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A JÓZSEF ATTILA TUDOMÁNYEGYETEM KÖZLEMÉNYEI

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Compact restrictions of operators. II

By ARLEN BROWN and CARL PEARCY in Bloomington (Indiana, U. S. A.)

1. Introduction. As the title indicates, this paper is a continuation of [1], and, accordingly, we shall assume that the reader is familiar with the results and terminology of that note. In particular, it should be recalled that if T is an operator from a Hilbert space $\mathscr{H}$ to a Hilbert space $\mathscr{K}$, then T is said to be *affiliated with* a given ideal $\mathfrak{J}$ in $\mathscr{L}(\mathscr{H})$ if the operator $(T^*T)^{\frac{1}{2}}$ belongs to $\mathfrak{J}$. (In this paper, as in [1], all Hilbert spaces will be assumed to be *complex*, *separable*, and *infinite dimensional*, and all operators will be assumed to be *bounded* and *linear*. Furthermore, $\mathscr{L}(\mathscr{H})$ will denote the algebra of all operators on a Hilbert space $\mathscr{H}$, and all ideals in $\mathscr{L}(\mathscr{H})$ referred to will be *two-sided*.)

The following result is [1, Theorem 3. 1]. It is central to our present needs, and we restate it here for convenience of reference.

Theorem A. *Let $\mathfrak{J}$ be any ideal in $\mathscr{L}(\mathscr{H})$ other than the ideal of operators of finite rank, and let T be any operator on $\mathscr{H}$. Let λ be any fixed scalar in the boundary of the Calkin spectrum of T, and let ε be any positive number. Then there exists a decomposition of $\mathscr{H}$ into infinite dimensional subspaces $\mathscr{K}$ and $\mathscr{K}^{\perp}$ such that the restriction $(T-\lambda)|\mathscr{K}$ of $T-\lambda$ to $\mathscr{K}$ $((T-\lambda)|\mathscr{K}:\mathscr{K}\to\mathscr{H})$ is affiliated with the ideal $\mathfrak{J}$ and has norm less than ε.*

In [1], Theorem A was used to show that every operator in $\mathscr{L}(\mathscr{H})$ is unitarily equivalent to a particular kind of 2×2 operator matrix, and this result was then applied to obtain certain results in the theory of commutators. In this note, we again employ Theorem A, this time to show that every operator on a Hilbert space $\mathscr{H}$ is unitarily equivalent to a 3×3 operator matrix of a certain form (Theorem 2. 1); this result is then used to prove an interesting theorem concerning the ranges of derivations on $\mathscr{L}(\mathscr{H})$.

2. A matricial standard form. The purpose of this section is to prove the following rather surprising theorem.

Theorem 2. 1. *Let $\mathfrak{J}$ be any ideal in $\mathscr{L}(\mathscr{H})$ other than the ideal of operators of finite rank, and let T be any operator on $\mathscr{H}$. Let λ be any fixed scalar in the bound-*

ary of the Calkin spectrum of T, and let ε be any positive number. Then there exists a unitary isomorphism of $\mathscr{H}$ onto $\mathscr{H}\oplus\mathscr{H}\oplus\mathscr{H}$ which carries the operator $T-\lambda$ onto a 3×3 operator matrix (with entries from $\mathscr{L}(\mathscr{H})$) of the form

$$(*)\qquad \begin{pmatrix} J_{11} & A & B \\ J_{21} & C & D \\ J_{31} & J_{32} & J_{33} \end{pmatrix},$$

where $J_{11}, J_{21}, J_{31}, J_{32}$, and J_{33} all belong to the ideal $\mathfrak{J}$, and all have norm less than ε.

The proof of Theorem 2. 1 depends on Theorem A and the following elementary lemma.

Lemma 2. 2. *Let $\mathscr{M}$ and $\mathscr{N}$ be any two infinite dimensional subspaces of $\mathscr{H}$. Then there exist infinite dimensional subspaces $\mathscr{M}_1\subset\mathscr{M}$ and $\mathscr{N}_1\subset\mathscr{N}$ such that $\mathscr{M}_1$ and $\mathscr{N}_1$ are orthogonal.*

Proof. Let x_1 be any unit vector in $\mathscr{M}$. Then, since $\mathscr{N}$ has dimension greater than 1, there exists a unit vector y_1 in $\mathscr{N}$ that is orthogonal to x_1. Since $\mathscr{M}$ has dimension greater than 2, it follows that there exists a unit vector x_2 in $\mathscr{M}$ that is orthogonal to x_1 and to y_1. Continuing via an obvious induction argument, we obtain orthonormal sequences $\{x_n\}_{n=1}^{\infty}$ in $\mathscr{M}$ and $\{y_n\}_{n=1}^{\infty}$ in $\mathscr{N}$ such that for every pair j, k of positive integers, x_j is orthogonal to y_k. The proof is completed by taking for $\mathscr{M}_1$ and $\mathscr{N}_1$ the subspaces spanned by the sequences $\{x_n\}$ and $\{y_n\}$, respectively.

Proof of Theorem 2. 1. According to Theorem A, there exists an infinite dimensional subspace $\mathscr{M}$ of $\mathscr{H}$ such that $(T-\lambda)|\mathscr{M}$ is affiliated with the ideal $\mathfrak{J}$ and has norm less than ε. Furthermore, since $\bar{\lambda}$ belongs to the boundary of the Calkin spectrum of T^*, it also follows from Theorem A that there exists an infinite dimensional subspace $\mathscr{N}$ of $\mathscr{H}$ such that $(T^*-\bar{\lambda})|\mathscr{N}$ is affiliated with $\mathfrak{J}$ and has norm less than ε. If we now apply Lemma 2. 2 to $\mathscr{M}$ and $\mathscr{N}$, we obtain infinite dimensional subspaces $\mathscr{M}_1\subset\mathscr{M}$ and $\mathscr{N}_1\subset\mathscr{N}$ such that $\mathscr{M}_1$ and $\mathscr{N}_1$ are orthogonal. Furthermore, it is obvious that the choices of $\mathscr{M}_1$ and $\mathscr{N}_1$ can be made in such a way that $\mathscr{K}_1=(\mathscr{M}_1\oplus\mathscr{N}_1)^{\perp}$ is also infinite dimensional. Let $\tilde{\mathscr{H}}$ denote the threefold direct sum $\tilde{\mathscr{H}}=\mathscr{H}\oplus\mathscr{H}\oplus\mathscr{H}$, and let the subspaces $\mathscr{H}\oplus 0\oplus 0$, $0\oplus\mathscr{H}\oplus 0$, and $0\oplus 0\oplus\mathscr{H}$ of $\tilde{\mathscr{H}}$ be denoted by $\tilde{\mathscr{H}}_1$, $\tilde{\mathscr{H}}_2$, and $\tilde{\mathscr{H}}_3$, respectively. Choose φ to be any Hilbert space isomorphism of $\mathscr{H}$ onto $\tilde{\mathscr{H}}$ such that $\varphi(\mathscr{M}_1)=\tilde{\mathscr{H}}_1$, $\varphi(\mathscr{K}_1)=\tilde{\mathscr{H}}_2$, and $\varphi(\mathscr{N}_1)=\tilde{\mathscr{H}}_3$. Then it is clear that the operator $\tilde{T}=\varphi T\varphi^{-1}$ on $\tilde{\mathscr{H}}$ has the property that the restrictions $(\tilde{T}-\lambda)|\tilde{\mathscr{H}}_1$ and $(\tilde{T}-\lambda)^*|\tilde{\mathscr{H}}_3$ are both affiliated with the ideal $\mathfrak{J}$ and have norm less than ε. It follows easily (see, for example, Theorem 3. 1 of [1] and the remark following) that if $\tilde{T}-\lambda$ is written as a 3×3

matrix with entries from $\mathscr{L}(\mathscr{H})$ in the usual way, then all of the entries in the first column and third row of this matrix belong to the ideal $\mathfrak{J}$ and all have norm less than ε. Thus the proof is complete.

3. Application to derivations. In this section we apply Theorem 2. 1 to obtain a result concerning the ranges of derivations on $\mathscr{L}(\mathscr{H})$. Recall that such a derivation is a linear function D mapping $\mathscr{L}(\mathscr{H})$ into itself satisfying the equation $D(AB) = D(A)B+AD(B)$ for every pair A, B of operators on $\mathscr{H}$. It has been known for some time [4, Theorem 9] that every derivation on $\mathscr{L}(\mathscr{H})$ is an *inner derivation*; i.e., if D is such a derivation, then there exists an operator T on $\mathscr{H}$ such that $D(A) = TA-AT$ for every operator A in $\mathscr{L}(\mathscr{H})$. We shall indicate this relationship between a given derivation D and the operator T by writing $D=D_T$. (The operator T associated with D is not unique, since, if λ is any scalar, then $D_T=D_{T-\lambda}$.)

Theorem 3. 1. *Let $\mathfrak{J}$ be any ideal in $\mathscr{L}(\mathscr{H})$ other than the ideal $\mathfrak{F}$ of operators of finite rank. Then there exists no derivation D on $\mathscr{L}(\mathscr{H})$ whose range contains $\mathfrak{J}$.*

Proof. As noted above, we may assume that D is of the form $D=D_{T-\lambda}$, where T is some operator on $\mathscr{H}$ and λ is a fixed scalar in the boundary of the Calkin spectrum of T. It follows from [2, Theorem 4. 7] that there exists an ideal $\mathfrak{K}$ in $\mathscr{L}(\mathscr{H})$ such that $\mathfrak{F}\subsetneqq\mathfrak{K}\subsetneqq\mathfrak{J}$. Therefore, according to Theorem 2. 1, $T-\lambda$ is unitarily equivalent to a 3×3 operator matrix M acting on $\mathscr{H}\oplus\mathscr{H}\oplus\mathscr{H}$ with the property that all entries in the first column and all entries in the third row of M lie in $\mathfrak{K}$. Let J be an operator in $\mathfrak{J}$ that does not belong to $\mathfrak{K}$, and let J' be the operator on $\mathscr{H}$ whose image under the given unitary isomorphism between $\mathscr{H}$ and $\mathscr{H}\oplus\mathscr{H}\oplus\mathscr{H}$ is the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ J & 0 & 0 \end{pmatrix}.$$

Then clearly J' belongs to $\mathfrak{J}$, and, since the product (in either order) of M with every 3×3 matrix with entries from $\mathscr{L}(\mathscr{H})$ can be seen by calculation to have the property that its (3, 1) entry lies in the ideal $\mathfrak{K}$, it follows that the range of the derivation $D_{T-\lambda}$ does not contain J'; thus the proof is complete.

Note that the proof just concluded actually proves somewhat more than Theorem 3. 1. We include this stronger result as a proposition.

Proposition 3. 2. *Let $\mathfrak{J}$ be any ideal in $\mathscr{L}(\mathscr{H})$ other than the ideal of operators of finite rank, and let T be any operator on $\mathscr{H}$. Then for each fixed λ in the boundary of the Calkin spectrum of T, the linear manifold*

$$\{(T-\lambda)X-Y(T-\lambda): X,\quad Y\in\mathscr{L}(\mathscr{H})\}$$

fails to contain the ideal $\mathfrak{J}$.

4. Some comments. Although considerable progress has been made in commutator theory in the past few years, many questions concerning derivations remain unanswered. It is not known, for example, whether there exists a derivation on $\mathscr{L}(\mathscr{H})$ with the property that the identity operator lies in the (uniform) closure of its range. Furthermore, it is not known whether the ideal of finite rank operators is contained in the range of any derivation. (*Added in proof.* This point has also been settled in the negative by STAMPFLI.) Thus, it would appear that the topic of derivations on $\mathscr{L}(\mathscr{H})$ is an interesting area for continued investigation. In this connection it should be noted that J. G. STAMPFLI [5] has recently proved the pretty theorem that no derivation on $\mathscr{L}(\mathscr{H})$ has range that is norm dense in $\mathscr{L}(\mathscr{H})$. A different proof of this theorem can be given by using Theorem 2. 1 above. (It was known previously [3, Theorem 4] that derivation by the unilateral shift of multiplicity one has range dense in $\mathscr{L}(\mathscr{H})$ in the strong operator topology.)

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The infimum of two projections

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1. The purpose of this note is to give a closed formula for the infimum of two projections in Hilbert space. Heretofore, this infimum has been expressed by an iterative scheme as follows:[1]) If P and Q are projections on closed subspaces M and N, then the projection $P \wedge Q$ on $M \cap N$ is given by

(1) $$P \wedge Q = \lim_{n \to \infty} P(QP)^n.$$

We shall show that

(2) $$P \wedge Q = 2P(P+Q)^+ Q$$

if $M+N$ is closed, where $(P+Q)^+$ denotes the inverse of $P+Q$ restricted to its range. This formula is exactly that given by ANDERSON and DUFFIN [2] for matrices, and the result of this note is that it generalizes to infinite dimensions as stated. We also give an apparently heretofore unstated necessary and sufficient condition for $M+N$ to be closed (namely, that M and N make a positive angle modulo $M \cap N$).

The combination $A(A+B)^+B$, introduced in [2], is called the parallel sum, so named because of its origin in and application to electrical network theory ($(r_1^{-1}+r_2^{-1})^{-1} = r_1(r_1+r_2)^{-1}r_2$ is the resistance arising from resistors r_1 and r_2 in parallel). A second formulation of the parallel sum in finite dimensions is given by ANDERSON [1].

We shall use the following notations: P_M for the projection on M, $R(A)$ and $N(A)$ for range and nullity, M^- and $M^\perp$ for closure and orthocomplement, $A|M$ for restriction.

2. We begin by noting some facts about non-negative bounded operators.

For hermitian operators generally one knows that $R(A) = A\overline{R(A)}$, that A is one-one on $\overline{R(A)}$, and A is invertible on $R(A)$ if and only if $R(A) = \overline{R(A)}$ (closed graph theorem).

[1]) See [4] for background.

Hence for non-negative operators we have

(i) $R(A)$ closed if and only if A is bounded away from 0 on $R(A)$.

It follows that $R(A)$ is closed if and only if $R(A^{1/2})$ is closed. Since $N(A)=N(A^{1/2})$ (because by positivity $(Ax,x)=0$ if and only if $x=0$), and therefore $\overline{R(A)}=\overline{R(A^{1/2})}$, we have

(ii) $R(A)$ closed implies $R(A)=R(A^{1/2})$.

T. Crimmins has shown that $R(S)+R(T) = (SS^*+TT^*)^{1/2}$ for any bounded operators S and T.[2]) From this we get

(iii) If $R(A)$, $R(B)$, and $R(A+B)$ are closed, then

$$R(A)+R(B)=R(A+B), \tag{3}$$

for $R(A)+R(B) = R(A^{1/2})+R(B^{1/2}) = R((A+B)^{1/2}) = R(A+B)$.

As it follows from the preceding computation that $R(A)+R(B)$ is closed if and only if $R((A+B)^{1/2})$ is closed, we have

(iv) If $R(A)$ and $R(B)$ are closed, a necessary and sufficient condition that $R(A+B)$ be closed is that $R(A)+R(B)$ be closed.

Next, we introduce a generalized inverse (see [3] for background) and the parallel sum. If A has closed range, define A^+ by

$$A^+|R = (A|R)^{-1}, \quad A^+|N = 0. \tag{4}$$

If $A+B$ has closed range, define the parallel sum $A:B$ by

$$A:B = A(A+B)^+B. \tag{5}$$

This operation satisfies

$$A:B = B:A = (A:B)^*, \quad R(A:B) = R(A)\cap R(B) \tag{6}$$

if $R(A)$ and $R(B)$ are closed, as we shall presently show. The equations $R(A+B) = R(A)+R(B)$ and $R(A:B) = R(A)\cap R(B)$, valid when $R(A)+R(B)$ is closed, show that the set of $A\geqq 0$ with closed range is almost a lattice, in the sense that the sup and inf, inherited from the lattice of subspaces, exist whenever the sum of the ranges is closed. If we now think of projections P and Q in this context and ask what significance attaches to their sup and inf, we are led to the following theorem:

[2]) The result is unpublished and was told to us by J. P. Williams. Proof: Write $A=\begin{pmatrix} S & -T \\ O & O\end{pmatrix}$ on $H\oplus H$. Then $\{R(S)+R(T)\}\oplus\{0\} = R(A) = R((AA^*))^{1/2} = R((SS^*+TT^*))^{1/2}\oplus\{0\}$.

Theorem. *If P and Q are projections such that $R(P)+R(Q)$ is closed, then $2P:Q = P\wedge Q$.*

Note that by (iv) the hypothesis is equivalent to assuming $R(P+Q)$ closed.

To prove the theorem, we first prove the identities (6) (following [2]). By (iii) we have $R(A)$, $R(B) \subset R(A+B)$. Hence $B(A+B)^+(A+B) = B$, $(A+B)^+(A+B)A = A$ (because $(A+B)^+(A+B)$ is the projection on $R(A+B)$). Then $A:B = A(A+B)^+B = (A+B-B)(A+B)^+(A+B-A)$ easily reduces to $B(A+B)^+A = B:A$. Since $(A:B)^* = B:A$, we have $(A:B)^* = A:B$. By the commutativity again we have $R(A:B) \subset R(A)\cap R(B)$. If $x\in R(A)\cap R(B)$, then

$$(A:B)(A^+ + B^+)x = A(A+B)^+BB^+x + B(A+B)^+AA^+x =$$
$$= (A(A+B)^+ + B(A+B)^+)x = P_{R(A+B)}x = x.$$

Hence $R(A:B) = R(A)\cap R(B)$. This last computation simplifies, if A and B are projections P and Q, to $2P(P+Q)^+Qx = 2(P:Q)x = x$ for $x\in R(P)\cap R(Q)$. Thus $2(P:Q)$ is a hermitian idempotent with range $R(P)\cap R(Q)$, and this proves the theorem.

3. We conclude by showing that $R(P)+R(Q)$ is closed if and only if $R(P)$ and $R(Q)$ make a positive angle modulo $R(P)\cap R(Q)$. To establish this, we first consider the case when the two subspaces, which we denote by M and N, are disjoint. Then the assertion is that $M+N$ is closed if and only if

$$|(x, y)| \leqq \|x\|\|y\|(1-\delta) \quad (\delta > 0) \tag{7}$$

for all $x\in M$, $y\in N$.

Suppose (7) holds, and consider a Cauchy sequence $u_n = x_n + y_n$ in $M+N$. Write $u_{nm} \equiv u_n - u_m$, etc. Then

$$\|u_{nm}\|^2 = \|x_{nm}\|^2 + \|y_{nm}\|^2 + 2\mathrm{Re}\,(x_{nm}, y_{nm}) \geqq \|x_{nm}\|^2 + \|y_{nm}\|^2 -$$
$$-2(1-\delta)\|x_{nm}\|\,\|y_{nm}\| = \{\|x_{nm}\| - \|y_{nm}\|\}^2 + 2\delta\|x_{nm}\|\,\|y_{nm}\|.$$

It follows that $u_{nm}\to 0$ implies $x_{nm}\to 0$, $y_{nm}\to 0$, and therefore that u_n converges in $M+N$.[3])

Conversely, suppose $M+N$ is closed. It then follows that the "coordinate" map $T: M+N \to M$ given by $T(x+y) = x$ is continuous. The proof consists in verifying that T has closed graph and then applying the closed graph theorem. This argument is due to Kober [5], and as the verification is simple we omit it. Since T is bounded, we have

$$\|x\| \leqq A\,\|x\pm y\|$$

[3]) This formulation of the argument is due to B. Glickfeld.

for all $x \in M$, $y \in N$. Taking $\|x\| = \|y\| = 1$ and squaring, we get

$$1 \leqq A^2(2 \pm 2\,\mathrm{Re}\,(x, y))$$

whence $|\mathrm{Re}\,(x, y)| \leqq 1 - \frac{1}{2A^2}$, and (7) follows from this immediately.

If $M \cap N = I \neq \{0\}$, we pass to the quotient space H/I. If $M+N$ is closed, so is $(M+N)/I = M/I + N/I$ and so M and N make a positive angle modulo I. Conversely, if M/I and N/I make a positive angle H/I, then $M/I + N/I$ is closed, whence so is $M+N$.

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Operators with a norm condition

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1. Introduction

A bounded linear operator T on a Hilbert space is called, according to [1], *paranormal* if

(1) $$\|T^2x\|\cdot\|x\| \geqq \|Tx\|^2.$$

Paranormal operators are abundant: a *hyponormal* operator, i.e. $T^*T\geqq TT^*$, is paranormal, and every power of a paranormal operator is again paranormal. We show further that an invertible operator T is paranormal if $\log(T^*T)\geqq\log(TT^*)$.

Two bounded linear operators T and S *double commute*, by definition, if T commutes with both S and S^*. The sum and product of two double commuting hyponormal operators are hyponormal. The corresponding assertion is shown not to hold for paranormal operators. We prove, however, that the product of two double commuting operators, one of which is paranormal and the other is hyponormal, is paranormal.

Our central result is that a bounded linear operator T is normal if (and only if) both T and T^* are paranormal and if they have the common kernel. Finally we prove that a paranormal operator is normal if some of its powers is normal.

2. Paranormality

Throughout the paper, $\mathfrak{D}(T)$, $\mathfrak{R}(T)$ and $\mathfrak{N}(T)$ for a (bounded or unbounded) linear operator T denote its domain, range and kernel respectively. The *compression* of T to a closed subspace $\mathfrak{M}$ is the operator PT considered on $\mathfrak{M}$, where P is the projection on $\mathfrak{M}$.

For many purposes, paranormality is more conveniently handled when it is defined by an additive inequality.

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*) This paper was originally submitted to the Colloquium on Hilbert space operators and operator algebras held in Tihany (Hungary, 1970) to which the author had been invited but was unable to attend.

Theorem 1. *A bounded linear operator T is paranormal if and only if.*

$$T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \geqq 0 \qquad (\lambda > 0). \tag{2}$$

Proof. Equivalence of (1) and (2) is an immediate consequence of the relation:

$$\|T^2x\| \cdot \|x\| = \inf_{\lambda>0} \tfrac{1}{2}\{\lambda^{-1}\|T^2x\|^2 + \lambda\|x\|^2\}.$$

Since $\Re(T^*)+\mathfrak{N}(T)$ is dense, (1) is clearly equivalent to

$$\|T^2T^*x\| \cdot \|T^*x\| \geqq \|TT^*x\|^2, \tag{1'}$$

so that (2) is equivalent to

$$AB^2A - 2\lambda A^2 + \lambda^2 \geqq 0 \qquad (\lambda > 0), \tag{2'}$$

where $A=(TT^*)^{\frac{1}{2}}$ and $B=(T^*T)^{\frac{1}{2}}$.

In order to generalize the result that a hyponormal operator is paranormal, let us recall some definitions. For semi-bounded (from below) selfadjoint operators A and B, the order relation $A \geqq B$ means that

$$\int_{-\infty}^{\infty} t \cdot d(E(t)x, x) \geqq \int_{-\infty}^{\infty} t \cdot d(F(t)x, x),$$

where $\{E(t)\}$ and $\{F(t)\}$ are the resolutions of identity for A and B respectively. When A and B are semi-bounded from above, $A \geqq B$ means, by definition, $-A \leqq -B$. If both A and B are *positive*, i.e. $A, B \geqq 0$, then $A \geqq B$ is equivalent to that $\mathfrak{D}(A^{\frac{1}{2}}) \subset \subset \mathfrak{D}(B^{\frac{1}{2}})$ and

$$\|A^{\frac{1}{2}}x\| \geqq \|B^{\frac{1}{2}}x\| \qquad (x \in \mathfrak{D}(A^{\frac{1}{2}})).$$

The spectral theory shows that if $A \geqq B \geqq 0$ and if B has inverse, then A has inverse and $0 \leqq A^{-1} \leqq B^{-1}$.

Theorem 2. *A bounded linear operator T is paranormal if $\mathfrak{N}(T) \subset \mathfrak{N}(T^*)$ and $\log(A) \geqq \log(B)$ where A and B are respectively the compressions of T^*T and TT^* to $\overline{\Re(T)}$.*

Proof. We may assume, without loss of generality $\|T\| \leqq 1$. $\mathfrak{N}(T) \subset \mathfrak{N}(T^*)$ implies that both A and B have inverse. Take $x \in \mathfrak{D}(B^{-1})$. Then the function

$$\Phi(t) := (A^t x, x)(B^{-t}x, x) - (x, x)^2 \qquad (1 \geqq t \geqq 0)$$

is convex. Since $t^{-1}(\alpha^t - 1)$ converges monotonously to $\log(\alpha)$ for $\alpha > 0$, the spectral theory shows that

$$\Phi'(0) = -\left\|(-\log(A))^{\frac{1}{2}}\right\|^2 \cdot \|x\|^2 + \left\|(-\log(B))^{\frac{1}{2}}x\right\|^2 \cdot \|x\|^2 \geqq 0.$$

With $\Phi(0)=0$ and $\Phi'(0)\geqq 0$ the convexity yields

$$\Phi(1) = (Ax, x)(B^{-1}x, x) - (x, x)^2 \geqq 0,$$

so that

$$(BABy, y)(By, y) \geqq (B^2y, y)^2 \qquad (y\in\mathfrak{R}(T)).$$

Then (1′) results, because

$$PBABP=TT^{*2}T^2T^*, \quad PB^2P=(TT^*)^2 \quad \text{and} \quad PBP=TT^*$$

where P is the projection on $\overline{\mathfrak{R}(T)}$. This completes the proof.

If $(T^*T)^s\geqq(TT^*)^s$ for some $s>0$, the condition of Theorem 2 is fulfilled. In fact, $\mathfrak{N}(T)\subset\mathfrak{N}(T^*)$ is trivial. The LOEWNER—HEINZ—KATO theorem (cf. [3] V, § 4) guarantees $(T^*T)^t\geqq(TT^*)^t$ for $0\leqq t\leqq s$, hence $\log(A)\geqq\log(B)$ as in the proof of Theorem 2.

3. Sum and product

The sum and product of two double commuting hyponormal operators are easily shown to be hyponormal. We shall show that the corresponding assertion does not hold for paranormal operators.

If a bounded linear operator T is paranormal, the *tensor product* $T\otimes 1$ (and $1\otimes T$) is paranormal. In fact, for $\lambda>0$

$$(T\otimes 1)^{*2}(T\otimes 1)^2-2\lambda(T\otimes 1)^*(T\otimes 1)+\lambda^2(1\otimes 1) = (T^{*2}T^2-2\lambda T^*T+\lambda^2)\otimes 1 \geqq 0,$$

because the tensor product of two positive operators is positive. We prove, however, that the tensor product $T\otimes T$ is not necessarily paranormal. Then this will show that the product of two double commuting paranormal operators is not necessarily paranormal, because

$$T\otimes T = (T\otimes 1)(1\otimes T)$$

and $T\otimes 1$ double commutes with $1\otimes T$. The construction of such an operator will be based on the idea of P. R. HALMOS (cf. [1]).

When $\mathfrak{H}$ is a Hilbert space, $\mathbf{H}$ denotes the infinite direct sum of copies of H, i.e. $\mathbf{H}=\mathfrak{H}\oplus\mathfrak{H}\oplus\cdots$. $\mathfrak{H}$ itself is identified with the first summand. Given two bounded positive operators A and B on $\mathfrak{H}$, $\mathbf{T}_{A,B,n}$ is the operator on $\mathbf{H}$, which assigns to a vector $\mathbf{x}=\langle x_1, x_2, \dots\rangle$ the vector $\mathbf{y}=\langle y_1, y_2, \dots\rangle$ such that $y_1=0$, $y_j=Ax_{j-1}$ $(1<j\leqq n+1)$ and $y_j=Bx_{j-1}$ $(j>n+1)$. Computation shows that the operator $\mathbf{T}_{A,B,n}$ is paranormal if and only if

$$AB^2A-2\lambda A^2+\lambda^2 \geqq 0 \qquad (\lambda>0) \tag{3}$$

and that it is hyponormal if and only if $B^2\geqq A^2$.

Let now $\dim(\mathfrak{H})=2$. Then each linear operator on $\mathfrak{H}$ is represented by the corresponding matrix with respect to a fixed orthonormal basis. Consider the operators

$$C=\begin{pmatrix}1&1\\1&2\end{pmatrix}\quad\text{and}\quad D=\begin{pmatrix}1&2\\2&8\end{pmatrix}.$$

Then both C and D are positive and for every $\lambda>0$

$$D-2\lambda C+\lambda^2=\begin{pmatrix}(1-\lambda)^2 & 2(1-\lambda)\\ 2(1-\lambda) & (2-\lambda)^2+4\end{pmatrix}\geqq 0.$$

Observe the operator $T=\mathbf{T}_{A,B,1}$ with $A=C^{\frac{1}{2}}$ and $B=(C^{-\frac{1}{2}}DC^{-\frac{1}{2}})^{\frac{1}{2}}$. Then T is paranormal by (3), but the tensor product $T\otimes T$ is not paranormal. In fact, otherwise by (2)

$$(T\otimes T)^{*2}(T\otimes T)^2-2(T\otimes T)^*(T\otimes T)+1\otimes 1\geqq 0,$$

so that the compression of the left side to the canonical imbedding of $\mathfrak{H}\otimes\mathfrak{H}$ in $\mathbf{H}\otimes\mathbf{H}$ is also positive. However the compression coincides with

$$D\otimes D-2C\otimes C+1\otimes 1=\begin{pmatrix}0&0&0&2\\0&5&2&12\\0&2&5&12\\2&12&12&57\end{pmatrix},$$

which is not positive.

Theorem 3. *Let T and S be double commuting paranormal operators. Then the product TS is paranormal if there are a selfadjoint operator A and bounded positive Borel functions $f(t)$ and $g(t)$ such that*

$$(f(t)-f(s))(g(t)-g(s))\geqq 0\qquad(-\infty<s,t<\infty)$$

and one of the following holds:

(a) $f(A)=T^*T$ *and* $g(A)=S^*S$,

(b) $f(A)=T^{*2}T^2$ *and* $g(A)=S^*S$,

(c) $f(A)=T^{*2}T^2$ *and* $g(A)=S^{*2}S^2$.

Proof. Remark, first of all, that the assumption implies

(4) $$(f(A)g(A)x,x)\cdot(x,x)\geqq(f(A)x,x)\cdot(g(A)x,x).$$

In fact, let $\{E(t)\}$ be the resolution of identity for A. Then

$$\big(f(A)g(A)x, x\big)(x, x)-\big(f(A)x, x\big)\big(g(A)x, x\big) =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{f(t)g(t)-f(t)g(s)\}d\big(E(t)x, x\big)d\big(E(s)x, x\big) =$$

$$= \iint_{t>s}\big(f(t)-f(s)\big)\big(g(t)-g(s)\big)d\big(E(t)x, x\big)d\big(E(s)x, x\big) \geqq 0.$$

Double commutativity and (4), when applied to (a), (b) and (c), yield respectively

(a′) $\|TSx\|\cdot\|x\| \geqq \|Tx\|\cdot\|Sx\|$,

(b′) $\|T^2Sx\|\cdot\|x\| \geqq \|T^2x\|\cdot\|Sx\|$,

(c′) $\|T^2S^2x\|\cdot\|x\| \geqq \|T^2x\|\cdot\|S^2x\|$.

Let (a′) hold. Then since both T and S are paranormal,

$$\|T^2S^2x\|\cdot\|S^2x\|\cdot\|Sx\|^2\cdot\|x\| \geqq \|TS^2x\|^2\cdot\|Sx\|^2\cdot\|x\| \geqq$$

$$\geqq \|TSx\|^2\cdot\|S^2x\|^2\cdot\|x\| \geqq \|TSx\|^2\cdot\|S^2x\|\cdot\|Sx\|^2,$$

hence TS is paranormal.

Let (b′) hold. Then since T commutes with S,

$$\|T^2S^2x\|\cdot\|T^2x\|\cdot\|x\| \geqq \|ST^2x\|^2\cdot\|x\| \geqq \|T^2Sx\|\cdot\|T^2x\|\cdot\|Sx\| \geqq \|TSx\|^2\cdot\|T^2x\|,$$

hence TS is paranormal.

Finally let (c′) hold. Since T and S are double commuting paranormal operators, for every $\lambda, \varrho>0$

$$(TS)^{*2}(TS)^2+\lambda^2\varrho^2+\lambda^2S^{*2}S^2+\varrho^2T^{*2}T^2 =$$

$$= (S^{*2}S^2+\varrho^2)(T^{*2}T^2+\lambda^2) \geqq 4\lambda\varrho(TS)^*(TS),$$

so that

$$\|T^2S^2x\|^2+\lambda^2\varrho^2\|x\|^2+\lambda^2\|S^2x\|^2+\varrho^2\|T^2x\|^2 \geqq 4\lambda\varrho\|TSx\|^2.$$

It follows with $\lambda\varrho = \|x\|^{-1}\cdot\|T^2S^2x\|$ and $\lambda^{-1}\varrho = \|T^2x\|^{-1}\cdot\|S^2x\|$ that

$$\|T^2S^2x\|\cdot\|x\|+\|T^2x\|\cdot\|S^2x\| \geqq 2\|TSx\|^2.$$

Now the paranormality of TS results from (c′). This completes the proof.

Theorem 4. *If a paranormal operator T double commutes with a hyponormal operator S, then the product TS is paranormal.*

Proof. Let $\{E(t)\}$ be the resolution of identity for S^*S. By assumption both T^*T and $T^{*2}T^2$ commute with every $E(t)$. Since $S^*S \geqq SS^*$, it follows that for $\lambda > 0$

$$(TS)^{*2}(TS)^2 - 2\lambda(TS)^*(TS) + \lambda^2 \geqq (T^{*2}T^2)(S^*S)^2 - 2\lambda(T^*T)(S^*S) + \lambda^2 =$$

$$= \int_0^\infty (t^2 T^{*2}T^2 - 2\lambda t T^*T + \lambda^2)\, dE(t) \geqq 0,$$

hence TS is paranormal by (2).

Even if both the operators in Theorem 4 are hyponormal, double commutativity can not be replaced by commutativity. To see this, let us construct a hyponormal operator T such that $T^2 - \xi$ is not paranormal for some complex ξ. Then $T - \xi^{\frac{1}{2}}$ and $T + \xi^{\frac{1}{2}}$ are commuting hyponormal operators with non-paranormal product. Since T^2 is paranormal in this case, this example will show that the sum of a paranormal operator and a scalar is not necessarily paranormal.

First we prove that if $S - \xi$ is paranormal for every complex ξ, then

$$\|Sx\|^2 \geqq |(S^2x, x)|. \tag{5}$$

In fact, the assumption implies by (2) that for every $0 \leqq \theta < 2\pi$ and $r > 0$

$$(S^* - re^{-i\theta})^2(S - re^{i\theta})^2 - 2r^2(S^* - re^{-i\theta})(S - re^{i\theta}) + r^4 \geqq 0,$$

so that as $r \to \infty$

$$e^{-2i\theta}S^2 + e^{2i\theta}S^{*2} + 2S^*S \geqq 0.$$

Then (5) result from the arbitrariness of θ.

Let now

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

Then $D \geqq C \geqq 0$, but

$$26D^2 - 25C^2 = \begin{pmatrix} 105 & 130 \\ 130 & 160 \end{pmatrix}$$

is not positive, hence $4^{\frac{1}{n}} D^2 - C^2$ is not positive, where n is a positive integer so large that $4 \times 25^n < 26^n$. Observe the operators A and B on 2^n dimensional space $\mathfrak{H}$:

$$A = (C \otimes \cdots \otimes C)^{\frac{1}{2}} \quad \text{and} \quad B = (D \otimes \cdots \otimes D)^{\frac{1}{2}}.$$

Since $D \geqq C$ is equivalent to $B^2 \geqq A^2$, the operator $T = \mathbf{T}_{A,B,4}$ on $\mathbf{H}$ is hyponormal, but $4B^4 - A^4$ is not positive, for otherwise $4^{\frac{1}{n}} D^2 \geqq C^2$. Suppose that $T^2 - \xi$ is paranormal for every complex ξ. Then since for $y, z \in \mathfrak{H}$

$$\|T^2\mathbf{x}\|^2 = \|A^2y\|^2 + \|B^2z\|^2 \quad \text{and} \quad (T^4\mathbf{x}, \mathbf{x}) = (A^4y, z)$$

where $x_1=y$, $x_5=z$ and $x_k=0$ for other k, it follows from (5) that

$$\|A^2 y\|^2 + \|B^2 z\|^2 \geqq |(A^4 y, z)|.$$

Since y and z are arbitrary, this leads to

$$2\|A^2 y\| \cdot \|B^2 z\| \geqq |(A^4 y, z)|$$

and finally to $4B^4 \geqq A^4$, a contradiction.

4. Normality

Paranormality, when combined with other conditions, leads often to normality. For instance, it is known (cf. [2]) that a compact paranormal operator is normal. It is quite trivial that a hyponormal operator with hyponormal adjoint is normal. However, the generalization to paranormal case is not at all trivial. Our proof is based on the following lemma on positive operators.

Lemma. *If bounded positive operators A and B satisfy*

$$(6) \qquad 2\lambda A^2(A^2+\lambda^2)^{-1} \leqq B \leqq (2\lambda)^{-1}(A^2+\lambda^2) \qquad (\lambda > 0),$$

then they coincide with each other.

Proof. It suffices to prove that B commutes with the resolution of identity $\{E(t)\}$ for A, because under the commutativity assumption (6) and the relation

$$\sup_{\lambda>0} 2\lambda\xi^2(\xi^2+\lambda^2)^{-1} = \xi = \inf_{\lambda>0} (2\lambda)^{-1}(\xi^2+\lambda^2) \qquad (\xi \geqq 0)$$

yield $A=B$ by standard arguments in spectral theory. Since (6) implies $\mathfrak{N}(A)=\mathfrak{N}(B)$, it remains to show that

$$\big(1-E(t+s)\big)B\big(E(t)-E(s)\big) = 0 \qquad (t \geqq s > 0).$$

Take x and y such that

$$x = \big(E(t)-E(s)\big)x \quad \text{and} \quad y = \big(1-E(t+s)\big)y.$$

Consider a partition:

$$s = t_0 < t_1 < \cdots < t_n = t \quad \text{with} \quad t_j - t_{j-1} < n^{-1}t,$$

and let

$$x_j = \big(E(t_j)-E(t_{j-1})\big)x \quad \text{and} \quad A_j = 2t_j(A^2+t_j^2)^{-1}.$$

Then it follows from (6) that

$$0 \leqq B - A^2A_j \leqq A_j^{-1} - A^2A_j =$$
$$= (2t_j)^{-1}(A-t_j)^2(A+t_j)^2(A^2+t_j^2)^{-1} \leqq s^{-1}(A-t_j)^2.$$

Since by definition

$$(A^2 A_j x_j, y) = 0 \quad \text{and} \quad \|(A-t_j)x_j\| \leqq n^{-1}t\|x_j\|,$$

the Schwartz inequality shows that

$$|(Bx_j, y)|^2 = |((B-A^2A_j)x_j, y)|^2 \leqq ((B-A^2A_j)x_j, x_j)\cdot((B-A^2A_j)y, y) \leqq$$
$$\leqq s^{-1}\|B\|\cdot\|(A-t_j)x_j\|^2\cdot\|y\|^2 \leqq (n^2s)^{-1}t^2\|B\|\cdot\|x_j\|^2\cdot\|y\|^2,$$

so that

$$|(Bx, y)|^2 = \left|\sum_{j=1}^{n}(Bx_j, y)\right|^2 \leqq n\cdot\sum_{j=1}^{n}|(Bx_j, y)|^2 \leqq (ns)^{-1}t^2\|B\|\cdot\|x\|^2\|y\|^2.$$

Now $(Bx, y)=0$ results as $n\to\infty$. This completes the proof.

Theorem 5. *A bounded linear operator T is normal if both T and T^* are paranormal and if $\mathfrak{N}(T)=\mathfrak{N}(T^*)$.*

Proof. Let $A=(T^*T)^{\frac{1}{2}}$ and $B=(TT^*)^{\frac{1}{2}}$. Then we have to prove $A^2=B^2$. Since $\mathfrak{N}(A)=\mathfrak{N}(B)$ by assumption, we may assume without loss of generality that both A and B have inverse. Paranormality in assumption means by (2′) that

$$AB^2A-2\lambda A^2+\lambda^2 \geqq 0 \qquad (\lambda>0) \tag{7}$$

and

$$BA^2B-2\lambda B^2+\lambda^2 \geqq 0 \qquad (\lambda>0). \tag{8}$$

Let $S=(BA^2B)^{\frac{1}{2}}$. Since

$$\mathfrak{D}(S^{-1}) = \mathfrak{D}((BA)^{-1}) \quad \text{and} \quad \|S^{-1}x\| = \|(BA)^{-1}x\|,$$

the spectral theory and (7) shows that

$$\|\{(S^2+\lambda^2)S^{-2}\}^{\frac{1}{2}}x\|^2 = \|x\|^2+\lambda^2\|S^{-1}x\|^2 = \|x\|^2+\lambda^2\|A^{-1}B^{-1}x\|^2 \geqq 2\lambda\|B^{-1}x\|^2,$$

so that

$$(S^2+\lambda^2)S^{-2} \geqq 2\lambda B^{-2}.$$

Then, as remarked in § 2,

$$2\lambda S^2(S^2+\lambda^2)^{-1} \leqq B^2.$$

Since, on the other hand, (8) implies

$$B^2 \leqq (2\lambda)^{-1}(S^2+\lambda^2),$$

Lemma shows $B^2=S$, hence $B^2=A^2$. This completes the proof.

J. Stampfli [4] proved that a hyponormal operator is normal if one of its powers is normal. We can generalize this result to paranormal case.

Theorem 6. *A paranormal operator T is normal if some power T^k is normal.*

Proof. First of all, recall that the compression of a paranormal operator to an invariant subspace is again paranormal, that every power of a paranormal operator is paranormal (cf. [1]) and that the spectral radius of a paranormal operator is equal to its norm (cf. [2]). Let $\{E(t)\}$ be the resolution of identity for the positive operator $(T^{*k}T^k)^{\frac{1}{k}}$. Since T commutes with the normal operator T^k, each $E(t)$ commutes with both T and T^* by the commutativity theorem. Now $TE(0)=0$, because $TE(0)$ is paranormal and

$$(TE(0))^k = T^k E(0) = 0.$$

It remains then to prove that $T(1-E(t))$ is normal for every $t>0$. Given $\varepsilon>0$, take a partition:

$$t = t_0 < t_1 < \cdots < t_n = \|T^k\| \quad \text{with} \quad 1-\varepsilon \leqq t_j^{-1} \cdot t_{j-1},$$

and let $E_j = E(t_j) - E(t_{j-1})$. Since

$$t_{j-1}\|E_j x\| \leqq \|T^k E_j x\| = \|T^{*k} E_j x\| \leqq t_j \|E_j x\|$$

and since TE_j is paranormal, it follows that

$$\|TE_j\| = \|T^* E_j\| \leqq t_j^{1/k}.$$

We shall show that

$$(1-\varepsilon)\|TE_j x\| \leqq \|T^* E_j x\| \leqq (1-\varepsilon)^{-1}\|TE_j x\|.$$

Suppose, for instance, that there is $y=E_j y$ such that $\|y\|=1$ and

$$\|Ty\| < (1-\varepsilon)\|T^* y\|.$$

Then

$$t_{j-1} \leqq \|T^k y\| \leqq \|TE_j\|^{k-1}\|Ty\| < (1-\varepsilon)t_j,$$

contradicting the choice of t_j. Now let $x = (1-E(t))x$, then

$$(1-\varepsilon)^2\|Tx\|^2 = (1-\varepsilon)^2 \sum_{j=1}^{n} \|TE_j x\|^2 \leqq$$

$$\leqq \|T^* x\|^2 \leqq (1-\varepsilon)^{-2} \sum_{j=1}^{n} \|TE_j x\|^2 = (1-\varepsilon)^{-2}\|Tx\|^2,$$

so that $\|Tx\| = \|T^* x\|$ follows as $\varepsilon \to 0$. This completes the proof.

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On the essential numerical range, the essential spectrum, and a problem of Halmos

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Introduction. The first four sections of this paper are essentially a survey of what is known about the nature of the spectrum of a coset in the Calkin algebra. In these sections we show that a great deal of information can be obtained from a simple but not very well known theorem of F. WOLF [18]. That result (Theorem (1. 1)) gives several characterizations of those cosets that have a left inverse.

In section 2 we use Wolf's theorem to exhibit the relations between the spectrum and two different essential spectra of bounded operator.

Wolf's theorem immediately suggests introduction of the left essential spectrum of an operator A. This set turns out to coincide with the collection of Weyl limit points of the spectrum of A. It is thus of interest to know that it is non-empty as we show in section 3. In that section we also indicate the relations between the left essential spectrum, the boundary of the spectrum, and the approximate point spectrum of an operator.

In section 4 we use Wolf's theorem to obtain a description of the essential spectrum of a hyponormal coset in terms of eigenvalues.

In section 5 we obtain an analogue of Wolf's theorem for the numerical range of a coset. This result yields several new characterizations of the essential numerical range of an operator introduced in [17].

Finally, in section 6 we use the techniques of § 3 to show that the non-cyclic operators are norm-dense in $\mathfrak{B}(\mathfrak{H})$. This answers a question raised by HALMOS in [8].

Notation. In the following $\mathfrak{H}$ will denote a complex separable infinite-dimensional Hilbert space, $\mathfrak{B}(\mathfrak{H})$ denotes the algebra of all bounded linear operators on $\mathfrak{H}$, and $\mathfrak{K}$ denotes the ideal of compact operators on $\mathfrak{H}$. We shall let ν denote the canonical homomorphism from $\mathfrak{B}(\mathfrak{H})$ onto the *Calkin algebra* $\mathfrak{B}(\mathfrak{H})/\mathfrak{K}$. (See [2].) The range,

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null space, and spectrum of an operator A are denoted by $\mathfrak{R}(A)$, $\mathfrak{N}(A)$, and $\sigma(A)$ respectively.

If $\mathfrak{G}$ is a complex Banach algebra with an identity of norm 1 then a *state* on $\mathfrak{G}$ is by definition a linear functional f on $\mathfrak{G}$ with the property $f(1)=1=\|f\|$. States always separate points of $\mathfrak{G}$ and if $\mathfrak{G}$ is a C^*-algebra then every state on $\mathfrak{G}$ is positive, i.e., $f(x^*x)\geqq 0$ for all $x\in\mathfrak{G}$.

1. Operators with closed range. Let T be a closed linear transformation with domain $\mathfrak{D}_T$ dense in $\mathfrak{H}$. F. Wolf [18] has shown that the following four conditions are equivalent:

(1) There exists a sequence $\{x_n\}$ of unit vectors in $\mathfrak{D}_T$ such that $x_n\to 0$ weakly and $Tx_n\to 0$ strongly.

(2) There exists an orthonormal sequence $\{e_n\}$ in $\mathfrak{D}_T$ such that $Te_n\to 0$ strongly.

(3) $E[0,\delta]\mathfrak{H}$ is infinite-dimensional for all $\delta>0$, where E is the spectral resolution of $(T^*T)^{\frac{1}{2}}$.

(4) Either the range $\mathfrak{R}(T)$ of T is non-closed, or the null-space $\mathfrak{N}(T)$ is infinite dimensional.

Consider the further conditions:

(5) Either $\mathfrak{N}(T)$ is infinite dimensional or 0 is a cluster point of $\sigma((T^*T)^{\frac{1}{2}})$.

(6) There exists an infinite-dimensional projection P such that $P\mathfrak{H}\subset\mathfrak{D}_T$ and TP is compact.

(7) There does not exists $X\in\mathfrak{B}(\mathfrak{H})$ such that $XT-I$ is compact.

(8) For every $\delta>0$ there exists a closed infinite-dimensional subspace $\mathfrak{M}_\delta\subset\mathfrak{D}_T$ such that $\|Tx\|\leqq\delta\|x\|$ for all $x\in\mathfrak{M}_\delta$.

Theorem (1. 1). *Conditions* (1)—(8) *are equivalent.*[1])

Proof. That (1) implies (3) results from the following computation of Wolf's:

$$\|x_n-E[0,\delta]x_n\|^2=\int_\delta^\infty d(E(t)x_n,x_n)=\int_\delta^\infty (t/\delta)^2 d(E(t)x_n,x_n)\leqq$$

$$\leqq (1/\delta)^2\int_\delta^\infty t^2 d(E(t)x_n,x_n)\leqq (1/\delta)^2\|(T^*T)^{\frac{1}{2}}x_n\|^2=(1/\delta)^2\|Tx_n\|^2\to 0.$$

The implications (3)→(2)→(1) are elementary. To see that (3) implies (6), choose an orthonormal sequence $\{e_n\}$ with $e_n\in E[0,1/n]\mathfrak{H}$, and let P be the projection on the span of the e_n. Then

$$P\mathfrak{H}\subset E[0,1]\mathfrak{H}\subset\mathfrak{D}_{(T^*T)^{\frac{1}{2}}}=\mathfrak{D}_T;$$

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[1]) That (3) implies (6) was independently observed by C. Apostol (private communication).

if P_n is the projection on the span of $e_1, e_2, \ldots, e_n$, we also have

$$\|TP - TP_n\| \leqq 1/(n+1),$$

which implies that TP is compact.

Suppose that $K = XT - I$ is compact, $X \in \mathfrak{B}(\mathfrak{H})$. If P is a projection with $P\mathfrak{H} \subset \mathfrak{D}_T$ and TP compact, then because $XTP - P = KP$, P must be compact and therefore finite-dimensional. Thus (6) implies (7). To show that (7) implies (4), assume that (7) holds and that $\mathfrak{R}(T)$ is closed. Define the linear transformation $X: \mathfrak{R}(T) \to \mathfrak{N}(T)^\perp$ to be the inverse of T, and let $X = O$ on $\mathfrak{R}(T)^\perp$. Then X is closed and everywhere defined, hence bounded, and $I - XT$ is the projection on $\mathfrak{N}(T)$. It follows from (7) that $\mathfrak{N}(T)$ is infinite-dimensional.

Next, we show that (4) implies (3). This is clear if $\mathfrak{N}(T)$ is infinite-dimensional, so assume that $\mathfrak{R}(T)$ is not closed. If $T = U(T^*T)^{\frac{1}{2}}$ is the polar decomposition, it is well known that U carries $\mathfrak{R}((T^*T)^{\frac{1}{2}})$ isometrically onto $\mathfrak{R}(T)$, and so $\mathfrak{R}((T^*T))^{\frac{1}{2}}$ is non-closed. It follows that $E[0, \delta]\mathfrak{H}$ is is infinite-dimensional for every $\delta > 0$.

If (8) holds then for each integer $n \geqq 1$ there is a closed infinite-dimensional subspace $\mathfrak{M}_n \subset \mathfrak{D}_T$ such that $\|Tx\| \leqq n^{-1}\|x\|$ for all $x \in \mathfrak{M}_n$. By induction one can choose $e_n \in \in \mathfrak{M}_n$ such that $\|e_n\| = 1$, $0 = (e_n, P_{\mathfrak{m}_n} e_k) = (e_n, e_k)$ for $k < n$. Then $\{e_n\}$ is an orthonormal sequence and $\|Te_n\| \leqq 1/n$. Thus (8) implies (2).

The equivalence of (3) and (5) is a consequence of a well-known theorem of WEYL (Theorem (3. 4) below).

Finally, (3) implies (8) because $(T^*T)^{\frac{1}{2}}$, and therefore also T, is bounded by δ on $E[0, \delta]\mathfrak{H}$.

An operator $A \in \mathfrak{B}(\mathfrak{H})$ is called *semi-Fredholm* (or *Fredholm*) if the range $\mathfrak{R}(A)$ of A is closed and if at least one (both) of the subspaces $\mathfrak{N}(A)$, $\mathfrak{R}(A)^\perp$ is finite-dimensional. For a bounded operator A the equivalence of conditions (4) and (6) reduces to the assertion that A has closed range and finite-dimensional null space if and only if the coset $\nu(A)$ has a left inverse in $\mathfrak{B}(\mathfrak{H})/\mathfrak{K}$. The following are therefore immediate consequences:

Corollary 1 (Atkinson's Theorem). $A \in \mathfrak{B}(\mathfrak{H})$ *is a Fredholm operator if and only if* $\nu(A)$ *is invertible in* $\mathfrak{B}(\mathfrak{H})/\mathfrak{K}$.

Corollary 2. *If A is semi-Fredholm (or Fredholm) then $A + K$ is semi-Fredholm (or Fredholm) for any compact operator K.*

Corollary 3. *The semi-Fredholm (or Fredholm) operators form an open set in* $\mathfrak{B}(\mathfrak{H})$.

Proof. Since A is Fredholm if and only if both A and A^* are semi-Fredholm, it suffices to prove that the set of those A in $\mathfrak{B}(\mathfrak{H})$ for which $\mathfrak{R}(A)$ is closed and $\mathfrak{N}(\mathfrak{H})$ is finite-dimensional is open. This in turn is a consequence of continuity of the quotient map ν and the fact that the left-invertible elements form an open set in any Banach algebra. (See [11] for example.)

We conclude this section with two related results. (The first is easy to prove; the second may be found in [6].)

Theorem (1. 2). *Suppose that A is a bounded operator with finite-dimensional null space. Then $\mathfrak{R}(A)$ is closed if and only if A maps closed bounded sets onto closed bounded sets.*

Theorem (1. 3). *Let A be a bounded linear transformation from a Banach space $\mathfrak{X}$ into a Banach space $\mathfrak{Y}$. If the range of A is not closed in $\mathfrak{Y}$, then for each $\varepsilon>0$ there is an infinite-dimensional closed subspace $\mathfrak{M}(\varepsilon)$ of $\mathfrak{X}$ such that the restriction of A to $\mathfrak{M}(\varepsilon)$ is compact and has norm less than ε.*

2. Essential spectra. There are several distinct definitions of the essential spectrum of an operator $A\in\mathfrak{B}(\mathfrak{H})$. In this section we shall indicate the basic facts concerning two of these. By definition, the *Wolf* (or Fredholm or Calkin) *essential spectrum* of A is the complement of the set of λ for which $A-\lambda$ is a Fredholm operator. Atkinson's theorem implies that the Wolf essential spectrum of A is therefore $\sigma(\nu(A))$, the spectrum of the coset $\nu(A)$ that contains A in the Calkin algebra. The second notion of essential spectrum that we shall examine is by definition the largest subset of $\sigma(A)$ that is invariant under compact perturbations of A, i.e., the set $\bigcap_{K\in\mathfrak{K}}\sigma(A+K)$ sometimes called the *Weyl spectrum*).[2])

In order to describe the relation between these two concepts we need to recall that a Fredholm operator has an *index* given by

$$\operatorname{ind}(A)=\dim\mathfrak{N}(A)-\dim\mathfrak{R}(A)^{\perp}$$

and that the index is invariant under compact perturbations (see [5, 6]). The following theorem is due to M. SCHECHTER [13]:

Theorem (2. 1).

$$\bigcup_{K\in\mathfrak{K}}\sigma(A+K)=\sigma(\nu(A))\cup\{\lambda: A-\lambda \text{ is Fredholm and } \operatorname{ind}(A-\lambda)\neq 0\}.$$

Proof. The quotient map ν is an algebra homomorphism, hence $\sigma(\nu(A)) = =\sigma(\nu(A+K))\subset\sigma(A+K)$ for every compact operator K. Moreover, if $A-\lambda$ is Fredholm with nonzero index, then so is $A+K-\lambda$ for any compact operator K. In particular, $A+K-\lambda$ is not invertible. This proves that the set on the right in Theo-

²) For a nice discussion of this topic, see S. K. BERBERIAN, The Weyl spectrum of an operator, *Indiana Univ. Math. J.*, **20** (1970), 529—554.

rem (2. 1) is contained in the set on the left. On the other hand, if λ does not belong to the set on the right, then $A-\lambda$ is Fredholm with index 0. This implies that $A-\lambda$ is of the form $B+K$ where B is invertible and K is compact. Thus $\lambda \notin \sigma(A-K)$ and hence λ does not belong to the set on the left in Theorem (2. 1).

The next two results indicate the relation between the essential spectrum and spectrum of an operator. Here $\sigma_p(T)$ denotes the point spectrum (=eigenvalues) of T.

Theorem (2. 2). $\sigma(A) = \bigcap \sigma(A+K) \cup \sigma_p(A)$.

Proof. The set on the right is clearly contained in $\sigma(A)$. Suppose then that $\lambda \in \sigma(A)$ and that $\lambda \notin \sigma(A+K)$ for some compact K. Then

$$(A+K-\lambda)(1-(A+K-\lambda)^{-1}K) = A-\lambda$$

is not invertible, so that $1-(A+K-\lambda)^{-1}K$ is not invertible. Hence 1 is an eigenvalue of the compact operator $(A+K-\lambda)^{-1}K$. But if $(A+K-\lambda)^{-1}Kx = x$ with $x \neq 0$, then $Kx = (A+K-\lambda)x$, and so $0 = (A-\lambda)x$. In other words, $\lambda \in \sigma_p(A)$. This completes the proof.

Theorem (2. 3). $\sigma(A) = \sigma(\nu(A)) \cup \sigma_p(A) \cup \sigma_p(A^*)^-$, *where the bar denotes complex conjugate.*

Proof. Suppose that $\lambda \in \sigma(A)$ and $\lambda \notin \sigma_p(A) \cup \sigma_p(A^*)^-$. Then $A-\lambda$ is one-to-one with dense range. Since $A-\lambda$ is not invertible, it follows that $\Re(A-\lambda)$ is not closed. Therefore $A-\lambda$ is not Fredholm so that $\lambda \in \sigma(\nu(A))$.

Remark. It is easy to see that if U is the unilateral shift of multiplicity 1, then $\bigcap \sigma(U+K)$ is the closed unit disk and $\sigma(\nu(U))$ is the unit circle. The larger essential spectrum is therefore obtained from the smaller one by filling in the hole. This is a general fact:

Theorem (2. 4). $\bigcap \sigma(A+K)$ *consists of* $\sigma(\nu(A))$ *together with some of the holes in* $\sigma(\nu(A))$.

Proof. Recall first that by definition a hole in a compact set X is a bounded component of the complement of X. We will use the following elementary fact: *If E and F are compact subsets of the plane such that $E \subset F$ and $\partial F \subset E$, then F is the union of E and those holes of E that meet F.*

Now if $E=\sigma(\nu(A))$ and $F = \bigcap \sigma(A+K)$, then $F-E$ consists of those complex numbers λ for which $A-\lambda$ is Fredholm of index $\neq 0$. By continuity of the index (see [5, 6]) this is an open set. Hence $\partial F \subset E$.

Corollary. $\bigcap \sigma(A+K)$ *and* $\sigma(\nu(A))$ *have the same convex hull.*

3. The left essential spectrum. Wolf's theorem motivates consideration of the left essential spectrum $\sigma_l(\nu(A))$ of an operator $A \in \mathfrak{B}(\mathfrak{H})$. By definition, a complex number λ belongs to $\sigma_l(\nu(A))$ if and only if the coset $\nu(A-\lambda) = \nu(A)-\lambda$ fails to have a left inverse in $\mathfrak{B}(\mathfrak{H})/\mathfrak{K}$. Equivalently, λ belongs to the left essential spectrum of A if and only if there is a sequence $\{x_n\}$ of unit vectors such that $x_n \to 0$ weakly and $\|(A-\lambda)x_n\| \to 0$. Moreover, the x_n can even be chosen orthonormal (Theorem (1. 1).) In the special case in which A is self-adjoint the same concept was introduced by Weyl; accordingly such a complex number λ is also called a Weyl limit point of the spectrum of A (see [12]). The *right essential spectrum* $\sigma_r(\nu(A))$ is defined in the obvious way.

The concepts just introduced derive their usefulness from the following:

Theorem (3. 1). $\sigma_l(\nu(A)) \cap \sigma_r(\nu(A)) \supset \partial\sigma(\nu(A))$. *Hence* $\sigma_l(\nu(A))$ *is a non-empty compact subset of* $\sigma(\nu(A))$.

Proof. The theorem is an immediate consequence of two well-known facts about Banach algebras. First, the set $\mathfrak{G}_l$ of elements that have a left inverse is open and second, any point of the boundary of $\mathfrak{G}_l$ is a right topological divisor of 0. (See [11] for example.)

Theorem (3. 2). *If* $A \in \mathfrak{B}(\mathfrak{H})$, *then* $\pi(A)$, *the approximate point spectrum of* A, *consists of* $\sigma_l(\nu(A))$ *together with the eigenvalues of finite multiplicity.*

Proof. If $\lambda \in \pi(A)$ and $\lambda \notin \sigma_l(\nu(A))$, then $A-\lambda$ is not bounded below but has closed range and finite dimensional null space. Hence $\mathfrak{N}(A-\lambda) \neq 0$ so that λ is an eigenvalue of finite multiplicity.

The next result of PUTNAM [9] is much deeper: [3])

Theorem (3. 3). *If* $A \in \mathfrak{B}(\mathfrak{H})$ *and* $\lambda \in \partial\sigma(A)$, *then either* λ *is an isolated point of* $\sigma(A)$ *and an eigenvalue of finite multiplicity, or it belongs to* $\sigma_l(\nu(A))$, *that is there is an orthonormal sequence* $\{e_n\}$ *such that* $\|(A-\lambda)e_n\| \to 0$.

If A is an operator with no eigenvalues, then Theorem (2. 2) asserts that $\sigma(A) = \bigcap \sigma(A+K)$. In particular, each compact perturbation of A has a larger spectrum that that of A. There is a simple relationship between these spectra:

Corollary. *Let* $A \in \mathfrak{B}(\mathfrak{H})$ *and assume that* A *has no eigenvalues. Then for any compact operator* K

$$\sigma(A+K) = \sigma(A) \cup \mathfrak{J} \cup \textit{ some holes in } \sigma(A),$$

where $\mathfrak{J}$ *is the set of isolated eigenvalues of* $A+K$ *of finite multiplicity.*

---

[3]) Prof. PUTNAM has requested that we refer to this result as the Putnam—Schechter theorem.

Proof. We have $\sigma(A)\cup\mathfrak{J}\subset\sigma(A+K)$ (by the preceding corollary). Also, by Putnam's Theorem,

$$\partial\sigma(A+K)\subset\sigma_l(A+K)\cup\mathfrak{J}=\sigma_l(\nu(A))\cup\mathfrak{J}\subset\sigma(A)\cup\mathfrak{J}.$$

The proof is completed by an application of the topological fact used in the proof of Theorem (2. 4).

If A is a normal operator on $\mathfrak{H}$, then it is easy to see from Theorem (2. 1) that $\bigcap\sigma(A+K)=\sigma(\nu(A))$. Moreover, $\sigma_l(\nu(A))=\sigma(\nu(A))$, and if E is the spectral measure of A, then $\lambda\in\sigma(\nu(A))$ if and only if every neighborhood $\mathfrak{U}$ of λ has infinite spectral measure ($\dim\mathfrak{R}(E(\mathfrak{U}))=\infty$). From this it is easy to obtain WEYL's characterization of the essential spectrum of A (see [12, p. 367]).

Theorem (3. 4). *If A is normal, then $\sigma(\nu(A))=\sigma_l(\nu(A))$ consists of the cluster points of $\sigma(A)$ together with the isolated eigenvalues of A of infinite multiplicity.*

Weyl's theorem has recently been generalized to hyponormal operators by COBURN [3]:

Theorem (3. 5). *If A is hyponormal, then $\bigcap\sigma(A+K)$ consists of the cluster points of $\sigma(A)$ and the isolated eigenvalues of infinite multiplicity.*

4. Eigenvalues in the Calkin algebra. In this section we obtain more detailed information about the Wolf essential spectrum of special operators. For statements about elements in the Calkin algebra it is convenient to use lower case Latin letters a, p instead of the more cumbersome notation $\nu(A)$, $\nu(P)$ for the cosets containing the operator A, P.

We begin with a simple reformulation of part of Wolf's theorem:

Theorem (4. 1). *Let $a\in\mathfrak{B}(\mathfrak{H})/\mathfrak{K}$ and let $\lambda\in\sigma(a)$. Then there is a projection $p\neq 0$ such that either $ap=\lambda p$ or $pa=\lambda p$.*

Proof. Suppose λ belongs to $\sigma_l(a)$. If $A\in a$, then $\nu(A-\lambda)$ does not have a left inverse, hence (Theorem 1. 1) there is a compact operator K such that $\dim\mathfrak{N}(A-\lambda-K)=\infty$. Let P be the orthogonal projection onto $\mathfrak{N}(A-\lambda-K)$ and let $p=\nu(P)$. Then $(A-\lambda-K)P=O$ so that $(a-\lambda)p=0$. Moreover, $p\pm 0$ since P has infinite rank.

To complete the proof we must also consider the possibility that $\nu(A-\lambda)$ fails to have a right inverse. However on taking adjoints, this case reduces to the one just discussed.

Corollary. *If $A\in\mathfrak{B}(\mathfrak{H})$ then there are orthogonal projections P and Q of infinite rank and nullity and a complex number λ such that $(A-\lambda)P$ and $Q(A-\lambda)$ are compact.*

Proof. Any two projections of infinite rank contain orthogonal sub-projections of infinite rank and nullity. Projections with the asserted properties can therefore be found for any $\lambda\in\sigma_l(\nu(A))\cap\sigma_r(\nu(A))$.

Note that if P is as in the corollary, then $\nu(A)\nu(P)=\nu(P)\nu(A)\nu(P)$. Thus $AP-PAP$ is compact so that A has an invariant subspace "modulo the compacts".

Theorem (4. 1) shows that λ belongs to the Wolf essential spectrum of A if and only if either λ is an eigenvalue of (the regular representation of) $\nu(A)$ or $\bar{\lambda}$ is an eigenvalue of $\nu(A^*)$. We will shapren this assertion for hyponormal elements of the Calkin algebra. For this we need a lemma.

Lemma 4. 1. *Let $\mathfrak{G}$ be a C^*-algebra with unit and let A be a hyponormal element of $\mathfrak{G}$ (i.e., $A^*A\geqq AA^*$). If A has a right inverse, then A is invertible.*

Proof. Since A is hyponormal, $Z^*AA^*Z\leqq Z^*A^*AZ$ for any Z. Hence $AZ=O$ implies $A^*Z=O$. Suppose now that $AX=1$. Then $A(XA-1)=O$ so $A^*(XA-1)=O$ and thus $XA-1=X^*A^*(XA-1)=O$.

Theorem (4. 2). *Let a be a hyponormal element of $\mathfrak{B}(\mathfrak{H})/\mathfrak{K}$. Then*
(1) *$\lambda\in\sigma(a)$ if and only if there is a projection $p\neq 0$ such that $a^*p=\bar{\lambda}p$.*
(2) *If p is a projection such that $ap=\lambda p$, then $a^*p=\bar{\lambda}p$.*
(3) *If $ap_i=\lambda_i p_i$ for $i=1, 2$, and if $\lambda_1\neq\lambda_2$, then $p_1p_2=0$.*

Proof. The first assertion is an obvious consequence of Lemma (4. 1). To prove (2), note that the condition $(a-\lambda)p=0$ implies $(a-\lambda)^*p=0$ because $a-\lambda$ is hyponormal (see the proof of Lemma (4. 1)).

If $ap_1=\lambda_1p_1$ and $ap_2=\lambda_2p_2$ then $p_1a=\lambda_1p_1$ by (2) so that $\lambda_2p_1p_2=p_1ap_2=\lambda_1p_1p_2$.

Remarks. (1) It is easy to exhibit non-normal hyponormal elements of $\mathfrak{B}(\mathfrak{H})/\mathfrak{K}$. For example, if U is an isometry with infinite defect, then $\nu(U)$ is such an element. Or again, if B is a positive noncompact operator with $0\in\sigma(\nu(B))$, then (Radjavi [10]) there exists $A\in\mathfrak{B}(\mathfrak{H})$ such that $A^*A-AA^*=B$. The coset $\nu(A)$ is then hyponormal but not normal.

(2) It is not true that every hyponormal coset a contains an operator of the form hyponormal+compact. (Let $a=\nu(A)$ where A is the adjoint of the unilateral shift and compute the Fredholm index.)

It is well known that any eigenvector of an operator A corresponding to an eigenvalue λ of modulus $|\lambda|=\|A\|$ must reduce A, i.e., $Ax=\lambda x$ implies $A^*x=\bar{\lambda}x$. The next result is the analogue of this fact for the Calkin algebra:

Theorem (4. 3). *Let $a\in\mathfrak{B}(\mathfrak{H})/\mathfrak{K}$ and suppose that there is a $\lambda\in\sigma_l(a)$ with $|\lambda|=\|a\|$. If p is a projection such that $ap=\lambda p$, then $a^*p=\bar{\lambda}p$.*

Proof. Without loss of generality we may suppose that $\|a\|=1=\lambda$. Then $pa^*p=(pap)^*=p$, and so

$$0 \leqq (a^*p = p)^*(a^*p-p) = paa^*p-pap-pa^*p+p = paa^*p-p = p(aa^*-1)p \leqq 0.$$

Hence $a^*p=p$.

Corollary. *If $A\in\mathfrak{B}(\mathfrak{H})$ and if the coset $\nu(A)$ has norm equal to its spectral radius, then there exists a projection P of infinite rank and a complex number λ such that $(A-\lambda)P$ and $P(A-\lambda)$ are both compact.*

Remark. A coset $\nu(A)$ has norm equal to its spectral radius in each of the following cases:

(i) $\nu(A)$ is hyponormal.

(ii) $\nu(A)$ contains a Toeplitz operator.

(iii) A has norm equal to its spectral radius and A has no isolated eigenvalues of finite multiplicity.

(Sufficiency of (i) can be proved by a slight modification of the proof in [14] of the corresponding fact in $\mathfrak{B}(\mathfrak{H})$. Condition (iii) is sufficient by Putnam's Theorem (3. 3), and (ii) is a special case of (iii) since a Toeplitz operator has no isolated eigenvalues of finite multiplicity [3].)

The corollary therefore implies that if A is a compact perturbation of a hyponormal operator or a Toeplitz operator then there is an orthonormal sequence $\{e_n\}$ and a complex number λ such that

$$(A-\lambda)e_n \to 0, \quad (A^*-\bar{\lambda})e_n \to 0.$$

Consequently, A is uniformly approximable by operators with a reducing eigenvector. (See [15], Theorems 1, 2.)

5. The essential numerical range. The numerical range of a bounded operator A on $\mathfrak{H}$ is defined as

$$W(A) = \{(Ax, x): \ \|x\| = 1\}.$$

In [17] a generalized *numerical range* $W_0(a)$ was introduced for an element a of an arbitrary complex Banach algebra $\mathfrak{G}$ with norm 1 unit. By definition $W_0(a)$ consists of the complex numbers of the form $f(a)$ where f ranges over the states of $\mathfrak{G}$. The set $W_0(a)$ is convex, compact, and contains the spectrum of a. If $\mathfrak{G}$ is a subalgebra of $\mathfrak{B}(\mathfrak{H})$ then for $A\in\mathfrak{G}$ the numerical range $W_0(A)$ coincides with the closure $W(A)^-$ of the usual numerical range.

The *essential numerical* range [17] of an $A\in\mathfrak{B}(\mathfrak{H})$ is by definition the numerical range $W_0(\nu(A))$ of the coset in $\mathfrak{B}(\mathfrak{H})/\mathfrak{K}$ that contains A. We shall denote this set by $W_e(A)$ in the following. A more explicit identification is given by the formula

$$W_e(A) = \bigcap\{W(A+K)^-: K\in\mathfrak{K}\}.$$

(Thus, unlike the situation for the essential spectrum, there is only one "natural" definition of essential numerical range. See Theorem (2. 1).)

The preceding formula was proved in [17] by a convexity argument. A simpler proof is obtained by noting that a complex number λ belongs to either side if and only if $\|A+K-z\| \geqq |\lambda-z|$ for all complex numbers z and all compact operators K. (This is an immediate consequence of Theorem 4 of [17] and the definition of the norm of the coset $\nu(A-z)$.)

In this section we give several new descriptions of the essential numerical range. These are obtained from the following analogue of Theorem (1. 1):

Theorem (5. 1). *For $T\in\mathfrak{B}(\mathfrak{H})$, the following conditions are equivalent:*

(1) $0\in\bigcap\{W(T+F)^-: F$ is of finite rank$\}$.

(2) $0\in W_e(T)$.

(3) *There exists a sequence $\{x_n\}$ of unit vectors such that $x_n\to 0$ weakly and $(Tx_n, x_n)\to 0$.*

(4) *There exists an orthonormal sequence $\{e_n\}$ such that $(Te_n, e_n)\to 0$.*

(5) *There exists an infinite-dimensional projection P such that PTP is compact.*

Proof. The implications $(5)\to(4)\to(3)\to(2)\to(1)$ are clear. We first prove that (1) implies (4). Let $\varepsilon_k\to 0$, and assume that orthogonal unit vectors $e_1, e_2, \dots, e_n$ have been found so that $|(Te_k, e_k)|<\varepsilon_k$ for $k=k, 2, \dots, n$. Let $\mathfrak{M}$ be the subspace spanned by the e_k, and let P be the projection on $\mathfrak{M}$. In order to exhibit a unit vector e_{n+1} orthogonal to $\mathfrak{M}$ with $|(Te_{n+1}, e_{n+1})|<\varepsilon_{n+1}$, it is sufficient to show that $0\in W((I-P)T|\mathfrak{M}^\perp)^-$. To see that the latter condition holds, choose

$$\mu\in W((I-P)T|\mathfrak{M}^\perp),$$

and let

$$F=\mu P-PTP-(I-P)TP-PT(I-P).$$

Then F is of finite rank, and

$$T+F=\mu P+(I-P)\,T(I-P)=\mu I_{\mathfrak{M}}\oplus(I-P)T|\mathfrak{M}^\perp.$$

In general it is true that $W(A\oplus B)$ is the convex hull of $W(A)$ and $W(B)$; and thus it follows that

$$W(T+F)=W((I-P)T|\mathfrak{M}^\perp)$$

since $W((I-P)T|\mathfrak{M}^\perp)$ is convex and contains μ. Hence (1) implies $0\in W((I-P)T|\mathfrak{M}^\perp)$, as required.

To complete the proof we show that (4) implies (5). Let $\{e_n\}$ be an orthonormal sequence with $(Te_n, e_n)\to 0$. By passing to a subsequence we can assume that

$$(*)\qquad \sum_{n=1}^{\infty}|(Te_n, e_n)|^2<\infty.$$

Put $n_1=1$. Since

$$\sum_{n=1}^{\infty} |(Te_{n_1}, e_n)|^2 \leqq \|Te_{n_1}\|^2 \quad \text{and} \quad \sum_{n=1}^{\infty} |(Te_n, e_{n_1})|^2 \leqq \|T^* e_{n_1}\|^2$$

by Bessel's inequality, there is an integer $n_2>n_1$ such that

$$\sum_{n=n_2}^{\infty} |(Te_{n_1}, e_n)|^2 < \tfrac{1}{2} \quad \text{and} \quad \sum_{n=n_2}^{\infty} |(Te_n, e_{n_1})|^2 < \tfrac{1}{2}.$$

By iterating this procedure we generate a strictly increasing sequence $\{n_k\}$ of positive integers with the property

$$(**) \qquad \sum_{n=n_{k+1}}^{\infty} |(Te_{n_k}, e_n)|^2 < 2^{-k} \quad \text{and} \quad \sum_{n=n_{k+1}}^{\infty} |(Te_n, e_{n_k})|^2 < 2^{-k}$$

for all $k \geqq 1$. Inequalities $(*)$ and $(**)$ imply that

$$\sum_{i,j=1}^{\infty} |(Te_{n_i}, e_{n_j})|^2 < \infty.$$

If P is the projection on the span of the e_{n_k}, this means that PTP is a Hilbert—Schmidt operator, and therefore PTP is compact.

Corollary. *Each of the following conditions is necessary and sufficient in order that* $\lambda \in W_e(T)$:

(1) $(Tx_n, x_n) \to \lambda$ *for some sequence* $\{x_n\}$ *of unit vectors such that* $x_n \to 0$ *weakly.*

(2) $(Te_n, e_n) \to \lambda$ *for some orthonormal sequence* $\{e_n\}$.

(3) $PTP - \lambda P$ *is compact for some infinite-dimensional projection* P.

Remarks. (1) As we observed in § 4 a point λ belongs to the Wolf essential spectrum of A if and only if $\nu(A)\nu(P)=\lambda\nu(P)$ or $\nu(P)\nu(A)=\lambda\nu(P)$ for some non-zero projection $\nu(P)$. By Theorem (5. 1) the corresponding statement for the numerical range is: $\lambda \in W\,(A)$ if and only if $\nu(P)\nu(A)\nu(P)=\lambda\nu(P)$ for some non-zero projection $\nu(P)$; i. e., $pap=\lambda p$.

(2) The analogy between Theorems (1. 1) and (5. 1) is not complete. Call a sequence of vectors *non-compact* if it has no strongly convergent subsequence. Then Wolf's statement of condition (3) of Theorem (1. 1) requires only that $Ax_n \to 0$ strongly for some non-compact sequence $\{x_n\}$ of unit vectors. However, the corresponding reformulation of condition (3) of Theorem (5. 1) is not equivalent to the conditions of that theorem. For example, let A be the operator with matrix diag $(-1, 1, 1, \ldots)$ in an orthonormal basis $\{e_n\}$ and let $x_n = (e_1 + e_{n+1})/\sqrt{2}$. Then $0 \notin W_e(A)$, and yet $\{x_n\}$ is a non-compact sequence with $(Ax_n, x_n) \to 0$.

(3) If the space $\mathfrak{H}$ is finite-dimensional no sequence of unit vectors can converge weakly so that the conditions of the corollary of Theorem (5. 1) have no reasonable analogue. A possible replacement is the condition: $(Ae_i e_i)=\lambda$ for some orthonormal set $\{e_i: 1 \leqq i \leqq k\}$ where $k \leqq \dim \mathfrak{H}$. In [4] it is shown that the set of complex numbers λ with this property constitute the k-numerical range [7] of A.

(4) The preceding remark suggests the following question: If $A \in \mathfrak{B}(\mathfrak{H})$ for which complex numbers λ does there exist a projection P of infinite rank such that $P(A-\lambda)P = O$? A partial answer is given in [1].

(5) The Corollary of Theorem (5. 1) shows that if f is a state on $\mathfrak{B}(\mathfrak{H})$ that annihilates $\mathfrak{K}$, then for each $A \in \mathfrak{B}(\mathfrak{H})$ there is an orthonormal sequence $\{e_n\}$ such that $f(A) = \lim_n (Ae_n, e_n)$.

6. Non-cyclic operators. We conclude with an application to a problem recently proposed by HALMOS [8]: does the set of cyclic operators have a non-empty interior? Although this set is readily seen to be open when $\dim \mathfrak{H} < \infty$, we have:

Theorem (6. 1). *When $\mathfrak{H}$ is infinite-dimensional, the non-cyclic operators are norm-dense in $\mathfrak{B}(\mathfrak{H})$.*

Proof. We use the observation (HALMOS, op. cit.) that if $\mathfrak{N}(A^*)=\mathfrak{R}(A)^\perp$ has dimension at least two, then for each f the span of $f, Af, A^2f, \ldots$ has codimension at least one, and so A is non-cyclic. Now let $T \in \mathfrak{B}(\mathfrak{H})$, choose λ in the left essential spectrum of T^* (cf. Theorem (3. 1) above), and let $\varepsilon > 0$. Then there exist orthogonal unit vectors φ and ψ such that $\|(T^*-\lambda I)\varphi\| < \varepsilon$ and $\|(T^*-\lambda I)\psi\| < \varepsilon$. Let P be the projection on the span of φ and ψ, and set

$$S^* = \lambda P + T^*(I-P).$$

Then $T^*-S^* = (T^*-\lambda I)P$ has norm at most 2ε. But $S^*-\lambda I$ has nullity at least two, so $S-\bar{\lambda} I$ is non-cyclic, and therefore S is non-cyclic. Since $\|T-S\| \leqq 2\varepsilon$ and ε is arbitrary, the proof is complete.

Remark. Since the approximating operator differs from the given operator only on a finite-dimensional subspace, the proof shows that the non-cyclic operators are dense in any norm.

For another application of Theorem (3. 1) we refer the reader to [16] where it is shown that for any $A \in \mathfrak{B}(\mathfrak{H})$ the range of the inner derivation $X \to AX-XA$ is never dense in $\mathfrak{B}(\mathfrak{H})$, nor does it contain all finite dimensional operators.

Our final result is an application of Wolf's theorem.

Theorem (6. 2) *Let $T \in \mathfrak{B}(\mathfrak{H})$. Then $1-T^*T$ is compact if and only if $T=U+K$ where K is compact and U is either an isometry or a co-isometry with finite-dimensional null space.*

Proof. Sufficiency is trivial. To prove necessity suppose that $1-T^*T$ is compact. Since $1-T^*T=(1+\sqrt{T^*T})(1-\sqrt{T^*T})$ and the left factor is invertible, it follows that $1-\sqrt{T^*T}$ is compact.

If $x_n\to 0$ weakly and $Tx_n\to 0$ strongly then $x_n=(1-T^*T)x_n+T^*Tx_n\to 0$ strongly Hence by Theorem (1. 1) T has closed range and finite dimensional null space.

Suppose $\dim \mathfrak{N}(T)\leqq\dim \mathfrak{R}(T)^{\perp}$. Replacing T by $T+K$ for some compact K if necessary we may assume that T is one-to-one. If $T=U\sqrt{T^*T}$ is the polar decomposition of T, then U is an isometry and $T=U-U(1-\sqrt{T^*T})=U+$compact.

To complete the proof we must also consider the case $\dim \mathfrak{R}(T)^{\perp}<\dim \mathfrak{N}(T)$. The above argument shows tha $T^*=U+K$ where U is an isometry and K is compact. The null space of U^* is finite-dimensional by the hypothesis on T.

Corollary. $T=$ isometry+compact *if and only if* $1-T^*T$ *is compact and* T *is semi-Fredholm with* ind $(T)\leqq 0$.

Remarks. If $1-T^*T$ is compact and $1-TT^*$ is not compact then $\sigma(T)$ contains the unit disk. This is a consequence of the theorem and fact that in any C^*-algebra with identity the spectrum of a non-unitary isometry is the unit disk.

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Operators with essentially disconnected spectrum

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1. Introduction. Throughout this paper $\mathfrak{H}$ will denote an infinite dimensional complex Hilbert space, $\mathscr{L}(\mathfrak{H})$ will represent the algebra of all (bounded linear) operators on $\mathfrak{H}$, and by $\mathscr{K}$ we shall mean the ideal of all compact operators on $\mathfrak{H}$. Let π be the canonical projection from $\mathscr{L}(\mathfrak{H})$ onto the (Calkin) quotient algebra $\mathscr{L}(\mathfrak{H})/\mathscr{K}$. For every $T\in\mathscr{L}(\mathfrak{H})$ the spectrum $E(T)$ of $\pi(T)$ in $\mathscr{L}(\mathfrak{H})/\mathscr{K}$ will be called the Calkin essential spectrum of T.

Definition. We say that the spectrum $\Sigma(T)$ of an operator $T\in\mathscr{L}(\mathfrak{H})$ is essentially disconnected if the polynomial hull $\hat{\Sigma}(T)$ of $\Sigma(T)$ is disconnected and $E(T)$ intersects more than one component of $\hat{\Sigma}(T)$ (the polynomial hull $\hat{X}$ of a compact subset X of the complex plane $\mathbf{C}$ is the complement of the unbounded component of $\mathbf{C}-X$).

Our main purpose in this note is to initiate the study of the class of all operators whose spectrum is essentially disconnected, which we shall denote by (ED). Examples of operators having such a property are easy to come by, taking, for instance, the direct sum of two operators on $\mathfrak{H}$ whose spectra are far from each other. In particular, a self-adjoint operator has an essentially disconnected spectrum if and only if its essential spectrum is disconnected. Of course this is not the case for an arbitrary operator on $\mathfrak{H}$.

Operators in (ED) have many interesting properties, especially those concerned with perturbations by either small norm operators or compact ones. Thus, if $T\in(ED)$, then $T+K\in(ED)$ and $\hat{\Sigma}(T+K)$ is disconnected for every $K\in\mathscr{K}$. Furthermore, an operator $T\in(ED)$ if and only if $\hat{E}(T)$ is disconnected (Theorem 2). On the other hand, the class (ED) is open in the uniform topology of $\mathscr{L}(\mathfrak{H})$ (Theorem 7). We also prove (Theorem 8) that if $T\in(ED)$ and $\mathscr{I}_T$ denotes the lattice of invariant subspaces of T equipped with the topology induced by the distance between subspaces ([5]), then there exist two infinite dimensional subspaces $\mathfrak{M}_1, \mathfrak{M}_2\in\mathscr{I}_T$ which

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are isolated points of $\mathscr{I}_T$, such that $\mathscr{I}_T$ is homeomorphic to $\mathscr{I}_{T_1} \times \mathscr{I}_{T_2}$, where $T_j = T|\mathfrak{M}_j$, $j=1, 2$.

In proving the last result we need to show (see the proof of Theorem 8) that every $T \in (ED)$ is similar to the direct sum of two operators S_1, S_2 acting on infinite dimensional Hilbert spaces such that $\hat{\Sigma}(S_1) \cap \hat{\Sigma}(S_2) = \emptyset$. Therefore, up to similarity, every operator in (ED) looks like the example given above.

We begin our considerations (§ 2) by discussing some relations between the different kinds of essential spectra of an operator. One of our main results in this direction is proved in § 3 (Theorem 4) and states that each separated part of the different kind of essential spectra of an operator T is an upper-semicontinuous function of T.

Finally, in § 5 we enumerate some questions raised in the paper and we present partial answers to some of them. As an immediate byproduct of these results we derive interesting properties of hyponormal operators.

2. Some properties of the essential spectrum. To begin with we recall some facts from the theory of Fredholm operators ([8]). For $T \in \mathscr{L}(\mathfrak{H})$ we have that $\pi(T)$ is invertible in $\mathscr{L}(\mathfrak{H})/\mathscr{K}$ if and only if ran T is closed, $\alpha(T) = \dim \text{null } T$ is finite and $\beta(T) = \dim \text{null } T^* (= \dim (\text{ran } T)^{\perp} = \alpha(T^*))$ is also finite (Atkinson's theorem). In this case T is called a Fredholm operator and its index is defined by $j(T) = \alpha(T) - \beta(T)$. Thus, the set Φ of all Fredholm operators is an open subset of $\mathscr{L}(\mathfrak{H})$ in the uniform topology; its components are also open and they correspond to each value of the (integer valued) function $j(T)$. We shall denote by Φ_0 the component of Φ consisting of all Fredholm operators of index zero.

With the above notation the Calkin essential spectrum of an operator T can be expressed as $E(T) = \{\lambda \in \Sigma(T) : T - \lambda \notin \Phi\}$. Another important concept native to the theory of compact perturbation is the Weyl spectrum $\Omega(T)$ of T ([1], [3]) i.e. $\Omega(T) = \{\lambda \in \Sigma(T) : T - \lambda \notin \Phi_0\}$. Schechter proved ([13]) that $\Omega(T) = \bigcap_{K \in \mathscr{K}} \Sigma(T+K)$.

On the other hand, Browder introduced in [2] a third concept of essential spectrum, namely $B(T) = \Sigma(T) - \{\lambda \in \Sigma(T) : T - \lambda \in \Phi_0,\ \lambda \text{ is an isolated point of } \Sigma(T)\}$.

Clearly $E(T) \subset \Omega(T) \subset B(T)$.

It is easy to see that if λ is an isolated point of $\Sigma(T)$ and $T - \lambda \in \Phi$, then $T - \lambda \in \Phi_0$. Also, it is an immediate consequence of [8], Chapter 4, Theorem 5. 31 that if λ is a limit point of $b\Sigma(T)$ (here and in what follows bX denotes the boundary of the set X), then $\lambda \in E(T)$. Therefore we conclude that $bB(T) \subset E(T)$. Given a compact subset X of the plane, a hole of X is a component of $\hat{X} - X$. If Y is another compact set such that $b(X) \subset Y \subset X$, it follows that $b(X) \subset b(Y)$, $\hat{X} = \hat{Y}$ and X can be obtained from Y by filling in some holes of Y. We sumarize all the above discussion in the following theorem:

Theorem 1. *Let* $T\in\mathscr{L}(\mathfrak{H})$. *Then*

a) $E(T)\subset\Omega(T)\subset B(T)$,

b) $bB(T)\subset b\Omega(T)\subset bE(T)$,

c) $\hat{E}(T)=\hat{\Omega}(T)=\hat{B}(T)$,

d) $\Omega(T)(B(T))$ *can be obtained from* $E(T)(\Omega(T))$ *by filling in some holes of* $E(T)(\Omega(T))$.*)

Corollary 2. 1. *Let* $T\in\mathscr{L}(\mathfrak{H})$. *If* $E(T)$ *is connected,* $\Omega(T)$ *is connected, and if* $\Omega(T)$ *is connected,* $B(T)$ *is connected.*

One can construct very easily examples showing that none of the reverse implications in Corollary 2. 1 hold in general. Let $\mathfrak{G}$ be a separable Hilbert space, and let V be a unilateral shift of multiplicity one on $\mathfrak{G}$; also let $N\in\mathscr{L}(\mathfrak{G})$ be any quasinilpotent operator. If we denote by D the closed unit disc in $\mathbf{C}$ we have $B(V\oplus V^*\oplus N)=D$, while $\Omega(V\oplus V^*\oplus N)=E(V\oplus V^*\oplus N)=bD\cup\{0\}$ ([7], Problem 144). Furthermore, $\Omega(V\oplus N)=D$, but $E(V\oplus N)=bD\cup\{0\}$.

Theorem 2. *For* $T\in\mathscr{L}(\mathfrak{H})$, *the following statements are equivalent:*

a) $T\in(ED)$, *that is* $\hat{\Sigma}(T)$ *is disconnected and* $E(T)$ *intersects more than one component of* $\hat{\Sigma}(T)$,

b) $\hat{E}(T)$ *is disconnected.*

Proof. The proof is a consequence of the fact that $\hat{\Sigma}(T)-\hat{E}(T)(=\hat{\Sigma}(T)-\hat{B}(T))$ consists of isolated points λ such that $T-\lambda\in\Phi_0$.

Next we introduce the following terminology: given a compact subset X of the plane we will denote by rad X the radius of X, i.e. rad $X=\sup_{\lambda\in X}|\lambda|$. Theorem 1 tells us that rad $E(T)=$ rad $\Omega(T)=$ rad $B(T)$. Thus it is natural to call this common value the essential spectral radius of T, which shall be denoted by $r_e(T)$.

Nussbaum in [9] already observed that the radius of the different kinds of essential spectra are the same, but our argument is much simpler than that used by Nussbaum.

The next lemma makes the definition of the essential spectral radius even more natural.

Lemma 2. 2. *If* $T\in\mathscr{L}(\mathfrak{H})$, *then*

$$r_e(T)=\inf_{K\in\mathscr{K}} r(T+K),$$

*) This interesting relationship between the Calkin spectrum and the Weyl spectrum is also discussed by Fillmore, Stampfli and Williams in their recent paper "Essential numerical range, essential spectrum and a problem of Halmos", *Acta Sci. Math.*, **33** (1972), 179—192.

where $r(T+K)$ denotes the spectral radius of $T+K$. Moreover, if Q is any projection in $\mathcal{L}(\mathfrak{H})$ and T_Q denotes the compression of T to the range of Q, i.e. $T_Q=(QT)|\text{ran } Q$ then we also have

$$r_e(T) = \inf_{(1-Q)\in\mathcal{P}_f} r(T_Q),$$

where $\mathcal{P}_f$ is the set of all finite rank projections in $\mathcal{L}(\mathfrak{H})$.

Proof. Let $\lambda_0\in\Sigma(T)$ be an isolated point such that $T-\lambda_0\in\Phi_0$. Set $\Sigma_0 = \Sigma(T)-\{\lambda_0\}$ and E_{Σ_0} the idempotent associated with the clopen subset Σ_0 of $\Sigma(T)$ ([11], § 148). Also we denote by Q_0 the (orthogonal) projection onto ran E_{Σ_0}. It follows that $\Sigma_0=\Sigma(T_{Q_0})$; hence $\Sigma(TQ_0) = \Sigma(T_{Q_0})\cup\{0\} = \Sigma_0\cup\{0\}$. Therefore $\text{rad}\,(\Sigma(TQ_0))=\text{rad}\,(\Sigma_T Q_0)$. Since $1-Q_0$ is a finite rank projection we see that

$$\inf_{K\in\mathcal{K}} r(T+K) \leqq r(TQ_0) = \text{rad}\,\Sigma_0 \quad \text{and also} \quad \inf_{(1-Q)\in\mathcal{P}_f} r(T_Q) \leqq \text{rad}\,\Sigma_0 .$$

Now the same argument used for λ_0 can be applied to any set consisting of finitely many isolated $\lambda\in\Sigma(T)$ such that $T-\lambda\in\Phi_0$. In this way we conclude that

$$\inf_{K\in\mathcal{K}} r(T+K) \leqq \text{rad}\,B(T) = r_e(T), \quad \text{and} \quad \inf_{(1-Q)\in\mathcal{P}_f} r(T_Q) \leqq r_e(T),$$

proving half of the lemma. On the other hand, recalling that $r_e(T)=\text{rad}\,\Omega(T)$, $\Omega(T) = \bigcap_{K\in\mathcal{K}} \Sigma(T+K)$ and observing that $\Omega(T) \subset \bigcap_{(1-Q)\in\mathcal{P}_f} \Sigma(T_Q)$ we see that the other half is also valid.

Remark. We list below some other elementary properties of the essential spectral radius of an operator T.

i) It follows from $\text{rad}\,E(T)=r_e(T)$ that

$$r_e(T) = \lim_{n\to\infty} \|\pi(T^n)\|^{1/n}.$$

ii) From Lemma 2. 2, and [7], Problem 122 it is not hard to see that

$$r_e(T) = \inf_{S\in\Phi_0'} \|[\pi(S)]^{-1}\pi(T)\pi(S)\|.$$

iii) Let $w_e(T)$ be the essential numerical radius of T ([12], § 3) that is $w_e(T) = \text{rad}\,W_e(T)$, where $W_e(T)$ is the essential numerical range of T ([12], [16]). Then

$$(*) \qquad r_e(T)\leqq w_e(T)\leqq \|\pi(T)\|.$$

We recall that $W_e(T)$ can be defined by the following identities ([12, Lemma 3. 3):

$$(**) \qquad W_e(T) = \bigcap_{K\in\mathcal{K}} \overline{W(T+K)} = \bigcap_{(1-Q)\in\mathcal{P}_f} \overline{W(T_Q)},$$

where $W(S)$ represents the numerical range of S.

iv) It is easy to give examples of operators S for which $r_e(S) < w_e(S) < \|\pi(S)\|$. We can take, for instance, S to be the 2×2 scalar operator matrix

$$S = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

acting on $\mathfrak{H}\oplus\mathfrak{H}$ in the usual fashion. We see that $r_e(S)=0$, $w_e(S)=1$, $\|\pi(S)\|=2$. It is also easy enough to find an operator T for which $r_e(T)=w_e(T)<\|\pi(T)\|$. Let us observe first that for every $R_1, R_2 \in \mathscr{L}(\mathfrak{H})$ $W_e(R_1\oplus R_2)$ coincides with the convex hull of $W_e(R_1)\cup W_e(R_2)$ (this readily follows from $(**)$). Now, let R be any operator such that $r_e(R)=\|\pi(R)\|=1$ (take for instance $R=1$) and set $T = R\oplus S$, where S is the nilpotent operator defined previously. Then

$$\|\pi(T)\| = \max\left(\|\pi(R)\|, \|\pi(S)\|\right) = 2,$$

while, by the preceding comment, $w_e(T)=1$ and hence $1=r_e(T)=w_e(T)<\|\pi(T)\|=2$.

We are indebted to J. STAMPFLI who has pointed out to us that the remaining situation concerning the strict inequality in $(*)$ is impossible. The proof of this fact, that we present below, is a simplification of Stampfli's argument.

Lemma 2. 3. *Let $T\in\mathscr{L}(\mathfrak{H})$ and suppose that $\|\pi(T)\|=w_e(T)$. Then $w_e(T)=$ $=r_e(T)$.*

Proof. Let $\lambda\in W_e(T)$ be such that $|\lambda|=\|\pi(T)\|$. It can be easily proved ([12], Lemma 2. 1) that

$$(***) \qquad \|\pi(T)\| = \inf_{(1-Q)\in\mathscr{P}_f} \|TQ\|.$$

Define, inductively, an orthonormal sequence $\{x_n\}$ in $\mathfrak{H}$ and a decreasing sequence of projections $\{Q_n\}$ in $\mathscr{L}(\mathfrak{H})$ as follows: let x_0 be any unit vector in $\mathfrak{H}$ and Q_0 be any projection in $\mathscr{L}(\mathfrak{H})$ such that $Q_0x_0=0$ and $(1-Q_0)\in\mathscr{P}_f$; having defined x_k and Q_k for $0\leqq k\leqq n$, let $x_{n+1}\in Q_n\mathfrak{H}$ with $\|x_{n+1}\|=1$ and let $Q_{n+1}\leqq Q_n$ with $Q_{n+1}x_{n+1}=0$, $(1-Q_{n+1})\in\mathscr{P}_f$ such that $|(Tx_{n+1}, x_{n+1})-\lambda| \leqq 1/n$, $\|TQ_{n+1}\| \leqq |\lambda|+1/(n+1)$ (the existence of x_{n+1} and Q_{n+1} is guaranteed by conditions $(**)$ and $(***)$). Since $|(Tx_n, x_n)| \leqq \|Tx_n\| \leqq |\lambda|+1/(n-1)$, $n>1$ and $|(Tx_n, x_n)|\to|\lambda|$ it follows that $\|Tx_n\|\to$ $\to|\lambda|$. Also, we see that $\|(T-\lambda)x_n\|^2 = \|Tx_n\|^2-\bar\lambda(Tx_n, x_n)-\lambda\overline{(Tx_n, x_n)}+|\lambda|^2\to 0$. If $\pi(T-\lambda)$ were invertible, then there would exist $S\in\mathscr{L}(\mathfrak{H})$ such that $\pi(S)\pi(T-\lambda) =$ $=\pi(1)$ and hence $S(T-\lambda) = 1+K$ for some $K\in\mathscr{K}$; but $\left(S(T-\lambda)x_n, x_n\right)\to 0$ while $\left((1+K)x_n, x_n\right)\to 1$. Therefore $\lambda\in E(T)$ and hence $w_e(T)=|\lambda|\leqq r_e(T)$.

3. Upper-semicontinuity of the essential spectrum. Let $\mathscr{B}$ be a complex Banach algebra with identity. It is well known ([10]) that the spectrum $\Sigma(T)$ of $T\in\mathscr{B}$ is an upper-semicontinuous function of T. The next lemma shows that each separated part (closed and open subset) is also an upper-semicontinuous function of T.

Theorem 3.[2]) *For $T\in\mathscr{B}$, let Σ be a non-empty clopen (closed and open) subset $\Sigma(T)$, and set $\Sigma'=\Sigma(T)-\Sigma$. If V and V' are two disjoint neighborhoods of Σ and Σ' respectively, then there exists $\varepsilon>0$ such that for every $S\in\mathscr{B}$ with $\|T-S\|<\varepsilon$ the following conditions are satisfied:*

a) $\Sigma(S)\subset V\cup V'$, and the set $\Lambda=\Sigma(S)\cap V$ is not empty,

b) if E_Σ and E_Λ are the idempotents associated with Σ and Λ corresponding to T and S, then there exists a constant $k>0$ such that $\|E_\Sigma-E_\Lambda\|<k\|T-S\|<1$, $1\leqq j\leqq n$,

c) if **B** *is the Banach algebra of all bounded operators on a complex Banach space, then* ran E_Σ *is topologically isomorphic to* ran E_Λ.

Proof. Let W be an open subset of $\mathbf{C}$ such that $\Sigma\subset W$, $\overline{W}\subset V$ and $\Gamma=\overline{W}-W$ consisting of finitely many rectifiable closed Jordan curves. From the upper-semicontinuity of $\Sigma(T)$ there exists $\delta>0$ such that if $\|T-S\|<\delta$, then $\Sigma(S)\subset W\cup V'$. Thus $\Sigma(S)\cap V=\Sigma(S)\cap W=\Lambda$. Let $M=\sup_{\lambda\in\Gamma}\|(T-\lambda)^{-1}\|$. Since $M<\infty$ we can choose $0<\eta<\min(\delta,1/M)$ so that, if $\|T-S\|<\eta$, then $\sup_{\lambda\in\Gamma}\|(T-\lambda)-(S-\lambda)\|<$ $<\inf_{\lambda\in\Gamma}\left(1/\|(T-\lambda)^{-1}\|\right)$ and hence

$$\sup_{\lambda\in\Gamma}\|(S-\lambda)^{-1}\|\leqq\sup\frac{\|(T-\lambda)^{-1}\|}{1-\|T-S\|\,\|(T-\lambda)^{-1}\|}\leqq\frac{M}{1-\eta M}$$

for every $S\in\mathscr{B}$ with $\|T-S\|<\eta$. Employing the last inequality we obtain

$$\|E_\Sigma-E_\Lambda\|=\left\|(1/2\pi i)\int_\Gamma[(T-\lambda)^{-1}-(S-\lambda)^{-1}]\,d\lambda\right\|\leqq$$

$$\leqq(1/2\pi)\int_\Gamma\|(T-\lambda)^{-1}(T-S)(S-\lambda)^{-1}\|\,|d\lambda|\leqq k\|T-S\|,$$

with $k=\dfrac{M^2}{2\pi(1-\eta M)}\displaystyle\int_\Gamma|d\lambda|$. Choosing $0<\varepsilon<\min\left(\eta,\dfrac{1}{k}\right)$ part b) follows. Since the norm of any non-zero idempotent is never less than one, a) is also clear. Finally, part c) is a consequence of the following classical result ([11]).

Lemma (Sz.-Nagy). *If F and G are idempotents on a Banach space and $\|F-G\|<1$, then* ran F *and* ran G *are topologically isomorphic, and hence they have the same dimension.*

Theorem 4. *Each separated part (clopen subset) of a) $E(T)$, b) $B(T)$, c) $\Omega(T)$ is an upper-semicontinuous function of $T\in\mathscr{L}(\mathfrak{H})$, in the sense of Theorem 3.*

[2]) A slightly different version of this result in case $\mathfrak{B}$ is equal to the algebra of all bounded operators on a Banach space can be found in [8], Theorem 3. 16 (Chap. 4).

Proof. Part a) follows directly from Lemma 3. 1 taking $\mathscr{B}=\mathscr{L}(\mathfrak{H})/\mathscr{K}$, and the fact that $\pi:\mathscr{L}(\mathfrak{H})\to\mathscr{L}(\mathfrak{H})/\mathscr{K}$ is norm decreasing.

In order to prove b) assume that B is a non-void clopen subset of $B(T)$ and write $B'=B(T)-B$. Suppose also that V and V' are two disjoint open neighborhoods of B and B', respectively. We must show that there exists $\eta>0$ such that for every $S\in\mathscr{L}(\mathfrak{H})$ with $\|T-S\|<\eta$ we have $B(S)\subset V\cup V'$ and $B(S)\cap V\neq\emptyset$. In fact taking smaller neighborhoods if necessary we can assume $\Sigma(T)\cap b(V\cup V')=$ $=\emptyset$. Since $\Sigma(T)-[\overline{V\cup V'}]$ consists of finitely many points we can choose a neighborhood V'' of $\Sigma(T)-[\overline{V\cup V'}]$ such that $V''\cap(V\cup V')=\emptyset$. From Theorem 1 $E(T)\cap V\neq\emptyset$ and $E(T)\subset V\cup V'$. Now using Theorem 3 and part a) we conclude that there exists $\eta>0$ such that, if $S\in\mathscr{L}(\mathfrak{H})$ and $\|T-S\|<\eta$, then $E(S)\subset$ $\subset V\cup V'$, $\Sigma(S)\subset V\cup V'\cup V''$ and $E(S)\cap V\neq\emptyset$. It follows, again from Theorem 1, that $B(S)\cap V''=\emptyset$ and hence that $B(S)\subset V\cup V'$, $B(S)\cap V\neq\emptyset$.

Finally to prove c) suppose that Ω is a non-void clopen subset of $\Omega(T)$ and let $\Omega'=\Omega(T)-\Omega$. Also, let U and U' be two disjoint open neighborhoods of Ω and Ω' respectively. Furthermore, let W and W' be relatively compact open neighborhoods of Ω and Ω' such that $\overline{W}\subset U$, $\overline{W}'\subset U'$. Since the set Φ_0 of all Fredholm operators of index zero is open, there exists $\delta'>0$ such that for any $S\in\mathscr{L}(\mathfrak{H})$ with $\|T-S\|<\delta'$ we have $S-\lambda\in\Phi_0$, for every $\lambda\in b(W\cup W')$. On the other hand, from part a) there exists $\varepsilon>0$ such that $\|T-S\|<\varepsilon$ implies $E(S)\subset W\cup W'$ and $E(S)\cap W\neq\emptyset$. It follows from Theorem 1 that for every $S\in\mathscr{L}(\mathfrak{H})$ with $\|T-S\|<$ $<\delta=\min(\varepsilon,\delta')$ we have $\Omega(S)\subset W\cup W'$ and $\Omega(S)\cap W=\Omega(S)\cap U\neq\emptyset$. This completes the proof of the theorem.

Given $T\in\mathscr{L}(\mathfrak{H})$ we say that an invariant (closed) subspace of T is hyperinvariant ([5]) if it is invariant under every operator in the commutant $\mathscr{A}'_T$ of T (recall that $\mathscr{A}'_T=\{S\in\mathscr{L}(\mathfrak{H}):ST=TS\}$). Let $\mathscr{A}''_T$ be the double commutant of T, i.e. $\mathscr{A}''_T=$ $=\{R\in\mathscr{L}(\mathfrak{H}):RS=SR$, for all $S\in\mathscr{A}'_T\}$. Clearly $T\in\mathscr{A}''_T\subset\mathscr{A}'_T$.

It is easy to check that for every $R\in\mathscr{A}''_T$ the range of R and the null of R are hyperinvariant subspaces of T.

Theorem 5. *If $T\in\mathscr{L}(\mathfrak{H})$ and $E(T)$ is disconnected, then there exists $\varepsilon>0$ such that for every $R\in\mathscr{L}(\mathfrak{H})$, $\|R\|<\varepsilon$ and for every $K\in\mathscr{K}$, the operator $T+R+K$ has a non-trivial hyperinvariant subspace.*

Proof. The theorem is a direct consequence of Theorem 4-a) and the next lemma.

Lemma 3. 1. *For $T\in\mathscr{L}(\mathfrak{H})$, let Σ be a clopen subset $\Sigma(T)$ and let E_Σ be the associated idempotent, then* ran E_Σ *and* null E_Σ *are hyperinvariant subspaces of T. Furthermore, if $E(T)$ is disconnected then T has a non-trivial hyperinvariant subspace.*

Proof. The first part follows from the fact that E_Σ belongs to the rational algebra generated by T and hence to $\mathscr{A}_T''$. To prove the second part note that if $\Sigma(T)$ is connected, then there exists $\lambda_0 \in \Sigma(T) - E(T)$ such that either ran $(T-\lambda_0)$ is proper or null $(T-\lambda_0)$ is so. In any case T has a non-trivial hyperinvariant subspace, as asserted.

In section 4 we will see that if $T \in (ED)$, then a more precise version of Theorem 5 can be given.

Two subspaces $\mathfrak{M}$ and $\mathfrak{N}$ of $\mathfrak{H}$ are said to be complementary if there exists an idempotent $F \in \mathscr{L}(\mathfrak{H})$ such that $\mathfrak{M} = \text{ran}\, F$, $\mathfrak{N} = \text{null}\, F$.

Theorem 6. *Let m, n be two non-zero cardinal numbers such that $m+n = \dim \mathfrak{H}$, then the set of all operators in $\mathscr{L}(\mathfrak{H})$ having two complementary hyperinvariant subspaces of dimension m and n has a non-void interior.*

Proof. Let $P \in \mathscr{L}(\mathfrak{H})$ be an (orthogonal) projection such that $\dim \text{ran}\, P = m$, $\dim \text{null}\, P = n$. It is easy to see that $P = (-1/2\pi i) \int_{|\lambda-1|=1/2} (P-\lambda)^{-1}\, d\lambda$ (where the circle $|\lambda - 1| = 1/2$ is positively oriented). From Theorem 3 we can find an $\varepsilon > 0$ such that, if $\|S-P\| < \varepsilon$, then $\Lambda = \Sigma(S) \cap \{\lambda \in \mathbf{C} : |\lambda - 1| < 1/2\} \neq \emptyset$ is a proper clopen subset of $\Sigma(S)$ and $\dim \text{ran}\, E_\Lambda = m$, $\dim \text{null}\, E_\Lambda = n$, where

$$E_\Lambda = (-1/2\pi i) \int_{|\lambda-1|=1/2} (S-\lambda)^{-1}\, d\lambda.$$

The theorem follows, now, from Lemma 3. 1.

Theorem 7. *The class (ED) is uniformly open in $\mathscr{L}(\mathfrak{H})$.*

Proof. The following elementary topological lemma together with Theorem 4-a) show that each separated part of $\hat{E}(T)$ is an upper-semicontinuous function of T. Clearly, from this assertion, the theorem follows.

Lemma 3. 2. *Let X be a compact subset of the plane and let U be an open neighborhood of $\hat{X}$, then there exists an open neighborhood V of X such that if Y is any compact subset of V, it follows that $\hat{Y} \subset U$.*

4. On invariant subspaces of operators in (ED). In this paragraph we turn our attention to the proof of a decomposition theorem for the invariant subspace lattice $\mathscr{I}_T$ of an operator $T \in (ED)$. The techniques provided in [4] and [5] are basic for our purposes. We start our discussion with a lemma which is useful for proving that certain operators lie in (ED).

Lemma 4. 1. *Let $T \in \mathscr{L}(\mathfrak{H})$ and assume that Σ is a clopen subset of $\Sigma(T)$. If E_Σ denotes the associated idempotent, then $E(T) \cap \Sigma \neq \emptyset$ if and only if* ran E_Σ *is* *infinite dimensional. In this case* $E(T)\cap\Sigma = E[T|\text{ran } E_\Sigma]$. *Thus, if* $E(T)\cap\Sigma=\emptyset$, Σ *consists only of finitely many points which are eigenvalues of finite multiplicity.*

Proof. Assume ran E_Σ is finite dimensional and let $\lambda_0\in\Sigma$, write $S=T-\lambda_0$, $S_1=S|\text{ran } E_\Sigma$ and $S_2=S|\text{null } E_\Sigma$. Since $\Sigma(S_2)=\Sigma(T)-\Sigma$, S_2 is invertible. Therefore ran S = ran S_1+ran S_2. But ran S_2=null E_Σ is closed and ran S_1 if finite dimensional, then ran S is also closed. Since null $S\subset$ ran E_Σ and ran $S\supset$ $\supset$null E_Σ we conclude that $S(=T-\lambda_0)$ is a Fredholm operator, and hence $\lambda_0\notin E(T)$. Conversely, suppose that ran E_Σ is infinite dimensional. If null E_Σ is finite dimensional, from the first part of the proof, $E(T)\subset\Sigma$ and there is nothing to prove. Thus, we can also assume that null E_Σ is infinite dimensional. At this point we need the following auxiliary construction: let $\mathfrak{H}_1=\text{ran } E_\Sigma$, $\mathfrak{H}_2=\mathfrak{H}_1^\perp$, also let $Z_1\in\mathscr{L}(\mathfrak{H}_1)$ be the identity operator on $\mathfrak{H}_1$, and let $Z_2\colon\mathfrak{H}_2\to\text{null } E_\Sigma$ be the bounded linear transformation given by $Z_2=(1-E_\Sigma)|\mathfrak{H}_2$. It is easy to see that Z_2 is invertible (it is bijective). Define now the invertible transformation $Z\colon \mathfrak{H}_1\oplus\mathfrak{H}_2\to\mathfrak{H}$ by $Z|\mathfrak{H}_1=Z_1$, $Z|\mathfrak{H}_2=Z_2$. Letting $T_1=T|\text{ran } E_\Sigma$, $T_2=T|\text{null } E_\Sigma$ and observing that $Z^{-1}TZ=Z_1^{-1}T_1Z_1\oplus$ $\oplus Z_2^{-1}T_2Z_2$ we have $E(T)=E(Z^{-1}TZ)=E(Z_1^{-1}T_1Z_1)\cup E(Z_2^{-1}T_2Z_2)=$ $=E(T_1)\cup E(T_2)$. Since $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are infinite dimensional we conclude that $E(T_j)=$ $=E(Z_j^{-1}T_jZ_j)\neq\emptyset$, $j=1,2$, and hence $E(T)\cap\Sigma=E(T_1)\neq\emptyset$, $E(T)\cap$ $\cap[\Sigma(T)-\Sigma]=E(T_2)\neq\emptyset$. The proof of the lemma is complete.

Given $T\in\mathscr{L}(\mathfrak{H})$, let $\mathscr{I}_T$ be the lattice of all invariant subspaces of T with the topology induced by the distance between subspaces, namely, if P, Q are the (orthogonal) projections onto the subspaces $\mathfrak{M},\mathfrak{N}\in\mathscr{I}_T$, then $\Theta(\mathfrak{M},\mathfrak{N})=\|P-Q\|$. It can be proved that ([5], Corollary 1. 2) if $\mathfrak{M}$ is an isolated point of $\mathscr{I}_T$, then it is a hyperinvariant subspace of T.

Theorem 8. *Let* $T\in(ED)$, *that is assume there exist two proper clopen subsets* Λ_1 *and* Λ_2 *of* $E(T)$ *such that* $E(T)=\Lambda_1\cup\Lambda_2$ *and* $\hat\Lambda_1\cap\hat\Lambda_2=\emptyset$. *Then there exist two infinite dimensional complementary subspaces* $\mathfrak{M}_1,\mathfrak{M}_2\in\mathscr{I}_T$ *which are isolated points of* $\mathscr{I}_T$, *such that if* $T_j=T|\mathfrak{M}_j$, $j=1,2$ *then* $\mathscr{I}_T$ *is homeomorphic to the topological product* $\mathscr{I}_{T_1}\times\mathscr{I}_{T_2}$ *and* $E(T_j)=\Lambda_j$, $j=1,2$.

Proof. From Theorem 1, $\hat\Sigma(T)-\hat E(T)$ is a set of isolated points, thus there exist two proper clopen subsets Σ_1, Σ_2 of $\Sigma(T)$ such that $\Sigma_1\cup\Sigma_2=\Sigma(T)$, $\hat\Sigma_1\cap$ $\cap\hat\Sigma_2=\emptyset$ and $\Lambda_j\subset\Sigma_j$, $j=1,2$. Now, let $\mathfrak{H}_1=\text{ran } E_{\Sigma_1}$, $\mathfrak{H}_2=\mathfrak{H}_1^\perp$. Furthermore, let $\mathfrak{M}_j=\text{ran } E_{\Sigma_j}$, and $T_j=T|\mathfrak{M}_j$, $j=1,2$. We deduce, as in the proof of Lemma 4. 1 that there exist an invertible transformation $Z\colon\mathfrak{H}_1\oplus\mathfrak{H}_2\to\mathfrak{H}$ such that $Z^{-1}TZ=$ $=Z_1^{-1}T_1Z_1\emptyset Z_2^{-1}T_2Z_2$. Using now Theorems 2 and 6 of [5] we obtain $\mathscr{I}_T\approx$ $\approx\mathscr{I}_{Z^{-1}TZ}\approx\mathscr{I}_{Z_1^{-1}T_1Z_1}\times\mathscr{I}_{Z_2^{-1}T_2Z_2}\approx\mathscr{I}_{T_1}\times\mathscr{I}_{T_2}$. On the other hand, using Theorem 1 in [4], it is easy to check that the subspaces $\mathfrak{H}_1\oplus\{0\}$ and $\{0\}\oplus\mathfrak{H}_2$ are isolated points of $\mathscr{I}_{Z^{-1}TZ}$ and hence $\mathfrak{M}_1,\mathfrak{M}_2$ are isolated points of $\mathscr{I}_T$. The last part of the theorem follows from Lemma 4. 1.

We close this section with a couple of results that illustrate how to produce non-trivial examples of operators in (ED).

Theorem 9. *Let P be any (orthogonal) projection in $\mathcal{L}(\mathfrak{H})$ and let V be any isometry. Then the* 2×2 *self-adjoint operator matrix*

$$T = \begin{bmatrix} P & V \\ V^* & 0 \end{bmatrix},$$

acting in the usual fashion on $\mathfrak{H}\oplus\mathfrak{H}$, is in (ED). Furthermore, if $\Lambda_1=[-1, 1/2(1-\sqrt{5})]$, $\Lambda_2 = [0, 1/2(1+\sqrt{5})]$, then $\Sigma(T) \subset \Lambda_1 \cup \Lambda_2$ and $E(T)\cap\Lambda_j \neq \emptyset$ $(j=1, 2)$.

Proof. Let p be the cubic polynomial $p(\lambda) = \lambda^3-\lambda^2-\lambda+1$. It can be checked that

$$p(T) = \begin{bmatrix} 1-(P-VV^*)^2 & 0 \\ 0 & V^*PV \end{bmatrix}.$$

Since it is clear that V^*PV and $(P-VV^*)^2$ are positive contractions, it follows that $p(T)$ enjoys the same property. Therefore from the spectral mapping theorem $\Sigma(T) \subset p^{-1}[0, 1] = \Lambda_1 \cup \Lambda_2$. Now, let $\mathfrak{M}_1(\mathfrak{M}_2)$ be the range of the map $y\to\begin{bmatrix} Vy \\ y\end{bmatrix}\left(y\to\begin{bmatrix} -Vy \\ y\end{bmatrix}\right)$ from $\mathfrak{H}$ into $\mathfrak{H}\oplus\mathfrak{H}$. It can be easily checked that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are infinite dimensional orthogonal subspaces and that the compression of T to $\mathfrak{M}_1(\mathfrak{M}_2)$ is a positive invertible (negative invertible) operator. We use this preceding remark to prove that $E(T)\cap\Lambda_j \neq \emptyset$. This will clearly complete the proof of the theorem. Assume for example that $E(T)\cap\Lambda_1 = \emptyset$. Then (Lemma 4. 1) $\Sigma(T)\cap\Lambda_1$ consists of finitely many points which are eigenvalues of finite multiplicity. Since T is self-adjoint there exists a finite rank projection $Q\in\mathcal{L}(\mathfrak{H}\oplus\mathfrak{H})$ such that $TQ(=QT)$ is positive and $T(1-Q)$ is negative. Let T' and T'' be the compressions of TQ and $T(1-Q)$, respectively to $\mathfrak{M}_1$. Then T' and $-T''$ are positive operators and $T' = T_1+(-T'')$ (where T_1 is the compression of T to $\mathfrak{M}_1$) is a positive invertible operator on the infinite dimensional space $\mathfrak{M}_1$. This is a contradiction since T' is a compact operator. An analogous reasoning shows that $E(T)\cap$ $\cap\Lambda_2 \neq \emptyset$.

Theorem 10. *Let $A\in\mathcal{L}(\mathfrak{H})$ be such that $\hat{E}(A)$ does not touch the segment $\Lambda=[-2, 2]$ of the real axis. Then the operator $T\in\mathcal{L}(\mathfrak{H}\oplus\mathfrak{H})$ defined by*

$$T = \begin{bmatrix} A & 1 \\ -1 & 0 \end{bmatrix}$$

is in (ED).

Proof. Using [7], Problem 55, it readily follows that $\Sigma(T)=f^{-1}[\Sigma(A)]$, where f is the analytic function defined on $\mathbf{C}-\{0\}$ by $f(\lambda) = \lambda+1/\lambda$. Furthermore, we see that $E(T)=f^{-1}[E(A)]$. Also it can be checked that $f^{-1\wedge}[E(A)]=f^{-1}[\hat{E}(A)]$. Therefore $f^{-1}[\hat{E}(A)]=\hat{E}(T)$. Since f maps the exterior of the closed unit disc D onto $\mathbf{C}-A$ and the interior of $D-\{0\}$ onto the same region, the theorem follows.

5. Open questions. As the reader may have noticed this paper raises some interesting questions. For example, from the proof of Lemma 2. 2, the following inclusion formula can be obtained:

$$\Omega(T)\subset\bigcap_{(1-Q)\in\mathscr{P}_f}\Sigma(T_Q)\subset B(T),$$

where $\mathscr{P}_f$ is the set of all finite rank projections and $T_Q=(QT)|\text{ran } Q$. Therefore it is natural to ask: does there exist $T\in\mathscr{L}(\mathfrak{H})$ for which the inclusions in the above chain of inequalities become proper?

Moreover, we can also ask the following related question: if $(1-Q)\in\mathscr{P}_f$ and $T\in\mathscr{L}(\mathfrak{H})$, does $\Sigma(T)-\Sigma(T_Q)$ consist only of isolated points?

Observe that if the last statement holds, then $B(T)=\bigcap_{(1-Q)\in\mathscr{P}_f}\Sigma(T_Q)$, as it is easy to verify.

In a different direction we may ask: do there exist compact operators K_j, $0\leqq j\leqq 4$, such that the following conditions are satisfied?

a) $\Sigma(T+K_0)=\Omega(T)$, b) $B(T+K_1)=\Omega(T)$, c) $\Sigma(T+K_2)=B(T)$, d) $\Sigma(T+K_3)=B(T+K_3)$, e) $\hat{\Sigma}(T+K_4)=\hat{B}(T+K_4)(=\hat{B}(T))$.

Furthermore,

f) if $T\in\mathscr{L}(\mathfrak{H})$ and $\hat{\Sigma}(T+K)$ is disconnected, for every $K\in\mathscr{K}$, is T in (ED)?

Note that a$\Rightarrow$b and (a *or* c)$\Rightarrow$d$\Rightarrow$e$\Rightarrow$f. On the other hand (b *and* c)$\Rightarrow$a.

Finally we give some fragmentary results in the positive direction concerning these last questions.

Theorem 11. *Let $T\in\mathscr{L}(\mathfrak{H})$ and assume that all points in $\Sigma(T)-B(T)$, except for a finite number of them, are reducing eigenvalues of T (i.e. the corresponding eigenspaces are reducing subspaces of T), then there exists $K\in\mathscr{K}$ such that $\Sigma(T+K)=B(T)$.*

Proof. If $\Sigma(T)-B(T)=\emptyset$, there is nothing to prove. Let λ_n, $n=1,2,\ldots$ be the points in $\Sigma(T)-B(T)$ and let ν_n, $n=1,2,\ldots$ be complex numbers such that the points $\mu_n=\nu_n+\lambda_n$ lie in the set $\Sigma'(T)$ of limit points of $\Sigma(T)$, and $\inf_{\mu\in\Sigma'(T)}|\lambda_n-\mu|=|\nu_n|$. Also, let $A_n=\Sigma(T)-\bigcup_{k=1}^{n}\{\lambda_k\}$ and $E_{\{\lambda_n\}}$, E_{A_n} be the idempotents associated with $\{\lambda_n\}$, A_n; define $K_m=\sum_{n=1}^{m}\nu_n E_{\{\lambda_n\}}$. Then

$$T+K_m=(T+K_m)E_{A_m}+(T+K_m)\left(\sum_{n=1}^{m}E_{\{\lambda_n\}}\right)=TE_{A_m}+\sum_{n=1}^{m}\{\mu_n+N_n\}E_{\{\lambda_n\}},$$

where N_n is a nilpotent operator acting on ran $E_{\{\lambda_n\}}$, $1 \leqq n \leqq m$. Therefore $\Sigma(T+K_m) = \Lambda_m \cup \bigcup_{n=1}^{m} \{\mu_n\} = \Lambda_m$. This completes the proof of the theorem in case $\Sigma(T)-B(T)$ is finite. By hypothesis there exists m_0, such that if $m > m_0$, then $E_{\{\lambda_m\}}$ is an orthogonal projection and since $E_{\{\lambda_m\}} E_{\{\lambda_n\}} = 0$, for $n \neq m$ and $\lim_{n\to\infty} \nu_n = 0$, we see that K_m converges, in the norm topology, to a compact operator K such that $TK = KT$. Since $T+K$ commutes with $T+K_m$, for each $m = 1, 2, \ldots$ we have ([8], Chap. IV, Theorem 3. 6) $\Sigma(T+K) = \lim_{m\to\infty} \Sigma(T+K_m) = \lim_{m\to\infty} \Lambda_m = B(T)$. Here the limits are taken in the Hausdorff metric topology for compact subsets of the plane.

Remark. i) We point out that the actual hypothesis needed to prove the preceding theorem is that the idempotent $E_{\{\lambda\}}$ be selfadjoint for all, but a finite number of $\lambda \in \Sigma(T)-B(T)$. On the other hand, BROWDER proved in [2], § 6, Lemma 17, that $\lambda \in \Sigma(T)-B(T)$ if and only if λ is an isolated eigenvalue of finite multiplicity of T which is a pole of the resolvent function $\mu \to (\mu - T)^{-1}$, $\mu \notin \Sigma(T)$. Also, it is shown in [6], Chap. 7, § 3, Theorem 18, that the residuum of the resolvent function around a pole $\lambda \in \Sigma(T)$ is $E_{\{\lambda\}}$. Therefore, the requirement stated at the begining of the present remark is equivalent to the following growth condition: if $\lambda \in \Sigma(T)-B(T)$ and m is the order of the pole λ,

$$\lim_{\mu\to\lambda} \left\| \frac{1}{(m-1)!} \frac{d^{(m-1)}}{d\mu^{(m-1)}} (\mu-\lambda)^m (\mu-T)^{-1} \right\| = 1 (= \|E_{\{\lambda\}}\|).$$

ii) All points in $B(T)-\Omega(T)$ are eigenvalues of finite multiplicity, but none of them are reducing. In fact, an elementary argument shows that λ is a reducing eigenvalue of T if and only if null $(T^* - \bar\lambda) \subset$ null $(T-\lambda)$. Suppose that λ is reducing and $(T-\lambda) \in \Phi_0$; since dim null $(T-\lambda) =$ dim null $(T^*-\bar\lambda)$, then null $(T-\lambda) =$ null $(T^*-\bar\lambda)$. This implies that $(T-\lambda)|[\text{null}\,(T-\lambda)]^\perp$ is invertible and hence λ is an isolated point of $\Sigma(T)$. We conclude that $\lambda \notin B(T)$, as asserted.

iii) Let X be a subset of the plane and let conh X denote its convex hull. From Theorem 1-c it readily follows that conh $E(T)=$ conh $\Omega(T)=$ conh $B(T)$.

As a consequence of the next theorem we shall obtain more information about the convex hull of the essential spectrum of T in case T is hyponormal (i.e. $\|T^*x\| \leqq \|Tx\|$, for all $x \in \mathfrak{H}$).

Theorem 12. *If $T \in \mathcal{L}(\mathfrak{H})$ is hyponormal, then there exists a normal compact operator K such that $T+K$ is hyponormal and $\Sigma(T+K) = \Omega(T)$.*

Proof. It is well known that if T is hyponormal, then every eigenvalue of T is reducing (null $(T^*-\bar\lambda) \subset$ null $(T-\lambda)$, for all $\lambda \in \mathbf{C}$). Therefore, from the preceding remark ii), $\Omega(T) = B(T)$. Let K be the compact operator constructed in the proof of Theorem 11. Arguing as in [3], Corollary 3. 3 we see that K is normal and $T+K$ is hyponormal. The second assertion follows directly from Theorem 11.

Corollary 5. 1. *If T is hyponormal, then* conh $E(T)=W_e(T)$. *Moreover, there exists* $K\in\mathcal{K}$ *such that* $\overline{W(T+K)}=W_e(T)$.

Proof. Let $K\in\mathcal{K}$ as in Theorem 12. Then from [15], Theorem 2 we have $\overline{W(T+K)}=\text{conh}\,\Sigma(T+K)=\text{conh}\,\Omega(T)=\text{conh}\,E(T)\subset W_e(T)$. Since $W_e(T)$ is clearly contained in $\overline{W(T+K)}$, the proof is complete.

Corollary 5. 2. *If T is hyponormal, then* $r_e(T)=\|\pi(T)\|$.

Proof. Let K as in Theorem 12, then from [14], Theorem 1, $\|\pi(T)\|\geqq r_e(T)=$ $=\text{rad}\,\Omega(T)=\text{rad}\,\Sigma(T+K)=\|T+K\|\geqq\|\pi(T)\|$, and the assertion follows.

Added in proof. The first question in Sec. 5 has a negative answer while the second question has an affirmative one. The solution to these problems are included in the author's recent paper "A characterization of the Browder spectrum", to appear in the *Proc. Amer. Math. Soc.* Problems a) and b) (and hence problems c), d), e) and f)) have been answered in the affirmative by J. G. STAMPFLI.

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On a property of approximate derivatives

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Introduction

An interesting property of the derivative f' of a continuous function f is that for every open interval (α, β) the inverse $f'^{-1}(\alpha, \beta)$ of (α, β) under f' is either void or of positive measure [1, 2]. This property of the derivative is also shared by the approximate derivative f'_{ap} [6, 11] and the nth Peano derivative f_n [8, 11]. Recently a necessary and sufficient condition is obtained under which a function of Baire class 1 will have the above property [7]. WEIL [12] proved that if f'_{ap} (resp. f_n) exists finitely and if the inverse $f'^{-1}_{ap}(\alpha, \beta)$ (resp. $f_n^{-1}(\alpha, \beta)$) is non-empty then the set $f'^{-1}(\alpha, \beta)$ (resp. $f^{(n)-1}(\alpha, \beta)$, $f^{(n)}$ being the nth derivative of f) is also of positive measure on the sets where it exists. It is also known that if f'_{ap} (resp. f_n) exists at each point then f' (resp. $f^{(n)}$) also exists on an everywhere dense set of intervals [3] (resp. [8]). The purpose of the present note is to establish certain results which will imply the above results as well as the results of WEIL [12].

Definitions and notations

We shall follow the standard definition of approximate derivatives (see [9]). For the definition of nth Peano derivative we refer to [8].

Throughout, all functions considered are real valued defined on the real line. To save space we shall use the following notations:

$$E(*, \alpha) = \{x : f'_{ap}(x) \leqq \alpha\}, \quad E(\beta, *) = \{x : f'_{ap}(x) \geqq \beta\},$$

$$E_0(*, \alpha) = \{x : f'(x) \text{ exists and } f'(x) \leqq \alpha\}, E_0(\beta, *) = \{x : f'(x) \text{ exists and } f'(x) \geqq \beta\}.$$

Main results

Theorem 1. *Let f be approximately continuous and let the approximate derivative f'_{ap} — finite or infinite — exits at each point. If for any two reals α, β, $\alpha < \beta$, ($\pm\infty$ admitted), the set*

$$E = \{x : \alpha < f'_{ap}(x) < \beta\}$$

is non-empty, then for every interval I where $I\cap E\neq 0$, there is a sub-interval $J\subset I$ such that

$$J\cap E\neq 0,\quad J\cap E=J\cap E_0,$$

where

$$E_0=\{x:f'(x)\ \text{exist and}\ \ \alpha<f'(x)<\beta\}.$$

Proof. We prove the theorem in two steps. In the first step we show that if for an interval I, $I\cap E\neq 0$, then $I\cap E_0\neq 0$. In the second step we complete the proof.

Step I. Suppose the contrary. Then there is an interval I such that

$$(1)\qquad I\cap E\neq 0,\quad I\cap E_0=0.$$

Let $x'\in I\cap E$. Then $\alpha<f'_{ap}(x')<\beta$. Choose $\alpha<\alpha_1<f'_{ap}(x')<\beta_1<\beta$. Then α_1, β_1 are finite. Let $\{Q\}_1$ and $\{Q\}_2$ be the collection of all non-degenerate components of $I\cap E_0(*,\alpha_1)$ and $I\cap E_0(\beta_1, *)$ respectively. Let $Q\in\{Q\}_1$. Then Q is an interval. Also $Q\subset E(*,\alpha_1)$. Since f'_{ap} possesses Darboux property [5], $\bar{Q}\subset E(*,\alpha_1)$, where $\bar{Q}$ is the closure of Q relative to I. Thus $f'_{ap}(x)\leqq\alpha_1$ for all $x\in\bar{Q}$ and hence $f'(x)$ exists and $f'(x)\leqq\alpha_1$ for all $x\in\bar{Q}$ [4]. Since Q is a component, $Q=\bar{Q}$. Thus every member of $\{Q\}_1$ is an interval which is closed relative to I. Similarly every member of $\{Q\}_2$ is also an interval which is closed relative to I.

Let $\{Q\}=\{Q\}_1\cup\{Q\}_2$. Let $P=I-\bigcup_{Q\in\{Q\}}Q^0$, where Q^0 denotes the interior of Q relative to I. Then P is non-void, since $x'\in P$. Clearly two distinct members of $\{Q\}$ cannot have a common end point. Hence the set P is perfect in I. We shall show that under the hypothesis f'_{ap}/P has no point of continuity in P. Since f'_{ap} is a function of Baire class 1 [10], it will lead to a contradiction.

Let $x_0\in P$ and let J be any open interval containing x_0. Then $J\cap I\cap E(*,\alpha_1)$ and $J\cap I\cap E(\beta_1, *)$ are non-void. For, if $J\cap I\cap E(*,\alpha_1)=0$, then $f'_{ap}(x)>\alpha_1$ for all $x\in J\cap I$ and hence $f'(x)$ exists and $f'(x)>\alpha_1$ for all $x\in J\cap I$. Since from (1) $I\cap E_0=0$, we conclude $f'(x)\geqq\beta_1$ for all $x\subset J\cap I$ and hence $J\cap I\subset Q^0$ for some $Q\in\{Q\}_2$, which is contrary to the fact that $x_0\in J\cap I$ and $x_0\in P$. Similar arguments hold for $J\cap I\cap E(\beta_1, *)$. From the above conclusion we assert that $J\cap P\cap E(*,\alpha_1)$ and $J\cap P\cap E(\beta_1, *)$ are also non-void. For, let $\xi\in J\cap I\cap E(*,\alpha_1)$. If $\xi\in P$ then $\xi\in J\cap P\cap E(*,\alpha_1)$ and the assertion follows. If $\xi\notin P$ then $\xi\in Q^0$ for some $Q\in\{Q\}_1$. Let η be the end point of Q which lies between x_0 and ξ. Then $\eta\in J\cap P\cap E(*,\alpha_1)$. Similar arguments hold for $J\cap P\cap E(\beta_1, *)$. Hence

$$\inf_{x\in J\cap P} f'_{ap}(x)\leqq\alpha_1,\qquad \sup_{x\in J\cap P} f'_{ap}(x)\geqq\beta_1$$

showing that the saltus of the function f'_{ap}/P at x_0 is at least $\beta_1-\alpha_1$. Hence f'_{ap}/P cannot be continuous at x_0. Since $x_0\in P$ is arbitrary, this completes the first step of our proof.

Step II. If possible, suppose that there is an interval I such that $I\cap E\neq 0$ but for every sub-interval $J\subset I$ satisfying $J\cap E\neq 0$ the relation

$$I\cap E\neq J\cap E_0 \tag{2}$$

holds. Let $F=I\cap E_0$. Then by Step I F is non-void. Also F is dense in itself. For, if $\varkappa_0\in F$ and G is any open interval containing $\varkappa_0$ then since f'_{ap} has Darboux property, there is $\varkappa'\neq\varkappa_0$ such that $\varkappa'\in G\cap I\cap E$. If G_1 is any open interval contained in G and containing $\varkappa'$ but not $\varkappa_0$ then $G_1\cap I\cap E\neq 0$ and hence by Step I $G_1\cap I\cap \cap E_0\neq 0$. So, there is $\varkappa''\in F$, $\varkappa''\neq\varkappa_0$ and $\varkappa''\in G$. Thus $\varkappa_0$ is a limit point of F. Let $\bar{F}$ denote the closure of F relative to I. Since F is dense in itself, $\bar{F}$ is perfect in I. We shall show that f'_{ap} has no point of continuity in $\bar{F}$ relative to $\bar{F}$.

Let $\varkappa_0\in\bar{F}$ and let H be any open interval containing $\varkappa_0$. Then there is $\varkappa'\in H\cap F$. Hence $\varkappa'\in H\cap I$ and $\alpha<f'(\varkappa')<\beta$. Choose $\alpha<\alpha_1<f'(\varkappa')<\beta_1<\beta$. Then the sets $H\cap I\cap E(*,\alpha_1)$ and $H\cap I\cap E(\beta_1,*)$ are non-void. For, if $H\cap I\cap E(*,\alpha_1)=0$ then $f'_{ap}(\varkappa)>\alpha_1$ for all $\varkappa\in H\cap I$ and hence $f'(\varkappa)$ exists for all $\varkappa\in H\cap I$. So, $H\cap I\cap E_0=H\cap I\cap E$ and $\varkappa'\in H\cap I\cap E$, which is contrary to (2). Similar arguments hold for $H\cap I\cap E(\beta_1,*)$.

Now for arbitrary $k_1,k_2,\alpha_1<k_1<k_2<\beta_1$, the sets $H\cap F\cap E(*,k_1)$ and $H\cap F\cap E(k_2,*)$ are non-empty. For, by the Darboux property of f'_{ap} there is $\xi\in H\cap I$ such that $\alpha_1<f'_{ap}(\xi)<k_1$. Hence as in Step I, there is $\xi_0\in H\cap I$ such that $f'(\xi_0)$ exists and $\alpha_1<f'(\xi_0)<k_1$ and so $\xi_0\in H\cap F$ showing that the set $H\cap F\cap \cap E(*,k_1)$ is non-empty. Similarly for the set $H\cap F\cap E(k_2,*)$. Hence $H\cap\bar{F}\cap E(*,k_1)$ and $H\cap\bar{F}\cap E(k_2,*)$ are also non-empty. Since k_1 and k_2 can be taken very near to α_1 and β_1 respectively, we conclude

$$\inf_{\varkappa\in H\cap F} f'_{ap}(\varkappa)\leqq\alpha_1,\qquad \sup_{\varkappa\in H\cap F} f'_{ap}(x)\geqq\beta_1$$

which shows that the saltus of $f'_{ap}/\bar{F}$ at $\varkappa_0$ is at least $\beta_1-\alpha_1$. Hence $f'_{ap}/\bar{F}$ is not continuous at $\varkappa_0$. Since $\varkappa_0$ is an arbitrary point of $\bar{F}$, it follows that $f'_{ap}/\bar{F}$ has no point of continuity in $\bar{F}$. Since f'_{ap} is a function of Baire class I, it provides with a contradiction. This completes the proof.

Putting $\alpha=-\infty$, $\beta=\infty$ we deduce the following known corollary.

Corollary [3]. *Let f have a finite approximate derivative f'_{ap} everywhere. Let $E=\{\varkappa\colon f'(\varkappa)$ exists$\}$. Then for every interval I, $I\cap E$ contains an interval.*

Remarks. It is known that a finite nth Peano derivative f_n possesses Darboux property, is of Baire class I, and is such that if it is bounded below or above in an interval then it is the ordinary nth derivative in that interval [8]. Since only these three properties of f'_{ap} are used in Theorem 1, the conclusion of Theorem 1 is valid if f'_{ap} is replaced by the finite nth Peano derivative f_n and f' is replaced by the ordinary nth derivative $f^{(n)}$.

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Uniformly closed Fourier algebras

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§ 0

At the end of his paper *The class of functions which are absolutely convergent Fourier transforms* [7], SEGAL remarks without proof that he and KAPLANSKY have proved the following theorem:

Theorem A. *If G is a locally compact group with the property that $L^1(G)$ is closed in the norm* $\|f\| = \sup\{\|f*g\|_2 : g \in L^2(G)\}$, *then G is finite.*

Theorem A is offered as a non-commutative version of the main result of Segal's paper which we state as follows:

Theorem B. *If G is a locally compact abelian group with the property that $L^1(G)$ is closed in the norm* $\|f\| = \sup\{|\hat{f}(\chi)| : \chi \in \hat{G}\}$, *then G is finite.*

The Plancherel theorem provides the perspective which assures us that for abelian G, the two norms are the same.

It is easy to see that the following is equivalent to Theorem A:

Theorem A′. *If the $*$-algebra $L^1(G)$ can be renormed as a C^*-algebra, G is finite.*

A different generalization of Theorem B to non-commutative groups is stimulated by recent advances in non-commutative harmonic analysis, especially the contributions of EYMARD [3] and WALTER [9]. In fact, since for abelian G, G is finite if and only if $\hat{G}$ is finite, we can rewrite Theorem B as follows:

Theorem B′. *If G is a locally compact abelian group such that* $A(G) = \{\hat{f} : f \in L^1(\hat{G})\}$ *is uniformly closed, G is finite.*

For a general locally compact group, $B(G)$, the Fourier—Stieltjes algebra of G, is the complex linear span of the continuous, positive definite functions on G.

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Normed as the dual of $C^*(G)$, and with pointwise algebraic operations, $B(G)$ is a commutative Banach algebra with identity, closed with respect to complex conjugation. $A(G)$, the Fourier algebra of G, is the closed ideal in $B(G)$ composed of precisely those functions of the form $f*g^{\sim}$, $f, g \in L^2(G)$. It is far from clear that this set of functions should have any of the algebraic or topological properties here ascribed to it. But see EYMARD [3].

At any rate, if G is abelian,

$$A(G)=\{\hat{f}: f \in L^1(\hat{G})\} \quad \text{and} \quad B(G)=\{\hat{\mu}: \ \mu \in M(\hat{G})\},$$

with the norms transported from $L^1(\hat{G})$, $M(\hat{G})$ respectively, which accounts for the names introduced above, and we can ask for the extension of Theorem B′ to non-abelian G:

Theorem C. *If $A(G)$ is uniformly closed, G is finite.*

A consequence of Theorem C would be

Theorem D. *If $B(G)$ is uniformly closed, G is finite.*

For, by uniqueness of complete norm topology, since A is closed in B, A would be uniformly closed.

Analogously, one could rewrite Theorem A (or A′) with $M(G)$ in place of $L^1(G)$.

We will prove all of these theorems, and prove their equivalence. More precisely, we will prove

Main Theorem. *Let G be a locally compact Hausdorff topological group. The following are equivalent*

i) *$A(G)$ is uniformly closed.*
ii) *The $*$-algebra $A(G)$ can be renormed as a C^*-algebra.*
iii) *$B(G)$ is uniformly closed.*
iv) *The $*$-algebra $B(G)$ can be renormed as a C^*-algebra.*
v) *$L^1(G)$ is uniformly closed in the left regular representation.*
vi) *The $*$-algebra $L^1(G)$ can be renormed as a C^*-algebra.*
vii) *$M(G)$ is uniformly closed in the left regular representation.*
viii) *The $*$-algebra $M(G)$ can be renormed as a C^*-algebra.*
ix) *The $*$-algebra $M(G)$ can be renormed as a W^*-algebra.*
x) *G is finite.*

(In references to $A(G)$ or $B(G)$ as $*$-algebras, complex conjugation is to be used as involution.)

Our interest in the question arose from WALTER's work [9], which shows $A(G)$ {respectively $B(G)$} to be a complete Banach-algebra invariant of the locally compact group G. This leads us to expect much more of the commutative picture to persist into the non-commutative setting than previously seemed reasonable.

Along the way we prove another result of interest: a locally compact group which is extremely disconnected must be discrete.

All the topological groups considered are assumed Hausdorff.

§ 1. Preliminaries

The following well-known lemma was first explicitely stated and proved in [8], for which reference I thank the referee. Professor I. HALPERIN has kindly pointed out that it can be proved as an easy consequence of [5], Theorem 1. We include a short proof.

Lemma 1. *A norm-separable W^*-algebra is finite dimensional.*

Proof. Let A be such an algebra. If A contains infinitely many mutually orthogonal projections, A contains a copy of l^∞ which is not norm separable. Therefore A has no type II or type III part, and A is a finite sum of n-homogeneous type I summands, each of the form $D\otimes L(H)$, with D commutative [1]. But since D (respectively $L(H)$ can have only finitely many mutually orthogonal projections, D and $L(H)$ must be finite dimensional. The lemma follows.

Lemma 2. *An extremely disconnected compact topological group is finite.*

Proof. Since an extremely disconnected space (one in which the closure of each open set is open) is totally disconnected, we have at once that the underlying space of G is $\{0, 1\}^m$, where m is a suitable cardinal number [4], Theorem 9. 15. We need only show that such a space is extremely disconnected only if it is finite.

First, assume $m=\aleph_0$. Then $X=\{0, 1\}^m$ is metric, with $d(x,0)=\sum_{i=1}^{\infty} x_i 2^{-i}$. Especially, $x\to d(x, 0)$ maps X onto $[0, 1]$.

Let $r\in[0, 1]$ be not a dyadic rational, and consider the open r-ball B about 0 in X. Since the dyadic representation of r is unique, and there exists a point $p\in X$ at distance r from 0, p is in the closure $\bar{B}$ of B, while no ball about p is in $\bar{B}$. Thus $\bar{B}$ is not open.

Now for arbitrary infinite cardinal m, keeping the notation X, B as above, we can write $\{0, 1\}^m = X\times\{0, 1\}^d$. Then $U = B\times\{0, 1\}^d$ is open, and $\bar{U} = \bar{B}\times\{0, 1\}^d$ cannot be open, or its projection $\bar{B}$ would be open in X. This concludes the proof.

Now we generalize to non-compact groups.

Definition. A topological space is *locally extremely disconnected* (L.E.D.) if every open subset with compact closure has open closure.

Lemma 3. *An open subspace A of L.E.D. Hausdorff space X is L.E.D.*

Proof. If A is open in X, and U is open in A with $\overline{U}\cap A$ compact, then U is open in X, and $\overline{U}\cap A$ is closed, so $\overline{U}=\overline{U}\cap A$ is compact, so open in X, so in A.

Proposition 4. *A locally extremely disconnected, locally compact topological group is discrete.*

Proof. Let V be a relatively compact, open neighborhood of e in G. Then $\overline{V}$ is a compact open neighborhood of e, and contains a compact, open subgroup H [4], Theorem 7. 5. Then H is an extremely disconnected compact group, so by Lemma 2, H is finite. Since H is open, G is discrete.

Corollary 5. *An extremely disconnected locally compact topological group is discrete.*

§ 2. Proof of the main theorem

Terminology. The left regular representation of a locally compact group G {or of $L^1(G)$, or of $M(G)$} is the restriction to G {or to $L^1(G)$, or to $M(G)$} of the action of M on $L^2(G)$ by left convolution: If $L_\mu(g)=\mu * g$, $(g\in L^2, \mu\in M)$ $\mu\to L_\mu$ is the left regular representation.

The weakly closed algebra generated by the image of L (this is unambiguous) is the left regular von Neumann algebra of G.

To avoid artificiality, we prove more than a minimal chain of implications. From those we prove the schemes

$$\begin{array}{ccc} \text{ii)} & & \\ \Updownarrow & & \\ \text{i)} \Leftrightarrow \text{iii)} \Rightarrow \text{x)}, \end{array} \quad \text{and} \quad \begin{array}{c} \qquad\qquad \text{vii)} \quad \text{v)} \\ \qquad\qquad \Updownarrow \qquad \Updownarrow \\ \text{ix)} \Leftrightarrow \text{viii)} \Rightarrow \text{vi)} \Rightarrow \text{iii)} \Rightarrow \text{x)} \end{array}$$

are easily extracted, which together with the fact that x) implies all the others serve to establish the equivalence.

The following implications are trivial: i)⇒ii); iii)⇒iv); v)⇒vi), vii)⇒viii), ix)⇒viii), x)⇒all others.

The implications ii)⇒i), iv)⇒iii), vi)⇒v), viii)⇒vii) all follow from the fact that the image of a C^*-algebra under any $*$-representation on a Hilbert space is again a C^*-algebra. For the first two implications, take as the representation $f\mapsto M_f\in L(L^2(G))$, where $M_f(g)=fg$ $(g\in L^2)$.

vi)⇒iii): If the $*$-algebra L^1 can be renormed as a C^*-algebra, the new norm $\|\cdot\|$ is smaller than the $\|\cdot\|_1$-norm, since the identity map is norm-reducing on $(L^1, \|\ \|_1)$ to $(L^1, \|\ \|)$. Therefore the two norms are equivalent. L^∞ thus becomes, with a change of norm, but not of norm topology, the dual space of a C^*-algebra, and hence the linear span of the positive linear functionals. But these can be chosen in L^∞ to be continuous positive definite functions. So L^∞ consists entirely of continuous functions, $L^\infty \cong B$, and B is uniformly closed.

The implications iii)$\Rightarrow$i), iv)$\Rightarrow$ii), vii)$\Rightarrow$v), viii)$\Rightarrow$vi) follow immediately from the fact that A is closed in B (respectively L^1 is closed in M), together with the equivalence of norms proved in the preceding paragraph.

vii)$\Rightarrow$ix) follows from SAKAI's characterization [6] of W^*-algebras as C^*-algebras which are dual spaces, since M is the dual of C_0. Again we use the equivalence of the original norm and the C^*-norm on M.

i)$\Rightarrow$ix): Since $A(G)$ is dense in $C_0(G)$ [3], i) is equivalent to $A(G)=C_0(G)$ as a topological vector space. But Eymard has shown that A^d=the left regular von Neumann algebra of G, while $C_0^d=M$. This gives us ix).

iii)$\Rightarrow$x): We have established that i)$\Rightarrow$ix)$\Rightarrow$viii)$\Rightarrow$vi)$\Rightarrow$iii)$\Rightarrow$i), so that iii)$\Rightarrow$vi). Therefore, if G satisfies iii), $L^\infty=B$, as shown in the proof that vi)$\Rightarrow$iii). Since L^∞ is a W^*-algebra, its spectrum is its $*$-spectrum (the set of $*$-homomorphisms onto $\mathbf{C}$, with the w^*-topology), and is an extremely disconnected compact Hausdorff space. Thus the $*$-spectrum Δ of B is extremely disconnected. But Δ is a compactification of G, and G, being locally compact, is open in every compactification. Thus by Lemma 3, G is locally extremely disconnected, and finally, by Proposition 4, G is discrete.

Now supposing G is infinite, we derive a contradiction. In fact, suppose H is a denumerably infinite subgroup of G. Then $M(H)=L^1(H)$ is a closed subalgebra of $M(G)=L^1(G)$, and so (iii)$\Rightarrow$vi)) can be renormed as a C^*-algebra, which is then a W^*-algebra (see viii)$\Rightarrow$ix)).

But $L^1(H)$ is norm separable, so by Lemma 1 finite dimensional, making H finite. This contradiction completes the proof.

In [2], R. E. EDWARDS proved that if G is a locally compact abelian group, and A a commutative C^*-algebra, $\varphi: A \to M(G)$ an algebraic isomorphism into $M(G)$, then A is finite dimensional.

We remark finally that the following theorem, which is a non-commutative version of Edwards' theorem, can be used to secure the implication vi)$\Rightarrow$x), replacing the argument involving extremely disconnected groups.

Theorem 6. *If A is a closed $*$-subalgebra of $M(G)$ which can be renormed as a C^*-algebra, then A is finite dimensional.*

The proof is an easy consequence of Ogasawara's Theorem 1, op. cit., together with Edvards' arguments. We omit the detail.

Problem. Is Theorem 6 valid with $B(G)$ in place of $M(G)$?

Added in proof. The problem is solved affirmatively by C. A. AKEMANN in private correspondence. Akemann alerts us to Sakai's theorem (*Proc. Japan Acad.*, **33** (1957), 439—444) asserting that the predaul of a W^*-algebra is sequentially weakly complete. Then the argument sketched above for $M(G)$ applies.

We thus obtain the stronger form of the main theorem:

If A is a subalgebra of $B(G)$ (a $*$-subalgebra of $M(G)$ which is ($*$-) isomorphic to a C^*-algebra, then A is finite dimensional.

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On a certain converse of Hölder's inequality. II

By L. LEINDLER in Szeged

A. PRÉKOPA [2] proved the integral inequality

$$(1)\qquad \int_{-\infty}^{\infty} \sup_{x+y=t} f(x)g(y)\,dt \geqq 2\Big[\int_{-\infty}^{\infty} f^2(x)\,dx\Big]^{1/2}\Big[\int_{-\infty}^{\infty} g^2(y)\,dy\Big]^{1/2},$$

for arbitrary Lebesgue measurable non-negative functions $f(x)$ and $g(y)$.

In [1] we proved the inequality of similar type

$$(2)\qquad \int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty} (f(x)g(t-x))^{\gamma}\,dx\Big]^{1/\gamma} dt \geqq \Big[\int_{-\infty}^{\infty} f^r(x)\,dx\Big]^{1/r}\Big[\int_{-\infty}^{\infty} g^s(x)\,dx\Big]^{1/s},$$

where $1\leqq r, s, \gamma\leqq\infty$ and $\frac{1}{r}+\frac{1}{s}=1+\frac{1}{\gamma}$. The proof of (2) is much easier than that of (1), but (2) does not include (1) because of the lack of the factor 2.

In the present paper we prove the inequality, more general than (1),

$$(3)\qquad \int_{-\infty}^{\infty} \sup_{x+y=t} f(x)g(y)\,dt \geqq p^{1/p}q^{1/q}\Big[\int_{-\infty}^{\infty} f^p(x)\,dx\Big]^{1/p}\Big[\int_{-\infty}^{\infty} g^q(x)\,dx\Big]^{1/q},$$

where $1\leqq p, q\leqq\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Here the constant factor at the second member is best possible (if $p=1$ then by this constant we understand 1).

This inequality can be generalized to any finite number of functions as follows:

Theorem. *Suppose that* $1\leqq p_i\leqq\infty$ $(i=1, 2, \dots, n)$ *and* $\sum_{i=1}^{n}\frac{1}{p_i}=1$. *Then for arbitrary non-negative Lebesgue measurable functions* $f_1(x^1), f_2(x^2), \dots, f_n(x^n)$ *we have*

$$(4)\qquad \int_{-\infty}^{\infty}\Bigg[\sup_{\sum_{i=1}^{n}\frac{x^i}{p_i}=t}\ \prod_{i=1}^{n} f_i(x^i)\Bigg]dt \geqq \prod_{i=1}^{n}\|f_i\|_{p_i}$$

and this inequality is best possible.

For $n=2$ inequality (4) reduces to (3) by the substitution $x^1=px$ and $x^2=qy$. If $n=1$, then (4) reduces to the obvious equality:

$$\int_{-\infty}^{\infty} \sup_{x=t} f_1(x)\,dt = \int_{-\infty}^{\infty} f_1(x)\,dx = \|f_1\|_1. \tag{5}$$

We also obtain (5) if n is arbitrary, but one of the numbers p, say p_1, is equal to 1; for in this case $p_2=p_3=\dots=p_n=\infty$, and thus we can divide (4) by $\prod_{i=2}^{n} \|f_i\|_\infty$ assuming this product is positive and finite (otherwise both sides of (4) are zero or infinite).

For similar reason, if one of the numbers p_i, say p_n, is equal to infinity, then we can divide (4) by $\|f_n\|_\infty$.

Therefore we may assume that $1<p_i<\infty$ for all i.

Proof of the Theorem. We may assume, as already explained, that $1<p_i<\infty$ for $i=1, 2, \dots, n$; for $n\geqq 2$, the integral on the left-hand side of (4) has finite value, and the functions $f_i(x^i)$ do not vanish almost everywhere. We prove (4) for step functions with integer points of discontinuity only, the transition to arbitrary Lebesgue measurable functions follows as in [2]. Moreover, it suffices to consider step functions which at their points of discontinuity are equal to the larger one of the values taken on the adjoining intervals (this latter convention will be important technically later, see (10), (11) and (12)).

First we set down some notations and definitions. Let

$$F_i(x^i) = \frac{1}{\max f_i} f_i(x^i) \qquad (i = 1, 2, \dots, n),$$

and N be an integer such that if $|x^i|>N$ then $f_i(x^i)=0$ for all i; furthermore let

$$F_i(x^i) = a_k^i \quad \text{if} \quad x^i \in (k-1, k), \qquad k = -N+1, -N+2, \dots, N-1, N.$$

Let v_i denote a fixed index with $a_{v_i}^i=1$. Finally we define the following auxiliary function:

$$G_i(x^i) = \begin{cases} F_i(x^i) & \text{if} \quad x^i \notin (v_i-1, v_i), \\ 2^{1/p_i} & \text{if} \quad x^i \in [v_i-1, v_i]. \end{cases}$$

Denoting the values of $G_i(x^i)$ on $(k-1, k)$ by b_k^i, it is clear that $b_k^i=a_k^i$ if $k\neq v_i$ and for $k=v_i$ $b_{v_i}^i=2^{1/p_i}$.

By means of these functions $G_i(x^i)$ we want to give a decomposition of the interval $(-\infty<t<\infty)$ such that the sum of the lower estimations to be given on the subintervals for the left-hand side of (4) be greater than

$$\left[\prod_{i=1}^{n}(\max f_i)\right]\sum_{i=1}^{n}\frac{1}{p_i}\int_{-\infty}^{\infty}F_i^{p_i}(x^i)\,dx^i.$$

From this point the proof of (4) will already be easy.

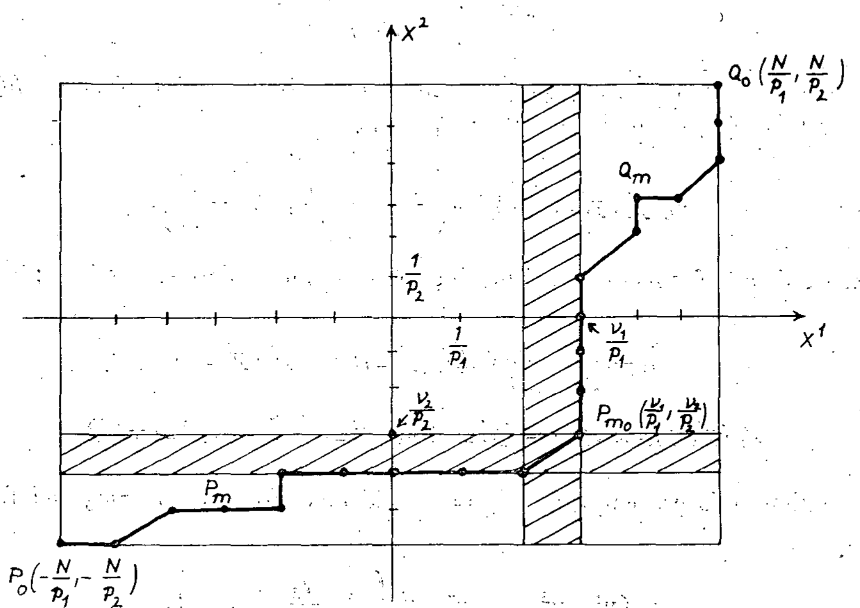
By the definition of N we have

$$S\equiv\int_{-\infty}^{\infty}\left[\sup_{\sum_{i=1}^{n}\frac{x^i}{p_i}=t}\prod_{i=1}^{n}F_i(x^i)\right]dt=\int_{-N}^{N}\left[\sup_{\sum_{i=1}^{n}\frac{x^i}{p_i}=t}\prod_{i=1}^{n}F_i(x^i)\right]dt\equiv S_N, \tag{6}$$

thus it is enough to decompose the interval $[-N, N]$.

First we sketch the idea of the decomposition in the case $n=2$.

We want to choose a path from the point $P_0\left(-\frac{N}{p_1}, -\frac{N}{p_2}\right)$ to the point $Q_0\left(\frac{N}{p_1}, \frac{N}{p_2}\right)$ such that by means of the "break points" of this path the required decomposition of the interval $[-N<t<N]$ could be given. See the following figure:



From a break point, e.g. from the point $P_m\left(\frac{k-1}{p_1}, \frac{-1}{p_2}\right)$ we go a step to the right or upward according as $(b_l^2)^{p_2}$ or $(b_k^1)^{p_1}$ is the larger number, that is, we go toward the direction where the smaller value of the functions $(G_1(x^1))^{p_1}$ and $(G_2(x^2))^{p_2}$ in the following step can be found; if $(b_l^2)^{p_2}=(b_k^1)^{p_1}$ then we go obliquely upward to the point $\left(\frac{k}{p_1}, \frac{l}{p_2}\right)$. We continue this procedure till one of the break points P_m, say $m=m_0$, comes up to the point $\left(\frac{\nu_1}{p_1}, \frac{\nu_2}{p_2}\right)$. This ensues necessarily since one of the break points "knocks against the lined wall" and after this the points go along the wall to the point $\left(\frac{\nu_1-1}{p_1}, \frac{\nu_2-1}{p_2}\right)$ (in fact, the functions take the largest value on the wall), and from the point $\left(\frac{\nu_1-1}{p_1}, \frac{\nu_2-1}{p_2}\right)$, by $(b_{\nu_1}^1)^{p_1}=(b_{\nu_2}^2)^{p_2}=$ $=2$, we jump to the point $\left(\frac{\nu_1}{p_1}, \frac{\nu_2}{p_2}\right)$. For similar reasons and by an analogous method we can come back from the point Q_0 to the point $\left(\frac{\nu_1}{p_1}, \frac{\nu_2}{p_2}\right)$ along the points Q_m. If we join the points Q_m to the points P_m in reverse order as we obtained them, then we will get the required path from P_0 to Q_0.

Now we construct such a path in the n-dimensional case. Let

$$s(t)=\begin{cases} 1, & \text{if } t\geqq 0, \\ 0, & \text{if } t<0, \end{cases}$$

and we denote, as usual, by $h_+(u_0)$ the limit from the right of the function $h(u)$ at u_0, and by $h_-(u_0)$ the limit from the left. We put

$$P_0(y_0^1, y_0^2, \dots, y_0^n) = \left(-\frac{N}{p_1}, -\frac{N}{p_2}, \dots, -\frac{N}{p_n}\right).$$

Next we define, for $m\geqq 1$, the following numbers and points successively:

$$u_m^i = \frac{1}{p_i} s\left(\min_{j\neq i} G_{j+}^{p_j}(p_j y_{m-1}^j) - G_{i+}^{p_i}(p_i y_{m-1}^i)\right)$$

and

$$P_m(y_m^1, y_m^2, \dots, y_m^n) = (y_{m-1}^1+u_m^1, y_{m-1}^2+u_m^2, \dots, y_{m-1}^n+u_m^n).$$

We continue this procedure till $y_{m_0}^i = \frac{\nu_i}{p_i}$ will hold, for some $m=m_0$ and for all i, i.e.

$$P_{m_0}(y_{m_0}^1, y_{m_0}^2, \dots, y_{m_0}^n) \equiv \left(\frac{\nu_1}{p_1}, \frac{\nu_2}{p_2}, \dots, \frac{\nu_n}{p_n}\right).$$

This follows necessarily by the same reason as we explained it in the case of two functions.

Then we define a sequence of points $Q_m(z_m^1, z_m^2, \ldots, z_m^n)$ in an analogous way coming back from the point $\left(\frac{N}{p_1}, \frac{N}{p_2}, \ldots, \frac{N}{p_n}\right)$. Let

$$Q_0(z_0^1, z_0^2, \ldots, z_0^n) = \left(\frac{N}{p_1}, \frac{N}{p_2}, \ldots, \frac{N}{p_n}\right).$$

Similarly as before, we define the following numbers and points, for $m \geqq 1$, successively:

$$v_m^i = \frac{1}{p_i} s\left(\min_{j \neq i} G_{j-}^{p_j}(p_j z_{m-1}^j) - G_{i-}^{p_i}(p_i z_{m-1}^i)\right)$$

and

$$Q_m(z_m^1, z_m^2, \ldots, z_m^n) = (z_{m-1}^1 - v_m^1, z_{m-1}^2 - v_m^2, \ldots, z_{m-1}^n - v_m^n).$$

In the n-dimensional case we also "knock against the wall', therefore it is clear that in a finite number of steps, say in m_1, we come to the point P_{m_0}, i.e. $P_{m_0} = Q_{m_1}$. For each i $(i=1, 2, \ldots, n)$ we put

$$y_{m_0+l}^i = z_{m_1-l}^i \qquad (l=0, 1, \ldots, m_1),$$

hereby we arranged the points in a sequence $P_m(y_m^1, y_m^2, \ldots, y_m^n)$ $(m=0, 1, \ldots, m_0+m_1)$, which gives the required path from P_0 to Q_0.

Now we prove that by means of the obtained path, i.e. by means of the sequence $P_m(y_m^1, y_m^2, \ldots, y_m^n)$ $(m=0, 1, \ldots, m_0+m_1)$, we can give the decomposition of the interval $[-N, N]$ we have required. First we set for each i $(i=1, 2, \ldots, n)$

$$\text{(7)} \qquad I_m^i = y_m^i - y_{m-1}^i, \qquad (m = 1, 2, \ldots, m_0+m_1),$$

furthermore denote by c_m^i the value of $F_i(x^i)$ on the interval $(p_i y_{m-1}^i, p_i y_m^i)$ if $I_m^i = \frac{1}{p_i}$, and at the point $x^i = p_i y_m^i$ if $I_m^i = 0$.

Let

$$\text{(8)} \qquad t_k = \sum_{i=1}^{n} y_k^i \qquad (k = 0, 1, \ldots, m_0+m_1).$$

It is easy to see that $t_0 = -N$ and $t_{m_0+m_1} = N$, furthermore for $k \geqq 1$

$$t_k = t_{k-1} + t_k - t_{k-1} = t_{k-1} + \sum_{i=1}^{n} I_k^i$$

also holds. Thus we can decompose each interval $[t_{k-1}, t_k]$ by the points

$$\text{(9)} \qquad \tau_{k,0} = t_{k-1} \quad \text{and} \quad \tau_{k,j} = t_{k-1} + \sum_{i=1}^{j} I_k^i \qquad (j = 1, 2, \ldots, n).$$

On such an interval $[\tau_{k,j-1}, \tau_{k,j}]$ for any k and j ($k=1, 2, \ldots, m_0+m_1$; $j=1, 2, \ldots, n$) we have the following lower estimate:

(10)
$$S_{k,j} \equiv \int_{\tau_{k,j-1}}^{\tau_{k,j}} \left[\sup_{\sum_{i=1}^{n} \frac{x^i}{p_i}=t} \prod_{i=1}^{n} F_i(x^i) \right] dt \geqq I_k^j \prod_{i=1}^{n} c_k^i.$$

To verify (10) we put $x^i=y_k^i p_i$ for $i<j$ and $x^i=y_{k-1}^i p_i$ for $i>j$, and we have x^j run from $y_{k-1}^j p_j$ to $y_k^j p_j$, then t goes from $\tau_{k,j-1}$ to $\tau_{k,j}$; in fact, by (7), (8) and (9) we have

$$t = \sum_{i=1}^{n} \frac{x^i}{p_i} \geqq \sum_{i=1}^{j-1} y_k^i + \sum_{i=j}^{n} y_{k-1}^i = t_{k-1} + \sum_{i=1}^{j-1} I_k^i = \tau_{k,j-1}$$

and

$$t = \sum_{i=1}^{n} \frac{x^i}{p_i} \leqq \sum_{i=1}^{j} y_k^i + \sum_{i=j+1}^{n} y_{k-1}^i = t_{k-1} + \sum_{i=1}^{j} I_k^i = \tau_{k,j}.$$

Choosing x^i in such a way as we mentioned above, we have

(11)
$$S_{k,j} = \int_{\tau_{k,j-1}}^{\tau_{k,j}} \left[\sup_{\sum_{i=1}^{n} \frac{x^i}{p_i}=t} \prod_{i=1}^{n} F_i(x^i) \right] dt \geqq$$
$$\geqq \prod_{i<j} F_i(y_k^i p_i) \prod_{i>j} F_i(y_{k-1}^i p_i) c_k^j \int_{\tau_{k,j-1}}^{\tau_{k,j}} dt,$$

and hence, by the definition of our step functions (see their definition at the points of discontinuity), (10) obviously follows.

By (9) and (10) we obtain

(12)
$$\sigma_k = \sum_{j=1}^{n} S_{k,j} = \int_{t_{k-1}}^{t_k} \left[\sup_{\sum_{i=1}^{n} \frac{x^i}{p_i}=t} \prod_{i=1}^{n} F_i(x^i) \right] dt \geqq \left(\sum_{j=1}^{n} I_k^j \right) \prod_{i=1}^{n} c_k^i.$$

By the definitions of y_k^i and I_k^i, furthermore taking into account that the functions at their points of discontinuity are equal to the larger one of the values taken on the adjoining intervals, it can be seen that I_k^j differs from zero only for such indices j for which $(c_k^j)^{p_j} \leqq (c_k^i)^{p_i}$ holds for all i ($i=1, 2, \ldots$). Thus we obtain from (12) that

(13)
$$\sigma_k \geqq \sum_{j=1}^{n} I_k^j (c_k^j)^{p_j},$$

since if $I_k^j \neq 0$ then

$$\prod_{i=1}^{n} c_k^i \geqq \prod_{i=1}^{n} (c_k^j)^{\frac{p_j}{p_i}} = (c_k^j)^{\sum_{i=1}^{n} \frac{p_j}{p_i}} = (c_k^j)^{p_j},$$

whence, by (12), inequality (13) follows obviously.

Since

$$S = S_N = \sum_{k=1}^{m_0+m_1} \sigma_k,$$

by (13), we have

$$\text{(14)} \qquad S \geqq \sum_{k=1}^{m_0+m_1} \sum_{j=1}^{n} I_k^j (c_k^j)^{p_j} = \sum_{j=1}^{n} \sum_{k=1}^{m_0+m_1} I_k^j (c_k^j)^{p_j}.$$

By the definition of I_k^j and c_k^j it is clear that

$$\text{(15)} \qquad \sum_{k=1}^{m_0+m_1} I_k^j (c_k^j)^{p_j} = \frac{1}{p_j} \sum_{l=-N+1}^{N} (a_l^j)^{p_j},$$

thus, by (14) and (15), using the following well-known inequality $\prod_{i=1}^{n} \varrho_i \leqq \sum_{i=1}^{n} \frac{1}{p_i} (\varrho_i)^{p_i}$ ($\varrho_i > 0$), we get

$$\text{(16)} \qquad S \geqq \sum_{j=1}^{n} \frac{1}{p_j} \sum_{l=-N+1}^{N} (a_l^j)^{p_j} \geqq \prod_{j=1}^{n} \left(\sum_{l=-N+1}^{N} (a_l^j)^{p_j} \right)^{1/p_j} = \prod_{j=1}^{n} \|F_j\|_{p_j}.$$

Multiplying both sides of (16) by $\prod_{i=1}^{n} (\max f_i)$ we obtain the required inequality (4).

To prove that the inequality (4) is best possible we define the following functions:

$$f_i^0(x^i) = \begin{cases} 1 & \text{on } (0, 1), \\ 0 & \text{otherwise}, \end{cases} \qquad i = 1, 2, \ldots, n.$$

Then

$$\int_{-\infty}^{\infty} \left[\sup_{\sum_{i=1}^{n} \frac{x^i}{p_i} = t} \prod_{i=1}^{n} f_i^0(x^i) \right] dt = \int_0^1 1\, dt = \prod_{i=1}^{n} \|f_i^0\|_{p_i}.$$

The proof is thus completed.

My grateful acknowledgement is due to Professor BÉLA SZŐKEFALVI-NAGY for stimulating conversations.

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On a generalization of Bernoulli's inequality

By L. LEINDLER in Szeged

At the Oberwolfach Conference on "Linear Operators and Approximation" in August, 1971, H. S. SHAPIRO [1] presented a lecture on "Fourier multipliers whose multiplier norm is an attained value". On this occasion he mentioned that the elementary inequalities

(1) $$|1+z|^4 \geqq 1+4\,\mathrm{Re}\, z+|z|^2+\frac{1}{5}|z|^4,$$

(2) $$|1+z|^4 \leqq 1+4\,\mathrm{Re}\, z+8|z|^2+3|z|^4$$

played an important role in the proof of his Theorem 2. (See Lemma 5 and 6 in [1].) He also stated (see also [2]) that to prove an analogue of his result for $p \geqq 2$ inequalities of the following type are needed:

(3) $$|1+z|^p \geqq 1+p\,\mathrm{Re}\, z+a_p|z|^2+b_p|z|^p,$$

(4) $$|1+z|^p \leqq 1+p\,\mathrm{Re}\, z+A_p|z|^2+B_p|z|^p,$$

where a_p, b_p, A_p, B_p are positive constants depending only on p.

The proof of (3) and (4) given in [2] does not seem to yield optimal values for these positive constants. In connection with this fact H. S. SHAPIRO raised the problem to find the best possible constants, i.e. the exact range of (a_p, b_p) such that (3) holds, and similarly for (4).

In the present paper we are going to give a proof of these inequalities which exhibits best possible constants. In fact we prove:

Theorem. *For any complex number z and for any $p \geqq 2$ the inequalities*

(5) $$|1+z|^p \geqq 1+p\,\mathrm{Re}\, z+a_p|z|^2+b_p|z|^p,$$

(6) $$|1+z|^p \leqq 1+p\,\mathrm{Re}\, z+A_p|z|^2+B_p|z|^p$$

hold with any positive a_p, b_p, A_p, B_p satisfying

(7) $$0 < a_p < \frac{p}{2},$$

(8) $$0 < b_p \leqq \mu_1(p) = \min_{t \geqq 2} \frac{(t-1)^p + pt - 1 - a_p t^2}{t^p},$$

(9) $$1 < B_p < \infty,$$

(10) $$M_1(p) = \sup_{t>0} \frac{(t+1)^p - 1 - pt - B_p t^p}{t^2} \leqq A_p < \infty,$$

or

(7′) $$0 < b_p < \mu_2(p) = \min_{t \geqq 2} \frac{(t-1)^p + pt - 1}{t^p},$$

(8′) $$0 < a_p \leqq \mu_3(p) = \min_{t \geqq 2} \frac{(t-1)^p + pt - 1 - b_p t^p}{t^2},$$

(9′) $$\frac{p(p-1)}{2} < A_p < \infty,$$

(10′) $$M_2(p) = \sup_{t>0} \frac{(t+1)^p - 1 - pt - A_p t^2}{t^p} \leqq B_p < \infty.$$

These ranges of (a_p, b_p) *and* (A_p, B_p) *are best possible.*

Remarks. For some special p the exact values or good approximation of the numbers μ_i and M_i can be gotten by an easy computation.

For instance if $p=2$ then $\mu_1(2) = 1-a_2$, $M_1(2) = 1-B_2$,

$$\mu_2(2) = 1, \quad \mu_3(2) = 1-b_2 \quad \text{and} \quad M_2(2) = 1-A_2.$$

If $p=4$ and we choose $a_4=1$ then $\mu_1(4)=\frac{1}{5}$, and, for $B_4=3$, $M_1(4)=8$; i.e. the constants $\frac{1}{5}$ and 8 appearing in (1) and (2) are optimal.

The following estimates of the numbers μ_i and M_i can be obtained by a standard computation:

$$\mu_1(p) \leqq \frac{2p-4a_p}{2^p}; \quad \mu_2(p) \leqq \frac{2p}{2^p}; \quad \mu_3(p) \leqq \frac{2p-2^p b_p}{4};$$

$$M_1(p) \geqq \max\left(2^p-1-p-B_p,\ \frac{p(p-1)}{2}\right); \quad M_2(p) \geqq 2^p-1-p-A_p;$$

$$\mu_1(p) \geqq \frac{p-2a_p}{2}\left(\frac{a_p}{p}\right)^{p-2}\left(1-\frac{1}{p-1}\right)^{p-1};$$

$$\mu_2(p) \geqq \frac{1}{2^p} \quad \text{and} \quad \mu_3(p) \geqq \frac{1-2^p b_p}{4}.$$

One more remark: inequality (5) can be slightly generalized as follows.

For any $p \geqq q \geqq 2$ *the inequality*

$$|1+z|^p \geqq 1+p \operatorname{Re} z + a_p(q)|z|^q + b_p(q)|z|^p \tag{11}$$

holds, where

$$0 < a_p(q) < \min_{t \geqq 2} \frac{(t-1)^p + pt - 1}{t^q},$$

$$0 < b_p(q) \leqq \min_{t \geqq 2} \frac{(t-1)^p + pt - 1 - a_p(q) t^q}{t^p}.$$

The proof of these inequalities is the same as that of (7) and (8).

Such a generalization of (6) is impossible. This fact can be seen easily if p and q are integers greater than two and z is a real number tending to zero.

Proof of (5). Denote $z = x+iy$ and $r=|z|$. For the sake of brevity we write a for a_p and b for b_p.

In the first step we fix p, q, and r. Then we have to prove the inequality

$$(1+2x+r^2)^{\frac{p}{2}} \geqq 1+px+ar^2+br^p \tag{12}$$

for all x lying in $[-r, r]$ with positive a and b. Put $R = \frac{1}{2}(1+r^2)$, $C = 1+ar^2+br^p$,

$$f(x) = 2^{\frac{p}{2}}(R+x)^{\frac{p}{2}} \quad \text{and} \quad g(x) = px+C.$$

Drawing the graphs of these functions it is easy to see that inequality (12) will be satisfied if the graph of $y=g(x)$ lies under the graph of $y=f(x)$ on the interval $[-r, r]$. We obtain the best possible result in respect to a and b if $y=g(x)$ is tangent to the graph of $y=f(x)$ inside of $[-r, r]$ or, when this is not the case, if $y=g(x)$ passes through the point $P(-r, f(-r))$. (See Fig. 1 and Fig. 2.)

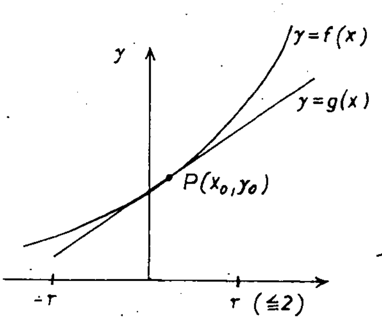

Fig. 1

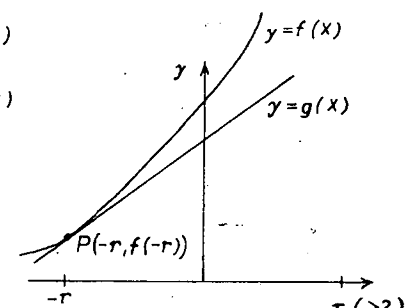

Fig. 2

In the first case our task reduces to find the point $P(x_0, y_0)$ on the graph $f(x)$ at which $f'(x_0)=p$,

In order to find $P(x_0, y_0)$ we calculate the derivative of $f(x)$ and solve the equation

$$f'(x) = p2^{\frac{p}{2}-1}(R+x)^{\frac{p}{2}-1} = p.$$

Thus we get $x_0 = -\frac{r^2}{2}$ and this x_0 lies in $[-r, r]$ if $r \leqq 2$.

In this case, i.e. if $r \leqq 2$, we obtain the best possible constants a and b if $f(x_0) = g(x_0)$, i.e. if

$$f\left(-\frac{r^2}{2}\right) = 1 = p\left(-\frac{r^2}{2}\right) + 1 + ar^2 + br^p = g\left(-\frac{r^2}{2}\right)$$

holds. Hence we obtain the following conditions on a and b:

$$0 < a < \frac{p}{2} \quad \text{and} \quad 0 < b \leqq \frac{p-2a}{2^{p-1}}.$$

If $r \geqq 2$ we have the following equation as a condition on a and b:

$$f(-r) = (r-1)^p = p(-r) + 1 + ar^2 + br^p = g(-r).$$

Hence we get

$$0 < a < u_1 = \min_{r \geqq 2} u_1(r) \equiv \min_{r \geqq 2} \frac{(r-1)^p + pr - 1}{r^2} \leqq \frac{p}{2}$$

and

$$0 < b \leqq u_2 = \min_{r \geqq 2} u_2(r) \equiv \min_{r \geqq 2} \frac{(r-1)^p + pr - 1 - ar^2}{r^p} \leqq \frac{p-2a}{2^{p-1}}.$$

It is easy to see that u_1 and u_2 are positive. In fact, $u_1(r) \geqq \frac{p}{r}$ for any $r \geqq 2$ and

$$u_1(r) \to \begin{cases} \infty & \text{for} \quad p > 2 \\ 1 & \text{for} \quad p = 2 \end{cases} \quad \text{as} \quad r \to \infty.$$

Similarly, $u_2(r) \geqq \frac{u_1 r^2 - ar^2}{r^p} = \frac{u_1 - a}{r^{p-2}} > 0$ for any $r \geqq 2$; furthermore

$$u_2(r) \to \begin{cases} 1 & \text{for} \quad p > 2 \\ 1-a & \text{for} \quad p = 2 \end{cases} \quad \text{as} \quad r \to \infty.$$

To complete the proof of (5) we have only to show that $u_1 = \frac{p}{2}$. To verify this we compute

$$(u_1(r))' = r^{-4}[(p(r-1)^{p-1}+p)r^2-2r((r-1)^p+pr-1)].$$

Let $h(r) = (p(r-1)^{p-1}+p)r-2((r-1)^p+pr-1)$. Now $h(2)=0$ and

$$h'(r) = p[(p-2)(r-1)^{p-1}+(p-1)(r-1)^{p-1}-1] \geqq 0$$

for all $r \geqq 2$; thus $h(r) \geqq 0$, which implies that $u_1'(r) \geqq 0$, i.e. $u_1(r)$ is an increasing function, hence $u_1 = \min_{r \geqq 2} u_1(r) = u_1(2) = \frac{p}{2}$ in accordance with our statement.

Setting $\mu_1(p)=u_2$ and collecting our results the proof of (5) is complete.

Proof of (6). We use the same notations as before except that we write a for A_p and b for B_p. We distinguish the cases $0 \leqq r \leqq 1$ and $r>1$. If $r \leqq 1$ then let $h_1(r) = (r+1)^p-(1-r)^p-2rp$. Since $h_1(0)=0$ and $h_1'(0)=0$, furthermore $h_1''(r) \geqq 0$ for all $0 \leqq r \leqq 1$, we have

$$h_1(r) \geqq 0 \quad \text{for all} \quad 0 \leqq r \leqq 1. \tag{13}$$

If $r \geqq 1$ let $h_2(r) = (r+1)^p-(r-1)^p-2rp$. As before, since $h_2(1) \geqq 0$, $h_2'(1) \geqq 0$ and $h_2''(r) \geqq 0$ for all $r \geqq 1$, we have

$$h_2(r) \geqq 0 \quad \text{for all} \quad r \geqq 1. \tag{14}$$

(13) and (14) imply that

$$\frac{f(r)-f(-r)}{2r} = \frac{(r+1)^p-|r-1|^p}{2r} \geqq p \tag{15}$$

for all $r>0$.

By (15) it is evident that inequality (6) is satisfied if $g(r) \geqq f(r)$ and we obtain the optimal constants if $g(r)=f(r)$.

Hence we get the following condition on a and b:

$$1+pr+ar^2+br^p = (r+1)^p.$$

It is easy to see that b must be greater than 1, and if b is fixed then the best possible value of a is

$$a = \sup_{r>0} \frac{(r+1)^p-1-pr-br^p}{r^2}.$$

To complete the proof of (6) we have only to prove that $\sup_{r>0} v(r) < \infty$, where

$$v(r) = \frac{(r+1)^p-1-pr-br^p}{r^2}.$$

In fact, since $b>1$,

$$v(r) \to \begin{cases} -\infty & \text{for} \quad p > 2 \\ 1-b & \text{for} \quad p = 2 \end{cases} \quad \text{as} \quad r \to \infty$$

and

$$v(r) \to \begin{cases} \dfrac{p(p-1)}{2} & \text{for} \quad p > 2 \\ 1-b & \text{for} \quad p = 2 \end{cases} \quad \text{as} \quad r \to 0,$$

and these statements imply the desired conclusion.

The proof of (6) is complete.

Inequalities (5) and (6) with (7′), (8′), (9′) and (10′) can be proved similarly, and therefore these proofs are omitted.

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The non-orthogonal Menchoff—Rademacher theorem

By A. SZÉP in Budapest

1. Introduction. A well-known result of RADEMACHER [1] and MENCHOFF [2] states that if $\varphi_1, \varphi_2, \ldots, \varphi_N$ is a finite set of orthogonal functions in some L_2 space of finite measure, then there exists a $\delta \in L_2$ such that

$$\left|\sum_{k=1}^{n} \varphi_k(x)\right| \leqq \delta(x) \quad \text{for} \quad 1 \leqq n \leqq N$$

and

$$\|\delta\| \leqq \log_2 4N \left(\sum_{k=1}^{N} \|\varphi_k\|^2\right)^{1/2}. \tag{1.1}$$

This result implies the convergence a.e. of any orthogonal series $\sum_{k=1}^{\infty} \varphi_k(x)$ satisfying

$$\sum_{k=1}^{\infty} \|\varphi_k\|^2 \log^2 k < +\infty. \tag{1.2}$$

According to results of MENCHOFF [2] and TANDORI [3], condition (1. 1) cannot be weakened in general, i.e. the factor $\log 4N$ cannot be replaced by $o(\log N)$. I. S. GÁL [4], [5], [6] and more recently R. I. SERFLING [7], [8] have generalized the above statements for spaces L_p with $p \geqq 2$ and for non-orthogonal series. In the present paper we shall give a new, simpler proof of Serfling's theorem, published in [7].

The method of proof is similar to that of Rademacher and Menchoff. In Ch. 2 we shall derive some remarkable consequences of Serfling's theorem.

2. Let L_p be a Lebesgue space ($2 \leqq p < \infty$) on some measure space with integral $M(.)$. Let $g(i, j)$ be a non-negative function defined for integers i, j ($1 \leqq i \leqq j \leqq N$) such that

$$g(i, j) + g(j+1, k) \leqq g(i, k) \quad \text{for} \quad 1 \leqq i \leqq j < k \leqq N. \tag{2.1}$$

Furthermore, for any $\varphi_i \in L_p$ ($1 \leqq i \leqq N$), set

$$s_{ij}(x) = \sum_{i<k\leqq j} \varphi_k(x) \quad \text{for} \quad 0 \leqq i < j \leqq N$$

and

$$s_j(x) = s_{0j}(x) \quad \text{for} \quad 1 \leqq j \leqq N.$$

Theorem 1. *If*

$$\|s_{ij}\|_p \leqq g(i+1, j) \quad \text{for} \quad 0 \leqq i < j \leqq N \tag{2.2}$$

then there exists a function $\delta \in L_p$ *such that*

$$|s_j(x)| \leqq \delta(x) \quad \textit{for} \quad 1 \leqq j \leqq N \tag{2.3}$$

and

$$\|\delta\|_p \leqq (\log_2 4N)^2 g(1, N). \tag{2.4}$$

Proof. Suppose first that $N=2^r$ with some integer $r>0$ and define

$$q_k^{(j)} = s_{2^{r-k}j,\, 2^{r-k}(j+1)} \quad \text{for} \quad 0 \leqq j \leqq 2^{k-1},\ 0 \leqq k \leqq r$$

and

$$Q_k = \sum_{j=0}^{2^k-1} |q_k^{(j)}|^p \quad \text{for} \quad 0 \leqq k \leqq r.$$

Every n satisfying $1 \leqq n \leqq N$ can be uniquely represented in the form

$$n = \alpha_0 + \alpha_1 \cdot 2 + \cdots + \alpha_r \cdot 2^r,$$

where $\alpha_l = 0$ or 1 for $0 \leqq l \leqq r$. Accordingly we shall have

$$s_n = \sum_{k=0}^{r} \alpha_k q_k,$$

where $q_k = q_k^{(j)}$ for suitable $j = j(n, k)$. Applying Hölder's inequality we get

$$|s_n(x)| \leqq \sum_{k=0}^{r} \alpha_k |q_k(x)| \leqq \left(\sum_{k=0}^{r} |q_k(x)|^p\right)^{1/p} \left(\sum_{k=0}^{r} 1\right)^{1/q}$$

where $1/p + 1/q = 1$. Clearly

$$Q_k(x) \geqq |q_k(x)|^p \qquad \text{for} \quad 1 \leqq k \leqq r,$$

hence

$$|s_n(x)|^p \leqq (r+1)^{p-1} \sum_{k=0}^{r} Q_k(x).$$

Obviously $\delta(x) = (r+1)^{1/q} \left(\sum_{k=0}^{r} Q_k(x)\right)^{1/p}$ statisfies condition 2.4 and $\delta \in L_p$.

Furthermore, we have

$$M(\delta^p) \leqq (r+1)^{p-1} \sum_{k=0}^{r} M(Q_k) = (r+1)^{p-1} \sum_{k=0}^{r} \sum_{j=0}^{2^k-1} M(|q_k^{(j)}|^p) \leqq$$

$$\leqq (r+1)^{p-1} \sum_{k=0}^{r} \sum_{j=0}^{2^k-1} \left[g(2^{r-k}j+1,\, 2^{r-k}(j+1))\right]^{p/2}.$$

Here, in the last step, we used (2. 2). Since $p/2\geqq 1$, Jensen's inequality yields:

$$\sum_{j=0}^{2^k-1}[g(2^{r-k}j+1,2^{r-k}(j+1))]^{p/2}\leqq\left[\sum_{j=0}^{2^k-1}g(2^{r-k}j+1,2^{r-k}(j+1))\right]^{p/2}\leqq[g(1,N)]^{p/2},$$

hence

$$M(\delta^p)\leqq(r+1)^p[g(1,N)]^{p/2},$$

i.e.

$$\|\delta\|^2\leqq(r+1)^2\cdot g(1,N).$$

As $r+1\leqq\log_2 N+1=\log_2 2N$ we have

$$\|\delta\|^2\leqq(\log_2 2N)^2\cdot g(1,N)$$

Let now N be arbitrary and denote by N' the first 2-power exceeding N. Define $\varphi_i\equiv 0$ for $N<i\leqq N'$, and

$$\bar g(i,j)=\begin{cases}g(i,j) & \text{if}\quad 1\leqq i\leqq j\leqq N,\\ g(i,N) & \text{if}\quad 1\leqq i\leqq N\leqq j\leqq N',\\ 0 & \text{if}\quad N<i\leqq j\leqq N'.\end{cases}$$

The extended system $\{\varphi_i\}$ obviously satisfies the conditions of the theorem for $\bar g$; hence there exists a $\delta\in L_p$ such that, in particular,

$$|s_n(x)|\leqq\delta(x)\quad for\quad 1\leqq n\leqq N$$

and $\|\delta\|^2\leqq(\log_2 2N')^2\bar g(1,N')$. As $N'<2N$, we have $\log_2 2N'\leqq\log_2 4N$ which completes the proof.

Corollary. *If $\varphi_1,\dots,\varphi_N$ are arbitrary elements of an abstract L_2-space then there exists a $\delta\in L_2$ such that*

$$\left|\sum_{k=1}^{n}\varphi_k(x)\right|\leqq\delta(x)\quad for\quad 1\leqq n\leqq N$$

and

$$\|\delta\|^2\leqq(\log_2 4N)^2\cdot\sum_{i,j=1}^{N}|(\varphi_i,\varphi_j)|.$$

(Here (,) denotes the scalar product in L_2.)

Proof. Apply Theorem 1 with

$$g(i,j)=\sum_{k=i}^{j}\sum_{l=1}^{j}|(\varphi_k,\varphi_l)|.$$

3. Non-orthogonal series. In the sequel we shall deal with L_2 spaces only.

Theorem 2. *Let $\{\varphi_i\}_{i=1}^{\infty}$ be a sequence of elements of L_2, $\{\alpha_i\}_{i=1}^{\infty}$ a nondecreasing sequence of positive real numbers with $\alpha_i \to \infty$ as $i \to \infty$, and suppose that*

$$\sum_{i,j=1}^{\infty} |(\varphi_i, \varphi_j)| \alpha_i \alpha_j < +\infty. \tag{3.1}$$

If $n(k)$ denotes the first integer m for which $k \leqq \alpha_m^2$, then $s_{n(k)} = \sum_{k=1}^{n(k)} \varphi_l$ converges a.e.

Proof. Obviously condition (3. 1) implies the convergence of the series $\sum_{k=1}^{\infty} \varphi_k$ in L_2.

Denote by f the sum of this series and define

$$d_n = \left| f - \sum_{k=1}^{n} \varphi_k \right|^2 = \sum_{i,j=n+1} (\varphi_i, \varphi_j).$$

If N is an arbitrary integer, $N > 1$, then

$$\sum_{k=1}^{N} d_{n(k)} = \sum_{k=1}^{N} [k-(k-1)] d_{n(k)} = \sum_{k=1}^{N-1} k(d_{n(k)} - d_{n(k-1)}) + N \cdot d_{n(N)} =$$

$$= \sum_{k=1}^{N-1} k \sum_{(n(k), n(k+1)]} (\varphi_i, \varphi_j) + N d_{n(N)};$$

here $(n(k), n(k+1)]$ denotes the set of all pairs (i, j) of integers for which $n(k) < \min(i, j) \leqq n(k+1)$. From the definition of α_i and $n(k)$:

$$|k \sum_{(n(k), n(k+1)]} (\varphi_i, \varphi_j)| \leqq \alpha_{n(k)}^2 \sum_{(n(k), n(k+1)]} |(\varphi_i, \varphi_j)| \leqq \sum_{(n(k), n(k+1)]} |(\varphi_i, \varphi_j)| \alpha_i \alpha_j;$$

hence

$$\sum_{k=1}^{N} d_{n(k)} \leqq \sum_{(n(1), n(N)]} |(\varphi_i, \varphi_j)| \alpha_i, \alpha_j + \alpha_{n(N)}^2 \sum_{i,j \geqq n(N)+1} |(\varphi_i, \varphi_j)| \leqq$$

$$\leqq \sum_{i,j \geqq n(1)} |(\varphi_i, \varphi_j)| \alpha_i \alpha_j < +\infty.$$

Hence the series $\Sigma d_{n(k)}$ converges, and this implies by Beppo Levi's theorem that $s_{n(k)} \to f$ a.e. as $k \to \infty$.

Theorem 3. *If $\{\varphi_i\}_{i=1}^{\infty} \subset L_2$ and*

$$\sum_{i,j=1}^{\infty} |(\varphi_i, \varphi_j)| \log i \log j < +\infty$$

then $s_n(x) = \sum_{i=1}^{n} \varphi_i(x)$ converges a.e. (cf. [8]).

Proof. Theorem 2 with $\alpha_i = \log i$ yields the convergence of the sequence $s_{2^m}(x)$ as $m \to \infty$. It is sufficient to show that

$$\max_{2^m < n \leq 2^{m+1}} |s_n(x) - S_{2^m}(x)| = \sigma_n(1) \quad \text{a.e. as} \quad n \to \infty,$$

Applying Corollary of Theorem 1 with $\varphi_{2^m+1}, \varphi_{2^m+2}, \ldots, \varphi_{2^{m+1}}$, we have

$$|s_n(x) - s_{2^m}(x)| \leqq \delta_{2^m}(x) \quad \text{for} \quad 2^m < n \leqq 2^{m+1}.$$

$$M(\delta_{2^m}^2) = o\big((\log 2^m)^2\big) \sum_{i,j=2^m+1}^{2^{m+1}} |(\varphi_i, \varphi_j)| =$$

$$= o(1) \sum_{i,j=2^m+1}^{2^{m+1}} |(\varphi_i, \varphi_j)| \log i \log j,$$

whence our assertion follows by Beppo Levi's theorem.

Corollary. *If* $\{\varphi_i\}_{i=1}^{\infty} \subset L_2$, $\varrho(k)$ *is a non-negative real valued function satisfying*

$$\sum_{k=1}^{\infty} k\varrho(k) < +\infty \quad \textit{and} \quad |(\varphi_i, \varphi_j)| \leqq \frac{\varrho(i+j)}{\log i \log j},$$

then the series $\sum_{i=1}^{\infty} \varphi_i$ *converges a.e.*

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Remark to a paper of Gaposhkin

By J. KOMLÓS and P. RÉVÉSZ in Budapest

In [1] GAPOSHKIN proved the following:

Theorem A. *Let* $\{\varphi_i\}$ *be a sequence of measurable functions defined on a measure space* $\{X, S, \mu\}$ *of finite measure for which*

(1) $$\int \varphi_i^4 \leqq K,$$

(2) $$\int \varphi_i \varphi_j \varphi_k \varphi_l = 0,$$

(3) $$\int \varphi_i^2 \varphi_j \varphi_k = 0$$

where i, j, k, l are different integers. Then $\{\varphi_i\}$ *is a convergence system, i.e. every series*

$$\sum_i c_i \varphi_i \quad \text{statisfying} \quad \sum_i c_i^2 < \infty$$

is convergent almost everywhere in every arrangement of its terms.

The proof was based on the following result of STEČKIN:

Theorem B. (See [2], p. 27—31.) *Let* $\{\varphi_k\}$ *be a sequence of measurable functions defined on a measure space of finite measure for which*

(4) $$\int \left(\sum_{k=1}^{N} c_k \varphi_k \right)^4 \leqq A \left(\sum_{k=1}^{N} c_k^2 \right)^2 \qquad (N = 1, 2, \ldots)$$

(for any sequence $\{c_k\}$ *of real numbers) where A is a positive constant. Then* $\{\varphi_k\}$ *is a convergence system.*

Making use of Theorem B in order to prove Theorem A it was enough to show that (1), (2) and (3) imply (4).

The aim of the present paper is to prove that condition (3) can be omitted, i.e. we prove the following:

Theorem. *Let* $\{\varphi_k\}$ *be a sequence of measurable functions defined on a measure space of finite measure, for which* (1) *and* (2) *hold. Then* $\{\varphi_k\}$ *is a convergence system.*

The proof of this theorem is also based on Theorem B, i.e. we prove that (1) and (2) (without (3)) imply (4). Notably we prove the following:

Lemma.[1]) *Let* $\{\varphi_k\}$ *be a sequence of measurable functions defined on a measure space of finite measure obeying conditions* (1) *and* (2). *Then*

$$\int\left(\sum_{i=1}^{N} c_i\varphi_i\right)^4 \leqq 17^2 K\left(\sum_{i=1}^{N} c_i^2\right)^2 \qquad (N = 1, 2, \ldots)$$

for any sequence $\{c_i\}$ *of real numbers.*

Proof. Introduce the notations

$$A = A_n = \left|\sum_{i,j,k} c_i^2 c_j c_k \int \varphi_i^2 \varphi_j \varphi_k\right|, \qquad B = B_n = \left|\sum_{i,j} c_i^3 c_j \int \varphi_i^3 \varphi_j\right|,$$

$$M = \max(A, B), \quad \text{and} \quad \psi_i = c_i\varphi_i,$$

where i, j, k take all (not necessarily different) integers between 1 and n.

We have

$$(\sum_i \psi_i)^4 = (\sum_{i,j} \psi_i\psi_j)^2 = \sum_{i=1}^{n} \psi_i^4 + 6\sum_{1\leqq i<j\leqq n} \psi_i^2\psi_j^2 + 4\sum_{1\leqq i<j\leqq n} (\psi_i^3\psi_j + \psi_i\psi_j^3) +$$
$$+12\sum_{1\leqq i<j<k\leqq n} (\psi_i^2\psi_j\psi_k + \psi_i\psi_j^2\psi_k + \psi_i\psi_j\psi_k^2) + 4!\sum_{1\leqq i<j<k<l\leqq n} \psi_i\psi_j\psi_k\psi_l,$$

$$\sum_{i,j,k} \psi_i^2\psi_j\psi_k = \sum_{i=1}^{n} \psi_i^4 + 2\sum_{1\leqq i<j\leqq n} (\psi_i^3\psi_j + \psi_i\psi_j^3) + 2\sum_{1\leqq i<j\leqq n} \psi_i^2\psi_j^2 +$$
$$+2\sum_{1\leqq i<j<k\leqq n} (\psi_i^2\psi_j\psi_k + \psi_i\psi_j^2\psi_k + \psi_i\psi_j\psi_k^2)$$

and

$$\sum_{i,j} \psi_i^3\psi_j = \sum_{i=1}^{n} \psi_i^4 + \sum_{1\leqq i<j\leqq n} (\psi_i^3\psi_j + \psi_i\psi_j^3);$$

hence

$$\left(\sum_{i=1}^{n} \psi_i\right)^4 = 6\sum_{i,j,k} \psi_i^2\psi_j\psi_k - 8\sum_{i,j} \psi_i^3\psi_j + 3\sum_{i=1}^{n} \psi_i^4 - \tag{5}$$
$$-6\sum_{1\leqq i<j\leqq n} \psi_i^2\psi_j^2 + 4!\sum_{1\leqq i<j<k<l\leqq n} \psi_i\psi_j\psi_k\psi_l.$$

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[1]) This lemma was essentially formulated previously by R. Y. Serfling [3] but his proof is not quite complete.

Now we have

$$A=\Big|\sum_{i,j,k}\int\psi_i^2\psi_j\psi_k\Big|=\Big|\sum_{i=1}^{n}\int\psi_i^2\sum_{j,k}\psi_j\psi_k\Big|\leqq$$

$$\leqq\sum_{i=1}^{n}\sqrt{\int\psi_i^4}\sqrt{\int(\sum_{j,k}\psi_j\psi_k)^2}\leqq\sqrt{K}\left(\sum_{i=1}^{n}c_i^2\right)\sqrt{\int(\sum_{j,k}\psi_j\psi_k)^2}\leqq$$

$$\leqq\sqrt{K}\left(\sum_{i=1}^{n}c_i^2\right)\sqrt{6A+8B+3K\left(\sum_{i=1}^{n}c_i^2\right)^2}$$

and

$$A^2\leqq 8K\left(\sum_{i=1}^{n}c_i^2\right)^2\left(A+B+K\left(\sum_{i=1}^{n}c_i^2\right)^2\right). \tag{6}$$

Similarly

$$B=\Big|\int\sum_{i,j}\psi_i^3\psi_j\Big|=\Big|\sum_{i=1}^{n}\int\psi_i^2\sum_{j=1}^{n}\psi_i\psi_j\Big|\leqq\sum_{i=1}^{n}\sqrt{\int\psi_i^4}\sqrt{\int\left(\sum_{j=1}^{n}\psi_i\psi_j\right)^2}\leqq$$

$$\leqq\sqrt{K}\sum_{i=1}^{n}c_i^2\sqrt{\int\left(\sum_{j=1}^{n}\psi_i\psi_j\right)^2}\leqq\sqrt{K}\sqrt{\sum_{i=1}^{n}c_i^4}\sqrt{\sum_{i=1}^{n}\int\psi_i^2\left(\sum_{j=1}^{n}\psi_j\right)^2}=$$

$$=\sqrt{K}\sqrt{\sum_{i=1}^{n}c_i^4}\sqrt{\int\sum_{i,j,k}\psi_i^2\psi_j\psi_k}=\sqrt{K}\sqrt{\sum_{i=1}^{n}c_i^4}\sqrt{A}$$

and

$$B^2\leqq 8K\left(\sum_{i=1}^{n}c_i^2\right)^2\left(A+B+K\left(\sum_{i=1}^{n}c_i^2\right)^2\right). \tag{7}$$

(6) and (7) together imply:

$$M^2\leqq 8K\left(\sum_{i=1}^{n}c_i^2\right)^2\left(2M+K\left(\sum_{i=1}^{n}c_i^2\right)^2\right),$$

i.e.

$$\left(M-8K\left(\sum_{i=1}^{n}c_i^2\right)^2\right)^2\leqq\left(\sum_{i=1}^{n}c_i^2\right)^4(64K^2+8K^2)$$

and

$$M\leqq\left(\sum_{i=1}^{n}c_i^2\right)^2K(\sqrt{72}+8K)\leqq 17K\left(\sum_{i=1}^{n}c_i^2\right)^2.$$

(5) and (8) immediately prove our Lemma, and hence our Theorem too.

Let us mention that our condition (2) cannot be replaced by the condition $\int\psi_i\psi_j\psi_k=0$. More precisely, we make the following

Remark 1. *Let* $\{c_k\}$ *be a decreasing sequence of real numbers for which*

$$\sum_k c_k^2 \log^2 k = \infty,$$

then one can construct a uniformly bounded sequence $\{\varphi_k\}$ *of measurable functions for which*

$$\int \varphi_k^2 = 1, \quad \int \varphi_i = \int \varphi_i \varphi_j = \int \varphi_i \varphi_j \varphi_k = 0$$

(where the indices i, j, k are different), and the series

$$\sum c_k \varphi_k$$

is nowhere convergent.

The proof of this remark is based on the following theorem of Tandori ([4]):

Theorem C. *If* $c_1, c_2, \ldots$ *is a decreasing sequence of real numbers for which*

$$\sum_{k=1}^{\infty} c_k^2 \log^2 k = \infty,$$

then there exists a uniformly bounded sequence $\{\Phi_k\}$ *of Lebesgue measurable functions on the interval* $[0, 1]$ *such that*

$$\int \Phi_i = \int \Phi_i \Phi_j = 0, \quad \int \Phi_i^2 = 1,$$

and the series

$$\sum_{k=1}^{\infty} c_k \Phi_k$$

is nowhere convergent.

Now the proof of our remark runs by the following construction:

Let $\{\Phi_k\}$ be a sequence with the properties mentioned in Theorem C. Define the sequence $\{\varphi_k\}$ on the interval $[0, 2]$ by

$$\varphi_k(x) = \begin{cases} \dfrac{\Phi_k(x)}{\sqrt{2}} & \text{for} \quad x \in [0, 1), \\[2ex] \dfrac{-\Phi_k(x-1)}{\sqrt{2}} & \text{for} \quad x \in [1, 2]. \end{cases}$$

This sequence obviously satisfies all conditions of our remark.

Remark 2. A very easy evalution shows that our theorem remains correct if condition (2) is replaced by the following one: there exists a function $B(k)$ $(k=1, 2, \ldots)$ for which

$$\left|\int \varphi_i \varphi_j \varphi_k \varphi_l\right| \leqq \min\big(B(l-k),\ B(j-i)\big) \quad (1 \leqq i<j<k<l)$$

and

$$\sum_{k=1}^{\infty} kB(k) < \infty.$$

This result is much stronger than that of RÉVÉSZ [5].

Added in proof: Prof. V. F. GAPOSHKIN informed us in a letter that he also proved the above theorem.

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О точных неравенствах между нормами функций и их производных

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1. Пусть k, n $(0 \leqq k < n)$ — целые числа; $1 \leqq p, q, r \leqq \infty$; I — числовая ось $(-\infty, \infty)$ или полуось $[0, \infty)$; пространства $C = C(I)$ и $L_p = L_p(I)$ определены обычным образом; $L^n_{p,r}$ — множество функций $f \in L_r$, у которых $n-1$-ая производная локально абсолютно непрерывна на I, а $f^{(n)} \in L_p$; $L^n_{V,r}$ — класс таких функций $f \in L^n_{\infty,r}$, что производная $f^{(n)}$ почти всюду на I совпадает с некоторой функцией φ ограниченной вариации $\mathrm{V}\varphi = \mathrm{V}_I\varphi$; в частном случае $r = \infty$ положим $L^n_{p,\infty} = L^n_p$ и $L^n_{V,\infty} = L^n_V$.

Настоящая работа посвящена точным неравенствам вида

$$\|f^{(k)}\|_{L_q} \leqq D \|f\|^{\alpha}_{L_r} \|f^{(n)}\|^{\beta}_{L_p}, \quad f \in L^n_{p,r}, \tag{1}$$

где α, β, D — постоянные, не зависящие от f. Константа D может быть конечной лишь при (см. [13])

$$\alpha = \frac{n-k-p^{-1}+q^{-1}}{n-p^{-1}+r^{-1}}, \quad \beta = 1-\alpha. \tag{2}$$

Если в неравенстве (1) константа D наилучшая, т.е.

$$D = \sup_{f \in L^n_{p,r}} \|f^{(k)}\|_{L_q} \|f\|^{-\alpha}_{L_r} \|f^{(n)}\|^{-\beta}_{L_p}, \tag{3}$$

то отличную от постоянной функцию $u \in L^n_{p,r}$, на которой неравенство (1) обращается в равенство, называют экстремальной; будем говорить что она единственная, если любая другая экстремальная функция имеет вид $c\,u(\nu t+\mu)$, где $\nu > 0$, μ, c — некоторые числа, причем $\mu = 0$ если $I = [0, \infty)$.

Неравенства вида (1) впервые изучались Харди, Литтлвудом [1], Ландау [2] и Адамаром [3]. С.-Надь [7] доказал справедливость неравенства (1), получил наилучшую константу и выписал экстремальную функцию в случае $n = 1$, $p \geqq 1$, $q > r > 0$. Это единственный случай, когда неравенство (1) исследовано для всех возможных значений параметров p, q, r. В следующую таблицу сведены те случаи, когда в (1) вычислена наилучшая константа; в некоторых из этих случаев исследовано множество экстремальных функций.

n	k	q	r	p	Авторы
			$I=(-\infty, \infty)$		
2	1	∞	∞	∞	Ландау [2]
$2<n\leq 5$		∞	∞	∞	Боссэ (Шилов) [5]
произвольное	$0<k<n$	∞	∞	∞	Колмогоров [6]
1	0	$\geq r$	>0	≥ 1	С.-Надь [7]
произвольное	$0<k<n$	2	2	2	Харди, Литтлвуд, Полиа [4]
произвольное	$0<k<n$	1	1	1	Штейн [9]
произвольное	$0\leq k<n$	∞	2	2	Тайков [16]
2	1	2	≥ 1	$\frac{r}{r-1}$	Харди, Литтлвуд, Полиа [4]
2	0,1	∞	≥ 1	∞	Габушин [15]
			$I=[0,\infty)$		
2	1	∞	∞	∞	Ландау [2]
3	1,2	∞	∞	∞	Маторин [8]
2	1	2	2	2	Харди, Литтлвуд, Полиа [4]
произвольное	$0\leq k<n$	∞	2	2	Габушин [18]
2	1	1	1	1	Бердышев [19]

Окончательные условия существования неравенства (1) получены Габушиным [13]; он доказал, что (1) с конечной константой D имеет место в том и только в том случае, если выполнены равенства (2) и неравенство

$$\frac{n-k}{r}+\frac{k}{p}\geq\frac{n}{q}. \tag{4}$$

В дальнейшем, когда будет идти речь о задаче (1), будем предполагать, что эти условия имеют место.

Наряду с задачей (1) мы будем рассматривать следующий частный случай задачи Стечкина [10, 11] о наилучшем приближении неограниченного оператора линейными ограниченными операторами.

Пусть $K^n_{p,r}$ — класс функций $f\in L^n_{p,r}$, у которых $\|f^{(n)}\|_{L_p}\leq 1$; а $\Sigma(N)$ — множество линейных ограниченных операторов S действующих из L_r в L_q с нормой

$$\|S\|=\|S\|^{L_q}_{L_r}\leq N.$$

Положим

$$\varrho(S)=\sup_{f\in K^n_{p,r}}\|f^{(k)}(x)-S(x,f)\|_{L_q},\quad E(N)=\inf_{S\in\Sigma(N)}\varrho(S); \tag{5}$$

в частном случае $q=r=\infty$ будем считать

$$(6)\qquad E(N)=\inf_{\|S\|_C^C\leq N}\ \sup_{\substack{\|f^{(n)}\|_{L_p}\leq 1\\ f\in C}}\|f^{(k)}(x)-S(x,f)\|_C=E_\infty(N).$$

Задача состоит в вычислении величины $E(N)$ и определении экстремального оператора S_N, удовлетворяющего условиям

$$S_N\in\Sigma(N),\quad \varrho(S_N)=E(N).$$

Для произвольных $h>0$, $\bar N>0$, справедливо равенство

$$(7)\qquad E(h^{-k+q^{-1}-r^{-1}}\bar N)=h^{n-k+q^{-1}-p^{-1}}E(\bar N);$$

причем, если при $N=\bar N$ экстремальный оператор $S_{\bar N}$ существует, то оператор $h^{-k}S_{\bar N}(xh^{-1},f(ht))$ будет экстремальным при $N=h^{-k+q^{-1}-r^{-1}}\bar N$. В случае $p=q=r$ это утверждение доказано Стечкиным [11], для произвольных значений параметров оно доказывается аналогично (см. [17]).

В некоторых случаях решение задачи (5) позволяет получить решение задачи (1) (см. работы [10], [11], [12], [14], [16], [18], [19]).

В данной работе найдена наилучшая константа и описано множество экстремальных функций неравенства (1) при $n=2$, $r=\infty$, $p\geq 1$, $q\geq 2p$ и $n=3$, $q=r=\infty$, $1\leq p<\infty$ для $I=[0,\infty)$ и $I=(-\infty,\infty)$; в случае $n=2$ существенно используется результат С.-Надя [7], а в случае $n=3$ вначале приводится решение задачи (6), и как следствие выписывается точное неравенство (1).

2. *Величина* $E(N)$ *и константа* (3) *связаны неравенством*

$$(8)\qquad D\leq\left(\frac{1}{\beta}E(N)\right)^{\beta}\left(\frac{1}{\alpha}N\right)^{\alpha}.$$

В случае $p=q=r$ это неравенство доказал Стечкин [10, 11] и сейчас мы фактически повторим его рассуждения.

Для любой функции $f\in L^n_{p,r}$ и оператора $S\in\Sigma(N)$ имеем

$$\|f^{(k)}\|_{L_q}\leq\|f^{(k)}-S(f)\|_{L_q}+\|S(f)\|_{L_q}\leq\varrho(S)\|f^{(n)}\|_{L_p}+N\|f\|_{L_r}.$$

Отсюда

$$\|f^{(k)}\|_{L_q}\leq E(N)\|f^{(n)}\|_{L_p}+N\|f\|_{L_r}.$$

Заменив здесь N на $h^{-k+q^{-1}-r^{-1}}N$, в силу (7), получим неравенство

$$\|f^{(k)}\|_{L_q}\leq h^{n-k-p^{-1}+q^{-1}}E(N)\|f^{(n)}\|_{L_p}+h^{-k+q^{-1}-r^{-1}}N\|f\|_{L_r}.$$

Минимизируя его правую часть по $h>0$, придем к соотношению

$$(9)\qquad \|f^{(k)}\|_{L_q}\leq\bar D\|f\|^{\alpha}_{L_r}\|f^{(n)}\|^{\beta}_{L_p}\leq E(N)\|f^{(n)}\|_{L_p}+N\|f\|_{L_r},$$

в котором

$$\bar{D} = \left(\frac{1}{\beta} E(N)\right)^{\beta} \left(\frac{1}{\alpha} N\right)^{\alpha} = \frac{(E(1))^{\beta}}{\alpha^{\alpha}\beta^{\beta}}, \tag{10}$$

а отсюда следует (8).

Пусть при $q=\infty$ *существует функция* $u \in L^n_{p,r}$ *и линейный оператор* $\bar{S}$ *из* L_r *в* C, *удовлетворяющие равенству*

$$\varrho(\bar{S}) = \left(u^{(k)}(0) - \|\bar{S}\| \, \|u\|_{L_r}\right) \|u^{(n)}\|^{-1}_{L_p}. \tag{11}$$

Тогда $\varrho(\bar{S}) = E(\|\bar{S}\|)$ *и оператор* $\bar{S}$ *— экстремальный.*

Действительно

$$\varrho(\bar{S}) \geqq E(\|\bar{S}\|) \geqq \inf_{\|S\| \leqq \|\bar{S}\|} \left(u^{(k)}(0) - S(0, u)\right) \|u^{(n)}\|^{-1}_{L_p} \geqq$$
$$\geqq \left(u^{(k)}(0) - \|\bar{S}\| \, \|u\|_{L_r}\right) \|u^{(n)}\|^{-1}_{L_p} = \varrho(\bar{S}).$$

Этими соображениями мы будем пользоваться при вычислении величины $E_\infty(N)$.

Теорема 1. *Пусть* $q=\infty$, $1<p<\infty$, $1 \leqq r \leqq \infty$. *Тогда* $E(N) < \infty$ *и существует экстремальный оператор* S_N. *В случае* $I = (-\infty, \infty)$ *экстремальный оператор единственный и имеет вид свертки;* в случае $I=[0, \infty)$ *однозначно определяется функционал* $S_N(0, f)$, *а одним из экстремальных является оператор* $S_N(0, f(x+t))$.

В тех же предположениях неравенство (1) (с константой (3)) *имеет единственную экстремальную функцию.*

Доказательство этой теоремы содержится в [17].

3. В этом параграфе будет получено решение задачи (1) в случае $n=2$, $r=\infty$. При этом будет использоваться частный результат С.-Надя [7], который мы приведем в следующей форме.

Пусть $1 \leqq p < \infty$, $1 < q \leqq \infty$, $I = (-\infty, \infty)$. *Тогда любая локально абсолютно непрерывная функция* φ, *обладающая свойствами* $\varphi \in L$, $\varphi' \in L_p$, *удовлетворяет неравенству*

$$\|\varphi\|_{L_q} \leqq Q \, \|\varphi\|^{\alpha}_{L} \|\varphi'\|^{\beta}_{L_p}, \tag{12}$$

в котором $\alpha = 1 - \beta$, $\beta = (1-q^{-1})/(2-p^{-1})$,

$$Q = \left\{\frac{2p-1}{2p} H\left(\frac{2p-1}{p(q-1)}, \frac{p-1}{p}\right)\right\}^{\beta},$$

$$H(x, y) = \frac{z(x+y)}{z(x)z(y)}, \quad z(x) = x^{-x}\Gamma(1+x)$$

для $x>0$ *и* $z(0)=1$. *Неравенство точное. При* $p>1$ *оно имеет единственную экстремальную функцию* ψ, *которая при* $q=\infty$ *задана равенствами* $\psi(t)=(1-|t|)^{\frac{p}{p-1}}$ *для* $|t|\leqq 1$, $\psi(t)=0$ *для* $|t|>1$, *а при* $1<q<\infty$ *определена неявно уравнением* $|t|=s(\psi)$ *для* $|t|\leqq T=s(0)$, *где*

$$s(\psi)=\int_{\psi}^{1}\frac{d\tau}{\tau^{\frac{1}{p}}(1-\tau^{q-1})^{\frac{1}{p}}}\qquad (0\leqq\psi\leqq 1),$$

и равна нулю для $|t|>T$. *В случае* $p=1$, $1<q<\infty$, *неравенство* (12) *строгое, а в случае* $q=\infty$, $p=1$ *обращается в равенство на любой функции* φ, *у которой* $|\varphi|$ *возрастает на некотором интервале* $(-\infty, t_0)$ *и убывает на* (t_0, ∞).

Если $I=[0, \infty)$, то функции $\varphi\in L^1_{p,1}$ удовлетворяют неравенству

$$(12')\qquad \|\varphi\|_{L_q}\leqq 2^{\beta}Q\|\varphi\|^{\alpha}_{L}\|\varphi'\|^{\beta}_{L_p}\qquad (q>1),$$

причем при $p>1$ экстремальными являются лишь функции $c\psi(\nu t)$ $(t\geqq 0)$, при $p=1$, $1<q<\infty$ неравенство строгое, а при $p=1$, $q=\infty$ экстремальной является любая монотонная функция. Чтобы в этом убедиться достаточно каждую функцию $\varphi\in L^1_{p,1}$ продолжить четно на всю ось и применить приведенный результат С.-Надя.

Для $1\leqq p<\infty$, $1<q\leqq\infty$ обозначим через g функцию, удовлетворяющую условиям: $-\inf g=\sup g$; $g'(t)=\psi(t)$ для $t\in I$, если $p>1$; $g'(t)=1$ для $|t|<1$, $g'(t)=0$ для $|t|>1$, если $p=1$.

Теорема 2. *При* $q\geqq 2p$ *справедливо неравенство*

$$(13)\qquad \|f'\|_{L_q}\leqq D_{p,q}\|f\|^{\alpha}_{C}\|f''\|^{\beta}_{L_p},\qquad f\in L^2_p,$$

в котором $\alpha=1-\beta$, $\beta=(1-q^{-1})/(2-p^{-1})$,

$$D_{p,q}=\begin{cases}2Q, & \text{если}\quad I=[0,\infty),\\ 2^{\beta}Q, & \text{если}\quad I=(-\infty,\infty).\end{cases}$$

При $p=1$, $q\geqq 2$ *имеет место также неравенство*

$$(14)\qquad \|f'\|_{L_q}\leqq D_{1,q}\|f\|^{\frac{1}{q}}_{C}(Vf')^{1-\frac{1}{q}},\qquad f\in L^1_V,$$

Оба неравенства точные. Более того, если $I=[0,\infty)$, $q\geqq 2p$ *или* $I=(-\infty,\infty)$, $q>2p$, *то при* $1<p<\infty$ *функция* g *является единственной экстремальной в* (13), *а при* $p=1$, $q<\infty$ *— в неравенстве* (14).

Напомним, что в случае $p=q=\infty$ неравенство (13) получено Ландау [1].

Отметим, что условие $q \geqq 2p$ является естественным, так как совпадает в данном случае с (4).

Одновременно с доказательством теоремы будет установлено, что если $I=(-\infty, \infty)$, $q=2p$, то при $1<p<\infty$ множество F_p экстремальных функций неравенства (13), а при $p=1$ — неравенства (14), состоит из функций f, для каждой из которых найдется быть может счетное число непересекающихся интервалов $\Delta_i=(a_i, b_i)$ со свойствами $|f(t)|=\|f\|_C$ вне интервалов Δ_i, и

$$(15)\qquad f(t)=\theta_i\|f\|_C\|g\|_C^{-1} g\left(T \frac{2t-(a_i+b_i)}{b_i-a_i}\right)$$

для $t \in \Delta_i$, где $|\theta_i|=1$, $(-T, T)$ — носитель функции g'; условие $f'' \in L_p$ при $p>1$ и $Vf'<\infty$ при $p=1$ эквивалентно неравенству $\Sigma|b_i-a_i|^{1-2p}<\infty$, откуда, в частности, следует, что на любом интервале конечной длины может быть лишь конечное число интервалов Δ_i. Если $q=\infty$, то неравенство (14) обращается в равенство на каждой функции f, у которой производная монотонная в случае $I=[0, \infty)$ и монотонная на интервалах $(-\infty, t')$, (t', ∞) для некоторого t' в случае $I=(-\infty, \infty)$; множество F' таких функций в данном случае описывает весь класс экстремальных функций. Экстремальные функции неравенства (13) при $p=1$, $q=\infty$ исчерпываются множеством $F=F'L_p^2$. При $p=1$, $q<\infty$ неравенство (13) строгое.

Докажем утверждения теоремы при $p<\infty$. Нетрудно проверить, что функции, перечисленные только что и в формулировке теоремы, обращают соответствующие неравенства в равенства.

Если $f \in L_p^2$ монотонная функция, то $\|f'\|_L \leqq 2\|f\|_C$ и f' удовлетворяет (12), если $I=(-\infty, \infty)$ и (12′) если $I=[0, \infty)$; поэтому f удовлетворяет (13). Из утверждения С.-Надя об экстремальных функциях неравенства (12) следует, что на подмножестве монотонных функций множества L_p^2 неравенство (13) при $p=1$, $q<\infty$ строгое, при $p=1$, $q=\infty$ оно обращается в равенство лишь на функциях класса F, а при $p>1$ монотонные функции $f \in L_p^2$, обращающие (13) в равенство имеют вид

$$(16)\qquad f(t)=cg(\nu t+\mu), \qquad t \in I,$$

($\mu=0$, если $I=[0, \infty)$).

Пусть теперь f — произвольная функция класса L_p^2. Множество Ω точек $t \in I$, в которых $f'(t) \neq 0$, состоит не более чем из счетного числа непересекающихся интервалов $\Delta_i=(a_i, b_i)$ и, быть может, еще одного интервала $\Delta_0=[0, b_0)$ в случае $I=[0, \infty)$. При каждом i определим функцию $f_i \in L_p^2$, положив $f_i(t)=f(t)$ для $t \in \Delta_i$ и доопределив ее константами на все I до непрерывной функции. Функции f_i монотонные и значит удовлетворяют (13).

Рассмотрим вначале случай $q=\infty$. Из того факта, что (13) (при $q=\infty$, $I=[0,\infty)$) с какой-то конечной константой D имеет место, легко следует (см. [7]), что $f'(t)\to 0$ при $|t|\to\infty$ для любой функции $f\in L_p^2$ $(p<\infty)$. Поэтому существует интервал Δ_j, на котором $\|f'\|_{C(\Delta_j)} = \|f'\|_{C(I)}$. Отсюда

$$\|f'\|_C=\|f_j'\|_C\leqq D_{p,\infty}\|f_j\|_C^\alpha\|f_j''\|_{L_p}^\beta\leqq D_{p,\infty}\|f\|_C^\alpha\|f''\|_{L_p}^\beta,$$

т. е. при $q=\infty$ любая функция класса L_p^2 удовлетворяет (13). Очевидно экстремальной может быть только монотонная функция. Все утверждения теоремы, относящиеся к этому случаю, доказаны.

При $2p\leqq q<\infty$, записав (13) для функций f_i, получаем соотношения

$$\|f'\|_{L_q(\Delta_i)}^q\leqq D_{p,q}^q\|f\|_{C(\Delta_i)}^{\alpha q}\|f''\|_{L_p(\Delta_i)}^{p\gamma}, \tag{17}$$

где $\gamma=(q-1)/(2p-1)\geqq 1$; просуммировав их, будем иметь

$$\|f'\|_{L_q}^q=\sum\|f'\|_{L_q(\Delta_i)}^q\leqq D_{p,q}^q\|f\|_C^{\alpha q}\sum\|f''\|_{L_p(\Delta_i)}^{p\gamma}\leqq D_{p,q}^q\|f\|_C^{\alpha q}\|f''\|_{L_p}^{p\gamma}. \tag{18}$$

Таким образом и в этом случае (13) справедливо для произвольной функции $f\in L_p^2$.

Очевидно при $q<\infty, p=1$ неравенство (13) строгое. Пусть f — экстремальная функция в (13) при $q<\infty$, $p>1$. Тогда на ней (18), а значит и (17) обращаются в равенства. Из этого в частности следует, что каждая функция f_i — экстремальная (монотонная) и $\|f_i\|_C=\|f\|_C$. Поэтому в случае $I=[0,\infty)$ для каждого i имеет место соотношение

$$f(t)=\theta_i\|f\|_C\|g\|_C^{-1}g\left(T\frac{t-a_i}{b_i-a_i}\right),\qquad t\in\Delta_i, \tag{19}$$

где $|\theta_i|=1$. Отсюда $|f(a_i)|=\|f\|_C$, но $f'(a_i)\neq 0$ вместе с $g'(0)$, а поскольку f' — непрерывная функция, то $a_i=0$ и, следовательно, $f(t)=cg(tTb_0^{-1})$ для $t\in[0,\infty)$. То есть в данном случае функция g — единственная экстремальная.

Аналогично, в случае $I=(-\infty,\infty)$ экстремальная функция f на интервалах Δ_i задается формулой (15), а так как при $2p<q$ $(\gamma>1)$ последнее неравенство (18) обращается в равенство, если в сумме только одно слагаемое отлично от нуля, то f совпадает с одной из функций f_i, или, что тоже самое, имеет вид (16). При $q=2p$ $(\gamma=1)$ получаем, что $f\in F_p$, но и обратно, каждая функция этого класса — экстремальная.

Обратимся к неравенству (14). Повторяя доказательство С.-Надя неравенства (12) при $p=1$ убеждаемся, что если $\varphi\in L$ имеет ограниченное изменение $\mathrm{V}\varphi$ на I, то

$$\|\varphi\|_{L_q}\leqq Q_1\|\varphi\|_L^{\frac{1}{q}}(\mathrm{V}\varphi)^{1-\frac{1}{q}}\qquad(1<q\leqq\infty), \tag{20}$$

где $Q_1=1$, если $I=[0,\infty)$ и $Q_1=2^{\frac{1}{q}-1}$ если $I=(-\infty,\infty)$. Причем при $q<\infty$ экстремальными являются только функции, равные константе на некотором интервале (a,b) ($[0,b)$, если $I=[0,\infty)$) и нулю на множестве $I\setminus[a,b]$. При $q=\infty$ экстремальными будут лишь производные функций $f\in F'$.

Отсюда следует, что монотонные функции класса L^1_V удовлетворяют (14). Если теперь для произвольной функции $f\in L^1_V$ вместо множества Ω рассмотреть множество точек $t\in I$, в которых $f'(t+0)f'(t-0)>0$, то так же как и раньше убеждаемся, что f удовлетворяет (14). Множество экстремальных функций исследуется аналогично тому, как это было сделано для неравенства (13).

Сгладив экстремальную функцию неравенства (14), легко убедиться, что (13) при $p=1$, $q<\infty$ точное.

Теорема доказана.

Стечкин [10, 11] решил задачу (6) в случае $p=\infty$, $n=2, 3$; в частности, он доказал, что для $n=2$ операторы

$$S_h(x,f)=\frac{1}{2h}\{f(x+h)-f(x-h)\}, \tag{21}$$

$$S_h^+(x,f)=\frac{1}{h}\{f(x+h)-f(x)\} \tag{22}$$

являются экстремальными, соответственно, при $N=h^{-1}$, $I=(-\infty,\infty)$ и $N=2h^{-1}, I=[0,\infty)$.

Теорема 3. *Если* $n=2$, $k=1$, $1\leqq p\leqq\infty$, $h>0$, *то*

$$\begin{aligned} E_\infty(h^{-1}) &= \frac{1}{2}h^{\frac{p-1}{p}}\left(\frac{2p-2}{2p-1}\right)^{\frac{p-1}{p}} \quad \text{для} \quad I=(-\infty,\infty),\\ E_\infty(2h^{-1}) &= h^{\frac{p-1}{p}}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}} \quad \text{для} \quad I=[0,\infty). \end{aligned} \tag{23}$$

Соответствующими экстремальными операторами являются (21) *и* (22).

Этот результат, в силу (8), влечет, что для любой функции $f\in L_p^2$ выполняется неравенство

$$\|f'\|_C\leqq D_I\|f\|_C^{\frac{p-1}{2p-1}}\|f''\|_{L_p}^{\frac{p}{2p-1}}, \tag{24}$$

в котором

$$D_{[0,\infty)} = 2^{\frac{p-1}{2p-1}}\left(\frac{2p-1}{p}\right)^{\frac{p}{2p-1}}, \quad D_{(-\infty,\infty)} = 2^{-\frac{1}{2p-1}}\left(\frac{2p-1}{p}\right)^{\frac{p}{2p-1}}. \tag{25}$$

Мы получаем еще одно доказательство неравенства (13) при $q=\infty$.

В случае $p=\infty$ теорема 3 доказана Стечкиным [10, 11]. Докажем ее для $p<\infty$. Очевидно

$$\|S_h\|=h^{-1}, \quad \|S_h^+\|=2h^{-1}.$$

При $I=[0, \infty)$ для произвольной функции $f\in L_p^2$ имеем равенство

$$f'(x)-S_h^+(x,f) = -\frac{1}{h}\int_0^h (h-t)f''(x+t)\,dt.$$

Откуда

$$|f'(x)-S_h^+(x,f)| \leqq \frac{1}{h}\|h-t\|_{L_{p'}(0,h)}\|f''\|_{L_p},$$

где $p' = p/(p-1)$, и, следовательно,

$$E_\infty(2h^{-1}) \leqq \varrho(S_h^+) \leqq h^{\frac{p-1}{p}}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}.$$

Неравенство (8) приводит теперь к (24) с константой (25). На функции

$$f(t)=\begin{cases} 1, & \text{если} \quad t\geqq 1\\ 1-2(1-t)^{\frac{2p-1}{p-1}}, & \text{если} \quad 0\leqq t<1,\end{cases}$$

при $p>1$ и на произвольной функции $f\in L_1^2$ с монотонной производной при $p=1$ неравенство (24) обращается в равенство. Следовательно формула (23) имеет место и оператор S_h^+ — экстремальный.

Если $I = (-\infty, \infty)$, то для $f\in L_p^2$

$$f'(x)-S_h(x,f) = \frac{1}{2h}\int_{-h}^{h} \sigma(h, t)f''(x+t)\,dt, \tag{26}$$

где

$$\sigma(h, t) = \begin{cases} h+t & \text{при} \quad t\in(-h, 0),\\ t-h & \text{при} \quad t\in(0, h);\end{cases}$$

дальнейшие рассуждения проводится так же, как в случае $I=[0, \infty)$.

Теорема доказана.

4. Перейдем к случаю $n=3$. В силу (7) задачу (6) достаточно решить для некоторого конкретного значения N.

Пусть вначале $1<p<\infty$. Прежде чем сформулировать и доказать относящиеся к этому случаю результаты, мы докажем несколько вспомогательных утверждений.

При $\alpha\in[1, \infty)$, $t\in(-\infty, \infty)$ положим

$$\varphi(t) = \varphi(t, \alpha) = t^2 - 2t\frac{\alpha^{3p-2}}{\alpha^{3p-2}+1} + \frac{\alpha^{3p-3}(\alpha^{3p-2}-1)}{(\alpha^{3p-3}+1)(\alpha^{3p-2}+1)}, \tag{27}$$

$$\psi(t) = \psi(t, \alpha) = |\varphi(t, \alpha)|^{\frac{1}{p-1}} \operatorname{sign} \varphi(t, \alpha)$$

и рассмотрим на интервале $[1, \infty)$ свойства непрерывных функций

$$R(\alpha) = \int_0^1 \left(t - \frac{1}{1+\alpha^2}\right)\psi(t, \alpha)\, dt, \quad r(\alpha) = \int_0^1 \psi(t, \alpha)\, dt. \tag{28}$$

Лемма 1. *Функция R имеет хотя бы один нуль на интервале* $(1, \infty)$, *наибольший нуль α_p этой функции конечен и справедливо неравенство*

$$r(\alpha_p) > 0. \tag{29}$$

Доказательство. Непосредственные вычисления показывают, что

$$\varphi(1) = -\alpha^{3(1-p)}\varphi(0), \quad \varphi_t'(1) = -\alpha^{2-3p}\varphi_t'(0) \tag{30}$$

и наименьший нуль $t(\alpha)$ полинома $\varphi(t)$ имеет вид

$$t(\alpha) = \frac{\alpha^{3p-2}}{\alpha^{3p-2}+1}\left\{1 - \left(\frac{\alpha^{3p-1}+1}{\alpha^{3p-1}+\alpha^{6p-4}}\right)^{\frac{1}{2}}\right\}.$$

Так как функция $s(\alpha) = t(\alpha) - \frac{1}{1+\alpha^2}$ непрерывна при $\alpha \geqq 1$, причем $s(1) = -\frac{1}{2}$, $s(\infty) = 1$, то найдется такое число $\bar\alpha > 1$, что $\frac{1}{1+\bar\alpha^2} = t(\bar\alpha)$.

В силу (30), полином $\varphi(t) = \varphi(t, \bar\alpha)$ имеет на интервале $[0, 1]$ лишь один нуль $t(\bar\alpha)$, но $\varphi(0) > 0$, поэтому полниом φ меняет знак с плюса на минус при переходе через точку $t(\bar\alpha)$. Отсюда

$$R(\bar\alpha) = \int_0^1 [t - t(\bar\alpha)]\psi(t, \bar\alpha)\, dt < 0.$$

С другой стороны

$$R(\infty) = \int_0^1 t(1-t)^{\frac{2}{p-1}}\, dt > 0.$$

Следовательно, уравнение $R(\alpha)=0$, имеет корень на полуоси $(\bar{\alpha}, \infty)$ и наибольший из этих корней α_p конечен.

Если $\mu>1$ — нуль функции r, то поскольку $\psi(t,\mu)>0$, при $0\leqq t<t(\mu)$ и $\psi(t,\mu)<0$ при $1\geqq t>t(\mu)$, имеем

$$R(\mu)=\frac{\mu^2}{1+\mu^2}r(\mu)+\int_0^1(t-1)\psi(t,\mu)\,dt=0=$$

$$=\int_0^1(t-1)\psi(t,\mu)\,dt<[t(\mu)-1]r(\mu)=0.$$

Таким образом, любой нуль функции r (если таковой существует) лежит левее точки α_p, но $r(\infty)>0$, поэтому и $r(\alpha_p)>0$, ч. т. д.

В дальнейшем положим $\alpha=\alpha_p$,

$$a_{-1}=0,\quad a_\mu=\sum_{i=0}^{\mu}\alpha^i\qquad(\mu=0,1,\ldots).$$

Определим затем при $t\geqq 0$ функцию $z(t)$ следующим образом

$$z(t)=(-1)^\mu\alpha^{3\mu(1-p)}\varphi\left(\frac{t-a_{\mu-1}}{\alpha^\mu}\right),\qquad t\in[a_{\mu-1},a_\mu),\quad \mu=0,1,\ldots. \tag{31}$$

Согласно (30), она непрерывно дифференцируема. Кроме того

$$z''(t)=(-1)^\mu 2\alpha^{\mu(1-3p)}\quad t\in(a_{\mu-1},a_\mu). \tag{32}$$

Нам будет удобно считать, что функция z'' непрерывна слева и $z''(0)=0$; в этом предположении

$$\begin{gathered}z''(+0)-z''(0)=2,\\ z''(a_\mu+0)-z''(a_\mu)=(-1)^\mu 2\alpha^{\mu(1-3p)}(1+\alpha^{1-3p}),\qquad \mu=0,1,\ldots,\end{gathered} \tag{33}$$

$$\mathbf{V}z''=\mathop{\mathbf{V}}_0^\infty z''=\frac{4}{1-\alpha^{1-3p}}. \tag{34}$$

Функция

$$\theta(t)=|z(t)|^{\frac{1}{p-1}}\operatorname{sign}z(t) \tag{35}$$

абсолютно интегрируема на $[0,\infty)$, поэтому функция

$$\eta(t)=\int_0^t\int_0^{t_1}\int_{t_2}^\infty\theta(\tau)\,d\tau\,dt_2\,dt_1 \tag{36}$$

определена и трижды непрерывно дифференцируема при $t\geqq 0$.

Докажем, что функция η обладает также следующими свойствами.

Лемма 2. *Функция* η *неотрицательна, ограничена и при любом* $\mu=0, 1, \ldots$

$$\eta(a_{2\mu-1})=0, \quad \eta(a_{2\mu})=\|\eta\|_{C[0,\infty)}.$$

Доказательство. Справедливы равенства

$$\eta''(a_{\mu-1}) = (-1)^{\mu}\frac{\alpha^{2-2\mu}}{1+\alpha^2}\int_0^1 \psi(t)\,dt. \tag{37}$$

В самом деле

$$\eta''(a_{\mu-1}) = \int_{a_{\mu-1}}^{\infty} \theta(\tau)\,d\tau = \sum_{\nu=\mu}^{\infty}\int_{a_{\nu-1}}^{a_\nu} \theta(\tau)\,d\tau;$$

если в каждом интеграле под знаком суммы сделать замену $\tau = a_{\nu-1}+\alpha^{\nu}t$ и использовать (31), (35), то получим (37).

Докажем теперь, что

$$\eta'(a_{\mu-1})=0 \qquad (\mu=0, 1, \ldots). \tag{38}$$

Имеем

$$\eta'(a_\mu)-\eta'(a_{\mu-1}) = \int_{a_{\mu-1}}^{a_\mu} \eta''(t)\,dt = \int_{a_{\mu-1}}^{a_\mu} \{\eta''(t)-\eta''(a_{\mu-1})\}\,dt+(a_\mu-a_{\mu-1})\eta''(a_{\mu-1}) =$$

$$= \alpha^{\mu}\eta''(a_{\mu-1})-\int_{a_{\mu-1}}^{a_\mu}(a_\mu-\tau)\theta(\tau)\,d\tau.$$

Заменив здесь $\eta''(a_{\mu-1})$ по формуле (37) и введя новую переменную $t\in[0, 1]$ формулой $\tau = a_{\mu-1}+\alpha^{\mu}t$, получим

$$\eta'(a_\mu)-\eta'(a_{\mu-1}) = (-1)^{\mu}\alpha^{-\mu}R(\alpha).$$

Поскольку $R(\alpha)=0$, то $\eta'(a_\mu)=\eta'(a_{\mu-1})$, но $\eta'(0)=0$, следовательно, равенства (38) проверены.

Покажем, что

$$\operatorname{sign}\eta'(t) = (-1)^{\mu} \quad \text{для} \quad t\in(a_{\mu-1}, a_\mu), \qquad \mu = 0, 1, \ldots. \tag{39}$$

В силу (37), функция η'' в точках $a_{\mu-1}$, a_μ принимает значения разных знаков. Согласно (38), производная η' на концах интервала $[a_{\mu-1}, a_\mu]$ равна нулю, поэтому если бы она имела еще нуль внутри него, то функция η'' имела бы там уже не менее чем три нуля, а функция η''' — по крайней мере два. Но функция $\eta''' = -\theta$ имеет на интервале $[a_{\mu-1}, a_\mu]$ лишь один нуль $a_{\mu-1}+\alpha^{\mu}t(\alpha)$. Следовательно, $\operatorname{sign}\eta'(t)=\operatorname{sign}\eta''(a_{\mu-1})$ для $t\in(a_{\mu-1}, a_\mu)$, а, в силу (37) и (29), это утверждение совпадает с (39).

Имеем, далее

$$\eta(a_\mu)-\eta(a_{\mu-1}) = \int\limits_{a_{\mu-1}}^{a_\mu} \eta'(t)\,dt = \int\limits_{a_{\mu-1}}^{a_\mu} \int\limits_{a_{\mu-1}}^{t} \eta''(\xi)\,d\xi\,dt =$$

$$= \int\limits_{a_{\mu-1}}^{a_\mu} (a_\mu-\xi)\eta''(\xi)\,d\xi = \int\limits_{a_{\mu-1}}^{a_\mu} (a_\mu-\xi)\{\eta''(\xi)-\eta''(a_{\mu-1})\}\,d\xi +$$

$$+ \int\limits_{a_{\mu-1}}^{a_\mu} (a_\mu-\xi)\eta''(a_{\mu-1})\,d\xi = \frac{1}{2}\alpha^{2\mu}\eta''(a_{\mu-1}) - \frac{1}{2}\int\limits_{a_{\mu-1}}^{a_\mu} (a_\mu-\tau)^2\theta(\tau)\,d\tau.$$

Отсюда, тем же путем, что и при доказательстве равенств (38), находим, что $\eta(a_\mu)-\eta(a_{\mu-1}) = (-1)^\mu d$, где число d не зависит от μ. Поскольку $\eta(0)=0$, то $\eta(a_{2\mu-1})=0$, $\eta(a_{2\mu})=d$. Теперь из (38) и (39) следует, что функция η возрастает от нуля до значения d на интервалах $[a_{2\mu-1}, a_{2\mu}]$ и убывает до нуля на интервалах $[a_{2\mu}, a_{2\mu-1}]$.

Все утверждения леммы доказаны.

Положим

(40) $$u(t) = \eta(|t|) - \frac{1}{2}\|\eta\|_{C[0,\infty)}, \qquad t\in(-\infty,\infty).$$

Из леммы 2 вытекает

Следствие. *Функция u четная, принадлежит классу* L_p^3 *и удовлетворяет соотношениям*

(41) $$u(a_{\mu-1}) = (-1)^{\mu-1}\|u\|_C, \qquad \mu=0,1,\ldots,$$

(42) $$u'''(t) = -|z(|t|)|^{\frac{1}{p-1}}\operatorname{sign} tz(|t|), \qquad t\neq 0.$$

Теорема 4. *Пусть* $n=3$, $k=2$, $1<p<\infty$, $I=(-\infty,\infty)$, *функция* φ *определена формулой* (27), *а функция* R — *формулой* (28), α — *наибольший нуль функции* R. *Тогда*

(43) $$E_\infty\left(\frac{4}{\varphi(0)(1-\alpha^{1-3p})}\right) = \frac{1+\alpha^{1-3p}}{\varphi(0)}\left\{\frac{1}{1-\alpha^{1-3p}}\int\limits_0^1 |\varphi(t)|^{\frac{p}{p-1}}\,dt\right\}^{\frac{p-1}{p}},$$

а экстремальным будет оператор

(44) $$S_2(x,f) = \frac{1+\alpha^{1-3p}}{\varphi(0)}\sum_{\mu=0}^{\infty}(-1)^\mu\alpha^{\mu(1-3p)}\{f(x+a_\mu)-2f(x)+f(x-a_\mu)\}.$$

Доказательство. Соотношения (32), (33) и $z(0)=\varphi(0)>0$ позволяют записать оператор S_2 в виде

(45) $$S_2(x,f) = -\frac{1}{2z(0)}\int_0^\infty [f(x+t)+f(x-t)]\,dz''(t).$$

Из этого представления и формулы (34) получаем оценку

$$\|S_2\| \leqq \frac{1}{z(0)}\,\mathrm{V}z'' = \frac{4}{\varphi(0)(1-\alpha^{1-3p})} = N_2.$$

С другой стороны, в силу (41) и (44),

(46) $$S_2(0,u) = N_2\|u\|$$

и значит $\|S_2\| = N_2$.

Если функция $f\in L_p^3$, то $\|f^{(i)}\|_C<\infty$ $(i=0, 1, 2)$. Кроме того, $z(\infty)=z'(\infty)=z''(\infty)=0$. Поэтому, взяв в правой части (45) интеграл три раза по частям, получим тождество

$$f''(x)-S_2(x,f) = -\frac{1}{2z(0)}\int_{-\infty}^{\infty} f'''(x+t)z(|t|)\,\mathrm{sign}\,t\,dt.$$

Применяя затем неравенство Гельдера, находим

$$\varrho(S_2) \leqq \frac{2^{-\frac{1}{p}}}{\varphi(0)}\|z\|_{L_{p'}(0,\infty)} = \varrho_2 \qquad \left(p' = \frac{p}{p-1}\right).$$

Свойство (42) функции u влечет равенство

$$u''(0)-S_2(0,u) = \frac{1}{\varphi(0)}\int_0^\infty |z(t)|^{p'}\,dt = \varrho_2\|u'''\|_{L_p}.$$

Следовательно

(47) $$\varrho(S_2) = \varrho_2 = u_2''(0)-S_2(0,u_2),$$

где

$$u_2(t) = u(t)\|u'''\|_{L_p}^{-1}.$$

Равенства (46) и (47) показывают, что оператор S_2 и функция u_2 удовлетворяют условию (11), поэтому $E(N_2)=\varrho_2$ и оператор S_2 — экстремальный.

Осталось доказать, что ϱ_2 совпадает с правой частью равенства (43). Имеем

$$\int_0^\infty |z(t)|^{p'}\,dt = \sum_{\nu=0}^{\infty}\int_{a_{\nu-1}}^{a_\nu}|z(t)|^{p'}\,dt.$$

Сделав в каждом интеграле замену $t = a_{\nu-1}+\alpha^{\nu}\tau$, получим

$$\int_0^\infty |z(t)|^{p'}\,dt = \frac{1}{1-\alpha^{1-3p}}\int_0^1 |\varphi(\tau)|^{p'}\,d\tau. \tag{48}$$

Теорема доказана.

Функция z была определена (31) лишь для $t \geqq 0$; распространим ее на полуось $(-\infty, 0)$, положив

$$z(b, t) = z(t) = bt^2+\varphi(t), \qquad t\in(-\infty, 0), \tag{49}$$

где $b<-1$ пока произвольное число.

Определим далее, на всей числовой оси трижды непрерывно дифференцируемую функцию $v(t)=v(b, t)$ таким образом, чтобы

$$v(t) = -u(t) \quad \text{при} \quad t\in[0, \infty), \tag{50}$$

$$v'''(t) = |z(t)|^{\frac{1}{p-1}}\operatorname{sign} z(t) \quad \text{при} \quad t\in(-\infty, \infty). \tag{51}$$

Лемма 3. *Существуют такие числа* $c_m<0, b_m<-1$ $(m=0, 1, 2)$, *что функции*

$$z_m(t)=z(b_m, t) \quad \text{и} \quad v_m(t)=v(b_m, t) \tag{52}$$

удовлетворяют условиям

$$v_0(c_0)=0, \quad v_0''(c_0)=0, \tag{53}$$

$$v_1(c_1) = -v(0), \quad z_1(c_1) = 0, \tag{54}$$

$$v_2(c_2) = -v(0), \quad z_2'(c_2) = 0. \tag{55}$$

Эти числа определяются единственным образом. Кроме того,

$$v_m'(t)>0 \quad \text{для} \quad t\in[c_m, 0), \tag{56}$$

$$z_0'(c_0)>0, \quad z_1'(c_1)>0, \quad z_2(c_2)>0. \tag{57}$$

Доказательство. Согласно (27) полином φ имеет вид $t^2-2\gamma_1 t+\gamma_2$, где $\gamma_i>0$. Отсюда, в силу (49) и (51), следует, что при любом фиксированном значении $b<-1$ функции z и v''' имеют единственный отрицательный нуль

$$c_1(b) = \frac{\gamma_1+\sqrt{\gamma_1^2-(1+b)\gamma_2}}{1+b},$$

а функция z' —

$$c_2(b) = \frac{\gamma_1}{1+b};$$

нетрудно проверить, что функции c_1 и c_2 непрерывные и убывают от 0 до $-\infty$, если $b\in(-\infty, -1)$.

При каждом значении $b<-1$ непрерывная функция

$$g(t)=g(b,t)=v''(0)-v''(t)=\int_t^0 v'''(\tau)\,d\tau$$

возрастает от $-\infty$ в интервале $(-\infty, c_1(b))$ и убывает до нуля в интервале $(c_1(b), 0]$. Но, в силу (50), (40), (37) и (29), число $v''(0)$ отрицательное и не зависит от b. Поэтому при любом $b<-1$ существует единственное решение $c_0(b)<0$ уравнения $g(b, c_0(b))=v''(0)$ или $v''(c_0(b))=0$; очевидно

(58) $$(c_0(b)<c_1(b)<c_2(b)<0.$$

Если $t<0$ фиксировано, то v''' как функция переменного $b\in(-\infty, -1)$ непрерывна и возрастает от $-\infty$ до $(\gamma_2-2\gamma_1 t)^{\frac{1}{p-1}}>0$. Из этого легко вывести, что функция $c_0(b)$ непрерывна и убывает от 0 до $-\infty$.

Далее, поскольку

(59) $$v''(t)=v''(0)-\int_t^0 v'''(\tau)\,d\tau,\quad v'(t)=-\int_t^0 v''(\tau)\,d\tau,$$

то

(60) $$v''(t)<0,\quad v'(t)>0\quad \text{для}\quad t\in(c_0(b), 0).$$

В силу (59), при каждом $t<0$ функция v'' переменного $b\in(-\infty, -1)$ убывает, а функция v' — возрастает; кроме того, функции $|c_m(b)|$ возрастают от 0 до ∞ и, согласно (58), (60), $v'(t)>0$ для $t\in(c_m(b), 0)$. Поэтому функции

$$G_m(b)=v(0)-v(c_m(b))=\int_{c_m(b)}^0 v'(t)\,dt$$

возрастают на $(-\infty, -1)$; очевидно эти функции непрерывные.

Используя соотношения (58)—(60), получаем

$$0\leqq G_m(b)=\int_{c_m(b)}^0\int_t^0\left\{\int_\xi^0 v'''(\tau)\,d\tau-v''(0)\right\}d\xi\,dt\leqq |c_m^3(b)|\,\mu(b)+|v''(0)|\,c_m^2(b),$$

где

$$\mu(b)=\max_{\tau\leqq 0} v''(\tau)=\left(\gamma_2-\frac{\gamma_1^2}{1+b}\right)^{\frac{1}{p-1}},$$

и, поскольку $c_m(b)\to 0$ при $b\to-\infty$, то $G_m(-\infty)=0$.

С другой стороны функция $v''(t)$ убывает от нуля в интервале $[c_0(b), c_1(b)]$ и возрастает до значения $v''(0)<0$ в интервале $[c_1(b), 0]$, поэтому, если положить

$$\omega(t) = \begin{cases} 0 & \text{для} \quad t \leqq c_2(b), \\ v''(0) & \text{для} \quad t > c_2(b), \end{cases}$$

то, в силу (58), $v''(t) \leqq \omega(t)$ для $t \leqq 0$. Отсюда

$$G_m(b) = -\int\limits_{c_m(b)}^{0} \int\limits_{\tau}^{0} v''(t)\,dt\,d\tau \geqq -\int\limits_{c_m(b)}^{0} \int\limits_{\tau}^{0} \omega(t)\,dt\,d\tau \geqq$$

$$\geqq -\int\limits_{c_2(b)}^{0} \int\limits_{\tau}^{0} \omega(t)\,dt\,d\tau = \frac{1}{2}|v''(0)|\,c_2^2(b),$$

и, так как $c_2(b) \to -\infty$ при $b \to -1$, то $G_m(-1) = \infty$.

Следовательно существуют решения $b_m < -1$ уравнений

(61) $$G_0(b_0) = v(0), \quad G_1(b_1) = G_2(b_2) = 2v(0)$$

(здесь $v(0)>0$). Но тогда числа b_m и $c_m = c_m(b_m)$ удовлетворяют системам (53)—(55) и выполняются соотношения (56), (57).

Осталось проверить, что числа $c_m < 0, b_m < -1$ единственные. Пусть $\bar{c}_m < 0, \bar{b}_m < -1$ другие решения систем (53)—(55). Второе уравнение каждой из этих систем, определение и свойства монотонности функций $c_m(b)$ показывают, что $\bar{c}_m = c_m(\bar{b}_m)$. В силу монотонности функций $G_m(b)$, уравнения (61) имеют единственные решения, следовательно $\bar{b}_m = b_m$, а значит и $\bar{c}_m = c_m$.

Все утверждения леммы доказаны.

Пусть $c_0<0, b_0 < -1$ — решения системы (53), а функции z_0 и v_0 определены в (52). Положим

$$y_0(t) = z_0(t+c_0), \qquad t \in [0, \infty),$$

(62) $$u_0(t) = v_0(|t|+c_0)\,\mathrm{sign}\, t, \qquad t \in (-\infty, \infty).$$

Функция u_0 нечетная и, в силу (53), $u_0 \in L_p^3$. Далее из (56), (53) и (41) вытекают соотношения

$$u_0(a_i - c_0) = (-1)^i \|u_0\|_{C(-\infty, \infty)}, \qquad i = 0, 1, \ldots,$$

а согласно (50)—(52) и (62), имеем

$$u_0'''(t) = |y_0(|t|)|^{\frac{1}{p-1}}\,\mathrm{sign}\, y_0(|t|), \qquad t \in (-\infty, \infty).$$

На интервале $[0, -c_0)$ функция $y_0''(t)$ постоянная и равна $2b_0+2 < 0$, кроме того $y_0(t) = z(t+c_0)$ для $t \geqq -c_0$, а следовательно $y_0''(-c_0+0) = 2$.

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Отсюда, используя (34), получаем

$$\bigvee_0^\infty y_0'' = 2\frac{1+b_0+(1-b_0)\alpha^{3p-1}}{\alpha^{3p-1}-1}.$$

Введем обозначения

$$N_1 = (\varphi'(c_0)+2b_0c_0)^{-1}\bigvee_0^\infty y_0'',$$

$$e_0 = 2^{-\frac{1}{p}}(\varphi'(c_0)+2b_0c_0)^{-1}\left\{\int_{c_0}^0 |\varphi(t)+b_0t^2|^{\frac{p}{p-1}}dt+\right.$$

$$\left.+\frac{1}{1-\alpha^{1-3p}}\int_0^1 |\varphi(t)|^{\frac{p}{p-1}}dt\right\}^{\frac{p-1}{p}},$$

где, согласно (57),

$$\varphi'(c_0)+2b_0c_0 = y_0'(0) = z_0'(c_0) > 0.$$

Наконец докажем следующее утверждение.

Теорема 5. *Пусть* $n=3$, $k=1$, $1<p<\infty$, $I=(-\infty,\infty)$, *функция* φ *определена формулой* (27), *числа* $c_0<0, b_0<-1$ *являются решением уравнений* (53), $\alpha>1$ — *наибольший нуль функции* R, *заданной формулой* (28). *Тогда*

$$E_\infty(N_1)=e_0, \tag{63}$$

а экстремальным будет оператор

$$S_1(x,f) = -b_0(\varphi'(c_0)+2b_0c_0)^{-1}\{f(x-c_0)-f(x+c_0)\}-$$

$$-(\alpha^{3p-1}+1)(\varphi'(c_0)+2b_0c_0)^{-1}\sum_{i=0}^\infty(-1)^i\alpha^{i(1-3p)}\{f(x-c_0+a_i)-f(x+c_0-a_i)\}.$$

Это предложение доказывается также как теорема 4. Действительно, оператор S_1 можно записать в виде

$$S_1(x,f) = \frac{1}{2y_0'(0)}\int_0^\infty [f(x+t)-f(x-t)]\,dy_0''(t).$$

Из этого представления следует, что для функций $f\in L_p^3$ справедливо тождество

$$f'(x)-S_1(x,f) = \frac{1}{2y_0'(0)}\int_{-\infty}^\infty y_0(|t|)f'''(x+t)\,dt.$$

Отсюда

$$E_\infty(\|S_1\|) \leqq \varrho(S_1) \leqq \frac{2^{-\frac{1}{p}}}{y_0'(0)}\|y_0\|_{L_{p'}(0,\infty)} = \varrho_1,$$

где $p' = p\diagdown(p-1)$.

С другой стороны,

$$u_0'(0)-S_1(0,u) = \frac{1}{y_0'(0)}\int_0^\infty |y_0(t)|^{p'}\,dt = \varrho_1\|u_0'''\|_{L_p}.$$

Таким образом оператор S_1 и функция u_0 удовлетворяют условию (11). Следовательно

$$E_\infty(\|S_1\|)=\varrho(S_1)=\varrho_1 \tag{64}$$

и оператор S_1 — экстремальный.

Далее имеем

$$\|S_1\|\leqq N_1,\quad S_1(0,u_0)=N_1\|u_0\|_C,$$

поэтому $\|S_1\|=N_1$. Наконец, из определения функции y_0 получаем

$$\|y_0\|^{p'}_{L_{p'}(0,\infty)} = \int_{c_0}^0 |\varphi(t)+b_0t^2|^{p'}\,dt+\int_0^\infty |z(t)|^{p'}\,dt.$$

Откуда, в силу (48), следует, что (63) и (64) совпадают.

Теорема доказана.

Перейдем к случаю $I=[0,\infty)$. Пусть числа $c_k<0$, $b_k<-1$ $(k=1,2)$ удовлетворяют уравнениям (54) и (55), а функции z_k и v_k определены равенствами (52). Положим для $t\in[0,\infty)$

$$y_k(t)=z_k(t+c_k),\quad u_k(t)=(-1)^{k-1}v_k(t+c_k). \tag{65}$$

Тогда

$$u_k\in{}_p^3,$$

$$u_k(a_{i-1}-c_k)=(-1)^{i+k+1}\|u_k\|_{C[0,\infty)}\qquad (i=0,1,\ldots),$$

$$u_k'''(t)=(-1)^{k-1}|y_k(t)|^{\frac{1}{p-1}}\operatorname{sign}y_k(t)\qquad (t\geqq 0),$$

$$y_1(0)=0,\quad y_2'(0)=0.$$

Кроме того, $y_k''(t)=2b_k+2<0$ для $t\in(0,-c_k)$, $y_k(t)=z(t+c_k)$ для $t\geqq -c_k$ и в частности, $y_k''(-c_k+0)=z''(+0)=2$. Поэтому, если положить $y_k''(0)=0$, то, с помощью (34), получаем

$$\bigvee_0^\infty y_k''=4\left(1+b_k+\frac{1}{1-\alpha^{1-3p}}\right).$$

Положим еще

$$\overline{N}_k=\frac{1}{y_k^{(2-k)}(0)}\bigvee_0^\infty y_k'',$$

$$e_k=\frac{1}{y_k^{(2-k)}(0)}\left\{\int_{c_k}^0 |\varphi(t)+b_kt^2|^{\frac{p}{p-1}}\,dt+\frac{1}{1-\alpha^{1-3p}}\int_0^1|\varphi(t)|^{\frac{p}{p-1}}\,dt\right\}^{\frac{p-1}{p}}.$$

Напомним, что функция φ определена формулой (27), функция R — формулой (28), а α — наибольший корень уравнения $R(\alpha)=0$, и докажем, наконец, такое предложение.

Теорема 6. *Если* $n=3$, $k=1, 2$, $1<p<\infty$, $I=[0, \infty)$, то $E_\infty(\bar{N}_k)=e_k$, *а операторы*

$$\bar{S}_k(x,f) = 2\frac{(-1)^{k-1}}{y_k^{(2-k)}(0)}\{(1+b_k)f(x)-b_kf(x-c_k)-$$

$$-(\alpha^{1-3p}+1)\sum_{i=0}^{\infty}(-1)^i\alpha^{i(1-3p)}f(x+a_i-c_k)\}$$

являются экстремальными, соответственно, при $k=1, 2$.

Доказательство. Так же как и раньше (теоремы 4, 5) убеждаемся, что

$$S_k(x,f) = \frac{(-1)^{k-1}}{y_k^{(2-k)}(0)}\int_0^\infty f(x+t)\,dy_k''(t),$$

$$\|\bar{S}_k\|=\bar{N}_k, \quad \bar{S}_k(0, u_k)=\bar{N}_k\|u_k\|_C.$$

Далее для функций $f\in L_p^3$ справедливы тождества

$$f^{(k)}(x)-\bar{S}_k(x,f) = \frac{(-1)^{k-1}}{y_k^{(2-k)}(0)}\int_0^\infty f'''(x+t)y_k(t)\,dt,$$

отсюда

$$\varrho(\bar{S}_k) = \frac{\|y_k\|_{L_{p'}(0,\infty)}}{y_k^{(2-k)}(0)} = \left(u_k^{(k)}(0)-\bar{S}_k(0, u_k)\right)\|u_k'''\|_{L_p}^{-1}.$$

Таким образом операторы $\bar{S}_k$ и функции u_k удовлетворяют равенству (11), а это доказывает теорему 6.

Замечание. В силу теоремы 1 экстремальные операторы, выписанные в теоремах 4, 5, 6 и теореме 3 при $1<p<\infty$, будут единственными в случае $I=(-\infty, \infty)$, а в случае $I=[0, \infty)$ единственными являются значения этих операторов при $x=0$.

Теорема 7. *При* $n=3$, $k=1, 2$, $1<p<\infty$ *неравенство*

$$\|f^{(k)}\|_C\leq \bar{D}\|f\|_C^\alpha\|f'''\|_{L_p}^\beta, \tag{66}$$

с константой $\bar{D}$, *определенной формулой* (10), *точное на классе* L_p^3. *Функции* u_0, u, u_1, u_2, *определенные равенствами* (62), (40), (65) *являются единственными экстремальными, соответственно, при* $k=1, 2$, $I=(-\infty, \infty)$ *и* $k=1, 2$, $I=[0, \infty)$.

Действительно, из доказательства теорем 4—6 видно, что каждая из перечисленных функций при соответствующем N удовлетворяет равенству

$$w^{(k)}(0) = N\|w\|_C + E_\infty(N)\|w'''\|_{L_p}.$$

В силу (9), из этого вытекает, что соответствующая функция обращает (66) в равенство. Единственность экстремальной функции содержится в теореме 1.

Рассмотрим случай $p=1$. Очевидно, из того, что $f\in L^n_{1,r}$ следует, что $f\in L^{n-1}_{V,r}$ и $\mathrm{V}f^{(n-1)}=\|f^{(n)}\|_L$.

Если $f\in L^{n-1}_{V,r}$, то $f^{(n-1)}(\pm\infty)=0$. Откуда

$$\|f^{(n-1)}\|_{L_\infty}=Y_I\mathrm{V}f^{(n-1)}, \tag{67}$$

где $Y_{[0,\infty)}=1$, $Y_{(-\infty,\infty)}=\frac{1}{2}$. В случае $I=[0,\infty)$ в (67) знак равенства достигается на любой функции g $(\in L^n_{1,r})$, у которой $|g^{(n-1)}|$ убывает. В случае $I=(-\infty,\infty)$ легко построить последовательность таких функций $g_\nu\in L^n_{1,r}$, что производные $g^{(n)}_\nu$ нечетные, $g^{(n-1)}_\nu(t)=1$ для $t\in(0,1)$ и $\|g^{(n)}_\nu\|_{L(1,\infty)}\to 0$ при $\nu\to\infty$. Следовательно неравенство (67) точное на классе $L^n_{1,r}$.

При $k=n-1$, $p=1$, $q=\infty$ для оператора $S\equiv 0$ имеем $E(N)\leqq\varrho(S)=Y_I$ и поскольку в этом случае константа $D=Y_I$ в неравенстве (1) наилучшая, то, в силу (8), $E(N)=Y_I$ и оператор $S\equiv 0$ является экстремальным. Он, вообще говоря, не единственный, поскольку, согласно теореме 3, операторы (21), (22) являются экстремальными при $n=2$ $(k=1)$, $p=1$, $q=r=\infty$.

Таким образом если $k=n-1$, $p=1$, $q=\infty$, то решения задач (1) и (5) тривиальные; в частности, это верно при $n=3$, $k=2$, $r=\infty$.

Нам осталось рассмотреть случай $n=3$, $k=1$, $p=1$.

Теорема 8. *Пусть* $n=3$, $k=1$, $p=1$, $h>0$. *Тогда*

$$E_\infty(h^{-1})=\frac{h}{8}, \quad \text{если} \quad I=(-\infty,\infty), \tag{68}$$

и

$$E_\infty(2h^{-1})=\frac{h}{2}, \quad \text{если} \quad I=[0,\infty),$$

операторы (21) *и* (22) *являются экстремальными, соответственно, при* $I=[0,\infty)$ *и* $I=(-\infty,\infty)$.

Функции $f\in L^2_V$ *удовлетворяют неравенствам*

$$\|f'\|_C\leqq Z_I\sqrt{\|f\|_C\mathrm{V}f''}, \tag{69}$$

где $Z_{[0,\infty]}=2$, $Z_{(-\infty,\infty)}=\frac{1}{\sqrt{2}}$. *В случае* $I=[0,\infty)$ *единственной экстремальной будет функция*

$$w(t)=\begin{cases}1-2(1-t)^2 & \text{для}\quad 0\leqq t\leqq 1,\\ 1 & \text{для}\quad t>1,\end{cases}$$

а в случае $I=(-\infty,\infty)$ *таковой будет лишь нечетная функция* w_1, *определенная при* $t\geqq 0$ *формулой* $w_1=1+w$.

Если $f\in L_1^3$, *то*

(70) $$\|f'\|_C<Z_I\sqrt{\|f\|_C\|f'''\|_L}$$

и константа Z_I *неуменьшаема.*

Доказательство. Рассмотрим вначале случай $I=(-\infty,\infty)$. Для любой функции $f\in L_V^2$ имеет место тождество (26), т. е.

$$f'(x)-S_h(x,f)=-\frac{1}{2h}\int_0^h(h-t)\{f''(x+t)-f''(x-t)\}\,dt.$$

Поскольку $f''(\pm\infty)=0$, то при $t\in(0,h)$

$$\left|f''(x+t)-\frac{1}{2}f''(x)\right|\leqq\frac{1}{2}\mathop{\mathrm{V}}_x^{\infty}f'',\quad \left|f''(x-t)-\frac{1}{2}f(x)\right|\leqq\frac{1}{2}\mathop{\mathrm{V}}_{-\infty}^{x}f''.$$

Следовательно

(71) $$f'(x)-S_h(x,f)\leqq\frac{h}{8}\mathrm{V}f''.$$

Отсюда приходим к неравенству

$$\|f'\|_C\leqq h^{-1}\|f\|_C+\frac{h}{8}\mathrm{V}f''.$$

Минимизируя его правую часть по $h>0$, получаем (69). Легко проверить, что на функции w_1 неравенство (69) обращается в равенство.

Функции класса L_1^3 также удовлетворяют (69). Причем нетрудно построить последовательность функций $f_\nu\in L_1^3$ со свойствами:

$$\|f_\nu-w_1\|\to 0,\quad f_\nu'(0)=w_1'(0),\quad \|f_\nu'''\|_L=\mathrm{V}w_1''.$$

Это будет означать, что (69) точное и на классе L_1^3.

Если мы установим, что функция w_1 единственная экстремальная в (69), то этим будет в частности доказано, что неравенство (70) строгое.

Пусть v — произвольная экстремальная функция неравенства (69). Выберем числа c, ν, μ так, чтобы функция $g(t) = cv(\nu t+\mu)$ удовлетворяла условиям

$$\mathrm{V}g''=1, \quad \|g\|_C=\frac{h^2}{8}, \quad g'(0)=\|g'\|_C.$$

Так как функция g экстремальная, то $g'(0)=\frac{h}{4}$. В силу (71), имеем

$$\frac{h}{4} = g'(0) \leqq S_h(0, g)+\frac{h}{8} \leqq h^{-1}\|g\|_C+\frac{h}{8} = \frac{h}{4}.$$

Следовательно

$$g'(0)-S_h(0, g) = \frac{h}{8}\mathrm{V}g'', \quad S_h(0, g) = \|S_h\|\,\|g\|_C.$$

Из приведенных оценок следует, что первое равенство имеет место лишь в том случае, если при $t\in(0, h)$

$$g''(t) = \frac{1}{2}\left(g''(0)-\mathop{\mathrm{V}}_{\infty}^{0} g''\right) = j_1 \leqq 0,$$

$$g''(-t) = \frac{1}{2}\left(g''(0)+\mathop{\mathrm{V}}_{0}^{\infty} g''\right) = j_2 \geqq 0,$$

и, кроме того, при $t\in(0, \infty)$ функции $g''(t)$, $-g''(-t)$ возрастают (до нуля). А поскольку $g'(\pm\infty) = 0$, то $g'\geqq 0$ и функция g возрастает.

Аналогично, условие $S_h(0, g)=\|S_h\|\,\|g\|_C$ можно записать в виде равенств $g(h) = -g(-h) = \|g\|_C$. Следовательно $g(t) = -g(-t) = g(h)$ для $t\geqq h$.

Так как $g\in C$ и $Vg''=1$, то $j_2 = -j_1 = \frac{1}{4}$. Таким образом $g(t) = \frac{h^2}{16}w_1(th^{-1})$.

Из неравенства (71) следует, что

$$E_\infty(h^{-1}) \leqq \varrho(S_h) \leqq \frac{h}{8},$$

а, в силу (8) и (70), $E_\infty(h^{-1}) \geqq \frac{h}{8}$, следовательно (68) имеет место.

Все утверждения теоремы для случая $I = (-\infty, \infty)$ доказаны.

Если $I=[0, \infty)$, то для $f\in L_V^2$ (или, тем более $f\in L_1^3$) имеем

$$f'(x)-S_h^+(x,f) = -\frac{1}{h}\int_0^h (h-t)f''(x+t)\,dt \leqq \frac{h}{2}\mathop{\mathrm{V}}_{0}^{\infty} f''.$$

Остальные рассуждения проводятся так же как в случае $I = (-\infty, \infty)$.

Теорема 8 полностью доказана.

В случае $n=3, p=\infty$ задача (6) решена Стечкиным [10, 11]. Здесь экстремальные операторы — конечно разностные: для $k=1, 2$, соответственно,

$$T_1(x,f) = \frac{1}{2h}\{f(x+h)-f(x-h)\},$$

$$T_2(x,f) = \frac{1}{h^2}\{f(x+h)-2f(x)+f(x-h)\}$$

при $I=(-\infty, \infty)$,

$$T_1(x,f) = \frac{1}{6h}\{-8f(x)+9f(x+h)-f(x+3h)\},$$

$$T_2(x,f) = \frac{1}{3h^2}\{2f(x)-3f(x+h)+f(x+3h)\}$$

при $I=[0, \infty)$.

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Stability and law of large numbers for sums of a random number of random variables

By D. SZÁSZ in Budapest*)

In [1] BIKELIS and MOGYORÓDI gave sufficient conditions for the stability and the law of large numbers in case of sums of a random number of random variables. They dealt with the case when the random indices have finite mathematical expectations and asserted that the given sufficient conditions are also necessary. However, we shall give a counterexample which shows that the mentioned conditions are not necessary. Moreover, without the condition of the existence of expectations, we can give necessary and sufficient conditions both for the stability and the law of large numbers for sums of a random number of random variables. These conditions follow from the results of the author on convergence of distributions of sums of a random number of random variables [2].

1. The results. For every n, let $\xi_{n1}, \dots, \xi_{nk}, \dots$ be a sequence of independent random variables and ν_n be a random index which is independent of the sequence $\{\xi_{nk}\}_k$. Suppose that

$$\lim_{n\to\infty} \sup_k P(|\xi_{nk}| > \varepsilon) = 0$$

for every $\varepsilon > 0$ and that

$$\text{P-}\lim_{n\to\infty} \nu_n = \infty. \text{ [1])}$$

Let

$$S_k^{(n)} = \xi_{n1} + \cdots \xi_{nk}.$$

Definitions. We say that the sequence of sums $S_{\nu_n}^{(n)}$ is *stable* if there exists a double sequence of constants $A_k^{(n)}$ $(n, k = 1, 2, \dots)$ such that

$$\text{P-}\lim_{n\to\infty} (S_{\nu_n}^{(n)} - A_{\nu_n}^{(n)}) = 0.$$

We say that the sequence of sums $S_{\nu_n}^{(n)}$ satisfies the *law of large numbers* if there exists a sequence of constants C_n such that

$$\text{P-}\lim_{n\to\infty} (S_{\nu_n}^{(n)} - C_n) = 0.$$

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*) This work was done at the Department of Probability, Lomonosov University, Moscow.

[1]) $\text{P-}\lim_{n\to\infty}$ denotes stochastic convergence.

In [1] it is supposed that $E\nu_n<\infty$ for every n and is proved that the sequence $\{S_{\nu_n}^{(n)}\}_n$ is stable, if

(i)$^+$
$$\lim_{n\to\infty} E\sum_{k=1}^{\nu_n}\int_{|x|>1} dF_{nk}(x+m_{nk}) = 0,$$

(ii)$^{++}$
$$\lim_{n\to\infty} E\sum_{k=1}^{\nu_n}\int_{|x|\le 1} x^2 dF_{nk}(x+m_{nk}) = 0,$$

where $F_{nk}(x)=P(\xi_{nk}<x)$ and m_{nk} denotes the median of ξ_{nk}.

If in addition to (i)$^+$ and (i)$^{++}$ there exists a sequence of constants C_n such that

(iii)
$$\text{P-}\lim_{n\to\infty}\left(\sum_{k=1}^{\nu_n}\left\{\int_{|x|\le 1} x\,dF_{nk}(x+m_{nk})+m_{nk}\right\}-C_n\right) = 0,$$

then it is proved that the sequence $\{S_{\nu_n}^{(n)}\}_n$ satisfies the law of large numbers.

Our assertion that the above conditions are not necessary is based on the following example: for every $n>4$ let

$$P(\nu_n = 2^k) = \frac{1}{n} \qquad (1\le k\le n).$$

Clearly $E\nu_n<\infty$ for every n. Further let

$$\xi_{nk}=0$$

if $1\le k\le 2^{n-1}$ or $k>2^n$ and

$$P(\xi_{nk}=0) = 1-\frac{2}{n},$$

$$P(\xi_{nk}=1) = P(\xi_{nk}=a_n) = \frac{1}{n},$$

if $2^{n-1}<k\le 2^n$ where $|a_n|>1$.

It is obvious that

$$\text{P-}\lim_{n\to\infty} S_{\nu_n}^{(n)}=0,$$

but

$$E\sum_{k=1}^{\nu_n}\int_{|x|>1} dF_{nk}(x+m_{nk}) = E\sum_{k=1}^{\nu_n}\int_{|x|\le 1} x^2 dF_{nk}(x+m_{nk}) = \frac{2^n-2^{n-1}}{n^2} = \frac{2^{n-1}}{n^2}\to\infty$$

as $n\to\infty$. So the sequence $\{S_{\nu_n}^{(n)}\}_n$ satisfies the law of large numbers (and is stable), but not conditions (i)$^+$ and (i)$^{++}$.

Our results do not suppose the finiteness of expectations. We prove:

Theorem 1. *The sequence* $\{S_{\nu_n}^{(n)}\}_n$ *is stable iff*

(i)
$$\text{P-}\lim_{n\to\infty} \sum_{k=1}^{\nu_n} \int_{|x|>1} dF_{nk}(x+m_{nk}) = 0$$

and

(ii)
$$\text{P-}\lim_{n\to\infty} \sum_{k=1}^{\nu_n} \int_{|x|\le 1} x^2\, dF_{nk}(x+m_{nk}) = 0.$$

Theorem 2. *The sequence* $\{S_{\nu_n}^{(n)}\}_n$ *satisfies the law of large numbers iff conditions* (i), (ii), *and for a suitable sequence of constants* C_n *condition* (iii) *are satisfied.*

We remark that conditions (i) and (ii) are respectively equivalent to the following conditions: for every $q\in[0, 1)$,

(1.1)
$$\lim_{n\to\infty} \sum_{k\le l_n(q)} \int_{|x|>1} dF_{nk}(x+m_{nk}) = 0$$

and

(1.2)
$$\lim_{n\to\infty} \sum_{k\le l_n(q)} \int_{|x|\le 1} x^2\, dF_{nk}(x+m_{nk}) = 0.$$

Here $l_n(q)$ denotes the lower bound of q-quantiles of the random variable ν_n.

We remark that an analogous result for the stability of sums $S_{\nu_n}^{(n)}$ was recently reached in a different way by A. V. PETCHINKIN (to be published later).

2. Preliminaries to the proofs. In [2] we have proved

Proposition 1 ([2], Corollary to Theorem 1). *Suppose that for almost every* $q\in[0, 1]$ *there exists a distribution function* $\Phi_q(x)$ *such that*

$$\lim_{n\to\infty} P(S_{l_n(q)}^{(n)} < x) = \Phi_q(x).$$

Then there exists a measurable stochastic process $\{\chi(t): t\in[0, 1)\}$ *with independent increments such that for almost every* t

$$P(\chi(t)<x) = \Phi_t(x)$$

and

$$\lim_{n\to\infty} P(S_{\nu_n}^{(n)} < x) = P(\chi(\pi)<x).$$

Here the random variable π *is uniformly distributed in* [0, 1] *and is independent of the process* $\{\chi(t): t\in[0, 1)\}$.

For an arbitrary random variable X there exists a unique real number $\Delta(X)$ such that

$$E \operatorname{arc\,tg}(X-\Delta(X)) = 0.$$

This number $\Delta(X)$ is called Doob's center or briefly center. A stochastic process $\{\chi(t):t\in T\}$ is centered, if $\Delta(\chi(t))=0$ for every $t\in T$. [3].

Proposition 2 ([2], Corollary 1 of Theorem 2). *Suppose that the random variables ξ_{nk} are symmetric for every n and k, and the sums $S_{\nu_n}^{(n)}$ have a limit distribution as $n\to\infty$. Then there exist a decomposition of indices to a finite or infinite number sequences: $N_1,\dots,N_j$, and centered, measurable stochastic processes $\{\chi^{(i)}(t):t\in[0,1)\}$ with independent increments $(1\leqq i\leqq j)$ such that for almost every t*

(2.1) $$\lim_{\substack{n\to\infty\\ n\in N_i}} P(S_{l_n(t)}^{(n)}<x)=P(\chi^{(i)}(t)<x)$$

and also

(2.2) $$\lim_{\substack{n\to\infty\\ n\in N_i}} P(S_{\nu_n}^{(n)}<x)=P(\chi^{(i)}(\pi)<x) \qquad (1\leqq i\leqq j).$$

Lemma ([2], Lemma 4). *If the distributions of the random variables X_n weakly converge to the distribution of the random variable X as $n\to\infty$, then*

$$\lim_{n\to\infty} \Delta(X_n)=\Delta(X).$$

2. The proofs. Let for every n $\{\eta_{nk}\}_k$ be a sequence of independent random variables such that it is independent of the sequence $\{\xi_{nk}\}_k$ and of the random variable ν_n and that

$$P(\xi_{nk}<x)=P(\eta_{nk}<x) \qquad (n,k=1,2\,\dots).$$

Let

$$\zeta_{nk}=\xi_{nk}-\eta_{nk},\quad Z_k^{(n)}=\zeta_{n1}+\dots+\zeta_{nk}.$$

Proof of Theorem 1. *Necessity.* If the assertion is not true then there exist some $q_0<1$ and a subsequence $\{n'\}$ of indices such that either

(3.1) $$\liminf_{n'\to\infty}\sum_{k\leqq l_{n'}(q_0)}\int_{|x|>1} dF_{n'k}(x+m_{n'k})>0$$

or

(3.2) $$\liminf_{n'\to\infty}\sum_{k\leqq l_{n'}(q_0)}\int_{|x|\leqq 1} x^2\,dF_{n'k}(x+m_{n'k})>0.$$

We can suppose and for the sake of simplicity we do suppose that the subsequence $\{n'\}$ coincides with the entire sequence $\{n\}$ of positive integers. Thus instead of n' we can always write n.

We remark that from the stability of sequence $\{S_{\nu_n}^{(n)}\}_n$ it follows that

(3.3) $$\operatorname{P-lim}_{n\to\infty} Z_{\nu_n}^{(n)}=0.$$

Let us use Proposition 2 for the sums $Z_{\nu_n}^{(n)} = \zeta_{n1}+\dots+\zeta_{n\nu_n}$. In consequence of (3. 3) and (2. 2), every process $\{\chi^{(i)}(t): t\in[0, 1)\}$ appearing in Proposition 2 vanishes, i.e. for all i and t $\left(1\leqq i\leqq j,\ t\in[0, 1)\right)$

$$P\left(\chi^{(i)}(t)=0\right)=1.$$

Because of (2. 1), for a suitable q_1 $(q_0\leqq q_1<1)$ we have

$$\operatorname*{P-lim}_{\substack{n\to\infty\\ n\in N_1}} Z_{l_n(q_1)}^{(n)}=0.$$

Here we can use the following simple remark: if for every n X_n and Y_n are independent random variables and

$$\operatorname*{P-lim}_{n\to\infty} (X_n+Y_n) = 0,$$

then there exists a sequence of constants a_n such that

$$\operatorname*{P-lim}_{n\to\infty} (X_n-a_n)=0.$$

On the basis of this remark, there exist constants a_n $(n\in N_1)$ such that

$$\operatorname*{P-lim}_{\substack{n\to\infty\\ n\in N_1}} (S_{l_n(q_1)}^{(n)}-a_n) = 0.$$

On the basis of the theorem of § 22 [4], we get that

$$\lim_{\substack{n\to\infty\\ n\in N_1}} \sum_{k\leqq l_n(q_1)} \int_{|x|>1} dF_{nk}(x+m_{nk}) = 0,$$

$$\lim_{\substack{n\to\infty\\ n\in N_1}} \sum_{k\leqq l_n(q_1)} \int_{|x|\leqq 1} x^2 dF_{nk}(x+m_{nk}) = 0.$$

These relations contradict both (3. 1) and (3. 2).

Sufficiency. From conditions (1. 1) and (1. 2) it follows that for every $q\in[0, 1)$ there exist sequences $\{m_n(q)\}_n$ of constants such that

$$\operatorname*{P-lim}_{n\to\infty} \left(S_{l_n(q)}^{(n)}-m_n(q)\right) = 0. \tag{3. 4}$$

Our lemma asserts that $m_n(q)$ can be chosen as

$$m_n(q)=\Delta(S_{l_n(q)}^{(n)}).$$

So if

$$A_k^{(n)}=m_n\left(A_n(k)\right),$$

where $A_n(x)=P(\nu_n<x)$, then Proposition 1 and (3. 4) give that

$$\text{P-}\lim_{n\to\infty} (S_{\nu_n}^{(n)} - A_{\nu_n}^{(n)}) = 0.$$

We remark that here Proposition 1 is applied instead of the summands ξ_{nk} to the summands $\xi_{nk}-(A_k^{(n)}-A_{k-1}^{(n)})$.

Proof of Theorem 2. Necessity of conditions (i) and (ii) follows from the preceding theorem. If (i) and (ii) are satisfied, then in (3. 4) $m_n(q)$ can be chosen also in the following manner

$$m_n(q) = \sum_{k\leqq l_n(q)} \left\{ \int_{|x|\leqq 1} x dF_{nk}(x+m_{nk}) + m_{nk} \right\},$$

and so in consequence of Proposition 1,

$$\text{P-}\lim_{n\to\infty} \left(S_{\nu_n}^{(n)} - \sum_{k=1}^{\nu_n} \left\{ \int_{|x|\leqq 1} x dF_{nk}(x+m_{nk}) + m_{nk} \right\} \right) = 0. \tag{3. 5}$$

However, at the same time

$$\text{P-}\lim_{n\to\infty} (S_{\nu_n}^{(n)} - C_n) = 0 \tag{3. 6}$$

and from (3. 5) and (3. 6) condition (iii) follows obviously.

To prove the sufficiency we remark that (3. 6) and (iii) together give (3. 6).

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Independence of the conditions of associativity in ternary operations

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1. Let S be a set of elements $\{a, b, c, \ldots\}$. A ternary operation f is a mapping of $S\times S\times S$ to S. The operation f is said to be associative, if for every $a, b, c, d, e \in S$ we have

$$(*) \qquad [a, b, (c, d, e)_f]_f = [a, (b, c, d)_f, e]_f = [(a, b, c)_f, d, e]_f,$$

where $(x, y, z)_f$ denotes the image by f of the ordered triplet (x, y, z) in S. The equalities in (1. 1) are called *associativity conditions* for the elements a, b, c, d, e.

Associativity conditions in a set are said to be independent if any of them is not implied by the rest [1]. Thus, for ternary operations in S, the associativity conditions are independent if for every sequence $\{a, b, c, d, e\}$ of five elements in S it is possible to define a ternary operation f in S in such a way that the equalities $(*)$ hold for every sequence of elements $\{p, q, r, s, t\}$ of S different from $\{a, b, c, d, e\}$ whereas for this latter sequence $(*)$ does not hold.

2. G. SZÁSZ [2] has investigated the independence of associativity conditions for binary operations and has established the following theorem:

Theorem. *If the number of elements in a set S is greater than three, then the associativity conditions in S are independent.*

In what follows we study the same problem for ternary operations and prove:

Theorem. *If the number of elements in a set S is greater than five, then the associativity conditions for ternary operations in S are independent.*

3. Proof. For simplicity we drop the mapping letter f for the ternary operation.

Let $\{a, b, c, d, e\}$ be an arbitrary sequence of five elements of S. We shall prove the theorem by defining ternary operations over S such that the associativity conditions hold in all other cases except for this sequence. In what follows we consider separately the various alternatives for the elements of the sequence $\{a, b, c, d, e\}$ to prove the theorem.

3. 1. Let $a=b=c=d=e$. Since S contains more than five elements, we can choose three more distinct elements u, v, w different from a, and define the following operation in S:

$$(a, a, a)=u, \quad (a, a, u)=v, \quad \text{and} \quad (x, y, z)=w \quad \text{in all other cases.}$$

Clearly, the associativity conditions do not hold for the given sequence of elements, as

$$[(a, a, a), a, a]=(u, a, a)=w, \quad [a, (a, a, a), a]=(a, u, a)=w,$$

$$\text{but} \quad [a, a, (a, a, a)]=(a, a, u)=v.$$

We show that for every sequence $\{p, q, r, s, t\}$ different from $\{a, a, a, a, a\}$, the associativity conditions do hold.

Since (p, q, r) and (q, r, s) are never a, we have $[(p, q, r), s, t]=w$ and $[p, (q, r, s), t]=w$.

Now, if $(r, s, t)\neq u$, then as $(r, s, t)\neq a$, $[p, q, (r, s, t)]=w$. And if $(r, s, t)=u$ but $p\neq a$, even then $(p, q, u)=w$. And even if $(r, s, t)=u$, $p=a$, then $(a, q, u)=w$, except when $q=a$, in which case $\{p, q, r, s, t\}$ is the same as $\{a, a, a, a, a\}$, the given sequence. This proves the contention.

3. 2. Let $a\neq b$ and $b=c=d=e$ (all but one element being the same).

Since S contains more than five elements, we can choose an element w different from a and b, and define the following operation in S:

$$(a, b, b)=a \quad \text{and} \quad (x, y, z)=w \quad \text{in all other cases.}$$

We have for the given sequence $\{a, b, b, b, b\}$:

$$[a, b, (b, b, b)]=w \quad \text{and} \quad [a, (b, b, b), b]=w, \quad \text{but} \quad [(a, b, b), b, b]=(a, b, b)=a,$$

so that the associativity conditions do not hold in this case.

For any sequence $\{p, q, r, s, t\}$ different from $\{a, b, b, b, b\}$ we show that the associativity conditions do hold.

Since $(q, r, s)\neq b$ and $(r, s, t)\neq b$,

$$[p(q, r, s), t]=w \quad \text{and} \quad [p, q, (r, s, t)]=w.$$

Now, if $(p, q, r)\neq a$, then $[(p, q, r), s, t]=w$. And if $(p, q, r)=a$, but $s\neq b$, then $[(p, q, r), s, t]=(a, s, t)=w$. And even if $(p, q, r)=a$, $s=b$, then $[(p, q, r), s, t]= =(a, b, t)=w$ except when $t=b$, in which case $\{p, q, r, s, t\}$ is the same as $\{a, b, b, b, b\}$, the given sequence.

3. 2.1. For another permutation of the same sequence of 3. 2 we prove the theorem as follows:

When the sequence is $\{b, a, b, b, b\}$, we can choose three more distinct elements u, v and w different from a and b (as S contains more than five elements) and then define the following operation in S:

$$(b, b, b)=u, \ (b, a, u)=v, \ \text{and} \ (x, y, z)=w \ \text{in all other cases.}$$

As $[(b, a, b), b, b]=w$, $[b, (a, b, b), b]=w$, and $[b, a, (b, b, b)]=(b, a, u)=v$, we see that the associativity conditions do not hold for the given sequence $\{b, a, b, b, b\}$.

For any sequence $\{p, q, r, s, t\}$ different from $\{b, a, b, b, b\}$ we show that the conditions do hold.

Since (p, q, r) and (q, r, s) can never be equal to a or b,

$$[(p, q, r), s, t]=w \ \text{and} \ [p, (q, r, s), t]=w.$$

Now, if $(r, s, t)\neq u$, then $[p, q, (r, s, t)]=w$, as $(r, s, t)\neq b$. And, if $(r, s, t)=u$, but $p\neq b$, then $[p, q, (r, s, t)]=w$. And even if $(r, s, t)=u$, $p=b$, $[p, q, (r, s, t)] = (b, q, u) = w$, except when $q=a$, in which case $\{p, q, r, s, t\}$ is the same as $\{b, a, b, b, b\}$, the given sequence.

By defining a similar operation in S, we can demonstrate the truth of the theorem in the same way for every other sequence of five elements in which four elements are equal but the fifth is different*).

3. 3. Let $a=b$, $c=d=e$ and $a\neq c$.

Since S contains more than five elements we can choose three more distinct elements u, v, and w different from a and c and define the operation as follows:

$$(c, c, c)=u, \ (a, a, u)=v, \ \text{and} \ (x, y, z)=w \ \text{in all other cases.}$$

We have for the given sequence

$$[(a, a, c), c, c]=w \ \text{and} \ [a, (a, c, c), c]=w, \ \text{but} \ [a, a, (c, c, c)]=(a, a, u)=v,$$

so that the associativity conditions do not hold, whereas we show that for any sequence $\{p, q, r, s, t\}$ different from $\{a, a, c, c, c\}$ the conditions do hold.

Since (p, q, r) and (q, r, s) can never be equal to a or c, we get

$$[(p, q, r), s, t]=w \ \text{and} \ [p, (q, r, s), t]=w.$$

Now if $(r, s, t)\neq u$, then $[p, q, (r, s, t)]=w$ as $(r, s, t)\neq c$. Also, if $(r, s, t)=u$, but $p\neq a$, $[p, q, (r, s, t)]=(p, q, u)=w$, and even if $(r, s, t)=u$ and $p=a$, $[p, q, (r, s, t)]=$

*) The authors have investigated each of these permutations also, but for brevity all these have not been incorporated here.

$=(a, q, u)=w$ except when $q=a$, in which case $\{p, q, r, s, t\}$ is the same as the given sequence $\{a, a, c, c, c\}$.

We can similarly prove the statement of the theorem for other arrangements of the elements in the sequence $\{a, a, c, c, c\}$ such as $\{a, c, a, c, c\}$, $\{a, c, c, a, c\}$ etc. by defining similar operations.

3. 4. Let $a=b$, $c=d$ and let e be different from a and c.

Since, under the hypothesis, S has more than five elements, we can choose three more distinct elements u, v, and w different from a, c, and e and define the following operation:

$$(c, c, e)=u, \quad (a, a, u)=v, \quad \text{and} \quad (x, y, z)=w \quad \text{in all other cases.}$$

We have for the given sequence

$$[(a, a, c), c, e]=w, \quad [a, (a, c, c,), e]=w, \quad \text{but} \quad [a, a, (c, c, e)]=(a, a, u)=v,$$

so that the associativity conditions do not hold, whereas we show that for any sequence $\{p, q, r, s, t\}$ different from $\{a, a, c, c, e\}$ the conditions do hold.

Since (p, q, r) and (q, r, s) can never be equal to a or c, we have $[(p, q, r), s, t]=w$ and $[p, (q, r, s), t]=w$. Now if $(r, s, t)\neq u$, then $[p, q, (r, s, t)]=w$ as $(r, s, t)\neq e$. Also, if $(r, s, t)=u$, but $p\neq a$, then $[p, q, (r, s, t)]=w$. And even if $(r, s, t)=u$, $p=a$, then $[p, q, (r, s, t)]=w$ except when $q=a$, in which case $\{p, q, r, s, t\}$ is the same as the given sequence $\{a, a, c, c, e\}$.

We can have a similar proof for other arrangements of the elements in the sequence $\{a, a, c, c, e\}$.

3. 5. Let $a=b=c$, $d\neq e$, and let d, e be different from a.

Since S contains more than five elements, we can choose three distinct elements u, v, w different from a, d, e and then define the following operation:

$$(a, d, e)=u, \; (a, a, u)=v, \quad \text{and} \quad (x, y, z)=w \quad \text{in all other cases.}$$

Here we see that the associativity conditions do not hold for the given sequence. For,

$$[(a, a, a), d, e]=w, \; [a, (a, a, d), e]=w, \quad \text{but} \quad [a, a, (a, d, e)]=(a, a, u)=v.$$

For any sequence $\{p, q, r, s, t\}$ different from $\{a, a, a, d, e\}$, we show that the conditions do hold.

Since (p, q, r) can never be equal to a and (q, r, s) can never be equal to a or d, we have

$$[(p, q, r)s, t]=w \quad \text{and} \quad [p, (q, r, s), t]=w.$$

Now if $(r, s, t)\neq u$, then $[p, q, (r, s, t)]=w$ as $(r, s, t)\neq e$. Also if $(r, s, t)=u$, but

$p \neq a$, then $[p, q, (r, s, t)] = w$. And even if $(r, s, t) = u$, $p = a$, then $[p, q, (r, s, t)] = (a, q, u) = w$, except when $q = a$, in which case $\{p, q, r, s, t\}$ is the same as the given sequence $\{a, a, a, d, e\}$.

We can have a similar proof for other arrangements of the elements in the sequence $\{a, a, a, d, e\}$.

3. 6. Let $a = c$, and let b, d, e be all distinct and different from a.

Since S contains more than five elements we can choose two distinct elements u and w in S, different from a, b, d, and e, and define the operation as follows:

$$(a, b, a) = a, \quad (b, a, b) = b, \quad (a, d, e) = u,$$

and

$$(x, y, z) = w \quad \text{in all other cases.}$$

We see that the associativity conditions do not hold for the given sequence, for

$$[a, b, (a, d, e)] = (a, b, u) = w, \quad [a, (b, a, d), e] = (a, w, e) = w,$$

but

$$[(a, b, a), d, e] = (a, d, e) = u.$$

For any sequence $\{p, q, r, s, t\}$ different from $\{a, b, a, d, e\}$ and containing an element different from a, b, d, e, the conditions do hold, as in all such cases

$$[(p, q, r), s, t] = w, \quad [p, (q, r, s), t] = w, \quad [p, q, (r, s, t)] = w.$$

Hence taking p, q, r, s, t all from a, b, d, e only, we show that the associativity conditions do hold in each case. We distinguish five cases, denoted by the letters (A)—(E).

(A) When $p \neq a$, we have $[(p, q, r), s, t] = (x, s, t)$, say, where $x = b$ or $x = w$.

(i) Let $x = b$ i.e. $(p, q, r) = (b, a, b)$. Then $[(b, a, b), s, t] = (b, s, t) = b$ when $s = a$, $t = b$ in which case

$$[b, (a, b, a), b] = (b, a, b) = b \quad \text{and} \quad [b, a, (b, a, b)] = (b, a, b) = b.$$

And when $s \neq a$ or $t \neq b$,

$$[(b, a, b)s, t] = [b, (a, b, s), t] = [b, a, (b, s, t)] = w.$$

(ii) Let $x = w$. Then

$$[(p, q, r), s, t] = (w, s, t) = w;$$

and

$$[p, (q, r, s), t] = (p, a, t) \quad \text{or} \quad (p, b, t), \quad \text{or} \quad (p, u, t), \quad \text{or} \quad (p, w, t).$$

If $(q, r, s)=a$ i.e. $(q, r, s)=(a, b, a)$, p cannot be equal to b, for otherwise $(p, q, r)=b$. Then

$$(p, a, t)=(p, b, t)=(p, u, t)=(p, w, t)=w$$

and $[p, q, (r, s, t)]=(p, q, a)$ or (p, q, b) or (p, q, u) or (p, q, w).

Now, if $(r, s, t)=b$, p and q can never have values b and a, respectively, for $(p, q, r)==w$. Therefore

$$(p, q, a)=(p, q, b)=(p, q, u)=(p, q, w)=w,$$

Hence

$$[(p, q, r), s, t]=[p, (q, r, s), t]=[p, q, (r, s, t)]=w.$$

(B) When $p=a$ but $q'\neq b$, we have

$$[(a, q, r), s, t]=[a, (q, r, s), t]=[a, q, (r, s, t)]=w,$$

for $(a, q, r)\neq a$ or b, $(q, r, s)\neq b$ or d, $(r, s, t)\neq e$.

(C) When $p=a$, $q=b$ but $r\neq a$, we have

$$[(a, b, r), s, t]=[a, (b, r, s), t]=[a, b, (r, s, t)]=w.$$

(D) When $p=a$, $q=b$, $r=a$ but $s\neq d$, we have

$$[(p, q, r), s, t]=[(a, b, a), s, t]=(a, s, t)=a \quad (\text{when } s=b,\ t=a)$$

in which case

$$[a, (b, a, b), a]=(a, b, a)=a \text{ and } [a, b, (a, b, a)]=(a, b, a)=a, \text{ and (when } s\neq b)$$

in all other cases

$$[(a, b, a), s, t]=[a, (b, a, s), t]=[a, b, (a, s, t)]=w.$$

(E) When $p=a$, $q=b$, $r=a$, $s=d$ but $t\neq e$, we have

$$[(a, b, a), d, t]=[a, (b, a, d), t]=[a, b, (a, d, t])=w.$$

This proves the contention.

3. 7. Let a, b, c, d, e be all different from one another.

As S contains more than five elements, we can choose an element w different from the given five elements and then define the operation in S as follows:

$$(c, d, c)=c, \quad (a, b, e)=a, \quad (d, e, b)=d,$$
$$(d, c, d)=d, \quad (b, e, b)=b, \quad (e, b, c)=c,$$
$$(c, d, e)=e, \quad (e, b, e)=e, \quad (b, c, d)=b, \text{ and}$$
$$(x, y, z)=w \text{ in all other cases.}$$

For the given sequence

$$[a, b, (c, d, e)]=(a, b, e)=a, \quad [a, (b, c, d), e]=(a, b, e)=a,$$

but

$$[(a, b, c), d, e]=(w, d, e)=w$$

so that the associativity conditions do not hold. For any sequence $\{p, q, r, s, t\}$ different from $\{a, b, c, d, e\}$ and containing an element different from a, b, c, d and e, we have

$$[(p, q, r), s, t]=[p, (q, r, s), t]=[p, q, (r, s, t)]=w.$$

Hence taking p, q, r, s, t all from a, b, c, d, e only, we demonstrate that the associativity conditions do hold in each case as follows:

(A) When $p \neq a$, $[(p, q, r), s, t]=(x, s, t)$, say, then $x \neq a$.

(i) Let $x=b$ i.e. $(p, q, r)=(b, e, b)$ or $(p, q, r)=(b, c, d)$. Then in the first case

$$[(p, q, r), s, t]=[(b, e, b), s, t]=(b, s, t)=b \text{ (when } s=e,\ t=b \text{ or } s=c,\ t=d),$$

and hence

$$[b, (e, b, e), b]=(b, e, b)=b \quad\text{and}\quad [b, e, (b, e, b)]=(b, e, b)=b,$$

$$[b, (e, b, c), d]=(b, c, d)=b \quad\text{and}\quad [b, e, (b, c, d)]=(b, e, b)=b,$$

while in the second case

$$[(p, q, r), s, t]=[(b, c, d), s, t]=(b, s, t)=b \text{ (when } s=e,\ t=b, \text{ or } s=c,\ t=d)$$

and hence

$$[b, (c, d, e), b]=(b, e, b)=b \quad\text{and}\quad [b, c, (d, e, b)]=(b, c, d)=b,$$

$$[b, (c, d, c), d]=(b, c, d)=b \quad\text{and}\quad [b, c, (d, c, d)]=(b, c, d)=b,$$

and $[(p, q, r), s, t]=[p, (q, r, s), t]=[p, q, (r, s, t)]=w$ in other cases.

(ii) Let $x=c$, i.e. $(p, q, r)=(c, d, c)$ or $(p, q, r)=(e, b, c)$.

Then

$$[(p, q, r), s, t]=[(c, d, c), s, t]=(c, s, t)=\begin{cases} c, & \text{if } s=d,\ t=c, \\ e, & \text{if } s=d,\ t=e; \end{cases}$$

in these cases

$$[c, (d, c, d), c]=(c, d, c)=c \quad \text{and} \quad [c, d, (c, d, c)]=(c, d, c)=c, \quad \text{and}$$

$$[c, (d, c, d), e]=(c, d, e)=e \quad \text{and} \quad [c, d, (c, d, e)]=(c, d, e)=e,$$

and $[(p, q, r), s, t]=[(e, b, c), s, t]=(c, s, t)=c$ (when $s=d$ and $t=c$)

or $=e$ (when $s=d$ and $t=e$), in which case

$$[e, (b, c, d), c]=(e, b, c)=c \quad \text{and} \quad [e, b, (c, d, c)]=(e, b, c)=c,$$

$$[e, (b, c, d), e]=(e, b, e)=e \quad \text{and} \quad [e, b, (c, d, e)]=(e, b, e)=e,$$

and $[(p, q, r), s, t]=[p, (q, r, s), t]=[p, q, (r, s, t)]=w$ in other cases.

(iii) Let $x=d$, i.e. $(p, q, r)=(d, c, d)$ or $(p, q, r)=(d, e, b)$. Then $[(p, q, r), s, t]= =(d, s, t)=d$ (when $s=c$, $t=d$, or, when $s=e$, $t=b$),

in which case

$$[d, (c, d, c), d]=(d, c, d)=d \quad \text{and} \quad [d, c, (d, c, d)]=(d, c, d)=d,$$

$$[d, (c, d, e), b]=(d, e, b)=d \quad \text{and} \quad [d, c, (d, e, b)]=(d, c, d)=d,$$

$$[d, (e, b, c), d]=(d, c, d)=d \quad \text{and} \quad [d, e, (b, c, d)]=(d, e, b)=d,$$

$$[d, (e, b, e), b]=(d, e, b)=d \quad \text{and} \quad [d, e, (b, e, b)]=(d, e, b)=d,$$

and $[(p, q, r), s, t]=[p, (q, r, s), t]=[p, q, (r, s, t)]=w$ in all the other cases.

(iv) Let $x=e$, i.e. $(p, q, r)=(c, d, e)$ or $(p, q, r)=(e, b, e)$.

Then

$$[(p, q, r), s, t]=(e, s, t) = \begin{cases} c, & \text{if } s=b, \ t=c, \\ e, & \text{if } s=b, \ t=e, \end{cases}$$

and in each case

$$[c, (d, e, b), c]=(c, d, c)=c \quad \text{and} \quad [c, d, (e, b, c)]=(c, d, c)=c,$$

$$[c, (d, e, b), e]=(c, d, e)=e \quad \text{and} \quad [c, d, (e, b, e)]=(c, d, e)=e,$$

$$[e, (b, e, b), c]=(e, b, c)=c \quad \text{and} \quad [e, b, (e, b, c)]=(e, b, c)=c,$$

$$[e, (b, e, b), e]=(e, b, e)=e \quad \text{and} \quad [e, b, (e, b, e)]=(e, b, e)=e,$$

and $[(p, q, r,)s, t]=[p, (q, r, s), t]=[p, q, (r, s, t)]=w$ in all other cases.

(v) Let $x=w$, i.e. $(p, q, r)=w$. Then $[(p, q, r), s, t]=(w, s, t)=w$. Furthermore, $[p, (q, r, s), t]=w$, for

$$(p, a, t)=w \text{ for all values of } p \text{ and } t,$$

$$(p, b, t)=w \text{ as } p\neq a \text{ and } p\neq e,$$

$$(p, c, t)=w \text{ as } p\neq b \text{ and } p\neq d,$$

$$(p, d, t)=w \text{ as } p\neq c,$$

$$(p, e, t)=w \text{ as } p\neq b \text{ and } p\neq d,$$

$$(p, w, t)=w \text{ for all values of } p \text{ and } t.$$

Again, in each case $[p, q, (r, s, t)]=w$, because

$$(p, q, a)=w \text{ for all values of } p \text{ and } q,$$

$$(p, q, b)=w \text{ as } (p\neq b, q\neq e) \text{ and } (p\neq d, q\neq e),$$

$$(p, q, c)=w \text{ as } (p\neq c, q\neq d) \text{ and } (p\neq e, q\neq b),$$

$$(p, q, w)=w \text{ for all values of } p \text{ and } q,$$

$$(p, q, d)=w \text{ as } (p\neq d, q\neq c) \text{ and } (p\neq b, q\neq c),$$

$$(p, q, e)=w \text{ as } (p\neq e, q\neq b) \text{ and } (p\neq a, q\neq b), (p\neq c, q\neq d).$$

(B) When $p=a$, but $q\neq b$, then

$$[(a, q, r), s, t]=(w, s, t)=w \quad \text{and} \quad [a, (q, r, s), t]=w, [a, q, (r, s, t)]=w.$$

(C) When $p=a$, $q=b$, but $r\neq c$, then

$$[(p, q, r), s, t]=[(a, b, r), s, t]=(a, s, t)=a \quad \text{if} \quad r=e,\ s=b,\ t=e,$$

in which case

$$[a, (b, e, b), e]=(a, b, e)=a \quad \text{and} \quad [a, b, (e, b, e)]=(a, b, e)=a,$$

and $[(a, b, r), s, t]=[a, (b, r, s), t]=[a, b, (r, s, t)]=w$ in all other cases.

(D) When $p=a$, $q=b$, $r=c$, but $s\neq d$, then

$$[(a, b, c), s, t]=[a, (b, c, s), t]=[a, b, (c, s, t)]=w.$$

(E) When $p=a$, $q=b$, $r=c$, $s=d$, but $t\neq e$, then

$$[(a, b, c), d, t]=[a, (b, c, d), t]=[a, b, (c, d, t)]=w.$$

Combining 3. 1 and 3. 7 together completes the proof of the theorem.

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Regressive and divergent functions on ordered and well-ordered sets

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0. Introduction

This paper may be divided into three loosely connected, independently readable parts, though each part relies on a few concepts introduced at the beginning of the previous one(s). In the first one we shall extend two well-known theorems of W. NEUMER and G. FODOR describing stationary subsets of well-ordered sets, in that we shall consider subsets of an ordered set that is not necessarily well-ordered. The second part gives a simple and coherent proof of a theorem of R. M. SOLOVAY asserting that every stationary subset of an uncountable regular cardinal[1]) $\varkappa$ can be split into $\varkappa$ mutually disjoint stationary sets. Finally, the third part considers a property closely connected with stationarity of subsets of a singular cardinal.

1. Extension of the theorems of Neumer and Fodor

Consider an ordered set S. S will be called *Dedekind-complete* if each of its nonempty subsets that is bounded from above has a supremum in S. This is clearly equivalent to saying that every nonempty subset bounded from below of S has an infimum in S. It is well known that to every ordered set S there corresponds a set $S' \supseteqq S$ that is Dedekind-complete. There is such an S' minimal with respect to inclusion; this set is uniquely determined up to isomorphism. It is called the *Dedekind completion* of S and is denoted by $\mathbf{Dc}(S)$. A convenient way to describe the Dedekind completion of a set can be given in terms of the familiar Dedekind cuts. Well-known examples for Dedekind-complete sets are the set of all real numbers and any well-ordered set.

Assume V is an ordered set and denote the ordering of V simply by $<$. Call a subset X of $\mathbf{Dc}(V)$ a *V-band* if X is not bounded in $\mathbf{Dc}(V)$ from above and X is

[1]) Cardinals are identified with their initial ordinals, and ordinals with the sets of all their predecessors.

closed upward in $\mathbf{Dc}(V)$, i.e. for any subset X' bounded from above of X, we have $\sup X' \in X$, where the supremum is taken in $\mathbf{Dc}(V)$. Call a subset X of V *V-stationary* if X meets every V-band. Call a function f *V-regressive* if f maps a subset of V into V and, with some v_0 depending on f, we have $f(x) < x$ whenever x belongs to the domain of f and $x \geqq v_0$. Call f *V-divergent* if f maps a subset of V into V and for no $v \in V$ is the set

$$\{x \in V : f(x) < v\}$$

cofinal to V. Then our generalization of Neumer's theorem runs as follows (cf. [5, Sätze 2 and 4 on p. 257], [2, Theorem on p. 204], and [1, Sätze 1 and 2 on p. 46]):

Theorem 1. 1. *Assume V is not cofinal to any ordinal $\leqq \aleph_0$.*[2]) *Then $X \subseteqq V$ is not V-stationary if and only if there is a V-regressive and V-divergent function f with domain X.*

The validity of the theorem can be extended to the pathological case in which V is cofinal to $\aleph_0$ if we redefine (as it is often useful to do) the notion of stationarity in such a way that, for V cofinal to $\aleph_0$, we call every subset of V V-nonstationary (i.e. not V-stationary).

Proof. We may assume that X is cofinal to V; otherwise the assertion would be ovious. First we establish the

"if" part. Assume that f is a V-regressive and V-divergent function on X. Put

$$g(u) = \inf \{f(x) : x \in X \,\&\, u \leqq x\}$$

for any $u \in \overline{V}$, where $\overline{V}$ Denotes the Dedekind completion $\mathbf{Dc}(V)$ of V and the infimum is taken in $\overline{V}$. Since f is V-divergent, for large $u \in \overline{V}$[3]) the set on the right-hand side in the last centred line is bounded from below; so the infimum exists. For small (i.e. not large) $u \in \overline{V}$ we may achieve that this infimum exist by changing the values of f assumed for small $x \in X$ (this does not affect the relevant properties of f).

The function g is defined for all elements of V and is obviously monotonic. For any element x of X we have

$$g(x) \leqq f(x).$$

So the set

$$B = \{u \in \overline{V} : g(u) \geqq u\}$$

intersects X only in a set not cofinal to $\overline{V}$. In view of the monotonity of g, B is obviously closed upward. Now, we have two alternatives: either

a) B is cofinal to $\overline{V}$, or *b)* B is not cofinal to $\overline{V}$.

---

[2]) I.e. V neither contains a last element nor is cofinal to $\aleph_0$.

[3]) I.e. for any $u > u_0$ with a fixed $u_0 \in V$, possibly depending on f.

In case *a)*, B is a $\bar{V}$-band (or, what is the same, a V-band) that is disjoint from X (in essence, i.e. if we disregard a set not cofinal to $\bar{V}$). So X is not V-stationary, which was to prove.

In case *b)*, g is a $\bar{V}$-regressive and $\bar{V}$-divergent function defined on the whole of $\bar{V}$. Assume $v_0 \in \bar{V}$ is such that for any $u \geqq v_0$ we have $g(u) < u$. If, for an integer k, v_k has already been defined, then choose v_{k+1} such that $v_k < g(v_{k+1})$. Put

$$v_\omega = \sup\{v_k : k < \aleph_0\} (= \sup\{g(v_k) : k < \aleph_0\}).$$

(The definition of v_ω is sound since $\bar{V}$ is Dedekind-complete and is not cofinal to $\aleph_0$.) Then the monotonity of g implies $g(v_\omega) \geqq v_\omega$, which is a contradiction. This completes the proof of the "if" part of the theorem. Now we are proving the

"only if" part. For any elements u and v of $\bar{V}$ choose $w = h(u, v)$ such that

$$w \in V \quad \text{and} \quad u \leqq w < v \quad \text{whenever} \quad u < v$$

(otherwise put e.g. $h(u, v) = v$). This is possible: Indeed assume $u < v$. If $u \in V$ then we may simply take $w = u$; and if $u \notin V$ then, u being an element of the Dedekind completion $\bar{V}$ of V, it can be represented as the infimum (in $\bar{V}$) of a suitable subset W of V. A small enough element of this set can be chosen as w.

Now, assuming that $B \subseteq \bar{V}$ is a V-band disjoint from the V-nonstationary set $X \subseteq V$, put

$$g(x) = \sup\{b \in B:\ b < x\},$$

for any $x \in X$. Then g is obviously $\bar{V}$-regressive and $\bar{V}$-divergent on X, the only trouble being that its values are not necessarily in V. So, putting

$$f(x) = h(g(x), x)$$

for every $x \in X$, the function f is V-regressive and V-divergent, completing the proof.

The next theorem is a generalization of an important theorem of Fodor (see [3, Satz 2 on p. 141]):

Theorem 1.2. *Assume that V is not cofinal to any ordinal $\leqq \aleph_0$, X is a V-stationary subset of V, and f is a V-regressive function on X. Then, for some $v \in V$ the set*

$$f^{-1}(\leqq v) \stackrel{\text{def}}{=} \{x \in X : f(x) \leqq v\}$$

is V-stationary.

Proof. A well-known theorem of Hausdorff says that every ordered set is cofinal to one of its well-ordered subsets; this is actually an easy consequence of Zermelo's Well Ordering Theorem. So, take an increasing sequence $\{v_\xi : \xi < \alpha\}$ that is cofinal to V; here choose α the least possible ordinal, which then may be called the *cofinality number* of V.

Assuming that for no v_ξ $(\xi<\alpha)$ is the set $X_\xi=f^{-1}(\leqq v_\xi)$ V-stationary, the preceding theorem ensures the existence of a V-regressive and V-divergent function f_ξ with domain X_ξ. Here, without any restriction of generality, we may assume that e.g.

(1. 1) $$f_\xi(x)<x \quad \text{whenever} \quad x>v_0 \quad \text{and} \quad \xi<\alpha.$$

Now, for any $\xi<\alpha$ put

$$Y_\xi = X_\xi - \bigcup_{\eta<\xi} X_\eta \; (=\{x\in X\colon (\forall\eta<\xi)(v_\eta<f(x)\leqq v_\xi)\}),$$

and, for $x\in Y_\xi$ write

$$g(x)=\max(f(x), f_\xi(x)).$$

Since $\bigcup_{\xi<\alpha} Y_\xi=X$, the domain of g is X. g is obviously V-regressive (cf. (1. 1)). We are going to show that it is also V-divergent. Indeed, for any $\lambda<\alpha$ we have:

$$\{x\in X\colon g(x)\leqq v_\lambda\} = \{x\in X\colon f(x)\leqq v_\lambda\}\cap \bigcup_{\xi<\alpha}\{x\in Y_\xi\colon f_\xi(x)\leqq v_\lambda\} = \\ = \bigcup_{\xi\leqq\lambda}\{x\in Y_\xi\colon f_\xi(x)\leqq v_\lambda\}.$$

By the divergence of f_ξ, none of the sets on the rightmost side here is cofinal to V. Since their number is less than α, which was chosen to be the cofinality number of V, neither is so their union. Since the sequence $\{v_\xi\colon \xi<\alpha\}$ is cofinal V, this shows that g is V-divergent. So the preceding theorem implies that X is V-nonstationary, in contradiction with our assumptions. The theorem is proved.

The above extended notion of stationary sets is closely connected with the classical one. To locate a point of contact we derive the following.

Theorem 1. 3. *Assume the cofinality of V is $>\aleph_0$. Assume, further, that X is a subset of V and $B\subseteqq \overline{V}(=\mathbf{Dc}(V))$ is a V-band. Then X is V-stationary if and only if $X\cap B$ is B-stationary.*

Proof. We only have to prove that the set $Y=\overline{V}-X$ includes a V-band if and only if $Y\cap B$ includes a B-band. Now, if $C\subseteqq Y\cap B$ is a B-band then C is obviously also a V-band included in Y. To see this we only have to observe that C is closed upward in $\overline{V}$. Conversely, assume that $D\subseteqq Y$ is a V-band. Then $D\cap B\subseteqq$ $\subseteqq Y\cap B$ is clearly closed upward in B. So it is a B-band, since it is also unbounded; this latter can be seen directly, but also follows immediately from the fact that the union of two $\overline{V}$-nonstationary sets (namely $\overline{V}-B$ and $\overline{V}-D$) is also V-nonstationary, which can be seen e.g. by invoking Theorem 1. 1. The proof is complete.

Now the theorem of Hausdorff mentioned at the beginning of the proof of Theorem 1. 2 entails the existence of a well-ordered V-band; so our last result establishes a simple connection between stationary sets in the extended sense and those in well-ordered sets.

2. Solovay's decomposition theorem

One of the nicest results in the theory of stationary sets is the following one conjectured by FODOR and proved by SOLOVAY (see [8, p. 418]; cf. also [6] and [7]):

Theorem 2. 1. *Assume $\varkappa$ is a regular cardinal $>\aleph_0$. Then every $\varkappa$-stationary set can be split into $\varkappa$ mutually disjoint $\varkappa$-stationary sets.*

Several weaker results had previously been obtained by G. FODOR, A. HAJNAL, A. LÉVY, and R. M. SOLOVAY. Here the definition of *$\varkappa$-stationary* sets can be obtained from the definition given in the preceding section if we take $V=\varkappa$ as ordered by the natural ordering of ordinals. Every well-ordered set being Dedekind-complete, the situation is simplified by the fact that in this case we have $\mathbf{Dc}(V)=V$. Therefore we do not have to distinguish between V- and $\mathbf{Dc}(V)$-regressive functions; so it will not lead to confusion if we simply speak of regressive functions.

To prove this theorem we need a few concepts. First of all, call a set $\mathscr{I}$ of subsets of $\varkappa$ a *$\varkappa$-complete ideal* carried by $\varkappa$ if any subset of an element of $\mathscr{I}$ as well as any union of a number less than $\varkappa$ of elements of $\mathscr{I}$ belongs to $\mathscr{I}$. $\mathscr{I}$ is said to be *proper* if $\varkappa\notin\mathscr{I}$. Call $\mathscr{I}$ *$\varkappa$-saturated* if there are no $\varkappa$ mutually disjoint subsets of $\varkappa$ none of which belongs to $\mathscr{I}$. Call $\mathscr{I}$ a *normal* ideal carried by $\varkappa$ if it is a $\varkappa$-complete proper ideal carried by $\varkappa$ that contains each one-element subset of $\varkappa$ and is such that every regressive function defined on a subset of $\varkappa$ not in $\mathscr{I}$ is constant on a set outside $\mathscr{I}$. Finally, for a class A of ordinals denote by $\mathbf{nst}(A)$ the class of all ordinals *nonstationary* with respect to A; i.e. $\alpha\in\mathbf{nst}(A)$ if and only if either the cofinality of α is $\aleph_0$ or its cofinality is $>\aleph_0$ and $A\cap\alpha$ is not α-stationary.[4])

Solovay's proof of Theorem 2. 1 is based on Lemmas 2. 2—2. 4 below. In their formulations, $\varkappa$ denotes a fixed regular cardinal $>\aleph_0$. The proofs of the first two of these given here are due to SOLOVAY; our innovation is only in the proof of the third one; our proof is in effect based on a reduced-product argument.

Lemma 2. 2. *If $A\subseteq K$ is $\varkappa$-stationary then so is $A\cap\mathbf{nst}(A)$.*

Proof.[5]) Assuming that the set $A\cap\mathbf{nst}(A)$ is $\varkappa$-nonstationary, there exists $\varkappa$-band B that is disjoint from it. The set $B'\subseteq B$ of all limit points of B is also $\varkappa$-band, so it intersects A. The first point α in $B'\cap A$ belongs to $\mathbf{nst}(A)$; indeed, if α is cofinal to $\aleph_0$ then this is by definition so; if not, then $B'\cap\alpha$ is an α-band disjoint from $A\cap\alpha$, showing that $A\cap\alpha$ is α-nonstationary. Since $\alpha\in B$, this contradicts the assumption that B and $A\cap\mathbf{nst}(A)$ are disjoint, completing the proof.

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[4]) Clearly, $\mathbf{nst}(A)$ is always a real class.

[5]) The proofs of this and the next lemma are reproduced here with R. M. SOLOVAY's permission.

Lemma 2. 3. *If $\mathscr{I}$ is a $\varkappa$-saturated normal ideal carried by $\varkappa$, then every regressive function f defined on a subset $X \notin \mathscr{I}$ of $\varkappa$ is bounded on the whole of its domain with the possible exception of a set $N \in \mathscr{I}$. (Shortly: f is essentially bounded.)*

Proof. Let Z be the set of all ordinals $\zeta < \varkappa$ such that

$$X_\zeta = \{\alpha \in X \colon f(\alpha) = \zeta\} \notin \mathscr{I}.$$

Then, by the $\varkappa$-saturatedness of $\mathscr{I}$, the cardinality of Z is less than $\varkappa$. On the other hand, by the normality of $\mathscr{I}$ we have $X - X' \in \mathscr{I}$ with $X' = \bigcup_{\zeta \in Z} X_\zeta$. For $\alpha \in X'$ we have $f(\alpha) \leqq \bigcup Z < \varkappa$. This completes the proof.

Lemma 2. 4. *Assume $\mathscr{I}$ is a $\varkappa$-saturated normal ideal carried by $\varkappa$, and $A \notin \mathscr{I}$. Then we have* $\mathbf{nst}(A) \cap \varkappa \in \mathscr{I}$.

Proof. By Neumer's theorem (cf. Theorem 1. 1 and the remark thereafter), for any α in $S \overset{\text{def}}{=} \mathbf{nst}(A) \cap \varkappa$ there exists an α-divergent regressive function f_α on $A \cap \alpha$. We may assume that $f_\alpha(\gamma) < \gamma$ holds for any $\gamma \neq 0$ in $A \cap \alpha$. Assuming that S does not belong to $\mathscr{I}$, for any $\gamma \in A$ write

$$f(\gamma) = \min\{\beta \colon \{\alpha \in S \colon \alpha > \gamma \,\&\, f_\alpha(\gamma) = \beta\} \notin \mathscr{I}\}.$$

Since $\mathscr{I}$ is a $\varkappa$-complete ideal, this definition is sound, and, obviously, f is regressive. So, by the preceding lemma, we have

$$f(\gamma) < \xi \quad \text{whenever} \quad \gamma \in A - M, \tag{2.1}$$

with a suitable ordinal $\xi < \varkappa$ and a set $M \in \mathscr{I}$.

Now, for any $\alpha \in S$ with $\alpha > \xi$ put

$$g(\alpha) = \bigcup \{\gamma \in A \cap \alpha \colon \; f_\alpha(\gamma) < \xi\}.$$

The α-divergence of f_α implies that $g(\alpha) < \alpha$ for any α in the domain of g. So, again by the preceding lemma, we have

$$g(\alpha) < \lambda \quad \text{whenever} \quad \alpha \in S - N,$$

with a suitable $\lambda < \varkappa$ and $N \in \mathscr{I}$. Taking the definition of g also into account, we see from here that $f_\alpha(\gamma) \geqq \xi$ whenever $\alpha \in S - N$, $\gamma \in A$, and $\lambda \leqq \gamma < \alpha$. Therefore, the definition of f implies that $f(\gamma) \geqq \xi$ for all $\gamma \in A$ with $\gamma \geqq \lambda$. This contradicts (2. 1), proving the lemma.

Theorem 2. 1 is a simple consequence of Lemmas 2. 2 and 2. 4. Indeed, in the latter lemma take $\mathscr{I}$ as the set of all subsets X of $\varkappa$ such that $X \cap A$ is $\varkappa$-nonstationary. Then $\mathscr{I}$ is a normal ideal in view of Fodor's theorem (cf. Theorem 1. 2); and, if we assume that Theorem 2. 1 fails for A, then $\mathscr{I}$ is also $\varkappa$-saturated. Now, the assertions of the two lemmas contradict each other, proving Theorem 2. 1.

With the aid of Theorem 1. 3, from Theorem 2. 1 we may easily derive the following

Corollary 2. 5. *Assume V is an ordered set, and* $\varkappa > \aleph_0$ *is the cofinality number of its order type. Then any V-stationary subset of V can be split into* $\varkappa$ *mutually disjoint V-stationary sets.*

Another result, in a sense running counter to Theorem 1. 3, may also be derived:

Corollary 2. 6. *Assume* $\varkappa$ *is an ordinal,* $\mathrm{cf}(\varkappa) > \aleph_0$.[6]) *Then there exist sets S and T with* $S \subseteqq T \subseteqq \varkappa$ *such that S is T-stationary, T is* $\varkappa$*-stationary, and yet S is not* $\varkappa$*-stationary.*

Proof. Choose $T \subseteqq \varkappa$ such that T and $\varkappa - T$ are $\varkappa$-stationary; this is possible e.g. by the preceding corollary.[7]) For any $\alpha \in T$ put

$$h(\alpha) = \min \{\beta : \forall \xi (\beta \leqq \xi < \alpha \rightarrow \xi \notin T)\}.$$

Then we obviously have $h(\alpha) \leqq \alpha$, and h is a $\varkappa$-divergent function. Set

$$S = \{\alpha \in T : h(\alpha) < \alpha\}.$$

In view of the divergence of h, S is $\varkappa$-nonstationary by Neumer's theorem (cf. Theorem 1. 1).

We are going to show that S is also T-stationary. Indeed, assume, on the contrary, that there exists a regressive $\varkappa$-divergent function f mapping S into T. Then the function

$$g(\gamma) = f(\eta), \quad \text{where} \quad \eta = \min \{\vartheta \in S : \vartheta > \gamma\},$$

defined for every $\gamma \in \varkappa - T$, is also regressive and $\varkappa$-divergent, in contradiction with the $\varkappa$-stationarity of this set. The proof is complete.

3. Solid sets

Assume $\varkappa$ is a singular cardinal of cofinality $> \aleph_0$. Call a set $X \subseteqq \varkappa$ $\varkappa$-solid if there is no $\varkappa$-band of cardinality $\varkappa$ disjoint from X. It is easy to see that a $\varkappa$-solid set is not necessarily $\varkappa$-stationary; but how can such sets be characterized? A kind of answer to this question is given by

[6]) $\mathrm{cf}(\varkappa)$ denotes the cofinality of $\varkappa$, i.e. the least cardinal cofinal to $\varkappa$.

[7]) The existence of such a T can also be shown by much simpler arguments.

Theorem 3. 1. *Assuming that* $\varkappa$ *is a singular cardinal with* $\mathrm{cf}(\varkappa) > \aleph_0$, *a set* $X \subseteqq \varkappa$ *is* $\varkappa$*-solid if and only if either* X *is* $\varkappa$*-stationary or*

$$\mathbf{U}\{\mathrm{cf}(\xi) : \xi \in \mathbf{nst}(X)\} < \varkappa. \tag{3. 1}$$

Proof. "*If*". Assume that X is not $\varkappa$-solid, and choose a $\varkappa$-band B of cardinality $\varkappa$ that is disjoint from X. Then, for any regular cardinal $\alpha < \varkappa$, the αth element of B, which obviously belongs to $\mathbf{nst}(X)$ (all limit points of B do), is of cofinality α; so (3. 1) does not hold. This completes the proof of the "if" part.

"*Only if*". Assuming that is X not $\varkappa$-stationary, there exists a $\varkappa$-band

$$B = \{\beta_\gamma\}_{\gamma < \mathrm{cf}(\varkappa)}$$

included in $\varkappa - X$. If, furthermore, we assume that (3. 1) does not hold, we see the existence of a sequence

$$\{\xi_\gamma\}_{\gamma < \mathrm{cf}(\varkappa)} \subseteqq \mathbf{nst}(X)$$

of ordinals ξ_γ such that

$$\eta_\gamma = \mathrm{cf}(\xi_\gamma) \text{ tends to } \varkappa \ \left(\gamma < \mathrm{cf}(\varkappa)\right).$$

We may assume that the sequence $\{\eta_\gamma\}_{\gamma < \mathrm{cf}(\varkappa)}$ is increasing and $\eta_0 > \aleph_0$.

So, for each $\gamma < \mathrm{cf}(\varkappa)$ there exists a ξ_γ-band

$$S_\gamma = \{\sigma_{\gamma,\delta}\}_{\delta < \eta_\gamma}$$

included in $\xi_\gamma - X$. Put

$$S'_\gamma = \{\sigma_{\gamma+1,\delta}\}_{\delta \leqq \eta_\gamma}.$$

Then S'_γ is obviously a closed set.

Now, for any ordinal $\gamma < \mathrm{cf}(\varkappa)$ denote by λ_γ the least ordinal $\lambda < \mathrm{cf}(\varkappa)$ such that $\sigma_{\lambda+1,\eta_\lambda} > \beta_\gamma$, and set

$$S = B \cup \mathbf{U}\{S'_{\lambda_\gamma} - \beta_\gamma : \gamma < \mathrm{cf}(\varkappa)\}.$$

Then S is obviously a $\varkappa$-band of cardinality $\varkappa$ that is included in $\varkappa - X$. This shows that X is not $\varkappa$-solid, completing the proof.

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On N consecutive integers in an arithmetic progression

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Let $B_N(d)$ denote a block $\{c+d, c+2d, \ldots, c+Nd\}$ of N consecutive integers in an arithmetic progression. It is known [1] that for each $N \geqq 17$ there exists a block $B_N(1)$ containing no integer relatively prime to each of the others. One might ask whether a similar result holds for blocks $B_N(2)$ of odd integers, or in general for blocks $B_N(d)$. We shall prove that in fact for any positive integers c and d and for all $N > N_0(d)$, there exists a block $B_N(d)$ whose integers are congruent to $c \pmod d$ which contains no integer relatively prime to each of the others. (This is of course trivial if $(c, d) > 1$.)

As the assertion is known for $d=1$, assume $d \geqq 2$ and let $t_1 < t_2 < \cdots < t_k$ be the prime divisors of d. Let $r(N)$ be the number of integers $b = t_1^{a_1} t_2^{a_2} \ldots t_k^{a_k}$ $(a_i = 0, 1, 2, \ldots)$ for which $b < N$. For a given i, the number of powers $t_i^{a_i}$ for which $t_i^{a_i} < N$ is $\leqq 1 + \frac{\log N}{\log t_i}$. Hence $r(N) \leqq \prod_{i=1}^{k} \left(1 + \frac{\log N}{\log t_i}\right) \leqq \left(1 + \frac{\log N}{\log 2}\right)^k$. Thus for all sufficiently large N, $r(N) < (\log N)^{k+1}$. By well-known theorems on distribution of primes, we conclude that for large N,

$$\pi(N/2) - \pi(N/4) > 2r(N), \tag{1}$$

$$\pi(3N/4) - \pi(N/2) > 4r(N). \tag{2}$$

There exists a prime t such that for all large N,

$$t_k < t < N/4. \tag{3}$$

Choose an integer $N_0(d)$ so large that (1), (2), and (3) hold for all $N > N_0(d)$. Fix $N > N_0(d)$ and let $r = r(N)$.

Let $b_1, \ldots, b_r$ denote the integers $b = t_1^{a_1} t_2^{a_2} \ldots t_k^{a_k}$ for which $b < N$. By (1), we can choose $2r$ distinct primes q_i such that

$$N/4 < q_i < [N/2] \qquad (i = 1, 2, \ldots, 2r). \tag{4}$$

By (2), we can choose $4r$ distinct primes p_i such that

$$N/2 < p_i < [3N/4] \qquad (i = 1, 2, \ldots, 4r). \tag{5}$$

Now let x be a solution of the system

(6) $$x \equiv c \pmod{d}$$

(7) $x \equiv 0 \pmod{p}$ for each prime $p \leqq N/2$ such that $p \notin \{t_1, \ldots, t_k, q_1, \ldots, q_{2r}\}$.

(8) $$x+db_i \equiv 0 \pmod{q_i} \qquad (i=1, 2, \ldots, r)$$

(9) $$x-db_i \equiv 0 \pmod{q_{r+i}} \qquad (i=1, 2, \ldots, r)$$

(10) $$x+dq_i \equiv 0 \pmod{p_i} \qquad (i=1, 2, \ldots, 2r)$$

(11) $$x-dq_i \equiv 0 \pmod{p_{2r+i}} \qquad (i=1, 2, \ldots, 2r).$$

(A solution exists as the moduli are relatively prime in view of (3), (4), and (5).)

We shall now show that the block $B_N(d) = \{x-d(N-[N/2]-1), \ldots, x+d[N/2]\}$ has the desired properties. That its integers are congruent to $c \pmod{d}$ follows from (6). To see that $B_N(d)$ contains no integer relatively prime to each of the others, we will produce, for each $u \in B_N(d)$, a corresponding $v \in B_N(d)$ such that $v \neq u$ and $(u, v) > 1$.

If $u=x$, we may choose $v = x+dt$ by (3) and (7). If $u = x+db_i$, we may choose $v = x+d(b_i-q_i)$ by (4) and (8). If $u = x-db_i$, we may choose $v = x+d(q_{r+i}-b_i)$ by (4) and (9). If $u = x+dq_i$, we may choose $v = x+d(q_i-p_i)$ by (4), (5) and (10). If $u = x-dq_i$, we may choose $v = x+d(p_{2r+i}-q_i)$ by (4), (5), and (11). Every other $u \in B_N(d)$ has the form $x \pm dm$, where m is divisible by a prime $p \leqq N/2$ such that $p \notin \{t_1, \ldots, t_k, q_1, \ldots, q_{2r}\}$. Hence by (7), we may choose $v=x$ for each of these u.

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Schwach distributive Verbände. I

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1. Einführung

Mehrere Arbeiten in der Verbandstheorie zeigen, daß es zwischen den Theorien

a) der projektiven Geometrien,

b) der Untergruppenverbände der Gruppen,

c) der Kongruenzklassengeometrien, und

d) der primitiven Klassen (Varietäten) der Verbände

gewisse Verbindungen gibt. (Siehe z. B. die Arbeiten von BAKER [1], JÓNSSON [3], WILLE [7]). In der vorliegenden Arbeit wird ein solcher verbandstheoretische Begriff eingeführt und untersucht, der die erwähnten Gebiete mit der Theorie der distributiven Verbände verbindet und dadurch zwischen diesen Gebieten weitere Zusammenhänge aufdeckt.

1. 1. Definition. Ein Verband L wird *n-distributiv* genannt, wenn er modular ist und in ihm die Identität

$$D_n: x \cup \bigcap_{i=0}^{n} y_i = \bigcap_{j=0}^{n} \left[x \cup \bigcap_{\substack{i=0 \\ i \neq j}}^{n} y_i \right]$$

für beliebige $x, y_0, \ldots, y_n \in L$ gilt. Mit der dualen Identität D_n^* von D_n erhält man die Definition der *dualen n-Distributivität*. (Später wird man sehen, daß die primitive Klasse $\mathbf{D}_n$ der n-distributiven Verbände mit der primitiven Klasse $\mathbf{D}_n^*$ der dual n-distributiven Verbände identisch ist, ferner, daß $\mathbf{D}_n$ echter Teil von $\mathbf{D}_{n+1}$ ist.) Wegen dieser Relationen werden diejenigen Verbände, die für irgendein n n-distributiv sind, *schwach distributiv* genannt.

Als Beispiel für eine Anwendung der n-distributiven Verbände schicken wir einige grundlegende Ergebnisse aus den Gebieten a)—d) voraus.

ad a) Für einen beliebigen Schiefkörper D ist die $(n-1)$-dimensionale projektive Geometrie (d. h. der Unterraumverband des n-dimensionalen Vektorraumes) $PG_{n-1}(D)$ über D ein n-distributiver, aber nicht $(n-1)$-distributiver Verband.

ad b) Für eine Abelsche Gruppe G ist der Untergruppenverband von G dann und nur dann n-distributiv, wenn der endliche Rang von G kleiner oder gleich n

ist (d. h., wenn jede endlich erzeugte Untergruppe von G auch durch n (nicht unbedingt verschiedene) Elemente erzeugbar ist). Für n = 1 siehe ORE [5].

ad c) Für Kongruenzklassengeometrien kann man das folgende Analogon des Satzes von HELLY[1]) beweisen: Ist der Kongruenzverband $\Theta(A)$ der universellen Algebra A normal und n-distributiv, so gilt die folgende Behauptung: Wenn $\mathfrak{R} = \langle Ki \rangle_{i=1}^{m}$ ein mindestens $(n+1)$-elementiges endliches System der Kongruenzklassen von A ist, so daß jedes $(n+1)$-elementige Teilsystem von $\mathfrak{R}$ ein gemeinsames Element enthält, dann gilt $\bigcap \mathfrak{R} \neq \emptyset$. Für $n=1$ siehe GRÄTZER [2].

ad d) Mit Hilfe der Ergebnisse über die n-Distributivität kann man solche primitive Klassen konstruieren, die durch ihre endlichen Elemente nicht erzeugt sind, ferner solche, die unendlich viele obere Nachbarn haben in dem Verband der primitiven Klassen der Verbände. Für die letzteren sind auch die primitiven Klassen $\mathbf{D}_n$ ($n \geqq 2$) Beispiele. (Die Frage der Existenz solcher primitiver Klassen war ein Problem von JÓNSSON [4]. Frühere Lösungen mit anderen Methoden kann man in den Arbeiten von BAKER [1], und WILLE [6] finden.)

Im Teil I der Arbeit werden die wichtigsten verbandtheoretischen Eigenschaften der n-distributiven Verbände angegeben, mit deren Hilfe wir die Zusammenhänge mit den Gebieten a)—d) im Teil II untersuchen werden.

2. Kriterien der n-Distributivität

In diesem Punkt wird die Verallgemeinerung des Distributivitätskriteriums von Birkhoff bewiesen, auf Grund dessen die weiteren Eigenschaften der n-distributiven Verbände leicht zu ermitteln sind.

2. 1. Satz. *Es sei L ein modularer Verband. Für eine beliebige natürliche Zahl n sind die folgenden beiden Aussagen äquivalent:*

(A) *L ist nicht n-distributiv.*

(B) *L enthält eine $(n+1)$-dimensionale Boolesche Algebra B als Teilverband und ein Element w mit der Eigenschaft $K(B, w)$: w ist relatives Komplement aller Atome von B im Intervall* $[\inf B, \sup B]$.

(Ein Verband heißt eine $(n+1)$-dimensionale Boolesche Algebra, wenn er zum Verband $\mathbf{2}^{n+1}$ isomorph ist, wobei $\mathbf{2}$ die zwei-elementige Kette bedeutet.)

Bemerkung. Für $n=1$ gibt der Satz das Distributivitätskriterium von BIRKHOFF.

[1]) Der Satz von Helly: Ist $\mathfrak{R}$ ein mindestens $(n+1)$-elementiges endliches System von konvexen Mengen im n-dimensionalen euklidischen Raum, dessen jedes $(n+1)$-elementiges Teilsystem einen gemeinsamen Punkt enthält, so gilt $\cap \mathfrak{R} \neq \emptyset$.

Beweis. Zum Beweis der Behauptung (B)⇒(A) seien b_i $(i=0, 1, \dots, n)$ die Atome von B und es bezeichne b_i' ihre Komplemente in B. Gemäß der Eigenschaft $K(B, w)$ gilt

$$w \cup \bigcap_{i=0}^{n} b_i' = w \cup \inf B = w < \sup B = \bigcap_{j=0}^{n} (w \cup b_j) = \bigcap_{j=0}^{n} \left[w \cup \bigcap_{\substack{i=0 \\ i \neq j}}^{n} b_i' \right],$$

d.h. L ist nicht n-distributiv.

Umgekehrt, wenn D_n in L nicht erfüllt ist, existieren Elemente $x, y_0, \dots, y_n \in L$, für die $w_0 = x \cup \bigcap_{i=0}^{n} y_i \neq \bigcap_{j=0}^{n} \left[x \cup \bigcap_{\substack{i=0 \\ i \neq j}}^{n} y_i \right] = u_0$, d.h. $w_0 < u_0$ gilt. Offenbar gelten für die Elemente $a_j = \bigcap_{\substack{i=0 \\ i \neq j}}^{n} y_i$ die Relationen

$$\bigcap_{j=0}^{n} a_j = \bigcap_{i=0}^{n} y_i \quad \text{und} \quad w_0 \cup a_i = \left(x \cup \bigcap_{j=0}^{n} y_j \right) \cup a_i = \left(x \cup \left(\bigcap_{j=0}^{n} y_j \cup a_i \right) \right) = x \cup a_i.$$

Mit Berücksichtigung der vorigen Ungleichung erhält man:

(1) $$\bigcap_{i=0}^{n} a_i \leqq w_0 < u_0 = \bigcap_{i=0}^{n} (w_0 \cup a_i).$$

Wir führen die folgenden Bezeichnungen ein:

$$v = \bigcup_{i=0}^{n} (w_0 \cap a_i), \quad b_i = (a_i \cup v) \cap u_0, \quad u = \bigcup_{i=0}^{n} b_i, \quad b_i' = \bigcup_{\substack{j=0 \\ j \neq i}}^{n} b_j, \quad w = u \cap w_0.$$

Weil $v \leqq w_0$ gilt, gilt auch $v < u_0$; wegen der Modularität kann man also die Elemente b_i auch in die Form $(a_i \cap u_0) \cup v$ schreiben. So gilt: $v \leqq b_i \leqq u \leqq u_0$. Da $\bigcap_{i=0}^{n} a_i \leqq w_0$ gilt, folgen die Relationen

$$\bigcap_{i=0}^{n} a_i = \bigcap_{i=0}^{n} a_i \cap a_j \leqq w_0 \cap a_j, \quad \text{d. h.} \quad \bigcap_{i=0}^{n} a_i \leqq \bigcup_{j=0}^{n} (w_0 \cap a_j) = v.$$

Es folgt also

(2) $$\bigcap_{i=0}^{n} a_i \leqq v.$$

Wir werden beweisen, daß der durch die Elemente b_i erzeugte Teilverband eine $(n+1)$-dimensionale Boolesche Algebra ist, ihre Atome die Elemente b_i sind und ihr kleinstes und größtes Element v bzw. u sind, ferner, daß w gemeinsames relatives Komplement der Elemente b_i im Intervall $[v, u]$ ist. Dazu werden wir folgendes zeigen:

1. Da $b_i \cup w_0 = [(a_i \cap u_0) \cup v] \cup w_0 = (a_i \cap u_0) \cup w_0 =$ (wegen der Modularität) $= (a_i \cup w_0) \cap u_0 = u_0$ gilt, erhält man, daß $b_i \cup w = b_i \cup (u \cap w_0) =$ (wegen der Modularität und der Relation $b_i \leqq u$) $= u \cap (b_i \cup w_0) = u \cap u_0 = u$ gilt.

2. $$b_i \cap w = b_i \cap (u \cap w_0) = b_i \cap w_0 = [(a_i \cup v) \cap u_0] \cap w_0 =$$
$$= (a_i \cup v) \cap u_0 \cap w_0 = (a_i \cup v) \cap w_0 = (a_i \cap w_0) \cup v = v.$$

3. $$b_i \cap b_i' = b_i \cap \bigcup_{\substack{j=0 \\ j \neq i}}^{n} b_j = [(a_i \cap u_0) \cup v] \cup \bigcup_{\substack{j=0 \\ j \neq i}}^{n} [(a_j \cap u_0) \cup v] =$$
$$= \left[(a_i \cap u_0) \cup \bigcup_{j=0}^{n} (w_0 \cap a_j)\right] \cap \left[\bigcup_{\substack{j=0 \\ j \neq i}}^{n} (a_j \cap u_0) \cup \bigcup_{j=0}^{n} (w_0 \cap a_j)\right] =$$
$$= \text{(mit der Anwendung der Ungleichung } w_0 \cap a_j \leqq u_0 \cap a_j) =$$
$$= \left[(a_i \cap u_0) \cup \bigcup_{\substack{j=0 \\ j \neq i}}^{n} (w_0 \cap a_j)\right] \cap \left[\bigcup_{\substack{j=0 \\ j \neq i}}^{n} (a_j \cap u_0) \cup (w_0 \cap a_i)\right].$$

Für den Ausdruck in der ersten eckigen Klammer gilt $(a_i \cap u_0) \cup \bigcup_{i \neq j=0}^{n} (w_0 \cap a_j) \geqq$ $\geqq a_i \cap u_0 \geqq a_i \cap w_0$. Durch Anwendung der Modularität erhalten wir:

$$b_i \cap b_i' = \left\{\left[(a_i \cap u_0) \cup \bigcup_{\substack{j=0 \\ j \neq i}}^{n} (w_0 \cap a_j)\right] \cap \bigcup_{\substack{j=0 \\ j \neq i}}^{n} (a_j \cap u_0)\right\} \cup (a_i \cap w_0).$$

Wendet man wieder die Modularität gemäß der Relation

$$\bigcup_{\substack{j=0 \\ j \neq i}}^{n} (w_0 \cap a_j) \leqq \bigcup_{\substack{j=0 \\ j \neq i}}^{n} (a_j \cap u_0)$$

an, so bekommt man:

$$b_i \cap b_i' = \left[(a_i \cap u_0) \cap \bigcup_{\substack{j=0 \\ j \neq i}}^{n} (a_j \cap u_0)\right] \cup \bigcup_{\substack{j=0 \\ j \neq i}}^{n} (w_0 \cap a_j) \cup (a_i \cap w_0) =$$

$$= \left[(a_i \cap u_0) \cap \bigcup_{\substack{j=0 \\ j \neq i}}^{n} (a_j \cap u_0)\right] \cup \bigcup_{j=0}^{n} (w_0 \cap a_j) = \left[(a_i \cap u_0) \cap \bigcup_{\substack{j=0 \\ j \neq i}}^{n} (a_j \cap u_0)\right] \cup v = v,$$

weil

$$(a_i \cap u_0) \cap \bigcup_{\substack{j=0 \\ j \neq i}}^{n} (a_j \cap u_0) \leqq a_i \cap \bigcup_{\substack{j=0 \\ j \neq i}}^{n} a_j \leqq \bigcap_{\substack{j=0 \\ j \neq i}}^{n} y_j \cap y_i = \bigcap_{i=0}^{n} a_i$$

gilt und gemäß (2) dieser letzte Ausdruck kleiner oder gleich als v ist. Die Elemente $b_0, b_1, \ldots, b_n$ bilden also eine unabhängige Menge, und somit erzeugen sie eine Boolesche Algebra. Infolge der Relationen $b_i \cap b_i' = v$ und $\bigcup_{i=0}^{n} b_i = u$ ist v das kleinste, u das größte Element dieser Booleschen Algebra.

4. Es bleibt zu beweisen, daß die durch die Elemente b_i erzeugte Boolesche Algebra $(n+1)$-dimensional ist. Wäre $b_i \leqq b_j$, so würde wegen der Relation $b_i \cap b_i' \geqq \geqq b_i \cap b_j = v$ folgen: $b_i = v$, d. h. $b_i \cup w_0 = v \cup w_0 = w_0$, aber es war im Punkt 1 bewiesen, daß $b_i \cup w_0 = u_0$ gilt; also hätte die Annahme $u_0 = w_0$ zur Folge, und das widerspricht der Bedingung (1).

Diesen Satz kann man auch in der folgenden symmetrischen Form aussagen.

2. 2. Korollar. *Der modulare Verband L ist dann und nur dann n-distributiv, wenn er keine Teilmenge $\langle x_0, x_1, \ldots, x_{n+1} \rangle$ enthält, für die die folgende Eigenschaft gilt:*

$$(i)\begin{cases} x_{i_0} \cap x_{i_1} \cap \cdots \cap x_{i_n} = x_{j_0} \cap x_{j_1} \cap \cdots \cap x_{j_n}, \\ x_{i_0} \cup (x_{i_1} \cap \cdots \cap x_{i_n}) = x_{j_0} \cup (x_{j_1} \cap \cdots \cap x_{j_n}) \end{cases}$$

für jede Wahl von $i_0, \ldots, i_n$; $j_0, \ldots, j_n$, wobei $i_0, \ldots, i_n$ und ebenso $j_0, \ldots, j_n$ alle verschieden sind.

Beweis. Die Notwendigkeit ist offenbar. Zum Beweis der Hinlänglichkeit der Bedingung sei L nicht n-distributiv und seien (mit den früheren Bezeichnungen) $x_i = b_i'$ $(i = 0, 1, \ldots, n)$ und $x_{n+1} = w$. Auf Grund des Satzes 2. 1 erhält man, daß

$$\begin{aligned} b_{i_0}' \cup (b_{i_1}' \cap \cdots \cap b_{i_{n-1}}' \cap w) &= b_{i_0}' \cup (b_{i_1}' \cap \cdots \cap b_{i_{n-1}}' \cap w) \cup (b_{i_0}' \cap \cdots \cap b_{i_{n-1}}') = \\ &= b_{i_0}' \cup \{(b_{i_1}' \cap \cdots \cap b_{i_{n-1}}') \cap [w \cup (b_{i_0}' \cap \cdots \cap b_{i_{n-1}}')]\} = \\ &= b_{i_0}' \cup \{(b_{i_1}' \cap \cdots \cap b_{i_{n-1}}') \cap \sup B\} = \sup B \end{aligned}$$

gilt, wenn $i_0, i_1, \ldots, i_{n-1}$ alle verschieden sind. Die Kombination des Ergebnisses des Satzes 2. 1 mit dieser Relation gibt die Behauptung (i) auf der Menge $\langle x_0, \ldots, x_{n+1} \rangle$.

Der folgende Satz verallgemeinert den Begriff des Mediums.

2. 3. Korollar. *Ein modularer Verband L ist dann und nur dann n-distributiv, wenn für seine beliebigen Elemente die Identität gilt:*

$$\bigcap_{k=0}^{n+1} \bigcup_{\substack{j=0 \\ j \neq k}}^{n+1} \bigcap_{\substack{i=0 \\ i \neq j,k}}^{n+1} x_i = \bigcup_{i=0}^{n+1} \bigcap_{\substack{i=0 \\ i \neq j}}^{n+1} x_i.$$

Beweis. Es ist bekannt, daß für die Elemente eines modularen Verbands auch die Identität

$$(1) \qquad \bigcap_{i=1}^{r} p_i \cup \bigcup_{i=1}^{r} q_i = \bigcap_{i=1}^{r} (p_i \cup q_i) \qquad \text{(wenn } p_i \geqq q_j \text{ für } i \neq j \text{ besteht)}$$

gültig ist. So erhält man, daß

$$\bigcup_{\substack{j=0\\ j\neq k}}^{n+1} \bigcap_{\substack{i=0\\ i\neq j,k}}^{n+1} x_i = \bigcap_{\substack{j=0\\ j\neq k,l}}^{n+1} \left[x_j \cup \bigcap_{\substack{i=0\\ i\neq j,k}}^{n+1} x_i\right] \quad \text{(für alle } l \neq k\text{), d. h.}$$

$$\bigcup_{\substack{j=0\\ j\neq k}}^{n+1} \bigcap_{\substack{i=0\\ i\neq j,k}}^{n+1} x_i = \bigcap_{\substack{l=0\\ l\neq k}}^{n+1} \bigcap_{\substack{j=0\\ j\neq k,l}}^{n+1} \left[x_j \cup \bigcap_{\substack{i=0\\ i\neq j,k}}^{n+1} x_i\right] = \bigcap_{\substack{j=0\\ j\neq k}}^{n+1} \left[x_j \cup \bigcap_{\substack{i=0\\ i\neq j,k}}^{n+1} x_i\right];$$

daher gilt

(2)
$$\bigcap_{k=0}^{n+1} \bigcup_{\substack{j=0\\ j\neq k}}^{n+1} \bigcap_{\substack{i=0\\ i\neq j,k}}^{n+1} x_i = \bigcap_{k=0}^{n+1} \bigcap_{\substack{j=0\\ j\neq k}}^{n+1} \left[x_j \cup \bigcap_{\substack{i=0\\ i\neq j,k}}^{n+1} x_i\right].$$

Wenn L n-distributiv ist, dann folgt aus (2), D_n und (1) die Relation

$$\bigcap_{k=0}^{n+1} \bigcup_{\substack{j=0\\ j\neq k}}^{n+1} \bigcap_{\substack{i=0\\ i\neq j,k}}^{n+1} x_i = \bigcap_{k=0}^{n+1} \bigcap_{\substack{j=0\\ j\neq k}}^{n+1} \left[x_j \cup \bigcap_{\substack{i=0\\ i\neq j,k}}^{n+1} x_i\right] = \bigcap_{j=0}^{n+1} \left[x_j \cup \bigcap_{\substack{i=0\\ i\neq j}}^{n+1} x_i\right] =$$

$$= \bigcap_{j=0}^{n+1} x_j \cup \bigcup_{k=0}^{n+1} \bigcap_{\substack{i=0\\ i\neq k}}^{n+1} x_i = \bigcup_{k=0}^{n+1} \bigcap_{\substack{i=0\\ i\neq k}}^{n+1} x_i.$$

Umgekehrt, wenn L nicht n-distributiv ist, dann seien $x_0, \ldots, x_{n+1}$ so gewählt, wie in 2. 2. Mit den früheren Bezeichnungen erhält man dann auf Grund von (2) und 2. 2, daß

$$\bigcap_{k=0}^{n+1} \bigcup_{\substack{j=0\\ j\neq k}}^{n+1} \bigcap_{\substack{i=0\\ i\neq j,k}}^{n+1} x_i = \sup B > \inf B = \bigcup_{j=0}^{n+1} \bigcap_{\substack{i=0\\ i\neq j}}^{n+1} x_i$$

gilt, und damit ist auch das Hinreichen der Bedingung bewiesen.

3. Weitere Eigenschaften der n-distributiven Verbände

Nach diesen Sätzen kann man zeigen, daß für n-distributive Verbände Dualitätsprinzipien gelten.

3. 1. Satz. *Die Klasse* $\mathbf{D}_n$ *ist mit der Klasse* $\mathbf{D}_n^*$ *identisch.*

Beweis. Nach Satz 2. 1 und seinem Dualen genügt es die folgende Behauptung zu zeigen: *Enthält ein modularer Verband eine* $(n+1)$*-dimensionale Boolesche Algebra* B *und ein Element* w *mit der Eigenschaft* $K(B, w)$, *so enthält er auch ein Element* w^* *mit der dualen Eigenschaft* $K^*(B, w^*)$ *von* $K(B, w)$ (und umgekehrt, aber die umgekehrte Behauptung folgt aus dieser auf Grund des allgemeinen Dualitätsprinzips unmittelbar).

In der Tat, es seien B und w mit der obigen Eigenschaft gegeben, bezeichne v die Infima und u die Suprema, b_i ($i=0, 1, \ldots, n$) die Atome von B, ferner sei $b_{ij} = b_i \cup b_j$ und bezeichne b_i' und b_{ij}' die Komplemente von b_i und b_{ij} in B. Es sei endlich $w^* = \bigcap_{i=0}^{n-1} t_{in}$, wobei $t_{ij} = (w \cap b_{ij}) \cup b_{ij}'$ ist. Wir werden zeigen, daß w^* relatives Komplement aller dualen Atome b_i' von B im Intervall $[v, u]$ ist.

1. Wenn $i \neq j$ gilt, dann erhalten wir:

$$t_{ij} \cup b_j = [(w \cap b_{ij}) \cup b_{ij}'] \cup b_j = [(w \cap b_{ij}) \cup b_j] \cup b_{ij}' =$$

$$= \text{(wegen der Modularität und der Relation } b_j < b_{ij}) =$$

$$= [(w \cup b_j) \cap b_{ij}] \cup b_{ij}' = (u \cap b_{ij}) \cup b_{ij}' = b_{ij} \cup b_{ij}' = u,$$

$$t_{ij} \cap b_j' = [(w \cap b_{ij}) \cup b_{ij}'] \cap b_j' =$$

$$= \text{(wegen der Modularität und der Relation } b_{ij}' < b_j') =$$

$$= [(w \cap b_{ij}) \cap b_j'] \cup b_{ij}' = (w \cap b_i) \cup b_{ij}' = v \cup b_{ij}' = b_{ij}'.$$

Aus Symmetriegründen folgen:

$$(1) \quad t_{ij} \cup b_i = t_{ij} \cup b_j = u \qquad (i \neq j;\ i, j = 0, 1, \ldots, n).$$

$$(2) \quad t_{ij} \cap b_i' = t_{ij} \cap b_j' = b_{ij}' \qquad (i \neq j;\ i, j = 0, 1, \ldots, n).$$

2. Da $t_{in} = (w \cap b_{in}) \cup b_{in}' \geqq b_{in}' \geqq b_j$ (wenn $j \neq i, n$) gilt, erhält man mit Hilfe von 2. 3 (1) und (1) die Relation

$$w^* \cup b_n' = \bigcap_{i=0}^{n-1} t_{in} \cup \bigcup_{i=0}^{n-1} b_i = \bigcap_{i=0}^{n-1} (t_{in} \cup b_i) = \bigcap_{i=0}^{n-1} u = u.$$

Für $j \neq n$ gilt: $t_{jn} \geqq b_i$ (wenn $i \neq j, n$), d. h. $t_{jn} \geqq \bigcup_{\substack{i=0 \\ i \neq j}}^{n-1} b_i$.

Wendet man gemäß dieser Relation die modulare Identität an, so erhält man:

$$w^* \cup b_j' = \bigcap_{i=0}^{n-1} t_{in} \cup \bigcup_{\substack{i=0 \\ i \neq j}}^{n} b_i = \left[\bigcap_{\substack{i=0 \\ i \neq j}}^{n-1} t_{in} \cap t_{jn}\right] \cup \bigcup_{\substack{i=0 \\ i \neq j}}^{n-1} b_i \cup b_n =$$

$$= \left\{\left[\bigcap_{\substack{i=0 \\ i \neq j}}^{n-1} t_{in} \cup \bigcup_{\substack{i=0 \\ i \neq j}}^{n-1} b_i\right] \cap t_{jn}\right\} \cup b_n.$$

Mit der Anwendung von 2. 3 (1) bekommt man, daß der Ausdruck in der eckigen Klammer mit u gleich ist, woraus mit Hilfe von (1)

$$w^* \cup b_j' = (u \cap t_{jn}) \cup b_n = t_{jn} \cup b_n = u \qquad (j = 0, 1, \ldots, n-1)$$

folgt.

3. Mit der Anwendung von (2) ergibt sich:

$$w^* \cap b_n' = \bigcap_{i=0}^{n-1} t_{in} \cap b_n' = \bigcap_{i=0}^{n-1} (t_{in} \cap b_n') = \bigcap_{i=0}^{n-1} b_{in}' = v.$$

Wenn $j \neq n$ gilt, so erhält man mit wiederholter Anwendung von (2):

$$w^* \cap b_j' = \bigcap_{\substack{i=0\\ i\neq j}}^{n-1} t_{in} \cap t_{jn} \cap b_j' = \bigcap_{\substack{i=0\\ i\neq j}}^{n-1} t_{in} \cap b_{jn}' = \bigcap_{\substack{i=0\\ i\neq j}}^{n-1} t_{in} \cap b_n' \cap b_j' =$$

$$= \bigcap_{\substack{i=0\\ i\neq j}}^{n-1} (t_{in} \cap b_n') \cap b_j' = \bigcap_{\substack{i=0\\ i\neq j}}^{n-1} b_{in}' \cap b_j' = b_j \cap b_j' = v.$$

Damit ist der Satz bewiesen.

3. 2. Satz. *Für eine beliebige natürliche Zahl $n(>1)$ ist die Klasse $\mathbf{D}_{n-1}$ echter Teil von $\mathbf{D}_n$.*

Beweis. Ist $L \notin \mathbf{D}_n$, so enthält er eine $(n+1)$-dimensionale Boolesche Algebra B und ein Element w mit der Eigenschaft $K(B, w)$. Es seien (mit der Bezeichnungen des Beweises des Satzes 3. 1) $B_* = B \cap b_0'$, $w_* = w \cap b_0'$. Dann ist B_* eine n-dimensionale Boolesche Algebra, deren Atome und Schrankenelemente $b_1, b_2, \dots, b_n$ bzw. v und b_0 sind. Für $i=1, \dots, n$ gilt ferner $w_* \cap b_i = w \cap b_0' \cap b_i = (w \cap b_i) \cap \cap b_0' = v$ und $w_* \cup b_i = (w \cap b_0') \cup b_i = (w \cup b_i) \cap b_0' = u \cap b_0' = b_0'$, d. h. Eigenschaft $K(B_*, w_*)$. Folglich ist $L \notin \mathbf{D}_{n-1}$. Also gilt: $\mathbf{D}_{n-1} \subseteqq \mathbf{D}_n$.

Den Beweis des Satzes kann man mit dem folgenden Beispiel beenden:

Beispiel. Jeder der Verbände $PG_{n-1}(D)$ ist n-distributiv, aber nicht $(n-1)$-distributiv.

Beweis. Es ist bekannt, daß die Länge von $PG_{n-1}(D)$ gleich n ist; da die Länge einer $(n+1)$-dimensionalen Booleschen Algebra $n+1$ ist, kann man sie in $PG_{n-1}(D)$ nicht einbetten, somit (gemäß 2. 1) ist $PG_{n-1}(D)$ n-distributiv. Um die andere Behauptung zu beweisen, betrachten wir $PG_{n-1}(D)$ als den Unterraumverband des n-dimensionalen Vektorraumes über D. Die Verbandsoperationen in diesem Verband sind

$A \cap B =$ der gemeinsame Teil von A und B, bzw.

$A \cup B = \{A, B\}$, wobei $\{H\}$ den durch H erzeugten Unterraum bezeichnet.

Mit den Bezeichnungen $e_i = (0, 0, \dots, 0, \overset{i}{1}, 0, \dots, 0)$, $e = (1, 1, \dots, 1)$ und $o = (0, 0, \dots, 0)$ bzw. $E_i = \{e_i\}$, $E = \{e\}$, $O = \{o\}$,

$$E_i' = \{\langle e_j \rangle_{j=1,\dots,n;\, j\neq i}\}, \quad I = \{\langle e_i \rangle_{i=1,\dots,n}\}$$

ist offenbar

$$E \cap \bigcup_{i=1}^{n} E_i = E \cap I = E > O = \bigcup_{j=1}^{n} (E \cap E'_j) = \bigcup_{j=1}^{n} \left(E \cap \bigcup_{\substack{i=1 \\ i \neq j}}^{n} E_i \right).$$

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Bi-ideals in associative rings and semigroups

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The concept of a quasi-ideal in an associative ring was introduced by OTTO STEINFELD in [18, 20]. He has developed an extensive theory concerning quasi-ideals in rings and semigroups. Bi-ideals were introduced in semigroups by GOOD and HUGHES [2], further treated by LAJOS [10, 12] and the author [6] among others. An explicit treatment has recently been given for bi-ideals in rings by LAJOS and SZÁSZ [13, 14]. We continue to develop some of the theory of bi-ideals in rings here.

In [20] STEINFELD showed that each minimal quasi-ideal of a ring R is either null or a division ring of the form eRe. We consider here bi-ideals of a ring and show that an analogous result is also true. In a regular ring the sets of bi-ideals and quasi-ideals coincide [10]. However as LUH points out [15] a ring may have these sets coincide without being regular. In general, a quasi-ideal is a bi-ideal. We will investigate minimal bi-ideals in arbitrary rings and determine several conditions under which such bi-ideals are quasi-ideals.

§ 1. Preliminaries, bi-ideals and regularity

We begin by recalling the following definitions for rings. For semigroups (written multiplicatively) one obtains from the following the corresponding definition of quasi-ideal, bi-ideal etc. by considering only the multiplicative requirement. In the sequel, when a definition or proposition holds for semigroups with this obvious modification we will place ($\mathscr{S}$) following the number of the statement.

We will assume that the semigroup has a zero, 0, since a zero element can always be adjoined (cf. [1] p. 4) and we will write $S=S^0$ to denote such a semigroup. When the corresponding result for semigroups is known we will cite the appropriate reference by: (1. 3) ($\mathscr{S}$-[1] p. 85 ex. 18b).

(1. 1) ($\mathscr{S}$) Definition. A subgroup $(A, +)$, of a ring, R, is a *quasi-ideal* of R if $RA \cap AR \subseteqq A$. (As usual $CD=\{\sum^{n} c_i d_i \mid c_i \in C, d_i \in D\}$ for subgroups $(C, +)$ and $(D, +)$ of a ring R.) For the semigroup S, we require $A \neq \emptyset$, the empty set. and $SA \cap AS \subseteqq A$.

(1. 2) ($\mathscr{S}$) Definition. A subring B, of a ring, R is a *bi-ideal* of R if $BRB \subseteqq B$. For the semigroup S, a non-empty subset, B, is a *bi-ideal* if $B^2 \cup BRB \subseteqq B$.

(1. 3) ($\mathscr{S}$-[1] p. 85 ex. 18b) Proposition. *Let B be a bi-ideal of a ring R. Then $J = B+BR$ and $L = B+RB$ are respectively right and left ideals of R and $JL \subseteqq B \subseteqq \subseteqq J \cap L$.*

Proof. A straightforward check shows that J and L are indeed right and left ideals of R. Clearly $B \subseteqq J \cap L$. On the other hand $JL = (B+BR)(B+RB) \subseteqq B^2 + +BRB+BR^2B \subseteqq B$ since B is a bi-ideal and the result follows.

We have the following partial converse of (1. 3).

(1. 4) ($\mathscr{S}$-[1] p. 85 ex. 18c) Proposition. *Let J and L be respectively right and left ideals of a ring R. Then any subgroup $(B, +)$ of R such that $JL \subseteqq B \subseteqq J \cap L$ is a bi-ideal of R.*

Proof. B is already by hypothesis a subgroup of R.

Since $B^2 \subseteqq JL$ and $JL \subseteqq B$ it follows that B is a subring. Moreover $BRB \subseteqq \subseteqq (J \cap L)R(J \cap L) \subseteqq JRL \subseteqq JL \subseteqq B$ so that B is indeed a bi-ideal.

Unlike the semigroup case, the additional assumption that B is a subgroup is necessary in (1. 4) as the following example shows.

(1. 5) Example. Let $R = \{\alpha | \alpha: Z \to Z$, Z the set of integers with $(n)\alpha = n(2k)$, k fixed$\}$ with the functional compositions defined in the usual fashion: $(n)[\alpha+\beta] = = (n)\alpha + (n)\beta$, $(n)[\alpha \cdot \beta] = ((n)\alpha)\beta$. Let $J = L = R^2$ and $B = \{\beta \in R |\ |(n)\beta| > 4n$ for each $n \neq 0$, or $(n)\beta = 4n$, or $\beta = 0\}$. Clearly $JL \subseteqq B \subseteqq R^2 = J \cap L$ and yet with $(n)\beta = 4n$, $\beta \in B$but $-\beta \notin B$ so that B is not even a subgroup no less a bi-ideal.

In the remaining part of this section we will consider bi-ideals which are either themselves regular rings [semigroups] or which are subrings [subsemigroups] of a regular ring [semigroup].

(1. 6) ($\mathscr{S}$) Definition. An element a of a ring R is *regular* if $a \in aRa$. R is *regular* if each element in it is regular.

We now have the following proposition.

(1. 7) ($\mathscr{S}$-[11] Theorem 10) Proposition. *Let $a \in R$, a ring. Then aRa is a bi-ideal. Indeed, a is regular if and only if aRa is the smallest bi-ideal containing a.*

Proof. By [12] Theorem 8, aRa is a bi-ideal. Suppose now that a is regular. Then $a \in aRa$. Let B be a bi-ideal containing a. We have $aRa \subseteqq BRB \subseteqq B$ so that aRa is indeed the smallest bi-ideal containing a. Conversely if aRa contains a then a is regular.

(1. 8) ($\mathscr{S}$) Proposition. *Let B be a bi-ideal of a ring R. If B is itself a regular ring then any bi-ideal of B is a bi-ideal of R.*

Proof. Let A be a bi-ideal of B. Then A is also a subring of R. Since B is regular we have $A \subseteqq AB$ and $A \subseteqq BA$ so that $ARA \subseteqq (AB)R(BA) \subseteqq A(BRB)A \subseteqq ABA \subseteqq A$. Thus A is a bi-ideal of R.

The following two propositions are generalizations of [1] ex. 18d, p. 85, [6] (2. 15) and [10] Theorem 1.

(1. 9) Proposition. *Let S be a semigroup and B a bi-ideal of S. If the elements of B are regular then B is a quasi-ideal.*

Proof. If $bs = rb^* \in BS \cap SB$ then there is a $b' \in S$ such that $bb'\, b = b$. Thus $bs = bb'bs = b(b'r)b^* \in BSB \subseteqq B$. Whence $BS \cap SB \subseteqq B$ and B is a quasi-ideal.

(1. 10) Proposition. *Let R be a ring and B a bi-ideal of R. If every element of B is regular then B is a quasi-ideal.*

Proof. Let $x \in BR \cap RB$. Then $x = \sum_{i=1}^{n} b_i r_i = \sum_{j=1}^{m} s_j b_j (*)$ where $b_i, b_j \in B$, $r_i, s_j \in R$. We procede inductively: $b_1 = b_1 t_1 b_1$ for some $t_1 \in R$ so $b_1 r_1 = b_1 t_1 b_1 r_1 = -\sum_{2}^{n} b_1 t_1 b_i r_i + b_1 t_1 x = -\sum_{2}^{n} b_i' r_i + b_1'$ where $b_1' \in B$ since $x \in RB$. Substituting back in $(*)$ $b_1' + \sum_{2}^{n} b_i'' r_i = x$. Again for b_2 we have a $t_2 \in R$ with $b_2 = b_2 t_2 b_2$ so that

$b_2 r_2 = -\sum_{3}^{n} b_i'' r_i + x - b_1'$ and

$$b_2 r_2 = b_2 t_2 b_2 r_2 = -\sum_{3}^{n} b_2 t_2 b_i'' r_i + b_2 t_2 (x - b_1') = -\sum_{3}^{n} b_2 t_2 b_i'' r_i + b_2'.$$

Substituting again we have $b_1' + b_2' + \sum_{3}^{n} b_i'' r_i = x$. We continue in the above fashion to obtain $\sum_{1}^{n} b_i' = x$. Since $(B, +)$ is a group the result follows.

It is possible to combine this with (1. 8) to obtain:

(1. 11) Corollary. *Let B be a bi-ideal of a ring R. If B is itself a regular ring then any bi-ideal of B is a quasi-ideal of R as well as B. If Q is a quasi-ideal of R which is itself regular then any quasi-ideal of Q is also a quasi-ideal of R.*

§ 2. General results on minimal bi-ideals

We gather in this section several general results concerning minimal bi-ideals. We have first the following definition.

(2. 1) $(\mathscr{S})$ Definition. A non-zero quasi-ideal [bi-ideal] U of a ring R is a *minimal* quasi-ideal [bi-ideal] if there is no quasi-ideal [bi-ideal], T, with $\{0\} \subset T \subset U$.

(We use $\subset$ for proper containment.) A similar definition is given for a semigroup $S=S^0$.

(2. 2) ($\mathscr{S}$-[6] (1. 8)) Proposition. *Let B be a minimal bi-ideal of a ring R. Then B is nilpotent if and only if $B^2=\{0\}$.*

Proof. Let $n\geqq 2$. Then since the product of two bi-ideals is a bi-ideal B^{n-1} is a bi-ideal which is clearly contained in B and we have $B^{n-1}=B$ if $B^{n-1}\neq\{0\}$. Thus $B^n=B^2=\{0\}$ precisely when B is nilpotent.

(2. 3) ($\mathscr{S}$) Proposition. *Let B be a minimal bi-ideal of a ring R with $B^2\neq\{0\}$. If $b_1\ Bb_1=\{0\}$ and $b_2\ Bb_2=\{0\}$ $[b_1Rb_1=\{0\}$ and $b_2Rb_2=\{0\}]$ for fixed $b_1, b_2\in B$ then $b_1\ Bb_2=\{0\}$ and $b_2\ Bb_1=\{0\}$ $[b_1Rb_2=\{0\}$ and $b_2Rb_1=\{0\}]$.*

Proof. If $b_1Bb_2\neq\{0\}$ then $b_1Bb_2=B$ by the minimality of B and [12] Theorem 8. We have $B^2=(b_1Bb_2b_1)Bb_2\subseteqq(b_1Bb_1)Bb_2=\{0\}$ a contradiction. Thus $b_1Bb_2=\{0\}$ and similarly $b_2Bb_1=\{0\}$.

The proof of the alternate reading is similar. Here we would have $B^2= =(b_1Rb_2b_1)Rb_2\subseteqq(b_1Rb_1)Rb_2=\{0\}$.

We remark that the above proposition is also valid with any bi-ideal T in place of R provided either $Tb_2\subseteqq T$ or $b_1T\subseteqq T$.

(2. 4) ($\mathscr{S}$-[6] (1. 8)) Theorem. *Let B be a minimal bi-ideal of a ring R. If $B^2\neq\{0\}$ then B is a division ring and a minimal quasi-ideal. Indeed B is of the form $B=eRe=eBe$ where e is the identity of B.*

Proof. Let $C=\{b\in B\mid bB=\{0\}\}$. It follows immediately that C is a subring since B is. Moreover for $c_1, c_2\in C$, $r\in R$, $c_1rc_2\in B$ and hence $c_1rc_2\in C$ so that C is a bi-ideal of R. By the minimality of B we must have $C=\{0\}$ since $B^2\neq\{0\}$ by hypothesis. Thus for $b\in B\setminus\{0\}$, $b\ B\neq\{0\}$. Since bB is a bi-ideal ([12] Theorem 8) it follows that $bB=B$. Similarly $Bb=B$. Thus for $b\in B\setminus\{0\}$ we have $Bb=bB=B$ and it follows that B is a division ring. Clearly B is thus regular so that B is a quasi-ideal by (1. 10). Since B is minimal as a bi-ideal it is surely minimal as a quasi-ideal. It now follows immediately from [20] Theorem 3 that $B=eRe=eBe$ where e is the identity of B.

§ 3. Nilpotent minimal bi-ideals

We have seen in the last section that minimal bi-ideals which are not nilpotent are quasi-ideals and moreover division rings. We will now consider the alternative case when the bi-ideal is nilpotent (recall (2. 2)!).

(3. 1) Definition. We will call a minimal bi-ideal [quasi-ideal] B a *nilpotent minimal bi-ideal* [quasi-ideal] if B is a zero subring, i.e., $B^2=\{0\}$.

As the following example show, even in a commutative ring, a nilpotent minimal bi-ideal need not be a (minimal) quasi-ideal. Thus the sets of minimal bi-ideals and minimal quasi-ideals for a given ring need not coincide.

(3. 2) Example. Let $S=Z/(6)$ where Z is the ring of integers and set $R==R[\bar{x}]=S[x]/(x^4)$ where x is transcendental over S. Let $B=\{0, 2\bar{x}^2, 4\bar{x}^2\}$. Clearly B is a subring of R. Since $B^2=\{0\}$ and R is commutative $BRB=B^2R=\{0\}\subseteq B$ so that B is a bi-ideal of R. However $4\bar{x}^3 = \bar{x}(4\bar{x}^2) = (4\bar{x}^2)\bar{x}\in BR\cap RB$ but $4\bar{x}^3\notin B$ so that B is not a quasi-ideal. It is easy to see that B is also a minimal bi-ideal.

We note that a similar statement is also true in the case of a commutative semigroup. It suffices to consider $(R, \cdot)$ above as our semigroup and $B'=\{0, 4\bar{x}^2\}$. Then $(B')^2=\{0\}$ so that $\{0\}=B'RB'\subseteq B'$ while again $4\bar{x}^3\notin B'$.

(3. 3) Theorem. *Let B be a nilpotent minimal bi-ideal of a semigroup $S=S^0$. Then the following sets of equivalent statements are mutually exclusive.*

1. *some non-zero element of B is irregular (iff),*
2. *no non-zero element of B is regular (iff),*
3. *for some $b\in B\setminus\{0\}$, $bSb=\{0\}$ (iff),*
4. *for each $b\in B$, $bSb=\{0\}$*

[*in any of the above cases $B=\{b, 0\}$*];

5. *each element in B is regular (iff),*
6. *some non-zero element of B is regular (iff),*
7. *$bSb\neq\{0\}$ for each $b\in B\setminus\{0\}$ (iff),*
8. *$bSb\neq\{0\}$ for some $b\in B$*

[*in any of these cases B is a quasi-ideal*].

Proof. In any of the above cases one need consider only bSb for $b\in B$. We observe that bSb is a bi-ideal contained in B. Thus by the minimality of B either $bSb=\{0\}$ or $bSb=B$. In cases 1 or 2 if b is irregular then $b\notin bSb$ so $bSb\subset B$ and hence $bSb=\{0\}$. Clearly $\{b, 0\}$ is then a bi-ideal and hence $B=\{b, 0\}$. The equivalence of statements 1—4 should now be obvious.

Indeed, it is now clear that a non-zero element of B can be regular precisely when each element in B is regular. Furthermore $b\neq 0$ is regular iff $bRb\neq\{0\}$ since in such a case $bRb=B$. It follows that each of the statements 5—8 are equivalent and for any of these cases B is a quasi-ideal by (1. 9).

We give the analogous result for nilpotent minimal bi-ideals in rings.

(3. 4) Theorem. *Let B be a nilpotent minimal bi-ideal of a ring R. Then the following sets of equivalent statements are mutually exclusive.*

1. *some non-zero element of B is irregular (iff),*
2. *no non-zero element of B is regular (iff),*
3. *for some $b\in B\setminus\{0\}$, $bRb=\{0\}$ (iff),*

4. *for each* $b \in B$, $bRb = \{0\}$

[*in any of the above cases* $B = ([b], +)$ *where* b *is of prime order*];

5. *each element in* B *is regular (iff),*
6. *some non-zero element of* B *is regular (iff),*
7. $bRb \neq \{0\}$ *for each* $b \in B \setminus \{0\}$ *(iff),*
8. $bRb \neq \{0\}$ *for some* $b \in B$

[*in any of these cases* B *is a quasi-ideal*].

Proof. The additive operation of R does not enter into consideration unti the final conclusion is approached. The proof of (3. 3) can be repeated intact. Now however if $bRb = \{0\}$, b will generate an additive subgroup $[b]$ which is a bi-ideal. Since $(nb)r(mb) = b(nmr)b = 0$ any subgroup of $([b], +)$ will also be a bi-ideal. Thus $B = ([b], +)$ and the order of b must clearly be finite (else take $[2b]$ etc.) and prime. If any of the conditions 5—8 hold B will be a quasi-ideal by (1. 10).

If B is a subgroup of a ring R, with $B^2 = \{0\}$ and the order of B prime, then it is clear that if B is a bi-ideal it must be minimal. It suffices to have either R commutative or B contained in the center of R to have this be the case. As the following example shows it is possiblet o have a subgroup $(B, +)$ of prime order and $B^2 = \{0\}$ without B being a bi-ideal.

(3. 5) Example. Let R be the ring of 4×4 matrices over $Z/(p)$, where Z is the ring of integers and p is a prime number. Let

$$B = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \end{pmatrix} \middle| \quad a \in Z/(p) \right\}.$$

It is easy to check that $B^2 = \{0\}$ but that B is not a bi-ideal of R. Moreover if we take here

$$S = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & y & 0 \\ 0 & 0 & 0 & 0 \\ u & 0 & v & 0 \end{pmatrix} \middle| \quad x, y, u, v \in Z/(p) \right\}$$

then S is a bi-ideal of R and B a bi-ideal of S since $BS = \{0\}$. Thus the regularity condition of (1. 8) is in one sense necessary for the middle subring. Here R is, as is well known, a regular ring.

It is easy to observe from the above two theorems ((3. 3) and (3. 4)), (2. 4) and [20] Theorem 3 that if a minimal bi-ideal (or quasi-ideal) is either a division ring (or group union $\{0\}$ in the semigroup case) or nilpotent and possesses no non-zero regular element (regular in the original ring or semigroup) then the bi-ideal (or quasi-ideal) considered itself as a ring (or semigroup) contains no non-trivial bi-ideals (or quasi-ideals).

In the first case the bi-ideal is also a quasi-ideal. This situation is altered in the remaining case when the elements of the nilpotent minimal bi-ideal are regular (the second set of conditions in (3. 4) or (3. 5)). Here there may be many proper bi-ideals of the given minimal bi-ideal. We conclude with the following examples which illustrate this situation.

(3. 6) Example. Let S be a completely 0-simple semigroup over a non-trivial group, G, (cf. [1], [4]) where S is not a completely simple semigroup with a adjoined 0, i.e., S has at least one non-zero nilpotent $\mathcal{H}$-class. It is easy to see that the minimal bi-ideals of S are just individual non-zero $\mathcal{H}$-classes union $\{0\}$. Since S is regular these are also the minimal quasi-ideals of S (cf. [19], [22], [5]). Let B denote a non-zero nilpotent $\mathcal{H}$-class union $\{0\}$. Then B is a minimal bi-ideal satisfying the second set of conditions in (3. 3). Since G is non-trivial, $|B|>2$. It is easy to see since $B^2=\{0\}$ that any proper subset of B which contains 0 will be a bi-ideal (quasi-ideal) of B.

(3. 7) Example. Let Q denote the rational numbers and let R by the complete ring of 2×2 matrices over Q. R is a regular ring. Let $B=\left\{\begin{pmatrix}0 & 0\\ q & 0\end{pmatrix} \middle| q\in Q\right\}$. Then one readily checks that B is a nilpotent minimal bi-ideal (quasi-ideal) of R which falls under the second set of conditions in (3. 4). Again since $B^2=\{0\}$ any non-trivial subgroup (under addition) of B, and there are many, will be a bi-ideal (quasi-ideal) of B.

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A note on semilattices of groups

By SÁNDOR LAJOS in Budapest

Let S be a semigroup. Following the notation and terminology of A. H. CLIFFORD and G. B. PRESTON [2] we shall say that S is a semilattice of groups if it is the set-theoretical union of a set of mutually disjoint subgroups G_α $(\alpha \in \Lambda)$:

(1) $$S = \bigcup_{\alpha \in \Lambda} G_\alpha$$

such that, for any couple α, β in Λ, the products $G_\alpha G_\beta$ and $G_\beta G_\alpha$ are both contained in the same G_γ $(\gamma \in \Lambda)$.

Recently the author proved several ideal-theoretic characterizations of semigroups that are semilattices of groups (see [3]—[10]). In this note we shall prove two new criteria for a semigroup S to be a semilattice of groups.

Theorem 1. *A semigroup S is a semilattice of groups if and only if the set of all bi-ideals of S is a semilattice under the multiplication of subsets.*

Proof. First suppose that S is a semigroup being a semilattice of groups. Then, by a recent result of the author [5]

(2) $$B_1 \cap B_2 = B_1 B_2$$

for any couple of bi-ideals of S. This implies that every bi-ideal B of S is globally idempotent and the condition

(3) $$B_1 B_2 = B_2 B_1$$

holds for any two bi-ideals B_1, B_2 of S. Hence the set of all bi-ideals of S becomes a multiplicative semilattice.

Conversely, let S be a semigroup where bi-ideals form a commutative idempotent semigroup. Then $L^2 = L$ and $R^2 = R$ for any left ideal L, and any right ideal R of S, respectively. Furthermore, the condition

(4) $$LR = RL$$

holds for every left ideal L and every right ideal R of S. Then, by a result of J. CALAIS

[1], S is a regular semigroup. Finally, a recent criterion of the author [6] guarantees that S is a semilattice of groups. Our Theorem 1 is completely proved.

The proof of the following result is quite similar to that of Theorem 1.

Theorem 2. *A semigroup S is a semilattice of groups if and only if all the quasi-ideals of S form a semilattice under the multiplication of subsets.*

Theorem 1, Theorem 2, and the author's earlier results imply the following statement.

Theorem 3. *For a semigroup S the following conditions are pairwise equivalent:*

(A) *S is a semilattice of groups.*

(B) *$L\cap R=LR$ for every left ideal L and every right ideal R of S.*

(C) *$B\cap L=LB$ for every bi-ideal B and left ideal L of S.*

(D) *$B\cap R=BR$ for every bi-ideal B and right ideal R of S.*

(E) *$L\cap Q=LQ$ for every left ideal L and quasi-ideal Q of S.*

(F) *$Q\cap R=QR$ for every quasi-ideal Q and right ideal R of S.*

(G) *$B\cap Q=BQ$ for every bi-ideal B and quasi-ideal Q of S.*

(H) *$B\cap Q=QB$ for every bi-ideal B and quasi-ideal Q of S.*

(I) *$Q_1\cap Q_2=Q_1Q_2$ for every two quasi-ideals of S.*

(J) *$B_1\cap B_2=B_1B_2$ for every couple of bi-ideals of S.*

(K) *$L_1\cap L_2=L_1L_2$ and $R_1\cap R_2=R_1R_2$ for every two left ideals L_1, L_2 and for every two right ideals R_1, R_2 of S.*

(L) *$I\cap L=LI$ and $I\cap R=IR$ for every left ideal L, right ideal R, and two-sided ideal I of S.*

(M) *$L\cap R=LSR$ for every left ideal L and right ideal R of S.*

(N) *$Q_1\cap Q_2=Q_1SQ_2$ for every two quasi-ideals of S.*

(O) *$B_1\cap B_2=B_1SB_2$ for every couple of bi-ideals of S.*

(P) *The intersection of every k quasi-ideals of S is equal to their product (k is an arbitrary fixed positive integer >1).*

(Q) *The intersection of every k bi-ideals of S is equal to their product (k is an arbitrary fixed positive integer >1).*

(R) *The set of all quasi-ideals of S is a multiplicative semilattice.*

(S) *The set of all bi-ideals of S is a commutative band under the multiplication of subsets.*

(T) *S is centric and every principal left ideal of S is globally idempotent.*

(U) *The intersection of every k left ideals of S is equal to their product and the same holds for right ideals too ($k>1$).*

(V) *S is a completely regular*[1]) *inverse semigroup.*

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[1]) A semigroup S is said to be *completely regular* if to every element a in S there exists x of S such that $axa=a$ and $ax=xa$.

(W) *S is a regular duo*[2]) *semigroup.*

(X) $A \cap B = AB$ *for every two* (m, n)*-ideals of* S *(*m*,* n *are arbitrary positive integers).*

(Y) *The intersection of every* k (m, n)*-ideals of* S *is equal to their product (*k*,* m*,* n *are arbitrary fixed positive integers,* $k>1$*).*

(Z) $A \cap B = AB$ *for every* $(0, n)$*-ideal* A *of* S *and every* $(m, 0)$*-ideal* B *of* S *(*m*,* n *are arbitrary fixed positive integers).*

For the definition and fundamental properties of (m, n)-ideals of semigroups, see the author's papers [11] and [12].

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[2]) A semigroup S is called a *duo semigroup* if every one-sided (left or right) ideal of S is two-sided.

Direct product in locally finite categories

By L. LOVÁSZ in Budapest

A category will be called *locally finite* if only a finite number of morphisms joins any two objects of it. We are going to study "cancellation" properties of direct products in such categories. The results are generalizations of those of [1, 2] and no essentially new idea is used; however, the questions and results extend to categories in such a natural and general form that it seems to be worth stating them in a short note.[1])

Given a category $\mathcal{K}$, we denote by $H(A, B)$ the set of all morphisms of $\mathcal{K}$ from A to B.

Lemma 1. *Let A, B be objects of the locally finite category $\mathcal{K}$, and assume that there are monomorphisms φ from A to B and η from B to A. Then both φ and η are isomorphisms.*

Proof. Consider the morphisms $(\varphi\eta)^n$ $(n=1, 2, \ldots)$. Since $\mathcal{K}$ is locally finite there exist $k, m>0$ such that

$$(\varphi\eta)^k = (\varphi\eta)^{k+m}.$$

Now $\varphi\eta$ being a monomorphism, this implies

$$(\varphi\eta)^m = id_A, \tag{1}$$

i.e. putting $\eta' = \eta(\varphi\eta)^{m-1}$, we have $\varphi\eta' = \mathrm{id}_A$. Multiplication of (1) from the left by η gives $(\eta\varphi)^m\eta = \eta \cdot (\varphi\eta)^m = \eta$; since η is a monomorphism, it follows that

$$(\eta\varphi)^m = \eta'\varphi = id_B.$$

Hence η' is the inverse of φ and thus φ is an isomorphism. Similarly η is an isomorphism.

Remark. Obviously, Lemma 1 remains true if we consider only the subcategory determined by A and B.

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[1]) Recently A. Pultr (Prague) informed me that he also remarked the possibility of this generalization and obtained similar results.

Let $M(A, B)$ be the set of all monomorphisms of A to B. If P is an equivalence relation on the set $S(X)$ of all morphisms to X, we denote by $H(P, A)$ the set of those morphisms $\varphi \in H(X, A)$ which satisfy $\alpha\varphi = \alpha'\varphi$ for every $(\alpha, \alpha') \in P$.

Lemma 2. *Let A, B, X be objects in the locally finite category $\mathcal{K}$ and assume $|H(P, A)| = |H(P, B)|$ for every equivalence-relation P on $S(X)$. Then $|M(X, A)| = |M(X, B)|$.*

Proof. We may assume $\mathcal{K}$ and thus $S(X)$ are finite. Obviously, $\varphi \in M(X, A)$ iff $\varphi \notin H(P, A)$ except P is the identity relation j on $S(X)$. Hence, by sieving we get

$$(2) \quad |M(X, A)| = \sum_{k \geqq 0} (-1)^k \sum_{P_1, \ldots, P_k \neq j} |H(P_1, A) \cap \cdots \cap H(P_k, A)| = \\ = \sum_{k \geqq 0} (-1)^k \sum_{P_1, \ldots, P_k \neq j} |H(P_1 \vee \cdots \vee P_k, A)|,$$

where $P_1 \vee \cdots \vee P_k$ means the least equivalence-relation containing $P_1 \cup \cdots \cup P_k$ (the member corresponding to $k=0$ is $|H(j, A)| = |H(X, A)|$).

Now $|M(X, B)|$ can also be expressed by a formula like (2) and the two right hand sides are equal by assumption. Hence the statement follows.

Remark. If the condition of the lemma holds for $X=A$ and $X=B$ then $|M(A, B)| = |M(A, A)| > 0$ and $|M(B, A)| = |M(B, B)| > 0$, thus by Lemma 1, A and B are isomorphic.

Lemma 3. *Let (π_1, π_2) be a (projective) direct product; $\pi_1 \in H(AB, A)$, $\pi_2 \in H(AB, B)$. Then for any object X and equivalence-relation P on $S(X)$, $|H(P, AB)| = |H(P, A)| \cdot |H(P, B)|$.*

Proof. It is easy to verify that a $\varphi \in H(X, AB)$ belongs to $H(P, AB)$ iff $\varphi\pi_1 \in H(P, A)$ and $\varphi\pi_2 \in H(P, B)$. Hence the proposition follows.

We prove now that the kth root is unique in any locally finite category.

Theorem 1. *Let $(\pi_1, \ldots, \pi_k)$ and $(\varrho_1, \ldots, \varrho_k)$ be two (projective) direct decompositions of the same object C of the locally finite category $\mathcal{K}$ and let $\pi_1, \ldots, \pi_k \in H(C, A)$, $\varrho_1, \ldots, \varrho_k \in H(C, B)$ (Fig. 1). Then A and B are isomorphic.*

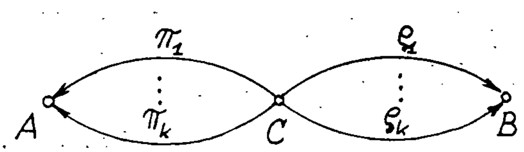

Fig. 1

Proof. By Lemma 3 we have $|H(P, C)|=|H(P, A)|^k=|H(P, B)|^k$ for any equivalence relation P on any $S(X)$. Hence $|H(P, A)|=|H(P, B)|$. By the remark following Lemma 2, this implies that A, B are isomorphic.

A question analogous to Theorem 1 is whether the following diagram (Fig. 2) implies that A and B are isomorphic (ι is isomorphism, $\pi_1, \pi_2, \varrho_1, \varrho_2$ direct products).

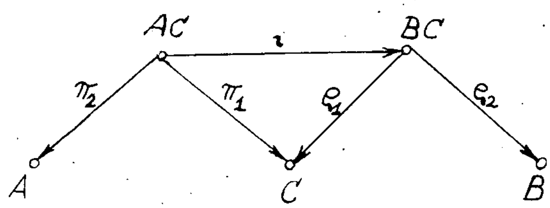

Fig. 2

This is not the case in general but we have

Theorem 2. *If in Fig. 2 both A and B have morphisms into C then they are isomorphic.*

Proof. Let P be an equivalence relation on $S(X)$ where X is some object. By Lemma 3,

$$|H(P, A)|\cdot|H(P, C)| = |H(P, AC)| = |H(P, BC)| = |H(P, B)|\cdot|H(P, C)|.$$

Now if $H(P, C)\neq\emptyset$ then $|H(P, A)|=|H(P, B)|$. But this also follows if $|H(P, C)|= =\emptyset$, since then both $H(P, A)$ and $H(P, B)$ are empty. Hence by Lemma 2, A and B are isomorphic.

Now we consider the case when Theorem 2 cannot be applied.

Theorem 3. *In the diagram of Fig. 3, AD and BD are isomorphic* ((π_1, π_2), (ϱ_1, ϱ_2), (σ_1, σ_2), (τ_1, τ_2) *are direct products, ι is an isomorphism*).

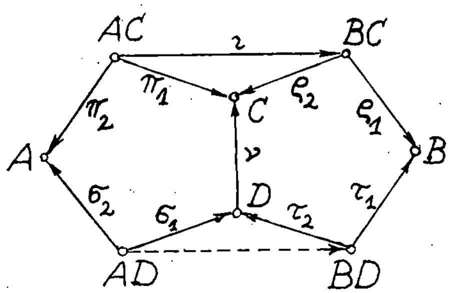

Fig. 3

The proof is very similar to that of Theorem 2, therefore we omit it.

Theorem 4. *In the diagram of Fig. 2 one can always find an isomorphism ι such that the diagram commutes.*

Proof. Denote by $\mathcal{K}^*$ the category of those morphisms φ which are in $H(AC, AC)$ and satisfy $\varphi\pi_1=\pi_1$; or in $H(AC, BC)$ and satisfy $\varphi\varrho_1=\pi_1$; or in $H(BC, AC)$ and satisfy $\varphi\pi_1=\varrho_1$; or in $H(BC, BC)$ and satisfy $\varphi\varrho_1=\varrho_1$. It is easily seen that these morphisms form a category indeed. We denote the set of morphisms, monomorphisms etc. in $\mathcal{K}^*$ by $H^*(X, Y)$, $M^*(X, Y)$ etc.

Let P be an equivalence relation on $S(X)$ where $X=AC$ or BC. It is enough to show

$$|H^*(P, AC)=|H^*(P, BC)|, \tag{3}$$

since then by the remark after Lemma 2 the statement follows.

Consider the pairs (δ, φ) where $\delta\in H(X, C)$ and $\varphi\in H^*(P, AB)$. Their number is $|H(X, C)|\cdot|H^*(P, AC)|$. We attach to every such (δ, φ) an (ε, ψ) where $\varepsilon\in H(X, C)$ and $\psi\in H^*(P, BC)$. Let φ^* be defined by

$$\varphi^*\pi_1=\delta, \quad \varphi^*\pi_2=\varphi\pi_2,$$

and set

$$\psi^*=\varphi^*\iota, \quad \varepsilon=\psi^*\varrho_1.$$

Define ψ by

$$\psi\varrho_1=\begin{cases}\pi_1 & \text{if } X=AC,\\ \varrho_1 & \text{if } X=BC,\end{cases} \qquad \psi\varrho_2=\psi^*\varrho_2.$$

Then, obviously, $\psi\in H^*(P, BC)$ and $\varepsilon\in H(X, C)$, and the correspondence is one-to-one since the argument defining (ε, ψ) can be converted. Hence the number of pairs (ε, ψ) $(\varepsilon\in H(X, C)$, $\psi\in H^*(P, BC)$ is also $|H(X, C)|\cdot|H^*(P, AC)|$, and since $H(X, C)\neq\emptyset$, (3) follows.

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Ein Satz vom Jacksonschen Typ für algebraische Polynome

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1. Einleitung

Sei f eine stetige Funktion auf $[-1, 1]$ mit dem gewöhnlichen Stetigkeitsmodul $\omega(f; h)$. Der Satz von D. JACKSON [11] sagt aus, daß es ein algebraisches Polynom $P_n(x)$ vom Grade $\leqq n$ gibt, so daß gilt

$$(1.1) \qquad |f(x)-P_n(x)| \leqq M_1 \omega\left(f; \frac{1}{n}\right) \qquad (-1 \leqq x \leqq 1).$$

Dieses Ergebnis wurde von A. F. TIMAN [29] wie folgt verbessert: Es gibt ein algebraisches Polynom $P_n(x)$ vom Grade $\leqq n$, so daß gilt

$$(1.2) \qquad |f(x)-P_n(x)| \leqq M_2 \omega(f; \Delta_n(x)) \qquad (-1 \leqq x \leqq 1),$$

wobei $\Delta_n(x)$ definiert ist durch

$$(1.3) \qquad \Delta_n(x) = \frac{1}{n}\left(\sqrt{1-x^2}+\frac{1}{n}\right).$$

G. G. LORENTZ [16, p. 65] wählt an Stelle von $\Delta_n(x)$ den Ausdruck

$$\Delta_n^*(x) = \max\left(\frac{\sqrt{1-x^2}}{n}, \frac{1}{n^2}\right).$$

Wegen $\Delta_n(x) \leqq 2\Delta_n^*(x) \leqq 2\Delta_n(x)$ kann man in (1.2) Δ_n durch Δ_n^* ersetzen und umgekehrt. Außerdem ist die unten erwähnte Verallgemeinerung auf L^p-Räume leichter mit der Funktion Δ_n durchzuführen.

V. K. DZYADYK [8] und A. F. TIMAN [30] haben folgende Umkehrung als Verschärfung eines Ergebnisses von S. N. BERNSTEIN [2] bewiesen: Gibt es für eine Funktion f eine Folge von algebraischen Polynomen P_n, die die Bedingung

$$(1.4) \qquad |f(x)-P_n(x)| \leqq M\omega^*(\Delta_n(x)) \qquad (-1 \leqq x \leqq 1)$$

erfüllen, wobei ω^* eine Funktion ist, die die Eigenschaften eines Stetigkeitsmoduls besitzt (s. z. B. [16, p. 43]), dann existiert eine Konstante $C>0$, so daß gilt

$$\omega(f; h) \leqq C \cdot h \cdot \int_h^1 \frac{\omega^*(u)}{u^2} du. \tag{1.5}$$

Diese Ergebnisse wurden von M. K. POTAPOV [22] auf die Approximation von L^p-Funktionen in der L^p-Norm verallgemeinert.

In der Bedingung (1. 2) hängt die Approximationsordnung von dem betreffenden Punkt x ab. Sie ist in den Endpunkten des Intervalls $[-1, +1]$ besser als im Innern. Man kann nun versuchen, den Stetigkeitsmodul ω in (1. 1) so zu einer Funktion Ω zu verallgemeinern, daß die Abschätzung (1. 1) noch für den verallgemeinerten Stetigkeitsmodul gültig ist. Dies wurde schon von G. V. ŽIDKOV [32] bei der Approximation in der L^2-Norm getan und von A. S. DŽAFAROV [7] für die C-Norm durchgeführt, wobei er Ergebnisse von G. G. KUŠNIRENKO [13, 14] für Funktionen, die auf der Einheitskugel im $\mathbf{R}^3$ definiert sind, angewandt hat. Verallgemeinerungen der Ergebnisse auf gewichtete L^2-Räume stammen von S. Z. RAFALSON [23, 24, 25]. Für beliebige L^p-Räume $(p \geqq 1)$ gibt es Ergebnisse von M. K. POTAPOV [22].

Das Ziel dieser Arbeit ist es, mehrere verallgemeinerte Stetigkeitsmoduln einzuführen und Sätze vom Jacksonschen Typ in der C-Norm und der L^p-Norm zu beweisen. Beim Beweis des Jacksonschen Satzes wird an Stelle eines gewöhnlichen Faltungsintegrals ein Integral vom ultrasphärischen Faltungstyp benutzt. Man erhält damit einen neuen direkten Beweis des Jacksonschen Satzes für algebraische Polynome (vergleiche R. DE VORE [31] und R. BOJANIC [4]), wobei in dieser Arbeit eine schwächere Form des Stetigkeitsmoduls benutzt wird.

Im fünften Abschnitt werden Approximationssätze für Funktionen bewiesen, die gewissen Differenzierbarkeitseigenschaften genügen. Nach einem Beispiel im 6. Abschnitt folgt im 7. Abschnitt eine Verallgemeinerung der Ergebnisse auf Funktionen, die auf der Oberfläche der Einheitskugel im k-dimensionalen euklidischen Raum definiert sind.

2. Bezeichnungen

Es sei C die Menge der auf dem Intervall $[-1, 1]$ stetigen Funktionen mit der üblichen Norm. Mit L^p_λ, $1 \leqq p < \infty$, $\lambda \geqq 0$, wird die Menge der auf $(-1, 1)$ Lebesgue-messbaren Funktionen bezeichnet, für die das Integral

$$\|f\|_{p,\lambda} = \left\{ \int_{-1}^1 |f(x)|^p w_\lambda(x)\, dx \right\}^{1/p} \tag{2.1}$$

existiert und endlich ist, wobei die Gewichtsfunktion w_λ gegeben ist durch

$$(2.2) \qquad w_\lambda(x) = (1-x^2)^{\lambda-\frac{1}{2}} \cdot \Big(\int_{-1}^{1} (1-y^2)^{\lambda-\frac{1}{2}}\, dy \Big)^{-1}.$$

Mit X ist immer einer der Räume C oder L_λ^p gemeint. Orthogonalisiert man die Funktionen $1, x, x^2, \ldots$ bezüglich der Gewichtsfunktion w_λ, dann erhält man die Gegenbauer- oder ultrasphärischen Polynome $P_n^\lambda(x)$. Normalisiert man sie durch die Bedingung

$$(2.3) \qquad P_n^\lambda(1) = 1,$$

dann gilt

$$(2.4) \qquad |P_n^\lambda(x)| \leqq 1 \qquad (-1 \leqq x \leqq 1),$$

$$(2.5) \qquad \int_{-1}^{1} P_n^\lambda(x) P_m^\lambda(x) w_\lambda(x)\, dx = \frac{\delta_{nm}}{c(n, \lambda)} \qquad (\lambda > 0),$$

$$(2.6) \qquad c(n, \lambda) = \frac{n+\lambda}{\lambda} \frac{\Gamma(n+2\lambda)}{\Gamma(2\lambda) n!} \qquad (\lambda > 0),$$

$$(2.7) \qquad \lim_{\lambda \to 0} c(n, \lambda) P_n^\lambda(\cos \vartheta) = \frac{2}{n} \cos n\vartheta \qquad (n \geqq 1).$$

Einer Funktion f aus X kann man die Entwicklung

$$(2.8) \qquad f(x) \sim \sum_{n=0}^{\infty} c(n, \lambda) f^\wedge(n) P_n^\lambda(x),$$

$$(2.9) \qquad f^\wedge(n) = \int_{-1}^{1} P_n^\lambda(y) f(y) w_\lambda(y)\, dy$$

zuordnen, wobei die Funktion f durch die Koeffizienten $f^\wedge(n)$ eindeutig bestimmt ist.

Die ultrasphärische Faltung zweier Funktionen f, g aus L_λ^1 ist für $\lambda > 0$ definiert durch

$$(2.10) \qquad (f * g)_\lambda(x) = \int_{-1}^{1} \int_{-1}^{1} f(xy + \sqrt{1-x^2} \cdot \sqrt{1-y^2} \cdot z) \cdot v_\lambda(z)\, dz\, g(y) w_\lambda(y)\, dy$$

$$(2.11) \qquad v_\lambda(z) = (1-z^2)^{\lambda-1} \Big\{ \int_{-1}^{1} (1-u^2)^{\lambda-1}\, du \Big\}^{-1}.$$

Für $\lambda = 0$ gilt

$$(2.12) \qquad (f * g)_0(x) = \int_{-1}^{1} \tfrac{1}{2} \{ f(xy + \sqrt{1-x^2} \cdot \sqrt{1-y^2}) +$$

$$+ f(xy - \sqrt{1-x^2} \cdot \sqrt{1-y^2}) \} g(y) w_\lambda(y)\, dy.$$

Es gelten folgende Eigenschaften:

$$(f*g)_\lambda=(g*f)_\lambda, \tag{2.13}$$

$$(f*g)_\lambda^\wedge(n)=f^\wedge(n)\cdot g^\wedge(n). \tag{2.14}$$

Ist $f\in L_\lambda^1$, $g\in L_\lambda^p$ oder $g\in C$, dann gilt

$$\|(f*g)_\lambda\|_{p,\lambda}\leqq\|f\|_{1,\lambda}\cdot\|g\|_{p,\lambda}, \tag{2.15}$$

$$\|(f*g)_\lambda\|_C\leqq\|f\|_{1,\lambda}\cdot\|g\|_C. \tag{2.16}$$

Man definiert nun eine verallgemeinerte Translation $T_h^\lambda f$ von f für $h>0$ durch

$$(T_h^\lambda f)(x)=\begin{cases}\int\limits_{-1}^{1}f\big(x\cos h+z\sqrt{1-x^2}\sin h\big)v_\lambda(z)\,dz & (\lambda>0),\\ \frac{1}{2}\big\{f\big(x\cos h+\sqrt{1-x^2}\sin h\big)+f\big(x\cos h-\sqrt{1-x^2}\sin h\big)\big\} & (\lambda=0).\end{cases} \tag{2.17}$$

$T_h^\lambda f$ ist für alle $f\in X$ definiert, und es gilt

$$\|T_h^\lambda f\|_X\leqq\|f\|_X, \tag{2.18}$$

$$(T_h^\lambda f)^\wedge(n)=f^\wedge(n)\,P_n^\lambda(\cos h), \tag{2.19}$$

$$\lim_{h\to 0}\|T_h^\lambda f-f\|_X=0. \tag{2.20}$$

Die Faltung kann damit in der Form

$$(f*g)_\lambda(x)=\int\limits_0^\pi(T_h^\lambda f)(x)g(\cos h)w_\lambda(\cos h)\sin h\,dh \tag{2.21}$$

geschrieben werden.

Die Eigenschaften (2. 3)—(2. 9) sind bekannt [28]. Der Begriff der ultrasphärischen Faltung wurde von S. Bochner [3] eingeführt. In der Arbeit [3] sind auch die Eigenschaften (2. 13)—(2. 16) und (2. 18)—(2. 20) bewiesen worden.

Bemerkung 2. 1. Man kann die verallgemeinerte Translation auch durch die Formel (2. 19) definieren. In Fall der Jacobi-Polynome wurden die so entstandenen Translationen von R. Askey und St. Wainger [1] und von C. Ganser [10] untersucht.

3. Der verallgemeinerte Stetigkeitsmodul

In der angestrebten Verallgemeinerung des in der Einleitung zitierten Satzes von Jackson wird der Stetigkeitsmodul $\omega(f, h)$ im wesentlichen durch die Differenz $\|T_h f-f\|$ ersetzt. Man definiert den verallgemeinerten Stetigkeitsmodul von f durch

$$\Omega_\lambda(f; h) = \sup_{0<t\leq h} \|T_t^\lambda f-f\| \qquad (0 < h \leqq \pi). \tag{3.1}$$

Es gilt: $\Omega_\lambda(f; h)$ ist eine auf $0<h\leqq\pi$ stetige, monoton wachsende Funktion mit

$$\lim_{h\to 0} \Omega_\lambda(f; h)=0 \qquad (f\in X). \tag{3.2}$$

Außerdem gilt:

Lemma 3. 1. *Zu jedem $f\in X$ gibt es eine für $0<t<\infty$ definierte, monoton wachsende, konkave (daher stetige) Funktion $K(t;f)$ mit den folgenden Eigenschaften:*

$$K(t;f)\leqq \max\left(1, \frac{t}{s}\right) K(s;f) \qquad (s, t>0), \tag{3.3}$$

$$\Omega_\lambda(f; h)\leqq C_1 \cdot K(h^2;f) \leqq C_2 \Omega_\lambda(f; h) \qquad (0 < h \leqq \pi), \tag{3.4}$$

wobei C_1 und C_2 (von λ abhängige) positive Konstanten sind.

$K(t,f)$ ist das von J. Peetre 1963 eingeführte K-Funktional. Der Beweis ist in [5, ch. III], [20] und [15] zu finden. Aus der Ungleichung (3. 3) folgt

$$K(\alpha\cdot t;f) \leqq (\alpha+1)\cdot K(t;f) \qquad (\alpha, t > 0). \tag{3.5}$$

4. Ein Satz vom Jacksonschen Typ

Der folgende Satz ist eine Verallgemeinerung des Jacksonsatzes.

Satz 4. 1. *Es sei $f\in X$ und Ω_λ wie oben definiert. Dann gibt es eine Folge von algebraischen Polynomen $I_n^\lambda f$, so daß gilt*

$$\|f-I_n^\lambda f\| \leqq M\Omega_\lambda\left(f; \frac{1}{n}\right), \tag{4.1}$$

wobei die Folge $I_n^\lambda f$ und die Konstante M von λ abhängen.

Zum Beweis des Satzes wird eine Folge von Polynomen $I_n^\lambda f$ angegeben, die sich als ultrasphärische Faltung von f mit nichtnegativen Polynomen darstellen läßt. Diese Polynome wurden von D. J. NEWMAN und H. S. SHAPIRO [18] zum Beweis des Jacksonschen Satzes für Funktionen auf der Einheitskugel benutzt. Wie schon oben erwähnt, verwenden R. DE VORE [31] und R. BOJANIC [4] gewöhnliche Faltungsintegrale. In allen Fällen wurde der gewöhnliche Stetigkeitsmodul benutzt. Dieser Beweis beruht wesentlich auf der Ungleichung (3. 5). Man definiert das Polynom k_{2m} vom Grade $2m$ durch

$$k_{2m}(x) = \{P^{\lambda}_{m+1}(x)/(x-x_{m+1})\}^2, \tag{4. 2}$$

wobei x_{m+1} die größte Nullstelle von $P^{\lambda}_{m+1}(x)$ ist. Dieses Polynom hat folgende Eigenschaften.

Lemma 4. 2. *Unter allen Polynomen $T(x)$, $T\not\equiv 0$, vom Grade $\leqq 2m$, die nicht negativ sind für $-1\leqq x\leqq 1$, wird der Quotient*

$$\int_{-1}^{1} xT(x)w_{\lambda}(x)\,dx\cdot\Big\{\int_{-1}^{1} T(x)w_{\lambda}(x)\,dx\Big\}^{-1} \tag{4. 3}$$

maximal, falls $T(x)=ck_{2m}(x)$ ist, wobei $c\neq 0$ eine beliebige Konstante ist. Das Maximum hat den Wert x_{m+1}.

Dieses Lemma stammt von D. J. NEWMAN und H. S. SHAPIRO [18]. Außerdem wird noch eine Abschätzung für x_{m+1} benötigt.

Lemma 4. 3. *Für die größte Nullstelle x_{m+1} von $P^{\lambda}_{m+1}(x)$ gilt die Abschätzung*

$$1-x_{m+1}\leqq c_{\lambda}\cdot\frac{1}{m^2} \tag{4. 4}$$

für genügend große m, wobei $c_{\lambda}>0$ nur von λ abhängt.

Beweis. Aus der Monotonie der Nullstellen $x_{m+1}(\lambda)$ als Funktion von λ (s. [28, ch. 6. 21]) folgt $x_{m+1}(\lambda)\geqq x_{m+1}(\{\lambda\})$, wobei $\{\lambda\}$ der kleinste halbzahlige Wert größer gleich λ ist. Für die Werte $\{\lambda\}$ ist (4. 4) ebenfalls von D. J. NEWMAN und H. S. SHAPIRO [18] bewiesen worden. Damit gilt für beliebige $\lambda\geqq 0$

$$1-x_{m+1}(\lambda)\leqq 1-x_{m+1}(\{\lambda\})\leqq c_{\lambda}\cdot\frac{1}{m^2},$$

womit die Behauptung bewiesen ist.

Bemerkung 4. 4. Für $0\leqq\lambda\leqq 1$ gewinnt man diese Abschätzung direkt aus den bekannten Ungleichungen für die Nullstellen der ultrasphärischen Polynome (s. [28, ch. 6. 21]).

Beweis von Satz 4. 1. Wir definieren eine Folge von Polynomen vom Grade $n=2m$ durch

$$(I_n^\lambda f)(x) = c_n(k_n * f)_\lambda(x) \tag{4. 5}$$

$$c_n^{-1} = \int_{-1}^{1} k_n(x) w_\lambda(x)\, dx. \tag{4. 6}$$

Wegen (2. 14) ist klar, daß $I_n^\lambda f$ ein Polynom vom Grade $\leqq n$ ist. Weiter gilt wegen (2. 21)

$$(I_n^\lambda f)(x) - f(x) = c_n \int_0^\pi \{(T_n^\lambda f)(x) - f(x)\} k_n(\cos h) w_\lambda(\cos h) \cdot \sin h\, dh.$$

woraus mit (3. 1) und (3. 4) folgt

$$\|I_n^\lambda f - f\| \leqq c_n \int_0^\pi \|T_h^\lambda f - f\| k_n(\cos h) w_\lambda(\cos h) \sin h\, dh$$

$$\leqq c_n \cdot C_1 \int_{-1}^{\pi} K(h^2; f) k_n(\cos h) w_\lambda(\cos h) \sin h\, dh.$$

Die Ungleichung (3. 5) ergibt nun

$$K(h^2; f) = K\left(h^2 \cdot n^2 \cdot \frac{1}{n^2}; f\right) \leqq (1 + n^2 h^2) \cdot K\left(\frac{1}{n^2}; f\right)$$

und damit wieder mit (3. 4)

$$\|T_n^\lambda f - f\| \leqq C_2 \Omega_\lambda\left(f; \frac{1}{n}\right)\left\{1 + n^2 \cdot c_n \int_0^\pi h^2 k_n(\cos h) w_\lambda(\cos h) \sin h\, dh\right\}.$$

Mit

$$h^2 \leqq \frac{\pi^2}{2}(1 - \cos h) \qquad (0 \leqq h \leqq \pi),$$

der Substitution $x = \cos h$, Lemma 4. 2 und Lemma 4. 3 folgt

$$\|I_n^\lambda f - f\| \leqq C_2 \Omega_\lambda\left(f; \frac{1}{n}\right)\left\{1 + n^2 \cdot \frac{\pi^2}{2} \cdot c_n \int_{-1}^{1} (1 - x) k_n(x) w_\lambda(x)\, dx\right\} \leqq$$

$$\leqq C_2 \Omega_\lambda\left(f; \frac{1}{n}\right)\left(1 + n^2 \cdot \frac{\pi^2}{2}(1 - x_{m+1})\right) \leqq C_2 \Omega_\lambda\left(f; \frac{1}{n}\right)\left(1 + n^2 \frac{\pi^2}{2} c_\lambda \cdot \frac{1}{m^2}\right) \leqq$$

$$\leqq M \Omega_\lambda\left(f; \frac{1}{n}\right),$$

womit Satz 4. 1 bewiesen ist.

Bemerkung 4. 5. Die von D. J. Newman und H. S. Shapiro [18] eingeführten Polynome $k_n(t)$ sind eine Verallgemeinerung der Polynome von L. Fejér [9] und P. P. Korovkin [12]. Im Falle $\lambda=0$ sind sie mit diesen identisch, wie die Darstellung von I. M. Petrov [21] zeigt.

Bemerkung 4. 6. Für die Funktion $\Omega(h)=h^\alpha$, $0<\alpha<1$, wurde das Ergebnis von Satz 4. 1 im Falle $X=L^2_{1/2}$ von G. V. Židkov [32] bewiesen, im Falle $X=L^2_\lambda$, $\lambda>0$, von S. Z. Rafalson [23, 24, 25], im Falle $X=C$ und $\lambda=\frac{1}{2}$ von A. S. Džafarov [7], für $X=L^p_0$, $1\leqq p\leqq\infty$, von M. K. Potapov [22].

5. Funktionen mit Differenzierbarkeitseigenschaften

Es sollen jetzt mit dem obigen zusammenhängende Folgerungen aus Differenzierbarkeitseigenschaften von f gezogen werden. Dazu benutzen wir den Differentialoperator Δ_λ, der formal gegeben ist durch

$$\Delta_\lambda=(1-x^2)^{-\lambda+\frac{1}{2}}\frac{d}{dx}(1-x^2)^{\lambda+\frac{1}{2}}\frac{d}{dx}.$$

Er hat die Eigenwerte $-n(n+2\lambda)$ und die Eigenfunktionen $P^\lambda_n(x)$, d.h. es gilt

$$\Delta_\lambda P^\lambda_n(x)=-n(n+2\lambda)P^\lambda_n(x)\qquad(n=0,1,2,\ldots),$$

wobei $P^\lambda_n(x)$ die einzigen polynomialen Eigenfunktionen des Operators Δ_λ sind (s. [28, ch. 4. 2]).

Sein Definitionsbereich wird durch das folgende Lemma charakterisiert.

Lemma 5. 1. *Für eine Funktion $f\in X$ sind folgende Aussagen äquivalent:*

a) *es existiert eine Funktion $g\in X$, so daß gilt*

$$\lim_{h\to 0}\left\|\frac{1}{2(2\lambda+1)h^2}\cdot(T^\lambda_h f-f)-g\right\|=0;$$

b) *es existiert eine Funktion $g\in X$, so daß gilt*

$$-n(n+2\lambda)f^\wedge(n)=g^\wedge(n);$$

c) *$f(x)$ ist lokal absolut stetig auf $(-1,1)$, die Funktion $(1-x^2)^{\lambda+\frac{1}{2}}f'(x)$ ist absolut stetig auf $[-1,1]$, verschwindet für $x=\pm1$, und es existiert eine Funktion $g\in X$ so daß für $-1\leqq x\leqq 1$ gilt*

$$(1-x^2)^{-\lambda+\frac{1}{2}}\frac{d}{dx}\left\{(1-x^2)^{\lambda+\frac{1}{2}}\frac{d}{dx}f(x)\right\}=g(x).$$

Dieses Lemma ist in [19, Sätze 4. 2. 5 und 5. 4. 3] bewiesen. Der Definitionsbereich des Operators Δ_λ ist in Teil c) genau angegeben.

Definition 5. 2. Man sagt, $f \in X$ ist aus dem Definitionsbereich von Δ_λ, $f \in D(\Delta_\lambda)$, falls f der Bedingung c) in Lemma 5. 1 genügt.

Δ_λ ist eine echte Verallgemeinerung der zweiten Ableitung, wie man am Beispiel der Funktion $f(x) = (1-x^2)^{\frac{3}{2}}$ und $X=C$ sieht. Es wird nun eine einfache Folgerung aus Satz 4. 1 angegeben.

Folgerung 5. 3.

a) *Ist* $f \in X$, *und gilt* $\Omega_\lambda(f; h) \leqq Mh^\alpha$ *mit* $0 < \alpha \leqq 2$, *dann folgt*

$$\|I_n^\lambda f - f\| = O\left(\frac{1}{n^\alpha}\right) \qquad (n \to \infty).$$

b) *Ist* $f \in D(\Delta_\lambda)$, *dann folgt*

$$\|I_n^\lambda f - f\| = O\left(\frac{1}{n^2}\right) \qquad (n \to \infty).$$

Beweis. Teil a) folgt direkt aus Satz 4. 1. Teil b) folgt aus der Beziehung (s. [19, Lemma 4. 2. 3])

$$T_h^\lambda f - f = \int_0^h (\sin t)^{-2\lambda} \int_0^t T_u \Delta_\lambda f \cdot (\sin u)^{2\lambda}\, du\, dt, \tag{5. 1}$$

woraus sich $\|T_h^\lambda f - f\| \leqq \|\Delta_\lambda f\| \cdot h^2$ ergibt.

Bemerkung 5. 4. Im klassischen Satz von Jackson ergibt sich die Approximationsordnung n^{-2} im Raume C nur, falls man voraussetzt, daß die 2. Ableitung von f wesentlich beschränkt ist[1]). Die hier verlangten Voraussetzungen sind an den Endpunkten des Intervalls schwächer, wie man an der Bedingung c) in Lemma 5. 1 sieht.

Beziehungen zwischen dem Verhalten von $\Omega_\lambda(f; h)$ für $h \to 0$ und den durch X und dem Definitionsbereich von Δ_λ erzeugten intermediären Räumen geben J. Löfström und J. Peetre [15]. Eines ihrer wesentlichsten Ergebnisse ist hier in (3.4) von Lemma 3. 1 formuliert.

Man definiert höhere Potenzen von Δ_λ in der üblichen Weise:

Definition 5. 5. Man sagt, ein Element $f \in X$ ist aus $D(\Delta_\lambda^r)$, $r > 1$, falls $f \in D(\Delta_\lambda^{r-1})$ und für die Funktion $g = \Delta_\lambda^{r-1} f$ gilt $g \in D(\Delta_\lambda)$. $\Delta_\lambda^r f$ ist dann definiert durch $\Delta_\lambda^r f = \Delta_\lambda(\Delta_\lambda^{r-1} f)$.

[1]) Die Verschärfung von A. F. Timan (s. Einleitung) ergibt die Approximationsordnung $\{\Delta_n(x)\}^2$, wobei $\Delta_n(x)$ in (1. 3) definiert ist.

Um das Ergebnis aus Abschnitt 4 und Folgerung 5. 3 auf höhere Potenzen von n zu übertragen, benutzen wir den Begriff der besten Approximation. Es sei

$$(5.2)\qquad E_n(X,f) = \min_{T_n \in \mathfrak{T}_n} \|f - T_n\|,$$

wobei $\mathfrak{T}_n$ die Menge der algebraischen Polynome vom Höchstgrad n ist. Für $f \in X$ mit $f^\wedge(0)=0$ definiert man noch

$$(5.3)\qquad E_n^*(X,f) = \min_{\substack{T_n \in \mathfrak{T}_n \\ T_n^\wedge(0)=0}} \|f - T_n\|.$$

Da für $f \in X$ gilt $E_n(X,f) \leqq \|I_n^\lambda f - f\|$ und, falls $f^\wedge(0)=0$ ist, auch $E_n^*(X,f) \leqq \leqq \|I_n^\lambda f - f\|$, ergibt Satz 4. 1. die Aussage $E_n(X,f) \leqq M\Omega_\lambda\left(f; \frac{1}{n}\right)$ bzw. $E_n^*(X,f) \leqq \leqq M\Omega_\lambda\left(f; \frac{1}{n}\right)$. Es wird nun die folgende Abschätzung bewiesen.

Satz 5. 6. *Es sei $f \in X$ und $f \in D(\Delta_\lambda^r)$, $r \geqq 1$, dann gilt*

$$E_n(X,f) \leqq M^r \cdot n^{-2r} E_n^*(X, \Delta_\lambda^r f) \leqq M^{r+1} n^{-2r} \Omega_\lambda\left(\Delta_\lambda^r f; \frac{1}{n}\right).$$

Beweis. Er verläuft wie üblich. Zunächst sei $r=1$ und U_n das Polynom, für das gilt

$$E_n^*(X, \Delta_\lambda f) = \|\Delta_\lambda f - U_n\|$$

und V_n das Polynom mit $V_n^\wedge(0)=0$, für das gilt $\Delta_\lambda V_n = U_n$, dann ergibt sich mit Satz 4. 1 und Folgerung 5. 3

$$E_n(X,f) = E_n(X, f - V_n) \leqq Mn^{-2} \|\Delta_\lambda f - \Delta_\lambda V_n\| \leqq$$

$$\leqq Mn^{-2} \|\Delta_\lambda f - U_n\| = Mn^{-2} E_n^*(X, \Delta_\lambda f) \leqq M^2 n^{-2} \Omega_\lambda\left(\Delta_\lambda f; \frac{1}{n}\right).$$

Den Beweis für $r > 1$ erhält man durch vollständige Induktion.

Bemerkung 5. 7. G. G. Kušnirenko [13, 14] hat einen ähnlichen Satz für stetige Funktionen auf der Einheitskugel im $\mathbf{R}^3$ bewiesen. Als Anwendung dieses Satzes erhält A. S. Džafarov [7] das Ergebnis von Satz 5. 6 für den Raum $X=C$ und $\lambda = \frac{1}{2}$.

Folgerung 5. 8. *Es sei $f \in X$, $g = \Delta_\lambda^r f \in X$ und $\Omega_\lambda(g; h) \leqq h^\alpha$, $0 < \alpha \leqq 2$, dann folgt*

$$E_n(X,f) \leqq C \cdot \frac{1}{n^{2r+\alpha}}.$$

6. Ein Beispiel

Die Ergebnisse werden nun an einem einfachen Beispiel erläutert. Es sei $X=C$, $\lambda=0$ und $f_\alpha(x) = (1-x^2)^{\alpha/2}$, $0<\alpha\leqq 1$. Für diese Funktion gilt wegen $f_\alpha(\cos\vartheta) = |\sin\vartheta|^\alpha$

$$\|T_h^0 f_\alpha - f_\alpha\| \leqq M h^\alpha.$$

Also gilt nach Folgerung 5. 3

$$\|I_n^0 f_\alpha - f_\alpha\| \leqq K \frac{1}{n^\alpha}.$$

Die Funktion f_α gehört auf dem Intervall $[-1, 1]$ höchstens der Klasse Lip $\frac{\alpha}{2}$ an, da für $x=1$ und $0<h<1$ gilt

$$|f_\alpha(1-h) - f_\alpha(1)| = \left(1-(1-h)^2\right)^{\alpha/2} = (2h-h^2)^{\alpha/2} > h^{\alpha/2}$$

(s. [17, ch. VI, 3]). Man würde nach dem klassischen Satz von Jackson höchstens erhalten

$$E_n(C, f_\alpha) = 0(n^{\alpha/2}) \qquad (n\to\infty).$$

Es ist aber gezeigt worden [17, ch. VI, 1; VII, 1; VIII, 3], daß im Falle $\alpha=1$ sogar die Abschätzung

$$\frac{1}{2\pi(2n+1)} < E_n(C, f_1) \leqq \frac{2}{\pi n} \qquad (n = 1, 2, \ldots)$$

gültig ist. Für $0<\alpha<1$ erhält man mit einem Satz von S. B. Stečkin [27], daß zwei positive Konstanten K_1 und K_2 existieren, so daß gilt

$$K_1 n^{-\alpha} \leqq E_n(C, f_\alpha) \leqq K_2 n^{-\alpha}.$$

Mit der hier bewiesenen Version des Satzes von Jackson (Satz 4. 1) erhält man die rechte Seite der Ungleichung ebenfalls, wie oben gezeigt wurde.

7. Funktionen auf der Kugel

Das Ergebnis von Satz 4. 1 läßt sich leicht auf Funktionen übertragen, die auf der Oberfläche S^k der Einheitskugel des k-dimensionalen euklidischen Raumes $\mathbf{R}^k$ definiert sind. Mit $C(S^k)$ bezeichnen wir die Menge der auf S^k stetigen Funktionen, $L^p(S^k)$, $1\leqq p<\infty$, die Menge der auf S^k definierten und dort bezüglich des Oberflächenelementes ds zur p-ten Potenz integrierbaren Funktionen mit der Norm

$$\|f\|_p = \left\{\frac{1}{O_k}\int_{S^k} |f(x)|^p\, ds(x)\right\}^{1/p}; \tag{7. 1}$$

wobei O_k die Oberfläche der Einheitskugel ist. Zonale Funktionen sind Funktionen, die invariant sind gegenüber Drehungen um eine Achse durch einen festgelegten Nordpol auf S^k. Es besteht ein isometrischer Isomorphismus zwischen der Menge der zonalen Funktionen aus $L^p(S^k)$ und dem Banachraum L^p_λ für $2\lambda = k-2$.

Man definiert die verallgemeinerte Translation von $f \in L^1(S^k)$ oder das sphärische Mittel von f durch

(7.2) $$(T_h f)(x) = \frac{1}{O_{k-1}(\sin h)^{2\lambda}} \int_{(x,y)=\cos h} f(y)\,dt(y),$$

wobei dt das $(k-2)$-dimensionale Oberflächenelement der Fläche $(x, y)=\cos h$ auf S^k und (x, y) das euklidische Skalarprodukt ist. Die Integration wird also auf dem Rand der Kugelkappe $D(x, h)$ mit Mittelpunkt x ausgeführt, wobei $D(x, h)$ gegeben ist durch

(7.3) $$D(x, h) = \{y;\ y \in S^k,\ (x,y) \geqq \cos h\}.$$

Für zonale Funktionen gilt $T_h f = T_h^\lambda f$ mit $\lambda = (k-2)/2$. Damit hat man eine anschauliche Definition der Translationen T_h^λ aus Abschnitt 2.

Nun sei X einer der Räume $C(S^k)$ oder $L^p(S^k)$, $1 \leqq p < \infty$. Für die Translationen T_h gilt ebenfalls für alle $f \in X$

(7.4) $$\|T_h f\| \leqq \|f\| \qquad (h > 0),$$

(7.5) $$\lim_{h \to 0} \|T_h f - f\| = 0.$$

Der Beweis ist in [19, Abschnitt 4. 2] zu finden.

Definiert man nun $\Omega(f; h)$ durch

(7.6) $$\Omega(f; h) = \sup_{0 < t \leqq h} \|T_h f - f\|,$$

dann gilt ein Analogon zu Lemma 3. 1 und

Satz 5. 1. *Zu jeder Funktion $f \in X$ gibt es eine Linearkombination $(V_n f)(x)$ von Kugelfunktionen $\{Y_m(x),\ m=0, 1, \ldots, n\}$, so daß gilt*

$$\|f - V_n f\| \leqq K \cdot \Omega\left(f; \frac{1}{n}\right),$$

wobei K eine positive Konstante ist, die nicht von f abhängt.

Beweis. Analog zu Satz 4. 1 definiert man für gerade n die Funktion $V_n f$ durch

$$(V_n f)(x) = \frac{1}{O_k} \int_{S^k} k_n[(x, y)] f(y)\,ds(y),$$

wobei k_n durch (4. 2) definiert ist. Außerdem benutzt man die Beziehung

$$(V_n f)(x) = \frac{O_{k-1}}{O_k} \int_0^{\pi} k_n(\cos u)(T_u f)(x)(\sin u)^{2\lambda}\, du \qquad (\lambda = (k-2)/2),$$

wobei gilt

$$O_k/O_{k-1} = \int_{-1}^{1} (1-t^2)^{\lambda-\frac{1}{2}}\, dt.$$

Der weitere Teil des Beweises verläuft analog zu dem von Satz 4. 1. Dieser Satz wurde auf S_3 und $X=C$ mit anderen Methoden von A. S. Džafarov [6] und G. G. Kušnirenko [13, 14] bewiesen. Ein Analogon zu Satz 5. 6 für Funktionen auf der Kugel ist ebenfalls gültig. Ein Satz vom Jacksonschen Typ für Funktionen auf kompakten Mannigfaltigkeiten und Liegruppen stammt von D. L. Ragozin [26].

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LEHRSTUHL FÜR NUMERISCHE UND ANGEW. MATH.
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A radical class which is fully determined by a lattice isomorphism

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In this note we introduce the concept of a quasi-semi-prime ideal in an associative ring as a generalization of the notion of a semi-prime ideal. We consider the class of rings in which all ideals have this property. This class is shown to be a non-hereditary radical class in the sense of Kurosh. As an application we show that the existence of a lattice isomorphism between the lattice of (two-sided) ideals in a ring R and the lattice of ideals in the ring R_n of $n \times n$ matrices over R, is equivalent to the fact that R belongs to the mentioned radical class.

1. The λ-radical of a ring

Definition 1. An ideal Q in a ring R may be called a *quasi-semi-prime ideal* if from $RAR \subseteqq Q$, where A is an ideal in R, it follows that $A \subseteqq Q$.

The following two theorems are easy consequences of this definition.

Theorem 2. *For an ideal Q of a ring R the following statements are equivalent:*

(i) *Q is a quasi-semi-prime ideal in R.*

(ii) *If a is an element of R such that $RaR \subseteqq Q$, then $a \in Q$.*

Theorem 3. *For a ring R the following statements are equivalent.*

(i) *All ideals in R are quasi-semi-prime.*

(ii) *Each element a of R belongs to the corresponding ideal RaR.*

(iii) *Each ideal A of R satisfies the relation $RAR = A$.*

We shall make particular use of the second condition of the latter theorem. For convenience we introduce the following terminology: An element a of a ring R is called a λ-element if $a \in RaR$. A ring R is called a λ-ring if every element of R is a λ-element. An ideal A of a ring is called a λ-ideal if A is a λ-ring. A ring is said to be λ-semi-simple if it contains no non-zero λ-ideals. We show that the class of

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λ-rings is a radical class according to the definition by KUROSH as adapted by DIVINSKY (cf. [2]).

Firstly, we note that the homomorphic closure of this class of rings follows directly by applying the operation preserving properties of ring homomorphisms.

Secondly, it is obvious that the zero ideal of an arbitrary ring is a λ-ideal. To show that every ring contains a unique maximal λ-ideal, we verify the following

Lemma 4. *The union $\lambda(R)$ of all the λ-ideals of a ring R is a λ-ideal in R.*

Proof. Let A and B be λ-ideals in R and let s be an arbitrary element of $A+B$. Then $s=a+b$, where $a\in A$ and $b\in B$. Since A is a λ-ideal in R, there exist elements x_i, y_i in A such that $a=\Sigma x_i a y_i$. Denoting the element $\Sigma x_i(a+b)y_i$ by c, we can write:

$$a+b-c = a+b-\Sigma x_i(a+b)y_i = b-\Sigma x_i b y_i.$$

Hence it follows that $a+b-c\in B$. Since B is a λ-ideal in R, there exist elements u_j, v_j in B such that

$$a+b-c = \Sigma u_j(a+b-c)v_j.$$

It follows that

$$a+b = \Sigma x_i(a+b)y_i+\Sigma u_j(a+b)v_j-\Sigma u_j[\Sigma x_i(a+b)y_i]v_j.$$

Since clearly $x_i, y_i, u_j, v_j, u_j x_i, y_i v_j\in A+B$, we have that s is a λ-element in $A+B$. Therefore $A+B$ is a λ-ideal in R.

Finally, since each element of the union of all λ-ideals of R belongs to the sum of a finite number of these ideals, it is clear that every such element is a λ-element in $\lambda(R)$. Therefore $\lambda(R)$ is a λ-ideal in R. This completes the proof of the lemma.

There remains to show that the factor ring $R/\lambda(R)$ is λ-semi-simple.

Lemma 5. *The factor ring $R/\lambda(R)$ contains no non-zero λ-ideals.*

Proof. Let $H/\lambda(R)$ be a λ-ideal in $R/\lambda(R)$ and let $h+\lambda(R)$ be an arbitrary element of $H/\lambda(R)$. Then there are elements x_i, y_i in H such that

$$h+\lambda(R) = \Sigma\big(x_i+\lambda(R)\big)\big(h+\lambda(R)\big)\big(y_i+\lambda(R)\big) = \Sigma x_i h y_i+\lambda(R).$$

This implies that $h-\Sigma x_i h y_i\in\lambda(R)$, and since $\lambda(R)$ is a λ-ideal in R, it follows that $h-\Sigma x_i h y_i = \Sigma u_j(h-\Sigma x_i h y_i)v_j$ for some $u_j, v_j\in\lambda(R)$. Therefore

$$h = \Sigma x_i h y_i+\Sigma u_j h v_j-\Sigma u_j[\Sigma x_i h y_i]v_j.$$

Since $u_j, v_j\in\lambda(R)\subseteqq H$, it follows that $x_i, y_i, u_j, v_j, u_j x_i, y_i v_j\in H$. The last equality therefore shows that H is a λ-ideal in R, and accordingly it must be contained in $\lambda(R)$. Therefore $H=\lambda(R)$, and $H/\lambda(R)$ is the zero ideal in $R/\lambda(R)$. This completes the proof of the lemma.

Theorem 6. *The property λ is a radical property.*

Being a radical property, λ satisfies the relation $\lambda(I) \subseteqq I \cap \lambda(R)$ for an arbitrary ideal I in any ring R. The reverse inclusion, however, does not hold in general; for instance, $E \cap \lambda(Z) = E \nsubseteq \lambda(E) = 0$, where E denotes the ring of even integers and Z the ring of all integers. Thus it follows that λ *is not hereditary.*

We conclude this section by comparing the λ-radical property with those between the Baer—McCoy radical property β and the upper radical property φ determined by the class of all fields.

Theorem 7. *The λ-radical property is independent of all radical properties χ such that $\beta \leqq \chi \leqq \varphi$.*

Proof. Every ideal of a λ-radical ring R is quasi-semi-prime. Therefore $R^3 \subseteqq R^2$ implies that $R \subseteqq R^2$, so that $R^2 = R$. Thus it follows that a nilpotent ring is not λ-radical, and consequently $\beta \nleqq \lambda$. On the other hand all fields are χ-semi-simple and at the same time λ-radical, so that $\lambda \nleqq \chi$. The proof is completed.

This independence was to be expected since the Baer—McCoy radical, for instance, is a measure for the presence of nilpotent ideals in a ring, while the λ-radical measures the presence of "well-behaved" ideals such as regular ideals and simple non-trivial ones. Where semi-simplicity with respect to χ is of special interest from a structural point of view, the emphasis must therefore be placed on radicality with respect to λ. The following section deals with an application in this respect.

2. Rings of $n \times n$ matrices over a ring

Although the ring R under consideration needs not possess a unit element, we still use the matrix units E_{ij} in a formal way: If $x \in R$, then xE_{ij} is to be interpreted as the matrix in R_n with the element x at the intersection of the i^{th} row and j^{th} column and the zero element of R elsewhere.

Theorem 8. *An ideal Q in a ring R is quasi-semi-prime if and only if Q_n is a quasi-semi-prime ideal in R_n.*

Proof. Suppose that Q is a quasi-semi-prime ideal in R and let $\alpha = \Sigma a_{ij} E_{ij}$ be any element of R_n such that $R_n \alpha R_n \subseteqq Q_n$. If $\alpha \notin Q_n$, then $a_{km} \notin Q$ for some $k, m \in \in \{1, 2, \dots, n\}$. Since Q is a quasi-semi-prime ideal in R, we have that $R a_{km} R \nsubseteq Q$, that is, there exist elements x and y in R such that $x a_{km} y \notin Q$. But if this was the case it would follow that $(xE_{kk})\alpha(yE_{mm}) = x a_{km} y E_{km} \notin Q_n$. However, this is impossible, since $R_n \alpha R_n \subseteqq Q_n$. Therefore $\alpha \in Q_n$, and we have that Q_n is quasi-semi-prime in R_n.

Conversely, suppose that Q_n is a quasi-semi-prime ideal in R_n, and let $a \in R$ such that $RaR \subseteqq Q$. We shall show that $R_n \gamma R_n \subseteqq Q_n$, where $\gamma = \Sigma a E_{ij}$. An arbitrary element of $R_n\gamma R_n$ is a finite sum of elements of the form

$$\gamma' = (\Sigma x_{ij}E_{ij})(\Sigma aE_{ij})(\Sigma y_{ij}E_{ij}),$$

which is the sum of $n\times n$ matrices of the form $x_{pr}ay_{sq}E_{pq}$. Since $RaR\subseteq Q$, it follows that $x_{pr}ay_{sq}\in Q$, and hence $\gamma'\in Q_n$. Therefore $R_n\gamma R_n\subseteq Q_n$. Since Q_n is a quasi-semi-prime ideal in R_n, it follows that $\gamma=\Sigma aE_{ij}\in Q_n$, and thus that $a\in Q$. Therefore Q is a quasi-semi-prime ideal in R. This completes the proof of the theorem.

To prove our final result, we shall need the following fact (cf. [3]).

Lemma 9. *If $\mathcal{M}$ is an ideal in the ring R_n then the set M of all elements at the intersections of the first rows and first columns of matrices in $\mathcal{M}$ is an ideal in R.*

Theorem 10. *The ideals of the ring R_n are of the form M_n, where M is an ideal in R, if and only if R is a λ-radical ring.*

Proof. Suppose that R is a λ-radical ring. Let $\mathcal{M}$ be an arbitrary ideal in R_n, and let M be the ideal in R associated with $\mathcal{M}$ as in Lemma 9. We show that $\mathcal{M}=M_n$. Let $\alpha=\Sigma a_{ij}E_{ij}\in\mathcal{M}$. Then, for arbitrary $x, y\in R$, one has

$$xa_{rs}yE_{11} = (xE_{1r})(\Sigma a_{ij}E_{ij})(yE_{s1})\in\mathcal{M}.$$

Thus, by definition of M, we have that $xa_{rs}y\in M$. Since this is true for arbitrary $x, y\in R$, it follows that $Ra_{rs}R\subseteq M$, and the fact that M is a quasi-semi-prime ideal in R ensures that $a_{rs}\in M$. The latter relationship, being true for all $r, s\in\{1, 2, \ldots, n\}$, yields the fact that $\alpha\in M_n$. Therefore $\mathcal{M}\subseteq M_n$.

If, on the other hand, m is any element of M, then there exists a matrix $\Sigma m_{ij}E_{ij}$ in $\mathcal{M}$ with $m_{11}=m$ and it follows that

$$xm_{11}yE_{pq} = (xE_{p1})(\Sigma m_{ij}E_{ij})(yE_{1q})\in\mathcal{M},$$

that is, $xmyE_{pq}\in\mathcal{M}$, where x and y are arbitrary elements of R and $p, q\in\{1, 2, \ldots, n\}$. Thus every finite sum of the form $\Sigma x_imy_iE_{pq}$, $(x_i, y_i\in R)$, belongs to $\mathcal{M}$. Since R is λ-radical, it follows that $mE_{pq}\in\mathcal{M}$ for every $m\in M$. Consequently $M_n\subseteq\mathcal{M}$, and we have that $\mathcal{M}=M_n$.

Conversely, suppose that every ideal in R_n has the form M_n, where M is an ideal in R, and let A be any ideal in R. Then the sets $\mathcal{L}$ and $\mathcal{R}$ of matrices in R_n with entries running through the ideals A, RA and AR as indicated in

$$\begin{bmatrix} A & A\ldots A \\ RA & RA\cdots RA \\ \cdots & \cdots \\ RA & RA\cdots RA \end{bmatrix} \quad\text{and}\quad \begin{bmatrix} A & AR\ldots AR \\ A & AR\ldots AR \\ \cdots & \cdots \\ A & AR\ldots AR \end{bmatrix}$$

respectively, are obviously ideals in R_n. By the hypothesis on R_n it follows that

$\mathcal{L}=\mathcal{R}=A_n=(RA)_n=(AR)_n$. Hence we have $A=RA=AR$, and it follows that $RAR=A$. The required result follows from Theorem 3 (iii).

By the preceding two theorems we obtain the following

Corollary 11. *The ring R_n is λ-radical if and only if R is λ-radical.*

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Distributivity and modularity in varieties of algebras

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1. Introduction. A variety $\mathfrak{V}$ of algebras is called *distributive*, if for every algebra $A \in \mathfrak{V}$ the lattice $\Theta(A)$ of all congruence relations over A is distributive. Distributivity of a variety will be denoted by $\Delta(\mathfrak{V})$. B. JÓNSSON [1] has shown the following theorem:

Let $\mathfrak{V}$ be a variety of algebras. $\Delta(\mathfrak{V})$ is valid if and only if for some integer $n \geqq 2$ the following condition holds. $\Delta_n(\mathfrak{V})$: There exist ternary (derived) operations $\tau_0, \tau_1, \ldots, \tau_n$, such that for $i=0, 1, \ldots, n-1$ the identities

$$\tau_0(x, y, z)=x, \quad \tau_n(x, y, z)=z, \quad \tau_i(x, y, x)=x$$

$$\tau_i(x, x, z)=\tau_{i+1}(x, x, z) \quad (i \text{ even}), \qquad \tau_i(x, z, z)=\tau_{i+1}(x, z, z) \quad (i \text{ odd})$$

hold in every member of $\mathfrak{V}$.

If $\Delta_n(\mathfrak{V})$ is valid, we say that the variety $\mathfrak{V}$ is *n-distributive*. Evidently, $\Delta_n(\mathfrak{V})$ implies $\Delta_{n+1}(\mathfrak{V})$, because we can define $\tau_{n+1}=\tau_n$. B. JÓNSSON has shown in [1] that $\Delta_3(\mathfrak{V})$ does not imply $\Delta_2(\mathfrak{V})$, and G. GRÄTZER asks in [2] for examples which show that $\Delta_n(\mathfrak{V})$ does not imply $\Delta_{n-1}(\mathfrak{V})$ for $n \geqq 3$. We prove this suggestion by methods previously applied by the author in [3].

We can answer also the analogous question for A. DAY's characterization of modularity [4].

Our terminology and notation are essentially those of [5].

2. Distributivity. We have the following:

Theorem. *For each integer $n \geqq 2$ there exists a variety which is $(n+1)$-distributive, but not n-distributive.*

Let the variety $\mathfrak{V}$ be defined by ternary basic operations $\tau_1, \tau_2, \ldots, \tau_n$ and the following identities

(1) $$\tau_i(x, y, x)=x \quad (i=1, 2, \ldots, n),$$

(2) $$x=\tau_1(x, x, z),$$

$$(3)\qquad \begin{cases} \tau_i(x, z, z) = \tau_{i+1}(x, z, z) & (i=1, 3, 5, \ldots), \\ \tau_i(x, x, z) = \tau_{i+1}(x, x, z) & (i=2, 4, 6, \ldots), \end{cases}$$

$$(4)\qquad \tau_n(x, x, z) = z \,(\text{if } 2/n) \quad \text{or} \quad \tau_n(x, z, z) = z \,(\text{if } 2\nmid n).$$

We prove that $\mathfrak{B}$ is $(n+1)$-distributive, but not n-distributive.

By *word* we will mean always a $\langle \tau_1, \ldots, \tau_n \rangle$-word in the alphabet $\langle x, y, z \rangle$. Two words are called *equal* if they coincide as rows. A word u is a *subword* of v if u is an interval in v; if u is not equal to v, then it is a *proper* subword. A word w of the form $w = \tau_i(u, \bar{u}, \bar{\bar{u}})$, $(1 \leqq i \leqq n)$ is called a τ_i*-word.* We say that the word w_1 is a *reduction* of the word $w_2 = \tau_i(u_2, \bar{u}_2, \bar{\bar{u}}_2)$, if $w_1 = w_2$ is an identity in $\mathfrak{B}$ (i.e., $w_1 = w_2$ holds in $\mathfrak{B}$) and w_1 is equal to at least one of the words u_2, $\bar{\bar{u}}_2$. A word $w = \tau_i(u, \bar{u}, \bar{\bar{u}})$ is called *reduced,* if neither $w = u$ nor $w = \bar{\bar{u}}$ are identities in $\mathfrak{B}$.

Let (w_1, w_2) be a pair of words such that w_1 changes into w_2 by a single application of any of the identities from (1)—(4) to the whole word w_1 or to a subword of w_1. Then we can distinguish between three types of such pairs:

Type 1: w_1 is a reduction of w_2.

Type 2: w_2 is a reduction of w_1.

Type 3: all other cases.

Proposition. *The pair of words is of Type* 3 *means:*

(i) w_1 *and* w_2 *are either both reduced or none of them is reduced,*

(ii) w_1 *moves into* w_2, *if we apply one of the identities* (3) *to the whole word* w_1 *or any identity from* (1)—(4) *to one of the proper subwords of* w_1.

The sequence of words $w_1, w_2, \ldots, w_m$ is called *simple,* if every term of the sequence is either equal to the next one or changes into it by a single application of one of the identities (1)—(4). A simple sequence $w_1, \ldots, w_m$ of words is called *minimal simple,* if every simple sequence which begins with w_1 and ends with w_m has at least m terms.

Lemma 1. *Let* $w_1, \ldots, w_s$ $(s>2)$ *be such a simple sequence of words that* w_1 *is a reduction of* w_2, *the word* w_s *a reduction of* w_{s-1} *and every pair* (w_{j-1}, w_j) $(j=3, \ldots, s-1)$ *is of Type 3. Then this sequence is not minimal simple.*

Proof. The word w_1 changes into w_2 by one of the identities (1), (2) or (4). By one of the same identities w_{s-1} changes into w_s. For $j=2, 3, \ldots, s-1$ the w_j are τ_{i_j}-words: $w_j = \tau_{i_j}(u_j, \bar{u}_j, \bar{\bar{u}}_j)$. In all the possible cases we can find a simple sequence beginning with w_1, ending with w_s and consisting of $s-2$ terms. The table below gives such a sequence in the following way: Let for instance w_1 move into w_2 by (1) and w_{s-1} into w_s by (2). Then w_1 is equal to u_2, and u_{s-1} is equal to w_s. The sequence $S_1: w_1=u_2, u_3, \ldots, u_{s-2}, u_{s-1}=w_2$ evidently consists of $s-2$ terms, and by (ii) in the proposition above it is simple because the (w_{j-1}, w_j) are of Type 3.

$w_{s-1}\to w_s$ by / $w_1\to w_2$ by	(1)	(2)	(4)
(1)	S_1	S_1	S_2
(2)	S_1	S_1	S_3
(4)	S_2	S_4	S_2

$S_1: w_1, u_3, u_4, \ldots, u_{s-2}, w_s$.

$S_2: w_1, \bar{\bar{u}}_3, \bar{\bar{u}}_4, \ldots, \bar{\bar{u}}_{s-2}, w_s$.

$S_3: w_1, \bar{u}_3, \bar{u}_4, \ldots, \bar{u}_{s-2}, w_s$ (n odd).

$w_1, \bar{u}_3, \ldots, \bar{u}_k, \bar{\bar{u}}_{k+1}, \ldots, \bar{\bar{u}}_{s-2}, w_s$ (n even).

Here k is an integer such that $1\leqq k<s$, $i_k=n-1$, $i_{k+1}=n$. This means $w_k=\tau_{n-1}(u_k, \bar{u}_k, \bar{\bar{u}}_k)$, $w_{k+1}=\tau_n(u_{k+1}, \bar{u}_{k+1}, \bar{\bar{u}}_{k+1})$ and $\bar{u}_k, \bar{\bar{u}}_k, \bar{u}_{k+1}, \bar{\bar{u}}_{k+1}$ are equal. S_4 is the reverse of S_3, where S_3 is formed from $w_s, \ldots, w_1$ instead of $w_1, \ldots, w_s$.

Lemma 2. *If $w=\tau_i(u, \bar{u}, \bar{\bar{u}})$ and $w'=\tau_{i'}(u', \bar{u}', \bar{\bar{u}}')$ are reduced words such that $w=w'$ is an identity in $\mathfrak{B}$, then* (i) *the difference between indices i and i' is at most* 1, (ii) *the identities $u=u'$, $\bar{u}=\bar{u}'$, $\bar{\bar{u}}=\bar{\bar{u}}'$ hold in $\mathfrak{B}$, too.*

Proof. Let $w=w_1, w_2, \ldots, w_m=w'$ be a minimal simple sequence. Such a sequence exists for any words w, w' provided $w=w'$ is an identity in $\mathfrak{B}$.

Suppose all pairs (w_j, w_{j+1}), $(j=1, 2, \ldots, m-1)$ are of Type 3. Then it follows from (ii) in the proposition above that for every j the equations

$$(5)\qquad u_j=u_{j+1}, \quad \bar{u}_j=\bar{u}_{j+1}, \quad \bar{\bar{u}}_j=\bar{\bar{u}}_{j+1} \qquad (j=1, 2, \ldots, m-1)$$

are identities in $\mathfrak{B}$. Assertion (ii) of the lemma follows obviously. To prove (i) we show that $|i-i'|>1$ implies that $u=\bar{\bar{u}}$ is an identity in $\mathfrak{B}$. Then with respect to (1), w cannot be reduced in contradiction to the supposition.

Let for any j the pair $(w_j, w_{j+1})=\big(\tau_{i_j}(u_j, \bar{u}_j, \bar{\bar{u}}_j), \tau_{i_{j+1}}(u_{j+1}, \bar{u}_{j+1}, \bar{\bar{u}}_{j+1})\big)$ be of Type 3 and the indices i_j and i_{j+1} be different. By definition of Type 3 this difference is 1. If the smaller one of these two integers i_j, i_{j+1} is odd, it follows by (3) that $\bar{u}_j=\bar{\bar{u}}_j$, which by (5) implies that $\bar{u}=\bar{\bar{u}}$ are identities in $\mathfrak{B}$. If the smaller one of i_j, i_{j+1} is even, we can see in the same way that $u=\bar{u}$ is an identity in $\mathfrak{B}$. If now $|i-i'|>1$, then there are pairs (w_j, w_{j+1}) of the first and of the second kind as well. Hence $u=\bar{u}=\bar{\bar{u}}$ holds in $\mathfrak{B}$.

To complete the proof, it is enough to show that all pairs (w_j, w_{j+1}) $(1\leqq j<m)$ are of Type 3. If there exists a pair (w_j, w_{j+1}) of Type 2, then, by definition, w_j is not reduced. But w is reduced; hence there are pairs of Type 1, too. Let r be the largest number such that (w_r, w_{r+1}) is of Type 1 and s the smallest number for which (w_{s-1}, w_s) is of Type 2 and $r<s\leqq m$. By Lemma 1 the sequence $w_r, \ldots, w_s$ is not

minimal simple in contradiction to the minimal simplicity of $w_1, \ldots, w_s$. Similarly we arrive at a contradiction if we suppose the existence of a pair of Type 1 in our sequence.

For any sequence of words $w_1, \ldots, w_m$ we will use the following notations:

(6) $\quad w_j^* = w_j(x, x, z) \quad (j=1, 3, 5, \ldots), \qquad w_j^* = w_j(x, z, z) \quad (j=2, 4, 6, \ldots),$

(7) $\quad w_j^{**} = w_j(x, z, z) \quad (j=1, 3, 5, \ldots), \qquad w_j^{**} = w_j(x, x, z) \quad (j=2, 4, 6, \ldots),$

(8) $\quad w_j^{***} = w_j(x, y, x) \quad (j=1, 2, \ldots, m).$

Lemma 3. *Let w be a τ_i-word: $w=\tau_i(u, \bar{u}, \bar{\bar{u}})$, $(1 \leqq i \leqq n)$. If one of the following identities on the left side hold in $\mathfrak{B}$, the corresponding identities on the right side hold in $\mathfrak{B}$, too:*

$$\text{(i)}\quad w^* = x \Rightarrow \begin{cases} u^* = x & (i = 1, 2, \ldots, n-1), \\ \bar{\bar{u}}^* = x & (i = 2, 3, \ldots, n). \end{cases}$$

$$\text{(ii)}\quad w^{**} = z \Rightarrow \begin{cases} u^{**} = z & (i = 1, 2, \ldots, n-1), \\ \bar{\bar{u}}^{**} = z & (i = 2, 3, \ldots, n). \end{cases}$$

$$\text{(iii)}\quad w^{***} = x \Rightarrow \begin{cases} u^{***} = x & (i = 1, 2, \ldots, n-1), \\ \bar{\bar{u}}^{***} = x & (i = 2, 3, \ldots, n). \end{cases}$$

Proof. (i) If $w^*=x$ is an identity in $\mathfrak{B}$, there is a minimal simple sequence $w^*=w_1, w_2, \ldots, w_r=x$. The form of w_1 and w_r involves that in this sequence there exists a term w_s such that w_{s+1} is a reduction of w_j: let s be the smallest index with this property. With respect to Lemma 1, all pairs (w_j, w_{j+1}) for $1 \leqq j < s$ must be of Type 3 and therefore the identities $u_1=u_2=\cdots=u_s$, $\bar{\bar{u}}_1=\bar{\bar{u}}_2=\cdots=\bar{\bar{u}}_s$ hold in $\mathfrak{B}$. By definition of s, the word w_{s+1} is a reduction of w_s. If $1 \leqq i < n$, it follows by (1), (2) that u_s and w_{s+1} are equal. If $1 < i \leqq n$, it follows by (1), (4) that $\bar{\bar{u}}_s$ and w_{s+1} are equal. In the first case $u^*=u_1=u_2=\cdots=u_s=w_{s+1}=x$ and in the second case $\bar{\bar{u}}^*=$ $=\bar{\bar{u}}_0=\cdots=\bar{\bar{u}}_s=w_{s+1}=x$ hold in $\mathfrak{B}$.

If $i_s=i$, assertion (i) is shown. If $i_s \neq i$, there is at least one $j<s$ such that one of the identities (3) applied to the whole word w_j yields w_{j+1}. Together with the move of w_s to w_{s+1}, this fact implies that for every $j \leqq s$ the equation $u_j=\bar{u}_j=\bar{\bar{u}}_j$ is an identity and therefore $u^*=x=\bar{\bar{u}}^*$ is an identity in $\mathfrak{B}$, too. Assertion (i) is proved.

To prove (ii) we replace in the proof of (i) x by z and all signs with one star by the same sign with two stars. Correspondingly, to prove (iii) we replace in the proof of (i) all signs with one star by the same with three stars.

Proof of theorem. It is clear by (1)—(4) that $\mathfrak{B}$ is $(n+1)$-distributive. We will prove that if $\mathfrak{B}$ is $(m+1)$-distributive then $m \geqq n$. Using the notations (6)—(8) given above, $(m+1)$-distributivity means that there is a sequence of words $v_1, v_2, \ldots, v_m$ such that

(9) $v_j^{***} = x \qquad (j=1, 2, \ldots, m),$

(10) $x = v_1^*,$

(11) $v_j^{**} = v_{j+1}^* \qquad (j = 1, m-1),$

(12) $v_m^{**} = z$

are identities in $\mathfrak{V}$.

Suppose that for a fixed integer $m \geqq 1$ among all such sequences the sequence $w_1, w_2, \ldots, w_m$ has the smallest total number of operator symbols τ_i $(1 \leqq i \leqq n)$. Let r be the largest index such that $1 \leqq r \leqq m$ and w_r^* has a reduction. Further, let s be the smallest index such that $r \leqq s \leqq m$ and w_s^{**} has a reduction. It is clear that such indices r, s exist, because w_1^* and w_m^{**} have reductions.

We distinguish 3 cases:

(α) there is no τ_n-word among $w_r, \ldots, w_s$,

(β) there is no τ_1-word among $w_r, \ldots, w_s$,

(γ) among $w_r, \ldots, w_s$ there are τ_1-words and τ_n-words as well.

In case (α) and (β) we take the sequences

$$w_1, \ldots, w_{r-1}, \quad \dot{u}_r, \ldots, \dot{u}_s, \quad w_{s+1}, \ldots, w_m$$

and

$$w_1, \ldots, w_{r-1}, \quad \overline{\overline{u}}_r, \ldots, \overline{\overline{u}}_s, \quad w_{s+1}, \ldots, w_m, \text{ respectively.}$$

In both cases the new sequence evidently consists of m words and has less operator symbols τ_i than the sequence $w_1, \ldots, w_m$. Moreover, the identities (9)—(12) hold for the new sequence, too. Indeed, the identity (9) follows from (iii) in Lemma 3. If $r=1$, then (10) follows from (i) in Lemma 3, and if $r>1$, then (10) is the same as in the given sequence. The identities (11) follows from the definition of the reduction and from Lemma 2. Finally, (12) follows either from (ii) of Lemma 3) (if $s=m$) or it is the same as in the sequence $w_1, \ldots, w_m$ (if $s<m$). Thus we got a contradiction to the minimum property of $w_1, \ldots, w_m$.

In case (γ) the words $w_{r+1}^*, w_{r+2}^*, \ldots, w_s^*$; $w_r^{**}, w_{r+1}^{**}, \ldots, w_{s-1}^{**}$ are reduced by supposition, and (11) states that the identities $w_j^{**} = w_{j+1}^*$ $(j=r, r+1, \ldots, s-1)$ hold in $\mathfrak{V}$. By (i) in Lemma 2 it follows that if w_j is an τ_{i_j}-word and w_{j+1} is an $\tau_{i_{j+1}}$-word, then $|i_j - i_{j+1}| \leqq 1$. But among the words $w_r, \ldots, w_s$ there are τ_1-words and τ_n-words as well. Hence there are τ_i-words for $i=1, 2, \ldots, n$. It follows that $s-r+1 \geqq n$ and therefore $m \geqq n$, q.e.d.

3. Modularity. Let $\Sigma(\mathfrak{V})$ denote the property of the variety $\mathfrak{V}$ that for every algebra $A \in \mathfrak{V}$ the lattice $\Theta(A)$ of all congruence relations over A is modular. A. DAY [4] has shown the following theorem:

Let $\mathfrak{V}$ be a variety of algebras. $\Sigma(\mathfrak{V})$ is valid if and only if for some integer $n \geqq 2$ the following holds. $\Sigma_n(\mathfrak{V})$: There exist 4-ary operations $\mu_0, \mu_1, \ldots, \mu_n$ such that for $i=0, 1, \ldots, n-1$ the identities

$$\mu_0(x, y, z, w) = z, \quad \mu_n(x, y, z, w) = w, \quad \mu_i(x, y, y, x) = x,$$

$$\mu_i(x, y, y, w) = \mu_{i+1}(x, y, y, w) \qquad (i \text{ odd}),$$

$$\mu_i(x, x, w, w) = \mu_{i+1}(x, x, w, w) \qquad (i \text{ even})$$

hold in every member of $\mathfrak{V}$.

If $\Sigma_n(\mathfrak{V})$ is valid, the variety $\mathfrak{V}$ is called n-modular. It is evident that $\Sigma_n(\mathfrak{V})$ implies $\Sigma_{n+1}(\mathfrak{V})$. The fact that $\Sigma_{n+1}(\mathfrak{V})$ does not imply $\Sigma_n(\mathfrak{V})$ can be proved in a way analogous to that of the proof concerning distributivity. In other words we have the following

Theorem. *For each integer $n \geqq 2$ there exists a variety which is $(n+1)$-modular, but not n-modular.*

We omit the proof, because it coincides essentially with the proof of the preceding theorem. Of course, there are some differences which are due to the parity of the operations μ_i, but they are only of formal nature.

Problems. Find finite algebras A_m $(m=2, 3, \ldots)$ such that any variety $\mathfrak{V}$ which contains A_m can be $(m+1)$-distributive but not m-distributive. Solve the same problem for modularity. Probably, by such examples the proofs in this paper may be shortened. In the case of varieties with ideals and varieties which are $(m+1)$-permutable but not m-permutable such finite examples are given by A. F. Mutylin [6] and E. T. Schmidt [7], respectively.

The results were reported in October 1970 in Szeged. I am grateful to colleagues of the University in Szeged for helpful discussions.

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Sur les opérateurs de classe $\mathscr{C}_\varrho$

Par GH. ECKSTEIN à Timişoara (Roumanie)

1. Soit $\mathfrak{H}$ un espace de Hilbert, T un opérateur borné dans $\mathfrak{H}$. Rappelons (voir [1]) que T est de classe $\mathscr{C}_\varrho$ ($\varrho>0$) s'il existe un espace de Hilbert $\mathfrak{K}\supset\mathfrak{H}$ et un opérateur unitaire U dans $\mathfrak{K}$, tel qu'on a

$$T^n P=\varrho P U^n P \quad \text{pour} \quad n\in N=\{1, 2, \ldots\},$$

où P est le projecteur orthogonal de $\mathfrak{K}$ sur $\mathfrak{H}$. L'opérateur U est appelé *ϱ-dilatation unitaire* de T. On a évidemment $\|T^n\|\leqq\varrho$, donc le spectre de T est situé dans le disque $\{\lambda:|\lambda|\leqq 1\}$.

On sait que la classe $\mathscr{C}_1$ est celle des contractions (SZ.-NAGY) et la classe $\mathscr{C}_2$ est celle des opérateurs T dont le rayon numérique $w(T)$ est $\leqq 1$ (BERGER). De plus on sait que si $\varrho_1>\varrho_2$, on a $\mathscr{C}_{\varrho_1}\supset\mathscr{C}_{\varrho_2}$.

Dans [2], B. SZ.-NAGY et C. FOIAŞ ont prouvé que les opérateurs de classe $\mathscr{C}_\varrho$ sont similaires à des contractions. Dans [4], CH. DAVIS et C. FOIAŞ ont trouvé une autre classe d'opérateurs similaires à des contractions. En étudiant ces deux classes, CH. DAVIS a conjecturé (communication orale, Congrès de Nice) le suivant

Théorème. *Pour tout opérateur T de classe $\mathscr{C}_\varrho$, la suite $\|T^n h\|$ ($n=1, 2, \ldots$) est convergente pour chaque $h\in\mathfrak{H}$.*

Pour la classe $\mathscr{C}_1$ la propriété est évidente car la suite $\|T^n h\|$ est décriossante; pour les opérateurs T avec $w(T)\leqq 1$, elle a été établie d'une manière directe par CRABB [6].

Dans la présente Note on prouve ce théorème à l'aide d'une construction qui réduit le problème au cas des translations pondérées.

2. Soit S une contraction et V une dilatation unitaire de S. On a:

$$(1)\qquad \|S^*h-\bar{v}h\|^2 = \|P(V^*-\bar{v}I)h\|^2 \leqq \|Vh-vh\|^2 \leqq$$

$$\leqq 2\|Vh-Sh\|^2+2\|Sh-vh\|^2 = 2(\|h\|^2-\|Sh\|^2)+2\|Sh-vh\|^2.$$

Soit T de classe $\mathscr{C}_\varrho$, où $\varrho>2$ (ce qui n'est pas une restriction, $\mathscr{C}_\varrho$ étant une fonction croissante de ϱ). On sait que si μ vérifie

$$1<|\mu|<\frac{\varrho-1}{\varrho-2},$$

l'opérateur $R_\mu=(|\mu|-1)(\mu I-T)^{-1}$ est une contraction (cf. [1], Ch. I. 11, remarque 3). Soit λ avec $|\lambda|=1$, et soit $1<r<\frac{\varrho-1}{\varrho-2}$. Posons $\mu=r\lambda$. On a:

$$\text{(2)}\quad \lambda(\mu I-T)(R_\mu-\bar{\lambda}I)=\lambda(r\lambda I-T)[(r-1)(r\lambda I-T)^{-1}-\bar{\lambda}I]=\lambda(\bar{\lambda}T-I)=T-\lambda I.$$

Il en résulte, en passant aux adjoints,

$$\text{(3)}\qquad \|(T^*-\bar{\lambda}I)h\|^2=\|\bar{\lambda}(\bar{\mu}I-T^*)(R_\mu^*-\lambda I)h\|^2\leqq(r+\varrho)^2\|R^*h-\lambda h\|^2.$$

En utilisant (1) pour $S=R_\mu$, on obtient

$$\text{(4)}\qquad \|(T^*-\bar{\lambda}I)h\|^2\leqq 2(r+\varrho)^2[\|h\|^2-\|R_\mu h\|^2+\|R_\mu h-\bar{\lambda}h\|^2].$$

Supposons maintenant que $\lambda\in\sigma(T)$ et que $\{h_n\}$ est une suite borné dans $\mathfrak{H}$ telle qu'on ait $\|Th_n-\lambda h_n\|\to 0$. La relation (2) implique

$$R_\mu h_n-\bar{\lambda}h_n=\bar{\lambda}(\mu I-T)^{-1}(Th_n-\lambda h_n),$$

donc $\|R_\mu h_n-\bar{\lambda}h_n\|^2\to 0$, et par (4) il en résulte $T^*h_n-\bar{\lambda}h_n\to 0$. On a donc démontré le suivant

Lemme. *Si $T\in\mathscr{C}_\varrho$ et si $\{h_n\}$ est une suite bornée dans $\mathfrak{H}$ telle que $Th_n-\lambda h_n\to 0$ pour un certain λ avec $|\lambda|=1$, on a aussi $T^*h_n-\bar{\lambda}h_n\to 0$.*

3. Soient $T\in\mathscr{C}_\varrho(\mathfrak{H})$, U une ϱ-dilatation unitaire de T dans l'espace de Hilbert $\mathfrak{K}$ et $\{m_i\}_{i\in N}$ une suite arbitraire de nombres naturels. Soient

$$\hat{\mathfrak{H}}=\bigoplus_{i=1}^{\infty}\mathfrak{H}_i\ \text{ et }\ \hat{\mathfrak{K}}=\bigoplus_{i=-\infty}^{\infty}\mathfrak{K}_i\quad\text{où}\quad\mathfrak{H}_i=\mathfrak{H}\ \text{ et }\ \mathfrak{K}_i=\mathfrak{K}.$$

On peut identifier les éléments

$$\hat{h}=(\overset{(1)}{h_1},\overset{(2)}{h_2},\ldots,\overset{(i)}{h_i},\ldots)\in\hat{\mathfrak{H}}\quad\text{et}\quad(\ldots,\overset{(-i)}{0},\ldots,\overset{(0)}{0},\overset{(1)}{h_1},\overset{(2)}{h_1},\ldots)\in\hat{\mathfrak{K}}.$$

Soient

$$\hat{T}(\overset{(1)}{h_1},\overset{(2)}{h_2},\ldots,\overset{(i)}{h_i},\ldots)=(\overset{(1)}{0},\overset{(2)}{T^{m_1}h_1},\overset{(3)}{T^{m_2}h_2},\ldots,\overset{(i+1)}{T^{m_i}h_i},\ldots),$$

$$\hat{U}(\ldots,\overset{(-i)}{k_{-i}},\ldots,\overset{(0)}{k_0},\overset{(1)}{k_1},\ldots,\overset{(i)}{k_i},\ldots)=(\ldots,\overset{(-i+1)}{k_{-i}},\ldots,\overset{(1)}{k_0},\overset{(2)}{U^{m_1}k_1},\ldots,\overset{(i+1)}{U^{m_i}k_i},\ldots),$$

et $\hat{P}$ le projecteur de $\hat{\mathfrak{K}}$ sur $\hat{\mathfrak{H}}$. $\hat{U}$ est évidemment un opérateur unitaire dans $\hat{\mathfrak{K}}$ et on vérifie directement que

$$\varrho \hat{P}\hat{U}^n\hat{h} = \hat{T}^n\hat{h} \quad \text{pour tous } \hat{h}\in\hat{\mathfrak{H}} \text{ et } n\in N,$$

donc $\hat{T}$ est un opérateur de classe $\mathscr{C}_\varrho$.

Soit maintenant $h_1\in\mathfrak{H}$ tel que $\|T^nh_1\|\not\to 0$. Posons

$$\hat{f}_1 = (\overset{(1)}{h_1}, \overset{(2)}{0}, \ldots, \overset{(i)}{0}, \ldots), \quad \hat{f}_i = \hat{T}^{i-1}\hat{f}_1 \quad \text{et} \quad \hat{e}_i = \hat{f}_i\|\hat{f}_i\|^{-1} \quad (i\in N).$$

Evidemment

$$\hat{f}_i = (\overset{(0)}{0}, \ldots, \overset{(i-1)}{0}, \overset{(i)}{T^{m_1+m_2+\cdots+m_{i-1}}h_1}, \overset{(i+1)}{0}, \ldots).$$

Soit $\hat{\mathfrak{H}}_0$ le sous-espace de $\hat{\mathfrak{H}}$ engendré par les $\hat{f}_i$ $(i\in N)$. $\hat{\mathfrak{H}}_0$ est évidemment invariant pour $\hat{T}$, donc la restriction $T_0=\hat{T}|_{\hat{\mathfrak{H}}_0}$ est de classe $\mathscr{C}_\varrho$ (car $\hat{U}$ est une ϱ-dilatation unitaire de T_0).

Remarquons que $\{\hat{e}_i\}_{i\in N}$ est une base orthonormale de $\hat{\mathfrak{H}}_0$ et que T_0 est la *translation pondérée* (voir [5])

$$\hat{e}_i \mapsto p_i\hat{e}_{i+1} \tag{5}$$

où

$$p_i = \frac{\|\hat{f}_{i+1}\|}{\|\hat{f}_i\|} = \frac{\|T^{m_1+m_2+\cdots+m_i}h_1\|}{\|T^{m_1+m_2+\cdots+m_{i-1}}h_1\|}. \tag{6}$$

T_0^* est évidemment l'opérateur

$$\hat{e}_i \mapsto 0, \quad \hat{e}_{i+1}\mapsto p_i\hat{e}_i \qquad (i\in N).$$

4. Les préparatifs étant terminés, on passe à la démonstration du théorème. Supposons que pour un certain $h_1\in\mathfrak{H}$ la suite $\{\|T^nh_1\|\}$ ne soit pas convegente. Soit $a=\inf\|T^nh_1\|$. Si $a=0$, on peut trouver une suite croissante $\{n_i\}$ de nombres naturels, telle qu'on a $\|T^{n_i}h_1\|\to 0$, mais alors, comme

$$\|T^{n_i+p}h_1\| \leqq \|T^p\|\cdot\|T^{n_i}h_1\| \leqq \varrho\|T^{n_i}h_1\|,$$

on a $\|T^nh_1\|\to 0$, en contradiction avec notre hypothèse. On a donc

$$\inf\|T^nh_1\| = a > 0. \tag{7}$$

La suite $\{\|T^nh_1\|\}$ n'étant pas convergente, on peut trouver $\varepsilon>0$ et une suite croissante $\{n_i\}$ de nombres naturels, telle qu'on ait $|\|T^{n_i}h_1\|-\|T^{n_{i-1}}h_1\||\geqq\varepsilon$ (où $n_0=0$), d'où on déduit

$$\left|\frac{\|T^{n_i}h_1\|}{\|T^{n_{i-1}}h_1\|}-1\right| \geqq \frac{\varepsilon}{\varrho\cdot\|h_1\|} = \varepsilon_1 > 0. \tag{8}$$

Posons $m_i = n_i-n_{i-1}$ $(i\in N)$, et avec h_1 et la suite $\{m_i\}$ construisons l'opérateur

T_0 dans $\mathfrak{H}_0$ (cf. le n° précédent). Nous obtenons ainsi la translation pondérée de classe $\mathscr{C}_\varrho$

$$\hat{e}_i \mapsto p_i \hat{e}_{i+1}$$

dont les poids sont

$$p_i = \frac{\|T^{m_1+\cdots+m_i} h_1\|}{\|T^{m_1+\cdots+m_{i-1}} h_1\|} = \frac{\|T^{n_i} h_1\|}{\|T^{n_{i-1}} h_1\|}.$$

Par (8), nous avons

$$|p-1| \geqq \varepsilon_1 > 0 \quad i \in N, \tag{9}$$

et à l'aide de (7),

$$\varrho \geqq p_1 p_2 \cdots p_i = \frac{\|T^{n_i} h_1\|}{\|h_1\|} \geqq \frac{a}{\|h_1\|}. \tag{10}$$

Soit

$$\hat{k}_i = \frac{\hat{e}_1 + T_0 \hat{e}_1 + \cdots + T_0^i \hat{e}_1}{\|\hat{e}_1 + T_0 \hat{e}_1 + \cdots + T_0^i \hat{e}_1\|} = \frac{\hat{e}_1 + p_1 \hat{e}_2 + \cdots + p_1 p_2 \cdots p_i \hat{e}_{i+1}}{\|\hat{e}_1 + p_1 \hat{e}_2 + \cdots + p_1 p_2 \cdots p_i \hat{e}_{i+1}\|}.$$

Nous avons, à l'aide de la relation (10),

$$\|T_0 \hat{k}_i - \hat{k}_i\|^2 = \frac{\|T_0^{i+1} \hat{e}_1 - \hat{e}_1\|^2}{\|\hat{e}_1\|^2 + \|T_0 \hat{e}_1\|^2 + \cdots + \|T_0^i \hat{e}_1\|^2} \leqq \frac{(\varrho+1)^2 \|h_1\|^2}{(i+1) a^2} \to 0.$$

Mais

$$\|T_0^* \hat{k}_i - \hat{k}_i\|^2 = \frac{(p_1^2-1)^2 + p_1^2 (p_2^2-1)^2 + \cdots + p_1^2 p_2^2 \cdots p_{i-1}^2 (p_i^2-1)^2 + p_1^2 p_2^2 \cdots p_i^2}{1 + p_1^2 + \cdots + p_1^2 p_2^2 \cdots p_i^2},$$

et par (9) et (10)

$$\|T_0^* \hat{k}_i - \hat{k}_i\|^2 \geqq \varepsilon_1^2 \frac{1 + p_1^2 + \cdots + p_1^2 p_2^2 \cdots p_{i-1}^2}{1 + p_1^2 + \cdots + p_1^2 p_2^2 \cdots p_i^2} \geqq \varepsilon_1^2 \frac{i \cdot a^2}{\|h_1\|^2 (i+1) \varrho^2} \not\to 0,$$

en contradiction avec le lemme. La démonstration est achevée.

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(Reçu le 18. décembre 1971),

On convergence properties of operators of class $\mathscr{C}_\varrho$

By W. MLAK in Kraków (Poland)

In the preceding paper[1]) G. ECKSTEIN proved that if a (bounded, linear) operator T on a (complex) Hilbert space $\mathfrak{H}$ belongs to a class $\mathscr{C}_\varrho$ $(\varrho>0)$[2]) then $\|T^{*n}f\|$ converges as $n\to\infty$ for every $f\in\mathfrak{H}$. We are going to give another proof of the same statement, and of some related convergence properties.

We assume once for all that T is of a class $\mathscr{C}_\varrho$ $(\varrho>0)$ so that it has a unitary ϱ-dilation on some Hilbert space $\mathfrak{K}(\supset\mathfrak{H})$, i.e. a unitary operator U such that

$$T^n f=\varrho PU^n f \quad \text{for} \quad f\in\mathfrak{H} \quad \text{and} \quad n=1,2,\ldots,$$

P being the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$. We set $\mathfrak{M}_+ = \bigvee_{n=0}^{\infty} U^n\mathfrak{H}$.

The following lemma is crucial for our purposes:

Lemma. *If $h\in U^{n+1}\mathfrak{M}_+$ for some $n\geqq 0$ then $Ph=T^nPU^{-n}h$.*

Proof. Since $U^{n+1}\mathfrak{M}_+$ is spanned by the elements of the form $h=U^{n+i}f$ $(f\in\mathfrak{H};\ i\geqq 1)$ it suffices to consider such an h. Then,

$$Ph = PU^{n+i}f = \frac{1}{\varrho}T^{n+i}f = \frac{1}{\varrho}T^nT^if = \frac{1}{\varrho}T^n\cdot\varrho PU^if =$$

$$= \frac{1}{\varrho}T^n\cdot\varrho PU^{-n}U^{n+i}f = T^nPU^{-n}h,$$

and the proof is done.

Denote by Q_n the projection onto $U^{n+1}\mathfrak{M}_+$. Then $Q=\lim Q_n$ exists and is the projection onto $\bigcap_{n=1}^{\infty} U^n\mathfrak{M}_+$. It follows from the lemma that $PQ_nh=T^nPU^{-n}Q_nh$ for every $h\in\mathfrak{K}$. Consequently, if $f\in\mathfrak{H}$, then $(h,Q_nf)=(PQ_nh,f)=(T^nPU^{-n}Q_nh,f)=$ $=(h,Q_nU^nT^{*n}f)$. It results that

$$Q_nf=Q_nU^nT^{*n}f \quad \text{for} \quad f\in\mathfrak{H} \quad \text{and} \quad n=1,2,\ldots.$$

[1]) Sur les opérateurs de classe $\mathscr{C}_\varrho$, *Acta Sci. Math.*, **33** (1972), 345—352.

[2]) For references on $\mathscr{C}_\varrho$ classes see: B. SZ.-NAGY and C. FOIAŞ, *Harmonic Analysis of Operators on Hilbert Space* (London—Amsterdam—Budapest, 1970).

Now since $U^n T^{*n} f \in U^n \mathfrak{M}_+$ $(n \geqq 0)$ we have for $g \in \mathfrak{K}$:

$$(U^n T^{*n} f, g) = (Q_n f, g) + \big(U^n T^{*n} f, (Q_{n-1} - Q_n) g\big).$$

The sequence $T^{*n} f$ is bounded and $Q_{n-1} - Q_n \to 0$. Hence,

(1) $$U^n T^{*n} f \to Qf \quad \textit{as} \quad n \to \infty, \quad \textit{weakly for every} \quad f \in \mathfrak{H}.$$

As $(U^n T^{*n} f, f) = \varrho\ (T^n T^{*n} f, f)$ for $n \geqq 1$, we deduce from (1) that $\|T^{*n} f\|^2 \to \to \varrho \|Qf\|^2$. Thus we have proved:

(2) $$\|T^{*n} f\|^2 \to \varrho \|Qf\|^2 \quad \textit{as} \quad n \to \infty, \quad \textit{for every} \quad f \in \mathfrak{H}.$$

As T^* is of class $\mathscr{C}_\varrho$ whenever T is, we have got a sharpening of ECKSTEIN's result.

As weak convergence $u_n \to u$ implies strong convergence if and only if $\|u_n\| \to \|u\|$, we infer from (1) and (2) that the convergence (1) holds for some f in the strong sense too if and only if $\|Qf\|^2 = \varrho \|Qf\|^2$. This is the case for every f if $\varrho = 1$, and for f satisfying $Qf = 0$ if $\varrho \neq 1$.

It is easy to give an example of operator T in $\mathscr{C}_\varrho$ $(\varrho > 1)$ for which $Qf = 0$ if $f = 0$. This is indeed the case for every unitary T since then $\|f\|^2 = \lim \|T^{*n} f\|^2 = = \varrho \|Qf\|^2$. Thus, in general, (1) does not hold true for strong convergence.

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Echelles continues de sous-espaces invariants. II

Par BÉLA SZ.-NAGY à Szeged et CIPRIAN FOIAŞ à Bucarest

On dit que la famille $\{H(\lambda)\}$ $(0 \leqq \lambda \leqq 1)$ de sous-espaces d'un espace H de Hilbert forme une *échelle continue* si elle vérifie les conditions suivantes:

$$H(0)=\{0\}, \quad H(\lambda) \subset H(\mu) \quad \text{pour} \quad 0 \leqq \lambda < \mu \leqq 1, \quad H(1)=H,$$

$$\bigvee_{\varkappa<\lambda} H(\varkappa)=H(\lambda) \quad \text{pour} \quad 0<\lambda \leqq 1, \quad \text{et} \quad \bigcap_{\mu>\lambda} H(\mu)=H(\lambda) \quad \text{pour} \quad 0 \leqq \lambda < 1.$$

En désignant par $E(\lambda)$ la projection orthogonale de H dans $H(\lambda)$, ces conditions veulent dire que $\{E(\lambda)\}$ $(0 \leqq \lambda \leqq 1)$ est une famille spectrale continue.

Dans la Note précédente [3] on a démontré un théorème dont il s'ensuit immédiatement que tout opérateur (linéaire borné) T dans un espace de Hilbert H de dimension infinie peut être prolongé à un opérateur S dans un espace de Hilbert $K(\supset H)$ tel que S admette dans K une échelle continue $\{K(\lambda)\}$ de sous-espaces invariants (c'est-à-dire $SK(\lambda) \subset K(\lambda)$); de plus on peut supposer que $K = H \oplus H'$ et $S = T \oplus T'$ où H' est un espace de même dimension que H, T' est un opérateur normal dans H', et $\|T'\| \leqq \|T\|$.

Dans la présente Note nous démontrons le suivant:

Théorème. *Tout opérateur T dans un espace H de dimension infinie est la limite forte d'une suite d'opérateurs S_n de H $(n=1, 2, \ldots)$ telle que chaque S_n admet une échelle continue de sous-espaces invariants et que de plus* $\|S_n\| \leqq \|T\|$.

Dans la démonstration nous utiliserons une méthode analogue à celle qui a été employée dans [2] pour démontrer, entre autres, la proposition suivante: Toute contraction T dans un espace de Hilbert H de dimension infinie est la limite faible d'une suite d'opérateurs unitaires dans le même espace. (D'ailleurs la méthode de [2] est voisine de celle de [1] où au lieu de limites faibles de suites il s'agit de points d'accumulation au sens de la topologie faible des opérateurs.)

Démonstration. Puisque la dimension de H est un nombre cardinal d infini, il existe dans H une suite de sous-espaces orthogonaux L_k $(k=1, 2, \ldots)$ tels que

$$\dim L_k=d \quad (k=1, 2, \ldots) \quad \text{et} \quad \bigoplus_{1}^{\infty} L_k=H.$$

En posant, pour $n=1, 2, \dots,$

$$H_n = \bigoplus_1^n L_k \quad \text{et} \quad H'_n = \bigoplus_{n+1}^\infty L_k$$

on aura

$$\dim H_n = \dim H'_n = d \quad \text{et} \quad H = H_n \oplus H'_n.$$

Soit P_n la projection orthogonale de H à H_n, et posons $T_n = P_n T | H_n$: T_n est un opérateur dans H_n. D'après le résultat cité de [3] il existe un opérateur (normal) T'_n dans H'_n tel que $\|T'_n\| \leqq \|T_n\|$ et que l'opérateur $S_n = T_n \oplus T'_n$ ait une échelle continue de sous-espaces invariants. Comme de plus $\|S_n\| = \|T_n\| \leqq \|T\|$ pour tout n, il ne nous reste qu'à montrer que S_n tend fortement vers T lorsque $n \to \infty$.

Soit $h \in H_m$. On a pour $n \geqq m$:

$$S_n h = T_n P_n h + T'_n (I - P_n) h = T_n h$$

parce que $h \in H_m \subset H_n$ donc $P_n h = h$. Par suite,

$$\|Th - S_n h\| = \|Th - T_n h\| = \|(I - P_n) Th\| \to 0 \quad \text{pour} \quad m \leqq n \to \infty.$$

Ainsi la convergence forte $S_n h \to Th$ est vérifiée par tous les vecteurs h dans $\bigcup_1^\infty H_m$, ce qui est une variété linéaire dense dans H. Vu que les S_n sont uniformément bornés, cela entraîne la convergence forte $S_n h \to Th$ pour tout $h \in H$. QED.

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(Reçu le 10 septembre 1972)

Bibliographie

Wolfgang Walter, Differential and Integral Inequalities, X+352 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1970. — DM 74,—.

This excellently written book is not only a translation from the German original published in 1964, but it also incorporates new results on differential and integral inequalities. This is shown e.g. by the fact that the Bibliography has been almost doubled in size; now it contains about 500 items. The new material got incorporated in a way which left the construction of the German edition unchanged.

The chapter headings are: I. Volterra Integral Equations, II. Ordinary Differential Equations, III. Volterra Integral Equations in Several Variables, Hyperbolic Differential Equations, IV. Parabolic Differential Equations.

As the author says in the Preface (to the English edition), the most substantial additions are in the field of existence theory. E.g., an existence theory for the general nonlinear parabolic equation in one space variable, based on the line method, is given in Section 36. This theory is considered by the author as one of the most significant recent applications of inequality methods.

A survey of the most important theorems on elliptic differential inequalities, with brief proofs, is given in an appendix.

L. Leindler (Szeged)

Murray Rosenblatt, Markov Processes. Structure and Asymptotic Behavior (Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, 184) XIII+268 Seiten, Berlin—Heidelberg—New York, Springer-Verlag, 1971. — DM 68, —.

In den letzten fünfzehn Jahren wurde die Theorie der Markovschen Prozesse intensiv erforscht. Die Ergebnisse dieser raschen Entwicklung wurden in mehreren Handbüchern zusammengefasst, über das wichtige Gebiet des asymptotischen Verhaltens von Markov-Prozessen kann man jedoch in diesen Monographien nur wenig finden. Der Verfasser des vorliegenden Buches hat sich u.a. das Ziel gesetzt, diese Lücke zu schließen.

In dem ersten Kapitel werden die grundlegenden Begriffsbildungen der Theorie, wie Markov-Ketten, Übergangsfunktionen, unabhängige Zufallsfolgen, Wiener- und Poisson-Prozesse, zufälliges Wandern, dargelegt. Besonderer Nachdruck wird auf die Prozesse mit diskreter Zeit gelegt; in dem weiteren Teil des Buches wird diese Klasse von Prozessen ausführlich untersucht. Das zweite Kapitel zeigt einige Anwendungen der Markov-Prozesse in der statistischen Mechanik, Lerntheorie und Ökonometrie. Das dritte Kapitel ist den Funktionen von Markovschen Prozessen gewidmet. Es wird der Zusammenhang zwischen der Markov-Eigenschaft und der Chapman—Kolmogorov-Gleichung untersucht, und werden Bedingungen für die Markovität der Funktionen von Markov-Prozessen angegeben. Im vierten Kapitel werden die grundlegenden Ergebnisse über das asymptotische Verhalten von Prozessen mit diskreter Zeit zusammengestellt. Neben einer verallgemeinerten Form des inviduellen Ergodensatzes werden Bedingungen für die Existenz von invarianten Maßen angegeben, und das in der Prädiktionstheorie wichtige asymptotische Verhalten der Potenzen des Übergangsoperators untersucht. Das fünfte Kapitel befaßt sich mit der Faltung von Maßen, definiert an topologischen Gruppen und Halbgruppen. Es wird die Rolle der idempotenten Maße, als Grenzverteilungen von Faltungspotenzen regulärer Maße herausgehoben.

Im sechsten Kapitel wird der Problemkreis untersucht, inwieweit man einen Markovschen Prozeß als (i.A. nichtlineare) Funktion einer Folge unabhängiger Zufallsgrößen darstellen kann. Die Bedeutung von Ergebnissen dieser Art in der nichtlinearen Prädiktionstheorie ist ähnlich der Rolle der Wold-Zerlegung in der linearen Theorie. In dem letzten Kapitel werden die Begriffe der starken Mischung und der gleichmässigen Ergodizität eigeführt, sowie ein zentraler Grenzwertsatz für Markov-Ketten bewiesen.

Die Darstellungen des Buches werden durch einen, die wichtigsten topologischen, funktionalanalytischen und wahrscheinlichkeitstheoretischen Grundlagen enthaltenen Anhang, ein ausführliches Literaturverzeichnis und einen Index sowie durch bibliographische Notizen abgerundet.

Das Buch ist in klarer mathematischer Form geschrieben, die konsequente funktionalanalytische Betrachtungsweise leiht ihm Übersichtlichkeit und Eleganz.

D. Vermes (Szeged)

K. Diederich—R. Remmert, Funktionentheorie. I (Heidelberger Taschenbücher, Bd. 103), XIII+246 Seiten, Berlin—Heidelberg—New York, Springer-Verlag, 1972. — DM 14.80.

Dieses Taschenbuch ist der erste Teil einer zweibändigen Darstellung der Grundlagen der Funktionentheorie, die auf Vorlesungen an der Universität Münster zurückgeht. Kap. I betrachtet die Grundlagen der Cauchyschen Theorie, während Kap. II die Grundlagen der Weierstraßschen Theorie darstellt. Kap. III behandelt Laurentreihen, Singularitäten und Fortsetzbarkeit. Normale Familien und Montelscher Satz folgen in Kap. IV. — Anwendungen und Beispiele gibt es praktisch keine. (Vielleicht die einzigen Beispiele sind die Berechnung der Integrale von $1/(1+x^2)$ und $(\sin x)/x$ auf $(-\infty, \infty)$ mit Hilfe des Residuensatzes; merkwürdigerweise bemerkt man nicht, daß im ersten Fall das Integral sich auch mit Hilfe der elementaren Stammfunktion $\operatorname{arc\,tg} x$ berechnen läßt.)

Béla Sz.-Nagy (Szeged)

A. V. Balakrishnan, Introduction to Optimization Theory in a Hilbert Space (Lecture Notes in Operations Research and Mathematical Systems, 42) IV+153 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1971. — DM 16,—.

The book is a work-out of lectures given by the author in a one-quarter course at the University of California, Los Angeles. The development of optimization theory in finite dimensions arose the question, which are the results that can be generalized for the infinite case. The present book is a guide through the results and difficulties related to this question in the case of a Hilbert space.

In the first chapter the basic properties of Hilbert spaces are sketched with special respect to the concept of weak convergence and the properties of convex sets. As a demonstration of the general theory, the basic results of convex programming, game theory and network flow optimization are presented. The second chapter deals with the fundamental concepts of the theory of linear operators and their spectral properties, while the third one presents the foundations of the theory of semi-groups of operators and their applications to problems of physics (heat, wave, Schrödinger equation) and of control (abstract Cauchy problem, controllability, observability, time-optimal control). The last chapter points out the difficulties concerning the definition of probability measures and random variables on Hilbert spaces.

The reader is expected to be familiar with the problems and results of optimization theory in a finite dimensional euclidean space as well as with the foundations of the theory of Hilbert spaces.

D. Vermes (Szeged)

Jean Céa, Optimisation: théorie et algorithmes (Méthodes mathématiques de l'informatique, vol. 2), IX+227 pages, Paris, Dunod, 1971. — 88 F.

Du point de vue mathématique, le problème de l'optimisation est celui de la minimisation d'une fonctionnelle avec contraintes ou non, en dimension finie ou non. Le but de ce livre est de donner une classification et l'exposé systématique des méthodes et des algorithmes qui s'imposent.

Les deux premiers chapitres traitent d'une manière rapide des éléments de l'analyse fonctionnelle et de la dérivation au sens de Gâteaux et Fréchet. Dans le chapitre 3 il est question de la minimisation sans contraintes. L'auteur expose ici les méthodes qui lui paraissent les plus intéressantes, en s'efforçant de les „unifier". Le chapitre 4 est sur la minimisation avec contraintes. Dans ces deux chapitres, on expose de nombreuses méthodes itératives. Le dernier chapitre fait une étude rapide de la dualité, basée sur les théorèmes de Hahn-Banach et du Min-max.

Quoique le contenu mathématique du livre n'est pas toujours très bien organisé (ce qui fait des sérieuses difficultés au lecteur), la diversité des méthodes exposées et des exemples étudiés fait ce livre une lecture intéressante et utile.

Béla Sz.-Nagy (Szeged)

Constructive Theory of Functions, Approximation Theory. Proceedings of the Conference held at Budapest, August 24 — September 3, 1969. Edited by **G. Alexits** and **S. B. Stechkin,** 538 pages, Akadémiai Kiadó, Budapest, 1972.

The Conference was organized by the Academies of Sciences of Hungary and of the USSR. The volume contains 52 papers each in the original language chosen by the author. (26 English, 12 German, 9 Russian and 5 French.)

The first two papers ,written by G. Szegő and I. I. Ibragimov, give appreciations of the works in approximation theory of the two late masters of this field, L. Fejér (1880—1959) and S. Bernstein (1880—1968).

Most of the further fifty papers deal with the theoretical aspects of approximation and contain new results. Some of them, besides the new results, have also a certain survey character and give useful references.

The topics treated cover different problems on approximation theory such as results in the classical theory of polynomial approximation, approximation by rational functions and spline functions, interpolation, approximation in normed vector spaces, theory of interpolation spaces, orthogonal series, etc.

The following authors' papers are presented in the volume: G. Albinus, H. J. Albrand and H. Kiesewetter, G. Alexits, L. Alpár, O. V. Besov, R. Bojanic, J. Brenner, Z. Ciesielski. L. Collatz, R. De Vore, Z. Ditzian, A. V. Efimov, P. Erdős, M. Frenkel, G. Freud and P, Popov, T. Ganelius, K. M. Garg, G. Goldner, M. Golitschek, E. Görlich and E. L. Stark, J. I. Ibragimov, Yu. A. Kazmin, O. Kis, Il. Kolumbán, Th. Kreutzkamp, A. F. Leontev, P. Lesky, I. Marusciac, M. Mikolás, J. Musielak, M. W. Müller, M. Z. Nashed, R. J. Nessel and W. Trebels, L. G. Pál and F. Schipp, J. Peetre, E. Popoviciu, T. Popoviciu, M. K. Potapov, G. Róna, I. J. Schoenberg (two papers), A. Sharma, D. D. Stancu, G. Sunouchi, J. Szabados, G. Szegő, K. Szilárd, S. A. Telyakovskii, P. Turán, A. K. Varma and P. O. H. Vértesi.

In addition there is a small report on unsolved problems proposed by R. Askey, R. De Vore, G. Freud, J. Musielak, J. J. Peetre, and T. Popoviciu.

The book is of great value for the experts of this field. The exposition is nice.

L. Leindler (Szeged)

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