ACTA UNIVERSITATIS SZEGEDIENSIS

ACTA SCIENTIARUM MATHEMATICARUM

TOMUS 36
FASC. 1—2

SZEGED 1974

INSTITUTUM BOLYAIANUM UNIVERSITATIS SZEGEDIENSIS

A JÓZSEF ATTILA TUDOMÁNYEGYETEM KÖZLEMÉNYEI

ACTA SCIENTIARUM MATHEMATICARUM

36. KÖTET

1—2. FÜZET

SZEGED, 1974. JÚNIUS

JÓZSEF ATTILA TUDOMÁNYEGYETEM BOLYAI INTÉZETE

The convolution theorems of Dieudonné

By J. T. BURNHAM in Heslington (England) and R. R. GOLDBERG in Iowa City (Iowa, USA)

Let G denote an arbitrary locally compact abelian group that is not discrete. In [1], DIEUDONNÉ showed that if $f \in L^1(G)$ then $f * L^1(G)$ must be a proper subset of $L^1(G)$. His proof involves a rather complicated construction of some particular functions on G.

In this note we prove a general result (Theorem B) about Banach algebras with elements which are generalized divisors of zero, and deduce Dieudonné's result as an easy corollary (Theorem D). An extension to Banach modules yields his result that $f * L^q(G) \neq L^q(G)$ if $f \in L^1(G)$, $1 < q < \infty$.

Definition A. Let B be a commutative normed algebra. The element $f \in B$ is said to be a *generalized divisor of zero* (gdz) if there exists a sequence $\{g_n\}$ in B such that

$$\|g_n\| = 1 \quad (n = 1, 2, \ldots), \quad \text{but} \quad fg_n \to 0.$$

This is equivalent to the definition in [2, p. 69] which treats only algebras with unit.

Theorem B. *Let B be a commutative, semi-simple Banach algebra. If $f \in B$ is a* gdz *then $fB \neq B$.*

Proof. We may assume that the Gelfand transform $\hat{f}$ of f never vanishes. For, if $\hat{f}(\lambda)=0$ for some λ, then $\hat{g}(\lambda)=0$ for every $g \in fB$ so that $fB \neq B$

Define $T: B \to B$ by $Tg = fg \quad (g \in B)$. Then T is $1-1$. For if $Tg=fg=0$, then $\hat{f}\hat{g}=0$. Since $\hat{f}$ never vanishes this implies $\hat{g}=0$, and so $g=0$ since B is semi-simple. T is clearly continuous. We wish to show that T is not onto.

Assume the contrary. Then T is $1-1$, continuous, and onto. The inverse function theorem for Banach spaces then implies that T^{-1} is continuous. Since f is a gdz there exists $\{g_n\}$ with $fg_n \to 0$ and

$$\|g_n\| = 1 \quad (n = 1, 2, \ldots). \tag{1}$$

Since T^{-1} is continuous we have $T^{-1}(fg_n) \to 0$. But $fg_n = Tg_n$ so that $T^{-1}(fg_n) = g_n$. Hence $g_n \to 0$ which contradicts (1). The contradiction shows that T is not onto which is what we wished to show.

We next show that every element f in $L^1(R)$ is a gdz. Let

$$g_n(t) = e^{int}\delta(t) \quad (-\infty < t < \infty; \quad n = 1, 2, \ldots)$$

where δ is any bounded L^1 function with $\|\delta\|_1 = 1$. Then $\|g_n\|_1 = 1$. Now for each t the function $f(u)\delta(t-u)$ is in L^1. Also

$$f * g_n(t) = e^{int} \int_{-\infty}^{\infty} e^{-inu} f(u)\delta(t-u)\,du$$

which tends to zero for each t as $n \to \infty$ by the Riemann—Lebesgue theorem. Moreover, $f * g_n$ is dominated by

$$\int_{-\infty}^{\infty} |f(u)\delta(t-u)|\,du$$

which is integrable since $f, \delta \in L^1$. Hence

$$\|f * g_n\|_1 \to 0$$

by the Lebesgue dominated convergence theorem. Thus $\|g_n\|_1 = 1$ and $f * g_n \to 0$ which shows that f is a gdz. (Note, too, that $\{g_n\}$ is independent of f.)

The above argument extends easily to any non-discrete G. Simply replace the functions e^{int} by characters $\chi_n(t)$ on G where $\{\chi_n\}$ tends to infinity on $\hat{G}$ (which is not compact). Thus we have

Theorem C. *If G is a locally compact abelian group that is not discrete, then every element of $L^1(G)$ is a* gdz.

Here is Dieudonné's result.

Theorem D. *If G is a locally compact abelian group that is not discrete, then $f * L^1(G) \neq L^1(G)$ for all $f \in L^1(G)$.*

Proof. The space $L^1(G)$ is a commutative semi-simple Banach algebra. If $f \in L^1(G)$ then, by Theorem C, f is a gdz. Hence, by Theorem B, $f * L^1(G) \neq L^1(G)$.

Dieudonné actually proved that $f * L^q \neq L^q$ for every $f \in L^1$, $1 < q < \infty$. To include this result we generalize to modules. See [3, p. 263] for the definition of a Banach A-module. The example that will interest us is L^q which is a Banach L^1-module.

Definition E. Let A be a commutative Banach algebra and let B be a Banach A-module. The element $f \in A$ is said to be a *generalized divisor of zero with respect to B* (abbreviate gdz-B) if there exists a sequence $\{g_n\}$ in B such that

$$\|g_n\|_B = 1 \quad (n = 1, 2, \ldots), \quad \text{but} \quad \|fg_n\|_B \to 0.$$

Completeness is not essential in the above definition but it is in what follows.

Because B in the following theorem need not be an algebra, we do not have the Gelfand transform available. We introduce a $1-1$ hypothesis that was not necessary in Theorem B, and we will handle the non $1-1$ case specially when we come to L^q.

Theorem F. *Let A be a commutative Banach algebra and let B be a Banach A-module. If $f \in A$ is a* gdz-B, *and if $T: B \to B$ defined by $Tg = fg$ $(g \in B)$ is $1-1$, then T is not onto. That is, $fB \neq B$.*

Proof. Same as that of Theorem B.

Corresponding to Theorem C we have

Theorem G. *If G is as in Theorem* C *then every $f \in L^1(G)$ is a* gdz-$L^q(G)$.

Proof. Again, the case $G=R$ tells all. Take any $f \in L^1(R)$. Let

$$g_n(t) = e^{int}\delta(t) \quad (-\infty < t < \infty; \quad n = 1, 2, \dots)$$

where δ is a bounded L^q function with $\|\delta\|_q = 1$. Then $f_n * g(t) \to 0$ for all t, as before. Moreover, $|f * g_n|^q$ is dominated by $(|f| * |\delta|)^q$. But $|f| * |\delta| \in L^q$ since $f \in L^1$, $\delta \in L^q$, so that $(|f| * |\delta|)^q$ is integrable. Thus $\|f * g_n\|_q \to 0$ by the dominated convergence theorem. Since $\|g_n\|_q = 1$, the proof is complete.

Finally,

Theorem H. *If G is as in Theorem* C, *then*

$$f * L^q(G) \neq L^q(G) \quad \text{for all} \quad f \in L^1(G).$$

Proof. For $f \in L^1(G)$ let $E \subset \hat{G}$ be the set where $\hat{f} = 0$. We consider two cases.

a) If $mE = 0$ then the map $T: g \to f * g$ $(g \in L^q)$ is $1-1$. For if $Tg = f * g = 0$ then $\hat{f}\hat{g} = 0$ almost everywhere on $\hat{G}$. Since $mE = 0$ this implies $\hat{g} = 0$ a.e. and hence $g = 0$. The desired conclusion then follows from Theorems F and G.

b) If $mE > 0$ then, since the transform of every function in $f * L^q(G)$ vanishes a.e. on E, it is clear that $f * L^q(G) \neq L^q(G)$.

This argument is valid for $1 < q \leqq 2$. For $q > 2$ an easy adjoint argument applies.

References

[1] J. Dieudonné, Sur le produit de composition. II, *J. Math. Pures Appl.*, **39** (3) (1960), 275—292.

[2] I. Gelfand, D. Raikov, G. Shilov, *Commutative normed rings* (New York, 1964).

[3] E. Hewitt and K. Ross, *Abstract harmonic analysis*. II (Berlin, 1970).

UNIVERSITY OF YORK
HESLINGTON, ENGLAND

UNIVERSITY OF IOWA
IOWA CITY, IOWA 52240, U.S.A.

(Received February 7, 1973)

On models for noncontractions

By DOUGLAS N. CLARK in Los Angeles (California, U.S.A.)*)

1. Introduction

1.1. Characteristic functions. The characteristic operator function Θ_T of a bounded linear operator T on a Hilbert space $\mathfrak{H}$ is by definition the operator-valued analytic function

$$(1) \qquad \Theta_T(z) = TJ_T - zQ_*(I-zT^*)^{-1}Q$$

where $J_T = \operatorname{sgn}(I-T^*T)$, $Q=|I-T^*T|^{\frac{1}{2}}$ and $Q_*=|I-TT^*|^{\frac{1}{2}}$, in the sense of the self-adjoint operator calculus (here $\operatorname{sgn} 0=1$), and where Θ_T acts from $\bar{R}(Q)$, the closure of the range of Q, to $\bar{R}(Q_*)$.

If T is a contraction, so that the operator J_T (and the absolute value signs) disappear from (1), Θ_T has been studied quite a bit and is fairly well understood. Sz.-Nagy and Foiaş, for example, in their book [6], study the relationship of T and Θ_T. Basic to their theory is the construction of a "canonical model" — a contraction operator T of a canonical type — such that $\Theta = \Theta_T$, for a given analytic operator function Θ with $\|\Theta(z)\| \leqq 1$ for $|z|<1$.

Several recent papers have concerned more general $\Theta(z)$; see, for example, Kužel' [4] and Davis and Foiaş [3]. Brodskii, Gohberg and Krein [2], working with a characteristic operator function somewhat different from (1), have given necessary and sufficient conditions that an analytic operator-valued function Θ should have the form $\Theta=\Theta_T$, for some bounded (invertible) operator T. Their condition translates into Theorem 1 below. Their proof uses Neumark's Theorem and does not appear to provide a clear analogue of the Sz.-Nagy—Foiaş model theory.

In this paper we give a construction (Theorem 2 below) which, although less geometrical than that of Sz.-Nagy and Foiaş, does yield a model analogous to theirs and also contains the theorem of Brodskii, Gohberg and Krein (Theorem 1) as a corollary.

---

*) Partially supported by NSF Grant GP 29011.

1.2. Statement of results. More precisely, let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be Hilbert spaces of the same dimension, let $B(z)$ be a function whose values are bounded operators from $\mathfrak{H}_1$ to $\mathfrak{H}_2$, and let $J=\operatorname{sgn}(I-B(0)^*B(0))$ and $J_*=\operatorname{sgn}(I-B(0)B(0)^*)$. The conditions of Brodskii, Gohberg and Krein, applied to our characteristic operator function become

Theorem 1. ([2], Theorem 6.1.) *Suppose $B(z)$ is analytic in some neighborhood D of 0. Then B is the characteristic operator function of some invertible operator if and only if B satisfies*

(i) *$B(0)$ is invertible,*

(ii) *the operator valued function*

$$G(z)=[\mathfrak{U}^*+B(z)]^{-1}[\mathfrak{U}^*-B(z)]J,$$

where $\mathfrak{U}\colon \mathfrak{H}_2\to\mathfrak{H}_1$ is a unitary operator satisfying $\mathfrak{U}J_*=J\mathfrak{U}$, extends to be analytic in $|z|<1$ with positive real part there:

$$\operatorname{Re}(G(z)\,x,x)\geqq 0 \quad \textit{if} \quad |z|<1 \quad \textit{and} \quad x\in\mathfrak{H}_1.$$

The existence of the unitary operator $\mathfrak{U}$ in (ii) comes from the polar representation of the (invertible) operator $B(0)$.

Theorem 1 will be seen to follow from

Theorem 2. *Let $B(z)$ be analytic and invertible in an open set D, with $0\in D\subset\subset\{|z|<1\}$. Extend $B(z)$ to the reflection $\tilde{D}$ of D by defining*

$$B(z)=J_*B(\bar{z}^{-1})^{*-1}J.$$

Then $B(z)$ is a characteristic operator function if and only if

$$b(w,z)=(1-\bar{w}z)^{-1}[J_*-B(z)JB(w)^*]$$

is a positive definite operator function on $\mathfrak{H}_2$.

The condition on $b(z,w)$ means that for $z_1,\dots,z_n\in D\cup\tilde{D}$ and for $x_1,\dots,x_n\in\mathfrak{H}_2$, not all 0, we have

$$\sum\left(b(z_i,z_j)x_i,x_j\right)>0. \tag{2}$$

1.3. Remarks on the theorems. The proofs of (the sufficiency parts of) the theorems will be given in Section 2 (Theorem 1) and Sections 3—5 (Theorem 2). The necessity parts are less difficult and will be proved in the next section.

We shall continually use the following fact about the Q's and J's. Since $(I-T^*T)T^*=T^*(I-TT^*)$ it follows that $f(I-T^*T)T^*=T^*f(I-TT^*)$ for any (bounded, Borel) function f. From this there follow relations of the form $JB(0)^*=B(0)^*J_*$, $Q_*B(0)=B(0)Q$, etc.

As we have pointed out, a different characteristic function is used in [2]. Let $K=(\Theta_T(0)^*\Theta_T(0))^{\frac{1}{2}}$, so that $\Theta_T(0)=\mathfrak{U}^*K$. Then

$$\Theta_T(z)=\Theta(z)=\Theta(0)^{*-1}J[J\Theta(0)^*\Theta(z)]=\mathfrak{U}^*K^{-1}J[\Theta(0)^*J_*\Theta(z)],$$

and from a relation of KUŽEL' [4], this is

$$=\mathfrak{U}^*K^{-1}J[J-Q(I-zT^*)^{-1}Q]=\mathfrak{U}^*\Theta_N(z)$$

where Θ_N is the characteristic function of the „node" $(\mathfrak{H},\mathfrak{H}_1;T,Q,J)$; [2].

As with contractions, if $\Theta_1=U\Theta V$, where U and V are constant isometries, then Θ_1 and Θ_2 are considered the same, as characteristic functions. Thus, given $B(z)$, one need only prove the existence of a T such that $B=U\Theta_T V$. In an appendix (Section 6) we have included our own proof that if S and T are (invertible) bounded operators and $\Theta_S=U\Theta_T V$, then S and T are unitarily equivalent.

1.4. Proofs of necessity. The proof of necessity in Theorem 2 follows easily from a relation of KUŽEL' [4]:

$$J_*-\Theta_T(z)J\Theta_T(w)^*=(1-z\bar{w})Q_*(I-zT^*)^{-1}(I-\bar{w}T)^{-1}Q_* \tag{3}$$

so that

$$b(w,z)=Q_*(I-zT^*)^{-1}(I-\bar{w}T)^{-1}Q_*$$

and this implies that $b(w,z)$ is a positive definite operator function.

To prove necessity in Theorem 1, we refer to the corresponding proof in [2]. Actually (i) is evident from (1); only (ii) needs attention. We have that $\Theta_T(z)=\mathfrak{U}^*\Theta_N(z)$, as in Section 1.3 above. Now, in the notation of [2, Section 6], it is easily seen that $\Theta_N(0)=K$ and so $H_0=K$, $U_0=I$. Thus

$$G(z)=[\mathfrak{U}^*+\Theta_T(z)]^{-1}[\mathfrak{U}^*-\Theta_T(z)]J=[I+\mathfrak{U}\Theta_T(z)]^{-1}[I-\mathfrak{U}\Theta_T(z)]J=$$
$$=[I+U_0^{-1}\Theta_N(z)]^{-1}[I-U_0^{-1}\Theta_N(z)]J=J\Omega(z)J$$

and the necessity part of Theorem 1 follows from that of [2, Theorem 6.1].

Finally, it is a pleasure to thank Chandler Davis, T. L. Kriete and Nhan Levan for discussions of various parts of this paper. I am also grateful to the referee for pointing out the paper [2] which I had overlooked.

2. Proof of Theorem 1, assuming Theorem 2

2.1. Integral representation. The function $G(z)$ is analytic for $|z|<1$ and Re $G(z)\geqq 0$ in $|z|<1$. Thus it follows from the operator-valued Riesz—Herglotz Theorem [1, p. 84] that there is a positive, operator valued measure dF such that

$$G(z)=\int_0^{2\pi}[e^{i\theta}+z]/[e^{i\theta}-z]\,dF(\theta). \tag{4}$$

Using this, we obtain

$$G(z)+G(w)^* = \int_0^{2\pi} \{[e^{i\theta}+z]/[e^{i\theta}-z]+[e^{-i\theta}+\bar{w}]/[e^{-i\theta}-\bar{w}]\}\, dF(\theta) =$$

$$= \int_0^{2\pi} [1-z\bar{w}][(1-e^{-i\theta}z)(1-\bar{w}e^{i\theta})]^{-1} dF(\theta).$$

Computing $G(z)+G(w)^*$ another way, using the definition of $G(z)$, we get

$$[\mathfrak{U}^*+B(z)][G(z)+G(w)^*][\mathfrak{U}+B(w)^*] =$$

$$= [\mathfrak{U}^*-B(z)]J[\mathfrak{U}+B(w)^*]+[\mathfrak{U}^*+B(z)]J[\mathfrak{U}-B(w)^*] =$$

$$= 2[\mathfrak{U}^*J\mathfrak{U}-B(z)JB(w)^*] = 2[J_*-B(z)JB(w)^*].$$

Combining this with the first expression for $G(z)+G(w)^*$ gives

$$b(w,z) = \frac{1}{2}[\mathfrak{U}^*+B(z)]\Big(\int_0^{2\pi} [(1-e^{-i\theta}z)(1-\bar{w}e^{i\theta})]^{-1} dF(\theta)\Big)[\mathfrak{U}+B(w)^*].$$

2.2. Operator integrals. We have thus far integrated only scalars against operator measures; we need now some notation for the integration of operator valued functions against them. Let $E(t)$ and $H(t)$ be operator-valued functions and $dF(t)$ a positive operator-valued measure on $[0, 2\pi]$. Suppose that $E(t)$ and $H(t)$ are the boundary values of operator-valued functions, holomorphic in $|z|\leqq 1$ or, more generally, that $H(t)$ is holomorphic and $E(t)$ is equal to a continuous (scalar-valued) function times an analytic function. Then, according to LANGER [5, Lemma 1″], the integral

$$\int_0^{2\pi} E(t)\big(dF(t)\big)H(t), \tag{5}$$

defined in terms of the convergence of Riemann sums of the form

$$\sum E(\xi_i)[F(x_i)-F(x_{i-1})]H(\xi_i),$$

exists. We shall use the integral (5) in case $E(t)$ is a linear combination of continuous functions times constant operator functions. Clearly one has:

i) For T a constant operator,

$$T\int_0^{2\pi} E(t)\,dF(t)H(t) = \int_0^{2\pi} TE(t)\,dF(t)H(t),$$

$$\Big[\int_0^{2\pi} E(t)\,dF(t)H(t)\Big]T = \int_0^{2\pi} E(t)\,dF(t)[H(t)T].$$

ii) $$\int_0^{2\pi} E(t)\,dF(t)E(t)^* \geqq 0.$$

It follows that we may rewrite the last integral in Section 2.1 as

$$b(w, z) = \frac{1}{2}\int_0^{2\pi} [\mathfrak{U}^* + B(z)][1 - e^{-i\theta} z]^{-1} dF(\theta)[\mathfrak{U} + B(w)^*][1 - e^{i\theta}\bar{w}]^{-1},$$

for $|z|, |w| < 1$.

2.3. Positive definiteness of $b(w, z)$ **in** $|z|, |w| < 1$**.** Let $z_1, \dots, z_n$ be complex numbers in D, $j = 1, \dots, n$ and let $x_1, \dots, x_n \in \mathfrak{H}_2$. Let $x_i^* x$ denote the linear functional $x \to (x, x_i)$ on $\mathfrak{H}_2$. We have

$$\sum_{i,j} (b(z_i, z_j)x_i, x_j) =$$

$$= \frac{1}{2}\sum_{i,j} x_j^* \int_0^{2\pi} (\mathfrak{U}^* + B(z_j))(1 - e^{-i\theta} z_j)^{-1} dF(\theta)(\mathfrak{U} + B(z_i)^*)(1 - e^{i\theta}\bar{z}_i)^{-1} x_i =$$

$$= \frac{1}{2}\sum_{i,j} \int_0^{2\pi} [(\mathfrak{U} + B(z_j)^*)(1 - e^{i\theta}\bar{z}_j)^{-1} x_j]^* dF(\theta)(\mathfrak{U} + B(z_i)^*)(1 - e^{i\theta}\bar{z}_i)^{-1} x_i =$$

$$= \frac{1}{2}\int_0^{2\pi} K(\theta)^* dF(\theta) K(\theta) \geqq 0,$$

where $K(\theta) = \sum_i (\mathfrak{U} + B(z_i)^*)(1 - e^{i\theta}\bar{z}_i)^{-1} x_i$. This proves $b(w, z)$ is a positive definite operator function in D.

Actually there is one difficulty here: the fact that the above sum has only been proved to be nonnegative. The reader will see, however, that he may easily divide out any elements of 0 norm in the proof of Theorem 2.

2.4. Positive definiteness of $b(w, z)$ **in general.** To complete the proof of the positive definiteness of $b(w, z)$, we shall prove that the integral representation (4) persists in $|z| > 1$. Once this is done, the proof in Sections 2.2 and 2.3 will apply *verbatim* to the case where one or both of the variables z, w has modulus > 1.

To extend (4) to $|z| > 1$, we shall prove that $G(z)$ satisfies

$$G(\bar{z}^{-1})^* = -G(z). \tag{6}$$

Since the right side of (4) obviously satisfies the analogous functional equation, the proof will be complete when (6) is verified.

To prove (6), we substitute in the definition of G and use the extension of $B(z)$ to $|z| > 1$.

$$G(\bar{z}^{-1})^* = \{[\mathfrak{U}^* + B(\bar{z}^{-1})]^{-1}[\mathfrak{U}^* J - B(\bar{z}^{-1})J]\}^* = [J\mathfrak{U} - JB(\bar{z}^{-1})^*][\mathfrak{U} + B(\bar{z}^{-1})^*]^{-1} =$$

$$= [J\mathfrak{U} - B(z)^{-1}J_*][\mathfrak{U} + JB(z)^{-1}J_*]^{-1} = [\mathfrak{U}B(z) - I]B(z)^{-1}J_* J_* B(z)[J\mathfrak{U}B(z) + J]^{-1} =$$

$$= [\mathfrak{U}B(z) - I][\mathfrak{U}B(z) + I]^{-1}J = [\mathfrak{U}B(z) + I]^{-1}[\mathfrak{U}B(z) - I]J =$$

$$= [B(z) + \mathfrak{U}^*]^{-1}[B(z) - \mathfrak{U}^*]J = -G(z)$$

and this completes the proof.

3. The Hilbert space H and the operator S

3.1. Definition of H. We now suppose only that $B(z)$ is analytic in a neighborhood D of 0, and that $b(w, z)$ is a positive definite operator function. Let $\tilde{D}$ denote the reflection of D, i.e. $\tilde{D}=\{\bar{z}^{-1}: z\in D\}$, and let H^0 be a set indexed by $(D\cup\tilde{D})\times \times\mathfrak{H}_2$; elements of H^0 are written $k_z f$ where $z\in D\cup\tilde{D}$ and $f\in\mathfrak{H}_2$. H^1 is defined to be the set of all finite linear combinations of elements of H^0.

We give H^1 the structure of a pre-Hilbert space by defining

$$(k_z f, k_w g) = \big(b(z, w)f, g\big) = (1-w\bar{z})^{-1}\big([J_* - B(w)JB(z)^*]f, g\big).$$

The positive-definiteness of $b(z, w)$ implies that this is a *bona fide* inner product on H^1. The Hilbert space H is the completion of H^1 in this norm.

3.2. The subspace $h_0 H$. Let R_0 be a real number, so large that $\{x: x\geqq R_0\}$ lies in $\tilde{D}$. Let $f\in\mathfrak{H}_1$. We claim that

$$h_0 f = \lim_{R_0\leqq R\to\infty} Rk_R J_* B(0)f \tag{7}$$

exists in H.

To prove the claim, pick $M, R\geqq R_0$ and compute

$$\|Rk_R J_* B(0)f - Mk_M J_* B(0)f\|^2 =$$
$$= R^2\big(b(R, R)J_* B(0)f, J_* B(0)f\big) - 2RM\,\mathrm{Re}\,\big(b(R, M)J_* B(0)f, J_* B(0)f\big) +$$
$$+ M^2\big(b(M, M)J_* B(0)f, J_* B(0)f\big) =$$
$$= [R^2(1-R^2)^{-1} - 2RM(1-RM)^{-1} + M^2(1-M^2)^{-1}]\big(J_* B(0)f, B(0)f\big) -$$
$$- R^2(1-R^2)^{-1}\big(JB(R)^* J_* B(0)f, B(R)^* J_* B(0)f\big) +$$
$$+ 2RM(1-RM)^{-1}\big(JB(R)^* J_* B(0)f, B(M)^* J_* B(0)f\big) -$$
$$- M^2(1-M^2)^{-1}\big(JB(M)^* J_* B(0)f, B(M)^* J_* B(0)f\big).$$

Elementary calculus shows that the first term tends to 0 as $M, R\to\infty$. Since $B(R)$ tends uniformly to $B(\infty)=J_* B(0)^{*-1}J$ as $R\to\infty$, it is not hard to see that the sum of the last three terms tends to 0 and we have proved the existence of the limit (7).

Now we want to find $(h_0 f, k_z g)$ for $z\in D\cup\tilde{D}$. All this takes is an application of (7). In fact

$$(h_0 f, k_z g) = \lim_{R\to\infty} R\big(k_R J_* B(0)f, k_z g\big) =$$
$$= \lim_{R\to\infty} R(1-Rz)^{-1}\big([J_* - B(z)JB(R)^*]J_* B(0)f, g\big) =$$
$$= \lim_{R\to\infty} (R^{-1}-x)^{-1}\big([B(0) - B(z)JB(R)^* J_* B(0)]f, g\big) =$$
$$= (-z)^{-1}\big([B(0) - B(z)JB(\infty)^* J_* B(0)]f, g\big)$$

and we have

$$(8)\qquad (h_0 f, k_z g) = z^{-1}([B(z)-B(0)]f, g).$$

3.3. The operator S. We define an operator S on the dense subset H^1 of H by

$$Sk_z f = \bar{z}k_z f - h_0 JB(z)^* f,$$

for $z\in D\cup\tilde{D}$ and $f\in\mathfrak{H}_2$. In part 4 of this paper, we shall show that S extends (uniquely) to a bounded operator (also denoted S) on H and that the operator S^* has $B(z)$ as its characteristic operator function.

4. Boundedness of S

4.1. Proposition. *For $z, w\in D\cup\tilde{D}$, S satisfies*

$$(Sk_z f, k_w g) = (k_z f, \bar{w}^{-1}[k_w - k_0]g).$$

Thus, the domain of the adjoint S^* of S contains H^1 (and hence is dense in H) and satisfies

$$(9)\qquad S^* k_z f = \bar{z}^{-1}[k_z - k_0]f.$$

In particular, S has a closure.

Proof.

$$(Sk_z f, k_w g) = (\bar{z}k_z f - h_0 JB(z)^* f, k_w g) =$$
$$= \bar{z}(1-\bar{z}w)^{-1}([J_* - B(w)JB(z)^*]f, g) - w^{-1}([B(w)-B(0)]JB(z)^* f, g) =$$
$$= w^{-1}[(1-\bar{z}w)^{-1}-1]([J_* - B(w)JB(z)^*]f, g) - w^{-1}([B(w)-B(0)]JB(z)^* f, g) =$$
$$= w^{-1}(k_z f, k_w g) - w^{-1}([J_* - B(w)JB(z)^*]f, g) -$$
$$- w^{-1}([B(w)JB(z)^* - B(0)JB(z)^*]f, g) =$$
$$= w^{-1}(k_z f, k_w g) - w^{-1}([J_* - B(0)JB(z)^*]f, g) = w^{-1}(k_z f, k_w g) - w^{-1}(k_z f, k_0 g).$$

This proves the first part of the proposition. All the other parts follow at once.

Henceforth, S will denote the closure of the operator S in Section 3.3 above.

4.2. Proposition. *For $z\in D\cup\tilde{D}$ and $f\in\mathfrak{H}_2$,*

$$(I-SS^*)k_z f = \bar{z}^{-1}h_0 J[B(z)^* - B(0)^*]f.$$

Thus $I-SS^*=0$ on $(h_0\mathfrak{H}_1)^\perp$, so that $(h_0\mathfrak{H}_1)^\perp$ is contained in the domain of S^*.

Proof.

$$(I-SS^*)k_z f = k_z f - S\bar{z}^{-1}[k_z - k_0]f =$$
$$= k_z f - \bar{z}^{-1}[\bar{z}k_z f - h_0 JB(z)^* f + h_0 JB(0)^* f] = \bar{z}h_0 J[B(z)^* - B(0)^*]f.$$

Again, all the other claims are obvious.

4.3. Proposition. *$h_0\mathfrak{H}_1$ is contained in the domain of S^* and*

$$S^*h_0f = -k_0J_*B(0)f.$$

Proof. Fix $f\in\mathfrak{H}_1$. We know h_0f is the limit of $Rk_RJ_*B(0)f$. Let us compute

$$\lim_{R\to\infty} S^*Rk_RJ_*B(0)f = \lim_{R\to\infty} RR^{-1}[k_R-k_0]J_*B(0)f =$$
$$= \lim_{R\to\infty} [k_R-k_0]J_*B(0)f = -k_0J_*B(0)f,$$

where $k_RJ_*B(0)f$ converges to 0 since the limit in (7) exists. Now the proposition follows from the fact that S^* is closed.

4.4. Lemma. *The following three operators are isometries:*

$$L_1: \mathfrak{H}_2\to k_0\mathfrak{H}_2, \quad L_2: \mathfrak{H}_1\to h_0\mathfrak{H}_1, \quad L_3: k_0\mathfrak{H}_2\to k_0\mathfrak{H}_2$$

where

$$L_1(|J_*-B(0)JB(0)^*|^{1/2}f) = k_0f,$$
$$L_2(f) = -h_0|J-B(0)^*J_*B(0)|^{-1/2}f.$$
$$L_3(k_0f) = k_0J_*f.$$

Proof. For $f\in\mathfrak{H}_2$,

$$\||J_*-B(0)JB(0)^*|^{1/2}f\|^2 = ([J_*-B(0)JB(0)^*]f,f) = \|k_0f\|^2,$$

so L_1 is an isometry.

For L_2, we have to determine the norm of h_0g, for $g\in\mathfrak{H}_1$. We have

$$\|h_0g\|^2 = \lim_{R\to\infty} \|Rk_RJ_*B(0)g\|^2 =$$
$$= \lim_{R\to\infty} R^2(1-R^2)^{-1}([J_*-B(R)JB(R)^*]J_*B(0)g, J_*B(0)g) =$$
$$= ([J-B(0)^*J_*B(0)]g, g).$$

Now we can compute

$$\|L_2f\|^2 = ([J-B(0)^*J_*B(0)][J-B(0)^*J_*B(0)]^{-1/2}f, [J-B(0)^*J_*B(0)]^{-1/2}f) = \|f\|^2$$

so L_2 is an isometry.

Finally, let $f\in\mathfrak{H}_2$. We have

$$\|k_0J_*f\|^2 = ([J_*-B(0)JB(0)^*]J_*f, J_*f) =$$
$$= ([J_*-J_*B(0)JB(0)^*J_*]f,f) = ([J_*-B(0)JB(0)^*]f,f) = \|k_0f\|^2,$$

and this completes the proof.

4.5. Proposition. *For $F\in h_0\mathfrak{H}_1$, we have*

$$L_3L_1B(0)L_2^{-1}F = S^*F. \tag{10}$$

In particular the domain of S^* contains the closure of $h_0\mathfrak{H}_1$, so S^* (and hence S) is bounded.

Proof. We have

$$B(0)L_2^{-1}h_0f = -B(0)[J-B(0)^*J_*B(0)]^{1/2}f = -[J_*-B(0)JB(0)^*]^{1/2}B(0)f$$

and so

$$L_3L_1B(0)L_2^{-1}h_0f = -L_3k_0B(0)f = -k_0J_*B(0)f.$$

Thus (10) follows from Proposition 4.3.

Now from Lemma 4.4, the operators L_1, L_2^{-1} and L_3 are bounded, and so the boundedness of S^* on the closure of $h_0\mathfrak{H}_1$ follows from (10). This and Proposition 4.2 show that the domain of S^* contains all of H, and the proposition follows from the Closed Graph Theorem.

5. Characteristic operator function of S^*

5.1. Proposition. *For* $f\in\mathfrak{H}_2$,

$$(I-S^*S)k_0f = k_0[I-B(0)B(0)^*]f, \quad |I-S^*S|k_0f = k_0[J_*-B(0)JB(0)^*]f,$$
$$\operatorname{sgn}(I-S^*S)k_0f = k_0J_*f, \quad |I-S^*S|^{1/2}k_0f = k_0[J_*-B(0)JB(0)^*]^{1/2}f.$$

Proof. The first relation is immediate, since

$$(I-S^*S)k_0f = k_0f+S^*h_0JB(0)^*f = k_0f-k_0J_*B(0)JB(0)^*f = k_0[I-B(0)B(0)^*]f,$$

by Proposition 4.3.

Lemma 4.4 shows L_3 is isometric, and a slight modification of its proof shows L_3 is self adjoint. The above computation shows $L_3(I-S^*S)\geqq 0$ on $k_0\mathfrak{H}_2$ and this proves the second and third relations. The last relation follows from the fact that the operator on the right is positive and its square is $|I-S^*S|$.

5.2. Proposition. *For* $f\in\mathfrak{H}_1$,

$$(I-SS^*)h_0f = h_0[I-B(0)^*B(0)]f, \quad |I-SS^*|h_0f = h_0[J-B(0)^*J_*B(0)]f,$$
$$\operatorname{sgn}(I-SS^*)h_0f = h_0Jf, \quad |I-SS^*|^{1/2}h_0f = h_0[J-B(0)^*J_*B(0)]^{1/2}f.$$

For $0\neq z\in D\cup\tilde{D}$ *and* $f\in\mathfrak{H}_2$,

$$|I-SS^*|^{1/2}k_zf = \bar{z}^{-1}h_0[J-B(0)^*J_*B(0)]^{-1/2}[B(z)^*-B(0)^*]f.$$

Proof. The first four relations follow from a computation similar to the proof of Proposition 5.1, which will be omitted. The last relation follows from Proposition 4.2 and the fact that

$$|I-SS^*|^{1/2}k_zf = |I-SS^*|^{1/2}[\operatorname{sgn}(I-SS^*)](I-SS^*)k_zf.$$

One more lemma, and we shall be able to compute the characteristic operator function of S.

5.3. Lemma. *For* $\bar{z}\in D\cup\tilde{D}$, *we have* $(I-\bar{z}S^*)^{-1}k_0 f=k_z f$.

Proof. $(I-\bar{z}S^*)k_z f=k_z f-\bar{z}\bar{z}^{-1}[k_z-k_0 f]=k_0 f$.

5.4. Theorem. *Up to a constant, isometric multiple, we have* $\Theta_{S*}(z)=B(z)$.

Proof. We find $\Theta_S(z)$ easier to compute. The theorem will follow from a result of Kužel' [4] which states $\Theta_{S*}(z)=\Theta_S(\bar{z})^*$.

We need to know what the range of $I-S^*S$ is, and a computation copied from Proposition 4.2 (using Proposition 4.3) shows that the closures of $R(I-S^*S)$ and $k_0\mathfrak{H}_2$ coincide. What we want is therefore

$$\Theta_S(z)k_0 f = S\,\mathrm{sgn}\,(I-S^*S)k_0 f-z|I-SS^*|^{1/2}(I-zS^*)^{-1}|I-S^*S|^{1/2}k_0 f.$$

We first use Proposition 5.1 to get

$$\Theta_S(z)k_0 f = Sk_0J_*f-z|I-SS^*|^{1/2}(I-zS^*)^{-1}k_0[J_*-B(0)JB(0)^*]^{1/2}f.$$

Now Lemma 5.3 gives

$$\Theta_S(z)k_0 f = Sk_0J_*f-z|I-SS^*|^{1/2}k_{\bar{z}}[J_*-B(0)JB(0)^*]^{1/2}f =$$
$$= Sk_0J_*f-h_0|J-B(0)^*J_*B(0)|^{-1/2}[B(\bar{z})^*-B(0)^*][J_*-B(0)JB(0)^*]^{1/2}f$$

by Proposition 5.2. Now we have

$$h_0|J-B(0)^*J_*B(0)|^{-1/2}B(0)^*[J_*-B(0)JB(0)^*]^{1/2}f = h_0B(0)^*f = -Sk_0J_*f,$$

and so our estimate of Θ_s becomes

$$\Theta_S(z)k_0 f = -h_0|J-B(0)^*J_*B(0)|^{-1/2}B(\bar{z})^*[J_*-B(0)JB(0)^*]^{1/2}f = L_2B(\bar{z})^*L_1k_0 f$$

and since L_1 and L_2 are (constant) isometries, it follows that $\Theta_S(z)$ and $B(\bar{z})^*$ are the same, considered as characteristic functions (see Section 6 below).

6. Appendix

6.1. Uniqueness of characteristic function. Let S and T be invertible operators on Hilbert spaces H_1 and H_2, and assume S and T have no reducing subspaces on which they are unitary.

Theorem. *If there are constant unitaries U and V such that $\Theta_S=U\Theta_T V$, then S and T are unitarily equivalent.*

Proof. By the computation of Kužel', which we used in Section 1.4, we have

$$(1-z\bar{w})^{-1}[J_*-\Theta_T(z)J\Theta_T(w)^*] = Q_*(I-zT^*)^{-1}(I-\bar{w}T)^{-1}Q_* \tag{11}$$

for z and w in $D\cup\tilde{D}$, for D some neighborhood of 0 (where $(I-\bar{z}T)^{-1}$ and $(I-\bar{z}S)^{-1}$

exist). Now let K, K_*, P, P_* denote the analogues of the operators J, J_*, Q, Q_* corresponding to S. We have $K=V^*JV$, $K_*=UJ_*U^*$, and so

$$K_*-\Theta_S(z)K\Theta_S(w)^* = U[J_*-\Theta_T J\Theta_T^*]U^*.$$

Combining this with (11), we have

$$P_*(I-zS^*)^{-1}(I-\bar{w}S)^{-1}P_* = UQ_*(I-zT^*)^{-1}(I-\bar{w}T)^{-1}Q_*U^*.$$

For $g, h\in\bar{R}(P_*)$, we therefore have

$$((I-\bar{w}S)^{-1}P_*g, (I-\bar{z}S)^{-1}P_*h) = ((I-\bar{w}T)^{-1}Q_*U^*g, (I-\bar{z}T)^{-1}Q_*U^*h)$$

whence it follows that the map

(12) $$\mathfrak{U}: (I-\bar{w}S)^{-1}P_*g \mapsto (I-\bar{w}T)^{-1}Q_*U^*g, \quad w\in D\cup\tilde{D},$$

is an isometry, from some subspace of H_1 to some subspace of H_2. If we can prove that the subspace M of H_1 of elements of the form $f(S)P_*g$, where $g\in\bar{R}(P_*)$, and f is a rational function with poles in $D\cup\tilde{D}$, is dense in H_1, then we will have that $\mathfrak{U}$ has a dense domain.

To prove M is dense in H_1, we shall prove that $\bar{M}$ reduces S and that S is unitary on $M^\perp$. M is certainly S invariant. To prove M is S^* invariant, note

$$S^*f(S)P_*g = S^{-1}[(SS^*-I)f(S)P_*g]+S^{-1}f(S)P_*g.$$

Let $h = |I-SS^*|^{1/2}K_*f(S)P_*g$, and we have

$$S^*f(S)P_*g = -S^{-1}P_*h+S^{-1}f(S)P_*g\in M.$$

Now we want to show that M contains the range of $I-S^*S$, i.e. $(I-S^*S)H_1$. To do this, let $f=(I-S^*S)g$. We have

$$f = S^{-1}(I-SS^*)Sg = S^{-1}P_*h \quad\text{where}\quad h = P_*KSg\in R(P_*).$$

Now M reduces S and contains $R(P)$ and $R(P_*)$, so S must be unitary on $M^\perp$; i.e. $M^\perp=\{0\}$. A similar argument shows the range of $\mathfrak{U}$ is dense in H_2, so $\mathfrak{U}$ is unitary.

To complete the proof of the theorem, we must show $S=\mathfrak{U}^*T\mathfrak{U}$. To do this, just refer back to (12). It implies

$$\mathfrak{U}f(S)P_*g = f(T)Q_*U^*g.$$

Replacing $f(t)$ by $tf(t)$ here, we get

$$\mathfrak{U}Sf(S)P_*g = Tf(T)Q_*U^*g = T\mathfrak{U}f(S)P_*g,$$

so that $\mathfrak{U}Sx=T\mathfrak{U}x$ for $x\in M$, and this completes the proof.

A corollary of the theorem is that any bounded, invertible operator is unitarily equivalent to the operator S constructed in Section 3.3.

References

[1] R. BEALS, *Topics in operator theory*, Chicago Lectures in Mathematics, University of Chicago Press (Chicago, 1971).
[2] V. M. BRODSKII, I. C. GOHBERG and M. G. KREIN, On characteristic functions of invertible operators, *Acta Sci. Math.*, **32** (1971), 141—164.
[3] CH. DAVIS and C. FOIAŞ, Operators with bounded characteristic function and their J-unitary dilation, *Acta Sci. Math.*, **32** (1971), 127—139.
[4] O. V. KUŽEL', The characteristic operator function of an arbitrary bounded operator, *A. M. S. Translations* (2), **90** (1970), 225—228.
[5] H. LANGER, Über die Methode der richtenden Funktionale von M. G. Krein, *Acta Math. Acad. Sci. Hung.*, **21** (1970), 207—224.
[6] B. SZ.-NAGY and C. FOIAŞ, *Harmonic analysis of operators on Hilbert space*, Akadémiai Kiadó (Budapest, 1970).

(Received November 4, 1972, revised March 5, 1973)

On weak convergence of the empirical process with random sample size

By SÁNDOR CSÖRGŐ in Szeged

Introduction. Let $U_1, U_2, \ldots, U_n$ be a random sample taken from the uniform distribution on [0, 1] and let $F_n(t)$ be their empirical distribution function and $Y_n(t)=\sqrt{n}\,(F_n(t)-t)$ the empirical process. If $D=D_{[0,1]}$ denotes the space of functions on [0, 1] having discontinuities only of the first kind, endowed with the Skorohod topology (see [4]) and $\mathscr{D}$ denotes the σ-algebra generated by the open sets of this topology, then $Y_n(t)$ (defined on some probability space $\{\Omega, \mathscr{B}, P\}$) is a random function of $\{D, \mathscr{D}\}$. Here and throughout this paper we use the standard terminology and notation of BILLINGSLEY's book [4] (see also [10] in these *Acta*). As well known, [4] or [12], the empirical process Y_n weakly converges (as $n\to\infty$) to the Brownian Bridge W° with covariance function $s(1-t)$ for $0\leqq s\leqq t\leqq 1$:

$$Y_n \xrightarrow{\mathscr{D}} W^\circ. \tag{1}$$

Let us consider the following functionals of $Y_n(t)$:

$$\sup_{0\leqq t\leqq 1} |Y_n(t)| \text{ — Kolmogorov's statistic,} \tag{2}$$

$$\sup_{0\leqq t\leqq 1} Y_n(t) \text{ — Smirnov's statistic,} \tag{3}$$

$$\int_0^1 [Y_n(t)]^2 f(t)\, dt \text{ — Cramér—von Mises statistic,} \tag{4}$$

$$\sup_{\alpha\leqq t\leqq\beta} \frac{|Y_n(t)|}{g(t)} \tag{5}$$

Anderson and Darling—Rényi statistics,

$$\sup_{\alpha\leqq t\leqq\beta} \frac{Y_n(t)}{g(t)} \tag{6}$$

where $f(t)$ and $g(t)$ are some non-negative weight functions, and on the interval $[\alpha, \beta]$, $[\alpha, \beta]\subseteqq[0, 1]$, the function $g(t)$ is bounded away from zero. ANDERSON and DARLING [1] particularly dealt with $g(t)=\sqrt{t(1-t)}$ and RÉNYI [17] with $g(t)=t$ and $1-t$. Since all the above functionals are continuous in the Skorohod topology, we have, as a consequence of relation (1), that the distributions of the statistics (2)—(6) converge to the distributions of the appropriate functionals of W° (cf. [12] and [8]).

Let now, for each n, ν_n be a positive valued random variable defined on the same probability space $\{\Omega, \mathcal{B}, P\}$. In [16], PYKE explains the importance of dealing with the random sample size empirical process $Y_{\nu_n}(t)$, that is, when at each given time n, the size of the sample is the random ν_n. He proves that if the variables ν_n are such that ν_n/n converges in probability (denoted from now on by $\xrightarrow{P}$) to 1, then

$$(7) \qquad Y_{\nu_n} \xrightarrow{\mathcal{D}} W^\circ.$$

As to the behaviour of ν_n, in [9] we used the more general condition that $\nu_n/n \xrightarrow{P} \nu$, where ν is an arbitrary positive random variable. There, in [9], we constructed a partial sum type process $X_n(t)$, for which the distributions of

$$\sup_{0 \leq t \leq 1} |X_{\nu_n}(t)| \quad \text{and} \quad \sup_{0 \leq t \leq 1} X_{\nu_n}(t)$$

for large n are the same as those of

$$\sup_{0 \leq t \leq 1} |Y_{\nu_n}(t)| \quad \text{and} \quad \sup_{0 \leq t \leq 1} Y_{\nu_n}(t),$$

and proved that $X_{\nu_n} \xrightarrow{\mathcal{D}} W^\circ$. That is, we proved that the random sample size KOLMOGOROV—SMIRNOV statistics of (2) and (3) have the same limit distributions as those of the originals. The aim of the present paper is to prove directly relation (7) under this latter condition on ν_n, i.e. $\nu_n/n \xrightarrow{P} \nu$, which is also the most frequently used condition in the theory of limit distributions of sequences of random variables with random indices (cf. [13]) in general.

The Results. Theorem 1. *If Y_n denotes the empirical process and W° the Brownian Bridge, and if the sequence of positive integer valued random variables ν_n is such that*

$$\frac{\nu_n}{n} \xrightarrow{P} \nu,$$

where ν is a positive random variable, then

$$Y_{\nu_n} \xrightarrow{\mathcal{D}} W^\circ.$$

A considerable part of the literature dealing with the empirical process is devoted to finding representations (in distribution) of the empirical distribution function which would easier lend themselves to analysis. (See e.g. RÉNYI [17], BREIMAN [6], BRILLINGER [7], PYKE [16], MÜLLER [15]). One of these can be described as follows. Let $Z_k = \xi_1 + \ldots + \xi_k$ be the partial sum sequence of independent exponential random variables ξ_n with mean 1, and let $U_1^{(n)}, U_2^{(n)}, \ldots, U_n^{(n)}$ denote the order statistics of the sample $U_1, \ldots, U_n$ of the Introduction. Then the joint distribution of $U_1^{(n)}, U_2^{(n)}, \ldots, U_n^{(n)}$ is the same as that of $Z_1/Z_{n+1}, \ldots, Z_n/Z_{n+1}$ (see BREIMAN [6]). Consequently, if we define the random functions (of $\{D, \mathscr{D}\}$)

$$G_n(x) = \begin{cases} 0, & \dfrac{Z_1}{Z_{n+1}} > x, \\ \dfrac{k}{n}, & \dfrac{Z_k}{Z_{n+1}} \leqq x < \dfrac{Z_{k+1}}{Z_{n+1}}, \quad k = 1, \ldots, n-1, \\ 1, & \dfrac{Z_n}{Z_{n+1}} \leqq x, \end{cases}$$

and $X_n(t) = \sqrt{n}\,(G_n(t) - t)$, then, for each n, the process $X_n(t)$ (the BREIMAN—BRILLINGER representation of the empirical process) has the distribution of the empirical process $Y_n(t)$. The weak convergence of $Y_n(t)$ can be easily proved using the representations $X_n(t)$, while that of $Y_{\nu_n}(t)$ cannot be done the same way. However, the weak convergence of $X_{\nu_n}(t)$ itself is, perhaps, of some interest. In fact, the following theorem is true.

Theorem 2. *If $X_n(t)$ is as above and ν_n, ν, W° are as in Theorem 1, then*

$$X_{\nu_n} \xrightarrow{\mathscr{D}} W^\circ.$$

For the proof of our theorems we will need the following results.

Lemma 1 (Theorem 3 of GUIAŞU [13]). *Suppose that the sequence ν_n and ν are the same as in Theorem 1, and further suppose that the sequence ξ_n of random variables satisfies the following two conditions:*

(i) *For every event A in the σ-algebra, $\mathscr{K}_\nu$, generated by ν, $(P\{A\}>0)$,*

$$\lim_{n\to\infty} P\{\xi_n \leqq a_n x \mid A\} = F(x), \tag{8}$$

at every continuity point x of the distribution function F. Here a_n is some sequence of positive constants.

(ii) *For every positive ε and η and every A in $\mathscr{K}_\nu(P\{A\}>0)$, there exist a positive real number $c=c(\varepsilon, \eta)$ and a natural number $n_0 = n_0(\varepsilon, \eta, A)$ such that for every $n \geqq n_0$*

$$P\{\max_{n(1-c) \leqq m \leqq n(1+c)} |\xi_m - \xi_n| > a_n \varepsilon \mid A\} < \eta.$$

Then

$$\lim_{n\to\infty} P\left\{\frac{\xi_{\nu_n}}{a_{\nu_n}} \leqq x\right\} = F(x)$$

at every continuity point x of F.

We remark that Lemma 1, in the above form, differs slightly from the original Theorem 3 of GUIAŞU. The difference is that in his theorem $a_n=1$ for each n. The presence and the use of the sequence a_n in the sense of the above Lemma 1, i.e. a_n not being absorbed into the sequence ξ_n, is needed for the present application in order to make it easier to check for the fulfillment of condition (C6) of GUIAŞU [13], which is replaced here by condition (ii), the original condition of ANSCOMBE [2] if we take $A=\Omega$. For an easy proof adapt the technique of BARNDORFF—NIELSEN [3] to complete the proof of GUIAŞU's Theorem 3 thus modified.

Lemma 2 (Lemma 3 of BLUM, HANSON and ROSENBLATT [5]). *Let η_n be a sequence of independent random variables, further let k_n and m_n, $k_n \leqq m_n$, be two (not constant) sequences of natural numbers. If for each n, A_n is an event depending only on the random variables $\eta_{k_n}, \dots, \eta_{m_n}$ then for every event A, having positive probability:*

$$\limsup_n P\{A_n|A\} = \limsup_n P\{A_n\}.$$

At a crucial stage, a recent and very important result of J. KIEFER [14], is going to be used in the proof of Theorem 1. His result concerns the representation of the sample distribution function by a SKOROHOD-type embedding in the appropriate two dimensional Gaussian process. Let $\xi(.,.)$ be a Gaussian process on $[0, 1]\times[0, \infty)$ with continuous sample functions, zero expectation, and covariance function

$$E(\xi(s_1, t_1)\xi(s_2, t_2)) = \min(t_1, t_2)[\min(s_1, s_2) - s_1 s_2],$$

so that there are independent increments in t and a Brownian bridge in s for fixed t.

Theorem A (J. KIEFER [14]). *ξ can be defined on a probability space on which there is a defined a random function T: $[0, 1]\times[0, \infty)$ such that $\xi(s, T(s, n))$ has the same joint distribution as $\sqrt{n}\, Y_n(s)$ and, as $n\to\infty$,*

$$\frac{1}{\sqrt{n}} \sup_{0\leqq s\leqq 1} \left|\xi(s, T(s, n)) - \xi(s, n)\right| = O\left(n^{-1/6}(\log n)^{2/3}\right)$$

with probability 1.

From now on we assume that the probability space $\{\Omega, \mathcal{B}, P\}$ of the Introduction is already that of Theorem A.

Proof of Theorem 1. To verify that Y_{ν_n} converges weakly to W°, we have to show two things (see Theorem 15.1 in [4] or Theorem A in [10] in these *Acta*): 1) The finite dimensional distributions of Y_{ν_n} converge to those of W°, and 2) The sequence Y_{ν_n} is tight.

Ad 1) As a consequence of relation (1) and the Cramér—Wold device (p. 49 in [4]), if we take the time points $t_1, \dots, t_k$ and the real numbers $c_1, \dots, c_k$

($k=1, 2, \ldots$, fixed), then

$$\tilde{R}_n = \sum_{i=1}^{k} c_i Y_n(t_i) \xrightarrow{\mathcal{D}} \sum_{i=1}^{k} c_i W^\circ(t_i) = R.$$

Naturally, $\xrightarrow{\mathcal{D}}$ here stands for convergence in distribution on the real line. By the Cramér—Wold device again, it is enough to show that

$$\tilde{R}_{\nu_n} \xrightarrow{\mathcal{D}} R. \tag{9}$$

Let us introduce the function

$$\psi(x) = \begin{cases} 1, & \text{if } x \geqq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Clearly

$$Y_n(t_i) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (\psi(t_i - U_j) - t_i).$$

The random variables $\psi_{ji} = \psi(t_i - U_j) - t_i$, $i=1, \ldots, k$; $j=1, 2, \ldots$ are independent, $E(\psi_{ji})=0$, $E^2(\psi_{ji})=t_i(1-t_i)$. To verify relation (9), we show now, that the sequence

$$\tilde{R}_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} c_i \sum_{j=1}^{n} \psi_{ji} = \frac{R_n}{\sqrt{n}}$$

satisfies the conditions of Lemma 1. As to condition (i) we first remember a well known result of Rényi [18], which states that the necessary and sufficient condition for relation (8) to hold (in the case of $R_n/\sqrt{n}$ instead fo ξ_n/a_n), not only for A's in $\mathcal{K}_\nu$ but for all A in $\mathcal{B}$ (that is, that the sequence $R_n/\sqrt{n}$ should be mixing), is that it should hold for each A of the form

$$A_r = \{R_r \leqq \sqrt{r}\, x\}, \quad r = 1, 2, \ldots.$$

To show this, put

$$\frac{1}{\sqrt{n}} R_n^* = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} c_i \sum_{j=p_n}^{n} \psi_{ji},$$

where p_n is a sequence of natural numbers tending to infinity, but so slowly that $p_n/n \to 0$ (e.g. $p_n = [\log n]$). It is obvious, via the Chebishev inequality, that

$$\frac{1}{\sqrt{n}} (R_n - R_n^*) \xrightarrow{P} 0, \quad \text{as} \quad n \to \infty.$$

Thus it is enough to show that, as $n \to \infty$,

$$P\{R_n^* \leqq \sqrt{n}\, x \mid A_r\} \to P\{R \leqq x\}, \tag{10}$$

where r is fixed and not less than an integer n_0, for which it is true that if $n \geqq n_0$, then $P\{A_n\}>0$. But now relation (10) holds, because, if n is so large that $p_n > r \geqq n_0$, then the random variables R_n^* and R_r are independent and thus the conditional probability becomes unconditional.

Turning to the verification of condition (ii) of Lemma 1 we fix the positive ε and η arbitrarily. Clearly

$$
\text{(11)} \qquad P\{\max_{n(1-c)\leqq m \leqq n(1+c)} |R_m - R_n| > \sqrt{n}\,\varepsilon \mid A\} \leqq
$$

$$
\leqq \sum_{i=1}^{k} P\left\{\max_{n(1-c)\leqq m\leqq n}\left|\sum_{j=1}^{m}\psi_{ji} - \sum_{j=1}^{n}\psi_{ji}\right| > \sqrt{n}\,\frac{\varepsilon}{2k|c_i|}\,\Big|\, A\right\} +
$$

$$
+ \sum_{i=1}^{k} P\left\{\max_{n\leqq m\leqq n(1+c)}\left|\sum_{j=1}^{m}\psi_{ji} - \sum_{j=1}^{n}\psi_{ji}\right| > \sqrt{n}\,\frac{\varepsilon}{2k|c_i|}\,\Big|\, A\right\}.
$$

Now Lemma 2 ensures the existence of a natural number n_1 (which may depend on A) so that if $n \geqq n_1$ then the right-hand side of inequality (11) is not larger than (putting $\theta_i = \dfrac{\varepsilon}{2k|c_i|}$)

$$
\text{(12)} \qquad \frac{\eta}{2} + \sum_{i=1}^{k} P\left\{\max_{n(1-c)\leqq m\leqq n}\left|\sum_{j=m+1}^{n}\psi_{ji}\right| > \sqrt{n}\,\theta_i\right\} +
$$

$$
+ \sum_{i=1}^{k} P\left\{\max_{n\leqq m\leqq n(1+c)}\left|\sum_{j=n+1}^{m}\psi_{ji}\right| > \sqrt{n}\,\theta_i\right\}.
$$

Using the KOLMOGOROV inequality, we can now chosoe an integer n_0 $(n_0 \geqq n_1)$ and a real number c (which c does not depend on A) so that for this c and $n \geqq n_0$ the value of formula (12) is less than η. Thus by Lemma 1, the finite dimensional distributions of $Y_{\nu_n}(t)$ converge to those of W°.

Ad 2) As $Y_{\nu_n}(0)=0$, for the tightness of the sequence Y_{ν_n} it is enough to prove (cf. Theorem 15.5 of BILLINGSLEY [4] or Chapter 9, § 8 of GIHMAN and SKOROHOD [12]) that for each positive ε we have

$$
\text{(13)} \qquad \lim_{\delta\to 0}\varlimsup_{n\to\infty} P\{\sup_{|s-t|<\delta} |Y_{\nu_n}(s) - Y_{\nu_n}(t)| > \varepsilon\} = 0.
$$

Let θ and ϱ be arbitrary positive numbers and choose a and b, $0<a<b$, so that $P\{a<\nu\leqq b\}>1-\theta$, $\varrho<a$. Since

$$
\lim_{n\to\infty} P\left\{\left|\frac{\nu_n}{n} - \nu\right| > \varrho\right\} = 0,
$$

the left hand side of (13) is bounded above by

$$
\theta + \lim_{\delta\to 0}\varlimsup_{n\to\infty} P\{\max_{n(a-\varrho)\leqq m\leqq n(b+\varrho)} \sup_{|s-t|<\delta} |Y_m(s) - Y_m(t)| > \varepsilon\}.
$$

According to Theorem A we can replace here $Y_m(\cdot)$ by $1/\sqrt{m}\,\xi(\cdot, m)$ on neglectins terms of order $m^{-1/6}(\log m)^{2/3}$ with probability 1. The process $\xi(s, n)$, in turn, ig equivalent in distribution to $X(s, n)-sX(1, n)$, where $X(s, t)$ is a continuous Gaussian process on $[0, 1]\times[0, \infty)$ with zero expectations and independent increments in both directions:

$$E\big(X(s_1, t_1)X(s_2, t_2)\big) = \min(s_1, s_2)\min(t_1, t_2).$$

With this replacement our last expression is less than or equal to

$$\lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\, P\left\{\max_{n(a-\varrho)\le m\le n(b+\varrho)}\frac{1}{\sqrt{m}}\sup_{|s-t|<\delta}|sX(1, m)-tX(1, m)| > \frac{\varepsilon}{2}\right\}+$$

$$+\lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\sum_{m=[n(a+\varrho)]}^{[n(b+\varrho)]} P\left\{\sup_{|s-t|<\delta}\frac{1}{\sqrt{m}}|X(s, m)-X(t, m)| > \frac{\varepsilon}{2}\right\}+\theta.$$

The first term here is bounded above by

$$\lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\, P\left\{\max_{1\le m\le n(b+\varrho)}\delta|X(1, m)| > \frac{\varepsilon}{2}\sqrt{n(a+\varrho)}\right\}\le$$

$$\le\lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\frac{[n(b+\varrho)]\delta^2}{n(a-\varrho)\varepsilon^2} = 0,$$

with KOLMOGOROV's inequality, while the second one by

$$(14)\quad \lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\sum_{m=[n(a-\varrho)]}^{[n(b+\varrho)]}\sum_{i=1}^{\left[\frac{1}{\delta}\right]+1} P\left\{\sup_{(i-1)\delta<s<i\delta}\big|X(s, m)-X((i-1)\delta, m)\big| > \frac{\varepsilon}{6}\sqrt{m}\right\}.$$

Using Theorem 2.1 of DOOB [11, p. 392], the probability in (14) is equal to

$$2P\left\{\big|X(i\delta, m)-X((i-1)\delta, m)\big| > \frac{\varepsilon}{6}\sqrt{m}\right\} =$$

$$= 4\int_{\varepsilon\sqrt{m}}^{\infty}\frac{1}{\sqrt{2\pi\delta}}e^{-\frac{x^2}{2\delta}}dx \le \frac{4\sqrt{\delta}}{\varepsilon\sqrt{m}}\int_{\frac{\varepsilon\sqrt{m}}{\sqrt{\delta}}}^{\infty}\frac{1}{\sqrt{2\pi}}ye^{-\frac{y^2}{2}}dy = \frac{\sqrt{\delta}}{\varepsilon\sqrt{m}}\sqrt{\frac{8}{\pi}}e^{-\frac{\varepsilon^2 m}{2\delta}};$$

hence (14) is not larger than

$$\lim_{\delta\to 0}\sqrt{\frac{8}{\pi}}\sum_{m=1}^{\infty}\left(\frac{1}{\delta}+1\right)\frac{\sqrt{\delta}}{\varepsilon\sqrt{m}}e^{-\frac{\varepsilon^2 m}{2\delta}} = \sqrt{\frac{8}{\pi\varepsilon^2}}\int_0^{\infty}\lim_{\delta\to 0}\left(\frac{1}{\sqrt{\delta}}+\sqrt{\delta}\right)\frac{1}{\sqrt{x}}e^{-\frac{\varepsilon^2 x}{2\delta}}dx = 0.$$

As θ is arbitrary small, Theorem 1 is proved.

Proof of Theorem 2. Let us define the inverse process $X_{\nu_n}^{-1}$ of the process X_{ν_n} of Theorem 2:

$$X_{\nu_n}^{-1}(t) = \sqrt{\nu_n}\left(\frac{Z_i}{Z_{\nu_n+1}} - \frac{i}{\nu_n}\right) \quad \text{for} \quad 1 \leqq i \leqq \nu_n, \quad i-1 < \nu_n t \leqq i.$$

It follows from the definition of the process X_{ν_n} that $X_{\nu_n}^{-1} = -X_{\nu_n}(G_{\nu_n}^{-1})$, where $G_n^{-1}(t) = \inf\{x: G_n(x) \geqq t\}$ which is left continuous, is zero at zero and equals Z_i/Z_{n+1} at i/n. Now

$$X_{\nu_n}^{-1}(t) = \frac{\nu_n}{Z_{\nu_n+1}}\left(\frac{Z_i - i}{\sqrt{\nu_n}} - \frac{i}{\nu_n}\frac{Z_{\nu_n+1} - \nu_n}{\sqrt{\nu_n}}\right) \quad \text{for} \quad 1 \leqq i \leqq \nu_n, \quad i-1 < \nu_n t \leqq i.$$

Because $E(\xi_1) = 1$, $E((\xi_1 - 1)^2) = 1$, it follows that

$$\frac{\nu_n}{Z_{\nu_n+1}} \xrightarrow{P} 1$$

and

$$\frac{\xi_{\nu_n+1}}{\sqrt{\nu_n}} \xrightarrow{P} 0, \quad \text{as} \quad n \to \infty.$$

Consequently if we define $S_i = Z_i - i$ for $i = 1, 2, \ldots$; $S_0 = 0$, and $S^{(n)}(t) = S_{[nt]}/\sqrt{n}$ for $0 \leqq t \leqq 1$ and $n = 1, 2, \ldots$, then the distribution of $X_{\nu_n}^{-1}(t)$ for large n is equal to that of $S^{(\nu_n)}(t) - tS^{(\nu_n)}(1)$, that is, for large n, $X_{\nu_n}^{-1}$ is a partial sum type process of [10]. And, as X_n^{-1} is known to converge weakly to a Brownian bridge W_1° also

$$X_{\nu_n}^{-1} \xrightarrow{\mathcal{D}} W_1^\circ,$$

as a consequence of Theorem 1 of [10]. Because

$$\sup_{0 \leqq t \leqq 1} \left(G_{\nu_n}^{-1}(t) - t\right) \xrightarrow{P} 0,$$

it follows that

$$X_{\nu_n} \xrightarrow{\mathcal{D}} -W_1^\circ = W^\circ;$$

since the negative of a Brownian bridge is again a Brownian bridge, this also completes the proof of Theorem 2.

Acknowledgements. The results of this paper are partly based on my doctoral dissertation at the University of Szeged, 1971. I am indebted to Professor Károly Tandori for his constant encouragement and to my brother, Professor Miklós Csörgő for his useful advice.

References

[1] T. W. Anderson and D. A. Darling, Asymptotic theory of certain "goodness of fit" criteria based on stochastic processes, *Ann. Math. Statist.*, **23** (1952), 193—212.

[2] F. J. Anscombe, Large-sample theory of sequential estimation, *Proc. Cambridge Philos. Soc.*, **48** (1952), 600—607.

[3] O. Barndorff-Nielsen, On the limit distribution of the maximum of a random number of independent random variables, *Acta Math. Acad. Sci. Hung.*, **15** (1964), 399—403.

[4] P. Billingsley, *Convergence of probability measures* (New York, 1968).

[5] J. Blum, D. Hanson and J. Rosenblatt, On the central limit theorem for the sum of a random number of independent random variables, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, **1** (1963), 389—393.

[6] L. Breiman, *Probability* (Reading, Mass., 1968).

[7] D. R. Brillinger, An asymptotic representation of the sample distribution function, *Bull. Amer. Math. Soc.*, **75** (1969), 445—547.

[8] M. Csörgő, A new proof of some results of Rényi and the asymptotic distribution of the range of his Kolmogorov—Smirnov type random variables, *Can. J. Math.*, **19** (1967), 550—558.

[9] M. Csörgő and S. Csörgő, An invariance principle for the empirical process with random sample size, *Bull. Amer. Math. Soc.*, **76** (1970), 706—710.

[10] M. Csörgő and S. Csörgő, On weak convergence of randomly selected partial sums, *Acta Sci. Math.*, **34** (1973), 53—60.

[11] J. L. Doob, *Stochastic Processes* (New York, 1953).

[12] I. I. Gikhman and A. V. Skorokhod, *Introduction to the theory of random processes* (Philadelphia, 1969).

[13] S. Guiaşu, On the asymptotic distribution of the sequences of random variables with random indices, *Ann. Math. Statist.*, **42** (1971), 2018—2028.

[14] J. Kiefer, Skorohod embedding of multivariate RV's and the sample DF, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, **24** (1972), 4—35.

[15] D. W. Müller, On Glivenko—Cantelli convergence, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **16** (1970), 195—210.

[16] R. Pyke, The weak convergence of the empirical process with random sample size, *Proc. Cambridge Philos. Soc.*, **64** (1968), 155—160.

[17] A. Rényi, On the theory of order statistics, *Acta Math. Acad. Sci. Hung.*, **4** (1953), 191—227.

[18] A. Rényi, On mixing sequences of sets, *Acta Math. Acad. Sci. Hung.*, **9** (1958), 215—228.

(Received February 9, revised October 11, 1973)

On the nilstufe of homogeneous groups

By SHALOM FEIGELSTOCK in Ramat-Gan (Israel)

1. All groups in this paper are abelian, with addition the group operation. A homogeneous group is a group all of whose elements are of the same type (see [1, p. 147] for a definition of type). We define the type of a homogeneous group to be the type of any of its non-zero elements. For G a group, and $g \in G$, the type of g will be denoted by $T(g)$, and if G is homogeneous, the type of G will be denoted by $T(G)$.

SZELE [3] defined the nilstufe of a group G to be n, n a positive integer, if there exists an associative ring R with additive group G such that $R^n \neq 0$, but for every associative ring R with additive group G holds $R^{n+1}=0$. If no such positive integer n exists, we will say that G has nilstufe ∞. By considering not necessarily associative rings with additive group G, we may similarly define the strong nilstufe of G. The nilstufe of G will be denoted by $\nu(G)$ and the strong nilstufe by $N(G)$. A multiplication on a group G is meant to be the multiplication of a ring R with additive group G.

RÉDEI and SZELE [2] have shown that if G is a rank one, torsion-free group, and if the components of $T(G)$ are not all 0 and ∞, then $\nu(G)=1$. We will show more generally (corollary to theorem 1) that if G is a homogeneous group, and if the components of $T(G)$ are not all 0 and ∞, then $N(G)=1$. Under certain restrictions, the nilstufe of a direct sum of homogeneous groups will be computed.

2. Lemma. *Let G, H, and K be homogeneous groups. If $T(K) \not\geqq T(G)+T(H)$ then* $\mathrm{Hom}\,(G \otimes H, K)=0$.

Proof. Let $0 \neq g \in G$, $0 \neq h \in H$, and let $\varphi \in \mathrm{Hom}\,(G \otimes H, K)$. If $\varphi(g \otimes h) \neq 0$, then $T(K)=T[\varphi(g \otimes h)] \geqq T(g \otimes h) \geqq T(g)+T(h)=T(G)+T(H)$, a contradiction. Therefore $\varphi=0$, and $\mathrm{Hom}\,(G \otimes H, K)=0$.

Theorem 1. *Let $\{G_i, i \in I\}$ be a set of homogeneous groups and let $G=\sum_{i \in I}^{n} \oplus G_i$. If for all $i, j, k \in I$, $T(G_k) \not\geqq T(G_i)+T(G_j)$, then $N(G)=1$.*

Proof. Let Mult G be the group of multiplications on G. Then, by the lemma,

$$\operatorname{Mult} G \simeq \operatorname{Hom}(G\otimes G, G) = \operatorname{Hom}\Big(\sum_{i,j\in I}\oplus G_i\otimes G_j,\ \sum_{k\in I}\oplus G_k\Big) \subseteqq$$

$$\subseteqq \sum_{i,j,k\in I}\oplus \operatorname{Hom}(G_i\otimes G_j, G_k) = 0.$$

Corollary. *Let G be a homogeneous group. If the components of $T(G)$ are not all 0 and ∞, then $N(G)=1$.*

Theorem 2. *Let G_i be a homogeneous group, with the components of $T(G_i)$ not all 0 and ∞, $1\leqq i\leqq n$. Let $G=\sum_{i=1}^{n}\oplus G_i$. If for $i\neq j$, $T(G_i)\neq T(G_j)$, $1\leqq i,j\leqq n$, then*

$$\nu(G)\leqq 2^n-1.$$

Proof. For $n=1$ the theorem is true by theorem 1. Let $n>1$ and suppose the theorem is true for the direct sum of $n-1$ homogeneous groups. It may be assumed that G_n is such that $T(G_i)\not\geqq T(G_n)$ for $1\leqq i\leqq n-1$. Let $g\in G_n$, and let $h\in G$. For any multiplication on G, $T(gh)\geqq T(g)\geqq T(G_n)$. However $gh=\sum_{i=1}^{n} g_i$, $g_i\in G_i$, $1\leqq i\leqq n$. For $1\leqq i\leqq n-1$ if $g_i\neq 0$, then $T(G_i)=T(g_i)\geqq T(gh)\geqq T(G_n)$, a contradiction. Therefore:

(1) $$G_nG\subseteqq G_n,$$

and similarly

(2) $$GG_n\subseteqq G_n.$$

Let $g, g'\in G_n$. For any multiplication on G, $T(gg')>T(G_n)$. However, $gg'=\sum_{i=1}^{n} g_i$. If $g_i\neq 0$, $1\leqq i\leqq n$, then $T(G_i)=T(g_i)\geqq T(gg')>T(G_n)$, a contradiction. Therefore

(3) $$G_n^2=0.$$

(1) and (2) imply that G_n is an ideal in every ring on G. Therefore, every multiplication on G induces a multiplication on G/G_n. By the induction hypothesis $(G/G_n)^{2^{n-1}}=0$. Hence:

(4) $$G^{2^{n-1}}\subseteqq G_n.$$

(3) and (4) yield that $G^{2^n}=(G^{2^{n-1}})^2\subseteqq G_n^2=0$, and hence $\nu(G)\leqq 2^n-1$.

References

[1] L. Fuchs, *Abelian Groups*, Akadémiai Kiadó (Budapest, 1966).
[2] L. Rédei—T. Szele, Die Ringe ersten Ranges, *Acta Sci. Math.*, **12A** (1950), 18—29.
[3] T. Szele, Gruppentheoretische Beziehungen bei gewissen Ringkonstruktionen, *Math. Z.*, **54** (1951), 168—180.

(Received March 21, 1973)

The nilstufe of rank two torsion free groups

By SHALOM FEIGELSTOCK in Ramat-Gan (Israel)

1. It is well known [5] that a ring R whose additive group is a rank one torsion free group is either a zero-ring ($xy=0$ for all $x, y \in R$) or isomorphic to a subring of the field of rational numbers.

SZELE [6] introduced the notion of the nilstufe of an abelian group G. (In what follows a group is always meant to be an abelian group with addition the group operation.) Let n be a positive integer. The nilstufe of G is said to be n, denoted $N(G)=n$, if there exists a multiplication on G, not necessarily associative, such that $G^n \neq 0$, but $G^{n+1}=0$ under every multiplication on G. If, for every positive integer n, there exists a multiplication on G such that $G^n \neq 0$, then G is said to have nilstufe ∞, denoted $N(G)=\infty$. An immediate consequence of the result mentioned in the previous paragraph is that if G is a rank one torsion free group, then $N(G)=1$ or ∞. The objective of this paper is to show that if G is a rank two torsion free group, then $N(G)=1, 2$ or ∞. One is naturally led to conjecture that if G is a rank n torsion free group, then $N(G)=1, 2, \ldots, n$, or ∞.

The major tools used to compute $N(G)$ are results of BEAUMONT and WISNER [1] concerning multiplications on a rank two torsion free group. These results are introduced in section 2. In section 3, $N(G)$ is computed for G a rank two torsion free group. Sufficient conditions are given for G to be a nil-group (i.e., $N(G)=1$) if G is the direct sum of rank one torsion free groups in section 4. Quasi-equality and quasi-decomposability are discussed in section 5, and their effect on the nilstufe is considered.

2. Definition 1. If G is the additive group of a ring R, then R is called a ring over G.

Lemma 1. *Let G be a rank two torsion free group, and let R be a ring over G. If R is non-commutative, then x and x^2 are dependent for all $x \in G$. If $R \neq 0$, and R is commutative, then there exists an $x \in G$ such that x and x^2 are independent.*

Proof: [1, p. 108].

Lemma 2. *Let G be a rank 2 torsion free group, R a non-commutative ring over G. Then there exists an $x \in G$ such that $x^2 \neq 0$.*

Proof: [1, p. 109, lemma 2].

3. Theorem 1. *Let G be a rank two torsion free group. Then $N(G)=1, 2$ or ∞.*

Proof. Suppose there exists a non-commutative ring R over G. By lemma 2 there exists an $x \in G$ such that $x^2 \neq 0$. x and x^2 are dependent by lemma 1, therefore there exist non-zero integers n, m such that $nx^2 = mx$. Suppose $x^k \neq 0$ for some positive integer k. $nx^{k+1} = mx^k \neq 0$, since G is torsion free. Therefore $x^{k+1} \neq 0$, and we have shown inductively that $x^n \neq 0$ for all positive integers n, hence $N(G)=\infty$.

It therefore suffices to consider the case G a group over which there are no non-commutative rings. Let G be such a group and suppose $N(G)=n$, $2<n<\infty$. Let R be a ring over G, $R \neq 0$. R is commutative, so that by lemma 1 there is an $x \in G$ such that x and x^2 are independent.

Let $g_1, \dots, g_n \in G$. There exist integers k_i, l_i, m_i ($k_i \neq 0$), such that $k_i g_i = l_i x + m_i x^2$ for $1 \leqq i \leqq n$. $N(G)=n$; therefore

$$\left(\prod_{i=1}^{n} k_i\right)\left(\prod_{i=1}^{n} g_i\right) = \left(\prod_{i=1}^{n} l_i\right) x^n.$$

If we can show that $x^n=0$, then by virtue of the fact that $\prod_{i=1}^{n} k_i \neq 0$, and that G is torsion free, we will have that $\prod_{i=1}^{n} g_i = 0$, thus contradicting the fact that $N(G)=n$. There exist integers k, l, m, $k \neq 0$ such that $kx^n = lx + mx^2$. $N(G)=n$ therefore $0 = kx^{n+1} = lx^2 + mx^3$. If $l=0$ or $m=0$, then $x^3=0$. Since $n>2$, we have that $x^n=0$. If $l \neq 0$ and $m \neq 0$, then $lx^n = x^{n-2}(lx^2) = x^{n-2}(-mx^3) = -mx^{n+1} = 0$. G is torsion free, therefore $x^n=0$.

Corollary 1. *Let G be a rank two torsion free ring. If there exists a non-commutative ring R over G, then $N(G)=\infty$.*

4. Let H be a rank one torsion free group. All non-zero elements of H have the same type. We therefore denote by $T(H)$ the type of any non-zero element of H, and call $T(H)$ the type of H.

Lemma 3. *Let G_1 and G_2 be rank one torsion free groups, then $G_1 \otimes G_2$ is a rank one torsion free group, and $T(G_1 \otimes G_2) = T(G_1) + T(G_2)$.*

Proof. [4, p. 255 and p. 261].

Lemma 4. *Let H and K be rank one torsion free groups. If $T(H) \not\leqq T(K)$, then* Hom $(H, K)=0$.

Proof. Let $h\in H$, $h\neq 0$, and $\varphi\in \mathrm{Hom}(H, K)$. Put $k=\varphi(h)\neq 0$. By $T(H)\not\leqq T(K)$, there exist a prime p and a positive integer l such that $p^l|h$, $p^l\nmid k$. Hence $h=p^l h'$, $h'\in H$. Therefore $k=\varphi(h)=p^l\varphi(h')$, a contradiction.

Theorem 2. *Let $G=G_1\oplus G_2$, G_1 and G_2 rank one torsion free groups. If $2T(G_1)\not\leqq T(G_2)$ and $2T(G_2)\not\leqq T(G_1)$, then $N(G)=1$.*

Proof. Let Mult G be the group of multiplications which can be defined on G; $\mathrm{Mult}\, G \cong \mathrm{Hom}(G\otimes G, G) = \sum_{i,j,k=1}^{2} \mathrm{Hom}(G_i\otimes G_j, G_k) = 0$ by lemmata 3 and 4.

5. Definition 2. A group G is said to be *quasi-contained* in a group H, denoted $G\dot{\subset} H$, if there exists an integer $n\neq 0$ such that $nG\subset H$. If $G\dot{\subset} H$ and $H\dot{\subset} G$, then G is said to be quasi-equal to H, $G\doteq H$.

Theorem 3. *Let G and H be torsion free groups. If $G\doteq H$, then $N(G)=N(H)$.*

Proof. $G\doteq H$; therefore there exist non-zero integers k, l, $kG\subset H$, and $lH\subset G$. Suppose $N(G)=\infty$. Let n be an arbitrary positive integer. There exist a multiplication $\times_G$ on G and elements $g_i\in G$ $(1\leqq i\leqq n)$, such that $g_1\times_G g_2\times_G\dots\times_G g_n\neq 0$. Let $h_1, h_2\in H$. Define $h_1\times_H h_2=(lh_1)\times_G(lh_2)$. $\times_H$ is a multiplication on H. $(kg_1)\times_H\cdot(kg_2)\times_H\dots\times_H(kg_n)=(k^n l^n)g_1\times_G g_2\times_G\dots\times_G g_n\neq 0$ since G is torsion free. Therefore $N(H)=\infty$. Similarly, we may prove that if $N(H)=\infty$ then $N(G)=\infty$. We may therefore assume that $N(G)$ and $N(H)$ are both finite.

Let $N(G)=n$, and let $\times_H$ be a multiplication on H. Let $g_1, g_2\in G$. Define $g_1\times_G g_2=(kg_1)\times_H(kg_2)$. $\times_G$ is a multiplication on G. Let $h_1, \dots, h_n, h_{n+1}\in H$.

$$(k^{n+1}l^{n+1})(h_1\times_H h_2\times_H \dots \times_H h_n\times_H h_{n+1}) =$$

$$= (klh_1)\times_H(klh_2)\times_H\dots\times_H(klh_n)\times_H(klh_{n+1}) =$$

$$= (lh_1)\times_G(lh_2)\times_G\dots\times_G(lh_n)\times_G(lh_{n+1}) = 0.$$

H is torsion free, therefore $h_1\times_H\dots\times_H h_n\times_H h_{n+1}=0$, so that $N(H)\leqq n=N(G)$. Similarly we can prove that $N(G)\leqq N(H)$, so that $N(G)=N(H)$.

Definition 4. A group G is said to be *quasi-decomposable* if there exist non-zero groups A, B such that $G\doteq A\oplus B$.

Corollary 2. *Let G be a quasi-decomposable rank two torsion free group, $G\doteq G_1\oplus G_2$. If $2T(G_1)\not\leqq T(G_2)$ and $2T(G_2)\not\leqq T(G_1)$ then $N(G)=1$.*

Proof. By theorem 2, $N(G_1\oplus G_2)=1$, and by theorem 3, $N(G)=N(G_1\oplus G_2)$.

References

[1] R. A. BEAUMONT—R. J. WISNER, Rings with additive group which is a torsion free group of rank two, *Acta Sci. Math.*, **20** (1959), 105—116.

[2] S. FEIGELSTOCK, On the nilstufe of the direct sum of two groups, *Acta Math. Hung.*, **24** (1973) 269—272.

[3] L. FUCHS, *Abelian Groups*, Akadémiai Kiadó (Budapest, 1966).

[4] L. FUCHS, *Infinite Abelian Groups*, vol. 1, Academic Press (New York, 1970).

[5] L. RÉDEI—T. SZELE, Die Ringe ersten Ranges, *Acta Sci. Math.*, **12**A (1950), 18—29.

[6] T. SZELE, Gruppentheoretische Beziehungen bei gewissen Ring Konstruktionen, *Math. Z.*, **54** (1951), 168—180.

BAR-ILAN UNIVERSITY
RAMAT-GAN, ISRAEL

(Received September 7, 1972)

On subdirect representations of finite commutative unoids

By F. GÉCSEG in Szeged

In this paper we give a representation of finite commutative unoids as homomorphic images of subdirect products of very simple finite commutative unoids. Furthermore, using this representation, we present a full characterization of those finite commutative unoids $\mathfrak{A}$ which have the following property: if $\mathfrak{A}$ can be given as a homomorphic image of a subdirect product of two finite commutative unoids $\mathfrak{B}$ and $\mathfrak{C}$ then there exists a subunoid of $\mathfrak{B}$ or $\mathfrak{C}$ which can be mapped homomorphically onto $\mathfrak{A}$.

Let $\mathfrak{A}=\langle A; F\rangle$ be a unoid. (For the terminology, see [1].) We say that $\mathfrak{A}$ is commutative if $af_1f_2=af_2f_1$ for any $a\in A$ and $f_1, f_2\in F$. In this paper by a unoid we always mean a finite commutative unoid.

Take an arbitrary unoid $\mathfrak{A}=\langle A; F\rangle$, an element $a\in A$ and an operation $f\in F$. Then by the *cycle generated by* (a,f) in $\mathfrak{A}$ we mean the set of elements $af^0, af, \dots$ $\dots, af^k, \dots$, where $af^0=a$ and $af^k=(af^{k-1})f$ for any positive integer k. For this cycle we use the short notation (a,f). If $af^0, \dots, af^u$ are all different and u is the least exponent for which there exists a $w>u$ such that $af^w=af^u$ then $af^0, \dots, af^{u-1}$ is the *preperiod* of this cycle and u is the *length of this preperiod*. (When the preperiod is empty its length equals 0.) Furthermore, if $u+v$ is the minimal number for which $af^{u+v}=af^u$ holds then $af^u, af^{u+1}, \dots, af^{u+v-1}$ is the *period* of the cycle under question and v is the *length of this period*. In this case we say that (a,f) *is a cycle of type* (u, v).

A unoid $\mathfrak{A}=\langle A; F\rangle$ is called *f-cyclic* ($f\in F$) of type (k, l) if for some $a\in A$, the set A coincides with the cycle (a,f) in $\mathfrak{A}$ and this cycle is of type (k, l), while the operations different from f are identical mappings of A.

$\mathfrak{A}$ is called *prime-power unoid* (with respect to $f\in F$) if it is f-cyclic of type $(0, r^n)$ where r is a prime number. $\mathfrak{A}$ is an *elevator* (regarding $f\in F$) if it is f-cyclic of type $(k, 1)$. We say that $\mathfrak{A}$ is a prime-power unoid (resp. elevator) if it is prime-power unoid (resp. elevator) regarding one of its operations.

Now we are ready to state our

Theorem 1. *Every commutative unoid can be given as a homomorphic image of a subdirect product of finitely many elevators and prime-power unoids.*

Proof. Let $\mathfrak{A}=\langle A; F\rangle$ be an arbitrary commutative unoid. Denote by F^* the unoid of all polynomials over F of the form xp under a fixed variable x. We shall write $xp\equiv xq(\varrho)$ if and only if $xp=xq$ holds identically in $\mathfrak{A}$. It is obvious that the relation ϱ is a congruence on F^* (we say that $\mathfrak{A}$ induces ϱ), and the factor unoid $F^*/\varrho=\mathfrak{B}(=\langle B; F\rangle)$ is commutative. For elements of $\mathfrak{B}$ we shall apply the following notation: $C_\varrho(xp)$ means the class of the partition of F^* induced by ϱ containing xp.

Let us suppose that $F=\langle f_1, \dots, f_k\rangle$, and define the unoids $\mathfrak{B}_i=\langle B_i; F\rangle$ $(i=1, \dots, k)$ as follows: B_i is the cycle $(C_\varrho(x), f_i)$ in $\mathfrak{B}$ and f_i is the restriction of f_i (on B) to B_i, while the operations f_j are identical mappings of B_i for $j\neq i$.

Now take the mapping φ of the direct product $\mathfrak{B}_1\times\dots\times\mathfrak{B}_k$ into $\mathfrak{B}$ defined in the following way:

$$\varphi\big((C_\varrho(xf_1^{n_1}), \dots, C_\varrho(xf_i^{n_i}), \dots, C_\varrho(xf_k^{n_k}))\big) = C_\varrho(xf_1^{n_1}\dots f_i^{n_i}\dots f_k^{n_k})$$
$$(n_1, \dots, n_i, \dots, n_k = 0, 1, \dots).$$

Using commutativity of $\mathfrak{B}$ it can immediately be verified that φ is a homomorphism *onto* $\mathfrak{B}$.

Let us denote by ϱ_1 the relation induced by $\mathfrak{B}_1\times\dots\times\mathfrak{B}_k$ on F^*. Then $\varrho_1\leqq\varrho$ because $\mathfrak{B}$ is a homomorphic image of $\mathfrak{B}_1\times\dots\times\mathfrak{B}_k$. Observe that $\mathfrak{B}_i$ is f_i-cyclic for every i $(1\leqq i\leqq k)$. Let $\mathfrak{B}_i$ be of type (u_i, v_i). In the case $v_i=1$ let $\mathfrak{B}_i'$ be an f_i-cyclic unoid of type $(u_i, 2)$ and let $\mathfrak{B}_i'=\mathfrak{B}_i$ in the other case. It is obvious that $\mathfrak{B}_i'$ can be mapped homomorphically onto $\mathfrak{B}_i$. Therefore, $\mathfrak{B}_1\times\dots\times\mathfrak{B}_k$ is a homomorphic image of $\mathfrak{B}_1'\times\dots\times\mathfrak{B}_k'$. Denote by ϱ_2 the relation of F^* induced by $\mathfrak{B}_1'\times\dots\times\mathfrak{B}_k'$. Then we get that $\varrho_2\leqq\varrho_1$.

As it can be seen in [2], every equation of an equational class of commutative unoids can have one of the following two forms:

$$(1)\qquad xf_1^{m_1}\dots f_k^{m_k} = xf_1^{n_1}\dots f_k^{n_k}$$
$$(2)\qquad xf_1^{m_1}\dots f_k^{m_k} \doteq yf_1^{n_1}\dots f_k^{n_k} \qquad (m_1, \dots, m_k, n_1, \dots, n_k \geqq 0).$$

Equation (2) implies $xf_1^{m_1}\dots f_k^{m_k}=yf_1^{m_1}\dots f_k^{m_k}$. Choose an element b_i from every B_i' $(i=1, \dots, k)$. Then $(b_1, b_2, \dots, b_k)f_1^{m_1}\dots f_k^{m_k}\neq(b_1f_1, b_2, \dots, b_k)f_1^{m_1}\dots f_k^{m_k}$ showing that (2) fails to hold on $\mathfrak{B}_1'\times\dots\times\mathfrak{B}_k'$.

Therefore, we have got that every equation which holds on $\mathfrak{B}_1'\times\dots\times\mathfrak{B}_k'$ is of the form (1). Since $\varrho_2\leqq\varrho$ thus all equations holding on $\mathfrak{B}_1'\times\dots\times\mathfrak{B}_k'$ hold on $\mathfrak{A}$, too, i.e., $\mathfrak{A}$ is contained in the equational class generated by $\mathfrak{B}_1'\times\dots\times\mathfrak{B}_k'$. This means that $\mathfrak{A}$ can be given as a homomorphic image of a subunoid of a finite direct power of $\mathfrak{B}_1'\times\dots\times\mathfrak{B}_k'$ (see, e.g., the proof of the Theorem in [1]).

In order to end the proof of Theorem 1, it is enough to show that every $\mathfrak{B}_i'$ can be given as a subdirect product of finitely many elevators and prime-power unoids. Let $B_i'=\langle b_0, \ldots, b_{u_i}, \ldots, b_{u_i+v_i-1}\rangle$ and $v_i=r_1^{w_1}\ldots r_t^{w_t}$ where r_i are different prime numbers. Define the relations $\sigma_0, \sigma_1, \ldots, \sigma_t$ on B_i' as follows: $b_j\equiv b_k(\sigma_0)$ if and only if $j=k$ or $j, k\geqq u_i$, and for every l ($l=1, \ldots, t$), $b_j\equiv b_k(\sigma_l)$ if and only if $j\equiv k \pmod{r_l^{w_l}}$. It is clear that $\sigma_0, \ldots, \sigma_t$ are congruences of $\mathfrak{B}_i'$; moreover, their intersection is the identity relation. Indeed, from $b_j\equiv b_k(\sigma_0\cap\ldots\cap\sigma_t)$ it follows that $j=k$ or $j, k\geqq u_i$ and (by the Chinese Remainder Theorem) $j\equiv k \pmod{v_i}$. In both cases we have $b_j=b_k$. Thus $\mathfrak{B}_i'$ can be given as a subdirect product of $\mathfrak{B}_i'/\sigma_0, \ldots, \mathfrak{B}_i'/\sigma_t$. Moreover, $\mathfrak{B}_i'/\sigma_0$ is an elevator and each of $\mathfrak{B}_i'/\sigma_1, \ldots, \mathfrak{B}_i'/\sigma_t$ is a prime-power unoid. This ends the proof of Theorem 1.

A unoid $\mathfrak{A}=\langle A; F\rangle$ is called *homomorphically prime* if $|A|>1$ and the fact $\mathfrak{A}$ is a homomorphic image of a subdirect product of two unoids $\mathfrak{A}_1$ and $\mathfrak{A}_2$ implies that there exists a subunoid in $\mathfrak{A}_1$ or $\mathfrak{A}_2$ which can be mapped homomorphically onto $\mathfrak{A}$.

First we state the following simple

Theorem 2. *If $|F|=1$ then $\mathfrak{A}=\langle A; F\rangle$ is homomorphically prime if and only if $\mathfrak{A}$ is an elevator or prime-power unoid.*

Proof. The subunoids and homomorphic images of elevators are elevators, too. Similar statement is valid for prime-power unoids. Therefore, by Theorem 1, every homomorphically prime unoid should be either elevator or prime-power unoid. It can be shown, by an easy computation, that in the case $|F|=1$ all elevators and prime-power unoids are homomorphically prime.

This Theorem 2 and Theorem 1 of Yoeli in [3] show that the class of all homomorphically prime unary algebras and that of all connected subdirectly irreducible unary algebras coincide.

We now prove

Theorem 3. *If $|F|\geqq 2$ then a commutative unoid $\mathfrak{A}=\langle A; F\rangle$ is homomorphically prime if and only if $\mathfrak{A}$ is an elevator.*

Proof. The subunoids and homomorphic images of an elevator are elevators. Prime-power unoids have similar property. Thus, by Theorem 1, homomorphically prime unoids should be either elevators or prime-power unoids.

First we show that none of the prime-power unoids is homomorphically prime. Before proving this statement, let us introduce the notation $k \pmod n$ for the least non-negative residue of k modulo n.

For the sake of simplicity, let $\mathfrak{A}=\langle A; F\rangle$ be a prime-power unoid with respect to f_1 such that $A=\langle a_0, \ldots, a_{m^n-1}\rangle$ and

$$a_i f_1 = a_{i+1 \pmod{m^n}}$$

where m is a prime number. Take two different prime numbers $m_1(\neq m)$, $m_2(\neq m)$ such that $m^2 \nmid (m_1 - m_2)$, and let r be an integer with $r > n$. Let us define the unoids $\mathfrak{A}_i = \langle A_i; F \rangle$ $(i=1,2)$ in the following way:

$$A_i = \langle a_{i0}, \ldots, a_{im^r} \rangle, \quad a_{ij} f_1 = a_{i(j+1)(\mathrm{mod}\, m^r)},$$

$$a_{ij} f_2 = a_{i(j+m_i)(\mathrm{mod}\, m^r)}$$

and $a_{ij} f_l = a_{ij}$ if $l > 2$, where $i = 1, 2$ and $j = 0, \ldots, m^r - 1$. It is obvious that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are commutative.

Consider the subdirect product $\mathfrak{A}_1 \dot{\times} \mathfrak{A}_2$ of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ consisting of all elements (a_{1j}, a_{2j}) and $(a_{1j}, a_{2j}) f_2^k$ $(j = 0, \ldots, m^r - 1;\ k = 1, 2, \ldots)$. We show that the mapping φ defined by

$$\varphi((a_{10}, a_{20}) f_1^t) = a_0 f_1^t \quad \text{and} \quad \varphi((a_{10}, a_{20}) f_1^t f_2^k) = a_0 f_1^t \quad (t, k = 1, 2, \ldots)$$

is a homomorphism of $\mathfrak{A}_1 \dot{\times} \mathfrak{A}_2$ onto $\mathfrak{A}$.

It is enough to prove that φ is well defined. Let t_1, k_1 and t_2, k_2 be natural numbers such that

$$(a_{10}, a_{20}) f_1^{t_1} f_2^{k_1} = (a_{10}, a_{20}) f_1^{t_2} f_2^{k_2}. \tag{3}$$

We show that this implies $a_0 f_1^{t_1} = a_0 f_1^{t_2}$, i.e., $t_1 \equiv t_2 \pmod{m^n}$. The equality (3) means that

$$t_1 + m_i k_1 \equiv t_2 + m_i k_2 \pmod{m^r} \quad (i = 1, 2), \tag{4}$$

whence we get $m^r \mid (m_1 - m_2)(k_1 - k_2)$. But $m^2 \nmid (m_1 - m_2)$, thus $m^n \mid (k_1 - k_2)$ because $r > n$. From this, using anyone of the congruences (4) we have $t_1 \equiv t_2 \pmod{m^n}$. Thus we have shown that φ is well defined. Therefore, by definition, it is a homomorphism.

It remains to be shown that $\mathfrak{A}$ cannot be given as a homomorphic image of a subunoid of $\mathfrak{A}_1$ or $\mathfrak{A}_2$. Neither $\mathfrak{A}_1$ nor $\mathfrak{A}_2$ have any subunoid different from themself. Thus take a mapping φ_i of $\mathfrak{A}_i$ onto $\mathfrak{A}$ $(i = 1, 2)$ such that $\varphi_i(a_{ij}) = a_u$. If φ_i is a homomorphism then

$$\varphi_i(a_{ij} f_2) = \varphi(a_{ij} f_1^{m_i}) = a_{(u+m_i)(\mathrm{mod}\, m^n)} = a_u.$$

But $a_{(u+m_i)(\mathrm{mod}\, m^n)} \neq a_u$ because $m^n \nmid m_i$. Therefore, φ_i cannot be a homomorphism.

We now show that every elevator is homomorphically prime. Let $\mathfrak{A}_k = \langle A_k; F \rangle$ denote the elevator with

$$A_k = \langle a_0, \ldots, a_k \rangle \quad (k > 0), \quad a_i f_j = a_i \quad (a_i \in A_k, f_j \in F) \quad \text{if} \quad j > 1$$

and

$$a_i f_1 = \begin{cases} a_{i+1} & \text{if } i < k, \\ a_k & \text{if } i = k. \end{cases}$$

In the sequel by p and q with or without indices we denote polynomials in which f_1 does not occur.

Let us assume that $\mathfrak{A}_k$ can be given as a homomorphic image of a subdirect product of two unoids $\mathfrak{B}_1$ and $\mathfrak{B}_2$ under a homomorphism φ for which $\varphi((b_1, b_2))=a_0$ $((b_1, b_2)\in B_1\times B_2)$ holds. First we show that at least one of the unoids $\mathfrak{B}_i$ $(i=1, 2)$ has the following property P: for every p, the elements $b_i p, b_i pf_1, \ldots, b_i pf_1^{k-1}$ are all different and there exists no $b_i pf_1^u$ with $u\geqq k$ which is equal to one of them. Indeed, in the opposite case there exist polynomials p_1 and p_2, non-negative integers u_1, t_1 and u_2, t_2 such that $b_1p_1f_1^{u_1}=b_1p_1f_1^{t_1}$, $b_2p_2f_1^{u_2}=b_2p_2f_1^{t_2}$; $u_1, u_2<k$; $t_1>u_1$ and $t_2>u_2$. By the commutativity of $\mathfrak{B}_1$ and $\mathfrak{B}_2$, $b_1p_1p_2f_1^{u_1}=b_1p_1p_2f_1^{t_1}$ and $b_2p_1p_2f_1^{u_2}=b_2p_1p_2f_1^{t_2}$. Now let us suppose that $u_2\geqq u_1$. Then

$$b_1p_1p_2f_1^{u_2} = b_1p_1p_2f_1^{u_2+(t_1-u_1)(t_2-u_2)}$$

and

$$b_2p_1p_2f_1^{u_2} = b_2p_1p_2f_1^{u_2+(t_1-u_1)(t_2-u_2)}.$$

Therefore,

$$(b_1, b_2)p_1p_2f_1^{u_2} = (b_1, b_2)p_1p_2f_1^{u_2+(t_1-u_1)(t_2-u_2)}.$$

Since φ is a homomorphism thus we get

$$a_0f_1^{u_2} = a_0f_1^{u_2+(t_1-u_1)(t_2-u_2)}$$

which is impossible.

In the sequel we write simply $\mathfrak{B}$ instead of $\mathfrak{B}_i$ having the above property and b instead of b_i. Consider the cycle (bp, f_1) in $\mathfrak{B}$ with minimal preperiod d among all cycles generated by pairs of the form (bq, f_1). Then property P implies $d\geqq k$. Take the subunoid $\mathfrak{B}'$ of $\mathfrak{B}$ generated by bp. We show that $\mathfrak{B}'$ can be mapped homomorphically onto $\mathfrak{A}_k$, namely, the mapping φ defined by $\varphi(bpqf_1^l)=a_0f_1^l$ for all q and non-negative integer l will be such a homomorphism.

To prove that φ is well defined let us assume that $bpq_1f_1^{l_1}=bpq_2f_1^{l_2}$. We must have $a_0f_1^{l_1}=a_0f_1^{l_2}$, or, equivalently, $l_1=l_2$ provided $l_1, l_2\geqq k$ does not hold. Suppose $l_1<l_2, k$. Observe that, for any q, the preperiod of the cycle (bpq, f_1) cannot be longer than d and in fact, in view of the minimum property of bp, it coincides with d. Indeed, by the commutativity of $\mathfrak{B}$, $bpf_1^u=bpf_1^v$ implies $bpqf_1^u=bpqf_1^v$. Now we distinguish two cases:

1) $l_2<d$. If c is the length of the period of the cycle (bpq_2, f_1) then $bpq_1f_1^{l_1+d-l_2}=bpq_1f_1^{l_1+d-l_2+c}$. But this is incompatible with property P because $l_1+d-l_2<d$.

2) $l_2 \geqq d$. Then we get similarly $bpq_1 f_1^{l_1} = bpq_1 f_1^{l_1+c}$, which is again a contradiction.

Finally, a short computation shows that $\varphi(bpqf_1^l)f = \varphi(bpqf_1^l f)$ holds for any $f \in F$. This completes the proof of Theorem 3.

Acknowledgements. The author is grateful to B. CSÁKÁNY for his useful comments which helped to make the paper more concise.

References

[1] F. GÉCSEG—S. SZÉKELY, On equational classes of unoids, *Acta Sci. Math.*, **34** (1973), 99—101.
[2] А. И. Мальцев, *Алгебраические системы* (Москва, 1970).
[3] M. YOELI, Subdirectly irreducible unary algebras, *Amer. Math. Monthly*, **74** (1967), 957—960.

(Received May 30, 1973)

Bernstein-type inequalities for families of multiplier operators in Banach spaces with Cesàro decompositions. II. Applications*)

By E. GÖRLICH, R. J. NESSEL and W. TREBELS in Aachen (FRG)

In this paper a number of applications of the results in Part I is given by studying certain concrete instances of Banach spaces X and systems $\{P_k\}$ of orthogonal projections. Rather than to give a complete list of possible applications, our aim is to show how the general approach proposed in Part I yields Bernstein-type inequalities for classical orthonormal systems such as those concerned with Bessel, Laguerre, Hermite and ultraspherical polynomials, Walsh and Haar functions, and spherical harmonics. Let us mention that the present unifying approach covers certain classical as well as a number of new Bernstein-type inequalities.

In the following, $L^p_w(a, b)$, $1 \leqq p \leqq \infty$, $-\infty \leqq a < b \leqq +\infty$, denotes the usual Banach space of measurable functions, pth power integrable with respect to the weight $w(x) \geqq 0$:

$$\|f\|_{p,w} = \Big\{\int_a^b |f(x)|^p w(x)dx\Big\}^{1/p}, \quad \|f\|_{\infty,w} = \operatorname*{ess.\,sup}_{a \leqq x \leqq b} |f(x)|\, w(x);$$

in the case $w(x)=1$ we abbreviate to L^p, $\|f\|_p$.

4. Bessel series

Let $a=0$, $b=1$, and $w(x)=1$. Denoting by $J_\nu(x)$ the Bessel function of the first kind of order $\nu > -1$ and by $\{c_k\}_{k\in\mathbf{N}}$ the sequence of positive zeros of $J_\nu(x)$, arranged in ascending order of magnitude, the functions

$$\varphi_k^{(\nu)}(x) = (2x)^{1/2}(J_{\nu+1}(c_k))^{-1} J_\nu(c_k x) \quad (k \in \mathbf{N})$$

*) This paper is a sequel to Part I, which appeared in *Acta Sci. Math.*, **34** (1973), 121—130. The contents (and notations) of the first part are assumed to be known. References as well as sections are numbered consecutively throughout this series. The contribution of W. Trebels was supported by a DFG-fellowship.

form an orthonormal system on (0, 1). Thus the projections $P_k^{(\nu)}$, defined by

$$(P_k^{(\nu)}f)(x) = \Big[\int_0^1 f(u)\varphi_k^{(\nu)}(u)\,du\Big]\,\varphi_k^{(\nu)}(x) \quad (k\in\mathbf{N})$$

are mutually orthogonal. Wing [42] has shown that $\{\varphi_k^{(\nu)}\}$ forms a Schauder basis in $L^p(0, 1)$, $1<p<\infty$, for $\nu \geqq -1/2$, and BENEDEK and PANZONE [18] have extended this result to $-1<\nu<-1/2$ provided $1/(\nu+3/2)<p<1/(-\nu-1/2)$; moreover, these bounds are sharp.

By (3.6) one then has

Corollary 4.1. *Let $f\in L^p(0, 1)$ with ν, p specified as above. Then*

$$\Big\|\sum_{k=1}^{n} k^{\omega} P_k^{(\nu)} f\Big\|_p \leqq A n^{\omega} \Big\|\sum_{k=1}^{n} P_k^{(\nu)} f\Big\|_p \quad (\omega > 0), \tag{4.1}$$

the constant A being independent of $n\in\mathbf{N}$ and f.

In case $\nu=\pm 1/2$, this inequality reduces to the standard Bernstein inequality for trigonometric polynomials (cf. (3.15)) since $J_{1/2}(x)=[2/(\pi x)]^{1/2}\sin x$ and $J_{-1/2}(x)=[2/(\pi x)]^{1/2}\cos x$. Clearly, inequalities corresponding to (3.5), (3.7), (3.9)—(3.11) may also be formulated.

To give a classical interpretation of $B^{\{k^{\omega}\}}$ let us consider the differential operator $D_{(\nu)}$ defined by

$$(D_{(\nu)}f)(x) = f''(x)-[(\nu^2-1/4)/x^2]f(x).$$

Then the Liouville normal form of the Bessel differential equation

$$(xu'(x))'+(\lambda x-\nu^2/x)u(x) = 0 \quad (0 < x \leqq 1) \tag{4.2}$$

reads $D_{(\nu)}f+\lambda f=0$, and $\varphi_k^{(\nu)}$ is just the eigenfunction of $D_{(\nu)}$ corresponding to the eigenvalue $\lambda=-k^2$, $k\in\mathbf{N}$. Then (4.1) gives

$$\|D_{(\nu)}f\|_p \leqq An^2\|f\|_p \quad (n\in\mathbf{N}) \tag{4.3}$$

for all

$$f\in \bigoplus_{k=1}^{n} P_k^{(\nu)}(L^p(0, 1)).$$

Analogously, one may consider the system of eigenfunctions $\psi_k^{(\nu)}$ of (4.2), namely

$$\psi_k^{(\nu)}(x) = \sqrt{2}\,(J_{\nu+1}(c_k))^{-1}J_{\nu}(c_k x) \quad (k\in\mathbf{N}).$$

They form an orthonormal system on (0, 1) with respect to the weight $w(x)=x$. Thus the projections $\tilde{P}_k^{(\nu)}$, defined by

$$(\tilde{P}_k^{(\nu)}f)(x) = \Big[\int_0^1 f(u)\psi_k^{(\nu)}(u)\,u\,du\Big]\,\psi_k^{(\nu)}(x) \quad (k\in\mathbf{N}),$$

are mutually orthogonal. Now $\{\psi_k^{(\nu)}\}$ is a Schauder basis in $L_w^p(0, 1)$ for $4/3<p<4$ in case $\nu\geqq -1/2$ (see [42]) and for $2/(2+\nu)<p<-2/\nu$ in case $-1<\nu<-1/2$ (see [18]); again the bounds are sharp. Thus, letting

$$(\tilde{D}_{(\nu)}f)(x) = (xf'(x))' - (\nu^2/x)f(x),$$

(3.6) delivers

$$\|\tilde{D}_{(\nu)}f\|_{p,w} \leqq An^2\|f\|_{p,w} \quad (n\in\mathbf{N}) \tag{4.4}$$

for each $f\in \bigoplus_{k=1}^{n} \tilde{P}_k^{(\nu)}(L_w^p(0, 1))$. Clearly (4.3) and (4.4) are equivalent in case $p=2$.

Similarly, using results of BENEDEK and PANZONE [17] and GENEROZOV [30], Bernstein-type inequalities corresponding to the eigenfunctions of the equation $(x^{2\varkappa}v'(x))'+\lambda v(x)=0$ with $-\infty<\varkappa<1$ may be obtained. Moreover, results of RUTOVITZ and CRUM cited in [17] allow one to apply the present method to the eigenfunctions of a certain general class of Sturm—Liouville problems.

5. Laguerre and Hermite series

Let $a=0$, $b=\infty$, and $w(x)=1$. Consider the Laguerre polynomials $L_k^{(\alpha)}$ of order $\alpha>-1$ defined by

$$L_k^{(\alpha)}(x) = (k!)^{-1}e^x x^{-\alpha}(d/dx)^k(e^{-x}x^{k+\alpha}) \quad (k\in\mathbf{P}).$$

Setting

$$\varphi_k^{(\alpha)}(x) = \left\{\Gamma(\alpha+1)\binom{k+\alpha}{k}\right\}^{-1} x^{\alpha/2}e^{-x/2}L_k^{(\alpha)}(x),$$

the projections

$$(P_k^{(\alpha)}f)(x) = \Big[\int_0^\infty f(u)\varphi_k^{(\alpha)}(u)\,du\Big]\,\varphi_k^{(\alpha)}(x)$$

are mutually orthogonal. The system $\{P_k^{(\alpha)}\}_{k\in\mathbf{P}}$ satisfies (2.7) for $j=0$ in case $4/3<p<4$, $\alpha>-1$ (see ASKEY—WAINGER [15], MUCKENHOUPT [34]), and for $j=1$ in case $1\leqq p\leqq\infty$, $\alpha>0$ or $(1+\alpha/2)^{-1}<p<-2/\alpha$, $-1<\alpha\leqq 0$ (see POIANI [36]). Hence by (3.7)

Corollary 5.1. *Let $f\in L^p(0, \infty)$ with α, p specified as above for $j=1$, and $\omega>0$. Then*

$$\Big\|\sum_{k=0}^{n}\log(1+k^{\omega})P_k^{(\alpha)}f\Big\|_p \leqq A\log(1+n^{\omega})\Big\|\sum_{k=0}^{n}P_k^{(\alpha)}f\Big\|_p \quad (n\in\mathbf{P}).$$

Since the $\varphi_k^{(\alpha)}$ are eigenfunctions of the differential operator

$$D_{(\alpha)} = \frac{d}{dx}\left(x\frac{d}{dx}\right)+\frac{\alpha+1}{2}-\frac{x}{4}-\frac{\alpha^2}{4x}$$

with eigenvalues $-k$, $k\in\mathbf{P}$, one has by (3.6) that for all $f\in L^p(0, \infty)$, p, α as specified above ($j=1$),

$$\left\|D_{(\alpha)}\left(\sum_{k=0}^{n} P_k^{(\alpha)} f\right)\right\|_p \leqq An \left\|\sum_{k=0}^{n} P_k^{(\alpha)} f\right\|_p \quad (n \in \mathbf{P}).$$

A consideration of the $L_k^{(\alpha)}(x)$ themselves in the space $L_w^p(0, \infty)$ with weight $w(x)=x^\alpha e^{-x}$ does not apply here since the $L_k^{(\alpha)}$ do not yield a (C, j)-basis for any $j \in \mathbf{P}$ except for the case $p=2$ (see POLLARD [37], ASKEY—HIRSCHMAN [14]).

Now let $a=-\infty$, $b=+\infty$, and $w(x)=1$. Consider the Hermite polynomials

$$H_k(x) = (-1)^k e^{x^2}(d/dx)^k e^{-x^2} \quad (k \in \mathbf{P}).$$

Setting

$$\varphi_k(x) = \left(2^k k! \sqrt{\pi}\right)^{-1/2} e^{-x^2/2} H_k(x),$$

$\{\varphi_k\}$ is an orthonormal family of functions on $(-\infty, \infty)$. Thus the projections

$$(P_k f)(x) = \left\{ \int_{-\infty}^{\infty} f(u)\varphi_k(u)\, du \right\} \varphi_k(x)$$

are mutually orthogonal. The system $\{P_k\}_{k \in \mathbf{P}}$ satisfies (2.7) for $j=1$ in case $1 \leqq p \leqq \infty$ (see [28a], [36]). Since the φ_k are eigenfunctions of the differential operator $(d^2/dx^2)+$ $+(1-x^2)$ with eigenvalues $-2k$, $k \in \mathbf{P}$, one has by (3.6)

$$\|(d^2/dx^2)f + (1-x^2)f\|_p \leqq An \|f\|_p$$

for all $f \in \bigoplus_{k=0}^{n} P_k(L^p(-\infty, \infty))$, $1 \leqq p \leqq \infty$, $n \in \mathbf{P}$. This inequality is contained in a paper of FREUD [28].

6. Ultraspherical series

Let $a=-1$, $b=1$, and $w(x)=1$. The ultraspherical polynomials C_k^λ of order $\lambda \geqq 0$ are given by

$$(6.1) \qquad C_k^\lambda(x) = M_{k,\lambda}(1-x^2)^{-\lambda+1/2}(d/dx)^k[(1-x^2)^{k+\lambda-1/2}] \quad (k \in \mathbf{P}),$$

$M_{k,\lambda}$ being a suitable constant. They are orthonormal on $(-1, 1)$ with respect to the measure $(1-x^2)^{\lambda-1/2}\,dx$. Hence, setting

$$\varphi_k^\lambda(x) = (1-x^2)^{\lambda/2-1/4} C_k^\lambda(x),$$

$$(P_k^{(\lambda)} f)(x) = \left[\int_{-1}^{1} f(u)\varphi_k^\lambda(u)\, du \right] \varphi_k^\lambda(x),$$

the projections $P_k^{(\lambda)}$ are mutually orthogonal on $L^p(-1, 1)$. The sequence $\{\varphi_k^\lambda\}_{k \in \mathbf{P}}$ forms a Schauder basis in $L^p(-1, 1)$ for $4/3 < p < 4$, $\lambda \geqq 0$ (cf. WING [42] for Jacobi polynomials). The functions φ_k^λ are eigenfunctions of the operator

$$D_{(\lambda)} = \frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right] + \left(\lambda - \frac{1}{2}\right)\left[1 - \left(\lambda - \frac{1}{2}\right)\frac{x^2}{1-x^2}\right]$$

with corresponding eigenvalues $-k(k+2\lambda)$, $k \in \mathbf{P}$. Thus by (3.5)

Corollary 6.1. *For any* $f\in L^p(-1,1)$, $4/3<p<4$,

$$\left\|\sum_{k=0}^{n} e^{k(k+2\lambda)} P_k^{(\lambda)} f\right\|_p \leqq A e^{n(n+2\lambda)} \left\|\sum_{k=0}^{n} P_k^{(\lambda)} f\right\|_p. \tag{6.2}$$

Noting that the C_k^λ are orthonormal on $(-1,1)$ with respect to $w(x)=(1-x^2)^{\lambda-1/2}$, set

$$(\tilde{P}_k^{(\lambda)} f)(x) = \Big[\int_{-1}^{1} f(u) C_k^\lambda(u)(1-u^2)^{\lambda-1/2}\,du\Big] C_k^\lambda(x) \quad (k\in\mathbf{P}).$$

The $\tilde{P}_k^{(\lambda)}$ form a (C,j)-decomposition in $L_w^p(-1,1)$ provided

$$\begin{cases} \dfrac{2\lambda+1}{\lambda+1+j} < p < \dfrac{2\lambda+1}{\lambda-j} & \text{if} \quad 0\leqq j\leqq \lambda,\ 0\leqq\lambda<\infty \\ 1\leqq p\leqq\infty & \text{if} \quad 0\leqq\lambda<j \end{cases} \tag{6.3}$$

(see Pollard [37] for $j=0$, Askey—Hirschman [14] for $j>0$). The C_k^λ are eigenfunctions of

$$\tilde{D}_{(\lambda)} = (1-x^2)(d^2/dx^2) - (2\lambda+1)x(d/dx)$$

with eigenvalues $-k(k+2\lambda)$ so that by (3.6)

Corollary 6.2. *Let* $\mathcal{P}_n$ *denote the set of all algebraic polynomials of degree* $\leqq n$. *Then*

$$\|\tilde{D}_{(\lambda)} f\|_{p,w} \leqq A n^2 \|f\|_{p,w} \quad (f\in\mathcal{P}_n,\ n\in\mathbf{P}), \tag{6.4}$$

provided (6.3) *is satisfied with* $j=1$.

So far we have stated Bernstein inequalities of type (3.1), (3.3). However, those of Corollary 3.3 are valid as well. For example, by (3.11)

Corollary 6.3. *The Riesz means* (3.8) (iii) *of order* $\varkappa$, $\nu>0$ *satisfy*

$$\left\|\sum_{k=0}^{n} k^\omega \left(1-\left(\frac{k}{n+1}\right)^{\varkappa}\right)^{\nu} \tilde{P}_k^{(\lambda)} f\right\|_{p,w} \leqq A n^\omega \|f\|_{p,w} \quad (n\in\mathbf{P})$$

for arbitrary $\omega>0$, $f\in L_w^p(-1,1)$, *provided* (6.3) *holds for some* $0\leqq j\leqq\nu$.

Remark. It is possible to extend the above results to Jacobi polynomials. Indeed, (6.2) may immediately be restated since [42] includes $(C,0)$-summability for the Jacobi case for $4/3<p<4$. Also (C, 0)-summability for Jacobi series in the weight space $L_w^p(-1,1)$ with $w(x)=(1-x)^\alpha(1+x)^\beta$, $\alpha,\beta>-1$, is known (see Pollard [38] and Muckenhoupt [33]) so that the Jacobi analogue of (6.4) follows, namely

$$\|\tilde{D}_{(\alpha,\beta)} f\|_{p,w} \leqq A n(n+\alpha+\beta+1)\|f\|_{p,w} \quad (f\in\mathcal{P}_n,\ n\in\mathbf{P}), \tag{6.5}$$

where $\tilde{D}_{(\alpha,\beta)}$ is defined by

$$\tilde{D}_{(\alpha,\beta)} = (1-x^2)(d^2/dx^2) + \big(\beta-\alpha-(\alpha+\beta+2)x\big)(d/dx)$$

and p is restricted to

$$(\alpha+1)\left(\frac{1}{p}-\frac{1}{2}\right)<\min\left\{\frac{1}{4},\ \frac{1+\alpha}{2}\right\},\ (\beta+1)\left|\frac{1}{p}-\frac{1}{2}\right|<\min\left\{\frac{1}{4},\ \frac{1+\beta}{2}\right\}$$

A more general inequality, which contains (6.5) and the $w(x)=1$-analogue of (6.5) as particular instances, can be stated as well using the result of [42] and [33].

Concerning (C,j)-summability in $L^p_w(-1,1)$ one may proceed via POLLARD [38], STEIN [41], ASKEY—HIRSCHMAN [14], ASKEY—WAINGER [16], and GASPER [29], according to a written communication of R. Askey.

The paper of STEIN [41] should also be mentioned in connection with (6.5) since it contains a proof for all $1\leqq p\leqq\infty$, $\alpha,\beta>-1$. He also assumes condition (2.7) to be valid for some $j\in\mathbf{P}$ and obtains Bernstein-type inequalities for orthonormal systems, using a different method, namely interpolation in polynomial subspaces.

7. Walsh series

Let $a=0$, $b=1$, and $w(x)=1$, all functions in this section being assumed to have period 1. Defining the Rademacher functions by

$$\varphi_0(x)=\begin{cases}1, & 0\leqq x<1/2,\\ -1, & 1/2\leqq x<1,\end{cases}\qquad \varphi_0(x+1)=\varphi_0(x),$$

$$\varphi_k(x)=\varphi_0(2^k x)\quad (k\in\overset{0}{\mathbf{N}}),$$

the Walsh functions are given by

$$\psi_0(x)=1,\quad \psi_k(x)=\varphi_{k_1}(x)\varphi_{k_2}(x)\dots\varphi_{k_i}(x),$$

$$k=2^{k_1}+\dots+2^{k_i},\quad k_1>\dots>k_i\geqq 0,\quad k_i\in\mathbf{P}.$$

They form an orthonormal system in $L^p(0,1)$, $1\leqq p<\infty$, which is also fundamental. Thus the projections

$$(P_k f)(x)=\left[\int_0^1 f(u)\psi_k(u)\,du\right]\psi_k(x)\quad (k\in\mathbf{P})$$

are mutually orthogonal and total in $L^p(0,1)$, $1\leqq p<\infty$; PALEY [35] has shown that the P_k form a Schauder decomposition of $L^p(0,1)$ for $1<p<\infty$; for the proof that they also form a $(C,1)$-decomposition of $L^1(0,1)$ see e.g. FINE [27], MORGENTHALER [32]. Hence, by (3.9)

$$\left\|\sum_{k=0}^{\infty}k^{\omega}e^{-(k/\varrho)^{\varkappa}}P_k f\right\|_p\leqq A\varrho^{\omega}\|f\|_p\quad(\varkappa,\omega>0)\tag{7.1}$$

for any $f\in L^p(0,1)$, $1\leqq p<\infty$, $\varrho>0$.

As in the preceding sections we would like to interpret the case $\omega=1$ via some differential operator. Concerning its definition we follow up BUTZER—WAGNER [21, 22]: Let G denote the dyadic group consisting of all sequences $x=\{x_n\}_{n=1}^{\infty}$ such that $x_n=0$ or 1, the operation of G being termwise addition mod 2. A function $f\in L^p(0,1)$ is said to have a strong derivative $D_G f$ in $L^p(0,1)$ if there exists $g\in L^p(0,1)$ such that

$$\lim_{m\to\infty}\left\|\frac{1}{2}\sum_{i=0}^{m}2^i[f(\cdot)-f(\cdot\otimes 2^{-i-1})]-g(\cdot)\right\|_p=0,$$

and in this case $D_G f=g$. Here

$$x\otimes y=\frac{1}{2}\sum_{n=1}^{\infty}|x_n-y_n|,\quad x=\sum_{n=1}^{\infty}x_n 2^{-n},\quad y=\sum_{n=1}^{\infty}y_n 2^{-n},$$

finite expansions being used for dyadic rationals. The operator D_G is closed and linear (see [21, Prop. 4.4] and [22, Sec. 3], where further details are given). Since the Walsh functions satisfy $D_G\psi_k=k\psi_k$ for each $k\in\mathbf{P}$, i.e. the ψ_k are eigenfunctions of D_G, one has by (7.1), (3.8) (i)

$$\|D_G W_\varkappa(\varrho)f\|_p\leqq A\varrho\|f\|_p$$

for all $f\in L^p(0,1)$, $1\leqq p<\infty$, of period 1.

8. Haar series

Let $a=0$, $b=1$, and $w(x)=1$. In the notation of [8, p. 49] the orthonormal system $\{h_k(x)\}_{k=1}^{\infty}$ of Haar functions is defined on [0, 1] by

$$h_1(x)=\chi_{[0,1]}(x)$$
$$h_k(x)=2^{m/2}\{\chi_{[0,1]}(2^{m+1}x-2k+2)-\chi_{(0,1]}(2^{m+1}x-2k+1)\},$$

where $k=2^m+i$, $m\in\mathbf{P}$, $i=1,2,\ldots,2^m$, and $\chi_{[a,b]}(x)$ denotes the characteristic function of the interval $[a,b]$. Hence the projections P_k defined by

$$(P_k f)(x)=\Big[\int_0^1 f(u)h_k(u)\,du\Big]h_k(x)\quad(k\in\mathbf{N})$$

are mutually orthogonal. Moreover, the Haar functions form a Schauder basis in $L^p(0,1)$, $1\leqq p<\infty$ (see also [11, p. 13]) so that one has as an immediate consequence of Corollary 3.2

$$\left\|\sum_{k=1}^{n}\alpha_k P_k f\right\|_p\leqq A\alpha_n\left\|\sum_{k=1}^{n}P_k f\right\|_p\quad(f\in L^p(0,1)),\tag{8.1}$$

where α may be any of the examples (3.4).

An explicit definition of a genuine differential operator D satisfying $Dh_k = kh_k$ for all $k \in \mathbf{N}$ seems to be unknown. Nevertheless such an operator can be identified with the infinitesimal generator of a suitable semi-group of class $(\mathscr{C}_0)$. For example, let the Abel—Poisson means $W(t)$ of the Haar expansion of f be defined by

$$(W(t)f)(x) = \sum_{k=1}^{\infty} e^{-tk} P_k f \quad (t > 0);$$

then $\lim_{t \to 0+} \|W(t)f - f\|_p = 0$ for $1 \leqq p < \infty$ (cf. [2II]). Setting $W(0) = I$, it follows that $\{W(t), t \geqq 0\}$ is a semi-group of bounded linear operators on $L^p(0, 1)$ of class $(\mathscr{C}_0)$. Its infinitesimal generator $\mathscr{A}$ is easily seen to be represented by

$$\mathscr{A}f \sim \sum_{k=1}^{\infty} (-k) P_k f$$

for every f in the domain $D(\mathscr{A})$ of $\mathscr{A}$. Moreover (cf. [20, p. 9]), $D(\mathscr{A})$ is dense in X, and $\mathscr{A}$ is a closed linear operator. Thus $-\mathscr{A}$ is just the desired operator D. Differential operators corresponding to the logarithmic and exponential cases in (8.1) may be defined similarly (see [7]).

Clearly in certain instances the semi-group theory yields directly Bernstein-type inequalities in an arbitrary Banach space X. Indeed, for holomorphic semi-groups of class $(\mathscr{C}_0)$ on X with infinitesimal generator $\mathscr{A}$ one always has the inequality $\|\mathscr{A}T(t)f\|_X \leqq Mt^{-1}\|f\|_X$ for all $f \in X$, $t > 0$ by Cauchy's integral formula (see Butzer—Berens [20, Sec. 1.1.2]).

Along the present lines one may also treat generalized Schauder systems (cf. Canturija [23]), generalized Haar systems (cf. Golubov [31], Sox—Harrington [40]), the (orthonormalized) Franklin system (cf. Ciesielski [24, 25], Radecki [39]) as well as further spline function systems (cf. Ciesielski—Domsta [26]). The Bernstein-type inequalities obtained in [24, 39, 26] deal with ordinary derivatives which, however, are not covered by our approach.

9. Spherical harmonics

Let $\mathbf{R}^N$ be the N-dimensional Euclidean space ($N \geqq 2$) with elements $v = (v_1, \dots, v_N)$, inner product $v \cdot v^* = \sum_{k=1}^{N} v_k v_k^*$, and $|v|^2 = v \cdot v$. Denoting by S_N the surface of the unit sphere in R^N with elements y, z, content $\Omega_N = 2\pi^{N/2}/\Gamma(N/2)$ and surface element ds, let $X(S_N)$ be one of the spaces $L^p(S_N)$, $1 \leqq p < \infty$, or $C(S_N)$ with norms

$$\|f\|_p = \Big\{\Omega_N^{-1} \int_{S_N} |f(y)|^p \, ds(y)\Big\}^{1/p} (1 \leqq p < \infty), \quad \|f\|_C = \max_{y \in S_N} |f(y)|,$$

respectively. If $Y_k(v)$ is a homogeneous polynomial of degree k satisfying

$$\Delta Y_k(v) = 0, \quad \Delta = \sum_{k=1}^{N} (\partial/\partial v_k)^2 \quad (v \in \mathbf{R}^N),$$

then the restriction of Y_k to S_N, denoted by Y_k, too, is called a surface spherical harmonic of degree k. The Y_k satisfy the differential equation

$$\tilde{\Delta} Y_k(y) = -k(k+N-2)\, Y_k(y), \quad \tilde{\Delta} f(v) = |v|^2 \Delta f(v/|v|). \tag{9.1}$$

Let the orthonormal sequence of projections $\{P_k\}_{k\in\mathbf{P}}$ be defined by (cf. (6.1), $\lambda=(N-2)/2$)

$$(P_k f)(y) = \sum_{r=1}^{H(k,N)} \Big\{ \int_{S_N} f(z)\, Y_k^r(z)\, ds(z) \Big\}\, Y_k^r(y) = \frac{\Gamma(\lambda)(k+\lambda)}{2\pi^{\lambda+1}} \int_{S_N} C_k^\lambda(y\cdot z) f(z)\, ds(z),$$

where $H(k,N)$ denotes the number of linearly independent spherical harmonics of degree k. The P_k form a (C,j)-decomposition of $X(S_N)$ for $j>(N-2)/2$ (see [19] and the literature cited there).

Since, for $\varkappa>2$, $\{k(k+N-2)n^{-2}[1+(k/n)^2]^{-\varkappa/2}\}_{k\in\mathbf{P}} \in bv_{j+1}$ uniformly in $n\in\mathbf{P}$, it follows by (3.10), (9.1) that

Corollary 9.1. *Let $f\in X(S_N)$. Then for any $\varkappa>2$*

$$\|\tilde{\Delta} L_\varkappa(x) f\|_X \leqq A n^2 \|f\|_X.$$

References

[14] R. Askey—I. I. Hirschman, Jr., Mean summability for ultraspherical polynomials, *Math. Scand.*, **12** (1963), 167—177.

[15] R. Askey—S. Wainger, Mean convergence of expansions in Laguerre and Hermite series, *Amer. J. Math.*, **87** (1965), 695—708.

[16] R. Askey—S. Wainger, A convolution structure for Jacobi series, *Amer. J. Math.*, **91** (1969), 463—485.

[17] A. Benedek—R. Panzone, Note on mean convergence of eigenfunction expansions, *Rev. Un. Mat. Argentina.* **25** (1970), 167—184.

[18] A. Benedek—R. Panzone, On mean convergence of Fourier—Bessel series of negative order, *Studies in Applied Math.*, **50** (1971), 281—292.

[19] H. Berens—P. L. Butzer—S. Pawelke, Limitierungsverfahren von Reihen mehrdimensionaler Kugelfunktionen und deren Saturationsverhalten, *Publ. Res. Inst. Math. Sci.*, (Ser. A) **4** (1968), 201—268.

[20] P. L. Butzer—H. Berens, *Semi-Groups of Operators and Approximation*, Springer, (Berlin, 1967).

[21] P. L. Butzer—H. J. Wagner, Walsh—Fourier series and the concept of a derivative, *Applicable Anal.*, **3** (1973), 29—46.

[22] P. L. Butzer—H. J. Wagner, Approximation by Walsh polynomials and the concept of a derivative, *Applications of Walsh Functions (Proc. Sympos. Naval Res. Lab., Washington, D. C.,*

27—29. March 1972, Ed. R. W. Zeek—A. E. Showalter) Washington, D. C. 1972, pp. 388—392.

[23] Z. A. Canturija, Bases of the space of continuous functions, *Soviet Math. Dokl.*, **10** (1969), 862—864.

[24] Z. Ciesielski, Properties of the orthonormal Franklin system, *Studia Math.*, **23** (1963), 141—157.

[25] Z. Ciesielski, Properties of the orthonormal Franklin system II, *Studia Math.*, **27** (1966), 289—323.

[26] Z. Ciesielski—J. Domsta, Construction of an orthonormal basis in $C^m(I^d)$ and $W_p^m(I^d)$, *Studia Math.*, **41** (1972), 211—224.

[27] N. J. Fine, On the Walsh functions, *Trans. Amer. Math. Soc.*, **65** (1949), 372—414.

[28] G. Freud, On an inequality of Markov type, *Soviet Math. Dokl.*, **12** (1971), 570—573.

[28a] G. Freud—S. Knapovski, On linear processes of approximation. III, *Studia Math.*, **25** (1965), 374—383.

[29] G. Gasper, Positivity and the convolution structure for Jacobi series, *Ann. of Math.*, **93** (1971), 112—118.

[30] V. L. Generozov, The convergence in L_p of expansions in terms of the eigenfunctions of a Sturm—Liouville problem, *Math. Notes*, **3** (1968), 436—441.

[31] B. I. Golubov, A certain class of complete orthogonal systems, *Sibirsk. Mat. Ž.*, **9** (1968), 297—314. (Russian.)

[32] G. W. Morgenthaler, On Walsh—Fourier series, *Trans. Amer. Math. Soc.*, **84** (1957), 472—507.

[33] B. Muckenhoupt, Mean convergence of Jacobi series, *Proc. Amer. Math. Soc.*, **23** (1969), 306—310.

[34] B. Muckenhoupt, Asymptotic forms for Laguerre polynomials. *Proc. Amer. Math. Soc.*, **24** (1970), 288—292.

[35] R. E. A. C. Paley, A remarkable series of orthogonal functions. *Proc. London Math. Soc.*, (3) **34** (1932), 241—279.

[36] E. L. Poiani, Mean Cesàro summability of Laguerre and Hermite series, *Trans. Amer. Math. Soc.*, **173** (1972), 1—31.

[37] H. Pollard, The mean convergence of orthogonal series. II, *Trans. Amer. Math. Soc.*, **63** (1948), 355—367.

[38] H. Pollard, The mean convergence of orthogonal series. III, *Duke Math. J.*, **16** (1949), 189—191.

[39] J. Radecki, Orthonormal basis in the space $C_1[0, 1]$, *Studia Math.*, **35** (1970), 123—163.

[40] J. L. Sox—W. J. Harrington, A class of complete orthogonal sequence of step functions, *Trans. Amer. Math. Soc.*, **157** (1971), 129—135.

[41] E. M. Stein, Interpolation in polynomial classes and Markoff's inequality, *Duke Math. J.*, **24** (1957), 467—476.

[42] G. M. Wing, The mean convergence of orthogonal series, *Amer. J. Math.*, **72** (1950), 792—807.

(Received August 21, 1972)

On the greatest zero of an orthogonal polynomial. II

By GÉZA FREUD in Budapest

1. Introduction

In the present paper we are continuing our investigations initiated in [2]. (See also [3] and P. G. NÉVAI [5].)

As in the papers mentioned, we denote by $p_n(W; x)$ the nth degree orthogonal polynomial with respect to the weight $W(x)$ $(-\infty<x<\infty)$ and by $X_n(W)$ we denote the greatest zero of $p_n(W; x)$. Through all our present paper we assume that $W(x)$ is even.[1])

The most typical result of this paper is

Theorem 1. *Let*

(1) $$W_Q(x) = \exp\{-2Q(x)\}$$

where $Q(x)$ is an even differentiable function, increasing for $x>0$, for which $x^\varrho Q'(x)$ is increasing for some $\varrho<1$ then we have

(2) $$c_1 q_n \leqq X_n(W_Q) \leqq c_2 q_n$$

where c_1, c_2 do not depend on n, and q_s $(s>0)$ is determined by the equation

(3) $$q_s Q'(q_s) = s.$$

Let us observe that as a consequence of our assumption $xQ'(x)$ is also increasing for $x>0$ so that the sequence $\{q_s\}$ is well defined.

We obtain Theorem 1 as a consequence of far more general but slightly technical estimates (see Theorem 2 and Theorem 3).

Theorem 1 is applicable for the case $Q_\alpha(x)=\frac{1}{2}|x|^\alpha$ $(\alpha>0)$ and we obtain $X_n(W_{Q_\alpha})\sim n^{1/\alpha}$. This was proved earlier for positive even integer values of α by the author in [2] and for general $\alpha>0$ by G. P. NÉVAI [5].

---

[1]) Estimates for the zero with greatest modulus of $p_n(W; x)$ are obtained in the general case by combining the result of this paper by Lemma 7 of our preceding paper [3].

We conjectured in our paper [3] that an inequality similar to (2) might hold under the less restrictive condition that $xQ'(x)$ is increasing for $x>0$. [2]) This problem remains unsettled.

2. Preliminary estimates

In all our paper we assume that $W(x)$ is an even continuous positive function on the whole real line. It follows that

$$G_\xi(W) = \exp\left\{\frac{1}{\pi}\int_0^\pi \log[W(\xi\cos\theta)]\,d\theta\right\} \tag{4}$$

is well defined and finite for every $\xi\geqq 0$.

We need also the trunkated weights

$$W_\xi(x) = \begin{cases} W(x) & (|x|\leqq\xi) \\ 0 & (|x|>\xi). \end{cases} \tag{5}$$

Lemma 1. *The leading coefficients $\gamma_\nu(W)$ resp. $\gamma_\nu(W_\xi)$ of the orthogonal polynomials $p_\nu(W;x)$ resp. $p_\nu(W_\xi;x)$ satisfy for every $\xi>0$*

$$\gamma_\nu(W)\leqq\gamma_\nu(W_\xi)\leqq\sqrt{\frac{2}{\pi}}\,\xi^{-1/2}\left(\frac{2}{\xi}\right)^\nu[G_\xi(W)]^{-1/2}. \tag{6}$$

Proof. (See also [4], part 2.) We have

$$\int_{-1}^{1}|p_\nu(W_\xi;\xi t)|^2W(\xi t)\,dt = \xi^{-1}\int_{-\xi}^{\xi}p_\nu^2(W_\xi;x)W_\xi(x)\,dx = \xi^{-1}$$

and by setting $z=e^{i\theta}$, $t=\cos\theta=\frac{1}{2}(z+z^{-1})$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|z^\nu p_\nu\left[W_\xi;\frac{1}{2}\xi(z+z^{-1})\right]\right|^2 W(\xi\cos\theta)|\sin\theta|\,d\theta = \frac{1}{\pi\xi}. \tag{7}$$

Since $W(x)$ is positive and continuous we have $\log W(\xi\cos\theta)\in L$ for every

---

[2]) In general q_s and q_{2s} have not the same order of increase for $s\to\infty$. As a consequence of the reasonable hypothesis for the general case is that there exist positive numbers c_3, c_4, c_5 and c_6 depending only on the choice of W_Q for which we have

$$c_3 q_{c_4 n}\leqq X_n(W_Q)\leqq c_5 q_{c_6 n}.$$

As a consequence of our additional assumption that $x^\varrho Q'(x)$ is increasing for a $\varrho<1$ we have $q_{cn}\sim q_n$ for every fixed $c>0$. (See (31).)

fixed $\xi>0$. By a theorem of G. SZEGŐ (see e.g. [6], part 10) there exists a function $D(z)\in H_2$ satisfying a.e.

(8) $$|D(e^{i\theta})|^2 = W(\xi\cos\theta)\,|\sin\theta|$$

and

(9) $$D(0) = \exp\left\{\frac{1}{2\pi}\int_0^{\pi}\log[W(\xi\cos\theta)\,|\sin\theta|]\,d\theta\right\} = \frac{1}{\sqrt{2}}[G_\xi(W)]^{1/2}.$$

Now

(10) $$P_\nu(z) = z^\nu p_\nu\left[W_\xi;\ \frac{1}{2}\xi(z+z^{-1})\right]$$

is a polynomial of degree 2ν for which we have

(11) $$P_\nu(0) = \left(\frac{\xi}{2}\right)^\nu \gamma_\nu(W_\xi).$$

By construction $P_\nu D\in H_2$ so that by (9) and (11) resp. by (7), (8), and (10)

$$\frac{1}{\sqrt{2}}[G_\xi(W)]^{1/2}\left(\frac{\xi}{2}\right)^\nu \gamma_\nu(W_\xi) = D(0)P_\nu(0) \leqq \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|P_\nu(e^{i\theta})D(e^{i\theta})|^2\,d\theta\right\}^{1/2} = (\pi\xi)^{-1/2}.$$

By reshuffling terms in this inequality we obtain the second part of (6). The first part of (6) follows from $W_\xi(x)\leqq W(x)$, q.e.d.

We turn to investigate the Christoffel functions (see e.g. [1])

(12) $$\lambda_n(W;x) = \left\{\sum_{\nu=0}^{n-1} p_\nu^2(W;x)\right\}^{-1}.$$

Lemma 2. *We have for every pair every (x,ξ) with $|x|>\xi>0$*

(13) $$\lambda_n^{-1}(W;x) \leqq \lambda_n^{-1}(W_\xi;x) \leqq \frac{4}{3\pi}[\xi G_\xi(W)]^{-1}\left(\frac{2|x|}{\xi}\right)^{2n-2}.$$

Proof. The first part of (13) is a consequence of $W_\xi(x)\leqq W(x)$. To prove the second part first we observe that all zeroes $x_{k\nu}=x_{k\nu}(W_\xi)$ of $p_\nu(W_\varrho;x)$ are contained in $(-\xi,\xi)$ (since $W_\xi(x)=0$ for $|x|>\xi$) and the $x_{k\nu}$'s are distributed symmetrically around the origin (since W_ξ is even). Consequently we have for every natural ν

(14) $$|p_\nu(W_\xi;x)| = \gamma_\nu(W_\xi)\,|x|^{\nu-2[\nu/2]}\prod_{x_{k\nu}>0}(x^2-x_{k\nu}^2) \leqq$$
$$\leqq \gamma_\nu(W_\xi)\,|x|^\nu \leqq \sqrt{\frac{2}{\pi}}\left(\frac{2}{\xi}\right)^\nu \xi^{-1/2}[G_\xi(W)]^{-1/2}|x|^\nu$$

the last part in consequence of (6).

By (14) we have

$$(15)\quad \lambda_n^{-1}(W;x) \leqq \lambda_n^{-1}(W_\xi;x) = \sum_{\nu=0}^{n-1} p_\nu^2(W_\xi;x) \leqq \frac{2}{\pi}[\xi G_\xi(W)]^{-1}\sum_{\nu=0}^{n-1}\left(\frac{2|x|}{\xi}\right)^{2\nu} =$$

$$= \frac{2}{\pi}[\xi G_\xi(W)]^{-1}\frac{\left(\frac{2|x|}{\xi}\right)^{2n}-1}{\left(\frac{2|x|}{\xi}\right)^{2}-1}.$$

Under the assumption $|x|>\xi$ (15) implies (13), q.e.d.

Remark. We assumed in the proof only that $W(x)$ is nonnegative and that $G_\xi(W)$ is finite. The example of Chebychev polynomials and $|\xi|=1$, $|x|\to\infty$ shows that under this less stringent assumptions the factor $2/\pi$ in (15) is best possible for every $n\geqq 2$. By a continuity argument it follows that $2/\pi$ is the best possible factor even if only continuous positive weights are admitted.

3. The fundamental estimates for $X_n(W)$

Theorem 2. *We have for every $\xi>0$ and every $A\geqq 1$*

$$(16)\qquad X_n(W) \leqq A\xi + \frac{4}{3\pi}\left(\frac{2}{\xi}\right)^{2n-1}[G_\xi(W)]^{-1}\int_{A\xi}^{\infty} x^{2n-1}W(x)\,dx.$$

Proof. By Chebychev's theorem

$$(17)\qquad X_n(W) = \sup_{P\in\Phi_{n-1}} \int_{-\infty}^{\infty} x[P(x)]^2 W(x)\,dx$$

where Φ_{n-1} is the set of polynomials $P(x)$ of degree $n-1$ at most for which we have

$$(18)\qquad \int_{-\infty}^{\infty}[P(x)]^2W(x)\,dx \leqq 1.$$

As a consequence of (18) we have

$$(19)\qquad [P(x)]^2 \leqq \lambda_n^{-1}(W;x) \quad (-\infty<x<\infty).$$

By (19) and (13)

$$(20)\quad \left|\left\{\int_{-\infty}^{-A\xi}+\int_{A\xi}^{\infty}\right\}x[P(x)]^2W(x)\,dx\right| \leqq 2\,\frac{4}{3\pi}\,\xi[G_\xi(W)]^{-1}\int_{A\xi}^{\infty} x\left(\frac{2x}{\xi}\right)^{2n-2}W(x)\,dx =$$

$$= \frac{4}{3\pi}\left(\frac{2}{\xi}\right)^{2n-1}[G_\xi(W)]^{-1}\int_{A\xi}^{\infty} x^{2n-1}W(x)\,dx.$$

By (18) we have

$$\int_{-A\xi}^{A\xi} x[P(x)]^2 W(x)\,dx \leqq A\xi \quad (P\in\Phi_{n-1}). \tag{21}$$

Now (16) follows from (17), (20) and (21), q.e.d.

Theorem 3. *We have for every even nonnegative W and every $\xi>0$*

$$X_n(W) \geqq \frac{1}{2}\left\{\frac{2}{\pi}\int_{-\infty}^{\infty} W(x)\,dx\right\}^{-\frac{1}{2n-2}} \{\xi^{2n-1}G_\xi(W)\}^{\frac{1}{2n-2}}. \tag{22}$$

Proof. By [2] we have

$$X_n(W) \geqq \frac{\gamma_{\nu-1}(W)}{\gamma_\nu(W)} \quad (\nu = 1, 2, \ldots, n-1).$$

Consequently

$$[X_n(W)]^{n-1} \geqq \prod_{\nu=1}^{n-1} \frac{\gamma_{\nu-1}(W)}{\gamma_\nu(W)} = \frac{\gamma_0(W)}{\gamma_{n-1}(W)} = \frac{1}{\gamma_{n-1}(W)\left\{\int_{-\infty}^{\infty} W(x)\,dx\right\}^{\frac{1}{2}}}; \tag{23}$$

(22) is implied by (6) and (23), q.e.d.

4. Special cases of the fundamental estimates

Let us assume that $W(x)$ is even, continuous, positive, *decreasing for* $x>0$ and that for every natural ν

$$\lim_{x\to\infty} x^\nu W(x) = 0. \tag{24}$$

For such W and for every ν there exists a smallest ξ_ν satisfying

$$\xi_\nu^\nu W(\xi_\nu) = \max_{x\geqq 0} x^\nu W(x). \tag{25}$$

Theorem 4. *Under the stated assumptions concerning W we have*

$$\frac{1}{2}\left\{\frac{\pi}{2\int_{-\infty}^{\infty} W(x)\,dx}\right\}^{\frac{1}{2n-2}} [\xi_{2n-1}^{2n-1} W(\xi_{2n-1})]^{\frac{1}{2n-2}} \leqq X_n(W) \leqq \left(2+\frac{1}{3\pi}n^{-1}\right)\xi_{4n}. \tag{26}$$

Proof. Since $W(x)$ is decreasing for $x>0$ we have

$$G_\xi(W) \geqq W(\xi). \tag{27}$$

Inserting $\xi=\xi_{2n-1}$ in (22) and taking (27) in consideration we obtain the left hand side of (26). To obtain the right hand side, let us insert in (16) $A=2$ and $\xi=\xi_{4n}$, thus by (27)

$$(28)\qquad \int_{2\xi_{4n}}^{\infty} x^{2n-1}W(x)\,dx \leqq \xi_{4n}^{4n}W(\xi_{4n})\int_{2\xi_{4n}}^{\infty} x^{-2n-1}\,dx \leqq \frac{1}{2n\,2^{2n}}\xi_{4n}^{2n}G_{\xi}(W).$$

From (28) and (16) with $A=2$, $\xi=\xi_{4n}$ we get the second half of (26), q.e.d.

Proof of Theorem 1.[3]) By (1), (3) and (25) we have $\xi_{\nu}=q_{\nu/2}$. Since $x^{\varrho}Q'(x)$ is increasing we have by assumption for every $0<s<S$

$$\frac{S}{s}=\frac{q_S Q'(q_S)}{q_s Q'(q_s)} \geqq \left(\frac{q_S}{q_s}\right)^{1-\varrho},$$

i.e.

$$(29)\qquad q_s \leqq q_S \leqq \left(\frac{S}{s}\right)^{\frac{1}{1-\varrho}} q_s \quad (0<s<S).$$

Assuming $\xi_s \geqq 1$ we have by (3)

$$Q(\xi_s) \leqq Q(1)+\xi_s^{\varrho}Q'(\xi_s)\int_1^{\xi_s} t^{-\varrho}\,dt < Q(1)+\frac{s}{1-\varrho}$$

and this is valid also for $0<\xi_s<1$ since $Q(x)$ is increasing. Consequently

$$(30)\qquad W(\xi_{2n-1}) = e^{-2Q(\xi_{2n-1})} \geqq e^{-2Q(1)-2(1-\varrho)^{-1}(2n-1)} = W(1)e^{-2(1-\varrho)^{-1}(2n-1)}.$$

The estimates (2) follow from (26), (29) and (30), q.e.d.

References

[1] G. Freud, *Orthogonale Polynome,* Birkhäuser Verl. (Basel, 1969); English translation: Pergamon Press (New York, 1971).

[2] G. Freud, On the greatest zero of an orthogonal polynomial. I, *Acta Sci. Math.*, **34** (1973), 91—97.

[3] G. Freud, An estimation of the greatest zeroes of orthogonal polynomials, *Acta Math. Ac. Sci. Hung.*, **25** (1974), 99—107.

[4] G. Freud, On polynomial approximation with respect to general weight functions, *Proceedings of the International Conference on Functional Analysis and its Applications, Madras,* 1973. to appear.

[5] G. P. Névai, Многочлены, ортонормальные на вещественной оси с весом $|x|^{\alpha}e^{-|x|^{\beta}}$ *Acta Math. Ac. Sci. Hung.,* **24** (1973), 335—342.

[6] G. Szegő, *Orthogonal Polynomials,* Amer. Math. Soc. Coll. Publ., Vol. XXIII (1959).

(Received May 25, 1973)

[3]) See the Introduction.

On fractional powers of operators in Hilbert space

By W. M. GREENLEE in Tucson (Arizona, USA)*

0. The primary concern of this note is to give conditions (Theorem 1) such that if A and B are each self adjoint operators with positive lower bounds and $A+B$ is self adjoint, then for $0 \leqq \tau \leqq 1$, the domain $D((A+B)^\tau)$ equals $D(A^\tau) \cap D(B^\tau)$. A theorem of LIONS and MAGENES [19] on interpolation of intersections is then obtained as a corollary. It is then verified that for a large class of Schrödinger operators $-\Delta + q(x)$ on R^n, Δ = Laplacian, q real valued, the conditions are satisfied so that Theorem 1 is applicable if $D(-\Delta + q(x)) = D(-\Delta) \cap D(q(x))$ in the operator theoretic sense.

In addition a new sufficient condition (Theorem 2) for the equality of $D(C^{1/2})$ and $D(C^{*1/2})$, where C is a regularly accretive operator, is given. This condition is shown to be applicable if C arises as an elliptic partial differential operator with homogeneous Dirichlet boundary conditions over certain (possibly unbounded) domains admitting corners, the Lipschitzian graph domains.

1. Let H be a complex Hilbert space with norm $|u|$ and inner product (u, v). Further let V_a (resp. V_b) be a complex Hilbert space with $V_a \underset{c}{\subseteq} H$ (resp. $V_b \underset{c}{\subseteq} H$), i.e. V_a is a vector subspace of H and the injection of V_a into H is continuous. Also assume that V_a, V_b, and $V_a \cap V_b$ are dense in H and denote the inner product in V_a (resp. V_b) by $a(u, v)$ (resp. $b(u, v)$). To the inner product $a(u, v)$ there corresponds a linear operator A in H, *the operator in H associated with* $a(u, v)$, defined on

$$D(A) = \{u \in V_a : v \to a(u, v) \text{ is continuous on } V_a \text{ in the topology induced by } H\}$$

by

$$(Au, v) = a(u, v) \quad \text{for all} \quad v \in V_a.$$

A is a positive definite self adjoint operator in H and $D(A)$ is dense in V_a. For τ positive, denote by A^τ the positive τth power of A as defined by use of the spectral

* The research for this paper was supported in part by the National Science Foundation.

theorem; A^τ is a positive definite, self adjoint operator in H. Furthermore, $D(A^{1/2})$ is V_a and $a(u, v)=(A^{1/2}u, A^{1/2}v)$ for all $u, v\in V_a$.

For $0\leqq\tau\leqq 1$, the τth *interpolation space by quadratic interpolation* between V_a and H, $[V_a, H]_\tau$, is defined as the Hilbert space

$$[V_a, H]_\tau = D(A^{\tau/2})$$

with inner product $(A^{\tau/2}u, A^{\tau/2}v)$. Further for $\tau\in[0, \infty)$ let $[V_a, H]_\tau$ be the Hilbert space $D(A^{\tau/2})$ with inner product $(A^{\tau/2}u, A^{\tau/2}v)$.

Let B be the operator in H associated with $b(u, v)$, i.e.

$$(Bu, v) = b(u, v), \quad u\in D(B), \quad Bu\in H, \quad v\in V_b,$$

and for $\tau\in[0, \infty)$ denote by $[V_b, H]_\tau$ the Hilbert space $D(B^{\tau/2})$ with inner product $(B^{\tau/2}u, B^{\tau/2}v)$. Now $V_a\cap V_b$, provided with the inner product $a(u, v)+b(u, v)$, is a Hilbert space and, since $V_a\cap V_b$ is dense in H, we may let Σ be the operator in H associated with $a(u, v)+b(u, v)$, i.e.

$$(\Sigma u, v) = a(u, v)+b(u, v), \quad u\in D(\Sigma),$$
$$\Sigma u\in H, \quad v\in V_a\cap V_b.$$

Then for $\tau\in[0, \infty)$ let $[V_a\cap V_b, H]_\tau$ be the Hilbert space $D(\Sigma^{\tau/2})$ with inner product $(\Sigma^{\tau/2}u, \Sigma^{\tau/2}v)$. We wish to obtain relationships between the Hilbert spaces $[V_a\cap V_b, H]_\tau$ and $[V_a, H]_\tau\cap[V_b, H]_\tau$ (with the inner product $(A^{\tau/2}u, A^{\tau/2}v)+(B^{\tau/2}u, B^{\tau/2}v)$), without assuming that $A^{1/2}$ and $B^{1/2}$ commute as in [19], p. 95.

Proposition 1. *For each* $\tau\in[0, 1]$,

$$[V_a\cap V_b, H]_\tau \underset{c}{\subseteq} [V_a, H]_\tau\cap[V_b, H]_\tau,$$

and, if $\alpha, \beta \geqq 0$ *and* $\alpha+\beta = 1$,

$$\alpha|A^{\tau/2}u|+\beta|B^{\tau/2}u| \leqq |\Sigma^{\tau/2}u| \quad \textit{for all} \quad u\in[V_a\cap V_b, H]_\tau.$$

Proof. Obviously the identity mapping is continuous from $V_a\cap V_b$ into V_a with bound $\leqq 1$, continuous from $V_a\cap V_b$ into V_b with bound $\leqq 1$, and continuous from H into H with bound 1. The proposition is thus a trivial consequence of the quadratic interpolation theorem of Lions [16], pp. 431—432 (cf. also Adams, Aronszajn and Hanna [1], App. I).

Observe that $A+B$ is essentially self adjoint if and only if $D(A+B)=D(A)\cap D(B)$ is dense in $D(\Sigma)$, i.e. if and only if $[V_a, H]_2\cap[V_b, H]_2$ is dense in $[V_a\cap V_b, H]_2$. Further if $A+B$ is essentially self adjoint, then the closure of $A+B$ is Σ.

Proposition 2. *If $A+B$ is essentially self adjoint, then for each* $\tau\in[1, 2]$ *such that* $D(A)\cap D(B)$ *is dense in* $[V_a, H]_\tau\cap[V_b, H]_\tau$,

$$[V_a, H]_\tau\cap[V_b, H]_\tau \underset{c}{\subseteq} [V_a\cap V_b, H]_\tau,$$

and

(1) $$|\Sigma^{\tau/2}u| \leqq |A^{\tau/2}u| + |B^{\tau/2}u| \quad \textit{for all} \quad u \in [V_a, H]_\tau \cap [V_b, H]_\tau.$$

Proof. Let $u \in D(A) \cap D(B)$. Then since $D(\Sigma)$ is dense in $D(\Sigma^\theta)$ for all $\theta < 1$,

$$|\Sigma^{\tau/2}u| = \sup\{|(\Sigma^{\tau/2}u, \Sigma^{1-(\tau/2)}v)| : v \in D(A) \cap D(B) \text{ and } |\Sigma^{1-(\tau/2)}v| = 1\} =$$

$$= \sup\{|(A^{\tau/2}u, A^{1-(\tau/2)}v) + (B^{\tau/2}u, B^{1-(\tau/2)}v| : v \in D(A) \cap D(B) \text{ and } |\Sigma^{1-(\tau/2)}v| = 1\} \leqq$$

$$\leqq \sup\{|(A^{\tau/2}u, A^{1-(\tau/2)}v)| : v \in D(A) \cap D(B) \text{ and } |\Sigma^{1-(\tau/2)}v| = 1\} +$$

$$+ \sup\{|(B^{\tau/2}u, B^{1-(\tau/2)}v)| : v \in D(A) \cap D(B) \text{ and } |\Sigma^{1-(\tau/2)}v| = 1\}.$$

Since $2-\tau \in [0, 1]$ it now follows from Proposition 1 that

$$|\Sigma^{\tau/2}u| \leqq \sup\{|(A^{\tau/2}u, A^{1-(\tau/2)}v)| : v \in D(A) \text{ and } |A^{1-(\tau/2)}v| = 1\} +$$

$$+ \sup\{|(B^{\tau/2}u, B^{1-(\tau/2)}v)| : v \in D(B) \text{ and } |B^{1-(\tau/2)}v| = 1\} = |A^{\tau/2}u| + |B^{\tau/2}u|.$$

Thus (1) holds for all u in the closure of $D(A) \cap D(B)$ in $[V_a, H]_\tau \cap [V_b, H]_\tau$. The proposition follows.

Observe that $A+B$ is self adjoint if and only if $\Sigma = A+B$ and when this is the case the norms $|\Sigma u| = |(A+B)u|$ and $(|Au|^2 + |Bu|^2)^{1/2}$ are equivalent on $D(A) \cap D(B)$ (by the closed graph theorem). In this case $A+B$ is also a topological isomorphism of $D(A) \cap D(B)$ onto H.

Theorem 1. *If $A+B$ is self adjoint, then for each $\tau \in [0, 2]$,*

$$[V_a \cap V_b, H]_\tau \underset{c}{\subseteq} [V_a, H]_\tau \cap [V_b, H]_\tau.$$

Moreover, for each $\tau \in [0, 2]$ such that $D(A) \cap D(B)$ is dense in $[V_a, H]_\tau \cap [V_b, H]_\tau$,

$$[V_a \cap V_b, H]_\tau = [V_a, H]_\tau \cap [V_b, H]_\tau,$$

with equivalent norms.

Proof. The first assertion is obtained by the method of proof of Proposition 1, and the second assertion via the proof of Proposition 2.

Corollary 1. ([19], p. 95) *If H is separable and $A^{1/2}$ and $B^{1/2}$ commute, then for each $\tau \in [0, 2]$,*

$$[V_a \cap V_b, H]_\tau = [V_a, H]_\tau \cap [V_b, H]_\tau$$

with equivalent norms.

Proof. By simultaneous diagonalization of A and B (cf. Dixmier [6], p. 217) it follows in much the same fashion as in the proof of Théorème 13.1, p. 95, [19], that the hypotheses of Theorem 1 are satisfied.

2. In this section we wish to illustrate how the previous results apply to characterization of the domains of fractional powers of Schrödinger operators $-\Delta u + q(x)u$,

$x \in R^n$, Δ=Laplacian, q real and $\geqq 2\delta > 0$. We shall use the theory of Bessel potentials (cf. ARONSZAJN [3], ARONSZAJN and SMITH [5], ADAMS, ARONSZAJN and SMITH [2]).

The Bessel kernel of order $\alpha > 0$ on R^n is the function given by

$$G_\alpha(x) = G_\alpha^{(n)}(x) = \frac{1}{2^{(n+\alpha-2)/2}\pi^{n/2}\Gamma(\alpha/2)} K_{(n-\alpha)/2}(|x|)\,|x|^{(\alpha-n)/2}$$

where K_ν is the modified Bessel function of the 3rd kind. For $0<\alpha<1$, let

$$C(n,\alpha) = \frac{2^{-2\alpha+1}\pi^{(n+2)/2}}{\Gamma(\alpha+1)\Gamma(\alpha+(n/2))\sin\pi\alpha}.$$

Further let D be a domain in R^n and let u be a complex valued function in $C^\infty(D)$. The standard α-norm over D, $|u|_{\alpha,D}$, is defined as follows,

$$|u|_{0,D}^2 = \int_D |u(x)|^2\,dx,$$

and for $0<\alpha<1$,

$$|u|_{\alpha,D}^2 = |u|_{0,D}^2 + \frac{1}{C(n,\alpha)G_{2n+2\alpha}(0)} \int_D\int_D \frac{G_{2n+2\alpha}(x-y)}{|x-y|^{n+2\alpha}} |u(x)-u(y)|^2\,dx\,dy.$$

For arbitrary $\alpha \geqq 0$, let $m=[\alpha]$ be the greatest integer $\leqq \alpha$ and let $\beta=\alpha-m$. Then

$$|u|_{\alpha,D}^2 = \sum_{k=0}^{m}\binom{m}{k}\sum_{|i|\leqq k}|D_i u|_{\beta,D}^2.$$

The space $\check{P}^\alpha(D)$ is the perfect functional completion in the sense of ARONSZAJN and SMITH [4] of the functions in $C^\infty(D)$ for which $|u|_{\alpha,D}$ is finite. For $D=R^n$, $\check{P}^\alpha(D)$ is denoted simply by P^α and $|u|_{\alpha,R^n}$ by $\|u\|_\alpha$. $P^\alpha(D)$ is defined as the space of all restrictions to D of functions in P^α with the norm

$$\|u\|_{\alpha,D} = \inf\|\tilde{u}\|_\alpha$$

with the infimum taken over all $\tilde{u}\in P^\alpha$ such that $\tilde{u}=u$ except on a subset of D of 2α-capacity zero. For all domains D to be considered in the present work, $\check{P}^\alpha(D)=$ $=P^\alpha(D)$ with equivalent norms (cf. [2] or [3]). It should be noted that for such domains D, $\check{P}^\alpha(D)$ is the class of corrections (cf. [2], § 0) of functions in the class $W^{\alpha,2}(D)$ (cf. LIONS and MAGENES [18], § 2). Finally recall that $C_0^\infty(R^n)$ is dense in P^α.

Now for $u, v \in P^1$, let

$$a(u,v) = \sum_{|i|=1}\int_{R^n} D_i u\,\overline{D_i v}\,dx + \delta\int_{R^n} u\bar{v}\,dx$$

where $\delta>0$, and define V_a as the space P^1 with $a(u,v)$ as inner product. Letting $H=L^2(R^n)=P^0$ with the usual inner product, it follows by use of Fourier transforms that the operator A, defined by $a(u,v)=(Au,v)$ is given by $-\Delta u+\delta u$ for $u\in D(A)=P^2$

with an equivalent norm, and that for $0 \leqq \tau \leqq 2$, $D(A^{\tau/2}) = P^\tau$ with an equivalent norm.

Let $q \in L^1_{\mathrm{loc}}(R^n)$ be a real valued function with $q(x) \geqq 2\delta$ a.e. For $u \in L^2(R^n)$, let

$$b(u, u) = \int_{R^n} q(x)|u|^2\,dx - \delta \int_{R^n} |u|^2\,dx$$

and define V_b as the space of all $u \in L^2(R^n)$ such that $b(u, u) < \infty$, with the corresponding inner product $b(u, v)$. Then the operator B, defined by $b(u, v) = (Bu, v)$ is given by $qu - \delta u$ for

$$u \in D(B) = \{u \in L^2(R^n) : \int_{R^n} q^2 |u|^2\,dx < \infty\}$$

and, for $0 \leqq \tau \leqq 2$,

$$D(B^{\tau/2}) = \{u \in L^2(R^n) : \int_{R^n} q^\tau |u|^2\,dx < \infty\}.$$

Now if q also satisfies the condition that

$$M_{q^2}(x) = \int_{|x-y| \leqq 1} |x-y|^{2-n-\varrho} |q(y)|^2\,dy$$

is locally bounded for some constant $\varrho > 0$, it follows as in KATO [15], pp. 349—351, that each $u \in D(A) \cap D(B)$ can be "mollified", producing a sequence $\{u_n\} \subset C_0^\infty(R^n)$ converging to u in the intersection norm. Then, since the mollifying operation is linear, it follows by interpolation between $D(A)$ and H and between $D(B)$ and H separately, that for each $\tau \in [0, 2]$ and $u \in [V_a, H]_\tau \cap [V_b, H]_\tau$ the mollifiers $\{u_n\} \subset C_0^\infty(R^n)$ converge to u in $[V_a, H]_\tau \cap [V_b, H]_\tau$. Thus $D(A) \cap D(B)$ is dense in $[V_a, H]_\tau \cap [V_b, H]_\tau$ for all $\tau \in [0, 2]$.

Hence for $q \in L^1_{\mathrm{loc}}(R^n)$ such that $M_{q^2}(x)$ is locally bounded, the technical condition, "$D(A) \cap D(B)$ is dense in $[V_a, H]_\tau \cap [V_b, H]_\tau$", is always satisfied. To apply Proposition 2 one may then use criteria for essential self adjointness of $A+B$ to be found e.g. in HELLWIG [9], IKEBE and KATO [10], or JÖRGENS [11]. Conditions on q yielding self adjointness of $A+B$ have been given by TRIEBEL [23], § 6.

3. Let V_a, H be as in Section 1 and let $u, v \to c(u, v)$ be a continuous sesquilinear form on V_a. Further assume that there is a $\gamma > 0$ such that

$$\operatorname{Re} c(v, v) \geqq \gamma a(v, v) \quad \text{for all} \quad v \in V_a.$$

As previously, let C be the operator in H associated with $c(u, v)$, i.e. $(Cu, v) = c(u, v)$ for all $v \in V_a$ with $D(C) = \{u \in V_a : v \to c(u, v)$ is continuous on V_a in the topology induced by $H\}$. Then C is a closed densely defined operator whose domain is also dense in V_a. The adjoint form $c^*(u, v)$, is defined by

$$c^*(u, v) = \overline{c(v, u)}, \quad u, v \in V_a,$$

and if C^* is the operator in H associated with $c^*(u, v)$, i.e., $(C^* u, v) = c^*(u, v)$, $u \in D(C^*)$, $C^* u \in H$, $v \in V_a$, then C^* is the adjoint of C. C and C^* are *regularly accretive* operators in the terminology of KATO [12]. (Kato assumes only that $\operatorname{Re} c(v, v) + \lambda |v|^2 \geqq \gamma a(v, v)$ but replacing C by $C + \lambda$ yields the same results.) Fractional powers of these operators have been studied by various authors, a particularly useful reference being Chapter IV of SZ.-NAGY and FOIAŞ [21] (cf. also SZ.-NAGY and FOIAŞ [20] and [22]). In [17] LIONS has proven (cf. also KATO [13], KATO [14] and FOIAŞ and LIONS [7]) that for $0 \leqq \tau \leqq 1$, $D(C^\tau) = D(|C|^\tau)$ and likewise $D(C^{*\tau}) = D(|C^*|^\tau)$. It is known, [12] and [21], Theorem 5.1, that $D(C^\tau) = D(C^{*\tau})$ for $0 \leqq \tau < \frac{1}{2}$. In Théorème 6.1 of [17], LIONS has given conditions implying that $D(C^{1/2}) = D(C^{*1/2})$, and then shown that these conditions are satisfied for a large class of elliptic boundary value problems under sufficient regularity conditions.

In this section another sufficient condition for the equality $D(C^{1/2}) = D(C^{*1/2})$ will be proven. It will then be shown that the condition is satisfied in the case of the Dirichlet problem with homogeneous boundary data on Lipschitzian graph domains (cf. [2], § 11).

Theorem 2. *If there exists a Hilbert space W such that*

i) $W \underset{c}{\subseteq} D(C)$, $W \underset{c}{\subseteq} D(C^*)$, *and* ii) $V_a \subset [W, H]_{1/2}$,

then $D(C^{1/2}) = D(C^{*1/2}) = V_a$.

Proof. By i) the identity mapping is continuous from W into $D(C)$, continuous from W into $D(C^*)$, and continuous from H into H. Therefore the quadratic interpolation theorem of [16], pp. 431—432, yields $[W, H]_{1/2} \underset{c}{\subseteq} D(|C|^{1/2})$ and $[W, H]_{1/2} \underset{c}{\subseteq} D(|C^*|^{1/2})$. Thus ii) and the preceding remarks yield $V_a \subset D(C^{1/2})$ and $V_a \subset D(C^{*1/2})$. The theorem now follows from Corollaire 5.1 of [17] or the Corollary of page 243, [14].

Now let $D \subset R^n$ be a Lipschitzian graph domain and let m be a positive integer. Denote the closure of $C_0^\infty(D)$ in $\check{P}^m(D)$ by $\check{P}_0^m(D)$. For $u, v \in \check{P}_0^m(D)$ let

$$c(u, v) = \sum_{|i|, |j| \leqq m} \int_D c_{ij}(x) D_j u \overline{D_i v} \, dx$$

with $c_{ij} \in C^{|i|}(\bar{D})$ where $C^{|i|}(\bar{D})$ here means the class of functions with all partial derivatives of order $\leqq |i|$ continuous and bounded on $\bar{D}$. Further assume that there is a $\gamma > 0$ such that

$$\operatorname{Re} c(v, v) \geqq \gamma |v|_{m, D}^2 \quad \text{for all} \quad v \in \check{P}_0^m(D).$$

Now let $H = L^2(D)$, $V_a = \check{P}_0^m(D)$, and $W = \check{P}_0^{2m}(D)$. It is easily verified (as e.g. in GREENLEE [8], § 6) that $W \underset{c}{\subseteq} D(C)$ and $W \underset{c}{\subseteq} D(C^*)$. Moreover, by Theorem 5.2 of [8], $[\check{P}_0^{2m}(D), L^2(D)]_{1/2} = [W, H]_{1/2}$ and $\check{P}_0^m(D) = V_a$ coincide with equivalent norms. Thus by Theorem 2, $D(C^{1/2}) = D(C^{*1/2}) = \check{P}_0^m(D)$.

Bibliography

[1] R. D. ADAMS, N. ARONSZAJN and M. S. HANNA, Theory of Bessel potentials. III: Potentials on regular manifolds, *Ann. Inst. Fourier (Grenoble)*, **19** (1969), no. 2, 279—338.

[2] R. D. ADAMS, N. ARONSZAJN and K. T. SMITH, Theory of Bessel potentials. II, *Ann. Inst. Fourier (Grenoble)*, **17** (1967), no. 2, 1—135.

[3] N. ARONSZAJN, Potentiels Besseliens, *Ann. Inst. Fourier (Grenoble)*, **15** (1965), 43—58.

[4] N. ARONSZAJN and K. T. SMITH, Functional spaces and functional completion, *Ann. Inst. Fourier (Grenoble)*, **6** (1955—1956), 125—185.

[5] N. ARONSZAJN and K. T. SMITH, Theory of Bessel potentials. I, *Ann. Inst. Fourier (Grenoble)*, **11** (1961), 385—475.

[6] J. DIXMIER, *Les algèbres d'opérateurs dans l'espace hilbertien*, Gauthier-Villars, Paris, 1957.

[7] C. FOIAŞ and J. L. LIONS, Sur certains théorèmes d'interpolation, *Acta Sci. Math.*, **22** (1961), 269—282.

[8] W. M. GREENLEE, Rate of convergence in singular perturbations, *Ann. Inst. Fourier (Grenoble)*, **18** (1968), 135—191.

[9] G. HELLWIG, *Differentialoperatoren der mathematischen Physik*, Springer-Verlag, Berlin, 1964; *Differential operators of mathematical physics*, Addison-Wesley, Reading, Mass., 1967.

[10] T. IKEBE and T. KATO, Uniqueness of the self-adjoint extension of singular elliptic differential operators, *Arch. Rational Mech. Anal.*, **9** (1962), 77—92.

[11] K. JÖRGENS, Wesentliche Selbstadjungiertheit singulärer elliptischer Differentialoperatoren zweiter Ordnung in $C_0^\infty(G)$, *Math. Scand.*, **15** (1964), 5—17.

[12] T. KATO, Fractional powers of dissipative operators, *J. Math. Soc. Japan*, **13** (1961), 246—274.

[13] T. KATO, A generalization of the Heinz inequality, *Proc. Japan Acad.*, **37** (1961), 305—308.

[14] T. KATO, Fractional powers of dissipative operators, II, *J. Math. Soc. Japan*, **14** (1962), 242—248.

[15] T. KATO, *Perturbation theory for linear operators*, Springer-Verlag (Berlin, 1966).

[16] J. L. LIONS, Espaces intermédiaires entre espaces hilbertiens et applications, *Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine* (N. S.) **2 (50)**, (1958), 419—432.

[17] J. L. LIONS, Espaces d'interpolation et domains de puissances fractionnaires d'opérateurs, *J. Math. Soc. Japan.*, **14** (1962), 233—241.

[18] J. L. LIONS and E. MAGENES, Problemi ai limiti non homogenei. III, *Ann. Scuola Norm. Sup. Pisa*, **15** (1961), 41—103.

[19] J. L. LIONS and E. MAGENES, *Problèmes aux limites non homogènes et applications*, Vol. 1, Dunod (Paris, 1968).

[20] B. SZ.-NAGY and C. FOIAS, Sur les contractions de l'espace de Hilbert. VI. Calcul fonctionnel, *Acta Sci. Math.*, **23** (1962), 130—167.

[21] B. SZ.-NAGY and C. FOIAŞ, *Analyse harmonique des opérateurs de l'espace de Hilbert*, Akadémiai Kiadó and Masson (Budapest and Paris, 1967); *Harmonic analysis of operators on Hilbert space*, North-Holland (Amsterdam, 1970).

[22] B. SZ.-NAGY and C. FOIAŞ, Accretive operators: corrections, *Acta Sci. Math.* **33** (1972), 117—118.

[23] H. TRIEBEL, Singuläre elliptische Differentialgleichungen und Interpolationssätze für Sobolev—Slobodeckij-Räume mit Gewichtsfunktionen, *Arch. Rational Mech. Annal.*, **32** (1969), 113—134.

UNIVERSITY OF ARIZONA
TUCSON, ARIZONA

(Received May 12, 1972)

On operator radii

By K. GUSTAFSON* and B. ZWAHLEN in Lausanne (Switzerland)

For a bounded linear operator T on a (real or complex) Banach space X, one has the relation

$$|\sigma(T)| \leqq |W(T)| \leqq \|T\|$$

between the spectral radius $|\sigma(T)|$, the numerical radius $|W(T)|$, and the operator radius $\|T\|$ (see definitions below). In a complex Banach space one has additionally that

$$\|T\| \leqq c|W(T)|,$$

where $c=2$ for a complex Hilbert space X (e.g., see [6]), whereas $c=e$ (see [1], [4], [9]) for a complex Banach space.

In this note we will examine the relations between these three radii $|\sigma(T)|$, $|W(T)|$, and $\|T\|$ for an arbitrary densely defined operator T in X.

We recall the definitions:

$$|\sigma(T)| = \sup|\lambda|, \quad \lambda \text{ in the spectrum } \sigma(T),$$

$$|W(T)| = \sup|\lambda|, \quad \lambda \text{ in the numerical range } W(T),$$

$$\|T\| = \sup\|Tx\|, \quad x \text{ in the domain } D(T) \text{ of } T, \quad \|x\| = 1,$$

where $W(T)=\{x^*Tx \mid x\in D(T),\ \|x\|=1,\ x^*\in J(x)\}$ and

$$J(x) = \{x^*\in X^* \mid x^*x = \|x\|^2 = \|x^*\|^2\}.$$

$J(x)$ denotes the totality of the "Hahn—Banach" duality vectors $x^*\in X^*$ for a given x, whereas here the numerical range $W(T)$ is to be understood as defined in terms of a single x^* selected from $J(x)$ for each x. Sometimes (e.g., see [2]) the numerical range of T is defined by $V(T)=\{x^*Tx \mid x\in D(T),\ \|x\|=1, \text{ all } x^*\in J(x)\}$, i.e. $V(T)= =\bigcup W_\varphi(T)$, for all functions $\varphi\colon D(T)\to J(D(T))$. Each such function $\varphi\colon X\to J(X)$ defines a semi-inner product $[y,x]=x^*y$ on X, and conversely each semi-inner product consistent with the norm $\|x\|$ is given exactly by a φ. For further information con-

* Partially supported by NSF GP 15239 A-1.

cerning semi-inner products and numerical ranges for bounded operators and Banach-algebras see the recent book by BONSALL—DUNCAN [2].

The general situation for the four cases 1) X a real Hilbert space, 2) X a complex Hilbert space, 3) X a real Banach space, 4) X a complex Banach space, is summarized by the following theorem.

Theorem. *Let T be a densely defined linear operator in X; then in cases* 1), 3), 4)

$$|\sigma(T)| = \infty \Rightarrow \|T\| = \infty \Leftarrow |W(T)| = \infty$$

and in case 2)

$$|\sigma(T)| = \infty \Rightarrow \|T\| = \infty \Leftrightarrow |W(T)| = \infty.$$

In 1), 2), *and* 3) *all other implications are false in general. In* 4), *for T closed, $\|T\| = \infty$ implies that $|W(T)| = \infty$ or $|\sigma(T)| = \infty$.*

Proof. We will consider in turn the six possible implications between the three conditions

$$|\sigma(T)| = \infty, \quad \|T\| = \infty, \quad |W(T)| = \infty.$$

In all cases $|\sigma(T)| = \infty \Rightarrow \|T\| = \infty$ follows from the defect index theory, and $|W(T)| = \infty \Rightarrow \|T\| = \infty$ follows from the Schwarz inequality.

The possible implication $\|T\| = \infty \Rightarrow |\sigma(T)| = \infty$ in case 1) (and hence case 3)) is ruled out by the example $T_1 = \oplus \begin{pmatrix} 0 & n \\ -n & 0 \end{pmatrix}$ $(n = 1, 2, 3, \ldots)$, the direct sum operator in real $l_R^2 = X$ with $D(T_1) = M$, the subspace of l^2 consisting of all vectors which have only a finite number of nonzero components. T_1 is unbounded, $\sigma(T_1)$ is empty and $W(T_1) = \{0\}$. To obtain a closed counterexample, one may observe that the closed operator $T_2 = \hat{T}_1$, the closure of T_1, has the same properties. The derivative operator $T_3 u = u'$, $D(T_3) = \{u \mid u$ absolutely continous, $u' \in L^2$, $u(0) = 0\} \subset L_C^2(0, 1) = X$ has empty spectrum and is closed and unbounded, and hence serves to negate this implications also in the cases 2) and 4).

The implication $|W(T)| = \infty \Rightarrow |\sigma(T)| = \infty$ is ruled out for the cases 1) and 3) by the example $T_4 = \oplus \begin{pmatrix} 0 & n \\ -n^2 & 0 \end{pmatrix}$ $(n = 1, 2, 3, \ldots)$ in l_R^2 with $D(T_4) = M$, since $W(T_4)$ is unbounded but $\sigma(T_4)$ is empty. For a closed counterexample with the same properties as T_4, take $T_5 = \hat{T}_4$. The counterexample T_3 given above negates the complex cases 2) and 4), since $|W(T_3)| = \infty$.

The remaining two possible implications are

$$\|T\| = \infty \Rightarrow |W(T)| = \infty \quad \text{and} \quad |\sigma(T)| = \infty \Rightarrow |W(T)| = \infty.$$

The example T_2 rules out the first implication in the cases 1) and 3), since $W(T_2) = \{0\}$, and the following example T_6 negates both implications in the cases 1) and 3). Let $T_6 u = u'$ with $D(T_6) = \{u \mid u$ absolutely continuous, $u' \in L^2$, $u(0) =$

$=u(1)=0\}\subset L^2_R(0,1)=X$. Then T_6 is unbounded, but $W(T_6)=\{0\}$ because for $u\in D(T_6)$ one has $(T_6u,u)=\frac{1}{2}\int_0^1 [u^2(x)]'\,dx=0$; moreover $|\sigma(T_6)|=\infty$ because the residual spectrum $\sigma_r(T_6)$ is the whole real line (since $R(\lambda-T_6)\perp e^{\lambda x}$ for each real λ).

In the case 2) of X a complex Hilbert space both of the above mentioned remaining two implications are true. It suffices of course to demonstrate the first (perhaps known). Let T be unbounded and densely defined and suppose that $|W(T)|<\infty$. Then by polarization and the parallelogram law, one has for $x,y\in D(T)$, that

$$|(Tx,y)|\leqq |W(T)|\cdot 4^{-1}[\|x+y\|^2+\|x+iy\|^2+\|x-y\|^2+\|x-iy\|^2]=$$
$$=|W(T)|\cdot[\|x\|^2+\|y\|^2],$$

so that $|(Tx,y)|=\|x\|\cdot\|y\|\cdot|(\|x\|^{-1}Tx,\|y\|^{-1}y)|\leqq 2|W(T)|\cdot\|x\|\cdot\|y\|$. Since $D(T)$ is dense, $\|Tx\|\cdot\|x\|^{-1}\leqq 2|W(T)|<\infty$, and T is bounded. Finally, in case 4) of X a complex Banach space and T a closed operator, it is known (KATO [7, p. 176]) that if $|\sigma(T)|<\infty$ then $\|T\|=\infty$ if and only if the resolvent operator $(\lambda-T)^{-1}$ has an essential singularity at infinity. Hence if both $|\sigma(T)|<\infty$ and $|W(T)|<\infty$, by noting that the latter implies that $\|(\lambda-T)^{-1}\|\to 0$ as $|\lambda|\to 0$, one has $\|T\|<\infty$. This concludes the proof of the theorem.

Remarks. We conclude with the following remarks.

1. The implications $|\sigma(T)|=\infty\Rightarrow\|T\|=\infty\Leftarrow|W(T)|=\infty$ clearly hold in a normed linear space also.

2. A special situation arises when T is everywhere defined on a Banach space X, i.e. when $D(T)=X$. By a well-known "metatheorem", then almost any additional condition will make T bounded.*)

In this situation, when $|(W(T)|<\infty$, by the closeability of T (see remark 3 below) one knows that T is closed and hence bounded (by the closed graph theorem).

Moreover, by the following arguments (perhaps known) it follows that $|\sigma(T)|<\infty$ and $D(T)=X$ imply that T is bounded.

a) Let $D(T)=X$; then T^* is bounded. This can be seen by letting $z_n^*=T^*y_n^*$ for any sequence $\{y_n^*\}$ in $D(T^*)$, $\|y_n^*\|=1$; fixing x, one has $z_n^*(x)=T^*y_n^*(x)=$ $=y_n^*Tx\leqq\|Tx\|$ so that (by the uniform boundedness principle) $\{\|T^*y_n^*\|\}$ is a bounded set.

b) Let $|\sigma(T)|<\infty$, $D(T)=X$; then by a) T^* is bounded. For $|\lambda|>\|T^*\|$ one has $0=\text{codim}\,\overline{R(\lambda-T)^*}=\text{codim}\,\overline{R(\bar\lambda I|D(T^*))}=\text{codim}\,\overline{D(T^*)}$, so that $D(T^*)$ is dense, and hence $D(T^*)=X^*$, which holds if and only if T is bounded.

---

*) For example, this has been recently put on a logical basis by M. AJTAI, On the boundedness of definable linear operators, *Periodica Math. Hungarica* (to appear).

In summary, when $D(T)=X$ a real or complex Banach space, one has

$$|\sigma(T)| = \infty \Leftrightarrow \|T\| = \infty \Leftrightarrow |W(T)| = \infty.$$

3. It is known (see KATO [7, p. 268]) for a Hilbert space that if $W(T)$ is not the whole plane, then T is closeable. This generalizes (e.g., see [10], [11]) to a Banach space when $W(T)$ is in a half plane (or half line in the real case.) Let us observe here that one can say roughly that some $W_\varphi(T)$ "not the whole complex plane" implies that T is closeable in the Banach space also. In particular, this will be the case when $W(T)$ misses an external sector somewhere in the plane; other geometrical situations that are included will be evident from the proof.

More precisely, let there exist a sequence of scalars $\{\lambda_k\}$, $|\lambda_k|\to\infty$, such that $d(\lambda_k, W(T))/|\lambda_k|\not\to 0$, and let T be densely defined in a normed linear space X (X either real or complex); then T is closeable.

Suppose, to the contrary, that there exists a sequence $x_n \in D(T)$, $x_n\to 0$, $Tx_n\to y$, $\|y\|=1$. By hypothesis we may assume $d(\lambda_k, W(T))/|\lambda_k| \geqq \varepsilon > 0$, for some fixed ε. By $D(T)$ dense, there exists $z_\varepsilon \in D(T)$, $\|z_\varepsilon\|=1$, $\|z_\varepsilon - y\| < \varepsilon/2$. Let

$$g(n,k) = \|\lambda_k x_n + z_\varepsilon - y - \lambda_k^{-1} T z_\varepsilon\|;$$

then

$$\lim_{n\to\infty} g(n,k) = \|z_\varepsilon - y - \lambda_k^{-1} T z_\varepsilon\| < \varepsilon/2 + |\lambda_k|^{-1}\|Tz_\varepsilon\|,$$

for fixed k. On the other hand, letting

$$u_{nk} = (x_n + \lambda_k^{-1} z_\varepsilon)\,\|x_n + \lambda_k^{-1} z_\varepsilon\|^{-1},$$

one has by Schwarz's inequality that

$$g(n,k) = \|(\lambda_k - T)(x_n + \lambda_k^{-1} z_\varepsilon) + (Tx_n - y)\| \geqq \|(\lambda_k - T)(x_n + \lambda_k^{-1} z_\varepsilon)\| - \|Tx_n - y\| \geqq$$

$$\geqq |\lambda_k - [Tu_{nk}, u_{nk}]|\,\|x_n + \lambda_k^{-1} z_\varepsilon\| - \|Tx_n - y\| \geqq d(\lambda_k, W(T))\,\|x_n + \lambda_k^{-1} z_\varepsilon\| - \|Tx_n - y\|.$$

Hence

$$\lim_{n\to\infty} g(n,k) \geqq d(\lambda_k, W(T))/|\lambda_k| \geqq \varepsilon.$$

But from the first estimate above, noting that $\|Tz_\varepsilon\|$ does not depend on k, one has $\varepsilon > \lim g(n,k)$ for k sufficiently large, contradicting the second estimate.

We mention that for X such that J is single valued and continuous (e.g., see [1], [3], [8]), one has additionally for closeable T that $\overline{W(\hat{T})} = \overline{W(T)}$ as in the Hilbert space case, since $x_n \to x$, $Tx_n \to \hat{T}x$ imply that $x_n^* T x_n \to x^* \hat{T} x$.

4. Although we have not done so here, one can make $|\sigma_{ext}(T)|=\infty \Leftrightarrow \|T\|=\infty$ by using the notion of extended spectrum (e.g., see [7]).

5. Of course, not all of the considered implications are independent. In particular, one has $\{\|T\|=\infty \Rightarrow |W(T)|=\infty\} \Leftrightarrow \{|\sigma(T)|=\infty \Rightarrow |W(T)|=\infty\}$ in case 4): to the right, by the previously noted general implications; and to the left, by the following argument. Given $\|T\|=\infty$, if $|W(T)|<\infty$, then by the right hand implication we would have $|\sigma(T)|<\infty$, and then, using the result [7, p. 176] already used above, one has $\|T\|<\infty$, a contradiction.

6. To recapitulate, exactly the following situations occur:

a) $\|T\| < \infty$, $|\sigma(T)| < \infty$, $|W(T)| < \infty$ cases 1)—4)

b) $\|T\| = \infty$, $|\sigma(T)| = \infty$, $|W(T)| = \infty$ cases 1)—4)

c) $\|T\| = \infty$, $|\sigma(T)| < \infty$, $|W(T)| = \infty$ cases 1)—4)

d) $\|T\| = \infty$, $|\sigma(T)| = \infty$, $|W(T)| < \infty$ cases 1), 3), not 2)

e) $\|T\| = \infty$, $|\sigma(T)| < \infty$, $|W(T)| < \infty$ cases 1), 3), not 2), not 4) for T closed.

7. There remains the question of whether $\|T\|=\infty \Rightarrow |W(T)|=\infty$ in the case 4). An exception to this clearly cannot occur, for example, when any of the following conditions prevails: a) $|\sigma(T)|<\infty$; b) $\exists\lambda\in\varrho(T)$, $|\lambda|>|W(T)|$; c) $|W(T^*)|<\infty$; d) $J(D(T))$ contains an eigenvector of $(\bar{\lambda}-T^*)$, $|\lambda|>W(T)$.

8. Finally we mention that one can construct a proof in the case 2) different from that given above; this proof completely avoids both polarization and the parallelogram law but still requires a bilinear form. The argument is similar to that used in [5] to show that the cosine of an unbounded operator is always zero, and we omit the details.

References

[1] H. F. Bohnenblust and S. Karlin, Geometrical properties of Banach algebras, *Annals of Math.*, **62** (1955), 217—229.

[2] F. F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and of elements of normed algebras,* Cambridge University Press (1971).

[3] J. R. Giles, Classes of semi-inner product spaces, *Trans. Amer. Math. Soc.*, **129** (1967), 436—446.

[4] B. W. Glickfeld, On an inequality of Banach algebra geometry and semi-inner-product space theory, *Ill. J. Math.*, **14** (1970), 76—81.

[5] K. Gustafson and B. Zwahlen, On the cosine of unbounded operators, *Acta Sci. Math.*, **30** (1969), 33—34.
[6] P. Halmos, *A Hilbert space problem book*, Van Nostrand (1967).
[7] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag (1966).
[8] T. Kato, Some mapping theorems for the numerical range, *Proc. Jap. Acad.*, **41** (1965), 652—655.
[9] G. Lumer, Semi-inner-product spaces, *Trans. Amer. Math. Soc.*, **100** (1961), 29—43.
[10] G. Lumer and R. S. Phillips, Dissipative operators in a Banach space, *Pac. J. Math.*, **11** (1961), 679—698.
[11] Ken-iti Sato, On dispersive operators in Banach lattices, *Pac. J. Math.*, **33** (1970), 429—443.

UNIVERSITY OF COLORADO AND
ECOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

(Received December 15, 1972, revised October 15, 1973)

Об одном двустороннем итерационном методе решения краевой задачи с запаздыванием

Ю. И. КОВАЧ (Ужгород, СССР), J. HEGEDŰS (Szeged)

Краевые задачи типа Штурма—Лиувилля для дифференциальных уравнений с отклоняющимся аргументом представляют большой интерес (см. напр. [5], [6]).

Задачи такого рода рассматривались напр. в [1], [2], [3], а также и в [5], [6] (см. библиографию в этих книгах), где доказывались теоремы существования и единственности, теоремы о собственных значениях и собственных функциях. Для аналитического и численного решения этих задач и оценки решения и его производных удобным средством является метод двусторонних приближений, который применялся для разных задач с запаздыванием [7—10].

В этой работе изложен двусторонний итерационный метод решения краевой задачи для дифференциального уравнения n-го порядка ($n \geqq 2$) с запаздывающим аргументом такого типа, где первые $n-1$ граничных условий задаются в начальной точке, а последнее в правом конце рассматриваемого отрезка. Тем самым мы на эту задачу распространили результаты работы [9], где для краевой задачи иного типа построены двусторонние приближения в случае системы уравнений специального вида. Эта статья примыкает к работе [10], где строился итерационный метод решения начальной задачи для системы с запаздыванием.

Краевая задача, которую мы будем рассматривать, для обыкновенного уравнения n-го порядка рассматривалась в [4].

В конце работы мы коротко отметим, как переносятся основные результаты на системы уравнений n_i-го порядка ($n_i \geqq 2$, $i=1, 2, \ldots, r$).

1. Постановка задачи

В разделах 1—6 мы будем рассматривать следуюшую краевую задачу:

(1.1)

$$y^{(n)}(x) = f[y] \equiv f\big(x, y(x), \ldots, y^{(n-1)}(x), y(g_0(x)), \ldots, y^{(n-1)}(g_{n-1}(x))\big) \quad (0 \leqq x \leqq 1),$$

(1.2) $$y(0) = \ldots = y^{(n-2)}(0) = y^{(n-1)}(1) = 0 \quad (n \geqq 2),$$

(1.3) $$y|_E = Q(x),$$

где заданные функции f, g_i, Q и начальное множество E удовлетворяют следующим условиям:

а) $f(x, u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1})$ определена в одном из двух $(2n+1)$-мерных брусов D_K и D_∞:

$$\left.\begin{array}{l} D_K: 0 \leqq x \leqq 1,\ |u_i| \leqq K,\ |v_i| \leqq K, \text{ где } K \text{ постоянная, } K > 0, \\ D_\infty: 0 \leqq x \leqq 1,\ |u_i| < \infty,\ |v_i| < \infty \end{array}\right\} \quad (i = 0, \ldots, n-1),$$

f непрерывна и удовлетворяет условию Липшица:

$$|f(x, \ldots, \tilde{v}_{n-1}) - f(x, \ldots, v_{n-1})| \leqq N \sum_{i=0}^{n-1} (|\tilde{u}_i - u_i| + |\tilde{v}_i - v_i|),$$

причем если f определена только в D_K то $|f| \leqq K$.

б) $g_i \in C[0, 1]$, $\lambda \leqq g_i(x) \leqq x \quad (i = 0, \ldots, n-1)$,

где λ отрицательная постоянная, g_{n-1} не меняет знак на [0, 1] (см. Замечание 6.1),

в) $E = [\lambda, 0]$,

г) $Q(x)$ определена и $(n-1)$-раз непрерывно дифференцируема на E и

$$Q(0) = \ldots = Q^{(n-2)}(0) = 0,$$

причем в случае если f определена только в D_K, то

$$|Q^{(i)}| \leqq K \quad (i = 0, \ldots, n-1).$$

При этих условиях функцию $y(x)$ мы будем называть решением задачи (1.1), (1.2), (1.3) если она принадлежит классу $C^{n-2}[\lambda, 1]$ $\big(y^{(n-1)}(x)$ при $x=0$ может иметь разрыв первого рода$\big)$, а сужение ее на [0, 1] классу $C^n[0, 1]$ и если она удовлетворяет уравнению (1.1) на $[+0, 1]$ и условиям (1.2), (1.3).

Отметим, что краевая задача рассматриваемая на произвольном отрезке $[a, b]$, заменой $t = \dfrac{x-a}{b-a}$ сводится к краевой задаче на отрезке [0, 1], а условия

$$y(0) = y_0, \ldots, y^{(n-2)}(0) = y_{n-2}; \quad y^{(n-1)}(1) = y_{n-1}$$

подстановкой

$$z(x) = y(x) - \left[y_0 + y_1 x + \ldots + \frac{y_{n-2}x^{n-2}}{(n-2)!} + \frac{y_{n-1}x^{n-1}}{(n-1)!}\right]$$

сводятся к нулевым условиям (1.2).

Последний раздел (разд. 7) посвящен краевой задаче для системы уравнений типа (1.1).

2. Существование и единственность решения задачи

Очевидно, что задача (1.1), (1.2), (1.3) эквивалентна задаче:

$$(2.1)\qquad y(x) = \begin{cases} Q(x), & x \in E, \\ -\int_0^1 G(x,t)f[y(t)]\,dt, & x \in [0,1], \end{cases} \qquad y(x)|_{[0,1]} \in C^{n-1}[+0, 1],$$

где $G(x, t)$ есть функция Грина задачи

$$y^{(n)}(x) = h(x) \quad (0 \leqq x \leqq 1), \quad y(0) = \ldots = y^{(n-2)}(0) = y^{(n-1)}(1) = 0,$$

а именно

$$G(x,t) = \begin{cases} \dfrac{x^{n-1}}{(n-1)!} & (x \leqq t), \\ \dfrac{x^{n-1} - (x-t)^{n-1}}{(n-1)!} & (t \leqq x). \end{cases}$$

Отметим некоторые свойства функции G:

$$0 \leqq \frac{\partial^j G(x,t)}{\partial x^j} \leqq \frac{\partial^j G(1,t)}{\partial x^j} = \frac{1-(1-t)^{n-1-j}}{(n-1-j)!} \quad (j = 0, \ldots, n-2),$$

$$0 \leqq \frac{\partial^{n-1} G(x,t)}{\partial x^{n-1}} \leqq 1,$$

$$\frac{d^n}{dx^n}\left(-\int_0^1 G(x,t)F(t)\,dt\right) = F(x) \quad (F \in C[0,1]).$$

Введем обозначение

$$\mathcal{M}_i = \{x : g_i(x) > 0\} \quad (i = 0, \ldots, n-1).$$

Теорема 2.1. *Если выполнено условие*

$$(2.2)\qquad N\left[2 - \frac{1}{n!} + \max_i \int_{\mathcal{M}_i} \left(1 + \sum_{k=0}^{n-2} \frac{\partial^k G(1,t)}{\partial x^k}\right) dt\right] < 1,$$

то решение задачи (1.1), (1.2), (1.3) *существует и единственно.*

Доказательство. Рассмотрим множество

$$M=\begin{cases}\{z: z\in C^{n-2}[\lambda,1],\quad z|_E=Q,\quad z|_{[0,1]}\in C^{n-1}[0,1],\quad |z^{(i)}|\leqq K,\quad z^{(n-1)}(1)=0\}\\ \qquad (i=0,\dots,n-1),\\ \text{если } f \text{ определена только в } D_K,\\ \{z: z\in C^{n-2}[\lambda,1],\quad z|_E=Q,\quad z_{[0,1]}\in C^{n-1}[0,1],\quad z^{(n-1)}(1)=0\},\\ \text{если } f \text{ определена в } D_\infty.\end{cases}$$

Вводим норму в M по формуле

$$\|z\|=\sum_{i=0}^{n-1}\max_{x\in[0,1]}|z^{(i)}(x)|.$$

На M определим оператор A:

$$Az=\begin{cases}Q(x), & x\in E,\\ -\int\limits_0^1 G(x,t)f[z(t)]\,dt, & x\in[0,1].\end{cases}$$

Непосредственно можно убедиться в том, что оператор A множество M переводит в M, и что задача (1.1), (1.2), (1.3) эквивалентна уравнению

$$y=Ay.$$

Используя условие Липшица и свойства G, для любых $z, y\in M$ получаем

$$\begin{aligned}\|Ay-Az\|\leqq N\int\limits_0^1\sum_{i=0}^{n-1}\left[|y^{(i)}(t)-z^{(i)}(t)|+\left|y^{(i)}(g_i(t))-z^{(i)}(g_i(t))\right|\right]dt+\\ +N\sum_{k=0}^{n-2}\int\limits_0^1\frac{\partial^k G(1,t)}{\partial x^k}\sum_{i=0}^{n-1}\left[|y^{(i)}(t)-z^{(i)}(t)|+\left|y^{(i)}(g_i(t))-z^{(i)}(g_i(t))\right|\right]dt.\end{aligned}\tag{2.3}$$

Вместо суммы $\sum\limits_{i=0}^{n-1}|y^{(i)}(t)-z^{(i)}(t)|$ поставим число $\|y-z\|$, а вместо $\left|y^{(i)}(g_i(t))-z^{(i)}(g_i(t))\right|$ для тех $t\in[0,1]$, для которых $g_i(t)>0$ ставим число $\max\limits_{t\in[0,1]}|y^{(i)}(t)-z^{(i)}(t)|$, для остальных t это слагаемое обращается в нуль. Итак получим:

$$\|Ay-Az\|\leqq\|y-z\|N\left[2-\frac{1}{n!}+\max_i\int\limits_{M_i}\left(1+\sum_{k=0}^{n-2}\frac{\partial^k G(1,t)}{\partial x^k}\right)dt\right].$$

Значит при выполнении условия (2.2) A будет сжатием. В силу теоремы о сжатом отображении, решение уравнения $y=Ay$ существует, единственно и может быть найдено методом последовательных приближений.

Теорема доказана.

Если в (2.3) вместо всех производных G поставим единицу, то получаем более простое условие сжатости $2nN<1$.

В следующих разделах мы будем излагать итерационные методы. В разделах 3, 4, 5 фигурируют предположения о существовании некоторых функций $z_1(x)$, $w_1(x)$ с неположительной и неотрицательной невязкой соответственно. В случае бруса D_K, если $|f| \leqq K$, такие $z_1(x)$, $w_1(x)$ можно найти явно: именно можно взять функцию равную $Q(x)$ на E и некоторому многочлену на $[0, 1]$, а если условие (2.2) выполнено, то легко показать, что они существуют и для случая бруса D_∞. Аналогичное утверждение справедливо и для системы уравнений (разд. 7).

3. Нелинейное уравнение с неположительными частными производными

Поставленную задачу мы сейчас будем решать для следующего частного случая.

(i) Условия б), в), г) выполнены,

$$\text{(ii)}\quad \left.\begin{array}{l} f,\ \dfrac{\partial f}{\partial u_i},\ \dfrac{\partial f}{\partial v_i} \in C(D_K) \quad \text{или} \quad C(D_\infty) \\ \dfrac{\partial f}{\partial u_i},\ \dfrac{\partial f}{\partial v_i} \leqq 0 \end{array}\right\} \quad (i = 0, \dots, n-1),$$

(iii) A сжато отображает M в M,

(iv) Существуют две функции $z_1(x)$, $w_1(x) \in M$, n-раз непрерывно дифференцируемые на $[0, 1]$, для которых

$$\alpha_1(x) = z_1^{(n)}(x) - f[z_1] \leqq 0, \quad \beta_1(x) = w_1^{(n)}(x) - f[w_1] \geqq 0.$$

Двусторонние приближения к решению $y(x)$ строятся по формулам

$$z_{p+1} = Az_p, \quad w_{p+1} = Aw_p \quad (p = 1, 2, \dots).$$

Для них справедлива следующая

Теорема 3.1. *При $p \to \infty$ справедливо*

$$\left.\begin{array}{ll} w_p^{(i)}(x) \nearrow\!\!\nearrow y^{(i)}(x) \swarrow\!\!\swarrow z_p^{(i)}(x) & (i = 0, \dots, n-1) \\ z_p^{(n)}(x) \nearrow\!\!\nearrow y^{(n)}(x) \swarrow\!\!\swarrow w_p^{(n)}(x) & \end{array}\right\} \quad (0 \leqq x \leqq 1)\ ^*).$$

Доказательство. Учитывая, что A сжатие получаем: все z_p, w_p принадлежат M и n-раз непрерывно дифференцируемы на $[0, 1]$, кроме того

$$\|z_p - y\|, \quad \|w_p - y\| \to 0 \quad (p \to \infty),$$

*) Запись $w_p^{(i)}(x) \nearrow\!\!\nearrow y^{(i)}(x)$, $y^{(i)}(x) \swarrow\!\!\swarrow z_p^{(i)}(x)$ означает, что последовательности $\{w_p^{(i)}(x)\}$, $\{z_p^{(i)}(x)\}$ монотонно не убывают, соотв. не возрастают и равномерно сходятся к $y^{(i)}(x)$.

а это означает, что при $p\to\infty$

$$z_p^{(i)}(x)-y^{(i)}(x),\quad w_p^{(i)}(x)-y^{(i)}(x)\rightrightarrows 0\quad (i=0,\dots,n-1;\quad 0\leqq x\leqq 1).$$

Отсюда в силу непрерывности f получаем, что при $p\to\infty$

$$z_p^{(n)}(x)-y^{(n)}(x),\quad w_p^{(n)}(x)-y^{(n)}(x)\rightrightarrows 0\quad (0\leqq x\leqq 1).$$

Нам осталось доказать монотонность последовательностей $\{z_p^{(s)}(x)\}$, $\{w_p^{(s)}(x)\}$ $(s=0,\dots,n)$. Для этого заметим, что при всех p натуральных справедливы следующие формулы

$$z_{p+1}(x)=z_p(x)-\eta_p(x),\quad w_{p+1}(x)=w_p(x)-\vartheta_p(x), \tag{3.1}$$

где

$$\eta_p(x)=\begin{cases}0, & x\in E,\\ -\int\limits_0^1 G(x,t)\alpha_p(t)\,dt, & x\in[0,1],\end{cases}$$

$$\vartheta_p(x)=\begin{cases}0, & x\in E,\\ -\int\limits_0^1 G(x,t)\beta_p(t)\,dt, & x\in[0,1],\end{cases}$$

а невязки $\alpha_p(x)$, $\beta_p(x)$ определяются по формулам

$$\alpha_p(x)=z_p^{(n)}(x)-f[z_p],\quad \beta_p(x)=w_p^{(n)}(x)-f[w_p].$$

Непосредственной проверкой можно убедиться в том, что при всех p натуральных

$$\begin{aligned}\alpha_{p+1}(x)&=\sum_{i=0}^{n-1}\frac{\partial f}{\partial u_i}\bigg|\,\eta_p^{(i)}(x)+\frac{\partial f}{\partial v_i}\bigg|\,\eta_p^{(i)}(g_i(x)),\\ \beta_{p+1}(x)&=\sum_{i=0}^{n-1}\frac{\partial f}{\partial u_i}\bigg|\,\vartheta_p^{(i)}(x)+\frac{\partial f}{\partial v_i}\bigg|\,\vartheta_p^{(i)}(g_i(x)),\end{aligned}\tag{3.2}$$

где как и в дальнейшем через $\frac{\partial f}{\partial u_i}\Big|$, $\frac{\partial f}{\partial v_i}\Big|$ обозначаем промежуточное (по формуле Лагранжа) значения этих производных. Воспользуясь этой формулой, на основании (iv) получаем, что при всех p натуральных

$$\alpha_p(x)\leqq 0,\quad \beta_p(x)\geqq 0,$$

а отсюда по (3.1) получаем доказываемую монотонность.

Следствие 3.1. *Если* $Q(x)\equiv 0$ *и* $f(x,0,\dots,0)\geqq 0$, *то*

$$y^{(i)}(x)\leqq 0,\quad y^{(n)}(x)\geqq 0\quad (i=0,\dots,n-1;\quad 0\leqq x\leqq 1),$$

а если $Q(x) \equiv 0$ *и* $f(x, 0, \ldots, 0) \leqq 0$, *то*

$$y^{(i)}(x) \geqq 0, \quad y^{(n)}(x) \leqq 0 \quad (i = 0, \ldots, n-1; \quad 0 \leqq x \leqq 1).$$

Это сразу получается из теоремы 3.1, ведь в первом случае можно взять $z_1(x) \equiv 0$, а во втором $w_1(x) \equiv 0$.

Приведенные в этом разделе результаты справедливы в частности для линейных уравнений с неположительными коэффициентами.

4. Нелинейное уравнение с неотрицательными частными производными

Предположим, что все условия предыдущего раздела выполнены, кроме второй строки (ii), заменяющейся в этом разделе условием

$$\frac{\partial f}{\partial u_i}, \quad \frac{\partial f}{\partial v_i} \geqq 0 \quad (i = 0, \ldots, n-1).$$

При обозначениях предыдущего раздела для решения $y(x)$ поставленной задачи и последовательностей $\{z_p(x)\}$, $\{w_p(x)\}$, образованных как и прежде методом последовательных приближений, исходя из $z_1(x)$, $w_1(x)$, справедлива следующая

Теорема 4.1. *Если* $\alpha_1(x) + \alpha_2(x) \leqq 0$, *то при* $l \to \infty$ $(l = 1, 2, \ldots)$

$$\left.\begin{array}{ll} z_{2l}^{(i)}(x) \nearrow\!\!\nearrow y^{(i)}(x) \swarrow\!\!\swarrow z_{2l-1}^{(i)}(x) & (i = 0, \ldots, n-1) \\ z_{2l-1}^{(n)}(x) \nearrow\!\!\nearrow y^{(n)}(x) \swarrow\!\!\swarrow z_{2l}^{(n)}(x) & \end{array}\right\} \quad (0 \leqq x \leqq 1),$$

аналогично из $\beta_1(x) + \beta_2(x) \geqq 0$ *следует, что при* $l \to \infty$ $(l = 1, 2, \ldots)$

$$\left.\begin{array}{ll} w_{2l-1}^{(n)}(x) \nearrow\!\!\nearrow y^{(i)}(x) \swarrow\!\!\swarrow w_{2l}^{(i)}(x) & (i = 0, \ldots, n-1) \\ w_{2l}^{(n)}(x) \nearrow\!\!\nearrow y^{(n)}(x) \swarrow\!\!\swarrow w_{2l-1}^{(n)}(x) & \end{array}\right\} \quad (0 \leqq x \leqq 1).$$

Доказательство. Как и в доказательстве теоремы 3.1 по теореме о сжатом отображении получаем, что все $z_p, w_p \in M$ $(p = 1, 2, \ldots)$ и что они n-раз непрерывно дифференцируемы на $[0, 1]$, кроме того при $p \to \infty$

$$z_p^{(s)}(x) \rightrightarrows y^{(s)}(x), \quad w_p^{(s)}(x) \rightrightarrows y^{(s)}(x) \quad (0 \leqq x \leqq 1, \quad s = 0, \ldots, n).$$

Нам осталось доказать монотонность подпоследовательностей фигурирующих в теореме.

Применяя формулу Лагранжа, легко показать, что при всех $p = 1, 2, \ldots$

$$\alpha_{p+1}(x) = \sum_{i=0}^{n-1} \left[\frac{\partial f}{\partial u_i} \bigg| \eta_p^{(i)}(x) + \frac{\partial f}{\partial v_i} \bigg| \eta_p^{(i)}(g_i(x)) \right],$$

$$\beta_{p+1}(x) = \sum_{i=0}^{n-1} \left[\frac{\partial f}{\partial u_i} \bigg| \vartheta_p^{(i)}(x) + \frac{\partial f}{\partial v_i} \bigg| \vartheta_p^{(i)}(g_i(x)) \right],$$

а отсюда, поскольку $\alpha_1(x) \leqq 0$, $\beta_1(x) \geqq 0$ по формулам (3.1) получаем, учитывая неотрицательность частных производных у f, при $0 \leqq x \leqq 1$; $i=0, \ldots, n-1$; $l=1, 2, \ldots$

$$z_{2l-1}^{(i)}(x) \geqq z_{2l}^{(i)}(x), \quad z_{2l+1}^{(i)}(x) \geqq z_{2l}^{(i)}(x), \quad z_{2l}^{(n)}(x) \geqq z_{2l-1}^{(n)}(x), \quad z_{2l}^{(n)}(x) \geqq z_{2l+1}^{(n)}(x),$$

$$w_{2l}^{(i)}(x) \geqq w_{2l-1}^{(i)}(x), \quad w_{2l}^{(i)}(x) \geqq w_{2l+1}^{(i)}(x), \quad w_{2l-1}^{(n)}(x) \geqq w_{2l}^{(n)}(x), \quad w_{2l+1}^{(n)}(x) \geqq w_{2l}^{(n)}(x).$$

Из (3.1 легко получить, что

$$(4.1) \quad z_3(x)-z_1(x) = \begin{cases} 0, & x \in E, \\ \int_0^1 G(x, t)\,[\alpha_1(t)+\alpha_2(t)]\,dt & x \in [0, 1], \end{cases}$$

$$w_3(x)-w_1(x) = \begin{cases} 0, & x \in E, \\ \int_0^1 G(x, t)\,[\beta_1(t)+\beta_2(t)]\,dt, & x \in [0, 1], \end{cases}$$

откуда используя условие теоремы и свойства G получаем, что

$$(4.2) \quad \left.\begin{matrix} z_3^{(i)}(x) \leqq z_1^{(i)}(x), & z_1^{(n)}(x) \leqq z_3^{(n)}(x) \\ w_1^{(i)}(x) \leqq w_3^{(i)}(x), & w_3^{(n)}(x) \leqq w_1^{(n)}(x) \end{matrix}\right\} \quad (0 \leqq x \leqq 1, \quad i = 0, \ldots, n-1).$$

Очевидно, что при всех $p \geqq 2$ натуральных

$$z_{p+2}-z_p = Az_{p+1}-Az_{p-1}, \quad w_{p+2}-w_p = Aw_{p+1}-Aw_p.$$

Расписав эти формулы по самому определению A, дифференцируя их s-раз ($s=0, \ldots, n$), применяя формулу Лагранжа и используя свойства G, (4.2) и неотрицательность частных производных f, по индукции можно убедиться в том, что последовательности фигурирующие в теореме не возрастают, соотв. не убывают. Теорема доказана.

Следствие 4.1. *Если* $Q(x) \equiv 0$ *и* $f(x, u_0, \ldots, v_{n-1}) \geqq 0$ *при всех* $u_i \leqq 0$, $v_i \leqq 0$ ($i=0, \ldots, n-1$), *то*

$$y^{(i)}(x) \leqq 0, \quad y^{(n)}(x) \geqq 0 \quad (0 \leqq x \leqq 1; \quad i = 0, \ldots, n-1),$$

аналогично из того, что $Q(x) \equiv 0$ *и* $f(x, u_0, \ldots, v_{n-1}) \leqq 0$ *при всех* $u_i \geqq 0$, $v_i \geqq 0$ ($i=0, \ldots, n-1$) *следует, что*

$$y^{(i)}(x) \geqq 0, \quad y^{(n)}(x) \leqq 0 \quad (0 \leqq x \leqq 1; \quad i = 0, \ldots, n-1).$$

Для доказательства достаточно заметить, что в первом случае можно взять $z_1(x) \equiv 0$, а во втором случае $w_1(x) \equiv 0$, и что при таком выборе

$$\alpha_1(x)+\alpha_2(x) = -f[-\eta_1], \quad \beta_1(x)+\beta_2(x) = -f[-\vartheta_1].$$

Очевидно, что вместо неотрицательности (соотв. неположительности) f достаточно потребовать выполнение следующих неравенств

$$f(x, 0, \ldots, 0) \geqq 0, \quad f[-\eta_1] \geqq 0; \quad f(x, 0, \ldots, 0) \leqq 0, \quad f[-\vartheta_1] \leqq 0,$$

где η_1, ϑ_1 образуем соответственно из $z_1(x) \equiv 0$ и $w_1(x) \equiv 0$. Прежние выводы останутся верными и в этом случае.

Укажем теперь на один метод, с помощью которого можно найти такие функции z_1, w_1 из M, которые дают неположительные (соотв. неотрицательные невязки). Возьмем две произвольные функции z, w из M, n-раз непрерывно дифференцируемые на $[0, 1]$. Вычислим для них невязки

$$\alpha(x) = z^{(n)}(x) - f[z], \quad \beta(x) = w^{(n)}(x) - f[w].$$

Возьмем теперь какие-нибудь две непрерывные на $[0, 1]$ функции $\tilde{\alpha}(t)$, $\tilde{\beta}(t)$ для которых

$$\tilde{\alpha}(t) \leqq 0, \quad \tilde{\alpha}(t) + \alpha(t) \leqq 0; \quad \tilde{\beta}(t) \geqq 0, \quad \tilde{\beta}(t) + \beta(t) \geqq 0 \quad (0 \leqq t \leqq 1)$$

и пусть

$$\eta(x) = \begin{cases} 0, & x \in E, \\ -\int\limits_0^1 G(x, t)\tilde{\alpha}(t)\, dt, & x \in [0, 1], \end{cases}$$

$$\vartheta(x) = \begin{cases} 0, & x \in E, \\ -\int\limits_0^1 G(x, t)\tilde{\beta}(t)\, dt, & x \in [0, 1], \end{cases}$$

и

$$z_1(x) = z(x) + \eta(x), \quad w_1(x) = w(x) + \vartheta(x).$$

Очевидно, что $z_1(x)$ и $w_1(x)$ n-раз непрерывно дифференцируемы на $[0, 1]$ и что если их производные не превосходят числа K, то они принадлежат M в случае бруса D_K (в случае D_∞: $z_1, w_1 \in M$ безусловно). Справедливо поэтому для них

Лемма 4.1. *Если* $z_1, w_1 \in M$, *то* $\alpha_1(x) \leqq 0$, $\beta_1(x) \geqq 0$.

В справедливости леммы легко убедиться.

Отметим наконец, что приведенные в этом разделе результаты справедливы в частности для линейных уравнений с неотрицательными коэффициентами.

5. Нелинейное уравнение с непрерывными частными производными

Поставленную задачу в этом разделе мы будем рассматривать при следующих предположениях

(i) Условия б), в), г) выполнены,

$$\left.\begin{array}{l} \text{(ii)}\ f,\ \dfrac{\partial f}{\partial u_i},\ \dfrac{\partial f}{\partial v_i}\in C(D_K)\ \text{ или }\ C(D_\infty) \\ \left|\dfrac{\partial f}{\partial u_i}\right| \leqq N_i,\quad \left|\dfrac{\partial f}{\partial v_i}\right| \leqq N_i \end{array}\right\} \quad (i = 0, \ldots, n-1),$$

(iii) A сжато отображает M в M и условие (2.2) выполнено.

(iv) Существуют две функции $z_1(x)$, $w_1(x)\in M$, n-раз непрерывно дифференцируемые на [0, 1], для которых определенные ниже невязки α_1, β_1 неположительны, соотв. неотрицательны.

Итерационный процесс тут строится исходя из $z_1(x)$, $w_1(x)$ по формулам

$$\text{(5.1)}\qquad \left.\begin{aligned} z_{p+1}(x) &= \frac{1}{2}(Az_p+Aw_p)-\frac{1}{2}\Theta_p(x) \\ w_{p+1}(x) &= \frac{1}{2}(Az_p+Aw_p)+\frac{1}{2}\Theta_p(x) \end{aligned}\right\} \quad (p = 1, 2, \ldots),$$

где

$$\Theta_p(x) = \begin{cases} \int\limits_0^1 G(x, t)\,\Delta_p(t)\,dt & \text{если} \quad 0\leqq x\leqq 1, \\ 0 & \text{если} \quad x\in E, \end{cases}$$

$$\Delta_p(t) = \sum_{i=0}^{n-1} N_i\big[z_p^{(i)}(t)+z_p^{(i)}(g_i(t))-w_p^{(i)}(t)-w_p^{(i)}(g_i(t))\big] \quad (0 \leqq t \leqq 1).$$

Невязки $\alpha_p(x)$, $\beta_p(x)$ и функции $\eta_p(x)$, $\vartheta_p(x)$ определяются по формулам

$$\alpha_p(x) = z_p^{(n)}(x)-\frac{1}{2}f[z_p]-\frac{1}{2}f[w_p]-\frac{1}{2}\Delta_p(x),$$

$$\beta_p(x) = w_p^{(n)}(x)-\frac{1}{2}f[z_p]-\frac{1}{2}f[w_p]+\frac{1}{2}\Delta_p(x),$$

$$\eta_p(x) = \begin{cases} 0, & x\in E, \\ -\int\limits_0^1 G(x, t)\alpha_p(t)\,dt, & x\in[0, 1], \end{cases}$$

$$\vartheta_p(x) = \begin{cases} 0, & x\in E, \\ -\int\limits_0^1 G(x, t)\beta_p(t)\,dt, & x\in[0, 1]. \end{cases}$$

Условие (iv) при сказанных в конце раздела 2 отпадает. Из практических соображенний покажем еще один способ построения этих функций $z_1(x)$, $w_1(x)$. Возьмем какие-нибудь две функции $z(x)$, $w(x)\in M$, n-раз непрерывно дифференцируемые на $[0, 1]$. Вычислим для них невязки $\alpha(x)$, $\beta(x)$, которые мы получаем таким образом, что в формулах для $\alpha_1(x)$, $\beta_1(x)$ вместо $z_1(x)$, $w_1(x)$ подставляем $z(x)$, $w(x)$. Возьмем теперь какие-нибудь две непрерывные на $[0, 1]$ функции $\tilde{\alpha}$, $\tilde{\beta}$ для которых

$$\tilde{\alpha}(t) \leqq 0, \quad \tilde{\alpha}(t)+\alpha(t) \leqq 0; \qquad \tilde{\beta}(t) \geqq 0, \quad \tilde{\beta}(t)+\beta(t) \geqq 0,$$

и пусть

$$\eta(x) = \begin{cases} 0, & x\in E, \\ -\int\limits_0^1 G(x, t)\tilde{\alpha}(t)\, dt, & x\in[0, 1], \end{cases}$$

$$\vartheta(x) = \begin{cases} 0, & x\in E, \\ -\int\limits_0^1 G(x, t)\tilde{\beta}(t)\, dt, & x\in[0, 1], \end{cases}$$

и

$$z_1(x) = z(x)+\eta(x), \quad w_1(x) = w(x)+\vartheta(x).$$

Очевидно, что эти функции n-раз непрерыно дифференцируемы на $[0, 1]$ и что если их производные не превосходят числа K, то они принадлежат M в случае бруса D_K (в случае бруса D_∞: $z_1, w_1\in M$ безусловно). Справедлива поэтому для них

Лемма 5.1. *Если* $z_1, w_1\in M$, *то* $\alpha_1(x)\leqq 0$, $\beta_1(x)\geqq 0$.

В справедливости леммы легко убедиться.

Обозначим через $y(x)$ решение рассматриваемой задачи. Справедлива для него следующая

Теорема 5.1. *Если выполнены условия*

(i) z_2, w_2 *не выходят из* M*),

(ii) $\alpha_1(x)+\alpha_2(x) \leqq 0$, $\beta_1(x)+\beta_2(x) \geqq 0$,

то при $l\to\infty$ $(l = 1, 2, \ldots)$; $i = 0, \ldots, n-1$

$$\left.\begin{array}{ll} z_{2l}^{(i)}(x)\nearrow\nearrow y^{(i)}(x)\swarrow\swarrow z_{2l-1}^{(i)}(x), & w_{2l-1}^{(i)}(x)\nearrow\nearrow y^{(i)}(x)\swarrow\swarrow w_{2l}^{(i)}(x) \\ z_{2l-1}^{(n)}(x)\nearrow\nearrow y^{(n)}(x)\swarrow\swarrow z_{2l}^{(n)}(x), & w_{2l}^{(n)}(x)\nearrow\nearrow y^{(n)}(x)\swarrow\swarrow w_{2l-1}^{(n)}(x) \end{array}\right\} \quad (0 \leqq x \leqq 1).$$

Доказательство. Доказательство разбиваем на три части: А), Б), В).

А) Предположим, что все z_p, $w_p\in M$ и найдем связь между $z_p^{(s)}(x)$ и $z_{p+1}^{(s)}(x)$, $w_p^{(s)}(x)$ и $w_{p+1}^{(s)}(x)$ при $p=1, 2, \ldots$; $s=0, \ldots, n$; $0\leqq x\leqq 1$.

---

*) В случае бруса D_∞ это автоматически выполняется.

Используя свойства G легко показать, что

$$z_{p+1}(x) = z_p(x)-\eta_p(x), \quad w_{p+1}(x) = w_p(x)-\vartheta_p(x) \quad (p = 1, 2, \ldots). \tag{5.2}$$

Применяя формулу Лагранжа к разностям $f[z_p]-f[z_p-\eta_p]$ и $f[w_p]-f[w_p-\vartheta_p]$ мы находим при всех $p=1, 2, \ldots$, что

$$\alpha_{p+1}(x) = \frac{1}{2}\sum_{i=0}^{n-1}\left[\eta_p^{(i)}(x)\left(\frac{\partial f}{\partial u_i}\right| + N_i\right) + \eta_p^{(i)}(g_i(x))\left(\frac{\partial f}{\partial v_i}\right| + N_i\bigg)\right] +$$
$$+\frac{1}{2}\sum_{i=0}^{n-1}\left[\vartheta_p^{(i)}(x)\left(\frac{\partial f}{\partial u_i}\right| - N_i\right) + \vartheta_p^{(i)}(g_i(x))\left(\frac{\partial f}{\partial v_i}\right| - N_i\bigg)\right],$$
$$\beta_{p+1}(x) = \frac{1}{2}\sum_{i=0}^{n-1}\left[\eta_p^{(i)}(x)\left(\frac{\partial f}{\partial u_i}\right| - N_i\right) + \eta_p^{(i)}(g_i(x))\left(\frac{\partial f}{\partial v_i}\right| - N_i\bigg)\right] +$$
$$+\frac{1}{2}\sum_{i=0}^{n-1}\left[\vartheta_p^{(i)}(x)\left(\frac{\partial f}{\partial u_i}\right| + N_i\right) + \vartheta_p^{(i)}(g_i(x))\left(\frac{\partial f}{\partial v_i}\right| + N_i\bigg)\right].$$

Учитывая теперь неположительность (соотв. неотрицательность) множителей в скобках и то, что $\alpha_1(x)\leqq 0$, $\beta_1(x)\geqq 0$ легко убедиться по индукции в том, что при p нечетном $\alpha_p(x)\leqq 0$, $\beta_p(x)\geqq 0$, а при p четном $\alpha_p(x)\geqq 0$, $\beta_p(x)\leqq 0$. Отсюда по (5.2) получаем, что при $l=1, 2, \ldots$; $i=0, \ldots, n-1$; $0\leqq x\leqq 1$

$$\begin{gathered} z_{2l}^{(i)}(x) \leqq z_{2l-1}^{(i)}(x), \quad z_{2l}^{(i)}(x) \leqq z_{2l+1}^{(i)}(x); \quad w_{2l-1}^{(i)}(x) \leqq w_{2l}^{(i)}(x), \quad w_{2l+1}^{(i)}(x) \leqq w_{2l}^{(i)}(x), \\ z_{2l-1}^{(n)}(x) \leqq z_{2l}^{(n)}(x), \quad z_{2l+1}^{(n)}(x) \leqq z_{2l}^{(n)}(x); \quad w_{2l}^{(n)}(x) \leqq w_{2l-1}^{(n)}(x), \quad w_{2l}^{(n)}(x) \leqq w_{2l+1}^{(n)}(x). \end{gathered} \tag{5.3}$$

Из проведенных (пока формальных) рассуждений можно сделать следующие выводы. Если при некотором p натуральном $z_1, \ldots, z_p$; $w_1, \ldots, w_p$ принадлежат M, тогда z_{p+1}, w_{p+1} можно вычислить по закону (5.2) и будут справедливы неравенства (5.3) соответствующие четности p между $z_{p+1}^{(s)}(x)$ и $z_p^{(s)}(x)$, $w_{p+1}^{(s)}(x)$ и $w_p^{(s)}(x)$.

Б) Покажем теперь, что все z_p, w_p принадлежат M, за одно найдем связь между $z_{p+2}^{(s)}(x)$ и $z_p^{(s)}(x)$, $w_{p+2}^{(s)}(x)$ и $w_p^{(s)}(x)$ при $s=0, \ldots, n$; $0\leqq x\leqq 1$.

По условию (i) функции z_2, w_2 принадлежат M. Применим поэтому теперь последнее утверждение части А) доказательства в случае $p=2$. Как частный результат получаем формулы (5.2) для $p=1, 2$. Складывая, а потом дифференцируя их получаем при $s=0, \ldots, n$; $0\leqq x\leqq 1$:

$$\begin{gathered} z_1^{(s)}(x)-z_3^{(s)}(x) = -\frac{d^s}{dx^s}\int_0^1 G(x, t)\,[\alpha_1(t)+\alpha_2(t)]\,dt, \\ w_1^{(s)}(x)-w_3^{(s)}(x) = -\frac{d^s}{dx^s}\int_0^1 G(x, t)\,[\beta_1(t)+\beta_2(t)]\,dt, \end{gathered} \tag{5.4}$$

а из этих формул в силу условия (ii) и свойств G, учитывая неравенства (5.3) для z_1, z_2; w_1, w_2; и их производных получаем при $i=0, \dots, n-1$

$$(5.5)\quad \left.\begin{matrix} z_2^{(i)}(x) \leqq z_3^{(i)}(x) \leqq z_1^{(i)}(x), & w_1^{(i)}(x) \leqq w_3^{(i)}(x) \leqq w_2^{(i)}(x) \\ z_1^{(n)}(x) \leqq z_3^{(n)}(x) \leqq z_2^{(n)}(x), & w_2^{(n)}(x) \leqq w_3^{(n)}(x) \leqq w_1^{(n)}(x) \end{matrix}\right\} \quad (0 \leqq x \leqq 1).$$

По самому построению итераций $z_3|_E = w_3|_E = Q$, так что (5.5) обеспечено, что $z_3, w_3 \in M$. Отсюда в силу последнего утверждения части A) доказательства получаем при $0 \leqq x \leqq 1$, $i=0, \dots, n-1$ следующие неравенства

$$(5.6)\quad z_4^{(i)}(x) \leqq z_3^{(i)}(x), \quad z_3^{(n)}(x) \leqq z_4^{(n)}(x); \quad w_3^{(i)}(x) \leqq w_4^{(i)}(x), \quad w_4^{(n)}(x) \leqq w_3^{(n)}(x).$$

Из (5.1) получаем при $\lambda \leqq x \leqq 1$

$$(5.7)\quad z_4(x) - z_2(x) = \frac{1}{2}(Az_3 + Aw_3 - Az_1 - Aw_1) + \frac{1}{2}(\Theta_1(x) - \Theta_3(x)).$$

Отсюда используя формулу Лагранжа получаем при $0 \leqq x \leqq 1$

$$(5.8)\quad \begin{aligned} z_2(x) - z_4(x) = {} & \frac{1}{2}\int_0^1 G(x,t) \sum_{i=0}^{n-1} \left\{ \left(z_3^{(i)}(t) - z_1^{(i)}(t)\right) \left[\left| \frac{\partial f}{\partial u_i} \right| + N_i \right] + \right. \\ & \left. + \left(z_3^{(i)}(g_i(t)) - z_1^{(i)}(g_i(t))\right) \left[\left| \frac{\partial f}{\partial v_i} \right| + N_i \right] \right\} dt + \\ & + \frac{1}{2}\int_0^1 G(x,t) \sum_{i=0}^{n-1} \left\{ \left(w_3^{(i)}(t) - w_1^{(i)}(t)\right) \left[\left| \frac{\partial f}{\partial u_i} \right| - N_i \right] + \right. \\ & \left. + \left(w_3^{(i)}(g_i(t)) - w_1^{(i)}(g_i(t))\right) \left[\left| \frac{\partial f}{\partial v_i} \right| - N_i \right] \right\} dt. \end{aligned}$$

Совершенно аналогично можно показать, что при $0 \leqq x \leqq 1$ имеет место следующая формула

$$(5.9)\quad \begin{aligned} w_2(x) - w_4(x) = {} & \frac{1}{2}\int_0^1 G(x,t) \sum_{i=0}^{n-1} \left\{ \left(z_3^{(i)}(t) - z_1^{(i)}(t)\right) \left[\left| \frac{\partial f}{\partial u_i} \right| - N_i \right] + \right. \\ & \left. + \left(z_3^{(i)}(g_i(t)) - z_1^{(i)}(g_i(t))\right) \left[\left| \frac{\partial f}{\partial v_i} \right| - N_i \right] \right\} dt + \\ & + \frac{1}{2}\int_0^1 G(x,t) \sum_{i=0}^{n-1} \left\{ \left(w_3^{(i)}(t) - w_1^{(i)}(t)\right) \left[\left| \frac{\partial f}{\partial u_i} \right| + N_i \right] + \right. \\ & \left. + \left(w_3^{(i)}(g_i(t)) - w_1^{(i)}(g_i(t))\right) \left[\left| \frac{\partial f}{\partial v_i} \right| + N_i \right] \right\} dt. \end{aligned}$$

Учитывая теперь свойства G и неравенства (5.5) легко из (5.8) и (5.9) вывести неравенства справедливые при $0 \leqq x \leqq 1$, $i=0, \ldots, n-1$

$$(5.10)\quad z_2^{(i)}(x) \leqq z_4^{(i)}(x), \quad w_4^{(i)}(x) \leqq w_2^{(i)}(x); \quad z_4^{(n)}(x) \leqq z_2^{(n)}(x), \quad w_2^{(n)}(x) \leqq w_4^{(n)}(x),$$

которые вместе с (5.6) обеспечивают то, что $z_4, w_4 \in M$, ведь при $0 \leqq x \leqq 1$, $i=0, \ldots, n-1$

$$(5.11)\quad \begin{aligned} &z_2^{(i)}(x) \leqq z_4^{(i)}(x) \leqq z_3^{(i)}(x), \quad w_3^{(i)}(x) \leqq w_4^{(i)}(x) \leqq w_2^{(i)}(x),\\ &z_3^{(n)}(x) \leqq z_4^{(n)}(x) \leqq z_2^{(n)}(x), \quad w_2^{(n)}(x) \leqq w_4^{(n)}(x) \leqq w_3^{(n)}(x). \end{aligned}$$

Для доказательства того, что $z_p, w_p \in M$ и при любом $p \geqq 4$ натуральном, надо по методу математической индукции провести аналогичные только что проведенным рассуждения. Именно, из того, что $z_1, \ldots, z_{p-1}$; $w_1, \ldots, w_{p-1}$ принадлежат M констатируем выполнение неравенств (5.3) соответственно четности p между $p-1$-ыми и p-ыми приближениями и их производными i-го порядка ($i=0, \ldots, n-1$). После этого выписываем неравенства типа (5.8), (5.9) полученные с заменой в них индексов $1, \ldots, 4$ на $p-3, \ldots, p$. По этим равенствам используя неравенства между $z_{p-1}^{(s)}(x)$, $w_{p-1}^{(s)}(x)$ и $z_{p-3}^{(s)}(x)$, $w_{p-3}^{(s)}(x)$ при $0 \leqq x \leqq 1$, $s=0, \ldots, n$ получаем неравенства типа (5.10), а это вместе с неравенствами для z_p, z_{p-1}; w_p, w_{p-1} дает неравенства типа (5.11). Итак получаем, что $z_p^{(i)}(x)$, $w_p^{(i)}(x)$ заключены между $z_{p-1}^{(i)}(x)$ и $z_{p-2}^{(i)}(x)$, $w_{p-1}^{(i)}(x)$ и $w_{p-2}^{(i)}(x)$ соответственно. Поскольку все первые $p-1$ приближений принадлежат M, потому и $z_p, w_p \in M$. Имеют при этом место следующие неравенства при $0 \leqq x \leqq 1$; $l=1, 2, \ldots$; $i=0, \ldots, n-1$

$$(5.12)\quad \begin{aligned} &z_{2l}^{(i)}(x) \leqq z_{2l+1}^{(i)}(x) \leqq z_{2l-1}^{(i)}(x), \quad w_{2l-1}^{(i)}(x) \leqq w_{2l+1}^{(i)}(x) \leqq w_{2l}^{(i)}(x),\\ &z_{2l-1}^{(n)}(x) \leqq z_{2l+1}^{(n)}(x) \leqq z_{2l}^{(n)}(x), \quad w_{2l}^{(n)}(x) \leqq w_{2l+1}^{(n)}(x) \leqq w_{2l-1}^{(n)}(x),\\ &z_{2l}^{(i)}(x) \leqq z_{2l+2}^{(i)}(x) \leqq z_{2l+1}^{(i)}(x), \quad w_{2l+1}^{(i)}(x) \leqq w_{2l+2}^{(i)}(x) \leqq w_{2l}^{(i)}(x),\\ &z_{2l+1}^{(n)}(x) \leqq z_{2l+2}^{(n)}(x) \leqq z_{2l}^{(n)}(x), \quad w_{2l}^{(n)}(x) \leqq w_{2l+2}^{(n)}(x) \leqq w_{2l+1}^{(n)}(x). \end{aligned}$$

B) Докажем, что $z_p^{(s)}(x) \rightrightarrows y^{(s)}(x)$, $w_p^{(s)}(x) \rightrightarrows y^{(s)}(x)$ на отрезке $[0, 1]$ при $p \to \infty$, $s=0, \ldots, n$.

Заметим для этого, что из (5.1) получаем

$$(5.13)\quad z_{p+1}(x) - w_{p+1}(x) = -\int_0^1 G(x, t) \Delta_p(t)\, dt \quad (0 \leqq x \leqq 1).$$

Если теперь оценим правую часть по норме, аналогично тому, как это мы сделали при доказательстве теоремы 2.1, то в силу условия (2.2) получим

$$(5.14)\quad \|z_{p+1} - w_{p+1}\| \leqq \theta \|z_p - w_p\| \quad (p = 1, 2, \ldots),$$

где θ постоянная, $0 < \theta < 1$.

Из этого неравенства и из формул (5.12) выражающих монотонность последовательностей итераций z_p, w_p и их производных следует, что при $0 \leqq x \leqq 1$, $i=0, \ldots, n-1$, $p=1, 2, \ldots$

$$|z_{p+2}^{(i)}(x)-z_p^{(i)}(x)| \leqq \theta^{p-1}\|z_1-w_1\|, \quad |w_{p+2}^{(i)}(x)-w_p^{(i)}(x)| \leqq \theta^{p-1}\|z_1-w_1\|.$$

Отсюда учитывая (5.14) и то, что при $x \in E$ $z_p(x)=w_p(x)=Q(x)$ при всех $p=1, 2, \ldots$, сразу следует, что последовательности $\{z_p^{(i)}(x)\}$, $\{w_p^{(i)}(x)\}$ равномерно сходятся на $[\lambda, 1]$ при $p \to \infty$, $i=0, \ldots, n-1$ к общему пределу $y_i(x)=y_0^{(i)}(x)$, т.е. эти последовательности сходятся в M по норме, а в силу полноты M, $y_0 \in M$.

Перейдем теперь к пределу при $p \to \infty$ в формулах (5.1). Мы получим в силу непрерывности A, что $y_0=Ay_0$, откуда следует, что $y_0=y$ (через y мы обозначили единственное решение нашей задачи). Утверждение теоремы для n-ых производных отсюда и из (5.12) легко вывести; надо продифференцировать n-раз формулы (5.1), перейти к пределу при $p \to \infty$, принять во внимание неравенства (5.12) и то, что $y^{(n)}(x)=f[y]$. Теорема доказана.

Отметим, что как подробные вычисления показывают, условия (ii) теоремы эквивалентны некоторым условиям для z_1, w_1 похожим на условие сжатости (2.2).

6. Монотонные приближения в случае уравнения с непрерывными производными

Рассмотрим поставленную задачу при условиях фигурирующих в начале предыдущего раздела, с той разницей, что здесь будут использованы не числа N_i мажорирующие по модулю частные производные f, а лишь какая-нибудь верхняя граница этих производных $\tilde{N}$:

$$\frac{\partial f}{\partial u_i} \leqq \tilde{N}, \quad \frac{\partial f}{\partial v_i} \leqq \tilde{N} \quad (i = 0, \ldots, n-1;\ \tilde{N} \geqq 0),$$

кроме того функции Δ_p, α_p, β_p определяются здесь по следующим формулам:

$$\Delta_p(x) = \tilde{N} \sum_{i=0}^{n-1} \left[z_p^{(i)}(x)+z_p^{(i)}(g_i(x))-w_p^{(i)}(x)-w_p^{(i)}(g_i(x))\right],$$

$$\alpha_p(x) = z_p^{(n)}(x)-f[z_p]+\Delta_p(x), \quad \beta_p(x) = w_p^{(n)}(x)-f[w_p]-\Delta_p(x).$$

Функции η_p, ϑ_p, определяются по α_p, β_p а z_{p+1}, w_{p+1} по z_p и w_p так же, как в формулах (3.1). Предпологаем естественно, что существуют функции z_1, w_1 для которых $\alpha_1(x)\leqq 0$, $\beta_1(x)\geqq 0$. Очевидно, что z_{p+1}, w_{p+1} в этом случае можно вычислить и по закону

$$z_{p+1}(x) = Az_p+\Theta_p(x), \quad w_{p+1}(x) = Aw_p-\Theta_p(x), \tag{6.1}$$

где $\Theta_p(x)$ определяется по Δ_p так же, как в (5.1).

Монотонность последовательностей $\{z_p^{(s)}(x)\}$, $\{w_p^{(s)}(x)\}$ здесь доказывается следующим образом. Учитывая формулы для $\alpha_1(x)$, $\beta_1(x)$ получаем

$$z_1(x)-w_1(x) = \begin{cases} 0, & x\in E, \\ -\int\limits_0^1 G(x,t)\,[L_1(z_1-w_1)+\alpha_1(t)-\beta_1(t)]\,dt, & x\in[0,1], \end{cases}$$

где L_1 есть линейный дифференциальный оператор действующий на элементах специального множества M для случая бруса D_∞ и $Q(x)\equiv 0$ (см. разд. 2). Обозначим это множество через M_0. Оператор L_1 действует следующим образом:

$$L_1(z) = \sum_{i=0}^{n-1}\left[\left(\left.\frac{\partial f}{\partial u_i}\right| -2\tilde N\right) z^{(i)}(t)+\left(\left.\frac{\partial f}{\partial v_i}\right| -2\tilde N\right) z^{(i)}\big(g_i(t)\big)\right],$$

где $\left.\frac{\partial f}{\partial u_i}\right|$, $\left.\frac{\partial f}{\partial v_i}\right|$ равны функциям $a_i(t)$, $b_i(t)$ фигурирующим в формуле Лагранжа

$$f[z_1]-f[w_1] = \sum_{i=0}^{n-1}\left\{a_i(t)\,[z_1^{(i)}(t)-w_1^{(i)}(t)]+b_i(t)\,[z_1^{(i)}\big(g_i(t)\big)-w_1^{(i)}\big(g_i(t)\big)]\right\}.$$

Предположим, что оператор

$$A_1 z = \begin{cases} 0, & x\in E, \\ -\int\limits_0^1 G(x,t)\,[L_1(z)+\alpha_1(t)-\beta_1(t)]\,dt, & x\in[0,1], \end{cases}$$

сжато отображает M_0 в M_0, тогда в силу результатов раздела 3 получаем

$$w_1^{(i)}(x)\leqq z_1^{(i)}(x), \quad z_1^{(n)}(x)\leqq w_1^{(n)}(x) \quad (i=0,\dots,n-1;\quad 0\leqq x\leqq 1). \tag{6.2}$$

Оператор A_1 заведомо будет сжатием если $\left|\frac{\partial f}{\partial u_i}-2\tilde N\right|$, $\left|\frac{\partial f}{\partial v_i}-2\tilde N\right|$ при всех $i= =0,\dots,n-1$ не превосходят числа N удовлетворяющего условию (2.2).

Доказательство монотонности продолжаем методом математической индукции. Предположим, что $z_1,\dots,z_l$, $w_1,\dots,w_l$ принадлежат M. Тогда справедливы согласно обозначениям предыдущих разделов при $p=1, \ldots, l-1$ следующие формулы

$$(6.3)\quad \begin{aligned} \alpha_{p+1}(x) &= \sum_{i=0}^{n-1} \left\{\eta_p^{(i)}(x)\left[\frac{\partial f}{\partial u_i}\right| - \tilde{N}\right] + \eta_p^{(i)}(g_i(x))\left[\frac{\partial f}{\partial v_i}\right| - \tilde{N}\right]\right\} + \\ &\quad + \tilde{N}\sum_{i=0}^{n-1}\left[\vartheta_p^{(i)}(x) + \vartheta_p^{(i)}(g_i(x))\right], \\ \beta_{p+1}(x) &= \sum_{i=0}^{n-1} \left\{\vartheta_p^{(i)}(x)\left[\frac{\partial f}{\partial u_i}\right| - \tilde{N}\right] + \vartheta_p^{(i)}(g_i(x))\left[\frac{\partial f}{\partial v_i}\right| - \tilde{N}\right]\right\} + \\ &\quad + \tilde{N}\sum_{i=0}^{n-1}\left[\eta_p^{(i)}(x) + \eta_p^{(i)}(g_i(x))\right]. \end{aligned}$$

Отсюда видна знакопостоянность невязок, а поэтому справедливы в частности и неравенства при $p=1, \ldots, l$; $i=0, \ldots, n-1$; $0 \leqq x \leqq 1$

$$(6.4)\quad z_{p+1}^{(i)}(x) \leqq z_p^{(i)}(x), \quad w_p^{(i)}(x) \leqq w_{p+1}^{(i)}(x); \quad z_p^{(n)}(x) \leqq z_{p+1}^{(n)}(x), \quad w_{p+1}^{(n)}(x) \leqq w_p^{(n)}(x).$$

По формуле Лагранжа имеем при $2 \leqq p \leqq l$ натуральном

$$f[z_p] - f[w_p] = \sum_{i=0}^{n-1} \left\{a_{i,p}(t)\left[z_p^{(i)}(t) - w_p^{(i)}(t)\right] + b_{i,p}(t)\left[z_p^{(i)}(g_i(t)) - w_p^{(i)}(g_i(t))\right]\right\}.$$

По функциям $a_{i,p}$, $b_{i,p}$ определяем дифференциальные операторы L_p и операторы A_p действующие на элементах M_0.

$$L_p(z) = \sum_{i=0}^{n-1} \left[(a_{i,p}(t) - 2\tilde{N})z^{(i)}(x) + (b_{i,p}(t) - 2\tilde{N})z^{(i)}(g_i(t))\right],$$

$$A_p(z) = \begin{cases} 0, & x \in E, \\ -\int\limits_0^1 G(x,t)L_p(z)\,dt, & x \in [0,1]. \end{cases}$$

Поскольку при $2 \leqq p \leqq l$

$$z_{p+1}(x) - w_{p+1}(x) = A_p(z_p - w_p)$$

и коэффициенты L_p неположительны следует, что

$$(6.5)\quad z_{p+1}^{(i)}(x) - w_{p+1}^{(i)}(x) \geqq 0, \quad w_{p+1}^{(n)}(x) - z_{p+1}^{(n)}(x) \geqq 0 \quad (i = 0, \ldots, n-1; \ p \leqq l, \ 0 \leqq x \leqq 1),$$

а это вместе с неравенствами (6.4) обеспечивает, что $z_{l+1}, w_{l+1} \in M$. По индукции значит можно получить неравенства (6.4), (6.5) при всех p натуральных $(p \geqq 2)$ и утверждать, что $z_p, w_p \in M$ при всех p.

Если дополнительно предположить, что семейство операторов A_p ($p=2, 3, \ldots$) обладает свойством равностепенной сжатости, т.е.

$$\|A_p z\| \leqq \theta \|z\| \quad (p = 2, 3, \ldots; 0<\theta<1), \tag{6.6}$$

то таким же путем, как в теореме 5.1 можно доказать на основании неравенств (6.4), (6.5) монотонность последовательностей $\{z_p^{(s)}(x)\}$, $\{w_p^{(s)}(x)\}$ и их сходимость к $y^{(s)}(x)$ т.е. к производным решения рассматриваемой краевой задачи при $0 \leqq x \leqq 1$. Итак мы доказали следующую теорему:

Теорема 6.1. *Если A_1 сжатие и семейство операторов A_p ($p \geqq 2$) обладает свойством равностепенной сжатости, тогда при $0 \leqq x \leqq 1$, $i=0, \ldots, n-1$*

$$w_p^{(i)}(x) \nearrow\!\!\nearrow y^{(i)}(x) \swarrow\!\!\swarrow z_p^{(i)}(x), \quad z_p^{(n)}(x) \nearrow\!\!\nearrow y^{(n)}(x) \swarrow\!\!\swarrow w_p^{(n)}(x).$$

Отметим, что (6.6) будет выполнено например, если коэффициенты всех L_p не превосходят по модулю числа N удовлетворяющего условию (2.2).

Замечание 6.1. *Условие $g_{n-1}(x)$ не меняет знак при $0 \leqq x \leqq 1$ в рассматриваемых уравнениях обеспечивает непрерывность $y^{(n)}(x)$ на всем отрезке* [0, 1], *где $y(x)$ есть решение соответствующего интегрального уравнения* (2.1).

Пусть теперь имеется конечное число точек (их совокупность обозначим через S) в которых функция g_{n-1} меняет знак. Пусть $\widetilde{[0, 1]} = [0, 1] \setminus S$. Соответствующее интегральное уравнение (2.1) и в этом случае разрешимо в M при выполнений условия (2.2). Решение обозначаем через $y(x)$. Его можно принять за решение поставленной задачи, несмотря на то, что уравнение (1.1) выполняется вообще говоря только на $\widetilde{[0, 1]}$, поскольку в точках S функция $y^{(n)}(x)$ может иметь разрывы. Последовательности $\{z_p(x)\}$, $\{w_p(x)\}$ строятся так же, как и выше, но здесь $\alpha_p(x), \beta_p(x), \eta_p^{(n)}(x), \vartheta_p^{(n)}(x), z_p^{(n)}(x), w_p^{(n)}(x)$ непрерывны только на $\widetilde{[0, 1]}$ и поэтому $z_p^{(n)}(x), w_p^{(n)}(x) \rightrightarrows y^{(n)}(x)$ только при $x \in \widetilde{[0, 1]}$. Все неравенства для n-ых производных справедливы только на $\widetilde{[0, 1]}$. Все утверждения приведенные выше, связанные с $\eta_p^{(i)}(x)$, $\vartheta_p^{(i)}(x)$, $z_p^{(i)}(x)$, $w_p^{(i)}(x)$, $y^{(i)}(x)$ остаются в силе при $i=0, \ldots, n-1$.

Что касается случая, когда g_{n-1} меняет знак в бесконечно много точках, отметим, что мы построили пример, где это множество имеет меру $1-\varepsilon$ (ε произвольное число между 0 и 1) и поэтому поставленная задача не имеет решения в том смысле, как это говорилось в разделе 1, поскольку n-я производная решения интегрального уравнения (2.1) в точках этого множества терпит разрыв.

7. О системе уравнений типа (1.1)

Результаты изложенные выше легко распространяются на задачу (7.1), (7.2), (7.3):

$$(7.1)\qquad y_i^{(n_i)}(x) = f_i[\vec{y}] = f_i\big(x, y_1(x), \ldots, y_1^{(n_1-1)}(x), \ldots, y_r(x), \ldots, y_r^{(n_r-1)}(x);$$

$$y_1\big(\overset{(i)}{g}_{1,0}(x)\big), \ldots, y_1^{(n_1-1)}\big(\overset{(i)}{g}_{1,n_1-1}(x)\big), \ldots, y_r\big(\overset{(i)}{g}_{r,0}(x)\big), \ldots, y_r^{(n_r-1)}\big(\overset{(i)}{g}_{r,n_r-1}(x)\big)\big)$$

$$(0 \leqq x \leqq 1),$$

$$(7.2)\qquad y_i(0) = \ldots = y_i^{(n_i-2)}(0) = y_i^{(n_i-1)}(1) = 0 \quad (n_i \geqq 2, \; i = 1, \ldots, r),$$

$$(7.3)\qquad y_i|_E = Q_i \quad (i = 1, \ldots, r),$$

где заданные функции f_i, $\overset{(i)}{g}_{i,j}$, Q_i и начальное множество E удовлетворяют следующим условиям:

Область определения f_i есть или конечный брус: x меняется между 0 и 1, а остальные блоки переменных между числами $-K_i$ и K_i, или же все блоки меняются между $-\infty$ и $+\infty$, причем в первом случае множество значений f_i содержится в $[-K_i, K_i]$. Функции f_i непрерывны и удовлетворяют условию Липшица. Функции $\overset{(i)}{g}_{i,j}(x)$ принадлежат классу $C[0, 1]$, их значения содержатся при всех x между λ и x (λ постоянная, $\lambda < 0$), $\overset{(i)}{g}_{l,n_l-1}$ не меняют знака (см. замечание 6.1), а E есть отрезок $[\lambda, 0]$. Функции $Q_i(x)$ принадлежат классу $C^{(n_i-1)}(E)$ и все их производные до (n_i-2)-го в нуле равны нулю, а в случае конечных блоков их производные до (n_i-1)-го порядка включительно содержатся в $[-K_i, K_i]$. Вектор функцию $\vec{y}(x) = \big(y_1(x), \ldots, y_r(x)\big)$ мы будем называть решением нашей задачи при условиях аналогичных сказанным в разделе 1, при этом задача (7.1), (7.2), (7.3) эквивалентна операторному уравнению

$$(7.4)\qquad y_i(x) = (A\vec{y})_i = \begin{cases} Q_i(x), & x \in E, \\ -\int\limits_0^1 G_i(x, t) f_i[\vec{y}]\, dt, & x \in [0, 1] \end{cases} \quad (i = 1, \ldots, r),$$

где G_i получаем из G (см. разд. 2) с заменой n на n_i, а оператор A определен на множестве $M_1 \times \ldots \times M_r$ где $M_i = M$ в случае бесконечных блоков, только n заменяем на n_i, а при конечных блоках M_i получаем из M (см. разд. 2) с заменой n, K на n_i, K_i. Задача (7.4) имеет единственное решение если

$$\sum_{i=1}^{r} N_i \left[2 - \frac{1}{n_i!} + \max_{s=1,\ldots,r;\; l=0,\ldots,n_s-1} \int\limits_{\overset{(i)}{M}_{s,l}} \left(1 + \sum_{j=0}^{n_i-2} \frac{\partial^j G_i(x, t)}{\partial x^j} \right) dt \right] < 1$$

(здесь $\overset{(i)}{M}_{s,l} = \{x\colon \overset{(i)}{g}_{s,l}(x) > 0\}$, а N_i есть постоянная в условии Липшица для f_i).

Здесь строим последовательности вектор функций $\{\vec{z}_p(x)\}$, $\{\vec{w}_p(x)\}$; сходимость их производных к $\vec{y}^{(s)}(x)$ и все неравенства приведенные выше остаются справедливыми если их понимать для вектор фунций покомпонентно. Невязки $\vec{\alpha}_p(x)=(\alpha_{p,1}(x), \ldots, \alpha_{p,1}(x))$ определяются (соответственно разделам 3, 4; 5; 6) таким образом:

$$\alpha_{p,i}(x) = z_{p,i}^{(n_i)}(x) - f_i(x, z_{p,1}(x), \ldots, z_{p,r}^{(n_r-1)}(\overset{(i)}{g}_{r,n_r-1}(x))),$$

$$\alpha_{p,i}(x) = z_{p,i}^{(n_i)}(x) - \frac{1}{2} f_i[\vec{z}_p] - \frac{1}{2} f_i[\vec{w}_p] - \frac{1}{2} \Delta_{p,i}(x),$$

$$\alpha_{p,i}(x) = z_{p,i}^{(n_i)}(x) - f_i[\vec{z}_p] + \tilde{\Delta}_{p,i}(x),$$

где

$$\Delta_{p,i}(x) = N_i \sum_{s=1}^{r} \sum_{l=0}^{n_s-1} [z_{p,s}^{(l)}(x) - w_{p,s}^{(l)}(x) + z_{p,s}^{(l)}(\overset{(i)}{g}_{s,l}(x)) - w_{p,s}^{(l)}(\overset{(i)}{g}_{s,l}(x))],$$

а $\tilde{\Delta}_{p,i}(x)$ получается из $\Delta_{p,i}(x)$ заменой N_i на верхнюю границу производных f_i. Невязки $\vec{\beta}_p(x)=(\beta_{p,1}(x), \ldots, \beta_{p,r}(x))$ определяются аналогично, только члены содержащие $\Delta_{p,i}$, $\tilde{\Delta}_{p,i}$ берутся с обратным знаком. После этого ясно, как остальные формулы, напр. для $\vec{\eta}_p(x)$, $\vec{\vartheta}_p(x)$, $\vec{z}_{p+1}$, $\vec{w}_{p+1}$ выписываются в этом случае по компонентам.

Вышеизложенный метод применялся для решения конкретных задач и сходимость приближений оказалась достаточно быстрой. Его можно применить и в том случае, когда метод шагов не применим, а также и в том случае, когда все $g_i(x)$, $\overset{(l)}{g}_{i,j}(x) \equiv x$ т.е. для краевых задач без запаздывания. Погрешность вычислений при этом методе удобно оценивается.

Изложенный выше метод распространяется и на другие краевые задачи, отличные от указанных выше, а также и на задачи с запаздыванием, рассмотренные в работах [11—20].

Литература *)

[1] Л. Э. Эльсгольц, О краевых задачах для обыкновенных дифференциальных уравнений с отклоняющимся аргументом, *УМН*, **15**, 5 (93), (1960), 222—224.

[2] С. Б. Норкин, О краевой задаче типа Штурма—Лиувилля для дифференциального уравнения второго порядка с запаздывающим аргументом, *Изв. высш. учебн. завед., Математика*, **6** (7), (1958), 203—214.

[3] Г. А. Каменский, О единственности решения краевой задачи для нелинейного дифференциального уравнения второго порядка нейтрального типа с отклоняющимся

*) *УМН = Успехи Математических Наук*
УМЖ = Украинский Математический Журнал
ДАН УРСР = Доповiдi Академiї Наук УРСР

аргументом, *Труды семинара по теории дифференциальных уравнений с отклоняющимся аргументом, Ун-т дружбы народов,* **4** (1967), 275—277.

[4] А. Я. Ляпин, А. Д. Мышкис, Существование решения одной нелинейной краевой задачи для обыкновенного дифференциального уравнения n-го порядка, *Дифференциальные уравнения,* **4**: № 7 (1968), 1171—1188.

[5] С. Б. Норкин, *Дифференциальные уравнения второго порядка с запаздывающим аргументом* Изд-во Наука (М., 1965).

[6] Л. Э. Эльсгольц, С. Б. Норкин, *Введение в теорию дифференциальных уравнений с отклоняющимся аргументом,* Изд-во Наука (М., 1971).

[7] Г. М. Жданов, О приближенном решении системы дифференциальных уравнений первого порядка с запаздывающим аргументом, *УМН,* **16**: **1** **(97)** (1961), 143—148.

[8] Э. И. Клямко, Некоторые применения метода Чаплыгина к приближенному решению дифференциальных уравнений с запаздывающим аргументом, *УМН,* **12**; **4** **(76)** (1957), 305—312.

[9] Ю. И. Ковач, Л. И. Савченко, О краевой задаче для нелинейной системы дифференциальных уравнений с запаздывающим аргументом, *УМЖ,* **22**: **1** (1970), 12—21.

[10] Ю. И. Ковач, О приближенном решении нелинейной системы дифференциальных уравнений с запаздывающим аргументом, *УМЖ,* **22**: **3** (1970), 380—388.

[11] Ю. И. Ковач, Применение теоремы о дифференциальных неравенствах к задаче Гурса для линейной системы дифференциальных уравнений с частными производными, *Дифференциальные уравнения,* **1**: **3** (1965), 411—420.

[12] Ю. И. Ковач, Приближенное интегрирование задачи Гурса для общей 2n-волновой системы дифференциальных уравнений методом двустороннего приближения, *УМЖ,* **17**: **4** (1965), 37—45.

[13] Ю. И. Ковач, Доказательство теоремы существования и единственности решения задачи Коши методом двустороннего приближения для $2n$-волновой системы, *Ж. вычислит. матем. и матем. физики,* **5**: **3** (1965), 551—557.

[14] Ю. И. Ковач, Теорема о „вилке” в задаче Коши для нелинейного уравнения с частными производными высших порядков, *УМЖ,* **18**: **5** (1966), 28—35.

[15] Ю. И. Ковач, Об оценке решения задачи Коши для уравнения в частных производных $2n$-го порядка, *Вопросы теории и истории дифференциальных уравнений,* АН УССР, К. (1968), 158—165.

[16] Ю. И. Ковач, В. В. Маринец, А. И. Моца, О двустороннем итеративном методе интегрирования задачи Дирихле для нелинейной эллиптической системы дифференциальных уравнений, *Математическая физика,* АН УССР, К., **6**, (1969), 101—107.

[17] Ю. И. Ковач, Об оценке решения нелинейной системы с запаздыванием содержащей оператор m_i-го порядка параболического или гиперболического вида, *Численный анализ,* **2**, ИК АН УССР, К. (1969), 20—37 (труды семинара).

[18] Ю. И. Ковач, Л. И. Савченко, Об оценке решения некоторых нелинейных интегро-диффереииальных и операторных уравнений с запаздыванием, *Дифференциальные уравнения,* **6**: **8** (1970), 1496—1505.

[19] Ю. І. Ковач, В. В. Маринець, Про один метод наближеного інтегрування нелінійної системи диференціальних рівнянь із запізненням, *ДАН УРСР,* **11**, серія А (1970), 120—123.

[20] Ю. І. Ковач, І. В. Брич, Наближене інтегрування однієї крайової задачі для нелінійної системи диференціальних рівнянь з запізнюючим аргументом, *ДАН УРСР,* **11**, серія А (1970), 980—982.

(Поступило 1. XII 1972 г.)

Some integral operators of trace class

By R. A. H. LORENTZ in St. Augustin (GFR)*)
and P. A. REJTŐ in Minneapolis (Minn., USA)

With an Appendix due to JOACHIM WEIDMANN in Frankfurt (GFR)

1. Introduction

In a series of four papers, BIRMAN and SOLOMJAK [18], [19], [20], [21] formulated estimates for the characteristic values of an integral operator. More specifically they considered two measure spaces (X, ϱ) and (Y, τ) and an integral operator mapping $\mathfrak{L}_2(X, \varrho)$ into $\mathfrak{L}_2(Y, \tau)$. Then they formulated estimates for the characteristic values of this operator. Clearly, these estimates imply trace class criteria. According to the summaries of the papers [18], [19], [21] in their main theorems $X=Y=Q^m$, the m-dimensional unit cube and $\varrho(X)\leqq 1$ and $\tau(Y)\leqq 1$. In their third paper [20] they allow X and Y to be unbounded subsets of $\mathscr{R}^m$ provided that either $\varrho(X)\leqq 1$ or $\tau(Y)\leqq 1$. They show how this case can be reduced to their previously treated case. At the same time they give examples of operators which can be reduced to this case.

In this paper the question of trace class criteria is taken up again for the case of $X=Y=\mathscr{R}^+$ and for the case of both measures being the Lebesgue measure.

In Section 2 first we assign a bound to a given integral operator K acting in $\mathfrak{L}_2(\mathscr{R}^+)$. This bound depends on a given set of three positive constants (α, β, γ) and we denote it by $\|K\|(\alpha, \beta, \gamma)$. The first constant α measures, so to speak, the modulus of mean continuity of the kernel $K(\xi, \eta)$ with reference the second variable η. The second constant β, so to speak, measures an additional smallness of this modulus of mean continuity near infinity. The third constant γ measures the smallness of the kernel itself near infinity and for brevity we refer to it as the decay constant. Then in Theorem 2.1 we formulate a trace class criterion with the aid of the bound $\|K\|(\alpha, \beta, \gamma)$. More specifically it is a family of criteria depending on the parameters

*) Supported by N.S.F. grant GP 28933

(α, β, γ). The only needed restriction on these parameters is that they satisfy the three inequalities of assumption (2.5) of Theorem 2.1. The first two of these inequalities simply says that α and γ is greater than 1/2. The third inequality involves all of the three parameters (α, β, γ). Roughly speaking it says that for given α the modulus of mean continuity near infinity is small compared to the decay exponent γ. The bigger this decay exponent the less additional smallness of the modulus of mean continuity is required near infinity.

In Section 3 we derive Theorem 2.1 from a Corollary of an abstract Lemma of Gokhberg—Krein [15] which was formulated by Birman—Solomjak [18. a]. The method of our proof differs from theirs inasmuch as the construction of approximating operators does. Specifically with the aid of the set of three positive constants (α, β, γ) of Theorem 2.1 for each positive integer n a partion of $\mathscr{R}^+$ is defined. Then this partion is used to define a subspace of $\mathfrak{L}_2(\mathscr{R}^+)$ and we choose the n-th approximating operator to be the restriction of K to this subspace. In our proof this partition plays the same role as the Birman—Solomjak approximation theorem by piecwise polynomial functions [17] did play in theirs.

The delicate counter-example of the Appendix is due to Weidmann. It illustrates that in assumption (2.5) of Theorem 2.1 one needs a strict inequality.

2. Formulation of the result

Let K be a Hilbert—Schmidt operator acting in $\mathfrak{L}_2(\mathscr{R}^+)$ with kernel $K(\xi, \eta)$. In this section we formulate criteria for K to be in trace class.

To describe these criteria to the operator K and to a given set of three positive constants (α, β, γ) we assign a bound, $\|K\|(\alpha, \beta, \gamma)$. Using the well known formula for the Hilbert—Schmidt norm of an integral operator [4. d] [13. c] and a theorem of Fubini [4. b.] we see that for each bounded interval $\mathscr{I}$ the mean

$$(2.1) \qquad M(\mathscr{I})K(\xi) = \frac{1}{|\mathscr{I}|} \int_{\mathscr{I}} K(\xi, \eta)\, d\eta$$

is a square integrable function of the variable ξ. Here, of course, $|\mathscr{I}|$ denotes the lengt of the interval $\mathscr{I}$. The first constant α will measure the smallness of the modulus of mean continuity of the kernel $K(\xi, \eta)$ with reference to the second variable η. More specifically define a preliminary bound by

$$(2.2) \qquad \|K\|(\alpha) = \sup_{\mathscr{I}} \left(\frac{1}{|\mathscr{I}|}\right)^{\frac{2\alpha+1}{2}} \Big(\iint_{\mathscr{R}^+ \times \mathscr{I}} |K(\xi, \eta) - M(\mathscr{I})K(\xi)|^2\, d\xi\, d\eta \Big)^{\frac{1}{2}}.$$

Here the supremum is taken over the bounded subintervals of $\mathscr{R}^+$. Incidentally note that for continuous kernels with bounded support the finiteness of this norm is implied by

$$\sup_{\xi}\sup_{\eta_1,\eta_2}\left(\frac{1}{|\eta_2-\eta_1|}\right)^{\alpha}|K(\xi,\eta_2)-K(\xi,\eta_1)|<\infty.$$

This implication is an elementary consequence of the fact that to a given vector with reference to a given subspace the best approximation is the orthogonal projection [4. a] [11. b]. Hence

$$\inf_{\eta_1}\int_{\mathscr{I}}|K(\xi,\eta)-K(\xi,\eta_1)|^2\,d\eta=\int_{\mathscr{I}}|K(\xi,\eta)-M(\mathscr{I})K(\xi)|^2\,d\eta.$$

The second constant β will measure an additional smallness property of the modulus of mean continuity near infinity. More specifically for a given pair of positive constant (α,β) we define a bound by setting

(2.3)$_1$

$$\|K\|_1(\alpha,\beta)=\sup\left(\frac{1}{|\mathscr{I}|}\right)^{\frac{2\alpha+1}{2}}(1+\min\partial\mathscr{I})^{\beta}\Big(\iint_{\mathscr{R}^+\times\mathscr{I}}|K(\xi,\eta)-M(\mathscr{I})K(\xi)|^2\,d\xi\,d\eta\Big)^{\frac{1}{2}}$$

Here, as usual $\partial\mathscr{I}$ denotes the boundary points of $\mathscr{I}$ and supremum is taken over all compact subintervals of $\mathscr{R}^+$. The third constant γ will measure the smallness of the kernel $K(\xi,\eta)$ itself near infinity. For brevity we refer to it as the decay exponent. More specifically with the aid of γ we define a bound by setting

(2.3)$_2$
$$\|K\|_2(\gamma)=\sup_{\eta_1}(1+\eta_1)^{\frac{\gamma-1}{2}}\Big(\iint_{(\eta_1,\infty)\times\mathscr{R}^+}|K(\xi,\eta)|^2\,d\xi\,d\eta\Big)^{\frac{1}{2}}.$$

Finally define

(2.4)
$$\|K\|(\alpha,\beta,\gamma)=\max\big(\|K\|_1(\alpha,\beta),\|K\|_2(\gamma)\big).$$

The theorem that follows formulates a family of trace class criteria for the operator K with the aid of the bound $\|K\|(\alpha,\beta,\gamma)$.

Theorem 2.1. *Let K be a Hilbert—Schmidt operator acting on $\mathfrak{L}_2(\mathscr{R}^+)$. Suppose that to this operator there are three positive constants (α,β,γ) such that*

(2.5)
$$\alpha>1/2,\quad \gamma>1/2,\quad \text{and}\quad (2\alpha+1-2\beta)<(2\gamma-1)(2\alpha-1)$$

and

(2.6)
$$\|K\|(\alpha,\beta,\gamma)<\infty.$$

Then this operator is in trace class, specifically

(2.7)
$$K\in\mathfrak{S}_1\big(\mathfrak{L}_2(\mathscr{R}^+)\big).$$

We shall establish this theorem in the next section. At present let us consider the three inequalities of assumption (2.5) again. The second one, namely that $\gamma > 1/2$ is evident. For, only in this case does the bound $\|K\|_2(\gamma)$ measure smallness at infinity. In fact for $\gamma < 1/2$ this bound is finite for any Hilbert—Schmidt operator. Concerning the first inequality all that is evident is the positivity of α. Nevertheless a straightforward adaptation of the Weidmann example of the Appendix shows that it is possible for a non-trace-class operator to have a finite $\|K\|(\alpha)$ bound with $\alpha = 1/2$. The details of this adaptation were carried out elsewhere [22]. Concerning the third inequality of assumption (2.5) all that we know is that it cannot be sharpened according to the Appendix.

3. Proof of Theorem 2.1

In this section we derive Theorem 2.1 from the Birman—Solomjak Corollary [18. a]. We do not know whether the assumptions of Theorem 2.1 allow one to construct an operator $\tilde{K}$ which is unitarily equivalent to the original operator K and is such that the Birman—Solomjak results of [20] apply to it. We do know, however, that the three constants (α, β, γ) of Theorem 2.1 allow one to construct a partition of $\mathcal{R}^+$. This partition, in turn, allows one to define a sequence of approximating operators satisfying the assumptions of the Birman—Solomjak Corollary [18. a].

Our construction will depend on whether

$$2\alpha + 1 - 2\beta \leqq 0. \tag{3.1$_1$}$$

or

$$2\alpha + 1 - 2\beta > 0. \tag{3.1$_2$}$$

In case relation $(3.1)_1$ holds, first we choose a preliminary constant r so that

$$r(2\gamma - 1) > 1. \tag{3.2$_1$}$$

Then set

$$\nu = n^r, \tag{3.3}$$

and to this r we choose σ so large that

$$2\alpha - \frac{2\alpha + 1}{\sigma} r > 1 \quad \text{and} \quad \sigma - r > 1. \tag{3.4$_1$}$$

In case relation $(3.1)_2$ holds first we choose the preliminary constant r so that

$$r(2\gamma - 1) > 1 \quad \textit{and} \quad 2\alpha - r(2\alpha + 1 - 2\beta) > 1. \tag{3.2$_2$}$$

Then as before define ν be equation (3.3). At present we choose σ so large that

$$(3.4)_2 \qquad \max(2\alpha+1-2\beta), \ \frac{2\alpha+1}{\sigma} = 2\alpha+1-2\beta \quad \text{and} \quad \sigma - r > 1.$$

Note that the two inequalities in $(3.2)_2$ together with relation $(3.1)_2$ are equivalent to

$$\frac{1}{2\gamma-1} < r < \frac{2\alpha-1}{2\alpha+1-2\beta}.$$

Remembering assumption (2.5) and using relation $(3.1)_2$ again we see that this inequality does admit a solution r. Let us emphasize again that the definition of the constants ν, σ depends on whether relation $(3.1)_1$ or $(3.1)_2$ holds. Having defined these constants for each positive integer n we define a function $g_n(\nu, \sigma)$ by

$$(3.5) \qquad g_n(\nu, \sigma)(x) = \nu\left(\frac{x}{n}\right)^{\sigma}.$$

Finally with the aid of this function we define a family of intervals by

$$(3.6) \qquad \mathscr{I}_n(i, \nu, \sigma) = [g_n(\nu, \sigma)(i), g_n(\nu, \sigma)(i+1)), \quad i = 0, 1, 2, \ldots, n-1.$$

According to definitions $(3.5)_{1,2}$, it is no loss of generality to assume that $\sigma > 1$, which implies that this function is strictly increasing. Then clearly this family of intervals defines a partition of the interval $[0, \nu)$.

Next let $c_n(i, \nu, \sigma)$ denote the characteristic function of the interval $\vartheta_n(i, \nu, \sigma)$ and define the subspace $\mathfrak{K}_n(\alpha, \beta, \gamma)$ to be their linear span. Specifically

$$(3.7) \qquad \mathfrak{K}_n(\alpha, \beta, \nu) = \{c_n(i, \nu, \sigma); \quad i = 0, 1, 2, \ldots, n-1\}.$$

Note that this subspace depends on the constants α, β, γ inasmuch as ν and σ do. Clearly

$$(3.8) \qquad \dim \mathfrak{K}_n(\alpha, \beta, \gamma) = n.$$

Let P_n denote the ortho-projector on $\mathfrak{K}_n(\alpha, \beta, \gamma)$ and set

$$(3.9) \qquad K_n = KP_n.$$

Then for each positive integer n this equation defines an operator of rank n. This fact allows us to apply the Birman—Solomjak Corollary [18. a]. According to this corollary for the $(2n+1)$-st characteristic value of the Hilbert—Schmidt operator K, that is for the $(2n+1)$-st eigenvalue of the positive self-adjoint operator $(K^*K)^{1/2}$, we have

$$(3.10) \qquad \mu_{2n+1}(K) \leqq \left(\frac{1}{n}\right)^{\frac{1}{2}} \|K-K_n\| \, (\text{H.S}).$$

Here the second factor denotes the Hilbert—Schmidt norm. Actually this estimate holds for any operator of rank n. The lemma that follows will imply that for the operator K_n of definition (3.9) the right members of estimate (3.10) from a convergent series.

Lemma 3.1. *Suppose that the operator K satisfies the assumptions of Theorem 2.1 and define the operator K_n by equation (3.9). Then there are constants $\delta(\alpha, \beta, \gamma)$ and $\lambda(\alpha, \beta, \gamma)$ such that*

$$\delta(\alpha, \beta, \gamma) > 1/2 \tag{3.11}$$

and for every positive integer n we have

$$\|K - K_n\|(\text{H.S}) \leqq \lambda(\alpha, \beta, \gamma)\left(\frac{1}{n}\right)^{\delta(\alpha, \beta, \gamma)}. \tag{3.12}$$

To establish conclusion (3.11) set

$$\varkappa(\alpha, \beta, \sigma) = \max\left(2\alpha + 1 - 2\beta,\ \frac{2\alpha+1}{\sigma}\right) \tag{3.13}$$

and

$$\delta(\alpha, \beta, \gamma) = 1/2 \min\{2\alpha - \varkappa(\alpha, \beta, \sigma)r,\ (2\gamma - 1)r\}. \tag{3.14}$$

Here the constants r and σ are defined by equations $(3.2)_{1,2}$ and $(3.4)_{1,2}$. At the same time we see from these definitions that this constant $\delta(\alpha, \beta, \gamma)$ is greater than 1/2. That is to say, conclusion (3.11) holds.

To establish conclusion (3.12) first we introduce a notation for the difference in (3.12) by setting

$$D_n = K - K_n \tag{3.15}$$

Remembering definition (3.9) an elementary argument shows that the kernel of this operator is given by

$$D_n(\xi, \eta) = \begin{cases} K(\xi, \eta) - M(\mathscr{I}_n(i, \nu, \sigma))K(\xi) & \text{for} \quad \eta \in \mathscr{I}_n(i, \nu, \sigma) \\ K(\xi, \eta) & \text{for} \quad \eta \in [\nu, \infty). \end{cases} \tag{3.16}$$

Next we introduce two more operators by setting

$$D_{n,1}(\xi, \eta) = \begin{cases} D_n(\xi, \eta) & \eta \in [0, \nu) \\ 0 & \eta \in [\nu, \infty) \end{cases} \tag*{$(3.17)_1$}$$

and

$$D_{n,2}(\xi, \eta) = \begin{cases} 0 & \eta \in [0, \nu) \\ K(\xi, \eta) & \eta \in [\nu, \infty). \end{cases} \tag*{$(3.17)_2$}$$

Remembering definition (3.15) we see that

$$D_n = D_{n,1} + D_{n,2}. \tag{3.18}$$

To estimate the square of the Hilbert—Schmidt norm of the operator $D_{n,1}$ we need a notation. Specifically for each positive integer n set

$$(3.19) \qquad S_n(\alpha, \beta, g_n(v, \sigma)) = \sum_{i=0}^{n-1} \frac{|g_n(v, \sigma)(i+1) - g_n(v, \sigma)(i)|^{2\alpha+1}}{(1+g_n(v, \sigma)(i))^{2\beta}}.$$

Then we claim that

$$(3.20)_1 \qquad \|D_{n,1}\|^2(\text{H.S}) \leqq \|K\|_1^2(\alpha, \beta) S_n(\alpha, \beta, g_n(v, \sigma)).$$

For definition $(3.17)_1$ together with the partition property of the intervals $\{\mathcal{I}_n(i, v, \sigma)\}$ yields

$$(3.21) \qquad \|D_{n,1}\|^2(\text{H.S}) = \sum_{i=0}^{n-1} \iint_{\mathcal{R}^+ \times \mathcal{I}_n(i, v, \sigma)} |D_n(\xi, \eta)|^2 \, d\xi \, d\eta,$$

if we use the well known formula [4. d] [13. c] for the square of the Hilbert—Schmidt norm of an integral-operator. Definitions $(2.3)_1$, (3.7), and relation (3.16) together show that

$$\iint_{\mathcal{R}^+ \times \mathcal{I}_n(i, v, \sigma)} |D_n(\xi, \eta)|^2 \, d\xi \, d\eta \leqq \|K\|_1^2(\alpha, \beta) \frac{|g_n(v, \sigma)(i+1) - g_n(v, \sigma)(i)|^{2\alpha+1}}{(1+g_n(v, \sigma)(i))^{2\beta}}.$$

Inserting this estimate in equality (3.21) and remembering definition (3.19) we obtain the validity of estimate $(3.20)_1$.

In the technical lemma that follows we estimate this sum in terms of n and v. Actually this is slightly more general than what we need inasmuch as we do not assume that v is a given function of n.

Lemma 3.2. *For each positive integer n and pair of positive constants v, σ define the function $g_n(v, \sigma)$ by equation* (3.5). *Let α, β be a given pair of positive constants and define the sum $S_n(\alpha, \beta, g_n(v, \sigma))$ by equation* (3.19). *Then to each α, β and $\sigma > 1$ there is a constant $\gamma(\alpha, \beta, \sigma)$ such that defining the constant $\varkappa(\alpha, \beta, \sigma)$ by equation* (3.13) *for every (v, n) in $(1, \infty)\times(1, \infty)$ we have*

$$(3.22) \qquad S_n(\alpha, \beta, g_n(v, \sigma)) \leqq \gamma(\alpha, \beta, \sigma) \left(\frac{1}{n}\right)^{2\alpha} v^{\varkappa(\alpha, \beta, \sigma)} + \left(\frac{v}{n^\sigma}\right)^{2\alpha+1}.$$

The assumption $\sigma > 1$ together with definition (3.5) shows that the derivative of $g_n(v, \sigma)$ is increasing. This, in turn, together with the mean value theorem shows that for $i = 1, 2, \ldots, n-1$,

$$|g_n(v, \sigma)(i+1) - g_n(v, \sigma)(i)| \leqq 2^\sigma \left(\frac{v\sigma}{n^\sigma}\right) (i)^{\sigma-1}.$$

At the same time definition (3.5) shows that for $i = 0$,

$$|g_n(v, \sigma)(1) - g_n(v, \sigma)(0)| = v \left(\frac{1}{n}\right)^\sigma.$$

Inserting these inequalities and definition (3.5) in definition (3.19) yields

$$S_n(\alpha,\beta,g_n(v,\sigma)) \leqq \left(2^\sigma \frac{v\sigma}{n^\sigma}\right)^{2\alpha+1} \sum_{j=1}^{n-1} \frac{j^{(\sigma-1)(2\alpha+1)}}{\left(1+v\left(\frac{j}{n}\right)^\sigma\right)^{2\beta}} + \left(\frac{v}{n^\sigma}\right)^{2\alpha+1}. \tag{3.23}$$

To estimate this sum define the function $f_n(\alpha, \beta, v, \sigma)$ by

$$f_n(\alpha,\beta,v,\sigma)(x) = \frac{x^{(\sigma-1)(2\alpha+1)}}{\left(1+v\left(\frac{x}{n}\right)^\sigma\right)^{2\beta}}. \tag{3.24}$$

First we consider the case of

$$\varkappa(\alpha,\beta,\sigma) = 2\alpha+1-2\beta. \tag{3.25$_1$}$$

We claim that this implies that the function $f_n(\alpha, \beta, v, \sigma)$ of definition (3.24) is increasing on the positive real axis. For, suppose that its derivative does vanish at some point m. Then elementary algebra shows that m satisfies the equation

$$[(\sigma-1)(2\alpha+1)-2\beta\sigma]v\left(\frac{m}{n}\right)^\sigma = -(\sigma-1)(2\alpha+1). \tag{3.26}$$

By assumption the right member is strictly negative. It is an elementary consequence of definition (3.13) and relation $(3.25)_1$ that the expression in the bracket is positive, that is

$$(\sigma-1)(2\alpha+1)-2\beta\sigma \geqq 0. \tag{3.27$_1$}$$

Hence there is no point m on the positive axis where this derivative does vanish, and our claim follows. The increasing character of this function shows that

$$\sum_{j=1}^{n-1} \frac{j^{(\sigma-1)(2\alpha+1)}}{\left(1+v\left(\frac{j}{n}\right)^\sigma\right)^{2\beta}} < \int_1^n \frac{x^{(\sigma-1)(2\alpha+1)}}{\left(1+v\left(\frac{x}{n}\right)^\sigma\right)^{2\beta}}\,dx. \tag{3.28}$$

Clearly

$$\int_1^n \frac{x^{(\sigma-1)(2\alpha+1)}}{\left(1+v\left(\frac{x}{n}\right)^\sigma\right)^{2\beta}}\,dx \leqq n^{1+(\sigma-1)(2\alpha+1)} \int_0^1 \frac{y^{(\sigma-1)(2\alpha+1)}}{(1+vy^\sigma)^{2\beta}}\,dy \tag{3.29}$$

and

$$\int_0^1 \frac{y^{(\sigma-1)(2\alpha+1)}}{(1+vy^\sigma)^{2\beta}}\,dy = \left(\frac{1}{v}\right)^{2\beta} \int_0^1 \frac{y^{(\sigma-1)(2\alpha+1)}}{\left(\frac{1}{v}+y^\sigma\right)^{2\beta}}\,dy. \tag{3.30}$$

Remembering relation $(3.27)_1$ we see that the integral on the right is bounded independently of v. In fact

$$\int_0^1 \frac{y^{(\sigma-1)(2\alpha+1)}}{\left(\frac{1}{v}+y^\sigma\right)^{2\beta}}\,dy \leqq 1.$$

This relation together with relations (3.29) and (3.30) inserted in estimate (3.28) yields

$$\sum_{j=1}^{n-1} \frac{j^{(\sigma-1)(2\alpha+1)}}{\left(1+\nu\left(\frac{j}{n}\right)^{\sigma}\right)^{2\beta}} \leqq n^{1+(\sigma-1)(2\alpha+1)} \left(\frac{1}{\nu}\right)^{2\beta}. \tag{3.31}$$

Inserting this estimate, in turn, in estimate (3.23) we arrive at,

$$S_n(\alpha, \beta, g_n(\nu, \sigma)) \leqq (2^{\sigma}\sigma)^{2\alpha+1} \left(\frac{1}{n}\right)^{2\alpha} \nu^{2\alpha+1-2\beta} + \left(\frac{\nu}{n^{\sigma}}\right)^{2\alpha+1}.$$

Hence setting

$$\gamma_1(\alpha, \beta, \sigma) = (2^{\sigma}\sigma)^{2\alpha+1} \tag{3.32$_1$}$$

and remembering definition (3.13) we arrive at the validity of conclusion (3.22) in case relation $(3.25)_1$ holds.

Second we consider the case of

$$\varkappa(\alpha, \beta, \sigma) = \frac{2\alpha+1}{\sigma}. \tag{3.25$_2$}$$

If the two numbers in definition (3.13) are equal then relation $(3.25)_1$ also holds and we have just seen the validity of conclusion (3.22). Accordingly we assume that

$$2\alpha+1-2\beta < \frac{2\alpha+1}{\sigma}.$$

Clearly, this implies that

$$(\sigma-1)(2\alpha+1)-2\beta\sigma < 0, \tag{3.27$_2$}$$

which, in turn, implies that equation (3.26) does admit a positive solution m. That is to say the derivative of the function $f_n(\alpha, \beta, \nu, \sigma)$ does vanish at m. Hence this function is increasing on the interval $[0, m]$ and decreasing on the interval $[m, \infty)$. This fact together with definition (3.24) shows that

$$\sum_{j=1}^{n-1} \frac{j^{(\sigma-1)(2\alpha+1)}}{\left(1+\nu\left(\frac{j}{n}\right)^{\sigma}\right)^{2\beta}} \leqq \int_1^n \frac{x^{(\sigma-1)(2\alpha+1)}}{\left(1+\nu\left(\frac{x}{n}\right)^{\sigma}\right)^{2\beta}} dx + \frac{m^{(\sigma-1)(2\alpha+1)}}{\left(1+\nu\left(\frac{m}{n}\right)^{\sigma}\right)^{2\beta}}. \tag{3.33}$$

It is not difficult to estimate this integral. In fact we claim that relation $(3.27)_2$ implies

$$\int_1^n \frac{x^{(\sigma-1)(2\alpha+1)}}{\left(1+\nu\left(\frac{x}{n}\right)^{\sigma}\right)^{2\beta}} dx \leqq \left(\frac{n^{\sigma}}{\nu}\right)^{(2\alpha+1)-\frac{2\alpha}{\sigma}} (2\nu)^{\frac{1}{\sigma}}. \tag{3.34}$$

For, setting

$$\nu\left(\frac{x}{n}\right)^{\sigma}=t,$$

an elementary change of variables, yields

$$\int_1^n \frac{x^{(\sigma-1)(2\alpha+1)}}{\left(1+\nu\left(\frac{x}{n}\right)^{\sigma}\right)^{2\beta}}\,dx \leqq \frac{1}{\sigma}\left(\frac{n^{\sigma}}{\nu}\right)^{(2\alpha+1)-\frac{2\alpha}{\sigma}} \int_0^{\nu} \frac{t^{\frac{\sigma-1}{\sigma}2\alpha}}{(1+t)^{2\beta}}\,dt.$$

At the same time, we see from relation $(3.27)_2$ that

$$\frac{\sigma-1}{\sigma}2\alpha-2\beta<\frac{1}{\sigma}-1.$$

This shows that

$$\int_0^{\nu}(1+t)^{\frac{\sigma-1}{\sigma}2\alpha-2\beta}\,dt \leqq (2\nu)^{\frac{1}{\sigma}}.$$

if we remember that by assumption $\nu>1$. Inserting this estimate in the previous one we obtain the validity of estimate (3.34).

It is not difficult to estimate the second term in (3.33) either. To do this recall that the positive number m was defined by equation (3.26). This equation together with relation $(3.27)_2$ shows that setting

$$\tilde{\gamma}(\alpha,\beta,\sigma)=\frac{-(\sigma-1)(2\alpha+1)}{(\sigma-1)(2\alpha+1)-2\beta\sigma},$$

we have

$$\nu\left(\frac{m}{n}\right)^{\sigma}=\tilde{\gamma}(\alpha,\beta,\sigma)$$

Then elementary algebra shows that equation (3.26) implies

$$\frac{m^{(\sigma-1)(2\alpha+1)}}{\left(1+\nu\left(\frac{m}{n}\right)^{\sigma}\right)^{2\beta}} \leqq \left(\tilde{\gamma}(\alpha,\beta,\sigma)\right)^{\frac{(\sigma-1)(2\alpha+1)}{\sigma}-2\beta}\left(\frac{n^{\sigma}}{\nu}\right)^{(2\alpha+1)-\frac{2\alpha+1}{\sigma}} \tag{3.35}$$

Inserting estimates (3.35) and (3.34) in estimate (3.33) we obtain

$$\sum_{j=1}^{n-1}\frac{j^{(\sigma-1)(2\alpha+1)}}{\left(1+\nu\left(\frac{j}{n}\right)^{\sigma}\right)^{2\beta}} \leqq \left(\frac{n^{\sigma}}{\nu}\right)^{(2\alpha+1)-\frac{2\alpha}{\sigma}}(2\nu)^{\frac{1}{\sigma}}+ \tag{3.36}$$

$$+\left(\tilde{\gamma}(\alpha,\beta,\sigma)\right)^{\frac{(\sigma-1)(2\alpha+1)}{\sigma}-2\beta}\left(\frac{n^{\sigma}}{\nu}\right)^{(2\alpha+1)-\frac{2\alpha+1}{\sigma}}.$$

Inserting this estimate (3.36), in turn, in estimate (3.23) and setting

$$(3.32)_2 \qquad \gamma_2(\alpha,\beta,\sigma) = (2^\sigma\sigma)^{2\alpha+1}\left[2+\left(\tilde{\gamma}(\alpha,\beta,\sigma)\right)^{\frac{(\sigma-1)(2\alpha+1)}{\sigma}-2\beta}\right],$$

we arrive at the validity of conclusion (3.22) in case relation $(3.25)_2$ holds. This completes the proof of Lemma 3.2, if we remember that according to definition (3.13) either relation $(3.25)_1$ or relation $(3.25)_2$ holds.

Having established Lemma 3.2 we can easily derive conclusion (3.12) of Lemma 3.1 from it. For, insertion of conclusion (3.22) in estimate $(3.20)_1$ yields

$$(3.37)_1 \quad \|D_{n,1}\|^2(\mathrm{H.S}) \leqq \|K\|_1^2(\alpha,\beta)\,\gamma^2(\alpha,\beta,\sigma)\left(\frac{1}{n}\right)^{2\alpha-\varkappa(\alpha,\beta,\sigma)r}+\left(\frac{1}{n}\right)^{(\sigma-r)(2\alpha+1)}$$

if we remember definition (3.3). According to relation $(3.17)_1$

$$\|D_{n,2}\|^2(\mathrm{H.S}) = \iint_{(\nu,\infty)\times\mathcal{R}^+} |K(\xi,\eta)|^2\,d\xi\,d\eta.$$

This relation together with definitions $(2.3)_2$ and (3.3) yields

$$(3.37)_2 \qquad \|D_{n,2}\|^2(\mathrm{H.S}) \leqq \|K\|_2^2(\gamma)\left(\frac{1}{n}\right)^{(2\gamma-1)r}.$$

Thus setting

$$\lambda(\alpha,\beta,\gamma) = [\|K\|_1^2(\alpha,\beta)\left(1+\gamma(\alpha,\beta,\sigma)+\|K\|_2^2(\gamma)\right]^{1/2}$$

and remembering definition (3.14) we arrive at the validity of conclusion (3.12) of Lemma 3.1.

Finally we can easily derive Theorem 3.1 from Lemma 2.1. For, insertion of conclusion (3.12) in the Birman—Solomjak Corollary (3.10) yields

$$\mu_{2n+1}(K) \leqq \lambda(\alpha,\beta,\gamma)\left(\frac{1}{n}\right)^{1/2+\delta(\alpha,\beta,\gamma)}$$

According to conclusion (3.11) of Lemma 3.1 the right members form a convergent series. Hence

$$(3.38) \qquad \sum_{n=1}^{\infty}\mu_{2n+1}(K) < \infty.$$

Since the characteristic values were ordered in decreasing order

$$\sum_{n=1}^{\infty}\mu_n(K) \leqq \mu_1(K)+\mu_2(K)+2\sum_{n=1}^{\infty}\mu_{2n+1}(K).$$

Hence inserting estimate (3.38) in this inequality we arrive at the validity of conclusion (2.7), if we remember the definition of trace class [13. d]. This completes the proof of Theorem 2.1.

Appendix

A Hilbert—Schmidt operator with $||K||\,(1, 1, 1) < \infty$ which is not in trace class

By JOACHIM WEIDMANN

Before constructing such an operator note that for $\alpha=1$, $\beta=1$, $\gamma=1$ we have

$$2\alpha+1-2\beta = (2\gamma-1)(2\alpha-1).$$

In other words for these constants the third inequality in assumption (2.5) of Theorem 2.1 is replaced by an equality.

To construct such an operator we first define two ortho-normal sets of functions in $\mathfrak{L}_2(\mathfrak{R}^+)$ by setting

$$a_i(\xi) = \begin{cases} \sqrt{2}\sin(i\pi\xi) & \xi\in[0,1) \\ 0 & \xi\in[1,\infty) \end{cases} \tag{A-1}$$

and

$$b_i(\eta) = \begin{cases} \sqrt{2}\sin(\pi\eta) & \eta\in[i,i+1] \\ 0 & \eta\notin[i,i+1]. \end{cases} \tag{A-2}$$

It is an immediate consequence of this ortho-normality that setting

$$K(\xi,\eta) = \sum_{i=1}^{\infty}\frac{1}{i+2}a_i(\xi)b_i(\eta), \tag{A-3}$$

we have

$$\int_0^\infty\int_0^\infty |K(\xi,\eta)|^2\,d\xi\,d\eta = \sum_{i=1}^{\infty}\left(\frac{1}{i+2}\right)^2 < \infty. \tag{A-4}$$

That is to say this kernel defines a Hilbert—Schmidt operator K. At the same time it follows that

$$(K^*K)^{1/2}(\xi,\eta) = \sum_{i=1}^{\infty}\frac{1}{i+2}a_i(\xi)a_i(\eta), \tag{A-5}$$

and hence

$$\operatorname{tr}[(K^*K)^{1/2}] = \sum_{i=1}^{\infty}\frac{1}{i+2} = \infty. \tag{A-6}$$

In other words the operator K of definition (A-3) is not in trace class.

Next we maintain that its kernel is Hölder continuous. More specifically we maintain that

$$\|K\|(1,1,1) < \infty. \tag{A-7}$$

To establish this estimate first we claim that

$$\sup_{\eta_1}(1+\eta_1)^{1/2}\Big(\int_{\eta_1}^{\infty}\int_0^{\infty} K^2(\xi,\eta)\,d\xi\,d\eta\Big)^{1/2} < \infty. \tag{A-8}$$

For, in view of the ortho-normality of the system $\{a_i\}$ at each point η definition (A-3) yields

$$\int_0^\infty K^2(\xi,\eta)\,d\xi = \sum_{i=1}^\infty \left(\frac{1}{i+1}\right)^2 b_i^2(\eta).$$

At the same time we see from definition (A-2) that

$$i+1<\eta_1 \quad \text{implies} \quad \int_{\eta_1}^\infty b_i^2(\eta)\,d\eta = 0.$$

Hence

$$\int_{\eta_1}^\infty\int_0^\infty K^2(\xi,\eta)\,d\xi\,d\eta \leqq \sum_{i=\eta_1}^{\infty} \frac{1}{(i+1)^2} \leqq \frac{1}{1+\eta_1},$$

and the validity of (A-8) follows. Second we claim that for every (ξ,η_1) and (ξ,η_2) we have

(A-9) $$|K(\xi,\eta_1)-K(\xi,\eta_2)| \leqq 2\pi\frac{|\eta_2-\eta_1|}{(1+\min(\eta_1,\eta_2))}.$$

For, in case η_1 and η_2 are in the same interval, say

(A-10) $$\eta_1\in[m,m+1) \quad\text{and}\quad \eta_2\in[m,m+1),$$

definition (A-3) yields

(A-11) $$|K(\xi,\eta_2)-K(\xi,\eta_2)| = \left|\frac{a_m(\xi)}{m+2}(b_m(\eta_2)-b_m(\eta_1))\right|.$$

Remembering definition (A-2) we see from the mean value theorem that

(A-12) $$|b_m(\eta_2)-b_m(\eta_1)| \leqq \sqrt{2\pi}\,|\eta_2-\eta_1|.$$

Definition (A-1) together with assumption (A-10) yields

(A-13) $$\left|\left(\frac{1}{m+2}\right)a_m(\xi)\right| \leqq \frac{\sqrt{2}}{1+\eta}.$$

Hence in this case relation (A-11) together with estimates (A-12) and (A-13) yields the validity of estimate (A-9). In the general case let the integers $m_{1,2}$ be defined by

(A-14) $$m_1\leqq\eta_1\leqq m_1+1\leqq m_2\leqq\eta_2<m_2+1.$$

Then definition (A-3) yields

(A-15) $$|K(\xi,\eta_2)-K(\xi,\eta_1)| \leqq \sum_{i=1}^{2}\left|\frac{a_{m_i}(\xi)}{m_i+2}b_{m_i}(\eta_i)\right|,$$

if we use the triangle inequality.

Since

$$\sin((m_1+1)\pi)=0,$$

the mean value theorem implies for $i=1, 2$

$$|b_{m_i}(\eta_i) - \sin((m_1+1)\pi)| < \sqrt{2\pi}\, |\eta_i - (m_1+1)\eta|.$$

Assumption (A-14) clearly implies that

$$|\eta_1 - (m_1+1)| + |\eta_2 - (m_1+1)\eta| = \eta_2 - \eta_1.$$

At the same time, similarly to (A-13) we have for $i=1, 2$,

$$\left|\frac{a_{m_i}(\xi)}{m_i+2}\right| \leqq \frac{\sqrt{2}}{1+\min(\eta_1, \eta_2)}.$$

Inserting these three relations in estimate (A-15) we arrive at the validity of estimate (A-9). Remembering that for ξ in $[1, \infty)$ the kernel $K(\xi, \eta)$ vanishes, we see that estimate (A-9) implies

$$\sup\,(1+\min(\eta_1, \eta_2))\left(\frac{1}{|\eta_2-\eta_1|}\right)^{3/2} \Big(\int_{\eta_1}^{\eta_2}\int_0^{\infty} |K(\xi, \eta) - K(\xi, \eta_2)|^2\, d\xi\, d\eta\Big)^{1/2} < \infty.$$

Finally combining estimates (A-8) and (A-16) we arrive at the validity of estimate (A-7).

References

[1] T. LALESCO, *Introduction à la théorie des équations intégrales,* Hermann (Paris, 1912). See pp. 86—89.

[2] H. WEYL, Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen, *Math. Ann.,* **11** (1912), 441—479.

[3] E. HILLE, and J. D. TAMARKIN, On the characteristic values of linear integral equations, *Acta Math.,* **57** (1931), 1—76.

[4] F. RIESZ and B. SZ.-NAGY, *Leçons d'analyse fonctionnelle,* Akadémiai Kiadó (Budapest, 1952), English translation: *Functional Analysis,* Frederick Ungar Publishing Co. (New York, 1955). a. Section 33; b. Section 40; c. Section 95; d. Section 97.

[5] M. G. KREIN, On the trace formula in perturbation theory. *Mat. Sbor.,* **33** (1953), 597—626. (Russian).

[6] FORREST W. STINESPRING, A sufficient condition for an integral operator to have a trace, *J. reine angew. Math.,* **200** (1958), 200—207.

[7] R. B. LAVINE, *The Weyl Transform — Fourier analysis for operators in L_p-spaces,* Thesis, M.I.T., 1965.

[8] J. WEIDMANN, Integraloperatoren der Spurklasse, *Math. Ann.,* **163** (1966), 340—345.

[9] C. A. MCCARTHY, „C_p", *Israel J. Math.,* **5** (1967), 249—271.

[10] R. H. LORENTZ, *Gentleness versus trace class.* Thesis, University of Minnesota, 1969.

[11] R. COURANT, and D. HILBERT, *Methods of Mathematical Physics,* Vol. 1. Interscience, John Wiley (New York, 1953). a. Section I.4; b. Subsection II.1.3.

[12] K. O. FRIEDRICHS, *Perturbation of Spectra in Hilbert Space.* Amer. Math. Soc. (Providence, 1965). See Appandix 6.

[13] N. Dunford, and J. T. Schwartz, *Linear Operators*, Part II, Interscience, John Wiley (New York, 1963). a. Theorem X.4.3; b. Definition XI.6.1; c. Exercise XI.8.44; d. Definition XI.9.1; e. Section XI.9.32.

[14] T. Kato, *Perturbation Theory for Linear Operators*. Springer Verlag (1966). See Section X.4.

[15] I. C. Gohberg, and M. A. Krein, *Introduction to the theory of nonselfadjoint operators*. Volume Eighteen, Translations of Mathematical Monographs, Amer. Math. Soc. (Providence, Rhode Island, 1969). See Lemma III.6.1.

[16] J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien*. Gauthier-Villars (Paris, 1969) (Second Edition).

[17] M. Sh. Birman and M. Z. Solomjak, Piecewise polynomial approximations of functions of classes W_p^α, *Mat. Sbornik*, (N. S.) **73** (**115**) (1967), 331—335 (Russian), *Math. Rev.*, **36**, No. 576.

[18] M. Sh. Birman and M. Z. Solomjak, On estimates for singular values of integral operators. I, *Vestnik Leningrad Univ.*, (22) No. **7**, (1967), 43—53 (Russian), *Math. Rev.*, **35**, No. 7173. a. Corollary after Lemma 1.

[19] M. Sh. Birman and M. Z. Solomjak, On estimates for singular values of integral operators. II, *Vestnik Leningrad Univ.*, (22) No. **13**, (1967), 21—28 (Russian), *Math. Rev.*, **36**, No. 739.

[20] M. Sh. Birman and M. Z. Solomjak, On estimates for the singular values of integral operators. III. Operators in unbounded domains, *Vestnik Leningrad Univ.*, (24) No. **1** (1969), 35—48 (Russian), *Math. Rev.* **39**, No. 7468.

[21] M. Z. Solomjak, On estimates for singular values of integral operators. IV, *Vestnik Leningrad Univ.*, No. **1** (1970), 76—87 (Russian), Math. Rev., **41**, No. 9066.

[22] R. H. Lorentz and P. A. Rejtő, *Some integral operators of trace class*. Batelle Institute, Advanced Studies Center, Geneva, Switzerland. Technical Report 50, 1971. See Appendix I.

[23] M. Sh. Birman and M. Z. Solomjak, Remarks on the nuclearity of integoperators and the boundedness of pseudodifferential operators. *Izv. Vysš. Učebn. Zaved. Matematika 1969*, no 9 (**88**) 11—17. (Russian), *Math. Rev.*, **40**, No. 7877.

[24] Ph. Martin and B. Misra, On trace-class operators of scattering theory and the asymptotic behaviour of scattering cross section at high energy. *J. Math. Phys.*, **14** (1973), 997—1005.

[25] Eugène B. Fabes, Walter Littman and Nestor M. Riviere, Transformers of pseudodifferential operators, *Notices Amer. Math. Soc.*, **21** (1974).

RUDOLPH A. H. LORENTZ
GESELLSCHAFT FÜR MATHEMATIK
UND DATENVERARBEITUNG
5205 ST. AUGUSTIN 1
POSTFACH 1240
GERMAN FEDERAL REPUBLIC

PETER A. REJTŐ
SCHOOL OF MATHEMATICS
UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MINNESOTA 55455
USA

(Received June 10, 1973)

On the affine umbilical hypersurfaces

By P. T. NAGY in Szeged

In this paper we shall give an affine generalization of a theorem on the umbilical hypersurfaces of a Euclidean space ([1], p. 30).

Let A^{n+1} be the affine space of dimension $n+1$ and denote by R^{n+1} its coordinate space. Consider a smooth manifold M of dimension n, and an immersion $f: M \to A^{n+1}$ of M into A^{n+1}. Since our discussion is local, we may assume that M is a hypersurface imbedded in A^{n+1}.

Let be given an affine normalization $\xi: M \to R^{n+1}$ of the hypersurface M in A^{n+1}, that is a vector field ξ_x on M, the value of which is linearly independent from the tangent vectors of M at every point $x \in M$.

We denote the canonical covariant differentiation in A^{n+1} by ∇'. Let X and Y be tangent vector fields on M. Since $(\nabla'_X Y)_x$ is defined for each $x \in M$, we shall denote its tangential component with respect to the normalization ξ_x by $(\nabla_X Y)_x$, so we have

$$(1) \qquad (\nabla'_X Y)_x = (\nabla_X Y)_x + g_x(X, Y)\xi_x.$$

It is known that $\nabla_X Y$ defines a symmetric linear connection on M, called the induced connection of M, and $g_x(X, Y)$ is a "scalar product" on M.

We say that the affine normalization $\xi: M \to R^{n+1}$ is relative, if the covariant derivative $(\nabla_X \xi)_x$ of ξ for each tangent vector $X \in TM$ has only tangential component, that is

$$(2) \qquad (\nabla'_X \xi)_x = B_x(X),$$

where $B_x: T_x M \to T_x M$ is a linear operator of the tangent space of M.

We say that the normalized hypersurface (M, f, ξ) is affine umbilical at a point $x \in M$, if $B_x = \lambda_x I_x$, where λ_x is a scalar and I_x denotes the identity operator of the tangent space $T_x M$.

Lemma. *We have $(\nabla_X B)(Y) = (\nabla_Y B)(X)$ for any tangent vector field X and Y of M.*

Proof. Applying the equations (1) and (2) we get

$$B([X,Y]) = \nabla'_{[X,Y]}\xi = (\nabla'_X\nabla'_Y - \nabla'_Y\nabla'_X)\xi = \nabla'_X(B(Y)) - \nabla'_Y(B(X)) =$$
$$= \nabla_X(B(Y)) + g(X, B(Y))\xi - \nabla_Y(B(X)) - g(Y, B(X))\xi =$$
$$= (\nabla_X B)(Y) + B(\nabla_X Y) + g(X, B(Y))\xi - (\nabla_Y B)(X) - B(\nabla_Y X) - g(Y, B(X))\xi.$$

Since $\nabla_X Y$ is a symmetric covariant differentiation, we have $\nabla_X Y - \nabla_Y X = [X, Y]$, that is

$$B(\nabla_X Y) - B(\nabla_Y X) = B([X, Y]).$$

From this and the preceding calculation follows

$$\{(\nabla_X B)(Y) - (\nabla_Y B)(X)\} + \{g(X, B(Y)) - g(Y, B(X))\}\cdot\xi = 0,$$

which shows that the tangential component is $(\nabla_X B)(Y) - (\nabla_Y B(X) = 0$.

Thus the lemma is proved.

We say that the affine normalization $\xi\colon M \to R^{n+1}$ of the hypersurface (M, f) is radial affine, if there is a coordinate system of A^{n+1} in which the affine normal vector ξ_x and the position vector of the point $f_x \in A^{n+1}$ coincide for each point $x \in M$.

It is trivial that every radial affine normalization is relative affine.

We say that the affine normalization $\xi\colon M \to R^{n+1}$ is similar to a radial affine normalization, if there exists a nonzero constant τ such that $\tau\cdot\xi\colon M \to R^{n+1}$ is a radial affine normalization.

Theorem 1. *Let (M, f, ξ) be an affine normalized hypersurface in A^{n+1}. If the affine normalization $\xi\colon M \to R^{n+1}$ is similar to a radial affine normalization, then the hypersurface is affine umbilical at every point $x \in M$.*

Proof. Identifying $f_x \in A^{n+1}$ with the corresponding position vector in R^{n+1}, we have $\xi_x = 1/\tau\, f_x$, which shows that for every tangent vector field X on M $(\nabla'_X\xi)_x = 1/\tau\, X_x$.

It follows that $B_x = 1/\tau\, I_x$, that is the hypersurface is affine umbilical.

Theorem 2. *Let (M, f, ξ) be a relative affine normalized hypersurface in A^{n+1}. If every point $x \in M$ is affine umbilical, then either the vector function $\xi\colon M \to R^{n+1}$ is constant or the normalization ξ is similar to a radial affine normalization of the hypersurface $f\colon M \to A^{n+1}$.*

Proof. Since every point $x \in M$ is affine umbilical, there exists a scalar function λ_x on M such that $B_x = \lambda_x I_x$. We are going to prove that λ_x is a constant function.

For any tangent vector fields X and Y on M we have

$$(\nabla_X B)(Y) = \nabla_X(BY) - B(\nabla_X Y) = \nabla_X(\lambda Y) - \lambda\cdot\nabla_X Y = (X\lambda)Y.$$

Similarly,

$$(\nabla_Y B)(X) = (Y\lambda)X.$$

By Lemma we obtain

$$(X\lambda)Y = (Y\lambda)X.$$

For each $x \in M$ we may choose the tangent vector fields X and Y on M so that X_x and Y_x are linearly independent. It follows that $Z\lambda = 0$ for every $Z \in T_x M$, that is λ is equal to a constant on M.

Identifying $f_x \in A^{n+1}$ with its position vector in R^{n+1}, we consider $f_x + \xi_x$ as an R^{n+1}-valued vector function on M.

If X is an element of $T_x M$, we have

$$\nabla'_X(\lambda f - \xi) = \lambda X - \nabla'_X \xi = \lambda X - \lambda X = 0,$$

which shows that $\lambda f_x - \xi_x = \alpha$ is a constant vector in R^{n+1}.

If $\lambda = 0$, then ξ_x is a constant vector, and our assertion is proved.

If $\lambda \neq 0$, then $\lambda f_x - \xi_x = \alpha$ implies $f_x - 1/\lambda\, \alpha = 1/\lambda \cdot \xi_x$, which shows that in the coordinate system of A^{n+1} translated with the vector $1/\lambda \cdot \alpha$ the affine normalization $1/\lambda\, \xi$ is a radial affine normalization of the hypersurface (M, f). Thus our theorem is proved.

Reference

[1] S. KOBAYASHI, K. NOMIZU, *Foundations of differential geometry*. Vol. II, Interscience Publishers (New York—London—Sydney, 1969).

(Received July 9, 1973)

Normal extensions of subnormal operators

By C. R. PUTNAM in Lafayette (Indiana, USA)

1. Introduction. Only bounded operators on Hilbert spaces will be considered below. Let T be subnormal on $\mathfrak{H}$ and let N on $\mathfrak{K} \supset \mathfrak{H}$ denote the minimal normal extension of T. (Concerning subnormal operators and their basic properties, see HALMOS [6], pp. 100 ff.) It was shown by HALMOS [5] that $\operatorname{sp}(N)$ is a subset of $\operatorname{sp}(T)$ and by BRAM [1] that, in fact, $\operatorname{sp}(T)$ consists of $\operatorname{sp}(N)$ together with some of the holes of $\operatorname{sp}(N)$; cf. [6], p. 102. A subnormal T will be called completely subnormal if there exists no non-trivial reducing space on which it is normal.

It is known that if T is isometric ($T^*T=I$) then T is subnormal and if, in addition, T is completely subnormal, that it is the direct sum of (one or more) copies of the unilateral shift; cf. [6], p. 74. Since the bilateral shift is the minimal normal (here, unitary) extension of the unilateral shift, the minimal unitary extension of a completely subnormal isometry is the direct sum of bilateral shifts.

If A is self-adjoint on a Hilbert space with the spectral resolution $A=\int t\,dE_t$, then the set $\mathfrak{H}_a(A)$ of vectors x for which $\|E_t x\|^2$ is an absolutely continuous function of t is a reducing space of A. A similar statement holds for a unitary operator $U=\int_0^{2\pi} e^{it}\,dE_t$. (The usual assumptions are made here, namely, that E_t is continuous from the right and that, in the unitary case, $E_0=0$, hence E_t is continuous at $t=0$, and $E_{2\pi}=I$.) The operator A or U is said to be absolutely continuous if $\mathfrak{H}_a(A)$ or $\mathfrak{H}_a(U)$ is the entire Hilbert space.

It is well-known that the bilateral shift is absolutely continuous with spectrum $\{z:|z|=1\}$; for a proof using commutators, see PUTNAM [9], p. 22. It follows that the minimal unitary extension of a completely subnormal isometry has the same properties, a result which will be generalized below. Some preliminaries will first be needed.

Let N be a normal operator on a Hilbert space $\mathfrak{K}$ with the spectral resolution

$$N=\int z\,dK. \tag{1.1}$$

This work was supported by a National Science Foundation research grant.

For each subset A of the complex plane, $\mathbf{C}$, let $p(A)$ denote the "radial projection" of A into the circle $|z|=1$ defined by $p(A)=\{p(z): z\in A\}$, where $p(0)=1$ and $p(z)=e^{it}$ if $z\neq 0$ and $z=|z|e^{it}$. Call N radially absolutely continuous if $K(A)=0$ whenever A is a planar Borel set whose radial projection $p(A)$ has measure 0 on $|z|=1$, the measure being ordinary Lebesgue arc length. Let U denote the unitary operator defined by

$$(1.2)\qquad U=\int_0^{2\pi} e^{it}\,dE_t,\quad \text{where}\quad E_t=K(A_t),$$

with $A_t=\{z: z\neq 0,\ 0<\arg z\leqq t\}$ for $0<t<2\pi$ and $A_{2\pi}=\mathbf{C}$ (and $E_t=0$ or $E_t=I$ according as $t\leqq 0$ or $t>2\pi$). Then, to say that N is radially absolutely continuous means that U is absolutely continuous as defined earlier.

Theorem 1. *Let T be a completely subnormal operator on a Hilbert space $\mathfrak{H}$ with the minimal normal extension N on $\mathfrak{K}$ and let Q denote the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$. Suppose that*

$$(1.3)\qquad Q(N^*N)=(N^*N)Q.$$

Then N is radially absolutely continuous, that is, U defined by (1.1) *and* (1.2) *is absolutely continuous. Further,*

$$(1.4)\qquad \operatorname{sp}(U)=\{z:|z|=1\}.$$

It is seen that if N is normal on $\mathfrak{K}$ with spectral resolution (1.1) then N has a polar representation $N=UP(=PU)$, where

$$(1.5)\qquad P=(N^*N)^{1/2}$$

and U is defined by (1.2). If (1.3) holds, that is, if $QP^2=P^2Q$, then, since $P\geqq 0$, $QP=PQ$, so that $\mathfrak{H}$ is invariant under P.

If N is unitary, then (1.3) holds trivially. Further, $P=I$ and $N=UP=U$. Thus, it follows from Theorem 1 that $N(=U)$ is absolutely continuous and that its spectrum is the entire circle $|z|=1$. In fact, as previously noted, much more is known: U is a direct sum of bilateral shifts. That the minimal normal extension N of a completely subnormal T may fail to be radially absolutely continuous if (1.3) is not assumed is easy to show by examples; cf. section 4 below. Further, if (1.3) fails to hold for T, it may be possible to replace T by another completely subnormal operator T_1 on a Hilbert space $\mathfrak{H}_1$, in such a way that the minimal normal extension of T_1 is a part, N_1, of N and such that $\mathfrak{H}_1$ is invariant under N_1 and $N_1^*N_1$. Then (1.3) would hold with Q replaced by Q_1, the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}_1$. See the example in section 4 below.

Roughly speakly, condition (1.4) says that the spectrum of N surrounds the origin. More precisely, relation (1.4) holds if and only if there does not exist an open wedge

(1.6) $$W = \{z : z = re^{it},\ r > 0,\ a < t < b\},$$

for which

(1.7) $$\operatorname{sp}(N) \cap W \text{ is empty.}$$

This fact is easily deduced from the definition (1.2) of U. Note that (1.7) may hold even though 0 is in the spectrum of N, although (1.7) does imply, of course, that 0 cannot be an interior point of sp (N).

Theorem 2. *Let T be completely subnormal on $\mathfrak{H}$ with the minimal normal extension N on $\mathfrak{K}$. Suppose that there exists some wedge W of* (1.6) *satisfying* (1.7). *Then $\mathfrak{K}$ is the least space containing $\mathfrak{H}$ and invariant under N and N^*N.*

The proof of Theorem 1 will be given in section 2 and will depend on certain results on commutators; see [9], pp. 21—22. Theorem 2 will be proved in section 3 as a consequence of Theorem 1. Examples illustrating Theorems 1 and 2 will be given in sections 4 and 5. In particular, Theorem 3 of section 5 is an application of Theorem 2 to certain analytic position operators. Finally, some remarks relating absolute continuity of normal operators and second order commutators will be made in section 6.

2. Proof of Theorem 1. Since T is subnormal, it is also hyponormal and hence

(2.1) $$T^*T - TT^* = D, \quad \text{where} \quad D \geqq 0.$$

Further, for $x \in \mathfrak{H}$, one has $T^*x = QN^*x$ (cf. [6], p. 103), thus $QN^*Nx = NQN^*x = Dx$ for $x \in \mathfrak{H}$. Let now the corresponding equation be considered on $\mathfrak{K}$, so that

(2.2) $$QN^*N - NQN^* = D_1,$$

with $D_1x = Dx$ for x in $\mathfrak{H}$. In view of (1.3) it is seen that D_1 is self-adjoint. Further, $\mathfrak{H}$ (hence $\mathfrak{H}^\perp$) is invariant under D_1 and clearly

(2.3) $$D_1 = D \oplus 0 \quad \text{on} \quad \mathfrak{K} = \mathfrak{H} \oplus \mathfrak{H}^\perp.$$

In particular, $D_1 \geqq 0$ on $\mathfrak{K}$.

Since $N = UP = PU$, where U and P are defined in (1.2) and (1.5), it is seen that (2.2) becomes $QP^2 - UPQPU^* = D_1$. Since $QP = PQ$ (by (1.3)) this becomes

(2.4) $$QP^2 - U(QP^2)U^* = D_1 \quad (D_1 = D \oplus 0 \geqq 0),$$

where QP^2 is self-adjoint.

If Z denotes any Borel set on $|z|=1$ of measure 0, it follows from Theorem 2.3.2 of [9], p. 22, that $E(Z)D_1=0$ and hence, by (2.3), $E(Z)DQ=0$. Hence, for $k=0, 1, 2, \ldots$, $0=N^k E(Z)DQ=E(Z)N^k DQ=E(Z)T^k DQ$, and so $E(Z)x=0$ for any x in the least subspace of $\mathfrak{H}$ which is invariant under T and contains the range of D. Since T is completely subnormal, such a subspace must coincide with $\mathfrak{H}$, a fact observed by CLANCEY [2]. Thus $0=E(Z)Q=QE(Z)$ and hence $R_{E(Z)}\subset\mathfrak{H}^{\perp}=\mathfrak{K}\ominus\mathfrak{H}$. But $R_{E(Z)}$, hence also $\mathfrak{K}_1=\mathfrak{K}\ominus R_{E(Z)}$, reduces N. Since $\mathfrak{H}\subset\mathfrak{K}_1\subset\mathfrak{K}$ and since N is the minimal normal extension of T, it follows that $\mathfrak{K}_1=\mathfrak{K}$. Thus $E(Z)=0$, that is, U is absolutely continuous.

It remains to be shown that (1.4) holds. Suppose the contrary, that is, meas sp $(U)<2\pi$. It follows from (2.4) and Theorem 2.3.1 of [9], p. 21, that $\mathfrak{H}_a(QP^2)$ (note that $QP^2=P^2Q$ is self-adjoint) contains the least space, M, invariant under QP^2 and which also reduces U and contains $R_{D_1}(=R_D)$. Since $\mathfrak{H}$ (hence $\mathfrak{H}^{\perp}$) is invariant under QP^2 and $QP^2|\mathfrak{H}^{\perp}=0$, it follows that $\mathfrak{H}_a(QP^2)\subset\mathfrak{H}$ and hence $M\subset\mathfrak{H}$. Since $QP^2=P^2Q$, it is clear that M is invariant under P^2 and hence also under P. Since M also reduces U it follows that M reduces N. Further, since T is completely subnormal, hence not normal, $R_D\neq 0$ and, in particular, $M\neq 0$. Consequently, M is a non-trivial reducing space of T on which T is normal, so that T is not completely subnormal, a contradiction. Hence, meas sp $(U)=2\pi$, and the proof of Theorem 1 is now complete.

3. Proof of Theorem 2. Let $\mathfrak{H}_1$ denote the least subspace of $\mathfrak{K}$ containing $\mathfrak{H}$ and invariant under both N and N^*N, and let T_1 denote the restriction of N to $\mathfrak{H}_1$. Then T_1 is subnormal on $\mathfrak{H}_1$ with minimal normal extension N on $\mathfrak{K}$. It will be shown that $\mathfrak{H}_1=\mathfrak{K}$ (so that $T_1=N$). To see this, suppose, if possible, that $\mathfrak{H}_1$ is properly contained in $\mathfrak{K}$. Then T_1 is not normal and hence has a representation $T_1=T_{11}\oplus T_{12}$ on $\mathfrak{H}_1=\mathfrak{H}_{11}\oplus\mathfrak{H}_{12}$, where $\mathfrak{H}_{11}\neq 0$, T_{11} is completely subnormal on $\mathfrak{H}_{11}$, and, if $\mathfrak{H}_{12}$ is not empty, T_{12} is normal on $\mathfrak{H}_{12}$. Then $N=N_1\oplus T_{12}$ on $\mathfrak{K}=(\mathfrak{K}\ominus\mathfrak{H}_{12})\oplus\mathfrak{H}_{12}$, where N_1 is the minimal normal extension on $\mathfrak{K}\ominus\mathfrak{H}_{12}$ of T_{11} on $\mathfrak{H}_{11}$. Further, $\mathfrak{H}_{11}$ is invariant under N_1 and $N_1^*N_1$.

Clearly, sp $(N_1)\subset$ sp (N) and hence, by (1.7),

$$\text{sp}\,(N_1)\cap W \text{ is empty.} \tag{3.1}$$

It is seen that the relation corresponding to (1.3) of Theorem 1 now holds with T, N, $\mathfrak{H}$ and $\mathfrak{K}$ replaced by T_{11}, N_1, $\mathfrak{H}_{11}$ and $\mathfrak{K}\ominus\mathfrak{H}_{12}$ respectively. Hence sp $(U_1)=\{z: |z|=1\}$, where U_1 corresponds to N_1 as U does to N, in contradiction with (3.1). Consequently, $\mathfrak{H}_1=\mathfrak{K}$ and Theorem 2 is proved.

4. An example. Let m denote the measure on the set

$$S=\{z:|z|=1\}\cup\{0\}, \tag{4.1}$$

which is normalized Lebesgue measure on $|z|=1$ and is 1 at $z=0$. Let N be the position operator $(Nf)(z)=zf(z)$ on the Hilbert space $\mathfrak{K}=L^2(m)$ and let T denote the restriction of N to the space $\mathfrak{H}=H^2(m)$, the subspace of $L^2(m)$ spanned by $\{z^n\}$, $n=0, 1, 2, \ldots$. (This example is given in HALMOS [6], p. 309; see also STAMPFLI [11], p. 379. For a discussion of position operators see [9], pp. 15 ff.) Then T is subnormal with the minimal normal extension N. An orthonormal basis for $\mathfrak{H}=H^2(m)$ is $\{e_n(z)\}$, where $e_0(z)=1/2^{\frac{1}{2}}$ and $e_n(z)=z^n$ for $n=1, 2, \ldots$. Also $Te_0=(1/2^{\frac{1}{2}})e_1$ and $Te_n=e_{n+1}$ for $n=1, 2, \ldots$, so that T is the unilateral weighted shift with weights $\{1/2^{\frac{1}{2}}, 1, 1, \ldots\}$. Further, sp (T) is the closed unit disk while sp (N) is the set S of (3.1). In particular, 0 is in the point spectrum of N and hence N cannot be radially absolutely continuous.

It follows from Theorem 1 that (1.3) cannot hold. This fact is also easily verified directly (note that N^*N is the multiplication operator $|z|^2$). It is seen that the operator N can be written as the direct sum $N=0\oplus N_1$ on $\mathfrak{K}=\mathfrak{K}_0\oplus\mathfrak{K}_1$, where $\mathfrak{K}_0$ is the eigenspace of N for $z=0$. (A basis for $\mathfrak{K}_0$ is the function which equals 1 at $z=1$ and equals 0 on $|z|=1$.) Further, N_1 is unitary and is absolutely continuous on $\mathfrak{K}_1$. In the context of Theorem 1 this can be explained by noting that N_1 is the minimal (unitary) extension of T_1: $(T_1f)(z)=zf(z)$ on $\mathfrak{H}_1=H^2(m_1)$ where m_1 is normalized Lebesgue measure on $|z|=1$.

5. Another example. Let D be a domain (non-empty connected open subset of the plane) and consider the Hilbert space $\mathfrak{H}=A^2(D)$ of functions analytic on D and square integrable with respect to ordinary Lebesgue planar measure; cf. [9], p. 15. Let T denote the position operator $(Tf)(z)=zf(z)$ for $f\in\mathfrak{H}=A^2(D)$ and let N denote its (minimal) normal extension $(Nf)(z)=zf(z)$ for $f\in\mathfrak{K}=L^2(D)$. Then

$$\text{sp}\,(T) = \text{sp}\,(N) = \text{closure of } D, \tag{5.1}$$

and, in addition, N is radially absolutely continuous. In fact, N is even absolutely continuous in the (stronger) ordinary two-dimensional sense, that is, if N has the spectral resolution (1.1), then

$$K(Z) = 0 \quad \text{whenever } Z \text{ is a Borel set of planar measure } 0. \tag{5.2}$$

It is seen that condition (1.3) is not fulfilled, since if $f(z)$ is analytic on K, the function $|z|^2f(z)$ is not analytic unless $f(z)\equiv 0$. Nevertheless, Theorem 2 can be applied to yield

Theorem 3. *Let D be a domain for which there exists an open wedge of* (1.6) *satisfying*

$$D\cap W \quad \text{is empty.} \tag{5.3}$$

Let $\mathfrak{H}_0(D)$ *denote the Hilbert space obtained by taking the closure of the linear manifold of functions* $\left\{\sum_{k=0}^{N}|z|^{2k}f_k(z)\right\}$, $N=0, 1, \ldots,$ *where the* $f_k(z)$ *are in* $A^2(D)$. *Then* $\mathfrak{H}_0 = L^2(D)$.

In fact, $\mathfrak{H}_0(D)$ is clearly the least subspace of $L^2(D)$ containing $\mathfrak{H}=A^2(D)$ and invariant under $N=z$ and $N^*N=|z|^2$. (Note also that the space $\mathfrak{H}_0(D)$ remains unchanged if, in its definition, $|z|^2$ is replaced by $|z|$.)

If (5.3) is not satisfied, the assertion of Theorem 3 can be false. For instance, if $D=\{z: |z|<1\}$, then $\mathfrak{H}_0(D)$ is a proper subspace of $L^2(D)$. In fact, one can here restrict the $f_k(z)$ to be polynomials in z. It is then easily verified that the space spanned by $\{\bar{z}^{n}\}$, $n=1, 2, \ldots$, is contained in the orthogonal complement $\mathfrak{H}_0^\perp(D)= =L^2(D)\ominus\mathfrak{H}_0(D)$.

6. Remarks. As noted above, a normal operator N of (1.1) is absolutely continuous (in the two-dimensional sense) if (5.2) holds. The question arises as to what conditions might assure this type of absolute continuity of the minimal normal extension of a subnormal operator. One answer can be given as follows. As before, suppose that T is completely subnormal on $\mathfrak{H}$ with the minimal normal extension N on $\mathfrak{K}$, and suppose that (1.3) holds. This guarantees, in particular, that N is radially absolutely continuous. It turns out that if, for instance, in addition to (1.3),

$$NQ = NA-AN \tag{6.1}$$

holds for some bounded operator A on $\mathfrak{K}$, then N is necessarily absolutely continuous.

To see this, let $[A, B]=AB-BA$ for any pair of bounded operators A and B on a Hilbert space (here, $\mathfrak{K}$), so that (2.2) becomes $[QN^*, N]=D_1$. By (6.1), $QN^*= =[A^*, N^*]$ and so

$$[[A^*, N^*], N] = D_1 \geqq 0. \tag{6.2}$$

An argument similar to that of [9], pp. 24—25 (see also [8]) then shows that $K(Z)D_1=0$ where Z is a Borel set of planar measure 0 and D_1 is the non-negative operator of (2.3). An argument like that of section 2 above then implies (5.2).

Similar reasoning shows that, instead of (5.1), one could suppose

$$\text{either } NQ+B = NA-AN \quad \text{or} \quad QN^*+B_1 = NA_1-A_1N, \tag{6.3}$$

where A or A_1 denote arbitrary bounded operators and B or B_1 denote operators commuting with N (hence, by Fuglede's theorem, also with N^*).

That a second order commutator equation such as occurs in (6.2) with D_1 non-negative should arise in connection with two-dimensional absolute continuity of a normal operator is not unnatural. The situation is analogous to that of an ordinary commutator and one-dimensional absolute continuity of a self-adjoint or unitary operator; cf. [9], Chapter II, also KATO [7], p. 543. Concerning the solution of commutator equations $[A, X]=C$, where A is self-adjoint, see also ROSENBLUM [10], and, where A is normal or an infinitesimal generator of a certain strongly continuous semigroup, see FREEMAN [3], [4].

References

[1] J. BRAM, Subnormal operators, *Duke Math. J.*, **22** (1955), 75—94.

[2] K. CLANCEY, On the subnormality of some singular integral operators (*preprint*, 1971).

[3] J. M. FREEMAN, The perturbation theory of some Volterra operators, *Thesis, M.I.T.*, June, 1963.

[4] J. H. FREEMAN, The tensor product of semigroups and the operator equation $SX-XT=A$, *J. Math. and Mech.*, **19** (1970), 819—828.

[5] P. R. HALMOS, Spectra and spectral manifolds, *Ann. Soc. Pol. Math.*, **25** (1952), 43—49.

[6] P. R. HALMOS, *A Hilbert space problem book*, Van Nostrand (1967).

[7] T. KATO, Smooth operators and commutators, *Studia Math.*, **31** (1968), 535—546.

[8] C. R. PUTNAM, Commutators and absolutely continuous operators, *Trans. Amer. Math. Soc.*, **87** (1958), 513—525.

[9] C. R. PUTNAM, *Commutation properties of Hilbert space operators and related topics*, Ergebnisse der Math., **36**, Springer (1967).

[10] M. ROSENBLUM, The operator equation $BX-XA=Q$ with self-adjoint A and B, *Proc. Amer. Math. Soc.*, **20** (1969), 115—120.

[11] J. G. STAMPFLI, Which weighted shifts are subnormal, *Pacific J. Math.*, **17** (1966), 367—379.

PURDUE UNIVERSITY

(Received August 16, 1973)

Automorphism groups of von Neumann algebras and ergodic type theorems

By KAZUYUKI SAITÔ in Sendai (Japan) *)

A great deal of work has been done for groups of $*$-automorphisms of operator algebras. Especially, relations between groups of von Neumann algebras and their invariant normal states have been studied by many authors. Above all, KOVÁCS and SZŰCS [5] showed that for a von Neumann algebra M and a group G of automorphisms of M, to have a separating family of G-invariant normal states (that is, M is G-finite), it is necessary and sufficient that there is a unique faithful normal G-invariant projection of norm 1 from M onto the fixed algebra M^G under G. In [12], E. STØRMER introduced a new equivalence (G-equivalence) on the projections in a von Neumann algebra M on a Hilbert space $\mathfrak{H}$ with a unitarily implemented group G of $*$-automorphisms which coincides with the one defined by HOPF [4] in the σ-finite abelian case and, in the general case, it includes the one due to MURRAY and VON NEUMANN [6].

Let $g \to U_g$ be a unitary representation of G on $\mathfrak{H}$ such that $U_g M U_g^* = M$ for all $g \in G$. Størmer calls two projections e and f in M G-equivalent ($e \overset{G}{\sim} f$) if for each $g \in G$ there exists an element $a_g \in M$ such that

$$\sum_{g \in G} a_g^* a_g = e \quad \text{and} \quad \sum_{g \in G} U_g^* a_g a_g^* U_g = f.$$

By using the cross product of M and G, he showed that this relation is in fact an equivalence relation, and if M is σ-finite, then M is $G\sim$finite (that is, $1 \overset{G}{\sim} e$, then $e=1$) if and only if M has a G-invariant faithful normal trace and that M is $G\sim$semi-finite (M has sufficiently many $G\sim$finite projections) if and only if M has a faithful normal G-invariant semi-finite trace.

Recently, using Ryll-Nardzewski's fixed point theorem [3, 7], F. J. YEADON gave an elegant proof of the existence of a trace in a finite von Neumann algebra [15]. It is suggestive that this method of functional analysis may be very useful

*) The author is partially supported by The Sakkokai Foundation.

for further studies of automorphism groups and their invariant maps of von Neumann algebras also.

The present author [8] gave a characterization of the finiteness of von Neumann algebras using weakly relatively compact subsets of their preduals and STØRMER [13] showed, roughly speaking, that a von Neumann algebra with a group G of $*$-automorphisms is G-finite if and only if for each φ in M_* (the predual of M), the orbit of φ under G is weakly relatively compact in M_*.

In this paper, we shall give some kind of a Banach space like characterization of the $G\sim$finiteness of a von Neumann algebra M with a group G of $*$-automorphisms of M, using weakly relatively compact subsets of the predual M_* which is a generalization of a theorem of HAJIAN and KAKUTANI [18, 19], more precisely to say, we shall prove

Theorem 1. *Let M be a von Neumann algebra with a group G of $*$-automorphisms of M and let $\tilde{G}$ be the group generated algebraically by G and the group of inner automorphisms of M. Then M is $G\sim$finite (see the definition below) if and only if for every weakly relatively compact subset K of the predual M_*, the set $\{\varphi\circ g \mid \varphi\in K, g\in\tilde{G}\}$ is also weakly relatively compact.*

The plan of this paper is as follows. Section 1 is concerned with the comparability theorem for projections relative to the $\overset{G}{\sim}$ equivalence (the abelian case was proved by STØRMER [11, Lemma 2.7]). Section 2 is devoted to the proof of Theorem 1 and contains related corollaries.

1. $\overset{G}{\sim}$ equivalence and the comparability theorem. Let M be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$ and G be a group of $*$-automorphisms of M defined by $a\to a^g$, $a\in M$, $g\in G$. Since we can consider G as a discrete group, there is a faithful covariant representation (U,π) of (M,G) on a Hilbert space $\mathfrak{H}$ such that $\pi(a^g)=$ $=U_g^*\pi(a)U_g$ for each $g\in G$ and $a\in M$ [20, Definition 3.1]. Since our discussions on weakly relatively compact subsets of the predual of M and others are independent of representations of M, we may always assume without loss of generality that M acts on $\mathfrak{H}$ and there is a unitary representation of G on $\mathfrak{H}$ such that $a^g=U_g^*aU_g$ for each $g\in G$ and $a\in M$.

Definition ([12]). For any pair of projections e and f in M, we write $e\overset{G}{\sim}f$ if for each $g\in G$ there exists an $a_g\in M$ such that

$$e=\sum_{g\in G}a_g a_g^*,\quad f=\sum_{g\in G}U_g^*a_g^*a_gU_g.$$

We write $e\overset{G}{\precsim}f$ or $f\overset{G}{\succsim}e$, if $e\overset{G}{\sim}e'\leqq f$ for some projection e' in M.

For each $s\in G$, let $\mathfrak{H}_s$ be a Hilbert space with the same dimension as $\mathfrak{H}$ and J_s be a linear isometry of $\mathfrak{H}$ onto $\mathfrak{H}_s$. Let $\tilde{\mathfrak{H}}$ be the direct sum of $\mathfrak{H}_s$ for $s\in G$. Any bounded linear operator R on $\tilde{\mathfrak{H}}$ is represented by a matrix $(R_{s,t})_{s\in G, t\in G}$, where $R_{s,t}=J_s^* R J_t$ (bounded linear operator on $\mathfrak{H}$). For any $a\in M$, let $\pi(a)$ be the operator on $\tilde{\mathfrak{H}}$ such that $R_{s,t}=\delta_{s,t}a$ for all s and t in G where $\delta_{s,t}$ is Kronecker's symbol. π is a $*$-isomorphism of M onto the von Neumann algebra $\{\pi(a)|a\in M\}=\tilde{M}$. For any $g\in G$, let $\tilde{U}_g$ be the unitary operator on $\tilde{\mathfrak{H}}$ defined by the matrix $(R_{s,t})$ with $R_{s,t}=0$ if $st\neq g$, $R_{gt,t}=U_g$ for all $t\in G$. Then $g\to\tilde{U}_g$ is a unitary representation of G on $\tilde{\mathfrak{H}}$ such that $\pi(U_g^* a U_g)=\tilde{U}_g^*\pi(a)\tilde{U}_g$ for all $g\in G$ and $a\in M$. Now let $M\times G$ be the von Neumann algebra generated by $\tilde{M}$ and $\{\tilde{U}_g;\ g\in G\}$. STØRMER proved the following:

Lemma 1. ([12]) *For any pair of projections in M, $e\overset{G}{\sim}f$ if and only if $\pi(e)\sim\pi(f)$ in $M\times G$ ($\sim$ is the Murray—von Neumann equivalence relation). Thus the relation $\overset{G}{\sim}$ is an equivalence relation.*

By the above lemma, $\overset{G}{\sim}$ is additive, that is, for any pair of orthogonal families of projections $\{e_\alpha\}$ and $\{f_\alpha\}$ in M with $e_\alpha\overset{G}{\sim}f_\alpha$ for each α, we have $\Sigma_\alpha e_\alpha\overset{G}{\sim}\Sigma_\alpha f_\alpha$.

Next, we shall show the comparability theorem which plays a central role in our theory and whose proof is a modification of the one given by STØRMER in the abelian case.

Proposition 1. *Let M^G be the fixed subalgebra of M under G, then for any pair e and f of projections in M, there is a projection z in $M^G\cap Z$ (Z is the center of M) such that $ez\overset{G}{\succsim}fz$ and $e(1-z)\overset{G}{\precsim}f(1-z)$.*

In order to prove this, we need some lemmas.

Lemma 2. *For any projection e in M, let $z^G(e)$ be the smallest projection in $M^G\cap Z$ majorizing e. Then $z^G(e)$ is the maximal projection f such that $f=\Sigma_\alpha f_\alpha$, $f_\alpha f_{\alpha'}=0$ $(\alpha\neq\alpha')$, and for each α, there are a projection e_α in M with $e_\alpha\leqq e$ and $g_\alpha\in\tilde{G}$ such that $U_{g_\alpha}^* e_\alpha U_{g_\alpha}=f_\alpha$.*

Proof. Let $\{f_\alpha\}$ be a maximal orthogonal family of projections in M such that for each α, there exists a projection e_α in M with $e_\alpha\leqq e$ and $g_\alpha\in\tilde{G}$ such that $U_{g_\alpha}^* e_\alpha U_{g_\alpha}=f_\alpha$. Putting $f=\Sigma f_\alpha$, we shall show $z^G(e)=f$. First we claim that $U_g f U_g^*=f$ for all $g\in\tilde{G}$. In fact, if not, there is a $g\in\tilde{G}$ such that $U_g f U_g^*(1-f)\neq 0$. Since $1-f\geqq f\vee U_g f U_g^* - f\sim U_g f U_g^* - U_g f U_g^*\wedge f\neq 0$ in usual equivalence in M, by [16, Lemma 1.7], there exist $g_0\in\tilde{G}$ and a non-zero projection $\tilde{f}$ in M such that $\tilde{f}\leqq 1-f$ and $U_{g_0}\tilde{f}U_{g_0}^*\leqq f$. $f=\Sigma f_\alpha$ implies that there is an α such that $U_{g_0}\tilde{f}U_{g_0}^* f_\alpha\neq 0$. Thus $U_{g_0}\tilde{f}U_{g_0}^*\geqq U_{g_0}\tilde{f}U_{g_0}^* - U_{g_0}\tilde{f}U_{g_0}^*\wedge(1-f_\alpha)\sim(1-f_\alpha)\vee U_{g_0}\tilde{f}U_{g_0}^*-(1-f_\alpha)\leqq f_\alpha$ in M and $(1-f_\alpha)\vee U_{g_0}\tilde{f}U_{g_0}^*-(1-f_\alpha)\neq 0$. By the same reason as above, there are a projection $\tilde{\tilde{f}}$ with $\tilde{\tilde{f}}\leqq\tilde{f}$ in M and $g'\in\tilde{G}$ such that $1-f\geqq\tilde{f}\geqq\tilde{\tilde{f}}\neq 0$ and $U_{g'}\tilde{\tilde{f}}U_{g'}^*\leqq f_\alpha$. By the definition of f_α, there is a $g_\alpha \in \tilde{G}$ such that $U_{g_\alpha}^* e_\alpha U_{g_\alpha} = f_\alpha$ for some projection e_α in M with $e_\alpha \leqq e$. Hence if we put $g_\alpha g' = \tilde{g} \in \tilde{G}$, then $U_{\tilde{g}} \tilde{\tilde{f}} U_{\tilde{g}}^* \leqq U_{g_\alpha} f_\alpha U_{g_\alpha}^* \leqq e_\alpha \leqq e$. Since $\tilde{\tilde{f}} \leqq 1-f$ and $\tilde{\tilde{f}} \neq 0$, $\{\tilde{\tilde{f}}, f_\alpha\} \supsetneqq \{f_\alpha\}$, and this contradicts the maximality of $\{f_\alpha\}$. Thus $f = U_g f U_g^*$ for all $g \in \tilde{G}$ and $f \in M^G \cap Z$. Moreover, we have $f \geqq e$. In fact, if otherwise, $1-f \geqq e \vee f - f \sim e - e \wedge f \neq 0$ (in M). Therefore, by the same argument as above, there are non-zero projections $e_1 (\leqq e - e \wedge f)$ and $f_1 (\leqq 1-f)$ in M and g in $\in \tilde{G}$ such that $e_1 = U_g f_1 U_g^*$, $\{f_1, f_\alpha\} \supsetneqq \{f_\alpha\}$, and this is a contradiction. Thus $e \leqq f$ and $z^G(e) \leqq f$. The equality $z^G(e) = f$ is shown by the following:

$$z^G(e)f = \Sigma_\alpha z^G(e) f_\alpha = \Sigma_\alpha z^G(e) U_{g_\alpha}^* e_\alpha U_{g_\alpha} = \Sigma_\alpha U_{g_\alpha}^* z^G(e) e_\alpha U_{g_\alpha} = \\ = \Sigma_\alpha U_{g_\alpha}^* e_\alpha U_{g_\alpha} = \Sigma_\alpha f_\alpha = f.$$

This completes the proof.

Lemma 3. *Let e and f be projections in M. If $z^G(e) z^G(f) \neq 0$, then there exist projections e_1 and f_1 in M such that $e_1 \leqq e$, $f_1 \leqq f$ and $e_1 \overset{G}{\sim} f_1$.*

Proof. By Lemma 2, there are orthogonal families $\{f_\alpha\}$ and $\{g_\beta\}$ of projections in M such that for each pair of α and β, $f_\alpha \overset{G}{\precsim} e$, $g_\beta \overset{G}{\precsim} f$, $z^G(e) = \Sigma_\alpha f_\alpha$, $z^G(f) = \Sigma_\beta g_\beta$. The assumption $z^G(e) z^G(f) \neq 0$ implies $f_\alpha g_\beta \neq 0$ for some α and β. Therefore $g_\beta \geqq (1-g_\beta) \vee f_\alpha - (1-g_\beta) \sim f_\alpha - f_\alpha \wedge (1-g_\beta) \neq 0$ in the usual equivalence in M. Since $\sim$ equivalence implies $\overset{G}{\sim}$ equivalence, we have $e \overset{G}{\succsim} f_\alpha - f_\alpha \wedge (1-g_\beta) \overset{G}{\sim} (1-g_\beta) \vee f_\alpha - (1-g_\beta) \leqq g_\beta \overset{G}{\precsim} f$, which implies that there are non-zero projections $e_1 (\leqq e)$ and $f_1 (\leqq f)$ such that $e_1 \overset{G}{\sim} f_1$. This completes the proof.

Proof of Proposition 1. Certain standard arguments show the desired property [17, Theorem 3.1.1], but for the sake of completeness we sketch them. Let $\{e_\alpha, f_\alpha\}$ be a maximal family of projections in M such that $e_\alpha e_{\alpha'} = 0$, $f_\alpha f_{\alpha'} = 0$ $(\alpha \neq \alpha')$, $e_\alpha \leqq e$, $f_\alpha \leqq f$ and $e_\alpha \overset{G}{\sim} f_\alpha$ for each α. Define $e_0 = \Sigma_\alpha e_\alpha$ and $f_0 = \Sigma_\alpha f_\alpha$, then by Lemma 1, $e_0 \overset{G}{\sim} f_0$. Putting $e_1 = e - e_0$ and $f_1 = f - f_0$, then by the maximality of $\{e_\alpha, f_\alpha\}$ we have $z^G(e_1) z^G(f_1) = 0$ by Lemma 3. Now put $g = z^G(e_1)$. Then $g \in M^G \cap Z$ and $f_1 \leqq z^G(f_1) \leqq 1 - z^G(e_1) \leqq 1 - g$, which implies that $fg = f_0 g + f_1 g = f_0 g \overset{G}{\sim} e_0 g \leqq eg$ and $e(1-g) = e_0(1-g) + e_1(1-g) = e_0(1-g) \overset{G}{\sim} f_0(1-g) \leqq f(1-g)$. This completes the proof.

Using proposition 1, we have the following lemma the proof of which is an obvious modification of [10, 2.4.2].

Lemma 4. *If M is $G \sim$finite (that is, if $e \overset{G}{\sim} 1$ (e a projection), then $e = 1$), then for any pair of projections f and h in M with $f \overset{G}{\sim} h$ we have $1 - f \overset{G}{\sim} 1 - h$.*

Proof. For $1-f$ and $1-h$, there is a projection g in $M^G \cap Z$ such that $(1-f)g \overset{G}{\precsim} (1-h)g$ and $(1-f)(1-g) \overset{G}{\succsim} (1-h)(1-g)$. Thus there is a projection f_1 in M such that $(1-f)g \overset{G}{\sim} f_1 \leqq (1-h)g$. This implies that $g = (1-f)g + fg \overset{G}{\sim} f_1 + hg \leqq g$

and by the $G\sim$finiteness of g, we have $g=f_1+hg$, that is, $f_1=g(1-h)$ and $(1-f)g\overset{G}{\sim}(1-h)g$. By the same argument, we have $(1-f)(1-g)\overset{G}{\sim}(1-h)(1-g)$ and hence $1-f\overset{G}{\sim}1-h$. This completes the proof.

2. Proof of Theorem 1 and related corollaries. We begin with some ergodic type theorems needed in the proof of Theorem 1.

Theorem 2. *Let M be a von Neumann algebra M with a group of $*$-automorphisms and let $\tilde{G}$ be the group generated algebraically by G and the group of inner automorphisms. Then M is $G\sim$finite if and only if M is $\tilde{G}$-finite in the sense that there is a separating family of $\tilde{G}$-invariant normal functionals (G-invariant normal traces) on M, and consequently M is $G\sim$finite if and only if $M\times\tilde{G}$ is finite in the usual sense.*

The σ-finite case of the above theorem was proved by STØRMER [12, Lemma 11] by using "cross product" techniques. Using Lemma 4, the proof of Theorem 2 is a straightforward modification of the one given by YEADON [12] for finite W^*-algebras. However, due to our more complicated G-equivalence, we sketch it for the sake of completeness. First we present two lemmas.

Lemma 5. *Let $\{e_k\}$ be an increasing sequence of projections in M such that for some projection f in M, $e_k\overset{G}{\precsim}f$ for each k. Then, for $e=\sup\{e_k, k\geqq 1\}$, we have $e\overset{G}{\precsim}f$.*

Proof. Since $e_1\overset{G}{\precsim}f$, there is a projection $f_1(\leqq f)$ in M such that $e_1\overset{G}{\sim}f_1$. Now, by Lemma 4, $e_2\overset{G}{\precsim}f$ implies $1-e_2\overset{G}{\succsim}1-f$. By additivity of G-equivalence we get $1-e_2+e_1\overset{G}{\succsim}1-f+f_1$, thus again by Lemma 4, we have $e_2-e_1\overset{G}{\precsim}f-f_1$ which implies that there is a projection f_2 in M which is orthogonal to f_1 and is equivalent to e_2-e_1. Thus by mathematical induction we can choose an orthogonal sequence $\{f_i\}$ of projections in M majorized by f such that

$$\sum_{i=1}^{\infty} f_i \leqq f \quad \text{and} \quad e_{i+1}-e_i\overset{G}{\sim}f_{i+1}$$

for each i. It follows that

$$e = e_1+\sum_{i=1}^{\infty}(e_{i+1}-e_i)\overset{G}{\sim}\sum_{i=1}^{\infty}f_i\leqq f$$

and $e\overset{G}{\precsim}f$. This completes the proof.

Let M_* be the predual of M, that is the set of all ultra-weakly continuous functionals on M and let $(T_g\varphi)(a)=\varphi(U_g^*aU_g)$ $\varphi\in M_*$, $a\in M$ and $g\in\tilde{G}$. Then T_g is a linear isometry of M_* onto M_*.

Lemma 6. *For any element φ in M_*, the set $K=\{T_g\varphi\,|\,g\in\tilde{G}\}$ is a weakly relatively compact subset of M_*.*

Proof. If not, by [1, theorem II.2], there are an orthogonal sequence $\{e_n\}$ of projections in M, a positive real number ε and a sequence $\{g_n\}$ in $\tilde{G}$ such that

$$(*) \qquad |\varphi(U_{g_n}^* e_n U_{g_n})| \geqq \varepsilon \quad \text{for} \quad n = 1, 2, 3, \dots .$$

Now put $f_n = U_{g_n}^* e_n U_{g_n}$. Then f_n is a projection in M such that $f_n \overset{G}{\sim} e_n$. Next we shall show that $f_n \to 0$ $(n \to \infty)$ strongly. Let $p_n = \sum_{m=n}^{\infty} e_m$ and $q_n = \bigvee_{m=n}^{\infty} f_m$, then $\{p_n\}$ and $\{q_n\}$ are monotone decreasing sequences of projections in M. For any n, put $r_k = \bigvee_{i=n}^{n+k} f_i$ for each k. Then $r_{k-1} \wedge f_{n+k} - r_{k-1} \overset{G}{\sim} f_{n+k} - r_{k-1} \wedge f_{n+k} \leqq f_{n+k} \overset{G}{\sim} e_{n+k}$ implies that $r_k = r_{k-1} \vee f_{n+k} \overset{G}{\precsim} \sum_{i=n}^{n+k} e_i \leqq p_n$ for all k, thus, by Lemma 5, $q_n = \bigvee_{k=1}^{\infty} r_k \overset{G}{\precsim} p_n$. Since M is $G\sim$finite, $1 - p_n \overset{G}{\precsim} 1 - q_n \leqq 1 - \bigwedge_{n=1}^{\infty} q_n$ for each n. Hence, again by Lemma 5, $1 = \bigvee_{n=1}^{\infty} (1 - p_n) \overset{G}{\precsim} \left(1 - \bigwedge_{n=1}^{\infty} q_n\right)$, that is $\bigwedge_{n=1}^{\infty} q_n = 0$. Thus $q_n \to 0$ $(n \to \infty)$ strongly and $q_n \geqq f_n$ for each n implies that $f_n \to 0$ (strongly) as $n \to \infty$. Hence this contradicts the inequality $(*)$, and thus K is weakly relatively compact. This completes the proof.

Proof of Theorem 2. By Krein—Šmulian's theorem [2, Theorem V.6.4], $Q(\varphi)$, the strongly closed convex hull of the weakly relatively compact set K, is weakly compact. Since the group $\{T_g : g \in \tilde{G}\}$ acting on $Q(\varphi)$ is noncontracting, that is, 0 does not belong to the closure of $\{T_g \psi_1 - T_g \psi_2 | g \in \tilde{G}\}$ whenever $\psi_1 \neq \psi_2$ and $\psi_1, \psi_2 \in Q(\varphi)$, by the Ryll-Nardzewski fixed point theorem [3, 7], there is an element $\psi \in Q(\varphi)$ such that $T_g \psi = \psi$ for all $g \in \tilde{G}$. If $\varphi \geqq 0$, then ψ can be chosen to be non-negative. For any $t \in M^{\tilde{G}} (= M^G \cap Z)$ with $t \geqq 0$, there is φ in M such that $\varphi(t) \neq 0$. For this φ, we can choose, by the above arguments, a $\psi \in Q(\varphi)$ invariant under $\tilde{G}$ such that $\psi(t) = \varphi(t) \neq 0$. Thus by [5, Proposition 1] M is $\tilde{G}$-finite, that is, M has sufficiently many $\tilde{G}$-invariant normal states. The converse assertion is clear from the definitions, so the proof is complete.

Before going into the proof of Theorem 1, we state here an ergodic type theorem.

Theorem 3. ([5]) *A von Neumann algebra M with a group G of its $*$-automorphisms is G-finite (that is, there is a separating set of G-invariant normal states on M) if and only if there is a unique faithful normal G-invariant projection of norm one from M onto the fixed algebra M^G under G.*

Proof. Let M_* be the predual of M. For every $g \in G$ and $\varphi \in M_*$, if we put $(T_g \varphi)(a) = \varphi(a^g)$ for all $a \in M$, then T_g is an order-preserving, linear isometry of M_* onto M_*. Let $Q(\varphi) = \overline{co}(T_g \varphi | g \in G)$ be the closed convex hull of $(T_g \varphi | g \in G)$ in M_*. First we shall show that $Q(\varphi)$ is weakly compact. By the Krein—Šmulian theorem

([2, Theorem V.6.4]), it suffices to prove that the bounded set $K=\{T_g\varphi \mid g\in G\}$ is weakly relatively compact. We claim that for any orthogonal sequence $\{p_n\}$ of non-zero projections in M, $\lim_{n\to\infty} (T_g\varphi)(p_n)=0$ uniformly for $g\in G$. In fact, if not, there is a sequence of mutually orthogonal non-zero projections $\{e_n\}$ in M, a positive number ε and a sequence $\{g_n\}$ in G such that $|\varphi(e_n^{g_n})|\geqq\varepsilon$ for all n. First of all, we shall show that $\{e_n^{g_n}\}$ is an infinite set. If not, there is a sequence of positive integers $\{n_k\}$ such that $e_{n_k}^{g_{n_k}}=e_{n_0}^{g_{n_0}}$ $(k=1, 2, 3, \ldots)$ for some n_0. For any G-invariant normal state ψ,

$$\psi(e_{n_k}) = \psi(e_{n_k}^{g_{n_k}}) = \psi(e_{n_0}^{g_{n_0}}) = \psi(e_{n_0}).$$

By the orthogonality of $\{e_{n_k}\}$, $\psi(e_{n_k})\to 0$ $(k\to\infty)$, which implies that $\psi(e_{n_0})=0$ for every G-invariant normal state ψ. Thus G-finiteness of M implies $e_{n_0}=0$, this is a contradiction and $\{e_n^{g_n}\}$ is a relatively σ-weakly compact infinite subset of M. Let a be a σ-weak cluster point of $\{e_n^{g_n}\}$, then for each positive integer i and any G-invariant normal state ϱ, there is an increasing sequence $\{n_i\}$ of positive integers such that $e_{n_i}^{g_{n_i}}\neq e_{n_j}^{g_{n_j}}$ $(i\neq j)$, and

$$|\varrho(a)-\varrho(e_{n_i}^{g_{n_i}})| < \frac{1}{i}$$

for each i. Since $\varrho(e_{g_i}^{g_{n_i}})=\varrho(e_{n_i})\to 0$ $(i\to\infty)$, we have $\varrho(a)=0$ for each G-invariant normal state ϱ, which implies, by the G-finiteness of M, that $a=0$. Since 0 is the only cluster point of $\{e_n^{g_n}\}$, we can take an element $e_{n'}^{g_{n'}}$ of $\{e_n^{g_n}\}$ such that $|\varphi(e_{n'}^{g_{n'}})|<\varepsilon$ and this is a contradiction, so $Q(\varphi)$ is weakly compact in M_*. The argument followed in the proof of Theorem 2 shows that there exists an element $\tilde{\varphi}$ in $Q(\varphi)$ such that $T_g\tilde{\varphi}=\tilde{\varphi}$ for all $g\in G$. For any $\psi\in M_*^G$ (predual of M^G) let $\varphi\in M_*$ be chosen so that $\varphi(c)=\psi(c)$ for $c\in M^G$ and $\|\varphi\|=\|\psi\|$. For this φ, as above, we can choose $\tilde{\psi}$ in $Q(\varphi)$ such that $T_g\tilde{\psi}=\tilde{\psi}$ for all $g\in G$. Next we shall show that the $\tilde{\psi}$ is uniquely determined by ψ. In fact, let ϱ be in M_* such that $\varrho(x)=\psi(x)$ for all $x\in M^G$ and $\varrho(x^g)=\varrho(x)$ $(g\in G, x\in M)$. Let $\sigma=\varrho-\tilde{\psi}$, and let $\sigma=v|\sigma|$ be the polar decomposition of σ ([10, 1.14.4]). Then $(T_g\sigma)(x)=\sigma(x^g)=|\sigma|(x^g v)=|\sigma|((xv^{g^{-1}})^g)=(T_g|\sigma|)(xv^{g^{-1}})=$ $=v^{g^{-1}}T_g|\sigma|(x)$ for all $x\in M$. On the other hand, since $\|T_g|\sigma|\|=\||\sigma|\|=\|\sigma\|=\|T_g\sigma\|$ and $T_g|\sigma|\geqq 0$, we have $|T_g\sigma(x)|^2\leqq(T_g|\sigma|)(xx^*)\|T_g|\sigma|\|$. By the unicity of polar decomposition, $T_g\sigma=\sigma$ $(g\in G)$ implies, that $T_g|\sigma|=|\sigma|$, $v^{g^{-1}}=v$ for all $g\in G$ and $\|\sigma\|=\||\sigma|\|=|\sigma|(1)=\sigma(v^*)=\varrho(v^*)-\psi(v^*)=0$ $(v\in M^G)$. Thus $\varrho=\psi$. If we apply the above argument with ψ, then we have $\|\psi\|=\|\tilde{\psi}\|$ and if $\psi\geqq 0$, then $\|\psi\|=\psi(1)=\tilde{\psi}(1)=$ $=\|\tilde{\psi}\|$ which implies $\tilde{\psi}\geqq 0$. Put $\Phi_*\psi=\psi$. Then Φ_* is a positive, linear isometric mapping of M_*^G to M_* such that

(a) $$T_g(\Phi_*\psi) = \Phi_*\psi \quad \text{for all} \quad g\in G,$$

(b) $$(\Phi_*\psi)(x) = \psi(x) \quad \text{for all} \quad x\in M^G.$$

Now denote the transposed mapping of Φ_* by Φ_G, then we can easily show that Φ_G is a faithful G-invariant normal projection of norm one from M onto M^G. It is easy to show the converse assertion. This completes the proof.

Remark. The last half part of the proof of the above theorem is a slight modification of [15, Theorem].

Now we are in the position to prove Theorem 1. The following lemma is essential for our discussions.

Lemma 8. *Keeping the notations and assumptions in Theorem 1 in mind, let $\{\varphi_i\}$ be a sequence in M_* such that φ_i converges weakly to φ_0 in M_*. If M is $G\sim$finite, then for arbitrary sequence $\{a_n\}$ in the unit sphere S of M such that $a_n \to 0$ $(n\to\infty)$ strongly, we have $\lim_{n\to\infty} \varphi_i(a_n^g)=0$ uniformly for $i=1, 2, 3, \ldots$ and for $g\in\tilde{G}$.*

Proof. Put $\varphi=\sum_{i=1}^{\infty}\frac{|\varphi_i|}{2^i\|\varphi_i\|}$ (where $|\varphi_i|$ is the absolute value of φ_i [10, 1.14.4]). Then $\varphi\in M_*$. Let $z^G(e_\varphi)$ be the least projection in $M^G\cap Z$ majorizing e_φ (the support projection of φ), then $Mz^G(e_\varphi)$ is σ-finite. In fact, since $z^G(e_\varphi)\in M^G\cap Z$, we may suppose $\tilde{G}$ is a group of $*$-automorphisms of $Mz^G(e_\varphi)$ and the fixed subalgebra of $Mz^G(e_\varphi)$ under $\tilde{G}$ is $(M^G\cap Z)z^G(e_\varphi)$. By Theorem 3, the $\tilde{G}$-finiteness of $Mz^G(e_\varphi)$ implies that there is a faithful normal $\tilde{G}$-invariant projection of norm one from $Mz^G(e_\varphi)$ onto $(M^G\cap Z)z^G(e_\varphi)$. Thus to prove the above assertion, we only have to show that $(M^G\cap Z)z^G(e_\varphi)$ is σ-finite. To do this, let e' be the support projection of the restriction of φ on $M^G\cap Z$, then $e'\geqq z^G(e_\varphi)$, which implies the σ-finiteness of $(M^G\cap Z)z^G(e_\varphi)$. An easy calculation shows that $\varphi_i(a)=\varphi_i(az^G(e_\varphi))$ for all i and $a\in M$, and hence to prove the lemma, we may assume that M is σ-finite. Thus by the $\tilde{G}$-finiteness of M (Theorem 2), there is a faithful $\tilde{G}$-invariant normal state (G-invariant normal trace) τ on M. We define a metric $d(x,y)$ on S as $d(x,y)==\tau((x-y)^*(x-y))^{1/2}$. Then (S, d) is a complete metric space which is equivalent to (S, strong topology). For each positive integer i, let

$$H_i=\{a\,|\,a\in S, |\varphi_j(a)-\varphi_0(a)|\leqq\varepsilon \quad \text{for all} \quad j\geqq i\}.$$

Then H_i is a closed subset of S for each i and $S=\bigcup_{i=1}^{\infty} H_i$. Baire's category theorem implies that there exists $a_0\in S$, a positive number β and a positive integer j_0 such that $\{a\,|\,\|a\|\leqq 1,\ \tau((a-a_0)^*(a-a_0))^{1/2}\leqq\beta\}\subset H_{j_0}$. Since $a_n\to 0$ $(n\to\infty)$ strongly and M is finite, the self-adjoint and skew-adjoint parts of $\{a_n\}$ both converge strongly to 0, so that we can suppose that each a_n is self-adjoint. Thus by spectral theory, there is a sequence $\{e_n\}$ of projections in M such that $e_n\to 1$ $(n\to\infty)$ strongly and $\|a_ne_n\|\leqq\varepsilon/6$ for each n. Thus, by the fact that $\|(a_ne_n)^g\|=\|a_ne_n\|$ for each $g\in\tilde{G}$, we get that

$$|(\varphi_j-\varphi_0)(a_n^g)| \leqq |(\varphi_j-\varphi_0)(e_n^g a_n^g e_n^g)|+|(\varphi_j-\varphi_0)(e_n^g a_n^g(1-e_n^g))|+$$
$$+|(\varphi_j-\varphi_0)((1-e_n^g)a_n^g e_n^g)|+|(\varphi_j-\varphi_0)((1-e_n^g)a_n^g(1-e_n^g))| \leqq$$
$$\leqq \varepsilon(\sup_j \|\varphi_j\|)+|(\varphi_j-\varphi_0)((1-e_n^g)a_n^g(1-e_n^g)).$$

Now put $b_n(g)=e_n^g a_0 e_n^g+(1-e_n^g)a_n^g(1-e_n^g)(\in S)$. Then by the $\tilde{G}$-invariant of τ, it follows that

$$\tau((b_n(g)-a_0)^*(b_n(g)-a_0))^{1/2} \leqq \tau((1-e_n^g)a_n^g(1-e_n^g)a_n^g(1-e_n^g))^{1/2}+$$
$$+\tau((1-e_n^g)a_0^*(1-e_n^g)a_0(1-e_n^g))^{1/2}+\tau((1-e_n^g)a_0^* e_n^g a_0(1-e_n^g))^{1/2}+$$
$$+\tau(e_n^g a_0^*(1-e_n^g)a_0 e_n^g)^{1/2} \leqq 3\tau(1-e_n^g)^{1/2}+\tau(e_n^g a_0^*(1-e_n^g)a_0 e_n^g)^{1/2}=$$
$$=3\tau(1-e_n^g)^{1/2}+\tau((1-e_n^g)a_0 e_n^g a_0^*(1-e_n^g))^{1/2} \leqq 4\tau(1-e_n^g)^{1/2}=4\tau(1-e_n)^{1/2}.$$

Since $e_n \to 1$ $(n\to\infty)$ strongly, there is a natural number $n_0(\beta)$ (independent of $g\in\tilde{G}$) such that $\tau((b_n(g)-a_0)^*(b_n(g)-a_0))^{1/2} \leqq 4\tau(1-e_n)^{1/2}<\beta$ for all $n\geqq n(\beta)$. Thus we have that

$$|(\varphi_j-\varphi_0)(e_n^g a e_n^g)+(\varphi_j-\varphi_0)((1-e_n^g)a_n^g(1-e_n^g))| \leqq \varepsilon$$

for all $j\geqq j_0$, for all $g\in\tilde{G}$ and for all $n\geqq n_0(\beta)$. By the same argument as above it follows that

$$\tau((e_n^g a_0 e_n^g-a_0)^*(e_n^g a_0 e_n^g-a_0))^{1/2} \leqq 3\tau(1-e_n)^{1/2},$$

so that there exists a natural number $n_1(\beta)$ (independent of $g\in\tilde{G}$) such that

$$\tau((e_n^g a_0 e_n^g-a_0)^*(e_n^g a_0 e_n^g-a_0))^{1/2} \leqq 3\tau(1-e_n)^{1/2}<\beta$$

for all $n\geqq n_1(\beta)$. Hence, observing that $e_n^g a_0 e_n^g\in S$, we get that

$$|(\varphi_j-\varphi_0)(e_n^g a_0 e_n^g)| \leqq \varepsilon$$

for all $j\geqq j_0$, all $g\in\tilde{G}$ and for all $n\geqq n_1(\beta)$. Thus combining the above estimates, we get

$$|(\varphi_j-\varphi_0)((1-e_n^g)a_n^g(1-e_n^g))| \leqq 2\varepsilon$$

for all $j\geqq j_0$, $g\in\tilde{G}$ and $n\geqq\max(n_0(\beta), n_1(\beta))$, which implies that

$$|(\varphi_j-\varphi_0)(a_n^g)| \leqq \varepsilon(\sup_i \|\mathring{\varphi}_i\|)+2\varepsilon$$

for all $j\geqq j_0$, $g\in\tilde{G}$ and $n\geqq\max(n_0(\beta), n_1(\beta))$. Now let $K=\{T_g(\varphi_j-\varphi_0), T_g\varphi_0 \mid j=1,2,3,\dots,j_0-1; g\in\tilde{G}\}$. Then by the proof of Theorem 3, K is a weakly relatively compact bounded subset of the predual M_* of M. Then by [1, Theorem II.3], there is a normal functional ψ on M such that for every positive number δ, there exists a positive number γ such that the inequality $\psi(a^*a+aa^*)<\gamma$ $(a\in S)$ implies that

$$|T_g(\varphi_j-\varphi_0)(a)|<\delta$$

and

$$|(T_g\varphi_0)(a)| < \delta$$

for all $g\in\tilde{G}$ and $j=1, 2, \ldots, j_0-1$. Since $a_n\to 0$ $(n\to\infty)$ strongly, take $\delta=\varepsilon$. Then for γ determined above, there is a natural number $n_2(\varepsilon)$ such that $\psi(a_n^*a_n+a_na_n^*)<\gamma$ for all $n\geqq n_2(\varepsilon)$. Thus it follows that

$$|(\varphi_j-\varphi_0)(a_n^g)| < \varepsilon, \quad |\varphi_0(a_n^g)| < \varepsilon$$

for all $g\in\tilde{G}$, $j=1, 2, \ldots, j_0-1$ and for all $n\geqq n_2(\varepsilon)$. Hence combining the above estimates, we have

$$|\varphi_j(a_n^g)| \leqq \varepsilon(\sup_i \|\varphi_i\|)+3\varepsilon$$

for all $g\in\tilde{G}$ and $j=1, 2, 3, \ldots$ and for all $n\geqq\max\big(n_0(\beta), n_1(\beta), n_2(\beta)\big)$, which implies that $\varphi_j(a_n^g)\to 0$ $(n\to\infty)$ uniformly for $g\in\tilde{G}$ and j. This completes the proof.

P r o o f o f T h e o r e m 1. Suppose M is $G\sim$finite. If K is any weakly relatively compact subset of M_*, then $\{T_g\varphi\,|\,\varphi\in K, g\in\tilde{G}\}$ is also weakly relatively compact. In order to prove this, we only have to prove that for every orthogonal sequence $\{e_n\}$ of projections, $\lim_{n\to\infty}\varphi(e_n^g)=0$ uniformly for $g\in\tilde{G}$ and $\varphi\in K$. If not, there is a positive number ε such that for each positive integer k, there are a natural number n_k $(n_k\uparrow\infty)$, $g_k\in\tilde{G}$ and $\varphi_k\in K$ such that

$$(*) \qquad |\varphi_k(e_{n_k}^{g_{n_k}})| \geqq \varepsilon.$$

By the Eberlein—Šmulian theorem ([2], Theorem V.6.1) there is a subsequence $\{\varphi_{k_p}\}$ of $\{\varphi_k\}$ $(k_p\uparrow\infty)$ such that $\varphi_{k_p}\to\varphi_0$ weakly $(p\to\infty)$ for some $\varphi_0\in M_*$. Now $e_{n_{k_p}}\to 0$ $(p\to\infty)$ strongly implies, by Lemma 8, that

$$\varphi_{k_{p'}}(e_{n_{k_p}}^g) \to 0 \quad (p\to\infty)$$

uniformly for $p=1, 2, 3, \ldots$ and $g\in\tilde{G}$ and that

$$|\varphi_{k_p}(e_{n_{k_p}}^{g_{k_p}})| \to 0 \quad (p\to\infty)$$

contradicting the inequality $(*)$. Hence $\{T_g\varphi\,|\,\varphi\in K, g\in\tilde{G}\}$ is weakly relatively compact. The converse is clear from Theorems 2 and 3. This completes the proof of Theorem 1.

The following corollary concerns another characterization of finiteness of a von Neumann algebra ([8], Theorem 1 and the Remark following Theorem 3, more precisely, a von Neumann algebra M is finite if and only if for every weakly relatively compact subset K of the predual M_* of M, the set $\{a\varphi\,|\,a\in S, \varphi\in K\}$ is also weakly relatively compact (where $(a\varphi)(x)=\varphi(xa)$ for all $x\in M$). In the proof of this assertion, the equivalence of the Mackey topology and the strong topology on S for finite algebras [9] plays an essential rôle.

Corollary 1. *For a von Neumann algebra M to be finite, it is necessary and sufficient that for every weakly relatively compact subset K of M_* (predual of M) the set*

$$\{u\varphi u^* \mid \varphi \in K, \ u \text{ is a unitary operator of } M\}$$

(where $(u\varphi u^*)(x) = \varphi(u^* x u)$ $x \in M$) is also weakly relatively compact.

Proof. Let $\mathfrak{U}$ be the group of inner automorphisms of M. Then if we apply Theorem 1 to M with $\mathfrak{U}$, we can easily show the above statement.

Corollary 2. *Let M be a finite von Neumann algebra with a group G of $*$-automorphisms of M. Then M is $G\sim$finite if and only if M is G-finite and if and only if for every weakly relatively compact subset K of M_*, $\{T_g\varphi \mid \varphi \in K, g \in G\}$ is also weakly relatively compact.*

Proof. If M is G-finite, then there is a separating set $\{\varphi_\alpha\}$ of G-invariant normal states on M. Since $\varphi_\alpha | Z$ (the restriction of φ_α to Z) is G-invariant for each α, Z is also G-finite. Let Φ be the center valued trace on M, then by unicity it follows that $\Phi(a^g) = \Phi(a)^g$ for all $g \in G$ and $a \in M$. Hence $\{(\varphi_\alpha | Z) \circ \Phi\}$ is a separating set of G-invariant normal traces on M and $G\sim$finiteness of M follows. The rest of the above assertions follows from Theorems 1, 2 and 3. The proof is now completed.

In particular, in the abelian case, we have a necessary and sufficient condition for the existence of invariant measures in ergodic theory, closely related to the theorem of HAJIAN and KAKUTANI [18, 19].

Let $(X, \mathfrak{M}, \mu)$ be a σ-finite measure space and let G be a group operating on the left on X by $\zeta \to s\zeta$, $\zeta \in X$ and suppose μ is quasi-invariant, that is, $\mu(sE) = 0$ if and only if $\mu(E) = 0$ for $E \in \mathfrak{M}$. Let $r_s(\cdot)$ be the Radon—Nikodym derivative such that $d\mu(s\zeta) = r_s(\zeta)\, d\mu(\zeta)$.

Corollary 3. *In the above notations, for $(X, \mathfrak{M}, \mu)$ to have a faithful G-invariant measure, it is necessary and sufficient that for every weakly relatively compact subset K of $L^1(X, \mathfrak{M}, \mu)$ (the set of all μ-integrable complex-valued functions on X)*

$$\{r_{s^{-1}}(\cdot) f(s^{-1}\cdot) \mid f \in K, g \in G\}$$

is also weakly relatively compact in $L^1(X, \mathfrak{M}, \mu)$.

Remark. HOPF [4] proved that $(X, \mathfrak{M}, \mu)$ has a faithful G-invariant measure if and only if $(X, \mathfrak{M}, \mu)$ is Hopf-finite in the sense that if for $E \in \mathfrak{M}$ there are countable families $\{E_i\}$ in $\mathfrak{M}$ and $\{g_i\}$ in G such that

$$X = \bigcup_{i=1}^{\infty} E_i \quad \text{and} \quad E = \bigcup_{i=1}^{\infty} g_i E_i, \quad \text{then} \quad \mu(X - E) = 0.$$

References

[1] C. A. AKEMANN, The dual space of an operator algebra, *Trans. Amer. Math. Soc.*, **126** (1967), 286—302.

[2] N. DUNFORD and J. T. SCHWARTZ, *Linear operators.* I, Interscience (New York, 1963).

[3] F. P. GREENLEAF, *Invariant means on topological groups,* Van Nostrand (New York, 1969).

[4] E. HOPF, Theory of measures and invariant integrals, *Trans. Amer. Math. Soc.*, **34** (1932), 373—393.

[5] I. KOVÁCS and J. SZŰCS, Ergodic type theorems in von Neumann algebras, *Acta Sci. Math.*, **27** (1966), 233—246.

[6] F. J. MURRAY and J. VON NEUMANN, On rings of operators, *Ann. of Math.*, **37** (1936), 116—229.

[7] I. NAMIOKA and E. ASPLUND, A geometric proof of Ryll-Nardzewski's fixed point theorem, *Bull. Amer. Math. Soc.*, **73** (1967), 443—445.

[8] K. SAITÔ, On the preduals of W^*-algebras, *Tôhoku Math. J.*, **19** (1967), 324—331.

[9] S. SAKAI, On topologies of finite W^*-algebras, *Ill. J. Math.*, **9** (1965), 236—241.

[10] S. SAKAI, *C^*-algebras and W^*-algebras,* Springer (Berlin—Göttingen—Heidelberg, 1971).

[11] E. STØRMER, Large groups of automorphisms of C^*-algebras, *Comm. Math. Phys.*, **5** (1967), 1—22.

[12] E. STØRMER, Automorphisms and equivalence in von Neumann algebras, *Pacific J. Math.*, **44** (1973), 371—383.

[13] E. STØRMER, Invariant states of von Neumann algebras, *Math. Scand.*, **30** (1972), 253—256.

[14] M. TAKESAKI, On the conjugate space of an operator algebra, *Tôhoku Math. J.*, **10** (1958), 194—203.

[15] F. J. YEADON, A new proof of the existence of a trace in a finite von Neumann algebra, *Bull. Amer. Math. Soc.*, **77** (1971), 257—260.

[16] J. DIXMIER, Sur la réduction des anneaux d'opérateurs, *Ann. Ecol. Norm. Sup.*, **68** (1951), 185—202.

[17] DIXMIER, Les algèbres d'opérateurs dans l'espace hilbertien, Gauthier-Villars (Paris, 1957).

[18] A. B. HAJIAN and S. KAKUTANI, Weakly wandering sets and invariant measures, *Trans. Amer. Math. Soc.*, **110** (1964), 136—151.

[19] A. B. HAJIAN and Y. ITÔ, Weakly wandering sets and invariant measures for a group of transformations, *J. Math. and. Mech.*, **18** (1969), 1203—1216.

[20] M. TAKESAKI, Covariant representations of C^*-algebras and their locally compact automorphism groups, *Acta Math.*, **119** (1967), 273—303.

MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN

(Received December 4, 1972)

Ideals of commutators of compact operators

By NORBERTO SALINAS in Princeton (N. J., USA)*)

1. Introduction

Throughout this paper $\mathfrak{H}$ will denote an infinite dimensional separable complex Hilbert space and $\mathscr{L}(\mathfrak{H})$ will represent the ring of all (bounded linear) operators on $\mathfrak{H}$.

Let $\mathscr{I}$ be a (two sided) ideal of $\mathscr{L}(\mathfrak{H})$. Following [5] we shall denote by $C(\mathscr{I})$ the set $C(\mathscr{I})=\{AB-BA: A, B\in\mathscr{I}\}$ and by $[\mathscr{I},\mathscr{I}]$ the linear span of $C(\mathscr{I})$. Clearly, the following chain of inclusions holds:

$$C(\mathscr{I})\subset[\mathscr{I},\mathscr{I}]\subset\mathscr{I}^2\subset\mathscr{I}.$$

One of the most interesting problems of the structure theory of compact operator is to determine, for a given ideal $\mathscr{I}$, which of the above inclusions are proper, and which are not (see [5, Problems 1, 2, 3, and 3′]). If $\mathscr{I}^2$ is a norm ideal [3; 8], then there is still another intermediate inclusion to be considered, i.e.,

$$\overline{[\mathscr{I},\mathscr{I}]}\subset\mathscr{I}^2$$

(where the closure is taken in the topology of $\mathscr{I}^2$).

In the present paper we make some remarks concerning the above-mentioned problems. In particular, we prove that if $\mathscr{I}^2$ is a separable norm ideal containing a positive operator whose sequence of eigenvalues (counted according to multiplicity) is a regularly decreasing sequence (see Definition 4.1), then $\overline{[\mathscr{I},\mathscr{I}]}=\mathscr{I}^2$ (Theorem 4.8). Moreover, we give (Corollary 4.7) sufficient conditions for the ideal $\mathscr{I}$ so that $[\mathscr{I},\mathscr{I}]=\mathscr{I}^2$, generalizing the results of [5].

Let $\mathscr{C}_p$, $p\geqq 1$ be the p-Schatten ideals [8], and let $\mathscr{C}_1^0$ be the set of operators in $\mathscr{C}_1$ of trace zero. If $\mathscr{I}$ is an ideal of $\mathscr{L}(\mathfrak{H})$ such that $\mathscr{I}^2\subset\mathscr{C}_1$, then necessarily $[\mathscr{I},\mathscr{I}]\subset\mathscr{C}_1^0$ and hence $[\mathscr{I},\mathscr{I}]$ is properly contained in $\mathscr{I}$. It will follow from our

*) The author was supported by National Science Foundation grant GP 7952X3.

discussion (Theorem 4.10) that if $\mathcal{J}^2$ is a norm ideal, then $\mathcal{C}_1^0 \subset \overline{C(\mathcal{J})}$; thus, in particular, we have (Corollary 4.11) $\mathcal{C}_1^0 = \overline{C(\mathcal{C}_2)}$.

In our work we shall make repeated use of a theorem of CALKIN [2, § 1] which establishes a lattice preserving bijective correspondence between the ideals of $\mathcal{L}(\mathfrak{H})$ and certain subsets of the set of all bounded sequences of non-negative real numbers called ideal sets. Denote by Z^+ the set of positive integers. J is an ideal set if it satisfies the following conditions:

i) if $\{\lambda_n\} \in J$, then $\{\lambda_{\pi(n)}\} \in J$ for every bijection π

ii) if $\{\lambda_n\} \in J$, and $\mu_n \leqq \lambda_n$ for every $n \in Z^+$, then $\{\mu_n\} \in J$;

iii) if $\{\lambda_n\} \in J$, and $\{\mu_n\} \in J$, then $\{\lambda_n + \mu_n\} \in J$.

Given a compact operator T on $\mathfrak{H}$, by the sequence of s-numbers $\{s_n(T)\}$ of T is usually meant [3, Chapt. 2] the sequence of eigenvalues (counted according to multiplicity) of the operator $(T^* T)^{1/2}$, arranged in decreasing order. If $\mathcal{J}$ is a proper ideal of $\mathcal{L}(\mathfrak{H})$, then the ideal set J that corresponds to $\mathcal{J}$ under the above mentioned lattice isomorphism is the set of all sequences of the form $\{s_{\pi(n)}(T)\}$, where $T \in \mathcal{J}$ and π is a bijection of Z^+. Thus, the ideal set C corresponding to the maximal ideal $\mathcal{C}$ of all compact operators is the set of all sequences of non-negative real numbers tending to zero, and the ideal set corresponding to the minimal ideal $\mathcal{F}$ of all finite rank operators is the subset of C consisting of all those sequences that have only finitely many terms different from zero. If C_p is the ideal set corresponding to $\mathcal{C}_p$, $p > 0$, then

$$C_p = \left\{ \{\lambda_n\} \in C : \sum_{n=1}^{\infty} \lambda_n^p < \infty \right\}.$$

As a natural generalization of the classes C_p, BROWN, PEARCY and the author introduced in [1] the ideal sets $S(f)$ and $D(f)$, where f is an admissible function. Following [1] a non-negative non-decreasing function f defined on the non-negative real line will be called an *admissible function* if $f(x) > 0$, for every $x > 0$, and $f(0) = 0 = \lim_{x \to 0} f(x)$. The ideal sets $S(f)$ and $D(f)$ are defined as follows:

$$\left.\begin{matrix} S(f) \\ D(f) \end{matrix}\right\} = \left\{ \{\lambda_n\} \in C : \sum_{n=1}^{\infty} f(\alpha\lambda_n) < \infty, \quad \text{for } \begin{matrix} \text{some} \\ \text{every} \end{matrix} \quad \alpha > 0 \right\}.$$

We shall denote by $\mathcal{S}(f)$ and $\mathcal{D}(f)$ the ideals of $\mathcal{L}(\mathfrak{H})$ corresponding to $S(f)$ and $D(f)$, respectively.

In § 2 we shall make some observations concerning the inclusion $\mathcal{J}^2 \subset \mathcal{J}$ giving especial attention to the cases $\mathcal{J} = \mathcal{S}(f)$ and $\mathcal{J} = \mathcal{D}(f)$. Our main theorem is proved in § 4 (Theorem 4.5). As a consequence of this result and the discussion of § 3 we shall conclude (Corollary 4.6) that if f is a sub-multiplicative admissible function and $S(f)$ ($D(f)$, reps.) contains a regularly decreasing sequence, then $[\mathcal{S}(f), \mathcal{S}(f)] = \mathcal{S}^2(f)$ ($[\mathcal{D}(f), \mathcal{D}(f)] = \mathcal{D}^2(f)$, reps.). We shall also give concrete examples showing that this result improves [5, Theorem 2].

2. Comparison between the ideals $\mathcal{I}$ and $\mathcal{I}^2$

Throughout the rest of the paper the set of all admissible functions will be denoted by $\mathfrak{A}$. If J is an ideal set and, for $p>0$, $f_{(p)}$ denotes the admissible function $f_{(p)}(x)=x^p$, $x\geqq 0$, it readily follows that $\{\{\lambda_n\}\in C\colon \{f_{(p)}(\lambda_n)\}\in J\}$ is an ideal set. Following [6] we shall write $J^p=\{\{\lambda_n\}\in C\colon \{f_{(1/p)}(\lambda_n)\}\in J\}$. It is easy to see that $J^2=\{\{\nu_n\}\in C\colon \nu_n=\lambda_n\mu_n\ (n=1,2,\ldots),\ \{\lambda_n\}, \{\mu_n\}\in J\}$. If K is another ideal set, we define the product of J and K to be the ideal set

$$J\cdot K = \{\{\nu_n\}\in C\colon \nu_n = \lambda_n\mu_n\ (n=1,2,\ldots),\ \{\lambda_n\}\in J,\ \{\mu_n\}\in K\}.$$

Theorem 2.1. *Let $\mathcal{I}$ and $\mathcal{K}$ be ideals of $\mathcal{L}(\mathfrak{H})$, and let J and K be the corresponding ideal sets. Then $J\cdot K$ is the ideal set corresponding to $\mathcal{I}\mathcal{K}$. In particular if k is any positive integer, then J^k is the ideal set of $\mathcal{I}^k$. Thus, if T is a positive operator on $\mathfrak{H}$, then $T\in\mathcal{I}^k$ if and only if $T^{1/k}\in\mathcal{I}$.*

Proof. Let L be the ideal set corresponding to $\mathcal{I}\mathcal{K}$. Since the inclusion $J\cdot K\subset L$ is easily established, we prove the reverse inclusion. Let A, B be two compact operators on $\mathfrak{H}$, then the following inequalities between the s-numbers of A, B, $A+B$ and AB are valid [3, Chapt. 2, § 3].

$$s_{p+q+1}(A+B)\leqq s_p(A)+s_q(B),\quad s_{p+q+1}(AB) < s_p(A)s_q(B)\quad (p,q=1,2,\ldots).$$

Now let $T\in\mathcal{I}\mathcal{K}$. Then there exist a finite number of operators $A_j\in\mathcal{I}$, $B_j\in\mathcal{K}$, $1\leqq j\leqq m$ such that $T=\sum_{j=1}^{m} A_jB_j$. An application of the above-mentioned inequalities of s-numbers yields

$$s_{2m(n+1)-1}(T)\leqq \sum_{j=1}^{n} s_n(A_j)s_n(B_j)\quad (n=1,2,\ldots).$$

For each $1\leqq j\leqq m$, let $\{r_n(A_j)\}$ denote the sequence obtained from $\{s_n(A_j)\}$ by repeating $s_1(A_j)$ $2m$ times, then repeating $s_2(A_j)$ $2m$ times, etc. Also, for each $1\leqq j\leqq m$, let $\{r_n(B_j)\}$ be defined similarly. Then

$$s_{n+4m-2}(T)\leqq \sum_{j=1}^{m} r_n(A_j)r_n(B_j)\quad (n=1,2,\ldots).$$

Since $r_n(A_j)\in J$ and $r_n(B_j)\in K$, $1\leqq j\leqq m$, it follows that $s_n(T)\in J\cdot K$ and the first assertion is proved. The second statement follows from the fact that $J^k=J\cdot J\cdot\ldots\cdot J$.

Corollary 2.2. *Let $\mathcal{I}$ be an ideal of $\mathcal{L}(\mathfrak{H})$. Then, for every pair of positive integers j, k we have*

$$\mathcal{I}^j \mathcal{I}^k = \mathcal{I}^{j+k}, \quad (\mathcal{I}^j)^k = \mathcal{I}^{jk}.$$

Furthermore, $\mathcal{I} = \mathcal{I}^2$ if and only if $\mathcal{I} = \mathcal{I}^k$, for every positive integer k.

Proof. It is a direct consequence of the corresponding properties for the ideal sets of $\mathcal{I}$.

Remark 2.3. Let $f, g \in \mathfrak{A}$. It was shown in [6, § 2] that the following statements are equivalent.

a) $S(g) \subset S(f)$;

b) $D(g) \subset D(f)$;

c) there exist positive constants $\alpha, \beta, c, \varepsilon$ such that $f(\alpha x) < cg(\beta x)$, $0 \leqq x \leqq \varepsilon$.

Following [6] we define the order relation $\prec$ and the equivalence relation $\sim$ on $\mathfrak{A}$ by: $f \prec g$ if and only if f and g satisfy the above property c); $f \sim g$ if and only if $f \prec g$ and $g \prec f$. Therefore, $f \sim g$ if and only if $S(f) = S(g)$ and $D(f) = D(g)$. On the other hand, $S(f) = D(f)$ if and only if the admissible function f satisfies the condition: there exist $\alpha > 1$, $c > 0$, $\varepsilon > 0$ such that $f(\alpha x) < cf(x)$, $0 \leqq x \leqq \varepsilon$ [1]. Following [6] those functions $f \in \mathfrak{A}$ for which the above conditions hold will be called mono-generating functions.

For a given $f \in \mathfrak{A}$ we define $\tilde{f} \in \mathfrak{A}$ by $\tilde{f}(x) = f(\sqrt{x})$, $x \geqq 0$. It can be easily seen that $S^2(f) = S(\tilde{f})$ and $D^2(f) = D(\tilde{f})$.

The following theorem is a consequence of Theorem 2.1 and Remark 2.3.

Theorem 2.4. *Let $f \in \mathfrak{A}$, then the following assertions are equivalent.*

a) $\mathcal{S}(f) = \mathcal{S}^2(f)$;

b) $\mathcal{D}(f) = \mathcal{D}^2(f)$;

c) *there exist positive constants α, c and ε such that*

$$f(\alpha \sqrt{x}) \leqq cf(x), \quad 0 \leqq x \leqq \varepsilon.$$

Remark 2.5. Let $g \in \mathfrak{A}$ such that $g(x) = (-1)/\ln x$, $0 < x < 1/2$. It follows that g satisfies Condition c) of Theorem 2.4. It is worth observing that g is a mono-generating function and $f_{(p)} \prec g$, for every $p > 0$. The following two theorems show that these are not accidental properties.

Theorem 2.6. *Let $f \in \mathfrak{A}$. If $\mathcal{S}(f) = \mathcal{S}^2(f)$ (or equivalently if $\mathcal{D}(f) = \mathcal{D}^2(f)$), then f is a mono-generating function.*

Proof. From Theorem 2.4 there exist positive constants α, c, and δ such that $f(\alpha\sqrt{x})\leqq cf(x)$, $0\leqq x\leqq\delta$. Let $\varepsilon=\min(\delta, \alpha^2/4)$. Then for $0\leqq x\leqq\varepsilon$ we have $f(2x)= =f\left[\alpha(2/\alpha\sqrt{x})\sqrt{x}\right]\leqq f(\alpha\sqrt{x})\leqq cf(x)$. Therefore f is a mono-generating function.

Theorem 2.7. *Let $f\in\mathfrak{A}$. If $\mathcal{S}(f)=\mathcal{S}^2(f)$, then*

$$\mathcal{S}(f)\subsetneqq\bigcap_{p>0}\mathcal{C}_p.$$

Proof. Since $\mathcal{S}(f)=\mathcal{S}^2(f)$, we use Theorem 2.6 to deduce that f is a mono-generating function. From [6, Theorem 3.10] we see that there exists $p>0$ such that $\mathcal{S}(f)\subset\mathcal{C}_p$, or equivalently $f_{(p)}\prec f$. On the other hand, from Theorem 2.4, we also obtain $f\sim\bar{f}$. The last two conditions yield $f_{(p/2)}\prec f$ and hence $\mathcal{S}(f)\subset\mathcal{C}_{(p/2)}$. Iterating this process we conclude that $\mathcal{S}(f)\subset\bigcap_{p>0}\mathcal{C}_p$. Since $\bigcap_{p>0}\mathcal{C}_p\neq\mathcal{S}(f)$, for every $f\in\mathfrak{A}$ [6, Theorem 2.20], it follows that

$$\mathcal{S}(f)\subsetneqq\bigcap_{p>0}\mathcal{C}_p.$$

Remark 2.8. Let $\mathcal{J}$ be an ideal of $\mathcal{L}(\mathfrak{H})$ and let J be its ideal set. In view of Theorem 2.1 for each $p>0$ we denote by $\mathcal{J}^p$ the ideal corresponding to the ideal set J^p. It is natural to call an ideal of $\mathcal{L}(\mathfrak{H})$ idempotent whenever it coincides with its square. It readily follows that the smalles idempotent ideal containing a given ideal $\mathcal{J}$ is $\bigcup_{p>0}\mathcal{J}^p$. Likewise, the largest idempotent ideal contained in $\mathcal{J}$ is given by $\bigcap_{p>0}\mathcal{J}^p$.

We conclude this section with an application of the results of the present paragraph to the structure theory of polynomially compact operators. The following theorem can be proved with arguments similar to those of [4, § 3].

Theorem 2.9. *Let $\mathcal{J}$ be an ideal of $\mathcal{L}(\mathfrak{H})$ different from $\mathcal{J}^{1/2}$. Also, let T be a positive operator in $\mathcal{J}^{1/2}$ which is not in $\mathcal{J}$. Then for every $S\in\mathcal{J}$, $(T+S)^2\neq0$.*

3. The ideals $\mathcal{J}_{\{\alpha_n\}}$

In this section we construct, for each ideal $\mathcal{J}$ of $\mathcal{L}(\mathfrak{H})$ and each sequence $\{\alpha_n\}$ in the ideal set J of $\mathcal{J}$ another ideal $\mathcal{J}_{\{\alpha_n\}}$ which is contained in $\mathcal{J}$. Afterwards, in § 4 we shall prove that if $\{\alpha_n\}$ is of an especial type, then $\mathcal{J}^2_{\{\alpha_n\}}\subset[\mathcal{J},\mathcal{J}]$.

Remarks 3.1. Let $\mathfrak{G}=\mathfrak{H}\oplus\mathfrak{H}\oplus\dots$ be the direct sum of $\aleph_0$ copies of $\mathfrak{H}$, and let $\Phi\colon\mathfrak{H}\to\mathfrak{G}$ be a Hilbert space isomorphism. Also, let $\mathcal{J}$ be an ideal of $\mathcal{L}(\mathfrak{H})$ and let J be its ideal set. For each $\{\alpha_n\}\in J$; such that $\alpha_n\neq0$, for some $n\in Z^+$, we shall write

$$\mathcal{J}_{\{\alpha_n\}}=\left\{T\in\mathcal{L}(\mathfrak{H}):\Phi^{-1}\left(\bigoplus_{n=1}^{\infty}\alpha_n T\right)\Phi\in\mathcal{J}\right\}.$$

A simple verification shows that $\mathscr{I}_{\{\alpha_n\}}$ is an ideal of $\mathscr{L}(\mathfrak{H})$ contained in $\mathscr{I}$. It follows that the ideal set $J_{\{\alpha_n\}}$ of $\mathscr{I}_{\{\alpha_n\}}$ is the set of all $\{\lambda_n\}\in C$ such that the double sequence $\{\alpha_k\lambda_n\}$ arranged as a simple sequence is in J (the order of this arrangement is not relevant since J is invariant under permutations of Z^+). Incidentally, let us observe that $\mathscr{I}^2_{\{\alpha_n\}}=(\mathscr{I}_{\{\sqrt{\alpha_n}\}})^2$. It is easy to see that

$$\begin{Bmatrix} S_{\{\alpha_n\}}(f) \\ D_{\{\alpha_n\}}(f) \end{Bmatrix} = \left\{\{\lambda_n\}\in C:\ \sum_{k,n=1}^{\infty} f(\varrho\alpha_k\lambda_n)<\infty,\quad \text{for}\ \begin{matrix}\text{some}\\ \text{all}\end{matrix}\quad \varrho>0\right\}.$$

In our subsequent discussion it will be important to know when $J_{\{\alpha_n\}}=J$, especially when J coincides with either $S(f)$ or $D(f)$.

Definition 3.2. We shall say that an admissible function f is *almost sub-multiplicative* if there exist positive constants α, c, ε such that for every $0\leqq x,\ y\leqq\varepsilon$ we have $f(\alpha xy)\leqq cf(x)f(y)$. The class of all almost sub-multiplicative admissible functions will be denoted by $\mathfrak{B}$.

Theorem 3.3. *If $f\in\mathfrak{B}$, then for every $\{\alpha_n\}\in S(f)$, we have $S_{\{\alpha_n\}}(f)=S(f)$. Furthermore, if $\{\alpha_n\}\in D(f)$, then $D_{\{\alpha_n\}}(f)=D(f)$.*

Proof. Let $\{\alpha_n\}$, $\{\lambda_n\}\in S(f)$. Then there exist $\sigma>0$, $\tau>0$ such that $\sum_{k=1}^{\infty} f(\sigma\alpha_k)<\infty$, $\sum_{n=1}^{\infty} f(\tau\lambda_n)<\infty$. Since $f\in\mathfrak{B}$, there exist $\alpha>0$, $c>0$, $\varepsilon>0$ such that $f(\alpha xy)\leqq cf(x)f(y)$, $0\leqq x,\ y\leqq\varepsilon$. Let $0<\xi\leqq\max(\alpha\sigma^2, \alpha\tau^2, \alpha\varepsilon^2/\sup_k \alpha_k^2, \alpha\varepsilon^2/\sup_k \lambda_n^2)$. Then we have

$$\sum_{n,k=1}^{\infty} f(\xi\alpha_k\lambda_n \leqq c\sum_{n,k=1}^{\infty} f(\sigma\alpha_k)f(\tau\lambda_n)<\infty.$$

Therefore $\{\lambda_n\}\in S_{\{\alpha_n\}}(f)$ and the first assertion is proved. Now, let $\{\alpha_n\}$, $\{\lambda_n\}\in D(f)$. To prove the second statement it suffices to show that for every $\eta>0$ there exists a pair of positive integers k_0, n_0 such that

$$\sum_{k=k_0}^{\infty}\sum_{n=n_0}^{\infty} f(\eta\alpha_k\lambda_n)<\infty.$$

Let $k_0, n_0\in Z^+$ such that $\alpha_k\leqq\varepsilon$ for every $k\geqq k_0$, $\eta\lambda_n/\alpha\leqq\varepsilon$, for every $n\geqq n_0$. Then we have

$$\sum_{\substack{k\geqq k_0\\ n\geqq n_0}} f(\eta\alpha_k\lambda_n)\leqq c\sum_{k\geqq k_0} f(\alpha_k)\sum_{n\geqq n_0} f(\eta\lambda_n/\alpha)<\infty.$$

Remark 3.4. The simplest examples of functions in $\mathfrak{B}$ are the functions $f_{(p)}$. Let φ and ψ be two convex functions in $\mathfrak{A}$ such that $\varphi(x)=e^{-1/x}$, $\psi(x)=(-\ln x)^{\ln x}$, $0<x<1/8$. Then φ, $\psi\in\mathfrak{B}$, and we have $\varphi\prec\psi\prec f_{(p)}$, for every $p>0$, and $\varphi\not\sim\psi$ (also observe that φ and ψ are not mono-generating).

It follows immediately from the definition that the almost submultiplicativity property depends only on the equivalence classes of $\mathfrak{A}$, i.e., if $f\in\mathfrak{B}$, $g\in\mathfrak{A}$ and $f\sim g$, then $g\in\mathfrak{B}$. This observation will be used in the next theorem which gives a sufficient condition for almost sub-multiplicativity.

Theorem 3.5. *Let* $f\in\mathfrak{A}$ *and suppose there exists* $0<\varepsilon<1$ *such that the function* h *defined, for* $0<x<\varepsilon$, *by* $h(x)=\int_0^x f(t)\,dt/xf(x)$ *is non-decreasing. Then* $f\in\mathfrak{B}$.

Proof. Let $g(x)=\int_0^x f(t)\,dt$. By hypothesis $xf(x)/g(x)$ is non-increasing for $x\in(0,\varepsilon]$. Therefore, for every $y\in(0,1)$ we have $f(x)/g(x)\geqq yf(xy)/g(xy)$, $0<x\leqq\varepsilon$. Thus, for every $0<y<1$, $0<x\leqq\varepsilon$, $\dfrac{\partial \frac{g(xy)}{g(x)}}{\partial x}=yf(xy)/g(x)-g(xy)f(x)/g^2(x)\geqq 0$. We deduce, that, for every $0<y<1$, the function $g(xy)/g(x)$ is non-decreasing on $(0,\varepsilon]$. Therefore, we have $g(xy)/g(x)\leqq g(\varepsilon y)/g(\varepsilon)\leqq g(y)/g(\varepsilon)$, for every $0<x$, $y\leqq\varepsilon$, and consequently $g\in\mathfrak{B}$. Since $ff_{(1)}\sim g$ [6, Theorem 3.4] we conclude that $ff_{(1)}\in\mathfrak{B}$ and hence $f\in\mathfrak{B}$.

In the rest of this section we shall need to use the following standard terminology [9, Chapter 5, § 5].

Definition 3.6. Let $f\in\mathfrak{A}$ be a right continuous function (i.e., $\lim\limits_{t\to 0+} f(x+t)=f(x)$, for every $x\geqq 0$) such that $\lim\limits_{x\to\infty} f(x)=\infty$. The function $f^{(-1)}$ defined for every $x\geqq 0$ by $f^{(-1)}(x)=\sup\limits_{f(t)\leqq x} t$ will be called the *right inverse* of f.

Remark 3.7. It is worth noting that if $g\in\mathfrak{A}$ then there always exists a right continuous function $f\in\mathfrak{A}$ such that $\lim\limits_{x\to\infty} f(x)=\infty$ and $f\sim g$ (see [6, Theorem 3.3]). The following are some elementary properties of the right inverse of the function f:

a) $f^{(-1)}\in\mathfrak{A}$,

b) $f^{(-1)}$ is right continuous and $\lim\limits_{x\to\infty} f^{(-1)}(x)=\infty$,

c) $[f^{(-1)}]^{(-1)}=f$,

d) for every $x>0$ we have $f[f^{(-1)}(x)]\geqq x$, $f^{(-1)}[f(x)-\varepsilon]\leqq x$, $0<\varepsilon\leqq f(x)$,

e) if f is continuous, then $f^{(-1)}$ is the inverse function of f,

f) if g is a right continuous function in $\mathfrak{A}$ such that $\lim\limits_{x\to\infty} g(x)=\infty$ and $g\prec f$, then $f^{(-1)}\prec g^{(-1)}$.

The following theorem provides a necessary condition for a function to be in $\mathfrak{B}$.

Theorem 3.8. *Let f be a right continuous function in $\mathfrak{B}$ such that* $\lim_{x\to\infty} f(x)=\infty$. *Then $f^{(-1)}$ is a mono-generating function.*

Proof. Since $f\in\mathfrak{B}$ there exist positive numbers α, c, and ε such that $f(\alpha xy)\leqq \leqq cf(x)f(y)$, $0\leqq x,\ y\leqq\varepsilon$. Let δ be a positive number such that $\delta\leqq\varepsilon$ and $cf(\delta)\leqq 1/3$. Then, for every $0\leqq t\leqq\alpha\delta\varepsilon$ we have $f(t)\leqq\frac{1}{3}f(t/\alpha\delta)$. Therefore $f^{(-1)}[2f(t)]\leqq \leqq f^{(-1)}[\frac{2}{3}f(t/\alpha\delta)]\leqq t/\alpha\delta$, $0\leqq t\leqq\alpha\delta\varepsilon$. Now let $\varepsilon'>0$ such that $f^{(-1)}(\varepsilon')<\alpha\delta\varepsilon$. It follows that for every $0\leqq x\leqq\varepsilon'$, $f^{(-1)}(2x)\leqq f^{(-1)}(2f[f^{(-1)}(x)])\leqq(1/\alpha\delta)f^{(-1)}(x)$. We conclude that $f^{(-1)}$ is a mono-generating function, as desired.

Theorem 3.9. *Let $f\in\mathfrak{B}$, then there exists $p>0$ such that $f\prec f_{(p)}$.*

Proof. As observed in Remarks 3.7 we can assume, without loss of generality, that f is a right continuous function and $\lim_{x\to\infty} f(x)=\infty$. Since $f^{(-1)}$ is mono-generating it follows from [6, Theorem 3.10] that there exists $q>0$ such that $f_{(q)}\prec f^{(-1)}$. Using Remark 3.7—e), and f) we see that $f\prec[f_{(q)}]^{(-1)}=f_{(1/q)}$. Thus, the desired number p is obtained by setting $p=1/q$.

Theorem 3.10. *Let f be a convex admissible function such that $f_{(p)}\not\precsim f\not\precsim f_{(1)}$, for every $p>1$. Then f cannot be in $\mathfrak{B}$* (cf. Remark 4.9).

Proof. Suppose that $f\in\mathfrak{B}$. Since $f\not\precsim f_{(1)}$ it can be easily checked that the function h defined, for $x\geqq 0$, by $h(0)=0$, $h(x)=f(x)/x$, $x>0$ is in $\mathfrak{B}$. From Theorem 3.9 there exists $q>0$ such that $h\prec f_{(q)}$. We conclude that $f\prec f_{(1+q)}$, in contradiction to the fact $f_{(p)}\not\precsim f$, for every $p>1$.

4. Comparison between $\mathscr{I}^2_{\{\alpha_n\}}$ and $[\mathscr{I}, \mathscr{I}]$

In the present section we shall need to use certain sequences which have a peculiar behavior.

Definition 4.1. A decreasing sequence $\{\alpha_n\}\in C$ will be said to be *regularly decreasing* if there exists a positive integer m such that for every $k=1, 2, \ldots$ we have

$$\alpha_k \leqq \sum_{j=(k-1)m+2}^{km+1} \alpha_j .$$

The smallest positive integer m satisfying the above condition will be denoted by $i(\{\alpha_n\})$. We shall represent by Δ the set of all regularly decreasing sequences.

Remark 4.2. An example of a regularly decreasing sequence is the sequence $\{1/n\}$, $i(\{1/n\})=3$. The following are immediate consequences of Definition 4.1:

a) if $\{\alpha_n\}\in\Delta$, then $\{\alpha_n\}$ is non-summable;
b) if $\{\alpha_n\}\in\Delta$, then every tail of $\{\alpha_n\}$ is also in Δ; and
c) if $\{\alpha_n\}\in\Delta$, then $\{\alpha_n^p\}\in\Delta$, for every $0<p<1$.

The next elementary lemma is an integral test for regularly decreasing sequences.

Lemma 4.3. *Let ω be a non-negative non-increasing function defined on* $[1, \infty)$ *such that* $\lim_{x\to\infty}\omega(x)=0$. *If* $\omega(k)\leqq\int_{(k-1)m+2}^{km+1}\omega(t)\,dt$ *for some positive integer m and for* $k=1, 2, \dots$ *then* $\{\omega(n)\}\in\Delta$. *On the other hand, if* $\{\omega(n)\}\in\Delta$ *then we must have* $\omega(k)\leqq \leqq\int_{(k-1)m+1}^{km}\omega(t)\,dt$, *for some positive integer m and for* $k=1, 2, \dots$.

The following result exhibits the main property of regularly decreasing sequences that we shall use later.

Theorem 4.4. *Let* $\{\alpha_n\}\in\Delta$, $m=i(\{\alpha_n\})$ *and let* $\{\alpha_n'\}$ *be the sequence obtained from* $\{\alpha_n\}$ *by repeatinng* α_1 $(m+1)$ *times, then repeating* α_2 $(m+1)$ *times and so forth. Then the alternating sequence* $\{\beta_n\}$ *defined by* $\beta_1=\alpha_1$, $\beta_{2n+1}=\alpha_{n+1}$, $\beta_{2n}=-\alpha_{n+1}$, $n= =1,2,\dots$, *can be rearranged into a sequence* $\{\gamma_n\}$ *that satisfies*

$$\left|\sum_{j=1}^{n}\gamma_j\right| \leqq \alpha_n', \quad n = 1, 2, \dots .$$

Proof. For each $k=1, 2, \dots$ let m_k be the first positive integer such that

$$\sum_{j=1}^{k}\alpha_j \leqq \sum_{j=2}^{m_k+1}\alpha_j .$$

Since $\{\alpha_n\}\in\Delta$, $m_k\leqq km$, $k=1, 2, \dots$. On the other hand,

$$\alpha_{k+1} \leqq \sum_{j=km+2}^{(k+1)m+1}\alpha_j \leqq \sum_{j=h+2}^{h+m+1}\alpha_j, \quad k = 1, 2, \dots, 1 \leqq h \leqq km.$$

It follows that $m_{k+1}-m_k\leqq m$, $k=1, 2, \dots$. Define $\{\gamma_n\}$ by $\gamma_1=\alpha_1$, $\gamma_n=-\alpha_n$, $2\leqq n\leqq \leqq m_1+1$, and for each $k=2, 3, \dots$ $\gamma_{k+m_{k-1}}=\alpha_k$, $\gamma_n=-\alpha_{n+1-k}$, $(k+1)+m_{k-1}\leqq n\leqq \leqq k+m_k$.

It readily follows that the sequence $\{\gamma_n\}$ satisfies the required conditions.

The following is the main theorem of the present section. Essentially, its proof follows the same pattern of the proofs of [5, Theorems 1 and 2].

Theorem 4.5. *Let* $\mathscr{I}$ *be an ideal of* $\mathscr{L}(\mathfrak{H})$ *and let J be its ideal set. Also, suppose that there exists a regularly decreasing sequence* $\{\alpha_n\}\in J^2$. *Then the ideal* $\mathscr{I}^2_{\{\alpha_n\}}$ *is contained in* $[\mathscr{I}, \mathscr{I}]$.

Proof. Let $T\in\mathcal{I}^2_{\{\alpha_n\}}$. Then $T=A_1-A_2+i(A_3-A_4)$ where A_j is a positive operator in $\mathcal{I}^2_{\{\alpha_n\}}$, $1\leqq j\leqq 4$. Thus, without loss of generality, we can assume that T is a positive operator. Let $\mathfrak{G}=\mathfrak{H}\oplus\mathfrak{H}\oplus\dots$ and let $\Phi\colon\mathfrak{H}\to\mathfrak{G}$ be the Hilbert space isomorphism introduced in Remark 3.1. Also, let $\Theta\colon\mathfrak{H}\to\mathfrak{H}\oplus\mathfrak{G}$ be the composition of Φ with the identification map $\mathfrak{G}\to\mathfrak{H}\oplus\mathfrak{G}$ and let $\Psi=(1_{\mathfrak{H}}\oplus\Phi^{-1})\Theta\colon\mathfrak{H}\to\mathfrak{H}\oplus\mathfrak{H}$. Then $\Psi T\Psi^{-1}$ can be represented as a two by two operator matrix, acting on $\mathfrak{H}\oplus\mathfrak{H}$, of the form $\begin{bmatrix}Q & N\\ N^* & R\end{bmatrix}$, where Q and R are positive operators. Let $N=UP$ be the polar decomposition of N (where $P=(N^*N)^{1/2}$). Then we have

$$\begin{bmatrix}0 & N\\ 0 & 0\end{bmatrix}=\begin{bmatrix}U\sqrt{P} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & \sqrt{P}\\ 0 & 0\end{bmatrix}-\begin{bmatrix}0 & \sqrt{P}\\ 0 & 0\end{bmatrix}\begin{bmatrix}U\sqrt{P} & 0\\ 0 & 0\end{bmatrix},$$

$$\begin{bmatrix}-R & 0\\ 0 & R\end{bmatrix}=\begin{bmatrix}0 & \sqrt{R}\\ \sqrt{R} & 0\end{bmatrix}\begin{bmatrix}0 & \sqrt{R}\\ 0 & 0\end{bmatrix}-\begin{bmatrix}0 & \sqrt{R}\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & \sqrt{R}\\ \sqrt{R} & 0\end{bmatrix}.$$

Since $\sqrt{P}$, and hence $U\sqrt{P}$ are in $\mathcal{I}$, and

$$\begin{bmatrix}Q & N\\ N^* & R\end{bmatrix}=\begin{bmatrix}Q+R & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & N\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & N\\ 0 & 0\end{bmatrix}^*+\begin{bmatrix}-R & 0\\ 0 & R\end{bmatrix},$$

in order to prove the theorem it suffices to show that

$$\Psi^{-1}[(Q+R)\oplus 0]\Psi=\Theta^{-1}[(Q+R)\oplus 0\oplus 0\oplus\dots]\Theta=\Phi^{-1}[(Q+R)\oplus 0\oplus 0\oplus\dots]\Phi$$

is in $[\mathcal{I},\mathcal{I}]$. Now, we observe that $Q+R\in\mathcal{I}^2_{\{\alpha_n\}}$. Thus to complete the proof we shall show that

$$\Phi^{-1}(\alpha_1 S\oplus 0\oplus 0\oplus\dots)\Phi\in[\mathcal{I},\mathcal{I}],$$

for every positive operator $S\in\mathcal{I}^2_{\{\alpha_n\}}$. Let $\{\varrho_n\}$ be given by $\varrho_{2n-1}=\sqrt{\alpha_n}$, $\varrho_{2n}=0$, $n=1,2,\dots$ and let $\{\beta_n\}$ be as in Theorem 4.4. Then

$$\alpha_1 S\oplus 0\oplus 0\oplus\dots=\left\{\varrho_1^2 S\oplus\left[\bigoplus_{n=1}^{\infty}(\varrho_{n+1}^2-\varrho_n^2)S\right]\right\}+\left[0\oplus\left(\bigoplus_{n=1}^{\infty}\beta_n S\right)\right].$$

We divide the rest of the proof in two cases considering separately each summand on the right hand side of the last identity.

Case I:

$$\Phi^{-1}\left\{\varrho_1^2 S\oplus\left[\bigoplus_{n=1}^{\infty}(\varrho_{n+1}^2-\varrho_n^2)S\right]\right\}\Phi$$

is in $C(\mathcal{I})$ (the set of commutators of operators in $\mathcal{I}$). Let V be the unilateral shift on $\mathfrak{G}$ with base space $\mathfrak{H}$, i.e., V can be represented as the infinite operator matrix

$$\begin{bmatrix} 0 & 0 & 0 & \dots \\ 1_{\mathfrak{H}} & 0 & 0 & \dots \\ 0 & 1_{\mathfrak{H}} & 0 & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}.$$

Also, let

$$A = \Phi^{-1}\left(\bigoplus_{n=1}^{\infty} \varrho_n \sqrt{S}\right) V^* \Phi, \quad \text{and} \quad B = \Phi^{-1} V \left(\bigoplus_{n=1}^{\infty} \varrho_n \sqrt{S}\right) \Phi.$$

Since $S \in \mathscr{I}^2_{\{\alpha_n\}}$, and hence $\sqrt{S} \in \mathscr{I}_{\{\sqrt{\alpha_n}\}}$, we see that $A, B \in \mathscr{I}$. A simple calculation shows that

$$\Phi^{-1}\left\{\varrho_1^2 S \oplus \left[\bigoplus_{n=1}^{\infty} (\varrho_{n+1}^2 - \varrho_n^2) S\right]\right\} \Phi = AB - BA,$$

and the first case is established.

Case II:

$$\Phi^{-1}\left[0 \oplus \left(\bigoplus_{n=1}^{\infty} \beta_n S\right)\right] \Phi \in C(\mathscr{I}).$$

Let $\{\gamma_n\}$ and $\{\alpha_n'\}$ be as in Theorem 4.4. Also, let W be a unitary operator on $\mathfrak{G}$ such that

$$W\left(\bigoplus_{n=1}^{\infty} \beta_n S\right) W^* = \bigoplus_{n=1}^{\infty} \gamma_n S,$$

and define the coisometry $\Xi : \mathfrak{H} \to \mathfrak{G}$ by $\Xi = W V^* \Phi$. Then

$$\Phi^{-1}\left[0 \oplus \left(\bigoplus_{n=1}^{\infty} \beta_n S\right)\right] \Phi = \Xi^* \left(\bigoplus_{n=1}^{\infty} \gamma_n S\right) \Xi.$$

Let $\{\delta_n\}$ be a sequence of complex numbers such that

$$\delta_n^2 = \sum_{j=1}^{n} \gamma_j, \quad n = 1, 2, \dots .$$

It follows that

$$\Xi^* \left(\bigoplus_{n=1}^{\infty} \gamma_n S\right) \Xi = \Xi^* \left\{\delta_1^2 S \oplus \left[\bigoplus_{n=1}^{\infty} (\delta_{n+1}^2 - \delta_n^2) S\right]\right\} \Xi.$$

Since

$$|\delta_n| \leqq \sqrt{\alpha_n'}, \; n = 1, 2, \dots \quad \text{and} \quad \mathscr{I}_{\{\sqrt{\alpha_n}\}} = \mathscr{I}_{\{\sqrt{\alpha_n'}\}}$$

we see that

$$\Xi^* \left(\bigoplus_{n=1}^{\infty} \delta_n \sqrt{S}\right) \Xi \in \mathscr{I}.$$

Define

$$A' = \Xi^*\left(\bigoplus_{n=1}^{\infty} \delta_n \sqrt{S}\right) V^*\Xi, \quad B' = \Xi^* V\left(\bigoplus_{n=1}^{\infty} \delta_n \sqrt{S}\right)\Xi.$$

It follows that $A', B' \in \mathcal{I}$ and

$$\Phi^{-1}\left[0\oplus\left(\bigoplus_{n=1}^{\infty} \beta_n S\right)\right]\Phi = \Xi^*\left\{\delta_1^2 S \oplus \left[\bigoplus_{n=1}^{\infty} (\delta_{n+1}^2 - \delta_n^2) S\right]\right\}\Xi = A'B' - B'A'.$$

The proof of the theorem is complete.

The following corollary is a consequence of Theorem 3.3 and Theorem 4.5.

Corollary 4.6. *Let* $f \in \mathfrak{B}$ *and suppose there exists a regularly decreasing sequence* $\{\alpha_n\} \in S^2(f)$. *Then* $[\mathcal{S}(f), \mathcal{S}(f)] = \mathcal{S}^2(f)$. *Moreover, if* $\{\alpha_n\} \in D(f)$, *then* $[\mathcal{D}(f), \mathcal{D}(f)] = \mathcal{D}^2(f)$.

Corollary 4.7. *Let* $\mathcal{I}$ *be an ideal of* $\mathcal{L}(\mathfrak{H})$, *and let* J *be its ideal set. If* $\mathcal{I}^2 = \bigvee_{\{\alpha_n\} \in \Delta \cap J^2} \mathcal{I}^2_{\{\alpha_n\}}$, *then* $[\mathcal{I}, \mathcal{I}] = \mathcal{I}^2$.

Theorem 4.8. *If* $\mathcal{I}^2$ *is a separable norm ideal and* $\Delta \cap J^2 \neq \Phi$, *then* $[\overline{\mathcal{I}, \mathcal{I}}] = \mathcal{I}^2$.

Proof. Let $\{\alpha_n\} \in \Delta \cap J^2$. Then, from Theorem 3.5, $\mathcal{I}^2_{\{\alpha_n\}} \subset [\mathcal{I}, \mathcal{I}]$. Since $\mathcal{I}^2_{\{\alpha_n\}}$ is an ideal of $\mathcal{L}(\mathfrak{H})$ it contains the ideal $\mathcal{F}$ of all finite rank operators. On the other hand, since $\mathcal{I}^2$ is separable $\overline{\mathcal{F}} = \mathcal{I}^2$ [3, Chapt. 3, § 6], and hence $[\overline{\mathcal{I}, \mathcal{I}}] = \mathcal{I}^2$.

Remark 4.9. From Remark 4.2 $\{1/n\} \in \Delta$. Since $\{1/n\} \in C_p$ for every $p > 1$, [5, Theorem 2] is a consequence of Corollary 4.6. Moreover, that corollary is also applicable to the functions φ, ψ of Remark 3.4 with $\{\alpha_n\} = \{1/n\}$. Since φ and ψ are convex admissible functions, it follows from [7] that the ideals they generate are norm ideals. We obtain the following inclusion formulas:

$$\bigcup_{p>0} \mathcal{C}_p \subsetneqq \mathcal{D}^2(\psi) \subsetneqq \mathcal{S}^2(\psi) \subsetneqq \mathcal{D}(\psi) \subsetneqq \mathcal{S}(\psi),$$

$$\mathcal{S}(\psi) \subsetneqq \mathcal{D}^2(\varphi) \subsetneqq \mathcal{S}^2(\varphi) \subsetneqq \mathcal{D}(\varphi) \subsetneqq \mathcal{S}(\varphi).$$

The above chains of inclusions follow from the more general fact that if $f \in \mathfrak{A}$ and $\mathcal{S}^2(f) \neq \mathcal{S}(f)$, then $\mathcal{D}^2(f) \subsetneqq \mathcal{S}^2(f) \subsetneqq D(f) \subsetneqq \mathcal{S}(f)$ (which in turn is a consequence of Theorem 2.6 and [6, Theorem 2.12]). Let χ be a convex function in $\mathfrak{A}$ such that $\chi(x) = (x/\ln x)^2$, $0 < x < 1/2$. Then χ is a mono-generating function which is not in $\mathfrak{B}$. On the other hand, since $\mathcal{S}^2(\chi) = \mathcal{S}(\bar{\chi}) = \mathcal{D}(\bar{\chi})$ (Remarks 2.3 and 2.5) and $\bar{\chi}$ is a convex function, it follows from [7] that $\mathcal{S}^2(\chi)$ is a separable norm ideal. Thus Theorem 4.8 tells us that $[\overline{\mathcal{S}(\chi), \mathcal{S}(\chi)}] = \mathcal{S}^2(\chi)$. (Observe that $\{1/n\} \in S^2(\chi)$). Also, we have $\mathcal{C}_1 \subsetneqq \mathcal{S}^2(\chi) \subsetneqq \bigcap_{p>1} \mathcal{C}_p$.

Theorem 4.10. *Let $\mathcal{I}$ be an ideal of $\mathcal{L}(\mathfrak{H})$ such that $\mathcal{I}^2$ is a norm ideal. Then $\mathcal{C}_1^0 \subset \overline{C(\mathcal{F})}$ (where the closure is taken in the topology of $\mathcal{I}^2$).*

Proof. Let $T \in \mathcal{C}_1^0$. Since the real and imaginary parts of T are in $\mathcal{C}_1^0$ and $C(\mathcal{F}) = [\mathcal{F}, \mathcal{F}]$, without loss of generality we can suppose that T is Hermitian. Then there exists a basis $\{e_n: n=1, 2, \dots)$ of $\mathfrak{H}$ such that $Te_n = \lambda_n e_n$, $n=1, 2, \dots$, where $\{\lambda_n\}$ is an absolutely summable sequence such that $\sum_{n=1}^{\infty} \lambda_n = 0$. Since $\overline{\mathcal{F}}$ is a minimal norm ideal, it follows from the results of [8, Chapt. 5, §§ 6, 7, 8] that the inclusion map $\mathcal{C}_1 \to \overline{\mathcal{F}}$ is norm decreasing. Thus, $|||S||| \leqq \|S\|_1$, for every $S \in \mathcal{C}_1$, where $|||\cdot|||$ and $\|\cdot\|_1$ stand for the norms in $\mathcal{I}^2$ and $\mathcal{C}_1$, respectively. Let $\varepsilon > 0$ be given and let $n_0 > 1$ such that $\sum_{n=n_0}^{\infty} |\lambda_n| < \varepsilon/2$. Now define X on $\mathfrak{H}$ by $Xe_n = \lambda_n e_n$ $1 \leqq n < n_0$,

$$Xe_{n_0} = \left(\sum_{n=n_0}^{\infty} \lambda_n \right) e_{n_0}, \quad Xe_n = 0, \quad n > n_0.$$

Then we have $|||T-X||| \leqq \|T-X\|_1 \leqq 2 \sum_{n>n_0}^{\infty} |\lambda_n| < \varepsilon$. Since X is a finite rank operator of trace zero, we conclude that $X \in C(\mathcal{F})$ and the theorem follows.

Corollary 4.11. *$\mathcal{C}_1^0 = \overline{C(\mathcal{C}_2)}$ (in the topology of $\mathcal{C}_1$).*

Theorem 4.12. *Let $\mathcal{I}(\lambda)$, $\lambda \in \Lambda$ be a family of ideals of $\mathcal{L}(\mathfrak{H})$ and let $J(\lambda)$, $\lambda \in \Lambda$ be the corresponding ideal sets. If for each $\lambda \in \Lambda$ there exists a regularly decreasing sequence $\{\alpha_{n,\lambda}\} \in \mathcal{I}^2(\lambda)$, then*

$$\bigvee_{\lambda \in \Lambda} \mathcal{I}^2_{\{\alpha_{n,\lambda}\}}(\lambda) \subset [\bigvee_{\lambda \in \Lambda} \mathcal{I}(\lambda), \bigvee_{\lambda \in \Lambda} \mathcal{I}(\lambda)].$$

Furthermore, if there is a regularly decreasing sequence $\{\alpha_n\} \in \bigcap_{\lambda \in \Lambda} J^2(\lambda)$, then

$$\bigcap_{\lambda \in \Lambda} \mathcal{I}^2_{\{\alpha_n\}}(\lambda) \subset [\bigcap_{\lambda \in \Lambda} \mathcal{I}(\lambda), \bigcap_{\lambda \in \Lambda} \mathcal{I}(\lambda)]$$

Proof. It is easy to see that

$$\bigvee_{\lambda \in \Lambda} [\mathcal{I}(\lambda), \mathcal{I}(\lambda)] \subset [\bigvee_{\lambda \in \Lambda} \mathcal{I}(\lambda), \bigvee_{\lambda \in \Lambda} \mathcal{I}(\lambda)].$$

This fact together with Theorem 4.5 proves the first assertion. To prove the second statement, we first observe that

$$\bigcap_{\lambda \in \Lambda} \mathcal{I}^2_{\{\alpha_n\}}(\lambda) = [\bigcap_{\lambda \in \Lambda} \mathcal{I}(\lambda)]^2_{\{\alpha_n\}}.$$

This is a consequence of the following chain of identities:

$$\bigcap_{\lambda \in \Lambda} J^2_{\{\alpha_n\}} = \bigcap_{\lambda \in \Lambda} (J_{\{\sqrt{\alpha_n}\}})^2 = (\bigcap_{\lambda \in \Lambda} J_{\{\sqrt{\alpha_n}\}})^2 = (\bigcap_{\lambda \in \Lambda} J(\lambda))^2_{\{\alpha_n\}}.$$

Now the proof of the theorem is completed after applying Theorem 4.5 to the ideal $\bigcap_{\lambda\in\Lambda} \mathscr{I}(\lambda)$.

Corollary 4.13.

$$\bigcup_{p>0} \mathscr{C}_p = [\bigcup_{p>0} \mathscr{C}_p, \bigcup_{p>0} \mathscr{C}_p], \quad \bigcap_{p>1} \mathscr{C}_p = [\bigcap_{p>2} \mathscr{C}_p, \bigcap_{p>2} \mathscr{C}_p].$$

Proof. Since $\{1/n\}\in C_p$, for every $p>1$ the corollary is a consequence of Corollary 4.6 and Theorem 4.12.

References

[1] A. Brown, C. Pearcy and N. Salinas, Ideals of compact operators on Hilbert space, *Mich. Math. J.*, **18** (1971), 373—384.

[2] J. W. Calkin, Two-sided ideals and congruences in the ring of bounded operators in Hilbert space, *Ann. of Math.*, (2) **42** (1941), 839—873.

[3] I. C. Gohberg and M. G. Krein, *Introduction to the theory of linear nonself-adjoint operators.* Translated from the Russian. (*Translations of Mathematical Monographs*, vol. 18), American Mathematical Society (Providence, R. I., 1969).

[4] C. L. Olsen, A structure theorem for polynomially compact operators, *Amer. J. Math.*, **93** (1971), 686—698.

[5] C. Pearcy and D. Topping, On commutators in ideals of compact operators, *Mich. Math. J.*, **18** (1971), 247—252.

[6] N. Salinas, Ideal sets and ideals of compact operators on Hilbert space. *(To appear in Trans. Amer. Math. Soc.)*

[7] N. Salinas, Symmetric norm ideals and relative conjugate ideals. *(To appear.)*

[8] R. Schatten, *Norm ideals of completely continuous operators*, Springer-Verlag (Berlin, 1960).

[9] A. Zaanen, *Linear Analysis*, Interscience, North Holland, Noordhoff (New York—Amsterdam—Groningen, 1953).

THE INSTITUTE FOR ADVANCED STUDY
PRINCETON, NEW JERSEY 08540

(Received September 8, 1972)

Bands of monoids

By B. M. SCHEIN in Saratov (USSR)

To my colleagues in the city of Szeged where this paper has been written

Let S be a semigroup which is the union of a family $(S_i)_{i\in I}$ of subsemigroups which are classes of a congruence relation on S. Then I may be endowed with a binary operation $ij=k \leftrightarrow S_i S_j \subset S_k$ for all $i, j, k \in I$. Under this operation I is a *band* (i.e. an idempotent semigroup) and S is called an *I-band* (or merely a band) of subsemigroups $(S_i)_{i\in I}$.

In this paper we present a new method of constructing bands of semigroups. This method permits to build up all bands of unipotent monoids (a *monoid* is a semigroup with identity, a monoid is called *unipotent* if it contains the only idempotent — its identity). In particular, we obtain a simple construction for orthodox bands of arbitrary monoids. Our method is a generalization of Clifford's sums of direct systems of groups [1] (called also rigid or strong semilattices of groups).

In our paper [2] we introduced a class of semigroups with the weak involutory property (WIP-semigroups). A semigroup S is a WIP-semigroup if for any $s, t \in S$ and any $\bar{s}, \bar{t} \in S$ such that $s\bar{s}s=s$, $\bar{s}s\bar{s}=\bar{s}$, $t\bar{t}t=t$, $\bar{t}t\bar{t}=\bar{t}$ (i.e. $\bar{s}$ and $\bar{t}$ are inverses for s and t respectively), $\bar{t}\bar{s}$ is an inverse for st. Among other properties it was proved that S is a WIP-semigroup if and only if the idempotents of S form a (possibly empty) subsemigroup [2]. Regular WIP-semigroups were considered also in [3] where they were called orthodox semigroups. So we call the WIP-semigroups *orthodox* (notice that an orthodox semigroup in our sense need not be regular).

Let $(S_i)_{i\in I}$ be a family of semigroups with pairwise disjoint sets of elements. Suppose $\leqq$ is a quasiorder (i.e. reflexive and transitive) binary relation on I. A family $\Phi=(\varphi_{ij})_{i\leqq j}$; $i, j \in I$ is called a *direct system of homomorphisms over* $\leqq$ if for every $i, j \in I$ such that $i \leqq j$ φ_{ij} is a homomorphism of S_j into S_i and the following two properties holds:

1) for every $i \in I$ φ_{ii} is the identical automorphism of S_i;

2) for every $i, j, k \in I$ if $i \leqq j \leqq k$ then $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$.

If S_i are monoids and e_i denotes the identity of S_i then we demand that $\varphi_{ij}(e_j)=e_i$, i.e. identities are preserved under homomorphisms of monoids.

Let I be endowed with an associative and idempotent binary operation $\cdot$, i.e. let $(I, \cdot)$ be a band. Define the following binary relations $\leqq_1$ and $\leqq_2$ on I: $i \leqq_1 j \leftrightarrow ji=i$; $i \leqq_2 j \leftrightarrow ij=i$. Clearly, both $\leqq_1$ and $\leqq_2$ are quasiorder relations on I. Suppose $\Phi=(\varphi_{ij})$ and $\Psi=(\psi_{ij})$ are direct systems of homomorphisms over $\leqq_1$ and $\leqq_2$ respectively. Φ and Ψ are called *commuting* if for all $i, j, k \in I$ such that $j \leqq_1 i$, $k \leqq_2 i$ the following diagram is commutative:

$$\begin{array}{ccc} S_i & \rightarrow & S_j \\ \downarrow & & \downarrow \\ S_k & \rightarrow & S_{kj} \end{array}$$

where the horizontal arrows represent homomorphisms from Φ and vertical arrows represent homomorphisms from Ψ (i.e. $\psi_{kj,j} \circ \varphi_{ji}=\varphi_{kj,k} \circ \psi_{ki}$). Clearly, $kj \leqq_1 k$ and $kj \leqq_2 j$ so that all homomorphisms mentioned do exist.

If $a_i \in S_i$ then ϱ_{a_i} and λ_{a_i} denote the right and left translations of S_i corresponding to a_i, i.e. $\varrho_{a_i}(s)=sa_i$ and $\lambda_{a_i}(s)=a_i s$ for all $s \in S_i$.

Suppose there are given two direct systems of homomorphisms Φ and Ψ over $\leqq_1$ and $\leqq_2$ respectively and an $(I \times I)$-matrix $A=(a_{ij})$ over $S=\bigcup_{i\in I} S_i$ such that $a_{ij} \in S_{ij}$ for all $i, j \in I$. We call the triple (Φ, Ψ, A) *balanced* if $a_{ii}=e_i$ for any $i \in I$ and

$$\varrho_{a_{ij,k}} \circ \varphi_{ijk,ij} \circ \lambda_{a_{ij}} \circ \psi_{ij,j} = \lambda_{a_{i,jk}} \circ \psi_{ijk,jk} \circ \varrho_{a_{jk}} \circ \varphi_{jk,j}$$

for all $i, j, k \in I$.

If $a_{ij}=e_{ij}$ for all $i, j \in I$ then the triple (Φ, Ψ, A) is balanced precisely if the direct systems Φ and Ψ commute.

A band S of monoids $(S_i)_{i\in I}$ is called *proper* if the identities of the monoids form a subsemigroup of S.

Theorem 1. *Let $(S_i)_{i\in I}$ be a family of pairwise disjoint unipotent monoids, $(I, \cdot)$ be a band, $\Phi=(\varphi_{ij})$ and $\Psi=(\psi_{ij})$ be direct systems of homomorphisms over $\leqq_1$ and $\leqq_2$ respectively, A be an $(I \times I)$-matrix over $S=\bigcup_{i\in I} S_i$ and the triple (Φ, Ψ, A) be balanced.*

Define a binary multiplication $\Box$ on S as follows: if $s_i \in S_i$ and $s_j \in S_j$ then $s_i \Box s_j=\varphi_{ij,i}(s_i) a_{ij} \psi_{ij,j}(s_j)$ where the right-hand side product is taken in the monoid S_{ij}. Then $(S, \Box)$ is an I-band of monoids $(S_i)_{i\in I}$ and every I-band of monoids $(S_i)_{i\in I}$ can be constructed in this way. Moreover, the triple (Φ, Ψ, A) is defined uniquely for any I-band of $(S_i)_{i\in I}$.

Theorem 2. *Let $(S_i)_{i\in I}$ be a family of pairwise disjoint semigroups, $(I, \cdot)$ be a band, and Φ and Ψ be commuting direct systems of homomorphisms over $\leqq_1$ and*

$\leqq_2$, respectively. Define a binary multiplication $\square$ on $S=\bigcup_{i\in I} S_i$ as follows: if $s_i\in S_i$ and $s_j\in S_j$ then $s_i\square s_j=\varphi_{ij,i}(s_i)\psi_{ij,j}(s_j)$ where the right-hand side product is taken in the semigroup S_{ij}. Then $(S, \square)$ is an I-band of semigroups $(S_i)_{i\in I}$. Moreover, if S_i are monoids then $(S, \square)$ is a proper I-band of the monoids $(S_i)_{i\in I}$ and every proper I-band of the monoids $(S_i)_{i\in I}$ can be constructed in the above fashion, the direct systems Φ and Ψ being determined in the unique way. $(S, \square)$ is orthodox if and only if all the monoids S_i are orthodox.

Some corollaries will follow after the proofs.

Proof of Theorem 1. Suppose $s_i\in S_i$, $s_j\in S_j$ and $s_k\in S_k$. Then

$$(s_i\square s_j)\square s_k = (\varphi_{ij,i}(s_i)a_{ij}\psi_{ij,j}(s_j))\square s_k = \varphi_{ijk,ij}(\varphi_{ij,i}(s_i)a_{ij}\psi_{ij,j}(s_j))a_{ij,k}\psi_{ijk,k}(s_k) =$$
$$= [\varphi_{ijk,ij}\circ\varphi_{ij,i}(s_i)][\varrho_{a_{ij},k}\circ\varphi_{ijk,ij}\circ\lambda_{a_{ij}}\circ\psi_{ij,j}(s_j)]\psi_{ijk,k}(s_k) =$$
$$= \varphi_{ijk,i}(s_i)[\lambda_{a_{i,jk}}\circ\psi_{ijk,jk}\circ\varrho_{a_{jk}}\circ\varphi_{jk,j}(s_j)][\psi_{ijk,jk}\circ\psi_{jk,k}(s_k)] =$$
$$=\varphi_{ijk,i}(s_i)a_{i,jk}[\psi_{ijk,jk}(\varphi_{jk,j}(s_j)a_{jk}\psi_{jk,k}(s_k))]=\varphi_{ijk,i}(s_i)a_{i,jk}\psi_{ijk,jk}(s_j\square s_k)=s_i\square(s_j\square s_k),$$

i.e. $(S, \square)$ is a semigroup. If $i=j$ then $s_i\square s_j=\varphi_{ii}(s_i)a_{ii}\psi_{ii}(s_j)=s_ie_is_j=s_is_j$. Thus, $(S, \square)$ is an I-band of the family $(S_i)_{i\in I}$ of monoids.

Now $e_i\square e_j=\varphi_{ij,i}(e_i)a_{ij}\psi_{ij,j}(e_j)=e_{ij}a_{ij}e_{ij}=a_{ij}$ so that the matrix A is determined in the unique way — $A=(e_i\square e_j)$. Using this fact we obtain

$$a_{i,ij} = e_i\square e_{ij} = e_i\square(e_i\square e_{ij}) = e_i\square(e_{ij}\square(e_i\square e_{ij})) = (e_i\square e_{ij})^2,$$

i.e. $a_{i,ij}$ is an idempotent from S_{ij}. Since S_{ij} is unipotent, $a_{i,ij}=e_{ij}$. Thus,

$$s_i\square e_{ij} = \varphi_{ij,i}(s_i)a_{i,ij}\psi_{ij,ij}(e_{ij}) = \varphi_{ij,i}(s_i)\cdot a_{i,ij}e_{ij} = \varphi_{ij,i}(s_i)$$

i.e. the direct system Φ of homomorphisms is determined in the unique way. Analogously we may prove that $\psi_{ij,j}(s_j)=e_{ij}\square s_j$ for any $s_j\in S_j$.

To prove the second part of Theorem 1 suppose $(S, \cdot)$ is a band of a family $(S_i)_{i\in I}$ of unipotent monoids. Let $a_{ij}=e_ie_j$ for any $i, j\in I$, $\varphi_{ij,i}(s_i)=s_ie_{ij}$ and $\psi_{ij,j}(s_j)=e_{ij}s_j$ for all $i, j\in I$ and $s_i\in S_i$, $s_j\in S_j$. Then $a_{ij}\in S_{ij}$ and if s_i, $t_i\in S_i$ then

$$\varphi_{ij,i}(s_it_i) = s_it_ie_{ij} = s_i(e_{ij}(t_ie_{ij})) = \varphi_{ij,i}(s_i)\varphi_{ij,i}(t_i),$$

i.e. $\varphi_{ij,i}$ is a homomorphism of S_i into S_{ij}. Since S_{ij} is unipotent, $\varphi_{ij,i}(e_i)=e_{ij}$. Clearly $\varphi_{ii}(s_i)=s_ie_i=s_i$. Now

$$\varphi_{ijk,ij}\circ\varphi_{ij,i}(s_i) = \varphi_{ijk,ij}(s_ie_{ij}) = (s_ie_{ij})e_{ijk} = s_i(e_{ij}e_{ijk}) = s_ie_{ijk} = \varphi_{ijk,i}(s_i)$$

so that $\Phi=(\varphi_{ij})$ forms a direct system of homomorphisms over $\leqq_1$. In the same way we may prove that $\Psi=(\psi_{ij})$ forms a direct system of homomorphisms over $\leqq_2$.

Now $a_{ii}=e_ie_i=e_i$ and

$$\varrho_{a_{ij},k}\circ\varphi_{ijk,ij}\circ\lambda_{a_{ij}}\circ\psi_{ij,j}(s_j) = \varrho_{a_{ij},k}\circ\varphi_{ijk,ij}\circ\varrho_{a_{ij}}(e_{ij}s_j) = \varrho_{a_{ij},k}\circ\varphi_{ijk,ij}(a_{ij}e_{ij}s_j) =$$
$$= \varrho_{a_{ij},k}(a_{ij}e_{ij}s_je_{ijk}) = a_{ij}e_{ij}s_je_{ijk}a_{ij,k} = a_{ij}s_je_{ijk}a_{ij,k} = a_{ij}s_ja_{ij,k} =$$
$$= e_ie_js_ja_{ij,k} = e_is_ja_{ij,k} = e_is_je_{ij}e_k = e_is_je_k = e_ie_{jk}s_je_k = a_{i,jk}s_je_k =$$
$$= a_{i,jk}s_je_je_k = a_{i,jk}s_ja_{jk} = a_{i,jk}e_{ijk}s_ja_{jk} = a_{i,jk}e_{ijk}s_je_{jk}a_{jk} =$$
$$= \lambda_{a_{i,jk}}(e_{ijk}s_je_{jk}a_{jk}) = \lambda_{a_{i,jk}}\circ\psi_{ijk,jk}(s_je_{jk}a_{jk}) =$$
$$= \lambda_{a_{i,jk}}\circ\psi_{ijk,jk}\circ\varrho_{a_{jk}}(s_je_{jk}) = \lambda_{a_{i,jk}}\circ\psi_{ijk,jk}\circ\varrho_{a_{jk}}\circ\varphi_{jk,j}(s_j),$$

i.e. the triple (Φ, Ψ, A) is balanced. Finally

$$s_i\Box s_j = \varphi_{ij,i}(s_i)a_{ij}\psi_{ij,j}(s_j) = s_ie_{ij}a_{ij}e_{ij}s_j = s_ia_{ij}s_j = s_ie_ie_js_j = s_is_j.$$

This fact completes the proof of Theorem 1.

Proof of Theorem 2. Suppose $s_i\in S_i$, $s_j\in S_j$ and $s_k\in S_k$. Then

$$(s_i\Box s_j)\Box s_k = (\varphi_{ij,i}(s_i)\psi_{ij,j}(s_j))\Box s_k = \varphi_{ijk,ij}(\varphi_{ij,i}(s_i)\psi_{ij,j}(s_j))\psi_{ijk,k}(s_k) =$$
$$= [\varphi_{ijk,ij}\circ\varphi_{ij,i}(s_i)][\varphi_{ijk,ij}\circ\psi_{ij,j}(s_j)]\psi_{ijk,k}(s_k) =$$
$$= \varphi_{ijk,i}(s_i)[\psi_{ijk,jk}\circ\varphi_{jk,j}(s_j)][\psi_{ijk,jk}\circ\psi_{jk,k}(s_k)] = \varphi_{ijk,i}(s_i)\psi_{ijk,jk}(s_j\Box s_k) = s_i\Box(s_j\Box s_k),$$

i.e. $(S, \Box)$ is a semigroup.

If $i=j$ then $s_i\Box s_j=\varphi_{ii}(s_i)\psi_{ii}(s_j)=s_is_j$. Thus, $(S, \Box)$ is an I-band of the family $(S_i)_{i\in I}$ of semigroups. Unicity of Φ and Ψ in case S are monoids for all $i\in I$ is proved in the same way as in the proof of Theorem 1.

If $(S, \cdot)$ is a proper I-band of monoids S_i then exactly in the same way as in the proof of Theorem 1 we may verify that $(S, \cdot)=(S, \Box)$ where Φ and Ψ are defined in the same way as in the proof of Theorem 1. Commutativity of Φ and Ψ follows readily.

If S_i are monoids then $e_i\Box e_j=\varphi_{ij,i}(e_i)\psi_{ij,j}(e_j)=e_{ij}e_{ij}=e_{ij}$. Therefore $(S, \Box)$ is a proper band of $(S_i)_{i\in I}$.

Clearly, if $(S, \Box)$ is orthodox then S_i are orthodox for all $i\in I$. Conversely, suppose S_i are orthodox and $s_i\in S_i$, $s_j\in S_j$ are idempotents of $(S, \Box)$. Then $s_i\Box s_j= =\varphi_{ij,i}(s_i)\psi_{ij,j}(s_j)$ and the right-hand side of the equality is a product of two idempotents of S_{ij} (since homomorphisms map idempotents onto idempotents). The orthodoxy of S_{ij} implies $s_i\Box s_j$ is an idempotent. Thus, $(S, \Box)$ is orthodox which completes the proof of Theorem 2.

Obviously, Theorem 2 in case of unipotent monoids is a particular case of Theorem 1.

Remark 1. Since every group is a unipotent and orthodox monoid, every band of groups may be constructed as in Theorem 1 and every orthodox band of groups may be constructed as in Theorem 2. Another construction for orthodox bands of groups has been given in [4]. A survey of constructions for orthodox unions of groups may be found in [5].

Remark 2. Suppose (Φ, Ψ, A) is a balanced triple and $k \leqq_1 j$, $i \leqq_2 j$. This being the case, $a_{ij} = e_i$ (which fact has been proved above) and analogously $a_{jk} = e_k$. Thus, the condition of balancedness may be written for these particular i, j, k as follows:

$$\varrho_{a_{ik}} \circ \varphi_{ik,i} \circ \psi_{ij} = \lambda_{a_{ik}} \circ \psi_{ik,k} \circ \varphi_{kj}. \tag{1}$$

If $i=k$ then we obtain $\varrho_{a_{ii}} \circ \varphi_{ii} \circ \psi_{ij} = \lambda_{a_{ii}} \circ \psi_{ii} \circ \varphi_{ij}$ or, equivalently, $\psi_{ij} = \varphi_{ij}$. Thus, if $i \leqq_1 j$ and $i \leqq_2 j$ (i.e. if $i = ij = ji$) then $\psi_{ij} = \varphi_{ij}$. In particular, if $(I, \cdot)$ is a semilattice then $\leqq_1$ coincides with $\leqq_2$ and Φ coincides with Ψ; in this case the construction of Theorem 2 turns out to be the well-known [1] construction for sums of direct systems of semigroups. Clearly, if $\Phi = \Psi$ then Φ and Ψ commute. Thus, *every proper semilattice of monoids is a sum of their direct system.*

Remark 3. Let the band $(I, \cdot)$ satisfy the pseudoidentity $xyx = xy \vee xyx = yx$ where $\vee$ is the disjunction sign. Let $x \leqq y$ mean that $x \leqq_1 y$ or $x \leqq_2 y$. Then $\leqq$ is a quasiorder relation on I. In effect, $\leqq$ is obviously reflexive. To show transitivity of $\leqq$, suppose $i \leqq j$ and $j \leqq k$ for some $i, j, k \in I$. Suppose $i \leqq_1 j$. If $j \leqq_1 k$ then $i \leqq_1 k$ and $i \leqq_1 k$, so let $j \leqq_2 k$. Then $ji = i$ and $jk = j$. Then $iki = ki$ or $iki = ik$. If $iki = ki$ then $i = ji = (jk)i = j(ki) = j(iki) = (ji)ki = iki = ki$ and $i \leqq_1 k$, whence $i \leqq k$. If $iki = ik$ then $i = ji = (jk)i = j(ki) = j(kiki) = (jk)iki = (jk)ik = jik = ik$ and $i \leqq_2 k$, whence $i \leqq k$. Analogously, $i \leqq_2 j$ implies $i \leqq k$. Therefore, $\leqq$ is a quasiorder relation.

Conversely, suppose $\leqq$ is a quasiorder relation. Then the band $(I, \cdot)$ satisfies the above pseudoidentity. In effect, for every two elements $x, y \in I$ the relations $xyx \leqq_1 xy$ and $xy \leqq_2 y$ hold in every band. Therefore, $xyx \leqq xy \leqq y$ and, since $\leqq$ is transitive, $xyx \leqq y$, i.e. $xyx \leqq_1 y$ or $xyx \leqq_2 y$. The latter means that $xyx = y(xyx) = (yx)^2 = yx$ or $xyx = (xyx)y = (xy)^2 = xy$, i.e. $xyx = xy \vee xyx = yx$.

Two quasiorder relations on a same set are called *compatible* if their set-theoretical union is a quasiorder relation. We have proved the following

Lemma 1. *A band satisfies the pseudoidentity $xyx = xy \vee xyx = yx$ if and only if its quasiorder relations $\leqq_1$ and $\leqq_2$ are compatible.*

Now if $i \leqq j$ then either $i \leqq_1 j$ or $i \leqq_2 j$ or both. Suppose two direct systems of homomorphisms Φ and Ψ over $\leqq_1$ and $\leqq_2$ respectively are given. Then φ_{ij} or ψ_{ij} is defined. If both homomorphisms are defined then $i \leqq_1 j$ and $i \leqq_2 j$ which implies, as we have seen in Remark 2, $\varphi_{ij} = \psi_{ij}$. Therefore, one may consider the system $X = (\chi_{ij})_{i \leqq j;\, i, j \in I}$ of homomorphisms: χ_{ij} coincides with that of homomorphisms φ_{ij}, ψ_{ij} which is defined.

Let the above pseudoidentity be satisfied and $(S, \cdot)$ be an I-band of the family $(S_i)_{i\in I}$ of monoids. If $i\leqq_1 j$, i.e. if $ji=i$, then, as we have seen above, $e_j e_i=e_j(e_j e_i)=$ $=e_j(e_i(e_j e_i))=(e_j e_i)^2$. Suppose now all S_i are unipotent. Then $e_j e_i=e_i$. Analogously $i\leqq_2 j$ implies $e_i e_j=e_i$. Now let i and j be arbitrary elements of I. Then either $iji=ij$ or $iji=ji$. In the first case $e_i e_j\in S_{ij}$, therefore $e_{ij}e_i e_j = e_i e_j$. Now

$$e_{ij}e_i\in S_{ij}S_i\subset S_{iji} = S_{ij},$$

therefore

$$e_{ij}e_i = (e_{ij}e_i)e_{ij} = e_{ij}(e_i e_{ij}) = e_{ij}e_{ij} = e_{ij},$$

since $ij \leqq_1 i$. Hence

$$e_i e_j = e_{ij}e_i e_j = e_{ij}e_j = e_{ij},$$

since $ij \leqq_2 j$.

Suppose now

$$iji = ji.$$

Then

$$ij = (ij)^2 = (iji)j = (ji)j \quad \text{and} \quad e_j e_{ij}\in S_{jij} = S_{ij}.$$

It follows that

$$e_j e_{ij} = e_{ij}(e_j e_{ij}) = (e_{ij}e_j)e_{ij} = e_{ij}e_{ij} = e_{ij}$$

and

$$e_i e_j = (e_i e_j)e_{ij} = e_i(e_j e_{ij}) = e_i e_{ij} = e_{ij}.$$

Thus,

$$a_{ij} = e_i e_j = e_{ij}$$

for any $i, j\in I$, i.e. $(S, \cdot)$ is a proper band of monoids. Then the direct systems Φ and Ψ commute.

Now let $i\leqq j\leqq k$. If $i\leqq_1 j\leqq_1 k$ then $\chi_{ik}=\varphi_{ik}=\varphi_{ij}\circ\varphi_{jk}=\chi_{ij}\circ\chi_{jk}$. Analogously, $\chi_{ik}=\chi_{ij}\circ\chi_{jk}$ in case when $i\leqq_2 j\leqq_2 k$. Now let $i\leqq_1 j\leqq_2 k$. Then, as we have seen above, $i\leqq k$, i.e. $i\leqq_1 k$ or $i\leqq_2 k$. If $i\leqq_1 k$ then $\chi_{ik}=\varphi_{ik}$ and for every $s_k\in S_k$

$$\chi_{ik}(s_k) = \varphi_{ik}(s_k) = s_k e_i = e_i(s_k e_i) = (e_j e_i)(s_k e_i) = e_j(e_i(s_k e_i)) = e_j(s_k e_i) =$$
$$= (e_j s_k)e_i = \varphi_{ij}\circ\psi_{jk}(s_k) = \chi_{ij}\circ\chi_{jk}(s_k),$$

i.e. $\chi_{ik}=\chi_{ij}\circ\chi_{jk}$. The same equality can be proved analogously if $i\leqq_2 j\leqq_1 k$. Since χ_{ii} is obviously the identical automorphism of S_i and X preserves identities of our monoids, X is a direct system of homomorphisms over $\leqq$.

The above argument together with Theorems 1 and 2 yields the following

Proposition 1. *Suppose $(I, \cdot)$ is a band satisfying the pseudoidentity $xyx=$ $=xy\vee xyx=yx$. Define $i\leqq j$ if and only if $i=iji$. Then $\leqq$ is a quasiorder relation, the set-theoretical union of the quasiorder relations $\leqq_1$ and $\leqq_2$ (i.e. $i\leqq j$ if and only if $i=ij$ or $i=ji$). Suppose $(S_i)_{i\in I}$ is a family of pairwise disjoint monoids and X is a direct system of homomorphisms over $\leqq$. Define a binary multiplication $\square$ on S as follows: if $s_i\in S_i$ and $s_j\in S_j$ then $s_i\square s_j=\chi_{ij,i}(s_i)\chi_{ij,j}(s_j)$ where the right-hand side product is taken inside the monoid S_{ij}. Then $(S, \square)$ is a proper I-band of the family*

$(S_i)_{i\in I}$ of monoids and conversely, every proper I-band of these monoids can be constructed in the above way, the direct system X being determined in the unique fashion for each proper I-band of $(S_i)_{i\in I}$. Moreover, every I-band of unipotent monoids is necessarily proper (and hence orthodox) and so it can be constructed in the above way.

In particular, Proposition 1 holds if $(I, \cdot)$ satisfies one of the following identities: $xyx=xy$, $xyx=yx$, $xyz=yxz$, $xyz=xzy$, $xy=x$, $xy=y$, $xy=yx$. In the latter case, i.e. for semilattices of unipotent monoids, this has been proved in [8].

It can be easily verified that $(I, \cdot)$ *satisfies the identity* $xyx=xy$ *[the identity* $xyx=yx$*] if and only if the quasiorder relation* $\leqq_1$ *[the quasiorder relation* $\leqq_2$*] is included into* $\leqq_2$ *[into* $\leqq_1$*]*. Every band is a semilattice of rectangular bands. Right zero and left zero bands are called singular. It can be trivially verified that *a band satisfies the pseudoidentity* $xyx=xy\vee xyx=yx$ *if and only if it is a semilattice of singular bands.*

Remark 4. Suppose $(I, \cdot)$ is a rectangular band and $i,j\in I$. Then $i\leqq_1 ij$ and $ij\leqq_1 i$, whence $\varphi_{i,ij}\circ\varphi_{ij,i}=\varphi_{ii}$ and $\varphi_{ij,i}\circ\varphi_{i,ij}=\varphi_{ij,ij}$. Therefore, $\varphi_{i,ij}$ is an isomorphism. In the same way we may prove that $\psi_{j,ij}$ is an isomorphism. It follows that S_i and S_j are isomorphic. Thus, all the monoids S_i are pairwise isomorphic. This fact permits us to give an alternative construction for rectangular bands of unipotent monoids.

Fix some element $o\in I$ and for every $i\in I$ fix an isomorphism α_i of S_i onto S_o, say, $\alpha_i=\psi_{o,io}\circ\varphi_{io,i}$. If $s_i\in S_i$ let $f(s_i)=(\alpha_i(s_i), i)$. Then f is a bijective mapping of $S=\bigcup_{i\in I}S_i$ onto the Cartesian product of the sets S_o and I. It remains to define a suitable multiplication in $S_o\times I$ in order f to be an isomorphism. It is clear that

$$\alpha_i(s_i) = \psi_{o,io}\circ\varphi_{io,i}(s_i) = \psi_{o,io}(s_ie_{io}) = e_o(s_ie_{io})$$

so that $f(s_i)=(e_os_ie_{io}, i)$. Now suppose $(s, i)\in S_o\times I$. Then $f^{-1}((s, i))=e_{io}se_i$. In effect,

$$e_{io}se_i\in S_{io}S_oS_i\subset S_{iooi} = S_i \quad\text{and}\quad f(e_{io}se_i) = (e_oe_{io}se_ie_{io}, i) = (s, i)$$

since

$$e_oe_{io} = e_{o(io)}e_{io} = e_{o(io)} = e_o \quad\text{and}\quad se_ie_{io} = se_{io} = (se_o)e_{io} = s(e_oe_{io}) = se_o = s$$

so that

$$e_oe_{io}se_ie_{io} = e_os = s.$$

Thus, we should define such a multiplication $\square$ on $S_o\times I$ that for any $s, t\in S_o$ and any $i, j\in I$

$$(s, i)\square(t, j) = f((e_{io}se_i)\cdot(e_{jo}te_j)) = (e_o(e_{io}se_i)(e_{jo}te_j)e_{(ij)o}, ij).$$

Now

$$e_oe_{io}s = e_os = s \quad\text{and}\quad (ij)o = io$$

so that

$$e_o(e_{io}se_i)(e_{jo}te_j)e_{(ij)o} = se_ie_{jo}te_je_{io} = [s(e_oe_ie_{jo})][t(e_oe_je_{io})] = (sb_{ij})(tb_{ji})$$

where

$$b_{ij} = e_oe_ie_{jo} \in S_{oijo} = S_o.$$

Now

$$b_{ij}b_{ji,k} = (e_oe_ie_{jo})(e_oe_{ji}e_{ko}) = e_oe_i(e_{jo}e_o)e_{ji}e_{ko} = e_oe_ie_{jo}e_{ji}e_{ko} =$$

$$= e_oe_ie_{jo}(e_je_{ji})e_{ko} = e_oe_i(e_{jo}e_j)e_{ji}e_{ko} = e_oe_ie_je_{ji}e_{ko} = e_oe_i(e_je_{ji})e_{ko} =$$

$$= e_oe_ie_{ji}e_{ko} = e_o(e_ie_{ji})e_{ko} = e_oe_ie_{ko} = b_{ik},$$

since

$$e_{jo}e_j = e_j \quad \text{and} \quad e_ie_{ji} = e_i$$

which may be proved in the same way as the above equality $e_{oi}e_o = e_o$.

Conversely, suppose a unipotent monoid S_o and a rectangular band I are given and $b_{ij}b_{ji,k} = b_{ik}$ for every $i, j, k \in I$. Then $b_{ii}b_{ii} = b_{ii}b_{ii,i} = b_{ii}$ which implies that $b_{ii} = e_o$ for every $i \in I$. Now

$$b_{ij,i} = b_{ij,i}e_o = b_{ij,i}b_{ij,ij} = b_{ij,i}b_{i(ij),ij} = b_{ij,ij} = e_o,$$

whence

$$b_{i,jk}b_{jki,j} = b_{ij} \quad \text{and} \quad b_{i,jk}b_{jki,j} = b_{i,jk}b_{ji,j} = b_{i,jk}e_o = b_{i,jk},$$

i.e. $b_{i,jk} = b_{ij}$. On the Cartesian product $S_o \times I$ define the following multiplication $\Box$: $(s, i)\Box(t, j) = (sb_{ij}tb_{ji}, ij)$. Then $(s, i)\Box(t, i) = (sb_{ii}tb_{ii}, ii) = (se_ote_o, i) = (st, i)$, i.e. $S_i = S_o \times \{i\}$ is isomorphic to S_o. Now

$$[(s, i)\Box(t, j)]\Box(u, k) = (sb_{ij}tb_{ji}, ij)\Box(u, k) = (sb_{ij}tb_{ji}b_{ij,k}ub_{k,ij}, ijk) =$$

$$= (sb_{ij}tb_{jk}ub_{ki}, ijk) = (sb_{i,jk}tb_{jk}ub_{kj}b_{jk,i}, ijk) =$$

$$= (s, i)\Box(tb_{jk}ub_{kj}, jk) = (s, i)\Box[(t, j)\Box(u, k)].$$

Thus, $(S_o \times I, \Box)$ is an I-band of monoids isomorphic to S_o, namely, of monoids S_i.

We have proved the following

Proposition 2. *Let S be a unipotent monoid, I be a rectangular band, $B = (b_{ij})$ be an $(I \times I)$-matrix over S such that $b_{ij}b_{ji,k} = b_{ik}$ for all $i, j, k \in I$. Define the following multiplication $\Box$ on the set $S \times I$: $(s, i)\Box(t, j) = (sb_{ij}tb_{ji}, ij)$. Then $(S \times I, \Box)$ is an I-band of monoids isomorphic to S and every I-band of monoids isomorphic to S can be constructed in the above way. In particular, there exists an I-band of a family $(S_i)_{i \in I}$ of unipotent monoids if and only if all the monoids are pairwise isomorphic.*

Another description of rectangular bands of unipotent monoids has been given in [9, Corollary 3.10].

In case of proper bands we have the following

Proposition 3. *Let $(S_i)_{i\in I}$ be a family of monoids and I be a rectangular band. There exists a proper I-band of $(S_i)_{i\in I}$ if and only if all the monoids are pairwise isomorphic, and every such band is isomorphic to a direct product of S_i for some fixed $i\in I$ and I. Conversely, every direct product of S_i and I is isomorphic to a proper I-band of $(S_i)_{i\in I}$.*

In effect, from Theorem 2 it follows that Φ and Ψ commute which implies easily our Proposition.

Another proof of Proposition 3 has been given in [6].

Since every band of semigroups is a semilattice of rectangular bands of semigroups [7], Proposition 2 gives some additional insight into the structure of bands of unipotent monoids and Proposition 3 — into the structure of proper bands of monoids.

In particular, if S is a *combinatorial* monoid (i.e. S has no invertible elements except 1 where 1 is the identity of S) then every I-band of monoids isomorphic to S is isomorphic to a direct product of S and I. This follows from the fact that b_{ij} is an invertible element of S for every $i;j\in I$. Moreover, $b_{ij}^{-1}=b_{ji,i}$. In effect, $b_{ij}b_{ji,i}=b_{ii}=1$ and $b_{ji,i}b_{ij}=b_{ji,i}b_{i(ji),j}=b_{ji,j}=b_{ji,ji}=1$.

It is a well-known fact that rectangular bands of groups are precisely the completely simple semigroups. Thus Proposition 2 gives, in particular, a new representation theorem for completely simple semigroups.

Remark 5. Suppose I is a band and $i\leqq j\leftrightarrow i=iji$. Then $\leqq$ is a quasiorder relation on I. Suppose $(S_i)_{i\in I}$ is a family of monoids and X is a direct system of homomorphisms over $\leqq$. Since $ij\leqq i$ and $ij\leqq j$ for every $i,j\in I$ we may define the following operation $\Box$ on $S=\bigcup_{i\in I}S_i$: if $s_i\in S_i$ and $s_j\in S_j$ then $s_i\Box s_j=\chi_{ij,i}(s_i)\chi_{ij,j}(s_j)$. Then $(S,\Box)$ is an I-band of our monoids. A particular case of this construction was used in Proposition 1. Clearly, $(S,\Box)$ is a proper I-band. Suppose $i\leqq j$. Then

$$e_i\Box s_j\Box e_i=\chi_{iji,i}(e_i)\chi_{iji,j}(s_j)\chi_{iji,i}(e_i)=\chi_{ii}(e_i)\chi_{ij}(s_j)\chi_{ii}(e_i)=e_i\chi_{ij}(s_j)e_i=\chi_{ij}(s_j)$$

for every $s_j\in S_j$.

Proposition 4. *Let I be a band satisfying the identity $xyxzx=xyzx$, $(S_i)_{i\in I}$ be a family of pairwise disjoint monoids and $i\leqq j\leftrightarrow i=iji$ for all $i,j\in I$. Define an operation $\Box$ on the set $S=\bigcup_{i\in I}S_i$ as follows: if $s_i\in S_i$ and $s_j\in S_j$ then $s_i\Box s_j=\chi_{ij,i}(s_i)\chi_{ij,j}(s_j)$. Then $(S,\Box)$ is a proper I-band of monoids $(S_i)_{i\in I}$ and every proper I-band of these monoids can be constructed in the above way, the direct system X being determined uniquely for every proper I-band of $(S_i)_{i\in I}$.*

Proof. Suppose $(S, \cdot)$ is a proper I-band of $(S_i)_{i\in I}$. Then for every $i, j \in I$ such that $i \leqq j$ and every $s_j, t_j \in S_j$

$$(e_i s_j e_i)(e_i t_j e_i) = e_i s_j e_i t_j e_i = (e_i s_j) e_{ij} e_i e_{ji} (t_j e_i) = e_i s_j e_j e_{ij} e_i e_{ji} e_j t_j e_i =$$
$$= e_i s_j e_j e_{ij} e_{ji} e_j t_j e_i = e_i s_j e_{ij} e_{ji} t_j e_i = e_i s_j t_j e_i,$$

i.e. the mapping $\chi_{ij}\colon S_j \to S_i$ such that $\chi_{ij}(s_j) = e_i s_j e_i$ is a homomorphism. Clearly, $\chi_{ij}(e_j) = e_i e_j e_i = e_{iji} = e_i$ and χ_{ii} is the identical automorphism of S_i. Now let $i \leqq j \leqq k$ and $s_k \in S_k$. Then

$$\chi_{ij} \circ \chi_{jk}(s_k) = \chi_{ij}(e_j s_k e_j) = e_i e_j s_k e_j e_i = (e_i e_j) e_{kji} s_k e_j e_i = e_{ijkji} s_k e_{ji} = e_{iji} s_k e_{ji} =$$
$$e_i s_k = e_{ji} = e_i s_k e_{ik} e_{ji} = e_i s_k e_{ikji} = e_i s_k e_i = \chi_{ik}(s_k),$$

i.e. (χ_{ij}) form a direct system of homomorphism. We used the fact that $ikji = i$. In effect,

$$ikji = ikiji = ijikiji = ijkiji = ijkji = iji = i.$$

References

[1] A. H. CLIFFORD, Semigroups admitting relative inverses, *Annals of Math.*, **42** (1941), 1037—1049.

[2] Б. М. Шайн, *К теории обобщенных групп и обобщенных груд*, В сборнике *Теория полугрупп и ее приложения*, Вып. **1** (Саратов, 1965), 286—324.

[3] T. E. HALL, On regular semigroups whose idempotents form a subsemigroup, *Bull. Austral. Math. Soc.*, **1** (1969), 195—208; **3** (1970), 287—280.

[4] M. YAMADA, Strictly inversive semigroups, *Bull. Shimane Univ. (Nat. Sci.)*, **13** (1963), 128—138.

[5] A. H. CLIFFORD, The structure of orthodox unions of groups, *Semigroup Forum*, **3** (1972), 283—337.

[6] M. PETRICH, The maximal matrix decomposition of a semigroup, *Portugal. Math.*, **25** (1966), 15—33.

[7] A. H. CLIFFORD, Bands of semigroups, *Proc. Amer. Math. Soc.*, **5** (1954), 499—504.

[8] M. PETRICH, *Introduction to semigroups*, Merrill Books (Columbus, Ohio, 1973).

[9] G. LALLEMENT and M. PETRICH, A generalization of the Rees theorem on semigroups, *Acta Sci. Math.*, **30** (1969), 113—132.

(Received September 20, 1972)

A spectral characterization of the maximal ideal in factors

By ŞERBAN STRĂTILĂ and LÁSZLÓ ZSIDÓ in Bucharest (Romania)

Introduction. In a recent paper ([3]) J. A. DYER, P. PORCELLI and M. ROSENFELD obtained a spectral characterization of the elements in the greatest proper ideal $\mathcal{J}$ of a properly infinite factor $\mathcal{M}$, namely that $x\in\mathcal{J}$ iff $\sigma(x+b)\cap\sigma(b)\neq\emptyset$ for every $b\in\mathcal{M}$. On the other hand, they proved that if $\mathcal{M}$ is a factor of type I_n, $n<\infty$, then for any $0\neq x\in\mathcal{M}$ there is $b\in\mathcal{M}$ such that $\sigma(x+b)\cap\sigma(b)=\emptyset$ and they conjectured that the same assertion is true if $\mathcal{M}$ is a factor of type II_1.

In the present paper we prove this conjecture by showing that if $\mathcal{M}$ is a factor of type II_1 and $0\neq x\in\mathcal{M}$, then there is a nilpotent element $b\in\mathcal{M}$ such that $x+b$ is invertible (Corollary 4), getting exactly the same result as for factors of type I_n, $n<\infty$. Moreover, the same result is established for elements in a properly infinite factor $\mathcal{M}$, which are not of the form $\lambda+a$ with $\lambda\in\mathbf{C}$ and $a\in\mathcal{J}$ (that is, for not "thin" elements; Corollary 5). This is done by proving Theorem 2 below, which allows us to represent every element x in $\mathcal{M}$ as a suitable operator matrix and then by using the trick of BROWN and HALMOS (cf. the proof of Theorem C in [3] and also below, Remark 3).

For results concerning operator algebras we refer to the treatise DIXMIER [2]. Two projections e, f in a C^*-algebra $\mathcal{M}$ are said to be *equivalent*, $e\sim f$, if there is an element $v\in\mathcal{M}$ such that $v^*v=e$, $vv^*=f$; then $v=ve=fv$ and e, f belong to the same (two sided) ideals of $\mathcal{M}$. If $\mathcal{M}$ is a W^*-algebra and $x\in\mathcal{M}$, then $LP(x)$ (resp. $RP(x)$) means the left projection (resp. the right projection) of x; it is known that $LP(x)\sim RP(x)$ ("$LP\sim RP$" theorem) (cf. [6]). By $\mathcal{B}(\mathfrak{H})$ we denote the algebra of all (bounded) operators on the Hilbert space $\mathfrak{H}$. As usually, for an element x in a Banach algebra we denote by $\sigma(x)$ its spectrum.

1. We begin with the following lemma which is surely known:

Lemma. *Let $e, f\in\mathcal{B}(\mathfrak{H})$ be two projections such that there is $0<\lambda\leqq 1$ with $f-fef\geqq\lambda f$. Then $e\wedge f=0$ and $(e\vee f)(\mathfrak{H})=e(\mathfrak{H})+f(\mathfrak{H})$.*

Proof. The inequality $f-fef\geqq\lambda f$ is equivalent to the conditions $\|(1-e)f\xi\|\geqq\sqrt{\lambda}\,\|f\xi\|$, $\xi\in\mathfrak{H}$. If $e\xi=\xi=f\xi$ then it follows that $\xi=0$, whence $e(\mathfrak{H})\cap f(\mathfrak{H})=0$

and $e\wedge f=0$. For $\xi,\eta\in\mathfrak{H}$ we have $\|e\eta+f\xi\|=\|(e\eta+ef\xi)+(1-e)f\xi\|\geqq\|(1-e)f\xi\|\geqq$ $\geqq\sqrt{\lambda}\,\|f\xi\|$, so that, if $\zeta\in(e\vee f)(\mathfrak{H})=\overline{e(\mathfrak{H})+f(\mathfrak{H})}$ and $\zeta=\lim_n(e\eta_n+f\xi_n)$ then the sequence $\{f\xi_n\}$ is convergent and $\xi=\lim_n f\xi_n\in f(\mathfrak{H})$. Then $\zeta-\xi=\lim_n e\eta_n\in e(\mathfrak{H})$, i.e. $\zeta\in e(\mathfrak{H})+f(\mathfrak{H})$. Hence $(e\vee f)(\mathfrak{H})=e(\mathfrak{H})+f(\mathfrak{H})$. Q.E.D.

Remark. The greatest λ satisfying the inequality $f-fef\geqq\lambda^2 f$ could be called "the sinus of the angle between the projections e and f". If $\lambda=1$, then e and f are orthogonal, and the lemma says that if "the angle between e and f" is not zero, then $(e\vee f)(\mathfrak{H})=e(\mathfrak{H})+f(\mathfrak{H})$.

2. Theorem. *Let $\pi\colon\tilde{\mathcal{M}}\to\mathcal{M}$ be a representation of the W^*-algebra $\tilde{\mathcal{M}}$ on the C^*-algebra $\mathcal{M}$, $\mathcal{Z}$ the center of $\mathcal{M}$ and $\mathcal{J}\subset\mathcal{M}$ a closed ideal in $\mathcal{M}$. For every $x\in\mathcal{M}$, $x\notin\mathcal{Z}+\mathcal{J}$ there are: an invertible element $u\in\mathcal{M}$, two equivalent orthogonal projections $e_1,e_2\in\mathcal{M}$, $e_1\notin\mathcal{J}\not\ni e_2$, and an element $y\in e_1\mathcal{M}e_2$ such that, putting $x_0=u^{-1}xu$, we have:*

(i) $x_0e_1=e_2x_0e_1,\ yx_0e_1=e_1,\ x_0y=e_2$;

(ii) *for every projection $e_1'\leqq e_1$ there is an equivalent projection $e_2'\leqq e_2$ and an element $y'\in e_1'\mathcal{M}e_2'$ such that*

$$x_0e_1'=e_2'x_0e_1',\quad y'x_0e_1'=e_1',\quad x_0y'=e_2'.$$

Proof. Suppose the W^*-algebra $\tilde{\mathcal{M}}$ is realized as a von Neumann algebra acting on the Hilbert space $\mathfrak{H}$, $\tilde{\mathcal{M}}\subset\mathcal{B}(\mathfrak{H})$ and put $\tilde{\mathcal{J}}=\pi^{-1}(\mathcal{J})$. Since the center of $\mathcal{M}/\mathcal{J}$ is the canonical image of $\mathcal{Z}$ and $x\notin\mathcal{Z}+\mathcal{J}$, there is a projection $p\in\mathcal{M}$ such that $(1-p)xp\notin\mathcal{J}$. Let $\tilde{x}\in\tilde{\mathcal{M}}$ and let $\tilde{p}\in\tilde{\mathcal{M}}$ be a projection such that $\pi(\tilde{x})=x$, $\pi(\tilde{p})=p$ and put $\tilde{a}=(\tilde{1}-\tilde{p})\tilde{x}\tilde{p}$, $|\tilde{a}|=\sqrt{\tilde{a}^*\tilde{a}}$. Since the support of $|\tilde{a}|$ is smaller than $\tilde{p}$ and $|\tilde{a}|\notin\tilde{\mathcal{J}}$, by using the spectral theorem we get a spectral projection $\tilde{e}\notin\tilde{\mathcal{J}}$, $\tilde{e}\leqq\tilde{p}$ such that

$$\|\tilde{a}\tilde{e}\xi\|\geqq\lambda\|\tilde{e}\xi\|;\quad \xi\in\mathcal{H};\quad \lambda>0.$$

In particular, for $\xi\in\mathfrak{H}$:

$$\|\tilde{x}\tilde{e}\xi\|\geqq\|(1-\tilde{e})\tilde{x}\tilde{e}\xi\|\geqq\|(1-\tilde{p})\tilde{x}\tilde{e}\xi\|=\|\tilde{a}\tilde{e}\xi\|\geqq\lambda\|\tilde{e}\xi\|.$$

We have $\tilde{e}=RP(\tilde{x}\tilde{e})$ and we put $\tilde{f}=LP(\tilde{x}\tilde{e})$. Then $\tilde{f}(\mathcal{H})=\overline{\tilde{x}\tilde{e}(\mathfrak{H})}=\tilde{x}\tilde{e}(\mathfrak{H})$ and $\|(1-\tilde{e})\tilde{x}\tilde{e}\xi\|\geqq(\lambda/\|\tilde{x}\|)\,\|\tilde{x}\tilde{e}\xi\|$, $\xi\in\mathfrak{H}$; that is $\tilde{f}-\tilde{f}\tilde{e}\tilde{f}\geqq(\lambda/\|\tilde{x}\|)^2\tilde{f}$. Hence $\tilde{e}\wedge\tilde{f}=0$ and $(\tilde{e}\vee\tilde{f})(\mathfrak{H})=\tilde{e}(\mathcal{H})+\tilde{f}(\mathcal{H})$, by Lemma 1.

The operator

$$(\tilde{1}-\tilde{e})\tilde{f}\colon\ \tilde{f}(\mathfrak{H})\to(\tilde{1}-\tilde{e})\tilde{f}(\mathfrak{H})$$

is invertible. We put $\tilde{e}_1=\tilde{e}$, $\tilde{e}_2=LP\big((\tilde{1}-\tilde{e})f\big)$ and we note that $\tilde{f}=RP\big((\tilde{1}-\tilde{e})\tilde{f}\big)$. Let $(\tilde{1}-\tilde{e})\tilde{f}=\tilde{w}\,|(\tilde{1}-\tilde{e})\tilde{f}|$ be the polar decomposition of $(\tilde{1}-\tilde{e})\tilde{f}$ and let $\tilde{g}\in\tilde{f}\tilde{\mathcal{M}}\tilde{f}$ be the inverse of $|(\tilde{1}-\tilde{e})\tilde{f}|$ in $\tilde{f}\tilde{\mathcal{M}}\tilde{f}$. Then the operator

$$\tilde{g}\tilde{w}^*\colon\ \tilde{e}_2(\mathfrak{H})\to\tilde{f}(\mathfrak{H})$$

is the inverse of the operator:

$$(\tilde{1}-\tilde{e})\tilde{f}:\ \tilde{f}(\mathfrak{H}) \to \tilde{e}_2(\mathfrak{H}).$$

Now define $\tilde{u}=\tilde{g}\tilde{w}^*\tilde{e}_2+(\tilde{1}-\tilde{e}_2)\in\tilde{\mathcal{M}}$. The operator $\tilde{u}$ is one-to-one and $\tilde{u}(\mathfrak{H})=$ $=\tilde{f}(\mathfrak{H})+\tilde{e}(\mathfrak{H})+(\tilde{1}-\tilde{e}_1-\tilde{e}_2)(\mathfrak{H})$. Since $(\tilde{e}\vee\tilde{f})(\mathfrak{H})=\tilde{e}(\mathfrak{H})+\tilde{f}(\mathfrak{H})$ it follows that $\tilde{u}(\mathfrak{H})=\mathfrak{H}$, whence $\tilde{u}$ is invertible and $\tilde{u}^{-1}\in\tilde{\mathcal{M}}$ by the closed graph theorem.

We put $\tilde{x}_0=\tilde{u}^{-1}\tilde{x}\tilde{u}$. The operator

$$\tilde{x}\tilde{e}:\ \tilde{e}(\mathfrak{H}) \to \tilde{f}(\mathfrak{H})$$

is invertible, thus so is the operator

$$\tilde{x}_0\tilde{e}_1:\ \tilde{e}_1(\mathfrak{H}) \to \tilde{e}_2(\mathfrak{H})$$

as well as any operator:

$$\tilde{x}_0\tilde{e}_1':\ \tilde{e}_1'(\mathfrak{H}) \to \tilde{e}_2'(\mathfrak{H})$$

where $\tilde{e}_1'\leqq\tilde{e}_1$ is a projection and $\tilde{e}_2'=LP(\tilde{x}_0\tilde{e}_1')\leqq\tilde{e}_2$. It follows that $\tilde{x}_0\tilde{e}_1'=\tilde{e}_2'\tilde{x}_0\tilde{e}_1'$ and that there is an element $\tilde{y}'\in\tilde{e}_1'\tilde{M}\tilde{e}_2'$, such that $\tilde{y}'\tilde{x}_0\tilde{e}_1'=\tilde{e}_1'$, $\tilde{x}_0\tilde{y}'=\tilde{e}_2'$. By the "$LP\sim RP$" theorem we have $\tilde{e}_1=\tilde{e}\sim\tilde{f}\sim\tilde{e}_2$ and $\tilde{e}_1'\sim\tilde{e}_2'$. Furthermore, since $\tilde{e}\notin\tilde{\mathcal{J}}$, we also have $\tilde{e}_1\notin\tilde{\mathcal{J}}\not\ni\tilde{e}_2$.

Putting $u=\pi(\tilde{u})$, $e_1=\pi(\tilde{e}_1)$, $e_2=\pi(\tilde{e}_2)$ and $y=\pi(\tilde{y})$ ($\tilde{y}$ is the corresponding $\tilde{y}'$ for $\tilde{e}_1'=\tilde{e}_1$) we obtain (i). If $e_1'\leqq e_1$ is a projection then there is a projection $\tilde{e}_1'\leqq\tilde{e}_1$ such that $\pi(\tilde{e}_1')=e_1'$ and (ii) follows. Q.E.D.

3. Remark. Brown and Halmos proved that for every $0\neq x\in\mathcal{B}(\mathfrak{H})$, $\dim\mathfrak{H}<\infty$, there is a nilpotent element $b\in\mathcal{B}(\mathfrak{H})$ such that $x+b$ is invertible (cf. the proof of Theorem C in [3]). The first step of their proof consists of finding an element in $\mathcal{B}(\mathfrak{H})$ similar to x and with a suitable matrix form; this suggested us Theorem 2. The second step of their proof is as follows. Let x be an operator $(n\times n)$-matrix of the "suitable form" $x=(x_{ij})$ with $x_{j,1}=0$ for $j<n$ and $x_{n,1}$ invertible. Consider the matrix $b=(b_{ij})$ with $b_{i,i+1}=1-x_{i,i+1}$, $b_{i,j}=-x_{i,j}$ for $j>i+1$ and $b_{i,j}=0$ for $i\geqq j$. For $n=3$ the picture is as follows

$$x=\begin{pmatrix}0 & x_{12} & x_{13}\\ 0 & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\end{pmatrix},\quad b=\begin{pmatrix}0 & 1-x_{12} & -x_{13}\\ 0 & 0 & 1-x_{23}\\ 0 & 0 & 0\end{pmatrix},\quad x+b=\begin{pmatrix}0 & 1 & 0\\ 0 & x_{22} & 1\\ x_{31} & x_{32} & x_{33}\end{pmatrix}.$$

It is obvious that b is nilpotent and $x+b$ is invertible.

Now, consider a C^*-algebra $\mathcal{M}$ with unit such that there are n mutually equivalent, mutually orthogonal projections $e_1,\dots,e_n$ in $\mathcal{M}$ such that $e_1+\dots+e_n=1$. Then every element $x\in\mathcal{M}$ can be represented as an operator $(n\times n)$-matrix whose components are in $e_1\mathcal{M}e_1$. Namely, let v_i be an element of $\mathcal{M}$, such that $v_i^*v_i=e_1$,

$v_i v_i^* = e_i$, $v_i = v_i e_1 = e_i v_i$, for $i = 1, \ldots, n$. Put $x_{ij} = v_i^* x v_j \in e_1 \mathcal{M} e_1$. Then:

$$x = \sum_{i,j} e_i x e_j = \sum_{i,j} v_i x_{ij} v_j^* = (x_{ij})$$

where the last equality is a notation. We say that x_{ij} is the (i, j)-th component in the matrix representation of x with respect to the "basis" $(e_1, \ldots, e_n)$. It is easy to see that $(x^*)_{i,j} = x_{j,i}^*$ and $(xy)_{i,j} = \sum_k x_{ik} y_{kj}$. In particular, if x has a "suitable" matrix representation then there is a nilpotent element b in $\mathcal{M}$ such that $x+b$ is invertible.

The method just explained and Theorem 2 allows us to settle affirmatively the conjecture of DYER, PORCELLI and ROSENFELD.

4. Corollary. *Let $\mathcal{M}$ be a finite factor and $0 \neq x \in \mathcal{M}$. There is a nilpotent element $b \in \mathcal{M}$ such that $x+b$ is invertible. In particular $\sigma(x+b) \cap \sigma(b) = \emptyset$.*

Proof. For factors of type I_n, $n < \infty$ the result is known. So, let $\mathcal{M}$ be a factor of type II_1 and denote by d its relative dimension function $(d(1)=1)$. If x is a scalar element, then we may take $b=0$. If not, x is not a central element. Since it suffices to prove the assertion of the corollary for an element similar (in $\mathcal{M}$) to x, from Theorem 2 it follows that we can suppose that there are: a positive integer n, two equivalent orthogonal projections $\bar{e}_1, \bar{e}_2 \in \mathcal{M}$ and an element $y \in \bar{e}_1 \mathcal{M} \bar{e}_2$ such that: $d(\bar{e}_1) = d(\bar{e}_2) = 1/n$, $x\bar{e}_1 = \bar{e}_2 x \bar{e}_1$, $yx\bar{e}_1 = \bar{e}_1$, $xy = \bar{e}_2$. Let $e_1, \ldots, e_n \in \mathcal{M}$ be mutually equivalent, mutually orthogonal projections with $e_1 + \ldots + e_n = 1$ and $e_1 = \bar{e}_1$, $e_n = \bar{e}_2$; then $xe_1 = e_n x e_1$, $y \in e_1 \mathcal{M} e_n$, $yxe_1 = e_1$, $xy = e_n$. The matrix representation of x with respect to the "basis" $(e_1, \ldots, e_n)$ is a "suitable" one. Indeed, for $j \neq n$:

$$x_{j,1} = v_j^* x v_1 = v_j^* e_j x e_1 v_1 = v_j^* e_j e_n x e_1 v_1 = 0$$

and $x_{n,1}$ is invertible in $e_1 \mathcal{M} e_1$ with the inverse $y_{1,n}$:

$$y_{1,n} x_{n,1} = (v_1^* y v_n)(v_n^* x v_1) = v_1^* y e_n x v_1 = v_1^* y x v_1 = v_1^* e_1 v_1 = e_1$$

Hence the corollary follows from Remark 3. Q.E.D.

We can also extend the result obtained in [3] for properly infinite factors to properly infinite "*C^*-factors*". In the following corollary "*large*" projections are those which are equivalent to 1 and "*small*" projections are those which are not equivalent to 1.

5. Corollary. *Let $\pi: \tilde{\mathcal{M}} \to \mathcal{M}$ be a representation of the properly infinite W^*-algebra $\tilde{\mathcal{M}}$ on the C^*-algebra $\mathcal{M}$ whose center reduces to the scalar elements. Then $\mathcal{M}$ has a greatest ideal $\mathcal{J}$ which is generated by the small projections in $\mathcal{M}$, and an element $x \in \mathcal{M}$ belongs to $\mathcal{J}$ iff $\sigma(x+b) \cap \sigma(b) \neq \emptyset$ for every $b \in \mathcal{M}$. Moreover, if x is not* "thin" *(i.e. x is not of the form $\lambda + a$ where $\lambda \in \mathbf{C}$ and $a \in \mathcal{J}$), then there is a nilpotent element $b \in \mathcal{M}$ such that $x+b$ is invertible.*

Proof. It is well known that in a properly infinite factor the small projections form a p-ideal and since any factor has a greatest ideal this is generated by the small projections. It is easy to imitate the above argument in the present situation and so, $\mathcal{M}$ has a greatest ideal $\mathcal{J}$ and $\mathcal{J}$ is generated by the small projections.

Since the spectral theorem holds in $\mathcal{M}$ and in $\mathcal{M}$ there are "large" and "small" projections, HALMOS's proof for the infinite dimensional case of the theorem of DYER, PORCELLI and ROSENFELD applies and so, $x \in \mathcal{J}$ iff $\sigma(x+b) \cap \sigma(b) \neq \emptyset$ for every $b \in \mathcal{M}$.

Now suppose $x \in \mathcal{M}$ and x is not "thin". By Theorem 2 we can suppose that there are two orthogonal equivalent projections $\bar{e}_1, \bar{e}_2 \in \mathcal{M}, \bar{e}_1 \notin \mathcal{J} \not\ni \bar{e}_2$ and an element $y \in \bar{e}_1 \mathcal{M} \bar{e}_2$ such that $x\bar{e}_1 = \bar{e}_2 x \bar{e}_1$, $yx\bar{e}_1 = \bar{e}_1$, $xy = \bar{e}_2$. Since $1-\bar{e}_1 \geqq \bar{e}_2 \sim 1$ and $\bar{e}_1 \sim 1 = \bar{e}_1 + (1-\bar{e}_1)$ there are two orthogonal large projection $\bar{e}_1'$, $\bar{e}''$ whose sum is $\bar{e}_1$. Again by Theorem 2, we find a projection $\bar{e}_2' \leqq \bar{e}_2$ and an element $y' \in \bar{e}_1' \mathcal{M} \bar{e}_2'$ such that $x\bar{e}_1' = \bar{e}_2' x \bar{e}_1'$, $y' x \bar{e}_1' = \bar{e}_1'$, $xy' = \bar{e}_2'$. We put $e_1 = \bar{e}_1'$, $e_2 = 1 - \bar{e}_1' - \bar{e}_2'$, $e_3 = \bar{e}_2'$. Then $\{e_1, e_2, e_3\}$ is a family of mutually orthogonal, mutually equivalent (large) projections, $e_1 + e_2 + e_3 = 1$, and the matrix representation of x with respect to this basis has the following properties: $x_{1,1} = 0 = x_{2,1}$ and $x_{3,1}$ is invertible in $e_1 \mathcal{M} e_1$. Hence the last assertion of the corollary follows from Remark 3. Q.E.D.

We note that the matrix representation of the not "thin" element x, obtained in the preceding proof, obviously implies a theorem of BROWN and PEARCY ([1], Theorem 2). So the proof of the commutator theorem may be shortened even in the case of a properly infinite C^*-factor (cf. also [4]).

We have also obtained an extension, to general W^*-algebras, of the theorem of DYER, PORCELLI and ROSENFELD, giving a spectral characterization of the strong radical ([5]).

References

[1] A. BROWN and C. PEARCY, Structure of commutators of operators, *Annals of Math.*, **82** (1965), 112—127.

[2] J. DIXMIER, *Les C^*-algèbres et leurs représentations*, Gauthier Villars (Paris, 1964).

[3] J. A. DYER, P. PORCELLI and M. ROSENFELD, Spectral characterization of two sided ideals in $\mathcal{B}(\mathcal{H})$, *Israel J. Math.*, **10** (1971), 26—31.

[4] H. HALPERN, Commutators in properly infinite von Neumann algebras, *Trans. Amer. Math. Soc.*, **139** (1969), 55—73.

[5] Ş. STRĂTILĂ and L. ZSIDÓ, An algebraic reduction theory for W^*-algebras. II *(to be published)*

[6] I. KAPLANSKY, *Rings of operators*, Benjamin (1968).

(Received February 21, 1972)

Fourier effective methods of summation

By GEN-ICHIRÔ SUNOUCHI in Sendai (Japan)

1. Let the Fourier expansion of $f(x)\in L(-\pi,\pi)$ be

$$f(x)\sim\sum_{n=-\infty}^{\infty}c_n e^{inx}=\sum_{n=0}^{\infty}a_n(x).$$

We consider now summation of the series at a given point x. The summation behaviour of this series at a point x is reduced to properties of the cosine expansion

$$\varphi(t)\sim\sum_{n=0}^{\infty}a_n\cos nt$$

at $t=0$, where

$$\varphi(t)=\{f(x+t)+f(x-t)\},\quad a_n=a_n(x).$$

By F_C we denote the class of all series $\sum a_n\cos nt$ for which $\varphi(t)$ is continuous at $t=0$, and by F_L the class of all series $\sum a_n\cos nt$ for which $t=0$ is a Lebesgue point of $\varphi(t)$, i.e.

$$\int_0^h|\varphi(t)-\varphi(0)|\,dt=o(h),\quad (0<h\to 0).$$

W. B. JURKAT and A. PEYERIMHOFF [3] considered general summation methods $B=(b_{n\nu})$ in the series to sequence form satisfying

$$b_{n\nu}\to 1\quad (n\to\infty,\ \nu\text{ fixed}),\quad b_{n\nu}\to 0\quad (n\text{ fixed},\ \nu\to\infty).$$

$$\sum_{\nu=0}^{\infty}b_{n\nu}a_\nu=\sigma_n(\varphi)\quad (C,1),$$

which means summable by the first Cesàro method. They called this the applicability condition. If for a method B satisfying the applicability condition

$$\sigma_n(\varphi)\to\varphi(0)\quad (n\to\infty)$$

for all φ corresponding to series F_C, respectively F_L, then we call the method B F_C-effective, respectively F_L-effective. Concerning F_C-effectiveness, they proved the following theorems.

Theorem A. *A method $B=(b_{n\nu})$ with the applicability property is F_C-effective if and only if*

$$\frac{1}{2}b_{n0}+\sum_{\nu=1}^{\infty} b_{n\nu}\cos \nu t \quad (n=0,1,\ldots)$$

are the cosine expansions of functions (which are called kernels) $b_n(t)\in L(0,\pi)$ satisfying for every δ in $0<\delta<\pi$,

(i) $$\operatorname*{ess\,sup}_{\delta\leqq t\leqq\pi}|b_n(t)|\leqq M_\delta \quad (n=0,1,\ldots),$$

(ii) $$\int_\delta^\pi b_n(t)\,dt\to 0,\quad \frac{2}{\pi}\int_0^\pi b_n(t)\,dt\to 1 \quad (n\to\infty),$$

(iii) $$\int_0^\pi |b_n(t)|\,dt\leqq M \quad (n=0,1,\ldots).$$

Theorem B. *Let $\sum a_\nu$ be summable to the same s by all F_C-effective methods B. Then the series $\sum a_\nu \cos \nu t$ is the cosine expansion of a function $\varphi(t)\in L(0,\pi)$ which is continuous at $t=0$. In other words, the intersection of summability fields of all F_C-effective methods is F_C.*

In the present note, we will give the complete analogues of Theorems A and B for F_L-effectiveness.

2. Theorem 1. *A method $B=(b_{n\nu})$ with the applicability property is F_L-effective if and only if*

$$\frac{1}{2}b_{n0}+\sum_{\nu=1}^{\infty} b_{n\nu}\cos \nu t \quad (n=0,1,\ldots)$$

are the cosine expansions of functions $b_n(t)\in L(0,\pi)$ satisfying for every δ $(0<\delta<\pi)$

(i) $$\operatorname*{ess\,sup}_{\delta\leqq t\leqq\pi}|b_n(t)|\leqq M_\delta \quad (n=0,1,\ldots),$$

(ii) $$\int_\delta^\pi b_n(t)\,dt\to 0,\quad \frac{2}{\pi}\int_0^\pi b_n(t)\,dt\to 1 \quad (n\to\infty),$$

(iii) $$\int_0^\pi m_n(t)\,dt\leqq M,\quad \text{where}\quad m_n(t)=\operatorname*{ess\,sup}_{t\leqq u\leqq\pi}|b_n(u)|.$$

In other words, the kernel $b_n(t)$ has hump-backed majorants with uniformly bounded integrals.

Proof. Since F_L-effectiveness implies F_C-effectiveness, $(b_{n\nu})$ has to satisfy the condition of Theorem A. We write the kernel $b_n(t)$ as

$$b_n(t)\sim\frac{1}{2}b_{n0}+\sum_{\nu=1}^{\infty} b_{n\nu}\cos \nu t \quad (n=0,1,\ldots).$$

If we can write

$$\sigma_n(\varphi) = \frac{2}{\pi}\int_0^\pi \varphi(t) b_n(t)\,dt \tag{1}$$

as a Lebesgue integral, a condition for F_L-effectiveness was given by D. FADDEEFF [2], see also S. G. KREIN—B. JA. LEVIN [4] and K. TANDORI [5]. The exposition is also given in ALEXITS' book [1].

For the representation (1), we proceed with Tandori's idea. Without loss of generality we can suppose $\varphi(0)=0$. Let us denote by L_0 the class of all functions $\varphi(t)\in L(0,\pi)$ satisfying $\varphi(0)=0$ and

$$\int_0^h |\varphi(t)|\,dt = o(h) \quad (0<h\to 0).$$

Then Tandori proved that with the norm

$$\|\varphi\|_0 = \sup_{0<h\leqq\pi}\left\{\frac{1}{h}\int_0^h |\varphi(t)|\,dt\right\}$$

L_0 is the Banach space. For any fixed n, $\sigma_n(\varphi)$ is evidently a linear functional on L_0. We consider all functions which belong to $L(0,\pi)$ and vanish near the origin. This class is a subspace of L_0 and denoted by L_0^*. For any $\varphi\in L_0^*$

$$\sigma_n(\varphi) = \frac{2}{\pi}\int_0^\pi \varphi(t) b_n(t)\,dt.$$

In particular for any fixed n we take

$$\int_{\pi 2^{-m-1}}^{\pi 2^{-m}} |\varphi_m(t)|\,dt = 1,$$

$$\int_{\pi 2^{-m-1}}^{\pi 2^{-m}} \varphi_m(t) b_n(t)\,dt \geqq \operatorname*{ess\,sup}_{\pi 2^{-m-1}\leqq t\leqq \pi 2^{-m}} |b_n(t)| - \frac{\varepsilon}{\pi}$$

for any given $\varepsilon>0$ and set

$$\varphi^*(t) = \pi 2^{-m}\varphi_m(t) \quad \text{in} \quad (\pi 2^{-m-1}, \pi 2^{-m}) \quad (m=0,1,\dots),$$

$$\varphi^*(t) = 0 \quad \text{in} \quad (0, \pi 2^{-N}) \quad \text{for some} \quad N>m+1.$$

If we take $\pi 2^{-k-1}<h\leqq \pi 2^{-k}$ $(N>k+1)$, then

$$\frac{1}{h}\int_0^h |\varphi^*(t)|\,dt \leqq \frac{2^{k+1}}{\pi}\int_{\pi 2^{-N}}^{\pi 2^{-k}} |\varphi^*(t)|\,dt = \frac{2^{k+1}}{\pi}\sum_{m=k}^{N-1}\frac{\pi}{2^m}\int_{\pi 2^{-m-1}}^{\pi 2^{-m}} |\varphi_m(t)|\,dt =$$

$$= 2^{k+1}\sum_{m=k}^{N-1} 2^{-m} = 2^{k+1}(2^{-k+1}-2^{-N+1}) = 4-2^{k-N+2} \leqq 4.$$

So $\|\varphi^*\|_0 \leqq 4$. On the other hand,

$$\sigma_n(\varphi^*) = \frac{2}{\pi}\int_0^\pi 2\varphi^*(t) b_n(t) \geqq \sum_{m=0}^{N} \frac{\frac{2}{\pi}}{2^m} \operatorname*{ess\,sup}_{\pi 2^{-m-1} \leqq t \leqq \pi 2^{-m}} |b_n(t)| - \varepsilon.$$

Hence we get

$$\sup_{\|\varphi^*\|_0 \leqq 1} |\sigma_n(\varphi^*)| \geqq \frac{1}{2} \sum_{m=0}^{N} \frac{1}{2^m} \operatorname*{ess\,sup}_{\pi 2^{-m-1} \leqq t \leqq \pi 2^{-m}} |b_n(t)|$$

and we have

$$\sum_{m=0}^{\infty} \frac{1}{2^m} \operatorname*{ess\,sup}_{\pi 2^{-m-1} \leqq t \leqq \pi 2^{-m}} |b_n(t)| \sim \int_0^\pi \{\operatorname*{ess\,sup}_{t \leqq u \leqq \pi} |b_n(u)|\}\, dt$$

is finite for any fixed n. Thus the integral

$$\int_0^\pi |\varphi(t) b_n(t)|\, dt \leqq \sum_{m=0}^{\infty} \left\{ 2^m \int_{\pi 2^{-m-1}}^{\pi 2^{-m}} |\varphi(t)|\, dt \right\} \left\{ \frac{1}{2^m} \operatorname*{ess\,sup}_{\pi 2^{-m-1} \leqq t \leqq \pi 2^{-m}} |b_n(t)| \right\}$$

exists in the Lebesgue sense for any $\varphi \in L_0$ as Tandori shows. We get the representation (1) by extension from L_0^* to L_0 and the conclusion is given by Faddeeff's theorem.

Theorem 2. *Let $\sum a_\nu$ be summable to the same s by all F_L-effective method B. Then $\sum a_\nu \cos \nu t$ is the cosine expansion of a function $\varphi \in L(0, \pi)$ which has $t=0$ as its Lebesgue point, i. e.*

$$\int_0^h |\varphi(t) - s|\, dt = o(h) \quad (0 < h \to 0).$$

In other words, the intersection of summability fields of all F_L-effective methods is F_L.

Proof. Fix the series

$$\frac{1}{2} a_0 + \sum_{\nu=0}^{\infty} a_\nu$$

and consider only kernels $b_n(t) \in C[0, \pi]$. By the same idea as in W. B. Jurkat and A. Peyerimhoff [3] we can prove that there exists a function $\varphi(t) \in L(0, \pi)$ such that

$$\frac{2}{\pi} \int_0^\pi \cos \nu t\, \varphi(t)\, dt = a_\nu \quad (\nu = 0, 1, 2, \ldots).$$

Hence for every bounded F_L-effective kernel $b_n(t)$ by Parseval's relation

$$\sigma_n(\varphi) = \frac{2}{\pi} \int_0^\pi b_n(t) \varphi(t)\, dt.$$

Next we have to show that

$$\int_0^h |\varphi(t)-s|\,dt = o(h) \quad (0<h\to 0). \tag{2}$$

We can suppose $s=0$. If (2) fails, we may assume that some $\varepsilon>0$ and $h_k\to 0$ exist such that

$$\frac{1}{h_k}\int_0^{h_k} |\varphi(t)|\,dt > \varepsilon.$$

Set
$E_k=[0,h_k]$, $E_k^+=\{t|0\leqq t\leqq h_k,\ \varphi(t)\geqq 0\}$, and $E_k^-=\{t|0\leqq t\leqq h_k,\ \varphi(t)<0\}$, then

$$E_k^+\cup E_k^-=E_k,\quad E_k^+\cap E_k^-=\varphi,\quad \text{and}\quad |E_k^+|+|E_k^-|=h_k.$$

We select a subsequence $\{n_k\}$ such that

$$\alpha=\lim_{n_k\to\infty} |E_{n_k}^-|/|E_{n_k}^+|$$

exists $(0\leqq\alpha\leqq\infty)$. Let us set

$$\Psi_{E_{n_k}}(t)=\operatorname{sign}\varphi(t)\quad\text{for}\quad t\in E_{n_k},\quad\text{and}\quad \Psi_{E_{n_k}}=0\quad\text{for}\quad t\notin E_{n_k},$$

and

$$d_{n_k}(t)=\frac{\pi}{2}\Psi_{E_{n_k}}(t)/(Ch_{n_k})$$

where C will be determined soon. Then,

$$\frac{2}{\pi}\int_0^\pi d_{n_k}(t)\,dt=\frac{2}{\pi}\int_0^{h_{n_k}} d_{n_k}(t)\,dt=\frac{1}{Ch_{n_k}}\int_0^{h_{n_k}}\Psi_{E_{n_k}}(t)\,dt$$

$$=\frac{|E_{n_k}^+|-|E_{n_k}^-|}{Ch_{n_k}}=\frac{1}{C}\frac{|E_{n_k}^+|-|E_{n_k}^-|}{|E_{n_k}^+|+|E_{n_k}^-|}\to\frac{1}{C}\frac{1-\alpha}{1+\alpha}\qquad(0\leqq\alpha\leqq\infty).$$

If $\alpha\neq 1$, we set $\dfrac{1}{C}=\dfrac{1+\alpha}{1-\alpha}$; then $d_{n_k}(t)$ satisfies (i) and (ii). The integral of the hump-backed majorant is

$$\frac{2}{\pi}\frac{\pi}{2}\int_0^\pi \frac{\chi_{E_{n_k}}(t)}{|C|h_{n_k}}\,dt=\frac{h_{n_k}}{|C|h_{n_k}}=\left|\frac{1+\alpha}{1-\alpha}\right|<\infty.$$

However,

$$\left|\frac{2}{\pi}\int_0^\pi \varphi(t)\,d_{n_k}(t)\,dt\right|=$$

$$=\frac{1}{|C|h_{n_k}}\int_0^{h_{n_k}}\varphi(t)\operatorname{sign}\varphi(t)\,dt=\frac{1}{|C|}\frac{1}{h_{n_k}}\int_0^{h_{n_k}}|\varphi(t)|\,dt>\frac{\varepsilon}{|C|}.$$

Now we approximate $d_{n_k}(t)$ by continuous $b_{n_k}(t)$ and obtain a contradiction.

When $\alpha=1$, the absolute values of both

$$\frac{1}{h_{n_k}}\int_0^{h_{n_k}} \varphi^+(t)\,dt \quad\text{and}\quad \frac{1}{h_{n_k}}\int_0^{h_{n_k}} \varphi^-(t)\,dt$$

are greater than $\varepsilon/2$, where $\varphi^+(t)$ and $\varphi^-(t)$ are the positive and negative parts of $\varphi(t)$ for large n_k. Since

$$|E_{n_k}^-|/|E_{n_k}^+| \to 1 \quad (n_k \to \infty, h_{n_k} \to 0),$$

the function

$$d_{n_k}(t) = \pi\chi_{E_{n_k}^+}(t)/h_{n_k}$$

satisfies conditions (i) and (ii) of Theorem 1. The integral of the hump-backed majorant is smaller than

$$\frac{2}{\pi}\pi\int_0^{h_{n_k}} \frac{1}{h_{n_k}}\,dt \leqq 2.$$

However, we also have

$$\frac{2}{\pi}\int_0^{\pi} \varphi(t)\,d_{n_k}(t)\,dt = \frac{2}{\pi}\pi\int_0^{\pi} \frac{\chi_{E_{n_k}^+}(t)}{h_{n_k}}\varphi(t)\,dt = \frac{2}{h_{n_k}}\int_0^{h_{n_k}} \varphi^+(t)\,dt > \varepsilon,$$

which is a contradiction. Hence we proved the theorem completely.

References

[1] G. Alexits, *Convergence problems of orthogonal series*, Pergamon Press (1961).

[2] D. K. Faddeeff, Sur la représentation des fonctions sommables au moyen d'intégrales singulières, *Mat. Sbornik*, **1** (1936), 351—368.

[3] W. B. Jurkat and A. Peyerimhoff, Fourier effectiveness and order summability, *J. Approximation Theory*, **4** (1971), 231—244.

[4] S. G. Krein and B. Ja. Levin, On the strong representation of functions by singular integrals, *Doklady Akad. Nauk USSR*, **60** (1948), 195—198. (Russian.)

[5] K. Tandori, Über die Konvergenz singulärer Integrale, *Acta Sci. Math.*, **15** (1954), 223—230.

MATHEMATICAL INSTITUTE
TOHOKU UNIVERSITY
SENDAI, JAPAN

(Received November 5, 1972)

On an ergodic type theorem for von Neumann algebras

By S. M. ABDALLA and J. SZŰCS in Szeged

Introduction

The main purpose of this paper is to give a new proof for a theorem of KOVÁCS and SZŰCS [4] concerning the ergodic behaviour of elements in a von Neumann algebra under certain groups of its automorphisms. We shall even point out that a more general result is true for semi-groups of normal endomorphisms instead of groups of automorphisms. In the original proof the Alaoglu—Birkhoff ergodic theorem played a key role, while in our present paper we shall use the Ryll-Nardzewski fixed point theorem [5]. We remark that a different proof also using Ryll-Nardzewski's fixed point theorem is given in [6].

Preliminaries

Let $\mathfrak{H}$ be a complex Hilbert space and denote by $\mathscr{B}(\mathfrak{H})$ the algebra of all bounded linear operators on $\mathfrak{H}$. Among the most often used topologies of $\mathscr{B}(\mathfrak{H})$ are the ultra-strong and ultra-weak topologies. The ultra-strong and the ultra-weak topologies are defined by the semi-norms of the form $T \to \left(\sum_{i=1}^{\infty} \|Tx_i\|^2\right)^{\frac{1}{2}}$, $T \to \left|\sum_{i=1}^{\infty} (Tx_i, x_i)\right|$, respectively, and where $x_i \in \mathfrak{H}$ and $\sum_{i=1}^{\infty} \|x_i\|^2 < +\infty$.

J. DIXMIER has proved [1] that every ultra-strongly continuous linear form is a linear combination of functionals of the form

$$T \to \sum_{i=1}^{\infty} (Tx_i, x_i), \quad \text{where} \quad x_i \in \mathfrak{H}, \; \sum_{i=1}^{\infty} \|x_i\|^2 < +\infty. \; ^{1)}$$

[1]) In particular, we can see that every ultra-strongly continuous linear form is ultra-weakly continuous, as well. Since the ultra-strong operator topology is obviously finer than the ultra-weak one, the converse of this is evident.

A subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathfrak{H})$ that is closed with respect to taking adjoint, contains the identity operator and is closed with respect to one (and then with respect to both)[2]) of the topologies just discussed is called a von Neumann algebra.[3])

By an endomorphism of a von Neumann algebra $\mathscr{A}$ we mean a mapping g of $\mathscr{A}$ into itself that is linear, multiplicative and adjoint preserving. We shall deal with ultra-weakly continuous endomorphisms only. The ultra-weakly continuous endomorphisms can be called normal[4]) endomorphisms, too, relying on a theorem of [2] (Chap. I, § 4. Th. 2, p. 56). In the sequel we shall do so, for the modifier "normal" is shorter than the modifier "ultra-weakly continuous". By the way, we shall never use the above mentioned theorem in our paper, however, the proof of part (iV) of Theorem 2 in [4], which is referred to in our present work, uses a generalization of it.

Let G now be a semi-group of normal endomorphisms of $\mathscr{A}$ and consider an arbitrary but fixed element T of $\mathscr{A}$. Denote by $\mathscr{K}_0(T, G)$ the convex hull of the set of all elements of the form $g(T)$ $(g \in G)$. Let $\mathscr{K}(T, G)$ denote the ultra-strong (and then the ultra-weak) closure of $\mathscr{K}_0(T, G)$. Furthermore, denote by $\mathscr{A}^G$ the set of all elements of $\mathscr{A}$ which are invariant with respect to all elements of G.[5]) Let us denote by $\mathscr{R}(\mathscr{A}, G)$ the set of all ultra-weakly continuous linear forms on $\mathscr{A}$ that are invariant with respect to G. We shall denote by $\mathscr{R}^+(\mathscr{A}, G)$ the positive portion of $\mathscr{R}(\mathscr{A}, G)$.

We shall use in our study the Ryll-Nardzewski fixed point theorem [5]. For the comfort of the reader we state this theorem as a lemma.

Lemma. *Let K be a non-empty weakly compact convex subset of a locally convex Hausdorff space E and let G be a non-contracting*[6]) *semi-group of weakly continuous affine maps of K into itself. Then there exists a common fixed point of the elements of G.*

The following definition of G-finiteness generalizes the one given in [4].

---

[2]) From the preceding footnote and from the separation theorem of convex sets ([3], 14.4, p. 119) it follows that every ultra-strongly closed convex subset of $\mathscr{B}(\mathfrak{H})$ is ultra-weakly closed, as well.

[3]) For the theory of von Neumann algebras we refer the reader to [2].

[4]) An endomorphism, or more generally an order preserving positive mapping g of a von Neumann algebra $\mathscr{A}$ into another von Neumann algebra $\mathscr{B}$ is said to be normal if $g(\sup \mathscr{F}) = \sup g(\mathscr{F})$ for any upward dierected bounded subset $\mathscr{F}$ of the positive portion of $\mathscr{A}$.

[5]) In general, $\mathscr{A}^G$ is not a von Neumann algebra but there exists a maximal (orthogonal) projection P in $\mathscr{A}^G$ such that $\mathscr{A}^G|P\mathfrak{H}$ is a von Neumann algebra.

[6]) By definition, G is non-contracting if for any two distinct elements x and y of K there exists a strongly continuous semi-norm p on E (depending on x and y) such that $\inf \{p(gx-gy): g \in G\} > 0$.

Definition. Let $\mathscr{A}$ be a von Neumann algebra and consider a semi-group G of normal endomorphisms of $\mathscr{A}$. The algebra $\mathscr{A}$ is said to be *G-finite* if for every non-zero element T of $\mathscr{A}^+$[7]) there exists an element σ of $\mathscr{R}^+(\mathscr{A}, G)$ such that $\sigma(T) \neq 0$.[8])

The theorems

KOVÁCS and SZŰCS [4] proved the following:[9])

Theorem 1. *Let $\mathscr{A}$ be a von Neumann algebra and consider a semi-group G of its normal endomorphisms. Suppose that $\mathscr{A}$ is G-finite. Then for every element T of $\mathscr{A}$ the set $\mathscr{K}(T, G) \cap \mathscr{A}^G$ consists of exactly one element.*

Proof. The von Neumann algebra $\mathscr{A}$ with the ultra-strong operator topology is a locally convex Hausdorff space. By Dixmier's result cited in Preliminaries, the weak topology of this locally convex space coincides with the ultra-weak operator topology. It is a well-known and easily provable fact that the unit ball of $\mathscr{B}(\mathfrak{H})$ is ultra-weakly compact. This implies that $\mathscr{K}(T, G)$ is compact in the ultra-weak operator topology for every element T of $\mathscr{A}$. For every $g \in G$ we obviously have $g(\mathscr{K}_0(T, G)) \subseteqq \mathscr{K}_0(T, G)$ and then by the ultra-weak continuity of the elements of G we have $g(\mathscr{K}(T, G)) \subseteqq \mathscr{K}(T, G)$. Ryll-Nardzewski's theorem shows that to prove $\mathscr{K}(T, G) \cap \mathscr{A}^G \neq \emptyset$ for any $T \in \mathscr{A}$ it is enough to show that G is non-contracting on every $\mathscr{K}(T, G)$ in the ultra-strong operator topology. To verify this, fix an element T of $\mathscr{A}$ and consider two distinct members A and B of $\mathscr{K}(T, G)$. From the G-finiteness of $\mathscr{A}$ there follows the existence of an element σ of $\mathscr{R}^+(\mathscr{A}, G)$ such that $\sigma((A-B)^*(A-B)) \neq 0$. For every element S of $\mathscr{A}$ put $p(S) = [\sigma(S^*S)]^{\frac{1}{2}}$. It is easy to see that p is a semi-norm on $\mathscr{A}$. Furthermore, for every element g of G, we have

$$p^2(g(A)-g(B)) = p^2(g(A-B)) = \sigma(g(A-B)^*(g(A-B))) =$$
$$= \sigma(g(A-B)^*(A-B)) = \sigma((A-B)^*(A-B)).$$

This shows that $\inf \{p(g(A)-g(B)) : g \in G\} > 0$. We shall show that p is ultra-strongly continuous. In fact, consider a net $\{S_\alpha\}$ of elements of $\mathscr{A}$ that tends to 0 in the ultra-strong topology. Then, by the definitions of the ultra-strong and ultra-weak topologies, $S_\alpha^* S_\alpha$ tends to 0 in the ultra-weak topology. Hence $p(S_\alpha) = [\sigma(S_\alpha^* S_\alpha)]^{\frac{1}{2}}$ tends to 0 wich shows that p is ultra-strongly continuous. Summarizing all our investigations,

[7]) $\mathscr{A}^+$ denotes the positive portion of $\mathscr{A}$.

[8]) If $\mathscr{A}$ as G-finite, then it is easy to see that $g(I) = I$ for every element g of G. Therefore in this case $\mathscr{A}^G$ is a von Neumann algebra (see the footnote on p. 000).

[9]) They supposed that G was a group.

the Ryll-Nardzewski fixed point theorem applies to every $\mathcal{K}(T, G)$ and to the semi-group G, so $\mathcal{K}(T, G)\cap\mathcal{A}^G\neq\emptyset$ for every element T of $\mathcal{A}$.

To accomplish the proof of Theorem 1 we have to show that $\mathcal{K}(T, G)\cap\mathcal{A}^G$ has only one element. To this effect denote by Q the set of all linear maps of $\mathcal{A}$ into itself of the form $\sum_{i=1}^{n}\alpha_i g_i$ $\left(\alpha_i>0,\ \sum_{i=1}^{n}\alpha_i=1,\ g_i\in G\right)$. Consider a fixed element T of $\mathcal{A}$ and suppose that S and R are two distinct elements of $\mathcal{K}(T, G)\cap\mathcal{A}^G$. Since $S\in\mathcal{K}(T, G)$, there exists a net $\{g_\alpha\}$ of elements of Q such that $\lim_\alpha g_\alpha(T)=S$ where the limit is taken in the ultra-weak topology. For every element σ of $\mathcal{R}^+(\mathcal{A}, G)$ we have

$$\sigma\big((S-R)^*S\big) = \lim_\alpha \sigma\big((S-R)^*g_\alpha(T)\big) = \lim_\alpha \sigma\big(g_\alpha((S-R)^*T)\big) = \sigma\big((S-R)^*T\big).$$

Similarly, for R in place of S we have

$$\sigma\big((S-R)^*R\big) = \sigma\big((S-R)^*T\big).$$

By subtraction we obtain

$$\sigma\big((S-R)^*(S-R)\big) = 0.$$

Since σ was an arbitrary element of $\mathcal{R}^+(\mathcal{A}, G)$, the G-finiteness of $\mathcal{A}$ implies that $S=R$. This completes the proof of Theorem 1.

In accordance with [4] let us denote the unique element of $K(T, G)\cap\mathcal{A}^G$ by T^G.

Relying on the previous theorem, KOVÁCS and SZŰCS [4] proved the following result stated for groups of automorphisms only.

Theorem 2. *Let $\mathcal{A}$ be a von Neumann algebra in a complex Hilbert space $\mathfrak{H}$ and let G be a semi-group of normal endomorphisms of $\mathcal{A}$. Suppose that $\mathcal{A}$ is G-finite. Then the mapping $T\to T^G$ possesses the following properties:*

(i) $\sigma(T)=\sigma(T^G)$ *for every* $\sigma\in\mathcal{R}(\mathcal{A}, G)$ *and* $T\in\mathcal{A}$;

(ii) $T\to T^G$ *is linear and strictly positive;*

(iii) $(ST)^G=ST^G$ *and* $(TS)^G=T^GS$ *for* $T\in\mathcal{A}$, $S\in\mathcal{A}^G$;

(iv) $T\to T^G$ *is ultra-weakly and ultra-strongly continuous;*

(v) $T=T^G$ *for every* $T\in\mathcal{A}^G$;

(vi) $\big(g(T)\big)^G=T^G$ *for every* $T\in\mathcal{A}$ *and* $g\in G$.

Conversely, if we do not suppose that $\mathcal{A}$ is G-finite but we know that there exists an ultra-weakly continuous strictly[10]) *positive linear mapping $T\to T'$ of $\mathcal{A}$ onto $\mathcal{A}^G$ such that*

[10]) In [4] the assumption of strictness does not occur.

a) *$T=T'$ for every $T\in\mathcal{A}^G$;*

b) *$(g(T))'=T'$ for every $T\in\mathcal{A}$, $g\in G$,*

then $\mathcal{A}$ is necessarily G-finite and $T'=T^G$ for every $T\in\mathcal{A}$.

Relying on our Theorem 1, in the more general situation of semi-groups of normal endomorphisms properties (i)—(vi) of the so-called G-canonical mapping $T\to T^G$ can be proved in the same way as they were in [4] with a minor modification in the proof of (vi) except for the first statement in property (ii) which asserts that the mapping $T\to T^G$ is linear. The proof of this fact in [4] relies not only on Theorem 1 of [4] but on its proof as well. Now we are going to show the linearity of the G-canonical mapping in the more general situation when G is a semi-group of normal endomorphisms of $\mathcal{A}$.

In fact, suppose that $\mathcal{A}$ is G-finite and use the notations of Theorem 1. Consider two elements, R and S, of $\mathcal{A}$. Since the G-canonical map is obviously homogeneous, it is enough to show that $(R+S)^G=R^G+S^G$. Since $(R+S)^G\in\mathcal{K}(R+S, G)$ we can find a net $\{g_\alpha\}$ of elements of Q such that

$$(R+S)^G = \text{ultra-weak}\lim_\alpha g_\alpha(R+S). \tag{1}$$

Since $\mathcal{K}(R, G)$ is ultra-weakly compact, we can find a subnet $\{h_\beta\}$ of the net $\{g_\alpha\}$ such that $h_\beta(R)$ is convergent in the ultra-weak topology. Then (1) shows that $h_\beta(S)$ is ultra-weakly convergent, too. Put $R_0=\lim_\beta h_\beta(R)$ and $S_0=\lim_\beta h_\beta(S)$. Then we have

$$R_0\in\mathcal{K}(R, G),\quad S_0\in\mathcal{K}(S, G)\quad\text{and}\quad (R+S)^G = R_0+S_0.$$

The fact $R_0\in\mathcal{K}(R, G)$ implies that $\mathcal{K}(R_0, G)\subseteq\mathcal{K}(R, G)$ and so, by uniqueness, $R_0^G=R^G$. Similarly, $S_0^G=S^G$. Choose a net $\{k_\gamma\}$ of elements of Q such that ultra-weak $\lim_\gamma k_\gamma(R_0)=R^G$. Then we have

$$k_\gamma(S_0) = k_\gamma((R+S)^G-R_0) = (R+S)^G-k_\gamma(R_0)$$

which shows that $k_\gamma(S_0)$ is convergent in the ultra-weak topology, too. Put $\lim_\gamma k_\gamma(S_0)=S_1$. Then we have

$$(R+S)^G = R^G+S_1,\quad S_1\in\mathcal{K}(S, G).$$

The fact $S_1\in\mathcal{K}(S, G)$ implies that $S_1^G=S^G$. Choose a net $\{l_\delta\}$ of elements of Q such that ultra-weak $\lim_\delta l_\delta(S_1)=S^G$. Then we have

$$(R+S)^G = \lim_\delta l_\delta((R+S)^G) = \lim_\delta l_\delta(R^G)+\lim_\delta l_\delta(S_1) = R^G+S^G.$$

This completes the proof of the linearity of the G-canonical mapping.

As far as the rest of Theorem 2 is concerned, in the case of semi-groups of normal endomorphisms we have to modify the proof of [4] in the following way:

Suppose that $\mathscr{A}$ admits an ultra-weakly continuous strictly positive linear mapping $T\to T'$ having properties a) and b) of Theorem 2. Consider an arbitrary non-vanishing element S of $\mathscr{A}^+$. It follows that S' is a non-vanishing positive element of $\mathscr{A}^G$. Then put $T_0=S'$ and define σ as in [4]. We have $\sigma\in\mathscr{R}^+(\mathscr{A}, G)$ and $\sigma(S)=\sigma(S')=$ $=\sigma(T_0)\neq 0$. Since S was an arbitrary non-vanishing element of $\mathscr{A}^+$, this shows that $\mathscr{A}$ is G-finite.

The equation $T'=T^G$ $(T\in\mathscr{A})$ can be proved in the same way as in [4].

We are now going to conclude with an example of a von Neumann algebra $\mathscr{A}$ and a cyclic semi-group $G=\{g^n\}_{n=1}^{\infty}$ of its normal endomorphisms such that $\mathscr{A}$ is G-finite and g is not an automorphism. In fact, let $\mathscr{A}$ be the von Neumann algebra of all multiplication operators generated by essentially bounded Lebesgue measurable functions on the complex Hilbert space $L^2[0, 1]$ and let g be the endomorphism of $\mathscr{A}$ generated by the point transformation $T: x\to 2x \pmod 1$ in the following way: $[g(f)](x)=f(Tx)$ $(f\in\mathscr{A})$ (here we identified the elements of $\mathscr{A}$ with any of the functions which generate them). It is immediate that g is normal, $\mathscr{A}$ is G-finite and g is not an automorphism. In this case the G-canonical mapping of $\mathscr{A}$ reduces to the mapping $f\to\left(\int_0^1 f(t)\,dt\right)e$ where e denotes the constant 1 function on [0, 1].

References

[1] J. Dixmier, Formes linéaires sur un anneau d'opérateurs, *Bull. Soc. Math. France,* **81** (1953), 9—39.

[2] J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien. Algèbres de von Neumann* (Paris, 1957).

[3] J. L. Kelley and J. Namioka, *Linear topological spaces* (Toronto, New York, London, 1963).

[4] I. Kovács and J. Szűcs, Ergodic type theorems in von Neumann algebras, *Acta Sci. Math.,* **27** (1966), 233—246.

[5] C. Ryll-Nardzewski, On fixed points of semigroups of endomorphisms of linear spaces, *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability, Berkeley, Calif. 1965/66,* vol. II, *Contributions to Probability Theory,* part I, Univ. of California Press (Berkeley, Cal., 1967), pp. 55—61.

[6] Kazuyuki Saitô, Automorphism groups of von Neumann algebras and ergodic type theorems, *Acta Sci. Math.,* **36** (1974), 119—130.

(Received May 30, 1973)

Об интегральном модуле непрерывности

Н. ТЕМИРГАЛИЕВ и П. Л. УЛЬЯНОВ (Москва, СССР)

Пусть 2π-периодическая функция $\varphi(t)\in L^p(0, 2\pi)$ при некотором $p\in[1, \infty)$, тогда

$$\omega_p(\delta; \varphi) = \sup_{|h|\leqq\delta}\left\{\int_0^{2\pi} |\varphi(t+h)-\varphi(t)|^p\,dt\right\}^{\frac{1}{p}} \quad (0 \leqq \delta \leqq 2\pi)$$

называют модулем непрерывности (в L^p) от φ.

Для случая непериодических функций $f(t)\in L^p(a, b)$, где $-\infty\leqq a<b\leqq+\infty$ модуль непрерывности определяется так:

$$\omega_p(\delta; f) = \sup_{0\leqq h\leqq\delta}\left\{\int_a^{b-h} |f(t+h)-f(t)|^p\,dt\right\}^{\frac{1}{p}} \quad (0 \leqq \delta \leqq 1),$$

где $b-h=\infty$, если $b=\infty$.

Отметим, что если в условиях утверждения (теоремы, леммы и т.п.) говорится о функциях из $L^p(a, b)$ (из $L^p(0, 2\pi)$), то подразумевается, что речь идет о непериодических (соответственно периодических) функциях, а встречающийся в утверждении (или же в его доказательстве) знак $\|f\|_p$ понимается в смысле нормы, т.е.

$$\|f\|_p \equiv \|f\|_{L^p(a,b)} \equiv \left\{\int_a^b |f(t)|^p\,dt\right\}^{\frac{1}{p}}.$$

Далее, всюду ниже через $|E|$ мы будем обозначать меру Лебега измеримого множества E.

Для доказательства точных теорем вложения H_p^ω (определение класса H_p^ω см. [5], стр. 649) в пространство L^ν существенно были иcпoльзованы (в случае $p=1$) одним из авторов (см. [5] и [6]) следующие установленные им утверждения о модулях непрерывности:

Лемма 1 (см. [6], стр. 108). *Если функция* $f(t)\in L(-\infty, \infty)$, *то*

$$\omega_1\left(\frac{1}{n}; f\right) \geqq \frac{1}{9} \sup_{\substack{E\subset(-\infty,\infty)\\ |E|\leqq\frac{1}{n}}} \int_E |f(t)|\,dt \quad (n = 1, 2, \ldots). \tag{1}$$

Лемма 2 (см. [6], стр. 107). *Если функция* $f(t)\in L(0, 1)$ *то*

$$(2)\qquad \omega_1\left(\frac{1}{n}; f\right) \geqq \frac{1}{9}\Big\{\sup_{\substack{E\subset[0,1]\\ |E|\leqq\frac{1}{n}}} \int_E |f(t)|\,dt - \inf_{\substack{E\subset[0,1]\\ |E|\geqq\frac{1}{n}}} \int_E |f(t)|\,dt\Big\} \quad (n = 1, 2, \ldots).$$

Пусть $F(z, |f|)$—невозрастающая на (0, 1) равноизмерим аяс $|f(t)|$ функция (см. [5], стр. 655). Тогда справедлива

Лемма 3 (см. [5], стр. 656). *Если неотрицательная функция* $f(t)\in L(0, 1)$, *а* $\alpha\in[0, 1]$—*некоторое число, то*

$$\text{а)}\qquad \sup_{\substack{M\subset[0,1]\\ |M|=\alpha}} \int_M f(t)\,dt = \int_0^\alpha F(z, f)\,dz,$$

$$\text{б)}\qquad \inf_{\substack{M\subset[0,1]\\ |M|=\alpha}} \int_M f(t)\,dt = \int_{1-\alpha}^1 F(z, f)\,dz.$$

Впоследствии стало известно, что в этом направлении уже имелись некоторые результаты о модулях непрерывности. Именно, пусть функция $f(t)\in L^p(0, 2\pi)$ при некотором $p\in[1, \infty)$ и f неэквивалентна постоянной, тогда

$$\int_a^{a+h} |f(t)|^p\,dt \leqq C_{f,p}\frac{\omega_p^p(h; f)}{h^{p-1}} \quad (-\infty < a < +\infty,\ 0 < h \leqq 2\pi),$$

где постоянная $C_{f,p}$ зависит только от f и p.

Это неравенство при $p=1$ было доказано Е. Хиллом и Дж. Клейном (см. [1]), а при $p>1$ — Идзуми (см. [2]).

В 1961 году венгерским математиком Й. Ципсером (см. [3]) при тех же условиях на функцию $f(t)$ было сформулировано следующее обобщение теоремы Хилла—Клейна—Идзуми:

$$(3)\qquad \sup_{\substack{E\subset[0,2\pi]\\ |E|=h}} \int_E |f(t)|^p\,dt \leqq C_{f,p}\,\omega_p(h; f) \quad (0 < h \leqq 2\pi).$$

Аналогичное неравенство сформулировано Й. Ципсером и для случая, когда $f(t)\in L^p(-\infty, \infty)$.

Однако доказательство Ципсера последних неравенство ошибочно, ибо он в своих рассуждениях существенно пользуется равенством:

$$(4)\qquad \int_0^h F(t)\,dt - \int_0^h (F(x+t)\,dt = \int_0^h [F(t) - F(t+h)]\,dt,$$

где $F(x)=\int_E |f(x+u)|^p du$, а измеримое множество E и переменная x взяты из области определения суммируемой в p-ой степени функции $f(u)$ и $|E|=h$.

Убедимся, что как в периодическом, так и в непериодическом случаях равенство (4) ошибочно. В самом деле, в силу аддитивности интеграла Лебега, соотношение (4) тождественно равенству

$$\int_0^h F(x+h)\,dt = \int_0^h F(t+h)\,dt. \tag{5}$$

Пусть $f(t)=t^{1/p}$ при $t\in[0, 2\pi)$. Тогда для числа $h\in(0, \pi)$ и множества $E=[0, h]$ имеем

$$F(x) = \int_0^h (x+t)\,dt = xh+\frac{h^2}{2} \quad (x\in[0, 2\pi-h]).$$

Отсюда

$$\int_0^h F(x+t)\,dt = xh^2+h^3 \not\equiv 2h^3 = \int_0^h F(h+t)\,dt \quad (0<h<\pi,\ 0<x<2\pi-h),$$

что противоречит (5).

Продолжая $f(t)$ с полуотрезка $[0, 2\pi)$ на всю ось периодически с периодом 2π в периодическом случае, и нулем в непериодическом случае, получаем, что равенство (4) действительно не имеет место.

Более того, если $f(t)\in L^p(-\infty, \infty)$, то из равенства (5) вытекает (при $x\to\infty$), что

$$\int_0^h F(h+t)\,dt = 0$$

и потому (опять в силу (5) и неотрицательности F) функция $F(x)\equiv 0$, т.е. $f(t)=0$ почти всюду.

Целью настоящей заметки является, с одной стороны, доказать, что неравенства вида (3) являются следствием лемм 1, 2 и 3 и, с другой стороны, сделать ряд замечаний по рассматриваемому вопросу (и в том числе по вопросу взаимосвязи модулей непрерывности функции f и ее равноизмеримой функции φ (см. об этом теорему 3)). Ниже нам понадобится следующее утверждение: если $f(t)\in L^p(a, b)$, то

$$\omega_1(\delta; |f|^p) \leqq p\cdot 2^{p-1}\|f\|_p^{p-1}\omega_p(\delta; f) \quad (0\leqq\delta\leqq 1). \tag{6}$$

Действительно, так как по теореме Лагранжа

$$|x^p-y^p| \leqq p(x+y)^{p-1}|x-y| \quad (p\geqq 1,\ x\geqq 0,\ y\geqq 0),$$

и

$$|x+y|^p \leqq 2^{p-1}(x^p+y^p) \quad (x\geqq 0,\ y\geqq 0),$$

то

$$\omega_1(\delta; |f|^p) = \sup_{0\le h\le\delta} \int_a^{b-h} \left||f(t+h)|^p - |f(t)|^p\right| dt \le$$

$$\le \sup_{0\le h\le\delta} p \int_a^{b-h} \left(|f(t+h)|+|f(t)|\right)^{p-1} \left||f(t+h)|-|f(t)|\right| dt \le$$

$$\le \sup_{0\le h\le\delta} p \left\{\int_a^b [|f(t+h)|+|f(t)|]^p\, dt\right\}^{\frac{p-1}{p}} \left\{\int_a^{b-h} \left||f(t+h)|-|f(t)|\right|^p dt\right\}^{\frac{1}{p}} \le$$

$$\le p2^{p-1}\|f\|_p^{p-1}\omega_p(\delta; f).$$

Перейдем к доказательству теорем.

Теорема 1. *Если* $f(t)\in L^p(-\infty, \infty)$ *при некотором* $p\in[1, \infty)$, *то*

$$\sup_{\substack{E\subset(-\infty,\infty)\\ |E|=h}} \int_E |f(t)|^p\, dt \le 18p2^{p-1}\|f\|_p^{p-1}\omega_p(h; f) \quad (0 \le h \le 1).$$

Доказательство. Действительно, пусть

(7) $$h\in\left(\frac{1}{n+1}, \frac{1}{n}\right] \quad (n = 1, 2, \dots).$$

В силу леммы 1 (см. также (6) и (7)) имеем:

$$\sup_{\substack{E\subset(-\infty,\infty)\\ |E|=h}} \int_E |f(t)|^p\, dt \le \sup_{\substack{E\subset(-\infty,\infty)\\ |E|=\frac{1}{n}}} \int_E |f(t)|^p\, dt \le$$

$$\le 9\omega_1\left(\frac{1}{n}; |f|^p\right) \le 18\omega_1\left(\frac{1}{n+1}; |f|^p\right) \le$$

$$\le 18\omega_1(h; |f|^p) \le 18p2^{p-1}\|f\|_p^{p-1}\omega_p(h; f),$$

что и требовалось доказать.

Теорема 2. *Если существенно неограниченная функция* $f(t)\in L^p(0, 1)$ *при некотором* $p\in[1, \infty)$ *и* $c>6$ — *произвольное, но фиксированное число, то найдется построянная* $h_0\equiv h_0(c, f, p)>0$, *для которой*

(8) $$\sup_{\substack{E\subset[0,1]\\ |E|=h}} \int_E |f(t)|^p\, dt \le cp2^{p-1}\|f\|_p^{p-1}\omega_p(h; f) \quad \text{при всех} \quad h\in[0, h_0].$$

Доказательство. В силу леммы 3 имеем

$$\sup_{\substack{E\subset[0,1]\\ |E|=h}} \int_E |f(t)|^p\, dt = \int_0^h F(z; |f|^p)\, dz, \qquad \inf_{\substack{E\subset[0,1]\\ |E|=h}} \int_E |f(t)|^p\, dt = \int_{1-h}^1 F(z; |f|^p)\, dz,$$

где $F(z; |f|^p)$ — невозрастающая на [0, 1] равноизмеримая с $|f(t)|^p$ функция.

Так как $f(t)$ существенно неограничена, то отсюда вытекает существование положительной постоянной $h_0 \equiv h_0(c, f, p)$ такой, что справедливо неравенство

$$\int_{1-h}^{1} F(z; |f|^p)\,dz \leqq \left(1-\frac{6}{c}\right)\int_0^h F(z; |f|^p)\,dz \quad (0 \leqq h \leqq h_0), \tag{9}$$

то есть

$$\frac{6}{c}\sup_{\substack{E\subset[0,1]\\|E|=h}}\int_E |f(t)|^p\,dt \leqq \sup_{\substack{E\subset[0,1]\\|E|=h}}\int_E |f(t)|^p\,dt - \inf_{\substack{E\subset[0,1]\\|E|=h}}\int_E |f(t)|^p\,dt \quad (0 \leqq h \leqq h_0). \tag{9'}$$

Пусть число n $(n=1, 2, \ldots)$ выбрано так, что

$$h \in \left(\frac{1}{n+1}, \frac{1}{n}\right]. \tag{10}$$

Имеем*) (см. (9′), (10) и (6)) при $h \in (0, h_0]$

$$\frac{6}{c}\sup_{\substack{E\subset[0,1]\\|E|=h}}\int_E |f(t)|^p\,dt \leqq \frac{6}{c}\sup_{\substack{E\subset[0,1]\\|E|=\frac{1}{n}}}\int_E |f(t)|^p\,dt \leqq$$

$$\leqq \sup_{\substack{E\subset[0,1]\\|E|=\frac{1}{n}}}\int_E |f(t)|^p\,dt - \inf_{\substack{E\subset[0,1]\\|E|=\frac{1}{n}}}\int_E |f(t)|^p\,dt \leqq 3\omega_1\left(\frac{1}{n}; |f|^p\right) \leqq$$

$$\leqq 6\omega_1\left(\frac{1}{n+1}; |f|^p\right) \leqq 6\omega_1(h; |f|^p) \leqq 6p - 2^{p-1}\|f\|_p^{p-1}\omega_p(h; f),$$

то есть неравенство (8) доказано.

Для случая ограниченных функций справедливо утверждение:

Если измеримая функция f существенно ограничена на отрезке [0, 1] и не эквивалентна постоянной, то

$$\sup_{\substack{E\subset[0,1]\\|E|=h}}\int_E |f(t)|^p\,dt \leqq c(f, p)\,\omega_p(h, f) \quad \text{при всех} \quad h \in [0, h_0], \tag{8'}$$

где постоянная $C(f, p)$ не зависит от $h \in [0, h_0]$.

Доказательство этого утверждения ведется совершенно также, как доказательство теоремы 2, только в неравенстве (9) постоянную $1-6/C$ следует заменить постоянной $1-\varepsilon(f, p)$, где $\varepsilon(f, p)$ достаточно малое положительное число (неравенство будет оставаться справедливым и для этого случая в силу того, что функция f не эквивалентна постоянной).

*) Здесь мы также пользуемся, данным в работе [4] К. И. Осколкова и С. А. Теляковского, следующим уточнеинем леммы 2: а неравенстве (2) вместо 1/9 можно поставить 1/3.

Значение постоянной $C(f, p)$ в неравенстве (8′) можно конкретизировать. Например, из оценок Ципсера (см. неравенства (2.9) и (3.1) в работе [3]) вытекает:

Если $f(t)$ существенно ограниченая на $[0, 2\pi]$ и не эквивалентная постоянной измеримая функция и $p\in[1, \infty)$—некоторое фиксированное число, то

$$\sup_{\substack{E\subset[0,2\pi]\\ |E|=h}} \int_E |f(t)|^p\,dt \leqq \sup_{t\in[0,2\pi]}\operatorname{vrai}|f(t)|^p \frac{8\pi}{\|f-m(f)\|_p}\,\omega_p(f,h) \quad (0<h\leqq 1),$$

где

$$m(f)=\frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx.$$

Из оценок (8) и (8′) непосредственно получаем:

Если функция $f\in L^p(0, 1)$ при некотором $p\in[1, \infty)$ и $f(t)$ не эквивалентна постоянной, то

$$(8'') \qquad \sup_{\substack{E\subset[0,1]\\ |E|=h}} \int_E |f(t)|^p\,dt \leqq c(f,p)\,\omega_p(h;f) \quad \text{при всех} \quad h\in[0, 1],$$

где $C(f, p)$ не зависит от $h\in[0, 1]$.

Замечание 1. Рассматривая линейную на $(0, 1)$ функцию нетрудно убедиться, что неравенство (8″) в смысле порядка неулучшаемо.

Замечание 2. Пусть $\varphi(t)$ неотрицательная, невозрастающая на $(0, 1)$ функция и $\varphi(t)\in L^p(0, 1)$ при некотором $p\in(1, \infty)$. Тогда

$$(11) \qquad \omega_p^p(\delta;\varphi) \leqq \int_0^\delta \varphi^p(t)\,dt \quad \text{при всех} \quad 0\leqq\delta\leqq 1.$$

Доказательство. Имеем

$$\int_0^{1-h} [\varphi(t)-\varphi(t+h)]^p\,dt \leqq \int_0^{1-h} \varphi^{p-1}(t)[\varphi(t)-\varphi(t+h)]\,dt \leqq$$

$$\leqq \int_0^{1-h} \varphi^p(t)\,dt - \int_0^{1-h} \varphi^p(t+h)\,dt = \int_0^h \varphi^p(t)\,dt - \int_{1-h}^1 \varphi^p(t)\,dt,$$

откуда и следует справедливость (11).

Отметим, что неравенство (11) неусиляемо. Чтобы убедиться в этом, достаточно рассмотреть функцию

$$\varphi(t)=\begin{cases} 1, & t\in\left[0, \frac{1}{2}\right] \\ 0, & t\in\left(\frac{1}{2}, 1\right]. \end{cases}$$

Замечание 3. Справедлива следующая

Теорема 3. *Если функция* $f \in L^p(0, 1)$ *при некотором* $p \in (1, \infty)$, *а* $\varphi(t)$ — *невозрастающая на* $(0, 1)$ *и равноизмеримая с* $|f(t)|$ *функция, то*

$$\omega_p^p(\delta; \varphi) \leqq c_{f,p}\omega_p(\delta; f) \tag{12}$$

при всех $\delta \in [0, 1]$, *где положительная постоянная* $C_{f,p}$ *зависит только от функции* f *и* p.

Доказательство следует из неравенств (8″) и (11) (см. также лемму 3). Любопытно отметить, что неравенство (12) является следствием двух неулучшаемых в смысле порядка неравенств (8″) и (11) (см. замечания 1 и 2).

Вопрос же о неусиляемости самого неравенства (12) связан с вопросом, ранее поставленным одним из авторов (см. [5], стр. 659): верно ли неравенство

$$\omega_p(\delta; f) \geqq c_p\omega_p(\delta; \varphi) \quad (c_p > 0,\ 1 \leqq p < \infty,\ 0 < \delta \leqq \delta_0(f) < 1), \tag{13}$$

где $\varphi(x)$ — функция, невозрастающая и равноизмеримая с неотрицательной функцей $f(t) \in L^p(0, 1)$; если же это неравенство верно, то каково наибольшее значение постоянной c_p?

Более того, неизвестно, будет-ли нижний предел положителен для каждой функции $f \in L^p(0, 1)$

$$\liminf_{\delta \to +0} \frac{\omega_p(\delta; f)}{\omega_p(\delta; \varphi)}$$

Одним из авторов (см. [5], стр. 652—658) была установлена справедливость неравенства (13) для $p=1$, а К. И. Осколков и С. А. Теляковский (см. [4]) показали, что в оценке

$$\omega_1\left(f; \frac{1}{n}\right) \geqq c\left\{\sup_{\substack{E \subset [0,1] \\ |E| = \frac{1}{n}}} \int_E |f(t)|\,dt - \inf_{\substack{E \subset [0,1] \\ |E| = \frac{1}{n}}} \int_E |f(t)|\,dt\right\} \quad (n = 1, 2, \ldots)$$

постоянная c заключена между 1/3 и 5/7.

Замечание 4. В работе [6] (см. стр. 114) было установлено, что если для достаточно малых δ справедлива оценка

$$\omega_1(\delta; f) \leqq c\delta \log\frac{1}{\delta}, \tag{14}$$

где $c < \frac{1}{9}$, то

$$\int_0^1 \exp|f|\,dt < \infty, \tag{15}$$

а при $c=1$ этом вывод уже сделать нельзя.

Здесь же был поставлен вопрос: для каких C, удовлетворяющих условию $1/9 \leqq c < 1$ из (14) следует (15)?

Из результатов К. И. Осколкова и С. А. Теляковского (см. [4]) следует, что из справедливости для функции $f(t)$ неравенства (14) с $c<1/3$ следует (15).

Для c, удовлетворяющих условиям $1/3 \leqq C < 1$, вопрос остается открытым.

В целом остается открытым вопрос о необходимом и достаточном условии на модуль непрерывности $\omega(\delta)$, чтобы имело место вложение

$$H_p^{\omega(\delta)} \subset e^L \quad (1 \leqq p < \infty)$$

(см. об этом [7], стр. 673).

Замечание 5. Нетрудно убедиться, что неравенства вида (3), (8), (8′) и (8″) не могут дать неулучшаемых предельных теорем вложения при $p>1$ (например, для случая $H_p^\omega \subset L^\nu$ с $1<p<\nu<\infty$ (см. [5] и [6])).

Литература

[1] E. Hille—G. Klein, Riemann's localization theorem for Fourier series, *Duke Math. J.*, **21** (1954), 587—591.

[2] S. Izumi, Some trigonometrical series. XIV, *Proc. Japan Acad.*, **31** (1955), 324—326.

[3] J. Czipszer, Sur le module de continuité intégrale, *Magyar Tud. Akad. Math. Kutató Int. Közl.*, **6** (1961), 393—398.

[4] К. И. Осколков—С. А. Теляковский, К оценкам П. Л. Ульянова для интегральных модулей непрерывности, *Изв. АН Арм. ССР*, серия математика, VI, № **5** (1971), 406—411.

[5] П. Л. Ульянов, Вложение некоторых классов функций H_p^ω, *Изв. АН СССР*, серия матем., **32**, № 3 (1968), 649—686.

[6] П. Л. Ульянов, Теоремы вложения и соотношения между наилучшими приближениями (модулями непрерывности) в разных метриках, *Матем. сборник*, **81** (123), № 1 (1970), 104—131.

[7] P. Uljanov, Allgemeine Entwicklungen und gemischte Fragen, *Actes Congrès Intern. Math. Nice*, **2** (1970), 667—678.

(Поступило 11/IV. 1973)

Sur les décompositions directes $C_0 - C_{11}$ des contractions

Par RADU I. TEODORESCU à Braşov (Roumanie)

Le but de cette Note est de donner une condition nécessaire et suffisante pour qu'une contraction complètement non-unitaire dont la fonction caractéristique admet un multiple scalaire, soit somme directe (pas nécessairement orthogonale) d'une contraction C_0 et d'une contraction de classe C_{11}.[1])

1. Soit $\{\mathfrak{E}, \mathfrak{E}_*; \Theta(\lambda)\}$ une fonction analytique contractive. On dit que $\Theta(\lambda)$ admet le multiple scalaire $\delta(\lambda)$, $\delta(\lambda)$ étant une fonction analytique scalaire, s'il existe une fonction analytique contractive $\{\mathfrak{E}_*, \mathfrak{E}, \Omega(\lambda)\}$ telle que

$$\Theta(\lambda)\Omega(\lambda) = \delta(\lambda)I_{\mathfrak{E}_*}, \quad \Omega(\lambda)\Theta(\lambda) = \delta(\lambda)I_{\mathfrak{E}}.$$

Rappelons que toute contraction complètement non-unitaire $T \in \mathscr{B}(H)$ admet des triangularisations de type

$$\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix} \quad \text{et} \quad \begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$$

par rapport aux décompositions

$$H = H_{0\cdot} \oplus H_{1\cdot} \quad \text{et} \quad H = H_{\cdot 1} \oplus H_{\cdot 0}.$$

On a donc en particulier $T|H_{0\cdot} \in C_{0\cdot}$ et $T|H_{\cdot 1} \in C_{\cdot 1}$. Dans le cas où la fonction caractéristique $\Theta(\lambda)$ de T admet un multiple scalaire, on a aussi

1) $T|H_{0\cdot} \in C_0, \quad T|H_{\cdot 1} \in C_{11}$;

2) $H_{0\cdot} = \{h \in H; m_{T_0}(T)h = 0\}$, $H_{\cdot 1} = \overline{m_{T_0}(T)H}$ où $m_{T_0}(\lambda)$ est la fonction minimale de $T_0 = T|H_{0\cdot}$;

3) $H_{0\cdot} \cap H_{\cdot 1} = \{0\}, \quad H = H_{0\cdot} \vee H_{\cdot 1}$;

4) Si L est un sous-espace de H, invariant à T et tel que $T|L \in C_0$ ou $T|L \in C_{11}$, alors $L \subset H_{0\cdot}$ ou $L \subset H_{\cdot 1}$, selon les cas.

[1]) Toutes les définitions et notations sont celles de [4].

Ces propriétés ont été démontrées dans [4] ch. VIII pour le cas où T est une contraction faible. Il est facile de vérifier que les démonstrations données là, conservent leurs validité pour toute contraction complètement non-unitaire T dont la fonction caractéristique admet un multiple scalaire. Pour cela nous ne répétons pas les démonstrations et nous utiliserons les mêmes notations que dans [4] ch. VIII. Notamment nous noterons, dans ce cas, les sous-espaces $H_{0\cdot}$ et $H_{\cdot 1}$ par H_0 et H_1 et nous les appellerons les sous-espaces C_0 et C_{11} de T.

Aussi nous utiliserons dans la suite le modèle fonctionnel (au sens de [4] ch. VI) de T, unitairement équivalent à T. Plus exactement, nous considérons l'opérateur T défini par

$$T^*(u \oplus v) = e^{-it}(u - u(0)) \oplus e^{-it} v$$

sur l'espace $H = K_+ \ominus G$ où $K_+ = H^2(\mathfrak{E}_*) \oplus \overline{\Delta L^2(\mathfrak{E})}$ et $G = \{\Theta w \oplus \Delta w : w \in H^2(\mathfrak{E})\}$.

De plus nous noterons $\Delta_*(t) = [I - \Theta(e^{it})\Theta^*(e^{it})]^{1/2}$ et rappelons que

$$\Theta(e^{it})\Delta(t) = \Delta_*(t)\Theta(e^{it}), \quad \Theta^*(e^{it})\Delta_*(t) = \Delta(t)\Theta^*(e^{it}).\ ^{2})$$

2. Pour le début nous déterminerons les espaces $H_{0\cdot}$ et $H_{\cdot 1}$. On sait que $H_{0\cdot} = \{u \oplus v \in H : T^n(u \oplus v) \to 0\}$, mais

$$T^n(u \oplus v) = P_+ U_+^n (u \oplus v) = (e^{int} u - \Theta w_n) \oplus (e^{int} v - \Delta w_n)$$

où $w_n \in H^2(\mathfrak{E})$ est déterminé par la condition

$$\text{(1)} \qquad e^{int}(\Theta^* u + \Delta v) - w_n \perp H^2(\mathfrak{E}).$$

Puisque $u \oplus v \in H$ nous aurons $\Theta^* u + \Delta v \perp H^2(\mathfrak{E})$. Soit

$$\Theta^* u + \Delta v = e^{-it} f_1 + e^{-2it} f_2 + \cdots + e^{-nit} f_n + \cdots .$$

De (1) on déduit que

$$w_n = f_n + e^{it} f_{n-1} + \cdots + e^{(n-1)it} f_1 = e^{nit}(e^{-it} f_1 + \cdots + e^{-nit} f_n)$$

donc $\|w_n\|^2 \to \|\Theta^* u + \Delta v\|^2$. D'autre part un calcul simple montre que

$$\|T^n(u \oplus v)\|^2 = \|u\|^2 + \|v\|^2 - \|w_n\|^2.$$

La condition $T^n(u \oplus v) \to 0$ étant équivalente à $\|w_n\|^2 \to \|u\|^2 + \|v\|^2$ nous avons obtenu l'équivalence $a \Leftrightarrow b$ dans la suivante

Proposition 1. *Soit $u \oplus v \in H$. Les conditions suivantes sont équivalentes:*

a) $u \oplus v \in H_{0\cdot}$; b) $\|u\|^2 + \|v\|^2 = \|\Theta^* u + \Delta v\|^2$; c) $\Theta v = \Delta_* u$.

---

[2]) Quand il n'y a pas de danger de confusion nous omettrons la variable.

Pour compléter la preuve observons que

$$\|u\|^2+\|v\|^2-\|\Theta^*u+\Delta v\|^2 = (u,u)+(v,v)-(\Theta^*u,\Theta^*u)-(\Theta^*u,\Delta v)-$$

$$-(\Delta v,\Theta^*u)-(\Delta v,\Delta v) = (u,u)+(v,v)-(\Theta\Theta^*u,u)-(\Delta\Theta^*u,v)-(\Theta\Delta v,u)-$$

$$-(\Delta^2 v,v) = (u,u)+(\Theta^*\Theta v,v)-(\Theta\Theta^*u,u)-(\Theta^*\Delta_* u,v)-(\Delta_*\Theta v,u) =$$

$$= (\Delta_*^2 u,u)+(\Theta v,\Theta v)-(\Delta_* u,\Theta v)-(\Theta v,\Delta_* u) = \|\Delta_* u-\Theta v\|^2$$

d'où l'équivalence $b \Leftrightarrow c$ est évidente.

Pour déterminer le sous-espace $H_{\cdot 1}$ partons de la relation

$$H_{\cdot 1} = H \ominus H_{\cdot 0} \tag{2}$$

dans laquelle

$$H_{\cdot 0} = \{u\oplus 0 \in H\} = \{u\oplus 0\colon\ u\in H^2(\mathfrak{E}_*),\ \Theta^*u\perp H^2(\mathfrak{E})\} =$$

$$= \{u\oplus 0\colon\ u\in H^2(\mathfrak{E}_*),\ u\perp \overline{\Theta H^2(\mathfrak{E})}\} = \{u\oplus 0\colon\ u\in H^2(\mathfrak{E}_*),\ u\perp \Theta_i H^2(\mathfrak{F})\}$$

où $\Theta(\lambda)=\Theta_i(\lambda)\Theta_e(\lambda)$ est la factorisation canonique avec $\{\mathfrak{E},\mathfrak{F},\Theta_e(\lambda)\}$ fonction extérieure et $\{\mathfrak{F},\mathfrak{E}_*,\Theta_i(\lambda)\}$ fonction intérieure. Ainsi nous pouvons affirmer la

Proposition 2. *Le sous-espace $H_{\cdot 1}$ de H est donné par*

$$H_{\cdot 1} = \{u\oplus v\in H\colon\ u\in \Theta_i H^2(\mathfrak{F})\}.$$

Proposition 3. *Soit T une contraction complètement non-unitaire dont la fonction caractéristique $\Theta(\lambda)$ admet un multiple scalaire. S'il existe des sous-espaces L_0 et L_1, invariants à T et tels que $T|L_0\in C_0$, $T|L_1\in C_{11}$ et*

$$H = L_0 \dotplus L_1, \tag{3}$$

alors $L_0=H_0$ et $L_1=H_1$.

Démonstration. De la propriété 4 de H_0 et H_1 il résulte que $L_0\subset H_0$ et $L_1\subset H_1$. Pour démontrer l'inclusion $H_0\subset L_0$ soit $h_0\in H_0$. En appliquant (3), h_0 se décompose sous la forme $h_0=l_0+l_1$ où $l_0\in L_0$ et $l_1\in L_1$; mais $T^n l_1=T^n h_0-T^n l_0\to 0$, donc $l_1=0$ et aussi $h_0=l_0\in L_0$ d'où $H_0\subset L_0$. Soit maintenant $h_1\in H_1$, $h_1=l_0'+l_1'$ où $l_0'\in L_0$ et $l_1'\in L_1$. Comme $L_0=H_0$, il résulte que $l_0'\in H_0$, donc $l_0'=h_1-l_1'\in H_0$. Mais comme $h_1-l_1'\in H_1$ et $H_0\cap H_1=\{0\}$ il résulte $l_0'=0$, d'où $h_1=l_1'$. Donc $H_1\subset L_1$.

Corollaire. *Soit T une contraction complètement non-unitaire dont la fonction caractéristique admet un multiple scalaire. S'il existe une décomposition de l'espace*

H en somme directe $H=L_0\dot{+}L_1$ telle que L_0 et L_1 soient des sous-espaces invariants à T et tels que $T|L_0\in C_0$ et $T|L_1\in C_{11}$, alors cette décomposition est unique.

3. Soit T une contraction complètement non-unitaire dont la fonction caractéristique admet un multiple scalaire; supposons que $H=H_0\dot{+}H_1$ et soit P la projection de H sur H_0 parallèle à H_1. Parce que H_0 et H_1 sont invariants à T et $H=H_0\dot{+}H_1$, il résulte que $TP=PT$; donc ([4], ch. II) il existe un opérateur lineaire borné $Y: K_+\to K_+$ tel que

$$\text{i) } YU_+ = U_+Y, \quad \|Y\| = \|P\|, \qquad \text{ii) } YG \subset G, \quad \text{iii) } P = P_+Y|H.$$

D'après [5], Y doit avoir la forme $Y=\begin{bmatrix} A(\cdot) & C(\cdot) \\ B(\cdot) & 0 \end{bmatrix}$ où $A(\cdot)$ est une fonction analytique bornée et $B(\cdot)$, $C(\cdot)$ sont fortement mesurables bornées telles que

$$A(\cdot): \mathfrak{E}_* \to \mathfrak{E}_*, \quad B(\cdot): \mathfrak{E}_* \to \overline{\Delta\mathfrak{E}}, \quad C(\cdot): \overline{\Delta\mathfrak{E}} \to \overline{\Delta\mathfrak{E}}$$

et de plus

$$A\Theta = \Theta A_0, \quad B\Theta + C\Delta = \Delta A_0 \tag{4}$$

où $A_0(\cdot): \mathfrak{E}\to\mathfrak{E}$ est une fonction analytique bornée.

Puisque $H=H_0\dot{+}H_1$ et $P=P_+Y|H$ les opérateurs P_+Y et $I-P_+Y$ doivent vérifier les conditions:

$$P_+YH \subset H_0, \quad P_+YH_1 = \{0\}; \quad (I-P_+Y)H \subset H_1, \quad (I-P_+Y)H_0 = \{0\}.$$

Puisque $P_+YG\subset P_+G=\{0\}$ il s'ensuit que

$$P_+YK_+ \subset H_0, \quad (I-P_+Y)K_+ \subset H_1\oplus G.$$

Nous continuons avec le suivant

Lemme 1. *Les fonctions $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ vérifient*

$$C(\cdot) = 0, \tag{5}$$

$$\Delta_* A = \Theta B. \tag{6}$$

Pour démontrer (5), soit $0\oplus v\in K_+$. Envisageons la décomposition

$$0\oplus v = \Theta(-w)\oplus(v-\Delta w) + \Theta w\oplus\Delta w$$

où $\Theta(-w)\oplus(v-\Delta w)\in H$ et $\Theta w\oplus\Delta w\in G$. En vertu de la proposition 2 on a de plus $\Theta(-w)\oplus(v-\Delta w)\in H_1$, d'où $P_+Y(0\oplus v)=0$ donc $0\oplus Cv=Y(0\oplus v)\in G$ quel que soit $v\in\overline{\Delta L^2(\mathfrak{E})}$. Il faut donc que pour tout $v\in\overline{\Delta L^2(\mathfrak{E})}$ il existe $w\in H^2(\mathfrak{E})$ tel que $\Theta w=0$

et $\Delta w = Cv$; mais comme $\Theta(\lambda)$ admet un multiple scalaire, en appliquant Ω à la relation $\Theta w = 0$ on trouve $w = 0$, d'où $Cv = 0$ pour tout $v \in \overline{\Delta L^2(\mathfrak{E})}$. Donc $C = 0$.

Pour démontrer la relation (6) nous utilisons les relations (4). Si l'on applique à la première relation Δ_* et à la seconde Θ, on obtient $(\Delta_* A - \Theta B)\Theta = 0$, ou $(\Delta_* A - \Theta B)\Theta\Omega = 0$, d'où $\Delta_* A = \Theta B$.

Lemme 2. *Il existe une fonction analytique bornée* $\{\mathfrak{E}_*, \mathfrak{F}, \Phi(\lambda)\}$ *telle que*

$$I - A = \Theta_i \Phi. \tag{7}$$

Démonstration. Pour $u \oplus v \in K_+$ on a

$$(I - P_+ Y)(u \oplus v) = (u - Au + \Theta w) \oplus (v - Bu + \Delta w)$$

avec un $w \in H^2(\mathfrak{E})$. Mais $(I - P_+ Y)K_+ \subset H_1 \oplus G$ donc

$$u - Au \in \Theta_i H^2(\mathfrak{F}) \quad \text{pour tout} \quad u \in H^2(\mathfrak{E}_*),$$

c'est-à-dire $(I - A)u = \Theta_i t$ avec $t = \Phi u$ où Φ est la multiplication par la fonction $\Phi(\cdot) = \Theta_i^*(\cdot)[I - A(\cdot)]$ qui est évidemment bornée et de plus analytique parce que $\Phi: H^2(\mathfrak{E}_*) \to H^2(\mathfrak{F})$.

Lemme 3. *Il existe une fonction analytique bornée* $\{\mathfrak{F}, \mathfrak{E}, \Psi(\lambda)\}$ *telle que*

$$A\Theta_i = \Theta \Psi. \tag{8}$$

Démonstration. Notons d'abord que pour $\Theta_i u \oplus 0 \in K_+$ on a $P_+ Y(\Theta_i u \oplus 0) = 0$. En effet, si $\Theta_i u \oplus 0 = (\Theta_i u - \Theta w_1) \oplus (-\Delta w_1) + \Theta w_1 \oplus \Delta w_1$ où $(\Theta_i u - \Theta w_1) \oplus (-\Delta w_1) \in H$ et $\Theta w_1 \oplus \Delta w_1 \in G$, il s'ensuit en vertu de la proposition 2 que $(\Theta_i u - \Theta w_1) \oplus$ $\oplus (-\Delta w_1) \in H_1$, d'où

$$P_+ Y(\Theta_i u \oplus 0) = P_+ Y[(\Theta_i u - \Theta w_1) \oplus (-\Delta w_1)] = 0.$$

Mais on a

$$P_+ Y(\Theta_i u \oplus 0) = P_+(A\Theta_i u \oplus B\Theta_i u) = (A\Theta_i u - \Theta w) \oplus (B\Theta_i u - \Delta w)$$

où $w \in H^2(\mathfrak{E})$. Donc on a

$$A\Theta_i u = \Theta w \quad \text{et} \quad B\Theta_i u = \Delta w,$$

et par conséquent $w = \Psi u$ où Ψ est la multiplication par la fonction

$$\Psi(\cdot) = \Theta^*(\cdot)A(\cdot)\Theta_i(\cdot) + \Delta(\cdot)B(\cdot)\Theta_i(\cdot)$$

qui est évidemment bornée et de plus analytique parce que $\Psi: H^2(\mathfrak{F}) \to H^2(\mathfrak{E})$.

Donc $A\Theta_i u = \Theta \Psi u$ pour tout $u \in H^2(\mathfrak{F})$ et alors $A\Theta_i = \Theta \Psi$. En tenant compte de (7) et (8) nous trouvons $\Theta_i \Phi \Theta_i + \Theta \Psi = \Theta_i$, mais comme Θ_i est une isométrie,

$$I = \Phi \Theta_i + \Theta_e \Psi.$$

Ainsi nous avons démontré la nécessité dans le suivant

Théorème. *Soit T une contraction complètement non-unitaire dont la fonction caractéristique admet un multiple scalaire et soient H_0, H_1 les sous-espaces C_0 et C_{11} de T, selon les cas. La condition nécessaire et suffisante pour que l'espace H se décompose en somme directe $H=H_0\dot{+}H_1$ est qu'il existe des fonctions analytiques bornées $\{\mathfrak{E}_*, \mathfrak{F}, \Phi(\lambda)\}$ et $\{\mathfrak{F}, \mathfrak{E}, \Psi(\lambda)\}$ telles que*

$$I = \Phi\Theta_i + \Theta_e\Psi \tag{9}$$

où $\Theta(\lambda)=\Theta_i(\lambda)\Theta_e(\lambda)$ est la factorisation canonique de $\Theta(\lambda)$ en ses facteurs intérieur et extérieur.

Il reste à démontrer la suffisance. Soit pour cela $A=I-\Theta_i\Phi$. Parce que $\Theta(\lambda)$ admet un multiple scalaire, le facteur $\{\mathfrak{F}, \mathfrak{E}_*, \Theta_i(\lambda)\}$ de la factorisation canonique admet aussi un multiple scalaire, donc il existe une fonction analytique contractive $\{\mathfrak{E}_*, \mathfrak{F}, \Omega_i(\lambda)\}$ telle que

$$\Theta_i(\lambda)\Omega_i(\lambda) = \delta_i(\lambda)I_{\mathfrak{E}_*}, \quad \Omega_i(\lambda)\Theta_i(\lambda) = \delta_i(\lambda)I_{\mathfrak{F}}$$

où $\delta_i(\lambda)$ est une fonction scalaire intérieure. Nous posons $B=\delta_i^*\Delta\Psi\Omega_i$ et observons que B satisfait à $\Delta_*A=\Theta B$. En effet,

$$\Theta B = \delta_i^*\Theta\Delta\Psi\Omega_i = \delta_i^*\Delta_*\Theta\Psi\Omega_i = \delta_i^*\Delta_*\Theta_i\Theta_e\Psi\Omega_i =$$
$$= \delta_i^*\Delta_*\Theta_i(I-\Phi\Theta_i)\Omega_i = \Delta_*(I-\Theta_i\Phi) = \Delta_*A.$$

Il est clair que A est analytique bornée et B mesurable bornée. Considérons l'opérateur

$$Y=\begin{bmatrix} A(\cdot) & 0 \\ B(\cdot) & 0 \end{bmatrix} \quad \text{et soit} \quad P = P_+Y|H.$$

Décomposons les éléments $u\oplus v\in H$ sous la forme

$$u\oplus v = P_+Y(u\oplus v)+(I-P_+Y)(u\oplus v).$$

Puisque $P_+Y(u\oplus v)=(Au-\Theta w)\oplus(Bu-\Delta w)$, on a

$$\Delta_*(Au-\Theta w) = \Theta(Bu-\Delta w), \quad \text{donc} \quad P_+Y(u\oplus v)\in H_0.$$

D'autre part, on a

$$(I-P_+Y)(u\oplus v) = (u-Au+\Theta w)\oplus(v-Bu+\Delta w) =$$
$$= (\Theta_i\Phi u+\Theta w)\oplus(v-Bu+\Delta w) = \Theta_i(\Phi u+\Theta_e w)\oplus(v-Bu+\Delta w),$$

donc

$$(I-P_+Y)(u\oplus v)\in H_1.$$

De plus cette décomposition est unique parce que $H_0\cap H_1=\{0\}$.

*

Nous remarquons que ce théorème est la généralisation du résultat obtenu par l'auteur dans [6] pour le cas des contractions complètement non-unitaires aux indices de défaut 1, 1.

Corollaire. *Soit T une contraction faible à fonction caractéristique $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ ayant les facteurs extérieur $\{\mathfrak{E}, \mathfrak{F}, \Theta_e(\lambda)\}$ et intérieur $\{\mathfrak{F}, \mathfrak{E}_*, \Theta_i(\lambda)\}$. Supposons de plus qu'il existe des fonctions analytiques bornées Φ et Ψ telles que*

$$I = \Phi\Theta_i + \Theta_e\Psi.$$

Alors T est un opérateur décomposable.

Démonstration. Il est suffisant d'observer que T peut être décomposé en somme directe $T = T_0 \dotplus T_1$ où $T_0 = T|H_0$ et $T_1 = T|H_1$, que T_0 est décomposable (cf. [2]), que T_1 est décomposable (cf. [1] ch. V) et que la somme directe d'opérateurs décomposables est aussi un opérateur décomposable.

Bibliographie

[1] I. Colojoară—C. Foiaş, *Theory of generalized spectral operators,* Gordon and Breach (New York, 1968).

[2] C. Foiaş, The class C_0 in the theory of decomposable operators, *Revue Roum. Math. Pures et Appl.*, **14** (1969), 1433—1440.

[3] K. Hoffman, *Banach spaces of analytic functions,* Prentice-Hall (Englewood Cliffs, 1962).

[4] B. Sz.-Nagy—C. Foiaş, *Harmonic Analysis of Operators on Hilbert space,* North Holland Publ. Co.—Akadémiai Kiadó (Amsterdam—Budapest, 1970).

[5] B. Sz.-Nagy—C. Foiaş, On the structure of intertwining operators, *Acta Sci. Math.*, **35** (1973), 225—254.

[6] I. R. Teodorescu, Décomposition $C_0 - C_{11}$ directe de certaines contractions faibles, *Revue Roum. Math. Pures et Appl.*, **19** (1974) (à apparaître).

UNIVERSITÉ DE BRAŞOV

(Reçu le 15 mai 1973)

Uniformly bounded groups in finite W^*-algebras

By F.-H. VASILESCU and L. ZSIDÓ in Bucharest (Romania)

1. This note contains a proof of the fact that every uniformly bounded group of elements in a finite W^*-algebra [6] is similar to a unitary group. As an application, we also get a generalization of a result of ARVESON and JOSEPHSON [2, Theorem 2.4].

The problem of similarity between a uniformly bounded group and a unitary one has been conjectured in [3] for linear operators on Hilbert spaces, where it is solved for amenable groups (i.e. groups having an invariant mean). As it is known, there are finite W^*-algebras whose unitary groups do not have any invariant mean in the sense of [3]. We give in what follows an answer for any uniformly bounded group in a finite W^*-algebra.

The proof we are going to give is an application of RYLL-NARDZEWSKI's fixed point theorem [5] but some ideas go back to [7] (see also [4, XV. 6]). Notice that the Ryll-Nardzewski fixed point theorem has been already used in finite W^*-algebras in order to give a simpler proof for the existence of a finite trace [8].

2. Let $\mathcal{M}$ be a W^*-algebra and $\mathcal{G}$ a uniformly bounded multiplicative group in $\mathcal{M}$, i.e. such that

$$\|x\| \leqq M \quad (x \in \mathcal{G}), \tag{1}$$

where $M > 0$ is independent of $x \in \mathcal{G}$.

Denote by $\overline{\mathcal{G}}^{co}$ the closure in the σ-topology (i.e. the weak topology induced in $\mathcal{M}$ by its predual $\mathcal{M}_*$ [6]) of the family

$$\left\{\sum_{j=1}^{n} \alpha_j x_j^* x_j \,|\, x_j \in \mathcal{G},\ \alpha_j \geqq 0 \quad (j = 1, \ldots, n), \quad \sum_{j=1}^{n} \alpha_j = 1\right\}. \tag{2}$$

It is obvious that $\overline{\mathcal{G}}^{co}$ is a convex set of positive elements in $\mathcal{M}$.

Lemma. *Let $\mathcal{G}$ be a uniformly bounded group in the W^*-algebra $\mathcal{M}$ and $M > 0$ a bound for it. Then for every $a \in \overline{\mathcal{G}}^{co}$ we have*

$$M^{-2} \leqq a \leqq M^2. \tag{3}$$

In particular, a is invertible in $\mathcal{M}$ for any $a \in \overline{\mathcal{G}}^{co}$.

Proof. As $\mathcal{G}$ is a group it is easy to see that

$$M^{-2} \leqq x^* x \leqq M^2 \quad (x \in \mathcal{G}). \tag{1'}$$

From this (3) follows immediately.

3. From now on we suppose that $\mathcal{M}$ is a finite W^*-algebra.

Theorem. *Let $\mathcal{G}$ be a uniformly bounded group in the finite W^*-algebra $\mathcal{M}$. Then there is a positive invertible element $b \in \mathcal{M}$, such that bxb^{-1} is unitary for any $x \in \mathcal{G}$.*

Proof. Let us define on $\mathcal{M}$ the mappings

$$T_x(a) = x^* a x \quad (a \in \mathcal{M}), \tag{4}$$

where x runs over the group $\mathcal{G}$. It is easy to see that $\{T_x\}_{x \in \mathcal{G}}$ is a group of operators on $\mathcal{M}$ and $T_x(\overline{\mathcal{G}}^{co}) \subset \overline{\mathcal{G}}^{co}$, for every $x \in \mathcal{G}$.

Let a_1, a_2 be two arbitrary elements of $\overline{\mathcal{G}}^{co}$, $a_1 \neq a_2$. Since $\mathcal{M}$ is finite, there is a normal finite trace τ on $\mathcal{M}$ [6] such that $\tau((a_1-a_2)^2) > 0$. As τ is a trace, we have for any $x \in \mathcal{G}$

$$\tau((a_1-a_2)^2) = \tau(x^{-1}(a_1-a_2)x^{*-1}x^*(a_1-a_2)x) = \tau(yx^*(a_1-a_2)x),$$

where $y = x^{-1}(a_1-a_2)x^{*-1}$. By the previous Lemma we have $\|y\| \leqq 2M^4$, where $M > 0$ is a bound for $\mathcal{G}$.

From the Schwarz inequality we get

$$\tau((a_1-a_2)^2) \leqq 2M^4 \|\tau\|^{1/2} \tau((x^*(a_1-a_2)x)^* x^*(a_1-a_2)x)^{1/2}. \tag{5}$$

If we set

$$|x|_\tau = |\tau(x^*x)|^{1/2} \quad (x \in \mathcal{M}),$$

then by (5) we obtain

$$\inf_{x \in \mathcal{G}} |T_x(a_1-a_2)|_\tau > 0. \tag{6}$$

Let us consider the locally convex topology of $\mathcal{M}$ given by the family of seminorms $|x|_\varphi = (\varphi(x^*x))^{1/2}$, where φ runs over the set of all σ-continuous positive functionals on $\mathcal{M}$. This is the s-topology of $\mathcal{M}$ and it is stronger than the σ-topology. Moreover, a linear function on $\mathcal{M}$ is σ-continuous if and only if it is s-continuous [6, Corollary 1.8.10]. By the preceding Lemma, $\overline{\mathcal{G}}^{co}$ is σ-compact, i.e. it is compact in the weak topology of $\mathcal{M}$ corresponding to the s-topology. Then by (6), $\{T_x\}_{x \in \mathcal{G}}$ is a non-contracting group of linear s-continuous operators on $\overline{\mathcal{G}}^{co}$ in the sense of RYLL-NARDZEWSKI [5]. By the Ryll-Nardzewski fixed point theorem [5], there is at least one $a \in \overline{\mathcal{G}}^{co}$ such that

$$x^* a x = a \quad (x \in \mathcal{G}). \tag{7}$$

According to the previous Lemma, a is positive and invertible, therefore if $b=a^{1/2}$ then b^{-1} exists. Let us set

$$u_x = bxb^{-1} \quad (x\in\mathscr{G}). \tag{8}$$

Then we may write on account of (7)

$$u_x^* u_x = b^{-1}x^*bbxb^{-1} = 1.$$

Analogously, from (7) we have $xa^{-1}x^*=a^{-1}$; hence

$$u_x u_x^* = bxb^{-1}b^{-1}x^*b = 1,$$

consequently u_x is unitary for any $x\in\mathscr{G}$.

4. Let us recall some definitions from [1]. Suppose that $\mathscr{M}$ is a W^*-algebra of operators acting on a separable Hilbert space. Let Φ be a faithful normal positive linear mapping of $\mathscr{M}$ into itself, such that $\Phi^2=\Phi$.

A subalgebra $\mathscr{S}$ of $\mathscr{M}$ is said to be *subdiagonal* (with respect to Φ) if it has the following properties:

(i) $\mathscr{S}+\mathscr{S}^*$ is σ-dense in $\mathscr{M}$.

(ii) $\Phi(ab)=\Phi(a)\Phi(b)$ $(a, b\in\mathscr{S})$.

(iii) $\Phi(\mathscr{S})\subset\mathscr{S}\cap\mathscr{S}^*$.

(iv) The nullspace of $(\mathscr{S}\cap\mathscr{S}^*)^2$ is trivial.

It is known that every subdiagonal subalgebra $\mathscr{S}$ of $\mathscr{M}$ is contained in a maximal subdiagonal subalgebra of $\mathscr{M}$ [1, Theorem 2.2.1].

Suppose now that $\mathscr{M}$ is finite.

A subdiagonal subalgebra $\mathscr{S}$ of $\mathscr{M}$ (with respect to Φ) is called *finite* if there is a faithful normal finite trace ϱ of $\mathscr{M}$ such that $\varrho(\Phi(x))=\varrho(x)$ for every $x\in\mathscr{M}$.

The next result is a generalization of Theorem 2.4 in [2].

Corollary. *Let $\mathscr{G}$ be a uniformly bounded group in a finite W^*-algebra $\mathscr{M}$ of operators acting on a separable Hilbert space. Let $\mathscr{S}$ be a finite maximal subdiagonal subalgebra in $\mathscr{M}$. Then there is an invertible element $a\in\mathscr{S}$ such that $a^{-1}\in\mathscr{S}$ and axa^{-1} is unitary for each $x\in\mathscr{G}$.*

Proof. According to the previous Theorem, there is a positive invertible element $b\in\mathscr{M}$, such that bxb^{-1} is unitary for every $x\in\mathscr{G}$. On account of [1, Theorem 4.2.1], $b=ua$, where u is unitary and $a\in\mathscr{S}\cap\mathscr{S}^{-1}$. Now, it is obvious that axa^{-1} is unitary when $x\in\mathscr{G}$.

References

[1] W. B. ARVESON, Analyticity in operator algebras, *Amer. J. Math.*, **89** (1967), 578—642.

[2] W. B. ARVESON—K. B. JOSEPHSON, Operator algebras and measure preserving automorphisms. II, *J. Funct. Analysis*, **4** (1969), 100—134.

[3] J. DIXMIER, Les moyennes invariantes dans les semi-groupes et leur applications, *Acta Sci. Math.*, **12** (1950), 213—227.

[4] N. DUNFORD and J. SCHWARTZ, *Linear operators*, Part 3, Wiley—Interscience (New York—London—Sydney—Toronto, 1971).

[5] CZ. RYLL-NARDZEWSKI, *On fixed points of semigroups of endomorphisms of linear spaces*, Fifth Berkeley Symposium on Mathematical Statistics and Probability. 2, Part 1, University of California Press (1967).

[6] S. SAKAI, *C*-algebras and W*-algebras*, Springer-Verlag (Berlin-Heidelberg-New York, 1971).

[7] B. SZ.-NAGY, On uniformly bounded linear transformations in Hilbert space, *Acta Sci. Math.*, **11** (1946), 152—157.

[8] F. J. YEADON, A new proof of the existence of a trace in a finite von Neumann algebra, *Bull. Amer. Math. Soc.*, **77** (1971), 257—260.

INSTITUTE OF MATHEMATICS
ACADEMY OF SCIENCES
BUCHAREST, ROMANIA

(Received December 19, 1972)

Diagonalization theorems for matrices over certain domains

By JÓZSEF SZŰCS in Szeged

Introduction

In [7] NORDGREN proved a diagonalization theorem for matrices over H^∞, the set of all bounded analytic functions on the unit disc. Making use of this result MOORE and NORDGREN gave, in [6], a new approach to the Jordan model theory of $\mathscr{C}_0$ contractions of finite defect [9—16] and established a conjecture of SZ.-NAGY and FOIAŞ [14]. In the present paper we prove an abstract algebraic generalization of Nordgren's diagonalization theorem.

0. Preliminaries

Let R be a *domain*, i.e. a commutative ring with identity 1 and without zero divisors.[1] Two $m \times n$ matrices A and B over R are said to be *equivalent* if there exist invertible $m \times m$ and $n \times n$ matrices X and Y over R such that $XAY = B$.

We set the following condition:

(*GCD*) In R any two elements have a greatest common divisor (g.c.d.).

It follows from (*GCD*) by induction that any finite system of elements $a_1, \ldots, a_n$ from R has a g.c.d. in R. This shall be denoted by $a_1 \wedge \ldots \wedge a_n$.[2] For any $m \times n$ matrix A over R and any integer k such that $1 \leqq k \leqq \min(m, n)$, $\mathscr{D}_k(A)$ will denote the g.c.d. of all minors of order k of A. Set $\mathscr{D}_0(A) = 1$. It is easy to see that if $\mathscr{D}_{k-1}(A) = 0$ for some k $(k \leqq \min(m, n))$ then $\mathscr{D}_k(A) = 0$ as well. For any k such that $1 \leqq k \leqq \min(m, n)$ we set $\mathscr{E}_k(A) = \mathscr{D}_k(A) / \mathscr{D}_{k-1}(A)$ with the convention that $\mathscr{E}_k(A) = 0$ if $\mathscr{D}_{k-1}(A) = 0$.[3] $\mathscr{E}_k(A)$ is called the *invariant factor* of kth order of A. Relying on elementary determinant theory one can easily see that if A and B are two equivalent matrices over R,

[1]) For the algebraic notions we refer the reader to [4].

[2]) $a_1 \wedge \ldots \wedge a_n$ is determined up to invertible factors.

[3]) The elementary theory of determinants shows that $\mathscr{D}_{k-1}(A) | \mathscr{D}_k(A)$ for $k = 1, \ldots, \min(m, n)$.

then $\mathscr{E}_k(A)$ equals $\mathscr{E}_k(B)$ up to invertible factors. The matrix A is said to be *normal* if all but possibly the diagonal entries of it vanish and each diagonal entry is a multiple of the preceding one. It is evident that the invariant factor of kth order of a normal matrix equals its diagonal entry in the kth row.

1. An equivalence theorem for matrices over certain domains

There is a classical result [4] which asserts that if R is a principal ideal domain, then every $m \times n$ matrix over R is equivalent to a normal one. Now we are going to prove the analogue of this theorem for domains R having property (GCD) and the following one:

(L) If a and b are relatively prime[4]) elements of R (in symbols $a \perp b$) then there exists an element y in R such that $a+by$ is invertible.

At the end of section 3 we shall see (cf. footnote [15])) that our theorem is not a special case of the classical one. On the other hand, the ring of all rational integers does not satisfy (L), so our theorem does not contain the classical one as a special case.

Theorem 1. *If a domain R has properties (GCD) and (L), then any $m \times n$ matrix A over R is equivalent to a normal one.*

Proof. Given any integer j, $1 \leqq j \leqq m$, there exists a matrix A' [5]) having the following properties:

(R_j) 1) a'_{j1} divides all entries in the jth row of A';
2) the g.c.d. of all entries in an arbitrary row of A equals the g.c.d. of all entries in the corresponding row of A';
3) A' is equivalent to A.

In fact, from (GCD) and (L) it follows by induction that there exist elements $r_2, \dots, r_n$ of R such that

$$a_{j1} + r_2 a_{j2} + \dots + r_n a_{jn} = a_{j1} \wedge \dots \wedge a_{jn}.$$

Let A' be the matrix obtained from A by adding to its first column the linear combination of its last $n-1$ columns with the coefficients $r_2, \dots, r_n$. Then the first two requirements in (R_j) are obviously fulfilled. On the other hand, it is an elementary fact that

[4]) I.e. $c \in R$ and $c|a$, $c|b$ imply that c^{-1} exists in R.

[5]) Matrices shall be denoted by Roman capital letters, their entries by the corresponding low case letters, with two subscripts the first one indicating the row.

A' can be obtained from A by multiplying A by a nonsingular matrix from the right, hence 3) in (R_j) also holds.

Now suppose that $j-1\geqq 1$. Then there exists a matrix A'' having the following properties:

(C_j) 1) $a''_{j-1,1}|a''_{j,1}$;

2) the jth and the subsequent rows are the same in A'' as in A;

3) A'' is equivalent to A.

In fact, (GCD) and (L) assure the existence of an element s of R such that $a_{j-1,1}+sa_{j1}=a_{j-1,1}\wedge a_{j1}$. Let A'' be the matrix obtained from A by adding s times its $(j-1)$th row to its jth row. Then 1) and 2) of (C_j) are obviously satisfied. On the other hand, A'' can be obtained from A by multiplying A by a non-singular matrix from the left, thus 3) in (C_j) also holds true.

Relying on the preceding observations, we are now going to show the existence of a matrix equivalent to the given matrix A and whose entry in the left upper corner divides all its other entries. To this effect, we replace the matrix A by a matrix A' having property (R_m) and denote this matrix A' again by A. Having done this, we replace the new A by a matrix A'' having property (C_m) and denote the replacing matrix again by A. Continuing, we alternately replace the current A by a matrix A' or A'' having successively the properties (R_{m-1}), (C_{m-1}), (R_{m-2}), (C_{m-2}), ..., (R_2), (C_2), (R_1). It is easy to see that the matrix A obtained in the last $((2m-1)$th$)$ step has the property $a_{11}|a_{ik}$ $(1\leqq i\leqq m,\ 1\leqq k\leqq n)$.

Subtracting appropriate scalar multiples of the first row (column) of A from the other rows (columns) of A, we can end up with a matrix, denoted by A again, all of whose entries in the first row and column except possibly the one in the left upper corner are zeros. It is an elementary fact that our new A is equivalent to the old one. On the other hand, it is obvious that $a_{11}|a_{ik}$ $(1\leqq i\leqq m,\ 1\leqq k\leqq n)$ is still true.

We can accomplish the proof of Theorem 1 in two ways. Either we use induction on m and n or we employ the preceding method min $(m-1, n-1)$ times more. This part of our proof is routine, so we omit it.

2. General diagonalization theorems

We want to prove a diagonalization theorem for matrices over some domains R having the property (GCD) and a property weaker than (L). For any fixed non-zero element ψ of R the following property is weaker than (L):

$(L\psi)$ If $a, b\in R$ and $a\perp b$, then there are elements x, y in R such that $xa+yb\perp\psi$ and $x\perp\psi$.

Putting $\psi=0$, we obtain a property equivalent to (L). It is obvious that property (L) implies property ($L\psi$) for any $\psi\in R$. On the other hand, the union of properties ($L\psi$) ($\psi\neq0$) is strictly weaker than property (L). In fact, it can be shown that every principal ideal domain, or more generally, every domain having properties (GCD), (ψ) ($RP\psi$) and (A) (see section 3) has property ($L\psi$) for each $\psi\in R$, $\psi\neq0$, however, for example, the domain of all rational integers does not satisfy property (L).

In the sequel we shall consider some quotient rings of R, $R\neq\{0\}$. Let us fix a non-zero element ψ of our domain R [6]) and suppose that R has property

($RP\psi$) For any elements a, b in R, $a\perp\psi$ and $b\perp\psi$ imply $ab\perp\psi$.

Consider the quotient field $\tilde{R}$ of R. Making use of ($RP\psi$) it can be easily verified that the set R_ψ of all elements t of R that can be written in the form $x=ab^{-1}$, $a, b\in R$, $b\perp\psi$ is a domain containing R. We can easily see that if R has property ($L\psi$), then R_ψ has property (L). If R also has property (GCD) then so does R_ψ. In fact, let $x=ab^{-1}$ and $y=cd^{-1}$ $\left(a, b, c, d\in R,\ b, d\overset{R}{\perp}\psi\right)$ [7]) be two elements of R_ψ. We shall show that $x\overset{R_\psi}{\wedge}y$ exists and equals $a\overset{R}{\wedge}c$. From ($L\psi$) there follows the existence of elements x, y, s of R such that $xa+yc=(a\overset{R}{\wedge}c)s$, $s\overset{R}{\perp}\psi$. Rewriting this as $(xs^{-1})a+(ys^{-1})c=a\overset{R}{\wedge}c$, we can see that if $t\in R_\psi$ is a common divisor of a and c in R_ψ then $t\overset{R_\psi}{|}(a\overset{R}{\wedge}c)$. This, together with the obvious relation $(a\overset{R}{\wedge}c)\overset{R_\psi}{|}a, c$ means that $a\overset{R_\psi}{\wedge}c$ exists and equals $a\overset{R}{\wedge}c$. Since x (resp. y) differs from a (resp. b) in an invertible factor only, this proves our assertion about $x\overset{R_\psi}{\wedge}y$. [8])

The preceding arguments show that if a domain R has property (GCD) and properties ($RP\psi$) and ($L\psi$) for some non-zero element ψ of R, then Theorem 1 can be applied to R_ψ. In particular, for every $m\times n$ matrix A over R, Theorem 1 assures the existence of two non-singular matrices X and Y over R_ψ and that of a normal matrix E over R_ψ such that $XA=EY$. From our reasoning about the g.c.d. in R_ψ and from Preliminaries it follows that E can be chosen to be equal to the diagonal

[6]) For $\psi=0$ property ($RP\psi$) always holds and our results in this section are still true but they are equivalent to those of the preceding section.

[7]) $d\overset{R}{\perp}\psi$ means that b is relatively prime to ψ over R. In the sequel, if misunderstanding were possible, we shall indicate by a superscript the domain over which the symbols $\perp$, $\wedge$ or $|$ shall be meant.

[8]) I am indebted to my colleague G. Pollák for a remark which enabled me to shorten the proof of the fact that R_ψ has property (GCD). Originally, I derived (GCD) for R_ψ from ($RP\psi$) and (GCD) only.

matrix of the invariant factors of A over R.[9]) The entries of X and Y are fractions whose numerators and denominatiors are elements of R, the latter ones can be supposed to be relatively prime to ψ. Denote by c the product of the denominators (supposed to be relatively prime to ψ) of all entries in X and Y. Then $(RP\psi)$ implies, by induction, that $c \perp \psi$. Furthermore, since the matrix X is invertible over R_ψ, there is an element r of R_ψ such that $r \det X = 1$. Put $\det X = \frac{x'}{x''}$ and $r = \frac{r'}{r''}$, $x', r', x'', r'' \in R$, $x'', r'' \overset{R}{\perp} \psi$. Then we have $x'r' = x''r''$, so $(RP\psi)$ shows that $x' \overset{R}{\perp} \psi$. Similarly, if $\det Y = \frac{y'}{y''}$, $y', y'' \in R$ and $y'' \overset{R}{\perp} \psi$, then $y' \overset{R}{\perp} \psi$. Put $X' = cX$, $Y' = cY$. Then the entries of X' and Y' are from R. Moreover, $\det X' = \frac{x'}{x''} c^m$, $\det Y' = \frac{y'}{y''} c^n$ so, on account of $(RP\psi)$ the relation $x', y', c \overset{R}{\perp} \psi$ implies that $\det X', \det Y' \overset{R}{\perp} \psi$. Since we also have $X'A = EY'$, our result can be summarized in the following:

Theorem 2. *Suppose that R has property (GCD), and for some non-zero*[10]) *element ψ of R, properties $(RP\psi)$ and $(L\psi)$. Then for every $m \times n$ matrix A over R we can find an $m \times m$ matrix X and an $n \times n$ matrix Y over R such that* $\det X, \det Y \perp \psi$ *and $XA = EY$, where E denotes the diagonal matrix of the invariant factors of A.*

We do not know in general whether the diagonal matrix of the invariant factors of A (even if R has properties (GCD) and $(RP\psi)$, $(L\psi)$ for some $\psi \in R$, $\psi \neq 0$) is normal or not. However, if the assumptions of Theorem 2 hold true for $\psi = \mathscr{E}_i(A)(\mathscr{E}_i(A) \wedge \mathscr{E}_{i+1}(A))^{-1}$ (provided that not both $\mathscr{E}_i(A)$ and $\mathscr{E}_{i+1}(A)$ vanish), then we have $\mathscr{E}_i(A) \overset{R}{|} \mathscr{E}_{i+1}(A)$. (Of course, if $\mathscr{E}_i(A) = \mathscr{E}_{i+1}(A) = 0$, then $\mathscr{E}_i(A) \overset{R}{|} \mathscr{E}_{i+1}(A)$ obviously holds.) In fact, the arguments preceding Theorem 2 show that for $\psi = \mathscr{E}_i(A)(\mathscr{E}_i(A) \wedge \mathscr{E}_{i+1}(A))^{-1}$ we have $\mathscr{E}_i(A) \overset{R_\psi}{|} \mathscr{E}_{i+1}(A)$, i.e. we have $\mathscr{E}_{i+1}(A) = \frac{r'}{r''} \mathscr{E}_i(A)$ $(r', r'' \in R,\ r'' \overset{R}{\perp} \psi (= \mathscr{E}_i(A)(\mathscr{E}_i(A) \wedge \mathscr{E}_{i+1}(A))^{-1})$. Since

$$\mathscr{E}_i(A)(\mathscr{E}_i(A) \wedge \mathscr{E}_{i+1}(A))^{-1} \overset{R}{\perp} \mathscr{E}_{i+1}(A)(\mathscr{E}_i)A) \wedge \mathscr{E}_{i+1}(A))^{-1}$$

[9]) In detail: On account of Preliminaries E necessarily equals one of the diagonal matrices of the invariant factors, over R_ψ, of A. If E' is one of the diagonal matrices of the invariant factors, over R, of A, then our comments on the g.c.d. in R_ψ show that $e_{ii} = \varphi_i e'_{ii}$ for some invertible elements φ_i of R_ψ and for $i = 1, \ldots, \min(m, n)$. Denoting by T the $n \times n$ diagonal matrix of these φ_i's, T is invertible over R_ψ, $E = E'T$, and $XA = E'TY$.

[10]) For $\psi = 0$ Theorem 2 reduces to Theorem 1.

the equality $r'' \mathscr{E}_{i+1}(A)(\mathscr{E}_i(A)\wedge\mathscr{E}_{i+1}(A))^{-1}=r'\mathscr{E}_i(A)(\mathscr{E}_i(A)\wedge\mathscr{E}_{i+1}(A))^{-1}$ shows, by $(RP\psi)$, that $\mathscr{E}_i(A)(\mathscr{E}_i(A)\wedge\mathscr{E}_{i+1}(A))^{-1}$ is invertible in R. i.e. $\mathscr{E}_i(A)\mid\mathscr{E}_{i+1}(A)$ [11]).

In order to state our general diagonalization theorem we need the following

Definition. Let A and B be two $m\times n$ matrices over a domain R. We say that A is *quasi-equivalent* to B if for any non-zero element ψ of R there exists an $m\times m$ matrix X and an $n\times n$ matrix Y over R such that $XA=BY$ and $\det X$, $\det Y\perp\psi$.

Suppose that R has property $(\psi)(RP\psi)$, i.e. property $(RP\psi)$ for *every* $\psi\in R$. Then the arguments used right before Theorem 2 show that A is quasi-equivalent to B over R if and only if A is equivalent to B over every R_ψ $(\psi\in R,\ \psi\neq 0)$. Hence in this case quasi-equivalence is an equivalence relation.

Theorem 2 and the remarks after it imply the following.

Theorem 3. *Suppose that the domain R has properties (GCD), $(\psi)(RP\psi)$ and $(\psi\neq 0)(L\psi)$* [12]). *Then, for matrices over R, quasi-equivalence is an equivalence relation. The diagonal matrix of the invariant factors of any matrix over R is normal and it is quasi-equivalent to the matrix considered.*

3. Examples for Theorems 1—3

For two elements f, g of a domain R we write $f\ll g$ if every non-invertible divisor of f has a non-invertible divisor that divides g. It is easy to see that the relation "$\ll$" is transitive. Let us consider the following property:

(A) For every two elements f and g of R with f non-vanishing there are two elements f_s and f_a in R such that $f_s\perp g$, $f_a\ll g$ and $f=f_sf_a$.

We are now going to give a slightly simpler proof for Lemma 3.1 of [7] in a more general situation.

Lemma. *If R has properties (GCD), (A) and $(\psi)(RP\psi)$, then R has property $(\psi\neq 0)(L\psi)$.*

Proof. Fix a non-vanishing ψ and consider two relatively prime elements a and b of R. Put $\psi=\psi_s\psi_a$, $\psi_a\ll a$, $\psi_s\perp a$ (property (A)!) and $\delta=a+b\psi_s$. We are going to prove that $\delta\perp\psi$, which will complete the proof of our lemma. For any $\omega\in R, \omega\neq 0$ put $\omega_s=\psi_s\wedge\omega$, $\omega_a=\dfrac{\omega}{\psi_s\wedge\omega}$. We have $\omega=\omega_a\omega_s$, $\omega_s\wedge\psi_s$ and $\omega_a\perp\psi_s\omega_s^{-1}$.

[11]) Professor B. Sz.-Nagy kindly called our attention to the paper [8] from which it follows that if $\bar{i}$ denotes the least common multiple of $1, \dots, i$, then $\mathscr{E}_i(A)|\bar{i}\,\mathscr{E}_{i+1}(A)$ for $i=1, \dots, \min(m, n)$ provided that R has properties (GCD) and $(RP\psi)$ for every $\psi\in R$.

[12]) $(\psi\neq 0)$ is a restricted quantifier and $(\psi\neq 0)(L\psi)$ means that $(L\psi)$ holds true for any $\psi\in R$, $\psi\neq 0$.

Suppose now that $\omega \mid \psi$. We prove that $\omega_a \ll \psi_a$. In fact, it is obvious that $\omega_a \mid \psi_a \psi_s \omega_s^{-1}$. For any c such that $c \mid \omega_a$ and c is not invertible, this divisibity relation, $\omega_a \perp \psi_s \omega_s^{-1}$ and $(\psi)\,(RP\psi)$ imply that c is not relatively prime to ψ_a, which means that $\omega_a \ll \psi_a$. If ω divides δ too, then from the equality $\delta = a + b\psi_s$ and from $\omega_s \mid \psi_s$ we deduce that $\omega_s \mid a$. But $\psi_s \perp a$ by the definition of ψ_s, so $\omega_s \perp a$ (since $\omega_s \mid \psi_s$) and hence ω_s is invertible. On the other hand, since $\omega_a \ll \psi_a$ and $\psi_a \ll a$, we have $\omega_a \ll a$. This means that if $c \mid \omega_a$ and c is not invertible, then there exists a non-invertible element d in R such that $d \mid c$ and $d \mid a$. Since $\delta = a + b\psi_s$ and ω is supposed to divide δ, we have $d \mid b\psi_s$. Furthermore, $d \mid a$ and $a \perp \psi_s$ imply $d \perp \psi_s$, which, together with $d \mid b\psi_s$ and $(\psi)\,(RP\psi)$, imply that d is not relatively prime to b. This contradicts the fact that $a \perp b$. Hence c and therefore ω_a are invertible. Since ω_s is also invertible, so is $\omega = \omega_a \omega_s$. This completes the proof of the fact that $\psi \perp \delta$.

Theorem 4. *If the domain R has properties (GCD), $(\psi)\,(RP\psi)$, and (A), then the conclusions of Theorem 3 are true for R.*

Proof. Our assumptions, by the Lemma and Theorem 3, immediately imply the conclusions of Theorem 3.

Let R now be a domain such that every non-zero element of R has a prime factorization. It is obvious that these prime factorizations are, up to invertible factors, uniquely determined by the elements factored and R has properties (GCD) and $(\psi)\,(RP\psi)$. Moreover, in R the relation $f \ll g$ is equivalent to the following: Every prime divisor of f is a divisor of g, too. For any two $f, g \in R$, $f \neq 0$ define f_s as the product of those prime factors of f (with multiplicity) which do not devide g. Putting $f_a = f f_s^{-1}$ we have $f = f_s f_a$ and $f_s \perp g$, $f_a \ll g$. This shows that R has property (A), too, and we have

Theorem 5. *If in the domain R every non-zero element has a prime factorization, then the conclusions of Theorem 3 are true for R.*

Remark. In this special case our lemma, that is property $(\psi \neq 0)\,(L\psi)$, could be proved more easily.

Since in any principal ideal domain every non-zero element has a prime factorization, we can consider this result a generalization of a weaker version of the classical theorem which asserts that every matrix over a principal ideal domain is equivalent to a normal one. Let us remark that there are numerous examples of domains which have the prime factorization property and which are not principal ideal domains [1].[13])

---

[13]) For example, the domain of all polynomials of n variables $(n \geqq 2)$ over an arbitrary field, or more generally over a domain which has the prime factorization property is not a principal ideal domain and has the prime factorization property [1].

An appropriate quotient domain of the domain H^∞ of all bounded analytic functions on the unit disc[14]) (from the study of which all our investigations have started) provides an example for a domain R having properties (GCD), (A) and (ψ)($RP\psi$) and not having the prime factorization property. In fact, using the properties of inner functions, it is easy to see that the domain $R=N^+$ of all analytic functions f on the unit disc, of the form $f=g/h$, where $g\in H^\infty$, $h\in H^\infty$ and h is outer, has properties (GCD), (A) and (ψ) ($RP\psi$), and N^+ does not have the prime factorization property[15]).

Let us now compare our results applied to the special case of the domain N^+ with those obtained by NORDGREN [7]. He states a theorem about H^∞ and not about the domain N^+ but his notion of divisibility in H^∞ coincides with the usual algebraic notion of divisibility over N^+. If in our Theorem 2 applied to the domain N^+ we multiply all entries of X and Y by the product of the denominators (which can be supposed to be outer functions) of all entries of X and Y, then we obtain Theorem 3.1 of [7]. Moreover, for matrices over H^∞, of equal size $m\times n$, one can introduce the following notion of quasi-equivalence: A is said to be quasi-equivalent to B if for any $\psi\in H^\infty$, $\psi\neq 0$ there exist square matrices X and Y over H such that $XA=BY$ and that $\det X$ and $\det Y$ are relatively prime to ψ over N^+. Theorem 3.1 of [7] asserts then that every matrix over H^∞ is quasi-equivalent to the matrix of its invariant factors. (See also [4].)

By the way, Theorems 2.1 and 3.1 of [7] together with the fact that quasi-equivalence in the above sense is an equivalence relation show that quasi-equivalence as defined in the present paper is the same as that defined in [7].

4. Additional remarks

After I had finished the studies contained in sections 1—3, J. ERDŐS called my attention to the paper [5] of I. KAPLANSKY. There it is proved among others a necessary and sufficient condition for a domain to be an *elementary divisor ring* (cf. Theorem 5.2), that is, a ring over which every $m\times n$ matrix is equivalent to a normal one. (Kaplansky studies non-abelian rings, too, in which case the definition of normality has to be modified.) It is easy to check that Kaplansky's conditions are satisfied in any domain having properties (GCD) and (L). So we could shorten the proof of Theorem 1 by a reference to Kaplansky's result. One of Kaplansky's conditions is that in the domain under consideration every finitely generated ideal is principal. It is easy to see that in the domain of all complex polynomials of two variables x and y the ideal generated by x^2, xy and y^2 is not principal, so we cannot generally have equivalence in Theorem 5 and so in Theorems 3—4, either.

---

[14]) For the notions about H^∞ we refer the reader to [3].

[15]) Neither does any of the domains R_ψ.

Finally, we mention that O. HELMER in his paper [2] introduces the notion of adequate rings. A ring is called an *adequate ring* if it is a domain, every finitely generated ideal in it is principal, and satisfies condition (A) of our paper. Helmer proves that every adequate ring is an elementary divisor ring. We have seen that the domain considered at the end of section 3 has property (A).

Problem. Is the domain N^+ considered at the end of section 3 adequate?

If it were, we would have equivalence in Nordgren's theorem on the domain N^+, instead of quasi-equivalence.

References

[1] H. HASSE, Über eindeutige Zerlegung in Primelemente oder in Primhauptideale in Integritätsbereichen, *J. reine angew. Math.*, **159** (1928), 3—12.

[2] O. HELMER, The elementary divisor theorem for certain rings without chain condition, *Bull. Amer. Math. Soc.*, **49** (1943), 225—236.

[3] K. HOFFMAN, *Banach spaces of analytic functions*, Prentice Hall (Englewood Cliffs, N. J., 1962).

[4] J. JACOBSON, *Lectures in abstract algebra. II*, Van Nostrand (Princeton, N. J., 1953).

[5] I. KAPLANSKY, Elementary divisors and moduls, *Trans. Amer. Math. Soc.*, **66** (1949), 464—491.

[6] B. MOORE, III and E. A. NORDGREN, On quasi-equivalence and quasi-similarity, *Acta Sci. Math.*, **34** (1973), 311—316.

[7] E. A. NORDGREN, On quasi-equivalence of matrices over H^∞, *Acta Sci. Math.*, **34** (1973), 301—310.

[8] L. RÉDEI, Über die Determinantenteiler, *Acta Math. Acad. Sci. Hungar.*, **3** (1952), 143—149.

[9] B. SZ.-NAGY and C. FOIAŞ, Sur les contractions de l'espace de Hilbert. VII. Triangulations canoniques. Fonction minimum, *Acta Sci. Math.*, **25** (1964), 12—37.

[10] B. SZ.-NAGY and C. FOIAŞ, Sur les contractions de l'espace de Hilbert, VIII. Fonctions caractéristiques. Modèles fonctionnels, *Acta Sci. Math.*, **25** (1964). 38—71.

[11] B. SZ.-NAGY and C. FOIAŞ, Sur les contractions de l'espace de Hilbert. XI. Transformations unicellulaires, *Acta Sci. Math.*, **26** (1965), 301—324 and **27** (1966), 265.

[12] B. SZ.-NAGY and C. FOIAŞ, Vecteurs cycliques et quasi-affinités, *Studia Math.*, **31** (1968), 35—42.

[13] B. SZ.-NAGY and C. FOIAŞ, Opérateurs sans multiplicité, *Acta Sci. Mat.*, **30** (1969), 1—18.

[14] B. SZ.-NAGY and C. FOIAŞ, Modèles de Jordan pour une classe d'opérateurs de l'espace de Hilbert, *Acta Sci. Math.*, **31** (1970), 91—115.

[15] B. SZ.-NAGY and C. FOIAŞ, Compléments à l'étude des opérateurs de classe C_0, *Acta Sci. Math.*, **31** (1970), 287—296.

[16] B. SZ.-NAGY and C. FOIAŞ, Local characterizations of operators of class C_0, *J. Functional Analysis*, **8** (1971), 76—81.

(Received March 26, 1973)

The ring N^+ is not adequate

By ERIC A. NORDGREN in Durham (N. H., USA)

O. HELMER [2] defined an integral domain R (see [3]) to be *adequate* in case 1) every finitely generated ideal in R is principal and 2) if $a, b \in R$ and $a \neq 0$, then $a = rs$ for $r, s \in R$ such that g.c.d. $(r, b) = 1$ and every nonunit divisor of s shares a nonunit divisor with b. The purpose of this note is to provide a negative answer to the question, raised by J. SZŰCS in the preceding paper [5], of whether or not the ring N^+ of quotients of H^∞ functions by bounded outer functions (see [1]) is adequate. That N^+ satisfies 2) is shown in [5]. That it does not satisfy 1) will be a consequence of the following fact.

Theorem. *There exist finitely generated weak* dense ideals of H^∞ that contain no outer functions.*

Proof. Let a be the atomic inner function

$$a(z) = \exp - [(1+z)/(1-z)],$$

and let b be the Blaschke product with zeros $z_n = 1 - 1/n^2$ $(n = 1, 2, \ldots)$. If I is the ideal in H^∞ generated by a and b, then since a and b have no nontrivial common inner divisors, and since weak* closed ideals of H^∞ have the form φH^∞ for φ inner [4], it follows that I is weak* dense in H^∞.

Suppose I contains an outer function c. Then there exist x and y in H^∞ such that

$$ax + by = c.$$

Letting u be the quotient of the outer factor of x by c, we would then have

$$|a(z_n)\, u(z_n)| \geqq 1 \tag{1}$$

for $n = 1, 2, \ldots$, which is not possible, as will be shown.

Let P be the Poisson kernel: $P(\theta; z) = \operatorname{Re}[(e^{i\theta}+z)/(e^{i\theta}-z)]$. Since

$$|u(z)| = \exp \frac{1}{2\pi} \int_0^{2\pi} P(\theta; z) \log |u(e^{i\theta})|\, d\theta,$$

taking logarithms converts inequality (1) to

$$(2)\qquad \frac{1}{2\pi}\int_0^{2\pi} P(\theta; z_n)\log|u(e^{i\theta})|\,d\theta-(1+z_n)/(1-z_n)\geqq 0.$$

Choosing $\delta>0$ so that

$$\frac{1}{2\pi}\int_{-\delta}^{\delta}\left|\log|u(e^{i\theta})|\right|d\theta<\frac{1}{2}$$

and denoting the left hand side of (2) by d_n, we have

$$d_n\leqq\frac{1}{2\pi}\int_{-\delta}^{\delta}P(\theta;z_n)\left|\log|u(e^{i\theta})|\right|d\theta+\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta}P(\theta;z_n)\log|u(e^{i\theta})|\,d\theta-(1+z_n)/(1-z_n)\leqq$$

$$\leqq\frac{1+z_n}{1-z_n}\frac{1}{2\pi}\int_{-\delta}^{\delta}\left|\log|u(e^{i\theta})|\right|d\theta+\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta}P(\theta;z_n)\log|u(e^{i\theta})|\,d\theta-(1+z_n)/(1-z_n)\leqq$$

$$<\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta}P(\theta;z_n)\log|u(e^{i\theta})|\,d\theta-\frac{1}{2}(1+z_n)/(1-z_n).$$

This implies that $d_n\to-\infty$, since the last integral tends to 0 as $n\to\infty$, contradicting (2), and the proof is complete.*)

To see that N^+ does not satisfy 1) consider an ideal I of H^∞ satisfying the conditions of the theorem and generated by functions a and b. The functions a and b then have no common inner divisor and consequently if the ideal they generate in N^+ were principal, then it would have to be all of N^+ since outer functions are units in N^+. Thus we could choose x and y in N^+ such that

$$ax+by=1,$$

and consequently it would follow that the product of the denominators of x and y is in I, which is impossible.

References

[1] P. L. Duren, *Theory of H^p spaces*, Academic Press (New York, 1970).

[2] O. Helmer, The elementary divisor theorem for certain rings without chain condition, *Bull. Amer. Math. Soc.*, **49** (1943), 225—236.

[3] N. Jacobson, *Lectures in abstract algebra*, Vol. I, Van Nostrand (Princeton, N. J., 1951).

[4] T. P. Srinivasan, Simply invariant subspaces and generalized analytic functions, *Proc. Amer. Math. Soc.*, **16** (1965), 813—818.

[5] J. Szűcs, Diagonalization theorems for matrices over certain domains, *Acta Sci. Math.*, **36** (1974), 193—201.

(Received October 9, 1973)

*) The same example of inner functions a, b for which $ax+by$ is not outer for any choice of $x, y\in H^\infty$, was contained, in connection with another problem, in an earlier letter of C. Foiaş to the Editor. *(The Editor)*

On a problem of Kátai

By JEAN-LOUP MAUCLAIRE in Paris

1. In his paper [1] I. KÁTAI set the following problem: Let f and g be two additive arithmetical functions; we suppose there exists a $l \in \mathbf{C}$ such that

$$\lim_{n \to +\infty} \{g(2n+1) - f(n)\} = l;$$

what can we deduce for the form of f or g?

The purpose of our paper is to prove the two following results:

(1) *Let g be an additive function; if f is a completely additive function, and if there exists an $l \in \mathbf{C}$ such that*

$$\lim_{n \to +\infty} \{g(2n+1) - f(n)\} = l \quad (\textit{hypothesis } \mathrm{H}),$$

then $f(n) = \dfrac{l}{\log 2} \log n$, *and for every odd prime p and every positive integer d we have* $g(p^\alpha) = f(p^\alpha)$.

(2) *Let g be an additive function; if f is a completely additive function and if there exists an $M \in \mathbf{R}^+$ such that*

$$|g(2m+1) - f(m)| \leqq M \quad \textit{for every} \quad m \in \mathbf{N}^*,$$

then $f(n) = C \log n$, where C is a constant.

1.1. Remarks. 1) It follows from the conclusion of (1) that $g(n) = f(n)$ for every odd $n \in \mathbf{N}^*$; of course, we cannot deduce anything about even n's, because only values of g on odd integers are involved in the hypothesis. 2) From the conclusion of (2), we can easily deduce that $f(2n+1) - g(2n+1)$ is bounded independently of n.

Let us verify assertion 2):

There is an A such that $|h(m)| \leqq A$; furthermore, we have $|g(2m+1) - h(m) - C \log m| \leqq M$ Then, it follows that $|g(2m+1) - C \log m| \leqq M + A$.

But there exists a B such that: $|C[\log(2m+1)-\log m]|\leqq B$. Then we have $|g(2m+1)-C\log(2m+1)|\leqq A+B+M$; and therefore

$$|g(2m+1)-\{C\log(2m+1)+h(2m+1)\}| \leqq 2A+B+M,$$

i.e.: $|g(2m+1)-f(2m+1)|\leqq 2A+M+B$.

2. Proof of (1). 2.1. First, we have $g(2n+1)-f(n)-l=o(1)$ and $g(2n-1)-f(n-1)-l=o(1)$ $(n\to+\infty)$; since $(2n+1, 2n-1)=1$, we also have $g(2n+1)+g(2n-1)=g(4n^2-1)$; hence

(A) $$g(4n^2-1)-f(n)-f(n-1)-2l = o(1) \quad (n\to+\infty).$$

Moreover, $g(4n^2-1)=g[2(2n^2-1)+1]$, and it follows from hypothesis (H) that:

(B) $$g(4n^2-1)-f(2n^2-1)-l = o(1) \quad (n\to+\infty).$$

We deduce from (A) and (B) that

(C) $$f(2n^2-1)-f(n)-f(n-1)-l = o(1) \quad (n\to+\infty).$$

2.2. Using hypothesis (H) we get:

$$g[(2n+1)^2]-f(2n(n+1))-l = o(1) \quad \text{and} \quad g[(2n-1)^2]-f(2n(n-1))-l = o(1) \quad (n\to+\infty).$$

But $(2n+1, 2n-1)=1$; hence $g[(2n+1)^2]+g[(2n-1)^2]=g[(4n^2-1)^2]$ and it follows that

(A′) $$g[(4n^2-1)^2]-f(2n(n-1))-f(2n(n+1))-2l = o(1) \quad (n\to+\infty).$$

Now we notice that $g[(4n^2-1)^2]=g[8n^2(2n^2-1)+1]$; using hypothesis (H) we get

(B′) $$g[(4n^2-1)^2]-f[4n^2(2n^2-1)]-l = o(1) \quad (n\to+\infty).$$

This, together with (A′), yields

$$f[4n^2\times(2n^2-1)]-f[2n(n+1)]-f[2n(n-1)]-l = o(1) \quad (n\to+\infty).$$

Since f is completely additive, we get

(C′) $$f(2n^2-1)-f(n-1)-f(n+1)-l = o(1) \quad (n\to+\infty).$$

2.3. We now replace $f(2n^2-1)$ in (C′) by its value obtained from (C); thus, we have $f(n+1)-f(n)=o(1)$ $(n\to+\infty)$. By a well-know theorem of Erdős ([2]), we have $f(n)=C\log n$, where C is a constant. Thus, (C) becomes:

$$\lim_{n\to+\infty} \{C\times(\log(2n^2-1)-\log n-\log(n-1))-l\} = 0,$$

which implies $C = \dfrac{l}{\log 2}$.

Now, hypothesis (H) becomes:

$$g(2n+1)-\frac{l}{\log 2}\log n-l=o(1),\quad \text{i.e.}\quad g(2n+1)-\frac{l}{\log 2}\log 2n=o(1)\quad (n\to+\infty).$$

But $\log(2n+1)-\log 2n=o(1)$ $(n\to+\infty)$. Therefore:

$$\text{(D)}.\qquad g(2n+1)-\frac{l}{\log 2}\log(2n+1)=o(1)\quad (n\to+\infty).$$

Now let $\alpha\in\mathbf{N}^*$ and let p be any odd prime; we take in (D) $2n+1=p^\alpha\times(2pm+1)$. We thus get

$$g(p^\alpha)+g(2pm+1)-\frac{l}{\log 2}\log p^\alpha-\frac{l}{\log 2}\log(2pm+1)=o(1)\quad (m\to+\infty).$$

But by (D) we have

$$g(2pm+1)-\frac{l}{\log 2}\log(2pm+1)=o(1)\quad (m\to+\infty).$$

It follows that

$$g(p^\alpha)-\frac{l}{\log 2}\log p^\alpha=o(1)\quad (m\to+\infty),$$

which implies

$$g(p^\alpha)=\frac{l}{\log 2}\log p^\alpha.$$

3. Proof of (2). Using the same method as for the proof of (1), we obtain:

$$\text{I.}\qquad |f(2n^2-1)-f(n)-f(n-1)|\leqq 3M.$$

$$\text{II.}\qquad |f(2n^2-1)-f(n-1)-f(n+1)|\leqq 3M.$$

$$\text{III.}\qquad |f(n+1)-f(n)|\leqq 6M.$$

By a result of Wirsing ([3]), we obtain: $f(n)=C\log n+h(n)$, where h is a bounded additive function. Since h is completely additive (because f is completely additive), h is identically zero.

References

[1] I. Kátai, Some results and problems in the theory of additive functions, *Acta Sci. Math.*, **30** (1969), 305—312.

[2] P. Erdős, On the distribution function of additive functions, *Ann. of Math.*, **47** (1946), 4—20.

[3] E. Wirsing, A characterization of the logarithm as an additive function, *Proc. Rome Conference of Number Theory*, 1968.

(Received November 25, 1972)

A note on non-quasitriangular operators*)

By L. A. FIALKOW in Stony Brook (N. Y., USA)

1. Introduction. Let $\mathfrak{H}$ be a fixed, separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathfrak{H})$ denote the algebra of all bounded linear operators on $\mathfrak{H}$. Let $\mathcal{P}$ denote the directed set of all finite rank projections in $\mathcal{L}(\mathfrak{H})$ under the usual ordering, and for each T in $\mathcal{L}(\mathfrak{H})$ define $q(T)=\liminf_{P\in\mathcal{P}}\|(1-P)TP\|$ and $Q(T)=\limsup_{P\in\mathcal{P}}\|(1-P)TP\|$. In [10], HALMOS initiated the study of quasitriangular operators and proved that an operator T is quasitriangular if and only if $q(T)=0$. In [7], DOUGLAS and PEARCY employed the η-function of BROWN and PEARCY (see [5], [12]) to prove that T is a *thin* operator (i.e., an operator that is the sum of a scalar and a compact operator) if and only if $Q(T)=0$. The functions q and Q were studied, respectively, by APOSTOL in [1] and by FOIAŞ and ZSIDÓ in [8]. We appreciatively acknowledge access to preliminary versions of [1] and [8].

In a preliminary version of [8], FOIAŞ and ZSIDÓ proved the following lemma.

Lemma F—Z. *Let T be in $\mathcal{L}(\mathfrak{H})$, $\|T\|=1$, and for $0\leqq t\leqq 1$, let E_t denote the spectral projection of $(T^*T)^{\frac{1}{2}}$ which corresponds to the interval* $[0, t]$. *The following implications are valid.*

i) *If* $q(T)=1$, *then* $\dim E_t\mathfrak{H}<\aleph_0$ *for all* $t<1$.

ii) *If* $q(T)\geqq 0.95$, *then there exists* $t>1-q(T)$ *such that* $\dim E_t\mathfrak{H}<\aleph_0$.

Because of its length and complexity, this writer could not see through the proof of Lemma F—Z. One purpose of this note is to provide (in section 3) a straightforward and short proof of a somewhat stronger version of Lemma F—Z. In particular, we prove that if $\|T\|=1$ and $q(T)>2/3$, then there exists $t>1-q(T)$ such that $\dim E_t\mathfrak{H}<\aleph_0$; an example shows that 2/3 is the best possible lower bound. We discuss the relationship between this result and a theorem of [8]. In section 2, values of q and q/Q are obtained for certain partial isometries. We also prove that if

*) This paper constitutes part of the author's Ph. D. thesis written at the University of Michigan under the direction of Prof. Carl Pearcy.

A is in $\mathcal{L}(\mathfrak{H})$ and $q(T+A)=0$ for each quasitriangular operator T in $\mathcal{L}(\mathfrak{H})$, then A is a thin operator.

The referee has kindly pointed out that several of the results in section two were proven independently by APOSTOL, FOIAŞ, and ZSIDÓ in [4], and by APOSTOL, FOIAŞ, and VOICULESCU in [2]. These papers followed [1] and [8] in a series of papers on non-quasitriangular operators. In an appendix we give the precise relationship between our results and those of the Rumanian mathematicians.

2. Partial isometries. Let (QT) and (N) denote, respectively, the subsets of quasitriangular and normal operators in $\mathcal{L}(\mathfrak{H})$.

In section 3 of [10], HALMOS proved $(N)\subset(QT)$. For each T in $\mathcal{L}(\mathfrak{H})$ we set $d(T)=\inf_{S\in(QT)}\|T-S\|$ and $d_N(T)=\inf_{S\in(N)}\|T-S\|$. Then clearly $d(T)\leqq d_N(T)$. The proofs of the following two lemmas are easy and will be omitted.

Lemma 2.1. (APOSTOL [1].) *If A and B are operators in $\mathcal{L}(\mathfrak{H})$, then* $|q(A)-q(B)|\leqq\|A-B\|$.

Remark. Lemma 2.1 implies that if T is in $\mathcal{L}(\mathfrak{H})$, then $q(T)\leqq d(T)$. Indeed, if $q(S)=0$, we have $q(T)\leqq\|T-S\|$, and therefore $q(T)\leqq\inf_{S\in(QT)}\|T-S\|$. We are also able to prove the reverse inequality $d(T)\leqq q(T)$ and to thereby conclude that $q(T)$ is the distance from T to the set (QT). This result is not used in this note and the proof will appear elsewhere.

Lemma 2.2. (FOIAŞ and ZSIDÓ [8].) *The following implications are valid.*

i) *If U is a non-unitary isometry, then $q(U)=1$.*

ii) *If T is in $\mathcal{L}(\mathfrak{H})$ and A is a thin operator, then $q(T)=q(T+A)$.*

The following proposition, which we believe to be new, is the converse of Lemma 2.2 ii).

Proposition 2.3. *If A is in $\mathcal{L}(\mathfrak{H})$ and $q(T+A)=0$ for each T in (QT), then A is a thin operator.*

Proof. If A is not thin, then Corollary 3.4 of [5] implies that A is similar to an operator $\mathfrak{H}\oplus\mathfrak{H}$ of the form

$$A_1=\begin{pmatrix} B & V \\ C & 0 \end{pmatrix},$$

where V is a non-unitary isometry. Let A_2 be the operator on $\mathfrak{H}\oplus\mathfrak{H}$ whose matrix is

$$\begin{pmatrix} B & 0 \\ C & 0 \end{pmatrix},$$

and choose an integer $n>1$ such that $n>\|A_2\|$. Let S denote the invertible operator on $\mathfrak{H}\oplus\mathfrak{H}$ of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix},$$

and let $A_3=S^{-1}A_1S$. Finally, let X and Y denote, respectively, the operators on $\mathfrak{H}\oplus\mathfrak{H}$ whose matrices are

$$\begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} 0 & nV \\ 0 & 0 \end{pmatrix}.$$

Theorem 6 of [6] implies that $q(X)=0$, and from Lemma 2.2 i), we have $q(X+Y)=n$ Since $|q(X+A_3)-q(X+Y)|\leqq\|A_3-Y\|<n$, it is clear that $q(X+A_3)>0$. Let $R\colon \mathfrak{H}\to\mathfrak{H}\oplus\mathfrak{H}$ be an invertible operator such that $A=R^{-1}A_3R$. Theorem 9 of [6] implies that $q(R^{-1}XR)=0$, and it follows that $q(R^{-1}XR+A)>0$. (Indeed, if $q(R^{-1}XR+A)=0$, another application of [6, Theorem 9] shows that

$$0=q\big(R(R^{-1}XR+A)R^{-1}\big)=q(X+RAR^{-1})=q(X+A_3),$$

which is a contradiction.)

Corollary 2.4. (Douglas and Pearcy [7]) *If A is in $\mathcal{L}(\mathfrak{H})$ and $\lim_{P\in\mathcal{P}}\|(1-P)AP\|=0$, then A is a thin operator.*

Proof. If $\lim_{P\in\mathcal{P}}\|(1-P)AP\|=0$, it is easy to prove that for each T in (QT), $q(A+T)=0$. Then, from Proposition 2.3, A is a thin operator.

Lemma 2.5. *If V is an isometry in $\mathcal{L}(\mathfrak{H})$, then $q(V^*)=0$.*

Proof. The proof is trivial if V is a unilateral shift of multiplicity one. If V is unitary, then V^* is in (N). The proof for an arbitrary isometry procedes from the above special cases via the von Neumann decomposition theorem and Theorem 4 of [10].

Proposition 2.6. *Let V be a partial isometry in $\mathcal{L}(\mathfrak{H})$ with nullity $V=\alpha$ and corank $V=\beta$. The following implications are valid.*

i) *If $\alpha=\beta<\aleph_0$, then $q(V)=0$.*

ii) *If $\alpha=\beta=\aleph_0$, then $q(V)\leqq 1/2$.*

iii) *If $\alpha<\beta$, then $q(V)=1$ and $q(V^*)=0$.*

Proof. i) If $\alpha=\beta<\aleph_0$, there is a finite rank operator F such that $V+F$ is unitary. Then $q(V)=q(V+F)=0$. ii) The proof of [9, Theorem 5] shows that if $\alpha=\beta$, then $d_N(V)\leqq 1/2$. Therefore $q(V)\leqq d(V)\leqq d_N(V)\leqq 1/2$. iii) If $\alpha<\beta$, there is a finite rank operator G such that $V+G$ is a non-unitary isometry. From Lemma 2.2 i), $q(V)=q(V+G)=1$, and from Lemma 2.5, $q(V^*)=q(V^*+G^*)=0$.

Lemma 2.7. *Let U denote a unilateral shift of multiplicity one in $\mathscr{L}(\mathfrak{H})$. If $T=U\oplus 0$ in $\mathscr{L}(\mathfrak{H}\oplus\mathfrak{H})$, then $q(T)=1/2$ and $Q(T)=1$.*

Proof. Let $S=T-1/2$. Since S is bounded below by $1/2$ and nullity $S^*\neq 0$, Lemma 2.1 of [6] implies that $q(T)=q(S)\geqq 1/2$. The reverse inequality follows directly from Proposition 2.6 ii).

Let $\mathscr{P}_1$ denote the directed set of all finite rank projections in $\mathscr{L}(\mathfrak{H}\oplus\mathfrak{H})$ under the usual ordering. To show that $Q(T)\geqq 1$, it suffices to prove that if P_0 is in $\mathscr{P}_1$, then there exists P_1 in $\mathscr{P}_1$ such that $P_1\geqq P_0$ and $\|(1-P_1)TP_1\|=1$. Now since P_0 is in $\mathscr{P}_1$, it is easy to prove that there exist projections R in $\mathscr{P}$ and P_1 in $\mathscr{P}_1$ such that $P_1=R\oplus R$ and $P_1\geqq P_0$. The proof of [6, Lemma 2.1] implies that R may be chosen so that $\|(1-R)UR\|=1$. Then $\|(1-P_1)TP_1\|=\|(1-R)UR\|=1$. Since $Q(T)\leqq\|T\|=1$, the proof is complete.

Proposition 2.8. *If $0\leqq r\leqq 1/2$, there exist partial isometries V and W in $\mathscr{L}(\mathfrak{H}\oplus\mathfrak{H})$ such that $q(V)/Q(V)=r$ and $q(W)=r$.*

Proof. Let U be a unilateral shift of multiplicity one in $\mathscr{L}(\mathfrak{H})$, and for $0\leqq t\leqq 1$ define $P(t)$ by the operator matrix

$$\begin{pmatrix} tU & 0 \\ \sqrt{1-t^2} & 0 \end{pmatrix}.$$

Then $P(t)$ is a norm continuous function on $[0, 1]$ whose values are partial isometries in $\mathscr{L}(\mathfrak{H}\oplus\mathfrak{H})$. It is easy to prove that if $0\leqq t\leqq 1$, then $Q(P(t))>0$. From Lemma 2.1 and an obvious analogue involving Q, the functions q and Q are continuous. If $f_1(t)=q(P(t))$ and $f_2(t)=f_1(t)/Q(P(t))$, then f_1 and f_2 are each continuous on $[0, 1]$ and therefore each has connected. range. The proof is completed by noting that $P(0)$ is quasitriangular [6, Theorem 6] and that $f_1(1)=f_2(1)=1/2$ by Lemma 2.7.

3. An improvement of Lemma F – Z. Theorem 3.1. *Let T be in $\mathscr{L}(\mathfrak{H})$, $\|T\|=1$, and for $0\leqq t\leqq 1$, let E_t denote the spectral projection for $(T^*T)^{\frac{1}{2}}$ which corresponds to the interval $[0, t]$. The following implications are valid.*

i) *If $0\leqq t_0<1/3$ and $\dim E_{t_0}=\aleph_0$, then $q(T)\leqq(3-t_0)/4$.*

ii) *If $1/3\leqq t_0<1$ and $\dim E_{t_0}=\aleph_0$, then $q(T)\leqq(1+t_0)/2$.*

Proof. i) Let $T=UP$ denote the polar decomposition of T. Since E_{t_0} reduces P, $P=P_1+P_2$, with P_1 in $\mathscr{L}((E_{t_0}\mathfrak{H})^{\perp})$ and P_2 in $\mathscr{L}(E_{t_0}\mathfrak{H})$. Clearly P_1 and P_2 are positive operators. The spectral theorem implies that $\|P_2\|\leqq t_0$ and that $t_0\leqq P_1\leqq 1$. If $V=U(1-E_{t_0})$, then V is a partial isometry such that nullity $V=\aleph_0$. Proposition 2.6 implies that $q(V)\leqq 1/2$, and therefore

$$q(T)\leqq q((1+t_0)/2V)+\|P-(1+t_0)/2(1-E_{t_0})\|\leqq(1+t_0)/4+\|P_1-(1+t_0)/2\oplus P_2\|.$$

Since

$$\|P_1-(1+t_0)/2\| \leqq \sup_{t_0 \leqq t \leqq 1} |t-(1+t_0)/2| = (1-t_0)/2$$

and

$$\|P_2\| \leqq t_0 \leqq (1-t_0)/2,$$

it follows that

$$q(T) \leqq (1+t_0)/4+(1-t_0)/2 = (3-t_0)/4.$$

ii) Proceeding as above, we have $q(T) \leqq q\big((1-t_0)V\big)+\|P-(1-t_0)(1-E_{t_0})\| \leqq$ $\leqq (1-t_0)/2+\|\big(P_1-(1-t_0)\big)\oplus P_2\|$. Now $\|P_1-(1-t_0)\| \leqq \sup_{t_0 \leqq t \leqq 1} |t-(1-t_0)|$, and an easy calculation shows that the supremum is less than or equal to t_0. Since $\|P_2\| \leqq t_0$, we have $q(T) \leqq (1-t_0)/2+t_0=(1+t_0)/2$.

Corollary 3.2. *Let T be as above. If $q(T)>2/3$, then there exists $t>1-q(T)$ such that* $\dim E_t \mathfrak{H} < \aleph_0$.

Proof. Suppose that for each $t>1-q(T)$, $\dim E_t \mathfrak{H}=\aleph_0$. Since $q(T)>2/3$, then $1-q(T)<1/3$, and therefore $\dim E_{\frac{1}{3}} \mathfrak{H}=\aleph_0$. Theorem 3.1 ii) implies that $q(T) \leqq$ $\leqq (1+1/3)/2=2/3$, which is impossible.

The following example shows that 2/3 is the best possible lower bound for a result like Corollary 3.2.

Example 3.3. Let U denote the unilateral shift of multiplicity one in $\mathscr{L}(\mathfrak{H})$ and let $A=U\oplus -1/3$ and $B=U\oplus 0$. Since $A-1/3$ is bounded below by 2/3 and nullity $(A-1/3)^* \neq 0$, Lemma 2.1 of [6] implies that $q(A)=q(A-1/3) \geqq 2/3$. Lemma 2.7 states that $q(B)=1/2$, and therefore $|q(A)-q(2/3B)|=|q(A)-1/3| \leqq \|A-2/3B\|=$ $=1/3$. Now $1-q(A)=1/3$ and $\dim E_{\frac{1}{3}} \mathfrak{H}=\aleph_0$. Therefore, for each $t>1/3$, $\dim E_t \mathfrak{H}=$ $=\aleph_0$. Since $\|A\|=1$, this example shows that Corollary 3.2 cannot be extended beyond those operators for which $q(T)>2/3\,\|T\|$.

Remark. In [8] FOIAŞ and ZSIDÓ used Lemma F—Z to prove that if T is in $\mathscr{L}(\mathfrak{H})$, $\|T\|=1$, and $q(T) \geqq 0.95$, then $T=U+S+K$, where U is a nonunitary isometry, S is an operator such that $\|S\|<q(T)$, and K is a finite rank operator. Corollary 3.2 extends this result to any operator T in $\mathscr{L}(\mathfrak{H})$ such that $q(T)>2/3$ and $\|T\|=1$. In particular, T is a semi-Fredholm operator with negative index. We furthur remark that if T is in $\mathscr{L}(\mathfrak{H})$, $\|T\|=1$, and T has the above structure, then $q(T)>1/2$. Indeed, since $T=U+S+K$, $q(T)=q(U+S)$ and therefore $|q(U)-q(T)| \leqq$ $\leqq \|S\|<q(T)$. Since $q(U)=1$, we have $1-q(T)<q(T)$, and the result follows. On the other hand, if $0<\varepsilon \leqq 2/3$, then there exists a Fredholm operator T_ε in $\mathscr{L}(\mathfrak{H}\oplus\mathfrak{H})$, such that $\|T_\varepsilon\|=1$, the index of T_ε is negative, and $q(T_\varepsilon)=\varepsilon$. For example, if V is the unilateral shift of multiplicity one in $\mathscr{L}(\mathfrak{H})$, then we may let T_ε be the operator in $\mathscr{L}(\mathfrak{H}\oplus\mathfrak{H})$ whose matrix is

$$\begin{pmatrix} 0 & V \\ \varepsilon & 0 \end{pmatrix}.$$

Finally, if $1/2<\varepsilon\leqq 2/3$, it is easy to prove that there exists $t>1-q(T_\varepsilon)$ such that $\dim E_t\mathfrak{H}<\aleph_0$. This proves that the converse of Corollary 3.2 is false.

4. Appendix. We wish to indicate that some of our results are related to results in [2] and [4]. (The results in [4] were announced in [3].) Proposition 2.6 is identical to Corollary 2.7 of [4]. The remark on page 3 is contained in Theorem 2.2 of [2], which proves, additionally, that the distance from an operator to the set (QT) is actually attained at some operator in (QT). Lemma 2.7 (about q) is contained in Corollary 4.3 of [2], and Proposition 2.8 (about q) is identical to Theorem 4.4 (about q) of [2]. In each of the above cases the proofs of the corresponding results differ somewhat from one another.

References

[1] C. Apostol, Quasitriangularity in Hilbert space (preprint).

[2] C. Apostol, C. Foiaş, and D. Voiculescu, Some results on non-quasitriangular operators. II, *Revue roumaine math. pures et appliquées,* **2** (1973), to appear.

[3] C. Apostol, C. Foiaş, and L. Zsidó, Sur les opérateurs non-quasitriangulaires, *Comptes Rendus Paris*, A **275** (1972), 501—503.

[4] C. Apostol, C. Foiaş, and L. Zsidó, Some results on non-quasitriangular operators, *Indiana J. Math.*, **22** (1973), 1151—1161.

[5] A. Brown and C. Pearcy, Structure of commutators of operators, *Ann. of Math.*, **82** (1965), 112—127.

[6] R. G. Douglas and C. Pearcy, A note on quasitriangular operators, *Duke Math. J.*, **37** (1970), 177—188.

[7] R. G. Douglas and Carl Pearcy, A characterization of thin operators, *Acta Sci. Math.*, **29** (1968), 295—297.

[8] C. Foiaş and L. Zsidó, Some results on non-quasitriangular operators (preprint).

[9] P. R. Halmos and J. E. McLaughlin, Partial isometries, *Pac. J. Math.*, **13** (1962), 585—596.

[10] P. R. Halmos, Quasitriangular operators, *Acta Sci. Math.*, **29** (1968), 283—294.

[11] C. Pearcy and N. Salinas, An invariant subspace theorem, *Mich. Math. J.*, **20** (1973), 21—31.

[12] N. Salinas, On the η-function of Brown and Pearcy and the numerical function of an operator, *Canad. J. Math.*, **23** (1971), 565—578.

(Received November 20, 1972)

Operators of class $C_0(N)$ and transitive algebras

By KEVIN CLANCEY[1]) in Athens (Ga., USA) and
BERRIEN MOORE, III[2]) in Durham (N.H., USA)

The recent remarkable result of V. I. LOMONOSOV [4], that if an operator (bounded linear transformation) T on a Banach space $\mathfrak{X}$ has a nonzero compact operator in its commutant then T has a nontrivial invariant subspace, has a beautiful and astonishingly simple proof. The proof establishes even stronger results than that stated. Lomonosov does mention one of these in a note at the end of his paper. Another and closely related result is that if $\mathscr{A}$ is a transitive algebra in the Banach algebra $\mathscr{B}(\mathfrak{H})$ of all operators on a separable complex Hilbert space $\mathfrak{H}$ which contains a nonzero compact operator, then $\mathscr{A}$ is weakly dense in $\mathscr{B}(\mathfrak{H})$; see [6].

By a transitive algebra $\mathscr{A}$ we mean a subalgebra of $\mathscr{B}(\mathfrak{H})$ for which there does not exist a nontrivial subspace which is invariant under each operator in $\mathscr{A}$. We should mention that a primary motivation for the study of transitive algebras is that if the only weakly closed transitive algebra is $\mathscr{B}(\mathfrak{H})$, then the invariant subspace conjecture is true, i.e. every operator on a separable complex Hilbert space has a nontrivial invariant subspace. For an excellent discussion of transitive algebras and the history of their development see the monograph by RADJAVI and ROSENTHAL [6; particularly Chapter 8 and 10].

In this paper, we establish that if T is a contraction on $\mathfrak{H}$ such that T^n and T^{*n} go strongly to zero as $n\to\infty$, and if the ranks of $I-T^*T$ and $I-TT^*$ are finite and equal (if N is this rank, then T is said to be of class $C_0(N)$, see [10; p. 350]; also finiteness implies their equality [10; Theorem VI.5.2]), then any transitive algebra that contains T is weakly dense in $\mathscr{B}(\mathfrak{H})$.

The essential underlying result for our study is that if T is in $C_0(N)$ then T commutes with a particularly simple nonzero compact operator, and this is established by working within the functional model $\mathbf{T}$ for T (see [8] or [10]) where the structure of commuting compacts is well understood (see [7] for $N=1$; [5] for $N\geqq 1$). Finally, the result is reached by using the transitive algebra result which followed from

1) Supported in part by NSF GRANT GP 38488.
2) Supported in part by NSF GRANT GP 14784.

Lomonosov's proof and noting that the specific nature of this commuting compact implies that it is in the weakly closed algebra $\mathscr{A}_{\mathbf{T}}$ generated by $\mathbf{I}$ and $\mathbf{T}$.

The functional model $\mathbf{T}$ of T in $C_0(N)$ on the space $\mathbf{H}$ is defined by

$$\mathbf{H} = H^2(\mathfrak{E}) \ominus \Theta H^2(\mathfrak{E}) \quad \text{and} \quad (\mathbf{T}u)(e^{it}) = \big(P_{\mathbf{H}}(\chi u)\big)(e^{it}) \quad (u \in \mathbf{H} \quad \text{and} \quad \chi(e^{it}) = e^{it}).$$

Here $\mathfrak{E}$ is N-dimensional complex Hilbert space, $H^2(\mathfrak{E})$ is the Hardy space of $\mathfrak{E}$-valued functions on the unit circle, $P_{\mathbf{H}}$ the orthogonal projection of $H^2(\mathfrak{E})$ onto $\mathbf{H}$, and Θ is a matrix-valued "analytic" function, in the sense that $\Theta H^2(\mathfrak{E}) \subseteq H^2(\mathfrak{E})$, on the unit circle which is inner from both sides, (i.e., unitary valued a.e. or equivalently, in this case, inner). Finally, the Banach algebras of matrix-valued "analytic" and continuous functions on the unit circle will be denoted by $H^\infty(\mathscr{B}(\mathfrak{E}))$ and $C(\mathscr{B}(\mathfrak{E}))$, respectively. When $\mathfrak{E}$ is simply the complex plane we shall use only H^∞ or C. For further discussion see [10; Chapter IV] and [1; Lectures VII and VIII].

In order to establish our Theorem we need the

Lemma. *If $\psi \in H^\infty$ is a nonconstant inner function which is not a finite Blaschke product then there exists $\varphi \in H^\infty$ such that*

$$\bar{\psi}\varphi \in H^\infty + C \quad \textit{and} \quad \bar{\psi}\varphi^p \notin H^\infty \quad \textit{for any positive integer } p.$$

Proof. This proof is similar to the proofs of Lemma 4 and Lemma 5 in [3]; however, there are some differences so we shall give the details for completeness.

Let $\beta\sigma = \psi$ be the factorization of ψ into a Blaschke product β and a singular inner function σ. If β is nontrivial, then let z_0 be a zero of β of multiplicity m. Define β_0 on the unit circle $\mathscr{T}$ by

$$\beta_0(z) = \left(\frac{z - z_0}{1 - \bar{z}_0 z}\right)^m.$$

Then $\varphi = \bar{\beta}_0 \psi \in H^\infty$, and $\bar{\psi}\varphi^p = \bar{\beta}_0 \varphi^{p-1}$, for any positive integer p. As β_0 does not divide φ^{p-1} we have $\bar{\psi}\varphi^p \notin H^\infty$.

The more difficult case occurs when ψ is purely singular, i.e.

$$\psi(z) = \exp\Big\{-\int_0^{2\pi} h(t, z)\, ds(t)\Big\} \quad (|z| = 1), \quad ^{3)}$$

where $h(t, z) = \dfrac{e^{it} + z}{e^{it} - z}$ and s is a singular, finite, positive Borel measure on $[0, 2\pi)$. We identify $[0, 2\pi)$ with $\mathscr{T}$.

Let $\mathscr{E}$ be a Borel set of Lebesgue measure zero such that $\mathscr{E}$ has full s-measure. By regularity, we can find a closed set $\mathscr{K}$ contained in $\mathscr{E}$ such that $s(\mathscr{K}) > 0$. Define the measure s_0 on the Borel sets $\mathscr{F}$ in $[0, 2\pi)$ by $s_0(\mathscr{F}) = s(\mathscr{K} \cap \mathscr{F})$. Clearly

[3]) Every integral with $h(t, z)$ is interpreted as a limit of the same integral with $h(t; rz)$ as $r \to 1-0$.

s_0 is supported on the closed set $\mathscr{K}$, and the nonconstant inner function

$$\psi_0(z) = \exp\left\{-\int_0^{2\pi} h(t,z)\,ds_0(t)\right\} \quad (|z|=1)$$

divides ψ. In fact, ψ_0 and $\psi/\psi_0=\gamma$ are relatively prime; therefore, ψ_0 does not divide γ^p for any positive integer p. Since s_0 is supported on $\mathscr{K}$, it follows that ψ_0 is continuous on the complement $\mathscr{T}\setminus\mathscr{K}$. Further, we can choose an outer function v which is continuous on $\mathscr{T}$ and vanishes on $\mathscr{K}$. This follows by applying the portion of the proof on page 80 of [2] in which a log-integrable function $y(\cdot)\geqq 0$ is constructed on $\mathscr{T}$ having the following properties: y is continuous on $\mathscr{T}$, continuously differentiable on $\mathscr{T}\setminus\mathscr{K}$, and vanishing precisely on $\mathscr{K}$. Then we define for $z\in\mathscr{T}$

$$v(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} h(t,z)\log y(e^{it})\,dt\right\};$$

v is an outer function in H^∞ which is continuous on $\mathscr{T}$ and vanishes precisely at the points of $\mathscr{K}$. Set $\varphi=v\gamma$. Again $\varphi\in H^\infty$, and $\bar\psi\varphi=\bar\psi_0 v$ is continuous. Further, for any positive integer p we have

$$\bar\psi\varphi^p = \bar\psi_0\gamma^{p-1}v^p,$$

but ψ_0 cannot divide γ^{p-1} because of being relatively prime to ψ, nor can ψ_0 divide v since v is outer; therefore, $\bar\psi\varphi^p\notin H^\infty$.

So in each case we have constructed $\varphi\in H^\infty$ such that $\bar\psi\varphi\in C$ but $\bar\psi\varphi^p\notin H^\infty$ for any positive integer p.

Theorem. *If a weakly closed transitive algebra $\mathscr{A}$ in $\mathscr{B}(\mathfrak{H})$ contains a nonzero $C_0(N)$ operator T, then it is $\mathscr{B}(\mathfrak{H})$.*

Proof. As stated, we shall work within the functional model $\mathbf{T}$; let Θ be the associated inner function. An operator $\mathbf{K}$ on $\mathbf{H}$ commutes with $\mathbf{T}$ if and only if there exists $\Phi\in H^\infty(\mathscr{B}(\mathfrak{E}))$ such that

$$\Phi\Theta H^2(\mathfrak{E}) \subseteqq \Theta H^2(\mathfrak{E})$$

and $\mathbf{K}=\Phi(\mathbf{T})$, where we define

$$\Phi(\mathbf{T})u = P_{\mathbf{H}}(\Phi u)$$

for every $u\in\mathbf{H}$. For the case $N=1$ see [7]; for the general case see [9] and within a functional model [10; in particular Theorem VI.3.6]. Since Θ is unitary valued and $\Phi\Theta H^2(\mathfrak{E})\subseteqq\Theta H^2(\mathfrak{E})$, it follows that $\Phi(\mathbf{T})$ is nonzero if and only if $\Theta^*\Phi\notin H^\infty(\mathscr{B}(\mathfrak{E}))$.

Let $\psi=\det\Theta$ and set $\Psi=\psi\cdot I$, where I is the identity matrix on $\mathfrak{E}$. If ψ is a finite Blaschke product, then $\mathbf{H}$ is finite dimensional and the result follows from Burnside's Theorem [6; Chapter 8]. If ψ is not a finite Blaschke product, then choose, by the lemma, a function $\varphi\in H^\infty$ such that $\psi\varphi\in C$ but $\bar\psi\varphi^p\notin H^\infty$ for $p=1, 2, \dots$. Set

$$\mathbf{H}' = H^2(\mathfrak{E})\ominus\Psi H^2(\mathfrak{E}),\quad \mathbf{T}'u = P_{\mathbf{H}'}(\chi u)\quad\text{and}\quad \Phi(\mathbf{T}')u = P_{\mathbf{H}'}(\Phi u)$$

where $u \in \mathbf{H}'$, $P_{\mathbf{H}'}$ is the orthogonal projection of $H^2(\mathfrak{E})$ onto $\mathbf{H}'$, and $\Phi = \varphi \cdot I$. By the choice of φ we have that

$$\Psi^* \Phi = \bar{\psi} \varphi I \in C(\mathscr{B}(\mathfrak{E})).$$

Further, it is obvious that $\Phi \Psi H^2(\mathfrak{E}) \subseteqq \Psi H^2(\mathfrak{E})$ since Φ and Ψ have diagonal matrices as values. Consequently, $\Phi(\mathbf{T}')$ is a compact operator. But $\Phi(\mathbf{T})$ is just the compression of $\Phi(\mathbf{T}')$ to the space $\mathbf{H}$. Hence $\Phi(\mathbf{T})$ is compact too. Further, since $\Phi = \varphi \cdot I$, $\Phi(\mathbf{T})$ is an H^∞ function of T, and hence it is in the weakly closed algebra $\mathscr{A}_{\mathbf{T}}$ generated by $\mathbf{I}$ and $\mathbf{T}$ (see [10; Theorem III.2.1]).

It remains only to show that $\Phi(\mathbf{T})$ is nonzero. This will follow if we can establish that $\Theta^* \Phi \notin H^\infty(\mathscr{B}(\mathfrak{E}))$. Assume the contrary, so that there exists $\Gamma \in H^\infty(\mathscr{B}(\mathfrak{E}))$ such that $\Phi = \Theta \Gamma$. Thus $\det \Phi = (\det \Theta)(\det \Gamma)$, so $\bar{\psi} \varphi^N = \det \Gamma \in H^\infty$, a contradiction to the choice of φ. Therefore, $\Phi(\mathbf{T})$ is a nonzero compact operator in $\mathscr{A}_{\mathbf{T}}$. Thus there is a nonzero compact in $\mathscr{A}_T \subseteqq \mathscr{A}$, so by Lomonosov $\mathscr{A} = \mathscr{B}(\mathfrak{H})$.

We would like to thank Professor Ronald Douglas for the suggestion to "take the determinant" in the proof of our Theorem, and we express our appreciation to Professor Béla Sz.-Nagy for his comments during revision which greatly aided in the improvement of our exposition.

References

[1] H. Helson, *Lectures on invariant subspaces*, Academic Press (New York, 1964).
[2] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice Hall (Englewood Cliffs, N. J., 1962).
[3] T. L. Kriete III, B. Moore III, and L. B. Page, Compact intertwining operators, *Michigan Math. J.*, **18** (1971), 115—119.
[4] V. I. Lomonosov, Invariant subspaces of a family of operators commuting with a compact operator, *Funkcional. Anal. Priložen.*, **3** (1973). (Russian)
[5] P. S. Muhly, Compact operators in the commutant of a contraction, *J. Funct. Anal*, **8** (1971), 197—224.
[6] H. Radjavi and P. Rosenthal, *Invariant Subspaces*, Springer-Verlag (Berlin, 1973).
[7] D. Sarason, Generalized interpolation in H^∞, *Trans. Amer. Math. Soc.*, **127** (1967), 179—203.
[8] B. Sz.-Nagy and C. Foiaş, Commutants de certains opérateurs, *Acta Sci. Math.*, **29** (1968), 1—17.
[9] B. Sz. -Nagy and C. Foiaş, Dilatation des commutants d'opérateurs, *C. R. Acad. Sci. Paris*, **266** (1968), 493—495.
[10] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*, Akadémiai Kiadó (Budapest, 1970).

(Received July 27, 1973, revised October 17, 1973)

On a property of operators of class C_0

By BÉLA SZ.-NAGY in Szeged

In their paper [1] CLANCEY and MOORE prove (as a step to their main result) that for any contraction T of class C_0 and with finite defect indices there exists a nonzero compact operator commuting with T.

Recall that T is of class C_0 if it is completely non-unitary and $\varphi(T)=0$ for some inner function φ; among these functions there is a minimal one (i.e. which is a divisor in H^∞ of all the others), denoted by m_T. To every given nonconstant inner function m there exist contractions T of class C_0 with m_T equal to m; the simplest example is the operator $T=S(m)$ on the function space $\mathfrak{H}(m)$, defined by

(1) $$\mathfrak{H}(m) = H^2 \ominus mH^2, \quad S(m)u = P_{\mathfrak{H}(m)}(\chi u) \quad \text{for} \quad u \in \mathfrak{H}(m),$$

where $\chi(e^{it})=e^{it}$ and H^2 is the Hardy—Hilbert space for the unit disc. See [3], Chapter III.

By theorems of SARASON [2] the operators Φ commuting with $S(m)$ are precisely those which can be written in the form

(2) $$\Phi u = P_{\mathfrak{H}(m)}(\varphi u) \quad (u \in \mathfrak{H}(m)),$$

where φ is any fixed function in H^∞. Moreover, Φ is compact if and only if φ/m is, on the unit circle, the sum of a continuous function and of an H^∞ function. From (1) and (2) it follows, finally, that $\Phi \neq 0$ if and only if $\varphi \notin mH^2$, i.e. if $\varphi/m \notin H^\infty$.

Now for every nonconstant inner m there exists even $\varphi \in H^\infty$ such that φ/m is continuous on the unit circle, but not belonging to H^∞. If m has at least one (simple) Blaschke factor b then an obvious choice is $\varphi = m/b$. If m is a purely singular inner function, such a φ was constructed in [1].

Thus every operator $S(m)$ has a nonzero compact operator in its commutant. This property is shared by all contractions of class C_0. Indeed, we have

Theorem. *For every contraction T of class C_0 on a Hilbert space $\mathfrak{H}$ there exists a nonzero compact operator commuting with T.*

Proof. By virtue of Proposition 2 in [4] we have $T \succ S(m) \oplus T_1$ [1]) for some contraction T_1 of class C_0 and for $m=m_T$. Applying this to T^* as well and taking adjoints it also follows that

$$S(m)\oplus T_2 \succ T \succ S(m)\oplus T_1 \tag{3}$$

with some contractions T_i on spaces $\mathfrak{H}_i$ $(i=1, 2)$, and with $m=m_T$. Hence there exist quasi-affinities

$$X_1 : \mathfrak{H}(m)\oplus\mathfrak{H}_1 \to \mathfrak{H}, \quad X_2 : \mathfrak{H} \to \mathfrak{H}(m)\oplus\mathfrak{H}_2$$

such that

$$TX_1 = X_1(S(m)\oplus T_1), \quad (S(m)\oplus T_2)X_2 = X_2 T. \tag{4}$$

Now choose a nonzero compact operator Φ commuting with $S(m)$ and define, for $h\in\mathfrak{H}$,

$$Fh = X_1(\Phi P_2 X_2 h \oplus 0_1), \tag{5}$$

where 0_1 denotes the zero vector in $\mathfrak{H}_1$ and P_2 is the orthogonal projection of $\mathfrak{H}(m)\oplus\mathfrak{H}_2$ onto its subspace $\mathfrak{H}(m)\oplus\{0\}$, which we identify with $\mathfrak{H}(m)$.

Clearly, $P_2(S(m)\oplus T_2) = S(m)P_2$ and by (4) we have for $h\in\mathfrak{H}$

$$\begin{aligned} FTh &= X_1(\Phi P_2 X_2 Th \oplus 0_1) = X_1(\Phi P_2(S(m)\oplus T_2)X_2 h \oplus 0_1) = \\ &= X_1(\Phi\, S(m) P_2 X_2 h \oplus 0_1) = X_1(S(m)\Phi P_2 X_2 h \oplus 0_1) = \\ &= X_1(S(m)\oplus T_1)(\Phi P_2 X_2 h \oplus 0_1) = TX_1(\Phi P_2 X_2 h \oplus 0_1) = TFh. \end{aligned}$$

Hence, T commutes with F. Since Φ is compact so is F by its definition (5). Moreover $F\neq 0$. For, $F=0$ implies $\Phi P_2 X_2 = 0$ because X_1 has zero kernel, $\Phi P_2 X_2 = 0$ implies $\Phi P_2 = 0$ because X_2 has dense range, and $\Phi P_2 = 0$ simply means $P_2 = 0$. This contradicts the fact that $\mathfrak{H}(m)\neq\{0\}$ for nonconstant inner m.

Thus F is a nonzero compact operator on $\mathfrak{H}$ commuting with T.

Remark. F is, in general, not included in the weakly closed algebra generated by T and $I_{\mathfrak{H}}$. Hence, we have no immediate generalization of the Theorem in [1].

References

[1] K. Clancey—B. Moore III, Operators of class $C_0(N)$ and transitive algebras, *Acta Sci. Math.*, **36** (1974), 215—218.

[2] D. Sarason, Generalized interpolation in H^∞, *Trans. Amer. Math. Soc.*, **127** (1967), 179—203.

[3] B. Sz.-Nagy—C. Foiaş, *Harmonic analysis of operators on Hilbert space*, Akadémiai Kiadó (Budapest, 1970).

[4] B. Sz.-Nagy—C. Foiaş, Compléments à l'étude des opérateurs de classe C_0, *Acta Sci. Math.*, **31** (1970), 287—296.

(Received October 29, 1973)

[1]) $A\succ B$ means that there exists a „quasi-affinity" X (i.e. an operator with zero kernel and dense range) such that $AX=XB$.

Ideals in pseudo-complemented lattices and semi-lattices*)

By P. V. VENKATANARASIMHAN in Trivandrum (Kerala, India)

Introduction. The object of this paper is to develop a theory of ideal-extension for lattices and semi-lattices.

The first section contains some basic results about prime ideals in pseudo-complemented semi-lattices. We prove that in a pseudo-complemented semi-lattice, the set-complement of a maximal dual ideal is a minimal prime ideal. This establishes the existence of prime ideals in pseudo-complemented semi-lattices. The second section deals with ideal-extensions. If A is an ideal of the Boolean algebra N of normal elements of a pseudo-complemented semi-lattice S, the extension A_e of A is defined to be the least ideal of S containing A. By the extensional envelope of an ideal A of S we mean the least ideal B of S containing A such that B is the extension of some ideal of N. We show that the map $A \to A_e$ is an embedding of the set of all ideals of N into the set of all ideals of S which preserves meets, pseudo-complements, prime property and normality. In the concluding section we prove that in a pseudo-complemented lattice, proper ideals of the form A_e are precisely meets of minimal prime ideals. This result is used to obtain a characterization of the extensional envelope of an ideal of a pseudo-complemented lattice.

The author is thankful to the referee for his various suggestions which helped to improve the earlier version of the paper.

1. Prime ideals and normal ideals. A subset A of a partially ordered set P is called an ideal of P if (i) $a \in A$, $b \leqq a (b \in P) \Rightarrow b \in A$ and (ii) the lattice-sum of any finite number of elements of A, whenever it exists, belongs to A. The concept of dual ideal is defined in the dual fashion. Maximal ideal and maximal dual ideal are defined as usual. An ideal A of a semi-lattice is said to be prime if $ab \in A \Rightarrow a \in A$ or $b \in A$. A prime ideal which does not contain any other prime ideal is called a minimal prime ideal.

We need the following two known results for frequent use.

---

*) This paper formed a part of the author's Ph. D. thesis (University of Madras) prepared under the guidance of Professors V. S. Krishnan and V. K. Balachandran.

Lemma I ([8]). *Any proper ideal (dual ideal) of a partially ordered set with* 1 (0) *is contained in a maximal ideal (dual ideal).*

Lemma II ([4]). *The normal elements of a pseudo-complemented semi-lattice form a Boolean algebra.*

We shall now obtain some basic results about prime ideals in a semi-lattice. In the sequel S will denote a pseudo-complemented semi-lattice. The lattice-product of two elements a, b of S is denoted by ab and the lattice-sum of a, b, whenever it exists, is denoted by $a+b$. The pseudo-complement of an element x of S is denoted by x^*. The principal ideal and the principal dual ideal generated by an element x of S are denoted by $(x]$ and $[x)$ respectively. Set-inclusion, set-union and set-intersection are denoted by $\subseteqq$, $\cup$ and $\cap$ respectively. The lattice-products in the lattice of all ideals of S and in the lattice of all dual ideals of S coincide with the corresponding set-intersections. The lattice-sums in these lattices are denoted by $\vee$. As S is closed for pseudo-complements, so is the lattice of all ideals of S (vide [7, Theorem 8]). The pseudo-complement of an ideal A of S is denoted by A^*. An ideal A of S is said to be normal if $A=A^{**}$.

Theorem 1. *A dual ideal A of S is maximal if and only if A contains precisely one of x, x^* for every x in S.*

Proof. First we observe that a proper dual ideal A of S contains at most one of x, x^*; for $x, x^* \in A \Rightarrow 0 = xx^* \in A$. If A is a maximal dual ideal and $x \notin A$, then $A \vee [x) = S$. Hence $ax = 0$ for some $a \in A$; $a \leqq x^*$ and so $x^* \in A$.

Conversely, if A is a dual ideal satisfying the given condition and $x \notin A$, then $x^* \in A$ and so $0 = xx^* \in [x) \vee A$. Hence $[x) \vee A = S$. Thus A is maximal.

Corollary. *If M is a maximal dual ideal of S, $x^{**} \in M \Rightarrow x \in M$.*

Theorem 2. *A subset A of S is a minimal prime ideal if and only if its set-complement cA is a maximal dual ideal. Consequently, every prime ideal of S contains a minimal prime ideal.*

Proof. Suppose cA is a maximal dual ideal. Then, clearly, $x \in A$, $y \leqq x \Rightarrow y \in A$. Let $x_1, x_2, \ldots, x_n \in A$. Then, by Theorem 1, $x_1^*, x_2^*, \ldots, x_n^* \in cA$ and so $x_1^* x_2^* \ldots x_n^* \in cA$. If $x_1 + x_2 + \ldots + x_n$ exists, then $x_1 + x_2 + \ldots + x_n \in A$; for, otherwise $x_1 + x_2 + \ldots + x_n \in cA$, so that, $0 = (x_1 + x_2 + \ldots + x_n) x_1^* x_2^* \ldots x_n^* \in cA$, which is absurd. Thus A is an ideal. Now $x, y \notin A \Rightarrow x, y \in cA \Rightarrow xy \in cA \Rightarrow xy \notin A$. Hence A is prime. Let B be any prime ideal contained in A. Then cB is a dual ideal containing cA. By the maximality of cA it follows that $cA = cB$ and so $A = B$. Hence A is a minimal prime ideal.

Conversely, suppose A is a minimal prime ideal. It is easily seen that cA is a proper dual ideal. By Lemma I, there exists a maximal dual ideal M containing cA.

Clearly A contains cM and, by the first part, cM is a prime ideal. By the minimality of A it follows that $A=cM$ and so $cA=M$. Thus cA is a maximal dual ideal.

Corollary 1. *A prime ideal of S is minimal prime if and only if it contains precisely one of x, x^* for every x in S.*

Corollary 2. *If M is a minimal prime ideal of S, $x \in M \Rightarrow x^{**} \in M$.*

Theorem 3. *If A is an ideal of S, then A^* is the product of all the (minimal) prime ideals not containing A.*

Proof. It is easily seen that if a prime ideal contains the product of two ideals, it contains at least one of them. Hence every (minimal) prime ideal which does not contain A contains A^*. Let $x \notin A^*$. Then $xy=0$ for some $y \in A$. By Lemma I, there exists a maximal dual ideal M containing $[xy)$. By Theorem 2, cM is a (minimal) prime ideal. Clearly $x \notin cM$ and $A \nsubseteq cM$ as $y \notin cM$. Hence the theorem.

Corollary 1. *Every normal ideal of S is the product of all the (minimal) prime ideals containing it. In particular, the product of all the (minimal) prime ideals of S is* (0].

Corollary 2. *Every normal ideal of S is minimal prime.*

Remark. Theorem 3 has been proved by BALACHANDRAN [2] for distributive lattices.

Theorem 4. *S is a Boolean algebra if and only if either of the following two conditions holds:*

(i) *Every principal ideal of S is the product of all the minimal prime ideals containing it.*

(ii) *Every principal dual ideal of S is the product of all the maximal dual ideals containing it.*

Proof. If S is a Boolean algebra, then every prime ideal (prime dual ideal) of S is minimal prime (maximal) (*vide* [1, Theorem 10]). Also, in a distributive lattice, every ideal (dual ideal) is the product of all the prime ideals (prime dual ideals) containing it. Hence (i) and (ii) hold in S.

Suppose S is a pseudo-complemented semi-lattice satisfying (i) and $a \in S$. Then, by Corollary 2 of Theorem 2, it follows that $a=a^{**}$. Hence, by Lemma II, S is a Boolean algebra. Proceeding on similar lines and using the corollary of Theorem 1 we can prove that if (ii) holds in S, then also S is a Boolean algebra.

2. Extension of an ideal. In this section we extend some results of KRISHNAN [5] on ideal-extension in distributive lattices to semi-lattices.

Throughout this section N denotes the Boolean algebra of normal elements of

S. The lattice-product in N is the same as that in S and the lattice-sum in N is denoted by $\oplus$. If X is a subset of S we write X_N for $X \cap N$.

If A is an ideal of N, the least ideal of S containing A is called the extension of A and is denoted by A_e. By the extensional envelope of an ideal A of S we mean the least ideal B of S containing A such that B is the extension of some ideal of N.

Lemma 1. *Let A be an ideal of N. Then an element a of S is in A_e if and only if there exists an element b in A such that $a \leqq b$.*

Proof. Let $a \in A_e$. Then there exist elements $b_1, b_2, \ldots, b_n$ in A such that $(a] \subseteqq (b_1] \vee (b_2] \vee \ldots \vee (b_n]$. Clearly $(b_1] \vee (b_2] \vee \ldots \vee (b_n] \subseteqq (b_1 \oplus b_2 \oplus \ldots \oplus b_n]$. Putting $b_1 \oplus b_2 \oplus \ldots \oplus b_n = b$, we see that $a \leqq b$.

On the other hand, if $a \leqq b$ for some $b \in A$, then $b \in A_e$ and so $a \in A_e$.

Lemma 2. *The extension of an ideal A of N is the least ideal of S meeting N in A.*

Proof. Let $a \in (A_e)_N$. Then $a \in A_e$ and so, by Lemma 1, there exists $b \in A$ such that $a \leqq b$. Also $a \in N$. It follows that $a \in A$. Thus $(A_e)_N \subseteqq A$. The reverse inequality is obvious. Hence $(A_e)_N = A$. The fact that A_e is the least such ideal follows by the very definition of A_e.

Theorem 5. *If A is an ideal of N, then $a \in A_e \Rightarrow a^{**} \in A_e$.*

Proof. If $a \in A_e$, by Lemma 1, there exists $b \in A$ such that $a \leqq b$. Since $b \in N$, $b = b^{**}$, so that $a^{**} \leqq b$. Hence $a^{**} \in A$ and so $a^{**} \in A_e$.

Theorem 6. *The map $A \to A_e$ is an embedding of the set of all ideals of N into the set of all ideals of S which preserves meets and pseudo-complements and in both directions the prime property and normality. Furthermore every normal ideal of S belongs to the range of the map.*

Proof. Let $\{A_i : i \in I\}$ be a family of ideals of N and $x \in \bigcap_{i \in I} (A_i)_e$. Then, by Theorem 5, $x^{**} \in \bigcap_{i \in I} (A_i)_e$. Hence $x^{**} \in \bigcap_{i \in I} ((A_i)_e)_N = \bigcap_{i \in I} A_i$ (by Lemma 2) $\subseteqq (\bigcap_{i \in I} A_i)_e$. Therefore $x \in (\bigcap_{i \in I} A_i)_e$, so that $\bigcap_{i \in I} (A_i)_e \subseteqq (\bigcap_{i \in I} A_i)_e$. The reverse inequality is obvious. Hence $(\bigcap_{i \in I} A_i)_e = \bigcap_{i \in I} (A_i)_e$.

Let A be a prime ideal of N and $ab \in A_e$. Then, by Theorem 5, $(ab)^{**} \in A_e$ and so $(ab)^{**} \in (A_e)_N$; that is $a^{**}b^{**} \in A$. Hence, as A is prime, $a^{**} \in A$ or $b^{**} \in A$, so that, by Lemma 1, $a \in A_e$ or $b \in A_e$. Thus A_e is prime.

Let A be a normal ideal of N. Then $A = \bigcap_{b \in B} (b^*]_N$ for some ideal B of N. (Here $(b^*]$ denotes the principal ideal of S generated by b^*. Clearly $(b^*]_N$ is the principal ideal of N generated by b^*). By the first para, $A_e = \bigcap_{b \in B} ((b^*]_N)_e = \bigcap_{b \in B} (b^*]$. Hence A_e is normal.

If A is an ideal of N such that A_e is prime (normal), it is easily seen that A is prime (normal).

From the above it follows that the map $A \to A_e$ preserves meets and in both directions the prime property and normality. By Lemma 2, the map is one-to-one. Using Theorem 3, we see that the map preserves pseudo-complements.

Let A be a normal ideal of S. Then $A = \bigcap_{b \in B} (b^*]$ for some ideal B of S. Also $(b^*] = ((b^*]_N)_e$. Hence $A = (\bigcap_{b \in B} (b^*]_N)_e$. Clearly $\bigcap_{b \in B} (b^*]_N$ is a normal ideal of N. Thus every normal ideal of S belongs to range of the map $A \to A_e$.

The proof of the theorem is now complete.

3. Ideals in pseudo-complemented lattices. In this section we shall obtain some results supplementing those in the previous sections. Throughout this section L denotes a pseudo-complemented lattice and N the Boolean algebra of normal elements of L. For a subset X of L we abbreviate $X \cap N$ to X_N.

It is easily seen that a subset of a lattice is a prime ideal if and only if its set-complement is a prime dual ideal (*vide* [3, p. 141, Theorem 8]). Hence by Theorem 2 it follows that every maximal dual ideal of L is prime. (This result has been proved by STONE [6] for distributive lattices). This result together with Theorem 2 establishes the existence of prime ideals and prime dual ideals in a pseudo-complemented lattice.

Theorem 7. *A prime ideal A of L is minimal prime if $a \in A \Rightarrow a^{**} \in A$.*

Proof. Let $x \in L$. Since A is prime, A contains at least one of x, x^*. Suppose $x \in A$. Then $(x+x^*)^{**} = 1 \notin A$ and so, by hypothesis, $x+x^* \notin A$. Consequently $x^* \notin A$. By Corollary 1 of Theorem 2, it follows that A is minimal prime.

Theorem 8. *The extension of a prime ideal of N is a minimal prime ideal of L.*

Proof. Let A be a prime ideal of N. Then, by Theorem 6, A_e is a prime ideal of L. By Theorems 5 and 7 it follows that A_e is minimal prime.

Theorem 9. *The following statements about a proper ideal A of L are equivalent:*

(i) *A is a product of minimal prime ideals of L.*

(ii) $a \in A \Rightarrow a^{**} \in A$.

(iii) *A is the extension of some ideal of N.*

Proof. By Corollary 2 of Theorem 2, any minimal prime ideal of L contains a^{**} whenever it contains a. Hence so does any product of minimal prime ideals of L. Therefore (i)$\Rightarrow$(ii).

(ii)$\Rightarrow$(iii): Suppose A satisfies (ii). Then, clearly, $A_N = \{a^{**} : a \in A\}$. Let $a^{**}, b^{**} \in A_N$. Then $a^{**}+b^{**} \in A$ and so $a^{**} \oplus b^{**} \in A_N$. Also, if $c \leqq a^{**}$ and $c \in N$, then $c \in A_N$. Hence A_N is an ideal of N. If B is an ideal of L such that $B_N = A_N$ and

$x \in A$, then $x^{**} \in A_N \subseteqq B$. It follows that $x \in B$. Thus A is the least ideal of L meeting N in A_N. Hence, by Lemma 2, $A = (A_N)_e$.

(iii)$\Rightarrow$(i): Suppose (iii) holds. Since N is a Boolean algebra, every ideal of N is a product of prime ideals of N. By Theorems 6 and 8 it follows that A is a product of minimal prime ideals of L.

Corollary. *The extensional envelope of an ideal A of L is the product of all the minimal prime ideals of L containing A or L according as there exist or do not exist minimal prime ideals of L containing A.*

References

[1] V. K. Balachandran, Prime ideals and the theory of last residue classes, *J. Indian Math. Soc.* (N. S.), **13** (1949), 31—40.

[2] V. K. Balachandran, Stone's topologization of the distributive lattice, *Doctoral thesis, Madras University,* 1950.

[3] G. Birkhoff, *Lattice theory,* Amer. Math. Soc. Colloq. Publ. Vol. 25, Amer. Math. Soc. (Providence, R. I., 1948).

[4] O. Frink, Pseudo-complements in semi-lattices, *Duke Math. J.,* **29** (1962), 505—514.

[5] V. S. Krishnan, The problem of the last residue class in the distributive lattice, *Proc. Indian Acad. Sci.,* Sect. A **16** (1942), 176—190.

[6] M. H. Stone, Topological representations of distributive lattices and Brouwerian logics, *Časopis Pest. Mat. Fys.,* **67** (1937), 1—25.

[7] P. V. Venkatanarasimhan, Ideals in semi-lattices, *J. Indian Math. Soc.* (N. S.), **30** (1966), 47—53.

[8] P. V. Venkatanarasimhan, Pseudo-complements in posets, *Proc. Amer. Math. Soc.,* **28** (1971), 9—17.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KERALA
KARIAVATTOM
TRIVANDRUM, INDIA

(Received May 30, 1972, revised January 10, 1973)

Bibliographie

S. K. Berberian, Baer $*$-Rings (Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, Bd. 195), XIII+296 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1972.

As the author states in the preface, this work is an elaboration of Irving Kaplansky's ideas introduced in his book *Rings of operators*. A Baer $*$-ring is a ring with involution in which the right annihilator of every subset is a principal right ideal generated by some projection. Baer $*$-rings are abstract generalizations of von Neumann algebras. Although these rings are much more general than von Neumann algebras, they still have some nice properties of these, for example the projection lattices of all members in a vast class of Baer $*$-rings are continuous geometries in the sense of von Neumann.

The book is a systematic exposition of the theory of Baer $*$-rings. It contains three parts and these parts are subdivided into chapters. The first part deals with the basic concepts and results while the second presents the structure theory. The most delicate and interesting part is the third one on finite Baer $*$-rings where one finds the following chapters: Dimension in finite Baer $*$-rings; Reductions of finite Baer $*$-rings; The regular ring of a finite Baer $*$-ring; Matrix rings over Baer $*$-rings. There are many exercises at the ends of the paragraphs. They are graded A, B, C, D; grade A problems are the easiest ones, grade D problems are open questions. There is a more than twenty page "Hints, Notes and References" part at the end of the book, to help the reader solve the exercises.

J. Szűcs (Szeged)

P. C. Clemmov, An introduction to electromagnetic theory, XI+297 pages, London, Cambridge University Press, 1973.

The book gives a detailed exposition of the basic concepts of the theory of electromagnetism. At the beginning the author points out one of the most fundamental problems of the theory: what is the connection between electromagnetism at microscopic and at macroscopic level. From the misunderstanding of this connection, it resulted a wrong notion, that of the magnetic charges, and led to the confusion of the role of the magnetic quantities B and H for a long time. Nowadays one begins to revide these notions. The author works with B as magnetic field strength and in the last chapter he gives an excellent and clear treatment of space averaging and electromagnetic media with special regard to magnetization and magnetic media. Magnetic charges, even virtual surface charge densities arising from a magnetic dipole density are ruled out.

The book, as its title shows, is introductory. Therefore complicated mathematics are avoided. It would be right, however, to call attention to some mathematical problems which are very elementary in nature and are connected with the relation between microscopic and macroscopic electromagnetism, based on the essential difference between the descriptions of point charges and spatially distributed charges. These mathematical difficulties show the fundamental problems not only of classical electromagnetic theory but also of quantum field theory. To mention the most simple, the equation $\oint E ds=0$ for static field is not valid if the curve of integration passes through a point charge.

Apart from this the book gives a good introduction to the modern theory of electromagnetism with special emphasis on the physical backgrounds. The material is well organized and many special problems are treated to illustrate the results of the theory.

T. Matolcsi (Szeged)

Computers and computation. Readings from Scientific American, 283 pages, San Francisco, W. H. Freeman & Co., 1970.

The book is a collection of papers on computers and computer science appeared during the past two decades in the columns of the *Scientific American* — one of the highest ranking popular magazines in this field. The 25 articles are ordered in five sections, each beginning with introductory remarks of the editors.

The First Section contains articles of primarily technical nature: fundamentals of logic circuitries, elementary programming, etc. The impression gained about the speed of progress, however, turns out to be more interesting than mere technicalities. In 1966 I. E. Sutherland writes: "New devices... are changing computers from hard-to-use consultants into ready tools to aid human thought" — and describes the brand-new graphic display as example. Four years later the same author speaks about a newly developed branch of science: computer graphics complaining about the difficulties in obtaining three-dimensional, colored drawings on the screen. Other articles about computer-aided molecular synthesis, thechnical drawing, etc. provide and abundant illustrative material confirming the correctness of Sutherland's wievs.

Artificial intelligence is the core of the Second Section. The authors are: C. E. Shannon, O. G. Selfridge, M. Minsky, and others. Different implementations as chess-playing algoritms, pattern recognition techniques, linguistic applications, etc., are richly illustrated in that chapter. Although progress in this field of computer science has not been so rapid as in technology, there is enough evidence that yesterday's research is becoming today's routine as, for example, in the printed character recognition. The presently developed new techniques ("interlevel communication", "hierarchically structured artificial intelligence" structures) can easily turn out to be the routines of tomorrow.

The few articles grouped in the Third Section (Mathematics) try to illustrate for the nonprofessional reader some problems, as the theoretical limit of addition speed available, remainder addition techniques, problems solved and unsolved in mathematical logic, in number theory, combinatorics, etc.. Among the authors are H. Wang, S. M. Ulam and D. D. McCraken.

Since the time of Turing much concern has been focused on determining the set of problems which computers cannot solve. The article by H. Wang shows that questions about the capability all lie in the overlapping area between computer theory and logic. Another question dealt with in the papers of both Ulam and McCraken is the nature of randomness. The problem is illustrated by describing the Monte Carlo technique invented by Ulam and first used to neutron shielding by J. von Neumann. McCraken's examples showing how one can rely on probability in solving real-to-life questions having apparently no relation to it — are really highlighting.

Chapter IV (Modelling) covers a wide-band spectrum of fascinating problems reaching from computer-aided molecular model building to system analysis of man as machine (including genetical reproduction) on the one side, fluid dynamics, meteorological structures and urban transportation systems on the other.

All the articles follow Turing in demonstrating that computers are universal in a nontrivial sense. The calculating power of the sequential machines is of course fundamentally limited in speed by that of light. Much attention is therefore given to parallel processing and examples are to be found illustrating contributions expectable if parallel processing is combined with modelling techniques.

I. Madarász (Szeged)

Siegfried Flügge, Practical quantum mechanics, I—II (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. 177—178), XIV+341, XII+287 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1971.

There are many books introducing to the theory of quantum mechanics but one hardly finds a monograph collecting its practical methods necessary for both theoretical and experimental physicists.

The fundamentally enlarged English edition of the author's book in German published in 1947 fills this gap. The book consists of 219 problems with solutions. It seems so that all important general problems, for instance momentum and angular momentum representations, scattering theory, partial wave expansion, spin functions for two and three particles, many body problems, approximation methods and others are included, followed by applications to diverse special problems. There are chapters devoted to areas beyond the quantum mechanics proper: they contain problems of relativistic quantum mechanics, radiation theory and field quantization. The often needed special functions and series are cited in a Mathematical Appendix.

It is a pity that, in the introductory chapter for general concepts and sometimes elsewhere too, Hilbert space and its operators are used in a formal way only and there are no hints to places where these notions and techniques can be found in an exact form.

Nevertheless, the book may prove to be a useful and perhaps indispensable tool for those who are applying quantum mechanics to various problems in physics.

T. Matolcsi (Szeged)

H. Heyer, Mathematische Theorie statistischer Experimente, XXII+209 Seiten, Berlin—Heidelberg—New York, Springer Verlag, 1973.

Ein statistisches Experiment ist gewisse Auswertung einer Stichprobe mit dem Ziel, auf Grund derselben eine Entscheidung zu treffen. Das Buch bietet eine sehr modrene Darstellung gewisser wichtiger Themakreise der Theorie statistischer Experimente. Die Behandlung beschränkt sich auf die wichtigsten Fragen der Test- und Schätztheorie im finiten Rahmen, also ohne Berücksichtigung der asymptotischen Theorie. Unter Heranziehung von Maßtheorie und Funktionalanalysis wird der allgemeine Erschöpftheitsbegriff (auch im nicht dominierten Fall) diskutiert, die Existenz trennscharfer Tests analysiert, der Begriff der Minimalschätzung eingeführt und die Fragen des Vergleiches von Experimenten ausführlich dargestellt. Die Betrachtungsweise setzt breite Vorkenntnisse aus der Wahrscheinlichkeitstheorie, mathematischer Statistik, und Maßtheorie voraus. Das Lesen wird mit Literaturhinweisen und mit einer Zusammenfassung der angewandten tiefliegenden Sätze erleichtert.

K. Tandori (Szeged)

A. P. Robertson—W. Robertson, Topological vector spaces, second edition (Cambridge Tracts in Mathematics and Mathematical Physics No. 53), VIII+172 pages, Cambridge, University Press, 1973.

The book gives an easily readable introduction to the theory of topological vector spaces. The authors assumed only a minimal knowledge of general topology and linear algebra and what is assumed is compiled at the beginning of the first chapter. After this, new topological concepts are introduced when needed. The first six chapters have supplementary sections containing illustrative examples and further results. The supplements sometimes use notions not, or not yet, defined in the text. The chapter headings are: I. Definitions and elementary properties; II. Duality and the Hahn—Banach theorem; III. Topologies on dual spaces and the Mackey—Arens theorem; IV. Barreled spaces and the Banach—Steinhaus theorem; V. Inductive and projective limits; VI. Completeness and the closed graph theorem; VII. Some further topics (1. Strict inductive limits. 2. Bilinear mappings and tensor products. 3. The Krein—Milman theorem); VIII. Compact linear mappings; Appendix.

There is a bibliography and an index at the end of the book.

The only changes in this second edition with respect to the first one are: 1) the inclusion of an appendix on spaces with webs in order to bring up to date the discussion of the closed graph theorem; 2) the removal of a number of errors and obscurities occurring in the first edition.

J. Szűcs (Szeged)

Derek J. S. Robinson, Finiteness Conditions and Generalized Soluble Groups, Part 1, XV+210 pages; Part 2, XIII+254 pages (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 62—63), Berlin—Heidelberg—New York, Springer-Verlag, 1972.

The simple but very fruitful idea of investigating those territories in the immense realm of infinite noncommutative groups that lie nearer to the more favourable regions of finite and commutative ones is a perpetual motive of group-theoretical research. It is a great and hard job to systematize the investigations in this area ramifying in many directions so that it is no wonder if till now there has not been any monograph on it. Of course, Part 4 of Kuroš's book "The Theory of Groups" is an excellent introduction to these topics, but it is clear that an independent comprehensive book is needed (e.g., from the fact that the bibliography of such a book must include more than a thousand items — as the considered work of D. J. S. Robinson testifies).

Robinson's book has got a lot of good properties. Its bulk is not very big but it contains a very large material owing to its carefully planned scheme and the many improved proofs. Authors writing in Russian played a definitive role in the development of the topic dealt with; their work is quoted here with exemplary completeness. Furthermore, the book is as up-to-date as possible; it contains references up to 1970. It is written in an informal, clear style; the reader is assumed to have a stable knowledge of the elements of group theory.

Let us, finally, give the list of chapters: 1. Fundamental Concepts in the Theory of Infinite Groups 2. Soluble and Nilpotent Groups. 3. Maximal and Minimal Conditions. 4. Finiteness

Conditions on Conjugates and Commutators. 5. Finiteness Conditions on the Subnormal Structure of a Group. 6. Generalized Nilpotent Groups. 7. Engel Groups. 8. Local Theorems and Generalized Soluble Groups. 9. Residually Finite Groups. 10. Some Topics in the Theory of Infinite Soluble Groups.

B. Csákány (Szeged)

S. Sakai, C^*-Algebras and W^*-Algebras (Ergebnisse der Mathematik und ihrer Grenzgebiete, Bd. 60), XII+256 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1971.

The theory of C^*- and W^*-algebras has grown so tremendously in the last two decades, that it seems to be very unlikely that anybody would attempt to write a complete monograph on the subject. Sakai says in his book's preface that he has "no intention of giving a complete coverage" and his "selection is concentrated heavily on the topics with which" he has been "more or less concerned". Comparing Sakai's treatise with Dixmier's two classics, one finds that Sakai's work covers the major part of the material presented in "Les algèbres d'opérateurs dans l'espace hilbertien (Algèbres de von Neumann)" but it provides considerably less information on C^*-algebras than "Les C^*-algèbres et leurs représentations". It also contains very important more recent results which were obtained after the publications of Dixmier's books. Among these are the commutation theorem of tensor products, decompositions of states, derivations of C^*- and W^*-algebras, uncountable families of type II_1, II_∞, and III factors and examples of global type II_1, II_∞, and III W^*-algebras.

Sakai's exposition is modern and concise. He defines W^*-algebra as a C^*-algebra that has a Banach-space predual and builds up the fundamentals of the theory in an abstract way. He introduces the notion of trace after the classification of W^*-algebras and he classifies W^*-algebras by relying on the comparability theorem of projections. The reduction theory of von Neumann is elaborated by using the recent decomposition theory of states. At the end of his book the author constructs uncountable families of type II_1, II_∞ and III factors and shows that von Neumann's reduction theory is not trivial.

Sakai's work contains four chapters: General theory; Classification of W^*-algebras; Decomposition theory; Special topics. The chapters are diveded into sections. Almost every section has a list of references and most of them have "concluding remarks". The book contains no exercises; however, some sections indicate unsolved problems and some "concluding remarks" give suggestions for research. It is a very valuable contribution to the literature on the subject.

J. Szűcs (Szeged)

K. Sarkady—I. Vincze, Mathematical Methods of Statistical Quality Control, 415 pages, Budapest, Akadémiai Kiadó, 1974.

The book consists of three chapters. The first one is a 12 page introduction. The second one, which is 215 pages long, contains the systematic exposition of the fundamentals of probability theory and mathematical statistics. The third, 147 pages, presents the methods of statistical quality control. In this chapter the following topics are considered: Statistical methods in the control of production processes, Process control by variables, Control by attributes, The choice of the interval between two consecutive controls, Acceptance sampling, Sampling by attributes, Sampling inspection by variables, Sequential sampling, Acceptance plans and production as a stochastic process, Reliability theory. The Appendix at the end of the book contains numerous useful tables. The book's exposition is simple, it avoids measure theoretic notions and does not provide proofs for the results considered, just as the earlier book of I. Vincze, "Mathematische Statistik mit industriellen Anwendungen" (Budapest, Akadémiai Kiadó, 1971). The book can be useful first of all for those who want to study or apply the basic results of mathematical statistics and those of statistical quality control.

K. Tandori (Szeged)

B. Segre, Some properties of differentiable varieties and transformations (Ergebnisse der Mathematik, und ihrer Grenzgebiete, Bd. 13), Second edition, IX+195 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1971.

The first edition of the book appeared in 1957 and it was a summary of a series of very new results as well as a good guide to the theory of several branches of mathematics whose chief characteristic is that of establishing suggestive and sometimes unforeseen relations between apparently diverse subjects.

The basic topics are investigations on differential, topological and projective invariants of differentiable and analytic point transformations between two portions of two Euclidean spaces. Some of these invariants are related to a class of important systems of curves, the study of which permits to extend in certain directions the projective-differential theory of surfaces, and it is shown how the theory of residues of analytic functions can be employed to the study of differential properties of analytic curves. Among many other interesting related investigations and results some aspects of linear partial differential equations are treated especially in Chapter IX added to the first edition.

At the end of each chapter there are brief "Historical Notes and Bibliography" which contain sufficient references to enable the reader to pursue his study further on this field.

T. Matolcsi (Szeged)

INDEX — TARTALOM

ACTA SCIENTIARUM MATHEMATICARUM

SZEGED (HUNGARIA), ARADI VÉRTANÚK TERE 1

On peut s'abonner à l'entreprise de commerce des livres et journaux „Kultúra" (1061 Budapest, I., Fő utca 32)

INDEX: 26 024

74-366 — Szegedi Nyomda

Felelős szerkesztő és kiadó: Szőkefalvi-Nagy Béla
A kézirat a nyomdába érkezett: 1974. január 23.
Megjelenés: 1974. június hó

Példányszám: 1200. Terjedelem: 20,3 (A/5) ív
Készült monószedéssel, íves magasnyomással,
az MSZ 5601-24 és az MSZ 5602-55 szabvány szerint