_{i0} on A in the following manner:

(i) for $a, b \in A_i$, $a \equiv b$ (π_{i0}) if and only if there exists a $p \in F(X)$ with $|p|=u \cdot t_i$ such that $ap=bp$,

(ii) if $a, b \notin A_i$ then $a \equiv b$ (π_{i0}),

(iii) $a \equiv b$ (π_{i0}) implies $a, b \in A_i$ or $a, b \notin A_i$. One can show, by a short computation, that π_{i0} has SP.

Now for any initial state a_{ij}, let π_{ij} be the following partition of A: the elements in the preperiod of (a_{ij}, x) as well as the elements in all preperiods having common elements with the preperiod of (a_{ij}, x) form one-element blocks of π_{ij}, and all other elements of A are in the same block of π_{ij}. Again, a short computation shows that π_{ij} has SP. Moreover, the intersection $\bigcap(\pi_{ij} | i=1, \ldots, k;\, j=0, \ldots, l_i)$ is the trivial partition having one-element blocks only. Therefore, $\mathbf{A}$ can be given as a subdirect product of the quotient automata $\mathbf{A}/\pi_{ij}$ $(i=1, \ldots, k;\, j=0, \ldots, l_i)$.

Let us consider a quotient automaton $\mathbf{A}/\pi_{ij}$ with $j>0$. Then $\mathbf{A}/\pi_{ij}$ is either a one-state automaton or it satisfies Lemma 3. If $\mathbf{A}/\pi_{ij}$ has only one state then it can be represented isomorphically by an α_0-power (having a single factor) of any automaton in $\sum_P$. In the other case, by Lemma 3, $\mathbf{A}/\pi_{ij}$ can be represented isomorphically by an α_0-power of $\mathbf{E}_{(2)}$.

Now let us investigate the quotient automaton $\mathbf{A}/\pi_{i0}$. Obviously, $(\pi_{i0}(a_{ij}), x)$ forms a cycle in $\mathbf{A}/\pi_{i0}$ of type $(0, t_i)$. (Note that this cycle is independent of j.) We distinguish the following three cases:

(1) $t_i=k=1$. Then $\mathbf{A}/\pi_{i0}$ is a one-state automaton. Therefore, it can be represented isomorphically by an α_0-power of any automaton from $\sum_P$.

(2) $t_i>1$ and $k=1$. In this case $\mathbf{A}/\pi_{i0}$ is isomorphic to $\mathbf{C}_{(t)}$. Let t_i be given in the form $t_i=r_1^{w_1} \ldots r_n^{w_n}$, where r_j are pairwise different prime numbers and $w_j>0$ $(j=1, \ldots, n)$. Then $\mathbf{C}_{(t)}$ is isomorphic to the direct product of $\mathbf{C}_{(s_1)}, \ldots, \mathbf{C}_{(s_n)}$, where $s_j=r_j^{w_j}$ (see the proof of Theorem 1 in [4]).

Take $\mathbf{C}_{(s)}$ such that $s=r^l$, where r is a prime number and $l>0$. We prove that $\mathbf{C}_{(s)}$ can be represented isomorphically by an α_0-power of $\mathbf{C}_{(r)}$. Obviously, it is enough to show that whenever $l>1$ then there exists an α_0-product of $\mathbf{C}_{(r^{l-1})}$ and $\mathbf{C}_{(r)}$ which is isomorphic to $\mathbf{C}_{(r^l)}$. Form the α_0-product $\mathbf{C}=(\mathbf{C}_{(r^{l-1})} \times \mathbf{C}_{(r)})[X, \varphi]$, where for any $y \in X$ and $(c_u^{(r^{l-1})}, c_v^{(r)})$ from $C_{r^{l-1}} \times C_r$, $\varphi_1(c_u^{(r^{l-1})}, c_v^{(r)}, y) = y$ and

$$\varphi_2(c_u^{(r^{l-1})}, c_v^{(r)}, y) = \begin{cases} x & \text{if} \quad u = r^{l-1}-1 \quad \text{and} \quad y = x, \\ x_e & \text{otherwise.} \end{cases}$$

By the definition of φ, $(c_0^{(r^{l-1})}, c_0^{(r)})x^z = (c_z^{(r^{l-1})}, c_0^{(r)})$ if $z < r^{l-1}$, and

$$(c_0^{(r^{l-1})}, c_0^{(r)})x^z = (c_0^{(r^{l-1})}, c_1) \quad \text{if} \quad z = r^{l-1}.$$

From this it can be seen immediately, that $(c_0^{(r^{l-1})}, c_0^{(r)})x^z \neq (c_0^{(r^{l-1})}, c_0^{(r)})$ if $z < r^l$, and $(c_0^{(r^{l-1})}, c_0^{(r)})x^z = (c_0^{(r^{l-1})}, c_0^{(r)})$ provided that $z = r^l$. Moreover, x_e induces the identity mapping of the state set of **C**. Therefore, **C** is x-cyclic of type $(0, r^l)$, showing that **C** is isomorphic to $\mathbf{C}_{(s)}$. Since the formation of the α_0-product is associative, thus we got that $\mathbf{A}/\pi_{i0}$ can be represented isomorphically by an α_0-product of automata from Σ_p.

(3) $k > 1$. Now if $t_i = 1$ then $\mathbf{A}/\pi_{i0}$ has two states and both input signals induce the identity mapping of its state set. Therefore, $\mathbf{A}/\pi_{i0}$ can be represented isomorphically by an α_0-power (with a single factor) of arbitrary automata from $\sum_P$. Thus, we may assume that $t_i > 1$ too. Then $\mathbf{A}/\pi_{i0}$ is isomorphic to the following automaton $\mathbf{C} = (X, C, \delta_{\mathbf{C}})$: $C = \{c, c_0, \ldots, c_{t_i-1}\}$, $\delta_{\mathbf{C}}(c, x) = \delta_{\mathbf{C}}(c, x_e) = c$, $\delta_{\mathbf{C}}(c_j, x) = c_{(j+1)(\mathrm{mod}\, t_i)}$ and $\delta_{\mathbf{C}}(c_j, x_e) = c_j$ $(0 \leqq j < t_i)$. We now prove that **C** can be represented isomorphically by an α_0-product of $\mathbf{E}_{(2)}$ and $\mathbf{C}_{(t_i)}$. Take $\mathbf{D} = (X, D, \delta_D) = \mathbf{E}_{(2)} \times \mathbf{C}_{(t_i)})\ [X, \varphi]$, where for any $(e_u, c_v^{(t_i)}) \in D$ and $y \in X$, $\varphi_1(e_u, c_v^{(t_i)}, y) = x_e$ and

$$\varphi_2(e_u, c_v^{(t_i)}, y) = \begin{cases} y & \text{if} \quad u = 2, \\ x_e & \text{if} \quad u = 1. \end{cases}$$

Then the mapping $\eta: C \to D$ with $\eta(c) = (e_1, c_0^{(t_i)})$ and $\eta(c_j) = (e_2, c_j^{(t_i)})$ $(0 \leqq j < t_i)$ is an isomorphism of **C** into **D**. Moreover, by the proof of (2), $\mathbf{C}_{(t_i)}$ can be represented isomorphically by an α_0-product of automata from $\sum_P$. Thus, we got that $\mathbf{A}/\pi_{i0}$ can be represented isomorphically by an α_0-product of automata in $\sum_P$. This completes the proof of Lemma 4.

Now we are ready to prove

Theorem 11. *A system $\sum$ of automata is isomorphically complete with respect to the (T, α_0)-product if and only if there exist an $\mathbf{A} = (X', A, \delta) \in \sum$, $a, b \in A$ and a word $p \in F(X')$ such that $a \neq b$ and $ap = bp = b$.*

Proof. The necessity of these conditions follows from Theorem 10.

Conversely, assume that in $\sum$ there is an automaton satisfying the above conditions. Again, let $\sum_1$, $\sum_2$ and $\sum_3$ denote those classes of automata as in the definition of the (T, α_0)-product.

Now take an automaton $\mathbf{C} = (Z, C, \delta_{\mathbf{C}})$ such that $Z = \{z, z_e\}$ and for any $c \in C$, $\delta_{\mathbf{C}}(c, z_e) = c$. By Lemma 4, **C** can be represented isomorphically by an α_0-product

$\mathbf{D}=(Z, D, \delta_{\mathbf{D}})=\prod_{i=1}^{n} \mathbf{B}_i[Z, \varphi]$ of automata from $\sum_P$. For any $i \leqq n$, define two automata in the following way:

(i) Assume that $\mathbf{B}_i$ is a prime automaton $\mathbf{C}_{(r)}$. Then let

$$\mathbf{C}'_{(r)} = (X, C'_r, \delta'_r), \quad \text{where} \quad X = \{x, x_e\},$$

$$C'_r = \{c_0^{(r)\prime}, c_0^{(r)*}, \ldots, c_{r-1}^{(r)\prime}, c_{r-1}^{(r)*}\},$$

$$\delta'_r(c_i^{(r)\prime}, x_e) = c_i^{(r)\prime},$$

$$\delta'_r(c_i^{(r)*}, x) = \delta'_r(c_i^{(r)*}, x_e) = c_i^{(r)*} \quad \text{and} \quad \delta'_r(c_i^{(r)\prime}, x) = c_i^{(r)*} \qquad (0 \leqq i < r).$$

Moreover, let $\mathbf{C}''_{(r)}=(X, C'_r, \delta''_r)$ be the automaton for which

$$\delta''_r(c_i^{(r)\prime}, x_e) = \delta''_r(c_i^{(r)\prime}, x) = c_i^{(r)\prime},\ \delta''_r(c_i^{(r)*}, x_e)=c_i^{(r)*}, \text{ and } \delta''_r(c_i^{(r)*}, x)=c_{(i+1)\,(\mathrm{mod}\, r)}^{(r)\prime}.$$

(ii) If $\mathbf{B}_i$ is the elevator $\mathbf{E}_{(2)}$ then we define the following two automata: $\mathbf{E}'_2=(X, E'_2, \delta'_{(2)})$ and $\mathbf{E}''_2=(X, E'_2, \delta''_{(2)})$, where $X=\{x, x_e\}$, $E'_2=\{e'_1, e^*_1, e'_2\}$ and

$\delta'_{(2)}$	x	x_e
e'_1	e^*_1	e'_1
e^*_1	e^*_1	e^*_1
e'_2	e'_2	e'_2

$\delta''_{(2)}$	x	x_e
e'_1	e'_1	e'_1
e^*_1	e'_2	e^*_1
e'_2	e'_2	e'_2

Let us form the α_0-products

$$\mathbf{D}' = (Z, D', \delta'_{\mathbf{D}}) = \prod_{i=1}^{n} \mathbf{B}'_i[Z, \varphi'] \quad \text{and} \quad \mathbf{D}'' = (Z, D', \delta''_{\mathbf{D}}) = \prod_{i=1}^{n} \mathbf{B}''_i[Z, \varphi'']$$

such that for any $(b_1, \ldots, b_n) \in D$ and $z' \in Z$,

$$\varphi'(d_1, \ldots, d_n, z') = \varphi''(d_1, \ldots, d_n, z') = \varphi(b_1, \ldots, b_n, z'),$$

where $d_i=b'_i$ or b^*_i $(i=1, \ldots, n)$. Moreover, take the temporal product $\mathbf{G}=(Z\times Z, G, \delta_{\mathbf{G}})$ of $\mathbf{D}'$ by $\mathbf{D}''$ with respect to the identity mapping γ' on $Z\times Z$. One can show that the mappings $\varkappa'\colon Z\to Z\times Z$ and $\eta\colon D\to D'$ with $\varkappa'(z')=(z', z')$ and $\eta((b_1, \ldots, b_n))=(b'_1, \ldots, b'_n)$ $(z'\in Z, (b_1, \ldots, b_n)\in D)$ is an isomorphism of $\mathbf{D}$ into $\mathbf{G}$.

It is obvious that $\mathbf{E}_{(2)}$ can be represented isomorphically by a temporal power of the automaton $\mathbf{A}$ satisfying the conditions of Theorem 11. Moreover, the well ordering $c_0^{(r)\prime}<c_0^{(r)*}<\ldots<c_{r-1}^{(r)\prime}<c_{r-1}^{(r)*}$ of the state set of $\mathbf{C}'_{(r)}$, and the well ordering $c_0^{(r)*}<c_1^{(r)\prime}<\ldots<c_{r-1}^{(r)*}<c_0^{(r)\prime}$ of the state set of $\mathbf{C}''_{(r)}$ satisfy the conditions of Lemma 4. Therefore, $\mathbf{C}'_{(r)}$ and $\mathbf{C}''_{(r)}$ can be represented isomorphically by an α_0-power of $\mathbf{E}_{(2)}$. Similarly, the well ordering $e'_1<e^*_1<e'_2$ of the state sets of $\mathbf{E}'_2$ and $\mathbf{E}''_2$ show that $\mathbf{E}'_2$ and $\mathbf{E}''_2$ can be represented isomorphically by α_0-powers of $\mathbf{E}_{(2)}$. Since the formation of the α_0-product is associative, thus we got that $\mathbf{D}', \mathbf{D}''\in \sum_2$.

Now let $\mathbf{B}=(Y, B, \delta')$ be an arbitrary automaton, and for every $y \in Y$ take $Z_y = \{y, y_e\}$ and denote by $\mathbf{B}_y=(Z_y, B, \delta_y)$ the automaton whose transition function is defined by $\delta_y(b, y)=\delta'(b, y)$ and $\delta_y(b, y_e)=b$ for any $b \in B$.

For all $\mathbf{B}_y$ take an α_0-product $\mathbf{D}_y=(Z_y, D_y, \bar{\delta}_y)=\prod_{i=1}^{n_y} \mathbf{B}_i^{(y)}[Z_y, \varphi_y]$ of prime automata $\mathbf{C}_{(r)}$ and $\mathbf{E}_{(2)}$ such that $\psi_y\colon B \to D_y$ is an isomorphism of $\mathbf{B}_y$ into $\mathbf{D}_y$. Without loss of generality we may assume that $D_y=D_{y'}(=D)$ and $\psi_y(b)=\psi_{y'}(b)(=\psi(b))$ for any $y, y' \in Y$ and $b \in B$. Indeed, if $\mathbf{C}_{(r)}$ is a factor in some $\mathbf{D}_{y'}$ with multiplicity m' and m_r is the maximal number of occurrences of $\mathbf{C}_{(r)}$ in the α_0-products $\mathbf{D}_{y'}$ then $\mathbf{D}_{y'}$ can be replaced by a suitable α_0-product of $\mathbf{D}_{y'}$ by $\mathbf{C}_{(r)}^{m_r-m'}$. Similar statement is valid for $\mathbf{E}_{(2)}$. (Observe that x_e always induces the identity mappings of the state sets of $\mathbf{C}_{(r)}$ and $\mathbf{E}_{(2)}$.) The requirement $\psi_y(b)=\psi_{y'}(b)$ can be satisfied by a suitable renaming of the elements of the D_y.

Now for all $y \in Y$ construct the α_0-products $\mathbf{D}'_y=(Z_y, D'_y, \delta'_y)$ and $\mathbf{D}''= (Z_y, D'_y, \delta''_y)$ (as for $\mathbf{D}$ at the beginning of the proof). It is obvious, by the construction of $\mathbf{D}'_y$ and $\mathbf{D}''_y$, that $|D'_y|=|D'_{y'}|$ for any $y, y' \in Y$. Moreover, these automata $\mathbf{D}'_y$ and $\mathbf{D}''_y$ are in $\sum_2$, and $\mathbf{D}_y$ is isomorphic to a subautomaton of the temporal product $\mathbf{G}_y$ of $\mathbf{D}'_y$ by $\mathbf{D}''_y$, under some mappings $\varkappa_y\colon Z_y \to Z_y \times Z_y$ and $\eta_y\colon D_y \to D'_y$. Again, by a suitable renaming of the elements of D'_y, we can achive that $D'_y=D'_{y'}$ $(=D')$ and $\eta_y(d)=\eta_{y'}(d)(=\eta(d))$ for all $y, y' \in Y$ and $d \in D_y$.

Assume that $Y=\{y_1, \ldots, y_k\}$. Take the temporal product $\mathbf{F}=(\bar{Z}, D', \bar{\delta})$ of the automata $\mathbf{D}'_{y_1}, \mathbf{D}''_{y_1}, \ldots, \mathbf{D}'_{y_k}, \mathbf{D}''_{y_k}$ with respect to $\bar{Z}$ and γ, where $\bar{Z}=Z_{y_1}\times Z_{y_1}\times \ldots \times Z_{y_k}\times Z_{y_k}$ and γ is the identity mapping of $\bar{Z}$. Define a mapping $\varkappa\colon Y \to \bar{Z}$ with

$$\varkappa(y_i) = ((y_1)_e, (y_1)_e, \ldots, (y_{i-1})_e, (y_{i-1})_e, \varkappa_{y_i}(y_i), (y_{i+1})_e, (y_{i+1})_e, \ldots, (y_k)_e, (y_k)_e)$$

for all $y_i \in Y$. A short computation shows that the pair $\varkappa\colon Y \to \bar{Z}$ and $\psi\eta\colon B \to D'$ is an isomorphism of $\mathbf{B}$ into $\mathbf{F}$. Moreover, $\mathbf{F} \in \sum_3$, which ends the proof of Theorem 11.

Corollary. *A system $\sum$ of automata is homomorphically complete with respect to the (T, α_0)-product if and only if it is isomorphically complete with respect to the (T, α_0)-product.*

Now we are ready to present a stronger result. First we prove

Lemma 5. *Let $\mathbf{B}=(Y, B, \delta)$ be an automaton with $Y=\{y, y_e\}$ such that y_e induces the identity mapping of B. If for any $b \in B$, the cycle (b, y) in $\mathbf{B}$ is of type $(0, t)$, where $t=1$ or t is a power of r and r is a fixed prime number, then $\mathbf{B}$ can be represented isomorphically by an α_0-power of $\mathbf{C}_{(r)}$.*

Proof. Like in the proof of Lemma 4, B can be given as a union of pairwise disjoint subsets $B_1, \ldots, B_k$ such that $\mathbf{B}_i=(Y, B_i, \delta_i)$ $(i=1, \ldots, k)$ are connected subautomata of $\mathbf{B}$. By our assumption, $\mathbf{B}$ has no initial states. Therefore, every B_i

is a cycle of type $(0, t_i)$, where $t_i=1$ or r^l. For any $i(=1, \ldots, k)$ define the partitions $\pi_i(=\pi_{i0})$ as in Lemma 4.

Let us distinguish the following three cases:

(1) $t_i=k=1$. Then **B** is a one-state automaton. Obviously, it can be represented isomorphically by an α_0-power of $\mathbf{C}_{(r)}$ (having a single factor).

(2) $t_i=r^l$ and $k=1$. Then, by the proof of Lemma 4, **B** is an α_0-power of $\mathbf{C}_{(r)}$.

(3) $k>1$. If $t_i=1$ then $\mathbf{B}/\pi_i$ has two states and both input signals induce the identity mapping of its state set. Therefore, $\mathbf{B}/\pi_i$ is isomorphic to an α_0-power of $\mathbf{C}_{(r)}$ (having one factor only). Now if $t_i=r^l$ then $\mathbf{B}/\pi_i$ can be represented isomorphically by an α_0-power of $\mathbf{C}_{(r)}$ having $l+1$ factors. This can be proved in the same way as the corresponding statement in Lemma 4. The only difference is that here we need $\mathbf{C}_{(r)}$ instead of $\mathbf{E}_{(2)}$.

Since the intersection $\bigcap(\pi_i|i=1, \ldots, k)$ is the trivial partition of B having one-element blocks only, thus **B** can be represented isomorphically by an α_0-power of $\mathbf{C}_{(r)}$.

Theorem 12. *Let $\sum$ be a system of automata. An automaton* **B** *can be represented homomorphically by a (T, α_0)-product of automata from $\sum$ if and only if* **B** *can be represented isomorphically by a (T, α_0)-product of automata from $\sum$.*

Proof. Assume that $\mathbf{B}=(Y, B, \delta')$ can be represented homomorphically by a (T, α_0)-product of automata from $\sum$. If there are $b\in B$ and $y\in Y$ such that for the type (u, v) of the cycle (b, y) in **B** we have $u>0$ then, by the proof of the necessity of Theorem 10, there exist $\mathbf{A}=(X, A, \delta)\in\sum$, $a_1, a_2\in A$ and $p\in F(X)$ with $a_1\neq a_2$ and $a_1p=a_2p=a_2$. Therefore, by Theorem 11, $\sum$ is isomorphically complete with respect to the (T, α_0)-product.

Thus, we may assume that for all $b\in B$ and $y\in Y$ the cycles (b, y) in **B** are of type $(0, t)$. If $t=1$ for all cycles in **B** and $|B|>1$ then there exists an $\mathbf{A}\in\sum$ having at least two states. Obviously, **B** can be represented isomorphically by an α_0-power of **A**. Furthermore, it is also obvious that if $|B|=1$ then **B** can be represented isomorphically by an α_0-power of any automaton from $\sum$.

Now we can suppose that there exists at least one cycle (b, y) in **B** of type $(0, t)$ such that $t>1$. Moreover, it can also be assumed that $\sum$ is not homomorphically complete with respect to the (T, α_0)-product. Thus, there exist an $\mathbf{A}=(X', A, \delta)\in\sum$, $a\in A$ and $x'\in X'$ such that the cycle (a, x') is of type $(0, l)$ with $l>1$.

Let $Y=\{y_1, \ldots, y_s\}$, and denote by $\mathbf{B}_i=(Z_i, B, \delta_i)$ the automaton for which $Z_i=\{y_i, z_e\}$, $\delta_i(b, y_i)=\delta'(b, y_i)$ for all $b\in B$, and z_e induces the identity mapping of B. Every $\mathbf{B}_i$ can be given as a union of pairwise disjoint connected subautomata $\mathbf{B}_{ij}=(Z_i, B_{ij}, \delta_{ij})$ $(j=1, \ldots, m_i)$ such that each $\mathbf{B}_{ij}$ is y_i-cyclic of type $(0, t_{ij})$. Set $m=\max\{m_i|i=1, \ldots, s\}$ and $t=\max\{t_{ij}|i=1, \ldots, s;\ j=1, \ldots, m_i\}$. We show that there are automata $\mathbf{D}_i'=(Z_i, D_i, \delta_i')$ and $\mathbf{D}_i''=(Z_i, D_i, \delta'')$ $(i=1, \ldots, s)$ in $\sum_2$ such that $\mathbf{B}_i$ is isomorphic to a subautomaton of a temporal product of $\mathbf{D}_i'$ by $\mathbf{D}_i''$.

For the sake of simplicity, assume that $m_i = u$ and $t_{ij} = v_j$. Moreover, let

$$B_{ij} = \{c_0^{(j)}, \ldots, c_{v_j-1}^{(j)}\} \quad \text{and} \quad \delta_{ij}(c_v^{(j)}, y_i) = c_{(v+1)\,(\mathrm{mod}\, v_j)}^{(j)}.$$

Take a prime r with $r|l$, and let w be a power of r such that $w \geqq 2t$. For every k $(k=1, \ldots, m)$ define an automaton $\mathbf{C}_k = (Z_i, C_k, \bar{\delta}_k)$, where

$$C_k = \{d_0^{(k)}, \ldots, d_{w-1}^{(k)}\}, \quad \bar{\delta}_k(d_v^{(k)}, y_i) = d_{(v+1)\,(\mathrm{mod}\, w)}^{(k)}$$

and

$$\bar{\delta}_k(d_v^{(k)}, z_e') = d_v^{(k)} \quad \text{for all} \quad v = (0, \ldots, w-1).$$

Assume that these sets C_k are pairwise disjoint. Define D_i by $D_i = \bigcup (C_k | k=1, \ldots, m)$,

$$\delta_i'(d_v^{(k)}, z) = \bar{\delta}_k(d_v^{(k)}, z) \quad \text{for all} \quad z \in Z_i.$$

$\mathbf{D}_i''$ is defined similarly. It differs from $\mathbf{D}_i'$ only in that for all $j=1, \ldots, u$, if $w > 2v_j$ then

$$\delta_i''(d_{2v_j-1}^{(j)}, y_i) = d_0^{(j)}, \quad \delta_i''(d_0^{(j)}, y_i) = d_{2v_j}^{(j)}, \quad \delta_i''(d_v^{(j)}, y_i) = d_{v+1}^{(j)}$$

whenever $2v_j \leqq v < w-1$, and $\delta_i''(d_{w-1}^{(j)}) = d_1^{(j)}$. In all other cases the transitions are the same as in $\mathbf{D}_i'$. By Lemma 5, both $\mathbf{D}_i'$ and $\mathbf{D}_i''$ are in $\sum_2$, since $\mathbf{C}_{(r)}$ is isomorphic to a subautomaton of an α_0-power of $\mathbf{A}$.

Now take the temporal product $\mathbf{D}_i = (V_i, D_i, \delta_i^*)$ of $\mathbf{D}_i'$ by $\mathbf{D}_i''$ with respect to V_i and γ_i, where $V_i = Z_i \times Z_i$ and γ_i is the identity mapping of V_i. A routine computation shows that the pair of mappings $\varkappa_i$: $z \to (z, z)$ $(z \in Z_i)$ and ψ_i: $c_v^{(j)} \to d_{2v}^{(j)}$ is an isomorphism of $\mathbf{B}_i$ into $\mathbf{D}_i$.

Observe that the cardinality of D_i is independent of i $(i=1, \ldots, s)$. Therefore, by a suitable renaming of the elements of D_i we can achive that $D_1 = \ldots = D_s$ $(=D)$ and $\psi_i(b) = \psi_j(b)$ for all $i, j = 1, \ldots, s$. Using the same idea as in the proof of Theorem 11, one can show that $\mathbf{B}$ is isomorphic to a subautomaton of a temporal product of $\mathbf{D}_1', \mathbf{D}_1'', \ldots, \mathbf{D}_s', \mathbf{D}_s''$. This ends the proof of Theorem 12.

We say that an automaton $\mathbf{A} = (X, A, \delta)$ is *completely isolated* if $\delta(a, x) = a$ for any $a \in A$ and $x \in X$.

Theorem 13. *A set $\sum$ of automata is homomorphically complete with respect to the T-product or (T, α_i)-product $(i=1, 2, \ldots)$ if and only if there is an automaton in $\sum$ which is not completely isolated.*

Proof. Since the products and temporal products of completely isolated automata are completely isolated thus the conditions of Theorem 13 are obviously necessary.

Conversely, assume that there exists an $\mathbf{A} = (X, A, \delta)$ in $\sum$ which is not completely isolated. Then the following two cases can occur:

(i) There are $a, b \in A$ and $p \in F(X)$ such that $a \neq b$ and $ap = bp = b$. Then, by Theorem 11, $\sum$ is isomorphically complete with respect to the (T, α_0)-product. Therefore, it is isomorphically complete with respect to the T-product or any (T, α_i)-product $(i=0, 1, \ldots)$.

(ii) There are $p\in F(X)$, $x\in X$ and $a_0, \ldots, a_{t-1}$ $(t>1)$ such that $a_j\neq a_k$ if $j\neq k$ $(0\leqq j, k<t)$, $a_j p=a_{(j+1)(\mathrm{mod}\, t)}$ and $\delta(a_j, x)=a_j$. Then the cyclic automaton $\mathbf{C}_{(t)}$ of type $(0, t)$ can be represented isomorphically by a temporal power of $\mathbf{A}$. Furthermore, it is obvious that the elevator $\mathbf{E}_{(2)}$ can be represented isomorphically by an α_1-power of $\mathbf{C}_{(t)}$. Therefore, since the α_0-product of α_1-products is an α_1-product, thus, by Theorem 11, we get that $\sum$ is isomorphically complete with respect to the (T, α_1)-product. This completes the proof of Theorem 13.

From the proof of Theorem 13 we get the following

Corollary. *A set $\sum$ of automata is homomorphically complete with respect to the T-product or (T, α_i)-products $(i>0)$ if and only if it is isomorphically complete with respect to the T-product or (T, α_i)-products with $i>0$.*

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On an asymptotic expansion for the von Mises ω^2 statistic

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In honour of Professor Károly Tandori on his 50th birthday

§ 1. Introduction. There are two classical types of statistics for testing the "goodness-of-fit" hypothesis that the distribution function of a statistical population coincides with the fully determined continuous distribution function $F(x)$. The defining statistics for one of them are those of Kolmogorov's $\sup |F_n(x)-F(x)|$ and Smirnov's $\sup (F_n(x)-F(x))$, while for the other $\omega_n^2=n\int\limits_{-\infty}^{\infty}(F_n(x)-F(x))^2 dF(x)$ of von Mises, where $F_n(x)$ denotes the empirical distribution function based on a random sample of size n. Considering the latter one, CRAMÉR [4] was the first in 1928, who proposed a statistic similar to ω_n^2, while ω_n^2 itself was proposed by von MISES [20] in 1931. He proved that for any complex λ

(1)
$$\lim_{n\to\infty} \mathbf{E} e^{-\lambda\omega_n^2} = \prod_{k=1}^{\infty}\left(1+\frac{2\lambda}{k^2\pi^2}\right)^{-1/2} = \left(\frac{\sqrt{2\lambda}}{\sinh\sqrt{2\lambda}}\right)^{1/2},$$

provided the null hypothesis holds true, and this we will assume throughout. The limiting Laplace—Stieltjes transform was first inverted by SMIRNOV [32] in 1937, who, in this way, proved the following limit distribution theorem

(2)
$$\lim_{n\to\infty} \mathbf{P}\{\omega_n^2 < x\} = \lim_{n\to\infty} V_n(x) = V(x),$$

where

$$V(x) = 1-\frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\int\limits_{(2k-1)\pi}^{2k\pi}(-u\sin u)^{-1/2}e^{-u^2x/2}du.$$

Another expression for $V(x)$, due to ANDERSON and DARLING [1] dates back to 1952:

$$V(x) = \frac{1}{\pi\sqrt{x}}\sum_{k=0}^{\infty}(-1)^k\binom{-1/2}{k}\sqrt{4k+1}\,e^{-\frac{(4k+1)^2}{16x}}B_{1/4}\left(\frac{(4k+1)^2}{16x}\right),$$

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where $B_{1/4}(y)$ is a standard Bessel function. As ω_n^2 (just like the Kolmogorov—Smirnov statistics) is distribution-free, it will not be a loss of generality to assume that the underlying population is uniformly distributed on the interval [0, 1] $(F(x)=x$ for $x\in[0, 1])$, when investigating the asymptotic behavior of its distribution. Then, introducing the empirical process $Y_n(t)=\sqrt{n}\,(F_n(t)-t)$, $0\leqq t\leqq 1$, we have $\omega_n^2=\int_0^1 Y_n^2(t)\,dt$. $Y_n(t)$ is a random element of Skorohod's space of functions on [0, 1], $D[0, 1]$, having discontinuities only of the first kind. It is known that Y_n converges weakly to the Brownian Bridge $B(t)$, a Gaussian process on [0, 1] with expectation 0 and covariance function $s(1-t)$ for $0\leqq s\leqq t\leqq 1$ (see [12], [13], [3] or [6] in these *Acta*). Introducing $\omega^2=\int_0^1 B^2(t)\,dt$, this latter result of DOOB and DONSKER immediately gives

$$V(x) = \mathbf{P}\{\omega^2 < x\}, \quad \mathbf{E}e^{-\lambda\omega^2} = \left(\frac{\sqrt{2\lambda}}{\sinh\sqrt{2\lambda}}\right)^{1/2},$$

since the square-integral is a continuous functional in the topology of $D[0, 1]$.

In the Kolmogorov—Smirnov case not only the exact rates of convergence $\left[O\left(\frac{1}{\sqrt{n}}\right)\right]$ for the appropriate limit distribution relations are known, but asymptotic expansions are also available from the first half of the 50's (see [14]). At the same time, such a complete set of results seemed to be far in the von Mises case; this was also emphasized by DURBIN and BICKEL in their recent survey papers [11] and [2]. This kind of asymptotic behavior of $V_n(x)=\mathbf{P}\{\omega_n^2<x\}$ is all the more important since, next to nothing is known about the exact distribution of ω_n^2. With the exception of the exact formulae for $n=1, 2, 3$ in [19] only an extreme lower tail of the distribution ([33]), the exponential decrease of the upper tail ([26]) and the first four moments of ω_n^2 ([24]) are known for any further n.

Put $\Delta_n=\sup_{-\infty<x<\infty}|V_n(x)-V(x)|$. The first estimate was given by KANDELAKI [15] in 1965, namely that $\Delta_n\leqq C(\log n)^{-1/4}$, with some absolute constant C. It was expected that Δ_n should be estimable the following way: for any $\varepsilon>0$ there should exist a constant $b(\varepsilon)$ such that for each n

$$\Delta_n \leqq b(\varepsilon)\frac{n^{\varepsilon}}{n^{a}}, \tag{3}$$

with some $a>0$. SAZONOV first proved (3) with $a=\frac{1}{10}$ ([28]) and then with $a=\frac{1}{6}$ ([29]). Using a Skorohod embedding (see [31]) ROSENKRANTZ [27] concluded in $\Delta_n=O((\log n)^{3/2}n^{-1/5})$, which is, of course, better than (3) with $a=\frac{1}{5}$. Next, by the same embedding KIEFER [17] proved $\Delta_n=O((\log n)^{3/2}n^{-1/4})$ and, independently, NIKITIN [21] announced $\Delta_n=O((\log n)^{5/4}n^{-1/4})$. Kiefer also proved (see also SAWYER [30]) that the Skorohod embedding cannot give more than $n^{1/4}$ in the denominator. Later ORLOV [22] increased a in (3) to $\frac{1}{3}$. Finally, in a new long paper [23] ORLOV proved that (3) does not hold with $a>1$ and holds with $a=\frac{1}{2}$.

In § 2 of the present paper a refinement of Orlov's estimate is given which turns out to be the best rate that can be achieved by all the previously existing methods. In § 3 a complete asymptotic expansion for the Laplace transform of ω_n^2 is given. (In this connection we have to mention an early result of DARLING [9], which he announced, without proof, in 1960. This is a one-term expansion for $\mathbf{E}e^{-\lambda\omega_n^2}$, but only for real positive λ, and so there is no hope to invert it). The first outline of its proof (without the estimation of the dependence on λ of the remainder term) was published in [7] and its details in [8]. The treatment of dependence on λ is new here. In § 4 the problem of inversion of this expansion is treated, without reaching the final answer. A few, not completely rigorous throughts on this inversion were also included in [8]. § 5 tries to motivate our conjecture concerning the final form the asymptotic expansion for $V_n(x)$ and the exact rate of convergence of Δ_n.

§ 2. A rate of convergence. The following theorem is true.

Theorem 1. $\Delta_n = \sup_{-\infty<x<\infty} |V_n(x)-V(x)| = O\left(\frac{\log n}{\sqrt{n}}\right)$.

The proof is entirely based on one of the recent and very important results of Komlós, Major and Tusnády.

Theorem A. (KOMLÓS, MAJOR and TUSNÁDI [18]) *For each n there exists an empirical distribution function $\tilde{F}_n(t)$ of independent, uniformly distributed random variables on* [0, 1] *and a Brownian Bridge $B_n(t)$ such that for the empirical process $\tilde{Y}_n(t)=\sqrt{n}\,(\tilde{F}_n(t)-t)$ we have, for each x*

$$\mathbf{P}\left\{\sup_{0\le t\le 1}|\tilde{Y}_n(t)-B_n(t)| > \frac{A\log n+x}{\sqrt{n}}\right\} < Be^{-Cx},$$

where A, B and C are positive absolute constants. Putting $x=K\log n$ so that $KC>1$ and using the Borel—Cantelli lemma one gets

$$\sup_{0\le t\le 1}|\tilde{Y}_n(t)-B_n(t)| = O\left(\frac{\log n}{\sqrt{n}}\right),$$

with probability 1.

Proof of theorem 1. Suppose that the random variables

$$\tilde{\omega}_n^2 = \int_0^1 \tilde{Y}_n^2(t)\,dt \quad \text{and} \quad \tilde{\omega}^2 = \int_0^1 B_n^2(t)\,dt$$

are built on the processes $\tilde{Y}_n(t)$ and $B_n(t)$ of Theorem A. Naturally, their distribution functions are $V_n(x)$ and $V(x)$, respectively. Then, by Theorem A, there exists with probability 1 a constant K such that

$$|\tilde{\omega}_n^2-\tilde{\omega}^2| = \left|\int_0^1 (\tilde{Y}_n(t)-B_n(t))(\tilde{Y}_n(t)+B_n(t))\,dt\right| \leqq K\frac{\log n}{\sqrt{n}} \int_0^1 |\tilde{Y}_n(t)+B_n(t)|\,dt \leqq$$

$$\leqq K\frac{\log n}{\sqrt{n}}\Big(\int_0^1 |\tilde{Y}_n(t)-B_n(t)|\,dt+2\int_0^1 |B_n(t)|\,dt\Big) \leqq K^2\frac{\log^2 n}{n}+2K\frac{\log n}{\sqrt{n}}\,\tilde{\omega},$$

where the last inequality follows from that of Buniakovsky—Schwarz. That is we have

$$\mathbf{P}\left\{|\tilde{\omega}_n^2-\tilde{\omega}^2| > \frac{1}{4}\varepsilon_n^2+\varepsilon_n\tilde{\omega}\right\} = 0, \tag{4}$$

where $\varepsilon_n = 2K\dfrac{\log n}{\sqrt{n}}$. Solving the corresponding quadratic inequalities for the sets $A_n=\left\{\tilde{\omega}^2<x-\frac{1}{4}\varepsilon_n^2-\varepsilon_n\tilde{\omega}\right\}$ and $B_n=\left\{\tilde{\omega}^2<x+\frac{1}{4}\varepsilon_n^2+\varepsilon_n\tilde{\omega}\right\}$ we find that

$$A_n = \left\{\tilde{\omega}^2 < x+\frac{1}{4}\varepsilon_n^2-\sqrt{\varepsilon_n^2 x}\right\} \quad \text{and} \quad B_n = \left\{\tilde{\omega}^2 < x+\frac{3}{4}\varepsilon_n^2+\left(\frac{1}{2}\varepsilon_n^4+\varepsilon_n^2 x\right)^{1/2}\right\}.$$

Consequently, from (4) one gets

$$V\left(x+\frac{1}{4}\varepsilon_n^2-\sqrt{\varepsilon_n^2 x}\right) \leqq V_n(x) \leqq V\left(x+\frac{3}{4}\varepsilon_n^2+\left(\frac{1}{2}\varepsilon_n^4+\varepsilon_n^2 x\right)^{1/2}\right),$$

and, a fortiori,

$$V\left(x-\frac{3}{4}\varepsilon_n^2-\left(\frac{1}{2}\varepsilon_n^4+\varepsilon_n^2 x\right)^{1/2}\right) \leqq V_n(x) \leqq V\left(x+\frac{3}{4}\varepsilon_n^2+\left(\frac{1}{2}\varepsilon_n^4+\varepsilon_n^2 x\right)^{1/2}\right).$$

This, with some constants A, B, C and D, implies

$$|V_n(x)-V(x)| \leqq \mathbf{P}\left\{x-\frac{3}{4}\varepsilon_n^2-\varepsilon_n\left(\frac{1}{2}\varepsilon_n^2+x\right)^{1/2} \leqq \omega^2 \leqq x+\frac{3}{4}\varepsilon_n^2+\varepsilon_n\left(\frac{1}{2}\varepsilon_n^2+x\right)^{1/2}\right\} \leqq$$

$$\leqq v(x)\left(A\frac{\log^2 n}{n}+B\frac{\log n}{\sqrt{n}}\left(C\frac{\log^2 n}{n}+x\right)^{1/2}\right) \leqq A\frac{\log^2 n}{n}v(x)+D\frac{\log n}{\sqrt{n}}v(x)\sqrt{x},$$

where $v(x)=\dfrac{d}{dx}V(x)$ is the density function of ω^2. Later (Lemma 8 in § 5) we will see, that $v(x)$ as well as $\sqrt{x}\, v(x)$ are bounded functions on the whole line, and thus the theorem is proved.

It is worthwhile here to remark that all the previous methods for getting a rate of convergence for Δ_n (just to mention the characteristic ones of ROSENKRANTZ [27], ORLOV [23] and the proof of the above Theorem 1) are based on some kind of approximation of the empirical process. From the nearness of the latter approximation then resulted a nearness of $V_n(x)$ and $V(x)$. Of course, the applied method in the proof of Theorem 1, i.e. the use of the $O\left(\frac{\log n}{\sqrt{n}}\right)$ approximation of Komlós, Major and Tusnády cannot give a better rate for Δ_n than $O\left(\frac{\log n}{\sqrt{n}}\right)$. But at the same time the Brownian Bridge of Komlós, Major and Tusnády is the best approximation for the empirical process (see also in a forthcoming book [5] of M. CSÖRGŐ and P. RÉVÉSZ). Therefore the following conclusion is true: one cannot get a better rate of convergence for Δ_n than $O\left(\frac{\log n}{\sqrt{n}}\right)$ of Theorem 1 via first approximating the empirical process.

We remark also that our rate was thought to be desirable (if not the best) by ROSENKRANTZ [27] and later by KIEFER [17] and BICKEL [2]. On the grounds of the following two paragraphs, however, one can even expect more, namely, that Δ_n has the order of $\frac{1}{n}$.

§ 3. Asymptotic expansion for the Laplace transform. In this section we prove:

Theorem 2. *For any complex λ, with $\operatorname{Re}\lambda \geqq 0$, natural s and real ε with $\varepsilon > 0$,*

$$\mathbf{E}e^{-\lambda\omega_n^2} - \mathbf{E}e^{-\lambda\omega^2} = \sum_{k=1}^{[s/2]} \left(\frac{1}{n}\right)^k a_k(\lambda) + h_s(\lambda)\, O(n^{\varepsilon-(s+1)/2}),$$

where

$$a_k(\lambda) = \sum_{(i_1, \dots, i_{2k})}' b_{i_1, \dots, i_{2k}} \lambda^{k+H_{2k}} \mathbf{E} \exp\left\{-\lambda \int_0^1 \alpha^2(t)\, dt\right\} \Pi_{i_1, \dots, i_{2k}}.$$

Here $H_n = \sum_{j=1}^{n} i_j$, $\alpha(t) = W(t) - \int_0^1 W(x)\,dx$ and $W(t)$ is the standard Brownian Motion. Summation $\sum'$ is taken over all non-negative integer solutions $(i_1, \dots, i_{2k})$ of the equation $i_1 + 2i_2 + \dots + 2ki_{2k} = 2k$. Further,

$$b_{i_1, \dots, i_{2k}} = \frac{1}{i_1! \dots i_{2k}} (-2)^{k+H_{2k}}$$

and

$$\Pi_{i_1, \dots, i_{2k}} = \prod_{m=1}^{2k} \left\{ \sum_{l=1}^{[(m+2)/2]} (-1)^{l-1}(l-1)! \sum_{(j_2, \dots, j_s)}'' d_{j_2, \dots, j_s} \prod_{r=2}^{s} \left(\int_0^1 \alpha^r(t)\, dt \right)^{j_r} \right\}^{i_m},$$

where

$$d_{j_2,\ldots,j_s} = 1/(j_2!\ldots j_s!(2!)^{j_2}\ldots(s!)^{j_s})$$

and summation $\sum''$ *is taken over all non-negative integer solutions* $(j_2, \ldots, j_s)$ *of the equations* $j_2+\ldots+j_s=l$ *and* $2j_2+\ldots+sj_s=m+2$. $O(n^{-(s+1)/2+\varepsilon})$ *does not depend on* λ *any more, and for the function* $h_s(\lambda)$ *the following estimate is valid:*

$$|h_s(\lambda)| \leqq |\lambda|^{(s+2)(s+4)/2} \text{ if } |\lambda| > 1, \text{ and } \quad |h_s(\lambda)| \leqq |\lambda|^{1/2} \text{ if } |\lambda| \leqq 1.$$

Proof. Let the standard Wiener process $W(t)$ be independent of the empirical process $Y_n(t)$ (which is based on uniform [0,1] random variables $U_1, \ldots, U_n$) for each n, and let $g_n(x)$ be a (nonrandom) sample function of $Y_n(t)$. The random variable $\int_0^1 g_n(x)\,dW(x)$ is normal with mean 0 and variance $\int_0^1 g_n^2(x)\,dx$ (see e.g. [31]). Therefore

$$\mathbf{E}\exp\left\{\sqrt{-2\lambda}\int_0^1 g_n(x)\,dW(x)\right\} = \exp\left\{-\lambda\int_0^1 g_n^2(x)\,dx\right\},$$

whence

$$\mathbf{E}\exp\{-\lambda\omega_n^2\} = \mathbf{E}\exp\left\{-\lambda\int_0^1 Y_n^2(x)\,dx\right\} = \mathbf{E}\exp\left\{\sqrt{-2\lambda}\int_0^1 Y_n(x)\,dW(x)\right\}.$$

If $g(x)$ is a continuous function on [0,1], then

$$\mathbf{E}\exp\left\{\sqrt{-2\lambda}\int_0^1 Y_n(x)\,dg(x)\right\} = \mathbf{E}\exp\left\{-\sqrt{-2\lambda}\int_0^1 g(x)\,dY_n(x)\right\} =$$

$$= \mathbf{E}\exp\left\{-\sqrt{-2\lambda/n}\sum_{k=1}^n g(U_k) + \sqrt{-2\lambda}\sqrt{n}\int_0^1 g(x)\,dx\right\} =$$

$$= \left\{\int_0^1 \exp\left(-\sqrt{-2\lambda/n}\left[g(t)-\int_0^1 g(x)\,dx\right]\right)dt\right\}^n.$$

Hence

$$\mathbf{E}\exp\{-\lambda\omega_n^2\} = \mathbf{E}\{1+\theta_n(\lambda)\}^n,$$

where

$$\theta_n(\lambda) = \int_0^1 \left(\exp\{-\sqrt{-2\lambda/n}\,\alpha(t)\}-1\right)dt.$$

Let a real ε be given with $0<\varepsilon<\dfrac{1}{6(s+1)}$, where s is an arbitrary natural number. If for a set B the indicator of B is denoted by χ_B then we have

$$\mathbf{E}\exp\{-\lambda\omega_n^2\} = \mathbf{E}\exp\{-\lambda\omega_n^2\}(1-\chi_{A_n^\varepsilon}) + \mathbf{E}\exp\{-\lambda\omega_n^2\}\chi_{A_n^\varepsilon} \tag{5}$$

where

$$A_n^\varepsilon = \{\sup_{0\leqq t\leqq 1}|W(t)| \leqq n^\varepsilon\}.$$

For the first teım in (5), using the Buniakovsky—Schwarz inequality and then an estimate (see e.g. in [10]) for the tail probability of a Brownian Motion, one gets

(6) $$\left|\mathbf{E}\exp\{-\lambda\omega_n^2\}(1-\chi_{A_n^\varepsilon})\right| \leqq (\mathbf{P}\{\bar{A}_n^\varepsilon\})^{1/2} \leqq \frac{\sqrt[4]{2/\pi}}{n^{\varepsilon/2}}\exp\{-n^{2\varepsilon}/4\}.$$

As, trivially,

$$\mathbf{P}\{|\theta_n(\lambda)| \geqq 1\} \leqq \mathbf{P}\left\{\exp\left(2|\sqrt{-2\lambda}|\frac{\sup|W(t)|}{\sqrt{n}}\right) \geqq 2\right\},$$

on the set A_n^ε we have

$$|\theta_n(\lambda)| \leqq K\sqrt{|\lambda|}\, n^{\varepsilon-1/2} < 1$$

with some constant K not depending on λ if n is large enough. For the same n (which we will take in the sequel as large as needed without any further mention of it) therefore

$$\left|\sum_{m=s+1}^{\infty}\frac{(-1)^{m+1}}{m}\theta_n^m(\lambda)\right| < K^{s+1}|\lambda|^{(s+1)/2}n^{(\varepsilon-1/2)(s+1)}.$$

It follows then

(7) $$\mathbf{E}e^{-\lambda\omega_n^2}\chi_{A_n^\varepsilon} = \mathbf{E}\exp\{n\log(1+\theta_n(\lambda))\}\chi_{A_n^\varepsilon} =$$

$$= \mathbf{E}\exp\left\{n\sum_{m=1}^{s}\frac{(-1)^{m+1}}{m}\theta_n^m(\lambda)\right\}\chi_{A_n^\varepsilon}[1+h_{s-2}^1(\lambda)O(n^{(\varepsilon-1/2)(s+1)}n)] =$$

$$= \mathbf{E}\exp\left\{n\sum_{m=1}^{s}\frac{(-1)^{m+1}}{m}\theta_n^m(\lambda)\right\}\chi_{A_n^\varepsilon}+h_{s-2}^1(\lambda)O(n^{(\varepsilon-1/2)(s-1)+2\varepsilon}),$$

where the function $h_{s-2}^1(\lambda)$ is such that

$$|h_{s-2}^1(\lambda)| \leqq |\lambda|^{(s+1)/2}.$$

At the last equality it was taken into account that the first term of the last row tends to $\mathbf{E}e^{-\lambda\omega^2}$ as $n\to\infty$, the absolute value of which is less than 1 by $\operatorname{Re}\lambda\geqq 0$. Now we compute the powers of $\theta_n(\lambda)$. Putting $\beta(t)=-\sqrt{-2\lambda}\,\alpha(t)$ (which is $-\sqrt{-2\lambda}\,O(n^\varepsilon)$ on A_n^ε) and using the MacLaurin formula and the fact that $\int_0^1\alpha(t)\,dt=0$ we find on A_n^ε that

(8) $$\theta_n(\lambda) = \sum_{j=2}^{s}\left(\frac{1}{\sqrt{n}}\right)^j\int_0^1\frac{\beta^j(t)}{j!}\,dt+h_{s-2}^2(\lambda)O(n^{(\varepsilon-1/2)(s+1)}),$$

where

$$|h_{s-2}^2(\lambda)| \leqq |\lambda|^{(s+2)/2} \text{ if } |\lambda|>1, \quad\text{and}\quad |h_{s-2}^2(\lambda)| \leqq |\lambda|^{1/2} \text{ if } |\lambda|\leqq 1.$$

There, for the estimation of the remainder, the simple fact that for $j_1 \leqq j_2$

$$(9) \quad |\lambda|^{k_1} O(n^{(\varepsilon-1/2)j_1}) + |\lambda|^{k_2} O(n^{(\varepsilon-1/2)j_2}) \leqq \begin{cases} |\lambda|^{\max(k_1,k_2)} O(n^{(\varepsilon-1/2)j_1}), & \text{if } |\lambda| > 1, \\ |\lambda|^{\min(k_1,k_2)} O(n^{(\varepsilon-1/2)j_1}), & \text{if } |\lambda| \leqq 1 \end{cases}$$

was used and will be often in the sequel. In what follows, all the figuring functions $h(\lambda)$ with different (lower and upper) indices are majorized in absolute value by $|\lambda|^{1/2}$ if $|\lambda| \leqq 1$ and if some assertion of the type of $|h(\lambda)| \leqq |\lambda|^r$ (with some r) appeais, then it refers to the case $|\lambda| > 1$. Using (9) several times with $|h_j(\lambda)| \leqq |\lambda|^{j/2}$ we get on A_n^ε from (8)

$$(10) \quad \theta_n^m(\lambda) = \left\{ \sum_{j=2}^{s} \left(\frac{1}{\sqrt{n}}\right)^j \int_0^1 \frac{\beta^j(t)}{j!} dt \right\}^m +$$

$$+ \sum_{\substack{k_1+k_2=m \\ (k_1,k_2)\neq(m,0)}} \left(\sum_{j=2}^{s} h_j(\lambda) O(n^{(\varepsilon-1/2)j}) \right)^{k_1} (h_{s-2}^2(\lambda) O(n^{(\varepsilon-1/2)(s+1)}))^{k_2} =$$

$$= \sum_{i_2+\dots+i_s=m} \frac{m!}{i_2! \dots i_s!} \left(\frac{1}{\sqrt{n}}\right)^{2i_2+\dots+si_s} \prod_{k=2}^{s} \left(\int_0^1 \frac{\beta^k(t)}{k!} dt \right)^{i_k} + h_{s,m}(\lambda) O(n^{(\varepsilon-1/2)[s+1+2(m-1)]}),$$

where

$$|h_{s,m}(\lambda)| \leqq |\lambda|^{(s+2)m/2}.$$

Multiplying by n and writing out explicitly the fiist term of the m-summation, and using again (9), we have

$$(11) \quad \mathbf{E} \exp \left\{ n \sum_{m=1}^{s} \frac{(-1)^{m+1}}{m} \theta_n^m(\lambda) \right\} \chi_{A_n^\varepsilon} =$$

$$= \mathbf{E} \exp \left\{ -\lambda \int_0^1 \alpha^2(t) dt + \sum_{j=1}^{s-2} \left(\frac{1}{\sqrt{n}}\right)^j \int_0^1 \frac{\beta^{j+2}(t)}{(j+2)!} dt + \right.$$

$$\left. + n \sum_{m=2}^{s} \frac{(-1)^{m+1}}{m} \sum_{j=2m}^{sm} \left(\frac{1}{\sqrt{n}}\right)^j (-\sqrt{-2\lambda})^j q_j^{(m)} + h_{s-2}^2(\lambda) O(n^{(\varepsilon-1/2)(s+1)+1}) \right\} \chi_{A_n^\varepsilon} =$$

$$= \mathbf{E} \exp \left\{ -\lambda \int_0^1 \alpha^2(t) dt + \sum_{m=1}^{s} \sum_{j=2m-2}^{sm-2} \left(\frac{1}{\sqrt{n}}\right)^j p_j^{(m)} \right\} \chi_{A_n^\varepsilon} + h_{s-2}^3(\lambda) O(n^{(\varepsilon-1/2)(s-1)+2\varepsilon}),$$

where

$$(12) \quad q_j^{(m)} = \sum_{\substack{2i_2+\dots+si_s=j \\ i_2+\dots+i_s=m}}'' \frac{m!}{i_2! \dots i_s!} \prod_{k=2}^{s} \left(\int_0^1 \frac{\alpha^k(t)}{k!} dt \right)^{i_k}$$

and $p_0^{(1)} = 0$, while for $j = 1, \dots, s^2-2$ and $m = 1, \dots, s$

$$(13) \quad p_j^{(m)} = p_j^{(m)}(\lambda) = \frac{(-1)^{m+1}}{m} (-\sqrt{-2\lambda})^{j+2} q_{j+2}^{(m)},$$

furthermore

$$|h_{s-2}^3(\lambda)| \leqq |\lambda|^{(s+2)s/2}.$$

Now we break up the double-sum in the exponent of the above expected value according to powers of $\frac{1}{\sqrt{n}}$ and estimate all the terms of this sum on A_n^ε where the power of $\frac{1}{\sqrt{n}}$ is greater than $s-2$. Since, on A_n^ε,

$$p_j^{(m)}(\lambda) = h_{j+2}(\lambda) O(n^{\varepsilon(j+2)})$$

with the functions $h_k(\lambda)$ as already introduced in connection with (10), it is easy to see that

$$\sum_{m=1}^{s} \sum_{j=2m-2}^{sm-2} \left(\frac{1}{\sqrt{n}}\right)^j p_j^{(m)} =$$

$$= \sum_{l=1}^{s-2} \left(\frac{1}{\sqrt{n}}\right)^l \eta_l(\lambda) + \sum_{k=2}^{s} \sum_{l=(k-1)s-1}^{ks-2} \left(\frac{1}{\sqrt{n}}\right)^l (p_l^{(k)} + \dots + p_l^{(\min([(l+2)/2], s))}) =$$

$$= \sum_{l=1}^{s-2} \left(\frac{1}{\sqrt{n}}\right)^l \eta_l(\lambda) + h_{s-2}^4(\lambda) O(n^{(\varepsilon-1/2)(s-1)+2\varepsilon}),$$

where

$$\eta_l(\lambda) = \sum_{m=1}^{[(l+2)/2]} p_l^{(m)}(\lambda) \tag{14}$$

and

$$|h_{s-2}^4(\lambda)| \leqq |\lambda|^{s^2/2}.$$

So we can continue our row (11) of equations the following way

$$= \mathbf{E} \exp\left\{-\lambda \int_0^1 \alpha^2(t)\,dt + \sum_{l=1}^{s-2} \left(\frac{1}{\sqrt{n}}\right)^l \eta_l(\lambda)\right\} \chi_{A_n^\varepsilon} + h_{s-2}^5(\lambda) O(n^{(\varepsilon-1/2)(s-1)+2\varepsilon}),$$

where

$$|h_{s-2}^5(\lambda)| \leqq |\lambda|^{s(s+2)/2}.$$

Putting now s instead of $s-2$, on the basis of (5), (6), (7) and (11), we have

$$\mathbf{E} \exp\{-\lambda\omega_n^2\} = \mathbf{E} \exp\left\{-\lambda \int_0^1 \alpha^2(t)\,dt + \sum_{l=1}^{s} \left(\frac{1}{\sqrt{n}}\right)^l \eta_l(\lambda)\right\} \chi_{A_n^\varepsilon} + h_s^6(\lambda) O(n^{(\varepsilon-1/2)(s+1)+2\varepsilon}), \tag{15}$$

where

$$|h_s^6(\lambda)| \leqq |\lambda|^{(s+2)(s+4)/2}.$$

Considering the sum expression in the exponent of (15), we have, again by the MacLaurin formula,

$$\exp\left\{\sum_{l=1}^{s} \left(\frac{1}{\sqrt{n}}\right)^l \eta_l(\lambda)\right\} = 1 + \sum_{k=1}^{s} \left(\frac{1}{\sqrt{n}}\right)^k \zeta_k(\lambda) + R_{s+1}\left(\frac{1}{\sqrt{n}}\right), \tag{16}$$

where, via the Faa di Bruno formula (see Lemma 1 and formula (1.6) on page 169 in [25]) for the differentiation of compound functions, we have

$$\zeta_k(\lambda)=\frac{1}{k!}\frac{d^k}{dx^k}\exp\left\{\sum_{l=1}^{s}x^l\eta_l(\lambda)\right\}\Bigg|_{x=0}=\sum_{(i_1,\dots,i_k)}{}' \prod_{m=1}^{k}\frac{1}{i_m!}(\eta_m(\lambda))^{i_m}. \tag{17}$$

As to the Lagrange remainder term we get, again by the Faa di Bruno formula,

$$\left|R_{s+1}\left(\frac{1}{\sqrt{n}}\right)\right|=\left(\frac{1}{\sqrt{n}}\right)^{s+1}\left|\exp\left\{\sum_{l=1}^{s}\left(\frac{\vartheta}{\sqrt{n}}\right)^{l}\eta_l(\lambda)\right\}\right|\times$$

$$\times\left|\sum_{(k_1,\dots,k_{s+1})}{}'\prod_{m=1}^{s+1}\frac{1}{k_m!}\left\{\prod_{k=0}^{s-m}\binom{m+k}{m}\eta_{m+k}(\lambda)\left(\frac{\vartheta}{\sqrt{n}}\right)^{k}\right\}^{k_m}\right|,$$

where ϑ is a random variable with $0<\vartheta<1$. We also recall that $3\varepsilon-\frac{1}{2}<0$. Then, on A_n^ε, the exponential factor is majorized by

$$1+|\lambda|^{(s+2)/2}O(n^{3\varepsilon-1/2}),$$

while the last factor by

$$|\lambda|^{(s+2)(s+1)/2}O(n^{\varepsilon(s+1)+2\varepsilon(s+1)})$$

on applying (9) several times and noticing that on A_n^ε

$$\eta_l(\lambda)=(-\sqrt{-2\lambda})^{l+2}O(n^{\varepsilon(l+2)}).$$

Consequently, on A_n^ε,

$$\left|R_{s+1}\left(\frac{1}{\sqrt{n}}\right)\right|\leqq|\lambda|^{(s+2)(s+2)/2}O(n^{(3\varepsilon-1/2)(s+1)}). \tag{18}$$

We get from (15) by (16) and (18) via (9) that

$$\mathbf{E}\exp\{-\lambda\omega_n^2\}=\mathbf{E}\exp\left\{-\lambda\int_0^1\alpha^2(t)\,dt\right\}\left(1+\sum_{k=1}^{s}\left(\frac{1}{\sqrt{n}}\right)^{k}\zeta_k(\lambda)\right)+ \tag{19}$$

$$+h_s(\lambda)O(n^{\delta-(s+1)/2}),$$

where $0<\delta=3\varepsilon(s+1)<\frac{1}{2}$, and for $h_s(\lambda)$ we already have

$$|h_s(\lambda)|\leqq|\lambda|^{(s+2)(s+4)/2}.$$

In these calculations we write $1+\chi_{A_n^\varepsilon}-1$ in place of the factor $\chi_{A_n^\varepsilon}$ in the expectation; then the new term with factor $\chi_{A_n^\varepsilon}-1$ decreases exponentially fast as $n\to\infty$. The latter can be shown exactly the same way as in (6).

From (1) and (19) it follows that

$$\mathbf{E}\exp\left\{-\lambda\int_0^1\alpha^2(t)\,dt\right\}=\mathbf{E}\exp\{-\lambda\omega^2\}. \tag{20}$$

Let us also observe that if λ is real with $\lambda \geqq 0$, then

$$\mathbf{E}\exp\{-\lambda\omega_n^2\} = \mathbf{E}\exp\left\{i\sqrt{2\lambda}\int_0^1 Y_n(x)\,dW(x)\right\}$$

is also real by the reflexivity of the Brownian Motion. From (19) and (20) it follows then that

$$\mathbf{E}\exp\{-\lambda\omega_n^2\} = \mathbf{E}\exp\{-\lambda\omega^2\} + \sum_{k=1}^{s}\left(\frac{1}{\sqrt{n}}\right)^k a_k^*(\lambda) + h_s(\lambda)\,O(n^{\delta-(s+1)/2}) \tag{21}$$

where $a_k^*(\lambda) = \operatorname{Re} C_k(\lambda)$, and from (12), (13), (14), (17) and (19) we have

$$C_k(\lambda) = \sum_{(i_1,\dots,i_k)}' \frac{(-\sqrt{-2\lambda})^{k+2H_k}}{i_1!\dots i_k!}\,\mathbf{E}\exp\left\{-\lambda\int_0^1 \alpha^2(t)\,dt\right\}\Pi_{i_1,\dots,i_k},$$

where $\Pi i_{i_1,\dots,i_k}$ is already as in the formulation of the theorem with k in place of $2k$. If λ is real and nonnegative, then $a_k^*(\lambda)=0$, if k is odd, and if $k=2\nu$, $\nu=1, 2, \dots$, then

$$a_{2\nu}^* = \sum_{(i_1,\dots,i_{2\nu})}' b_{i_1,\dots,i_{2\nu}}\lambda^{\nu+H_{2\nu}}\,\mathbf{E}\exp\left\{-\lambda\int_0^1 \alpha^2(t)\,dt\right\}\Pi_{i_1,\dots,i_{2\nu}}.$$

But a_k^* being an analytical function of λ, the same formulae hold true for any complex λ with $\operatorname{Re}\lambda \geqq 0$. Introducing $a_\nu(\lambda) = a_{2\nu}^*(\lambda)$, $\nu=1, 2, \dots$, we can rewrite (21) the following way

$$\mathbf{E}\exp\{-\lambda\omega_n^2\} = \mathbf{E}\exp\{-\lambda\omega^2\} + \sum_{k=1}^{[s/2]}\left(\frac{1}{n}\right)^k a_k(\lambda) + h_s(\lambda)\,O(n^{\delta-(s+1)/2})$$

valid for any complex λ with $\operatorname{Re}\lambda \geqq 0$. This was to be proved.

§ 4. **On the problem of inversion.** In the knowledge of the asymptotic expansion of Theorem 2 for the Laplace transform one should naturally like to invert it in order to get the corresponding form for the expanded distribution function. This task, unfortunately, is not accomplished here. A considerable work is done, however, towards this end. Our result is that the problem of the existence, of an (exactly computed) asymptotic expansion for $V_n(x) - V(x)$ is reduced to a qualitative problem concerning the behaviour of the characteristic function $f_n(t)$ of ω_n^2.

The following known results will be used.

Lemma A. (See e.g. [16]) *If the characteristic function $\tilde{f}(t)$ of an arbitrary distribution function $\tilde{F}(x)$ satisfies*

$$\int_{-\infty}^{\infty} |t|^p\,|\tilde{f}(t)|\,dt < \infty, \tag{22}$$

with some integer $p \geqq 0$, *then the* $(p+1)^{st}$ *derivative of* $\tilde{F}(x)$ *exists and*

$$\tilde{F}^{(p+1)}(x) \to 0 \quad \text{as} \quad |x| \to \infty.$$

Lemma B. (Esseen, see e.g. [25]) *Let* $\tilde{F}(x)$ *be a nondecreasing function and* $\tilde{G}(x)$ *a differentiable function of bounded variation on the real line,* $\tilde{f}(t)$ *and* $\tilde{g}(t)$ *the corresponding Fourier—Stieltjes transforms,* $\tilde{F}(-\infty)=\tilde{G}(-\infty)$, $\tilde{F}(\infty)=\tilde{G}(\infty)$ *and* T *an arbitrary positive number. Suppose* $\sup\limits_{-\infty<x<\infty} |\tilde{G}'(x)| \leqq C$ *with some constant* C. *Then for any number* $K_1 > \dfrac{1}{2\pi}$

$$\sup_{-\infty<x<\infty} |\tilde{F}(x)-\tilde{G}(x)| \leqq K_1 \int_{-T}^{T} \left| \frac{\tilde{f}(t)-\tilde{g}(t)}{t} \right| dt + K_2 \frac{C}{T},$$

where K_2 *is a positive constant depending only on* K_1.

First we prove some lemmas needed in the sequel.

Lemma 1. *The distribution function* $V(x)$ *of the random variable* ω^2 *is arbitrary many times differentiable and for an arbitrary integer* p

$$V^{(p)}(x) \to 0 \quad as \quad |x| \to \infty.$$

Proof. For the characteristic function $f(t)$ of ω^2 (see (1)) we have, by direct computation

$$|f(t)| = \left| \left(\frac{\sqrt{-2it}}{\sinh \sqrt{-2it}} \right)^{1/2} \right| \leqq 2^{3/4} |t|^{1/4} \frac{\exp\left\{-\frac{1}{2}\sqrt{|t|}\right\}}{\left(1-\exp\left\{-2\sqrt{|t|}\right\}\right)^{1/2}} \tag{23}$$

which shows that $f(t)$ satisfies condition (22) of Lemma A.

From inequality (23) we also have

Lemma 2. *For an arbitrary nonnegative real* p

$$|t|^p \, |f(t)| \to 0 \quad as \quad |t| \to \infty.$$

Now we show that our smooth distribution function $V(x)$ also rises smoothly from the point 0. Let (throughout the rest) $v(x) = \dfrac{d}{dx} V(x)$ be the density function of ω^2.

Lemma 3. *Denote (as before) by* $v^{(q)}(x)$ *the derivative of order* q *of* $v(x)$. *Then for an arbitrary* q $(=0, 1, 2, \ldots,)$ *we have*

$$v^{(q)}(0) = 0.$$

Proof. It vill be more comfortable to work now with the Laplace transform $Ee^{-\lambda\omega^2}$. By direct computation, again from $\left(\frac{\sqrt{2\lambda}}{\sinh\sqrt{2\lambda}}\right)^{1/2}$, we have

$$\int_0^\infty e^{-\lambda x}v(x)\,dx = 2^{3/4}\lambda^{1/4}\frac{\exp\left\{-\frac{1}{2}\sqrt{2\lambda}\right\}}{(1-\exp\{-2\sqrt{2\lambda}\})^{1/2}}.$$

Thus for real λ

$$\lim_{\lambda\to\infty}\lambda^q\int_0^\infty e^{-\lambda x}v(x)\,dx = 0 \quad (q = 0, 1, \ldots).$$

Using this we get, by integration by parts

$$0 = \lim_{\lambda\to\infty}\lambda\int_0^\infty e^{-\lambda x}v(x)\,dx = \int_0^\infty e^{-u}\left[\lim_{\lambda\to\infty} v\left(\frac{u}{\lambda}\right)\right]du = v(0),$$

$$0 = \lim_{\lambda\to\infty}\lambda^2\int_0^\infty e^{-\lambda x}v(x)\,dx = \int_0^\infty e^{-u}u\left[\lim_{\lambda\to\infty}\frac{v(u/\lambda)}{u/\lambda}\right]du = v'(0),$$

$$0 = \lim_{\lambda\to\infty}\lambda^3\int_0^\infty e^{-\lambda x}v(x)\,dx = \int_0^\infty e^{-u}u\left[\lim_{\lambda\to\infty}\frac{v'(u/\lambda)}{u/\lambda}\right]du = v''(0),$$

and so on. Hence the lemma is proved by induction.

It is very easy to see that $\alpha(t)=W(t)-\int_0^1 W(x)\,dx$ is a Gaussian process with $E\alpha(t)=0$ and continuous covariance function $\min(s,t)-s\left(1-\frac{s}{2}\right)-t\left(1-\frac{t}{2}\right)+\frac{1}{3}$. Therefore it can be expanded in the following form (see [12])

$$\alpha(t) = \sum_{k=1}^\infty \xi_k\varphi_k(t), \tag{24}$$

where ξ_k $(k=1,2,\ldots)$ is a normally distributed random variable with $E\xi_k=0$, $\{\varphi_k\}_{k=1}^\infty$ is an orthonormal system of continuous functions on [0, 1] and the series in (24) converges with probability 1. Let us denote

$$\alpha_r = \int_0^1 \alpha^r(t)\,dt, \quad r = 2, \ldots, s. \tag{25}$$

The products of different powers of these α_r figure in the coefficients of the asymptotic expansion for the Laplace transform and we will need the following

Lemma 4. *For arbitrary $k_2, \ldots, k_s \geqq 0$ the function* (of x) $E\{\alpha_2^{k_2}\alpha_3^{k_3}\ldots\alpha_s^{k_s}|\alpha_2=x\}$ *is differentiable as many times as we wish.*

Proof.

$$\mathbf{E}\{\alpha_2^{k_2}\alpha_3^{k_3}\dots\alpha_s^{k_s}|\alpha_2=x\}=$$

$$=x^{k_2}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}x_3^{k_3}\dots x_s^{k_s}\,d\mathbf{P}\{\alpha_3<x_3,\dots,\alpha_s<x_s|\alpha_2=x\}=$$

$$=\frac{x^{k_2}}{v(x)}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}x_3^{k_3}\dots x_s^{k_s}f(x,x_3,\dots,x_s)\,dx_3\dots dx_s,$$

where $f(x_2, x_3, \dots, x_s)$ is the common density of the variables $\alpha_2, \alpha_3, \dots, \alpha_s$ (which will be shown to exist below). There we used that

$$\mathbf{P}\left\{\int_0^1\alpha^2(t)\,dt<x\right\}=\mathbf{P}\{\omega^2<x\}=V(x),$$

a consequence of relation (20) and the uniqueness theorem for Laplace transforms. Using Lemma 1, it is enough to show that $f(x_2, x_3, \dots, x_s)$ is arbitrary many times differentiable in x_2. Towards this end we will give an expression for this density.

Relation (24) gives

$$\alpha_r=\sum_{k=1}^{\infty}\sum_{j_1+\dots+j_r=k}C_{j_1,\dots,j_r}\xi_{j_1}\cdot\dots\cdot\xi_{j_r}, \tag{26}$$

where

$$C_{j_1,\dots,j_r}=\int_0^1\varphi_{j_1}(t)\cdot\dots\cdot\varphi_{j_r}(t)\,dt.$$

Specifically, via the orthonormality of the φ_k system,

$$\alpha_2=\sum_{k=1}^{\infty}\xi_k^2, \tag{27}$$

with probability 1. We would like to get rid of the difficulty that in (26) an infinite number of Gaussian variables express α_r. Therefore we rewrite this expression the following way

$$\alpha_r=\gamma^{r,0}+\sum_{i=1}^{s-1}\gamma_i^{r,1}\xi_i+\sum_{i,j=1}^{s-1}\gamma_{i,j}^{r,2}\xi_i\xi_j+\dots+\sum_{i_1,\dots,i_r=1}^{s-1}\gamma_{i_1,\dots,i_r}^{r,r}\xi_{i_1}\cdot\dots\cdot\xi_{i_r}, \tag{28}$$

$r=2, \dots, s$, where

$$\vec{\gamma}^{(r)}=\{\gamma^{r,0};\ \gamma_i^{r,1},\ i=1,\dots,s-1;\ \dots;\ \gamma_{i_1,\dots,i_r}^{r,r},\ i_1,\dots,i_r=1,\dots,s-1\}$$

are random variables depending on $\xi_s, \xi_{s+1}, \dots$, but not on $\xi_1, \dots, \xi_{s-1}$. Specifically, on the basis of (27) we have

$$\alpha_2=\gamma^{2,0}+\sum_{k=1}^{s-1}\xi_k^2. \tag{29}$$

From the introduced random vectors $\vec{\gamma}^{(r)}$, having $Q_r = \sum_{j=0}^{r} (s-1)^j$ components, formulate the random vector $\vec{\gamma} = (\vec{\gamma}^{(2)}, \ldots, \vec{\gamma}^{(r)})$, having $Q = \sum_{r=2}^{s} Q_r$ components. For each $r = 2, \ldots, s$, let $\overrightarrow{h^{(r)}}$ be the same way indexed nonrandom real vector as $\overrightarrow{\gamma^{(r)}}$, having Q_r components and let $\vec{h} = (\overrightarrow{h^{(2)}}, \ldots, \overrightarrow{h^{(r)}})$ be the corresponding Q component real vector. Let us consider the following system of algebraic equations

$$a_r = g_r(y_1, \ldots, y_{s-1}) \quad (r = 2, 3, \ldots, s), \tag{30}$$

where $a_2, a_3, \ldots, a_s$ are arbitrarily fixed real numbers. Further (from (29))

$$g_2(y_1, \ldots, y_{s-1}) = h^{2,0} + \sum_{i=1}^{s-1} y_i^2,$$

and, for $3 \leqq r \leqq s$, $g_r(y_1, \ldots, y_{s-1})$ is the right hand side of (28), having written h's and y's respectively, in place of γ's and ξ's. It is clear that the number of such vectors $\vec{h}$ for which the system (30) has infinitely many solutions $(y_1, \ldots, y_{s-1})$ is finite. Similarly, the Jacobian

$$J(y_1, \ldots, y_{s-1}) = \frac{\partial(g_2, \ldots, g_s)}{\partial(y_1, \ldots, y_{s-1})} \tag{31}$$

can be equal to zero only on hypersurfaces of the $s-1$ dimensional Euclidean space $\mathbf{R}^{s-1}$, which are defined by different vectors $\vec{h}$, and the number of such vectors is also finite. Let $\vec{\gamma} = \vec{h}$ be fixed, so that the Jacobian (31) is not zero and the system (30) has only a finite number of solution vectors $(y_1, \ldots, y_{s-1})$. This latter number we denote by q. Divide $\mathbf{R}^{s-1}$ onto q subspaces $U_1, \ldots, U_q$, so that in the interior of each one of them the system (30) would have only one solution. Denote by G the transformation $(y_1, \ldots, y_{s-1}) \to (x_2, \ldots, x_s)$ of $\mathbf{R}^{s-1}$ onto itself, defined by

$$g_r(y_1, \ldots, y_{s-1}) = x_r \quad (r = 2, 3, \ldots, s), \tag{32}$$

and let us define the functions $g_{2,k}^{-1}, \ldots, g_{s,k}^{-1}$, on the images $G(U_k)$, $k = 1, \ldots, q$, satisfying

$$g_{r,k}^{-1}(x_2, \ldots, x_s) = y_{r-1} \quad (r = 2, 3, \ldots, s),$$

if (32) holds.

Let

$$p(y_1, \ldots, y_{s-1}) = \prod_{k=1}^{s-1} \frac{1}{\sqrt{2\pi}\, \sigma_k} \exp\left\{\frac{-y_k^2}{2\sigma_k^2}\right\}$$

denote the common density of the independent normal variables $\xi_1, \ldots, \xi_{s-1}$. Then, after dividing the domain of integration onto intersections with the U_k's, changing the variables on these intersections and substituting the corresponding Jacobian determinant with the reciprocal of its inverse, for the common distribution function of $\alpha_2, \ldots, \alpha_s$ given $\vec{\gamma}=\vec{h}$ we get the following form

$$F(a_2, \ldots, a_s|\vec{\gamma} = \vec{h}) = \underset{\substack{g_r(y_1, \ldots, y_{s-1})<a_r \\ r=2,\ldots,s}}{\int \cdots \int} p(y_1, \ldots, y_{s-1})\, dy_1 \ldots dy_{s-1} =$$

$$= \underset{\substack{z_r<a_r \\ r=2,\ldots,s}}{\int \cdots \int} \sum_{k=1}^{q} \chi_{U_k}(z_2, \ldots, z_s) \frac{p(g_{2,k}^{-1}(z_2, \ldots, z_s), \ldots, g_{s,k}^{-1}(z_2, \ldots, z_s))}{|J(g_{2,k}^{-1}(z_2, \ldots, z_s), \ldots, g_{s,k}^{-1}(z_2, \ldots, z_s))|}\, dz_2 \ldots dz_s,$$

where χ_{U_k} is the indicator of the set U_k. This means that the conditional common density of the variables $\alpha_2, \ldots, \alpha_s$ given $\vec{\gamma}=\vec{h}$ is

$$f(a_2, \ldots, a_s|\vec{h}) = \sum_{\substack{(y_1, \ldots, y_{s-1}) \\ g_r(y_1, \ldots, y_{s-1})=a_r \\ r=2,\ldots,s}} \frac{p(y_1, \ldots, y_{s-1})}{|J(y_1, \ldots, y_{s-1})|}, \tag{33}$$

where the summation extends over all solutions $(y_1, \ldots, y_{s-1})$ of (30), i.e. the sum consists of q terms. Hence the common density of $\alpha_2, \ldots, \alpha_s$, which we were to find, has the form

$$f(a_2, \ldots, a_s) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{Q \text{ times}} f(a_2, \ldots, a_s|\vec{h})\, d\mathbf{P}\{\vec{\gamma} < \vec{h}\}.$$

It follows that for proving the arbitrarily many times differentiability of $f(a_2, \ldots, a_s)$ in a_2, it is enough to prove this for $f(a_2, \ldots, a_s|\vec{h})$.

Introduce polar coordinates

$$\begin{aligned} y_1 &= r\varkappa_1(\theta) = r\cos\theta_1 \\ y_2 &= r\varkappa_2(\theta) = r\sin\theta_1\cos\theta_2 \\ &\vdots \\ y_{s-1} &= r\varkappa_{s-1}(\theta) = r\sin\theta_1\sin\theta_2 \ldots \sin\theta_{s-3}\sin\theta_{s-2}, \end{aligned}$$

then

$$f(a_2, \ldots, a_s|\vec{h}) = \sum_{(r, \theta_1, \ldots, \theta_{s-2})} \frac{p(r\varkappa_1(\theta), \ldots, r\varkappa_{s-1}(\theta))}{|J(r\varkappa_1(\theta), \ldots, r\varkappa_{s-1}(\theta))|},$$

where the summation extends over all solutions $(r, \theta_1, \ldots, \theta_{s-2})$ of the following system of equations

$$\begin{aligned} a_2 &= h^{2,0} + r^2 \\ a_3 &= h^{3,0} + l_1^{(3)}(\theta)r + l_2^{(3)}(\theta)r^2 + l_3^{(3)}(\theta)r^3 \\ &\vdots \\ a_s &= h^{s,0} + l_1^{(s)}(\theta)r + \ldots + l_s^{(s)}(\theta)r^s. \end{aligned}$$

Here $l_j^{(i)}(\theta)$, $j=1, \ldots, i$; $i=3, \ldots, s$, is a trigonometric polynomial of $\theta_1, \ldots, \theta_{s-2}$, already not depending on r. The sum now also consists of q terms, therefore it is enough to show the differentiability of the single summands. But differentiability in a_2 is equivalent to differentiability in r^2, and every summand is differentiable in r^2 as many times as we wish. Lemma 4 is proved.

As the random variable $\Pi_{i_1, \ldots, i_{2k}}$, figuring in the coefficients of the asymptotic expansion of Theorem 2, is a linear combination of variables of the form $\alpha_2^{k_2} \cdot \ldots \cdot \alpha_s^{k_s}$, this latter result implies

Lemma 5. *For any* $i_1, \ldots, i_{2k}$ *the function* $\mathbf{E}\left\{\Pi_{i_1, \ldots, i_{2k}} \Big| \int_0^1 \alpha^2(t)\, dt = x\right\}$ *has derivatives of arbitrary order.*

The following equation will be useful, by means of which the coefficients will be inverted.

Lemma 6. *For an arbitrary natural number* q,

$$\lambda^q \int_0^\infty e^{-\lambda x} \mathbf{E}\{\Pi_{i_1, \ldots, i_{2k}} | \alpha_2 = x\} v(x)\, dx = \int_0^\infty e^{-\lambda x} \frac{d^q}{dx^q} [\mathbf{E}\{\Pi_{i_1, \ldots, i_{2k}} | \alpha_2 = x\} v(x)]\, dx.$$

Proof. A row of q successive integrations by parts, where, at the k-th step we integrate the function $(-1)^{q-k} \dfrac{d^{q-k}}{dx^{q-k}} e^{-\lambda x}$ and differentiate the function $\varphi^{(k)}(x) = \dfrac{d^k}{dx^k} [\mathbf{E}\{\Pi_{i_2, \ldots, i_{2k}} | \alpha_2 = x\} v(x)]$, $k=0, 1, \ldots, q-1$. All the integrated out terms disappear, as by Lemma 3 we have $\varphi^{(k)}(0)=0$ for each k.

Since, by this Lemma 6, we have

$$\lambda^q \mathbf{E} \exp\{-\lambda \alpha_2\} \Pi_{i_1, \ldots, i_{2k}} =$$

$$= \lambda^q \int_0^\infty e^{-\lambda x} \mathbf{E}\{\Pi_{i_1, \ldots, i_{2k}} | \alpha_2 = x\} v(x)\, dx =$$

$$= -\frac{1}{\lambda} \int_0^\infty e^{-\lambda x} \frac{d^{q+1}}{dx^{q+1}} [\mathbf{E}\{\Pi_{i_1, \ldots, i_{2k}} | \alpha_2 = x\} v(x)]\, dx,$$

we also proved the following

Lemma 7. *For any natural* q *and* $i_1, \ldots, i_2 \geqq 0$ *we have, as* $|\lambda| \to \infty$,

$$|\lambda|^q \left| \mathbf{E} \exp\left\{-\lambda \int_0^1 \alpha^2(t)\, dt\right\} \Pi_{i_1, \ldots, i_{2k}} \right| \to 0.$$

We now start inverting the asymptotic expansion of Theorem 2. Integrating by parts we have

$$\mathbf{E}\exp\{-\lambda\omega_n^2\} = \lambda\int_0^\infty e^{-\lambda x}V_n(x)\,dx$$

and

$$\mathbf{E}\exp\{-\lambda\omega^2\} = \lambda\int_0^\infty e^{-\lambda x}V(x)\,dx.$$

This implies

$$\int_0^\infty e^{-\lambda x}V_n(x)\,dx = \int_0^\infty e^{-\lambda x}V(x)\,dx + \sum_{k=1}^{[s/2]}\left(\frac{1}{n}\right)^k\frac{a_k(\lambda)}{\lambda} + \frac{h_s(\lambda)}{\lambda}O(n^{\varepsilon-(s+1)/2}),$$

and, by Lemma 6, we have

$$\int_0^\infty e^{-\lambda x}V_n(x)\,dx = \int_0^\infty e^{-\lambda x}\left\{V(x) + \sum_{k=1}^{[s/2]}\left(\frac{1}{n}\right)^k\psi_k(x)\right\}dx + \frac{h_s(\lambda)}{\lambda}O(n^{\varepsilon-(s+1)/2})$$

where

$$\psi_k(x) = \sum_{(i_1,\ldots,i_{2k})}{}' b_{i_1,\ldots,i_{2k}}\frac{d^{k-1+H_{2k}}}{dx^{k-1+H_{2k}}}[\mathbf{E}\{\Pi_{i_1,\ldots,i_{2k}}|\alpha_2 = x\}v(x)].$$

The functions $V_n(x)$, $V(x)$ and $\psi_k(x)$, $k=1,\ldots,\left[\frac{s}{2}\right]$, are continuous and are of bounded variation on each finite interval from $[0,\infty)$; further, the Laplace transform itself here has abscissa of convergence 0. Therefore, the complex inversion formula can be applied for the left hand side, and also for the first term of the right hand side. We write formally

$$V_n(x) = V(x) + \sum_{k=1}^{[s/2]}\left(\frac{1}{n}\right)^k\psi_k(x) + A(\varepsilon, s, n, x), \tag{34}$$

where $A=A(\varepsilon, s, n, x)$ is a function, into which $\frac{h_s(\lambda)}{\lambda}O(n^{\varepsilon-(s+1)/2})$ is inverted.

For justification of an asymptotic expansion of type (34). i.e. to estimate the remainder term A here, usually Essen's result of Lemma B is applied. For doing this we rewrite Theorem 2 in terms of characteristic functions.

$$\mathbf{E}e^{it\omega_n^2} = \mathbf{E}e^{it\omega^2} + \sum_{k=1}^{[s/2]}\left(\frac{1}{n}\right)^k a_k(-it) + h_s(-it)O(n^{\varepsilon-(s+1)/2}).$$

Put

$$f_n(t) = \int_0^\infty e^{itx}\,dV_n(x) = \mathbf{E}e^{it\omega_n^2},$$

and define the functions $G_{n,s}$ by the following equation

$$g_{n,s}(t) = \int_{-\infty}^\infty e^{itx}\,dG_{n,s}(x) = \mathbf{E}e^{it\omega^2} + \sum_{k=1}^{[s/2]}\left(\frac{1}{n}\right)^k a_k(-it).$$

This means

$$G_{n,s}(x) = V(x)+\sum_{k=1}^{[s/2]}\left(\frac{1}{n}\right)^k \psi_k(x).$$

It is easy to see that $V_n(x)$ and $G_{n,s}(x)$ satisfy the conditions of Lemma B. Specifically, from Lemmas 1 and 5 it follows that $G_{n,s}$ is of bounded variation and, from the existence of the integrals of Lemma 6, that $\psi_k(0)=\psi_k(\infty)=0$; that is $V_n(-\infty)= =G_{n,s}(-\infty)=0$ and $V_n(\infty)=G_{n,s}(\infty)=1$. Put $C=\sup|G'_{n,s}(x)|$. Then, by Lemma B, we have

$$\sup_{-\infty<x<\infty}|V_n(x)-G_{n,s}(x)| = \sup_{-\infty<x<\infty}|A(\varepsilon, s, n, x)| \leqq$$

$$\leqq K_2\frac{C}{T}+K_1\int_{-T}^{T}\left|\frac{f_n(t)-g_{n,s}(t)}{t}\right|dt.$$

Put $T=n^\beta$ where $\beta=\frac{s+1}{2}-2\varepsilon$ and $\delta=\frac{2\varepsilon}{(s+2)(s+4)}$. Then

$$\int_{-n^\beta}^{n^\beta}\left|\frac{f_n(t)-g_{n,s}(t)}{t}\right|dt \leqq \int_{-n^\delta}^{n^\delta}\left|\frac{f_n(t)-g_{n,s}(t)}{t}\right|dt+$$

$$+\int_{n^\delta\leqq|t|\leqq n^\beta}\left|\frac{g_{n,s}(t)}{t}\right|dt+\int_{n^\delta\leqq|t|\leqq n^\beta}\left|\frac{f_n(t)}{t}\right|dt = I_1+I_2+I_3,$$

where for the first term we have by Theorem 2

$$I_1 = O(n^{\varepsilon-(s+1)/2})\int_{-n^\delta}^{n^\delta}\left|\frac{h_s(-it)}{t}\right|dt =$$

$$= O(n^{\varepsilon-(s+1)/2})\left(\int_{-1}^{1}|t|^{-1/2}dt+\int_{1\leqq|t|\leqq n^\delta}|t|^{(s+2)(s+4)/2-1}dt\right) =$$

$$= O(n^{\varepsilon-(s+1)/2+\delta(s+2)(s+4)/2}) = O(n^{2\varepsilon-(s+1)/2}).$$

For estimating the second one, let us observe that

$$|g_{n,s}(t)| \leqq |f(t)|+\sum_{k=1}^{[s/2]}\left(\frac{1}{n}\right)^k|a_k(-it)|,$$

and that $|a_k(-it)|$ is majorized by a linear combination of functions of the form

$$|t|^{k+H_{2k}}|\mathbf{E}e^{it\alpha_2}\Pi_{i_1,\dots,i_{2k}}|.$$

Therefore, by Lemmas 2 and 7, there exists (for any positive number m) a constant C_m such that

$$|g_{n,s}(t)| \leqq \frac{C_m}{|t|^m}.$$

This implies

$$I_2 \leqq \int_{n^\delta \leqq |t| < \infty} C_m |t|^{-(m+1)} dt = O(n^{-m\delta}) = O(n^{2\varepsilon-(s+1)/2})$$

as m was arbitrary. Unfortunately, we do not have any estimate for I_3. On this way there exists a constant K_s depending only on s, so that

(35) $$\sup_{-\infty < x < \infty} |A(\varepsilon, s, n, x)| \leqq K_s n^{2\varepsilon-(s+1)/2} + K_1 \int_{T_n(s, 2\varepsilon)} \left| \frac{f_n(t)}{t} \right| dt,$$

where

(36) $$T_n(s, \varepsilon) = \{t : n^{\varepsilon/(s+2)(s+4)} \leqq |t| \leqq n^{(s+1)/2-\varepsilon}\}.$$

By (34) and (35) we then have

Theorem 3. *For any natural s and real positive ε*

$$V_n(x) - V(x) = \sum_{k=1}^{[s/2]} \left(\frac{1}{n}\right)^k \psi_k(x) + O(n^{-(s+1)/2+\varepsilon}) + O\left(\int_{T_n(s, \varepsilon)} \left| \frac{f_n(t)}{t} \right| dt \right),$$

where

$$\psi_k(x) = \sum_{(i_1, \dots, i_{2k})}' b_{i_1, \dots, i_{2k}} \frac{d^{k-1+H_{2k}}}{dx^{k-1+H_{2k}}} \left[\mathrm{E}\left\{ \Pi_{i_1, \dots, i_{2k}} \Big| \int_0^1 \alpha^2(t)\, dt \right\} v(x) \right],$$

$\alpha(t)$, $b_{i_1, \dots, i_{2k}}$, H_{2k} *and* $\Pi_{i_1, \dots, i_{2k}}$ *are as in Theorem 2,* $v(x)$ *is the density of* ω^2, $f_n(t)$ *is the characteristic function of* ω_n^2 *and* $T_n(s, \varepsilon)$ *is as in* (36).

§ 5. **Remarks, conjectures.** Now by Theorem 3 the existence of an asymptotic expansion $\left(\text{surprisingly according to powers of } \frac{1}{n} \text{ instead of those of } \frac{1}{\sqrt{n}}\right)$ is reduced to the behaviour of $f_n(t)$. In this connection it is natural to make the following

Conjecture: $\int_{T_n(s, \varepsilon)} \left| \frac{f_n(t)}{t} \right| dt = O(n^{\varepsilon-(s+1)/2})$, *i.e. the asymptotic expansion*

$$V_n(x) = V(x) + \sum_{k=1}^{[s/2]} \left(\frac{1}{n}\right)^k \psi_k(x) + O(n^{-(s+1)/2+\varepsilon})$$

holds true.

For this, of course, it would be enough to prove that $|f_n(t)|$ decreases faster than any power of $|t|$, as $|t| \to \infty$, just like the limiting characteristic function (Lemma 2). Or, equivalently, it would be enough to prove that the sequence $f_n(t)$ of our characteristic functions converges uniformly *on the whole real line* to $f(t)$. As a matter of fact it would be enough to show that $|t|^S |f_n(t)| \to 0$, as $|t| \to \infty$, where $S = \left(\frac{s+1}{2} - \varepsilon\right) \frac{(s+2)(s+4)}{\varepsilon}$, or, equivalently, that $V_n(x)$ is $(S+1)$-times differentiable.

In the special case $s=2$, which would be important in practical applications, the coefficient $\psi_1(x)$ can easily be computed. One gets

(37)
$$V_n(x)-V(x)=\frac{1}{n}\left(-\frac{1}{2}x\,v(x)-\frac{1}{4}x^2v'(x)\right)+O(n^{-3/2+\varepsilon})+O\left(\int\limits_{T_n(2,\varepsilon)}\left|\frac{f_n(t)}{t}\right|dt\right),$$

where, specifically,

$$T_n(2,\varepsilon)=\{t\colon n^{\varepsilon/24}\leqq|t|\leqq n^{3/2-\varepsilon}\}.$$

It should be remarked that (37) can be proved without our general Lemmas 4, 5, 6 and 7 because, here, $a_1(\lambda)$ has the following simple form

$$a_1(\lambda)=-\frac{\lambda^2}{2}\mathbf{E}e^{-\lambda\alpha_2}\alpha_2^2,$$

and for instance we get consequently

$$g_{n,2}(t)=-\frac{t^2}{n}f''(t),\quad\text{where}\quad f(t)=\left(\frac{\sqrt{-2it}}{\sinh\sqrt{-2it}}\right)^{1/2},$$

and the corresponding estimates can be computed in a direct way.

In addition we prove the following simple fact.

Lemma 8. *For any real $p\geqq 0$ and integers $q=0,1,2,\dots$ the function $x^pv^{(q)}(x)$ is bounded on $(-\infty,\infty)$.*

Proof. It is enough to show this for $p>0$. From the inversion formula for Fourier transforms we have

$$v^{(q)}(x)=(-i)^q\int\limits_{-\infty}^{\infty}e^{-itx}t^qf(t)\,dt.$$

Integrating p times by parts

$$x^pv^{(q)}(x)=(-i)^{q+p}\int\limits_{-\infty}^{\infty}e^{-itx}\frac{d^p}{dt^p}(t^qf(t))\,dt,$$

whence

$$|x^pv^{(q)}(x)|\leqq\int\limits_{-\infty}^{\infty}\left|\frac{d^p}{dt^p}\left\{t^q\frac{\sqrt{-2it}}{\sinh\sqrt{-2it}}\right\}\right|dt<\infty.$$

From the special case $s=2$ of our Conjecture and this Lemma 8 we would have the rate of convergence

$$\Delta_n=\sup_{-\infty<x<\infty}|V_n(x)-V(x)|=O\left(\frac{1}{n}\right).$$

For the latter it would be enough to show only that, if $|t|\to\infty$, then $|t|^{48}|f_n(t)|\to 0$.

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On degree of approximation of a class of functions by means of Fourier series

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1. Let f be periodic with period 2π, and integrable in the Lebesgue sense. The Fourier series associated with f at the point x, is given by

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{1.1}$$

If $\{p_n\}$ is a sequence of positive constants, such that

$$P_n = p_0 + p_1 + \ldots + p_n \to \infty \quad \text{as} \quad n \to \infty$$

then a given series $\sum_{n=0}^{\infty} c_n$ with the sequence of partial sums $\{s_n\}$ is said to be Nörlund summable (N, p_n) to s, provided that

$$T_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k \to s \quad \text{as} \quad n \to \infty.$$

We call T_n the (N, p_n)-mean or Nörlund mean of $\sum c_n$. In the following we assume that the Nörlund means are regular, more precisely, we assume that

$$0 < np_n \leqq cP_n \quad \text{for} \quad n = 1, 2, \ldots, \text{ and } p_0 > 0. \tag{1.2}$$

2. The following theorem on the degree of approximation of a function $f \in \text{Lip}\, \alpha$, by the (C, δ)-means of its Fourier series, is due to G. Alexits [1].

Theorem A. *If a periodic function $f \in Lip\, \alpha$ for $0 < \alpha \leqq 1$, then the degree of approximation of the (C, δ)-means of its Fourier series for $0 < \alpha < \delta \leqq 1$ is given by*

$$\max_{0 \leqq x \leqq 2\pi} |f(x) - \sigma_n^{(\delta)}(x)| = O\left(\frac{1}{n^\alpha}\right)$$

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and for $0<\alpha\leqq\delta\leqq 1$, *is given by*

$$\max_{0\leqq x\leqq 2\pi} |f(x)-\sigma_n^{(\delta)}(x)| = O\left(\frac{\log n}{n^\alpha}\right)$$

where $\sigma_n^{(\delta)}$ *are the* (C, δ)*-means of the partial sums of* (1.1).

Let $C^*[0, 2\pi]$ denote the class of all continuous functions on $[0, 2\pi]$, periodic and of period 2π. The object of this paper is to prove the following theorem.

Theorem. *If* $\omega(t)$ *is the modulus of continuity of* $f\in C^*[0, 2\pi]$, *then the degree of approximation of* f *by the Nörlund means of the Fourier series for* f *is given by*

$$E_n \equiv \max_{0\leqq t\leqq 2\pi} |f(t)-T_n(t)| = O\left\{\frac{1}{P_n}\sum_{k=1}^{n}\frac{P_k\omega(1/k)}{k}\right\},$$

where T_n *are the* (N, p_n)*-means of the Fourier series for* f.

If we deal with Cesàro means of order δ and consider a function $f\in\operatorname{Lip}\alpha$, $0<\alpha\leqq 1$, then our Theorem reduces to Theorem A.

Proof.

$$T_n(x)-f(x) = \frac{1}{2\pi P_n}\int_0^\pi \{f(x+t)+f(x-t)-2f(x)\}\sum_{k=0}^{n} p_{n-k}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\,dt.$$

If we write $\varphi(t)=|\frac{1}{2}f(x+t)+\frac{1}{2}f(x-t)-f(x)|$ then it is clear that

$$\varphi(t)\leqq\omega(t),$$

and therefore,

$$|f(x)-T_n(x)| \leqq \frac{1}{\pi P_n}\int_0^{\pi/n}\frac{\omega(t)}{t/2}\left|\sum_{k=0}^{n} p_{n-k}\sin(k+1/2)t\right|dt+$$

$$+\frac{1}{\pi P_n}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t/2}\left|\sum_{k=0}^{n} p_{n-k}\sin kt\right|dt+\frac{1}{\pi P_n}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t/2}\left|\sum_{k=0}^{n} p_{n-k}\cos kt\right|dt =$$

$$= I_1+I_2+I_3, \text{ say.}$$

Now

$$I_1 = \frac{1}{\pi P_n}\int_0^{\pi/n}\frac{\omega(t)}{t/2}\left|\sum_{k=0}^{n} p_{n-k}\sin(k+1/2)t\right|dt =$$

$$= O\left(\frac{1}{P_n}\right)\int_0^{\pi/n}\frac{\omega(t)}{t}\sum_{k=0}^{n} p_{n-k}(k+1/2)t\,dt =$$

$$= O\left(\frac{1}{P_n}\right)\int_0^{\pi/n}\omega(t)\,dt\sum_{k=0}^{n} p_{n-k}(k+1/2) = O\left(\frac{1}{nP_n}\right)\omega\left(\frac{1}{n}\right)\sum_{k=0}^{n} kp_{n-k} = O(\omega(1/n)).$$

By (1.2),

$$\frac{1}{P_n}\sum_{k=0}^{n}\frac{P_k\omega(1/k)}{k} \geqq \frac{\omega(1/n)}{cP_n}\sum_{k=0}^{n}p_k = \frac{1}{c}\omega(1/n),$$

consequently,

$$I_1 = O\left\{\frac{1}{P_n}\sum_{k=0}^{n}\frac{P_k\omega(1/k)}{k}\right\}.$$

Now

$$I_2 \leqq \frac{2}{\pi P_n}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t}\left|\sum_{k=0}^{n}p_{n-k}\sin kt\right|dt = O\left\{\frac{1}{P_n}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t}P\left(\frac{1}{t}\right)dt\right\} =$$

$$= O\left\{\frac{1}{P_n}\int_{n/\pi}^{1/\pi}\frac{\omega(1/t)}{1/t}P(t)\left(-\frac{dt}{t^2}\right)\right\} = O\left\{\frac{1}{P_n}\int_{n/\pi}^{1/\pi}\frac{\omega(1/t)}{t}P(t)\,dt\right\} =$$

$$= O\left\{\frac{1}{P_n}\sum_{k=1}^{n}\frac{P(k)\omega(1/k)}{k}\right\}, \quad \text{where} \quad P(k) = P_{[k]}.$$

Similarly,

$$I_3 \leqq \frac{2}{\pi P_n}\int_{\pi/n}^{\pi}\omega(t)P\left(\frac{1}{t}\right)dt = O\left\{\frac{1}{P_n}\sum_{k=1}^{n}\frac{P(k)\omega(1/k)}{k^2}\right\}$$

which is dominated by the bound for I_2.

Adding the bounds for I_1, I_2, I_3 we have the desired result.

3. Remarks. It may be interesting to know the answers to the following questions:

(i) Can our Theorem be extended to matrix summability?

(ii) Can the result be extended to differentiated Fourier series?

(iii) Can the result be extended to some other series, viz. Legendre series, ultraspherical, Bessel series, etc?

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Zur Charakterisierung von Vereinigungserweiterungen von Halbgruppen durch partielle Morphismen

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In [4, 5] führte VERBEEK den Begriff der Vereinigungserweiterung von Halbgruppen ein. Er gewann diesen als eine Verallgemeinerung des Begriffs der Idealerweiterung. VERBEEK [4, 5] und der Verfasser [2, 3] zeigten, daß Idealerweiterungen und Vereinigungserweiterungen in vieler Hinsicht eng verwandt sind. Im folgenden befassen wir uns mit der bekannten teilweisen Charakterisierung von Idealerweiterungen durch partielle Morphismen [1, Satz 4.19.]. Wir geben eine Verallgemeinerung für Vereinigungserweiterungen an; interessant daran scheint uns insbesondere, daß der zweite der Teil der Aussage — jede Idealerweiterung eines Monoids wird durch einen partiellen Morphismus definiert — sich mit einer einfachen Zusatzbedingung auf den Fall der Vereinigungserweiterungen übertragen läßt.

Für in dieser Arbeit nicht definierte Begriffe verweisen wir auf [1, 2, 5].

Den in [1] eingeführten Begriff des partiellen Morphismus verallgemeinernd, definieren wir sogenannte i-partielle Morphismen.

Definition 1. A, S seien Halbgruppen, i ein idempotentes Element von S und $S_i := S \setminus \{i\}$. Eine Abbildung

$$f\colon \{s \mid s \in S_i,\ i \in S^1 s S^1\} \to A : s \mapsto \bar{s} := f(s)$$

mit den folgenden Eigenschaften heißt *i-partieller Morphismus von S in A:*

(P0) $\forall s, t \in S_i : st \neq i \wedge i \in S^1 s S^1 \wedge i \in S^1 t S^1 \to \bar{s}\,\bar{t} = \overline{st}$.

(P1) $\forall s \in S_i : is \neq i = isi \neq si \to \overline{is} = \overline{si}$ ist Nullelement.

(P2) $\forall s \in S_i : is \neq i = si \to \forall a, b \in A : a\bar{s}b = \overline{is}b$.

(P3) $\forall s \in S_i : is = i \neq si \to \forall a, b \in A : b\bar{s}a = b\overline{si}$.

(P4) $\forall s, t \in S_i : is \neq i = ist \to \forall a \in A : a\bar{s}\bar{t} = \overline{is}\,\bar{t}$.

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(P5) $\forall s, t \in S_i:\ ti \neq i = sti \rightarrow \forall a \in A:\ \bar{s}\bar{t}a = \bar{s}\overline{ti}.$

(P6) $\forall s, t \in S_i:\ si \neq i = sit \neq it \rightarrow \overline{si}\bar{t} = \bar{s}\overline{it}.$

Satz 1. *A, S seien Halbgruppen, i ein idempotentes Element von S, $S_i := S \setminus \{i\}$ und $s \mapsto \bar{s}$ ein i-partieller Morphismus von S in A. Durch die Abbildung $s \mapsto \bar{s}$ wird eine Vereinigungserwiterung $(E, *)$ von A mit S bezüglich i folgendermaßen festgelegt:*

$$\text{(M1)}\quad s*t = \begin{cases} st, & \text{falls } st \neq i, \\ \bar{s}\bar{t}, & \text{falls } st = i, \end{cases} \qquad \text{(M2)}\quad a*s = \begin{cases} is, & \text{falls } is \neq i, \\ a\bar{s}, & \text{falls } is = i, \end{cases}$$

$$\text{(M3)}\quad s*a = \begin{cases} si, & \text{falls } si \neq i, \\ \bar{s}a, & \text{falls } si = i, \end{cases} \qquad \text{(M4)}\quad a*b = ab,$$

mit $s, t \in S_i$ und $a, b \in A$.

Falls umgekehrt $(E, *)$ Vereinigungserweiterung von A mit S bezüglich i ist und eine Abbildung f: $\{s | s \in S_i,\ i \in S^1 s S^1\} \rightarrow A$ mit $\bar{s} := f(s)$ den Bedingungen P0 *und* M1—M4 *genügt, so hat f auch die Eigenschaften* P1 – P6.

Beweis. Zum Beweis des ersten Teils der Aussage ist die Assoziativität von $*$ zu zeigen. Dazu unterscheiden wir 8 Fälle, die wir mit AAA, AAS_i, ... bezeichnen — je nach der Herkunft der Elemente eines Tripels. Die Fälle AAA, AAS_i und S_iAA sind leicht nachzurechnen; P0—P6 werden dabei nicht benutzt. Sei nun $a, b \in A$, $s \in S_i$. Für AS_iA ergibt sich:

$$(a*s)*b = \begin{Bmatrix} is*b & \text{für} & is \neq i \\ & & \\ a\bar{s}*b & \text{für} & is = i \end{Bmatrix} = \begin{Bmatrix} isi & \text{für} & isi \neq i \\ \overline{is}b & \text{für} & is \neq i = isi \\ a\bar{s}b & \text{für} & is = i \end{Bmatrix} =$$

$$= \begin{Bmatrix} isi & \text{für} & isi \neq i \\ \overline{is}b & \text{für} & isi = i, is \neq i \neq si \\ \overline{is}b & \text{für} & is \neq i = si \\ a\bar{s}b & \text{für} & is = i \neq si \\ a\bar{s}b & \text{für} & is = i = si \end{Bmatrix} = \begin{Bmatrix} isi & \text{für} & isi \neq i \\ a\overline{si} & \text{für} & isi = i, is \neq i \neq si \\ a\bar{s}b & \text{für} & is \neq i = si \\ a\overline{si} & \text{für} & is = i \neq si \\ a\bar{s}b & \text{für} & is = i = si \end{Bmatrix} =$$

$$= \begin{Bmatrix} isi & \text{für} & isi \neq i \\ a\overline{si} & \text{für} & isi = i \neq si \\ a\bar{s}b & \text{für} & si = i \end{Bmatrix} = \begin{Bmatrix} a*si & \text{für} & si \neq i \\ & & \\ a*\bar{s}b & \text{für} & si = i \end{Bmatrix} = a*(s*b)$$

mit P1, P2 und P3. Analog erhält man AS_iS_i mit P0 und P4, S_iAS_i mit P6, S_iS_iA mit P0 und P5, $S_iS_iS_i$ mit P0.

Zu Beweis des zweiten Teiles des Satzes leitet man aus der Assoziativität von $*$ die Bedingungen P1—P6 her, und zwar P1, P2 und P3 aus AS_iA, P4 aus AS_iS_i usw. wie oben. Nicht offensichtlich ist darunter nur P1. Sei also $s \in S_i$ und $is \neq i \neq si$ und

$isi=i$. Aus der Assioziativität im Falle AS_iA folgt, daß füı alle $a, b\in A$ gelten muß: $a\,\overline{si}=\overline{is}\,b$. Wegen $isi=i$ sind is und si idempotent; wegen P0 sind es dann auch $\overline{is}$ und $\overline{si}$. Mit $a=\overline{si}$ und $b=\overline{is}$ folgt

$$\overline{si} = \overline{si}\,\overline{si} = \overline{is}\,\overline{is} = \overline{is},$$

$$\overline{si} = \overline{si}\,\overline{si} = \overline{is}\,A \quad \text{und} \quad \overline{is} = \overline{is}\,\overline{is} = A\,\overline{si}.$$

Also ist $\overline{is}=\overline{si}$ Nullelement von A. Falls umgekehrt $\overline{is}=\overline{si}$ Nullelement von A ist, gilt natürlich auch $a\,\overline{si}=\overline{is}\,b$ für alle $a, b\in A$. Q.e.d.

Im folgenden seien A, S, i, S_i wie in Satz 1. Falls i Nullelement von S ist, sind die i-partiellen Morphismen von S in A gerade die Abbildungen von S_i in A mit der Eigenschaft P0; die bekannte Aussage [1, Satz 4.19] über Idealerweiterungen ergibt sich also als ein Spezialfall von Satz 1.

Ein anderer interessanter Sonderfall liegt vor, wenn i Einselement von S ist. VERBEEK [5, Satz 2] zeigt, daß dabei das Folgende gilt: Falls $S_iS_i\subseteq S_i$ ist, gibt es genau eine Vereinigungserweiterung $(E, *)$ von A mit S bezüglich i, und diese ist durch $a*b=ab$, $a*s=s*a=s$, $s*t=st$ für $a, b\in A$ und $s, t\in S_i$ definiert. Falls $i\in S_iS_i$ ist, gibt es genau dann eine (und auch nur eine) Vereinigungserweiterung von A mit S bezüglich i, wenn A ein Nullelement 0 besitzt; diese Erweiterung ist dann durch $a*b=ab$, $a*s=s*a=s$, $s*t=st$ für $st\neq i$, $s*t=0$ für $st=i$ mit $a, b\in A$ und $s, t\in S_i$ eindeutig definiert. Offensichtlich ist $(E, *)$ in beiden Fällen durch einen i-partiellen Morphismus festgelegt — den leeren bzw. denjenigen mit Bild 0.

Die Bedingungen P0—P6 beschreiben gewissermaßen eine Homomorphie über das ausgelassene idempotente Element i hinweg. In P2 ist $\overline{is}$ und damit $\overline{is}b$ für alle $b\in A$ Rechtsnull; $A\bar{s}$ ist dort Linksideal und Rechtsfastideal, also Fastideal. In P4 ist $\overline{is}\,\bar{t}$ Rechtsnull. In P6 gilt $\overline{si}\,\bar{t}=\overline{si}\,\overline{it}=\bar{s}\,\overline{it}$. Interessant sind natürlich auch die Fälle, in denen die Prämissen für mehrere der Bedingungen P1—P6 erfüllt sind: So garantieren die Kombinationen $P2\wedge P3$, $P2\wedge P5$, $P3\wedge P4$, $P4\wedge P5$ jeweils die Existenz eines Nullelementes 0 in A zusammen mit Beziehungen der Form $\overline{is}=\overline{ti}=0$.

Falls A ein Einselement 1_A besitzt, kann man P0—P6 einfacher formulieren. Es gilt

Satz 2. *S sei eine Halbgruppe mit idempotentem Element i, A ein Monoid. Eine Abbildung*

$$f\colon \{s \mid \{\in S_i,\ i\in S^1sS^1\} \to A$$

mit $\bar{s} := f(s)$ ist genau dann ein i-partieller Morphismus von S in A, wenn sie den Bedingungen genügt: $P0'=P0$, $P1'=P1$, $P6'=P6$ *und*

(P2′) $\forall s\in S_i : is \neq i = si \rightarrow \bar{s} = \overline{is}$ *ist Rechtsnull,* (P3′) $\forall s \in S_i : is = i \neq si \rightarrow \bar{s} = \overline{si}$ *ist Linksnull,*

(P4′) $\forall s, t \in S_i : is \neq i = ist \rightarrow \bar{s}\bar{t} = \overline{is}\bar{t}$ *ist Rechtsnull,*

(P5′) $\forall s, t \in S_i : ti \neq i = sti \rightarrow \bar{s}\bar{t} = \bar{s}\overline{ti}$ *ist Linksnull.*

Beweis. f sei ein i-partieller Morphismus von S in A. Sei $s \in S_i$ und $is \neq i = si$; mit P2 folgt $a\ \bar{s} = a\bar{s}\ 1_A = \overline{is}\ 1_A = \overline{is}$ für alle $a \in A$, insbesondere also auch für $a = 1_A$; damit gilt P2′. Dual erhält man P3′ aus P3. Ähnlich ergibt sich P4′ aus P4 und P5′ aus P5. f erfülle nun umgekehrt die Bedingungen P0′—P6′. Sei $s \in S_i$ und $is \neq i = si$; wegen P2′ folgt für alle $a, b \in A$: $a\,\bar{s}\,b = \bar{s}\,b = \overline{is}\,b$, also P2. Analog erhält man P3—P5. Q.e.d.

Falls A Monoid ist und die Vereinigungserweiterung $(E, *)$ von A mit S bezüglich i durch einen i-partiellen Morphismus von S in A festgelegt wird, gilt

$$\bar{s} = 1_A * s * 1_A$$

für alle $s \in S_i$ mit $isi = i$. Diese Beobachtung führt zu einer Verallgemeinerung des zweiten Teils von [1], Satz 4.19. Zuvor stellen wir einige Fakten zusammen.

Lemma 1. *A sei ein Monoid, S eine Halbgruppe, i ein idempotentes Element von S, $(E, *)$ Vereinigungserweiterung von A mit S bezüglich i. Die Abbildung $S_i \rightarrow E$: $s \mapsto 1_A * s * 1_A$ erfüllt* P1′—P6′.

Beweis. Sei $s \in S_i$ und $is \neq i$; dann ist

$$(1_A * s) * 1_A = is * 1_A = 1_A * is * 1_A;$$

für $a \in A$ folgt

$$a * (1_A * s * 1_A) = (a * s) * 1_A = is * 1_A = 1_A * s * 1_A.$$

Für $t \in S_i$ mit $ti \neq i$ ist analog

$$1_A * t * 1_A = 1_A * ti * 1_A = 1_A * t * 1_A * a.$$

Für $s = t$ und $isi = i$ ergibt sich P1′, für $si = i$ P2′, für $it = i$ P3′, usw. Q.e.d.

Man beachte, daß $1_A * s * 1_A$ genau dann in A liegt, wenn $isi = i$ ist. Daher kann man auch nur für den Fall, daß $iSi = i$ gilt, erwarten, daß eine zum Beweis von [1], Satz 4.19, analoge Konstruktion sämtliche Vereinigungserweiterungen durch partielle Morphismen gewinnen läßt. Im Hinblick auf diesen Beweis ist weiter zu bemerken, daß die für Idealerweiterungen gültige Beziehung $1_A * s = s * 1_A$ für Vereinigungserweiterungen allgemein nicht gültig ist.

Lemma 2. *A sei ein Monoid, S eine Halbgruppe, i ein idempotentes Element von S, $(E, *)$ Vereinigungserweiterung von A mit S bezüglich i. Falls $iSi = i$ gilt und*

$(E, *)$ durch einen i-partiellen Morphismus von S in A festgelegt ist, gilt

$$(1_A * s * 1_A) * (1_A * t * 1_A) = 1_A * s * t * 1_A$$

für alle $s, t \in S_i$.

Beweis. Die Bedingung $iSi=i$ bewirkt, daß der i-partielle Morphismus $s \mapsto \bar{s}$ überall auf S_i definiert ist und daß $1_A * s * 1_A$ für alle $s \in S_i$ in A liegt. Es folgt $\bar{s} = 1_A * s * 1_A$ und daher

$$(1_A * s * 1_A) * (1_A * t * 1_A) = \bar{s}\bar{t} = \begin{Bmatrix} \overline{s*t} & \text{für} & st \neq i \\ s*t & \text{für} & st = i \end{Bmatrix} = 1_A * s * t * 1_A.$$

Q.e.d.

Im Hinblick auf Lemma 1 besteht das angekündigte Ergebnis in einer Umkehrung von Lemma 2.

Satz 3. *A sei ein Monoid, S eine Halbgruppe mit idempotentem Element i, für das $iSi=i$ gilt. Jede Vereinigungserweiterung $(E, *)$ von A mit S bezüglich i wird durch einen i-partiellen Morphismus $s \mapsto \bar{s}$ von S in A festgelegt.*

Beweis. Falls $(E, *)$ durch einen i-partiellen Morphismus $s \mapsto \bar{s}$ von S in A festgelegt wird, so muß $\bar{s} = 1_A * s * 1_A$ wegen $iSi=i$ gelten. Sei also die Abbildung $s \mapsto \bar{s}$ in dieser Weise gegeben. Es gilt $S_i = \{s \mid s \in S_i,\ i \in S^1 s S^1\}$. Nach Lemma 1 sind P1′—P6′ erfüllt. Wir zeigen P0′: Sei $s, t \in S_i$; es gilt

$$\bar{s}\bar{t} = (1_A * s * 1_A) * (1_A * t * 1_A) = \begin{cases} is * 1_A * ti & \text{für} \quad is \neq i \neq ti \\ is * (t * 1_A) & \text{für} \quad is \neq i = ti \\ (1_A * s) * ti & \text{für} \quad is = i \neq ti \\ (1_A * s) * (t * 1_A) & \text{für} \quad is = i = ti. \end{cases}$$

Falls $is=i$ oder $ti=i$ ist, folgt

$$\bar{s}\bar{t} = 1_A * s * t * 1_A.$$

Damit dies auch für $is \neq i \neq ti$ gilt, ist notwendig und hinreichend, daß dann $is * ti = is * 1_A * ti$ ist; diese Gleichung folgt aus der Assoziativität von $*$ folgendermaßen:

$$is * 1_A * ti = ((is * ti) * is) * 1_A * ti = is * (ti * (is * 1_A * ti)) = is * ti.$$

Für $st \neq i$ ergibt sich insbesondere $\bar{s}\bar{t} = \overline{st}$. Damit ist die Abbildung $s \mapsto \bar{s}$ ein i-partieller Morphismus von S in A. Es bleibt zu zeigen, daß $(E, *)$ durch diese Abbildung bestimmt wird: Sei $s, t \in S_i$, $a \in A$. Für M1 sei $st=i$; dann ist mit dem Obigen

$$s * t = 1_A * s * t * 1_A = \bar{s}\bar{t}.$$

Für M2 sei $is=i$; dann ist

$$a * s = a * 1_A * s * 1_A = a\bar{s}.$$

M3 folgt dual. M4 ist klar. Q.e.d.

Wir bemerken noch die folgende interessante Tatsache, die aus den Sätzen 2 und 3 folgt: Falls es — unter den Voraussetzungen von Satz 3 — in S_i ein s mit $is \neq i$ oder ein t mit $ti \neq i$ gibt, so besitzt A eine Rechtsnull, nämlich $is * 1_A$, bzw. eine Linksnull, nämlich $1_A * ti$; falls also s und t wie oben beide existieren, so ist das Element $is * ti = is * 1_A * ti = is * 1_A = 1_A * ti$ Nullelement von A.

Aus dem Beweis von Satz 3 kann man weiter für den Fall $iSi \neq i$ schließen: Es gibt nur dann eine Vereinigungserweiterung $(E, *)$ eines Monoids A mit S bezüglich i, wenn es eine Abbildung $s \mapsto \bar{s}$ von $S_i' = \{s \mid s \in S_i, isi = i\}$ in A gibt, die den Bedingungen P0′—P6′ und M1—M4 — jeweils mit S_i' statt S_i — genügt (in P0′ muß dabei zusätzlich $st \in S_i'$ gefordert werden).

Zusatz während der Korrektur: Die Vereinigungserweiterungen durch Halbgruppen S bezüglich i mit $iSi = i$ wurden von uns inzwischen vollständig beschrieben (Vereinigungserweiterungen durch vollständig O-einfache Halbgruppen, *Semigroup Forum,* **11** (1975/76), 185—188).

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Unitary dilations and C*-algebras

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The purpose of this Note is to present a "global", i.e., point-free characterization of the $\mathcal{C}_a$-classes of operators introduced as generalizations of the $\mathcal{C}_\varrho$-classes of SZ.-NAGY and FOIAŞ [2] (for the details see **1**). This permits us to define analogous classes in an arbitrary C^*-algebra. Certain properties of the $\mathcal{C}_a$-classes derive in a simpler way in this more general setting.

1.

Let H be a complex Hilbert space, $B(H)$ the C^*-algebra of all bounded linear operators of H. Denote by $B^+(H)$ the convex cone of positive elements of $B(H)$. Consider a boundedly invertible element a of $B^+(H)$. An element $\mathbf{x}$ of $B(H)$ is said to admit a *unitary a-dilation* if there exists a Hilbert space K containing H as its subspace and a unitary element $\mathbf{u}$ of $B(K)$ such that

$$a^{-1/2}x^n a^{-1/2} = \mathrm{pr}_H u^n \quad (n = 1, 2, \ldots).$$

The set of elements of $B(H)$ which admit unitary a-dilations is denoted by $\mathcal{C}_a$. LANGER characterized the $\mathcal{C}_a$-classes in the following manner (cf. [3], pp. 53—54):

An element x of $B(H)$ belongs to $\mathcal{C}_a$ if and only if:

(i) *the spectrum $\sigma(x)$ of x is contained in the closed disc $\mathbf{C}_1$ of the complex number field $\mathbf{C}$, and*

(ii) *for every $\xi \in H$ and $\mu \in \mathbf{C}_1$,*

$$\langle a\xi, \xi\rangle - 2\,\mathrm{Re}\,\langle \mu(a-e)x\xi, \xi\rangle + |\mu|^2 \langle (a-2e)x\xi, x\xi\rangle \geqq 0$$

(e denotes the identity operator of H).

Since

$$\mathrm{Re}\,\langle y\xi, \xi\rangle = \langle (\mathrm{Re}\, y)\xi, \xi\rangle, \qquad (y \in B(H),\ \xi \in H)$$

the preceding inequality can be written as

$$\langle (a - 2\,\mathrm{Re}\,\mu(a-e)x + |\mu|^2 x^*(a-2e)x)\xi, \xi\rangle \geqq 0,$$

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i.e.,

$$a-2\,\mathrm{Re}\,\mu(a-e)x+|\mu|^2x^*(a-2e)x \geqq 0. \tag{1}$$

But

$$a-2\,\mathrm{Re}\,\mu(a-e)x+|\mu|^2x^*(a-2e)x =$$

$$= a-\mu ax-\bar{\mu}x^*a+\mu x+\bar{\mu}x^*+(\mu x)^*a(\mu x)-(\mu x)^*(\mu x)-|\mu|^2x^*x+e-e =$$

$$= [(\mu x)^*a-a-(\mu x)^*+e](\mu x)-[(\mu x)^*a-a-(\mu x)^*+e]+e-|\mu|^2x^*x =$$

$$= (\mu x-e)^*(a-e)(\mu x-e)+e-|\mu|^2x^*x.$$

Thus (1) is equivalent to

$$|\mu|^2x^*x \leqq e+(\mu x-e)^*(a-e)(\mu x-e)$$

or, putting $\mu=1/\lambda$, to

$$x^*x \leqq |\lambda|^2e+(x-\lambda e)^*(a-e)(x-\lambda e).$$

Thus condition (ii) is equivalent to condition

(iii) $\qquad x^*x \leqq |\lambda|^2e+(x-\lambda e)^*(a-e)(x-\lambda e)\quad$ *for all* $\quad\lambda\in\mathbf{C},\ |\lambda|\geqq 1.$

Summing up the results, we obtain

Proposition 1. *Let* $a\in B^+(H)$ *be arbitrary. For an element* x *of* $B(H)$ *conditions* (ii) *and* (iii) *are equivalent.*

2.

Let A be an arbitrary complex C^*-algebra with unity e. Denote by A^+ the convex cone of positive elements of A. Let $a\in A^+$ be arbitrary. Denote by C_a the set of the elements x of A which satisfy condition (iii).

Proposition 2. C_a *is an increasing function of* a *in the sense that* $a_1, a_2\in A^+$, $a_1\leqq a_2$ *imply* $C_{a_1}\subseteqq C_{a_2}$.

Proof. This is a consequence of the fact that for every $y\in A$ we have $y^*a_1y\leqq$ $\leqq y^*a_2y$.

Proposition 3. *If* $\|a\|<2$, *then for* $x\in C_a$ *we have*

$$\|x\| \leqq (\|a\|/(2-\|a\|))^{1/2}. \tag{2}$$

In particular, $\|a\|<1$ *implies* $\|x\|<1$ *for every* $x\in C_a$.

Proof. For $\lambda=\pm 1$, condition (iii) takes the forms

$$x^*x \leqq e+(x-e)^*(a-e)(x-e),\quad x^*x \leqq e+(x+e)^*(a-e)(x+e).$$

By adding up these two inequalities, we obtain

$$2x^*x \leqq x^*ax + a.$$

Now, it is known that in a C^*-algebra $u \geqq 0$, $v \geqq 0$, $u \leqq v$ imply $\|u\| \leqq \|v\|$. Hence,

$$2\|x^*x\| = 2\|x\|^2 \leqq \|x\|^2\|a\| + \|a\|,$$

i.e.

$$(2 - \|a\|)\|x\|^2 \leqq \|a\|,$$

which is equivalent to (2). The rest of the proof is obvious.

Theorem 1. *If $x \in C_a$, then the spectrum $\sigma(x)$ of x is contained in $\mathbf{C}_1$.*

Proof. We know that $\sigma(x)$ is the union of the left-spectrum $\sigma_l(x)$ and the right-spectrum $\sigma_r(x)$ of x. Furthermore,

$$\sup_{\lambda \in \sigma_l(x)} |\lambda| = \sup_{\lambda \in \sigma_r(x)} |\lambda|.$$

Thus, it suffices to show that, for instance, we have $\sigma_l(x) \subset \mathbf{C}_1$. Let S_x^l be the set of all left x-multiplicative states of A: $s \in S_x^l$ if s is a state and if $s(yx) = s(y)s(x)$ for all $y \in A$. It is known ([1]) that

$$\sigma_l(x) = \{s(x) : s \in S_x^l\}.$$

Let μ now be an arbitrary element of $\sigma_l(x)$ and s an element of S_x^l for which $\mu = s(x)$. Apply s to the inequality in (iii). We get

$$s(x^*x) = s(x^*)s(x) = |\mu|^2 \leqq |\lambda|^2 + |\mu - \lambda|^2(s(a) - 1) \tag{3}$$

for each $\lambda \in \mathbf{C}$, $|\lambda| \geqq 1$. Assume that $|\mu| > 1$ and consider a real number ϱ such that $0 < \varrho < 1 - 1/|\mu|$. Put $\lambda = \mu - \varrho\mu$. Then $|\lambda| > 1$. For this particular λ, relation (3) gives

$$|\mu|^2 \leqq |\mu|^2(s - \varrho)^2 + \varrho^2|\mu|^2(s(a) - 1) = |\mu|^2 - 2\varrho|\mu|^2 + \varrho^2|\mu|^2 s(a).$$

This leads us to the inequality $0 \leqq -2 + \varrho s(a)$. If we let ϱ tend to zero, we obtain $0 \leqq -2$: a contradiction. Thus Theorem 1 is proved.

Consider now the case $A = B(H)$. Theorem 1 allows us to formulate the following "global" characterization of the $\mathscr{C}_a$ classes.

Theorem 2. *An element x of $B(H)$ belongs to $\mathscr{C}_a$ if and only if it satisfies condition* (iii).

Proof. On account of Langer's result mentioned in **1**, the necessity part of the theorem follows from Proposition 1. The sufficiency part is a consequence of Theorem 1 and Proposition 1 (using Langer's result).

Corollary 1. *$\mathscr{C}_a$ is an increasing function of a.*

Corollary 2. *If* $\|a\| < 1$, *then the minimal unitary dilation of every element* x *of* $\mathscr{C}_a$ *is a bilateral shift with multiplicity equal to* dim H.

Proof. See proposition 3 of the present paper and [3], Cor. II. 7. 5.

Problem. We could not decide yet whether C_a is a strictly increasing function of a or not.

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Factorization of operator pencils

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Let L be the pencil

$$L(\lambda) = \lambda^n I + \lambda^{n-1} L_{n-1} + \ldots + \lambda L_1 + L_0 \tag{1}$$

where the coefficients $L_0, L_1, \ldots, L_{n-1}$ are bounded operators in a Banach space $\mathfrak{B}$. We set

$$\mathbf{L} = \begin{bmatrix} -L_{n-1} & -L_{n-2} & \ldots & -L_1 & -L_0 \\ I & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & I & 0 \end{bmatrix}; \tag{2}$$

this is an operator in $\mathbf{B} = \mathfrak{B}^n$. In the papers [1]—[5] it was shown that there is a close connection between certain invariant subspaces of $\mathbf{L}$ and the representation of $L(\lambda)$ as the product of a pencil of degree $n-1$ and of another of degree 1 with leading coefficients I. The aim of this note is the study of a similar connection for other types of factorizations, e.g. for those into factors of degree >1 each.

If the underlying space is a Hilbert space, $\mathfrak{B} = \mathfrak{H}$, and the coefficients in (1) are selfadjoint, then the invariant subspaces of $\mathbf{L}$ we shall treat are maximal nonnegative or maximal nonpositive with respect to some indefinite scalar product on $\mathbf{H} = \mathfrak{H}^n$.

Some of the results may turn out to be new even in the case of a matrix pencil. Theorem 4, for instance, states that in a unitary space a pencil of degree n with hermitean matrix coefficients can be written as the product of two pencils of degrees $\left[\frac{n}{2}\right]$ and $\left[\frac{n+1}{2}\right]$, such that one factor is invertible in the open upper and the other is invertible in the open lower half plane. [1])

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1. Preliminaries

In the following all the operators are bounded. If $\mathfrak{H}$ is a Hilbert space with scalar product $(.,.)$ and G is a selfadjoint operator in $\mathfrak{H}$, we define the G-scalar product $[.,.]$ by the equation

$$[x, y] = (Gx, y) \quad (x, y \in \mathfrak{H}). \tag{3}$$

We only need the case where G is indefinite and boundedly invertible, that is, in the terminology of [6] $\mathfrak{H}$ is a Krein space with respect to the scalar product (3). A subspace $\mathfrak{L} \subset \mathfrak{H}$ is called G-nonnegative, G-nonpositive or G-neutral according as $[x, x] \geqq 0$, $\leqq 0$ or $= 0$ for all $x \in \mathfrak{L}$; it is called uniformly G-positive if $[x, x] \geqq \gamma \|x\|^2$ for all $x \in \mathfrak{L}$ with some $\gamma > 0$. A G-nonnegative subspace which is not properly contained in any other G-nonnegative subspace is called maximal G-nonnegative. An operator A in $\mathfrak{H}$ is said to be G-selfadjoint, if $GA = (GA)^*$, or equivalently, if

$$[Ax, y] = [x, Ay] \quad \text{for all} \quad x, y \in \mathfrak{H}.$$

If $\mathfrak{L} \subset \mathfrak{H}$, we write

$$\mathfrak{L}^{[\perp]} = \{x : [x, \mathfrak{L}] = \{0\}\} \tag{4}$$

and call $\mathfrak{L}^{[\perp]}$ the G-orthogonal companion of $\mathfrak{L}$.

With the pencil (1) in the Banach space $\mathfrak{B}$ we associate the operators $\mathbf{L}$ (see (2)) and

$$\mathbf{G} = \begin{bmatrix} 0 & 0 & \dots & 0 & I \\ 0 & 0 & \dots & I & L_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & I & \dots & L_3 & L_2 \\ I & L_{n-1} & \dots & L_2 & L_1 \end{bmatrix} \tag{5}$$

in $\mathbf{B} = \mathfrak{B}^n$. Evidently, $\mathbf{G}$ is boundedly invertible. If $\mathfrak{B} = \mathfrak{H}$ is a Hilbert space and the L_j are selfadjoint, then $\mathbf{G}$ is also selfadjoint; moreover, $\mathbf{G}$ is indefinite and $\mathbf{L}$ is $\mathbf{G}$-selfadjoint.

In the next section together with the Banach space $\mathfrak{B}$ we consider its dual $\mathfrak{B}^*$. Then (f^*, x) denotes the value of $f^* \in \mathfrak{B}^*$ at the point $x \in \mathfrak{B}$, and for a subspace $\mathfrak{L} \subset \mathfrak{B}$ we write

$$\mathfrak{L}^{\perp} = \{f^* \in \mathfrak{B}^* : (f^*, \mathfrak{L}) = \{0\}\}.$$

2. Invariant subspaces of L and factorizations of L.

Theorem 1. *The pencil* (1) *admits a factorization*

$$L(\lambda) = \tilde{M}(\lambda) K(\lambda) \tag{6}$$

with a pencil K: $K(\lambda) = \lambda^k I + \lambda^{k-1} K_{k-1} + \dots + \lambda K_1 + K_0$ *of degree* $k (< n)$ *and a pencil* $\tilde{M}$

of degree $n-k$ if and only if the operator $\mathbf{L}$ in $\mathfrak{B}$ has an invariant subspace $\mathfrak{K}$ of the form

$$\mathfrak{K}=\left\{\begin{bmatrix} K_{11}x_1+K_{12}x_2+\dots+K_{1k}x_k \\ K_{21}x_1+K_{22}x_2+\dots+K_{2k}x_k \\ \vdots \\ K_{n-k,1}x_1+K_{n-k,2}x_2+\dots+K_{n-k,k}x_k \\ x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} : x_1, x_2, \dots, x_k \in \mathfrak{B}\right\} \tag{7}$$

with bounded linear operators K_{ij} in $\mathfrak{B}$ such that $-K_{n-k,j}=K_{k-j}$ $(j=1, 2, \dots, k)$. In this case also the operators K_{ij} $(1\leqq i\leqq n-k-1;\ 1\leqq j\leqq k)$ are uniquely determined by the operators $K_0, K_1, \dots, K_{k-1}$. Moreover, $\sigma(K)=\sigma(\mathbf{L}|\mathfrak{K})$.[2])

Proof. (α) In order to show that the invariance of the subspace (7) implies the existence of a factorization (6), we consider the resolvent $(\mathbf{L}-\lambda\mathbf{I})^{-1}$ for sufficiently large λ. It has the following matrix form:

$$-\begin{bmatrix} \lambda^{n-1} \\ \lambda^{n-2} \\ \vdots \\ \lambda \\ 1 \end{bmatrix} L(\lambda)^{-1}[1, \lambda, \dots, \lambda^{n-2}, \lambda^{n-1}] \begin{bmatrix} I & L_{n-1} & \dots & L_2 & L_1 \\ 0 & I & \dots & L_3 & L_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & I & L_{n-1} \\ 0 & 0 & \dots & 0 & I \end{bmatrix} +$$

$$+\begin{bmatrix} 0 & I & \lambda I & \dots & \lambda^{n-3}I & \lambda^{n-2}I \\ 0 & 0 & I & \dots & \lambda^{n-4}I & \lambda^{n-3}I \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & \lambda I \\ 0 & 0 & 0 & \dots & 0 & I \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

which can be verified by multiplication with the matrix of $\mathbf{L}-\lambda\mathbf{I}$ from the left. Applying $(\mathbf{L}-\lambda\mathbf{I})^{-1}$ to an element of (7) with $x_2=x_3=\dots=x_k=0$, the $(n-k)$-th component is

$$\lambda^k L(\lambda)^{-1}\tilde{M}(\lambda)x_1+x_1, \tag{8}$$

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2) For the definition of the spectrum of a pencil see e.g. [3].

where

$$(9)\qquad \tilde{M}(\lambda) = [1, \lambda, \dots, \lambda^{n-2}, \lambda^{n-1}] \begin{bmatrix} I & L_{n-1} \dots L_2 & L_1 \\ 0 & I \quad \dots L_3 & L_2 \\ \vdots & \vdots \qquad \vdots & \vdots \\ 0 & 0 \quad \dots \; I & L_{n-1} \\ 0 & 0 \quad \dots \; 0 & I \end{bmatrix} \begin{bmatrix} K_{11} \\ K_{21} \\ \vdots \\ K_{n-k,1} \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

By the invariance of $\mathfrak{R}$, the element (8) equals $K_{n-k,1}\, y_1 + \dots + K_{n-k,k}\, y_k$ with

$$y_j = -\lambda^{k-j} L(\lambda)^{-1} \tilde{M}(\lambda) x_1, \quad j = 1, 2, \dots, k.$$

This gives $(\lambda^k I - \lambda^{k-1} K_{n-k,1} - \dots - K_{n-k,k}) L(\lambda)^{-1} \tilde{M}(\lambda) = I$, and the factorization (6) follows.

(β) Suppose, conversely, that $L(\lambda)$ has a factorization of the form (6). Define matrices $\mathcal{K}_j$ $(k \leqq j \leqq n-1)$ by the equations

$$\mathcal{K}_j = \underbrace{\begin{bmatrix} -K_{k-1} & -K_{k-2} \dots -K_1 & -K_0 & 0 \dots 0 \\ I & 0 \quad \dots \quad 0 & 0 & 0 \dots 0 \\ 0 & I \quad \dots \quad 0 & 0 & 0 \dots 0 \\ \vdots & \vdots \qquad\quad \vdots & \vdots & \vdots \quad \vdots \\ 0 & 0 \quad \dots \quad 0 & 0 & 0 \dots I \end{bmatrix}}_{j \text{ columns}} \left.\vphantom{\begin{bmatrix} a\\a\\a\\a\\a \end{bmatrix}}\right\} j+1 \text{ rows}$$

Here $K_0, K_1, \dots, K_{k-1}$ are the coefficients of the factor $K(\lambda)$ appearing in (6). The first step in the partial division of a polynomial $\lambda^l B_l + \lambda^{l-1} B_{l-1} + \dots + \lambda B_1 + B_0$ by $K(\lambda)$ $(l \geqq k)$ from the right gives a remainder whose coefficients are the entries of the product

$$[B_l, B_{l-1}, \dots, B_1, B_0]\mathcal{K}_l.$$

Therefore, the factorization (6) yields

$$[L_{n-1} - K_{k-1}, L_{n-2} - K_{k-2}, \dots, L_{n-k} - K_0, L_{n-k-1}, \dots, L_0]\mathcal{K}_{n-1} \dots \mathcal{K}_k = 0.$$

This is equivalent to

$$(10)\qquad \mathbf{L}\mathcal{K}_{n-1} \dots \mathcal{K}_k = \begin{bmatrix} -K_{k-1} \dots -K_0 & 0 \dots 0 & 0 \\ I \quad \dots \quad 0 & 0 \dots 0 & 0 \\ \vdots \qquad\quad \vdots & \vdots \quad \vdots & \vdots \\ 0 \quad \dots \quad 0 & 0 \dots I & 0 \end{bmatrix} \mathcal{K}_{n-1} \dots \mathcal{K}_k =$$

$$= \mathcal{K}_{n-1} \dots \mathcal{K}_k \begin{bmatrix} -K_{k-1} \dots -K_1 & -K_0 \\ I \quad \dots \quad 0 & 0 \\ \vdots \qquad\quad \vdots & \vdots \\ 0 \quad \dots \quad I & 0 \end{bmatrix},$$

which implies the invariance of the subspace $\mathcal{K}_{n-1}\dots\mathcal{K}_k\mathfrak{B}^k$ under **L**. On the other hand, it is easy to see that this subspace has the form (7) with $K_{n-k,j}=-K_{k-j}$ $(j=1, 2, \dots, k)$.

(γ) Suppose now again that the subspace $\mathfrak{K}$ of the form (7) is invariant under **L**, and let $\mathbf{y}\in\mathfrak{K}$. Considering the components with index $n-k, n-k-1, \dots, 2$ of $\mathbf{Ly}$, one finds easily that the operators K_{ij} $(1\leq i\leq n-k-1;\ 1\leq j\leq k)$ are uniquely determined by $K_{n-k,1}, \dots, K_{n-k,k}$. Therefore,

$$\mathfrak{K} = \mathcal{K}_{n-1}\dots\mathcal{K}_k\mathfrak{B}^k. \tag{11}$$

From (10), using the notation

$$\mathbf{K} = \begin{bmatrix} -K_{k-1} \dots -K_1 & -K_0 \\ I \quad \dots \quad 0 & 0 \\ \vdots \qquad\quad \vdots & \vdots \\ 0 \quad \dots \quad I & 0 \end{bmatrix},$$

we get

$$(\mathbf{L}-\lambda\mathbf{I})\mathcal{K}_{n-1}\dots\mathcal{K}_k = \mathcal{K}_{n-1}\dots\mathcal{K}_k(\mathbf{K}-\lambda\mathbf{I}),$$

and the last statement of the theorem follows. In the case $k=1$ Theorem 1 coincides with Lemma 2 of [5].

Taking adjoints [3]) in (6), we find

$$L^*(\lambda) = K^*(\lambda)\tilde{M}^*(\lambda),$$

where e.g. L^* denotes the pencil

$$L^*(\lambda) = \lambda^n I+\lambda^{n-1}L_{n-1}^*+\dots+\lambda L_1^*+L_0^*$$

in $\mathfrak{B}^*$. Therefore, by Theorem 1, the factor $\tilde{M}^*(\lambda)$ of $L^*(\lambda)$ corresponds to a subspace $\tilde{\mathfrak{M}}^*\subset\mathfrak{B}^*$, of the form

$$\tilde{\mathfrak{M}}^* = \left\{\begin{bmatrix} \tilde{M}_{11}^*f_1^*+\dots+\tilde{M}_{1,n-k}^*f_{n-k}^* \\ \vdots \\ \tilde{M}_{k1}^*f_1^*+\dots+\tilde{M}_{k,n-k}^*f_{n-k}^* \\ f_1^* \\ \vdots \\ f_{n-k}^* \end{bmatrix} : f_1^*, \dots, f_{n-k}^*\in\mathfrak{B}^*\right\}, \tag{12}$$

which is invariant under the operator

$$\tilde{\mathbf{L}} = \begin{bmatrix} -L_{n-1}^* & -L_{n-2}^* \dots -L_1^* & -L_0^* \\ I & 0 \quad \dots \quad 0 & 0 \\ 0 & I \quad \dots \quad 0 & 0 \\ \vdots & \vdots & \\ 0 & 0 \quad \dots \quad I & 0 \end{bmatrix}$$

[3]) If $\mathfrak{B}$ is a Hilbert space, we also have to make a substitution $\lambda\to\lambda^*$ (λ^* — complex conjugate of λ).

in $\mathfrak{B}^* = (\mathfrak{B}^*)^n$. We are going to prove that the pair of subspaces $\mathfrak{K} \subset \mathfrak{B}$, $\tilde{\mathfrak{M}}^* \subset \subset \mathfrak{B}^*$, associated with the factorization (6), satisfies the relation

$$\tilde{\mathfrak{M}}^* = (\mathbf{G}\mathfrak{K})^\perp \quad \text{(see (5))}. \tag{13}$$

To this end first observe that, as one easily checks, $(\mathbf{GL})^* = \mathbf{G}^*\tilde{\mathbf{L}}$. Therefore, $(\mathbf{G}\mathfrak{K})^\perp$ is invariant under $\tilde{\mathbf{L}}$:

$$(\tilde{\mathbf{L}}(\mathbf{G}\mathfrak{K})^\perp, \mathbf{G}\mathfrak{K}) = ((\mathbf{GL})^*(\mathbf{G}\mathfrak{K})^\perp, \mathfrak{K}) = ((\mathbf{G}\mathfrak{K})^\perp, \mathbf{GL}\mathfrak{K}) \subset ((\mathbf{G}\mathfrak{K})^\perp, \mathbf{G}\mathfrak{K}) = \{0\}.$$

Further, an element

$$\begin{bmatrix} v_1^* \\ \vdots \\ v_k^* \\ f_1^* \\ \vdots \\ f_{n-k}^* \end{bmatrix} \in \mathfrak{B}^*; \quad v_1^*, \dots, v_k^*, f_1^*, \dots, f_{n-k}^* \in \mathfrak{B}^*,$$

belongs to $(\mathbf{G}\mathfrak{K})^\perp$ if and only if

$$\begin{bmatrix} K_{11}^* \dots K_{n-k,1}^* & I & 0 \dots 0 \\ K_{12}^* \dots K_{n-k,2}^* & 0 & I \dots 0 \\ \vdots \quad\quad \vdots & \vdots & \vdots \quad \vdots \\ K_{1k}^* \dots K_{n-k,k}^* & 0 & 0 \dots I \end{bmatrix} \mathbf{G}^* \begin{bmatrix} v_1^* \\ \vdots \\ v_k^* \\ f_1^* \\ \vdots \\ f_{n-k}^* \end{bmatrix} = 0. \tag{14}$$

From (14), in view of (5), the vectors $v_1^*, \dots, v_k^*$ can be expressed through $f_1^*, \dots, f_{n-k}^*$, i.e., $(\mathbf{G}\mathfrak{K})^\perp$ has the form given on the right hand side of (12) with bounded operators $\tilde{M}_{ij}$ $(1 \leqq i \leqq k;\ 1 \leqq j \leqq n-k)$. It remains to show that $\tilde{M}_{k1}^*, \dots, \tilde{M}_{k,n-k}^*$ are the coefficients of the pencil $\tilde{M}^*(\lambda)$. The first row in (14) expresses v_k^* through $f_1^*, \dots, f_{n-k}^*$, thus it has the form

$$v_k^* - \tilde{M}_{k1}^* f_1^* - \dots - \tilde{M}_{k,n-k}^* f_{n-k}^* = 0.$$

In order to get the factor of $L^*(\lambda)$, belonging to the subspace $(\mathbf{G}\mathfrak{K})^\perp$, we have to make the formal substitutions $v_k^* \to \lambda^{n-k}, f_1^* \to \lambda^{n-k-1}, \dots, f_{n-k}^* \to 1$, so this factor is given by

$$[K_{11}^*, \dots, K_{n-k,1}^*, I, 0, \dots, 0]\, \mathbf{G}^* \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda^{n-k} \\ \vdots \\ \lambda \\ 1 \end{bmatrix}. \tag{15}$$

But it is easy to see that the first $n-k+1$ components of

$$\mathbf{G}^* \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda^{n-k} \\ \vdots \\ \lambda \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & 0 & \dots & 0 & 0 \\ L_{n-1}^* & I & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ L_2^* & L_3^* & \dots & I & 0 \\ L_1^* & L_2^* & \dots & L_{n-1}^* & I \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix}$$

are the same, that is, the pencil (15) coincides with $\tilde{M}^*(\lambda)$ of (9), and (13) is proved.

In the rest of the paper $\mathfrak{B}$ will be a Hilbert space, $\mathfrak{B}=\mathfrak{H}$, and the operators L_j will be selfadjoint, $L_j=L_j^*$ $(0\leqq j\leqq n-1)$. Then the operator $\mathbf{L}$ is $\mathbf{G}$-selfadjoint [cf. (2) and (5)]. Further, the orthogonal complement appearing in (13) is to be taken with respect to the natural scalar product of $\boldsymbol{\mathfrak{H}}=\mathfrak{H}^n$, and (13) can also be written as as $\tilde{\mathfrak{M}}^*=\mathfrak{K}^{[\perp]}$ (see (4)), i.e. the subspaces $\mathfrak{K}$ and $\tilde{\mathfrak{M}}^*$ associated with the factorization (6) are $\mathbf{G}$-orthogonal companions of each other.

Theorem 2. *The pencil* (1) *with selfadjoint coefficients $L_0, L_1, \dots, L_{n-1}$ in the Hilbert space $\mathfrak{H}$ admits a factorization*

$$L(\lambda) = M^*(\lambda)R(\lambda)K(\lambda)$$

into three pencils with leading coefficient I and of degree m, r, k $(m+r+k=n)$ if and only if the operator $\mathbf{L}$ in $\boldsymbol{\mathfrak{H}}$ has a pair of invariant subspaces $\mathfrak{K}$ and $\mathfrak{M}\subset\mathfrak{K}^{[\perp]}$ of the form (7) *and*

$$\mathfrak{M} = \left\{ \begin{bmatrix} M_{11}x_1 + \dots + M_{1m}x_m \\ \vdots \\ M_{n-m,1}\,x_1 + \dots + M_{n-m,m}\,x_m \\ x_1 \\ \vdots \\ x_m \end{bmatrix} : x_1, x_2, \dots, x_m \in \mathfrak{H} \right\}$$

with bounded linear operators M_{ij} in $\mathfrak{H}$. The subspace $\mathfrak{M}$ and the pencil M are connected in the same way as $\mathfrak{K}$ and K.

Proof. Suppose $\mathbf{L}$ has a pair of invariant subspaces $\mathfrak{K}$, $\mathfrak{M}$ with the properties mentioned in the theorem. The condition $\mathfrak{M}\subset\mathfrak{K}^{[\perp]}$ implies

$$\text{(16)} \qquad [K_{11}^*, K_{21}^*, \dots, K_{n-k,1}^*, I, 0, \dots, 0]\mathbf{G} \begin{bmatrix} M_{11} & M_{12} & \dots & M_{1m} \\ \vdots \\ M_{n-m,1} & M_{n-m,2} & \dots & M_{n-m,m} \\ I & 0 & \dots & 0 \\ 0 & I & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & I \end{bmatrix} = 0.$$

If the matrices $\mathcal{M}_j$ $(m \leqq j \leqq n-1)$ correspond to the subspace $\mathfrak{M}$ in the same way as the matrices $\mathcal{K}_j$ correspond to $\mathfrak{K}$ (cf. (11)), we get from (16)

$$[K_{11}^*, K_{21}^*, \dots, K_{n-k,1}^*, I, 0, \dots, 0]\mathbf{G}\mathcal{M}_{n-1} \dots \mathcal{M}_m = 0. \tag{17}$$

But the components of the vector

$$[K_{11}^*, K_{21}^*, \dots, K_{n-k,1}^*, I, 0, \dots, 0]\mathbf{G}$$

are the coefficients of $\tilde{M}^*(\lambda)$ (see (9)). Therefore (17) yields that the subspace $\mathfrak{M} = \mathcal{M}_{n-1} \dots \mathcal{M}_m \mathfrak{B}^m$ is invariant under the operator $\tilde{\mathbf{M}}$ in $\mathfrak{B}^{n-k}$ corresponding to the pencil $\tilde{M}^*(\lambda)$ (cf. (10)). Hence, by theorem 1, the polynomial $\tilde{M}^*(\lambda)$ has the right hand factor $M(\lambda)$. This reasoning can be reversed, and the statement follows from Theorem 1.

3. Maximal G-nonnegative invariant subspaces and factorizations

Let $\mathfrak{H}$ be a Hilbert space, and let $L_0, L_1, \dots, L_{n-1}$ be selfadjoint operators in $\mathfrak{H}$. In order to show the existence of invariant subspaces of $\mathbf{L}$ of the form (7), we use results from operator theory in spaces with an indefinite metric. The key is

Theorem 3. *A maximal* **G**-*nonnegative* (**G**-*nonpositive*) *subspace* $\mathfrak{K} \subset \mathfrak{H}$ *which is invariant under the operator* $\mathbf{L}$ *has the form* (7) *with* $k = \left[\frac{n+1}{2}\right]$ $\left(k = \left[\frac{n}{2}\right]\right)$.

Proof. We restrict ourselves to the case $n = 2k$. Let $\mathfrak{K}$ be maximal **G**-nonnegative and invariant under $\mathbf{L}$. Suppose $\mathfrak{K}$ contains a sequence of elements

$$x^{(r)} = \begin{bmatrix} x_1^{(r)} \\ x_2^{(r)} \\ \vdots \\ x_n^{(r)} \end{bmatrix}; \quad r = 1, 2, \dots,$$

with $x_j^{(r)} \to 0$ $(r \to \infty;\ j = k+1, \dots, n)$ and $\sum_{j=1}^{k} \|x_j^{(r)}\|^2 = 1$ $(r = 1, 2, \dots)$. Then we have $[\mathbf{x}^{(r)}, \mathbf{x}^{(r)}] \to 0$ and, by Schwarz's inequality, $[\mathbf{L}^\varkappa \mathbf{x}^{(r)}, \mathbf{x}^{(r)}] \to 0$ $(r \to \infty;\ \varkappa = 1, 2, \dots)$. Now from the matrix representation of $\mathbf{GL}, \mathbf{GL}^3, \dots, \mathbf{GL}^{n-1}$ it easily follows that $x_j^{(r)} \to 0$ $(j = k, k-1, \dots, 1)$. Contradiction. Therefore, if $\mathbf{x} \in \mathfrak{K}$, the relations $x_j = 0$ $(k+1 \leqq j \leqq n)$ imply $x_j = 0$ $(1 \leqq j \leqq k)$. Now it is easy to see that for an $\mathbf{x} \in \mathfrak{K}$ the first k components are uniquely defined by the last k components. Moreover, from the linearity of $\mathfrak{K}$ and the foregoing consideration, they are even bounded linear functions of the last k components. Therefore, with $\mathbf{x}' = \begin{bmatrix} x_{k+1} \\ \vdots \\ x_{2k} \end{bmatrix} \in \mathfrak{H}^k$ and a bounded linear operator $\mathbf{K}'$ in $\mathfrak{H}^k$, the subspace $\mathfrak{K}$ can be written as

$$\mathfrak{K} = \left\{\begin{pmatrix}\mathbf{K}'\,\mathbf{x}' \\ \mathbf{x}'\end{pmatrix} : \mathbf{x}' \in \mathfrak{D}'\right\},$$

where $\mathfrak{D}'$ is evidently a closed subspace of $\mathfrak{H}^k$. It remains to show that $\mathfrak{D}' = \mathfrak{H}^k$. Suppose $\mathfrak{D}' \neq \mathfrak{H}^k$. Then there exists an $\mathbf{y}_0' \in \mathfrak{H}^k$ with the property

$$(\mathbf{G}_{12}'\mathfrak{D}', \mathbf{y}_0') = \{0\}, \quad \text{where} \quad \mathbf{G}_{12}' = \begin{bmatrix} 0 & \dots & 0 & I \\ 0 & \dots & I & L_{n-1} \\ \vdots & & \vdots & \vdots \\ I & \dots & L_{n-k+1} & L_{n-k} \end{bmatrix}.$$

This is equivalent to

$$\left[\mathfrak{K}, \begin{pmatrix}\mathbf{y}_0' \\ 0\end{pmatrix}\right] = \{0\}, \quad \text{that is} \quad \begin{pmatrix}\mathbf{y}_0' \\ 0\end{pmatrix} \in \mathfrak{K}^{[\perp]}.$$

But $\begin{pmatrix}\mathbf{y}_0' \\ 0\end{pmatrix}$ is **G**-neutral and does not belong to $\mathfrak{K}$, so that $\mathfrak{K}$ cannot be maximal **G**-nonnegative. Contradiction.

Theorems 1 and 3 yield a factorization of a selfadjoint pencil (1) as soon as the existence of a maximal **G**-nonnegative subspace $\mathfrak{K}$ of the corresponding operator **L** is known. From a result of PONTRJAGIN [6; Theorems IX. 7.2 and VIII. 3.2] we immediately obtain the following

Theorem 4. *Let L be a pencil in the finite dimensional unitary space $\mathfrak{H}$. Suppose that the coefficients of L are symmetric matrices. Decompose the nonreal spectrum σ_0 of L into two disjoint parts σ_I and σ_{II} such that $\sigma_I \cap \sigma_I^* = \emptyset$ [4]), $\sigma_{II} = \sigma_I^*$. Then L admits a factorization*

(18) $$L(\lambda) = \tilde{L}_{II}(\lambda) L_I(\lambda),$$

where L_I and $\tilde{L}_{II}$ are pencils of degree $\left[\frac{n}{2}\right]$ and $\left[\frac{n+1}{2}\right]$. In addition, we may require that the nonreal spectrum of L_I and $\tilde{L}_{II}$ is σ_I and σ_{II} respectively.

Now let L be a pencil of the form (1) with selfadjoint coefficients L_j $(0 \leqq j \leqq n-1)$ in the infinite dimensional Hilbert space $\mathfrak{H}$. Suppose L has only real zeros (in the terminology of [3]), that is, each polynomial $p_x(\lambda) = (L(\lambda)x, x)$ $(x \in \mathfrak{H};\ x \neq 0)$ has n real zeros $\lambda_1(x) \geqq \lambda_2(x) \geqq \ldots \geqq \lambda_n(x)$. Then the spectral zones

$$\Lambda_j = \{\lambda_j(x) : x \in \mathfrak{H}, x \neq 0\} \quad (j = 1, 2, \ldots, n)$$

[4]) σ_I^* denotes the set of complex conjugates to the points of σ_I.

are intervals of the real axis, and the intersection of two different zones consists of no more than one point. Define

(19) $$\Lambda_+ = \bigcup_{j=0}^{\left[\frac{n-1}{2}\right]} \overline{\Lambda_{2j+1}}, \quad \Lambda_- = \bigcup_{j=1}^{\left[\frac{n}{2}\right]} \overline{\Lambda_{2j}}.$$

By [3; Theorem 2] the operator $\mathbf{L}$ is definitizable in $\mathfrak{H}$, so that [7; Theorem 3.2] it has a maximal $\mathbf{G}$-nonnegative and a maximal $\mathbf{G}$-nonpositive invariant subspace, say $\mathfrak{R}_+$ and $\mathfrak{R}_-$, with $\sigma(\mathbf{L}|\mathfrak{R}_+)\subset\Lambda_+$ and $\sigma(\mathbf{L}|\mathfrak{R}_-)\subset\Lambda_-$. From theorems 1 and 3 we obtain

Theorem 5. *Suppose that the pencil L of degree n in the Hilbert space $\mathfrak{H}$ has only real zeros. Then L admits a factorization of the form*

(20) $$L(\lambda) = \tilde{L}_-(\lambda)L_+(\lambda),$$

where L_+ and $\tilde{L}_-$ are pencils of degree $\left[\frac{n+1}{2}\right]$ and $\left[\frac{n}{2}\right]$, respectively, and $\sigma(L_+)\subset\Lambda_+$, $\sigma(\tilde{L}_-)\subset\Lambda_-$.

Taking adjoints in (20), we get a factorization

$$L(\lambda) = \tilde{L}_+(\lambda)L_-(\lambda),$$

where $L_-(\lambda)=\tilde{L}_-^*(\lambda)$, $\tilde{L}_+(\lambda)=L_+^*(\lambda)$. Evidently, degrees and spectra of L_- and $\tilde{L}_+$ have the same properties as those of $\tilde{L}_-$ and L_+. In general, the factorizations (18) and (20) are not uniquely determined by the properties listed in Theorems 4 and 5. A similar remark holds for Theorem 7 below.

4. A theorem on maximal *G*-nonnegative invariant subspaces

The following theorem is a slight generalization of [6, Theorem VIII. 3.2] as applied to bounded operators.

In the sequel, $\mathscr{S}_\infty$ denotes the set of compact operators in the Hilbert space $\mathfrak{H}$. If $B_2-B_1\in\mathscr{S}_\infty$ for two (bounded linear) operators B_1, B_2 in $\mathfrak{H}$, we write $B_1\sim B_2$. Clearly, $B_1\sim B_2$ is equivalent to $B_1^*\sim B_2^*$.

Theorem 6. *Suppose the operators A_1 and A_2 in $\mathfrak{H}$ have the following properties:*

(i) *A_1 is G_1-selfadjoint, A_2 is G_2-selfadjoint;*

(ii) *A_1 has a maximal uniformly G_1-positive invariant subspace $\mathfrak{L}_+$;*

(iii) $A_1\sim A_2$, $G_1\sim G_2$.

Then the nonreal spectrum σ_0 of A_2 is discrete. If the sets σ_I, σ_{II} form a partition of σ_0 with the properties $\sigma_I\cap\sigma_I^*=\emptyset$, $\sigma_{II}=\sigma_I^*$, then there exists a maximal G_2-nonnegative subspace $\mathfrak{R}_I$ which is invariant under A_2 and satisfies the conditions $\sigma(A_2|\mathfrak{R}_I)\cap\sigma_0=\sigma_I$,

$\sigma_{ess}(A_2|\mathfrak{K}_I)=\sigma_{ess}(A_1|\mathfrak{L}_+)$. *A similar statement holds with "G_2-nonnegative" replaced by "G_2-nonpositive".*

Proof. We start with four simple remarks which can be checked easily.

(a) If $(.,.)$ and $(.,.)_0$ are two equivalent Hilbert scalar products on $\mathfrak{H}$ and we have

$$(G_1x, y) = (G_1'x, y)_0, \quad (G_2x, y) = (G_2'x, y)_0$$

for all $x, y \in \mathfrak{H}$, then $G_1 \sim G_2$ implies $G_1' \sim G_2'$.

(b) If G_1', G_2' are boundedly invertible selfadjoint operators, then $G_1' \sim G_2'$ implies $|G_1'| \sim |G_2'|$ and sgn $G_1' \sim$ sgn G_2'.

(c) Let A be a G-selfadjoint operator in the Hilbert space $\mathfrak{H}_0$ with scalar product $(.,.)_0$. Define a second Hilbert scalar product $(.,.)_1$ on $\mathfrak{H}_0$ by the equation

$$(x, y)_1 = (|G|x, y)_0 \quad (x, y \in \mathfrak{H}_0).$$

Denote the adjoints of A with respect to $(.,.)_0$ and $(.,.)_1$ by $A^{\otimes}$ and A^*, respectively. Then the condition $A \sim A^*$ is equivalent to $|G|A \sim A^{\otimes}|G|$.

(d) If A is G-selfadjoint, $G=P_+-P_-$ with two orthogonal projectors P_+, P_- $(P_++P_-=I)$ and $P_+AP_- \in \mathscr{S}_\infty$, then for each maximal G-nonnegative invariant subspace $\mathfrak{L}$ we have $\sigma_{ess}(A|\mathfrak{L})=\sigma_{ess}(P_+A|P_+\mathfrak{H})$.

Having made these remarks, consider the decomposition $\mathfrak{H}=\mathfrak{L}_+\dotplus\mathfrak{L}_-$, where $\mathfrak{L}_-$ denotes the G_1-orthogonal companion of $\mathfrak{L}_+$. Introduce the Hilbert scalar product

$$(x, y)_0 = (G_1x_+, y_+)-(G_1x_-, y_-);$$

$$x = x_+ + x_-, \; y = y_+ + y_-; \quad x_+, y_+ \in \mathfrak{L}_+; \quad x_-, y_- \in \mathfrak{L}_-.$$

Conditions (i) and (ii) imply

$$A_1 = A_1^{\otimes}, \tag{21}$$

where $^{\otimes}$ denotes the adjoint with respect to the scalar product $(.,.)_0$. Define operators G_1', G_2' by the equations

$$(G_1x, y) = (G_1'x, y)_0, \quad (G_2x, y) = (G_2'x, y)_0 \quad (x, y \in \mathfrak{H}).$$

Then G_1' is the difference of two complementary projectors which are orthogonal with respect to the scalar product $(.,.)_0$. From (a) and (b) we have $G_1' \sim G_2'$, $|G_2'| \sim I$, therefore in view of (iii) and (21),

$$|G_2'|A_2 - A_2^{\otimes}|G_2'| \sim A_2 - A_2^{\otimes} \sim A_1 - A_1^{\otimes} = 0,$$

i.e.

$$|G_2'|A_2 \sim A_2^{\otimes}|G_2'|. \tag{22}$$

Introducing the scalar product $(x, y)_1=(|G_2'|x, y)_0$ in $\mathfrak{H}$, we have $(G_2x, y)=(G_2'x, y)_0=$ $=(\text{sgn } G_2'x, y)_1$. From remark (c) and (22) it follows that A_2 satisfies the conditions of [6; Theorem VIII. 3.2] relative to the decomposition of $\mathfrak{H}$ into eigenspaces of sgn G_2'. Therefore A_2 has a maximal G_2-nonnegative invariant subspace $\mathfrak{K}_I$ with the property $\sigma(A_2|\mathfrak{K}_I)\cap\sigma_0=\sigma_I$.

Moreover, for the projectors $P_1=\frac{1}{2}(I+G_1')$, $P_2=\frac{1}{2}(I+\operatorname{sgn} G_2')$ from (b) we obtain $P_1\sim P_2$ and $P_1A_1P_1\sim P_2A_2P_2$. By well-known results of perturbation theory, this implies that the operators $P_1A_1P_1$ and $P_2A_2P_2$ have the same essential spectrum. Then, with possible exception of the point zero, the same is true for the restrictions $P_1A_1|_{P_1\mathfrak{H}}$ and $P_2A_2|_{P_2\mathfrak{H}}$. All these considerations are invariant with respect to a shift $A_1\to A_1+\alpha I$, $A_2\to A_2+\alpha I$, α real. Therefore we find $\sigma_{\text{ess}}(P_1A_1|P_1\mathfrak{H})=\sigma_{\text{ess}}(P_2A_2|P_2\mathfrak{H})$. Now the last assertion of the theorem follows from (*d*).

5. Further factorization theorems

As an immediate consequence of Theorems 1, 3 and 6 we have

Theorem 7. *Let* L *be a pencil of the form* (1) *in the Hilbert space* $\mathfrak{H}$. *Assume that the following conditions are fulfilled:*

1) $L_j=L_j'+L_j''$, *where the operators* L_j', L_j'' *are selfadjoint* $(j=0, 1, \ldots, n-1)$.

2) $L_j''\in\mathcal{S}_\infty$ $(j=0, 1, \ldots, n-1)$.

3) *The pencil* $L^{(1)}: L^{(1)}(\lambda) = \lambda^n I+\sum_{j=0}^{n-1}\lambda^j L_j'$ *has only real zeros.*

4) *For the spectral zones* $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ *of* $L^{(1)}$ *we have*

$$\bar{\Lambda}_i\cap\bar{\Lambda}_j=\emptyset \quad (i\neq j;\ i,j=1,2,\ldots,n).$$ [5]

Then the nonreal spectrum σ_0 *of* L *is discrete. If the sets* σ_I, σ_{II} *form a partition of* σ_0 *with the properties* $\sigma_I\cap\sigma_I^*=\emptyset$, $\sigma_{II}=\sigma_I^*$, *there exists a factorization* $L(\lambda)=\tilde{L}_{II}(\lambda)L_I(\lambda)$ *with two pencils* L_I, $\tilde{L}_{II}$ *of degree* $\left[\frac{n}{2}\right]$ *and* $\left[\frac{n+1}{2}\right]$ *such that the nonreal spectrum of* L_I *and* $\tilde{L}_{II}$ *is* σ_I *and* σ_{II}, *respectively, while* [6] $\sigma_{\text{ess}}(L_I)=\sigma_{\text{ess}}(L)\cap\Lambda_+$, $\sigma_{\text{ess}}(\tilde{L}_{II})=\sigma_{\text{ess}}(L)\cap\Lambda_-$ (*cf.* (19)).

[5]) Condition 4) can be replaced by the weaker condition 4'): *The corresponding operator* $\mathbf{L}^{(1)}$ *in* $\mathfrak{H}$ *has no singular critical points* (see [2]).

[6]) The essential spectrum of a pencil L is by definition the essential spectrum of the corresponding operator $\mathbf{L}$ (see (2)).

Proof. We have to show that there exists a maximal **G**-nonnegative subspace invariant under **L**. But the operators

$$\mathbf{A}_2 = \mathbf{L}, \quad \mathbf{A}_1 = \mathbf{L}^{(1)} = \begin{bmatrix} -L'_{n-1} & \dots & -L'_1 & -L'_0 \\ I & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & I & 0 \end{bmatrix},$$

$$\mathbf{G}_2 = \mathbf{G} = \begin{bmatrix} 0 & 0 & \dots 0 & I \\ 0 & 0 & \dots I & L_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ I & L_{n-1} & \dots L_1 & L_1 \end{bmatrix}, \quad \mathbf{G}_1 = \begin{bmatrix} 0 & 0 & \dots 0 & I \\ 0 & 0 & \dots I & L'_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ I & L'_{n-1} & \dots L'_2 & L'_1 \end{bmatrix}$$

satisfy the conditions of theorem 6 as soon as $\mathfrak{L}_+$ is chosen to be the Riesz spectral subspace of $\mathbf{L}_1$ belonging to Λ_+.

Obviously, Theorem 7 contains Theorem 4. Another example for the application of Theorem 7 is the following. Take $n=2$, $L'_0=\alpha I$ (α a real number), and suppose that conditions 1), 2) of theorem 7 are fulfilled. Then conditions 3) and 4) are also fulfilled if and only if

$$(L'_1x, x)^2 - 4\alpha \|x\|^4 \geqq \gamma \|x\|^4 \quad (x \in \mathfrak{H})$$

with some $\gamma > 0$.

Theorem 8. *Let L be a pencil of odd degree $n=2k+1$ with selfadjoint coefficients $L_0, L_1, \dots, L_{2k}$. Suppose there exists a closed subset Ω of the open upper half plane such that each polynomial $p_x(\lambda)=(L(\lambda)x, x)$ $(x\in\mathfrak{H},\ x\neq 0)$ has one zero on the real axis and the other $2k$ zeros are in $\Omega\cup\Omega^*$. Then L admits a factorization of the form*

$$L(\lambda) = K^*(\lambda)(\lambda I - Z_0)K(\lambda), \tag{23}$$

where K is a pencil of degree k with leading coefficient I, the operator Z_0 is selfadjoint, and $\sigma(K)=\sigma(L)\cap\Omega$, $\sigma(Z_0)=\sigma(L)\cap R^1$.

Proof. The Riesz spectral subspace of **L** belonging to $\Omega\cap\sigma(L)$ is maximal **G**-neutral and maximal **G**-nonpositive. Therefore (23) follows from Theorems 2 and 3.

The existence of a factorization (6) with $k=1$, $K(\lambda)=\lambda I-K_0$ implies that K_0 is a solution of the equation

$$L(K_0) \equiv K_0^n + L_{n-1}K_0^{n-1} + \dots + L_1K_0 + L_0 = \frac{1}{2\pi i}\int_{\mathscr{C}_0} L(\lambda)(\lambda I - K_0)^{-1}\,d\lambda = 0$$

($\mathscr{C}_0$ — contour surrounding $\sigma(K_0)$). Conversely, every solution K_0 of $L(K_0)=0$ evidently gives rise to a factorization (6) with $k=1$ and $K(\lambda)=\lambda I-K_0$.

In the case $n=3$, under the hypotheses of Theorem 8 the equation $L(Z)=0$ has three solutions Z_1, Z_2, Z_3. They are uniquely determined by the properties

$$\sigma(Z_1) = \sigma(L)\cap\Omega,\ \sigma(Z_2) = \sigma(L)\cap\Omega^*,\ \sigma(Z_3) = \sigma(L)\cap R^1.$$

The operator Z_3 is similar to a selfadjoint operator. Indeed, the existence and the properties of Z_1 and Z_2 follow from Theorem 8, the existence of Z_3 from [8] (see also [9; Theorem 6]). We have the factorizations

$$L(\lambda) = (\lambda I - Z_1^*)(\lambda I + Z_1^* + Z_1 + L_2)(\lambda I - Z_1) = (\lambda I - Z_3^*)(\lambda I + Z_3^* + Z_1 + L_2)(\lambda I - Z_1)$$

and their analogs with Z_2 instead of Z_1. If L is of even degree $n=2k$ and $\sigma(L)\cap R^1=\emptyset$, then from Theorems 2 and 3 we get the well-known factorization

$$L(\lambda) = K^*(\lambda)K(\lambda),$$

where K is a pencil of degree k and $\sigma(K)=\sigma_0$, the part of $\sigma(L)$ located in the upper half plane. Moreover, K is uniquely determined by these properties as the corresponding subspace $\mathfrak{K}$ is uniquely determined by the properties $\sigma(\mathbf{L}|\mathfrak{K})=\sigma_0$, $\sigma(\mathbf{L}|\mathfrak{K}^{(\perp)})=\sigma_0^*$. Under the weaker condition $L(\lambda)\geqq 0$ $(\lambda=\bar\lambda)$ a similar factorization was proved in [10; Theorem 3.3].

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Rings with e as a radical element

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In [4] rings with identity e, having e as their radical element, were introduced. Here e is said to be a radical element of the ring R, if for every $x, y \in R$ there exists an element b in R such that $xy = bxy$.

Rings having this property are close to commutative rings, but still different. In [4], some properties of these rings are established and it is shown that for primitive rings "every left ideal is a two-sided ideal" is equivalent to "there exists an identity e and e is a radical element".

A primitive ring with e as a radical element is a division ring. In general: e is a radical element in a ring R if and only if $Rxy = Ryx$ for all $x, y \in R$. In § 1 we show that for a simple primering S the property $Sxy = Syx$ for all $x, y \in S$ implies that S has an identity and is a division ring (Theorem 3).

An easy application of the Wedderburn-Artin structure theorem gives that a nil-semisimple artinian ring R with e as a radical element is a finite direct sum of division rings (Theorem 2). We give a general structure theorem for rings R with e as a radical element and having no proper nilpotents (Theorem 4). This last theorem is analogous to a similar theorem of Reid for subcommutative rings [3]. Therefore we investigate the relationship between rings with e as a radical element and subcommutative rings in § 2. Using a fundamental result of LAWVER, we are able to give a counter-example to a conjecture in [4]. It is here that the ring $\mathbf{D}_2$ of rational quaternions with denominators prime to 2 is used. This ring has a proper Jacobson radical ($\neq 0$, $\neq \mathbf{D}_2$) which has some interest in its own and is investigated in § 3. The Jacobson radical $\mathbf{J}(\mathbf{D}_2)$ of $\mathbf{D}_2$ is a ring such that every l-ideal or r-ideal is two-sided, but it does not have an identity.

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§ 1.

Definition. Let R be a ring with identity e. Then e is called a *radical element in* R if for every $x, y \in R$ there exists a non-zero element $b \in R$ such that $xy = byx$.

Evidently any commutative ring with identity e and any division ring has e as a radical element.

Following THIERRIN [5] an ideal I of the ring R is called completely prime if $ab \in I$ implies that $a \in I$ or $b \in I$ for any two elements a and b of R, and completely semi-prime if $a^n \in I$ implies that $a \in I$ for any element a of R. Furthermore, R is called completely prime (completely semi-prime) if the zero-ideal of R is completely prime (completely semi-prime). Clearly R is completely prime if and only if R has no zero divisors i.e. if R is a domain, and completely semi-prime if and only if R has no nonzero nilpotents.

Lemma 1. *Let R be a ring with e as a radical element. Then*

$$R \text{ is prime} \Leftrightarrow R \text{ is completely prime},$$

$$R \text{ is semi-prime} \Leftrightarrow R \text{ is completely semi-prime}.$$

Proof. In [4] Proposition 4.3 it is shown that R is a prime ring implies R has no zero-divisors, hence R is completely prime. The converse is clear. Now let R be semi-prime and let $a \in R$ with $a^n = 0$. If $x \in (Ra)^n$ then x is a sum of elements of the form

$$r_1 a \cdot r_2 a \cdots a \cdot r_n a = r_1 a \cdot r_2 a \cdots a \cdot r_{n-1}[c_n(r_n a)]a = \cdots = r_1 c_2 \cdot r_2 c_3 \cdots r_{n-1} c_n \cdot r_n a^n = 0.$$

So $(Ra)^n = 0$. But R has no nonzero nilpotent l-ideals, hence $Ra = 0$. Then $a = 0$, since $a = ea \in Ra$. So R has no nonzero nilpotents and R is completely semi-prime. The converse is again clear. In the same way it may be shown that an ideal in R is a prime (semi-prime) ideal if and only if it is completely prime (completely semi-prime). This means, in particular, that the intersection of all prime ideals in R coincides with the intersection of all completely prime ideals.

In [4] it is shown that in a ring R with radical element e: $\sqrt{(0)} = \{x \in R : x^n = 0$ for some natural number $n\}$ is the intersection of all completely prime ideals not containing e i.e. the intersection of all completely prime ideals. Hence the intersection of the completely prime ideals is the set of nilpotent elements in a ring R with radical element e.

Now THIERRIN [5] has defined the so-called generalized nil-radical $\mathbf{N}_g$, which is the upper radical determined by the class of all rings without zero-divisors. $\mathbf{N}_g$ is a special radical and for any ring R one has: $\mathbf{N}_g(R) =$ intersection of all ideals I in R such that R/I has no zero-divisors i.e. I is a completely prime ideal in R. Hence for a ring R with e as a radical element, the radical $\mathbf{N}_g$ coincides with the intersection of all prime ideals which is the lower nil radical. Since the upper nil-radical $\mathbf{N} \subseteq \mathbf{N}_g$ for any ring R, one has that for a ring R with radical element e the following ideals coincide:

a) Lower nil radical β=intersection of all prime ideals,
b) Upper nil radical $\mathbf{N}$,
c) Generalized nil radical $\mathbf{N}_g$=intersection of all completely prime ideals,
d) The ideal of all nilpotent elements.

Next we show

Theorem 2. *Let R be a nil-semisimple Artinian ring with e as a radical element. Then R is a direct sum of a finite number of division rings.*

Proof. By the Wedderburn—Artin theorem $R=Re_1\oplus\ldots\oplus Re_n$, where the Re_i are minimal left ideals in T and the $e_i\in R$ satisfy $e_ie_j=e_i$ if $i=j$ and $e_ie_j=0$ if $i\neq j$ $(i,j=1,\ldots,n)$. Also $e=e_1+\ldots+e_n$ is an identity for R. We claim that the Re_i are division rings. Let $ae_i\neq 0$. Then $(Re_i)(ae_i)\neq 0$, since $(Re_i)(ae_i)=0$ would imply $(ae_i)^2=0$, hence $ae_i=0$, since R has no nonzero nilpotents. Also $(Re_i)(ae_i)\subseteqq Re_i$ and since Re_i is minimal, this implies $(Re_i)(ae_i)=Re_i$. So for any $be_i\in Re_i$, there exists $xe_i\in Re_i$ with $(xe_i)(ae_i)=be_i$. Then Re_i is a division ring.

One might expect that full matrix rings over division rings can occur as rings with e as a radical element. Our next theorem shows that this cannot happen.

Theorem 3. *Let S be a simple prime ring with $Sxy=Syx$ for all $x, y\in S$. Then S has an identity e, e is a radical element for S and S is a division ring.*

Proof. From $Sxy=Syx$ for all $x, y\in S$ and S is a prime ring, one can conclude that S has no zero-divisors in the same way as in the proof of Proposition 4.3 [4]. Now let $x\neq 0$ in S. Then Sx is a non-zero ideal in S, since $(sx)y=byx$ for $y\in S$ and some $b\in S$. Hence $Sx=S$. Thus, S has no proper left ideals and so it is a division ring. The rest of the theorem follows obviously.

Let R be a ring with radical element e. If N is the ideal of nilpotent elements of R then the ring $\bar{R}=R/N$ is a ring without nilpotent elements and with radical element $\bar{e}=e+N$, the identity of R/N. To state our theorem on such rings, we use the following:

Definition. Let D be a division ring. We call a subring S of D a *commutator subring* if, given $s_1\neq 0$, $s_2\neq 0$ in S, the element $s_1s_2s_1^{-1}s_2^{-1}\in S$.

Theorem 4. *Let R be a ring with e as a radical element. Then R has no nilpotents if and only if R is a subdirect sum of commutator subrings of division rings.*

Proof. Let R be a ring with radical element e and without nilpotent elements. Then the intersection of the prime ideals P in $R=(0)$, so that R is a subdirect sum of the rings R/P, P a prime ideal in R. The rings R/P are prime rings and have no divisors of zero. Being homomorphic images of R they have the property that $\bar{e}=e+P$ is a radical element for R/P. This last condition implies that any pair $\bar{x}$, $\bar{y}$ of non-zero elements of R/P has a non-zero common left multiple i.e. there exists an element $\bar{d}\neq\bar{o}$ in R/P such that $\bar{x}\bar{y}=(\bar{d}\bar{y})\bar{x}$. Hence by a well-known theorem of Ore there exists a division ring Δ_p containing R/P. For any pair $\bar{a}$, $\bar{b} \in R/P$, $\bar{a} \neq \bar{o}$, $\bar{b} \neq \bar{o}$, the equation $\bar{a}\bar{b} = \bar{c}\bar{b}\bar{a}$ has a unique solution in Δ_p, namely $\bar{a}\bar{b}\bar{a}^{-1}\bar{b}^{-1}$. The fact that $\bar{e}$ is a radical element for R/P implies that this solution must lie in R/P. Hence R/P is a commutator subring of Δ_p as required. The converse is obvious.

From the proof it follows that a prime ring having e as a radical element is a commutatoı subring of a division ring. This implies, in particular, Proposition 4.3 [4].

Remark. By Theorem 4 the rings with e as a radical element which are nil-semisimple (or β-semisimple) are characterized.

Corollary. *Let Δ be a division ring with identity e and let R be a subring* ($\neq 0$) *of Δ. Then R is a commutator subring of Δ if and only if e is a radical element for R.*

§ 2. Subcommutative rings

Definition. A ring R is said to be *r-subcommutative* if for every $a, b \in R$ there is an element $c \in R$ such that $ab = bc$. Similarly R is said to be *l-subcommutative* if for every $a, b \in R$ there is an element $d \in R$ such that $ab = da$.

Subcommutative rings have been introduced by Bucur [1], using the first part of the definition. This is also used by Lawver [2]. On the contrary, Reid [3] uses the second part of the definition, and calls such rings subcommutative. We shall use the terms r- and l-subcommutative respectively, according to the above definition. Now let R be a ring with identity e. It can be easily seen that R is r-subcommutative if and only if every r-ideal of R is two-sided and R is l-subcommutative if and only if every l-ideal of R is two-sided. So a ring R is both r- and l-subcommutative if and only if any one-sided ideal is two-sided. Such rings have been considered by Thierrin [6] and are called duo rings.

The following result is due to Reid [3].

Theorem 5. *Any l-stable subring of a direct product of division rings is l-subcommutative and has no proper nilpotent elements. Every l-subcommutative ring without proper nilpotent elements is a subdirect sum of l-stable subrings of division rings.*

Here an l-stable subring is defined as follows:

Let I be an index set and for each $i \in I$, Δ_i a division ring. For $a \in \pi\Delta_i$ (the ring direct product), define a' by

$$(a')_i = \begin{cases} 0 & \text{if } a_i = 0 \\ a_i^{-1} & \text{if } a_i \neq 0. \end{cases}$$

A subring A of $\pi\Delta_i$ is called an l-stable subring if $aAa' \subseteq A$ for each $a \in A$. Similarly, a subring A of $\pi\Delta_i$ is called an r-stable subring if $a'Aa \subseteq A$ for each $a \in A$, and an analogous theorem holds for r-stable subrings of $\pi\Delta_i$ and r-subcommutative rings.

Clearly, a commutator subring of a division ring Δ is an l-stable subring of Δ.

We shall give an example which shows that an ℓ- and r-stable subring of a division ring Δ need not be a commutator subring.

Let R be a ring with identity e, which is a radical element. For given $a, b \in R$ we have: $ab = e(ab) = c(ba)$ for some $c \in R$. Hence the equation $ab = xa$ always has a solution in R for given $a, b \in R$, so R is ℓ-subcommutative and every ℓ-ideal in R is two-sided. In [4] it is conjectured that the converse also holds, i.e. if R is an ℓ-subcommutative ring with identity e, then R has e as a radical element. We will now give a counterexample to this conjecture.

Let $\mathbf{Q}_2$ be the rational numbers with denominators prime to 2. Let $\mathbf{D}$ be the division algebra of rational quaternions. We will use the notation: $\mathbf{D} = \{(a, b, c, d)\colon a, b, c, d \in \mathbf{Q}\}$, where $(a, b, c, d) = a + bi + cj + dk$ and $\mathbf{Q}$ is the set of rational numbers.

In [2] LAWVER characterizes r-stable subrings of $\mathbf{D}$. In the main theorem it is said, among others, that an r-stable non-commutative subring R of $\mathbf{D}$ with identity has rank 4 and has one of the following forms: $R = \mathbf{D}$, $R = \mathbf{D}_2 = \{(a, b, c, d)\colon a, b, c, d \in \mathbf{Q}_2\}$ or $R = R(m) = \{(a, b, c, d)\colon a \in \mathbf{Q}_2, b, c, d \in 2^m\mathbf{Q}_2\}$ for some positive integer m.

In [3] it is shown that $\mathbf{D}_2$ is ℓ-stable in $\mathbf{D}$, hence $\mathbf{D}_2$ is ℓ-subcommutative. Therefore $\mathbf{D}_2$ is both ℓ- and r-stable in $\mathbf{D}$, so both ℓ- and r-subcommutative, or $\mathbf{D}_2$ is a duo ring.

We want to show that the identity $(1, 0, 0, 0) \in \mathbf{D}_2$ is not a radical element for $\mathbf{D}_2$. Choose $x = (0, 2, 0, 2)$ and $y = (0, 0, 2, 2)$ in $\mathbf{D}_2$. Then $xy = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\, yx$, but $(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}) \notin \mathbf{D}_2$. So for $x = (0, 2, 0, 2)$, $y = (0, 0, 2, 2) \in \mathbf{D}_2$ there does not exist an element $b \in \mathbf{D}_2$ such that $(1, 0, 0, 0)\, xy = byx$ or $(1, 0, 0, 0)$ is not a radical element. Since $\mathbf{D}_2$ is ℓ-subcommutative, this provides the counterexample.

This also shows that, although $\mathbf{D}_2$ is an ℓ-and r-stable subring of $\mathbf{D}$, it is not a commutator subring, since $xyx^{-1}y^{-1} = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ is not in $\mathbf{D}_2$.

In fact, we have the following result:

Theorem 6. *Let R be a subring of $\mathbf{D}$ $(\neq 0, \neq \mathbf{D})$. The the following are equivalent:*

a) *R is a commutator subring of $\mathbf{D}$,*

b) *R is commutative and $e \in R$ (e = identity of $\mathbf{D}$,*

c) *e is a radical element in R.*

Proof.

a)$\Rightarrow$b). Let R be a commutator subring of $\mathbf{D}$. Then $e \in R$ by the definition of commutator subring. Also R is an ℓ-stable subring of $\mathbf{D}$. Although in [2] r-stable subrings of $\mathbf{D}$ are characterized (main theorem), it is clear that the class of ℓ-stable subrings of $\mathbf{D}$ with e coincides with the class of r-stable subrings of $\mathbf{D}$ with e. Suppose that R is non-commutative. Then either $R = \mathbf{D}_2$ or $R = R(m)$ for some positive integer m. But $\mathbf{D}_2$ is not a commutator subring of $\mathbf{D}$, as we have seen, and the same argument can be used with respect to $R(m)$ for any positive integer m. This contradiction implies that R must be commutative.

b) → c). Clear from the definition of radical element.

c) → a). See the corollary of Theorem 4. In fact, the equivalence of a) and c) is true for any division ring Δ, which is the content of the corollary of Theorem 4.

§ 3. The Jacobson radical

Our next object is to consider the Jacobson radical of the ring $\mathbf{D}_2$. Let K be the set of all elements in $\mathbf{D}_2$ which do not have inverses in $\mathbf{D}_2$. It can easily be seen that the element $(a, b, c, d)\in\mathbf{D}_2$ $(\neq 0)$ does not have an inverse in $\mathbf{D}_2$ if and only if an even number (0, 2 or 4) of the rationals a, b, c, d have the form $\frac{2p}{q}$, with $p, q\in Z$, q odd, i.e. belong to $2\mathbf{Q}_2$. A straightforward calculation shows that these elements form an ideal in $\mathbf{D}_2$. Then it is well known, that K is the Jacobson radical $\mathbf{J}(\mathbf{D}_2)$ of $\mathbf{D}_2$. As the elements not in K all have inverses in $\mathbf{D}_2$, it follows that $\mathbf{D}_2/\mathbf{J}(\mathbf{D}_2)$ is a division ring and $\mathbf{D}_2$ is a local ring with $\mathbf{J}(\mathbf{D}_2)$ as its unique maximal ideal. In fact, $\mathbf{D}_2/\mathbf{J}(\mathbf{D}_2)\cong$ $\cong\mathbf{Z}_2$, as can easily be checked. It is easy to see that $\mathbf{J}(\mathbf{D}_2)$ can be also characterized as the set of those elements (a, b, c, d) which have a norm $N(a, b, c, d)=a^2+b^2+c^2+d^2$ with even numerator: $\mathbf{J}(\mathbf{D}_2)=\{x\in\mathbf{D}_2\colon N(x)=\frac{2p}{q}, p, q\in\mathbf{Z},\ q \text{ odd}\}$. Now let $a, b\in\mathbf{J}(\mathbf{D}_2)$ with $a\neq 0$. Then $N(a^{-1}ba)=N(b)=\frac{2p}{q}$, hence $a^{-1}ba\in\mathbf{J}(\mathbf{D}_2)$. Therefore $a^{-1}\mathbf{J}(\mathbf{D}_2)a\subseteqq\mathbf{J}(\mathbf{D}_2)$ and similarly $a\mathbf{J}(\mathbf{D}_2)a^{-1}\subseteqq\mathbf{J}(\mathbf{D}_2)$, So $\mathbf{J}(\mathbf{D}_2)$ is an ℓ-and κ-stable subring of $\mathbf{D}$ and Theorem 5 implies that $\mathbf{J}(\mathbf{D}_2)$ is ℓ- and κ-subcommutative. Since both $x=$ $=(0, 2, 0, 2)$ and $y=(0, 0, 2, 2)$ are in $\mathbf{J}(\mathbf{D}_2)$, but $xyx^{-1}y^{-1}\notin\mathbf{J}(\mathbf{D}_2)$, $\mathbf{J}(\mathbf{D}_2)$ is not a commutator subring of $\mathbf{D}_2$. Also a commutator subring of a division ring must have an identity and $\mathbf{J}(\mathbf{D}_2)$ does not have an identity.

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An integrability theorem for power series

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In [6] we proved the following

Theorem A. *Let $\lambda(t)>0$ be a nonincreassing, integrable function on the interval $0<t\leqq 1$ such that $\lambda(1/n+1)=O(\lambda(1/n))$, and let $A(x)$ be defined on the interval $0\leqq x<1$ by the series $\sum_{k=0}^{\infty} a_k x^k$ with $a_k\geqq 0$. Furthermore let $0<p\leqq\infty$. Then $\lambda(1-x)(A(x))^p\in L(0,1)$ if and only if*

$$\sum_{n=1}^{\infty}\lambda(1/n)n^{-2}\left(\sum_{k=0}^{n}a_k\right)^p<\infty. \tag{1}$$

If $\lambda(t)=t^{-\gamma}(\gamma<1)$, Theorem A reduces to a theorem of KHAN [5], which in its turn includes a theorem of ASKEY ([1], $\gamma=0$) and a theorem of HEYWOOD ([2], $p=1$).

In [6], Theorem A was stated for $p\geqq 1$ only, but it is easy to see that the proof actually holds for $0<p<1$, too.

Recently JAIN [4] obtained

Theorem B. *Let*

$$B(x)=\sum_{n=0}^{\infty}b_n x^n,\quad 0\leqq x\leqq 1\quad \textit{and}\quad \gamma<1.$$

Suppose that there is a positive number ε such that

$$b_n>\frac{-K}{n^{(\gamma/p)+1+\varepsilon-1/p}}\quad (0<p<\infty,\ K\ \textit{constant})$$

for all sufficiently large values of n. Then

$$(1-x)^{-\gamma}|B(x)|^p\in L(0,1)$$

if and only if

$$\sum_{n=1}^{\infty}n^{\gamma-2}\left(\sum_{k=1}^{n}|c_k|\right)^p<\infty.$$

In the particular case $p=1$ Theorem B was proved by HEYWOOD [3].

In the present paper Theorem B will be generalized as follows:

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Theorem. *Let $\lambda(t)>0$ be a nonincreasing function on the interval $0<t\leqq 1$ such that*

$$\sum_{n=k}^{\infty} \lambda(1/n)n^{-2} \leqq M\lambda(1/k)/k \tag{2}$$

and let

$$F(x) = \sum_{n=0}^{\infty} c_n x^n, \quad 0 \leqq x < 1.$$

Suppose there is a positive monotonic sequence $\{\varrho_n\}$ with $\sum_{n=1}^{\infty} 1/n\varrho_n<\infty$ such that

$$c_n > \frac{-K}{(\varrho_n\lambda(1/n))^{1/p}\cdot n^{1-1/p}} \quad (0<p<\infty,\ K>0) \tag{3}$$

for all sufficiently large values of n. Then $\lambda(1-x)|F(x)|^p\in L(0,1)$ if and only if

$$\sum_{n=1}^{\infty} \lambda(1/n)n^{-2}\left(\sum_{k=0}^{n}|c_k|\right)^p < \infty. \tag{4}$$

It is clear that if $\lambda(t)=t^{-\gamma}$ $(\gamma<1)$ then our Theorem reduces to Theorem B.

Proof. Let

$$A(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{for} \quad 0\leqq x<1$$

with $a_0=0$ and

$$a_n = \frac{K}{(\varrho_n\lambda(1/n))^{1/p}n^{1-1/p}} \quad \text{for} \quad n\geqq 1.$$

First we show that these coefficients a_n satisfy condition (1). If $p\geqq 1$ then we use the inequality

$$\sum_{n=1}^{\infty}\lambda_n\left(\sum_{k=1}^{n}a_k\right)^p \leqq p^p\sum_{n=1}^{\infty}\lambda_n^{1-p}\left(\sum_{k=n}^{\infty}\lambda_k\right)^p a_n^p, \tag{5}$$

which holds for any $\lambda_n>0$ and $a_n\geqq 0$ (see [7], inequality (1′)), with $\lambda_n=\lambda(1/n)n^{-2}$. Using (5), by (2), we have

$$\sum_{n=1}^{\infty}\lambda(1/n)n^{-2}\left(\sum_{k=1}^{n}a_k\right)^p \leqq O(1)\sum_{n=1}^{\infty}\lambda(1/n)n^{-2+p}a_n^p \leqq$$

$$\leqq O(1)\sum_{n=1}^{\infty}\lambda(1/n)n^{-2+p}(\varrho_n\lambda(1/n)n^{p-1})^{-1} \leqq O(1)\sum_{n=1}^{\infty}1/n\varrho_n<\infty.$$

If $0<p<1$, using some elementary estimates and (2), we obtain

$$\sum_{n=2}^{\infty}\lambda(1/n)n^{-2}\left(\sum_{k=1}^{n}a_k\right)^p \leqq \sum_{m=0}^{\infty}\sum_{n=2^m+1}^{2^{m+1}}\lambda(1/n)n^{-2}\left(\sum_{a=1}^{2^{m+1}}a_k\right)^p \leqq$$

$$\leqq O(1)\sum_{m=0}^{\infty}\lambda(1/2^{m+1})2^{-m}\left(\sum_{k=1}^{m+1}(2^k)^{1/p}(\lambda(1/2^k)\varrho_{2^k})^{-1/p}\right)^p \leqq$$

$$\leqq O(1)\sum_{k=1}^{\infty}(2^k/\varrho_{2^k}\lambda(1/2^k))\sum_{m=k}^{\infty}\lambda(1/2^m)2^{-m} \leqq O(1)\sum_{k=1}^{\infty}1/\varrho_{2^k} < \infty.$$

Hereby we proved that the coefficients of the function $A(x)$ satisfy condition (1), so by Theorem A

(6) $$\lambda(1-x)(A(x))^p \in L(0,1).$$

By (3) the coefficients a_n+c_n are positive for all sufficiently large values of n, thus the function

$$A(x)+F(x) = \sum_{n=0}^{\infty}(a_n+c_n)x^n$$

has the property

(7) $$\lambda(1-x)(A(x)+F(x))^p \in L(0,1)$$

if and only if

(8) $$\sum_{n=1}^{\infty}\lambda(1/n)n^{-2}\left(\sum_{k=0}^{n}(a_k+c_k)\right)^p < \infty.$$

Hence we obtain the statement of Theorem easily. Indeed, if $\lambda(1-x)|F(x)|^p \in L(0,1)$ then (6) implies (7), which implies (8). But by (3) we have

$$|c_n| \leqq 2a_n + c_n$$

whence, by (8), (4) follows. If (4) holds, then this implies (8) and equivalently (7). From (6) and (7), $\lambda(1-x)|F(x)|^p \in L(0,1)$ follows obviously.

Thus Theorem is proved.

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Generalization of a converse of Hölder's inequality

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In [1] we proved the integral inequality

$$\int_{-\infty}^{\infty} \sup_{\sum_{i=1}^{n} x_i = t} \prod_{i=1}^{n} f_i(x_i)\, dt \geqq \prod_{i=1}^{n} (p_i)^{1/p_i} \Big(\int_{-\infty}^{\infty} f_i^{p_i}(x_i)\, dx_i \Big)^{1/p_i} \tag{1}$$

for nonnegative step functions $f_i(x_i)$ $(i=1, 2, \dots, n)$ and exponents p_i satisfying the conditions $1 \leqq p_i \leqq \infty$ and $\sum_{i=1}^{n} 1/p_i = 1$.

In the course of the proof of (1) we implicitly also proved the inequality

$$\int_{-\infty}^{\infty} \sup_{\sum_{i=1}^{n} x_i = t} \prod_{i=1}^{n} F_i(x_i)\, dt \geqq \sum_{i=1}^{n} \int_{-\infty}^{\infty} F_i^{p_i}(x_i)\, dx_i, \tag{2}$$

where

$$F_i(x_i) = (\max f_i)^{-1} f_i(x).$$

In the present paper inequality (2) will be generalized in two directions.

Let H_n denote the set of nonnegative and continuous functions $H(x_1, x_2, \dots, x_n)$ of n variables such that $H(0, 0, \dots, 0) = 0$ and

$$H(x_1, \dots, x_n) \geqq \min(|x_1|, |x_2|, \dots, |x|_n \quad \text{at any point} \quad (x_1, x_2, \dots, x_n). \tag{3}$$

Furthermore, let $S(M)$ denote the set of nonnegative step functions $f(x)$ with $\max_x f(x) = M$.

We prove the following

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Theorem. *Suppose $H(x_1, \dots, x_n) \in H_n$ and $f_i(x) \in S(M)$ $(i=1, 2, \dots, n)$. Then we have for any $\Delta \geqq 0$*

$$(4) \qquad \int_{-\infty}^{\infty} \sup_{t \leqq \sum_{i=1}^{n} x_i \leqq t+\Delta} H(f_1(x_1), f_2(x_2), \dots, f_n(x_n))\, dt \geqq$$

$$\geqq \sum_{i=1}^{n} \int_{-\infty}^{\infty} f_i(x)\, dx + \Delta \cdot \max_{x_1, x_2, \dots, x_n} H(f_1(x_1), \dots, f_n(x_n)).$$

In the particular case $\Delta = 0$ and $H(x_1, \dots, x_n) = \prod_{i=1}^{n} |x_i|^{1/p_i}$ we obtain inequality (1) by replacing $f_i(x)$ by $f_i^{p_i}(x)$ and using the well-known inequality

$$\prod_{i=1}^{n} \varrho_i \leqq \sum_{i=1}^{n} \frac{1}{p_i} (\varrho_i)^{p_i} \quad \text{for} \quad \varrho_i \geqq 0, \quad \sum_{i=1}^{n} \frac{1}{p_i} = 1.$$

Next we remark that if one of the functions $f_i(x)$ belongs to $S(M')$, where $M' \neq M$, then inequality (4) does not necessarily hold.

Finally we mention that from our theorem we can deduce an inequality concerning series of positive terms.

Let $s^+(M)$ denote the set of sequences $a = \{a_n\}$ with $a_n \geqq 0$ and $\max_n a_n = M$. Furthermore let

$$\|a\|_\infty = \sup_n a_n \quad \text{and} \quad \|a\|_p = \left\{ \sum_{n=-\infty}^{\infty} a_n^p \right\}^{1/p}.$$

Corollary. *Suppose $H(x_1, \dots, x_n) \in H_n$ and $a^{(i)} \in s^+(M)$ $(i=1, 2, \dots, n)$. Then*

$$(5) \quad (n-1) \sup_{k_1, \dots, k_n} H(a_{k_1}^{(1)}, a_{k_2}^{(2)}, \dots, a_{k_n}^{(n)}) + \sum_{k=-\infty}^{\infty} \sup_{k \leqq k_1 + k_2 + \dots + k_n \leqq k+l} H(a_{k_1}^{(1)}, \dots, a_{k_n}^{(n)}) \geqq$$

$$\geqq \sum_{i=1}^{n} \sum_{k=-\infty}^{n} a_k^{(i)} + l \sup_{k_1, \dots, k_n} H(a_{k_1}^{(1)}, \dots, a_{k_n}^{(n)})$$

holds for any nonnegative integer l.

Hence, taking $H(x_1, \dots, x_n) = \prod_{i=1}^{n} |x_i|^{1/p_i}$ $\left(\sum_{i=1}^{n} 1/p_i = 1\right)$ and replacing $a_k^{(i)}$ by $(a_k^{(i)})^{p_i}$ we obtain the inequality

$$(6) \quad \sum_{k=-\infty}^{\infty} \sup_{k \leqq k_1 + \dots + k_n \leqq k+l} a_{k_1}^{(1)} a_{k_2}^{(2)} \dots a_{k_n}^{(n)} \geqq \prod_{i=1}^{n} \|a^{(i)}\|_\infty \left\{ \sum_{i=1}^{n} \|a^{(i)}\|_\infty^{-p_i} \|a^{(i)}\|_{p_i}^{p_i} + l - n + 1 \right\},$$

where $a^{(i)}$ denotes an arbitrary nonnegative sequence. Inequality (6) was proved by B. Uhrin [3], the special case $l=0$ of (6) can be found in [2], too.

Proof of the theorem. The way of our proof is similar to the proof given by us in [1]. We may assume that the step functions $f_i(x)$ have integer points of discontinuity and have at their points of discontinuity the larger one of the values taken on the adjoining intervals (this convention will be of technical importance).

Let N be an integer such that if $|x|>N$ then $f_i(x)=0$ for all i; furthermore let

$$f_i(x) = a_k^i \quad \text{if} \quad x\in(k-1,k), \quad k=-N+1, -N+2, \dots, N-1, N.$$

Let v_i denote a fixed index for which $a_{v_i}^i=M$. Furthermore we define the following auxiliary function:

$$F_i(x) = \begin{cases} f_i(x) & \text{if} \quad x\notin(v_i-1, v_i), \\ M+1 & \text{if} \quad x\in[v_i-1, v_i]. \end{cases}$$

It is clear that if b_k^i denote the values of $F_i(x)$ then $b_k^i=a_k^i$ if $k\neq v_i$ and $b_{v_i}^i=M+1$.

By means of these functions $F_i(x)$ we shall give a decomposition of the interval $(-\infty<t<\infty)$ such that the sum of the lower estimations to be given on the subintervals for the left-hand side of (4) be already greater than the right-hand side of (4).

First we consider the special case $\Delta=0$.

By the definition of N we have

$$S \equiv \int_{-\infty}^{\infty} \sup_{\sum_{i=1}^{n} x_i=t} H(f_1(x_1), f_2(x_2), \dots, f_n(x_n))\, dt =$$

$$= \int_{-nN}^{nN} \sup_{\sum_{i=1}^{n} x_i=t} H(f_1(x_1), f_2(x_2), \dots, f_n(x_n))\, dt \equiv S_N,$$

thus it is enough to decompose the interval $[-nN, nN]$.

Let

$$s(t) = \begin{cases} 1 & \text{if} \quad t \geqq 0, \\ 0 & \text{if} \quad t<0, \end{cases}$$

and we denote, as usual, by $h_+(u_0)$ the limit from the right of the function $h(u)$ at u_0, and by $h_-(u_0)$ its limit from the left. We put

$$P_0(y_1^0, y_2^0, \dots, y_n^0) \equiv (-N, -N, \dots, -N)$$

and define, for $m\geqq 1$, the following numbers and points successively:

$$u_i^m = s\big(\min_{j\neq i} F_{j+}(y_j^{m-1}) - F_{i+}(y_i^{m-1})\big)$$

and

$$P_m(y_1^m, y_2^m, \dots, y_n^m) = (y_1^{m-1}+u_1^m, y_2^{m-1}+u_2^m, \dots, y_n^{m-1}+u_n^m).$$

By the definition of the points P_m it is clear that starting from the point P_0 we go from a point P_m one step on the axis x_i where the minimum of the values $F_{i+}(y_i^m)$ $(i=$

$=1, 2, \ldots, n$) is taken if it is reached only at one j; otherwise we go simultaneously one-one step on all of the axes where the value of $F_{i+}(y_i^m)$ equals the minimum value. We continue this procedure till $y_i^{m_0}=v_i$ will hold for some $m=m_0$ and for all i, i.e.

$$P_{m_0}(y_1^{m_0}, y_2^{m_0}, \ldots, y_n^{m0}) \equiv (v_1, v_2, \ldots, v_n).$$

This necessarily follows because of the definition of the functions $F_i(x)$ on the stripes $[v_i+1, v_i]$.

Then we define a sequence of points $Q_m(z_1^m, z_2^m, \ldots, z_n^m)$ in an analogous way comming back from the point $Q_0(z_1^0, z_2^0, \ldots, z_n^0)\equiv(N, N, \ldots, N)$. Similarly as before, we define, for $m\geqq 1$, the following numbers and points successively:

$$v_i^{(m)} = s\bigl(\min_{j\neq i} F_{j-}(z_j^{m-1})-F_{i-}(z_i^{m-1})\bigr)$$

and

$$Q_m(z_1^m, z_2^m, \ldots, z_n^m) = (z_1^{m-1}-v_1^m, z_2^{m-1}-v_2^m, \ldots, z_n^{m-1}-v_n^m).$$

For similar reasons as in the case of the points P_m, we come in a finite number, say m_1, steps to the point $P_{m_0}=(v_1, v_2, \ldots, v_n)$, i.e. $P_{m_0}=Q_{m_1}$. Now we can give a path going from the point P_0 to the point Q_0 such that by means of the "break points" of this path the required decomposition of the interval $[-nN<t<nN]$ can be given.

For each i ($i=1, 2, \ldots, n$) we put

$$y_i^{m_0+l} = z_i^{m_1-l} \quad (l = 0, 1, \ldots, m_1);$$

hereby we arranged the points in a sequence $P_m(y_1^m, y_2^m, \ldots, y_n^m)$ ($m=0, 1, \ldots, m_0+m_1$), which gives the required path from P_0 to Q_0.

Next we give the required decomposition of the interval $[-nN, nN]$. First we set for each i ($i=1, 2, \ldots, n$)

$$I_i^m = y_i^m-y_i^{m-1} \quad (m = 1, 2, \ldots, m_0+m_1), \tag{7}$$

furthermore denote by c_i^m the value of $f_i(x_i)$ on the interval (y_i^{m-1}, y_i^m) if $I_i^m=1$, and at the point $x_i=y_i^m$ if $I_i^m=0$.

Let

$$t_k = \sum_{i=1}^{n} y_i^k \quad (k = 0, 1, \ldots, m_0+m_1). \tag{8}$$

It is easy to see that $t_0=-nN$ and $t_{m_0+m_1}=nN$, furthermore for any $k\geqq 1$

$$t_k = t_{k-1}+t_k-t_{k-1} = t_{k-1}+\sum_{i=1}^{n} I_i^k.$$

Thus we can decompose each interval $[t_{k-1}, _k]$ by the points

$$\tau_{k,0} = t_{k-1} \quad \text{and} \quad \tau_{k,j} = t_{k-1}+\sum_{i=1}^{j} I_i^k \quad (j = 1, 2, \ldots, n) \tag{9}$$

into subintervals. On such a subinterval $[\tau_{k,j-1}, \tau_{k,j}]$ we have for any k and j ($k = 1, 2, \ldots, m_0+m_1$; $j=1, 2, \ldots, n$) the following lower estimate:

$$S_{k,j} \equiv \int_{\tau_{k,j-1}}^{\tau_{k,j}} \sup_{\sum_{i=1}^{n} x_i=t} H(f_1(x_1), \ldots, f_n(x_n))\, dt \geqq I_j^k c_j^k. \tag{10}$$

To verify this inequality we put $x_i=y_i^k$ for $i<j$ and $x_i=y_i^{k-1}$ for $i>j$, and let x_j run from y_j^{k-1} to y_j^k, then t goes from $\tau_{k,j-1}$ to $\tau_{k,j}$; in fact we have then, by (7), (8) and (9)

$$t = \sum_{i=1}^{n} x_i \geqq \sum_{i=1}^{j-1} y_i^k + \sum_{i=j}^{n} y_i^{k-1} = t_{k-1} + \sum_{i=1}^{j-1} I_i^k = \tau_{k,j-1}$$

and

$$t = \sum_{i=1}^{n} x_i \leqq \sum_{i=1}^{j} y_i^k + \sum_{i=j+1}^{n} y_i^{k-1} = t_{k-1} + \sum_{i=1}^{j} I_i^k = \tau_{k,j}.$$

Choosing the values of x_i as above and taking into account that I_j^k differs from zero only for such subscripts j for which $c_j^k \leqq c_i^k$ holds for all i ($i=1, 2, \ldots, n$), we obtain by (3) inequality (10) immediately.

By (9) and (10),

$$\sigma_k = \sum_{j=1}^{n} S_{k,j} = \int_{t_{k-1}}^{t_k} \sup_{\sum_{i=1}^{n} x_i=t} H(f_1(x_1), \ldots, f_n(x_n))\, dt \geqq \sum_{j=1}^{n} I_j^k c_j^k,$$

and hence

$$S = S_N = \sum_{k=1}^{m_0+m_1} \sigma_k \geqq \sum_{k=1}^{m_0+m_1} \sum_{j=1}^{n} I_j^k c_j^k = \sum_{j=1}^{n} \sum_{k=1}^{m_0+m_1} I_j^k c_j^k = \sum_{j=1}^{n} \int_{-\infty}^{\infty} f_j(x)\, dx,$$

which proves inequality (4) if $\Delta=0$.

Next we consider the case $\Delta>0$.

Let $(x_1^0, x_2^0, \ldots, x_n^0)$ denote such a point where $H(f_1(x_1), f_2(x_2), \ldots, f_n(x_n))$ takes its maximum value and x_i^0 is the right-hand side end point of one of constant intervals of $f_i(x)$ on the axis x_i. The fact that $\sum_{i=1}^{n} x_i$ can be chosen from an interval $(t, t+\Delta)$ can be considered so that one of the intervals $[x_i^0-1, x_i^0]$ ($i=1, 2, \ldots, n$) is enlarged, e.g. for $i=1$, to $[x_1^0-1, x_1^0+\Delta]$ and on this enlarged interval we set $f_1(x_1)=f_1(x_1^0)$, furthermore everything is shifted by Δ to the right on $[x_1^0, \infty)$; and we estimate a similar integral as before. If we take the integral

$$\int_{x_1^0}^{x_1^0+\Delta} H(f_1(x_1), f_2(x_2^0), \ldots, f_n(x_n^0))\, dx_1,$$

this is obviously equal to

$$\Delta \cdot \max_{x_1, \ldots, x_n} H(f_1(x_1), f_2(x_2), \ldots, f_n(x_n));$$

and the rest of the integral

$$\int_{-\infty}^{\infty} \sup_{t \leq \sum_{i=1}^{n} x_i \leq t+\Delta} H(f_1(x_1), f_2(x_2), \ldots, f_n(x_n))\, dt$$

is not less than

$$\int_{-\infty}^{\infty} \sup_{\sum_{i=1}^{n} x_i = t} H(f_1(x_1), f_2(x_2), \ldots f_n(x_n))\, dt.$$

Hence and from the result proved in the case $\Delta=0$, (4) follows for $\Delta>0$, too. The proof is thus completed.

The Corollary can be deduced easily, we have just to note that considering the series as step functions $f_i(x_i)$, the left-hand side of (5) is not less than the left-hand side of (4) with $\Delta=l$.

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Finite type representations of infinite symmetric groups

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The group $G(S)$ of all permutations of a set S of infinite cardinality d is a topological group in the topology of pointwise convergence on S. The author has shown [3] that any continous unitary representation of $G(S)$ is the direct sum of irreducible representations, each (non-trivial) of which acts on a Hilbert space of dimension d. The author, on account of Theorem 1 below, conjectures that any unitary representation of $G(S)$ on a Hilbert space of dimension d is continuous (and thus any unitary representation on a Hilbert space of dimension less than d is trivial). Our results and conjecture seem analogous to certain theorems for Lie groups [2 and 5].

A representation of a group is of finite type if the von Neumann algebra generated by the range of the representation is of finite type [1, definition 5, p. 97].

Theorem. *Let S be an infinite set and let $G(S)$ be the group of all permutations of S. Any non-trivial unitary representation of $G(S)$ of finite type acts on a Hilbert space of dimension greater than the cardinal of S. In particular, the permutation group of the integers has no non-trivial unitary representation of finite type on separable Hilbert space.*

Proof. Let S_1, S_2, and S_3 be pairwise disjoint subsets of S, with cardinal $(S_i)=$ $=$cardinal $(S)=d$ for $i=1, 2, 3$. Let φ be a 1—1 correspondence between S_1 and S_2. If $s\in S_1$, let $p(s)$ be the permutation which interchanges s with $\varphi(s)$ and leaves all other members of S fixed. Let Z be the set of all subsets of S_1 which have cardinality d. If $Y\in Z$ let $p(Y)=\pi_{s\in Y}p(s)$. Define an equivalence relation $\sim$ on Z by $Y_1\sim Y_2$ iff the cardinal of the symmetric difference $Y_1\Delta Y_2$ is less than d. Let T be a subset of Z which contains exactly one member of each equivalence class of Z under $\sim$; then cardinal$(T)=2^d$. If $t_1, t_2\in T$, then $p(t_1)$ is contained in no proper normal subgroup of G [4, p. 306], $p(t_1)=p(t_1)^{-1}$, $p(t_1)p(t_2)=p(t_1\Delta t_2)$, and $p(t_1)$ and $p(t_1\Delta t_2)$ are conjugate to each other since each of these permutations is the product of d 2-cycles and also leaves S_3 elementwise fixed.

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Let U be a unitary representation of G on a Hilbert space H such that the von Neumann algebra $U(G)''$ generated by $U(G)$ is of finite type [1, definition 5, p. 97]. Assume that no subrepresentation of U is trivial. There is an ultraweakly continuous trace tr on $U(G)''$ such that $\mathrm{tr}(I_H)=1$, where I_H is the identity operator on H. Without loss of generality, we may assume that there is a vector $v\varepsilon H$, $\|v\|=1$, such that $tr(W)==(Wv, v)$ for all $W\in U(G)''$. (If not, replace $U(g)$ by the representation $g\to U(g)\otimes I_K$, where K is an infinite dimensional separable Hilbert space and $U(g)\otimes I_K$ acts on $H\otimes K$. [See 1, Theorem 1, p. 51].) We may assume that tr is faithful. (If not, replace U by the restriction of U to the closed subspace spanned by $U(G)v$.)

Let V be the representation of $U(G)''$ determined by the vector state $W\to(Wv, v)$ of $U(G)''$ by the Gelfand—Segal construction. If $g\in G$, let $V_1(g)=V(U(g))$. Then V_1 is a representation of G and is unitarily equivalent to the subrepresentation of U on the closure of $U(G)''v$. V_1 acts on the Hilbert space which is the completion of $U(G)''$ with respect to the inner product $\langle W_1, W_2\rangle=tr(W_2*W_1)$.

If $g\in G$, then $\langle U(g), U(g)\rangle=tr(U(g)*U(g))=tr(I_H)=1$. If $t_1, t_2\in T$ with $t_1\neq t_2$, then $\langle U(p(t_1)), U(p(t_2))\rangle=tr(U(p(t_2))^* U(p(t_1)))=tr(U(p(t_2)) U(P(t_1)))==tr(U(p(t_2)p(t_1)))=tr(U(p(t_1\Delta t_2)))$. Let $tr(U(p(t_1)))=\alpha$. We have $\alpha\neq 1$ since for $g\in G$, the equality $tr(U(g))=1$ is equivalent to $U(g)=I_H$, and thus $\alpha=1$ would imply the triviality of U, because $p(t_1)$ is contained in no proper normal subgroup of G.

Since $p(t_1)$ is conjugate to $p(t_1\Delta t_2)$, we have $tr(U(p(t_1\Delta t_2)))=\alpha$. A simple computation shows that $\|U(p(t_1))-U(p(t_2))\|=(2(1-\alpha))^{1/2}$; the norm on the Hilbert space on which V_1 acts. Consequently, the open balls of radius $((1-\alpha)/2)^{1/2}$ centered at the $U(p(t))$ with $t\in T$ are mutually disjoint open sets in the Hilbert space on which V_1 acts. Consequently, the dimension of this Hilbert space, and therefore the dimension of H, is at least G^d.

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A forbidden substructure characterization of Gauss codes

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GAUSS [2; pp. 272, 282—286] considered the following problem. Given a closed curve in the plane which is *normal*, i.e., lies in general position. Label the crossing points of the curve. The *Gauss code* of the curve is the word obtained by proceeding along the curve and noting each crossing point label as it is traversed. In the resulting word, every label occurs exactly twice. The problem is to characterize those words which are Gauss codes. Such words will be called here *realizable*. For a brief history of the work on the problem see [3; pp. 71—73]. In that reference, GRÜNBAUM says, "Solutions of the characterization problem have been found recently (TREYBIG [6], MARX [4]); however, they are of the same aesthetically rather unsatisfactory character as MacLane's criterion for the planarity of graphs. A characterization of Gauss codes in the spirit of the Kuratowski criterion for planarity of graphs is still missing." This work is an attempt to supply the "missing" criterion. The reader must be the judge of the aesthetic merits. Note that our characterization does meet EDMONDS' criterion [1] for a "good characterization".

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In the sequel, the symbols that make up a word are called *letters* and are denoted by capital Roman letters. Words and sequences of consecutive letters within a word are denoted by lower case Greek letters. For our purposes, two cyclic rearrangements of a word are equivalent. Given a word α, $|\alpha|$ is the number of letters in α.

Lemma 1. *Let A, A', B, B' be non-vertices on a normal planar curve G. Suppose no pair of edges (different from those containing A, A', B, B') separates A from A' and B from B'. Suppose G can be imbedded so that A and A' are on the boundary of the same face; similarly assume that G can be imbedded with B and B' on the same face.*

Then G has an imbedding with A and A' on the same face boundary and B and B' on the same face boundary. Also, the directions of the curve at A and A', relative to each other, are the same as in the hypothesized imbedding; similarly for B and B'.

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Proof. We use induction on the number of vertices.

I. Assume no two vertices of G can be separated by two edges. Then it is well known (see [5]) that the imbedding of G is essentially unique in the sense that the boundaries of faces are uniquely determined. Thus the assertion is trivial.

II. Assume there is a pair e, f of edges separating two vertices. There are three cases to consider.

IIa. Suppose e, f separate no two of A, A', B, B'. Let G_1 be the component of $G-e-f$ containing A, A', B, B' and G_2 the other. Let us replace G_2, e, f by an edge j connecting the endpoints of e and f in G_1. This way a new normal planar curve G', containing A, A', B, B', arises. Moreover, G' has imbeddings with A and A' (B and B') on one face. Hence by induction, G' has an imbedding with A and A' on the same face and B and B' on the same face. Clearly we can replace j by $e \cup G_2 \cup f$, obtaining an imbedding of G with the same property; further, the directions are as desired.

IIb. Suppose e, f separate e.g. A from A', B, B'. Let G_1 be the component of $G-e-f$ containing A', B, B' and G_2 the other. Replace G_2, e, f by an arc j connecting the endpoints of e and f in G_1 as above to obtain a curve G'. Select a point A'' on j. Then the hypothesized imbeddings of G yield an imbedding of G' with A' and A'' on the same face and another one with B and B' on the same face. It is easy to see that no pair of edges separates A' from A'' and B from B'. Therefore, G' has an imbedding with A' and A'' on the same face and, with B and B' on the same face, by the induction hvpothesis. In this imbedding A must be on the boundary of one of the two faces adjacent to e.f. Thus, replacing j by $e \cup G_2 \cup f$ (and "flipping over" G_2 if necessary) we obtain the desired imbedding of G. The directions are again easily seen to be as desired.

IIc. Assume e, f separate two of A, A', B, B' from the other two. By the assumption, they must separate $\{A, A'\}$ from $\{B, B'\}$. Then we can imbed the component of $G-e-f$ containing A and A' as in the hypothesized imbedding with A and A' on one face. We can imbed the other component as in the other hypothesized imbedding, and thus obtain the required imbedding of G.

Definition.

(1) For a word $\omega = A\alpha A\beta$ we define the *vertex split at A* to be the word $\omega_A = \alpha^{-1}\beta$.

(2) For a word $\omega = A\alpha A\beta$ we define the *loop removal at A* to be the word obtained by deleting A and both occurrences of the letters in α.

(3) A *subword* of a word ω is any word obtained by a sequence of vertex splits and loop removals.

(4) A word ω has the *parity condition* if between the two occurrences of any letter there are an even number of letters.

(5) A word ω has the *biparity condition* if given any unlinked vertices A, B with $\omega = A\alpha A\mu B\beta B\gamma$, α and β have an even number of letters in common.

The parity and biparity conditions are independent, necessary for planarity, but not sufficient (e.g. consider the word ABCDEFBADCFE).

Lemma 2. *Suppose $\omega = A\alpha A\mu B\beta B\gamma$ has ω_A and ω_B realizable and α and β have an even number of letters in common. If we cannot factor $\alpha = \alpha_1\alpha_2$ and $\beta = \beta_1\beta_2$ (the factors are assumed non-empty) so that α_1, β_2, μ have no letters in common with α_2, B_1, γ, then ω is realizable.*

Proof. Realize $A\alpha A\mu\beta^{-1}\gamma$ ($\alpha^{-1}\mu B\beta B\gamma$, respectively) and then split B (A, respectively). We get two realizations of $A\alpha^{-1}A'\mu B\beta^{-1}B'\gamma$, one with A and A' on the same face, one with B and B' on the same face. If some pair of edges separated A from A' and B from B', we could get a factorization such as we have ruled out. Thus, Lemma 1 applies and there is a realization with A and A' on the same face boundary; also B and B'. The directions are also proper for reconnection.

Let Γ be an arc from A to A' and Δ an arc from B to B', each spanning the face of which the points in question are boundary points. We may assume Δ and Γ intersect in at most one point. We show they are disjoint. Let Γ_1 be the arc from A to A' corresponding to α^{-1}; similarly Δ_1. Then $\Gamma \cup \Gamma_1$ and $\Delta \cup \Delta_1$ are closed curves intersecting in an even number of points. The intersections correspond to the common letters in α and β — even in number — and the intersections of Γ_1 and Δ_1. Thus $\Gamma_1 \cap \Delta_1 = \varphi$. Reconnect A and A' along Γ_1 and B and B' along Δ_1, giving a realization of ω.

Corollary. *Suppose $\omega = A\alpha A\mu B\beta B\gamma$ is not realizable but ω_A and ω_B are. Also α and β have an even number of letters in common. The following are ruled out*

(1) $\omega = AX—A—BX—B—$,

(2) $\omega = A—A—X—B—B—X—$,

(3) $\omega = AX—A—B—B—X—$,

(4) $\omega = AX—Y—Z—A—BY—X—Z—B—$.

Definition. We say ω is *critical* if it fails to be realizable, but its every vertex split is realizable.

Lemma 3. *Suppose $\omega = A\alpha B\beta A\gamma B\delta$ is critical, has biparity, and all letters of α and γ are contained in δ. Then α is empty if and only if γ is empty.*

Proof. Suppose e.g. $\alpha = \varnothing$ and $\gamma \neq \varnothing$. We have $\omega = AB\beta A\gamma_1 XB\delta_1 X\delta_2$. Apply Lemma 2 (1) to A and X, and obtain a contradiction. The case $\gamma = \varnothing$ is similar.

Lemma 4. *Suppose $\omega = A\alpha B\beta A\gamma B\delta$ is critical and satisfies the parity and biparity conditions and all letters of α and γ are contained in δ. Then α and γ are both empty.*

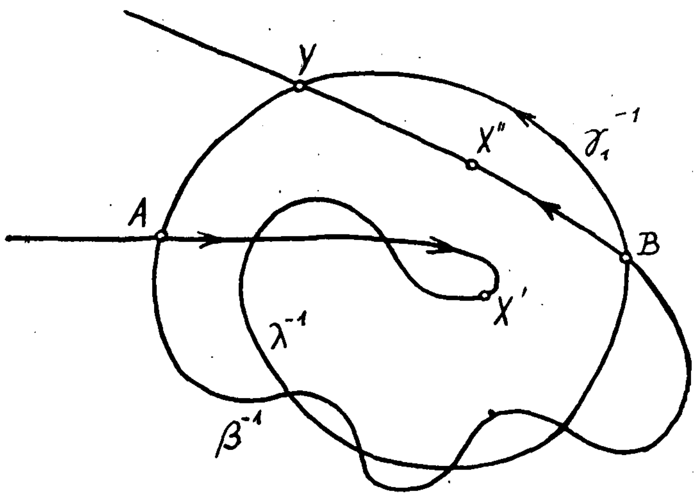
Proof. We may assume from Lemma 3 that both α and γ are non-empty. We shall obtain the contradiction that ω is realizable.

Let X and Y be two adjacent letters of α: by Corollary (2) of Lemma 2 the common letters of α and δ occur in the same order. Hence we can write $\omega = A\alpha_1 XY\alpha_2 B\beta A\gamma B\delta_1 X\delta_2 Y\delta_3$. We show $\delta_2 = \emptyset$. To this end note that δ_2 has no letters in common with α_1 or δ_3, again by Corollary (2). Thus, Lemma 3 applies to ω written in the form $X\delta_2 Y\delta_3 A\alpha_1 XY\alpha_2 B\beta A\gamma B\delta_1$, and so $\delta_2 = \emptyset$. We conclude $\omega = A\alpha B\beta A\gamma B\varepsilon_1\alpha\varepsilon_2$. Further, by Corollary (3) to Lemma 2, γ and ε_1 have no letters in common.

By a similar argument, or by applying the results of the preceding paragraph to ω^{-1}, we can write $\omega = A\alpha B\beta A\gamma B\lambda\alpha\zeta\gamma\mu$. From Corollary (2) we know that every letter of ζ occurs in β. Then Corollary (4) applies, where X is the first letter of α, B plays the role of Y, Z is a letter in ζ and β, and the last letter of γ plays the role of B; hence $\zeta = \emptyset$. Let $\alpha = \alpha_1 X$, $\gamma = Y\gamma_1$. We write $\omega = A\alpha_1 XB\beta AY\gamma_1 B\lambda\alpha_1 XY\gamma_1\mu$.

From the parity conditions on A, B, X, and Y, we can deduce that α_1 has even length. From the biparity condition on A and Y we deduce that β and λ have an even number of letters in common.

Now realize $\omega_X = A\alpha_1 X'\alpha_1^{-1}\lambda^{-1}B\gamma_1^{-1}YA\beta^{-1}BX''Y\gamma_1\mu$. Because of the above parity arguments the curve $\alpha_1 X'\alpha_1^{-1}$ and the arc $X''Y$ are inside the A loop and the directions are proper for reconnection (see figure).



We can then reconnect X' and X'', getting a relatization of ω. This contradiction completes the proof of the lemma.

Theorem. *Suppose ω is critical and satisfies the biparity condition. Then $\omega = A_1 A_2 \dots A_n A_1 A_2 \dots A_n$; the A_j are distinct, and n is even.*

Proof. First we show that either ω is in the desired form or it satisfies the parity condition. To this end note that for linking vertices A, X, parity on one implies parity on the other. Write $A\alpha X\beta A\gamma X\delta$. Split X and obtain $A\alpha\gamma^{-1}A\beta^{-1}\delta$ realizable; whence (i) $|\alpha|\equiv|\gamma|$. The parity condition for A is (ii) $|\alpha|\equiv|\beta|+1$; for X, (iii) $|\beta|+1\equiv|\gamma|$. Now (i) and (ii) are equivalent to (i) and (iii). Let A be an arbitrary vertex of ω. If any vertex B fails to link A, split it, and the parity for A is immediate. So consider the case where every vertex links A, and, even more in view of the preceding, every vertex must link every vertex that links A. The only such words have the form $A_1\dots$ $\dots A_n A_1\dots A_n$. If n is odd, ω is realizable; so we conclude n is even.

Now consider a critical ω with parity and biparity. We show this must lead to a contradiction. First, we recognize that ω must have at least two unlinked vertices; otherwise it has the form $A_1\dots A_n A_1\dots A_n$, n odd, and is realizable. We can select two such vertices so that $\omega=A\alpha AB\beta B\gamma$. Next we establish that α and β have no common letters. Let X be the first letter of β also in α; we can write $\omega=$ $=A\alpha_1 X\alpha_2 AB\beta_1 X\beta_2 B\gamma$. By choice of X, α_1 and α_2 have no letters in common with β_1; by Corollary (2) to Lemma 2, α_1 and α_2 are disjoint. Thus Lemma 4 applies to A and X, and we get the contradiction that $B\beta_1$ is empty. Finally, we can write $\omega=AY\alpha_1 AB\beta B\gamma_1 Y\gamma_2$, but this is ruled out by Corollary (3) of Lemma 2.

Theorem. *A word ω is realizable if and only if it contains no subword of the form $A_1A_2\dots A_nA_1A_2\dots A_n$, n even.*

Proof. If ω is realizable, it is easy to show it has no subword of the above form.

So suppose ω is not realizable. We proceed by induction on the number of vertices in ω. If this number is 2, $\omega=A_1A_2A_1A_2$, the desired conclusion. So suppose the theorem true for all words of $<N$ vertices and let ω have N vertices.

By the induction hypothesis, we can assume ω is critical. If ω has biparity, apply the previous theorem, and the conclusion follows. If ω does not have biparity, then $\omega=A\alpha A\beta C\gamma C\delta$, where α and γ have an odd number of points in common. The realizable vertex split ω_A tells us that γ is even. From this, we see that the loop removal of A leaves us with a word without parity. Again apply the induction hypothesis.

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J-symmetric canonical models

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On a Hilbert space K to be specified below, we consider a bounded operator J such that $J=J^*=J^{-1}$. This implies there exist two orthogonal projections P_+ and P_- for which $I=P_++P_-$, $J=P_+-P_-$, and $P_+P_-=0$. Hence, we can write $K=K_+\oplus K_-$, where $K_\pm=P_\pm K=\{x\in K|Jx=\pm x\}$. A bounded operator A is called J-symmetric iff $A=JA^*J$. These operators have been widely studied and [3, 4] give references to the literature. Recently, P. A. FUHRMANN [2] characterized the J-symmetric restricted shifts T_φ acting on $(\varphi H^2)^\perp$, where φ is a scalar inner function, as those generated by φ having real Taylor coefficients. In this note, we extend Fuhrmann's results to a more general class of operators which have applications in linear systems theory.

Let C and C_* be separable Hilbert spaces and let $L^2(C)$, $L^2(C_*)$, $H^2(C)$, and $H^2(C_*)$ denote the standard vector-valued Lebesgue and Hardy spaces defined on the unit circle. (See [6] for a general reference.) We use "t" to denote the argument of a function defined on the unit circle, and for analytic functions (vector or operator valued), we freely identify $h(t)$ on the circle with $h(z)$, its extension to the disc. Let φ denote a fixed purely contractive analytic operator-valued function from C to C_*, i.e. $\varphi(z)\colon C\to C_*$ with $\|\varphi(z)\|\leqq 1$, $\varphi(z)c\in H^2(C_*)$ for all $c\in C$, and $\|\varphi(0)c\|<\|c\|$ for all $c\in C$, $c\neq 0$. Let $\Delta(t)=(I-\varphi(t)^*\varphi(t))^{1/2}$ and let $H=H^2(C_*)\oplus\overline{\Delta L^2(C)}$. Then $M=\{(\varphi(z)f(z), \Delta(t)f(t))|f\in H^2(C)\}$[1]) is invariant under U_+, the unilateral shift on H defined by $U_+(f,g)=(zf, e^{it}g)$, so $K=H\ominus M$ is invariant under U_+^*. Let P denote the projection of H onto K, and let T be the compression of U_+ onto K; thus, $T(f,g)=P(zf, e^{it}g)$ for $(f,g)\in K$. In this context, K is called the Sz.-Nagy—Foiaş space generated by φ, and T is called a canonical model. The Sz.-Nagy—Foiaş model theorem states that any completely non-unitary contraction S is unitarily equivalent to the canonical model on the space generated by a contractive operator-valued analytic function which coincides with the characteristic function of S [6, Chapter VI].

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[1]) Since we shall not use inner products in this paper, we write (f, g) for $f\oplus g$.

($\varphi_1(z)$ and $\varphi_2(z)$ coincide iff $\varphi_1(z)=A\varphi_2(z)B$ for some constant unitary A and B; characteristic functions are necessarily purely contractive [6, p. 239].) Note that if φ is inner,[2]) i.e. $\varphi(t)$ is unitary a.e., then $\Delta(t)=0$ a.e. so $H=H^2(C_*)\ominus\varphi H^2(C)$, and if dim $C=\dim C_*=1$, then $\varphi(z)$ is a scalar-valued function acting by multiplication, so restricted shifts are special cases of canonical models.

Given φ, define $\tilde{\varphi}(z)=\varphi(\bar{z})^*$, an analytic purely contractive function mapping C_* to C. Note that φ is inner iff $\tilde{\varphi}$ is inner. Analogously to above, let $\tilde{\Delta}(t)=(I-\tilde{\varphi}(t)^*\tilde{\varphi}(t))^{1/2}$, $\tilde{H}=H^2(C)\oplus\overline{\tilde{\Delta}L^2(C_*)}$, $\tilde{K}=\tilde{H}\ominus\{(\tilde{\varphi}f,\tilde{\Delta}f)|f\in H^2(C_*)\}$, and $\tilde{T}(f,g)=\tilde{P}(zf,e^{it}g)$ for $(f,g)\in\tilde{K}$, where $\tilde{P}$ projects $\tilde{H}$ onto $\tilde{K}$. We define τ on K by

$$(1)\qquad \tau(f,g)=e^{-it}(\varphi(-t)^*f(-t)+\Delta(-t)g(-t),\ \tilde{\Delta}(t)f(-t)-\varphi(-t)g(-t))),$$

one can show that τ is a unitary map of K onto $\tilde{K}$ for which $\tilde{T}\tau=\tau T^*$, and $\tau^{-1}=$ $=\tau^*=\tilde{\tau}$ mapping $\tilde{K}$ onto K is defined by a formula analogous to (1) for $\tilde{\varphi}$ in place of φ [1]. Thus, if $\varphi=\tilde{\varphi}$, then τ is a J-operator on K and T is J-symmetric. We see below that for scalar functions, $\varphi=\tilde{\varphi}$ is also necessary for T to be J-symmetric, provided we normalize φ by requiring its first non-vanishing Taylor coefficient to be positive. We get similar results in the vector case.

Before proceeding to the main theorem, we establish two lemmas. The first relies on the following theorem of B. Sz.-Nagy and C. Foiaş.

Theorem. (i) (The lifting theorem, [6, II. 2]). *Let T_i be the canonical model on $K_i\subset H_i$, $i=1,2$. If V: $K_1\to K_2$ such that $VT_1=T_2V$ (i.e. V intertwines T_1 and T_2), then $V=PY|K_1$ for some Y: $H_1\to H_2$ such that $U_{+_2}Y=YU_{+_1}$, $PYM=0$, and $\|Y\|=$ $=\|V\|$.*

(ii) [7, p. 235] *The map Y above has the form*

$$(Y(f,g))(t)=Y(t)\begin{bmatrix}f(t)\\ g(t)\end{bmatrix}\quad where\quad Y(t)=\begin{bmatrix}A(t) & 0\\ B(t) & C(t)\end{bmatrix}$$

for some bounded analytic $A(t)$: $C_{*_1}\to C_{*_2}$ and bounded measurable $B(t)$: $C_{*_1}\to\overline{\Delta_2(t)C_2}$, $C(t)$: $\overline{\Delta_1(t)C_1}\to\overline{\Delta_2(t)C_2(t)}$ such that $A\varphi_1(t)=\varphi_2(t)A_*(t)$ and $B(t)\varphi_1(t)+C(t)\Delta_1(t)=$ $=\Delta_2(t)A_*(t)$ a.e., for some bounded analytic $A_*(t)$: $C_1\to C_2$.

Lemma 1. *V: $K_1\to K_2$ intertwining T_1 and T_2 is unitary if and only if $V=\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix}$ for some unitary maps $\alpha: C_{*_1}\to C_{*_2}$, $\beta: C_1\to C_2$ such that*

$$\alpha\varphi_1(t)=\varphi_2(t)\beta\quad and\quad \alpha^*\varphi_2(t)=\varphi_1(t)\beta^*\ \ a.e.$$

Proof. We have $V=PY$ with Y as in the previous theorem, but since $\|Y\|=1$ and V is unitary, $Y=V$ in K_1. On K_2, $Y^*=V^*$ is also unitary so $Y^*=\begin{bmatrix}A(t)^* & B(t)^*\\ 0 & C(t)^*\end{bmatrix}$,

[2]) 'Inner from both sides' in the sense of [6], p. 190.

which implies $B(t)=0$ and $A(t)=\alpha$ is constant a.e. Clearly, $\alpha\colon C_{*_1}\to C_{*_2}$ is unitary and $C(t)\colon \overline{\Delta_1(t)L^2(C_1)}\to\overline{\Delta_2(t)L^2(C_2)}$ is unitary a.e., and by the theorem applied to Y and Y^*, we have

$$(2)\qquad \begin{matrix}\alpha\varphi_1(t)=\varphi_2\beta(t)\\ C(t)\Delta_1(t)=\Delta_2(t)\beta(t)\end{matrix}\quad\text{and}\quad\begin{matrix}\alpha^*\varphi_2(t)=\varphi_1(t)\gamma(t)\\ C(t)^*\Delta_2(t)=\Delta_1(t)\gamma(t)\end{matrix}\quad\text{a.e.,}$$

for some analytic $\beta(t)$, $\gamma(t)$. Using (2), we have

$$\varphi_1(t)^*\varphi_1(t)=(\varphi_1(t)^*\alpha^*)(\alpha\varphi_1(t))=\beta(t)^*\varphi_2(t)^*\varphi_2(t)\beta(t)$$

and

$$\Delta_1(t)^2=(I-\varphi_1(t)^*\varphi_1(t))=(\Delta_1(t)C(t)^*)(C(t)\Delta_1(t))=\beta^*(t)\Delta_2^2(t)\beta(t),$$

so $\beta(t)^*\beta(t)=I$ a.e. Similarly we see $\beta(t)\gamma(t)=I$ a.e. and hence $\gamma(t)=\beta(t)^{-1}=\beta(t)^*$ a.e. is analytic so $\beta(t)=\beta$ is constant a.e. Since $\beta=\gamma^*$, (2) yields $C(t)\Delta_1^2(t)=\Delta_2^2C(t)$, which implies that $C(t)\Delta_1(t)=\Delta_2(t)C(t)$ a.e. since Δ_i is a positive contraction. Consequently, $\beta C(t)^*=I$ on $\Delta_2L^2(C_2)$, so $C(t)=\beta$ a.e. The converse follows immediately. Note that if φ is a scalar function, then $\alpha=\beta$ is a complex number of modulus one and V is multiplication by a scalar.

Lemma 2. *For $|w|<1$, $x\in C_*$, $y\in C$, define*

$$d_{w,x}=\left(\frac{I-\varphi(z)\varphi(w)^*}{1-z\overline{w}}x,\ -\frac{\Delta(t)\varphi(w)^*}{1-e^{it}\overline{w}}x\right)$$

and

$$D_{w,y}=\left(\frac{\varphi(z)-\varphi(\overline{w})}{z-\overline{w}}y,\ -\frac{\Delta(t)}{e^{it}-\overline{w}}y\right).$$

Then $d_{w,x}$ and $D_{w,y}$ are in K and

(i) $d_{w,x}=P(x/(1-z\overline{w}),0)$ *and* $D_{w,y}=P(\varphi(t)y/(e^{it}-\overline{w}),\ \Delta(t)y/(e^{it}-\overline{w})$.

(ii) *if we define $\tilde{d}_{w,y}$ and $\tilde{D}_{w,x}$ analogously for $\tilde{\varphi}$, then*

$$\tau d_{w,x}=\tilde{D}_{w,x}\quad\text{and}\quad\tau D_{w,y}=\tilde{d}_{w,y}.$$

(iii) *For $F=(f,g)\in K$ and (τ_1F) the first coordinate of τF, $(F,d_{w,x})_K=(f(w),x)_{C*}$ and $(F,D_{w,y})_K=((\tau_1F)(w),y)_C$.*

(iv) *The linear span of $\{d_{w,x}+D_{w,y}\,|\,|w|<1,\ x\in C_*,\ y\in C\}$ is dense in K.*

Proof. These all follow from straightforward computations and are found in [1]. The duality in (ii) is helpful for showing (iii) and (iv).

Theorem 1. *T is J-symmetric if and only if $\tilde{\varphi}(z)=A\varphi(z)A$ (i.e. $\tilde{\varphi}$ coincides with φ), where A is an arbitrary unitary map from C_* to C. In this case,*

$$J=\pm\begin{bmatrix}A^* & 0\\ 0 & A\end{bmatrix}\tau.$$

Proof. If $T^*=JTJ$ for some J, then $V=\tau J$ is unitary and $\tilde{T}V=VT$. Thus by Lemma 1, $J=\tau^*\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix}$ with α, β unitary, $\alpha\varphi=\tilde{\varphi}\beta$, $\alpha^*\tilde{\varphi}=\varphi\beta^*$. Using these properties, we have

$$Jd_{w,x}=\tau^*\tilde{d}_{w,\alpha x}=D_{w,\alpha x};\quad JD_{w,y}=\tau^*\tilde{D}_{w,\beta y}=d_{w,\beta y};$$

$$J^*d_{w,x}=\begin{bmatrix}\alpha^* & 0\\ 0 & \beta^*\end{bmatrix}\tilde{D}_{w,x}=D_{w,\beta^*x};\quad\text{and}\quad J^*D_{w,y}=\begin{bmatrix}\alpha^* & 0\\ 0 & \beta^*\end{bmatrix}\tilde{d}_{w,y}=d_{w,\alpha^*y}.$$

Since $J=J^*$, we have $\alpha=\beta^*$, so $\alpha\varphi=\tilde{\varphi}\alpha^*$, and $\alpha\varphi\alpha=\tilde{\varphi}$.

Conversely, if $\tilde{\varphi}=A\varphi A$, $\pm\begin{bmatrix}A^* & 0\\ 0 & A\end{bmatrix}\tau$ is a J-operator since $\{d_{w,x}+D_{w,y}\}$ spans K. Using this J, T is symmetric.

In the scalar case, α and β are complex numbers of modulus one. If we normalize φ by requiring the first nonvanishing Taylor coefficient to be positive, then $\alpha^2=1$ and we have the following

Corollary 1. *If T is a scalar canonical model, i.e., $\varphi(z)$ is a (normalized) scalar function, then T is J-symmetric if and only if all the Taylor coefficients of φ are real, and $J=\pm\tau$.*

Theorem 2. *Let $\tilde{\varphi}=A\varphi A$ as in Theorem 1, so T is J-symmetric for $J=\begin{bmatrix}A^* & 0\\ 0 & A\end{bmatrix}\tau$. Let K_+ and K_- be defined by*

$$K_\pm=\text{closed span }\{d_{w,x}\pm D_{w,Ax}\,|\,x\in C_*,\ |w|<1\}.$$

Then $K_\pm=\{f\in K\,|\,Jf=\pm f\}$.

Proof. $Jd_{w,x}=\begin{bmatrix}A^* & 0\\ 0 & A\end{bmatrix}\tilde{D}_{w,x}=$

$$=\big(A^*(z-\bar{w})^{-1}(\tilde{\varphi}(z)-\tilde{\varphi}(\bar{w}))x,\ A(e^{it}-\bar{w})^{-1}\tilde{\Delta}(t)x\big)=$$

$$=\big((z-w)^{-1}(\varphi(z)-\varphi(\bar{w}))Ax,\ (e^{it}-\bar{w})^{-1}\tilde{\Delta}(t)Ax\big)=D_{w,Ax}.$$

Similarly, $JD_{w,y}=d_{w,A^*y}$, so $J=\pm I$ on $K_\pm$. The subspaces are clearly orthogonal since $F\in K_+$, $G\in K_-$ implies $(F,G)=(JF,G)=(F,JG)=-(F,G)$, and by lemma 2, $K_+\oplus K_-$ spans K.

Corollary 2. *If* $\dim C<\infty$, *then K_+ is finite dimensional if and only if $\varphi(z)$ is of finite Blaschke type. The same holds for K.*

Proof. We note that if $\dim C=n$, then $\dim C_*=n$ since φ and $\tilde{\varphi}$ coincide, and $\varphi(z)$ can be realized as an $n\times n$ matrix whose entries are scalar H^∞ functions. We say φ is of finite Blaschke type iff $\det(\varphi(z))$ is a finite Blaschke product. Alternatively, the structure of contractive functions of finite-dimensional spaces is described in great detail in [5]; in that context the terminology is self-evident.

If $\varphi(z)$ is of finite Blaschke type, then K is finite-dimensional, and thus so are K_+ and K_-. Conversely, suppose $\dim(K_+)=N<\infty$. Since the second coordinate of $(d_{w,x}+D_{w,Ax})$ is $\Delta(t)\big(-(1-e^{it}\bar{w})^{-1}\varphi(w)^* x+e^{-it}(1-e^{-it}\bar{w})^{-1}Ax\big)$, it follows that $\Delta(t)=0$ a.e., so φ must be inner. (Note that this is still true if $\dim C=\infty$.) For w_j, $j=1,\dots,N+1$ distinct points in D, there exist constants a_j such that

$$\sum_{j=1}^{N+1} a_j(d_{w_j,x}+D_{w_j,Ax}) = 0.$$

Rearranging terms yields

$$\varphi(z)p(z)x = q(z)x, \quad \text{where} \quad p(z) = \sum_{j=1}^{N+1} a_j\big((1-z\bar{w}_j)^{-1}\varphi(w_j)^* - (z-\bar{w}_j)^{-1}A\big)$$

and

$$q(z) = \sum_{j=1}^{N+1} a_j\big((1-z\bar{w}_j)^{-1}I-(z-\bar{w}_j)^{-1}\varphi(\bar{w}_j)A\big).$$

Taking determinants shows that $\det(\varphi(z))$ is a rational function, so $\varphi(z)$ is of finite Blaschke type. A similar argument holds for K_-.

If [5, p. 212], $\varphi(z)=B(z)D$ where $B(z)$ is a diagonal matrix whose jth entry is $b_j(z)$, a scalar finite Blaschke product, and D is a constant unitary matrix, we can normalize $B(z)$ by requiring that each component be normalized in the scalar sense. Recall we have $\tilde{B}(z)=AB(z)A$; it is now easy to see that if the $b_j(z)$ are distinct, then (A) must be a diagonal matrix with entries ± 1 on the diagonal. If some $b_j(z)$ coincide, then (A) can be a block diagonal matrix, with blocks corresponding to coinciding $b_j(z)$, and each diagonal block a J-matrix. In any case, we have $\tilde{B}(z)=$ $=B(z)$, so we may take $J=\tau_B$ in theorem 2, where τ_B: $[BH^2(C)]^\perp \to [BH^2(C)]^\perp$. Clearly,

$$[BH^2(C)]^\perp = \oplus \sum_{j=1}^{N} (b_j H^2)^\perp, \quad \tau_B = \oplus \sum_{j=1}^{N} \tau_{b_j}, \quad \text{and} \quad K_\pm = \oplus \sum_{j=1}^{N} (K_\pm)_j.$$

Fuhrmann showed [2] that $\dim(K_+)_j=\left[\dfrac{n_j+1}{2}\right]$ and $\dim(K_-)_j=\left[\dfrac{n_j}{2}\right]$, where n_j is the number of factors in $b_j(z)$, and "[ ]" denotes the greatest integer function. Thus, we have determined the signature of $K_\pm$ in this special case. In general with $\dim C<\infty$, a finite Blaschke type inner function has the representation

$$B(z) = \prod_{k=1}^{n} B_k(z)U_k, \text{ where } B_k(z) = \begin{bmatrix} I_1 & 0 & \\ 0 & b_k(z) & I_2 \end{bmatrix}, \; b_k(z) = (z_k-a_k)(1-\bar{a}_k z)^{-1},$$

I_1 and I_2 are appropriate identity matrices, and U_k is a constant unitary matrix [5]. In this case, the signature is more difficult to determine. If $\dim C=\infty$, then $\dim(K)=\infty$ so either $\dim(K_+)$ or $\dim(K_-)$ (and in fact usually both) will be infinite. However, if $\varphi(z)$ is an infinite diagonal matrix whose first entry is a finite Blaschke product and all of whose remaining diagonal entries are $(z-\lambda)(1-\lambda z)^{-1}$, $-1<\lambda<1$, then we see $\dim(K_+)=\infty$ and $\dim(K_-)$ can be finite.

Corollary 3. *Let φ be a contractive operator-valued function. Then T on K is self-adjoint if and only if*

$$\varphi(z) = A^*(z + A\varphi(0))(I + z\varphi(0)^*A^*)^{-1},$$

where A is an arbitrary unitary matrix such that $A\varphi(0) = \varphi(0)^*A^*$.

Proof. If T is self-adjoint, then it is J-symmetric for $J=I$. The corollary follows from the computations in the proof of Theorem 1. Note that $\varphi(z)$ above is inner.

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On the convergence properties of weakly multiplicative systems

F. MÓRICZ

To my teacher Professor K. Tandori on his 50th birthday

§ 1. Results

In this paper $(X, \mathcal{A}, \mu)$ will be a measure space with a σ-finite [1]) non-negative measure μ, unless otherwise stated. Let $\{\varphi_i\}$ be a system of measurable functions defined on X and taking on real values. The crucial property of the system $\{\varphi_i\}$ which will be used in the proofs is the fact that it is "weakly multiplicative" in the sense that the integrals $\int \varphi_{i_1}\varphi_{i_2} \dots \varphi_{i_r} d\mu$ [2]) are small if $i_1, i_2, \dots, i_r$ are different integers for a fixed even integer r, $r \geqq 4$. More exactly, set

$$\beta_{i_1, i_2, \dots, i_r} = \int \varphi_{i_1}\varphi_{i_2} \dots \varphi_{i_r} d\mu$$

and denote by B_r the infinite vector whose components are $\beta_{i_1, i_2, \dots, i_r}$, where $i_1, i_2, \dots, i_r$ simultaneously run over the integers satisfying only the condition $1 \leqq i_1 < i_2 < \dots < i_r$. The notion of *weak multiplicity* is understood in the sense that the symmetric and absolute norm of B_r in l_q is finite:

$$\|B_r\|_q = \Big[\sum_{1 \leqq i_1 < i_2 < \dots < i_r} |\beta_{i_1, i_2, \dots, i_r}|^q\Big]^{1/q} < \infty,$$

where q is a fixed number, $1 \leqq q < \infty$. The purpose of the present paper is to obtain

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1) If the measure μ is not σ-finite, then instead of the original measure space $(X, \mathcal{A}, \mu)$ consider its restriction to the union X_1 of the supports of the integrable functions φ_i $(i = 1, 2, \dots)$ of the system in question. It is clear that μ is σ-finite on X_1 and concerning the problem of convergence the set $X \setminus X_1$ is irrelevant.

2) For the sake of simplicity we do not indicate the arguments of functions; we write φ, f etc. instead of $\varphi(x), f(x)$, etc., unless this causes any confusion; we write $\int \varphi d\mu$ and L_r instead of $\int_X \varphi d\mu$ and $L_r(X, \mathcal{A}, \mu)$, respectively; we also say "almost everywhere" (in abbreviation: a.e.) instead of "μ-almost everywhere".

somewhat stronger results than those of GAPOŠKIN [6], KOMLÓS and RÉVÉSZ [9] under less restrictive conditions.

Throughout the paper r will denote an even integer, $r \geqq 4$, p will denote a real number, $1 < p \leqq 2$, while q will denote the "complementary" exponent, i.e., $1/p + 1/q = 1$. Besides them, C, C_r, $C_{r,p}$, $C^*_{r,p}$, etc. will denote positive constants, not necessarily the same at each occurrence. Furthermore, K, K_1, and K_2 will denote positive numbers, which are (upper or lower) bounds of the integrals of the appropriate power of functions φ_i in question.

We recall here the well-known notion of $\mathcal{S}_r$ system [7, pp. 243—246]: a system $\{\varphi_i\}$ belonging to L_r is said to be an *$\mathcal{S}_r$ system* if for every sequence $\{c_i\}$ of real numbers and for every positive integer n the inequality

$$\int \left(\sum_{i=1}^{n} c_i \varphi_i \right)^r d\mu \leqq C_r \left(\sum_{i=1}^{n} c_i^2 \right)^{r/2} \tag{1.1}$$

holds.[3]) Let us introduce the following generalization of this notion. We say that $\{\varphi_i\}$ is an *$\mathcal{S}_{r,p}$ system* if for every sequence $\{c_i\}$ and for every integer n we have

$$\int \left(\sum_{i=1}^{n} c_i \varphi_i \right)^r d\mu \leqq C_{r,p} \left(\sum_{i=1}^{n} |c_i|^p \right)^{r/p}.$$

In the study of the convergence of series $\sum c_i \varphi_i$, where $\{\varphi_i\}$ is a weakly multiplicative system ("*direct theorems*") a result of Erdős—Stečkin (as far the proof, see GAPOŠKIN [4, pp. 28—31]) and its generalization, due to TJURNPÜ [15], play a key role: *If $\{\varphi_i\}$ is an $\mathcal{S}_{r,p}$ system and if $r > p > 1$, then there exists another constant $C^*_{r,p}$ such that for every sequence $\{c_i\}$ and for every integer n the inequality*

$$\int \max_{1 \leqq k \leqq n} \left(\sum_{i=1}^{k} c_i \varphi_i \right)^r d\mu \leqq C^*_{r,p} \left(\sum_{i=1}^{n} |c_i|^p \right)^{r/p} \tag{1.2}$$

also holds true.[4])

Making use of this result we can arrive in a routine way at the following assertion: *Every $\mathcal{S}_{r,p}$ system is an unconditional convergence system* (UCS) *for l_p if $r > p > 1$.* This means that every series $\sum c_i \varphi_i$ with $\sum |c_i|^p < \infty$ is convergent a.e. in every arrangement of its terms. Furthermore, (1.2) yields also the slightly stronger assertion that, under the above conditions, the maximum of the moduli of the partial sums of $\sum c_i \varphi_i$ belongs to L_r in every arrangement of the terms.

Our main direct theorem reads as follows.

[3]) The notion of $\mathcal{S}_r$ system is defined for any positive number r, but when r is not an even integer, on the left-hand side of (1.1) we must have $\int \left| \sum_{i=1}^{n} c_i \varphi_i \right|^r d\mu$.

[4]) The case $p=2$ is due to Erdős ($r=4$) and Stečkin ($r>2$), while the general case $r>p>1$ was treated by Tjurnpü.

Theorem 1. *Let r be an even integer, $r \geqq 4$, let p be a real number, $1<p\leqq 2$, and let q be defined by $1/p+1/q=1$. Let $\{\varphi_i\}$ be a system of functions in L_r for which*

$$\int \varphi_i^r \, d\mu \leqq K \quad (i = 1, 2, \ldots) \tag{1.3}$$

and

$$\|B_r\|_q^q = \sum_{1 \leqq i_1 < i_2 < \ldots < i_r} \left| \int \varphi_{i_1} \varphi_{i_2} \ldots \varphi_{i_r} \, d\mu \right|^q < \infty. \tag{1.4}$$

Then $\{\varphi_i\}$ is an $\mathcal{S}_{r,p}$ system.

Consequently, $\{\varphi_i\}$ is an UCS *for l_p and the maximum of the moduli of the partial sums of $\sum c_i\varphi_i$ with $\sum |c_i|^p < \infty$ belongs to L_r in every arrangement of the terms.*

We point out that in Theorem 1 the stipulation on p is essential. In other words, if condition (1.4) is required to hold for a q such that $1<q<2$, this stronger condition does not imply the a.e. convergence of $\sum c_i\varphi_i$ for any $\{c_i\} \in l_p \setminus l_2$ in the case when $p>2$. The reason is that the converse of Theorem 1, under a natural further assumption on the lower boundedness of $\int \varphi_i^2 d\mu$ $(i=1, 2, \ldots)$, is also true. If the series $\sum c_i\varphi_i$ converges at the points of a set of positive measure, then $\sum c_i^2$ is finite. We shall prove much more general theorems, too.

In the sequel we restrict ourselves to the case $r=4$. This case illuminates the general situation well enough.

In the study of the divergence of series $\sum c_i\varphi_i$, where $\{\varphi_i\}$ is a weakly multiplicative system ("*converse theorems*") the following inequality if of basic importance.

Theorem 2. *Let $\{\varphi_i\}$ be a system of functions in L_4 for which*

$$\int \varphi_i^4 \, d\mu \leqq K \quad (i = 1, 2, \ldots), \tag{1.5}$$

$$\|B_4\|_2^2 = \sum_{1 \leqq i < k < l < m} \left(\int \varphi_i \varphi_k \varphi_l \varphi_m \, d\mu \right)^2 < \infty, \tag{1.6}$$

and

$$K_1 \leqq \int_F \varphi_i^2 \, d\mu \leqq K_2 \quad (i > i_0), \tag{1.7}$$

where F is a set of positive and finite measure, and let δ be a positive number. Then there exists an integer n_0 such that for any sequence $\{c_i\}$ of numbers and for any integer $n \geqq n_0$ we have

$$(1-\delta) K_1 \sum_{i=n_0}^{n} c_i^2 \leqq \int_F \left(\sum_{i=n_0}^{n} c_i \varphi_i \right)^2 d\mu \leqq (1+\delta) K_2 \sum_{i=n_0}^{n} c_i^2. \tag{1.8}$$

We note that the second inequality of (1.7) is a consequence of (1.5) with $K_2 = [K\mu(F)]^{1/2}$, because of $\mu(F) < \infty$.

We shall consider an arbitrary linear method of summation defined by a doubly infinite matrix $T^* = (\alpha_{mn})$, whose elements satisfy the first and third conditions of regularity:[5])

(1.9)
$$\lim_{m\to\infty} \alpha_{mn} = 0 \quad (n = 1, 2, \dots)$$

and

(1.10)
$$\lim_{m\to\infty} \sum_{n=1}^{\infty} \alpha_{mn} = 1.$$

All linear methods of summation used in analysis are T^* methods. Set

$$t_m = \sum_{n=1}^{\infty} \alpha_{mn} s_n, \quad s_n = \sum_{i=1}^{n} c_i \varphi_i.$$

We say that the series $\sum c_i \varphi_i$ is T^* *summable* to a limit s if the T^* *mean* t_m tends to s as $m \to \infty$.

Theorem 3. *Let* $\{\varphi_i\}$ *be a system of functions in* L_4 *satisfying conditions* (1,5), (1.6), *and*

(1.11)
$$\liminf_{i\to\infty} \int_E \varphi_i^2 \, d\mu > 0,$$

where E is a set of positive measure. If a series $\sum c_i \varphi_i$ *is* T^* *summable or, more generally, its* T^* *means are bounded on E, then* $\sum c_i^2$ *is finite.*

The following proposition immediately follows from Theorems 1 and 3.

Corollary 1. *If the system* $\{\varphi_i\}$ *satisfies* (1.5), (1.6), *and* (1.11) *holds for every set E of positive measure, then any series* $\sum c_i \varphi_i$ *is a.e. convergent or a.e. not* T^* *summable in any arrangement of its terms, according as the series* $\sum c_i^2$ *is finite or not.*

In probability theory this fact is called the *law of zero or unity.*

For certain problems it is desirable to have a similar result in the case, when only one-sided boundedness of the T^* means is supposed. Before stating our next result in an explicit form, we introduce the following notation. Set

$$R_{mi} = \sum_{n=i}^{\infty} \alpha_{mn} \quad (i = 1, 2, \dots).$$

It is obvious that the mean t_m can be rewritten into the form

$$t_m = \sum_{n=1}^{\infty} \alpha_{mn} s_n = \sum_{i=1}^{\infty} R_{mi} c_i \varphi_i.$$

[5]) The second condition of regularity, which is neglected in our paper reads as follows: the sums $\sum_{n=1}^{\infty} |\alpha_{mn}|$ are bounded ($m = 1, 2, \dots$). As to the notion of regularity, see, e.g., Zygmund [16, p. 74].

It can be easily seen from (1.9) and (1.10) that

$$\lim_{m\to\infty} R_{mi} = 1 \quad (i = 1, 2, \ldots). \tag{1.12}$$

Theorem 4. *Let $\{\varphi_i\}$ be a system of functions in L_4 satisfying conditions* (1.5) *and* (1.6); *furthermore, assume that* (1.11) *holds for every set E of positive measure. If $\sum c_i^2$ is not finite, then the set of points x at which* [6])

$$\lim_{m\to\infty} \frac{t_m^+(x)}{\left[\sum_{i=1}^{\infty} R_{mi}^2 c_i^2\right]^{1/2}} = 0 \tag{1.13}$$

holds, is of measure zero.

We remark that the sum in brackets is finite by virtue of Theorem 3 provided that the series defining $t_m(x)$ converges on a set of positive measure. From (1.12) it follows immediately that the denominator of (1.13) tends to ∞ as $m\to\infty$. Hence Theorem 4 implies

Corollary 2. *Under the conditions of Theorem* 4, *and if the T^* means of $\sum c_i\varphi_i$ are bounded from above (or from below) on a set of positive measure, then $\sum c_i^2$ is finite.*

§ 2. Historical comments

Let $\{\varphi_i\}$ be a system of measurable functions on $(X, \mathcal{A}, \mu)$, $\mu(X)<\infty$, such that $\varphi_i\in L_q$ for every $q\geqq 2$, or, in particular, let φ_i be essentially bounded $(i=1, 2, \ldots)$. In this section we assume that

$$\int \varphi_i\, d\mu = 0 \quad \text{and} \quad \int \varphi_i^2\, d\mu = 1 \quad (i = 1, 2, \ldots). \tag{2.1}$$

The following definitions [7]) were introduced by ALEXITS [1, pp. 186—187]: $\{\varphi_i\}$ is said to be

(i) a *multiplicative system* (MS) if

$$\int \varphi_{i_1}\varphi_{i_2}\ldots\varphi_{i_k}\, d\mu = 0;$$

(ii) a *strongly multiplicative system* (SMS) if

$$\int \varphi_{i_1}^{\alpha_1}\varphi_{i_2}^{\alpha_2}\ldots\varphi_{i_k}^{\alpha_k}\, d\mu = 0,$$

[6]) Here $t_m^+=\max(0, t_m)$.

[7]) In earlier papers the underlying measure space $(X, \mathcal{A}, \mu)$ was a special probability space: $X=[0,1]$, $\mathcal{A}$ is the class of the Borel subsets of $[0,1]$, and μ is the Lebesgue measure on it.

where $\alpha_1, \alpha_2, \dots, \alpha_k$ can be equal to 1 or 2 but at least one of them is equal to 1;

(iii) an *equinormed strongly multiplicative system* (ESMS) if

$$\int \varphi_{i_1}^{\alpha_1} \varphi_{i_2}^{\alpha_2} \dots \varphi_{i_k}^{\alpha_k} \, d\mu = \int \varphi_{i_1}^{\alpha_1} \, d\mu \int \varphi_{i_2}^{\alpha_2} \, d\mu \dots \int \varphi_{i_k}^{\alpha_k} \, d\mu,$$

where $\alpha_1, \alpha_2, \dots, \alpha_k$ can be equal to 1 or 2. In all these three definitions: $1 \leqq i_1 < i_2 < \dots < i_k$, $k = 2, 3, \dots$.

Making use of the method of the Lebesgue functions, ALEXITS [1a] (see also ALEXITS and TANDORI [3]) proved the following

Theorem A. *If* $\{\varphi_i\}$ *is a uniformly bounded* ESMS, *then the condition*

$$\sum_{i=1}^{\infty} c_i^2 < \infty \tag{2.2}$$

implies the a.e. convergence of the series

$$\sum_{i=1}^{\infty} c_i \varphi_i. \tag{2.3}$$

Later ALEXITS and SHARMA [2] showed that Theorem A remains valid in the case when $\{\varphi_i\}$ is only a uniformly bounded MS. A simpler proof of this assertion was found by PRESTON [12].

Obviously any independent system of random variables defined on a probability space $(X, \mathcal{A}, \mu)$ and satisfying (2.1) is an ESMS. A classical Kolmogorov theorem states that if the random variables $\varphi_1, \varphi_2, \dots$ are independent with expectation 0 and variance 1, then condition (2.2) implies the a.e. convergence of (2.3). Therefore, even the theorem of Alexits and Tandori would be much stronger than Kolmogorov's theorem if the condition of uniform boundedness could be dropped.

The first step toward this direction was made by RÉVÉSZ [13].

Theorem B. *Suppose that*

$$\int \varphi_i^4 \, d\mu \leqq K \quad (i = 1, 2, \dots) \tag{2.4}$$

and

$$\int \varphi_i^2 \varphi_k \varphi_l \, d\mu = \int \varphi_i^2 \varphi_k \, d\mu = \int \varphi_i \varphi_k \varphi_l \varphi_m \, d\mu = \int \varphi_i \varphi_k \varphi_l \, d\mu = \int \varphi_i \varphi_k \, d\mu = 0,$$

where i, k, l, m are different integers. Furthermore, let $\{c_i\}$ be a sequence for which there exists an integer s such that

$$\sum_{i=1}^{\infty} c_i^2 l_s^2(i) < \infty, \tag{2.5}$$

where $l_s(i)$ means the sth iterate of $\log i$.[8]) *Then the series* (2.3) *converges a.e.*.

[8]) I.e., $l_s(t)$ is defined by the following recurrence relation: $l_s(t) = l(l_{s-1}(t))$ if $s \geqq 2$, where $l(t) = l_1(t) = \log t$ if $t \geqq 2$, and $=1$ if $0 < t < 2$.

Condition (2.5) is not very far from condition (2.2). This fact suggested the conjecture that (2.5) can be replaced by (2.2). This was shown by GAPOŠKIN [5], under weaker assimptions on $\{\varphi_i\}$.

Theorem C. *Suppose that condition* (2.4),

$$\int \varphi_i \varphi_k \varphi_l \varphi_m \, d\mu = 0, \tag{2.6}$$

and

$$\int \varphi_i^2 \varphi_k \varphi_l \, d\mu = 0 \tag{2.7}$$

hold, where i, k, l, m are different integers. Then $\{\varphi_i\}$ *is an* $\mathcal{S}_4$ *system.*

KOMLÓS and RÉVÉSZ [9] observed that condition (2.7) can be omitted.

Theorem D. *Under conditions* (2.4) *and* (2.6), $\{\varphi_i\}$ *is an* $\mathcal{S}_4$ *system.*

We note that this fact was essentially formulated previously by SERFLING [14], but we think his proof is not complete. At the same time, independently of the above authors, GAPOŠKIN [6] also obtained similar results.

Theorem E. *If*

$$\int \varphi_i^r \, d\mu \leqq K \quad (i = 1, 2, \ldots) \tag{2.8}$$

and

$$\int \varphi_{i_1} \varphi_{i_2} \ldots \varphi_{i_r} \, d\mu = 0 \quad (1 \leqq i_1 < i_2 < \ldots < i_r), \tag{2.9}$$

where r is an even integer, $r \geqq 4$, *then* $\{\varphi_i\}$ *is an* $\mathcal{S}_r$ *system.*

In addition, Gapoškin pointed out that the vanishing of the integrals in (2.9) is of no relevance, only their "relative smallness" is needed.

Theorem F. *Suppose that* (2.8) *holds and there exists a non-negative function* $f(i)$ $(i=1, 2, \ldots)$ *such that*

$$\left|\int \varphi_{i_1} \varphi_{i_2} \ldots \varphi_{i_r} \, d\mu\right| \leqq \min \{f(i_2-i_1), f(i_4-i_3), \ldots, f(i_r-i_{r-1})\}$$

for every $1 \leqq i_1 < i_2 < \ldots < i_r$ *and*

$$\sum_{i=1}^{\infty} i^{(r-2)/2} f(i) < \infty, \tag{2.10}$$

where r is an even integer, $r \geqq 4$, *then* $\{\varphi_i\}$ *is an* $\mathcal{S}_r$ *system.*

We mention that in [9] KOMLÓS and RÉVÉSZ also stated this result for $r=4$.

Our Theorem 1 evidently contains Theorem D and Theorem E even in the special case $p=2$. Theorem 1 and Theorem F are incomparable, as no one of the conditions (1.4) and (2.10) implies the other.

Inequality (1.1) expressing the $\mathcal{S}_r$ property of a system is valid for a large class of independent random variables and is a classical result of probability theory. Furthermore, it is well-known for lacunary trigonometric series[9]) (cf. [16, p. 215]). In the case of multiplicative systems, inequality (1.1) was proved first by the present author [10].

Theorem G. *Let* $\{\varphi_i\}$ *be a uniformly bounded* SMS *and let* q *be any positive number. Then for every sequence* $\{c_i\}$ *and for every integer* n *we have*

$$C_q'\left(\sum_{i=1}^{n} c_i^2\right)^{q/2} \leqq \int_0^1 \left|\sum_{i=1}^{n} c_i\varphi_i\right|^q d\mu = C_q\left(\sum_{i=1}^{n} c_i^2\right)^{q/2}.$$

Now we provide a brief review on the converse results. The first such result is also due to ALEXITS [1, p. 194].

Theorem H.[10]) *Suppose that conditions* (2.6), (2.7), *and*

$$\int \varphi_i^2\varphi_k^2\, d\mu = 1 \tag{2.11}$$

are satisfied, where i *and* k *are different integers, furthermore, for every set* E *of positive measure the relation*

$$\int_E \varphi_i^2\, d\mu \geqq K_1\mu(E) \quad (i > i_0) \tag{2.12}$$

holds. If the series (2.3) *is summable on a set of positive measure by a regular summation method that is finite with respect to the rows, then its coefficients satisfy condition* (2.2).

Later ALEXITS and SHARMA [2] showed that Theorem H is true if condition (2.11) is replaced by the condition of uniform boundedness of $\{\varphi_i\}$.

The present author proved [11] that if (2.4) holds, then condition (2.11) yields (2.12) with a constant $K_1 \sim 1$. More precisely, our result reads as follows.

Theorem I. *Suppose we are given a set* E *of positive measure and a positive number* δ. *Under conditions* (2.4), (2.6), (2.7), *and* (2.11) *there exists an integer* n_0 *such that for any sequence* $\{c_i\}$ *and for any integer* $n \geqq n_0$ *we have*

$$(1-\delta)\mu(E)\sum_{i=n_0}^{n} c_i^2 \leqq \int_E \left(\sum_{i=n_0}^{n} c_i\varphi_i\right)^2 d\mu \leqq (1+\delta(\mu(E)\sum_{i=n_0}^{n} c_i^2.$$

---

[9]) The trigonometric series $\sum_{k=1}^{\infty}(a_k\cos n_k t + b_k \sin n_k t)$ is said to be lacunary if n_k's are integers and $n_{k+1}/n_k \geqq q > 1$ $(k = 1, 2, \ldots)$.

[10]) Here we give the original theorem of Alexits with a slight modification. It is evident from his proof that this modification also holds true. This remark relates also to Theorem I.

Furthermore, if the T^* means of the series (2.3) *are bounded on a set of positive measure, then condition* (2.2) *holds.*

KOMLÓS [8] observed that conditions (2.7) and (2.11) are superfluous.

Theorem J. *Suppose that* $\{\varphi_i\}$ *satisfies conditions* (2.4), (2.6), *and*

$$\liminf_{i\to\infty} \int_E \varphi_i^2 \, d\mu > 0$$

for every set E of positive measure, then the convergence of (2.3) *on any set of positive measure implies* (2.2).

Obviously, Theorem 3 contains Theorem J even in the special case of convergence, and Theorem 4 is a generalization of a result of ZYGMUND [16, p. 205]. We note that an intermediate step of generalization of Zygmund's theorem referred to above appeared in [11].

We remark that all the theorems mentioned, except Theorem J, was originally stated for finite measure spaces, in spite of the fact that finiteness is essential only in the proof of Theorem A.

§ 3. Proof of Theorem 1

The following lemma is of fundamental significance in establishing direct theorems of convergence.

Lemma 1. *Let* $a_1, a_2, \ldots, a_n$ *be real numbers, let r be an integer,* $r \geqq 2$, *and let p be a positive real number,* $p \leqq 2$. *Set*

$$S = \sum_{i=1}^n a_i, \quad S_p = \left(\sum_{i=1}^n |a_i|^p\right)^{1/p},$$

and

$$T_r = \sum_{1 \leqq i_1 < i_2 < \ldots < i_r \leqq n} a_{i_1} a_{i_2} \ldots a_{i_r}.$$

Then

$$|S^r - r!\, T_r| \leqq C_r \{S_p^r + S_p |S|^{r-1}\}.$$

This lemma immediately follows from that of GAPOŠKIN [6] if we take into consideration that

$$S_2 \leqq S_p \quad (0 < p \leqq 2)$$

and that for any positive numbers a and b the inequality

$$a^{r-1} b + a^{r-2} b^2 + \ldots + a^2 b^{r-2} \leqq \frac{1}{2}(r-2)(a^r + ab^{r-1})$$

holds.

Proof of Theorem 1. By virtue of Lemma 1 we have

$$\int S^r\,d\mu \leqq C_r\left\{\int S_p^r\,d\mu+\int S_p|S|^{r-1}\,d\mu\right\}+r!\left|\int T_r\,d\mu\right|,$$

where A, S_p, and T_r are defined as follows:

$$S=\sum_{i=1}^n c_i\varphi_i,\quad S_p=\left(\sum_{i=1}^n |c_i\varphi_i|^p\right)^{1/p},$$

and

$$T_r=\sum_{1\leqq i_1<i_2<\ldots<i_r\leqq n} c_{i_1}c_{i_2}\ldots c_{i_r}\int \varphi_{i_1}\varphi_{i_2}\ldots\varphi_{i_r}\,d\mu.$$

Using Minkowski's inequality and (1.3), we obtain:

$$\int S_p^r\,d\mu=\int\left(\sum_{i=1}^n |c_i\varphi_i|^p\right)^{r/p}d\mu\leqq\left[\sum_{i=1}^n\left(\int|c_i\varphi_i|^r\,d\mu\right)^{p/r}\right]^{r/p}\leqq$$

$$\leqq\left[\sum_{i=1}^n |c_i|^p\left(\int\varphi_i^r\,d\mu\right)^{p/r}\right]^{r/p}\leqq K\left(\sum_{i=1}^n|c_i|^p\right)^{r/p}.$$

Now Hölder's inequality gives that

$$\int S_p|S|^{r-1}\,d\mu\leqq\left(\int S_p^r\,d\mu\right)^{1/r}\left(\int S^r\,d\mu\right)^{(r-1)/r}\leqq K^{1/r}\left(\sum_{i=1}^n|c_i|^p\right)^{1/p}\left(\int S^r\,d\mu\right)^{(r-1)/r}.$$

Finally, we can estimate $\left|\int T_r d\mu\right|$ in the following way:

$$\left|\int T_r\,d\mu\right|\leqq\sum_{1\leqq i_1<\ldots<i_r\leqq n}\left|c_{i_1}\ldots c_{i_r}\int\varphi_{i_1}\ldots\varphi_{i_r}\,d\mu\right|\leqq$$

$$\leqq\Big(\sum_{1\leqq i_1<\ldots<i_r\leqq n}|c_{i_1}|^p\ldots|c_{i_r}|^p\Big)^{1/p}\Big(\sum_{1\leqq i_1<\ldots<i_r\leqq n}\left|\int\varphi_{i_1}\ldots\varphi_{i_r}\,d\mu\right|^q\Big)^{1/q}\leqq$$

$$\leqq\|B_r\|_q\left(\sum_{i=1}^n|c_i|^p\right)^{r/p}.$$

Putting this all together we obtain:

$$\int S^r\,d\mu\leqq C_r\left\{K\left(\sum_{i=1}^n|c_i|^p\right)^{r/p}+K^{1/r}\left(\sum_{i=1}^n|c_i|^p\right)^{r/p}\left(\int S^r\,d\mu\right)^{(r-1)/r}\right\}+$$

$$+r!\,\|B_r\|_q\left(\sum_{i=1}^n|c_i|^p\right)^{r/p}.$$

Setting

$$z=\frac{\left(\int S^r\,d\mu\right)^{1/r}}{\left(\sum_{i=1}^n|c_i|^p\right)^{1/p}},$$

provided that $\sum|c_i|^p\neq 0$, we arrive at the inequality

$$z^r\leqq C_r(K+K^{1/r}z^{r-1})+r!\,\|B_r\|_q.$$

Using the elementary fact that if for positive z, a, and b we have

$$z^r \leqq az^{r-1}+b$$

then

$$z \leqq a+b^{1/r},$$

we get the desired inequality

$$\left(\int S^r\,d\mu\right)^{1/r} \leqq \{C_r K^{1/r}+(C_r K+r!\,\|B_r\|_q)^{1/r}\}\left(\sum_{i=1}^{n}|c_i|^p\right)^{1/p},$$

which expresses the $\mathcal{S}_{r,p}$ property of the system $\{\varphi_i\}$. Thus Theorem 1 is proved.

§ 4. Proof of two lemmas

We begin with proving a Bessel type inequality for weakly multiplicative systems. We consider the generalized Fourier coefficients of a function f in L_2 with respect to the system $\{\varphi_i\varphi_k\}$, defined as follows:

$$\gamma_{ik} = \int f\varphi_i\varphi_k\,d\mu \quad (i, k = 1, 2, \dots;\ i \neq k). \tag{4.1}$$

Lemma 2. *Let $\{\varphi_i\}$ be a system of functions in L_4 satisfying conditions* (1.5) *and* (1.6). *Then for any square integrable function f we have*

$$\sum_{1\leqq i<k}\gamma_{ik}^2 \leqq C\int f^2\,d\mu. \tag{4.2}$$

Proof of Lemma 2. The proof is similar to that of Bessel's inequality, well-known in the theory of orthogonal series. We note that this lemma has already been formulated and proved by Komlós [8] under more restricted conditions.

We shall use the elementary identity

$$\Big(\sum_{1\leqq i<k\leqq n} a_{ik}\Big)^2 = \sum_{i=1}^{n}\sum_{\substack{k=1\\k\neq i}}^{n}\sum_{\substack{l=i\\l\neq i}}^{n} a_{ik}a_{il} - \sum_{1\leqq i<k\leqq n} a_{ik}^2 + 2\sum_{1\leqq i<k<l<m\leqq n}(a_{ik}a_{lm}+a_{il}a_{km}+a_{im}a_{kl}).$$

Setting $a_{ik}=\gamma_{ik}\varphi_i\varphi_k$ and taking into account (4.1), we obtain the inequality

$$\begin{aligned}0 \leqq \int \Big(\lambda f-\sum_{1\leqq i<k\leqq n}\gamma_{ik}\varphi_i\varphi_k\Big)^2 d\mu &= \lambda^2\int f^2\,d\mu - 2\lambda\sum_{1\leqq i<k\leqq n}\gamma_{ik}^2 + \\ &+\sum_{i=1}^{n}\sum_{\substack{k=1\\k\neq i}}^{n}\sum_{\substack{l=1\\l\neq i}}^{n}\gamma_{ik}\gamma_{il}\int\varphi_i^2\varphi_k\varphi_l\,d\mu - \sum_{1\leqq i<k\leqq n}\gamma_{ik}^2\int\varphi_i^2\varphi_k^2\,d\mu + \\ &+2\sum_{1\leqq i<k<l<m\leqq n}(\gamma_{ik}\gamma_{lm}+\gamma_{il}\gamma_{km}+\gamma_{im}\gamma_{kl})\int\varphi_i\varphi_k\varphi_l\varphi_m\,d\mu,\end{aligned} \tag{4.3}$$

where λ denotes a parameter, whose value will be determined later, and n is fixed for temporarily.

Consider separately the third and the fifth sum on the right-hand side of (4.3). We remind that, under the conditions of Lemma 2, $\{\varphi_i\}$ is an $\mathcal{S}_4$ system $(p=2)$, by virtue of Theorem 1. Using the Buniakowski—Schwarz inequality, condition (1.5), and the $\mathcal{S}_4$ property of $\{\varphi_i\}$, we obtain that

$$(4.4)\qquad S_1 = \sum_{i=1}^{n}\sum_{\substack{k=1\\k\neq i}}^{n}\sum_{\substack{l=1\\l\neq i}}^{n}\gamma_{ik}\gamma_{il}\int\varphi_i^2\varphi_k\varphi_l\,d\mu = \sum_{i=1}^{n}\int\varphi_i^2\left(\sum_{\substack{k=1\\k\neq i}}^{n}\gamma_{ik}\,\varphi_k\right)^2 d\mu \leqq$$

$$\leqq \sum_{i=1}^{n}\left[\int\varphi_i^4\,d\mu\right]^{1/2}\left[\int\left(\sum_{\substack{k=1\\k\neq i}}^{n}\gamma_{ik}\varphi_k\right)^4 d\mu\right]^{1/2} \leqq$$

$$\leqq K^{1/2}C_4^{1/2}\sum_{i=1}^{n}\sum_{\substack{k=1\\k\neq i}}^{n}\gamma_{ik}^2 = 2K^{1/2}C_4^{1/2}\sum_{1\leqq i<k\leqq n}\gamma_{ik}^2.$$

Now applying the Cauchy inequality, from (1.6) it follows that

$$\sum_{1\leqq i<k<l<m\leqq n}\gamma_{ik}\gamma_{lm}\int\varphi_i\varphi_k\varphi_l\varphi_m\,d\mu \leqq \left[\sum\gamma_{ik}^2\gamma_{lm}^2\right]^{1/2}\left[\sum\left(\int\varphi_i\varphi_k\varphi_l\varphi_m\,d\mu\right)^2\right]^{1/2} \leqq$$

$$\leqq \|B_4\|_2\left[\sum_{1\leqq i<k<l<m\leqq n}\gamma_{ik}^2\gamma_{lm}^2\right]^{1/2} \leqq \|B_4\|_2\sum_{1\leqq i<k\leqq n}\gamma_{ik}^2.$$

Hence we find that

$$(4.5)\qquad S_2 = 2\sum_{1\leqq i<k<l<m\leqq n}(\gamma_{ik}\gamma_{lm}+\gamma_{il}\gamma_{km}+\gamma_{im}\gamma_{kl})\int\varphi_i\varphi_k\varphi_l\varphi_m\,d\mu \leqq$$

$$\leqq 6\|B_4\|_2\sum_{1\leqq i<k\leqq n}\gamma_{ik}^2.$$

Estimating the right-hand side of (4.3) by means of inequalities (4.4) and (4.5), we arrive at

$$0 \leqq \lambda^2\int f^2\,d\mu - 2\lambda\sum_{1\leqq i<k\leqq n}\gamma_{ik}^2 + S_1 - \sum_{1\leqq i<k\leqq n}\gamma_{ik}^2\int\varphi_i^2\varphi_k^2\,d\mu + S_2 \leqq$$

$$\leqq \lambda^2\int f^2\,d\mu - 2(\lambda - K^{1/2}C_4^{1/2} - 3\|B_4\|_2)\sum_{1\leqq i<k\leqq n}\gamma_{ik}^2,$$

where the fourth sum on the right-hand side was simply omitted, being always non-negative. Choosing

$$\lambda = 2(K^{1/2}C_4^{1/2} + 3\|B_4\|_2),$$

we get that

$$\sum_{1\leqq i<k\leqq n}\gamma_{ik}^2 \leqq \lambda\int f^2\,d\mu = 2(K^{1/2}C_4^{1/2} + 3\|B_4\|_2)\int f^2\,d\mu.$$

Since this is true for all n, the assertion of Lemma 2 follows.

In the proof of Theorem 4 we need

Lemma 3. *Let* $\{\varphi_i\}$ *be a system of functions in* L_4 *satisfying conditions* (1.5) *and* (1.6), *and let F be a measurable set of finite measure. Then*

$$\sum_{i=1}^{\infty}\Big(\int_F \varphi_i\,d\mu\Big)^2 < \infty. \quad (4.6)$$ [11]

Proof of Lemma 3. The proof can be carried out by using an argument similar to that used in the proof of Lemma 2. For the sake of brevity, set

$$\gamma_i = \int_F \varphi_i\,d\mu \quad (i = 1, 2, \dots).$$

Let us start again with the inequality

$$0 \leqq \int_F \Big(\lambda - \sum_{i=1}^{n} \gamma_i\varphi_i\Big)^2 d\mu = \lambda^2\mu(F) - 2\lambda\sum_{i=1}^{n}\gamma_i^2 + $$
$$+ \sum_{i=1}^{n}\gamma_i^2\int_F \varphi_i^2\,d\mu + 2\sum_{1\leqq i<k\leqq n}\gamma_i\gamma_k\int_F \varphi_i\varphi_k\,d\mu,$$

where λ is a parameter and n is a fixed positive integer.

The last sum on the right-hand side of this inequality can be estimated as follows. Using the Cauchy inequality we get that

$$S = \sum_{1\leqq i<k\leqq n}\gamma_i\gamma_k\int_F \varphi_i\varphi_k\,d\mu \leqq \big[\sum \gamma_i^2\gamma_k^2\big]^{1/2}\Big[\sum\Big(\int_F \varphi_i\varphi_k\,d\mu\Big)^2\Big]^{1/2} \leqq$$
$$\leqq \Big[\sum_{1\leqq i<k\leqq n}\Big(\int_F \varphi_i\varphi_k\,d\mu\Big)^2\Big]^{1/2}\sum_{i=1}^{n}\gamma_i^2.$$

By virtue of Lemma 2 we have

$$\sum_{1\leqq i<k}\Big(\int_F \varphi_i\varphi_k\,d\mu\Big)^2 \leqq \int \chi_F^2\,d\mu = C\mu(F),$$

which, combined with the preceding inequality, gives that

$$S \leqq C^{1/2}\mu^{1/2}(F)\sum_{i=1}^{n}\gamma_i^2.$$

Hence we find that

$$0 \leqq \lambda^2\mu(F) - 2\lambda\sum_{i=1}^{n}\gamma_i^2 + \sum_{i=1}^{n}\gamma_i^2\int_F \varphi_i^2\,d\mu + 2S \leqq$$
$$\leqq \lambda^2\mu(F) - 2\Big(\lambda - \frac{1}{2}K^{1/2}\mu^{1/2}(F) - C^{1/2}\mu^{1/2}(F)\Big)\sum_{i=1}^{n}\gamma_i^2,$$

[11]) Lemma 3 is true for any square integrable function whose support is of finite measure instead of the characteristic function χ_F of the set F.

where we took into account that by (1.5)

$$\int_F \varphi_i^2\,d\mu \leqq \Big(\int_F \varphi_i^4\,d\mu\Big)^{1/2}\Big(\int_F d\mu\Big)^{1/2} \leqq K^{1/2}\mu^{1/2}(F).$$

Choosing

$$\lambda = (K^{1/2}+2C^{1/2})\mu^{1/2}(F),$$

we get that

$$\sum_{i=1}^{n} \gamma_i^2 \leqq \lambda\mu(F) = (K^{1/2}+2C^{1/2})\mu^{3/2}(F),$$

and letting $n\to\infty$, we obtain (4.6), which was to be proved.

§ 5. Proofs of Theorems 2—4

Using Lemma 2 and Lemma 3, the proofs of our converse theorems follow a standard way.

Proof of Theorem 2. We start with the inequality

$$\int_F \Big(\sum_{i=n_0}^{n} c_i\varphi_i\Big)^2 d\mu = \sum_{i=n_0}^{n} c_i^2 \int_F \varphi_i^2\,d\mu + 2\sum_{n_0\leqq i<k\leqq n} c_i c_k \int_F \varphi_i\varphi_k\,d\mu, \tag{5.1}$$

where n_0 will be determined later. As for the first sum on the right-hand side of (5.1), by (1.7) we have

$$K_1\sum_{i=n_0}^{n} c_i^2 \leqq \sum_{i=n_0}^{n} c_i^2 \int_F \varphi_i^2\,d\mu \leqq K_2\sum_{i=n_0}^{n} c_i^2. \tag{5.2}$$

Let us estimate the second sum on the right-hand side of (5.1). Using the Cauchy inequality, the modulus of this sum does not exceed

$$2\Big[\sum_{n_0\leqq i<k\leqq n} c_i^2c_k^2\Big]^{1/2}\Big[\sum_{n_0\leqq i<k\leqq n}\gamma_{ik}^2\Big]^{1/2} \leqq 2\sum_{i=n_0}^{n} c_i^2\Big[\sum_{n_0\leqq i<k}\gamma_{ik}^2\Big]^{1/2}, \tag{5.3}$$

where

$$\gamma_{ik} = \int_F \varphi_i\varphi_k\,d\mu = \int \chi_F\varphi_i\varphi_k\,d\mu \quad (i\neq k).$$

Since the characteristic function χ_F is square integrable, F being of finite measure, in virtue of Lemma 2 there exists an integer n_0 such that

$$\sum_{n_0\leqq i<k}\gamma_{ik}^2 < \frac{1}{4}\delta^2K_I^2 \leqq \frac{1}{4}\delta^2K_2^2. \tag{5.4}$$

Hence if $n\geqq n_0$, from (5.1)—(5.4) we can conclude inequality (1.8), which was to be proved.

Proof of Theorem 3. We may suppose that E is a set of finite measure.[12]) By (1.11) there is a $K_1^*>0$, which can be taken, e.g., $\frac{1}{2}\liminf_{i\to\infty}\int_E \varphi_i^2 d\mu$, and a positive integer i_1 for which

(5.5) $$\int_E \varphi_i^2\, d\mu \geqq K_1^* \quad (i > i_1).$$

The hypothesis is that for almost every x in E each of the series $\sum_n \alpha_{mn} s_n$ converges to a sum t_m $(m=1, 2, \ldots)$, which tends to a finite limit or, more generally, bounded as $m\to\infty$. Therefore, we can find a subset F of E with $\mu(F)>0$ and a positive number M such that

(5.6) $$|t_m(x)| \leqq M \quad (x\in F;\ m = 1, 2, \ldots),$$

and, in addition, the relation

(5.7) $$\int_F \varphi_i^2\, d\mu \geqq K_1 \quad (i > i_1)$$

also holds. The latter relation readily follows from (5.5) if $\mu(E\setminus F)$ is sufficiently small, because

$$\int_F \varphi_i^2\, d\mu = \int_E \varphi_i^2\, d\mu - \int_{E\setminus F} \varphi_i^2\, d\mu \geqq K_1^* - K^{1/2}\mu^{1/2}(E\setminus F),$$

where we used (1.5) and the Buniakowskii—Schwarz inequality.

Firstly we deal with the case when the summation matrix T^* is row-finite. We apply Theorem 2 with $\delta=\frac{1}{2}$. Then there exists an integer n_0 $(\geqq n_1)$ such that (1.8) holds for every $n\geqq n_0$. Using the elementary inequality

$$(a+b)^2 \geqq \frac{1}{2}a^2 - b^2,$$

we get that

(5.8) $$\int_F t_m^2\, d\mu \geqq \frac{1}{2}\int_F \left(\sum_{i=n_0}^{\infty} R_{mi} c_i \varphi_i\right)^2 d\mu - \int_F \left(\sum_{i=1}^{n_0-1} R_{mi} c_i \varphi_i\right)^2 d\mu,$$

where the sum $\sum_{i=n_0}^{\infty} R_{mi} c_i \varphi_i$ now has only a finite number of terms different from zero. According to (1.8) we have

(5.9) $$\int_F \left(\sum_{i=n_0}^{\infty} R_{mi} c_i \varphi_i\right)^2 d\mu \geqq \frac{1}{2} K_1 \sum_{i=n_0}^{\infty} R_{mi}^2 c_i^2.$$

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[12]) Namely, let $E=\bigcup_{i=1}^{\infty} E_i$, where $\mu(E_i)<\infty$ $(i=1, 2, \ldots)$. If relation (1.11) is not true for any E_i, then, using the Cantor diagonal process, one can easily show that (1.11) is not true for E, either.

The second integral on the right-hand side of (5.8) can be estimated by using Minkowski's inequality as follows:

$$\int_F \left(\sum_{i=1}^{n_0-1} R_{mi} c_i \varphi_i\right)^2 d\mu \leqq \left[\sum_{i=1}^{n_0-1} |R_{mi}||c_i| \left(\int_F \varphi_i^2 \, d\mu\right)^{1/2}\right]^2 \leqq \left[\sum_{i=1}^{n_0-1} |R_{mi}||c_i| K_2^{1/2}\right]^2,$$

where we took into consideration that by (1.5)

$$\int_F \varphi_i^2 \, d\mu \leqq \left[\int_F \varphi_i^4 \, d\mu \int_F d\mu\right]^{1/2} \leqq [K\mu(F)]^{1/2} = K_2.$$

By virtue of (1.12) the inequality $|R_{mi}| \leqq 2$ holds for $i=1, 2, \ldots, i_0-1$ if m is large enough. Therefore, continuing the above argument, for such m's we have

$$\int_F \left(\sum_{i=1}^{n_0-1} R_{mi} c_i \varphi_i\right)^2 d\mu \leqq 4K_2 \left(\sum_{i=1}^{n_0-1} |c_i|\right)^2 = C. \tag{5.10}$$

Collecting (5.6), (5.8), (5.9), and (5.10) we obtain that

$$M^2 \mu(F) \geqq \int_F t_m^2 \, d\mu \geqq \frac{1}{2} K_1 \sum_{i=n_0}^{\infty} R_{mi}^2 c_i^2 - C.$$

Making here $m \to \infty$ and observing (1.12) we get the required result: $\sum c_i^2 < \infty$.

Now we remove the constraint on T^* to be row-finite. This can be done in the same way as in ZYGMUND's book [16, p. 205]. For the sake of completeness we give the proof here.

Let t_m^* be an expression analogous to t_m, except that the upper limit of summation is not ∞ but a number $N=N(m)$:

$$t_m^* = \sum_{n=1}^{N} \alpha_{mn} s_n.$$

We take N so large that the following conditions be satisfied:

(i) $$|t_m(x) - t_m^*(x)| \leqq \frac{1}{m} \quad \text{for} \quad x \in F \setminus F_m,$$

where

$$\mu(F_m) \leqq \frac{1}{2^{m+1}} \mu(F);$$

(ii) $$\lim_{m \to \infty} \sum_{n=1}^{N} \alpha_{mn} = 1.$$

Setting

$$F^* = \bigcup_{m=1}^{\infty} F_m,$$

we have

$$\mu(F^*) < \mu(F)$$

and on the set $F\setminus F^*$, which is of positive measure, the mean $t_m^*(x)$ tends to a finite limit or is bounded as $m\to\infty$, respectively. But condition (ii) ensures that the t_m^*'s are T^* means corresponding to a row-finite matrix. Thus the general case is reduced to the special case already dealt with.

This completes the proof of Theorem 3.

Proof of Theorem 4. The proof closely follows that of a similar theorem concerning lacunary trigonometric series in Zygmund's book [16, pp. 205—206].

In the course of the proof we assume that $c_i=0$ for some i, say $i<n_0$, where n_0 is determined by Theorem 2, since we may always omit a finite number of terms of $\sum c_i\varphi_i$ without influencing its T^* summability (although this can affect the value of the upper or lower bound of the T^* means).

Set

$$\Gamma_m^2=\sum_{i=1}^{\infty} R_{mi}^2 c_i^2 \quad (m=1,2,\ldots).$$

Suppose that we have (1.13) for every $x\in E$, $\mu(E)>0$, and that $\sum c_i^2$ diverges. Given any positive number ε, there exist an integer m_0 and a set $F\subset E$ with $\mu(F)\geqq\frac{1}{2}\mu(E)$ such

$$t_m(x)\leqq \varepsilon\Gamma_m \quad (x\in F;\ m\geqq m_0).$$

Then

$$\int_F |t_m|\,d\mu \leqq \int_F \{|t_m-\varepsilon\Gamma_m|+\varepsilon\Gamma_m\}\,d\mu = \tag{5.11}$$

$$=\int_F \{2\varepsilon\Gamma_m - t_m\}\,d\mu = 2\varepsilon\mu(F)\Gamma_m-\int_F t_m\,d\mu.$$

We are going to estimate the last integral on the right-hand side by applying Lemma 3. By the Cauchy inequality we get

$$\int_F t_m\,d\mu=\sum_{i=n_0}^{\infty} R_{mi}c_i\int_F \varphi_i\,d\mu \leqq \left[\sum_{i=n_0}^{\infty} R_{mi}^2 c_i^2\right]^{1/2}\left[\sum_{i=n_0}^{\infty}\Big(\int_F \varphi_i\,d\mu\Big)^2\right]^{1/2}\leqq \varepsilon\Gamma_m, \tag{5.12}$$

if $c_i=0$ for $i<n_0$ and n_0 is chosen so that

$$\sum_{i=n_0}^{\infty}\Big(\int_F \varphi_i\,d\mu\Big)^2\leqq\varepsilon^2.$$

This is possible because of (4.6).

Therefore, the right-hand side of (5.11) is less than $2\varepsilon\mu(F)\Gamma_m+\varepsilon\Gamma_m$. This shows that

$$\int_F |t_m|\,d\mu = o(\Gamma_m) \quad (m\to\infty). \tag{5.13}$$

On the other hand, consider the inequality

$$\int_F t_m^2\,d\mu \leqq \Big[\int_F |t_m|\,d\mu\Big]^{2/3}\Big[\int_F t_m^4\,d\mu\Big]^{1/3},$$

which is an immediate consequence of Hölder's inequality. By virtue of Theorem 2, the left-hand side here exceeds some fixed multiple of Γ_m^2. On account of Theorem 1 the integral $\int_F t_m^4\,d\mu$ ($\leqq \int t_m^4\,d\mu$) does not exceed some fixed multiple of Γ_m^4. Thus, $\int_F |t_m|\,d\mu$ exceeds some fixed multiple of Γ_m. This contradicts (5.13) and proves Theorem 4.

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An inequality for functions

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1. The main purpose of this note is to prove the following:

Theorem 1. *Let* (X, Σ, m) *be a probability measure space and* $0<p<q<\infty$. *Let* $f\in L^q(m)$, $\int |f|^q dm=1$ *and* $A_0^p=\int |f|^p dm$. *Suppose* $0<A<A_0$ *and let* $c>0$, $y>1$ *satisfy the equation*

(1)
$$\frac{1}{c}=\frac{y^q-A^q}{1-A^q}=\frac{y^p-A^p}{A_0^p-A^p}.$$

Then

$$m\{x\in X;\, |f(x)|>A\}\geqq c.$$

Equality holds if and only if there exists a measurable set S with $m(S)=1-c$ *and* $|f|=A$ *on S and* $|f|=y$ *on* $X\setminus S$.

This result shows that the above constant c is the best possible for the function class $\{f\in L^q(m);\, \|f\|_q=1 \text{ and } \|f\|_p=A_0\}$ and is a refinement of an inequality given in BURKHOLDER and GUNDY [1, *p.* 258, Lemma 2. 3]. Applications of inequalities of this type are also found in ZYGMUND [3, p. 216—p. 217]. Also this is a generalization of an inequality for analytic functions in KAMOWITZ [2, p. 236, Theorem B]. His result follows from the next theorem, which is an immediate corollary of his Lemma 3 in the case of non-atomic measure space, and which also in the general setting can be proved in the same way as in the proof of Theorem 1.

Theorem 2. *Let* (X, Σ, m) *be a probability measure space and* $0<p<\infty$. *Let* $f\in L^p(m)$, $\int |f|^p dm=1$ *and* $\log|f|\in L^1(m)$, $A_0=\exp\int \log|f|dm$. *Suppose* $0<A<A_0$ *and let* $c>0$, $y>1$ *satisfy the equation*

(2)
$$\frac{1}{c}=\frac{y^p-A^p}{1-A^p}=\frac{\log y-\log A}{\log A_0-\log A}.$$

Then

$$m\{x\in X;\, |f(x)|>A\}\geqq c.$$

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Equality holds if and only if there exists a measurable set S *with* $m(S)=1-c$ *and* $|f|=A$ *on* S *and* $|f|=y$ *on* $X\setminus S$.

As for Kamowitz's results we shall discuss them in the last section. To prove Theorems 1 and 2 we have improved the method of the proof of Kamowitz's theorem in [2]. We shall prove in the next section Theorem 1 only and omit the proof of Theorem 2.

2. We state first three elementary lemmas.

Lemma 1. *Given* $0<A<A_0<1$, $0<p<1$, *there exist unique* c *and* y *such that* $0<c<1$, $y>1$ *and*

$$1 = (1-c)A+cy \quad \text{and} \quad A_0^p = (1-c)A^p+cy^p. \tag{3}$$

Further y *satisfies the equation*

$$\frac{y-A}{1-A} = \frac{y^p-A^p}{A_0^p-A^p}. \tag{4}$$

Also, for fixed A, *the solution* y *of* (4) $(y>1)$ *decreases when* A_0 *increases and, for fixed* A_0, *it increases when* A *increases.*

Proof. It is obvious that the equation (4) has a unique solution for $y>1$, since $y=A$ is a solution of (4) and $1<(1-A^p)(A_0^p-A^p)^{-1}$. Let $c=(1-A)(y-A)^{-1}$, for this y. Then c and y satisfy equation (3) and $0<c<1$. Also by elementary calculation one sees that the last assertions hold.

Lemma 2. *Let* (X, Σ, m) *be a finite positive measure space. If* $0<p<1$, $0<A\leqq 1$, $G\in L^\infty(m)$ *and* $|G|\leqq A$, *then*

$$p\Big(m(X)A-\int_X |G|\,dm\Big) \leqq m(X)A^p-\int_X |G|^p\,dm.$$

Equality holds only when $|G|=A$.

Proof. By elementary computation one has the inequality $p(A-t)\leqq A^p-t^p$ for $0<t\leqq A$. Integrating the both sides of the inequality $p(A-|G|)\leqq A^p-|G|^p$, we have the desired one. It is then clear that the equality holds only when $|G|=A$.

Lemma 3. *Let* $0<A<1$, $0<p<1$, $\beta\geqq 1$, $y>1$ *and* $0\leqq pa\leqq b$. *Then*

$$b+\beta y^p-(\beta-1)A^p \geqq \big(a+\beta y-(\beta-1)A\big)^p.$$

Equality holds if and only if $\beta=1$ *and* $a=b=0$.

Proof. Note that

$$b+\beta y^p-(\beta-1)A^p = b+y^p+(\beta-1)(y^p-A^p),$$
$$a+\beta y-(\beta-1)A = a+y+(\beta-1)(y-A).$$

Let

$$g(y) = \{(y^p+b+(\beta-1)(y^p-A^p))^{1/p}-(y+a+(\beta-1)(y-A))\}/y.$$

Then we get

$$g(y) = \{1+s+(\beta-1)(1-B^p)\}^{1/p}-\{1+t+(\beta-1)(1-B)\},$$

where $t=a/y$, $s=b/y^p$ and $B=A/y$. Now we have clearly

$$g(y) \geqq 1+(s+(\beta-1)(1-B^p))/p-(1+t+(\beta-1)(1-B)) =$$
$$(s-pt)/p+(\beta-1)(1-B^p-p(1-B))/p.$$

Further by the assumption $b\geqq ap$ we have

$$s-pt = (by^{1-p}-ap)/y \geqq (b-ap)/y \geqq 0,$$

and as in Lemma 2 we see $1-B^p\geqq p(1-B)$. Hence we have $g(y)\geqq 0$. It is then obvious that $g(y)=0$ if and only if $\beta=1$ and $ap=b=0$.

3. Now we are in the position to prove Theorem 1 for $q=1$.

Proof. Assume $d=m\{|f|>A\}<c$. Let $S=\{|f|\leqq A\}$ and $S'=X\setminus S$. Then we have by Lemma 1

$$\int_S|f|+\int_{S'}|f| = 1 = (1-c)A+cy = (1-d)A+cy-(c-d)A, \tag{5}$$
$$\int_S|f|^p+\int_{S'}|f|^p = A_0^p = (1-c)A^p+cy^p = (1-d)A^p+cy^p-(c-d)A^p.$$

By Hölder's inequality one gets

$$\int_{S'}\frac{|f|^p}{d}\,dm \leqq \Big(\int_{S'}\frac{|f|}{d}\,dm\Big)^p. \tag{6}$$

Combining this with (5) we have

$$\frac{(1-d)A^p-\int_S|f|^p}{d}+\frac{c}{d}y^p-\left(\frac{c}{d}-1\right)A^p \leqq \left(\frac{(1-d)A-\int_S|f|}{d}+\frac{c}{d}y-\left(\frac{c}{d}-1\right)A\right)^p. \tag{7}$$

However by Lemmas 2 and 3 we have the converse inequality and hence the equality, which implies $c=d$, a contradiction. Next suppose the equality holds in (6) and let $S=\{|f|\leqq A\}$ and $S'=\{|f|>A\}$. Then we have $m(S)=1-c$ and we see by the above argument that the equality holds in (7), which implies $\int_S|f|dm=(1-c)\,A$ and that the equality holds in (6). Hence we get $|f|=A$ on S and $|f|$ is constant on S'. This value is y by (5). The proof is complete.

4. Now Theorem 1 follows immediately from the special case above. In fact, let $g=|f|^q$ in the setting of Theorem 1. Then

$$\int g\,dm = 1, \quad \int g^{p/q}dm = \int |f|^p dm = (A_0^q)^{p/q} \quad \text{and} \quad \{|f| > A\} = \{g > A^q\}.$$

Theorem 1 results if we replace the y of the case $q=1$, by y^q.

5. Application. Let $f(z)$ be an analytic function in the open unit disc in the complex plane which lies in the Hardy space H^p for some $0<p<\infty$, i.e., let

$$\|f\|_p = \sup_{0<r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta\right)^{1/p} < \infty, \quad \text{and} \quad F(\theta) = \lim_{r\to 1} f(re^{i\theta}).$$

Then $\log|F(\theta)|$ is integrable unless $F(\theta)\equiv 0$ and one has by Jensen's inequality for H^p functions

$$\log|f(0)| \leqq \frac{1}{2\pi}\int_0^{2\pi}\log|F(\theta)|\,d\theta.$$

One sees easily that the constant c in Theorem 2 is an increasing function of A_0 for fixed A. Hence applying Theorem 2 we have the following theorem of Kamowitz.

Theorem 3. *Let* $f\in H^p$, $0<p<\infty$ *and* $\|f\|_{H^p}=1$. *If* $0<A<|f(0)|$, *then* $m\{0\leqq\theta\leqq 2\pi;\ |F(\theta)|>A\}\geqq c$, *where* $c=(1-A^p)(y^p-A^p)^{-1}$ *and* y *is determined by the equation* (2) *in Theorem* 2 *for* $A_0=|f(0)|$. *This constant is the best possible. Here* m *denotes the normalized Lebesgue measure on* $[0, 2\pi]$.

That the constant c is the best possible is shown by the H^∞ outer function defined by

$$\exp\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log g(\theta)\,d\theta,$$

where $g(\theta)=A$ for $0<\theta\leqq 2\pi(1-c)$ and $=y$ for $2\pi(1-c)<\theta\leqq 2\pi$.

One can also formulate Theorem 1 for H^p functions, and also in this case i. is shown by the above function that the arising constant c is the best possiblet Finally we remark that Kamowitz uses the inner-outer factorization theorem for H^p functions and he states Theorem 3 only for $1\leqq p<\infty$.

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An inclusion theorem for normal operators

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1. Introduction

In the present remark we present a simple result relating, for normal operators, the spectrum of a submatrix to the spectrum of the whole matrix.

First some terminological conventions. Given a linear operator A on a Hilbert space H and a disc $D=\{z: |z-z_0|\leqq r\}$ in the complex plane we shall say that D is an inclusion disc for A if $D\cap\sigma(A)$ is nonvoid; here, of course, $\sigma(A)$ stands for the spectrum of A. Given an orthogonal projection P in H (a bounded linear operator P such that $P^2=P$ and $P^*=P$) with range H_0 we shall denote by A_P the restriction to H_0 of the operator PAP (or PA, which is the same).

We shall denote by Q the operator $I-P$; Q is the orthogonal projection whose range is $H_0^\perp$.

We shall prove two inclusion theorems, one for "matrices" and one for operators.

Although the main idea of the proof is the same, we prefer to treat the finite-dimensional and infinite-dimensional cases separately. First of all, the proof becomes simpler in finite-dimensional spaces since in this case the spectrum coincides with the point spectrum. Second, owing to a regrettable lack of communication between functional analysts and specialists in finite-dimensional problems the result could easily be overlooked if presented as a corollary of a result in functional analysis.

2. The inclusion theorems

We begin with the finite-dimensional case.

(2.1) Theorem. *Let A be a normal operator on a finite-dimensional Hilbert space H. Let P be an orthogonal projection in H. If $\lambda_P\in\sigma(A_P)$ then there exists a $\lambda\in\sigma(A)$ such that $|\lambda-\lambda_P|\leqq|QAP|$.*

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Proof. Since $\lambda_P \in \sigma(A_P)$ there exists a vector x such that $|x|=1$, $x=Px$ and $(PA-\lambda_P)x=0$. We have then

$$(A-\lambda_P)x = (P+Q)(A-\lambda_P)x = Q(A-\lambda_P)x = Q(A-\lambda_P)Px = QAPx,$$
$$((A-\lambda_P)^*(A-\lambda_P)x, x) = |(A-\lambda_P)x|^2 = |QAPx|^2 \leqq |QAP|^2.$$

Let ξ be the minimum of the quadratic form corresponding to $(A-\lambda_P)^*(A-\lambda_P)$ on the unit sphere of H. It follows that $\xi \leqq |QAP|^2$. Since ξ belongs to the spectrum of $(A-\lambda_P)^*(A-\lambda_P)$ and $A-\lambda_P$ is normal, there exists a proper value λ of A such that $|\lambda-\lambda_P|^2=\xi$ whence $|\lambda-\lambda_P| \leqq |QAP|$. The proof is complete.

To extend this result to the infinite-dimensional case small changes have to be made in the statement and in the proof. We need the notion of the approximate point spectrum $\sigma_a(T)$ of a linear operator T on a Banach space E. We say that λ belongs to the approximate point spectrum of T if $\inf \{|(T-\lambda)x|;\ x\in E,\ |x|=1\}=0$. Clearly $\sigma_a(T)\subset\sigma(T)$; if $\lambda\in\sigma_a(T)$, the equation

$$(T-\lambda)x = 0$$

need not have nontrivial solutions but does have approximate solutions: for each $\varepsilon>0$ there exists a vector x of norm one such that $|(T-\lambda)x|<\varepsilon$. Now we may state the inclusion theorem.

(2.2) Theorem. *Let A be a normal operator on a Hilbert space H. Let P be an orthogonal projection in H and set $Q=I-P$. Then each disc of diameter $|QAP|$ and centre in $\sigma_a(A_P)$ intersects the spectrum of A.*

Proof. Suppose that $\lambda\in\sigma_a(A_P)$ and that $x=Px$. We have then

$$(A-\lambda)x = (P+Q)(A-\lambda)Px = P(AP-\lambda)x+Q(AP-\lambda)Px = P(AP-\lambda)x+QAPx$$

and, since P and Q are projections on H_0 and $H_0^\perp$

$$|(A-\lambda)x|^2 = |P(AP-\lambda)x|^2+|QAPx|^2 = |(A_P-\lambda)x|^2+|QAPx|^2.$$

Since $\lambda\in\sigma_a(A_P)$, it follows that the infimum of $|(A-\lambda)x|^2$ on the unit sphere of H_0 is $\leqq |QAP|^2$. Consequently, the infimum of $|(A-\lambda)x|$ on the unit sphere of H is $\leqq |QAP|$. It follows that the disc $|z|\leqq|QAP|$ must contain a point of $\sigma(A-\lambda)$ so that the disc $|z-\lambda|\leqq|QAP|$ must contain a point of $\sigma(A)$. The proof is complete.

Let us add a few remarks concerning applications of the preceding theorem. In order to obtain inclusion discs for the operator A, we must have some information about the approximate point spectrum of the "smaller" operator $T=PAP$ restricted to the range of P. It is a well known fact that, in general, the approximate point spectrum $\sigma_a(T)$ of an operator T, although always nonempty, may differ considerably from the whole spectrum. If T is normal, we have $\sigma_a(T)=\sigma(T)$. However, the restriction of PAP to the range of P need not be normal if A is normal; for the restriction of PAP to be normal it suffices that P commute with A. If A happens to be symmetric then T is symmetric as well. The only other case likely to be of use is that of a finite-dimensional projection P; then T is finite-dimensional so that $\sigma_a(T)=\sigma(T)$.

Using these remarks, we may now state a corollary of the theorem with approximate point spectrum replaced by the whole spectrum.

(2.3) *Let A be a normal operator on a Hilbert space H. Let P be an orthogonal projection in H. Denote by A_P the operator PAP restricted to the range of P. Then each disc of diameter $|(I-P)AP|$ and centre in $\sigma(A_P)$ is an inclusion disc for A provided one of the following conditions is satisfied:*

1° A_P is normal, 2° $PA=AP$, 3° A is symmetric, 4° P is finite-dimensional.

3. Some consequences

In this section we formulate three immediate consequences of the theorem in important particular cases.

First we investigate one-dimensional projections. Clearly each such projection is given by the formula $Px=(x, e)e$ where e is an arbitrary vector of norm one.

(3.1) *Let A be a normal operator on a Hilbert space H. Let e be a vector of norm one. Then the disc*

$$|z-(Ae, e)| \leqq \big|(A-(Ae, e))e\big|$$

contains at least one point of the spectrum of A.

Proof. Clearly,

$$PAPx = (x, e)(Ae, e)e, \quad QAPx = APx-PAPx = (x, e)\big(A-(Ae, e)\big)e,$$

and A_P has a one-point spectrum (Ae, e). The conclusion follows immediately from the theorem.

Another particular case of interest is that of projections onto a hyperplane.

(3.2) *Let A be a normal operator on a Hilbert space H. Let e be a vector of norm one. Let P be defined by*

$$Px = x-(x, e)e.$$

Then each disc of diameter $|(A-(Ae, e))e|$ and centre in $\sigma_a(A_P)$ intersects the spectrum of A.

Proof. It suffices to compute QAP. We have for every $x\in H$

$$QAPx = (APx, e)e = \big(A(x-(x, e)e), e\big)e =$$
$$= \big((Ax, e)-(x, e)(Ae, e)\big)e = \big(x, A^*e-(Ae, e)^*e\big)e.$$

It follows that

$$|QAP| = \big|(A-(Ae,e))^* e\big| = \big|(A-(Ae,e))e\big|.$$

The last equality is a consequence of the fact that, for a normal operator A, the operator $A-(Ae, e)$ is normal as well.

To conclude we present a result formulated in the classical language of "matrix theory".

(3.3) Theorem. *Let A be a complex n by n matrix with elements a_{ik}. Denote by $A^{(i)}$ the $n-1$ by $n-1$ matrix obtained by deleting the* ith *row and* ith *column of A. Suppose that A is normal so that the following equality holds*

$$\sum_{\substack{k=1\\k\neq i}}^{n} |a_{ik}|^2 = \sum_{\substack{j=1\\j\neq i}}^{n} |a_{ji}|^2 \text{ for each } i.$$

Denote by r_i the nonnegative square root of this number. Then:

1° *Each disc of the form $|z-a_{ii}|\leqq r_i$ contains at least one proper value of A.*

2° *Each disc of the form $|z-\alpha|\leqq r_i$, where α is a proper value of $A^{(i)}$, contains at least one proper value of A.*

Proof. An immediate consequence of the preceding two results.

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Uniformly distributed sequences in quotient groups

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Let G be a compact topological group with countable base, H a closed normal subgroup, $p\colon G \to G/H$ the canonical homomorphism. If a sequence (x_n) is uniformly distributed in G, then it is easy to prove that $p(x_n)$ is u.d. in G/H. If $G=K\times H$, and (y_n) is, u.d. in K, then as is proved in [1], for almost every sequence (z_n), $z_n\in H$ (with respect to the product-measure on H^∞) the sequence (y_n, z_n) is u.d. in G. We prove the following

Theorem 1. *If (y_n) is u.d. in G/H, then there exists a u.d. sequence (x_n) in G such that $p(x_n)=y_n$.*

Remark. The result in [1] is based on a Theorem of HLAWKA ([3], Th. 11) using a theorem of Hill on infinite matrices. Here we are going to use a different method.

The main result of this paper is the following

Proposition. *Let G be a locally compact group, H a closed normal amenable subgroup such that G/H is compact If (y_n) is u.d. in $G/H=K$, then for any $f\in L^1(G)$, $\int f(x)\,dx=0$ (dx = left Haar measure on G) there exists a sequence (x_n) in G, satisfying $p(x_n)=y_n$ and $\lim_{N\to\infty}\left\|\frac{1}{N}\sum_{n=1}^{N} x_n f\right\|_1=0$ ($\|g\|_1=\int|g(x)|\,dx$ and $yf(x)=f(y^{-1}x)$).*

Theorem 1 then follows from the following

Lemma 1. *Let G be a compact metric group, then there exists an $f\in L^1(G)$ such that $\int f(x)\,dx=0$ and $\frac{1}{N}\left\|\sum_{n=1}^{N} x_n f\right\|_1\to 0$ implies: (x_n) is u.d. in G.*

Proof. We may choose an $f\in L^2(G)$ such that $\int f(x)\,dx=0$ and $\int f(x)D(x)\,dx$ is a non-singular matrix for any non-trivial continuous irreducible unitary representation D of G (there are only countably many inequivalent ones) and then apply Th. 2 of [6].

If G is compact so is G/H and H, H is amenable ([4], Ch. 8) and Theorem 1 follows from the Proposition and Lemma 1.

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Proof of the Proposition

Lemma 2. *Given* $\varepsilon>0$, *there exists a sequence* (x_n) *such that* (i) $p(x_n)=y_n$ *and* (ii) $\limsup_{N\to\infty}\left\|\frac{1}{N}\sum_{n=1}^{N}x_n f\right\|_1 \leqq \varepsilon$.

Proof. Let $T_H\colon L^1(G)\to L^1(G/H)$, $[T_H f](p(x))=\int_H f(xy)\,dy$ be the canonical morphism onto $L^1(K)$, $dy=$ left Haar-measure on H, and put $g=T_H f$, then $\int_K g(x)\,dx=0$ by Weil's formula ([4], Ch. 3, §4. 4, 5). Choose a neighbourhood of U of the neutral element of G such that

(1) $\|xf-f\|_1<\varepsilon$ for all $x\in U$, put $V=p(U)$ ([4], §5.5).

There exist finitely many elements $b_1,\dots,b_m$ in $K=G/H$ such that

$$\bigcup_{i=1}^{m} b_i V = K. \quad \text{Put} \quad B_l = b_l V - \bigcup_{i=1}^{l-1} b_i V \quad (l=1,\dots,m).$$

Then $B_1,\dots,B_m$ constitute a partition of K into measurable sets. Let χ_i denote the characteristic function of B_i. Then we have

$$\sum_{i=1}^{m}\chi_i * g = 1*g = \int g(x)\,dx = 0. \tag{2}$$

If $v\in V$, choose $u\in U$ such that $p(u)=v$, then by means of the relation $T_H(uf)=vT_H f$ and by (1) we obtain

$$\|(b_i v)g-b_i g\|_1 = \|vg-g\|_1 = \|T_H(uf-f)\|_1 \leqq \|uf-f\|_1 \leqq \varepsilon,$$

thus we have

$$\left\|\chi_i * g-\left(\int\chi_i\right)b_i g\right\|_1 \leqq \int_{b_i V}\chi_i(y)\|yg-g\|_1\,dy < \varepsilon\left(\int\chi_i\right). \tag{3}$$

Choose elements $a_1,\dots,a_m$ from G in such a way that $p(a_i)=b_i$ and set $f_1=\sum_{i=1}^{m}\left(\int\chi_i\right)a_i f$. Then we have $\|T_H f_1\|_1<\varepsilon$ $\big((2)+(3)\big)$. We have assumed that H was amenable, therefore there exist elements $s_1,\dots,s_r\in H$ such that

$$\frac{1}{r}\left\|\sum_{k=1}^{r}s_k f_1\right\|_1 < \varepsilon \quad \text{([4], Ch. 8, §4.3, §6.5).} \tag{4}$$

We may suppose that the boundary of V has measure 0. (If not, replace V by a neighbourhood V' of the neutral element of K that is contained in V and whose boundary has measure 0, also replace U by $U\cap p^{-1}(V)$). Then we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_i(y_n) = \int_K \chi_i(x)\,dx \quad \text{([2], Th. 13).} \tag{5}$$

For $i=1, \dots, m$ and $j=1, 2, \dots$ let $n(j, i)$ be that index n of y for which $y_n \in B_i$ and exactly j members of the sequence $y_1, \dots, y_n$ belong to B_i. Then we have

(6) $$y_{n(j,i)} = b_i v_{n(j,i)}, \quad v_{n(j,i)} \in V.$$

Define the sequence (z_n) in G by $z_{n(j,i)} = s_k a_i$ if $j \equiv (k-1) \bmod r$, then by (4) and (5) we obtain:

(7) $$\limsup_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} z_n f \right\|_1 \leqq \varepsilon.$$

Choose finally $u_n \in U$ suh that $p(u_n) = v_n$, $x_n = z_n u_n$, then $\|z_n f - x_n f\|_1 = \|u_n f - f\|_1$ and by (1) we obtain that $\limsup_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} x_n f \right\|_1 \leqq 2\varepsilon$. This completes the proof of Lemma 2. Now let $x_{n,k}$ be the sequence obtained by Lemma 2 for $\varepsilon = 1/2k$, then we can find a strictly increasing sequence of positive integers N_k satisfying

(8) $$\text{a)} \quad \left\| \frac{1}{N} \sum_{n=1}^{N} x_{n,k} f \right\|_1 \leqq \frac{1}{k}, \quad N \geqq N_k, \quad \text{b)} \quad N_1 + \dots + N_k \leqq N_{k+1}.$$

We define: $x_n = x_{n,k+1}$ if $N_k < n \leqq N_{k+1}$; $k = 0, 1, \dots$, $N_0 = 0$, then (8) a) implies that $\left\| \sum_{n=M+1}^{N} x_{n,k} f \right\|_1 \leqq (N+M)/k$, $N > M \geqq N_k$, thus by (8) b) we obtain that for $N_k < N \leqq N_{k+1}$ we have

$$\left\| \sum_{n=1}^{N} x_n f \right\|_1 \leqq N_1 \|f\|_1 + (N_1 + N_2) + (N_2 + N_3)/2 + \dots + (N_{k-1} + N_k)/(k-1) +$$
$$+ (N_k + N)/k = o(N)$$

and the proof of the Proposition is complete.

As a further application of the Proposition we obtain

Theorem 2. *If (y_n) is a uniformly distributed sequence modulo* 1, *then there exists a sequence (x_n) such that $x_n \equiv y_n$ (mod* 1) *and (x_n) is u.d. modulo a for all $a > 0$.*

Proof. We apply the Proposition to $f \in L^1(R)$ satisfying $\hat{f}(t) \neq 0$ iff $t \neq 0$, then there exists a sequence (x_n) such that $p(x_n) = y_n$ and $\lim_N \left\| \frac{1}{N} \sum_{n=1}^{N} x_n f \right\|_1 = 0$, and by direct computation we obtain that $\lim \frac{1}{N} \sum_{n=1}^{N} \exp(iyx_n) = 0$ for all $y \neq 0$, which proves Theorem 2 by means of Weyl's criterion.

Example. If z is an arbitrary irrational number then the sequence (nz) is u.d. mod 1, therefore there exists a sequence (x_n) congruent to (nz) mod 1 and such that (x_n) is u.d. mod a for all $a > 0$, whereas (nz) is u.d. mod a iff a is an irrational multiple of z.

Remarks. A stronger version of the Proposition is true: there exists a single sequence that satisfies the relation in the Proposition for all $f\in L^1(G)$, $\int f=0$ (compare [7], Th. 1, Th. 2). This gives a partial answer to a question in [5], (starting from a countable dense set of $L^0(G)=\{f\colon f\in L^1(G).\ \int f=0\}$ a similar proof leads to this result. G must be second countable.) Theorem 1 remains valid if G is compact and H has a countable dense subset. It can be shown that there exists a sequence (s_n) such that $\lim \left\|\frac{1}{r}\sum_{n=1}^{r} s_n f\right\|_1 = \|T_H f\|_1$ for all $f\in L^1(G)$ (construction and proof as in [8]). The same proof as that of the Proposition (compare (4)!) shows that there exists a sequence (x_n), $p(x_n)=y_n$ such that $\lim \left\|\frac{1}{N}\sum_{n=1}^{N} x_n f\right\|_1 =0$ for all $f\in L^1(G)$, $\int f=0$, which implies that (x_n) is u.d. in G ([7], Th. 2).

Finally, it should be noted that the condition that H is amenable in the assumptions of the Proposition is necessary ([4], Ch. 8, § 4.3).

Additional Remark (by proof-reading). Th. 1 implies immediately: *Let G, G_1 be compact metric groups, $p\colon G\Rightarrow G_1$ a continuous homorphism. If (y_n) is u. d. in $p(G)$, then there exist x_n, $p(x_n)=y_n$, (x_n) is u. d. in G.*

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On radicals in lattices

M. STERN

To the memory of my teacher Professor Andor Kertész

§ 1. Introduction

In [2] B. STENSTRÖM has defined the radical of a complete lattice L as the meet of all dual atoms of L. Furthermore he has studied properties of the radical in several classes of complete modular lattices. Our aim in this note is to generalize some of the theorems of [2] to certain classes of lattices which are not modular in general but preserve some properties of modularity such as M-symmetry, cross-symmetry and the covering property. Applications are given to some classes of AC-lattices. Our main results are Theorems 3.8 and 3.14.

§ 2. Basic notions

The least element and the greatest element of a lattice, if they exist, are denoted by 0 and 1, respectively. In a lattice L, we say that the element $a \in L$ covers the element $b \in L$ and write $b \prec a$ in case $b < a$ and $b \leqq x \leqq a$ implies $x = b$ or $x = a$ .In a lattice L with 0 an element $p \in L$ is called an atom, if $0 \prec p$. In a lattice L with 1 an element $m \in L$ is called a dual atom if $m \prec 1$. $a \| b$ means that $a \in L$ and $b \in L$ are incomparable elements, that is neither $a \leqq b$ nor $b \leqq a$; $c = a + b$ stands for $c = a \cup b$ and $a \cap b = 0$; if $a \leqq b$, then $[a, b] = \{x \in L \colon a \leqq x \leqq b\}$.

Definition 2.1. Let L be a lattice and $a, b \in L$. We say that a, b is a *modular pair* and write $(a, b)M$ when $c \leqq b$ implies $(c \cup a) \cap b = c \cup (a \cap b)$ in L. We say that a, b is a *dual modular pair* and we write $(a, b)M^*$ when $c \geqq b$ implies $(c \cap a) \cup b = c \cap (a \cup b)$ in L.

Lemma 2.2 ([1, Lemma 1.3, p. 2]). *Let L be a lattice and $a, b \in L$. If both $(a, b)M$ and $(a, b)M^*$ then the sublattices $[a, a \cup b]$ and $[a \cap b, b]$ are isomorphic and*

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we write $[a, a\cup b]\cong[a\cap b, b]$. *An isomorphism is effected by the following mutual inverse mappings:* $\varphi(x)=x\cup a$ *and* $\psi(y)=y\cap b$.

From the isomorphic mappings of the preceding lemma one gets:

Corollary 2.3. (i) *If* m *is a dual atom of* $[a\cap b, b]$ *then* $\varphi(m)=m\cup a$ *is a dual atom of* $[a, a\cup b]$;

(ii) $\varphi(\cap m_\nu)=\cap\varphi(m_\nu)$ $(m_\nu\in[a\cap b, b])$ *if the meets exist,*

Proof. (i) is obvious. For (ii) we have

$$\psi[\cap\varphi(m_\nu)] = b\cap[\cap\varphi(m_\nu)] = \cap[b\cap\varphi(m_\nu)] = \cap\psi\varphi(m_\nu) = \cap m_\nu.$$

Hence

$$\cap\varphi(m_\nu) = \varphi\psi[\cap\varphi(m_\nu)] = \varphi(\cap m_\nu).$$

Lemma 2.4 ([1, Lemma 1.5, p. 2]) *Let* L *be a lattice and* $a, b\in L$. *If* $(a, b)M$ *then* $(a_1, b_1)M$ *for any* $a_1\in[a\cap b, a]$ *and* $b_1\in[a\cap b, b]$.

Definition 2.5 A lattice L is called *modular* when $(a, b)M$ holds for all $a, b\in L$. A lattice L is called *M-symmetric* when $(a, b)M$ implies $(b, a)M$ in L. A lattice L is called *cross-symmetric* if $(a, b)M$ implies $(b, a)M^*$ in L.

For a detailed treatment of symmetric lattices we refer to [1].

Theorem 2.6 ([1, Theorem 1.9, p. 3]). *A cross-symmetric lattice is M-symmetric.*

Corollary 2.7. *Let* L *be a cross-symmetric lattice and* $a, b\in L$. *If* $(a, b)M$ *then* $(b, a)M^*$, $(b, a)M$ *and* $(a, b)M^*$.

Proof. The assertion follows immediately from Definition 2.5 and Theorem 2.6.

The implication

$$N^*\colon a\prec a\cup b \quad \text{implies} \quad a\cap b\prec b$$

plays an important role in this paper. It is a sort of dual covering property and is satisfied in every modular lattice.

A lattice L with 0 is called atomistic if every element of L is the join of a family of atoms. An element of a lattice L with 0 is called a finite element when it is either 0 or the join of a finite number of atoms. The set of all finite elements of L is denoted by $F(L)$. The covering property is introduced as follows:

if p is an atom and $a\cap p=0$, then $a\prec a\cup p$. We call L an AC-lattice if it is an atomistic lattice with covering property.

For the theory of AC-lattices we refer to [1].

Definition 2.8 In a lattice L, an element $a\in L$ is called a *modular element* when $(x, a)M$ for every $x\in L$. A lattice L with 0 is called *finite-modular*, when every finite element of L is a modular element.

Theorem 2.9 ([1, Theorem 9.5, p. 42]). *Let L be an AC-lattice. The following two statements are equivalent:*

(i) *L is finite-modular;*

(ii) *in L the implication N^* holds true.*

A lattice L with 0 and 1 is called a *DAC-lattice* when both L and its dual are AC-lattices (cf. [1, p. 123]).

Theorem 2.10 ([1, Theorem 27.6, p. 123]). *Every DAC-lattice is finite-modular and M-symmetric.*

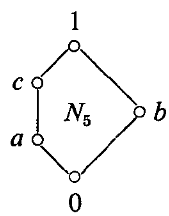
A matroid lattice may be defined as an upper continuous AC-lattice (cf. [1, p. 56]). Now we are ready to define the radical and to study its properties.

§ 3. The radical and its properties

In this paragraph L denotes always a complete lattice.

Definition 3.1. Let $[a, b]$ be an interval of a lattice L. The *radical* of $[a, b]$ is the meet of all dual atoms of $[a, b]$, and is denoted by $R[a, b]$. If $[a, b]$ has no dual atom, then $R[a, b]=b$. Instead of $R[0, 1]$ we shall write $R(L)$. A lattice L is called *radical free* if $R(L)=0$. A lattice L is called *strongly radical free* if $R[a, 1]=a$ for every $a\in L$.

By definition, a strongly radical free lattice is radical free. The converse is not true; consider the following lattice



We have $R(N_5)=b\cap c=0$ but $R[a, 1]=c>a$.

The following lemma gives an equivalent condition for an AC-lattice to be strongly radical free.

Lemma 3.2 *An AC-lattice L is strongly radical free if and only if L is relatively dually atomic (that is, for every $a>b$ there exists a dual atom m of L such that $a>a\cap m\geqq b$).*

Proof. By definition, L is strongly radical free if and only if every $a\in L$ is the meet of those dual atoms m_ν of L, for which $m_\nu\geqq a$ (in the terminology of [1]:

if and only if L is dually atomistic). By the dual of [1, Lemma 7.2, p. 30] this is the case if and only if L is relatively dually atomic.

Corollary 3.3. *Any DAC-lattice is strongly radical free.*

Proof. A *DAC*-lattice is by definition relatively dually atomic, Hence the assertion follows from the preceding lemma.

Corollary 3.4 *Any matroid lattice is strongly radical free.*

Proof. A matroid lattice is relatively dually atomic (cf. [1, Remark 13. 1, p. 56]). Hence the assertion follows from Lemma 3.2.

Corollary 3.5 ([2, Proposition 12]). *If L is a modular matroid lattice then* $R(L)=0$.

Now we are going on to study properties of the radical in lattices which need not be modular but satisfy certain conditions that are fulfilled in the presence of modularity.

Lemma 3.6. *Let L be a lattice in which N^* is satisfied. If $a \leqq b_1 \leqq b_2$ in L then*

$$R[a, b_1] \leqq R[a, b_2].$$

Proof. Let first $m \in [a, b_2]$ and $m \prec b_2$. If $b_1 \leqq m$, then $R[a, b_1] \leqq b_1 \leqq m$. Let now $b_1 \| m$. Then $b_1 \cup m = b_2 \succ m$ and hence by N^* one has $b_1 \cap m \prec b_1$. From this it follows that $R[a, b_1] \leqq b_1 \cap m \leqq m$. Therefore in any case $R[a, b_1] \leqq R[a, b_2]$. If $[a, b_1]$ or $[a, b_2]$ has no dual atoms then the assertion is obvious.

Corollary 3.7 ([2, Proposition 10]). *If L is a modular lattice and $a \in L$, then* $R[0, a] \leqq R(L)$.

Proof. N^* holds in every modular lattice. Applying the preceding lemma, we get the assertion.

It has been proved in [2, Proposition 11] that if in a modular lattice L, 1 is the direct sum of two elements, then $R(L)$ is the direct sum of the radicals of the two direct summands. This is generalized in the following

Theorem 3.8 *Let L be a lattice and let $a, b \in L$. Assume that the following three conditions are satisfied:*

(i) *N^* holds in L;*

(ii) *$(a, b)M$, $(b, a)M^*$ and $(b, a)M$, $(a, b)M^*$ hold in L;*

(iii) *$(b, R[a \cap b, a])M^*$ or $(a, R[a \cap b, b])M^*$ holds in L.*

Then

$$R[a \cap b, a] \cup R[a \cap b, b] = R[a \cap b, a \cup b].$$

Proof. Let L be a lattice and $a, b \in L$. Let (i), (ii) and

$$(1) \qquad (b, R[a\cap b, a])M^*$$

be satisfied in L. If instead of (1) the relation $(a, R[a\cap b, b])M^*$ is satisfied then the proof is similar to the now given one. By Lemma 3.6 we get

$$(2) \qquad R[a\cap b, a]\cup R[a\cap b, b] \leqq R[a\cap b, a\cup b].$$

Now we prove that the converse inequality holds true in L. By condition (ii) we get from lemma 2.2 the following isomorphisms:

$$[a, a\cup b] \cong [a\cap b, b] \quad \text{and} \quad [b, a\cup b] \cong [a\cap b, a].$$

Let $\varphi(x)=x\cup a$ and $\bar\varphi(x)=x\cup b$. By $\{m_\alpha : \alpha\in A\}$ we denote the set of the dual atoms of $[a\cap b, b]$ and by $\{n_\beta : \beta\in B\}$ we denote the set of the dual atoms of $[a\cap b, a]$. Then we have by Corollary 2.3

$$(3) \qquad \varphi(R[a\cap b, b]) = \varphi(\cap m_\alpha) = \cap\varphi(m_\alpha) = \cap(m_\alpha\cup a) \geqq R[a\cap b, a\cup b]$$

and in a similar manner

$$(4) \qquad \bar\varphi(R[a\cap b, a]) = \bar\varphi(\cap n_\beta) = \cap\bar\varphi(n_\beta) = \cap(n_\beta\cup b) \geqq R[a\cap b, a\cup b].$$

By (3) and (4) it follows that

$$R[a\cap b, b]\cup a, \quad R[a\cap b, a]\cup b \geqq R[a\cap b, a\cup b]$$

and hence

$$(5) \qquad R[a\cap b, a\cup b] \leqq (R[a\cap b, b]\cup a)\cap(R[a\cap b, a]\cup b).$$

From (1) and from $R[a\cap b, b]\cup a \geqq R[a\cap b, a]$ we get (cf. Definition 2.1)

$$(6) \qquad (R[a\cap b, b]\cup a)\cap(b\cup R[a\cap b, a]) = \{(R[a\cap b, b]\cup a)\cap b\}\cup R[a\cap b, a].$$

Since $R[a\cap b, b]\leqq b$ and $(a, b)M$ we get further (cf. Definition 2.1)

$$(7) \quad (R[a\cap b, b]\cup a)\cap b = R[a\cap b, b]\cup(a\cap b).$$

Now by (5), (6) and (7) it follows that

$$(8) \qquad R[a\cap b, a\cup b] \leqq R[a\cap b, a]\cup R[a\cap b, b]\cup(a\cap b).$$

(2) and (8) together prove the theorem.

Corollary 3.9. *Let L be a cross-symmetric lattice in which N^* is satisfied. If $(a, b)M$ then*

$$R[a\cap b, a]\cup R[a\cap b, b] = R[a\cap b, a\cup b].$$

Proof. We show that conditions (i)—(iii) of Theorem 3.8 are satisfied. Condition (i) is satisfied by assumption. Condition (ii) holds by Corollary 2.7. Since $(a, b)M$

and $a \cap b \leqq R[a \cap b, a] \leqq a$ one gets $(R[a \cap b, a], b)M$ by Lemma 2.4. From this it follows that $(b, R[a \cap b, a])M^*$ holds since L is cross-symmetric. This means that condition (iii) is likewise satisfied. Hence the assertion follows from the preceding theorem.

Corollary 3.10. *Let L be a modular lattice and $a, b \in L$. Then*

$$R[a \cap b, a] \cup R[a \cap b, b] = R[a \cap b, a \cup b].$$

Proof. A modular lattice is cross-symmetric and satisfies N^*. Furthermore $(a, b)M$ for all $a, b \in L$. Applying Corollary 3.9, we get the assertion.

Remark. By similar arguments as in Theorem 3.8 we are able to prove the following

Theorem. *Let L be a modular lattice and $a, b \in L$. Then*

$$R[a, a \cup b] \cap R([b, a \cup b] = R[a \cap b, a \cup b].$$

Specializing Corollary 3.10 we get

Corollary 3.11 ([2, Proposition 11]). *If L is a modular lattice and $1 = a + b$ then $R(L) = R[0, a] + R[0, b]$.*

Proof. From $1 = a + b$ we get $R[a \cap b, a] = R[0, a]$, $R[a \cap b, b] = R[0, b]$ and $R[a \cap b, a \cup b] = R[0, 1] = R(L)$. Since $0 \leqq R[0, a] \leqq a$ and $0 \leqq R[0, b] \leqq b$, we have $R[0, a] \cap R[0, b] = 0$. Now the assertion follows from Corollary 3.10.

In the following two corollaries we apply Theorem 3.8 to finite-modular *AC*-lattices.

Corollary 3.12. *Let L be a finite-modular AC-lattice and let $a, b \in L$. If $a \in F(L)$ then $R[a \cap b, b] = R[a \cap b, a \cup b]$. Similarly, if $b \in F(L)$ then $R[a \cap b, a] = R[a \cap b, a \cup b]$.*

Proof. We show that conditions (i)—(iii) of Theorem 3.8 hold in L. Condition (i) is satisfied by Theorem 2.9. Condition (ii) and condition (iii) hold by [1, Corollary 9.4, p. 42]. If now $a \in F(L)$, then $[0, a]$ is a matroid lattice by [1, Lemma 8.10, p. 37.] Hence $R[a \cap b, a] = a \cap b$ by Corollary 3.4. Applying Theorem 3.8 we get $R[a \cap b, b] = R[a \cap b, a \cup b]$. Similarly $R[a \cap b, a] = R[a \cap b, a \cup b]$ if $b \in F(L)$.

Corollary 3.13. *Let L be a finite-modular AC-lattice and let $a \in F(L)$. If $a \in F(L)$ has a complement in L then $a \cap R(L) = 0$.*

Proof. Let b be a complement of $a \in F(L)$. Since $a \in F(L)$, $a \cap b = 0$ and $a \cup b = 1$

we get by Corollary 3.12 that $R(L)=R[0, b]$. From this it follows that

$$a\cap R(L) = a\cap R[0, b] = a\cap b = 0,$$

which proves the corollary.

Now we put the question: under which conditions can we prove in the preceding corollary the converse implication? An answer to this question is given in

Theorem 3.14. *Let L be an M-symmetric finite-modular AC-lattice and let $a\in F(L)$. If $a\cap R(L)=0$, then $a\in F(L)$ has a complement in L.*

Proof. Let

$$a\cap R(L) = 0. \tag{9}$$

Assume that $a\in F(L)$ has no complement in L. From this assumption we shall derive a contradiction. Let $a_m(\leqq a)$ be a minimal element without complement in L. Such an a_m exists since $a\in F(L)$. Furthermore $a_m\neq 0$, since 0 has the complement 1 in L. From (9) it follows that

$$a_m\cap R(L) = 0.$$

Hence there exists a dual atom $n\in L$ such that $a_m\| n$. Then by N^* (cf. Theorem 2.9)

$$a_m\cap n \prec a_m. \tag{10}$$

By the minimality of a_m, it follows from (10) that $a_m\cap n$ has a complement $b\in L$. Let

$$d \overset{\text{def}}{=} b\cap n.$$

We show that d is a complement of a_m which is a contradiction to our assumption. Evidently

$$n\cap b < b \tag{11}$$

since from $n\cap b=b$ it follows that $b\leqq n$ and $1=(a_m\cap n)\cup b\leqq n$, a contradiction.

From (11) we get by N^* that $n\cap b\prec b$. By [1, Lemma 7.5, p. 31] it follows that $(n, b)M$. Since L is M-symmetric, we get $(b, n)M$. This means that

(12) $x\leqq n$ implies $(x\cup b)\cap n=x\cup(b\cap n)$ in L (cf. Definition 2.1).

Since $a_m\cap n\leqq n$, it follows from (12) that

$$\{(a_m\cap n)\cup b\}\cap n = (a_m\cap n)\cup(b\cap n). \tag{13}$$

Then by the definition of d and by (13)

$$a_m\cup d = a_m\cup (a_m\cap n)\cup d = a_m\cup [(a_m\cap n)\cup(b\cap n)] = a_m\cup[n\cap \{(a_m\cap n)\cup b\}] =$$
$$= a_m\cup(n\cap 1) = a_m\cup n = 1.$$

Furthermore

$$a_m \cap d = a_m \cap n \cap b = 0.$$

Hence d is a complement of a_m. This contradiction proves the theorem.

Corollary 3.15. *Let L be a DAC-lattice. If $a \in F(L)$ then it has a complement in L.*

Proof. A *DAC*-lattice is finite-modular and *M*-symmetric by Theorem 2.10. For a *DAC*-lattice L it follows by Corollary 3.3 that $R(L)=0$. Applying Theorem 3.14 we get the assertion.

We remark that Corollary 3.15 forms a part of [1, Theorem 27.10, p. 124]. Summarizing Corollary 3.13 and Theorem 3.14 we have

Corollary 3.16. *Let L be an M-symmetric finite-modular AC-lattice and let $a \in F(L)$. Then $a \cap R(L)=0$ if and only if $a \in F(L)$ has a complement in L.*

We conclude this paragraph by remarking that the preceding corollary is an extension of a part of [2, Theorem 14].

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Ringe, in welchen jedes Element ein Linksmultiplikator ist

F. SZÁSZ

Das Ziel dieser Note ist das im Buche [7] des Verfassers gestellte Problem 82 zu lösen: „In welchen Ringen ist jedes Element ein Linksmultiplikator?"

Unter einem Ring verstehen wir hierbei immer einen assoziativen Ring. Bezüglich der benötigten Grundbegriffe verweisen wir auf N. DIVINSKY [2], N. JACOBSON [3], L. RÉDEI [4] und auf [7].

Unter einem Linksmultiplikator l eines Ringes A verstehen wir ein Element von A, für das es eine ganze rationale Zahl n gibt, so dass $lx=nx$ für jedes $x\in A$ gilt. Im Ring Z der ganzen rationalen Zahlen ist also jedes Element ein Linksmultiplikator. Die Linksmultiplikatoren sind Verallgemeinerungen der Linkseinselemente eines Ringes. Bezüglich der Existenz des Einselementes eines Ringes spielen übrigens auch die Linksmultiplikatoren eine wichtige Rolle, wie es die Arbeiten B. BROWN—N. H. McCOY [1], F. SZÁSZ [5] und [6], witherin J. SZENDREI [8] zeigen. Andererseits ist jedes Element a^* des Linksannullatorideals A^* des Ringes A offenbar ein Linksmultiplikator, denn es gilt

$$a^*\cdot x = 0\cdot x = 0 \quad \text{für} \quad \text{jedes } x\in A.$$

Satz. *In einem Ring A ist jedes Element von A dann und nur dann ein Linksmultiplikator, wenn sich jedes Element $a\in A$ in der Gestalt einer Summe ka_0+a^* $(k\in Z)$ darstellen läßt, wobei $a_0x=n_0x$, $n_0\in Z$, $n_0>0$ und $a^*\in A^*$, also $a^*y=0$ für jede $x, y\in A$ bestehen.*

Bemerkung. Die zyklische additive Gruppe $(Za_0)^+$ erzeugt mod A^* den ganzen Ring, denn es gilt $(A/A^*)^+\cong(Za_0)^+$.

Beweis. Nehmen wir an, dass jedes Element des Ringes ein Linksmultiplikator ist. Gilt $A^*=A$, so ist jedes Element, als ein Linksannullator, auch ein Linksmultiplikator. Gilt aber $a\cdot A\neq 0$, so gibt es wegen der Definition eines Linksmultiplikators eine von Null verschiedene ganze Zahl $n\in Z$, so dass $(a-n)\cdot A=0$ gilt. Wählen wir

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jetzt das Element $a_0 \in A$ so, dass die enstprechende von Null verschiedene ganze Zahl $n_0 \in Z$ mit $(a_0 - n_0) \cdot A = 0$ in Absolutbetrag möglichst klein sei. Wir können voraussetzen, dass $n_0 > 0$ ist; sonst wählen wir $-a_0$ statt a_0.

Es sei nun $n = kn_0 + r$ mit $0 \leqq r < n_0$ und $b = a - ka_0$. Dann bestehen die Gleichungen:

$$bx = ax - ka_0x = nx - kn_0x = (n - kn_0)x = rx$$

für jedes $x \in A$, woraus man $(b - r) \cdot A = 0$ erhält. Wegen der Minimalität von n_0, wegen $0 \leqq r < n_0$ und $(b - r) \cdot A = 0$ ergibt sich $r = 0$, $rx = 0$, $bx = 0$ und somit ist $b = a^* \in A^*$ ein Linksannullator. Hiernach gilt aber $a = ka_0 + b = ka_0 + a^*$.

Umgekehrt nehmen wir an, daß $a = ka_0 + a^*$ gilt, wobei $a_0 \cdot x = n_0x$ und $a^* \cdot y = 0$ für jedes $x, y \in A$ bestehen. Dann erhält man für jedes $x \in A$ die Gleichungen:

$$ax = (ka_0 + a^*)x = ka_0x = (kn_0)x.$$

Damit haben wir den Satz bewiesen und auch Problem 82 gelöst.

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Bemerkungen zur Konvergenz der Reihen nach multiplikativen Funktionensystemen

KÁROLY TANDORI

1. Es sei (X, A, μ) ein Maßraum; die Klasse der A-meßbaren, in X fast überall endlichen reellen Funktionen bezeichnen wir mit $S(X)$.

Ein System $h=\{h_n(x)\}_0^\infty$ von Funktionen $h_n(x)\in S(X)$ ist ein *Konvergenzsystem für l^2 in $E(\in A)$*, wenn jede Reihe

$$\sum_{n=0}^{\infty} c_n h_n(x) \tag{1}$$

mit $c=\{c_n\}_0^\infty \in l^2$ in E fast überall konvergiert.

Das System h ist ein *Konvergenzsystem dem Maß nach für l^2*, wenn die Reihe (1) im Falle $c\in l^2$ zu einer Funktion aus $S(X)$ dem Maß nach konvergiert.

Für das System h und für eine Menge $E(\in A)$ mit endlichem Maß setzen wir

$$A_h(E) = \left\{\sum_{n=0}^{\infty}\Big(\int_E h_n(x)\,d\mu\Big)^2\right\}^{1/2}.$$

Wir sagen, daß *h die Eigenschaft B* besitzt, wenn die folgende Bedingung erfüllt ist: für jede positive Zahl ε und für jede meßbare Menge $E(\in A)$ mit endlichem Maß, gibt es eine meßbare Menge $F_\varepsilon(\subseteqq E)$ derart, daß $\mu(F_\varepsilon)\geqq\mu(E)-\varepsilon$ und $A_h(F_\varepsilon)<\infty$ bestehen.

Es sei $f=\{f_n(x)\}_1^\infty$ ein System von Funktionen aus $S(X)$. Das *Produktsystem* $\psi=\{\psi_n(x)\}_1^\infty$ von f definieren wir folgenderweise: es sei $\psi_0(x)\equiv 1$ $(x\in X)$ und für eine natürliche Zahl n mit der dyadischen Entwicklung $n=2^{n_1}+\ldots+2^{n_m}$ $(0\leqq n_1<\ldots<n_m)$ setzen wir $\psi_n(x)=\prod_{i=1}^{m} f_{n_i+1}(x)$.

Sei $E(\in A)$ von endlichem Maß $\mu(E)$. Das System f heißt *multiplikativ orthogonal in E*, wenn

$$\int_E \psi_n(x)\,d\mu = 0 \quad (n = 1, 2, \ldots),$$

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schwach multiplikativ in E, wenn

$$\sum_{n=0}^{\infty} \left| \int_E \psi_n(x)\, d\mu \right| < \infty,$$

und 2-*schwach multiplikativ in E*, wenn $A_\psi(E) < \infty$ gilt. Endlich heißt f *fast* 2-*schwach multiplikativ*, wenn das Produktsystem ψ von f die Eigenschaft B besitzt.

2. In dieser Note werden wir zuerst die folgende Behauptung beweisen:

Satz I. *Es sei (X, A, μ) ein σ-endlicher Maßraum. Ist f ein fast 2-schwach multiplikatives System, für welches $|f_n(x)| \leqq 1$ $(x \in X;\ n = 1, 2, \ldots)$ gilt, so ist ψ ein Konvergenzsystem für l^2 in X.*

Der Satz I folgt leicht aus dem folgenden

Satz A. *Ist f ein 2-schwach multiplikatives System in der Menge $E(\in A)$ von endlichem Maß, für welches $|f_n(x)| \leqq 1$ $(x \in X;\ n = 1, 2, \ldots)$ gilt, so ist ψ ein Konvergenzsystem für l^2 in E.*

Diese Behauptung hat F. Schipp und H. Türnpu [5] bewiesen. (Siehe noch F. Schipp und H. Türnpu [4].)

Beweis des Satzes I. Auf Grund der σ-Endlichkeit gibt es eine Folge von paarweise disjunkten A-meßbaren Mengen mit endlichem Maß derart, daß $\bigcup_{l=1}^{\infty} E_l = X$ ist. Es sei $\varepsilon (>0)$ beliebig vorgegeben. Dann gibt es für jeden Index l eine A-meßbare Menge $F_\varepsilon(l)$ derart, daß $\mu(F_\varepsilon(l)) \geqq \mu(E_l) - \varepsilon/2^l$ und $A_\psi(F_\varepsilon(l)) < \infty$ erfüllt sind. Nach dem Satz A ist ψ ein Konvergenzsystem für l^2 in $F_\varepsilon(l)$ $(l = 1, 2, \ldots)$. Es sei $F_\varepsilon = \bigcup_{l=1}^{\infty} F_\varepsilon(l)$. Da

$$u(X \setminus F_\varepsilon(l)) \leqq \sum_{l=1}^{\infty} \mu(E_l \setminus F_\varepsilon(l)) < \varepsilon$$

gilt und $\varepsilon (>0)$ beliebig ist, folgt die Behauptung.

3. Für eine Folge $c = \{c_n\}_1^\infty \in l^2$ setzen wir

$$a_0 = 1, \quad a_n = \begin{cases} c_k, & n = 2^{k-1} \quad (k = 1, 2, \ldots), \\ 0 & \text{sonst.} \end{cases}$$

Es ist klar, daß $a \in l^2$. Ist $M \geqq 1$, und f ein 2-schwach multiplikatives System in E, für welches $|f_n(x)| \leqq M$ $(x \in E;\ n = 1, 2, \ldots)$ gilt, dann ist $\{f_n(x)/M\}_1^\infty$ auch 2-schwach multiplikativ in E. Das Produktsystem des Systems $\{f_n(x)/M\}_1^\infty$ bezeichnen wir mit $\{\psi_n^*(x)\}_0^\infty$. Dann gilt $\psi_{2k-1}^*(x) = f_k(x)/M$ $(k = 1, 2, \ldots)$. Aus dem Satz A folgt also die folgende Behauptung:

Satz B. *Es sei E eine A-meßbare Menge von endlichem Maß. Ist f ein in E gleichmäßig beschränktes, 2-schwach multiplikatives System in E, dann ist f ein Konvergenzsystem für l^2 in E.*

Diese Behauptung haben G. ALEXITS und A. SHARMA [1] für in E multiplikativ orthogonale, dann G. ALEXITS [2] für in E schwach multiplikative Systeme und endlich F. SCHIPP und H. TÜRNPU [5] in E 2-schwach multiplikative Systeme sogar in etwas allgemeinerer Form bewiesen.

In den Beweisen dieser Behauptungen spielt die gleichmäßige Beschränkheit von f eine wesentliche Rolle. Doch kann man sie auf gewisse nicht gleichmäßig beschränkte Systeme übertragen. Es gilt nämlich der folgende Satz:

Satz II. *Es sei (X, A, μ) ein σ-endlicher Maßraum, und $F(x)$ eine in X fast überall endliche, A-meßbare, positive Funktion. Ist f ein fast 2-schwach multiplikatives System, für welches $|f_n(x)| \leqq F(x)$ $(x \in X;\ n=1, 2, \ldots)$ gilt, dann ist f ein Konvergenzsystem für l^2 in X.*

Beweis des Satzes II. Ohne Beschränkung der Allgemeinheit können wir $F(x) \geqq 1$ $(x \in X)$ voraussetzen.

Es sei $\{E_l\}_1^\infty$ eine Folge von A-meßbaren, paarweise disjunkten Mengen mit $\mu(E_l) < \infty$, und

$$\bigcup_{l=1}^{\infty} E_l = X.$$

Wir setzen

$$E_{l,N} = \{x \in E_l;\ N \leqq F(x) < N+1\} \quad (N = 1, 2, \ldots).$$

Dann ist

$$\mu\left(E_l \setminus \bigcup_{N=1}^{\infty} E_{l,N}\right) = 0 \quad (l = 1, 2, \ldots).$$

Es sei $\varepsilon(>0)$ beliebig vorgegeben. Dann gibt es eine A-meßbare Menge $F_\varepsilon(l, N)$ $(\subseteqq E_{l,N})$ mit $\mu(F_\varepsilon(l, N)) \geqq \mu(E_{l,N}) - \varepsilon/2^{l+N}$, und $A_\psi(F_\varepsilon(l, N)) < \infty$ $(l, N = 1, 2, \ldots)$. Man setze

$$F_\varepsilon(l) = \bigcup_{N=1}^{\infty} F_\varepsilon(l, N) \quad (l = 1, 2, \ldots) \quad \text{und} \quad F_\varepsilon = \bigcup_{l=1}^{\infty} F_\varepsilon(l).$$

Dann ist

$$\mu(E_l \setminus F_\varepsilon(l)) = \sum_{N=1}^{\infty} \mu(E_{l,N} \setminus F_\varepsilon(l, N)) < \varepsilon/2^l \quad (l = 1, 2, \ldots),$$

und

$$\mu(X \setminus F_\varepsilon) = \sum_{l=1}^{\infty} \mu(E_l \setminus F_\varepsilon(l)) < \varepsilon. \tag{2}$$

Nach dem Satz B ist f ein Konvergenzsystem für l^2 in $F_\varepsilon(l, N)$ $(l, N = 1, 2, \ldots)$, somit ist f auch ein Konvergenzsystem für l^2 in F_ε. Da $\varepsilon(>0)$ beliebig war, folgt die Behauptung aus (2).

4. Endlich werden wir zeigen, daß die fast 2-schwache Multiplikativität in gewissen Fällen eine natürliche Bedingung ist.

Es seien im Folgenden $X = (0, 1)$, A die Klasse der im Lebesgueschen Sinne meßbaren Teilmengen von $(0, 1)$, und μ das Lebesguesche Maß.

Das System h ist *fast orthonormiert*, wenn es für jede positive Zahl ε eine meßbare Menge F_ε $(\subseteqq (0, 1))$, eine positive Zahl M_ε und ein in $(0, 1)$ orthonormiertes System

$g_\varepsilon = \{g_n(\varepsilon; x)\}_0^\infty$ gibt derart, daß die Beziehungen $\mu(F_\varepsilon) \geqq 1-\varepsilon$ und

$$h_n(x) = M_\varepsilon g_n(\varepsilon; x) \quad (x \in F_\varepsilon;\ n = 0, 1, \ldots)$$

bestehen.

Wir beweisen zuerst den folgenden

Hilfssatz. *Ist h ein Konvergenzsystem dem Maß nach für l^2, dann besitzt h die Eigenschaft B.*

Ist nämlich h ein Konvergenzsystem dem Maß nach für l^2, dann ist nach einem Satz von E. M. Nikischin [3] h fast orthonormiert. Es sei $\varepsilon(>0)$ beliebig, und $E(\subseteqq (0, 1))$ eine meßbare Menge. Es seien $F_\varepsilon, M_\varepsilon, g_\varepsilon$ wie in der Definition der fast Orthonormalität. Es sei ferner $H_\varepsilon = E \cap F_\varepsilon$. Dann ist $\mu(H_\varepsilon) \geqq \mu(E) - \varepsilon$, und auf Grund der Besselschen Ungleichung:

$$A_h^2(H_\varepsilon) = M_\varepsilon^2 A_g^2(H_\varepsilon) \leqq M_\varepsilon^2 \mu(H_\varepsilon) \leqq M_\varepsilon^2.$$

Aus dem Hilfssatz bekommen wir die folgenden Sätze unmittelbar.

Satz III. *Ist das Produktsystem ψ des Systems f in* (0, 1) *ein Konvergenzsystem für l^2 (oder nur ein Konvergenzsystem dem Maß nach für l^2), dann ist f fast 2-schwach multiplikativ.*

Satz IV. *Es sei f ein 2-schwach multiplikatives System in* (0, 1), *für welches* $|f_n(x)| \leqq 1$ $(x \in (0, 1);\ n = 1, 2, \ldots)$ *gilt. Dann ist f fast 2-schwach multiplikativ.*

Wir erwähnen noch eine unmittelbare Folgerung, die zeigt, warum die Produktsysteme „gut" sind.

Satz V. *Ist ψ das Produktsystem eines Systems f mit* $|f_n(x)| \leqq 1$ $(x \in (0, 1);\ n = 1, 2, \ldots)$, *dann ist eine der notwendigen Bedingungen (nämlich die Eigenschaft B) für die Maßkonvergenz aller Reihen*

$$\sum_{n=0}^{\infty} c_n \psi_n(x) \quad (c \in l^2) \tag{3}$$

auch hinreichend defür, dass alle Reihen (3) *in* (0, 1) *fast überall konvergieren.*

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Idempotent reducts of abelian groups

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1. Introduction. The aim of this paper is to describe all idempotent reducts of abelian groups, in particular all minimal nontrivial idempotent reducts and to characterize the lattice of all subclones of the clone of the full idempotent reduct of abelian groups. These results extend a theorem of PŁONKA (see [2], [3]) which states that the clones of the idempotent reducts of a (not necessarily abelian) group form a chain if and only if the group is abelian and of prime power exponent. Moreover, if an abelian group is of exponent p^k for a prime p ($k \in N$) then this chain consists of $k+1$ elements. Our main result (Theorem 1) gives a representation for any idempotent reduct of the group of integers as a finite intersection of reducts of a very simple type. Hence the further results mentioned above can be deduced easily.

Basic universal algebraic concepts are from [1]. We are only interested in algebras up to equivalence. Let $\langle A; P \rangle$ be an algebra where P can be supposed to be the set of all polynomials. Reducts of $\langle A; P \rangle$ are defined to be algebras of the form $\langle A; R \rangle$ with $R \subseteq P$. By an idempotent reduct of $\langle A; P \rangle$ we mean a reduct $\langle A; J \rangle$ with all operations in J idempotent. The maximal idempotent reduct of $\langle A; P \rangle$, i.e. the reduct $\langle A; I \rangle$ where I contains all the idempotent operations of P, will be called the full idempotent reduct.

We adopt the definition of a *clone* due to TAYLOR. In [5] a clone is defined to be a heterogeneous algebra $\langle A_k;\ C_m^n,\ e_i^n \rangle_{k,m,n,i \in N, i \leqq n}$ with heterogeneous operations

$$C_m^n : A_n \times A_m^n \to A_m$$

called *substitutions* and

$$e_i^n : \{\emptyset\} \to A_n$$

called *projections*, satisfying the identities:

$$C_m^p(z, C_m^n(y_1, x_1, \ldots, x_n), \ldots, C_m^n(y_p, x_1, \ldots, x_n)) =$$
$$= C_m^n(C_n^p(z, y_1, \ldots, y_p), x_1, \ldots, x_n), \quad n, m, p \in N = \{1, 2, \ldots\};$$
$$C_m^n(e_i^n, x_1, \ldots, x_n) = x_i, \quad m, n, i \in N, \quad i \leqq n;$$
$$C_n^n(y, e_1^n, \ldots, e_n^n) = y, \quad n \in N.$$

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The concepts of isomorphism, subalgebra, subalgebra generated by a subset, etc. can naturally be generalized for heterogeneous algebras, in particular for clones, too (see [5]).

Note that for any algebra $\langle A; P\rangle$ the set of all polynomials P is a clone and the reducts of $\langle A, P\rangle$ are determined up to equivalence by the subclones of P.

The following notations will be used in the paper. N_0 or N will stand for the set of nonnegative or positive integers, respectively. Z and Z_m will mean the set of integers and the set of integers modulo m ($m\in N_0$), respectively. The greatest common divisor of natural numbers m and n will be denoted by (m, n). If e is an element of a lattice L we shall write $[e)_L$ for the dual principal ideal of L generated by e. The subclone of a clone C generated by the subset H of C will be denoted by $[H]$.

Any n-ary ($n\in N$) polynomial of an abelian group $\langle G; +, -, 0\rangle$ is of the form $\langle g_1, \ldots, g_n\rangle \mapsto c_1g_1+\ldots+c_ng_n$ where $c_1, \ldots, c_n\in Z$. It will be denoted by $c_1x_1+\ldots$ $\ldots+c_nx_n|_G$. Such a polynomial is idempotent if $c_1+\ldots+c_n=1$. In particular, $c_1x_1+\ldots+c_nx_n|_Z$ is idempotent if and only if $c_1+\ldots+c_n=1$.

2. The main theorem. Let n be a natural number. Consider the set of all idempotent polynomials $c_1x_1+\ldots+c_mx_m|_Z$ with the property that all the coefficients c_i but one are divisible by n. Obviously, they form a clone for which we shall write $Cl(n)$. In particular, the clone of the full idempotent reduct of $\langle Z; +, -, 0\rangle$ coincides with $Cl(1)$, while $Cl(0)$ is the clone consisting of all the projections only. Note that $Cl(n)$ consists exactly of those polynomials $c_1x_1+\ldots+c_mx_m|_Z$ for which $c_1x_1+\ldots$ $\ldots+c_mx_m|_{Z_n}$ is a projection.

Theorem 1. *For any clone C with $Cl(1)\supset C\supset Cl(0)$ there exist uniquely determined pairwise relatively prime numbers $p_1, \ldots, p_k>1$ such that*

$$C = \bigcap\big(Cl(p_i)|1 \leqq i \leqq k\big). \tag{1}$$

We prepare the proof of the theorem by stating several lemmas. For simplicity subscript Z in polynomials will be omitted.

Lemma 1. *If $(Cl(1)\supseteqq)C\ni x+(-n)y+nz$ ($n\in N_0$) then C together with any polynomial $c_1x_1+\ldots+c_mx_m$ contains each polynomial $(c_1+t_1n)x_1+\ldots+(c_m+t_mn)x_m$ with $t_1, \ldots, t_m\in Z$ and $t_1+\ldots+t_m=0$. In particular, $Cl(n)$ is generated by the polynomial $x+(-n)y+nz$ and, consequently,*

$$[Cl(m)\cup Cl(n)] = Cl\big((m, n)\big), \quad m, n\in N_0. \tag{2}$$

Proof. First we prove our claim for $C=[\{x+(-n)y+nz\}]$, i.e. we prove $[\{x+(-n)y+nz\}]=Cl(n)$. Inclusion $\subseteqq$ is obvious. Inclusion in the opposite direction follows in two steps. By induction on t we get

$$x+(-tn)y+tnz\in C,$$

then by induction on r we can prove that for any $d_1x_1+\ldots+d_rx_r\in Cl(n)$ and $i\neq j$

$$d_1x_1+\ldots+d_rx_r=\left(\sum_{\substack{k=1\\k\neq i}}^{r} d_kx_k+d_ix_j\right)+(-d_i)x_j+d_ix_i\in C,$$

as required.

Let C stand now for an arbitrary clone containing the polynomial $x+(-n)y+nz$. Obviously $C\supseteq Cl(n)$; hence if $c_1x_1+\ldots+c_mx_m\in C$ and $t_1,\ldots,t_m\in Z$ with $t_1+\ldots+t_m=0$ then

$$(c_1+t_1n)x_1+\ldots+(c_m+t_mn)x_m=$$

$$=(c_1x_1+\ldots+c_mx_m)+t_1nx_1+\ldots+t_mnx_m\in C$$

which was to be proved.

As for (2) we note that

$$x+(-(m,n))y+(m,n)z=(x+(-um)y+umz)+(-vn)y+vnz$$

where $u,v\in Z$ and $um+vn=(m,n)$. This implies inclusion $\supseteq$ in (2). Inclusion $\subseteq$ is obvious, thus the proof of the lemma is complete.

Lemma 2. *If* $(Cl(1)\supseteq)C\supset Cl(p)$, *where* p *is a prime, then* $C=Cl(1)$.

Proof. If C is properly contained in $Cl(1)$ then the polynomials $c_1x_1+\ldots+c_mx_m|_{Z_p}$ where $c_1x_1+\ldots+c_mx_m\in C$ constitute a proper subclone in the clone of the full idempotent reduct of $\langle Z_p;+,-,0\rangle$. This contradicts the theorem of Płonka quoted in the introduction.

Lemma 3. *Let* $n\in N$, $n\geqq 2$ *and let* $p_1,\ldots,p_n>1$ *be pairwise relatively prime numbers. If*

$$(Cl(1)\supseteq)C\supset Cl(p_1p_n)\cap\left(\bigcap(Cl(p_j)|2\leqq j\leqq n-1)\right)$$

and C contains a polynomial

$$d_1x_1+\ldots+d_mx_m\in\bigcap\left(Cl(p_j)|\,1\leqq j\leqq n\right)$$

such that there exist two coefficients in $\langle d_1,\ldots,d_m\rangle$ *not divisible by* p_1p_n *(for brevity we will say that this polynomial separates* p_1 *and* p_n*) then*

$$C\supseteq\bigcap\left(Cl(p_j)|\,1\leqq j\leqq n\right).$$

Proof. Set $p=p_1\ldots p_n$. By Lemma 1, we can assume $p\nmid d_i$, $i=1,\ldots,m$. Moreover, we can suppose $d_i=e_iq_i$, where

$$q_i=\frac{p}{p_{j_{i-1}+1}\cdots p_{j_i}}\quad(i=1,\ldots,m),$$

and $0=j_0<j_1<\ldots<j_m=n$. First we show that

$$\frac{p}{p_1}u_1x_1+\ldots+\frac{p}{p_n}u_nx_n\in C \tag{3}$$

whenever $\sum_{i=1}^{n}\frac{p}{p_i}u_i=1$. This is obvious for $n=2$. Suppose $n\geqq 3$. Let

$$f_iq_i=\sum_{j=j_{i-1}+1}^{j_i}\frac{p}{p_j}u_j \quad (i=1,\ldots,m).$$

We have $\sum_{i=1}^{m}e_iq_i=\sum_{i=1}^{m}f_iq_i=1$ and $(q_1,\ldots,q_{i-1},q_{i+1},\ldots,q_m)=\frac{p}{q_i}$, thus $p|(e_i-f_i)q_i$, $i=1,\ldots,m$. As $Cl(p)\subseteqq C$ we can apply Lemma 1 to have

$$f_1q_1x_1+\ldots+f_mq_mx_m\in C.$$

Choose integers v_1, v_2 such that $v_1p_1p_n+v_2p_2\ldots p_{n-1}=1$. Clearly

$$v_1p_1p_nx+v_2p_2\ldots p_{n-1}y\in C,$$

$$\left(\frac{p}{p_1}u_1+\frac{p}{p_n}u_n\right)x_1+\frac{p}{p_2}u_2x_2+\ldots+\frac{p}{p_{n-1}}u_{n-1}x_{n-1}\in C,$$

thus

$$\begin{aligned}
&v_2p_2\ldots p_{n-1}\big(f_1q_1x_1+(1-f_1q_1)x_n\big)+\\
&+v_1p_1p_n\left(\left(\frac{p}{p_1}u_1+\frac{p}{p_n}u_n\right)x_1+\frac{p}{p_2}u_2x_2+\ldots+\frac{p}{p_{n-1}}u_{n-1}x_{n-1}\right)=\\
&=\left(v_2p_2\ldots p_{n-1}\left(\frac{p}{p_1}u_1+\ldots+\frac{p}{p_{j_1}}u_{j_1}\right)+v_1p_1p_n\left(\frac{p}{p_1}u_1+\frac{p}{p_n}u_n\right)\right)x_1+\\
&+(1-v_2p_2\ldots p_{n-1})\frac{p}{p_2}u_2x_2+\ldots+(1-v_2p_2\ldots p_{n-1})\frac{p}{p_{n-1}}u_{n-1}x_{n-1}+\\
&+\left(v_2p_2\ldots p_{n-1}\left(\frac{p}{p_{j_1+1}}u_{j_1+1}+\ldots+\frac{p}{p_{n-1}}u_{n-1}\right)+(1-v_1p_1p_n)\frac{p}{p_n}u_n\right)x_n=\\
&=\left(\frac{p}{p_1}u_1+t_1p\right)x_1+\ldots+\left(\frac{p}{p_n}u_n+t_np\right)x_n\in C
\end{aligned}$$

where $t_1,\ldots,t_n$ are integers with $t_1+\ldots+t_n=0$. This implies (3) by Lemma 1.

Finally we drop the assumption $n\geqq 3$ and prove that

$$\left[Cl(p)\cup\left\{\frac{p}{p_1}u_1x_1+\ldots+\frac{p}{p_n}u_nx_n\right\}\right]=\bigcap\big(Cl(p_i)|1\leqq i\leqq n\big). \tag{4}$$

Let we denote the clone on the left by D. Suppose

$$d_1'x_1+\ldots+d_m'x_m\in\bigcap\big(Cl(p_i)|1\leqq i\leqq n\big).$$

Using the above notations we can suppose $d_i'=e_iq_i$ $(i=1, \dots, m)$, because (4) is symmetric in $p_1, \dots, p_n$. Applying $p|(e_i-f_i)q_i$ and

$$f_1q_1x_1+\dots+f_mq_mx_m\in D$$

we have

$$d_1'x_1+\dots+d_m'x_m\in D$$

proving that $D\supseteq\bigcap(Cl(p_i)|1\leqq i\leqq n)$. Inclusion $\subseteq$ is obvious. The proof of the lemma is complete.

Lemma 4. *Let $m\in N$ and let $q_1, \dots, q_m, q>0$ be pairwise relatively prime numbers. If*

(5) $$(Cl(1)\supseteq)C\supset Cl(q_1^{k_1}\dots q_m^{k_m}q)$$

and C contains the polynomial

$$vq_1^{j_1}\dots q_m^{j_m}qx+(1-vq_1^{j_1}\dots q_m^{j_m}q)y,$$

where $v\in Z$, $(v, q_1\dots q_m)=1$ and $1\leqq j_i<k_i$ $(i=1, \dots m)$, then

$$C\supseteq Cl(q_1^{j_1}\dots q_m^{j_m}q).$$

Proof. Let us introduce the notations

$$p=q_1^{j_1}\dots q_m^{j_m}q,\quad p'=q_1^{k_1}\dots q_m^{k_m}q,\quad t=\frac{p'}{p}.$$

First suppose $p'|p^2$. By induction on r we show that

$$P_r(x_0, \dots, x_r)=(1-rvp)x_0+vpx_1+\dots+vpx_r\in C.$$

$P_1(x_0, x_1)$ is the polynomial given above, and for $r\geqq 2$ we have

$$P_r(x_0, \dots, x_r)=((1-vp)P_{r-1}(x_0, \dots, x_{r-1})+vpx_r)+$$
$$+(1-r)v^2p^2x_0+v^2p^2x_1+\dots+v^2p^2x_{r-1},$$

where $p'|p^2$. Thus

$$x_0+vpx_1+(-vp)x_2=P_t(x_0, x_1, x_2, \dots, x_2)+vp'x_0+(-vp')x_2\in C.$$

Applying Lemma 1 and $(vp, p')=p$, we have $C\supseteq Cl(p)$, as was to be proved.

By the assumption of the lemma there exists a natural number k such that $p'|p^{2^k}$. We can choose k to be minimal with this property. We prove the lemma by induction on k. For $k=1$, the statement was proved in the preceding paragraph. Suppose $k\geqq 2$ and the lemma is true for $k-1$. Obviously

$$v^2p^2x+(1-v^2p^2)y=vp(vpx+(1-vp)y)+(1-vp)y\in C,$$

$(v^2, q_1\dots q_m)=1$ and $p'|(p^2)^{2^{k-1}}$, which implies

(5') $$C\supseteq Cl(q_1^{2j_1}\dots q_m^{2j_m}q).$$

We can now apply the lemma (case $k=1$) for (5) substituted by (5'), hence we have

$$C \supseteqq Cl(q_1^{j_1} \dots q_m^{j_m} q),$$

completing the proof of the lemma.

Lemma 5. *Let $p>2$, $q\geqq 1$ be relatively prime numbers. If*

$$(Cl(1) \supseteqq) C \supset Cl(pq)$$

and C contains a polynomial

$$vqx+(1-vq)y$$

with $v\in Z$, $(v, p)=(1-vq, p)=1$, then $C\supseteqq Cl(q)$.

Proof. Let φ denote Euler's function. Using congruences

$$(1-vq)^{\varphi(p)} \equiv 1 \pmod p$$

implied by $(1-vq, p)=1$ and

$$(1-vq)^{\varphi(p)} \equiv 1 \pmod q,$$

we have

$$(1-vq)^{\varphi(p)} = 1+v'pq, \quad v'\in Z.$$

Clearly

$$(1-vq)^{\varphi(p)-1}\big((1-vq)x+vqy\big)+\big(1-(1-vq)^{\varphi(p)-1}\big)z =$$

$$= (1+v'pq)x+(1-vq)^{\varphi(p)-1}vqy+\big(1-(1-vq)^{\varphi(p)-1}\big)z\in C$$

therefore by Lemma 1 and $Cl(pq)\subseteqq C$ we have

$$x+uqy+(-uq)z\in C,$$

where $u=(1-vq)^{\varphi(p)-1}v$ and $(u, p)=1$. Applying again Lemma 1 we conclude $C\supseteqq Cl(q)$, which completes the proof of the lemma.

Lemma 6. *Let $p_1, p_2, p_3\geqq 1$ be pairwise relatively prime odd numbers. If*

$$(Cl(1) \supseteqq) C \supset Cl(2p_1p_2p_3)$$

and C contains a polynomial

$$v_1p_2p_3x_1+v_2p_1p_3x_2+v_3p_1p_2x_3,$$

where v_i, $i=1, 2, 3$ are odd integers, then $C\supseteqq Cl(p_1p_2p_3)$.

Proof. We have

$$v_3p_1p_2(v_1p_2p_3x_1+v_2p_1p_3x_2+v_3p_1p_2x_3)+(1-v_3p_1p_2)x_3 =$$

$$= (v_1v_3p_2)p_1p_2p_3x_1+(v_2v_3p_1)p_1p_2p_3x_2+(1-2tp_1p_2p_3)x_3\in C$$

with $v_1v_3p_2$ and $v_2v_3p_1$ odd and $C\supseteqq Cl(2p_1p_2p_3)$, therefore by Lemma 1 we have

$$x_3+up_1p_2p_3x_1+(-up_1p_2p_3)x_2\in C,$$

where $u=v_1v_3p_2$. We can apply again Lemma 1 to complete the proof.

We remark that Lemma 2 for $p>2$ is the special case $q=1$ of Lemma 5 and Lemma 2 for $p=2$ is the special case $p_1=p_2=p_3=1$ of Lemma 6.

Lemma 7. *For any clone C with $Cl(1)\supseteqq C\supset Cl(0)$ there exists a natural number $n>0$ such that $C\supseteqq Cl(n)$.*

Proof. By assumption C does not coincide with the trivial clone containing projections only and thus contains a polynomial $(1-k)x+ky$ for an integer $k\geqq 2$. If $k=2$ then

$$C\supseteqq[\{(-1)x+2y\}]=[\{(-1)(2x+(-1)y)+2z\}]=Cl(2).$$

Suppose now $k\geqq 3$. By induction on r it follows that

$$P_r(x,y)=rk^{r-1}(1-k)x+\big(1-rk^{r-1}(1-k)\big)y\in C.$$

This is clear for $r=1$ and supposing to be true for r it is true also for $r+1$, because

$$P_{r+1}(x,y)=kP_r(x,y)+(1-k)\big(k^rx+(1-k^r)y\big)$$

and $k^rx+(1-k^r)y$ is obviously contained in C. Clearly $n=k^{k-2}(1-k)^2$ is even and

$$nx+(1-n)y=(1-k)^2\big(k^{k-2}x+(1-k^{k-2})y\big)+\big(1-(1-k)^2\big)y\in C,$$
$$(-n)x+(1+n)y=P_{k-1}(x,y)\in C.$$

To show the inclusion $C\supseteqq Cl(n)$ observe that

$$n\big(nx+(1-n)y\big)+(1-n)\big((1+n)x+(-n)z\big)=x+n(1-n)y+n(n-1)z\in C,$$

$$(-n)\big((-n)x+(1+n)z\big)+(1+n)\big((1-n)x+ny\big)=x+n(1+n)y+n(-n-1)z\in C,$$

and $\big(n(n-1),n(n+1)\big)=n$, which by Lemma 1 completes the proof of the lemma.

Proof of the theorem. By Lemma 7, there exists a natural number $n\geqq 1$ such that $C\supseteqq Cl(n)$. First we show the existence of $p_1,\dots,p_k$ in (1) under the assumption

$$C\subseteqq\bigcap\big(Cl(q_j^{t_j})\,|\,1\leqq j\leqq m\big)\tag{6}$$

where $q_1,\dots,q_m$ are distinct primes and the prime factorization of n is $n=q_1^{t_1}\dots q_m^{t_m}$. To show (1) it suffices to prove the following statement: if $p_1,\dots,p_k>1$ $(k\in N)$ are pairwise relatively prime numbers with $p_1\dots p_k=n$ and

$$C\supset\bigcap\big(Cl(p_j)\,|\,1\leqq j\leqq k\big),$$

then there exists an $i\in N$ with $1\leqq i\leqq k$ and integers $p_i', p_i''>1$ such that $(p_i', p_i'')=1$, $p_i'p_i''=p_i$ and

$$C\supseteqq\bigcap\big(Cl(p_j)\,|\,1\leqq j\leqq k, j\neq i\big)\cap Cl(p_i')\cap Cl(p_i'').$$

Suppose the conditions of this statement are satisfied by C and

$$d_1x_1+\ldots+d_rx_r\in C-\bigcap\big(Cl(p_j)\,|\,1\leqq j\leqq k\big)\quad (r\geqq 2).$$

This means that there exist two coefficients d_{i_1}, d_{i_2}, $1\leqq i_1<i_2\leqq r$ and an index i, $1\leqq i\leqq k$ such that $p_i\nmid d_{i_1}, d_{i_2}$. By symmetry we can assume $i_1=1$, $i_2=2$. Now (6) implies that for each $j\,(1\leqq j\leqq m)$ all the coefficients but one in $\langle d_1, \ldots, d_r\rangle$ are divisible by $q_j^{t_j}$. Consequently, (p_i, d_1) or (p_i, d_2) is greater than 1, say $(p_i, d_1)=p_i''>1$. Moreover, if we set $p_i'=\dfrac{p_i}{p_i''}$ and $d_1'=\dfrac{d_1}{p_i''}$, then we obtain $p_i'|d_j$ for $j=2, \ldots, r$, hence $p_i'|d_2+\ldots+d_r=1-d_1$ and obviously $p_i'>1$. Choose integers u, v such that $u\dfrac{n}{p_i}+vp_i=1$. Since

$$u\frac{n}{p_i}\big(d_1'p_i''x+(1-d_1)y\big)+vp_iy = ud_1'\frac{n}{p_i'}x+\left((1-d_1)u\frac{n}{p_i}+vp_i\right)y\in C$$

and this polynomial separates p_i' and p_i'', by Lemma 3 the proof of the statement is complete.

It has remained to prove that (6) holds if n is chosen to be minimal with the property $C\supseteqq Cl(n)$. Suppose, otherwise,

$$H = C-\bigcap\big(Cl(q_j^{t_j})\,|\,1\leqq j\leqq m\big)\neq\emptyset.$$

First we show that H contains a binary polynomial. Assume that either all primes q_i $(i=1, 2, \ldots, m)$ are odd or $4|q_1^{t_1}\ldots q_m^{t_m}$. Let $d_1x_1+\ldots+d_tx_t\in H$, $t\geqq 2$. Then there are at least two coefficients not divisible by one of the prime powers $q_i^{t_i}$, say by $q_1^{t_1}$. Suppose all the coefficients not divisible by $q_1^{t_1}$ are $d_1, \ldots, d_r$ $(r\geqq 2)$ and $d_1, \ldots, d_s$ $(s\geqq 1)$ are not even divisible by q_1. If $s=1$ then $q_1^{t_1}\nmid d_2$, $q_1\nmid 1-d_2$, hence $d_2x+(1-d_2)y\in H$. If $s>1$ and, say, $d_1\not\equiv 1 \pmod{q_1}$, then $q_1^{t_1}\nmid d_1$, $q_1\nmid 1-d_1$, thus $d_1x+(1-d_1)y\in H$, while if $d_i\equiv 1 \pmod{q_1}$ for all i, $1\leqq i\leqq s$. then $s\geqq q_1+1\geqq 3$, hence $d_1+d_2\equiv 2 \pmod{q_1}$ implies $(d_1+d_2)x+(1-d_1-d_2)y\in H$.

We reduce the remaining case $q_1^{t_1}=2$ to the one settled in the previous paragraph by proving that

$$(H\supseteqq)H' = C-\bigcap\big(Cl(q_j^{t_j})\,|\,2\leqq j\leqq m\big)\neq\emptyset.$$

Suppose that, in contrary to our claim, $q_1^{t_1}=2$ and $C\subseteqq\bigcap\big(Cl(q_j^{t_j})|2\leqq j\leqq m\big)$. Then H contains a ternary polynomial which can naturally be supposed to have form

$$u_1p_2p_3x+u_2p_1p_3y+u_3p_1p_2z$$

where u_i $(i=1, 2, 3)$ are odd integers and $p_i=q_{r_{i-1}+1}^{t_{r_{i-1}+1}} \dots q_{r_i}^{t_{r_i}}$ with $1=r_0\leqq r_1\leqq r_2\leqq r_3=$ $=m$. Applying Lemma 6 we have $C\supseteqq Cl(q_2^{t_2} \dots q_m^{t_m})$, contradicting the minimality of n.

Any binary polynomial in H can be written in the form

$$u_1 q_1^{s_1} \dots q_r^{s_r} x + u_2 q_{r+1}^{s_{r+1}} \dots q_k^{s_k} y \tag{7}$$

with $0\leqq r\leqq k\leqq m$, $(u_1, n)=(u_2, n)=1$ and $s_i\geqq 1$, $(i=1, 2, \dots, k)$. For brevity, we introduce the following notations:

$$p_1 = q_1^{t_1} \dots q_r^{t_r}, \quad p_2 = q_{r+1}^{t_{r+1}} \dots q_k^{t_k}, \quad p_3 = q_{k+1}^{t_{k+1}} \dots q_m^{t_m} = \frac{n}{p_1 p_2},$$

$$p_1' = q_1^{s_1} \dots q_r^{s_r}, \quad p_2' = q_{r+1}^{s_{r+1}} \dots q_k^{s_k}.$$

(a) If both $r=k=0$, i.e. $p_1'=p_2'=1$, then applying Lemma 5 we get $C=Cl(1)$ which contradicts the minimality assumption. Hence we can suppose $r\geqq 1$.

(b) If $k=m$, then there exists an index i with $s_i<t_i$, $1\leqq i\leqq m$. Set $v_j=\min(s_j, t_j)$ $(j=1, 2, \dots, m)$. Since

$$u_1 u_2 p_1' p_2' x + (1-u_1 u_2 p_1' p_2') y \in C,$$

by Lemma 4 we have $C\supseteqq Cl(q_1^{v_1} \dots q_m^{v_m})$, contradicting the minimality of n.

(c) Let $k<m$, i.e. $p_3>1$. Clearly,

$$p_2' p_3 \mid 1-(u_1 p_1')^{\varphi(p_3)}$$

for $u_1 p_1' + u_2 p_2' = 1$, thus (b) applies to polynomial

$$(u_1 p_1')^{\varphi(p_3)} x + \left(1-(u_1 p_1')^{\varphi(p_3)}\right) y \in C,$$

provided the product of the two coefficients are not divisible by n. In the opposite case by Lemma 3 we have

$$C \supseteqq Cl(p_1) \cap Cl(p_2 p_3). \tag{8}$$

If $s_i<t_i$ is satisfied for an i, $1\leqq i\leqq k$, say for $i=1$, then by (8) we can choose integers v_1, v_2 such that

$$v_1 p_2 p_3 (u_1 p_1' x + u_2 p_2' y) + v_2 p_1 y \in C,$$

hence again applies (b). (In case $i>r$ the role of p_1 and p_2 has to be interchanged in (8), too.)

Finally, if $p_i | p_i'$ $(i=1, 2)$ then we rewrite the polynomial (7) in the form $u_1' p_1 x+$ $+u_2' p_2 y$, where $(u_1', p_2 p_3)=1$, $(u_2', p_1 p_3)=1$. Applying again (8) we have

$$v_2 p_1 (u_1' p_1 x + u_2' p_2 y) + v_1 p_2 p_3 x \in C,$$

where $(v_2 p_1 u_1' p_1 + v_1 p_2 p_3, p_3)=(v_2, p_3)=1$, $(v_2 p_1 u_2' p_2, p_3)=1$, thus obviously $p_3\neq 2$. Hence by Lemma 5 we have $C\supseteqq Cl(p_1 p_2)$ which contradicts the minimality of n. The existence of $p_1, \dots, p_k$ in Theorem 1 is proved.

To prove uniqueness is suffices to show that for any two sequences of pairwise relatively prime numbers $p_1, \dots, p_k$ (k$\geqq$1) and $\bar{p}_1, \dots, \bar{p}_m$ ($m \geqq 1$) the inclusion

$$\bigcap (Cl(p_i) | 1 \leqq i \leqq k) \subseteqq \bigcap (Cl(\bar{p}_j) | 1 \leqq j \leqq m)$$

implies that each $\bar{p}_j$ divides one of the p_i-s.

Let we denote the clones above by C and $\bar{C}$, respectively. Set $p = p_1 \dots p_k$ and $\bar{p} = \bar{p}_1 \dots \bar{p}_m$. Choose integers u_i, $1 \leqq i \leqq k$ such that $\sum_{i=1}^{k} \frac{p}{p_i} u_i = 1$. Since

$$\frac{p}{p_1} u_1 x_1 + \dots + \frac{p}{p_k} u_k x_k \in C \subseteqq \bar{C}$$

all the coefficients but one of this polynomial are divisible by $\bar{p}_j$. Assume

(9) $$\bar{p}_j \left| \left(\frac{p}{p_2} u_2, \dots, \frac{p}{p_k} u_k \right) = u p_1. \right.$$

Applying the obvious inclusion $Cl(p) \subseteqq \bar{C}$ and the fact that $\bar{p}_1, \dots, \bar{p}_m$ are pairwise relatively prime we have

(10) $$\bar{p} | p.$$

Suppose $\bar{p}_j \nmid p_1$. Now (9) and (10) together with the equations $(p_1, p_i) = 1$, $(i = 2, \dots, k)$ imply u to have a prime factor v with $v \nmid p_1$ and $v | \bar{p}_j$ and hence with $v \left| \frac{p}{p_1} \right.$. Then

$$v \left| \frac{p}{p_1} u_1 + \left(\frac{p}{p_2} u_2 + \dots + \frac{p}{p_k} u_k \right) = 1. \right.$$

This contradiction implies $\bar{p}_j | p_1$, hence the proof of Theorem 1 is complete.

3. Applications. Next we state two propositions that reduce the problem of describing all idempotent reducts of an arbitrary abelian group to that of the infinite cyclic group $\langle Z; +, -, 0 \rangle$. Sometimes it will be convenient to use notation Z_0 instead of Z.

Proposition 1. *The clone of an abelian group $\langle G; +, -, 0 \rangle$ is isomorphic to that of $\langle Z_i; +, -, 0 \rangle$, where $i=0$ or $i=m (\geqq 1)$ according to whether $\langle G; +, -, 0 \rangle$ satisfies no nontrivial identity or is of exponent m. In both cases the following map is an isomorphism:*

$$c_1 x_1 + \dots + c_n x_n|_G \mapsto c_1 x_1 + \dots + c_n x_n|_{Z_i}$$

for every $\langle c_1, \dots, c_n \rangle \in Z_i^n$.

In particular, this map is an isomorphism between the clones of the full idempotent reducts of $\langle G; +, -, 0 \rangle$ and $\langle Z_i; +, -, 0 \rangle$.

Proposition 2. *For every $n \in N$ the lattice L_n of all subclones of the clone of the full idempotent reduct of $\langle Z_n; +, -, 0\rangle$ is isomorphic to the dual principal ideal $[Cl(n))_{L_0}$ of L_0 generated by $Cl(n)$, where L_0 is the lattice of all subclones of the clone of the full idempotent reduct of $\langle Z; +, -, 0\rangle$. The following map is an isomorphism:*

$$(Cl(n) \subseteqq)C \mapsto \{d_1x_1 + \ldots + d_mx_m|_{Z_n} \mid d_1x_1 + \ldots + d_mx_m|_Z \in C\}.$$

The proof of these propositions is straightforward and is therefore left to the reader.

Theorem 2. *Let $\langle G; I\rangle$ denote the full idempotent reduct of the abelian group $\langle G; +, -, 0\rangle$.*

(i) *If $\langle G; +, -, 0\rangle$ satisfies no nontrivial identity, then the lattice received from the lattice of all subclones of I by omitting the least element (i.e., the clone of projections) is dually isomorphic to the subdirect product of the partition lattice $E(N)$ of the set N and countably infinite samples of the chain $Q = \{0 \leqq 1 \leqq \ldots\}$ defined as follows:*

$$\mathscr{L}_0 = \{\langle \pi, s\rangle \mid \pi \in E(N),\ s \in Q^N, \textit{ all but a finite number of components of } s \textit{ equal } 0,$$
$$s(i) = 0 \textit{ implies that } i \textit{ constitutes in } \pi \textit{ a class in itself}\}$$

(ii) *If $\langle G; +, -, 0\rangle$ is of exponent $n(>1)$ with prime factorization $n = p_1^{t_1} \ldots p_m^{t_m}$, then the lattice of subclones of the clone I is dually isomorphic to the subdirect product of the partition lattice $E(m)$ of the set $\{1, 2, \ldots, m\}$ and the chains $Q_i = \{0 \leqq 1 \leqq \ldots \leqq t_i\}$, $i = 1, 2, \ldots, m$ defined as follows:*

$$\mathscr{L}_m^{(t_1, \ldots, t_m)} = \{\langle \pi, s_1, \ldots, s_m\rangle \mid \pi \in E(m),\ s_i \in Q_i,\ s_i = 0 \textit{ implies that } i \textit{ constitutes in } \pi \textit{ a class in itself}\}$$

Proof. First we prove (i). By Proposition 1 it suffices to prove it for the group $\langle Z; +, -, 0\rangle$. By Lemma 7 the subset obtained from L_0 by omitting its least element constitutes a sublattice of L_0 which will be denoted by $\tilde{L}_0$. Consider the following map:

$$\psi: \mathscr{L}_0 \to \tilde{L}_0$$

$$\langle \pi, s\rangle \mapsto \bigcap (Cl(\prod_{i \in c} p_i^{s(i)}) \mid c \in \mathfrak{C}(\pi)) \tag{11}$$

where $\mathfrak{C}(\pi)$ means the set of classes corresponding to π and $\{p_1, \ldots, p_k, \ldots\}$ is the set of all primes.

By the definition of $\mathscr{L}_0$ all but a finite number of terms in the meet in (11) equal $Cl(1)$, thus ψ is a map into $\tilde{L}_0$. Obviously, ψ is a monotone order reversing map. Theorem 1 implies ψ to be onto, moreover, in the proof of Theorem 1 we showed uniqueness just by proving that ψ is invertible and ψ^{-1} is also monotonic. Hence ψ is an isomorphism which was to be proved.

By Propositions 1 and 2 (ii) is an easy consequence of (i).

Theorem 3. *An abelian group satisfying no nontrivial identity has no minimal nontrivial idempotent reduct.*

Let $n>1$ be a natural number with prime factorization $n=p_1^{t_1}\dots p_m^{t_m}$. The clones of the minimal nontrivial idempotent reducts of any abelian group $\langle G;+,-,0\rangle$ of exponent n are the following ones:

(a) $[\{x+n_iy+(-n_i)z|_G\}]$ *for* $n_i=\frac{n}{p_i}$ $1\leqq i\leqq m$, *provided either* $p_i^2|n$ *or* $m=1$ *and* $n=p_1$;

(b) $[\{u_1q_1x+u_2q_2y|_G\}]$ *for all pairs of integers* $q_1,q_2>1$ *with* $(q_1,q_2)=1$, $q_1q_2=n$ *and* $u_1q_1+u_2q_2=1$ $(u_1,u_2\in Z)$.

Proof. Our first assertion is an obvious consequence of Theorem 2. To prove the second statement observe that by Theorem 2 and by Lemma 3 the clones covering $Cl(n)$ in L_0 (i.e. the clones $C\supset Cl(n)$ having the property that there does not exist any clone $C'\in L_0$ with $C\supset C'\supset Cl(n)$) are the following ones:

$$\text{(a)}' \qquad Cl(n_i)=[\{x+n_iy+(-n_i)z|_Z\}],$$

$$\text{(b)}' \qquad Cl(q_1)\cap Cl(q_2)=[Cl(n)\cup\{u_1q_1x+u_2q_2y|_Z\}].$$

We used the same notation as in (a) and (b). Now we can apply Propositions 1 and 2 to complete the proof of the theorem.

Remark. Since ψ given in the proof of Theorem 2 is an isomorphism, for any sequence $p_1,\dots,p_n$ of pairwise relatively prime numbers the interval $\left[Cl(\prod_{i=1}^{n}p_i),\ \bigcap(Cl(p_i)|1\leqq i\leqq n)\right]_{L_0}$ is dually isomorphic to the partition lattice $E(n)$. Hence we can apply the result of SACHS [4] to obtain that the lattice L_0 generates the variety of all lattices.

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Fonctions caractéristiques constantes

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1. Dans l'étude d'une contraction complètement non-unitaire (c.n.u.) $T\in\mathfrak{B}(\mathfrak{H})$ sur un espace de Hilbert séparable, il est important de connaître la fonction caractéristique, c'est-à-dire la fonction analytique contractive $\{\mathfrak{D}_T, \mathfrak{D}_{T^*}, \Theta_T(\lambda)\}$ où $\mathfrak{D}_T = (I-T^*T)^{1/2}$, $D_{T^*}=(I-TT^*)^{1/2}$ sont les opérateurs de défaut, $\mathfrak{D}_T=\overline{D_T\mathfrak{H}}$, $\mathfrak{D}_{T^*}=\overline{D_{T^*}\mathfrak{H}}$ sont les sous-espaces de défaut et $\Theta_T(\lambda)$ est donnée par

$$\Theta_T(\lambda) = [-T+\lambda\mathfrak{D}_{T^*}(I-\lambda T^*)^{-1}D_T]|\mathfrak{D}_T \quad (|\lambda| < 1).$$

Pour les concepts et notations employées dans cette Note cf. [3] ch. VI—VII.

Dans la présente Note on caractérise les contractions dont la fonction caractéristiqu est constante, en obtenant ainsi des indications supplémentaires concernant les factorisations «étranges».

2. Nous commençons par déterminer la fonction caractéristique $\Theta_T(\lambda)$ d'une contraction c.n.u. $T\in\mathfrak{B}(H)$ pour laquelle il existe un sous-espace $\mathfrak{H}_1\subset\mathfrak{H}$, invariant pour T, tel que les opérateurs T_1 et T_2 dans la triangulation correspondante $T= \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ sont une translation unilatérale et l'adjoint d'une telle translation.

Proposition *1. Pour que T admette une triangulation telle que T_1 est une translation unilatérale et T_2 est l'adjoint d'une translation unilatérale, il faut et il suffit que la fonction caractéristique $\Theta_T(\lambda)$ soit constante.*

La condition est nécessaire. En effet, soit $\Theta_T(\lambda)=\Theta_2(\lambda)\,\Theta_1(\lambda)$ la factorisation régulière correspondant au sous-espace invariant $\mathfrak{H}_1$; les parties pures des fonctions facteurs $\{\mathfrak{D}_T, \mathfrak{F}, \Theta_1(\lambda)\}$ et $\{\mathfrak{F}, \mathfrak{D}_{T^*}, \Theta_2(\lambda)\}$ coïncident alors avec les fonctions caractéristiques de T_1 et T_2 selon le cas. Comme T_1 est une translation unilatérale, sa fonction caractéristique est $\{\{0\}, \mathfrak{D}_{T^*}, 0\}$, et T_2 étant l'adjoint d'une translation unilatérale sa fonction caractéristique est $\{\mathfrak{D}_{T_2}, \{0\}, 0\}$. En tenant compte de [4], th. 2, on déduit que $\Theta_T(\lambda)$ est constante.

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La condition est suffisante. Pour démontrer cette affirmation notons que si une fonction analytique contractive $\{\mathfrak{E}, \mathfrak{E}_x, \Theta(\lambda)\}$ est constante, alors la fonction $\Delta(e^{it})=[I-\Theta^*(e^{it})\Theta(e^{it})]^{1/2}$ est aussi constante, d'où $\overline{\Delta L^2(\mathfrak{E})}=L^2(\overline{\Delta\mathfrak{E}})$. En désignant par P la projection orthogonale de $L^2(\overline{\Delta\mathfrak{E}})$ sur $H^2(\overline{\Delta\mathfrak{E}})$ notons que pour tout $v\in L^2(\overline{\Delta\mathfrak{E}})$ on a $P\Delta v=\Delta Pv$. Pour $u\oplus v\in\mathfrak{H}$ on a

$$(*) \qquad \Theta^*u+P\Delta v = 0.$$

En effet, vu que $u\oplus v\in\mathfrak{H}$, on a $\Theta^*u+\Delta v\perp H^2(\mathfrak{E})$ mais $\Theta^*u\in H^2(\mathfrak{E})$ parce que Θ est constante, d'où $(*)$. Nous décomposons les éléments $u\oplus v\in\mathfrak{H}$ sous la forme

$$u\oplus v = (u\oplus Pv)+(0\oplus(I-P)v).$$

Il est évident que $u\oplus Pv\in\mathfrak{H}$, $0\oplus(I-P)v\in\mathfrak{H}$ et aussi

$$\langle u\oplus Pv, 0\oplus(I-P)v\rangle = 0.$$

Donc l'espace $\mathfrak{H}$ se décompose en somme orthogonale $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2$ où

$$\mathfrak{H}_1 = \{u\oplus v;\, u\oplus v\in\mathfrak{K}_+,\, \Theta^*u+\Delta v = 0\},$$

$$\mathfrak{H}_2 = \{0\oplus v;\, 0\oplus v\in\mathfrak{H}\} = \{0\oplus v;\, v\in L^2(\overline{\Delta\mathfrak{E}}),\, v\perp H^2(\overline{\Delta\mathfrak{E}})\}.$$

Il est manifeste que l'espace $\mathfrak{H}_1$ est invariant à T et la restriction $T_1=T|\mathfrak{H}_1$ est une isométrie; de plus cette isométrie est une translation unilatérale parce que $\bigcap_{n=0}^{\infty} T_1^n\mathfrak{H}_1=\{0\}$.

Pour montrer que la compression T_2 de T à $\mathfrak{H}_2=\mathfrak{H}\ominus\mathfrak{H}_1$ est l'adjointe d'une translation unilatérale, il suffit de vérifier que $T^*|\mathfrak{H}_2$ est une telle translation, ce qui est évident en tenant compte des expressions de $\mathfrak{H}_2$ et de T^*.

Remarque. Si la fonction caractéristique d'une contraction T est constante, on a $\Theta_T(\lambda)=-T|\mathfrak{D}_T$.

3. Soit $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ une fonction analytique contractive. On dit que la factorisation $\Theta(\lambda)=\Theta_2(\lambda)\Theta_1(\lambda)$ est *étrange* si elle n'est pas régulière, mais il existe néanmoins un sous-espace fermé $\mathfrak{H}_1\subset\mathfrak{H}$ invariant pour T, tel que les parties pures de $\Theta_1(\lambda)$ et $\Theta_2(\lambda)$ coïncident avec les fonctions caractéristiques de la restriction T_1 de T à $\mathfrak{H}_1$ et de la compression T_2 de T à $\mathfrak{H}_2=\mathfrak{H}\ominus\mathfrak{H}_1$, selon le cas. Dans [1], C. Foiaş montre que la fonction analytique contractive pure $\{\mathfrak{E}, \mathfrak{E}, -{}^1/_2 I\}$, où $\mathfrak{E}$ est de dimension infinie, admet des factorisations étranges, an remarquant que l'opérateur T correspondant contient une translation unilatérale de multiplicité infinie et aussi l'adjoint d'un tel opérateur. Nous allons démontrer la suivante

Proposition 2. *Soit $T\in\mathfrak{B}(\mathfrak{H})$ une contraction c.n.u. telle que* $\dim\overline{T^*\mathfrak{D}_{T^*}}=\infty$. *La fonction analytique contractive pure $\{\mathfrak{D}_T, \mathfrak{D}_{T^*}, -T\}$ admet alors des factorisations étranges.*

Démonstration.[1]) Envisageons les fonctions analytiques contractives (cons-

[1]) Cette démonstration est inspirée de celle de [1], prop. 1.

tantes) $\{\mathfrak{D}_T, \mathfrak{D}_{T^*}\oplus\mathfrak{D}_T\oplus\mathfrak{D}_T, \Theta_1(\lambda)\}$ et

$$\{\mathfrak{D}_{T^*}\oplus\mathfrak{D}_T\oplus\mathfrak{D}_T, \mathfrak{D}_{T^*}, \Theta_2(\lambda)\}$$

où

$$\Theta_1 h = T^*h\oplus\frac{1}{\sqrt{2}}D_T h\oplus\frac{1}{\sqrt{2}}D_t h \quad \text{et} \quad \Theta_2(e\oplus f\oplus g) = -e.$$

Il est évident que $-T=\Theta_2\Theta_1$ donc il nous reste seulement à vérifier que cette factorisation est étrange. Pour cela notons que la partie pure de la fonction $\{\mathfrak{D}_T, \mathfrak{D}_{T^*}\oplus\oplus\mathfrak{D}_T\oplus\mathfrak{D}_T, \Theta_1\}$ est

$$\big\{\{0\}, \{e\oplus f\oplus g;\ e\in\mathfrak{D}_{T^*}, f, g\in\mathfrak{D}_T, \sqrt{2}\,T^*e+D_Tf+D_Tg = 0\}, 0\big\}$$

et, de même, la partie pure de $\{\mathfrak{D}_{T^*}\oplus\mathfrak{D}_T\oplus\mathfrak{D}_T, \mathfrak{D}_{T^*}, \Theta_2\}$ est $\big\{\{0\}, \{0\}\oplus\mathfrak{D}_T\oplus\mathfrak{D}_T, 0\big\}$. La fonction analytique contractive $\{\mathfrak{D}_T, \mathfrak{D}_{T^*}, -T\}$ étant constante, il existe d'après la proposition 1 un sous-espace $\mathfrak{H}_1$ tel que la restriction T_1 de T à $\mathfrak{H}_1$ soit une translation unilatérale et la compression T_2 de T à $\mathfrak{H}_2=\mathfrak{H}\ominus\mathfrak{H}_1$ soit l'adjoint d'une telle translation. Il est facile de vérifier que le sous-espace ambulant de T_1 est $\mathfrak{L}=\{e\oplus f\oplus\mathfrak{K}_+, e\in\mathfrak{H}_{T^*}, f\in\mathfrak{H}_T, T^*e=D_Tf\}$. En comparant le sous-espace $\{e\oplus f\oplus g;\ \sqrt{2}\,T^*e+D_Tf+ +D_Tg=0\}$ avec le sous-espace $\mathfrak{L}$ et en tenant compte de ce que $\dim\overline{T^*\mathfrak{D}_{T^*}}=\infty$ on trouve que la partie pure de la fonction $\{\mathfrak{D}_T, \mathfrak{D}_{T^*}\oplus\mathfrak{D}_T\oplus\mathfrak{D}_T, \Theta_1\}$ coïncide avec la fonction caractéristique de T_1. Vu que $\mathfrak{H}_T=\overline{T^*\mathfrak{D}_{T^*}}\oplus\ker T$ il résulte que $\dim\mathfrak{H}_T=\infty$ donc la partie pure de la fonction $\{\mathfrak{D}_{T^*}\oplus\mathfrak{D}_T\oplus\mathfrak{D}_T, \mathfrak{D}_{T^*}, \Theta_2\}$ coïncide avec la fonction caractéristique de T_2. Pour montrer que la factorisation en question est étrange il ne nous reste qu'à montrer que l'égalité

$$\overline{\Delta^2\Theta_1\mathfrak{D}_T}\oplus\overline{\Delta_1\mathfrak{D}_T} = \overline{\Delta_2(\mathfrak{D}_{T^*}\oplus\mathfrak{D}_T\oplus\mathfrak{D}_T)}\oplus\overline{\Delta_1\mathfrak{D}_T}$$

n'est pas vraie. Pour cela notons que

$$\overline{\Delta_2\Theta_1\mathfrak{D}_T}\oplus\overline{\Delta_1\mathfrak{D}_T} = \{0\oplus e\oplus e;\ e\in\mathfrak{D}_T\}\oplus\{0\}$$

et

$$\overline{\Delta_2(\mathfrak{D}_{T^*}\oplus\mathfrak{D}_T\oplus\mathfrak{D}_T)}\oplus\overline{\Delta_1\mathfrak{D}_T} = \big\{\{0\}\oplus\mathfrak{D}_T\oplus\mathfrak{D}_T\big\}\oplus\{0\},$$

d'où notre affirmation.

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A note on reductive operators

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A bounded linear operator T on a Hilbert space H is called a *reductive operator* if every invariant subspace of T reduces T. In this note, we shall study a local spectral theoretic condition which is satisfied by certain types of reductive operators. Consequently we shall obtain a set of conditions which are sufficient for a reductive operator to be equal to the sum of a normal operator and a commuting quasinilpotent operator. This will provide alternative proofs of some of the results of JAFARIAN [2], NORDGREN—RADJAVI—ROSENTHAL [3] and RADJABALIPOUR [4]. We shall be using the notation and terminology of DUNFORD and SCHWARTZ [1].

If an operator T has the single valued extension property, called property (A), then it satisfies Dunford's condition (B) if $\sigma(T, x)\cap\sigma(T, y)=\varphi$ implies that $\|x\|\leqq K\|x+y\|$, where K is independent of x and y. In [5], STAMPFLI introduced an orthogonality version of condition (B), that is, $\sigma(T, x)\cap\sigma(T, y)=\varphi$ implies that $(x, y)=0$, for all vectors x and y in the Hilbert space H. An operator T with property (A) satisfies Dunford's condition (C) if for each closed set δ, $H(\delta)=\{x\in H: \sigma(T, x)\subset\delta\}$ is closed subspace. A basic theorem of DUNFORD [1, page 2147] asserts that an operator, T on a Hilbert space H is spectral (i.e., $T=S+Q$ where S is similar to a normal operator, Q is quasinilpotent and $SQ=QS$) if and only if T satisfies conditions (A), (B), (C), and (D). It is easy to prove that if in this result the condition (B) is replaced by the orthogonality version of condition (B) then S will be a normal operator and conversely [5, Lemma 7].

Proposition 1. *If T is a reductive operator and if T satisfies* (A) *and* (C) *then T satisfies the orthogonality version of condition* (B).

Proof. For any closed set δ, since T is reductive, $TP_\delta=P_\delta T$ where P_δ denotes the projection of H onto $H(\delta)$. Thus $\sigma(T, P_\delta x)\subset\sigma(T, x)$ for all $x\in H$. If $\sigma(T, x)\cap\sigma(T, y)=\varphi$ then $\sigma(T, P_\delta y)\subset\sigma(T, y)\cap\delta$ where $\delta=\sigma(T, x)$. Thus $\sigma(T, P_\delta y)=0$, and hence $P_\delta y=0$.

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Corollary 1. *If T is a reductive spectral operator then $T=N+Q$ where N is normal, Q is quasinilpotent and $NQ=QN$.*

Corollary 2. [3] *If T is a reductive operator which is similar to a normal operator then T is normal.*

Corollary 3. *If T is reductive and if $\sigma(T)$ (the spectrum of T) is totally disconnected then $T=N+Q$.*

Proof. If $\sigma(T)$ is totally disconnected then T is spectral if and only if T satisfies the Dunford condition (B) [1, page 2149]. Since T is reductive, the result follows from the proposition.

If T is a reductive operator and if T satisfies conditions (A) and (C), then for each closed set δ, $\sigma(T, P_\delta x)\subset\sigma(T, x)$ for all $x\in H$, where P_δ is the projection of H onto $H(\delta)$. This is the condition (I), introduced by the author in [6], where in it was shown that a decomposable operator T which satisfies condition (I), (in particular if T is reductive decomposable operatoı), then T is the sum of a normal operator and a commuting quasinilpotent operator. The next theorem is a generalization of this result.

Theorem 1. *If T satisfies conditions* (A), (C), (I), *and if for each closed set δ, $H=H(\delta)+H(\bar{\delta}')$ [$\bar{\delta}'$ denotes the closure of the complement of δ] then $T=N+Q$ where N is normal, Q is quasinilpotent, and $NQ=QN$.*

Proof. By Dunford's theorem, we only need to show that T satisfies Dunford's condition (D). In order to show this, it is enough to show that for each closed set δ, $H=H(\delta)\oplus\overline{H(\delta')}$. Since T satisfies condition (I), it satisfies the orthogonality version of condition (B) and hence $H(\delta')\subset H^\perp(\delta)$ where $H^\perp(\delta)$ is the orthogonal complement of $H(\delta)$ in H. Now for any $x\in H^\perp(\delta)$, let $x=u+v$, where $u\in H(\delta)$ and $v\in H(\bar{\delta}')$. Thus $0=P_\delta x=u+P_\delta v$ and $\sigma(T, u)=\sigma(T, P_\delta v)\subset\delta\cap\sigma(T, v)$. Hence $x\in \in H(\bar{\delta}')$. Thus for any closed set δ, $\overline{H(\delta')}\subset H^\perp(\delta)\subset H(\bar{\delta}')$. Now for any open set G which contains δ, $H^\perp(\bar{G})\subset H(\delta')$ and hence $H^\perp(\delta')\subset\bigcap H(\bar{G})=H(\delta)$, where the intersection is taken over all open sets G which contain δ.

Corollary 4. *If T is a reductive operator and if T satisfies conditions* (A) *and* (C) *and if for each closed set δ, $H=H(\delta)+H(\bar{\delta}')$ then $T=N+Q$ where N is a normal operator, Q is quasinilpotent and $NQ=QN$.*

This result appears to be a generalization of Theorem 1.1 of JAFARIAN [2].

Let $g: s^1=\{z: |z|=1\}\to J$ be an arc length parametrization of a rectifiable Jordan curve. Since J is rectifiable, $g'(s)$ exists almost everywhere (with respect to Lebesgue measure) on the unit circle s^1. An operator T satisfies the growth condition (G_m) if

$\|(\lambda-T)^{-1}\| \leqq M[\text{dist}(\lambda, \sigma(T))]^{-m}$ for all $\lambda \in \varrho(T)$ and $|\lambda| \leqq \|T\|+1$, where $\varrho(T)$ denotes the resolvent set of T.

Theorem 2. *Let T be an operator such that $\sigma(T)$ is contained in a rectifiable Jordan curve J. If T satisfies the growth condition (G_m) and the condition* (I) *then $T=N+Q$ where N is normal operator, Q is quasinilpotent and $NQ=QN$.*

Proof. Since T satisfies the growth condition (G_m) and $\sigma(T)$ lies on J, it follows from [7, Theorem 11] that $H(\delta)$ is closed subspace of H for every closed set δ. Also from the proof of [7, Theorem 10], it follows that for any $w_1, w_2 \in J$, $w_i=g(s_i)$ are such that $g'(s_i)$ exists, $H=H[w_1, w_2]+H[w_2, w_1]$ where $[w_1, w_2]=\{g(s): s_1 \leqq s \leqq s_2\}$. By Theorem 1, we only need to show that for each closed set δ, $H=H(\delta)+H(\bar{\delta}')$, i.e., $H^{\perp}(\delta) \subset H(\bar{\delta}')$. Now suppose that there exist $x \in H^{\perp}(\delta)$ such that $\sigma(T, x) \cap \delta^0 \neq \varphi$ (δ^0 denotes the interior of δ). Then we can find $w_1, w_2 \in J$ such that $w_i=g(s_i)$ and $g'(s_i)$ exists, $\sigma(T, x) \cap [w_1, w_2] \neq \varphi$ and $[w_1, w_2] \subset \delta$. Let $x=x_1+x_2$ where $x_1 \in H[w_1, w_2]$ and $x_2 \in H[w_2, w_1]$, then $0=P_\delta x=x_1+P_\delta x_2$. Thus $\sigma(T, x_1)=\sigma(T, P_\delta x_2) \subset \delta \cap \sigma(T, x_2)$, i.e., $\sigma(T, x) \subset \delta$.

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Finite partitions of the real line consisting of similar sets

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In this note we shall prove and discuss a generalization of the theorem of WARREN PAGE ([3]) concerning partitions of the real line R and we shall study the Baire property of the sets in this partition.

It is not difficult to observe that if $\{A_1, \ldots, A_N\}$ is a partition of R (i.e. each A_i is nonempty, $\bigcup_{i=1}^{N} A_i = R$, and $A_i \cap A_j = \emptyset$ for $i \neq j$), then the set $G(A_1, \ldots, A_N)$ consisting of all numbers a such that $A_i + a = A_{k_i}$ for $i \in \{1, \ldots, N\}$ is an additive group. For $a \in G(A_1, \ldots, A_N)$ and $i \in \{1, \ldots, N\}$ let $f_i(a) = k_i$ if $A_i + a = A_{k_i}$. The following theorem holds:

Theorem. *If* $\{A_1, \ldots, A_N\}$, $N \geqq 2$ *is a partition of the real line such that* $G(A_1, \ldots, A_N)$ *fulfills the following conditions:*

(1) *for every* $i \in \{1, \ldots, N\}$, $f_i(G(A_1, \ldots, A_N)) = \{1, \ldots, N\}$;

(2) *for every* $i \in \{1, \ldots, N\}$, *every* $j \in f_i(G(A_1, \ldots, A_N))$, *and every* $\varepsilon > 0$ *there exists* $a \in G(A_1, \ldots, A_N)$ *such that* $|a| < \varepsilon$ *and* $A_i + a = A_j$,

then none of the sets A_i *is measurable or has the Baire property.*

Proof. Suppose that, for some $i_0 \in \{1, \ldots, N\}$, A_{i_0} is measurable. Then in virtue of (1) every A_i is measurable. Hence, similarly as in [3], from (2) we have $m(A_i \cap I) = m(A_j \cap I)$ for every i, j and for every interval I (m denotes Lebesgue measure). Then $m(A_i \cap I) = N^{-1} \cdot m(I)$ for every i and for every interval I: a contradiction with the Lebesgue density theorem. Hence each A_i is not measurable.

Suppose now that, for some $i \in \{1, \ldots, N\}$, A_{i_0} has the Baire property. Then in virtue of (1) every A_i has the Baire property. Obviously every A_i is of the second category. Let, for some i_0, $A_{i_0} = B \triangle C$, where B is open and nonempty, and C is of the first category. If (a, b) is a component of B, then $A_{i_0} \cap (a, b)$ is residual in (a, b). From (1) and (2) it follows that, for every i, $A_i \cap (a, b)$ is residual in (a, b): a contradiction. Hence none of the A_i has the Baire property. The Theorem is proved.

It is not difficult to show that for every natural $N \geqq 2$ there exists a partition $\{A_1, \ldots, A_N\}$ of R such that $G(A_1, \ldots, A_N)$ fulfills (1) and (2). We shall construct the example in a similar way as in [1]. $G_N = \{m \cdot (N+1)^{-k}: m, k$ — integers, $k \geqq 0\}$ is a group and $H_N = \{N \cdot m \cdot (N+1)^{-k}: m, k$ — integers, $k \geqq 0\}$ is a subgroup of G_N with

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index N. Let $\{C_1, \ldots, C_N\}$ be the family of all cosets of H_N in G_N. Let E be a set including exactly one number of each coset; then we set $A_i = \bigcup_{x \in E} (x + C_i)$ for $i \in \{1, \ldots, N\}$. $\{A_1, \ldots, A_N\}$ is a partition of R and $G(A_1, \ldots, A_N) = G_N$. Conditions (1) and (2) are obviously fulfilled.

If for some partition $\{A_1, \ldots, A_N\}$ the group $G(A_1, \ldots, A_N)$ fulfils only (2), then some of the sets $A_1, \ldots, A_N$, or even all of them, may be measurable and may have the Baire property. For example, if G_{N-1} and H_{N-1} are groups as above (for $N \geqq 3$), let $\{A_1, \ldots, A_{N-1}\}$ be the family of all cosets of H_{N-1} in G_{N-1} and let $A_N = R - G_{N-1}$. $\{A_1, \ldots, A_N\}$ is a partition of R, $G(A_1, \ldots, A_N) = G_{N-1}$, and all sets $A_1, \ldots, A_N$ are measurable (all but the last have measure 0) and all sets have the Baire property (all but the last are of the first category). However if we replace (1) by the following condition:

(1′) for every $i \in \{1, \ldots, N\}$, $f_i(G(A_1, \ldots, A_N))$ consists of at least two numbers,

then from (1′) and (2) it follows that in the family $\{A_1, \ldots, A_N\}$ there are at least two nonmeasurable sets and at least two sets which do not have the Baire property. The proof is similar to that of the theorem. In this case the partition may include simultaneously measurable (Baire) sets and nonmeasurable (not Baire) sets and the subfamilies of measurable sets and sets having the Baire property may be equal or not, as the following examples show: Let $\{A_1, A_2\}$ be a partition of the type constructed immediately after the proof of the theorem. Put $A_1' = H_2$, $A_2' = G_2 - H_2$, $A_3' = A_1 - G_2$, $A_4' = A_2 - G_2$. Then $\{A_1', A_2', A_3', A_4'\}$ is a partition of R fulfilling (1′) and (2), A_1' and A_2' are null sets of the first category, and A_3' and A_4' are not measurable sets which do not have the Baire property. Finally, let G_2, and H_2 be groups as above and let E be the set constructed after the proof of the theorem. If $R = A \cup B$, where A is a null set, B is of the first category, and $A \cap B = \emptyset$ (see for example [2]), set $E_1 = E \cap A$, $E_2 = E \cap B$. It is not difficult to see that E_1 and E_2 are nonempty. Let $A_1 = \bigcup_{x \in E_1} (x + H_2)$, $A_2 = \bigcup_{x \in E_1} (x + (G_2 - H_2))$, $A_3 = \bigcup_{x \in E_2} (x + H_2)$, $A_4 = \bigcup_{x \in E_2} (x + (G_2 - H_2))$. Then $\{A_1, A_2, A_3, A_4\}$ is a partition of R fulfilling (1′) and (2). A_1, A_2 do not have the Baire property, and A_3, A_4 are not measurable.

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Commutants of $C_0(N)$ contractions

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1. Introduction. Let $\mathfrak{H}$ be a complex separable Hilbert space and T a bounded linear operator on $\mathfrak{H}$. Let Lat T denote the lattice of all closed subspaces invariant under T. Let $\mathcal{A}_T$, $\{T\}''$, and $\{T\}'$ denote the smallest weakly closed subalgebra of $\mathcal{B}(\mathfrak{H})$ containing T and I, the double commutant of T, and the commutant of T, respectively. P. ROSENTHAL and D. SARASON, independently, asked the question: If $A\in\{T\}'$ and Lat $T\subset$ Lat A, is A in $\mathcal{A}_T$? An affirmative answer to this would imply affirmative answers to other unsolved problems (cf. [3]). BRICKMAN and FILLMORE [1] showed that this is true if T is an operator on a finite dimensional Hilbert space. Imitating their proof, it is not difficult to show that this also holds for algebraic operators. Recently, A. FEINTUCH [4] proved that if T is a compact operator with infinite spectrum then we also have the conclusion. In this paper we add one more class of operators to this list. We show that this holds for $C_0(N)$ contractions. We also show that such contractions are in class (dc) as defined in [14], that is, they satisfy $\mathcal{A}_T=\{T\}''$. Our proofs are largely dependent on the remarkable work of B. SZ.-NAGY and C. FOIAŞ on the structure of $C_0(N)$ contractions, namely, the functional models and Jordan models for such operators. A very brief description of these models will be given in § 2. The main reference for this part will be [13] and [11]. From time to time definitions and results will be taken from there without specification. § 3 contains the proofs of our main theorems.

An operator T is *reflexive* if Lat $T\subset$ Lat A implies $A\in\mathcal{A}_T$. The questions concerning reflexive operators asked by J. DEDDENS in [3] can now be answered for $C_0(N)$ contractions. These are contained in § 4, along with some characterizations for multiplicity-free contractions (cf. [10]). This provides more evidence of the analogy between $C_0(N)$ contractions and operators on finite dimensional spaces. We also give sufficient conditions for such contractions to be reflexive.

Finally, we conclude in § 5 with some remarks and open questions related to the previously given results.

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In the following $\mathbf{C}$ will denote the complex plane and $\mathbf{D}$ the open unit disk in $\mathbf{C}$.

2. Preliminaries. Let T be a contraction on the Hilbert space $\mathfrak{H}$. T is *of class* $C_0(N)$, $N\geqq 1$, if $T^n\to 0$ and $T^{*n}\to 0$ in the strong operator topology as $n\to\infty$, and the defect indices

$$d_T \equiv \operatorname{Rank}(I-T^*T)^{1/2} \quad \text{and} \quad d_{T^*} \equiv \operatorname{Rank}(I-TT^*)^{1/2}$$

are both equal to N. Let $\Theta_T(\lambda)$ denote the characteristic function of T. Note that if T is of class $C_0(N)$, $\Theta_T(\lambda)$ is an inner function ("inner from both sides" in the terminology of [13]), that is, $\Theta_T(e^{it})$ is a unitary operator on $\mathbf{C}^N$ for almost all t. With respect to a fixed orthonormal basis of $\mathbf{C}^N$, $\Theta_T(\lambda)$ can be represented as an N by N matrix over H^∞ (the space of complex bounded analytic functions defined on $\mathbf{D}$). Let H_N^2 denote the space of analytic functions from $\mathbf{D}$ to $\mathbf{C}^N$ which are square-integrable.

Now we assume T is a $C_0(N)$ contraction. Then T is unitarily equivalent to the compression of the shift on the space $H_N^2\ominus\Theta_T H_N^2$, that is, the operator $\mathbf{T}$ defined by

$$(\mathbf{T}^*f)(\lambda) = \frac{f(\lambda)-f(0)}{\lambda} \quad \text{for} \quad \lambda\in\mathbf{D} \quad \text{and} \quad f\in H_N^2\ominus\Theta_T H_N^2.$$

This will be called the *functional model* for T. From now on we will always consider the $C_0(N)$ contraction T as in its functional model. Moreover, to each factorization $\Theta_T(\lambda)=\Theta_2(\lambda)\Theta_1(\lambda)$ of $\Theta_T(\lambda)$ as a product of two inner functions there corresponds a subspace

$$\Theta_2 H_N^2\ominus\Theta_T H_N^2$$

invariant under T and all the invariant subspaces for T can be obtained in this way.

A contraction T is *of class* C_0 if T is completely non-unitary (c.n.u.) and there exists a function $u\neq 0$ in H^∞ such that $u(T)=0$; in this case u can be taken to be an inner function which is minimal in the sense that it will be a divisor of any inner function v for which $v(T)=0$. Such an inner function is unique up to a constant factor of modulus one; it will be called the *minimal function* of T and denoted by m_T. Note that a $C_0(N)$ contraction is of class C_0 and $\det\Theta_T$, the determinant of its characteristic function $\Theta_T(\lambda)$, is also an inner function; moreover m_T divides $\det\Theta_T$, and $\det\Theta_T$ divides m_T^N.

For a c.n.u. contraction T on $\mathfrak{H}$, a functional calculus can be defined for some functions. Indeed, let N_T denote the class of functions which are of the form $\varphi=v^{-1}u$ where u, $v\in H^\infty$ and $v(T)$ is an injective operator with dense range in $\mathfrak{H}$ (called a *quasi-affinity*); for such a function φ define

$$\varphi(T) = v(T)^{-1}u(T).$$

This definition does not depend on the particular choice of the representation $\varphi = u/v$ and, in general, $\varphi(T)$ may not be a bounded operator. If $\varphi(T)$ is a bounded operator, then $\varphi(T)$ is in the double commutant $\{T\}''$ of T. For $C_0(N)$ contractions, we have the converse:

Theorem 2.1. (see [11]) *If T is a $C_0(N)$ contraction for some $N \geqq 1$, then* $\{T\}'' \subset \{\varphi(T): \varphi \in N_T\}$.

Two operators T_1 and T_2 are *quasi-similar* if there exist quasi-affinities X and Y such that

$$T_1 X = X T_2 \quad \text{and} \quad T_2 Y = Y T_1.$$

A C_0 contraction T on H is called *multiplicity-free* if one of the following equivalent conditions holds (cf. [10] and [12]):

(i) T has a cyclic vector, i.e. a vector x_0 such that $\mathfrak{H}$ is spanned by $T^n x_0$ $(n = 0, 1, 2, \ldots)$;

(ii) T is quasi-similar to the operator $S(m_T)$ defined on $\mathfrak{H}(m_T) \equiv H^2 \ominus m_T H^2$ by

$$(S(m_T)^* f)(\lambda) = \frac{f(\lambda) - f(0)}{\lambda} \quad \text{for} \quad \lambda \in \mathbf{D} \quad \text{and} \quad f \in \mathfrak{H}(m_T).$$

Every $C_0(N)$ contraction T is quasi-similar to a uniquely determined operator of the form

$$S(m_1) \oplus S(m_2) \oplus \cdots \oplus S(m_k), \tag{2}$$

where $m_1, m_2, \ldots, m_k$ are nonconstant inner functions each of which is a divisor of its predecessor. This operator (2) is called the *Jordan model* of T. In the proof of our main theorem we will need another version of the Jordan model, which we state as

Theorem 2.2. (see [11]) *Let T be a contraction of class $C_0(N)$, $N \geqq 1$ on the space $\mathfrak{H}$. Then there exist invariant subspaces $\mathfrak{H}_1, \mathfrak{H}_2, \ldots, \mathfrak{H}_k$ for T such that $\mathfrak{H} = \bigvee_i \mathfrak{H}_i$,*

$$\Big(\bigvee_{i \in I} \mathfrak{H}_i\Big) \cap \Big(\bigvee_{j \in J} \mathfrak{H}_j\Big) = \{0\}$$

for any non-empty disjoint decomposition $\{I, J\}$ of the set $\{1, 2, \ldots, k\}$, and $T_i \equiv T | \mathfrak{H}_i$ is multiplicity-free. Moreover, if m_i is the minimal function of T_i, then m_i is a divisor of m_{i-1} for all i and m_1 coincides with the minimal function of T.

Another result needed in the sequel is the following.

Theorem 2.3. *Let T be a contraction of class $C_0(N)$ with the minimal function m_T. Let $u = u_i u_e$ be the canonical factorization of a function $u \in H^\infty$ as the product of its outer factor u_e and inner factor u_i. Then $u(T)$ is a quasi-affinity if and only if u_i and m_T have no non-trivial common inner factors.*

The proof of this theorem is essentially contained in [13] Prop. III. 4.7 (b) with minor changes; also compare [6] Theorem 2.5. We leave the details to the readers.

3. Main Theorems. A subspace $\mathfrak{K}$ is *bi-invariant* for T if $\mathfrak{K}$ is invariant under every operator in $\{T\}''$.

Theorem 3.1. *Let T be a contraction of class $C_0(N)$, $N\geqq 1$. Then every invariant subspace for T is bi-invariant.*

Proof. Let Θ_T be the characteristic function of T and consider T in its functional model as the compression of the shift on the space $\mathfrak{H}\equiv H_N^2\ominus\Theta_T H^2$. Let $\mathfrak{K}=\Theta_2 H^2\ominus\ominus\Theta_T H_N^2$ be an arbitrary invariant subspace for T, with the corresponding factorization

$$\Theta_T(\lambda) = \Theta_2(\lambda)\Theta_1(\lambda).$$

Let A be an operator in $\{T\}''$. Then $A=\varphi(T)=v(T)^{-1}u(T)$ for some $\varphi\in N_T$ (by Theorem 2.1).

Let $f=\Theta_2 g$ be a vector in $\mathfrak{K}$ and set $h=Af=v(T)^{-1}u(T)f$. (As all these vectors are contained in the space H_N^2 they can be considered as column N-vectors.)

We want to show that $h\in\mathfrak{K}$. Since $v(T)h=u(T)f=u(T)(\Theta_2 g)$, we have $P_H(vh)==P_H(u\Theta_2 g)$. If follows that $vh-u\Theta_2 g\in\Theta_T H_N^2$, and hence,

$$vh = \Theta_2 w \quad \text{for some} \quad w\in H_N^2. \tag{3}$$

Carrying out the matrix multiplication and using Cramer's rule we have

$$w_j \det\Theta_2 = v\det\Phi_j, \tag{4}$$

where Φ_j is the N by N matrix obtained from Θ_2 by replacing the j-th column by the column vector h. Note that $v(T)$ is a quasi-affinity. By Theorem 2.3, v_i and m_T have no non-trivial common inner factor. As $m_T|\det\Theta_T|m_T^N$, v_i and $\det\Theta_T$, and consequently v_i and $\det\Theta_2$, have no non-trivial common inner factor, either. From (4), we conclude that v is a divisor of w_j, for $j=1, 2, \ldots, N$. Say, $w_j=vx_j$. It is easily seen that $x_i\in H^2$ and equation (3) can be simplified to $h=\Theta_2 x$. Hence h is an element of $\Theta_2 H_N^2\ominus\Theta_T H_N^2=\mathfrak{K}$. This shows that $\mathfrak{K}$ is invariant under A, completing the proof.

Theorem 3.2. *Let T be a contraction of class $C_0(N)$, $N\geqq 1$. Then $\mathscr{A}_T=\{T\}''$.*

Proof. For any operator A, we denote the operator $\underbrace{A\oplus A\oplus\ldots\oplus A}_{n}$ by $A^{(n)}$. Let $A\in\{T\}''$. It is easily verified that $A^{(n)}\in\{T^{(n)}\}''$ for any $n=1, 2, \ldots$. Note that $T^{(n)}$ is a contraction of class $C_0(nN)$. If follows from Theorem 3.1 that any invariant subspace for $T^{(n)}$ is invariant under $A^{(n)}$, that is $\operatorname{Lat} T^{(n)}\subset\operatorname{Lat} A^{(n)}$, for any n. Hence A is in $\mathscr{A}_T$ ([8] Theorem 7.1). This shows that $\{T\}''\subset\mathscr{A}_T$. Since $\mathscr{A}_T\subset\{T\}''$ holds for any operator T, this completes the proof.

Now we are ready to prove our main theorem. The proof here is very similar to the one given by BRICKMAN and FILLMORE [1] for operators on finite dimensional spaces in that both use some kind of "Jordan model."

Theorem 3.3. *Let T be a contraction of class $C_0(N)$, $N\geqq 1$. If $A\in\{T\}'$ and* Lat $T\subset$ Lat A, *then $A\in\mathcal{A}_T$.*

Proof. Let $\mathfrak{H}_1, \mathfrak{H}_2, \dots, \mathfrak{H}_k$ be the invariant subspaces for T such that

$$(5)\qquad \text{(a)}\quad \mathfrak{H}=\bigvee_i \mathfrak{H}_i, \qquad \text{(b)}\quad \big(\bigvee_{i\in I}\mathfrak{H}_i\big)\cap\big(\bigvee_{j\in J}\mathfrak{H}_j\big)=\{0\},$$

for any decomposition $\{I, J\}$ of $\{1, 2, \dots, k\}$ and $T_i\equiv T|\mathfrak{H}_i$ is multiplicity-free with minimal function m_i satisfying $m_i|m_{i-1}$ for all i and $m_1=m_T$ (Theorem 2.2). Let $x_i\in H_i$ be a cyclic vector for T_i ($i=1, 2, \dots, k$). Consider the cyclic invariant subspace K generated by $x\equiv x_1+x_2+\dots+x_k$. We claim that the minimal function m_0 of $T_0\equiv T|\mathfrak{K}$ coincides with m_T. Indeed, since

$$m_0(T_0)x = m_0(T)x_1+\dots+m_0(T)x_k = 0,$$

by (5b) we have $m_0(T)x_1=0$. It follows that $m_0(T)\mathfrak{H}_1=0$. Hence $m_1=m_T$ is a divisor of m_0. On the other hand, since $m_T(T)\mathfrak{K}=0$, m_0 is a divisor of m_T. This shows that m_0 coincides with m_T, as asserted.

Since $\mathfrak{K}$ is invariant under T, it is also invariant under A. Let $A_0=A|\mathfrak{K}$. Since $A_0\in\{T_0\}'$ and T_0 is a multiplicity-free contraction, it is proved by SZ.-NAGY and FOIAŞ [10] that $A_0=\varphi(T_0)$ for some $\varphi\in N_{T_0}$. Say, $\varphi=\dfrac{u}{v}$, where $u, v\in H^\infty$ and $v(T_0)$ is a quasi-affinity. Hence $A_0=v(T_0)^{-1}u(T_0)$ on $\mathfrak{K}$. In particular, $v(T_0)A_0x=u(T_0)x$. Equivalently, we have

$$v(T)Ax_1+\dots+v(T)Ax_k = u(T)x_1+\dots+u(T)x_k.$$

By (5b), this implies that $v(T)Ax_i=u(T)x_i$ for all i. Hence we have $v(T)A=u(T)$ on $\mathfrak{H}_i$ for all i. It follows that $v(T)A=u(T)$ on $\mathfrak{H}$ (by (5a)). We want to show that $A\in\{T\}''$. For any $B\in\{T\}'$, we have

$$(6)\qquad v(T)AB = u(T)B = B\,u(T) = B\,v(T)A = v(T)BA.$$

Since $v(T_0)$ is a quasi-affinity on $\mathfrak{K}$, v_i and m_0 have no non-trivial common inner factor, where v_i denotes the inner factor of v (by Theorem 2.3). As shown before, m_0 coincides with m_T. Hence v_i and m_T have no non-trivial common inner factor. This implies that $v(T)$ is a quasi-affinity (by Theorem 2.3 again!). From (6), we conclude that $AB=BA$, that is $A\in\{T\}''$. On account of Theorem 3.2 the proof is done.

4. Miscellaneous results. Corollaries 4.1, 4.2 and 4.3 below answer DEDDENS' questions [3] positively for $C_0(N)$ contractions. The proofs are routine. We include them here for completeness.

Corollary 4.1. *If T is a $C_0(N)$ contraction contained in a commutative reflexive algebra $\mathscr{A}$, then T is reflexive.*

Note that a weakly closed algebra $\mathscr{A}$ is *reflexive* if $\mathscr{A}=\{A\colon \operatorname{Lat}\mathscr{A}\subset\operatorname{Lat}A\}$, where $\operatorname{Lat}\mathscr{A}$ denotes the lattice of subspaces invariant under every operator in $\mathscr{A}$.

Proof. Let S be an operator such that $\operatorname{Lat}T\subset\operatorname{Lat}S$. Since $T\in\mathscr{A}$, we have $\operatorname{Lat}\mathscr{A}\subset\operatorname{Lat}T$. Hence $\operatorname{Lat}\mathscr{A}\subset\operatorname{Lat}S$. The reflexivity of $\mathscr{A}$ implies that $S\in\mathscr{A}$. Hence $ST=TS$, that is, $S\in\{T\}'$. By Theorem 3.3, we conclude that $S\in\mathscr{A}_T$. This shows that T is reflexive.

Corollary 4.2. *Let T_1 and T_2 be $C_0(N)$ contractions. If T_1 and T_2 are reflexive then $T_1\oplus T_2$ is reflexive.*

Proof. Let S be an operator such that $\operatorname{Lat}(T_1\oplus T_2)\subset\operatorname{Lat}S$. It is easily seen that S must be of the form $S_1\oplus S_2$, where S_1 and S_2 are operators satisfying $\operatorname{Lat}T_1\subset\operatorname{Lat}S_1$ and $\operatorname{Lat}T_2\subset\operatorname{Lat}S_2$. The reflexivity of T_1 and T_2 implies that $S_1\in\mathscr{A}_{T_1}$ and $S_2\in\mathscr{A}_{T_2}$. We have $S_1\in\{T_1\}'$ and $S_2\in\{T_2\}'$. Hence $S=S_1\oplus S_2\in\{T_1\oplus T_2\}'$. By Theorem 3.3, we conclude that $S\in\mathscr{A}_{T_1\oplus T_2}$. Hence $T_1\oplus T_2$ is reflexive, as asserted.

Corollary 4.3. *If T is a $C_0(N)$ contraction, then $T^{(n)}$ is reflexive for any $n=2, 3, \ldots$.*

Proof. We first show that $T\oplus T$ is reflexive. Let S be an operator such that $\operatorname{Lat}(T\oplus T)\subset\operatorname{Lat}S$. It is easily seen that S must be of the form $S_1\oplus S_1$, where S_1 is an operator satisfying $\operatorname{Lat}T\subset\operatorname{Lat}S_1$. Note that for any two operators A, B, $AB=BA$ if and only if the graph of A is an invariant subspace for $B\oplus B$. Since $\operatorname{Lat}(T\oplus T)\subset\operatorname{Lat}(S_1\oplus S_1)$, we deduce that $S_1\in\{T\}''$. Hence $S_1\in\{T\}'$ and $S=S_1\oplus S_1\in\{T\oplus T\}'$. Using Theorem 3.3 we have $S\in\mathscr{A}_{T\oplus T}$, which shows that $T\oplus T$ is reflexive. Now we want to show that $T^{(n)}$ is reflexive for any $n\geqq 2$. Let V be an operator such that $\operatorname{Lat}T^{(n)}\subset\operatorname{Lat}V$. As before, we have $V=V_1^{(n)}$ for some operator V_1 satisfying $\operatorname{Lat}T\subset\operatorname{Lat}V_1$. From $\operatorname{Lat}T^{(n)}\subset\operatorname{Lat}V_1^{(n)}$ we deduce that $\operatorname{Lat}T^{(2)}\subset\operatorname{Lat}V_1^{(2)}$. By what we just proved, $V^{(2)}\in\mathscr{A}_{T^{(2)}}\subset\{T^{(2)}\}'$. Hence $V_1\in\{T\}'$ and $V_1^{(n)}\in\{T^{(n)}\}'$. It follows from Theorem 3.3 that $V=V_1^{(n)}\in\mathscr{A}_{T^{(n)}}$. Hence $T^{(n)}$ is reflexive, completing the proof.

It was proved by Sz.-Nagy and Foiaş that a $C_0(N)$ contraction T is multiplicity-free if and only if $\{T\}'$ is abelian, or equivalently, $\{T\}''=\{T\}'$. (This and other characterizations can be found in [10] and [12].) The next corollary gives some other equivalent conditions.

Corollary 4.4. *Let T be a contraction of class $C_0(N)$, $N\geqq 1$. Then the following are equivalent to each other:*

(i) *T is multiplicity-free;*

(ii) $\mathscr{A}_T=\{T\}'$;

(iii) $\{T\}'$ *is a singly generated algebra;*

(iv) $\{T\}'$ *is a maximal abelian algebra, that is,* $\{T\}'$ *is abelian and if* $\mathscr{A}$ *is a weakly closed abelian algebra containing* $\{T\}'$, *then* $\mathscr{A}=\{T\}'$;

(v) *Every invariant subspace for T is hyperinvariant, that is, invariant under every operator in* $\{T\}'$.

If this is the case, then $\mathscr{A}_T=\{T\}''=\{T\}'\subset\{\varphi(T)\colon \varphi\in N_T\}$.

Proof. That (i) implies (ii) follows from Theorem 3.2 and the remark given above; (v) implies (ii) follows from Theorem 3.3. Other implications are clear.

It seems to be unknown whether reflexive operators are preserved under quasi-similarities. (Note that they are preserved under similarities.) The next corollary makes a modest step in this direction.

Corollary 4.5. *Let T_1 and T_2 be $C_0(N)$ contractions which are multiplicity-free. Assume T_1 is quasi-similar to T_2. Then T_1 is reflexive if and only if T_2 is.*

Proof. By symmetry, we have only to show half of the assertion. Assume T_1 is reflexive. Let X and Y be quasi-affinities such that $T_1X=XT_2$ and $T_2Y=YT_1$. Let S be an operator with Lat $T_2\subset$ Lat S, and $\mathfrak{K}_1$ be an invariant subspace for T_1. Assume m is the minimal function of $T_1|\mathfrak{K}_1$. Let $\mathfrak{K}_2$ be the unique invariant subspace for T_2 for which $T_2|\mathfrak{K}_2$ has minimal function m (cf. [10]). Note that $\mathfrak{K}_1=\{x\colon m(T_1)x=0\}$ and $\mathfrak{K}_2=\{y\colon m(T_2)y=0\}$ ([10]). For any $x\in\mathfrak{K}_1$, we have $m(T_2)Yx=Ym(T_1)x=0$. This implies that $Yx\in\mathfrak{K}_2$. Since $\mathfrak{K}_2$ is invariant for S, we have $SYx\in\mathfrak{K}_2$. Hence $m(T_1)XSYx=Xm(T_2)SYx=0$. This shows that $XSYx\in\mathfrak{K}_1$, and hence $\mathfrak{K}_1$ is invariant under XSY. Since $\mathfrak{K}_1$ is arbitrary, we conclude that $XSY\in\mathscr{A}_{T_1}$ (by the reflexivity of T_1). In particular, XSY commutes with T_1. Since X, Y are quasi-affinities, it is easily seen that S must commute with T_2. Using Theorem 3.3, we have $S\in\mathscr{A}_{T_2}$. This shows that T_2 is reflexive, completing the proof.

As a special case, we have

Corollary 4.6. *Let φ_1, φ_2 be (scalar valued) inner functions with $(\varphi_1, \varphi_2)=1$, and $\varphi=\varphi_1\cdot\varphi_2$. Let $S(\varphi_1)$, $S(\varphi_2)$ and $S(\varphi)$ denote the corresponding compressions of the shift acting on $\mathfrak{H}(\varphi_1)$, $\mathfrak{H}(\varphi_2)$ and $\mathfrak{H}(\varphi)$, respectively. Then the following are equivalent to each other:*

(i) $S(\varphi_1)$ *and* $S(\varphi_2)$ *are reflexive;*

(ii) $S(\varphi_1)\oplus S(\varphi_2)$ *is reflexive;*

(iii) $S(\varphi)$ *is reflexive.*

Proof. The equivalence of (i) and (ii) is proved in [2]. The equivalence of (ii) and (iii) follows from Corollary 4.5 and the fact that $S(\varphi_1)\oplus S(\varphi_2)$ and $S(\varphi)$ are quasi-similar to each other for relatively prime inner functions φ_1, φ_2 (cf. [9], pp. 50—51).

J. ERDŐS has asked whether operators with the property that their invariant subspaces are all spanned by eigenvectors are necessarily reflexive. The next corollary answers the question positively for $C_0(N)$ contractions. Note that for such contractions, that all invariant subspaces are spanned by eigenvectors is equivalent to the fact that the minimal function is a Blaschke product with simple zeros (cf. [13], Prop. III. 7.2).

Corollary 4.7. *If T is a $C_0(N)$ contraction on $\mathfrak{H}$ whose minimal function m_T is a Blaschke product with simple zeros, then T is reflexive.*

Proof. Let S be an operator such that Lat $T \subset$ Lat S. Let $\{\lambda_i\}$ be the zeros of m_T. Then $\{\lambda_i\}$ are eigenvalues for T. If $\mathfrak{H}_i$ denotes the subspace of eigenvectors associated with λ_i, then $\mathfrak{H}_i$ $(i=1, 2, \ldots)$ span $\mathfrak{H}$ (cf. [13], Prop. III. 7.2). Each $\mathfrak{H}_i$, being invariant for T, is invariant under S. Hence for $x_i \in \mathfrak{H}_i$ we have

$$TSx_i = \lambda_i Sx_i = S\lambda_i x_i = STx_i.$$

This shows that T and S commute on $\mathfrak{H}_i$, $i=1, 2, \ldots$. It follows that $TS=ST$ on $\mathfrak{H}$. By Theorem 3.3, we have $S \in \mathscr{A}_T$. Hence T is reflexive, as asserted.

Note that the condition we give here is, in general, not necessary. As an example, consider the operator $S(\varphi) \oplus S(\varphi)$, which is reflexive for any inner function φ (by Corollary 4.3). However, for compressions of the shift we have

Corollary 4.8. *Let φ be a Blaschke product and $S(\varphi)$ the corresponding compression of the shift. Then $S(\varphi)$ is reflexive if and only if φ has only simple zeros.*

In the proof we will need the following simple fact, due to DEDDENS [3], concerning unicellular operators, the proof of which is included here for completeness. Recall that an operator T is *unicellular* if Lat T is totally ordered.

Lemma 4.9. *No unicellular operator on a space with dimension $\geqq 2$ is reflexive.*

Proof. Let T be a unicellular operator acting on $\mathfrak{H}$ which is reflexive. Let $\mathfrak{H}_1$, $\mathfrak{H}_2$ be invariant subspaces for T and P_1 the (orthogonal) projection onto $\mathfrak{H}_1$. We have $\mathfrak{H}_1 \subset \mathfrak{H}_2$ or $\mathfrak{H}_2 \subset \mathfrak{H}_1$. In either case P_1 will leave $\mathfrak{H}_2$ invariant. Since $\mathfrak{H}_2$ is arbitrary, by the reflexivity of T we have $P_1 \in \mathscr{A}_T$. Thus P_1 commutes with T. Hence both $\mathfrak{H}_1$ and $\mathfrak{H}_1^\perp$ are invariant under T. Then $\mathfrak{H}_1 \subset \mathfrak{H}_1^\perp$ or $\mathfrak{H}_1^\perp \subset \mathfrak{H}$ and we have $\mathfrak{H}_1 = \{0\}$ or $\mathfrak{H}_1 = \mathfrak{H}$. This shows that the only invariant subspaces for T are $\{0\}$ and $\mathfrak{H}$. Thus every operator on $\mathfrak{H}$ is in $\mathscr{A}_T$, hence commutes with T. A standard argument shows that T is a scalar multiple of the identity. Obviously, this cannot happen unless dim $\mathfrak{H}=0$ or 1, which proves our assertion.

Proof of Corollary 4.8. We have only to show that if $S(\varphi)$ is reflexive then φ has only simple zeros. Assume that λ_0 is a zero of φ with multiplicity $n_0 \geqq 2$. We have $\varphi(\lambda)=\varphi_0(\lambda)\varphi_1(\lambda)$, where

$$\varphi_0(\lambda) = \left(\frac{\lambda-\lambda_0}{1-\bar{\lambda}_0\lambda}\right)^{n_0} \quad \text{for} \quad \lambda \in \mathbf{D},$$

and $\varphi_1(\lambda)$ is a Blaschke product with $\varphi_1(\lambda_0)\neq 0$. Since $(\varphi_0, \varphi_1)=1$, the reflexivity of $S(\varphi)$ implies the reflexivity of $S(\varphi_0)$ (by Corollary 4.6). But it is easily seen that $S(\varphi_0)$ is a unicellular operator on a space with dimension $n_0 \geqq 2$. By Lemma 4.9 we have a contradiction, which proves our assertion.

Consider an inner function $\varphi(\lambda)=\psi(\lambda)\eta(\lambda)$ factored as the product of a Blaschke product $\psi(\lambda)$ and a singular inner function $\eta(\lambda)$. (For the structure of scalar valued inner functions consult [7].) Since $(\psi, \eta)=1$, we conclude, by Corollary 4.6, that $S(\varphi)$ is reflexive if and only if $S(\psi)$ and $S(\eta)$ both are reflexive. The preceding corollary gives a complete characterization for $S(\psi)$ being reflexive. As for the case of $S(\eta)$, we are not so fortunate. We have only the following partial result.

Recall that a *singular inner function* $\eta(\lambda)$ is a function of the form

$$\eta(\lambda) \equiv \eta(\mu; \lambda) = \exp\left(-\int \frac{e^{it}+\lambda}{e^{it}-\lambda}\, d\mu(t)\right),$$

where μ is a finite positive Borel measure on the unit circle C which is singular with respect to Lebesgue measure. The measure μ has an *atom* E, if E is a Borel subset with $\mu(E)>0$ and for any Borel subset F of E we have $\mu(F)=0$ or $\mu(E\setminus F)=0$.

Corollary 4.10. *If η is a singular inner function whose associated measure μ has an atom, then $S(\eta)$ is not reflexive.*

Proof. Let E be an atom of μ. Consider the functions $\eta_E(\lambda)=\eta(\mu_E; \lambda)$ and $\eta_{C\setminus E}(\lambda)=\eta(\mu_{C\setminus E}; \lambda)$, where μ_E and $\mu_{C\setminus E}$ are the restrictions of the measure μ to the sets E and $C\setminus E$, respectively. Note that $\eta(\lambda)=\eta_E(\lambda)\eta_{C\setminus E}(\lambda)$ and $(\eta_E, \eta_{C\setminus E})=1$. If $S(\eta)$ is reflexive, so is $S(\eta_E)$ (by Corollary 4.6). As any inner factor of $\eta_E(\mu; \lambda)$ must be of the form $\eta_E(a\mu; \lambda)$ for some $a\in[0, 1]$, the lattice of invariant subspaces of $S(\eta_E)$ is totally ordered, that is, $S(\eta_E)$ is unicellular. By Lemma 4.9 this can happen only when the space on which $S(\eta_E)$ is acting has dimension $\leqq 1$. However, this is impossible for a singular inner function η_E. This shows that $S(\eta)$ cannot be reflexive.

Note that the preceding result does not hold for $C_0(N)$ contractions with singular, atomic minimal functions. (Consider the direct sum of a compression of the shift with itself.) On the other hand, whether $C_0(N)$ contractions with singular, totally non-atomic minimal functions are indeed reflexive is still unknown. C. Foiaş [5] has shown that $S(\varphi)$ is reflexive for certain singular φ with totally non-atomic measures.

We remark that Corollaries 4.8 and 4.10 have been obtained earlier by J. CONWAY and, independently, by B. MOORE, III and E. NORDGREN (unpublished).

5. Concluding remarks. As the Jordan models for $C_0(N)$ contractions have been generalized to C_0 contractions with finite multiplicity (cf. [12]), it seems likely that our main theorems in § 3 hold in this more general context. However the proofs we gave do not seem to be readily extended to cover this case.

We also remark that if the answer to Rosenthal and Sarason's question (cf. § 1) is affirmative, most of the results we gave in § 4 will hold for arbitrary operators.

Finally, we raise the following question to conclude this paper: If T_1 and T_2 are $C_0(N)$ contractions which are quasi-similar to each other, is it true that T_1 is reflexive if and only if T_2 is? (The answer is "yes" for $C_0(N)$ contractions which are multiplicity-free.)

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On a convolution theorem

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Let G be a non-discrete locally compact Abelian group and let $1 \leqq q < \infty$. DIEUDONNÉ [3] showed that $f * L^q(G) \neq L^q(G)$ for every $f \in L^1(G)$. In [2], BURNHAM and GOLDBERG proved Dieudonné's result for $q=1$ by considering Banach algebras with elements which are "generalized divisors of zero", and an extension to Banach modules yields Dieudonné's result for $1 < q < \infty$. In this note we give a simple, elementary proof of the following result.

Theorem. *Let G be a non-discrete locally compact Abelian group and let $1 \leqq q \leqq \infty$. Then $L^1(G) * g \neq L^q(G)$ for every $g \in L^q(G)$.*

Proof. Suppose $L^1(G) * g = L^q(G)$ for some $g \in L^q(G)$. Then there exists $j \in L^1(G)$ such that $j * g = g$. Now if $h \in L^q(G)$, then $h = k * g$ for some $k \in L^1(G)$, and hence $j * h = j * k * g = k * j * g = k * g = h$. Thus $j * h = h$ for every $h \in L^q(G)$. Now for any $f \in L^1(G)$, choose a sequence $\{h_n\}$ in $L^1(G) \cap L^q(G)$ such that $\|h_n - f\|_1 \to 0$ as $n \to \infty$. Then $\|j * f - f\|_1 \leqq \|j * f - j * h_n\|_1 + \|j * h_n - f\|_1 \leqq \|j\|_1 \|f - h_n\|_1 + \|h_n - f\|_1 \to 0$. Thus j is an identity element for $L^1(G)$. But this is impossible, since G is non-discrete.

Remark 1. It is clear that the set $L^q(G)$ in the preceding theorem can be replaced by many other sets, and we mention some examples below.

(i) If B is any dense subset of $L^1(G)$ with $L^1(G) * B \subset B$, then $L^1(G) * g \subsetneqq B$ for every $g \in B$. In particular, if $S(G)$ is a Segal algebra in $L^1(G)$, then $L^1(G) * g \subsetneqq S(G)$ for every $g \in S(G)$.

(ii) If $g \in L^{pq}(G)$, then $L^1(G) * g \subsetneqq L^{pq}(G)$. See BLOZINSKI [1, 2.9] and YAP [5, (4.2)] for the relevant facts.

Remark 2. KROGSTAD [4] has used the above theorem (with $q=1$) to show that the union of all proper Segal algebras on G is $L^1(G)$. This answers a question of H. C. WANG.

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Correction

The paper by L. MÁRKI, On locally regular Rees matrix semigroups, *Acta Sci. Math.*, **37** (1975), 95—102, appeared with an unfortunate misprint. On p. 97, the Corollary after Lemma 1 is, in fact, a Corollary to Lemma 2, and should therefore be placed immediately after Lemma 2. On the other hand, Lemma 1 has the following Corollary, which was left out. It should be inserted after Lemma 1 in place of the (displaced) Corollary.

Corollary. *In a locally regular Rees matrix semigroup* $M^\circ(H; I, \Lambda; P)$ *there exist indices* $i \in I$ *and* $\lambda \in \Delta$ *such that* $M^\circ_{i\lambda} \cong H$.

Acta Sci. Math., 38 (1976), 207—220

Bibliographie

Friedrich Bachmann, Aufbau der Geometrie aus dem Spiegelungsbegriff, 2. ergänzte Auflage (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. 96), XVI+374 Seiten, Berlin—Heidelberg—New York, Springer-Verlag, 1973.

This book is the second enlarged edition of the first, which is already a classical textbook on the foundation of geometries based on reflections. This unified, group theoretical treatment shed new light on the systems of axioms in geometry and initiated new progress in this classical subject.

The original material (pp. 1—304) is supplemented with notes and references (pp. 305—310). The new Supplement (pp. 311—357) contains a detailed survey of the recent progress in "reflection geometry". A full bibliography from 1959 to 1972, consisting of 162 items, is also added.

P. T. Nagy (Szeged)

Walter Benz, Vorlesungen über Geometrie der Algebren. Geometrien von Möbius, Laguerre—Lie, Minkowski in einheitlicher und grundlagengeometrischer Behanglung (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. 197), XI+368 Seiten, Berlin—Heidelberg—New Yok, Springer-Verlag, 1973.

The geometries of Möbius, Laguerre, Lie and the pseudoeuclidean plane geometry can be treated in an analoguous way based on the geometry of the projective line over an algebra (the algebras of complex, dual complex and anormal complex numbers). The aim of this book is the systematic exposition of the geometry (called "chain geometry") of the projective line over an algebra and of its application to the study of classical geometries over an arbitrary field.

Let L be a commutative algebra over a field K, and consider the analytical projective lines $P(L)$ and $P(K)$ over L and K, respectively. We can regard the projective line $P(K)$ imbedded in a natural way in the line $P(L)$.

The "chain geometry" $\Sigma(K, L)$ is defined as follows: its points are the points of the projective line $P(L)$. The "chains" of $\Sigma(K, L)$ are defined as the set of points of $P(L)$ which are the images of the projective line $P(K)$ under the projectivities on $P(L)$. The incidence relation between points and chains is defined by inclusion. The notions of "tangence" and "angle" can be defined in a natural manner.

The book contains a detailed introduction to the geometries of Möbius, Laguerre, Lie, and to pseudoeuclidean geometry over the reals (Chapter I).

The general chain geometry is explained in Chapter II. There is a discussion of the problems: (i) Is every automorphism of a chain geometry $\Sigma(K, L)$ a projectivity on $P(L)$? (ii) Is each isomorphism of chain geometries induced by an isomorphism of the coordinate algebras? The answer in general is *not*, but in the case of the above-mentioned classical geometries the corresponding theorems are proved.

Chapter III is devoted to the study of questions of axiomatic nature.

Chapter IV deals with models of greater dimension for chain geometry. There is given a glance on the chain geometry over a noncommutative algebra.

The book is written in an always clear and well-readable way and only presupposes familiarity with the basic concepts of algebra and geometry.

P. T. Nagy (Szeged)

D. G. Douglas, Banach Algebra Techniques in Operator Theory (Pure and Applied Mathematics, A Series of Monographs and Textbooks, 49), XVI+216 pages, Academic Press, New York and London, 1972.

Operator theory includes the study of operators and collections of operators arising in mathematics, mechanics and other branches of physics. It is now sufficiently well developed to have a logic of its own.

This book presents a nice introduction to the study of bounded operators on Hilbert space based on powerful and interesting techniques drawn from functional analysis, from the theory of Banach spaces and Banach algebras. The author presumes only that the reader is familiar with general topology, measure theory, and algebra. He does not attempt completeness so that many elementary facts are either omitted or mentioned only in problems, which are of different character: either allow the reader to test his understanding, or indicate certain generalizations, or alert to certain important and related results, or point out open questions.

The book consists of seven chapters, references, and an index.

Chapter 1: *Banach Spaces.* Basic results along with many relevant examples. Discussion of theorems due to Alaoglu, Hahn and Banach, Riesz and Markov, and Banach. Lebesgue spaces L^1 and L^∞, and Hardy spaces H^1 and H^∞.

Chapter 2: *Banach Algebras.* Elementary theory of commutative Banach algebras, due essentially to Gelfand and Shilov, the technique of which is very essential in the subsequent chapters. The algebra of all continuous functions on some compact Hausdorff space is discussed here, including the Stone-Weierstrass theorem.

Chapter 3: *Geometry of Hilbert Space.* A short introduction with many examples.

Chapter 4: *Operators on Hilbert Space and C^*-algebras.* After the standard material the notion of a C^*-algebra is introduced and used throughout the rest of the chapter. The commutative Gelfand-Naimark theorem gives here an abstract spectral theorem and functional calculus. Commutative W^*-algebra theory is used to obtain an extended functional calculus. A theorem by Fuglede concludes the chapter.

Chapter 5.: *Compact Operators, Fredholm Operators, and Index Theory.* The approach is somewhat unorthodox: it gives the key results as quickly as possible and adds many examples. Certain ancillary results concerning ideals in C^*-algebras are also proved.

Chapter 6: *The Hardy Spaces.* Various properties of the spaces H^1, H^2 and H^∞ are derived, and results of Hartman, Wintner, Brown and Halmos, Coburn and Widom are treated.

The author adds short notes at the end of each chapter suggesting thus further reading.

The book is a very useful reading for anyone wishing to learn, or make further research in, the theory of operators.

Zoltán Sebestyén (Budapest)

Carl Faith, Algebra: Rings, Modules and Categories. I (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 190), XXIII+565 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1973.

This book is a survey of aspects of ring theory incorporating many of the now classical ring-theoretical ideas with homological ones. In Volume I the emphasis is equally divided between these two influences while Volume II — which has not appeared as yet — will be devoted to ring theory. The book is intended to contain everything that is worth knowing in this subject. There is no possibility here to give the details of the rich material covered in the book and we only give a short sketch of the four parts of Volume I.

After a short foreword on set theory in Part I (Chapters 1—6) the basic concepts and theorems of the theory of rings, modules and categories are presented. Part II (Chapters 7—10) is a discussion of the structure theory of Noetherian semiprime rings. Tensor algebra and the Morita theorems together with their application to the determination of the Picard group are developed in Part III (Chapers 11—13). Volume I is concluded in Part IV (Chapters 14—16) by the theory of Abelian categories including the theory of Grothendieck categories, the Mitchell-Gabriel embedding theorems and the Gabriel-Popesco theorem.

"This book is designed to introduce students to the basic ideas and operations of rings, modules and categories as patiently and as thoroughly as time and space permit, and then bring them to the frontiers of research as rapidly and as comprehendingly as their abilities permit." To this end it contains useful suggestions for reading. The description of the logical dependencies of the chapters makes it easier to peruse the book for those who are interested only in portions of it. A large bibliography of papers closely related to problems occurring in the book is also given. The book can be used as a reference book as well. It will be of great value in promoting and aiding further research on this subject.

Á. Szendrei (Szeged)

Wilhelm Flügge, Viscoelasticity, Second revised edition, VII+149 pages, Springer-Verlag Berlin—Heidelberg—New York, 1975.

The book presents an introductory course in the theory of viscoelasticity. The behaviour of viscoelastic material is described by a mixture of elementary models: the helical spring satisfying Hooke's law and the piston moving in a cylinder with a perforated bottom so that no air is trapped inside. The linear theory of viscoelasticity treated in this book presupposes that the differential equation expressing the connection of stresses, strains and displacements is linear. The reader is supposed to be familiar with some knowledge of Calculus only.

P. T. Nagy (Szeged)

Terence M. Gagen, Topics in Finite Groups (London Mathematical Society Lecture Note Note Series 16), VIII+85, Cambridge University Press, Cambridge—London—New York—Melbourne, 1976.

This book is a well-written explanation of H. Bender's theory of the classification of non-soluble groups with Abelian Sylow 2-subgroups and some related results. The topics covered in the book are of current research interest and were, as yet, accessible only to a very few specialists. The author's aim is to present this rich and original material to a wider community of group theorists.

A. P. Huhn (Szeged)

Richard B. Holmes, Geometric Functional Analysis and its Applications, X+246 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1975.

The main purpose of this book is to give a glimpse of applications of functional analysis to optimization theory and in particular to the theory of best approximation. In accordance with this objective it presents only parts of functional analysis having a geometrical nature. Most of the results of this book are based on the concept of convexity; the others generally use outgrowths concerning conjugate spaces or compactness properties, both of which topics are important for the proper setting and resolution of optimization problems. The book is divided into four chapters, all of them containing applications of functional analytic methods to the problems mentioned above.

Chapter I discusses convexity in linear spaces (using only the linear structure). The Hahn-Banach theorem appears in ten different (algebraic and geometric) but equivalent forms, some of which are optimality criteria for convex programs.

In Chapter II the concept of linear topological space is introduced. This chapter contains investigations concerning locally convex spaces, convexity and topology, weak topologies, extreme points, convex functions and optimizations, and some more applications.

Chapter III deals with Banach spaces, examines the questions of completion, congruence, reflexivity and gives some applications of category theorems.

Chapter IV is devoted to studying properties and characterizations of conjugate spaces and isomorphism of certain conjugate spaces. Universal spaces are also investigated.

All of the four chapters end with a rich selection of problems. Some are intended to be of a rather routine nature, many others, however, contain significant further results, converses or counter-examples.

The book is recommended to mathematicians doing research in functional analysis and in its applications, and to students whose mathematical background includes basic courses in linear algebra, measure theory, and general topology.

L. Gehér (Szeged)

C. Hooley, Applications of Sieve Methods to the Theory of Numbers (Cambridge Tracts in Mathematics, 70), XIV+122 pages, Cambridge University Press, Cambridge—London—New York—Melbourne, 1976.

This book, based on an Adams Prize winning essay of the author, presents some new and less known applications of sieve methods to additive and prime number theory. After a short survey of sieve methods, it proves a series of deep results, most of them due to the author. Emphasis is put on combination of sieve methods with each other and with further techniques. A number theory background is desirable on the reader's part.

L. Lovász (Szeged)

J. E. Humphreys, Introduction to Lie Algebras and Representation Theory (Graduate Texts in Mathematics, Vol. 9), XII+169 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1972.

The book intends "to introduce the reader to the theory of semisimple Lie algebras over an algebraically closed field of characteristic 0, with emphasis on representations".

The first four Chapters (I. Basic concepts, II. Semisimple Lie algebras, III. Root systems, IV. Isomorphism and conjugacy theorems) present the classical parts of the theory, and "might well be read by a bright undergraduate". Chapter V deals with the Poincaré—Birkhoff—Witt theorem and Serre's existence theorem and their consequences. Here a description of the classical simple Lie algebras is given. In Chapter VI representation theory is studied, especially finite dimensional Lie algebra modules. Chapter VII serves as an introduction to the theory of Chevalley algebras and groups, and their applications to Lie algebra representation theory.

Some standard topics are omitted (theorems of Levi and Ado, classification over reals etc.), which are better suited to a second course in the author's opinion.

Each chapter contains references and a lot of exercises. The reader is supposed to be familiar with linear algebra and with the elements of general algebra.

The book is written in a well-readable way. It will be useful to everyone wanting to get acquainted with the representation theory of Lie algebras.

P. T. Nagy (Szeged)

D. L. Johnson, Presentation of Groups, V+204 pages (London Mathematical Society Lecture Note Series 22), Cambridge University Press, Cambridge—London—New York—Melbourne, 1976.

This is a useful and easily readable book for those whishing to learn more group theory than the standard material of an ordinary undergraduate course. The book deals, among others, with free groups, free presentations of groups, Tietze transformations, van Kampen diagrams, coset enumerations, the elements of homological algebra, cohomology of groups, presentations of group extensions and presentations of direct products and wreath products.

A. P. Huhn (Szeged)

O. Neugebauer, A history of ancient mathematical astronomy. In 3 parts (Studies in the History of Mathematics and Physical Sciences, Vol. 1), XXIII, VII, V+1456 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1975.

The book contains the history of ancient mathematical astronomy up to late antiquity. Its objective is exclusively to make the reader familiar with the numerical, geometrical and graphical methods developed in the time period mentioned to describe the mechanism of the planetary system. The plan of the book does not follow strictly the chronological order of discoveries. It begins with a discussion of Almagest since "it is fully preserved and constitutes the keystone to the understanding of all ancient and mediaeval astronomy". Then it goes back in time to Babylonian astronomy for which a fair amount of contemporary original sources is available. Next come a short survey of Egyptian astronomy and then the most fragmentary and most complex section of the book: Greek astronomy and its relation to Babylonian methods. The concluding part of the main body of the work deals with Hellenistic astronomy as known from papyri, Ptolemy's minor works and the "Handy Tables". The material mentioned so far comprises two volumes. There is also a third volume which contains details concerning technical terminology, descriptions of chronological, astronomical and mathematical tools. An abundance of figures and plates, an extensive bibliography and subject index are also given in the third volume.

J. Szűcs (Szeged)

G. Ringel, Map Color Theorem (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, 209), IX+191 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1974.

If you find a problem too difficult to solve, generalize it... In 1890 P. J. Heawood stated a formula which expressed the maximum chromatic number $\chi(S)$ of graphs embeddable in any given surface S, thus generalizing the Four Color Conjecture. It happened that one could obtain a coloration with this many colors by an easy inductive argument — except if S was the sphere. So to completely prove Heawood's formula two tasks remained:

(A) to show that if S is the sphere then $\chi(S)=4$;

(B) to whow that there is a graph on each surface S whose chromatic number attains Heawood's bound.

Problem (A), the Four Color Conjecture, is still unsolved. But (B) did not turn out much simpler either. It suffices to show that the complete graph whose number of points is Heawood's bound can be embedded in *S*. After the pioneering work of Ringel, Gustin, and others, in 1968 Ringel and Youngs proved this conjecture. The main tool is the construction of certain combinatorial patterns called schemes. This goes quite differently for different residue classes mod 12. The difficulty varies a lot with the residue mod 12 and the different cases give a good picture of the evolution of ideas.

This book contains a very graphic presentation of the solution of this problem. It explains all of the history as well as the graph-theoretical and topological background of the problem. This not only makes it self-contained but also a very enjoyable reading, accessible to students and non-specialists. There are many exercises and open problems. The book is warmly recommended to those who want to get acquainted with topological graph theory.

L. Lovász (Szeged)

J. A. Rosanow, Stochastische Prozesse, eine Einführung (Mathematische Lehrbücher und Monographien, 28), Übersetzung aus dem Russischen, IX+288 Seiten, Akademie Verlag, Berlin, 1975.

Der Theoretiker betrachtet die stochastischen Prozesse als abstrakte Objekte der mathematischen Forschung, während für den Praktiker sind sie Werkzeuge zur Lösung praktischer Probleme. Das vorliegende Buch entstand aus den Vorlesungen des Verf., an dem Moskauer physikalisch-technischen Institut, wendet sich also an die Praktiker. Alle eingeführte Begriffe und erhaltene Ergebnisse werden anschaulich interpretiert, und die Auswahl des Stoffes wird durch die Erfordernisse der Anwendungen bestimmt. Praxisorientiertheit und begrenzter Umfang ziehen notwendigerweise etwas Grosszügigkeit bzgl. mathematische Genauigkeit nach sich, sie wirkt aber nicht störend aus.

Die erste Hälfte des Buches enthält eine übliche Einführung in die Wahrscheinlichkeitsrechnung von Grundbegriffen bis zum zentralen Grenzwertsatz (in der Ljapunowschen Form). Vorausgesetzt sind nur Grundkenntnisse aus der Differential- und Integralrechnung. Der Umfang des behandelten Stoffes von dem Gebiet der stoch. Prozesse illustrieren die Kapiteltitel der zweiten Hälfte des Buches: 1. Definitionen und Beispiele; 2. Markowsche Ketten, Klassifikation der Zustände, stationäre Verteilungen; 3. Markowsche Ketten mit stetiger Zeit; 4. Verzweigungsprozesse; 5. Einige stoch. Prozesse in der Bedienungstheorie und Irrfahrten; 6. Stoch. Prozesse in linearen Systemen; 7. Stationäre Prozesse; 8. Diffusionsprozesse; 9. Prognose und Filtration stoch. Prozesse.

Grosser Vorteil des Buches ist seine Kürze, Anschaulichkeit der Darlegungen und die enge Verbindung mit den Anwendungen. Schade, dass keine Literaturangaben die weitere Orientation helfen.

D. Vermes (Szeged)

Murray Rosenblatt, Random Processes (Graduate Texts in Mathematics 17), second edition, IX+228 pages, New York—Heidelberg—Berlin, Springer-Verlag, 1974.

Since the middle of the fifties the theory of random processes has been renewed by a revolutionary development, initiated by the works of Prohorov, Skorohod, Ito, Hunt, Dynkin, and others. The new point is the inclusion of sample space properties into the investigation, while in the classical theory only distributions were considered.

The present book, an enlarged edition of the original published in 1962 by Oxford University Press, aims to give a first introduction to the classical parts of the theory. To make the book understandable for students in lower semesters, a short introduction in probability theory is included while functional analytical methods and more complicated proofs are avoided. After the introduction and the basic definitions, Markov chains, ergodic theory of stationary sequences, Markov processes in continuous time (approach via Kolmogorov equations), the spectral decomposition of weakly stationary processes and convergence theorems for martingales are presented. Problems are included at the end of each chapter.

The book is very useful in giving a first insight into the classical theory of random processes and also as a textbook for a one or two semester course requiring only the elements of calculus and matrix algebra as background.

D. Vermes (Szeged)

C. L. Siegel—J. K. Moser, Lectures on Celestial Mechanics (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. 187), X+290 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1971.

The book is a revised and enlarged translation of the very successful "Vorlesungen über Himmelsmechanik" by the first author (Grundlehren der mathematischen Wissenschaften, Bd. 85, 1956). Although new sections have been added to reflect recent work in the field, the basic organization of Siegel's original book has not been altered. As in 1957 there appeared a review of the German original in the 18th volume of these *Acta* (pp. 145—146) this time we shall discuss only the new parts added in the English text. Nevertheless, let us quote only one sentence from the English translation of the preface to the first edition in which Siegel says that his aim was "to develop some of the ideas and results that have evolved over the period of the past 70 years in the study of solutions to differential equations in the large, in which of course applications to Hamiltonian systems and in particular the equations of motion for the three body problem occupy an important place". Just as in the German original the authors again did not attempt to give a complete presentation of the subject, for example they do not discuss the now revitalized measure-theoretical methods of mechanics. The new parts added to the first edition are the following: Two sections in the first chapter on triple collision in the three-body problem by Siegel. The only relevant difference in Chapter II is the addition of the convergence proof for the transformation into Birkhoff's normal form of an area-preserving map near a hyperbolic fixed point. The main additions can be found in Chapter III. One can find a new and simpler proof for Siegel's theorem on conformal mapping near a fixed point as well as 5 sections on stability theorems for systems of two degrees of freedom and the existence theorem for quasi-periodic solutions, these 5 sections being based on the work of Arnold, Moser, and Kolmogorov.

J. Szűcs (Szeged)

Allan M. Sinclair, Automatic continuity of linear operators (London Mathematical Society Lecture Note Series, 21), 92 pages, Cambridge—London—New York—Melbourne, Cambridge University Press, 1976.

These notes are based on postgraduate lectures given at the University of Edinburgh during the spring of 1974. They contain a good amount of the results on automatic continuity of intertwining operators of Banach space operators and on homomorphisms of Banach algebras that were obtained between 1960 and 1973. They do not deal with axiomatic results such as Wright's asserting that under some reasonable hypotheses on the system of axioms all linear operators are bounded.

In the study of the automatic continuity of a linear operator S from a Banach space X into a Banach space Y the separating space $\mathfrak{S}(S)$ of S plays an important rôle. By definition $\mathfrak{S}(S)= \{y\in Y\colon \text{there is a sequence } \{x_n\} \text{ in } X \text{ with } x_n\to 0 \text{ and } Sx_n\to y\}$. A very important result concerning

$\mathfrak{S}(S)$ is that if $\{T_n\}$ and $\{R_n\}$ are sequences from the sets $B(X)$ and $B(Y)$ of all bounded linear operators on X and Y, respectively, and if $ST_n=R_nS$ $(n=1, 2, \ldots)$, then there exists an index N such that the closure of $R_1\ldots R_n\mathfrak{S}(S)$ equals the closure of $R_1\ldots R_N\mathfrak{S}(S)$ for $n\geqq N$. This result implies necessary and sufficient conditions on the pair (T, R), $T\in B(X)$, $R\in B(Y)$ in order that the operator S satisfying $ST=RS$ be continuous, provided that R has countable spectrum. Another application of the above result on $\mathfrak{S}(S)$ reveals some properties of discontinuous homomorphisms from $C_0(\Omega)$ into a radical Banach algebra, where Ω is a locally compact Hausdorff space. It is also proved in the book that under additional hypotheses on the (bounded) operators T and R there exist discontinuous linear operators S satisfying the relation $ST=RS$. In case X and Y are Hilbert spaces, T and R are normal and again we have the intertwining relation $ST=RS$, then S is decomposed into continuous and highly discontinuous parts. The uniqueness of the complete norm topology of a semisimple Banach algebra is proved in a way that emphasizes its relation to other automatic continuity theorems. By using the properties of the separating space of a homomorphism the continuity of a homomorphism from a Banach algebra onto a dense subalgebra of a strongly semi-simple unital Banach algebra is proved. A new proof for the existence of a discontinuous derivation from the disc algebra into a Banach module over it is given. Bade and Curtis's theorem on the decomposition of a homomorphism from $C(\Omega)$, where Ω is a compact Hausdorff space, into continuous and discontinuous parts is proved. If a unital C^*-algebra has no closed cofinite ideals, then it is shown that any homomorphism from it into a Banach algebra is continuous.

A Bibliography and an Index facilitate the reading of the book. As each member of the London Mathematical Society Lecture Note Series, this work, too, is recommended to postgraduate students and to research workers.

József Szűcs (Szeged)

E. L. Stiefel—G. Scheifele, Linear and Regular Celestial Mechanics (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. 174), IX+301 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1971.

The modifier "linear" in the title refers to that the pure two body problem is treated in the book by means of linear differential equations which turn out to have constant coefficients. More precisely, the description of the pure resp. perturbed Keplerian motion reduces to that of a pure resp. perturbed harmonic oscillator. By harmonic oscillator the book means any physical system the behavior of which is described by the equations of the harmonic oscillation regardless the signs of the eigenvalues (i.e., an oscillator does not necessarily oscillate!). The adjective "regular" in the title is used in the usual sense: the differential equations describing the motion are regular, i.e., the highest order derivatives of the unknown functions are expressible in terms of regular functions of the lower order derivatives. The classical Newtonian equations are singular at the place where the attracting central mass is placed, and therefore they are not sufficient if collision occurs. On the other hand, during collision the mutual velocity is infinite. Consequently, from the point of view of numerical integration the Newtonian equations behave badly not only if the two particles collide but also when the two particles move too close with high speed. Such "nearcentre" cases are very important in space science for example when "a space vehicle is parked on an orbit about the earth and is then injected into its interplanetary orbit by a thrust of the engines". As the pure elliptic Kepler motion is described by the differential equations of a harmonic oscillator with negative eigenvalues (the one that really oscillates), the differential equations of the pure motion are stable in the sense of Ljapunov, a feature not occurring if we use the classical equations of Newton. Stability is of great significance if one wants to solve differential equations by numerical methods. This is not to say that one should solve the equations of the pure two body problem by numerical methods, since explicit formulae are available for that purpose. However, one cannot expect stability in the case of slight perturbations if the unperturbed equations are not stable.

The book could go under the subtitle "How should one calculate the motion of his space mobile?" It comprises three parts, the first and third being written by the first author, while the second part by the second author. Throughout the book particular attention is paid to numerical solutions of the problems discussed. Part I starts with the classical Newtonian equations of the two body problem and then these equations are linearized and regularized by means of the Levi-Cività transformation $x_1+ix_2=(u_1+iu_2)^2$ (squaring complex numbers), provided that the motion takes place in the (x_1, x_2)-plane. Although we know that pure gravitational motions are plane motions there is a need to extend the Levi-Cività theory to the three dimensional space. To explain this need it is enough to mention that the perturbed motion will not be a plane motion in general. The first author raised the problem of this extension of Levi-Cività's theory at an Oberwolfach meeting in 1964, where P. Kustaanheimo suggested using the ideas of spinor theory, in other words, employing a pair of complex numbers. In an 1965 paper Stiefel solved the problem in giving the theory of such a transformation, the so called KS-transformation. This transformation reminds us of squaring quaternions, and thus it increases the number of space coordinates by one, which, of course, causes some difficulties; however, the advantage gained turns out to be so valuable that one can allow such an increase of the number of parameters that proves to be harmless anyway. In spite of the close connection with queternions the book prefers the usage of usual real matrices to that of queternion formalism and the authors say that any attempt to use quaternions leads "to failure or at least to a very unwieldy formalism". The first part of the book proceeds with the properties of KS-transformation and the equations of motion in the new coordinates. One has to introduce a new independent variable, the so-called fictitious time s and thus the physical time t becomes a dependent variable (this is necessary in the case of plane motion, too). The relation between s and t is given by the equality $dt=r\,ds$, where r denotes the distance of the moving particle from the central mass. The differential equations of the motion after the transformation have total order ten, since we have to add an equation of order one that describes the variation of the total energy (and the above equation for the physical time). The solutions of this set of equations are discussed in detail in the first part of the book. The pure Kepler motion obtains a uniform treatment, regardless the shape of the orbit (ellipse, hyperbola, parabola), by using the Levi-Cività transformation and Stumpff functions. Next comes the initial value problem in *space* using the KS-transformation. The initial values in terms of the u coordinates are given in the critical case of ejection, too. Then the u-coordinates are given as functions of the 6 classical Lagrangian elements. (An element of a differential equation system is a quantity that varies linearly with respect to the independent variable.) Several aspects of the unperturbed and perturbed linear differential equations of motion are discussed. It is shown that in the case of elliptic motion the new equations are stable while the classical ones are not. In the case of elliptic motion a complete set of regular elements is given and the element equations are computed for the perturbed motion. The ten first order element equations describe the change of the elements with respect to the so-called generalized eccentric anomaly. One has his choice to solve the element equations (element method), or to solve the perturbed linear equations referred to above (u-method). The advantages and disadvantages of both methods are discussed. A chapter is devoted to gravitational perturbations (oblateness, third body attraction), with numerical examples. The last chapter of Part I bears the name "Refined Numerical Methods". It studies numerical methods for the solution of the perturbed problem that have the following property: "if the perturbing terms are switched-off at an arbitrary instant of the independent variable t (or s), then the numerical methods at hand should integrate without discretization error the subsequent unperturbed equation." It is mentioned that the classical Runge-Kutta method, the finite difference methods and the method of Encke do not satisfy this requirement if we want to integrate the classical Newtonian equations. Any element method satisfies the above requirement combined with any reasonable numerical method since during the unperturbed motion the elements vary linearly. The element- and the *u*-methods are discussed from several aspects of numerical integration, examples and good advices are given.

Part II is devoted to the canonical theory of the perturbed linear differential equations. In contrast to the classical canonical theory of Hamilton, transformations of the independent variable, the use of redundant variables as well as forces without potential are allowed and the form of the canonical differential equation system is different from the classical one, too. It is established that the differential equations in the first part of the book are all canonical and the basic equations are calculated by means of canonical transformations. Canonical equations are given that are closed in the sense that all the differential equations are incorporated in the canonical system which are needed for computing the motion; this is not an obvious problem, since, for example, if the fictitious time is the independent variable, then the system has to include the physical time as a dependent variable. A whole chapter is devoted also to the classical canonical theory of the perturbations of elements, the perturbation of the Delaunay and the classical Lagrangian elements is computed.

The very short third part is concerned with those geometrical properties of the KS-transformation that were not treated in the first two parts of the book.

The work of Stiefel and Schiefele is a very interesting reading. As it reflects some of the newest developments of the perturbed two body problem and pays special attention to the actual, numerical solution of the problem, it can and has to be recommended to anyone whose work is concerned with the calculation of the motion of artificial celestial bodies. Only basic calculus and familiarity with the elementary concepts of physics are supposed on the part of the reader (for example, the classical canonical theory is *not* a prerequisite). Thus graduate students of mathematics may enjoy its reading, too.

József Szűcs (Szeged)

Götz Uebe, Produktionstheorie (Unter Mitwirkung von Joachim Fischer) (Lecture Notes in Economics and Mathematical Systems, Bd. 114), X+301 Seiten, Berlin—Heidelberg—New York, Springer-Verlag, 1976.

Dieses Buch ist aus der Mitschrift einer Vorlesung und eines Seminars entstanden, die an der Universität Bonn gehalten wurden. Der Autor schreibt in dem Vorwort: "Hauptziel dieses Buches ist eine strenge Grundlegung zu geben. Die zweite Zielsetzung ist, die Produktionstheorie als Problem der Konkaven Programmierung zu sehen. ...Ein drittes Anliegen schliesslich ist, einige natürliche Erweiterungen aus der Theorie der Produktionsfunktion zu bringen." Das Buch verwirklicht diese Zielsetzungen. Diese Arbeit ist von einer modernen Betrachtungsweise durchgedrungen. Sie enthält Kenntnisse, die für die Leser die neuesten Ergebnisse der Literatur der Produktionstheorie zu verstehen erleichtern. Ein ausführliches Literaturverzeichnis ist beigefügt.

L. Megyesi (Szeged)

M. M. Wainberg und W. A. Trenogin, Theorie der Lösungsverzweigung bei nichtlinearen Gleichungen, XII+408 pages, Berlin, Akademie Verlag, 1973.

Summing in a few words the subject of the book it is the analysis of branching points of certain nonlinear functional equations depending on parameters. There are given methods for the description of all solutions of the equation which bifurcate from a known solution by the change of the parameters. Among others the book discusses the question of the theory of implicit functions (both in the finite dimensional case and in Banach space) the branching theory of periodic solutions of ordinary differential equations and contains an extensive account of certain classes of nonlinear integral and integro-differential equations, as well as applications of the theory to a number of practical problems. There is a valuable bibliography.

The topic is of fundamental importance for applied mathematics, especially in problems represented by a nonlinear system of parameters. It has been researched for a long time and numerous articles have been published. The present book gives a good summary of the researches up to the recent ones. If will be useful both for mathematicians as a monograph and for students as an introduction to bifurcation theory.

The original Russian work was published in 1969.

J. Terjéki (Szeged)

Garth Warner, Harmonic Analysis on Semi-Simple Lie Groups I, II (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. 188), XVI+529, VIII+461 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1972.

This two volume monograph is devoted to the representation theory of semi-simple Lie groups which, to a great extent, is the work of one single man: Harish-Chandra. Harmonic analysis on locally compact groups has been vigorously developed in the past 30 years. Although little is known about representations of a general locally compact group, two extreme cases have been studied in detail: Dixmier, Kirillov and Pukánszky have made great progress in the theory of nilpotent groups and, on the other hand, Harish-Chandra has created the representation theory of semi-simple Lie groups, the book under review is the first systematic exposition of his results. The author made a successful effort to give complete proofs of the basic theorems and thus to provide a reasonably self-contained introduction to Harish-Chandra's theory. The prerequisites vary in the following way: the part that is an introduction to general group representation theory and spherical functions (chapters 4, 5, 6, 7) requires a general knowledge of functional analysis, some distribution theory and elementary abstract harmonic analysis. The reader, if he wishes to, can start with this part of the book without the danger of encountering references to the first three chapters. The part on the structure theory of semi-simple Lie groups and algebras (chapters 1 and 2) requires a background equivalent to JACOBSON's *Lie algebras*. Interscience, New York, 1962. The part on the finite dimensional representations of semi-simple Lie groups (chapter 3) assumes a little sheaf theory, while the rest of the treatise (chapters 8, 9, 10) deals exclusively with semi-simple Lie groups and needs all the prerequisites mentioned above.

Let us say a few words about the subject of the book in a way understandable to every mathematician. To this end let G be a locally compact unimodular group satisfying the second axiom of countability. Assume, moreover, that G is postliminaire, i.e., all continuous representations of G are of type I (the von Neumann algebras generated by the images of G via the representations are discrete). Let us denote by G the topological space of unitary equivalence classes of irreducible (continuous) unitary representations of G equipped with the so-called hull-kernel topology. Take a Haar measure dx on G. Then a famous theorem of I. E. Segal asserts the existence of a unique positive measure μ defined on the Borel structure of $\hat{G}$ such that $\int_G |f(x)|^2\,dx = \int_{\hat{G}} \mathrm{tr}\,\big(U(f)\,U(f)^*\big)\;d\mu(\hat{U})$ for all $f \in L^1(G) \cap L^2(G)$. On the right side in the above equality tr means "trace of an operator" and for the representant U of any element $\hat{U}$ of $\hat{G}$ the operator $U(f)$ is defined as $\int_G f(x)\,U(x)\,dx$. The measure μ is called the Plancherel measure (associated with dx). One basic problem of harmonic analysis is the explicit determination of μ. It is the subject of the book to do this in the case of a semi-simple Lie group.

For those that have a solid background in abstract harmonic analysis we are now going to give more insight into the contents of the book by drawing freely form the chapter and section headings: The structure of real semi-simple algebras: Bruhat decomposition, parabolic subgroups, Cartan subalgebras and subgroups; The universal enveloping algebra of a semi-simple Lie algebra: invariant theory, reductive Lie algebras, representations of reductive Lie algebras, representations on cohomology groups; Finite dimensional representations of a semi-simple Lie groups; Infinite dimensional group representation theory: representations on a locally convex, Banach or Hilbert space, differentiable and analytic vectors, large compact subgroups; Induced representations: unitarily induced representations, quasi-invariant distributions, irreducibility of unitarily induced representations systems of imprimitivity ; The general theory of spherical functions; Topology on the dual, Plancherel measure; Analysis on a semi-simple Lie group: differential operators, central eigendistributions and invariant integral on reductive Lie groups and algebras; Spherical functions on a semi-simple Lie group: asymptotic behavior of μ-spherical functions and zonal spherical functions on a semi-simple Lie group, spherical functions and differential equations; the discrete series for a semi-simple Lie group — existence and exhaustion.

There are examples throughout the book that help the reader understand the abstract theory and acquire some knowledge of the applications. Altogether there are three appendices in the two volumes that compile notions and results concerning quasi-invariant measures, distributions on a manifold, and the theory of differential equations.

While the work cannot be recommended as a first introduction, those who have a solid background in the theory can learn from it most of what is presently known about semisimple Lie groups.

József Szűcs (Szeged)

J. H. Wells, L. R. William, Embeddings and Extensions in Analysis (Ergebnisse der Math., Band 84, VI+108 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1975.

The object of this book is a presentation of the main results of two geometrically inspired problems in analysis. The first is the problem of embedding metric spaces into Hilbert space or more generally into L^p spaces, the second is the problem of extending of continuous maps.

Chapter I deals with isometric embeddings into Hilbert space and characterization of subspaces of L^p; Chapter II is devoted to integral representations of functions of positive definite and negative type. Chapter III contains the main results of extension problems for contractions and isometries of Banach spaces. Chapter IV gives a glimpse on interpolation and L^p inequalities. The theme of Chapter V is the extension problem for Lipschitz—Hölder maps between L^p spaces.

The book is recommended to all mathematicians, who are interested in extension and embedding problems.

L. Gehér (Szeged)

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