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BIBLIOGRAPHIE

Some absolute topological properties under monotone unions

By TINUOYE M. ADENIRAN in Zaria (Nigeria)*)

1. Definition. A property P is said to be *absolute under monotone unions* (aumu) *in a class* $\mathscr{C}$ of topological spaces if, for any given Y and X_i $(i=1, 2, \ldots)$ in $\mathscr{C}$, with $X_i \subset Y$, $X_i \subset X_{i+1}$, the fact that each X_i has property P implies that $\bigcup_{i=1}^{\infty} X_i$ also has property P.

Connectedness and arcwise (path) connectedness are absolute under monotone unions in the class $\mathscr{C}_a$ of all topological spaces. But local connectedness and disconnectedness are not so; as an example illustrating the former, consider the Warsaw circle W, consisting of the curve $\sin\frac{\pi}{x}$ $(0<x\leqq 1)$, the interval $(-1, +1)$ of the y axis and a simple curve joining the points $(0, -1)$ and $(1, 0)$. Take as X_n the set

$$W-\left\{(x, y): y = \sin\frac{\pi}{x}, \quad 0 < x \leqq \frac{\pi}{n}\right\};$$

then each X_i is locally connected but $\bigcup_{i=1}^{\infty} X_i = W$ is not so. For an example illustrating the latter, let P be the set of irrationals in E^1 and let $Q=\{r_1, r_2, \ldots\}$ be an enumeration of E^1-P. Let $P_j = P \cup \{r_1, r_2, \ldots, r_j\}$. Each P_i is disconnected, but $\bigcup_{i=1}^{\infty} P_i = E^1$ is not. This last example also shows that the property of being 0-dimensional is not aumu in the class of all topological spaces.

By restricting $\mathscr{C}_a$ to the class $\mathscr{C}_0$ of countable metric spaces, disconnectedness is aumu in $\mathscr{C}_0$. This is a simple consequence of the well-known fact, that any non-void connected metric space has at least a continuum number of points.

A further example for a property which is aumu is the property of being F_σ in the class $\mathscr{C}_a$. But the property of being G_δ is not aumu in $\mathscr{C}_a$. This is well known, nevertheless we shall give a simple counter-example:

Consider the real line E^1. A finite set of rationals is trivially G_δ, but *the set Q of rationals is not G_δ in E^1*. This follows easily from the Baire category theorem and from the fact that Q is a set of first category in itself.

*) The paper was substantially revised, with the author's subsequent consent, by L. GEHÉR. *(The Editor.)*

2. The reader can easily see that a monotone union of T_0-spaces is a T_0-space. In this section we shall show that the property of being T_1 is also aumu in any class $\mathcal{C}$ while any separation axiom beyond this is not. We state the first assertion as

Theorem 1. *The property of being a T_1-space is aumu in any class $\mathcal{C}$.*

Proof. Let Y be a topological space with the sequence $\{X_i\colon X_i \subset X_{i+1}\}$ of subsets of Y such that each X_i is T_1. Let $X = \bigcup_{i=1}^{\infty} X_i$ and let x, y be two distinct points of X. Then there exists, for some $j \in Z^+$, $X_j \subset X$ such that $x, y \in X_j$. Since X_j is T_1 there exist open sets U', V' in X_j such that $x \in U'$, $x \notin V'$, $y \in V'$, $y \notin U'$. Furthermore there exist open sets U, V in X such that $U \cap X_j = U'$ and $V \cap X_j = V'$. Since $x \in U'$, $x \in U$, similarly $y \in V$. $x \in X_j$ and $x \notin V'$ imply that $x \notin V$, similarly $y \notin U$. We have thus found sets U, V open in X with $x \in U$, $y \in V$, $x \notin V$ and $y \notin U$. By definition, this entails that X is T_1 and the theorem is proved.

Corollary. *Let $\{X_i\colon X_i \subset X_{i+1}\}$ be a sequence of spaces such that each X_i is T_2 (regular, Tychonoff, normal). Then $\bigcup_{i=1}^{\infty} X_i$ is at least T_1.*

Theorem 2. *The property of being T_2 is not aumu in $\mathcal{C}_a$.*

Proof. Let I be the open unit interval (0, 1), and let

$$(1) \qquad X_k = (0 \times I) \cup (1 \times I) \cup \left(\frac{1}{2} \times I\right) \cup \cdots \cup \left(\frac{1}{k} \times I\right) \quad \text{for} \quad k = 1, 2, \ldots.$$

Each X_k is a finite union of open intervals in E^2 and since each I is T_2, each X_k is also T_2. So let $X = \bigcup_{i=1}^{\infty} X_i$. Topologize X as follows: On $X - (0 \times I)$ use the usual topology on E^1. For a neighbourhood of a point x in $0 \times I$, take an open interval in I of $0 \times I$ containing x and all the $\left(\frac{1}{j} \times I\right)$ with $\frac{1}{j} < \varepsilon$, where ε is an arbitrary positive real. Now let P_1 and P_2 be two distinct points of $0 \times I$. It is easy to see that any open subset of X containing P_1 meets any other containing P_2; we cannot therefore have two disjoint open sets containing P_1 and P_2 respectively; hence $X = \bigcup_{i=1}^{\infty} X_i$ is not T_2. — The same example shows:

Theorem 3. *Any separation axiom implying T_2 is not aumu in $\mathcal{C}_a$.*

Each X_k in (1) is metrizable, but X is not normal, and therefore not metrizable. So we have:

Theorem 4. *Metrizability is not aumu in $\mathcal{C}_a$.*

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Bemerkung zu einem von F. Szász angegebenen Ring

Von HANNS JOACHIM WEINERT in Clausthal (BRD)

In [3], Satz 3 gibt F. Szász einen assoziativen Ring A mit folgenden Eigenschaften an:

I) A hat zwei modulare nilpotente Rechtsideale R_1 und R_2, deren Durchschnitt nicht modular in A ist (vgl. [1], § 28, Seite 123).

II) Das Jacobsonsche Radikal J von A ist ein maximales modulares Rechtsideal von A mit $J^2 \neq 0$ und $J^3 = 0$.

Dabei definiert F. Szász diesen Ring A als Algebra über dem Primkörper K_2 der Charakteristik 2 durch folgende Multiplikationstafel der vier Basiselemente a, b, c und d:

(1)

$\cdot$	a	b	c	d
a	a	$a+b+c$	a	d
b	$a+b+d$	b	c	b
c	c	b	c	$a+c+d$
d	a	d	$b+c+d$	d .

Es wird behauptet, daß der Ring A nicht monomial im Sinne von Rédei [2], § 66 ist (vgl. auch [4], § 4). Gegenstand dieser kurzen Note ist zu zeigen, daß *dieser Ring doch eine monomiale Basis über K_2 besitzt und mit ihrer Hilfe die Behauptungen* I) *und* II) *und auch die Assoziativität von A sehr leicht nachzuweisen sind.*

Mit $\{a, b, c, d\}$ bilden auch die folgenden vier Elemente eine Basis des Vektorraumes A über K_2:

$$\alpha = a, \quad \beta = a+b+c+d, \quad \gamma = a+c, \quad \delta = a+d.$$

Aus (1) folgt die Multiplikationstafel

(2)

$\cdot$	α	β	γ	δ
α	α	β	0	δ
β	β	0	0	0
γ	γ	0	0	0
δ	0	0	β	0

und umgekehrt. Damit ist $\{\alpha, \beta, \gamma, \delta\}$ monomiale Basis[1]) von A über K_2. (2) ist sogar im wesentlichen die Strukturtafel einer Halbgruppe mit Nullelement $H=\{\alpha, \beta, \gamma, \delta, 0\}$, die etwa durch folgende Transformationen auf der Menge $X=\{1, 2, 3, 4\}$ realisiert werden kann:

$$\alpha=\begin{pmatrix}1&2&3&4\\1&2&4&4\end{pmatrix},\ \beta=\begin{pmatrix}1&2&3&4\\2&4&4&4\end{pmatrix},\ \gamma=\begin{pmatrix}1&2&3&4\\4&4&2&4\end{pmatrix},\ \delta=\begin{pmatrix}1&2&3&4\\3&4&4&4\end{pmatrix},\ 0=\begin{pmatrix}1&2&3&4\\4&4&4&4\end{pmatrix}.$$

Aber auch ohne eine solche Darstellung prüft man die Assoziativität bei (2) leichter als bei (1).

Beweis von I). $R_1=\{\gamma, 0\}$ und $R_2=\{\beta+\gamma, 0\}$ sind nach (2) Rechtsideale von A, mit α als Linkseinselement modulo R_1 und $\alpha+\delta$ als Linkseinselement modulo R_2. Weiter gilt $R_1^2=R_2^2=0$. Der Durchschnitt $R_1\cap R_2=0$ ist aber kein modulares Rechtsideal, da A wegen $A\gamma\not\ni\gamma$ kein Linkseinselement besitzt.

Beweis von II). Der von $\{\beta, \gamma, \delta\}$ erzeugte Unterraum J von R ist nach (2) zweiseitiges Ideal, aus Anzahlgründen maximales Rechtsideal und wegen $\alpha^2=\alpha$ modular. Wegen $A/J\approx K_2$ ist J das Radikal von A. Aus (2) ersieht man $J^2=\{0, \beta\}$ und $J^3=0$.

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[1]) Es existieren weitere monomiale Basen von A über K_2, doch ist es nicht möglich, die erzeugenden Elemente $\gamma=a+c$ und $\beta+\gamma=b+d$ der Rechtsideale R_1 bzw. R_2 (vgl. Beweis von I)) zusammen in eine monomiale Basis aufzunehmen.

D'Alembert's functional equation in Banach algebras

By JOHN A. BAKER in Waterloo (Ontario, Canada)

1. Suppose B is a Banach algebra and $f\colon R \to B$ (R denotes the field of real numbers) such that

(1) $$f(s+t)+f(s-t) = 2f(s)f(t)$$

for all $s, t \in R$. S. KUREPA [6] has shown that if B has identity e, $f(0)=e$, and f is measurable then there exists a unique $b \in B$ such that

$$f(s) = e+\frac{s^2b}{2!}+\frac{s^4b^2}{4!}+\cdots$$

for all $s \in R$. Note that if $b=a^2$ for some $a \in B$ then $f(s) = \frac{1}{2}\{\exp(sa)+\exp(-sa)\} = \cosh(sa)$ for all $s \in R$. In this paper we consider the problem of finding the solutions of (1) on $(0, \infty)$ and without the assumption that B has an identity. The main result is that if $f\colon (0, \infty) \to B$ satisfies (1) for $s>t>0$ and if $\lim_{t \to 0+} f(t)$ exists then there exists $j, b, c \in B$ such that $j^2=j$, $jb=bj=b$, $cj=c$, $jc=0$ and $f(s) = \left(j+\frac{s^2b}{2!}+\frac{s^4b^2}{4!}+\cdots\right)+c\left(sj+\frac{s^3b}{3!}+\frac{s^5b^2}{5!}+\cdots\right)$ for all $s>0$. This result is analogous to a result concerning the functional equation $f(s+t) = f(s)f(t)$ which can be found on page 283 of the book of HILLE and PHILLIPS [4]. Also included in the present paper are certain general results concerning (1) when the domain is an Abelian group and the range is an associative algebra over the rationals. Some regularity properties are also included in cases when topologies are present.

2. We begin by deriving some general properties of solutions of (1). Let G be an additive Abelian group, let B be an associative algebra over the field of rational numbers and suppose $f\colon G \to B$ satisfies (1) for all $s, t \in G$.

Let $j=f(0)$. Then, putting $s=t=0$ in (1) we find

(2) $$j^2=j.$$

With $t=0$ in (1) we have

(3) $$f(s)=f(s)j$$

for all $s \in G$.

Now let g and h be the even and odd parts of f respectively; that is, $2g(s) = f(s)+f(-s)$, $2h(s) = f(s)-f(-s)$ for all $s\in G$. Letting $s=0$ in (1) we find

$$g=jf. \tag{4}$$

Thus $g = jg+jh$ and so, since g and jg are even,

$$jh=0. \tag{5}$$

From (4) and (2) it follows that

$$jg=j^2f=jf=g. \tag{6}$$

Now (3) implies

$$gj=g \tag{7}$$

and

$$hj=h. \tag{8}$$

Thus, by (7) and (5),

$$g(s)h(t)=(g(s)j)h(t)=g(s)(jh(t))=0 \tag{9}$$

and similarly, by (5) and (8),

$$h(s)h(t)=0 \tag{10}$$

for all $s, t\in G$. Using (4), (1) and (9) we conclude that

$$\begin{aligned} g(s+t)+g(s-t) &= j(f(s+t)+f(s-t)) = 2jf(s)f(t) = 2g(s)f(t) = \\ &= 2g(s)g(t)+2g(s)h(t) = 2g(s)g(t) \quad \text{for all} \quad s, t\in G. \end{aligned} \tag{11}$$

If $f(0)=0$ then, by (3), $f\equiv 0$. If $j\neq 0$ then j is an identity for the subalgebra $B'=\{x\in B\colon jx=xj=x\}$ and, from (6) and (7), $g(s)\in B'$ for all $s\in G$. Thus g can be considered as a mapping of G into B' which is a solution of (11), or (1) and $g(0)=j$, the identity of B'.

From (9) and (10) we find

$$\begin{aligned} h(s+t)+h(s-t) &= f(s+t)+f(s-t)-g(s+t)-g(s-t) = \\ &= 2f(s)f(t)-2g(s)g(t) = 2h(s)g(t) \quad \text{for all} \quad s, t\in G. \end{aligned} \tag{12}$$

3. In this section we impose topologies on G and B and consider some regularity properties of solutions of (1).

Proposition 1. *Let G be a locally compact Abelian group, let B be a Banach algebra and suppose $f\colon G\to B$ satisfies* (1) *for all $s, t\in G$. If f is strongly measurable on a set of positive, finite Haar measure, then the mapping $t\to f(2t)$ is continuous at* 0.

Proof. Suppose f is strongly measurable on a measurable set A of positive finite Haar measure. Then f is the pointwise limit almost everywhere on A of a sequence of countably valued measurable functions (see [4] page 72). As in the complex valued case, the theorems of Egorov and Lusin can be proved (see [3] pages 158—160) and we conclude that there exists a compact subset K of A of positive Haar measure such that the restriction of f to K is continuous. It follows that f is uniformly continuous on K. (See [7] page 256.)

Since K has positive finite Haar measure there exists a neighborhood V of $0 \in G$ such that

$$K \cap (K+v) \cap (K-v) \neq \emptyset$$

whenever $v \in V$. (See [2] page 296.)

Let $\varepsilon > 0$ and $M = \max \{\|f(t)\| : t \in K\}$. Since f is uniformly continuous on K there exists a symmetric neighborhood U of $0 \in G$ such that $\|f(s) - f(t)\| < \varepsilon/4M$ provided $s, t \in K$ and $s - t \in U$. Now

$$f(2v) + f(2u) = 2f(u+v)f(u-v)$$

and so

$$\|f(2v) - f(0)\| = 2\|f(u+v)f(u+v) - f(u)f(u)\| \leqq$$

$$\leqq 2\|f(u+v)\| \, \|f(u-v) - f(u)\| + 2\|f(u)\| \, \|f(u+v) - f(u)\|.$$

If $v \in V \cap U$ then there exists $u \in K$ such that $u + v \in K$ and $u - v \in K$ so that $v \in V \cap U$ implies $\|f(2v) - f(0)\| < \varepsilon$.

Corollary. *If in addition to the hypotheses of Proposition 1 it is assumed that the mapping $t \to 2t$ is a bicontinuous automorphism of G, then f is continuous at* 0.

Proposition 2. *Let X be a Hausdorff linear topological space, B a Banach algebra and suppose $f: X \to B$ satisfies* (1) *for all $s, t \in X$. If f is continuous at* 0, *then f is continuous everywhere.*

Proof. Replace s by nt in (1) where n is a positive integer to find that

$$f((n+1)t) = 2f(nt)f(t) - f((n-1)t)$$

for all $t \in X$ and $n = 1, 2, \ldots$. Since f is continuous at 0, f is bounded on an open neighborhood U of $0 \in X$. Hence, by induction, f is bounded on nU for $n = 1, 2, 3, \ldots$. But $X = \bigcup_{n=1}^{\infty} nU$ and thus f is bounded in a neighborhood of each point of X since each nU is open. We know that

$$\lim_{t \to 0} \frac{f(s+t) + f(s-t)}{2} = \lim_{t \to 0} f(s)f(t) = f(s)f(0) = f(s)$$

for all $s \in X$ by (1) and (3). Suppose f is not continuous at some fixed $s \in X$. Then there exists $d>0$ and a net $\{t_\alpha\}\subset X$ such that $t_\alpha\to 0$ and

$$\|f(s+t_\alpha)-f(s)\| \geqq d \quad \text{for all} \quad \alpha.$$

But then, by (1) and (3),

$$\|f(s+2t_\alpha)-f(s)\| = \|f(s+2t_\alpha)+f(s)-2f(s+t_\alpha)-2f(s)+2f(s+t_\alpha)\| =$$
$$= \|\{2f(s+t_\alpha)f(t_\alpha)-2f(s+t_\alpha)f(0)\}-2\{f(s)-f(s+t_\alpha)\}\| \geqq$$
$$\geqq 2\|f(s)-f(s+t_\alpha)\|-2\|f(s+t_\alpha)\{f(t_\alpha)-f(0)\}\|$$

for all α. Since f is bounded in a neighborhood of s and f is continuous at 0, $\lim_\alpha \|f(s+t_\alpha)\{f(t_\alpha)-f(0)\}\| = 0$. Hence

$$\limsup_\alpha \|f(s+2t_\alpha)-f(s)\| \geqq 2d.$$

It follows by induction that

$$\limsup_\alpha \|f(s+2^k t_\alpha)-f(s)\| \geqq 2^k d$$

for each $k=1, 2, \ldots$ which contradicts the fact that f is bounded in a neighborhood of s. Thus, by contradiction, f is continuous at every $s\in X$.

Corollary. *If B is a Banach algebra, f: $R^n\to B$ satisfies* (1) *for all $s, t\in R^n$ and if f is measurable on a set of positive, finite, n-dimensional Lebesgue measure, then f is continuous.*

Proof. This follows from the corollary to Proposition 1 and Proposition 2.

4. The theorem of this section, which generalizes a theorem of S. Kurepa [6], is the main result of this paper. In its proof we use several properties of a Riemann-type integral for vector valued functions for which we omit the elementary proofs. If $[a, b]$ is a compact interval, if X is a Banach space and f: $[a, b]\to X$ is continuous, then f is uniformly continuous on $[a, b]$. As in the real valued case one can prove the existence of a unique $x\in X$ which has the following property: To each $\varepsilon>0$ there corresponds $\delta>0$ such that $\|x-\sum_{k=1}^{n}(t_k-t_{k-1})f(s_k)\| < \varepsilon$ provided $a=t\leqq s_1\leqq t_1\leqq s_2\leqq t_2\leqq\cdots\leqq t_{n-1}\leqq s_n\leqq t_n=b$ and $|t_k-t_{k-1}| < \delta$ for $k=1, 2, \ldots, n$. We write $x = \int_a^b f(t)dt$ and call this vector the integral of f over $[a, b]$.

Lemma. *Let X be a Banach space and let $0<a<\infty$. Suppose that φ: $(0,a)\to X$ is continuous, $\varphi'(t)$ exists, and $\|\varphi'(t)\|\leqq M<\infty$ for $0<t<a$. Then*

(i) $\lim_{t\to 0+}\varphi(t)=\alpha$ *exists;*

(ii) *if* $\lim_{t\to 0+}\varphi'(t)=\beta$ *exists, we have* $\beta = \lim_{t\to 0+}\frac{1}{t}(\varphi(t)-\alpha)$.

Proof. (i) Suppose $\{t_n\}_{n=1}^{\infty} \subseteq (0, a)$ and $t_n \to 0$ as $n \to \infty$. Then

$$\|\varphi(t_n) - \varphi(t_m)\| = \left\| \int_{t_n}^{t_m} \varphi'(t)\,dt \right\| \leqq M|t_n - t_m| \to 0 \quad \text{as} \quad n, m \to \infty.$$

Thus $\alpha = \lim_{n\to\infty} \varphi(t_n)$ exists since X is complete. If $\{s_n\}_{n=1}^{\infty} \subseteq (0, a)$ and $s_n \to 0$ as $n \to \infty$ then $\alpha' = \lim_{n\to\infty} \varphi(s_n)$ exists. Letting $u_n = t_n$ for n even and $u_n = s_n$ for n odd we find

$$\alpha = \lim_n \varphi(t_n) = \lim_n \varphi(u_n) = \lim_n \varphi(s_n) = \alpha'.$$

Hence $\lim_{t\to 0+} \varphi(t)$ exists and is equal to α.

(ii) Let $\Phi(t) = \begin{cases} \varphi'(t) & \text{if} \quad 0 < t < a, \\ \beta & \text{if} \quad t = 0. \end{cases}$ Then Φ: $[0, a) \to X$ is continuous and

$$\int_0^s \Phi(t)\,dt = \int_0^\varepsilon \Phi(t)\,dt + \int_\varepsilon^s \varphi'(t)\,dt = \int_0^\varepsilon \Phi(t)\,dt + \varphi(s) - \varphi(\varepsilon)$$

whenever $0 < \varepsilon < s < a$. Letting $\varepsilon \to 0+$ we conclude $\varphi(s) - \alpha = \int_0^s \Phi(t)dt$ for $0 < s < a$ and so

$$\frac{1}{s}(\varphi(s) - \alpha) = \frac{1}{s} \int_0^s \Phi(t)\,dt \to \Phi(0) = \beta \text{ as } s \to 0+.$$

Theorem. *Let B be a Banach algebra and let f: $(0, \infty) \to B$ be such that*

$$f(s+t) + f(s-t) = 2f(s)f(t)$$

whenever $s > t > 0$. If $\lim_{t\to 0+} f(t) = j$ exists then $j^2 = j$ and there exist elements $b, c \in B$ such that $jb = bj = b$, $cj = c$, $jc = 0$ and

$$f(s) = \left(j + \frac{s^2 b}{2!} + \frac{s^4 b^2}{4!} + \cdots +\right) + c\left(sj + \frac{s^3 b}{3!} + \frac{s^5 b^2}{4!} + \cdots\right) \tag{13}$$

for all $s > 0$. Conversely, with such j, b, and c, if f is defined by (13) *for all $s \in R$ then f satisfies* (1) *for all $s, t \in R$.*

Proof. We begin by proving the first assertion. Putting $s = 2t$ in (1) we find

$$f(3t) + f(t) = 2f(2t)f(t) \tag{14}$$

for all $t > 0$. If we let $t \to 0+$ in (14) we conclude that $j^2 = j$.

Since $\lim_{t\to 0+} f(t)$ exists, f is bounded on an interval of the form $(0, a)$ for some $a > 0$. But then (14) implies f is bounded on $(0, (3/2)a)$. By induction one can prove that f is bounded on any finite subinterval of $(0, \infty)$.

We now aim to show that

$$f(t)j=f(t) \tag{15}$$

for all $t>0$. To this end let $\varphi(t)=f(t)-f(t)j$ for $t>0$. Since $j^2=j=\lim_{t\to 0+} f(t)$ we have $\lim_{t\to 0+}\varphi(t)=0$. Also, whenever $s>t>0$, $\varphi(s+t)+\varphi(s-t)=2f(s)f(t)-2f(s)f(t)j=2f(s)\varphi(t)$. If $u>v>0$,

$$\varphi(u)+\varphi(v)=2f\left(\frac{u+v}{2}\right)\varphi\left(\frac{u-v}{2}\right).$$

Fix $a>0$ and let $M=\sup\{\|f(t)\|:0<t<a\}$. Let $\varepsilon>0$ and choose $\delta>0$ such that $0<t<\delta$ implies $\|\varphi(t)\|<\varepsilon/4M$. Then if $0<v<u<a$ and $u-v<2\delta$,

$$\|\varphi(u)+\varphi(v)\|\leqq 2M(\varepsilon/4M)=\varepsilon/2$$

so that

$$\|\varphi(u)-\varphi(v)\|=\left\|\varphi(u)+\varphi\left(\frac{u+v}{2}\right)-\varphi(v)-\varphi\left(\frac{u+v}{2}\right)\right\|\leqq$$

$$\leqq\left\|\varphi(u)+\varphi\left(\frac{u+v}{2}\right)\right\|+\left\|\varphi(v)+\varphi\left(\frac{u+v}{2}\right)\right\|<\varepsilon.$$

We have shown that φ is uniformly continuous on $(0, a)$ for any $a>0$ and hence φ is continuous. Thus, for any $s>0$,

$$2\varphi(s)=\lim_{t\to 0+}\varphi(s+t)+\varphi(s-t)=\lim_{t\to 0+}2f(s)\varphi(t)=0,$$

which proves (15).

The next step in the proof consists of showing that f is continuous. Let $a>0$ and $M=\{\|f(t)\|: 0<t<a\}$. If $0<v<u<a$ then by (1) and (15)

$$\left\|f(u)+f(v)-2f\left(\frac{u+v}{2}\right)\right\|=\left\|2f\left(\frac{u+v}{2}\right)f\left(\frac{u-v}{2}\right)-2f\left(\frac{u+v}{2}\right)j\right\|\leqq$$

$$\leqq 2M\left\|f\left(\frac{u-v}{2}\right)-j\right\|.$$

Thus for every $\varepsilon>0$ there exists $\delta>0$ such that

$$\left\|f(u)+f(v)-2f\left(\frac{u+v}{2}\right)\right\|<\varepsilon \tag{16}$$

whenever $0<u$, $v<a$ and $0<|u-v|<\delta$.

Now suppose f is not continuous at s where $0<s<a$. Then there exist $d>0$ and a sequence $\{t_n\}$ converging to 0 such that $\|f(s+t_n)-f(s)\|\geqq d$ for each $n=1, 2, \ldots$.

Hence

$$\|f(s+2t_n)-f(s)\| = \|f(s+2t_n)+f(s)-2f(s+t_n)+2f(s+t_n)-2f(s)\| \geqq$$
$$\geqq 2\|f(s+t_n)-f(s)\|-\|f(s+2t_n)+f(s)-2f(s+t_n)\|$$

for each $n=1, 2, \ldots$. But, by (16),

$$\lim_{n\to\infty} \|f(s+2t_n)+f(s)-2f(s+t_n)\| = 0$$

so that

$$\limsup_{n\to\infty} \|f(s+2t_n)-f(s)\| \geqq 2d.$$

As in the proof of Proposition 2, this contradicts the boundedness of f in a neighborhood of s. Thus f is continuous at s. Since a was arbitrary, f is continuous on $(0, \infty)$.

Now define $F(s)=\begin{cases} f(s) & \text{for } s>0 \\ j & \text{for } s=0 \end{cases}$. Then F is continuous on $[0, \infty)$,

(17) $$F(s+t)+F(s-t) = 2F(s)F(t)$$

whenever $s\geqq t\geqq 0$ and

(18) $$Fj=F.$$

Motivated by the consideration in section 2 we let $G=jF$ and $H = F-G$. Then

(19) $$G(0)=jF(0)=j^2=j \quad\text{and}\quad H(0) = F(0)-G(0) = 0,$$

G and H are continuous on $[0, \infty)$ and, by (18),

(20) $$jG=G=Gj, \quad jH=0 \quad\text{and}\quad Hj=H.$$

Therefore, by (20),

(21) $$G(s)H(t)=(G(s)j)H(t)=G(s)(jH(t))=0$$

and

(22) $$H(s)H(t)=(H(s)j)H(t)=H(s)(jH(t))=0 \quad\text{for all}\quad s, t\geqq 0.$$

Let $B'=\{x\in B: xj=jx=x\}$. Then B' is a closed subalgebra of B and is thus a Banach algebra. Furthermore, j is the identity of B'. Also note that, by (20), G: $[0, \infty)\to B'$ and, from (21),

(23) $$G(s+t)+G(s-t) = 2jF(s)F(t) = 2G(s)G(t)$$

provided $s\geqq t\geqq 0$.

Let $a>0$. If $0<\varepsilon<a<s$ then, by (23),

$$\int_0^\varepsilon G(s+t)+G(s-t)\,dt = 2G(s)\int_0^\varepsilon G(t)\,dt.$$

But $\lim_{\varepsilon \to 0+} (1/\varepsilon) \int_0^{\varepsilon} G(t)\,dt = G(0) = j$ so for sufficiently small $\varepsilon > 0$, $\int_0^{\varepsilon} G(t)\,dt$ has an inverse in B'. We fix $\varepsilon > 0$ and let $\gamma^{-1} = \int_0^{\varepsilon} G(t)\,dt$ to deduce that $G(s) = \frac{1}{2}\left\{\int_s^{s+\varepsilon} G(t)\,dt - \int_{s-\varepsilon}^{s} G(t)\,dt\right\}\gamma$ for all $s > a$. It follows that G has continuous derivatives of every order on (a, ∞) and, since a was arbitrary, G has continuous derivatives of every order on $(0, \infty)$.

Differentiating (23) with respect to t we find

$$G'(s+t) - G'(s-t) = 2G(s)G'(t)$$

whenever $s > t > 0$. With sufficiently small $s > 0$,

$$\lim_{t \to 0+} G'(t) = \lim_{t \to 0+} \frac{1}{2} G(s)^{-1}[G'(s+t) - G'(s-t)] = 0.$$

By the lemma,

$$G'(0) = \lim_{t \to 0+} \frac{G(t) - G(0)}{t} = 0. \tag{24}$$

From (23) it follows that

$$G''(s+t) + G''(s-t) = 2G(s)G''(t) \tag{25}$$

for $s > t > 0$. Thus for sufficiently small $s > 0$

$$\lim_{t \to 0+} G''(t) = \lim_{t \to 0+} \frac{1}{2} [G(s)^{-1}][G''(s+t) + G''(s-t)] = G(s)^{-1} G''(s).$$

It follows from the lemma that G' is continuously differentiable on $[0, \infty)$. If we let $b = G''(0) \in B'$ and let $t \to 0+$ in (25) we find that

$$G''(s) = G(s)b \tag{26}$$

for all $s > 0$. Since $b \in B'$, (26) also holds if $s = 0$.

From (26), (24) and (19) it follows that

$$G(t) = j + \int_0^t \int_0^u G(s)b\,ds\,du = j + \int_0^t (t-s)G(s)b\,ds$$

for all $t \geqq 0$. By iteration one finds

$$G(t) = j + \frac{t^2 b}{2!} + \cdots + \frac{t^{2n} b^n}{(2n)!} + \frac{1}{(2n+1)!} \int_0^t (t-s)^{2n+1} G(s) b^{n+1}\,ds$$

for all $t \geqq 0$. The last term on the right tends to 0 as $n \to \infty$ for any fixed $t > 0$, so

$$G(t) = j + \frac{t^2 b}{2!} + \frac{t^4 b^2}{4!} + \cdots \tag{27}$$

for all $t \geqq 0$ since this series converges absolutely. Also note that $bj = jb = b$ since $b \in B'$.

We now solve for H. From (17) and (23),

$$H(s+t) + H(s-t) = 2F(s)F(t) - 2G(s)G(t)$$

and then, in view of (21) and (22), we find

$$H(s+t) + H(s-t) = 2H(s)G(t) \quad \text{for} \quad s \geqq t \geqq 0. \tag{28}$$

As with G, we deduce from (28) that H has continuous derivatives of every order on $(0, \infty)$. Differentiating (27) twice with respect to t and letting $t \to 0+$ we find

$$H''(s) = H(s)b \quad \text{for all} \quad s > 0. \tag{29}$$

Now since $\lim_{s \to 0+} H''(s) = \lim_{s \to 0+} H(s)b = 0$ it follows from the lemma that $\lim_{s \to 0+} H'(s) = c$ exists. Another application of the lemma proves that $H'(0) = c$ exists and $c = \lim_{s \to 0+} H'(s)$.

As with G, we deduce from (28), (19), and the fact that $H'(0) = c$ that for all $s > 0$

$$H(s) = c\left(sj + \frac{s^3 b}{3!} + \frac{s^5 b^2}{5!} + \cdots\right). \tag{30}$$

From (20) we find that $jc = 0$ and $cj = c$.

We have thus shown that f satisfies (13) for all $s > 0$.

To prove the converse let $j, b, c \in B$ such that $jb = bj = b$, $cj = c$ and $jc = 0$. Define $G: R \to B$ by (27) and $H: R \to B$ by (30) and let $f(s) = G(s) + H(s)$ for all $s \in R$. Note that $bc = (bj)c = b(jc) = 0$ and thus

$$G(s)H(t) = H(s)H(t) = 0 \tag{31}$$

for all $s, t \in R$. It is not difficult to verify directly that G satisfies (23) for all $s, t \in R$. Note that $H = cG'$ so that

$$H(s+t) + H(s-t) = cG'(s+t) + cG'(s-t) = 2cG'(s)G(t) = 2H(s)G(t) \tag{32}$$

for all $s, t \in R$. Thus by (23), (31) and (32)

$$f(s+t) + f(s-t) = 2G(s)G(t) + 2H(s)G(t) =$$

$$= 2[G(s) + H(s)][G(t) + H(t)] = 2f(s)f(t) \quad \text{for all } s, t \in R.$$

This completes the proof of the theorem.

The following corollary follows directly from the corollary to Proposition 2 and the above theorem.

Corollary. *Let B be a Banach algebra and suppose f: $R \to B$ is such that* (1) *is true for all $s, t \in R$. Then f is measurable on a set of positive Lebesgue measure if and only if f has the form* (13) *for constants $j, b, c \in B$ satisfying $j^2 = j$, $jb = bj = b$, $cj = c$ and $cj = 0$.*

Remarks. Many authors have considered equation (1), often called D'Alembert's equation (see [1]). KANNAPPAN [5] has shown that the general solution of (1) among complex valued functions defined on an Abelian group G is of the form $f(s) = \frac{1}{2}\{m(s)+m(-s)\}$ where m is a complex valued function defined on G and satisfying $m(s+t) = m(s)m(t)$ for all $s, t \in G$. SOVA [8] has considered the strongly continuous solutions of (1) where f is defined on $(0, \infty)$ and has values in the Banach algebra of bounded operators on a Banach space and succeeded in proving an analogue of the Hille—Yosida theorem in the theory of semi-groups of operators.

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A connection between commutativity and separation of spectra of operators

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1. Introduction. Recent results indicate that there is a basic connection between the commutativity of certain operators on a Banach space and the spectra of those operators. In [2] it was shown that if A is an operator on a complex Banach space and $\sigma(A)\cap\sigma(e^{2\pi ik/n}A)=\emptyset$ for $k=1,\dots,n-l$, then A and A^n commute with the same operators. This result was strongly generalized in [3] as follows: if f is holomorphic on a neighborhood of $\sigma(A)$, f is $1-1$ on $\sigma(A)$ and $f'(z)\neq 0$ on $\sigma(A)$, then A and $f(A)$ commute with the same operators. In this paper we generalize the results of [2] for the case $n=2$ by considering two operators A and B such that $\sigma(A)\cap\sigma(B)=\emptyset$.

2. Notation and terminology. We shall consider a Banach algebra $\mathscr{B}$ with an identity element I and elements $A, B, X, \dots$; $\sigma(A)$ is the *spectrum* of A. In case $\mathscr{B}$ is the algebra of continuous linear operators on a Hilbert space we use the standard notation: if $A\in\mathscr{B}$, then A^* is the (Hilbert space) adjoint of A, Re $A=(A+A^*)/2$, and Im $A=(A-A^*)/2i$. In this case we say that A is *normal* if $AA^*=A^*A$ and A is *unitary* if $AA^*=A^*A=I$.

3. The theorem. In [4, Theorem 3.1] it was proved that if $\sigma(A)\cap\sigma(B)=\emptyset$, then for each Y in $\mathscr{B}$ there exists a unique solution to the equation $BX-XA=Y$. In particular, $BX-XA=0$ only in case $X=0$. We use this result to prove:

Theorem. *If $\sigma(A)\cap\sigma(B)=\emptyset$, then X commutes with each of A and B if and only if X commutes with each of $A+B$ and AB.*

Proof. One of the implications is obvious. Assume that X commutes with $A+B$ and AB. Then

$$A(AX-XA)-(AX-XA)B = A^2X-AX(A+B)+X(AB) =$$

$$= A^2X-A(A+B)X+(AB)X = 0.$$

Thus by [4, Theorem 3. 1], we have $AX-XA=0$. It is now obvious that $BX-XB=0$ also.

The hypothesis of the theorem, calling for a separation of the spectrum of A and the spectrum of B, is dictated by the example of the operators $A=-B=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ on two-dimensional complex Banach space. In this case $A+B=AB=0$.

4. Applications. We list below a few of the general applications of our theorem and then concentrate on the applications to operators on Hilbert space.

Corollary 1. *If $\sigma(A)\cap\sigma(B)=\emptyset$, then A and B commute if and only if $A+B$ and AB commute.*

Proof. A and B commute if and only if $A+B$ commutes with each of A and B. Apply the theorem with $X=A+B$.

Corollary 2. ([2] and [3]) *If $\sigma(A)\cap\sigma(-A)=\emptyset$, then X commutes with A if and only if X commutes with A^2.*

Proof. Apply the theorem with $B=-A$.

The next result is applicable to any invertible element of $\mathscr{B}$ of norm less than 1.

Corollary 3. *If A is invertible and $\sigma(A)\cap\sigma(A^{-1})=\emptyset$, then X commutes with A if and only if X commutes with $A+A^{-1}$.*

Proof. Apply the theorem with $B=A^{-1}$.

Other general algebraic applications are obvious.

In Corollaries 4—8 we assume that $\mathscr{B}$ is the Banach algebra of continuous linear operators on a Hilbert space.

The first application to operators on Hilbert space is obtained by choosing $B=A^*$.

Corollary 4. *If $\sigma(A)\cap\sigma(A^*)=\emptyset$, then X commutes with each of A and A^* if and only if X commutes with each of* $\operatorname{Re} A$ *and* AA^*.

A special result of Corollary 4 is obtained by choosing $X=\operatorname{Re} A$.

Corollary 5. *If $\sigma(A)\cap\sigma(A^*)=\emptyset$, then A is normal if and only if* $\operatorname{Re} A$ *commutes with AA^*.*

This last corollary is reminiscent of the result in [1, Theorem 1]: A is normal if and only if each of AA^* and A^*A commutes with $\operatorname{Re} A$. The restriction on the spectrum of A in Corollary 5 thus reduces the number of commutativity relations required to force A to be normal.

Another consequence of Corollary 4 is obtained by assuming that A is unitary.

Corollary 6. *If A is unitary and $\sigma(A)\cap\sigma(A^*)=\emptyset$, then X commutes with A if and only if X commutes with* $\operatorname{Re} A$.

If the Hilbert space under consideration is finite dimensional and A is any unitary operator, then it follows from Corollary 6 that some unit multiple of A, say $e^{i\theta}A$, is such that an operator X commutes with $\operatorname{Re}(e^{i\theta}A)$ if and only if X commutes with $\operatorname{Im}(e^{i\theta}A)$.

Corollary 7. *If either* $\operatorname{Re} A$ *or* $\operatorname{Im} A$ *is invertible, then X commutes with each of A and A^* if and only if X commutes with each of A and* $(\operatorname{Re} A)\cdot(\operatorname{Im} A)$.

Proof. Under the hypothesis, we have $\sigma(\operatorname{Re} A)\cap\sigma(i\operatorname{Im} A)=\emptyset$. Apply the theorem with $A_1=\operatorname{Re} A$ and $B_1=i\operatorname{Im} A$.

As a final application consider Corollary 7 with $X=A$ to give an equivalent condition for the normality of an operator A.

Corollary 8. *If either* $\operatorname{Re} A$ *or* $\operatorname{Im} A$ *is invertible, then A is normal if and only if A commutes with* $(\operatorname{Re} A)\cdot(\operatorname{Im} A)$.

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On the Suzuki structure theory for non self-adjoint operators on Hilbert space

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Throughout this paper all Hilbert spaces will be complex and all operators considered on them will be linear and bounded. Let A be an operator and $p(z, \bar{z})$ a complex non-commutative polynomial in z and $\bar{z}$. In Section 1 we shall give a complete structure theorem for the operator A whenever $p(A, A^*)$ is compact. The theorems in Section 1 are based on the structure of the W^*-algebra generated by A and they will include the results of N. SUZUKI [14], who developed this theory for an operator A with $\operatorname{Im} A$ compact, and also the generalizations of Suzuki's work by H. BEHNCKE [1] and [2] and the author [8]. In Section 2 we shall give an application of this theory to the study of non self-adjoint spectral operators on Hilbert space. By using C^*-algebra techniques, one can also obtain many of the results in this paper. In particular, Lemma 4 in [1] and its generalization to non-separable spaces play a role in the C^*-algebra development analogous to the role of Proposition 1 in the W^*-algebra approach presented here.

If A is an operator on a Hilbert space, we shall denote by $R(A)$ the W^*-or von Neumann algebra generated by A, that is, the smallest weakly closed algebra containing A and I and closed under the operation of taking adjoints. The set of all operators which commute with every operator in $R(A)$ is called the commutant of $R(A)$ and is denoted by $R(A)'$. N. SUZUKI [14] called an operator *primary* if $R(A)$ is a factor, that is, if its center $Z(A) \equiv R(A) \cap R(A)'$ consists of the scalar multiples of the identity. For the terminology, notation and basic results on von Neumann algebras we refer to J. DIXMIER [6].

[1]) This paper was prepared while the author was an Office of Naval Research Postdoctoral Associate at Indiana University (1969—70). This work represents generalizations of parts of the author's Ph. D. thesis which was directed by N. SUZUKI.

1. Structure theorems

In this section we prove the following main structure theorem.

Theorem 1. *Let A be an operator on a Hilbert space H and $p(z, \bar{z})$ be a non-commutative complex polynomial for which $p(A, A^*)$ is a compact operator. Then there exists a unique sequence of central projections $\{P_i\}_{i=0}^{n}$ $(n \leqq \infty)$ in $R(A)$ so that*

$$A = A_0 \oplus \sum_{i=1}^{n} \oplus A_i,$$

where $A_0 \equiv AP_0H$ satisfies $p(z, \bar{z})$[2]*), $A_i \equiv A|P_iH$ $(i \geqq 1)$ are primary operators with $p(A_i, A_i^*)$ compact and non-zero, and $K = H \ominus P_0H$ is separable.*

We are interested in studying this theorem in the special cases where $p(z, \bar{z})$ is one of the following polynomials: 1) $p(z, \bar{z}) = z - \bar{z}$, 2) $p(z, \bar{z}) = z\bar{z} - \bar{z}z$), 3) $p(z, \bar{z}) = z\bar{z}z - \bar{z}z^2$, 4) $p(z, \bar{z}) = 1 - \bar{z}z$, and 5) $p(z, \bar{z}) = z - z\bar{z}z$. Case 1) has been studied by M. S. Brodskiĭ and M. S. Livšic [3], and N. Suzuki's original work also concerns it. The cases 2) and 3) have been studied by H. Behncke [1] and [2]; and case 3) by A. Brown [4]. Behncke obtained his structure by using C^*-algebra methods while Suzuki's original work is based on W^*-algebraic techniques. Case 4) has been studied by B. Sz.-Nagy and C. Foiaş if T is a contraction and by the author [8], where results analogous to Theorem 1 appear.

The proof of the theorem will be based on a proposition from the theory of von Neumann algebras. Let M be a von Neumann algebra and T be an operator in M. The *support* of T is the projection P on $\overline{T^*H}$ and $P \in M$. The *central support* of T is the smallest projection $F \in Z \equiv M \cap M'$ which majorizes P. If $\mathscr{J}$ is a family of operators in M we say that F is the central support of $\mathscr{J}$ if it is the smallest projection in Z which majorizes the support of each $T \in \mathscr{J}$. A non-zero projection $Q \in M$ is called *minimal* if it is an atom in the lattice of projections in M, that is, whenever R is a non-zero projection in M such that $R \leqq Q$, then $R = Q$.

Proposition 1. *Let M be a von Neumann algebra such that $M \cap \mathscr{C} \equiv \mathscr{C}_M$ has central support I.*[3]*) Then the lattice of projections in Z (the center of M) is atomic, that is, each non-zero $P \in Z$ majorizes a non-zero minimal projection $Q \in Z$.*

Proof. Let $0 \neq P \in Z$. If $PT = 0$ for each $T \in \mathscr{C}_M$, then $(I-P)T = T$ for each $T \in \mathscr{C}_M$. Hence $I-P$ would majorize the central support of $\mathscr{C}_M$ and $I \leqq I-P$ which implies that $P = 0$. Thus there is a $T \in \mathscr{C}_M$ such that $PT \neq 0$. Furthermore we may assume that $T = T^*$ and $PT = T$. By the spectral decomposition of the compact

[2]) We say that the operator T satisfies $p(z, \bar{z})$ if $p(T, T^*) = 0$.

[3]) $\mathscr{C}$ is the two sided ideal of compact operators in H.

selfadjoint operator T, we may conclude that $E=PE\neq 0$, where E is the spectral projection on an eigenspace corresponding to a non-zero eigenvalue of T. E is finite dimensional since T is compact and the eigenvalue associated with E is non-zero. It is easy to show that $E\in M$ (Proposition 1 in [14]). Since E is a finite dimensional projection in M we may choose a projection $E_1\in M$ so that $0<E_1\leqq E$ and E_1 is a minimum non-zero projection in M (E_1 may be chosen so that $0\neq\dim(E_1H)=\min\{\dim(FH)\colon F\in M \text{ and } 0\neq F\leqq E\}$). If we let Q be the central support of $E_1\in M$, then we shall show that Q is a non-zero minimal projection in Z which is majorized by P. Since $P\geqq E_1$, it is clear that $P\geqq Q$. Let $R\in Z$ such that $R\leqq Q$. If $RE_1=0$, then $(Q-R)E_1=E_1$; hence $Q\leqq Q-R$, which implies that $R=0$. Since E_1 is a minimal projection in M, if $RE_1\neq 0$, then we have that $RE_1=E_1$. Because R is a central projection, we obtain the inequality $0\neq R\leqq Q\leqq R$, and hence $R=Q$. Therefore we have shown that Q is a minimal projection in Z.

Using this proposition we now prove Theorem 1.

Proof. First we describe the subspace $H\ominus P_0H$ which occurs in the statement of the theorem. Let $w(A,A^*)=\prod_{i=1}^{n}A^{k_i}A^{*m_i}$ be a word in A and A^*, that is, k_i and m_i are non-negative integers, possibly zero, and n is any positive integer. Denote by $\mathcal{M}$ the subspace of H generated by $\{w(A,A^*)x\colon x\in p(A,A^*)H$ and $w(A,A^*)$ is any word in A and $A^*\}$. The image of a compact operator is a separable subspace; hence $p(A,A^*)H$ is separable and thus the separability of $\mathcal{M}$ follows from the construction of $\mathcal{M}$. It is also clear that $\mathcal{M}$ is invariant under A and A^* and hence $\mathcal{M}$ reduces A, that is, if Q is the projection on $\mathcal{M}$, then $Q\in R(A)'$. Let T be an arbitrary operator in $R(A)'$ and $y\in\mathcal{M}$ be of the form $w(A,A^*)x$, where $x=p(A,A^*)z$. Then $Ty=Tw(A,A^*)x=w(A,A^*)Tx=w(A,A^*)p(A,A^*)Tz\in\mathcal{M}$; thus $\mathcal{M}$ is invariant under $T\in R(A)'$. Since $R(A)^*=R(A)$, we may conclude that $Q\in R(A)''=R(A)$ and therefore that $Q\in Z(A)\equiv R(A)\cap R(A)'$.

Denote by P_0 the central projection $I-Q$ and by A_0 the restriction of A to P_0. Next we shall show that $p(A_0,A_0^*)=0$. If $x\in H_0\equiv P_0H$, then $x=(I-Q)x$ and $p(A_0,A_0^*)x=Qp(A_0,A_0^*)x=Qp(A_0,A_0^*)(I-Q)x=Q(I-Q)p(A_0,A_0^*)x=0$. Furthermore, since Q is a central projection in $R(A)$, $\mathcal{M}$ is generated as before, with A replaced by AQ. If we denote by A_Q the operator $A|QH$, then $H\ominus H_0=\mathcal{M}$ is generated by words in A_Q and A_Q^* acting on $p(A_Q,A_Q^*)$.

The algebra $R(A)_Q=\{T|QH\colon T\in R(A)\}$ is equal to $R(A_Q)$ and $Z(A)_Q=Z(A_Q)$ [6]. By our remarks above, the identity operator on QH is the central support of the set of operators consisting of $p(A_Q,A_Q^*)$ multiplied by words in A_Q and A_Q^*. Each of these operators is compact and thus I_Q is the central support of $\mathscr{C}_{R(A_Q)}$. By Proposition 1, the lattice of projections in $Z(R(A_Q))$ is a complemented atomic lattice. By Zorn's lemma we may choose a maximal family $\{\tilde{P}_i\}_{i=1}^{n}$ ($n\leqq\infty$) of mutually orthogonal minimal projections in $Z(A_Q)$. This family is countable since

$QH=\mathcal{M}$ is separable and $\sum_{i=1}^{n}\tilde{P}_i=I_Q$ since the family is maximal. Because $Z(A_Q)=$ $=Z(A)_Q$ there are projections $\{Q_i\}_{i=1}^{n}\subset Z(A)$ such that $Q_i|QH=\tilde{P}_i$. If we define $P_i\equiv Q_iQ$, then $P_i|QH=\tilde{P}_i$ and $\{P_i\}_{i=1}^{n}$ is a family of mutually orthogonal minimal projections in $Z(A)$ with the property that $\sum_{i=1}^{n}P_i=Q$.

Since P_i is minimal projection in $Z(A)$, it follows that A_{P_i} is primary. Since $p(A_{P_i}, A^*_{P_i})=p(A, A^*)|P_iH$, it is clear that $p(A_{P_i}, A^*_{P_i})$ is compact; however, we must show that $p(A_{P_i}, A^*_{P_i})\neq 0$. If we assume that $p(A_{P_j}, A^*_{P_j})=0$ for some $j\geqq 1$, then $w(A_{P_j}, A^*_{P_j})p(A_{P_j}, A^*_{P_j})=0$, for any word $w(A_{P_j}, A^*_{P_j})=\prod_{i=1}^{n}A^{k_i}_{P_j}A^{*m_i}_{P_j}$. We would then have that $\{0\}=w(A_{P_j}, A^*_{P_j})p(A_{P_j}, A^*_{P_j})P_jH=P_j(w(A, A^*)p(A, A^*)H$; thus it would follow that $P_jQ=0$. Therefore $P_j\perp Q$, which is a contradiction, since P_j is non-zero and $P_j\leqq Q$.

Remark 1. The argument given in the paragraph above is valid if P_j is any projection in $R(A)'$. That is, if N is a reducing space of A on which $p(A|N, A^*|N)=0$, then $N\subset P_0H$.

Remark 2. The central support P of an operator $T\in M$ is also the central support of T^* and P also majorizes the projection on the smallest reducing space of T which contains TH. Thus we see that Q, as defined in the proof of Theorem 1, is the central support of $p(A, A^*)$.

For each $i\geqq 1$, we have that $0\neq p(A_i, A_i^*)$ and thus the dimension of $p(A_i, A_i^*)H_i$ $(H_i\equiv P_iH)$ is $\geqq 1$. Therefore, if $p(A, A^*)$ is itself of finite rank, then the decomposition given in Theorem 1 is finite.

Corollary 1. *Let A be an operator and $p(z, \bar{z})$ a nonmmutative polynomial for which $p(A, A^*)$ has finite rank. Then the decomposition in Theorem 1 is finite, that is, the index n in Theorem 1 is finite.*

Proof. The decomposition of A given by Theorem 1 has the property that $\dim(p(A, A^*)H)=\sum\dim(p(A_i, A_i^*)H_i)$ and for $i\geqq 0$ $\dim(p(A_i, A_i^*)H_i)\neq 0$.

In some cases we may wish to consider more than one non-commutative polynomial of z and $\bar{z}$. We can then extend the above idea so as to include this situation. For simplicity we shall only consider the case of two non-commutative polynomials.

Proposition 2. *Let A be an operator and $p(z,\bar{z})$ and $q(z, \bar{z})$ be commutative polynomials. Then there exists unique central projections, P_i $(i=1, 2, 3; 4)$ in $R(A)$ such that $A=A_1\oplus A_2\oplus A_3\oplus A_4$, where $A_1\equiv A|P_1H$ satisfies p and q, A_2 satisfies p and has no reducing subspace on which it satisfies q, A_3 satisfies q and has no reducing subspace on which it satisfies p, and A_4 has no reducing space on which it satisfies either p or q.*

Proof. Let Q_1 be the central support of $p(A, A^*)$ and Q_2 the central support of $q(A, A^*)$. Let $Q_1 \cdot Q_2 = P_4$; then by Remark 1 $A_4 \equiv A|P_4 H$ has no reducing space on which A_4 satisfies either p or q. Let Q_3 be the central support of the set $\{p(A, A^*), q(A, A^*)\}$, that is $Q_3 = Q_2 + Q_1 - Q_1 Q_2$. If $P_1 = I - Q_3$, $P_2 = Q_3 - Q_1$, and $P_3 = Q_3 - Q_2$, then $\{P_1, P_2, P_3, P_4\}$ satisfy the conclusion of the proposition.

Remark. As a special case of Proposition 2 we may consider only one polynomial $p(z, \bar{z})$. In this case we decompose A into $A_0 \oplus A_1$ where $p(A_0, A_0^*) = 0$. By the remark following the proof of Theorem 1, we note that H_1, the space on which A_1 is defined, is generated by $\{w(A, A^*)p(A, A^*)H$: $w(A, A^*)$ is any word in A and $A^*\}$. This result is known in some special cases. LIVŠIC and BRODSKIĬ [3] call an operator simple if it has no reducing space on which it is selfadjoint. In this case H_1 is generated by $\{A^n(A - A^*)H$: $n = 0, 1, 2, \ldots\}$ and $A|H_1$ is called the simple part of A. HALMOS [9] calls an operator abnormal if it has no reducing space on which it is normal. Finally B. SZ.-NAGY and C. FOIAŞ use the terminology completely non-unitary for contractions with no reducing spaces on which they are unitary. This latter notation seems the most descriptive of the situation.

If we combine Theorem 1 and Proposition 2 we obtain the form that the structure theorem takes in many of its applications.

Theorem 2. *Let A be an operator and $p(z, \bar{z})$ and $q(z, \bar{z})$ be two non-commutative polynomials such that $p(A, A^*)$ is compact. Then there exists unique central projections $\{P_i\}_{i=1}^n$ $(n \leqq \infty)$ in $R(A)$ so that*

$$A = A_1 \oplus A_2 \oplus A_3 \oplus \sum_{i \geqq 4} \oplus A_i,$$

where $A_i \equiv A|P_i H$, $p(A_1, A_1^*) = q(A_1, A_1^*) = 0$, $p(A_2, A_2^*) = 0$, A_2 has no reducing space on which $q(A_2, A_2^*) = 0$, $q(A_3, A_3^*) = 0$, and A_3 has no reducing subspaces on which $p(A_3, A_3^*) = 0$, A_i $(i \geqq 4)$ are primary operators with $p(A_i, A_i^*)$ compact, and each A_i $(i \geqq 4)$ has no reducing subspace on which $q(A_i, A_i^*) = 0$ or $p(A_i, A_i^*) = 0$.

Proof. From Proposition 2 we obtain the projections P_1, P_2, and P_3. Applying Theorem 1 to the operator A_{P_4} and the algebra $R(A_{P_4})$ we complete the decomposition of A.

We now turn to the structure of primary operators A for which $p(A, A^*)$ is compact and non-zero. Here the algebraic character of the operator plays the important role. This fact was first noticed by SUZUKI for primary operators with compact imaginary parts. The following proposition is essentially a restatement of Proposition 2 in [14]. Let A be a primary operator and $p(z, \bar{z})$ a non-commutative polynomial for which $p(A, A^*)$ is compact and non-zero. The projections on proper subspaces of $\mathrm{Re}\,(p(A, A^*))$ and $\mathrm{Im}\,(p(A, A^*))$ corresponding to non-zero proper values have finite rank and belong to $R(A)$. Since at least one such projection exists and is non-zero, we have that $R(A)$ contains finite dimensional projections and hence $R(A)$ contains minimal projections. Therefore the von Neumann algebra $R(A)$ is a factor of type I and the dimension n of a minimal projection in $R(A)$ is uniquely determined. The number n is a unitary invariant for A and is called the multiplicity of A.

Proposition 3. *Let A be a primary operator and $p(z, \bar{z})$ a non-commutative polynomial for which $p(A, A^*)$ is compact and non-zero. If n is the multiplicity of A, then $R(A)'$ (the commutant of $R(A)$) is of type I_n.*

The proof is similar to the proof of Proposition 2 in [14].

The type of algebra generated by an operator has been studied by many authors. As a corollary to Proposition 3, we have the following result.

Proposition 4. *If A is an operator and $p(z, \bar{z})$ is a non-commutative polynomial for which $p(A, A^*)$ is compact, then $R(A)$ is a type I algebra if and only if A_0 (given by Theorem 1) generates an algebra of type I.*

Now for special cases we can determine certain operators that generate type I algebras.

Corollary 2. *Let A be an operator for which $p(A, A^*)$ is compact. Then A generates a type I von Neumann algebra if*

i) $p(z, \bar{z}) = z - \bar{z}$, ii) $p(z, \bar{z}) = \bar{z}z - z\bar{z}$, *or* iii) $p(z, \bar{z}) = 1 - z\bar{z}$.

Proof. This result is known for case i) (Suzuki [14]) and case ii) (Behncke [1]). Case iii) follows since an isometry generates a type I von Neumann algebra.

Remark. Carl Pearcy gives examples of partial isometric operators which do not generate type I von Neumann algebras [10]. Hence for $p(z, \bar{z}) = z - z\bar{z}z$ and an operator A such that $p(A, A^*)$ is compact, the algebra $R(A)$ need not be type I.

Now we complete the algebraic structure of operators A for which $p(A, A^*)$ is compact and non-zero. We shall show that when the operator A is also primary, then it is just the direct sum of n copies of an irreducible operator V with the properties that $p(V, V^*)$ is compact and non-zero. The following theorem is similar to Theorem 3 in [14] where the case $p(z, \bar{z}) = z - \bar{z}$ was considered.

Theorem 3. *Let A be a primary operator and $p(z, \bar{z})$ a non-commutative polynomial such that $p(A, A^*)$ is compact and non-zero. If m is the multiplicity of A, then A is unitarily equivalent to an operator $V \otimes I_m$, where V is an irreducible operator with $p(V, V^*)$ compact and non-zero and I_m is the identity operator on an m-dimensional Hilbert space.*

Proof. A von Neumann algebra of type $I_{\alpha m}$ is spatially isomorphic to $\mathscr{L}(K)\otimes \otimes\{\lambda I_m\}$, where $\mathscr{L}(K)$ is the algebra of all bounded operators on an α-dimensional Hilbert space K and λI_m are the scalar multiples of the identity operator I_m on an m-dimensional Hilbert space [6]. Thus A is unitarily equivalent to an operator of the form $V\otimes I_m \in \mathscr{L}(K)\otimes\{\lambda I_m\}$. One can then show that V must be irreducible.

If $p(z, \bar{z})$ is a non-commutative polynomial, we say that the operator A has *p-rank* r if rank $p(A, A^*)$ is r. Using strictly algebraic ideas we obtain the following two corollaries of Theorem 3.

Corollary 3. *If A is a primary operator with p-rank r and multiplicity m then A is unitarily equivalent to $V\otimes I_m$ and the p-rank of V is n where $r = n\cdot m$.*

Corollary 4. *Let A be a primary operator with p-rank r. If the multiplicity of A is 1 and r is a prime number, then A is either irreducible or else A is unitarily equivalent to $V\otimes I_r$, in which case the p-rank of V is 1.*

We wish to illustrate this theory with examples using the specific non-commutative polynomials mentioned at the beginning of this section. Operators A with $A-A^*$ compact have been extensively studied by various authors; see [3] and [14]. In this case A is uniquely decomposed by central projections in $R(A)$ as

$$A = A_0 \oplus \sum_{i=1}^{n} \oplus A_i \qquad (n \leqq \infty),$$

where A_0 is a self adjoint operator and each A_i $(i \geqq 1)$ is a primary operator with $\operatorname{Im} A_i$ compact. By theorem 3 each $A_i = V_i\otimes I_{n_i}$, V_i is irreducible with $\operatorname{Im} V_i$ compact and non-zero, and $n_i < \infty$. These results are due to N. Suzuki [14].

Following Suzuki's original work, H. Behncke [1] used the theory of C^*-algebras to prove the analogous theorem when $p(z) = \bar{z}z - z\bar{z}$. If $A^*A - AA^*$ is compact then A is uniquely decomposed by central projections in $R(A)$ as

$$A_0 \oplus \sum_{i=1}^{n} \oplus A_i \qquad (n \leqq \infty),$$

where A_0 is normal, each A_i is primary with $A_i^*A_i - A_iA_i^*$ compact and each $A_i = V_i\otimes I_{n_i}$, where V_i is irreducible $V_i^*V_i - V_iV_i^*$ is compact and non-zero, and $n_i < \infty$.

Using the polynomial $p(z, \bar{z}) = z\bar{z}z - \bar{z}z^2$ and $q(z, \bar{z}) = \bar{z}z - z\bar{z}$ and Theorem 2, we can obtain the decomposition given by H. Behncke in [2] whenever $p(A, A^*)$ is compact.

For contraction operators A with $p(A, A^*) = I - A^*A$ compact the algebraic decomposition has been given by the author [8]. If we consider the polynomials $p(z, \bar{z}) = 1 - \bar{z}z$ and $q(z, \bar{z}) = 1 - z\bar{z}$ and an operator A for which $p(A, A^*)$ is compact, then Theorems 2 and 3 give the following unique central decomposition.

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \sum_{i=3}^{n} \oplus A_i \qquad (n \leqq \infty),$$

where A_0 is unitary, A_1 is a forward unilateral shift, A_2 is a backward unilateral shift and each A_1 ($i \geqq 3$) is a primary operator. Furthermore, for $i \geqq 3$, $A_i = V_i \otimes I_{n_i}$, where V_i is irreducible, $I - V_i^* V_i$ is compact and non-zero, $n_i < \infty$, and V_i is completely non-isometric.

2. Applications

In this section we give an application of Theorem 1 based on the theory of spectral operators [7]. The results of this section give striking examples of how the algebraic decomposition of an operator can be used to determine its exact structure.

J. Schwartz [12] and N. Suzuki [15] have determined a structure theorem for the spectral operator A whenever $A - A^*$ is compact. We will give the analogous result whenever: i) $A^*A - AA^*$, ii) $I - A^*A$, or iii) $AA^*A - A^*A^2$ is compact. This will correspond to three specific uses of Theorem 1.

In what follows we shall use several results concerning the Calkin algebra associated with $\mathscr{L}(H)$. The algebra $\mathscr{L}(H)/\mathscr{C}$ ($\mathscr{C}$ is the compact operators in $\mathscr{L}(H)$) is a B^* algebra with involution $*$ and it is called the Calkin algebra associated with H. If $\hat{A}$ denotes the image of A in $\mathscr{L}(H)/\mathscr{C}$, then $(\hat{A})^* = \hat{A}^*$ and $\sigma(\hat{A}) \subset \sigma(A)$. For details concerning this algebra we refer to [5].

The following lemma gives conditions on a spectral operator A which imply that the quasinilpotent part is compact or equivalently, that the operator $\hat{A}$ is a scalar type operator in $\mathscr{L}(H)/\mathscr{C}$.

Lemma 1. *Let A be a spectral operator with the canonical decomposition $A = S + N$, where S is a scalar type operator and N is a quasinilpotent operator. Then N is compact if any of the following operators* i) $A^*A - AA^*$, ii) $A^* - A$, iii) $I - A^*A$, *or* iv) $AA^*A - A^*A^2$ *is compact.*

Proof. Since $A = S + N$, we have $\hat{A} = \hat{S} + \hat{N}$ as the canonical decomposition of $\hat{A}$ in $\mathscr{L}(H)/\mathscr{C}$. In cases i) and ii) we clearly have that $\hat{A}$ is normal. Since the decomposition into scalar and quasinilpotent parts is unique, we may conclude that $\hat{N} = 0$ and therefore N is compact. Part i) was proven by Schwartz in [12].

In the case iii), $\hat{A}$ is an isometry. It can be shown directly that isometric spectral operators are normal.

In case iv) we are considering an operator $\hat{A} = B$ such that $BB^*B - B^*B^2 = 0$. A. Brown [4] has completely characterized these operators; he shows that $B = VD$, where V is an isometry, $D \geqq 0$ and $VD = DV$. Again one can directly show that a spectral operator B satisfying iv) is normal. However in case iii) and iv) the operator $\hat{A}$ is also subnormal.

J. Stampfli has shown [13] that in a separable Hilbert space every subnormal spectral operator is normal. His proof is independent of separability and hence can be used here. Hence in either iii) or iv) we may conclude that $\hat{N}=0$ and therefore N is compact.

Now we present the main theorem of this section.

Theorem 4. *Let A be a spectral operator on a Hilbert space H. Whenever at least one of the operators* i) $A^*A - AA^*$, ii) $A^* - A$, iii) $I - A^*A$, *or* iv) $AA^*A - A^*A^2$ *is compact, then A decomposes into the algebraic direct sum*

$$A = A_0 \dotplus \sum_{i=1}^{n} \dotplus (\lambda_i I_i + N_i) \quad (n \leqq \infty) \quad \text{on} \quad H = H_0 \dotplus \sum_{i=1}^{n} \dotplus H_i;$$

where $\{H_i\}_{i=0}$ are invariant subspaces for A, $A_0 \equiv A|H_0$ is scalar, I_i is the identity operator on H_i, $(\lambda_i I_i + N_i) \equiv A|H_i$, $\lambda_i \in \sigma(A)$, N_i is a compact quasinilpotent operator and $\|N_i\| \to 0$ if $n = \infty$. Furthermore in the cases ii) *and* iii) *we also have, that respectively, A_0 is similar to a self-adjoint operator with* $\operatorname{Im} \lambda_i \to 0$ *if $n = \infty$; and A_0 is similar to a unitary operator with $|\lambda_i| \to 1$ if $n = \infty$. Finally, the non-scalar summand $\sum_{i=1}^{n} \dotplus H_i$ is separable.*

Proof. Let A be a spectral operator with canonical decomposition $A = S + N$ where N is compact. Let R be the invertible operator for which RSR^{-1} is normal and let $B \equiv RAR^{-1}$, $T \equiv RSR^{-1}$, and $L \equiv RNR^{-1}$. Now L is also compact and T is normal, so that $\hat{B} = \hat{T}$ and $B^*B - BB^*$ is compact.

Using the polynomial $p(z, \bar{z}) = \bar{z}z - z\bar{z}$ in Theorem 1, the operator B decomposes as

$$B = B_0 \oplus \sum_{i=1}^{n} \oplus B_i \quad (n \leqq \infty), \quad \text{with} \quad H = H_0 \oplus \sum_{i=1}^{n} \oplus H_i,$$

where $B_0 \equiv B|H_0$ is normal and $B_i \equiv B|H_i$ is a primary operator ($i \geqq 1$).

Each B_i ($i \geqq 1$) is also a spectral operator and has the canonical decomposition $B_i = T_i + L_i$. Since T, $L \in R(B)'$ and the decomposition of B was by central projections in $R(B)$, the operator T_i is $T|H_i$ and L_i is $L|H_i$. Each T_i is normal and belongs to the center of the algebra $R(B_i)$. Since B_i is a primary operator, we may conclude that $T_i = \lambda_i I_i$ for some scalar λ_i (I_i is the identity operator on H_i). Because $\{\lambda_i\} = \sigma(T_i) \subset \sigma(T) = \sigma(B) = \sigma(A)$, we note that $\lambda_i \in \sigma(A)$. Therefore B is decomposed as $B = B_0 \oplus \sum_{i=1}^{n} \oplus (\lambda_i I_i + L_i)$ ($n \leqq \infty$); furthermore, since L is compact, $\|L_i\| \to 0$ if $n = \infty$.

If A satisfies any of the conditions i)—iv), we have by Lemma 1 that N is compact. Therefore, A has the decomposition given above. Now we shall discuss the special cases ii) and iii). In either case $\sigma(B)=\sigma(A)\supset\sigma(B_0)$ and $\sigma(\hat{A})=\sigma(\hat{B})\supset\sigma(\hat{B}_0)$. In case ii), $\sigma(\hat{A})$ is real and hence B_0 is a normal operator with $\sigma(\hat{B}_0)$ real, that is, $\hat{B}_0$ is self adjoint and Im (B_0) is compact. By reordering, in the above decomposition, and redenoting B_0 as the selfadjoint part of B_0, we obtain in case ii):

$$B = B_0 \oplus \sum_{i=1}^{n} \oplus (\lambda_i I_i + L_i) \qquad (n \leqq \infty),$$

where B_0 is selfadjoint, Im $\lambda_i \to 0$ and $\|L_i\| \to 0$ if $n=\infty$. In case a particular λ_i arises from the previous B_0 we simply define $L_i \equiv 0$. Now if we premultiply by R and postmultiply by R^{-1} we obtain the desired result

$$A = A_0 \dotplus \sum_{i=1}^{n} \dotplus (\lambda_i I_i + N_i) \quad (n \leqq \infty) \quad \text{on} \quad H = H_0 \dotplus \sum_{i=1}^{n} \dotplus H_i,$$

where A_0 is a scalar operator with real spectrum, Im $\lambda_i \to 0$ and $\|N_i\| \to 0$ if $n=\infty$.

In case iii) we may proceed as in case ii). Since spectral isometries are unitary, it follows that $\hat{A}$ is unitary; thus $\sigma(\hat{A})\subset\{z:|z|=1\}$ and $\sigma(\hat{B})\subset\{z:|z|=1\}$. Thus B_0 is a normal operator with $\sigma(\hat{B}_0)$ on the boundary of the unit disk. Hence $B_0 = U\oplus\sum\oplus\lambda_i I_i$, where U is a unitary operator, $\{\lambda_i\}=\sigma(B_0)\setminus\{z:|z|=1\}$, and I_i is the identity operator on the eigenspace corresponding to λ_i. We may redenote B_0 as the unitary part of B_0 and obtain the decomposition:

$$B = B_0 \oplus \sum_{i=1}^{n} \oplus (\lambda_i I_i + L_i) \qquad (n \leqq \infty),$$

where B_0 is unitary and $\|L_i\| \to 0$ if $n=\infty$. The set $\{\lambda_i\}$ does not have limit points in the set $\{z:|z|<1\}$, since $\partial\sigma(A)\subset\sigma(\hat{A})\cup\{$isolated eigenvalues of A of finite multiplicity$\}$ [11]; therefore, we conclude that $|\lambda_i|\to 1$ if $n=\infty$.

By premultiplying by R and postmultiplying by R^{-1} we finally obtain that:

$$A = A_0 \dotplus \sum_{i=1}^{n} \dotplus (\lambda_i I_i + N_i) \quad (n \leqq \infty) \quad \text{on} \quad H = H_0 \dotplus \sum_{i=1}^{n} \dotplus (\lambda_i I_i + N_i),$$

where A_0 is a scalar type operator with $\sigma(A_0)$ lying on the circumference of the unit circle, $|\lambda_i|\to 1$ and $\|N_i\|\to 0$ if $n=\infty$.

Remark. The use here of Theorem 1 is similar to that made by N. SUZUKI in the case ii) [15]. However, the use of the spectral properties of an operator A and $\hat{A}$ are details that differ from the proof of ii) in [15]. This Theorem for case ii) was originally given by J. SCHWARTZ using completely different methods.

Remark. By the argument given in the first part of the proof, we see that the decomposition in the theorem holds for any spectral operator with compact quasinilpotent part.

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Weighted bilateral shifts of class C_{01}

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In this paper all operators are bounded operators on separable Hilbert spaces. B. SZ.-NAGY and C. FOIAŞ have developed a classification theory for contraction operators ($\|T\| \leqq 1$) which is based on the asymptotic behavior of the operator and its adjoint [6; Chapter II, Section 4]. A contraction operator T on H is called type C_{01} if $T^n h \to 0$ for all $h \in H$ and $T^{*n} h \not\to 0$ for each $h \in H$, $h \neq 0$. For complete details of this classification theory we refer the reader to [6], Chapter II, Section 4.

Some properties of the operators in C_{01} are known. Whenever $T \in C_{01}$ and the rank of $I - T^*T$ is finite, then the rank of $I - TT^*$ is *strictly* smaller than the rank of $I - T^*T$; *cf.* [6], Proposition I. 2. 1 and Theorems II. 1. 1—2. Hence it follows from [6], Theorem VI. 4. 1, that $\sigma_p(T)$ includes the whole open unit disk D.

A contraction T is called a *weak contraction* if $I - T^*T$ is of trace class and if $\sigma(T) \neq \bar{D}$. In [6], Chapter VIII, the structure of weak contractions is extensively developed. Our examples shall show that this structure cannot be extended to the Schatten class $\mathfrak{S}_p$ for any $p > 1$; *cf.* [1], X. 1. 9.

In this note we present examples of contraction operators in the class C_{01} which have no point spectrum. Example 1 will show that the spectrum can lie on the circumference of the unit disk and the point spectrum can be empty even when $I - T^*T$ is an $\mathfrak{S}_p$ operator with $p > 1$. Furthermore the example will give realizations of C_{01} operators for which T has a cyclic vector. Examples will be in C_{01} with $\sigma(T) = \bar{D}$. Specifically all the examples will have in common the following properties:

(i) T is irreducible,

(ii) $\sigma_p(T^*) = \sigma_p(T) = \emptyset$,

(iii) T has a cyclic vector,

(iv) T^* has *no* invariant subspaces on which it is an isometry.

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[1]) This work was done while the author was an Office of Naval Research Postdoctoral Associate at Indiana University. The author acknowleges that these results are examples for questions raised by C. FOIAŞ.

The examples will be generated by weighted bilateral shifts. Let H be a separable Hilbert space and $\{e_n\}$ $(n=0, \pm 1, \pm 2, \ldots)$ an orthonormal basis. Let T be the operator which maps e_n onto $\omega_n e_{n+1}$ $(n=0, \pm 1, \pm 2, \ldots)$, where ω_n is a complex number. The set $\{\omega_n\}$ is called the *weights* of T. T is a contraction iff $|\omega_n| \leqq 1$ for every n. The following proposition determines the class to which T belongs.

Proposition. *Let T be a weighted bilateral shift with weights $\{\omega_n\}$ such that T is a contraction.*

a) $T \in C_{0\cdot}$ *if and only if either* (i) *for every positive integer N there exists an $n > N$ such that $\omega_n = 0$, or* (ii) *for some subsequence $\{n_i\}$ of positive integers with $\omega_{n_i} \neq 0$ the infinite product $\prod |\omega_{n_i}|$ diverges.*

b) $T \in C_{1\cdot}$ *if and only if each $\omega_i \neq 0$ and the infinite product $\prod_{i \geqq 0} |\omega_i|$ converges.*

The proof of this proposition is straightforward and appears in [2], Chapter II. As a corollary of this result we determine when T is a C_{01} contraction.

Corollary. *Let T be a weighted bilateral shift with weights $\{\omega_n\}$ such that T is a contraction. Then $T \in C_{01}$ if and only if, for all $n = 0, \pm 1, \ldots,$*

(i) $\prod_{i \geqq n} |\omega_i|$ *diverges, and* (ii) $\prod_{i \leqq n} |\omega_i|$ *converges.*

Remark. If we assume that $\omega_i \neq 0$ for all i, then $T \in C_{01}$ if and only if $\prod_{i \leqq 0} |\omega_i|$ converges and $\prod_{i \geqq 0} |\omega_i|$ diverges.

Now we shall present the first example.

Example 1. *Let T be the weighted bilateral shift with weights*

$$\omega_n \equiv \begin{cases} \left(\dfrac{n-1}{n}\right)^{\frac{1}{2}} & \text{if} \quad n > 1, \\[2ex] \dfrac{n^2-1}{n^2} & \text{if} \quad n < -1, \quad \text{and} \\[2ex] 1 & \text{otherwise.} \end{cases}$$

The operator T is in the class C_{01}, has properties (i)—(iv) *and furthermore $I - T^* T$ is an $\mathfrak{S}_p$ operator for $p > 1$.*

First we shall show that $T \in C_{01}$. Since all the weights are less than or equal to 1 we conclude that $\|T\| \leqq 1$. The infinite product $\prod \left(\dfrac{n-1}{n}\right)^{\frac{1}{2}}$ has its partial products converging to zero. By the proposition we can conclude that $T \in C_{0\cdot}$. The series $\sum \frac{1}{n^2}$ is convergent and hence the infinite product $\prod \frac{n^2-1}{n^2}$ does convergence. From our corollary and the remark following it, we have $T \in C_{01}$. Furthermore the products $\beta_k = \prod_{i \leqq k} \omega_i$ are convergent and have the property that $\beta_k \to 0$ as $k \to +\infty$.

Now we shall discuss the properties (i)—(iv). Properties (i) and (iv) are easily shown. That T is irreducible can be deduced from a result due to R. L. KELLEY [3], Problem 129. Assume that T^* has an invariant subspace on which it is an isometry and h is any non-zero vector in that subspace. Since $\{e_n\}$ is an orthonormal basis, we have, $h = \sum_{k=-\infty}^{\infty} \alpha_k e_k$ and $T^{*n} h = \sum_{k=-\infty}^{\infty} \left(\prod_{i=0}^{n-1} \omega_{i-n} \right) e_{k-n}$. For n large enough ($n \geqq 4$) and for some k with $\alpha_k \neq 0$, we will have $\left| \prod_{i=0}^{n} \omega_{i-n} \right| \neq 1$. When this happens, then $\|T^{*n} h\| \neq \|h\|$. Thus we reach a contradiction to our assumption that T^* had an invariant subspace on which T^* was an isometry. For $p > 1$ the sum $\sum_{-\infty}^{\infty} (1 - \omega_i^2)^p$ is just the sum $\sum_{-\infty}^{\infty} \|(I - T^* T) e_i\|^p$. By our choice of ω_i, this sum is finite whenever $p > 1$, and hence T belongs to the Schatten class $\mathfrak{S}_p$.

The convergence properties of the weights will enable us to show property (ii). As we mentioned in the introduction, of most interest is the property that $\sigma_p(T) = \emptyset$. It follows from [5], Theorem 5, that $\sigma(T) = \{\lambda : |\lambda| = 1\}$. Therefore since T is a completely non-unitary contraction, we have $\sigma_p(T) = \sigma_p(T^*) = \emptyset$. However this is easy to see by directly calculating the spectral radius of T^{-1}. From our definition of T it follows that $\|T^{-n}\| \leqq n$ ($n > 1$) and hence the spectral radius of T^{-1} is 1. Since the spectral radius of T and T^{-1} is 1 we must have that $\sigma(T) \subset \{\lambda : |\lambda| = 1\}$.

In order to show (iii) we shall construct the cyclic vector using the criterion for a cyclic vector of the simple bilateral shift (that is, all weights are 1 and the multiplicity is 1) [4], p. 114. In order to do this we first show that the simple bilateral shift is quasi affine to T. We have already mentioned that $\beta_n = \prod_{i \leqq n} \omega_i$ is defined for all n. If we define X to be the operator which maps e_n to $\beta_n e_n$, then X is an injective selfadjoint operator on H. For each vector e_n we have $T X e_n = T \beta_n e_n = \omega_n \beta_n e_{n+1} = \beta_{n+1} e_{n+1} = X e_{n+1} = X S e_n$, where S is the simple bilateral shift. Let f be a cyclic vector for S. Thus $\operatorname{span}\{T^n X f\} = \operatorname{span}\{X S^n f\} = X \operatorname{span}\{S^n f\} = H$ and Xf is a cyclic vector for T.

If we choose different weights we can construct an example of a C_{01} operator with properties (i)—(iv) and with the additional property that $\sigma(T) = \{\lambda : |\lambda| \leqq 1\}$.

Example 2. *Let T be the weighted bilateral shift with weights*

$$\omega_n \equiv \begin{cases} \dfrac{n^2-1}{n^2} & \text{if } n<-1, \\ \dfrac{1}{k} & \text{if } n=k^3,\ k>1, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

Then T belongs to C_{01}, has properties (i)—(iv) *and the property that $\sigma(T)$ is the closed unit disk.*

That T is in C_{01} and satisfies (i) and (iv) is clear. To see that T has a cyclic vector we proceed exactly as in the proof of Example 1. R. L. KELLEY has shown that $\sigma(T)$ is connected [5], p. 354. Since $\sigma(T)$ has circular symmetry [3], p. 75, we have that $\sigma(T)=\{\lambda:|\lambda|\leqq 1\}$. To show property (ii) let us assume that $\lambda\in\sigma_p(T)$. T is completely non-unitary, hence $|\lambda|\neq 1$. Since all the weights are non-zero we also know that $0\notin\sigma_p(T)$. Let $h=\Sigma\alpha_n e_n$ be an eigenvector for eigenvalue λ of T. By matching the corresponding Fourier coefficients of Th and λh, we obtain for all n

$$(*)\qquad \omega_{n-1}\alpha_{n-1}=\lambda\alpha_n.$$

If $\alpha_0=0$, then $h=0$ since our weights are all non-zero. For $n>0$ we obtain from $(*)$ that

$$\alpha_n=\lambda^{-n}\left(\frac{\beta_{n-1}}{\beta_0}\right)\alpha_0.$$

If we let $n+1=k^3$, then

$$\alpha_{n+1}=\lambda^{-k^3}(k!)^{-1}\beta_0^{-1}\alpha_0.$$

This sequence does *not* converge to zero whenever $|\lambda|<1$. Hence $\{\alpha_n\}$ *cannot* be the Fourier coefficients of a vector $h\in H$. By this contradiction we conclude that $\sigma_p(T)=\emptyset$.

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Degree of approximation by Cesàro means of Fourier—Laguerre expansions

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1. The Fourier—Laguerre expansion of a function $f(x) \in L[0, \infty]$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x), \tag{1.1}$$

where

$$\Gamma(\alpha+1)\binom{n+\alpha}{n} a_n = \int_0^{\infty} e^{-x} x^{\alpha} f(x) L_n^{(\alpha)}(x)\, dx, \tag{1.2}$$

and $L_n^{(\alpha)}(x)$ denotes the Laguerre polynomials of order $\alpha > -1$, defined by the generating function

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\omega^n = (1-\omega)^{-\alpha-1} \exp\left(-\frac{x\omega}{1-\omega}\right). \tag{1.3}$$

The nth Cesàro sum of order k of the series

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(t) \tag{1.4}$$

is, by definition, the coefficient of r^n in the expression

$$(1-r)^{-k-1} \sum_{n=0}^{\infty} L_n^{(\alpha)}(t) r^n = (1-r)^{-k-1}(1-r)^{-\alpha-1} \exp\left(-\frac{tr}{1-r}\right),$$

and is therefore equal to $L_n^{(\alpha+k+1)}(t)$.

In this paper we shall discuss the order of Cesàro means of the series (1. 1) at the point $x=0$. On account of the relation $L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}$, we have

$$\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(0) = \{\Gamma(\alpha+1)\}^{-1} \sum_{n=0}^{\infty} \int_0^{\infty} e^{-t} t^{\alpha} f(t) L_n^{(\alpha)}(t)\, dt \tag{1.5}$$

(see SZEGŐ [7], p. 269). Using the Cesàro means of the series (1. 4), we find that the

nth Cesàro means of order k of the series (1. 5) are given by

(1. 6)
$$\sigma_n^{(k)}(0) = \{A_n^{(k)}\Gamma(\alpha+1)\}^{-1}\int_0^\infty e^{-t}t^\alpha f(t)L_n^{(\alpha+k+1)}(t)\,dt,$$

where
$$A_n^{(k)} = \binom{n+k}{k}.$$

The Cesàro summability of the series (1. 5) has been studied by KOGBETLIANTZ [2] and SZEGŐ [6]. It has been shown by SZEGŐ [6] and [7], p. 270, that if $f(x)$ is continuous at $x=0$ and if

(1. 7)
$$\int_1^\infty e^{-x/2}x^{\alpha-k-1/3}|f(x)|\,dx < \infty,$$

then the series (1. 1) is (C, k)-summable at the point $x=0$ with the sum $f(0)$, provided that $k > \alpha+1/2$.

In Theorem I of this paper we estimate the order of Cesàro means of the series (1. 5) after replacing the continuity condition in Szegő's theorem by a much lighter condition. Similar results for Fourier-trigonometric series and for ultraspherical series on a sphere were established by OBRECHKOFF [3], [4]. In Theorem II we prove an extension of Theorem I by introducing a parameter p thus arriving at a deeper insight into the behaviour of Cesàro means. Such extensions in the case of Fourier-trigonometric series were given by WANG [8] and SUNOUCHI [5], while the author [1] has earlier studied such a problem for the ultraspherical series on a sphere.

Theorem I. *If*

(1. 8)
$$F(t) = \int_t^\delta \frac{|f(u)|}{u}\,du = o\left(\log\frac{1}{t}\right)$$

and

$$\int_1^\infty e^{-t/2}t^{\alpha-k-1/3}|f(t)|\,dt < \infty,$$

then

$$\sigma_n^{(k)}(0) = o(\log n),$$

provided that $k > \alpha+1/2$.

2. In the proof of the theorem we shall require the following order estimates and asymptotic values of the Laguerre functions given by SZEGŐ [7], pp. 175 and 239.

Order estimates. If α is an arbitrary real number, and c and ω are fixed positive constants, and $n\to\infty$, then.

(2. 1)
$$L_n^{(\alpha)}(x) = \begin{cases} x^{-\alpha/2-1/4}O(n^{\alpha/2-1/4}), & \text{if } c/n \leqq x \leqq \omega, \\ O(n^\alpha), & \text{if } 0 \leqq x \leqq c/n. \end{cases}$$

*Asymptotic property.**) If α and λ are arbitrary real numbers, $a>0$ and $0<\eta<4$, then for $n\to\infty$

$$\max e^{-x/2}x^{\lambda}|L_n^{(\alpha)}(x)| \sim n^{Q}, \tag{2.2}$$

where

$$Q=\begin{cases}\max(\lambda-1/2,\ \alpha/2-1/4) & \text{if } a\leqq x\leqq(4-\eta)n,\\ \max(\lambda-1/3,\ \alpha/2-1/4) & \text{if } x\geqq a,\end{cases} \tag{2.3}$$

and the maximum at the left hand member of (2. 2) is taken in the respective interval pointed out in (2. 3).

3. Proof of Theorem I. From (1. 6),

$$\sigma_n^{(k)}(0)=\{A_n^{(k)}\Gamma(\alpha+1)\}^{-1}\left[\int_0^{1/n}+\int_{1/n}^{1}+\int_1^{\infty}\right]=I_1+I_2+I_3. \tag{3.1}$$

Using the order estimate (2. 1) we find that**)

$$\begin{aligned} I_1&=O(n^{-k})\int_0^{1/n}e^{-t}t^{\alpha}|f(t)|n^{\alpha+k+1}\,dt=O(n^{\alpha+1})\int_0^{1/n}t|f(t)|\,dt=\\ &=O(n^{\alpha+1})[t^{\alpha}F(t)]_0^{1/n}+O(n^{\alpha+1})\int_0^{1/n}t^{\alpha-1}F(t)\,dt=\\ &=O(n^{\alpha+1})\left[t^{\alpha}o\left(t\log\frac{1}{t}\right)\right]_0^{1/n}+O(n^{\alpha+1})\int_0^{1/n}o\left(t^{\alpha}\log\frac{1}{t}\right)dt=\\ &=o(\log n)+O(n^{\alpha+1})\left[\log\frac{1}{t}\,\frac{t^{\alpha+1}}{(\alpha+1)}+\int\frac{t^{\alpha}}{\alpha+1}\,dt\right]_0^{1/n}=\\ &=o(\log n)+o(\log n)+o(1)=o(\log n).\end{aligned} \tag{3.2}$$

In I_2, we make use of the first estimate of $L_n^{\alpha}(x)$ given in (2. 1) and we obtain

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*) If $b_n\neq 0$ and the sequence $\frac{|a_n|}{|b_n|}$ has finite positive limits of determination, we write $a_n\sim b_n$.

**) Condition (1.8) implies that

$$F(t)=\int_0^t|f(u)|\,du=o\left(t\log\frac{1}{t}\right).$$

$$I_2 = O(n^{-k}) \int_{1/n}^{1} e^{-t} t^{\alpha} |f(t)| n^{(\alpha+k+1)/2-1/4} t^{-(\alpha+k+1)/2-1/4} \, dt =$$

(3. 3)
$$= O[n^{-k+(\alpha+k+1)/2-1/4}] \int_{1/n}^{1} t^{\alpha/2-k/2-3/4} |f(t)| \, dt =$$

$$= O[n^{\alpha/2-k/2+1/4} n^{-\alpha/2+k/2-1/4}] \int_{1/n}^{1} \frac{|f(t)|}{t} \, dt = O(1) \left(\int_{1/n}^{\delta} + \int_{\delta}^{1} \right) =$$

$$= O(1) o(\log n) + O(1) \int_{\delta}^{1} \frac{|f(t)|}{t} \, dt = o(\log n) + O(1) = o(\log n).$$

Finally, from (2. 2) and (1. 9),

$$I_3 = O(n^{-k}) \int_{1}^{\infty} e^{-t} t^{\alpha} |f(t)| \, |L_n^{(\alpha+k+1)}(t)| \, dt =$$

$$= O(n^{-k}) \int_{1}^{\infty} e^{-t/2} t^{k+1/3} |L_n^{(\alpha+k+1)}(t)| e^{-t/2} t^{\alpha-1/3-k} |f(t)| \, dt =$$

(3. 4)
$$= O(n^{-k}) \int_{1}^{\infty} e^{-t/2} t^{\alpha-k-1/3} |f(t)| O(n^k) \, dt = O(1) = o(\log n).$$

The theorem gets proved on account of (3. 1), (3. 2), (3. 3) and (3. 4).

4. An additional parameter p, $-1 < p < \infty$, may be introduced into the theorem proved above so as to obtain a still finer result:

Theorem II. *If*

$$\int_{t}^{\delta} \frac{|f(u)|}{u} \, du = o\left[\left(\log \frac{1}{t}\right)^{p+1}\right] \qquad (t \to 0, \ -1 < p < \infty),$$

and if

$$\int_{1}^{\infty} e^{-t/2} t^{\alpha-k-1/3} |f(t)| \, dt < \infty,$$

then $\sigma_n^{(k)}(0) = o[(\log n)^{p+1}]$, *provided that* $k > \alpha + 1/2$.

Proof. As in the proof of Theorem I, we break the integral into $I_1+I_2+I_3$. I_3 gets disposed off exactly as before. Coming to I_1, we have

$$I_1 = O(n^{-k})\int_0^{1/n} e^{-t}t^\alpha|f(t)|n^{\alpha+k+1}\,dt = O(n^{\alpha+1})\int_0^{1/n}\frac{|f(t)|}{t}t^{1+\alpha}\,dt =$$

$$= O(n^{\alpha+1})\left[-t^{\alpha+1}\int_t^1\frac{|f(u)|}{u}\,du\right]_{t=0}^{t=1/n} + O(n^{\alpha+1})(\alpha+1)\int_0^{1/n}t^\alpha\left(\int_t^1\frac{|f(u)|}{u}\,du\right)dt =$$

$$= o[(\log n)^{p+1}] + o(n^{\alpha+1})\int_0^{1/n}t^\alpha\left(\log\frac{1}{t}\right)^{p+1}dt = o[(\log n)^{p+1}].$$

The estimate for I_2 is immediately obtained from (3. 3). This completes the proof.

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Operators unitary in an indefinite metric and linear fractional transformations

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Introduction

There is a close connection [2] between unitary operators on a Hilbert space with an indefinite metric and linear fractional transformations defined on the unit ball of a certain operator algebra (general symplectic maps). Invariant subspace problems for indefinite metric-unitary operators are equivalent to fixed point problems for general symplectic maps. In this note we define three natural classes of general symplectic maps — elliptic, hyperbolic, and parabolic. A linear fractional transformation of the disk onto itself in the complex plane is elliptic if and only if it has a fixed point in the interior of the disk. We prove that this is true for general symplectic maps. We also prove a basic inequality (6). We illustrate the strength of these two fundamental facts by giving a new proof of a generalized version [1] of NAĬMARK's Theorem [3] that every commuting family of unitary operators on a Pontryagin space has an invariant maximal positive subspace.

Background

The notation to be used in this paper is the same as the notation in [1]. We describe it briefly in this section.

The bilinear form $Q(\ ,\ )$ on a complex Hilbert space H is called an indefinite inner product on H provided that H is the direct sum of two orthogonal subspaces H_+, H_- with respect to which $Q(\ ,\ )$ has the representation

$$Q(x, y) = (E_+ x, y) - (E_- x, y) \tag{1}$$

where $E_\pm$ are the orthogonal projections of H onto $H_\pm$, and x, y are two vectors in H. A closed subspace P of H which contains only vectors p for which $Q(p, p) \geqq 0$

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is called *positive*. A *maximal* positive subspace is positive and not properly contained in any positive subspace of H. If S is a subspace we let $S'=\{q|\,Q(s,q)=0 \text{ if } s\in S\}$. An operator U on H which satisfies $Q(Ux, Uy)=Q(x,y)$ for all x, y in H is called *Q-unitary*. Let $\mathbf{B}$ denote the set of operators from H_+ into H_- with norm $\leqq 1$.

The following facts are well known [1] [4] [5]. There is a natural one-one correspondence between maximal positive subspaces P of H and operators J in $\mathbf{B}$ such that $P=(I+J)H_+$; we write $P\sim J$.

If U is a Q-unitary operator, then the matrix for U with respect to H_+, H_- has the form

$$U=\begin{pmatrix}(1-J^*J)^{-\frac{1}{2}}\psi & J^*(1-JJ^*)^{-\frac{1}{2}}\varphi\\ J(1-J^*J)^{-\frac{1}{2}}\psi & (1-JJ^*)^{-\frac{1}{2}}\varphi\end{pmatrix}. \tag{2}$$

where ψ and φ are unitary operators on H_+ and H_- respectively and J is an operator from H_+ to H_- with norm $\leqq 1$. The map $\mathfrak{F}$: $\mathbf{B}\to\mathbf{B}$ defined by

$$\mathfrak{F}(K)=(1-JJ^*)^{-\frac{1}{2}}[J\psi+\varphi K][\psi+J^*\varphi K]^{-1}(1-J^*J)^{\frac{1}{2}} \tag{3}$$

for all $K\in\mathbf{B}$ has the property

$$\text{if}\quad P\sim K,\quad \text{then}\quad UP\sim\mathfrak{F}(K). \tag{4}$$

If U is any Q-unitary operator with this property we write $U\sim\mathfrak{F}$ and say that U corresponds to $\mathfrak{F}$. If $U\sim\mathfrak{F}$ and $V\sim\mathfrak{F}$, then V is a scalar multiple of U. Any map $\mathfrak{F}$ that arises from a Q-unitary operator in the manner described above is called general symplectic. The set of all general symplectic maps is a group under composition and is denoted by $\mathscr{G}_1$. Note that if $\mathfrak{F}$ is defined by equation (3), then $\mathfrak{F}(0)=J$.

A simple inequality

Suppose that $K\in\mathbf{B}$ with $\|K\|<1$, suppose that $\mathfrak{F}\in\mathscr{G}_1$ and set $J=\mathfrak{F}(0)$. The elementary identity $J=(1-JJ^*)^{-\frac{1}{2}}J(1-J^*J)^{\frac{1}{2}}$ combined with the definition (3) of $\mathfrak{F}$ yields

$$\begin{aligned}\mathfrak{F}(K)-J&=(1-JJ^*)^{-\frac{1}{2}}\{[J\psi+\varphi K][\psi+J^*\varphi K]^{-1}-J\}(1-J^*J)^{\frac{1}{2}}\\ &=(1-JJ^*)^{-\frac{1}{2}}\{J\psi+\varphi K-J\psi-JJ^*\varphi K\}[\psi+J^*\varphi K]^{-1}(1-J^*J)^{\frac{1}{2}}\end{aligned}$$

and hence

$$\mathfrak{F}(K)-\mathfrak{F}(0)=(1-JJ^*)^{\frac{1}{2}}\varphi K[\psi+J^*\varphi K]^{-1}(1-J^*J)^{\frac{1}{2}}. \tag{5}$$

Since $\|K\|<1$ the inequality $\|[\psi+J^*\varphi K^{-1}]\|\leqq\{1-\|K\|\}^{-1}$ is valid and equation (5) implies that

$$\begin{aligned}\|\mathfrak{F}(K)x-\mathfrak{F}(0)x\|&\leqq\|(1-JJ^*)^{\frac{1}{2}}\|\,\|K\|\{1-\|K\|\}^{-1}\|(1-J^*J)^{\frac{1}{2}}x\|\\ &\leqq\sqrt{2}\,\|K\|\{1-\|K\|\}^{-1}\{\|x\|^2-\|\mathfrak{F}(0)x\|^2\}^{\frac{1}{2}}.\end{aligned}$$

Now we extend this to a more general inequality. Suppose that $M\in\mathbf{B}$ and $\|M\|<1$. There is a map $\mathfrak{G}\in\mathcal{G}_1$ such that $\mathfrak{G}(0)=M$ (c.f. Lemma 1. 1 [6]) and it is easy to see that $\|\mathfrak{G}^{-1}(K)\|<1$ since $\|K\|<1$. Since $\mathcal{G}_1$ is a group, $\mathfrak{F}\circ\mathfrak{G}\in\mathcal{G}_1$; thus if we substitute $\mathfrak{F}\circ\mathfrak{G}$ for $\mathfrak{F}$ and $\mathfrak{G}^{-1}(K)$ for K into the above inequality we get

$$\|\mathfrak{F}(K)x-\mathfrak{F}(M)x\| \leqq \sqrt{2}\,\|\mathfrak{G}^{-1}(K)\|\{1-\|\mathfrak{G}^{-1}(K)\|\}^{-1}\{\|x\|^2-\|\mathfrak{F}(M)x\|^2\}^{\frac{1}{2}}.$$

In other words

$$\|\mathfrak{F}(K)x-\mathfrak{F}(M)x\| \leqq c\{\|x\|^2-\|\mathfrak{F}(M)x\|^2\}^{\frac{1}{2}} \tag{6}$$

where c is a constant independent of $\mathfrak{F}$ and of x.

Three classes of maps in $\mathcal{G}_1$

Let $\mathfrak{F}^N$ ($\mathfrak{F}^{-N}$) denote the N^{th} iterate of the map $\mathfrak{F}$ ($\mathfrak{F}^{-1}$) in $\mathcal{G}_1$, for $N=0, 1, 2, \dots$. The set $\mathbf{B}^0=\{M\in\mathbf{B}: \|M\|<1\}$ is called the interior of $\mathbf{B}$.

Definition. Suppose that $\mathfrak{F}$ is in $\mathcal{G}_1$. An operator M in $\mathbf{B}^0$ will be called a uniformly elliptic [E], a uniformly parabolic [P], or a uniformly hyperbolic [H] point for $\mathfrak{F}$ provided that

[E] there is a number $\alpha<1$ such that $\|\mathfrak{F}^{\pm N}(M)\|<\alpha$ for all N.

[P] $\mathfrak{F}^{\pm N}(M)$ is invertible for large N, $\|[\mathfrak{F}^{\pm N}(M)]^{-1}\|\to 1$, and $\|\mathfrak{F}^N(M)-\mathfrak{F}^{-N}(M)\|\to 0$.

[H] $\mathfrak{F}^{\pm N}(M)$ is invertible for large N, $\|[\mathfrak{F}^{\pm N}(M)]^{-1}\|\to 1$, and there is a $\delta>0$, so that $\|[\mathfrak{F}^N(M)-\mathfrak{F}^{-N}(M)]x\| \geqq \delta\|x\|$ for all N and all $x\in H_+$.

Theorem I. *A map $\mathfrak{F}\in\mathcal{G}_1$ has a uniformly elliptic, parabolic, or hyperbolic point if and only if every operator M in $\mathbf{B}^0$ is a uniformly elliptic, parabolic, or hyperbolic point for $\mathfrak{F}$.*

Proof. Elliptic case: Suppose that $M\in\mathbf{B}^0$ is not a uniformly elliptic point for $\mathfrak{F}$. Then there is a sequence of vectors $x_N\in H_+$ with $\|x_N\|=1$ such that $\|\mathfrak{F}^N(M)x_N\|\to 1$. If $K\in\mathbf{B}^0$, then inequality (6) implies that $\|\mathfrak{F}^N(K)x_N-\mathfrak{F}^N(M)x_N\|\to 0$. Therefore $\|\mathfrak{F}^N(K)x_N\|\to 1$, and so K is not an elliptic point for $\mathfrak{F}$.

Parabolic Case: Suppose that M is a uniformly parabolic point for $\mathfrak{F}$. If $K\in\mathbf{B}^0$, then inequality (6) implies that $\|\mathfrak{F}^{\pm N}(K)-\mathfrak{F}^{\pm N}(M)\|\to 0$ and thus $\mathfrak{F}^{\pm N}(K)$ is invertible for large N and $\|[\mathfrak{F}^{\pm N}(K)]^{-1}\|\to 1$. Furthermore,

$$\|\mathfrak{F}^N(K)-\mathfrak{F}^N(K)\| \leqq \|\mathfrak{F}^N(K)-\mathfrak{F}^N(M)\|+\|\mathfrak{F}^N(M)-\mathfrak{F}^{-N}(M)\|+ +\|\mathfrak{F}^{-N}(M)-\mathfrak{F}^{-N}(K)\|. \tag{7}$$

Inequality (6) and the fact that M is a uniformly parabolic point for $\mathfrak{F}$ imply that the right hand side of inequality (7) converges to 0. Therefore K is a uniformly parabolic point of F.

The Hyperbolic Case is proved similarly.

Definition. A map $\mathfrak{F}\in\mathcal{G}$ is called uniformly elliptic, parabolic or hyperbolic if and only if it has a uniformly elliptic, parabolic or hyperbolic point, respectively.

Fixed point theorems

Theorem II. *A map $\mathfrak{F}$ in $\mathcal{G}_1$ is uniformly elliptic if and only if $\mathfrak{F}$ has a fixed point in the interior of* **B**.

Proof. The following is a consequence of Theorem 6. 1 [5] due to R. S. PHILLIPS:

(8) If U is a Q-unitary operator, then $\|U^{\pm N}\| < M$ for all N if and only if U has an invariant maximal positive subspace P with the property $P+P' = H$.

We now prove the equivalence of the Theorem II and (8). Suppose that U corresponds to $\mathfrak{F}$ as in (4) with the matrix representation for U given by (2). Since U is Q-unitary, $U^{-1} = [E_+ - E_-]U^*[E_+ - E_-]$ and an easy computation shows that for $x\in H_+$ and $y\in H_-$ we have

$$\|U^{-1}[x+y]\|^2 = \|\psi^*(1-J^*J)^{-\frac{1}{2}}x - \psi^*(1-J^*J)^{-\frac{1}{2}}J^*y\|^2 + \\ + \|-\varphi^*(1-JJ^*)^{-\frac{1}{2}}Jx + \varphi^*(1-JJ^*)^{-\frac{1}{2}}y\|^2 = \\ = \|(1-J^*J)^{-\frac{1}{2}}[x-J^*y]\|^2 + \|(1-JJ^*)^{-\frac{1}{2}}[y-Jx]\|^2$$

where $J=\mathfrak{F}(0)$. Consequently

$$\frac{1}{1-\|J_N\|^2} \leq \|U^{-N}\|^2 \leq 8\frac{1}{1-\|J_N\|^2}.$$

Thus $U^{\pm N}$ is uniformly bounded if and only if $\|\mathfrak{F}^{\pm N}(0)\| \leq \alpha < 1$ and hence if and only if $\mathfrak{F}$ is uniformly elliptic. Now Lemma 6. 3 [5] says that a maximal positive subspace P has the property $P+P' = H$ if and only if $P\sim J$ and $\|J\|<1$. These last two facts when combined with the fact that the Q-unitary operator U corresponding to $\mathfrak{F}$ has an invariant maximal positive subspace P if and only if the contraction J corresponding to P is fixed by $\mathfrak{F}$ imply that Theorem II and statement (8) are equivalent.

It is not known if hyperbolic and parabolic maps have fixed points. We shall now consider commuting families of general symplectic maps. Suppose $\mathcal{S}$ is a subgroup of $\mathcal{G}_1$ and $\Gamma_{\mathcal{S}} = \{U\colon U \text{ corresponds to } \mathfrak{F} \text{ and } \mathfrak{F}\in\mathcal{S}\}$. The group $\mathcal{S}$ is commutative if and only if the group $\Gamma_{\mathscr{S}}$ is *scalar commutative* (cf. sec. Ia. [2]) i.e. if $U, V \in \Gamma_{\mathscr{S}}$ then there is a number β with $|\beta|=1$ such that $UV=\beta VU$. A scalar commutative group $\mathscr{S}$ of operators is called *full* if $\alpha U \in \mathscr{S}$ whenever $U \in \mathscr{S}$ and α is a scalar with $|\alpha|=1$. The *group* $\mathscr{S}$ will be called *elliptic* if for each $x \in H$ there is a number $a(x)<1$ such that $\|\mathfrak{F}(0)x\| \leqq a(x)\|x\|$ for all $\mathfrak{F} \in \mathscr{S}$; the group $\mathscr{S}$ will be called *uniformly elliptic* if $a(x)<a<1$ for all $x \in H$.

Theorem III. *A commuting group $\mathscr{S}$ of general symplectic maps is uniformly elliptic if and only if $\mathscr{S}$ has a fixed point in the interior of* **B**.

Proof. We must prove statement (8) not for a single Q-unitary map U but for a scalar commuting family $\Gamma_{\mathscr{S}}$ of Q-unitary operators. It is clear from the original proof of (8) in [5] that any group Γ of Q-unitary operators has an invariant positive subspace P with $P+P' = H$ if and only if there is a bounded invertible operator B such that BUB^{-1} is unitary for each $U \in \Gamma$. The proof of Theorem II implies that $\mathscr{S}$ is uniformly elliptic if and only if $\Gamma_{\mathscr{S}}$ is uniformly bounded. Thus we need only prove

Lemma. *If Γ is a full, scalar commuting group of operators which is uniformly bounded, then Γ is similar to a group of unitary operators.*

Proof. The proof in the case where Γ is commutative involves finding an invariant mean on Γ. The case at hand requires just a slight modification of this. Although Γ is not commutative, $\Gamma/T = \Gamma$ modulo the circle group T is commutative. Thus there is an invariant mean on Γ/T (for instance see [1]). For fixed $x, y \in H$ the function f on Γ/T defined by $f(\tilde{U})=(Ux, Uy)$ where $\tilde{U} \in \Gamma/T$ and $U \in \Gamma$ is any element in the equivalence class $\tilde{U}$ is bounded. Thus we may define a bilinear form $(\ ,\ )'$ on it by

$$(x, y)'=m(f).$$

Since m is an invariant mean each $U \in \Gamma$ is unitary with respect to $(\ ,\ )'$ and it is easy to see that $\|x\|'=\sqrt{(x,x)'}$ is equivalent to the original norm on H. The lemma is immediate from this.

Inequality (6) yields the following lemma for the non-elliptic case.

Theorem IV. *If $\mathscr{S}$ is a commutative group of maps in $\mathscr{G}_1$ which is not elliptic and if for each $\mathfrak{F} \in \mathscr{S}$ the operator $\mathfrak{F}(0)$ is compact, then $\Gamma_{\mathscr{S}}$ has a non-trivial positive invariant subspace.*

Proof. The condition $\mathfrak{F}(0)$ is compact is equivalent to $\mathfrak{F}$ being continuous in the weak operator topology (cf. [3] and the author's Stanford dissertation).

Since $\mathscr{S}$ is not elliptic there is a sequence $\mathfrak{F}_N \in \mathscr{S}$ and a vector x such that $\|\mathfrak{F}_N(0)x\| \to 1$. Since **B** is compact in the weak operator topology we may assume that $\mathfrak{F}_N(0) \to T$ in the weak operator topology. If $\mathfrak{G} \in \mathscr{S}$, then $\mathfrak{G}[\mathfrak{F}_N(0)] \to \mathfrak{G}(T)$

in the weak operator topology; however by inequality (6)

$$\|\mathfrak{G}(\mathfrak{F}_N(0))x - \mathfrak{F}_N(0)x\| = \|\mathfrak{F}_N(\mathfrak{G}(0))x - \mathfrak{F}_N(0)x\| \to 0.$$

Thus $\mathfrak{G}(T)x = Tx$. Let $p = x + Tx$, let U be a Q-unitary operator which corresponds to $\mathfrak{G}$, and let $P \sim T$. Then $p = x + Tx \in P$ and property (4) implies that $p = x + \mathfrak{G}(T)x \in UP$. Therefore $p \in S = \bigcap_{U \in \Gamma_{\mathcal{S}}} UP$ and S is non-trivial. The form of S implies that it is invariant under operators in $\Gamma_{\mathcal{S}}$ and that S is positive.

Spaces with H_+ finite dimensional (Pontryagin spaces)

We give a new proof of:

Theorem. *If H_+ is finite dimensional and if $\mathcal{S}$ is a commutative subgroup of $\mathcal{G}_1$, then $\mathcal{S}$ has a fixed point.*

Proof. Let P be a subspace of H which is maximal with respect to being positive and invariant under $\Gamma_{\mathcal{S}}$. By Naimark's arguments in [4] it suffices to prove that $\Gamma_{\mathcal{S}}$ restricted to P' or to an appropriate modification of P' has a non-trivial invariant positive subspace. In effect it suffices for us to prove that $\Gamma_{\mathcal{S}}$ has a non-trivial invariant positive subspace.

Since H_+ is finite dimensional either $\Gamma_{\mathcal{S}}$ is uniformly elliptic or $\Gamma_{\mathcal{S}}$ is not elliptic. Theorem III and Theorem IV imply that $\Gamma_{\mathcal{S}}$ has a non-trivial invariant positive subspace in either case.

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A remark on the cosine of linear operators

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1. In their recent note [2], K. GUSTAFSON and B. ZWAHLEN proved that an unbounded linear operator T acting in a pre-Hilbert space has cosine zero. It is our purpose to show that this statement can be extended to the case of unbounded linear mappings T from a complex (real) normed vector space X into a normed vector space Y, provided there is given a sesquilinear form $Q: X \times Y \to \mathbf{C}(\mathbf{R})$ such that

$$|Q(x, y)| \leq \|x\| \, \|y\| \tag{1}$$

for all $x \in X$, $y \in Y$. The *cosine* of a mapping T from X to Y with respect to Q is then defined by

$$\cos_Q(T) = \inf \frac{|Q(x, Tx)|}{\|x\| \, \|Tx\|},$$

where the infimum is taken over all x in the domain $D(T)$, with $x \neq 0$, $Tx \neq 0$.

Theorem. *If to the linear operator $T: D(T) \subset X \to Y$ there exists a sesquilinear form Q such that $\cos_Q(T) > 0$, then T is bounded.*

The proof of the theorem is devided into two parts. We first introduce the concept of quasi-boundedness, which is due to F. E. BROWDER and the writer, and which turned out to be extremely useful in the study of nonlinear mappings of monotone type [1]. The mapping T is said to be *quasi-bounded with respect to the form Q*, if from the boundedness of the sequence $\{x_n\} \subset D(T)$ together with the boundedness of the sequence $\{Q(x_n, Tx_n)\}$ it follows that $\{Tx_n\}$ remains bounded. We prove that for an operator T which is homogeneous of some positive degree k (i.e. $D(T)$ a cone and $T(\mu x) = \mu^k T(x)$ for $\mu > 0$, $x \in D(T)$), quasi-boundedness implies boundedness. This observation allows us to give a proof of the theorem which seems to be more transparent even in the particular situation discussed in [2].

A closing example shows that the existence of a form Q with $\cos_Q(T) > 0$ is *not* necessary for the boundedness of a linear mapping T.

2. We shall preface the proof of the theorem with the following

Lemma. *Let the mapping $T: D(T) \subset X \to Y$ be homogeneous of degree $k > 0$, and suppose there exists a sesquilinear form Q such that T is quasi-bounded with respect to Q. Then T maps bounded sets in X onto bounded sets in Y.*

Proof. For $\lambda > 0$, let

$$f(\lambda) = \sup \{\|Tu\|:\ u \in D(T),\ \|u\| \leqq 1,\ |Q(u, Tu)| \leqq \lambda\}.$$

Because of the quasi-boundedness of T, f is a well-defined increasing function. We observe that for $\lambda \geqq 1$,

$$f(\lambda) \leqq \lambda^{\frac{k}{1+k}} f(1).$$

Hence

$$f(\lambda) \leqq \lambda^{\frac{k}{1+k}} f(1) + f(1), \qquad \lambda > 0.$$

For $x \in D(T)$ with $\|x\| \leqq 1$ we set $\lambda = |Q(x, Tx)|$ and get

$$\|Tx\| \leqq f(|Q(x, Tx)|) \leqq \|Tx\|^{\frac{k}{1+k}} f(1) + f(1).$$

This estimate implies the boundedness of T, q.e.d.

Proof of the Theorem. In virtue of the lemma, it suffices to prove that T is quasi-bounded with respect to Q.

Assume that $\{x_n\} \subset D(T)$ is a sequence with $\|x_n\| \leqq c$, $|Q(x_n, Tx_n)| \leqq c$, but $\|Tx_n\| \to \infty$. Since $\|x_n\|\,\|Tx_n\| \cos_Q(T) \leqq |Q(x_n, Tx_n)| \leqq c$, we infer that[1] $x_n \to 0$. We construct a bounded sequence $\{u_n\} \subset D(T)$ such that $Q(u_n, Tx_n) = 0$ and $\{Tu_n\}$ is bounded. For this purpose, let a and b be linearly independent vectors of $D(T)$ with $\|a\| = \|b\| = 1$,[2] and for each n set $u_n = \alpha_n a + \beta_n b$, where α_n and β_n are solutions of the equations $|\alpha_n|^2 + |\beta_n|^2 = 1$, $\alpha_n Q(a, Tx_n) + \beta_n Q(b, Tx_n) = 0$. The function g: $[\alpha, \beta] \to g(\alpha, \beta) = \|\alpha a + \beta b\|$ is continuous, hence it admits its supremum and infimum on the (compact) unit sphere $|\alpha|^2 + |\beta|^2 = 1$. Because of the linear independence of a and b, the infimum is positive. Consequently there exists $\gamma > 0$ such that $\gamma^{-1} \leqq \|u_n\| \leqq \gamma$ for all n. In addition, $\|Tu_n\| \leqq \|Ta\| + \|Tb\|$. Setting $w_n = x_n + u_n \in D(T)$, we obtain

$$\frac{|Q(w_n, Tw_n)|}{\|w_n\|\,\|Tw_n\|} \leqq \frac{|Q(x_n, Tx_n)| + |Q(u_n, Tx_n)| + |Q(x_n, Tu_n)| + |Q(u_n, Tu_n)|}{|\|u_n\| - \|x_n\|| \cdot |\|Tx_n\| - \|Tu_n\||},$$

where the right hand side converges to 0 as $n \to \infty$, since the numerator remains

[1]) We use the symbols "$\to$" and "$\rightharpoonup$" to denote *strong* and *weak* convergence, respectively.

[2]) If $D(T)$ is one dimensional, the theorem is trivial.

bounded and the denominator tends to $+\infty$. We are thus led to a contradiction to the assumption $\cos_Q(T)>0$, q.e.d.

That $\cos_Q(T)>0$ for some form Q is not necessary for the boundedness of the linear mapping T, is shown by the following

Example. *Let T be a bounded linear operator from X to Y, and suppose there exists a sequence $\{x_n\}\subset D(T)$ with $\|x_n\|=1$, $x_n \rightharpoonup 0$, $Tx_n\neq 0$, such that the linear span of $\{Tx_n\}$ has finite dimension. Then* $\cos_Q(T)=0$ *for each form Q.*

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Compact restrictions of operators

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1. Introduction. The purpose of this note is to set forth a definitive version of a theorem concerning operators on Hilbert space, and to discuss some consequences of that theorem that seem not to have been noticed before now. The theorem asserts that, unless an operator is, in a sense, nearly invertible, then it is "very small" on an infinite dimensional subspace. This fact has already been noted several times in the literature in one form or another (see, for example, [15, § 1. 2]; the main special case is valid even on Banach spaces [9, III. 1. 9]; for a version of the theorem valid in an infinite factor see [6], and the only thing in § 2 that can claim to be new is the manner in which we construe the notion of "very small". The results recounted in §§ 3—5 have greater claim to novelty.

Throughout this paper all *Hilbert spaces* will be complex, separable, and, unless the contrary possibility is explicitly stated, infinite dimensional. Furthermore, *operators* are always bounded, linear transformations from one Hilbert space into another. If $\mathscr{H}$ is a Hilbert space, then the algebra of all operators T from $\mathscr{H}$ into $\mathscr{H}$ will be denoted by $\mathscr{L}(\mathscr{H})$. We shall have occasion to refer to various ideals of operators, and we take this opportunity to remind the reader of the basic facts concerning the ideal structure of $\mathscr{L}(\mathscr{H})$. (By *ideal* we shall always mean two-sided ideal. Recall that $\mathscr{H}$ is assumed to be infinite dimensional; otherwise $\mathscr{L}(\mathscr{H})$ is simple.)

In the first place, every ideal $\mathfrak{I}$ in $\mathscr{L}(\mathscr{H})$ satisfies the condition

$$\mathfrak{F} \subset \mathfrak{I} \subset \mathfrak{C},$$

where $\mathfrak{F}$ denotes the ideal of operators of finite rank and $\mathfrak{C}$ the ideal of all compact operators. From this it is immediately apparent that $\mathfrak{C}$ is the *only* proper norm-closed ideal in $\mathscr{L}(\mathscr{H})$. Non-closed ideals exist in great abundance, however, and have been completely described. Indeed, if C denotes the collection of all sequences $\{\lambda_n\}_{n=1}^{\infty}$ of non-negative real numbers that tend to zero, then there is a simple one-to-one, inclusion preserving correspondence between the ideals $\mathfrak{I}$ in $\mathscr{L}(\mathscr{H})$ and the subsets J of C, called *ideal sets*, that satisfy the following conditions:

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[1]) The research for this paper was supported in part by the National Science Foundation.

i) if $\{\lambda_n\}$ is a sequence in J, and if π is any permutation of the positive integers, then $\{\lambda_{\pi(n)}\}$ is also in J,
ii) if $\{\lambda_n\}$ and $\{\mu_n\}$ are in J, then so is $\{\lambda_n+\mu_n\}$,
iii) if $\{\lambda_n\}$ is in J, and if $0\leqq\mu_n\leqq\lambda_n$ for all n, then $\{\mu_n\}$ is also in J.

The precise nature of this correspondence is as follows: if T belongs to $\mathfrak{J}$ then $|T|=(T^*T)^{\frac{1}{2}}$ does too, and, since $|T|$ is compact, its eigenvalues (counting multiplicities) can be arranged in a sequence belonging to C. The set of all sequences $\{\lambda_n\}$ so obtained from the various operators $T\in\mathfrak{J}$ forms the *ideal set* J of $\mathfrak{J}$. Conversely, if J is an ideal set in C, and if we say of an operator T on $\mathscr{H}$ that it *belongs to* J if, when the eigenvalues of $|T|$ are arranged in a sequence, that sequence belongs to J, then the set of all operators belonging to J forms an ideal $\mathfrak{J}$, of which J is clearly the ideal set. (These results are due originally to VON NEUMANN; a good account of them may be found in [5] or [7].) Note that under this correspondence the entire set C is the ideal set of the maximum ideal $\mathfrak{C}$ of all compact operators, and that the ideal set of the ideal $\mathfrak{F}$ of operators of finite rank is the set F of finitely non-zero sequences. Note also that these facts free the discussion of ideals in $\mathscr{L}(\mathscr{H})$ from the Hilbert space $\mathscr{H}$. When, in the sequel, we refer to an ideal $\mathfrak{J}$ in $\mathscr{L}(\mathscr{H})$ and then to the "same" ideal on another space $\mathscr{K}$, what is meant, of course, is that ideal in $\mathscr{L}(\mathscr{K})$ having the same ideal set as $\mathfrak{J}$. Moreover, the correspondence between ideal sets and operators can be extended even to operators from one space to another. Let J be an ideal set of sequences and let $\mathfrak{J}$ be its associated ideal, and suppose given an operator T mapping one Hilbert space $\mathscr{H}$ into another space $\mathscr{K}$. Then we shall say that T is *affiliated with* $\mathfrak{J}$ if, when the eigenvalues of $|T|=(T^*T)^{\frac{1}{2}}$ are arranged in a sequence, that sequence belongs to J. (When $\mathscr{H}$ and $\mathscr{K}$ do coincide, affiliation reduces to set membership.) Note that if $T\colon\mathscr{H}\to\mathscr{K}$ is affiliated with $\mathfrak{J}$ in this sense, then it continues to be true that $T^*\ \mathscr{K}\to\mathscr{H}$ is also. Similarly, it is easy to show that if T_1 and T_2 both map $\mathscr{H}$ into $\mathscr{K}$ and if both are affiliated with $\mathfrak{J}$, then T_1+T_2 is too, and that if $T\colon\mathscr{H}\to\mathscr{K}$ is affiliated with $\mathfrak{J}$ and if $S_1\colon\mathscr{K}\to\mathscr{K}_1$, $S_2\colon\mathscr{H}_1\to\mathscr{H}$ so that the product S_1TS_2 is defined, then S_1TS_2 is also affiliated with $\mathfrak{J}$.

2. Operators with small restrictions. The following theorem is the central tool of the paper.

Theorem 2. 1. *Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces, and let T be an operator mapping $\mathscr{H}$ into $\mathscr{K}$. Suppose that there does not exist a finite dimensional subspace $\mathscr{D}\subset\mathscr{H}$ such that $T|\mathscr{D}^\perp$ is bounded below. Then for any prescribed ideal $\mathfrak{J}$ other than the ideal $\mathfrak{F}$ of operators of finite rank, and for any η greater than zero, there exists an infinite dimensional subspace $\mathscr{L}\subset\mathscr{H}$ such that the restriction $T_0=T|\mathscr{L}\,(T_0\colon\mathscr{L}\to\mathscr{K})$ is affiliated with $\mathfrak{J}$ and satisfies the condition $\|T_0\|<\eta$.*

Before proving the theorem, it is advantageous to establish a working criterion for determining when an operator is affiliated with a given ideal.

Lemma 2. 2. *Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces. Then a necessary and sufficient condition for an operator $T\colon \mathscr{H} \to \mathscr{K}$ to be affiliated with a given ideal $\mathfrak{I}$ is that there exist an orthonormal basis $\{e_n\}$ in $\mathscr{H}$, and an orthonormal sequence $\{f_n\}$ in $\mathscr{K}$ such that $Te_n = \lambda_n f_n$ for all n, where $\{|\lambda_n|\}$ belongs to the ideal set of $\mathfrak{I}$.*

Proof. If the criterion is satisfied, then $|T|e_n = |\lambda_n|e_n$ for all n, so the condition is clearly sufficient. On the other hand, if T is affiliated with $\mathfrak{I}$, then there exists an orthonormal basis $\{e_n\}$ in $\mathscr{H}$ such that $|T|e_n = \lambda_n e_n$ for all n, where $\{\lambda_n\}$ is in the ideal set of $\mathfrak{I}$. But then, if W denotes the partial isometry in the polar resolution of T, so that $T = W|T|$, and if we set $f_n = We_n$, then $\{f_n\}$ is an orthonormal sequence in $\mathscr{K}$, and $Te_n = \lambda_n f_n$. □

Proof of Theorem 2. 1. If T has an infinite dimensional null space, we may simply set $T_0 = 0$. Otherwise, let $T = W|T|$ be the polar resolution of T as above, and let E denote the spectral measure of $|T|$. Then, according to our assumptions, no projection $E([0, \varepsilon))$ $(\varepsilon > 0)$ has finite rank, while $E(\{0\})$ does have finite rank. Hence $E((0, \varepsilon))$ has infinite rank for every positive ε, and it follows at once that for every positive ε there exists δ, $0 < \delta < \varepsilon$, such that $E((\delta, \varepsilon))$ has rank greater than one.

Now let $\{\lambda_n\}$ be any one fixed sequence in the ideal set J of $\mathfrak{I}$ satisfying the conditions $0 < \lambda_{n+1} \leqq \lambda_n < \eta$ for every n. (Such sequences exist since $J \neq F$; see [4, Lemma 1. 1].) We set $\varepsilon_1 = \lambda_1$ and determine δ_1, $0 < \delta_1 < \varepsilon_1$ such that $E_1 = E((\delta_1, \varepsilon_1))$ has rank exceeding one. Next, define $\varepsilon_2 = \delta_1 \wedge \lambda_2$ and choose δ_2 so that $0 < \delta_2 < \varepsilon_2$ and so that $E_2 = E((\delta_2, \varepsilon_2))$ has rank exceeding one. Continuing in this fashion, we obtain an infinite sequence of spectral projections E_n such that, for every n, $\mathscr{M}_n = E_n(\mathscr{H})$ has dimension at least two and such that $\| \, |T| \, |\mathscr{M}_n\| \leqq \lambda_n < \eta$. In each subspace $\mathscr{M}_n$ we select a pair of orthogonal unit vectors e_n and f_n in such a way that the plane $[e_n, f_n]$ contains the vector $|T|e_n$, and write

$$|T|e_n = \alpha_n e_n + \beta_n f_n.$$

Then $0 < \alpha_n = (|T|e_n, e_n) \leqq \lambda_n$ and $|\beta_n| \leqq 2\lambda_n$ for all n.

Finally, let $\mathscr{L}$ denote the subspace spanned by the sequence $\{e_n\}$, and set $A = |T|\,|\mathscr{L}\colon \mathscr{L} \to \mathscr{H}$, so that $T_0 = T|\mathscr{L}$ is given by $T_0 = WA$. Since the vectors Te_n are all orthogonal and less than η in norm, it is obvious that $\|T_0\|$ is also less than η. On the other hand, if P denotes the (orthogonal) projection of $\mathscr{H}$ onto $\mathscr{L}$, then PA and $(1-P)A$, regarded as mappings from $\mathscr{L}$ to $\mathscr{K}$, both clearly satisfy the criterion of Lemma 2. 2. But then, of course, $A = PA + (1-P)A$ and $T_0 = WA$ are also affiliated with $\mathfrak{I}$. □

The hypotheses of Theorem 2. 1 are formulated as they are in order to facilitate the proof of the theorem, not with a view to applications. We pause to list some alternate versions of the condition imposed on T.

Lemma 2. 3. *The following conditions are equivalent for any operator* $T: \mathscr{H} \to \mathscr{K}$.[2])

i) *T is bounded below on the orthocomplement of some finite dimensional subspace.*
ii) *The null space of T is finite dimensional and the range of T is closed.*
iii) *There exists an operator $S: \mathscr{K} \to \mathscr{H}$ such that ST is a projection of finite co-rank.*
iv) *T is semi-Fredholm with index less than $+\infty$.*
v) *There exists no orthonormal sequence $\{e_n\}_{n=1}^{\infty}$ such that $\|Te_n\| \to 0$.*

In the special case $\mathscr{H} = \mathscr{K}$ the conclusion of the main theorem can also be reformulated in a useful manner. The following is an immediate consequence of Theorem 2. 1, from which, in turn, the latter may easily be deduced.

Corollary 2. 4. *Let T be an operator in $\mathscr{L}(\mathscr{H})$ and suppose that the range of T is not closed, or that the null space of T is infinite dimensional. Let $\mathfrak{J}$ be any ideal other than the ideal $\mathfrak{F}$, and let η be a positive number. Then there exists a decomposition $\mathscr{H} = \mathscr{L} \oplus \mathscr{L}^{\perp}$ of $\mathscr{H}$ into infinite dimensional subspaces with respect to which the matrix representation of T has the form*

$$\begin{pmatrix} K & * \\ L & * \end{pmatrix}$$

where K and L are both affiliated with $\mathfrak{J}$ and have norm less than η.

Proof. From the proof of Theorem 2. 1 it is clear that both the subspace $\mathscr{L}$ constructed there and its orthocomplement are infinite dimensional. Everything else is obvious. □

3. Subspaces that are nearly invariant. If $\mathfrak{J}$ is any ideal in $\mathscr{L}(\mathscr{H})$, then the quotient algebra $\mathscr{L}(\mathscr{H})/\mathfrak{J}$ is clearly a *-algebra. Moreover, for the norm-closed ideal $\mathfrak{C}$ of all compact operators the quotient algebra is even a C^*-algebra with respect to the quotient norm. As is customary, we shall refer to this algebra as the *Calkin algebra* over $\mathscr{H}$. If T is an operator in $\mathscr{L}(\mathscr{H})$, we denote by $\hat{T}$ the residue class of T in the Calkin algebra.

[2]) This lemma is but a part of a more encompassing theorem due to J. P. Williams [14, Theorem (1. 1)], which generalizes some results of Wolf [15]. The authors wish to take this opportunity to express this gratitude to Williams for a number of stimulating and enlightening conversations on this point as well as on other related subjects.

Theorem 3. 1. *Let T be an operator $\mathscr{L}(\mathscr{H})$, and let $\mathfrak{J}$ be any ideal other than $\mathfrak{F}$. Then there exists a scalar λ and a decomposition of $\mathscr{H}$ into infinite dimensional subspaces $\mathscr{L}$ and $\mathscr{L}^{\perp}$ such that the corresponding matrix representation of T has the form*

$$\begin{pmatrix} \lambda+K & * \\ L & * \end{pmatrix} \tag{1}$$

where K and L are both affiliated with $\mathfrak{J}$. Moreover, the decomposition can be so arranged that the norms of K and L are less than any prescribed positive η.

Proof. The residue class $\hat{T}$ of T in the Calkin algebra over $\mathscr{H}$ has non-empty spectrum σ by the Gelfand—Mazur Theorem, and in σ there are points λ such that $\hat{T}-\lambda$ has no left inverse. (These are the points of the *left essential spectrum* in the terminology of [14]. For example, any complex number in the topological boundary of σ is such a λ.) But then $T-\lambda$ fails to satisfy the criterion of Lemma 2. 3, and the theorem follows. □

As the proof of Theorem 3. 1 shows, the choice of λ is quite independent of $\mathfrak{J}$ and of η. It may be noted that λ can be taken to be any scalar in the boundary of the spectrum of T itself, other than an isolated eigenvalue of finite multiplicity, since such points automatically survive in the spectrum of $\hat{T}$; see, for instance, [10, Theorem 2]. It may also be noted that Theorem 3. 1, as well as Corollaries 3. 2, 3. 5, and 3. 6, are definitely false for $\mathfrak{J}=\mathfrak{F}$. Finally, if $\mathscr{L}$ and $\mathscr{L}^{\perp}$ are both identified with the same space $\mathscr{K}$ (as they may be whenever convenience so dictates), then the entries in (1) will all be in $\mathscr{L}(\mathscr{K})$, and K and L will be actual members of the ideal $\mathfrak{J}$ on $\mathscr{K}$.

Theorem 3. 1 may be paraphrased by saying that the residue class of T modulo $\mathfrak{J}$ has the form

$$\begin{pmatrix} \lambda & * \\ 0 & * \end{pmatrix}.$$

In this formulation, however, the matrix entries are to be interpreted merely as the components in the Pierce decomposition of the residue class of T relative to a non-zero, Hermitian idempotent; residue classes modulo $\mathfrak{J}$ cannot, in general, be realized spatially as operators.

Corollary 3. 2. *For any operator T in $\mathscr{L}(\mathscr{H})$, and for any ideal $\mathfrak{J}$ in $\mathscr{L}(\mathscr{H})$ other than $\mathfrak{F}$, there exists an infinite dimensional subspace $\mathscr{L}$ with infinite dimensional orthocomplement $\mathscr{L}^{\perp}$ such that $\mathscr{L}$ is invariant under T modulo $\mathfrak{J}$, i.e., such that $(1-P)TP\in\mathfrak{J}$, where P denotes the projection of $\mathscr{H}$ onto $\mathscr{L}$.*

Note, in particular, that Corollary 3. 2 solves in the affirmative the invariant subspace problem in the Calkin algebra. (For another representation of $\mathscr{L}(\mathscr{H})$

having the same property the reader may consult [1].) The following result exploits the metrical aspect of Theorem 3. 1.

Corollary 3. 3. *For any operator T in $\mathcal{L}(\mathcal{H})$ and any positive number η there exists an operator R such that $\|T-R\| < \eta$ and such that R possesses an infinite dimensional invariant subspace $\mathcal{L}$ having infinite dimensional orthocomplement. Likewise, for any positive integer p, there exists an operator R_p that is within η of T in norm and possesses a p-dimensional invariant subspace.*

Proof. By Theorem 3. 1 there exists an infinite dimensional subspace $\mathcal{L}$ with infinite dimensional orthocomplement such that the corresponding matrix representation has the form (1) with the property that $\|L\| < \eta$. To obtain a suitable operator R we have but to define

$$R = T - \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix}.$$

In order to construct R_p we choose bases $\{e_n\}$ and $\{f_n\}$ in $\mathcal{L}$ and $\mathcal{L}^\perp$, respectively. It is then a simple matter, since K and L are compact, to find p basis vectors e_n such that, if $\mathcal{P}$ denotes the subspace they span, then $\|(T-\lambda)|\mathcal{P}\| < \eta$. Then the matrix of R_p may be obtained by replacing all the off-diagonal entries in the correspondings columns by zero's. □

In the special case of a seminormal operator the preceding results can be improved in a natural but significant manner. First, a lemma.

Lemma 3. 4. *Let S and T be two operators from $\mathcal{H}$ into $\mathcal{K}$, and suppose that S is metrically dominated by T, i.e., that $\|Sx\| \leqq \|Tx\|$ for every x in $\mathcal{H}$. Then S is affiliated with every ideal with which T is.*

Proof. It is clear that $|S|$ is metrically dominated by $|T|$. The lemma follows via a straightforward application of the minimax principle, or alternatively, via [8, Theorem 1]. □

Theorem 3. 5. *Let T be a seminormal operator in $\mathcal{L}(\mathcal{H})$, and let $\mathfrak{J}$ be any ideal other than $\mathfrak{F}$. Then there exists a scalar λ and a decomposition of $\mathcal{H}$ into infinite dimensional subspaces $\mathcal{L}$ and $\mathcal{L}^\perp$ such that the corresponding matrix representation of T has the form*

$$\begin{pmatrix} \lambda + K & M \\ L & * \end{pmatrix} \tag{2}$$

where K, L, and M all are affiliated with $\mathfrak{J}$. Moreover, the decomposition can be so arranged that the norms of K, L, and M are all less than any prescribed positive η.

Proof. We may suppose that T is hyponormal. Let $\mathscr{H}$ be decomposed as in Theorem 3. 1, in such a way that, in the matrix representation (1), the operator

$$\begin{pmatrix} K & 0 \\ L & 0 \end{pmatrix}$$

has norm less than η. Since K and L are affiliated with $\mathfrak{J}$, it follows, as we have seen, that $(T-\lambda)|\mathscr{L}: \mathscr{L} \to \mathscr{H}$ is affiliated with $\mathfrak{J}$ and has norm less than η. Since $T-\lambda$ is hyponormal along with T, this implies in turn, by Lemma 3. 4, that $(T-\lambda)^*|\mathscr{L}$ is also affiliated with $\mathfrak{J}$ and has norm less than η. Since the matrix of $(T-\lambda)^*$ is

$$\begin{pmatrix} K^* & L^* \\ M^* & * \end{pmatrix}$$

it follows, finally, that M and M^* are affiliated with $\mathfrak{J}$ and have norm less than η. □

Here again, as was the case in Theorem 3. 1, the result may be interpreted matricially if we are careful not to attribute undue spatial significance to the matrix entries. It says that if $\mathfrak{J} \neq \mathfrak{F}$, and if T is seminormal, then the residue class of T modulo $\mathfrak{J}$ has the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & * \end{pmatrix}. \tag{3}$$

(In this connection see also [14, Theorem (4. 2)].)

Corollary 3. 6. *If T is a seminormal operator in $\mathscr{L}(\mathscr{H})$, and if $\mathfrak{J}$ is any ideal in $\mathscr{L}(\mathscr{H})$ other than $\mathfrak{F}$, then there exists an infinite dimensional subspace $\mathscr{L}$, with infinite dimensional orthocomplement, such that $\mathscr{L}$ is reducing for T modulo $\mathfrak{J}$, i.e., such that $TP-PT \in \mathfrak{J}$ where P denotes the projection of $\mathscr{H}$ onto $\mathscr{L}$.*

Corollary 3. 7. *For any seminormal operator T in $\mathscr{L}(\mathscr{H})$ and any positive number η there exists an operator R such that $\|T-R\| < \eta$ and such that R possesses an infinite dimensional reducing subspace with infinite dimensional orthocomplement. Likewise, for any positive integer p, there exists an operator R_p that is within η of T in norm and possesses a p-dimensional reducing subspace.*

The proofs of Corollaries 3. 6 and 3. 7 are straightforward analogs of those of Corollaries 3. 2 and 3. 3, and will be omitted. The finite dimensional part of Corollary 3. 7 is essentially due to Stampfli [12], who states the result in the case $p=1$. We owe to the same paper the observation that Corollary 3. 7 remains valid if T merely differs from a seminormal operator by a compact operator. (The same may also be said, of course, of Corollary 3. 3.)

Theorem 3. 5 yields at least one other interesting result. Indeed, a glance at (3) reveals the validity of the following assertion.

Corollary 3.8. *If T is a seminormal operator in $\mathcal{L}(\mathcal{H})$, and if $\mathfrak{I}$ is any ideal other than $\mathfrak{F}$, then there exists an infinite dimensional subspace $\mathcal{L}$ such that, for every X in $\mathcal{L}(\mathcal{H})$, the commutator $C = TX - XT$ has the property that its compression $PC|\mathcal{L}$ to $\mathcal{L}$ belongs to $\mathfrak{I}$.*

In particular, this shows that 0 belongs to the (essential) numerical range of C (see [13]), thus recapturing a result of C. R. PUTNAM [11].

4. Operators congruent to scalars. In this section we give several criteria for an operator in $\mathcal{L}(\mathcal{H})$ to be congruent to a complex number modulo one or another of the ideals in $\mathcal{L}(\mathcal{H})$.

Theorem 4.1. *Let T be an operator in $\mathcal{L}(\mathcal{H})$ and let $\mathfrak{I}$ be an ideal. Then a necessary and sufficient condition for T to be congruent to a scalar modulo $\mathfrak{I}$ is that, for any two orthogonal subspaces $\mathcal{M}$ and $\mathcal{N}$ in $\mathcal{H}$,*

(C) $P_{\mathcal{N}} T P_{\mathcal{M}} \in \mathfrak{I}$, *where $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ denote the (orthogonal) projections of $\mathcal{H}$ onto $\mathcal{M}$ and $\mathcal{N}$, respectively.*

Proof. The necessity of the condition is evident. To prove sufficiency, consider first the case $\mathfrak{I} \neq \mathfrak{F}$. According to Theorem 3.1, there exist subspaces $\mathcal{L}$ and $\mathcal{L}^{\perp}$, both infinite dimensional, with respect to which T has the form

$$\begin{pmatrix} \lambda + K & X \\ L & Y \end{pmatrix}$$

with K and L affiliated with $\mathfrak{I}$. Moreover, X is also affiliated with $\mathfrak{I}$ because of (C). Hence, T is congruent modulo $\mathfrak{I}$ to the matrix

$$T' = \begin{pmatrix} \lambda & 0 \\ 0 & Y \end{pmatrix}.$$

Now let V be an isometry of $\mathcal{L}^{\perp}$ onto $\mathcal{L}$, and use the map $1 \oplus V$ to identify $\mathcal{H}$ with $\mathcal{L} \oplus \mathcal{L}$. Under this unitary equivalence, T' is carried onto the operator

$$T'' = \begin{pmatrix} \lambda & 0 \\ 0 & Y_0 \end{pmatrix}$$

where $Y_0 = VYV^*$. Clearly T'' continues to satisfy (C), so that if $\mathcal{M}$ and $\mathcal{N}$ denote, respectively, the subspaces $\{(x, x): x \in \mathcal{L}\}$ and $\{(x, -x): x \in \mathcal{L}\}$, then $P_{\mathcal{N}} T'' P_{\mathcal{M}}$ must belong to $\mathfrak{I}$. But for any vector (x, y) in $\mathcal{L} \oplus \mathcal{L}$ we have $P_{\mathcal{N}}(x, y) = \frac{1}{2}(x - y, y - x)$, so that

$$P_{\mathcal{N}} T''(x, x) = \tfrac{1}{2}((\lambda - Y_0)x, (Y_0 - \lambda)x).$$

It follows at once that Y_0 is congruent to λ modulo $\mathfrak{I}$, and hence that T'' and T' are too.

It remains to consider the case $\mathfrak{I}=\mathfrak{F}$. If T satisfies (C) with $\mathfrak{I}=\mathfrak{F}$, then, by what has already been shown, T is congruent to some λ modulo every ideal $\mathfrak{I}\neq\mathfrak{F}$ (clearly the same λ in each case), so that $T-\lambda$ belongs to the intersection of all the ideals $\mathfrak{I}\neq\mathfrak{F}$. Since this intersection is known to be equal to $\mathfrak{F}$ (see [4]), the theorem follows. □

A second criterion is given by the following corollary.

Corollary 4. 2. *A necessary and sufficient condition for an operator T in $\mathscr{L}(\mathscr{H})$ to be congruent to some scalar modulo a given ideal $\mathfrak{I}$ is that for every infinite dimensional subspace $\mathscr{L}$ with infinite dimensional complement, the compression $P_{\mathscr{L}}T|\mathscr{L}$ of T to $\mathscr{L}$ should be congruent modulo $\mathfrak{I}$ to some scalar.*

Proof. Once again, it is clear that the condition is necessary. The proof will be completed by showing that an operator T satisfying the hypothesis of the corollary also satisfies condition (C) of Theorem 4. 1. Accordingly, let $\mathscr{M}$ and $\mathscr{N}$ be orthogonal subspaces of $\mathscr{H}$. Clearly we may assume both $\mathscr{M}$ and $\mathscr{N}$ to be infinite dimensional, since otherwise $P_{\mathscr{N}}TP_{\mathscr{M}}$ is automatically in $\mathfrak{F}$. Write $\mathscr{M}=\mathscr{M}_1\oplus\mathscr{M}_2$, where $\mathscr{M}_1$ and $\mathscr{M}_2$ are both infinite dimensional, and consider the compression of T to $\mathscr{M}_1\oplus\mathscr{N}$. The hypothesis assures us that this compression is congruent to some scalar modulo $\mathfrak{I}$, whence, by Theorem 4. 1, $P_{\mathscr{N}}TP_{\mathscr{M}_1}$ must belong to $\mathfrak{I}$. Similarly, $P_{\mathscr{N}}TP_{\mathscr{M}_2}$ belongs to $\mathfrak{I}$, from which it follows immediately that $P_{\mathscr{N}}TP_{\mathscr{M}}$ does so too. □

Our third and final criterion is one that has already essentially been noted by Calkin (see [5, Theorem 2. 9]) but our proof is completely different from his.

Theorem 4. 3. *A necessary and sufficient condition for an operator T in $\mathscr{L}(\mathscr{H})$ to be congruent to a scalar modulo an ideal $\mathfrak{I}$ is that $TX-XT$ should belong to $\mathfrak{I}$ for every X in $\mathscr{L}(\mathscr{H})$.*

Proof. As before, the condition is clearly necessary, and we verify its sufficiency by showing that an operator that satisfies it also satisfies condition (C). Let $\mathscr{M}$ and $\mathscr{N}$ be orthogonal subspaces of $\mathscr{H}$ (infinite dimensional as before), and let W be any partial isometry with initial space $\mathscr{N}$ and final space $\mathscr{M}$. Then $(TW-WT)P_{\mathscr{M}}$ belongs to $\mathfrak{I}$ along with $TW-WT$, and since $W|\mathscr{M}=0$, this implies that $WTP_{\mathscr{M}}$ belongs to $\mathfrak{I}$. But then so does $P_{\mathscr{N}}WTP_{\mathscr{M}}=WP_{\mathscr{N}}TP_{\mathscr{M}}$ and therefore, finally, $W^*WP_{\mathscr{N}}TP_{\mathscr{M}}=P_{\mathscr{N}}TP_{\mathscr{M}}$. □

It may be noted that in the special case $\mathfrak{I}=\mathfrak{C}$ all three of these results yield criteria for an operator not to be a commutator [3]. This observation, Theorem 4. 3, and also the final result of § 3 all suggest that the ideas of the present note have interesting ramifications into commutator theory. In the next and final section we explore these connections in some depth.

5. Applications to commutator theory. As has just been noted, it is shown in [3] that an operator T in $\mathscr{L}(\mathscr{H})$ is a commutator if and only if it is not congruent to a non-zero scalar modulo the ideal $\mathfrak{C}$. On the other hand, in the earlier paper [2] it was shown, using considerably more elementary techniques, that every operator on $\mathscr{H}\oplus\mathscr{H}$ of the form

$$\begin{pmatrix} * & K_1 \\ * & K_2 \end{pmatrix}$$

where K_1 and K_2 are compact operators, is a commutator. Considering this fact, together with Theorem 3. 1, and taking adjoints if necessary, we immediately obtain the following result.

Theorem 5. 1. *Every non-Fredholm operator in $\mathscr{L}(\mathscr{H})$ is a commutator.*

This theorem prompts the following question: how far is it possible to proceed with the solution of the commutator problem, using only the techniques of [2] and the results of § 2? In other words, how far can one proceed without use of the sophisticated results of [3]; in particular, without introduction of the η-function and the standard form for operators of class (F)?

It is almost certain that one should not expect much success with the Fredholm operators of index zero, since the non-commutators in $\mathscr{L}(\mathscr{H})$ are Fredholm of index zero, while, at the same time, there are many Fredholm operators of index zero that are commutators, e.g., the invertible operators of class (F). Thus it is reasonable to limit attention to Fredholm operators of index different from zero. Operating under the above named restrictions, we are able to prove the following suggestive result.

Theorem 5. 2. *Every partial isometry in $\mathscr{L}(\mathscr{H})$ that is a Fredholm operator of index different from zero is a commutator.*

Proof. Note first that consideration of adjoints shows that it suffices to deal with the case in which the given partial isometry W has negative index. In this case there exists an operator F of finite rank (possibly zero) such that $V+F$ is an isometry, and such that the ranges of F and W are orthogonal. The isometry $W+F$ can be written uniquely as $W+F = U\oplus S$, where U is a unitary operator on a k-dimensional subspace $\mathscr{K}$ of $\mathscr{H}$ $(0\leqq k\leqq\aleph_0)$, while S is a unilateral shift of multiplicity m $(0<m<\aleph_0)$ acting on the space $\mathscr{M} = \mathscr{H}\ominus\mathscr{K}$. Suppose, temporarily, that $m=1$, and let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in $\mathscr{M}$ such that $Se_n=e_{n+1}$ for all n. Reordering this basis as

$$\{e_1, e_3, \dots, e_{2n-1}, \dots;\ e_2, e_4, \dots, e_{2n}, \dots\}$$

we obtain a unitary isomorphism of $\mathscr{M}$ onto a Hilbert space $\mathscr{N} \oplus \mathscr{N}$, which carries S onto an operator matrix of the form

$$\begin{pmatrix} 0 & S_0 \\ 1 & 0 \end{pmatrix}, \tag{4}$$

where S_0 is unitarily equivalent with S. A similar device shows that, no matter what the multiplicity m may be, S is always unitarily equivalent with (4), where S_0 is unitarily equivalent with S itself. It follows easily that $W+F = U \oplus S$ is unitarily equivalent with an operator matrix

$$\begin{pmatrix} U_1 & S_1 \\ B_1 & 0 \end{pmatrix} \tag{5}$$

acting on a Hilbert space $\mathscr{P} \oplus \mathscr{P}$, where U_1 is the direct sum of a unitary operator and the zero operator on an infinite dimensional space, while S_1 is an isometry and B_1 is a co-isometry. (If $k=0$, then $U_1=0$, if k is finite, then U_1 has finite rank, and, if $k=\aleph_0$, then S_1 has infinite defect.) Now the unitary isomorphism φ of $\mathscr{H}$ onto $\mathscr{P} \oplus \mathscr{P}$ that carries $W+F = U \oplus S$ onto (5) also carries F onto some matrix, — say the matrix

$$\begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}.$$

Clearly each F_i $(i=1, 2, 3, 4)$ is of finite rank, and clearly also the given partial isometry W is unitarily equivalent via φ with an operator W_0 having the matrix

$$\begin{pmatrix} U_1-F_1 & S_1-F_2 \\ B_1-F_3 & -F_4 \end{pmatrix}. \tag{6}$$

Since the range of W is orthogonal in $\mathscr{H}$ to the range of F, it follows easily that the null space in $\mathscr{P}$ of S_1-F_1 is contained in the null space of F_4. Since S_1-F_1 is a semi-Fredholm operator, this implies that there exists an operator Y of finite rank in $\mathscr{L}(\mathscr{P})$ such that $Y(S_1-F_2) = F_4$ (see [8, Theorem 1]). We now apply a similarity transformation to (6) as follows:

$$\begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix} \begin{pmatrix} U_1-F_1 & S_1-F_2 \\ B_1-F_3 & -F_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -Y & 1 \end{pmatrix},$$

obtaining a matrix of the form

$$\begin{pmatrix} Z & * \\ * & 0 \end{pmatrix}, \tag{7}$$

where $Z = U_1-F_1-(S_1-F_2)Y$. Since U_1 has infinite dimensional null space (no matter what k is) and since $F_1+(S_1-F_2)Y$ has finite rank, it is easily seen that Z has an infinite dimensional null space too. Hence Z is a commutator (this follows, for instance, from Theorem 5. 1), say $Z=[A, B]$. Consider now the two operator matrices

$$(8) \qquad \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} B & T \\ R & 0 \end{pmatrix},$$

where R and T remain to be determined. Calculation shows that the commutator of the operators in (8) is the operator matrix

$$(9) \qquad \begin{pmatrix} Z & (A-1)T \\ R(1-A) & 0 \end{pmatrix}.$$

Since A may be replaced by any translate $A+\lambda$ without changing any of these calculations, we may certainly arrange for $A-1$ to be invertible, whereupon it becomes a triviality to solve for R and T in (9) so as to make (9) equal to (7). Thus W_0 is similar to a commutator, and the theorem is proved. □

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Extending mutually orthogonal partial latin squares

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1. Introduction

By an $n \times n$ (partial) *latin square* is meant an $n \times n$ array such that (in some subset of the n^2 cells of the array) each of the cells is occupied by an integer from the set $\{1, 2, \ldots, n\}$ and such that no integer from this set occurs in any row or column more than once. We will also refer to an $n \times n$ (partial) latin square as a *finite* (partial) latin square. By an *infinite* latin square is meant a countably infinite array of rows and columns such that each positive integer occurs exactly once in each row and column.

If P is a finite (partial) latin square we will denote by S_p the set of all cells which are occupied in P. If P and Q are (partial) latin squares of the same size, by (P, Q) is meant the set $\{(p_{ij}, q_{ij}): (i, j) \in S_P \cap S_Q\}$. If P and Q are finite (partial) latin squares and $|(P, Q)| = |S_P \cap S_Q|$ we say that P and Q are *orthogonal* and write $P \perp Q$. If P and Q are infinite latin squares we say that P and Q are orthogonal provided that $(P, Q) = Z \times Z$ (where Z is the set of all positive integers) and every pair of cells in different rows and columns are occupied by the same symbol in at most one of P and Q. As above if P and Q are orthogonal infinite latin squares we write $P \perp Q$.

In this paper the term latin square will mean either a finite or infinite latin square.

If $\{P_i\}_{i \in I}$ is a collection of mutually orthogonal latin squares of the same size we say that this collection is a *complete set* of mutually orthogonal latin squares provided that every pair of cells in different rows and columns are occupied by the same symbol in exactly one member of the collection. We note that if the latin squares in this collection are finite and based on $N = \{1, 2, \ldots, n\}$ then $I = \{1, 2, \ldots, n-1\}$. If the latin squares are infinite then I is the set of positive integers.

In this paper we prove the following theorem.

Theorem. *A finite collection of mutually orthogonal $n \times n$ partial latin squares can be embedded in a complete set of mutually orthogonal infinite latin squares.*

The following ideas are used in the proof.

By a *plane* we will always mean a set π which is the union of two disjoint sets $\mathscr{P}$ and $\mathscr{L}$ (the elements of which are called points and lines) and a relation I from $\mathscr{P}$ to $\mathscr{L}$ called *incidence*. If $(P, l) \in I$ we will say that the point P is on or belongs to the line l and that l contains P. If (P, l) and $(P, k) \in I$ we will say that the lines l and k intersect in the point P. With this convention we make the following definitions.

For the notion of a *partial plane*, *projective plane*, and *affine plane*, the reader is referred to [1].

If π_1 and π_2 are partial planes we say that π_1 is explicitly contained in π_2 and write $\pi_1 < \pi_2$ if and only if the following conditions are satisfied.

(i) The points and lines of π_1 are contained in π_2.

(ii) If the points P, Q and the line l are in π_1. and if P and Q belong to l in π_2 they belong to l in π_1.

(iii) If the lines l, k and the point P are in π_1 and the lines l and k intersect in P in π_2 they intersect in P in π_1.

2. Proof of the Theorem

Let $P_1, P_2, \ldots, P_t$ be a collection of mutually orthogonal $n \times n$ partial latin squares. We define a partial plane π_0 in which there are points P_{ij} $(i, j = 1, 2, \ldots, n)$ and lines l_{ij} $(i = 1, \ldots, t;\ j = 1, \ldots, n)$, where the point P_{rs} belongs to the line l_{ij} if and only if in P_i the cell (r, s) is occupied by j. We now successively define partial planes π_1, π_2, and π_3 so that $\pi_0 < \pi_1 < \pi_2 < \pi_3$ as follows.

The points of π_1 are the points of π_0 and the lines are those of π_0 along with the following lines. For each set of points $\{P_{i1}, P_{i2}, \ldots, P_{in}\}$ $(i = 1, 2, \ldots, n)$ we define a line h_i containing exactly these points. For each set of points $\{p_{1i}, P_{2i}, \ldots, P_{ni}\}$ $(i = 1, 2, \ldots, n)$ we define a line v_i containing exactly these points. For every pair of points not already belonging to one of the above lines we define a line containing exactly these two points.

The lines of π_2 are those in π_1 and the points are those in π_1 along with the following points. For the set of lines $\{h_1, \ldots, h_n\}$ define a point H belonging to exactly these lines. For the set of lines $\{v_1, \ldots, v_n\}$ define a point V belonging to exactly these lines. For each set of lines $\{l_{i1}, l_{i2}, \ldots, l_{in}\}$ $(i = 1, 2, \ldots, t)$ define a point L_i belonging to exactly these lines. For each pair of lines not intersecting in one of the above points define a point belonging to exactly these two lines.

The points of π_3 are those in π_2 and the lines of π_3 are the lines of π_2 along with the following lines. For the set of points $\{H, V, L_1, L_2, \ldots, L_t\}$ define a line p_∞ containing exactly these points. For every pair of points not contained in one of the above lines define a line containing exactly these two points.

From the definition of π_0, π_1, π_2, and π_3 it follows that $\pi_0 < \pi_1 < \pi_2 < \pi_3$. In [1] M. HALL has shown that if π is a partial plane there is a projective plane π' such that $\pi < \pi'$. In case π is finite, Hall's theorem leads to a countably infinite containing plane.

Let π be a countably infinite projective plane such that $\pi_3 < \pi$. Then $\pi_0 < \pi$. We now remove from π the line p_∞ along with the points belonging to this line to obtain an affine plane π^*. Among the points removed from π are the points $H, V, L_1, L_2, \ldots, L_t$ so that in π^* the lines $h_1, \ldots, h_n$; $v_1, \ldots, v_n$; and $l_{i1}, \ldots, l_{in}$ $(i=1, \ldots, t)$ are parallel. Let $\mathscr{H}$ denote the pencil of lines in π^* containing the h's, $\mathscr{V}$ the pencil containing the v's, and $\mathscr{P}_i$ $(i=1, 2, \ldots)$, the other pencils with the requirement that the lines $l_{i1}, \ldots, l_{in}$ belong to $\mathscr{P}_i$. Label the lines in each pencil with the positive integers with the additional proviso that in $\mathscr{H}$ the line h_i is labeled i, in $\mathscr{V}$ the line v_i is labeled i, and in $\mathscr{P}_K$, $k=1, 2, \ldots, t$ the line labeled l_{ki} is labeled i. Now construct a collection of infinite latin squares $C_1, C_2, \ldots, C_i, \ldots$ as follows. In C_k the cell (i, j) is occupied by x if and only if the line labeled x in $\mathscr{P}_K$ contains the point of intersection of the lines labeled i and j in $\mathscr{H}$ and $\mathscr{V}$ respectively. It is routine matter to check that the collection $C_1, C_2, \ldots$ obtained in this manner is in fact a complete set of mutually orthogonal infinite latin squares and P_i is embedded in the upper left-hand corner of C_i $(i=1, 2, \ldots, t)$.

This completes the proof of the theorem.

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On an extremum problem for polynomials

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Recently, P. TURÁN [8] treated the problem to determine lower bounds of the expression

$$M_n(p) = \inf_{\varrho \in P_{n-1}} \sup_{x \in [-1,+1]} |p(x)[x^n + \varrho(x)]|$$

for fixed but arbitrary values of the natural number n, where P_{n-1} is the set of polynomials of degree $n-1$ at most, and $p(x)$ is a given polynomial. In the present paper we consider the problem for arbitrary bounded functions $p(x) \geqq 0$; our estimates are sharper than those of TURÁN [8] and cover some of ELBERT's results [4], [5], too.

Theorem I. *For an arbitrary bounded*[1]) *function* $p(x) \geqq 0$

(1) $$2^n M_n(p) \geqq G(p^*) \qquad (n = 1, 2, \ldots)$$

and

(2) $$\overline{\lim_{n \to \infty}}\, 2^n M_n(p) \leqq 2G(p^*),$$

where p^* *is the upper limit function of* p *and*[2, 3])

(3) $$G(p^*) = \exp\left\{\frac{1}{\pi} \int_0^\pi \log p^*(\cos\theta)\, d\theta\right\}.$$

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[1]) If $p(x)$ is unbounded but $M_n(p)$ is finite for $n \geqq m$, then there exists a nonnegative polynomial of minimal degree $\pi_0(x) = x^m + \varrho(x)$ $(\varrho \in P_{m-1})$ for which $\pi_0 p$ is bounded. Clearly $M_n(p) = M_{n-m}(\pi_0 p)$ and we have[3]) $G(\pi_0 p^*) = G(\pi_0) G(p^*) = 2^{-m} G(p^*)$, so that (1) and (2) are valid even if $p(x)$ is unbounded.

[2]) The integral in (3) is defined, because p^* is bounded, positive and (as an upper limit function) semicontinuous from above, but it may take the value $-\infty$; in this case we set $G(p^*) = 0$.

[3]) If $p(x) = p^*(x) = |x - b_1|^{\beta_1} |x - b_2|^{\beta_2} \ldots |x - b_k|^{\beta_k}$, where $b_1, b_2, \ldots, b_k$ are arbitrary complex numbers, $\beta_1, \ldots, \beta_k$ are real numbers and $\beta_i \geqq 0$ if $b_i \in [-1, +1]$, then we have

$$G(p^*) = 2^{-k} \prod_{j=1}^{k} |b_j + \sqrt{b_j^2 - 1}|$$

(see BERNSTEIN [1]); this is the case treated by TURÁN [8] and ELBERT [4], [5].

Proof of (1). We have[4] $M_n(p)=M_n(p^*)$. If $\log p^*(\cos\theta)\notin L$, (1) is satisfied in a trivial way, for its right hand side is zero. So we may assume $\log p^*(\cos\theta)\in L$.

For an arbitrary but fixed $\varepsilon>0$ we take a $\psi_n(x)=x^n+\cdots\in P_n$ for which

(4) $$p^*(x)|\psi_n(x)| \leqq M_n(p^*)+\varepsilon = M_n(p)+\varepsilon.$$

By a well-known theorem of G. Szegő [7], the function

$$\varphi(z) = \exp\left\{\frac{1}{\pi}\int\limits_0^\pi \frac{e^{i\theta}+z}{e^{i\theta}-z}\log p^*(\cos\theta)\,d\theta\right\} \qquad (|z|\leqq 1)$$

belongs to H^1 and satisfies $|\varphi(e^{i\theta})|=p^*(\cos\theta)$ a.e. Applying (4) to $x=\cos\theta= =\frac{1}{2}(e^{i\theta}+e^{-i\theta})$ we find that

$$F(z) = 2^n z^n \psi_n\left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]\varphi(z)\in H^1$$

has for a.e. boundary values not exceeding $2^n[M_n(p^*)+\varepsilon]$ in modulus. As a consequence of $f\in H^1$ the maximum principle is applicable; so we obtain

$$G(p^*) = \varphi(0) = F(0) \leqq \operatorname{vrai\,max}_\theta |F(e^{i\theta})| \leqq 2^n[M_n(p^*)+\varepsilon] = 2^n[M_n(p)+\varepsilon]$$

and for $\varepsilon\to 0$ we get (1). Q.e.d.

Proof of (2). Since p^*, as an upper limit function, is bounded and semicontinuous from above, there exists a decreasing sequence $\{p_s(x)\}$ of nonvanishing continuous functions such that

$$\lim_{s\to\infty} p_s(x)=p^*(x) \qquad (x\in[-1,+1]).$$

Since $p^*(x)\leqq p_s(x)$, we have $M_n(p)=M_n(p^*)\leqq M_n(p_s)$, so that by a theorem of Bernstein [2]

$$\overline{\lim_{n\to\infty}}\, 2^n M_n(p) \leqq \lim_{n\to\infty} 2^n M_n(p_s) = 2G(p_s).$$

Now, if $\log p^*(\cos\theta)\in L$, we obtain (2) from (3) by an application of Lebesgue's theorem on bounded convergence, taking $s\to\infty$. If $\log p^*(\cos\theta)\notin L$, we get from (3) by an indirect application of Fatou's lemma $\lim_{s\to\infty} G(p_s)=0$; this completes the proof of (2).

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[4]) Proof: For an arbitrary $\varepsilon>0$ there exists a $\varrho\in P_{n-1}$ such that $\sup_{x\in[-1,1]} p(x)|x^n+\varrho(x)| \leqq \leqq M_n(p)+\varepsilon$; we conclude that for every sequence $x_k\to x$ ($x_k\in[-1,+1]$) we have

$$\overline{\lim_{k\to\infty}}\,\{p(x_k)\,|x_k^n+\varphi(x_k)|\} \leqq M_n(p)+\varepsilon,$$

i.e. by continuity of $x^n+\varrho(x)$, $p^*(x)\,|x^n+\varrho(x)| \leqq M_n(p)+\varepsilon$ so that $M_n(p^*)\leqq M_n(p)+\varepsilon$. In turn, from $p\leqq p^*$ it follows $M_n(p)\leqq M_n(p^*)$, and these two results imply $M_n(p)=M_n(p^*)$.

Theorem II. *For an arbitrary function $p(x)\geqq 0$ and an arbitrary pair of natural numbers $n<r$,*

(5) $$2^n M_n(p) \geqq \frac{1}{2} 2^r M_r(p).$$

Conversely, for an arbitrary $\delta>0$ and arbitrary natural number n there exists a continuous function $s(x)=s(n,\delta;x)>0$ such that

(6) $$2^n M_n(s) < \frac{1+\delta}{2} 2^r M_r(s) = (1+\delta)G(s) \qquad (r=n+1, n+2, \ldots).$$

Proof of (6). The Chebyshev polynomial T_{r-n} satisfies $|T_{r-n}(x)|\leqq 1$ ($x\in[-1,+1]$) and has the leading coefficient 2^{r-n-1}. So we have

$$M_r(p) \leqq \inf_{\varrho\in P_{n-1}} \sup_{x\in[-1,+1]} |p(x)2^{-r+n+1}T_{r-n}(x)[x^n+\varrho(x)]| = \\ = 2^{-r+n+1} M_n(p|T_{r-n}|) \leqq 2^{-r+n+1} M_n(p),$$

and multiplying by 2^{r-1} we get the desired inequality (4).

Proof of (6). Let $a>1$ and $s_a(x) = \left(1-\frac{x}{a}\right)^{-2n}$. By a result of BERNSTEIN ([3], pp. 11—14) we have $2^r M_r(s_a) = 2G(s_a)$ $(r=n+1, n+2, \ldots)$ and

$$2^n M_n(s_a) = \frac{2}{1+\left(a-\sqrt{a^2-1}\right)^{2n}} G(s_a).$$

To prove (6) we need only to observe that

$$\lim_{a\to 1+0} \frac{2}{1+\left(a-\sqrt{a^2-1}\right)^{2n}} = 1$$

and take $s=s_a$ for a sufficiently near to 1.

Theorem III. *For an arbitrary natural number n and arbitrary large $A>0$ there exists a continuous function $p_A(x)>0$ for which*

(7) $$2^n M_n(p_A) > A \lim_{r\to\infty} 2^r M_r(p_A).$$

Remark. This result is a consequence of an earlier theorem of ELBERT [5]. In the shorter proof what follows we make use of another idea of ELBERT, which is reproduced here with his permission.

Proof of Theorem III. Let $a=\frac{3}{2\sqrt{2}}>1$, $b=\frac{\sqrt{3}}{2}<1$, and $t(x)=\left(1-\frac{x}{a}\right)^m$, where m is a natural integer to be specified later. By Bernstein's theorem[3])

(8) $$\lim_{r\to\infty} 2^r M_r(t) = 2\left(\frac{a+\sqrt{a^2-1}}{2a}\right)^m = 2^{-m-1}\left(\frac{4}{3}\right)^m.$$

We have further by the transformation $x=b\xi$

$$M_n(t) = \min_{\varrho\in P_{n-1}} \max_{|x|\le 1}\left(1-\frac{x}{a}\right)^m |x^n+\varrho(x)| \geqq \min_{\varrho\in P_{n-1}} \max_{|x|\le b}\left(1-\frac{x}{a}\right)^m |x^n+\varrho(x)| =$$

$$= \min_{\varrho^*\in P_{n-1}} \max_{|\xi|\le 1}\left(1-\frac{b}{a}\xi\right)^m |b^n\xi^n+\varrho^*(\xi)| = b^n M_n(t_b),$$

where

$$t_b(x) = \left(1-\frac{b}{a}x\right)^m.$$

Applying Theorem II and then Bernstein's theorem [3]) we obtain

$$2^n M_n(t) \geqq b^n 2^n M_n(t_b) \geqq \frac{1}{2} b^n \lim_{r\to\infty} 2^r M_r(t_b) =$$

$$= \frac{1}{2} b^n \left(\frac{\frac{a}{b}+\sqrt{\frac{a^2}{b^2}-1}}{2\frac{a}{b}}\right)^m = 2^{-m-1} b^n \left(\frac{3}{2}\right)^m.$$

From (7) and (8) we get

$$\frac{2^n M_n(t)}{\lim\limits_{r\to\infty} 2^r M_r(t)} \geqq \frac{b^n}{4}\left(\frac{9}{8}\right)^m.$$

For a fixed value of n the right hand side exceeds by a suitable choice of m, any large $A>0$. Q.e.d.

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A new proof of the formulas involving the distributions δ^+ and δ^-

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1. Introduction Throughout in this paper, $(\mathcal{O}_{-\alpha})$ will mean for any fixed $\alpha > 0$ the linear space of all (C^∞)-functions φ on $\mathbf{R}$ such that $\varphi^{(p)}(t) = O\left(\frac{1}{|t|^\alpha}\right)$ for $p = 0, 1, \ldots$ (as $|t| \to \infty$). $(\mathcal{O}'_{-\alpha})$ will mean the space of all continuous linear functionals on $(\mathcal{O}_{-\alpha})$. For basic facts concerning the space (O_α) and its dual (O'_α) we refer to [2] and [7]..

The purpose of this note is to give a new proof of the formulas (4) (utilized constantly in quantum mechanics) by a direct and short method, based upon the well known formulas of J. PLEMELJ.

An entirely different technique is described in [2, pp. 60—66], and for other distributional spaces in [1, pp. 155—156], [3, pp. 49—50], [4, pp. 975—976], [5, pp. 426—427], and [9, pp. 85—86].

2. Lemmas. We begin with a lemma on the distribution $\mathrm{Vp}\frac{1}{t}$ and recall a theorem of Plemelj.

First of all let us observe that the linear form $\delta: \varphi \to \varphi(0)$ is continuous on $(\mathcal{O}_{-\alpha})$ since

$$|\langle \delta, \varphi \rangle| \leqq M \max_t \{(1+|t|)^\alpha |\varphi(t)|\}.$$

If φ_n converges in $(\mathcal{O}_{-\alpha})$ to zero as $n \to \infty$, then $\langle \delta, \varphi_n \rangle$ tends to zero. Thus δ is a distribution in $(\mathcal{O}'_{-\alpha})$.

In [2, p. 62] it is proved by means of the distribution δ^+ that $\mathrm{Vp}\frac{1}{t}$ is a distribution in $(\mathcal{O}'_{-\alpha})$. In the following this will be proved directly.

Lemma 1. *The linear form* $\mathrm{Vp}\frac{1}{t}$ *defined by*

$$\left\langle \mathrm{Vp}\frac{1}{t}, \varphi \right\rangle = \mathrm{Vp}\int_{-\infty}^{\infty} \frac{\varphi(t)}{t}\, dt = \lim_{\varepsilon \to 0} \int_{|t| \geqq \varepsilon} \frac{\varphi(t)}{t}\, dt \tag{1}$$

is a distribution in $(\mathcal{O}'_{-\alpha})$.

Proof. For each $\varphi\in(\mathcal{O}_{-\alpha})$ the limit (1) exists. The argument is the same as in the case of the test functions that belong to the space $(\mathcal{D})$. Observe that the integrand is $O\left(\frac{1}{|t|^{\alpha+1}}\right)$ for large $|t|$. On the other hand, for each $\varepsilon>0$ the linear form

(2) $$\varphi \to \int_{|t|\geqq\varepsilon} \frac{\varphi(t)}{t}\, dt = \left\langle \left(\frac{1}{t}\right)_\varepsilon, \varphi \right\rangle$$

is a distribution in $(\mathcal{O}'_{-\alpha})$ defined on $\mathbf{R}$. In fact, we can write

$$\left|\left\langle \left(\frac{1}{t}\right)_\varepsilon, \varphi \right\rangle\right| \leqq 2\int_\varepsilon^\infty \frac{|\varphi(t)|}{|t|}\, dt \leqq \left(2\int_\varepsilon^\infty \frac{dt}{|t|(1+|t|)^\alpha}\right) \max_t \{(1+|t|)^\alpha |\varphi(t)|\}.$$

Now suppose that φ_n converges in $(\mathcal{O}_{-\alpha})$ to zero as $n\to\infty$. Then the sequence of numbers $\left\langle \left(\frac{1}{t}\right)_\varepsilon, \varphi_n \right\rangle$ tends to zero.

By the theorem on the convergence of distributions in $(\mathcal{O}'_{-\alpha})$ it follows that the limit (1) defines a distribution, that is, $\left(\frac{1}{t}\right)_\varepsilon$ converges to $\mathrm{Vp}\frac{1}{t}$ in $(\mathcal{O}'_{-\alpha})$ as ε tends to zero.

Lemma 2 (J. Plemelj). *Let f be a function on $\mathbf{R}$ to $\mathbf{C}$ satisfying the (Hölder) condition H on every compact subset of $\mathbf{R}$, and with $f(t) = O\left(\frac{1}{|t|^\lambda}\right)$ for large $|t|$ for some $\lambda>0$. If z tends from $D^+=\{z|\operatorname{Im}(z)>0\}$ or from $D^-=\{z|\operatorname{Im}(z)<0\}$ to a point $a\in\mathbf{R}$, then the integral*

$$F(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{t-z}\, \mathrm{dt}$$

converges to the limits

$$F^\pm(a) = \pm\frac{1}{2}f(a) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{t-a}\, dt,$$

respectively, where the singular integral is taken as the Cauchy principal value (with respect to the point a).

3. The Theorem. *If*

(3) $$\langle\delta^\pm, \varphi\rangle = \lim_{\varepsilon\to+0} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\varphi(t)}{t\mp i\varepsilon}\, dt$$

then

(4) $$\delta^\pm = \pm\frac{\delta}{2} + \frac{1}{2\pi i}\mathrm{Vp}\frac{1}{t},$$

in the sense of $(\mathcal{O}'_{-\alpha})$.

Proof. First we prove, *independently* of the relations (4), that the linear forms $\delta^\pm$ are distributions in $(\mathcal{O}'_{-\alpha})$.

Note that for each $\varepsilon > 0$ the integrals in (3) converge because the integrands are $O\left(\frac{1}{|t|^{\alpha+1}}\right)$. Also, for each $\varepsilon > 0$, the linear forms

(5) $$\varphi \to \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\varphi(t)}{t \mp i\varepsilon}\,dt$$

are distributions. In fact, identifying the distributions with the functions

$$t \to \frac{1}{t \mp i\varepsilon}$$

to which they correspond, we have

$$\left|\left\langle\frac{1}{t\pm i\varepsilon},\varphi\right\rangle\right| \leqq \left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dt}{(1+|t|)^\alpha\sqrt{t^2+\varepsilon^2}}\right)\max_t\{(1+|t|)^\alpha|\varphi(t)|\}.$$

The integral being convergent, the rest of the argument is obvious from what has been shown in Lemma 1.

Now let us consider the integral of the Cauchy type

$$\hat{\varphi}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\varphi(t)}{t-z}\,dt, \quad \operatorname{Im}(z) \neq 0.$$

Note that $\hat{\varphi}$ is holomorphic in $D^+ = \{z \mid z = x + i\varepsilon\}$ $(\varepsilon > 0)$ and in $D^- = \{z \mid z = x - i\varepsilon\}$ $(\varepsilon > 0)$. Every function $\varphi \in (\mathcal{O}_{-\alpha})$ is bounded on $\mathbf{R}$ and, being a (C^∞)-function, satisfies with each of its derivatives condition H on every compact subset of $\mathbf{R}$. The range of the distributions (5) coincides with the range of the function $\hat{\varphi}$ for $z = \pm i\varepsilon$, respectively. The limits (3) are equal with the limits of $\hat{\varphi}(z)$ as z approaches to the point $a = 0$ along the imaginary axis from D^+ and D^-, respectively. By Lemma 2 the limits

$$\langle\delta^\pm,\varphi\rangle = \lim_{\varepsilon\to+0}\frac{1}{2\pi i}\left\langle\frac{1}{t\mp i\varepsilon},\varphi\right\rangle = \lim_{\varepsilon\to+0}\hat{\varphi}(\pm i\varepsilon)$$

exist for every $\varphi \in (\mathcal{O}_{-\alpha})$. Applying the theorem on the convergence of distributions, it follows that δ^+ and δ^- are actually distributions in $(\mathcal{O}'_{-\alpha})$.

At the same time we have

$$\langle\delta^\pm,\varphi\rangle = \pm\frac{\varphi(0)}{2} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\varphi(t)}{t}\,dt = \pm\frac{\langle\delta,\varphi\rangle}{2} + \frac{1}{2\pi i}\left\langle \mathrm{Vp}\frac{1}{t},\varphi\right\rangle.$$

This implies the relations (4). The proof is complete.

Remark 1. Let $\delta_{(a)}$ be a distribution defined by $\langle \delta_{(a)}, \varphi \rangle = \varphi(a)$, $a \in \mathbf{R}$ (for $a=0$, $\delta_{(a)} = \delta$). Let $\mathrm{Vp}\,\frac{1}{t-a}$, $\delta^+_{(a)}$, $\delta^-_{(a)}$ be the distributions deduced from (1) and (3) if in place of the terms t, $t-i\varepsilon$, $t+i\varepsilon$ we set $t-a$, $t-a-i\varepsilon$, $t-a+i\varepsilon$, respectively. In this case, the same method gives

$$\delta^{\pm}_{(a)} = \pm \frac{\delta_{(a)}}{2} + \frac{1}{2\pi i}\,\mathrm{Vp}\,\frac{1}{t-a}.$$

Remark 2. Since Plemelj's theorem is valid in a complex Banach space ([6]), it is possible to derive the same formulas if $\delta^{\pm}$ are vector-valued distributions (compare with [8, pp. 659—661]).

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Generalizations of the Hardy—Littlewood inequality

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1. G. H. HARDY (see for instance [3], p. 239) proved the following

Theorem A. *If* $p>1$, $a_n \geqq 0$ $(n=1, 2, \ldots)$ *and* $A_{1n} = \sum_{i=1}^{n} a_i$, *then*

$$\sum_{n=1}^{\infty} n^{-p} A_{1n}^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p \tag{1}$$

unless all a_n *vanish. The constant is best possible.*

This result was generalized by HARDY and LITTLEWOOD [2] as follows:

Theorem B. *Suppose* $p>0$, *c is real (but not necessarily positive), and* Σa_n *is a series of positive terms. Set*

$$A_{1n} = \sum_{k=1}^{n} a_k \quad \text{and} \quad A_{n\infty} = \sum_{k=n}^{\infty} a_k.$$

If $p>1$ *we have*

$$\sum_{n=1}^{\infty} n^{-c} A_{1n}^p \leqq K \sum_{n=1}^{\infty} n^{-c} (na_n)^p \quad \text{with} \quad c>1, \text{*)} \tag{2}$$

$$\sum_{n=1}^{\infty} n^{-c} A_{n\infty}^p \leqq K \sum_{n=1}^{\infty} n^{-c} (na_n)^p \quad \text{with} \quad c<1; \tag{3}$$

and if $p<1$ *we have*

$$\sum_{n=1}^{\infty} n^{-c} A_{1n}^p \geqq K \sum_{n=1}^{\infty} n^{-c} (na_n)^p \quad \text{with} \quad c>1, \tag{4}$$

$$\sum_{n=1}^{\infty} n^{-c} A_{n\infty}^p \geqq K \sum_{n=1}^{\infty} n^{-c} (na_n)^p \quad \text{with} \quad c<1. \tag{5}$$

Theorem A was generalized by HARDY ([4], p. 273—275), and then by G. M. PETERSON and G. S. DAVIES ([7], [8]), in such a way that the arithmetic means of a_n

*) K denotes a positive absolute constant, not necessarily the same at each occurrence.

in (1) are replaced by more general sums. M. Izumi, S. Izumi and G. M. Peterson ([5]) gave further generalizations, notably they proved inequalities of type

$$\sum_{n=1}^{\infty} c_{n,n} f(n) \left\{ \sum_{m=1}^{n} c_{n,m} a_m \right\}^p \leqq K \sum_{n=1}^{\infty} c_{n,n} f(n) a_n^p \tag{6}$$

under certain conditions on the matrix $(c_{m,n})$, the sequence $\{f(n)\}$, and p.

Theorem B was generalized by L. Leindler ([6]), who replaced in (2)—(5) the sequence $\{n^{-c}\}$ by an arbitrary sequence $\{\lambda_n\}$; for instance he proved the inequality

$$\sum_{n=1}^{\infty} \lambda_n A_{1n}^p \leqq p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \left(\sum_{m=n}^{\infty} \lambda_m \right)^p a_n^p \tag{7}$$

with $p \geqq 1$ and $\lambda_n > 0$.

In the present paper we intend to generalize and to combine these results.

2. We use the following definitions:

a) $C \in M_1$ denotes that the matrix $C = (c_{m,v})$ satisfies the conditions:

$$c_{m,v} > 0 \quad (v \leqq m), \qquad c_{m,v} = 0 \quad (v > m) \qquad (m, v = 1, 2, \ldots), \quad \text{and}$$

$$0 < \frac{c_{m,v}}{c_{n,v}} \leqq N_1 \,^{*)} \qquad (0 \leqq v \leqq n \leqq m). \tag{8}$$

b) $C \in M_2$ denotes that $c_{m,v} > 0$ $(v \geqq m)$, $c_{m,v} = 0$ $(v < m)$ $(m, v = 1, 2, \ldots)$, and

$$\frac{c_{m,v}}{c_{n,v}} \geqq N_2 \qquad (0 \leqq n \leqq m \leqq v). \tag{9}$$

c) $C \in M_3$ denotes that $c_{v,m} > 0$ $(v \geqq m)$, and $c_{v,m} = 0$ $(v < m)$ $(v, m = 1, 2, \ldots)$,

$$0 < \frac{c_{v,m}}{c_{v,n}} \leqq N_3 \qquad (v \geqq n \geqq m \geqq 0). \tag{10}$$

d) $C \in M_4$ denotes that $c_{v,m} > 0$ $(v \leqq m)$, and $c_{v,m} = 0$ $(v > m)$ $(v, m = 1, 2, \ldots)$,

$$\frac{c_{v,m}}{c_{v,n}} \geqq N_4 \qquad (0 \leqq v \leqq m \leqq n). \tag{11}$$

3. We prove the following

Theorem. *Let $a_n \geqq 0$ and $\lambda_n > 0$ $(n = 1, 2, \ldots)$ be given, and let $C = (c_{m,k})$ be a triangular matrix.*

(a) *If $C \in M_1$ and $p \geqq 1$, then*

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{m=1}^{n} c_{n,m} a_m \right)^p \leqq N_1^{p(p-1)} p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \left(\sum_{m=n}^{\infty} \lambda_m c_{m,n} \right)^p a_n^p. \tag{12}$$

*) N_i denote positive absolute constants $(i = 1, 2, 3, 4)$.

(b) *If* $C \in M_3$ *and* $p \geqq 1$, *then*

$$\sum_{m=1}^{\infty} \lambda_m \left(\sum_{n=m}^{\infty} c_{n,m} a_n \right)^p \leqq N_3^{p(p-1)} p^p \sum_{m=1}^{\infty} \lambda_m^{1-p} \left(\sum_{n=1}^{m} \lambda_n c_{m,n} \right)^p a_m^p. \tag{13}$$

(c) *If* $C \in M_2$ *and* $0 < p \leqq 1$, *then*

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{\nu=n}^{\infty} c_{n,\nu} a_\nu \right)^p \geqq N_2^{(1-p)p} p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \left(\sum_{k=1}^{n} c_{k,n} \lambda_k \right)^p a_n^p. \tag{14}$$

(d) *If* $C \in M_4$ *and* $0 < p \leqq 1$, *then*

$$\sum_{m=1}^{\infty} \lambda_m \left(\sum_{n=1}^{m} c_{n,m} a_n \right)^p \geqq N_4^{(1-p)p} p^p \sum_{m=1}^{\infty} \lambda_m^{1-p} \left(\sum_{n=m}^{\infty} \lambda_n c_{m,n} \right)^p a_m^p. \tag{15}$$

4. We remark that this theorem implies LEINDLER's theorem [6], further if $\lambda_m = c_{m,m} f_{(m)}^{1-p}$ and we write $c_{m,n} f_{(m)}$ instead of elements of the matrix C, then assertion (a) includes Theorem 3 of [5], and in the case $\lambda_n = f_{(n)}^{-p}$ and $c_{k,n} = f(k) a_{k,n}$, assertion (d) reduces to Theorem 5 of [7].

5. We require the following lemmas:

Lemma 1. ([7], Lemma 1) *If* $p > 1$ *and* $z_n \geqq 0$ $(n = 1, 2, \ldots)$ *then*

$$\left(\sum_{k=1}^{n} z_k \right)^p \leqq p \sum_{k=1}^{n} z_k \left(\sum_{\nu=1}^{k} z_\nu \right)^{p-1}.$$

The proofs of the following lemmas are similar to that of Lemma 1.

Lemma 2. *If* $0 < p < 1$ *and* $z_1 > 0$, $z_n \geqq 0$ $(n = 2, 3, \ldots)$ *then*

$$\left(\sum_{k=1}^{n} z_k \right)^p \geqq p \sum_{k=1}^{n} z_k \left(\sum_{\nu=1}^{k} z_\nu \right)^{p-1}.$$

Lemma 3. *If* $0 < p < 1$ *and* $z_n \geqq 0$ $(n = 1, 2, \ldots)$ *then for every natural number* N, *for which* $z_N > 0$,

$$\left(\sum_{k=n}^{N} z_k \right)^p \geqq p \sum_{k=n}^{N} z_k \left(\sum_{\nu=k}^{N} z_\nu \right)^{p-1}.$$

Lemma 4. *If* $p > 1$ *and* $z_n \geqq 0$ $(n = 1, 2, \ldots)$ *then for every natural number* N

$$\left(\sum_{k=n}^{N} z_k \right)^p \leqq p \sum_{k=n}^{N} z_k \left(\sum_{\nu=k}^{N} z_\nu \right)^{p-1}.$$

6. Proof of Theorem. For $p = 1$ the assertions are obvious; we have only to interchange the order of the summations. Further we may assume that not all a_n vanish. (Otherwise the theorem is evident.)

Proof of inequality (12). By Lemma 1 we obtain for $C=(c_{m,k})\in M_1$

$$\sum_{n=1}^{N}\lambda_n\left(\sum_{m=1}^{n}c_{n,m}a_m\right)^p \leqq p\sum_{n=1}^{N}\lambda_n\sum_{m=1}^{n}c_{n,m}a_m\left(\sum_{k=1}^{m}c_{n,k}a_k\right)^{p-1}\leqq$$

$$\leqq N_1^{p-1}p\sum_{n=1}^{N}\lambda_n\sum_{m=1}^{n}c_{n,m}a_m\left(\sum_{k=1}^{m}c_{m,k}a_k\right)^{p-1}=N_1^{p-1}p\sum_{m=1}^{N}\left(\sum_{k=1}^{m}c_{m,k}a_k\right)^{p-1}a_m\sum_{n=m}^{N}\lambda_n c_{n,m}.$$

Hence, using Hölder's inequality, we have

$$\sum_{n=1}^{N}\lambda_n\left(\sum_{m=1}^{n}c_{n,m}a_m\right)^p\leqq N_1^{p-1}p\left\{\sum_{m=1}^{N}\lambda_m\left(\sum_{k=1}^{m}c_{m,k}a_k\right)^p\right\}^{1/q}\left\{\sum_{m=1}^{N}\lambda_m^{1-p}\left(\sum_{n=m}^{N}\lambda_n c_{n,m}\right)^p a_m^p\right\}^{1/p}$$

which, by a standard computation, gives assertion (a).

Proof of inequality (13). By Lemma 4 we have for $C=(c_{m,k})\in M_3$

$$\sum_{m=1}^{N}\lambda_m\left(\sum_{n=m}^{N}c_{n,m}a_n\right)^p\leqq p\sum_{m=1}^{N}\lambda_m\sum_{n=m}^{N}c_{n,m}a_n\left(\sum_{\nu=n}^{N}c_{\nu,m}a_\nu\right)^{p-1}\leqq$$

$$\leqq N_3^{p-1}\cdot p\sum_{m=1}^{N}\lambda_m\sum_{n=m}^{N}c_{n,m}a_n\left(\sum_{\nu=n}^{N}c_{\nu,n}a_\nu\right)^{p-1}=N_3^{p-1}p\sum_{n=1}^{N}\left(\sum_{\nu=n}^{N}c_{\nu,n}a_\nu\right)^{p-1}a_n\sum_{m=1}^{n}c_{n,m}\lambda_m\leqq$$

$$\leqq N_3^{p-1}p\left\{\sum_{n=1}^{N}\lambda_n\left(\sum_{\nu=n}^{N}c_{\nu,n}a_\nu\right)^p\right\}^{1/q}\left\{\sum_{n=1}^{N}\lambda_n^{1-p}\left(\sum_{m=1}^{n}c_{n,m}\lambda_m\right)^p a_n^p\right\}^{1/p}.$$

This gives assertion (b).

Proof of inequality (14). Using Lemma 3 with an index N for which $a_N>0$ we obtain

$$\sum_{n=1}^{N}\lambda_n\left(\sum_{\nu=n}^{N}c_{n,\nu}a_\nu\right)^p\geqq p\sum_{n=1}^{N}\lambda_n\sum_{\nu=n}^{N}c_{n,\nu}a_\nu\left(\sum_{k=\nu}^{N}c_{n,k}a_k\right)^{p-1}\geqq$$

$$\geqq N_2^{1-p}\cdot p\sum_{n=1}^{N}\lambda_n\sum_{\nu=n}^{N}c_{n,\nu}a_\nu\left(\sum_{k=\nu}^{N}c_{\nu,k}a_k\right)^{p-1}=N_2^{1-p}\cdot p\sum_{\nu=1}^{N}\left(\sum_{k=\nu}^{N}c_{\nu,k}a_k\right)^{p-1}a_\nu\sum_{n=1}^{\nu}\lambda_n c_{n,\nu}.$$

Hence, using Hölder's inequality ([1], p. 19) we have

$$\sum_{n=1}^{N}\lambda_n\left(\sum_{\nu=n}^{N}c_{n,\nu}a_\nu\right)^p\geqq N_2^{1-p}p\left\{\sum_{\nu=1}^{N}\lambda_\nu\left(\sum_{k=\nu}^{N}c_{\nu,k}a_k\right)^p\right\}^{1/q}\left\{\sum_{\nu=1}^{N}\lambda_\nu^{1-p}\left(\sum_{n=1}^{\nu}\lambda_n c_{n,\nu}\right)^p a_\nu^p\right\}^{1/p}.$$

Hence we obtain (14).

Proof of inequality (15). We may assume that $a_1 \neq 0$. Using Lemma 2 we have

$$\sum_{m=1}^{N} \lambda_m \left(\sum_{n=1}^{m} c_{n,m} a_n \right)^p \geqq p \sum_{m=1}^{N} \lambda_m \sum_{n=1}^{m} c_{n,m} a_n \left(\sum_{k=1}^{n} c_{k,m} a_k \right)^{p-1} \geqq$$

$$\geqq N_4^{1-p} \cdot p \sum_{m=1}^{N} \lambda_m \sum_{n=1}^{m} c_{n,m} a_n \left(\sum_{k=1}^{n} c_{k,n} a_k \right)^{p-1} = N_4^{1-p} \cdot p \sum_{n=1}^{N} \left(\sum_{k=1}^{n} c_{k,n} a_k \right)^{p-1} a_n \sum_{m=n}^{N} \lambda_m c_{n,m} \geqq$$

$$\geqq N_4^{1-p} \cdot p \left\{ \sum_{n=1}^{N} \lambda_n \left(\sum_{k=1}^{n} c_{k,n} a_k \right)^p \right\}^{1/q} \left\{ \sum_{n=1}^{N} \lambda_n^{1-p} \left(\sum_{m=n}^{N} \lambda_m c_{n,m} \right)^p a_n^p \right\}^{1/p}.$$

Hence we get the required inequality (15), and we have completed our proof.

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Logarithmic concave measures with application to stochastic programming

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1. Introduction. The problem we are dealing with in the present paper arose in stochastic programming. A wide class of stochastic programming decision rules (see [8], [9]) lead to non-linear optimization problems where concavity or quasi-concavity of some functions is desirable. Let us consider the following special decision problem of this kind for illustration:

Minimize $f(\mathbf{x})$ subject to the constraints:

$$P\{g_1(\mathbf{x}) \geqq \beta_1, \ldots, g_m(\mathbf{x}) \geqq \beta_m\} \geqq p, \quad h_1(\mathbf{x}) \geqq 0, \ldots, h_M(\mathbf{x}) \geqq 0. \tag{1. 1}$$

Here $\beta_1, \ldots, \beta_m$ are random variables, p is a prescribed probability $(0<p<1)$ and $g_1(\mathbf{x}), \ldots, g_m(\mathbf{x}), h_1(\mathbf{x}), \ldots, h_M(\mathbf{x}), -f(\mathbf{x})$ are concave functions [1]) in the entire space R^n, where the vectors $\mathbf{x}$ are taken from. If we want to solve Problem (1. 1) numerically then the first thing is to discover the type of the function of the variable $\mathbf{x} \in R^n$:

$$h(\mathbf{x}) = P\{g_1(\mathbf{x}) \geqq \beta_1, \ldots, g_m(\mathbf{x}) \geqq \beta_m\}. \tag{1. 2}$$

If this is concave or at least quasi-concave then the numerical solution of Problem (1. 1) is hopeful. We are interested in random variables $\beta_1, \ldots, \beta_m$ having a continuous joint probability distribution. Examples show that in the most frequent and practically interesting cases we cannot expect that the function (1. 2) is concave. Surprisingly, however, a special kind of quasi-concavity holds for a wide class of joint probability distributions of the random variables $\beta_1, \ldots, \beta_m$. Notably, we show that under some conditions $\log h(\mathbf{x})$ is a concave function in the entire space R^n. This unexpectedly good behaviour of function (1. 2) and problem (1. 1) will result very likely in a frequent application of this and related models.

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[1]) From the point of view of numerical solution it is enough to suppose that $h_1(\mathbf{x}), \ldots, h_M(\mathbf{x})$ are quasi-concave. A function $h(\mathbf{x})$ defined in a convex set L is quasi-concave if for any $\mathbf{x}_1, \mathbf{x}_2 \in L$ and $0<\lambda<1$ we have $h(\lambda\mathbf{x}_1+(1-\lambda)\mathbf{x}_2) \geqq \min\{h(\mathbf{x}_1), h(\mathbf{x}_2)\}$.

The main theorem in our paper is Theorem 2 which is proved in Section 3. The basic tools for the proof of this theorem are an integral inequality and the Brunn—Minkowski theorem for convex combinations of two convex sets. The integral inequality states that for any measurable non-negative functions f, g we have

$$\int_{-\infty}^{\infty} \sup_{x+y=2t} f(x)g(y)\,dt \geqq \Big(\int_{-\infty}^{\infty} f^2(x)\,dx\Big)^{\frac{1}{2}} \Big(\int_{-\infty}^{\infty} g^2(y)\,dy\Big)^{\frac{1}{2}}. \tag{1.3}$$

This will be proved in Section 2.

Let A and B be two convex sets of the space R^n. The Minkowski combination $A+B$ of A and B, and the multiple λA of A (for a real number λ) are defined by $A+B = \{\mathbf{a}+\mathbf{b} \mid \mathbf{a}\in A, \mathbf{b}\in B\}$ and $\lambda A = \{\lambda \mathbf{a} \mid \mathbf{a}\in A\}$.

Theorem of Brunn. *If A and B are bounded convex sets in R^n and $0<\lambda<1$, then we have*

$$\mu^{\frac{1}{n}}\{\lambda A+(1-\lambda)B\} \geqq \lambda \mu^{\frac{1}{n}}\{A\}+(1-\lambda)\mu^{\frac{1}{n}}\{B\}, \tag{1.4}$$

where μ denotes Lebesgue measure.

This theorem is sharpened by the

Theorem of Brunn—Minkowski. *If the conditions of the theorem of Brunn are fulfilled, moreover both A and B are closed and have positive Lebesgue measures, then equality holds in* (1.4) *if and only if A and B are homothetic.*

Our main theorem contains an inequality similar to that of Brunn. Instead of Lebesgue measure more general measures are involved. Let P be a probability measure [2]) defined on the Borel sets of R^n. We say that the measure P is logarithmic concave if for every convex sets A, B of R^n we have

$$P\{\lambda A+(1-\lambda)B\} \geqq (P\{A\})^{\lambda}(P\{B\})^{1-\lambda} \qquad (0<\lambda<1). \tag{1.5}$$

In section 4 we show that many well-known multivariate probability distributions have this property because they satisfy the conditions of the main theorem.

Inequality (1.5) has an important consequence, namely that the P measure of the parallel shifts of a convex set is a logarithmic concave function of the shift vector. This will be shown in Section 3.

As for the practical applications of the theory presented in this paper the reader is referred to the detailed study [9].

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[2]) We restrict ourselves to finite measures and, having in mind the applications of our theory, we consider probability measures. The finiteness condition, however, can be dropped as it will be clear from the proofs.

2. An integral inequality. In this section we prove the inequality (1. 3). We formulate it now in the form of a theorem.

Theorem 1. *Let f, g be two non-negative Lebesgue measurable functions defined on the real line R^1. Then the function*

$$r(t)=\sup_{x+y=2t} f(x)g(y) \tag{2. 1}$$

is also measurable and we have

$$\int_{-\infty}^{\infty} r(t)\,dt \geqq \Big(\int_{-\infty}^{\infty} f^2(x)\,dx\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^{\infty} g^2(y)\,dy\Big)^{\frac{1}{2}} \tag{2. 2}$$

(where the value $+\infty$ is also allowed for the integrals).

Proof. First we prove the assertion for such functions f, g which are constant on the subintervals

$$\left[0,\frac{1}{n}\right),\quad \left[\frac{1}{n},\frac{2}{n}\right),\ \ldots,\ \left[\frac{n-2}{n},\frac{n-1}{n}\right),\quad \left[\frac{n-1}{n},1\right]$$

of the interval $[0, 1]$ and vanish elsewhere. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ denote the values of f and g on these subintervals, respectively. Then we have

$$\int_0^1 r(t)\,dt=[A_2+\max(A_2,A_3)+\cdots+\max(A_{2n-1},A_{2n})+A_{2n}]\frac{1}{2n},$$

where

$$A_m=\max_{\substack{i+j=m\\ 1\leqq i,\,j\leqq n}} a_ib_j \qquad (m=2,3,\ldots,2n), \tag{2. 3}$$

and

$$\int_0^1 f^2(x)\,dx=\frac{1}{n}\sum_{i=1}^n a_i^2,\qquad \int_0^1 g^2(y)\,dy=\frac{1}{n}\sum_{i=1}^n b_i^2.$$

Thus the inequality to be proved reduces to the inequality

$$\frac{1}{2}[A_2+\max(A_2,A_3)+\cdots+\max(A_{2n-1},A_{2n})+A_{2n}]\geqq\Big(\sum_{i=1}^n a_i^2\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^n b_i^2\Big)^{\frac{1}{2}} \tag{2. 4}$$

for any sequences of non-negative numbers $a_1, \ldots, a_n$; $b_1, \ldots, b_n$.

First we consider the case

$$a_1\geqq a_2\geqq\cdots\geqq a_n,\qquad b_1\geqq b_2\geqq\cdots\geqq b_n. \tag{2. 5}$$

This implies $A_2\geqq A_3\geqq\cdots\geqq A_{2n}$. It is enough to prove (2. 4) for the special case $a_1=b_1=1$. We prove then that

$$2A_2+A_3+\cdots+A_{2n-1}+A_{2n}\geqq\sum_{i=1}^n a_i^2+\sum_{i=1}^n b_i^2 \tag{2. 6}$$

which is stronger than the required inequality because

$$\frac{1}{2}\left(\sum_{i=1}^{n} a_i^2+\sum_{i=1}^{n} b_i^2\right) \geqq \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}.$$

Let us arrange the numbers $a_2, \ldots, a_n, b_2, \ldots, b_n$ according to their order of magnitude. We may suppose that the first number is a_2. If some a's are equal we keep among these the original ordering and the same is done to the b's. If $a_i = b_j$ for some $i > 2$ and $j > 1$ then the ordering between these two numbers is b_j, a_i. Let a_r be the first among $a_3, \ldots, a_n$ which is smaller than or equal to b_2. It is possible, of course, that such an a_r does not exist, i.e. $a_n > b_2$. In this case $a_n b_{m-n} \geqq \geqq b_2 b_{m-n} \geqq b_{m-n}^2$ $(m = n+2, \ldots, 2n)$, thus (2.6) follows then from the relations $A_2 = a_1 b_1 = 1$, $A_m \geqq a_{m-1} b_1 = a_{m-1} \geqq a_{m-1}^2$ $(m = 3, \ldots, n+1)$, $A_m \geqq a_n b_{m-n}$ $(m = n+2, \ldots, 2n)$. In the case that a_r exists the following reasoning applies. We associate with each b_j the nearest a to the left: let $a_{i(j)}$ be this number. Similarly, we associate with each a_p the nearest b to the left: let $b_{q(p)}$ be this number. We have

$$a_{i(j)} b_j \geqq b_j^2 \quad (j=2, \ldots, n), \qquad a_p b_{q(p)} \geqq a_p^2 \quad (p=r, \ldots, n).$$

It is easy to see that for any j and p satisfying $2 \leqq j \leqq n$, $r \leqq p \leqq n$, the relation $i(j)+j \neq \neq p+q(p)$ holds. In fact there is no a_p between $a_{i(j)}$ and b_j. Consequently a_p is either to the right from b_j in which case we have $q(p) \geqq j$, $p > i(j)$ or $p \leqq i(j)$ in which case $q(p) < j$. A second remark is that the numbers $i(j)+j$ $(j=2, \ldots, n)$ are different from each other and the same holds for the numbers $p+q(p)$ $(p=r, \ldots, n)$. From these we conclude that

$$A_3 + A_4 + \cdots + A_{2n} \geqq A_3^2 + \cdots + A_r^2 + A_{r+1} + \cdots + A_{2n} \geqq$$

$$\geqq a_2^2 + \cdots + a_{r-1}^2 + \sum_{p=r}^{n} a_p b_{q(p)} + \sum_{j=2}^{n} a_{i(j)} b_j \geqq a_2^2 + \cdots + a_{r-1}^2 + \sum_{p=r}^{n} a_p^2 + \sum_{j=2}^{n} b_j^2.$$

This proves (2.6) because $A_2 = a_1 b_1 = 1$.

Now we prove that if we perform independent permutations on the numbers (2.5) then the left hand side of (2.4) becomes the smallest at the original non-increasing ordering. Let us consider the following scheme (illustrated in the case $n=3$):

$$
\begin{array}{ccccccl}
 & & a_1 b_1 & & & & A_2 \\
 & a_1 b_2 & & a_2 b_1 & & & A_3 \\
a_1 b_3 & & a_2 b_2 & & a_3 b_1 & & A_4 \\
 & a_2 b_3 & & a_3 b_2 & & & A_5 \\
 & & a_3 b_3 & & & & A_6
\end{array}
\tag{2.7}
$$

with the row maxima at the right hand side. If in the sequence $a_1, \ldots, a_n$ we interchange a_i and a_j then this means in the scheme (2. 7) that the ith and jth northeast-southwest rows are interchanged. The situation is similar if we interchange b_i and b_j in the sequence $b_1, \ldots, b_n$. Under such transformations the horizontal rows interchange some elements. The following assertion is true, however. The kth largest horizontal row maximum in the original scheme is not larger then the kth largest horizontal row maximum of another scheme obtained from the original by some (independent) permutations of the skew rows. In other terms, if $B_2, \ldots, B_{2n}$ are the horizontal row maxima of the transformed scheme and $B_2^*, \ldots, B_{2n}^*$ denote the same numbers but arranged according to their magnitude, i.e. $B_2^* \geqq B_3^* \geqq \cdots \geqq B_{2n}^*$, then

(2. 8) $$A_i \leqq B_i^* \qquad (i=2, \ldots, 2n).$$

In (2. 8) we already took into account that $A_2 \geqq A_3 \geqq \cdots \geqq A_{2n}$. To prove this statement, suppose that the kth largest horizontal row maximum in the original scheme is realized by the element $a_p b_q$. Then in the rectangle

(2. 9) $$\begin{array}{ll} a_1 b_1 & a_2 b_1 \ldots a_p b_1 \\ a_1 b_2 & a_2 b_2 \ldots a_p b_2 \\ \multicolumn{2}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \\ a_1 b_q & a_2 b_q \ldots a_p b_q \end{array}$$

which stands skew in the scheme, all numbers are greater than or equal to $a_p b_q$. We remark that $k = p+q-1$. Now it is easy to see that under any permutations of the skew rows of the original scheme, the numbers (2. 9) cannot be condensed into less than $k = p+q-1$ rows. This means

$$B_{k+1}^* \geqq A_{k+1} (=a_p b_q) \qquad (k = 1, \ldots, 2n-1),$$

which are the required inequalities.

We arrived at the final step of the proof of the inequality (2. 4). From relation (2. 8) we conclude

$$A_2 + \sum_{i=2}^{2n} A_i \leqq B_2^* + \sum_{i=2}^{2n} B_i^* = B_2^* + \sum_{i=2}^{2n} B_i.$$

On the other hand we have for an arbitrary sequence of numbers $B_2, \ldots, B_{2n}$,

$$B_2^* + B_2 + \cdots + B_{2n} = \leqq B_2 + \max(B_2, B_3) + \cdots + \max(B_{2n-1}, B_{2n}) + B_{2n},$$

where B_2^* is the largest among $B_2, \ldots, B_{2n}$. Hence it follows for our non-negative numbers

$$\frac{1}{4}\left[A_2 + A_{2n} + \sum_{i=2}^{2n-1} \max(A_i, A_{i+1})\right]^2 = \frac{1}{4}\left[A_2 + \sum_{i=2}^{2n} A_i\right]^2 \leqq$$
$$\leqq \frac{1}{4}\left[B_2 + B_{2n} + \sum_{i=2}^{2n-1} \max(B_i, B_{i+1})\right]^2.$$

This means that the left hand side of (2. 4) is the smallest at the original permutations of the sequences $a_1, \ldots, a_n$; $b_1, \ldots, b_n$.

If f, g are continuous functions in some closed intervals and are equal to 0 elsewhere then these can be uniformly approximated by such functions for which we already proved the integral inequality (2. 2). Thus (2. 2) holds for these functions f, g too.

If f and g are continuous on the entire real line then first we define

$$f_T(x)=f(x) \quad \text{if} \quad |x| \leqq T, \quad \text{and} \quad f_T(x)=0 \quad \text{otherwise,}$$

$$g_T(y)=g(y) \quad \text{if} \quad |y| \leqq T, \quad \text{and} \quad g_T(y)=0 \quad \text{otherwise.}$$

It follows that

$$r(t) = \sup_{x+y=2t} f(x)g(y) \geqq \max_{x+y=2t} f_T(x)g_T(y) = r_T(t).$$

So we have

$$\int_{-\infty}^{\infty} r(t)\,dt \geqq \int_{-\infty}^{\infty} r_T(t)\,dt \geqq \left(\int_{-\infty}^{\infty} f_T^2(x)\,dx\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} g_T^2(y)\,dy\right)^{\frac{1}{2}},$$

and hence we infer that (2. 2) also holds.

Let us now prove the theorem for arbitrary non-negative Lebesgue measurable functions. It is enough to consider functions which are bounded and equal to zero outside the interval [0, 1]. We may also suppose that both f and g have a finite number of different values. In fact every measurable bounded function can be uniformly approximated by such functions with arbitrary precision.

The measurability of $r(t) = \sup_{x+y=2t} f(x)g(y)$ will be proved as follows. The space R^2 can be subdivided into a finite number of disjoint rectangular Lebesgue measurable sets $E_1, \ldots, E_N$ each of which has the property that the function of two variables $f(x)g(y)$ is constant on it. The sets

$$H_i = \{t \mid 2t = x+y,\ (x, y) \in E_i\} \qquad (i=1, \ldots, N)$$

are clearly measurable. If $E_1, \ldots, E_N$ are arranged so that the values of $f(x)g(y)$ follow each other according to the order of magnitude where the largest value is the first, then $r(t)$ is constant on the sets

$$H_i \setminus \bigcup_{j=i+1}^{N} H_j \quad (i = 1, \ldots, N-1), \quad \text{and} \quad H_N,$$

which proves the measurability of $r(t)$.

Let F be the class of functions defined on [0, 1] consisting of all non-negative step functions and all functions which can be obtained in the following way: take any non-negative step function $h(x)$, any sequence of intervals $I_1, I_2, \dots$ with finite sum of lengths and define

$$(2.10)\qquad k(x) = 0 \quad \text{if} \quad x \in \bigcup_{k=1}^{\infty} I_k, \quad \text{and} \quad k(x) = h(x) \text{ otherwise.}$$

This class of functions has the property that for any pair f, g in F, inequality (2. 2) holds. This statement is trivial for step functions. If f and g are in F and one of them or both are not step functions then

$$f(x) = \lim_{i\to\infty} f_i(x), \quad g(y) = \lim_{i\to\infty} g_i(y),$$

where f_i, g_i are defined so that on the right hand side of (2. 10) we put $h=f$ resp. $h=g$ and write $\bigcup_{k=1}^{i} I_k$ instead of $\bigcup_{k=1}^{\infty} I_k$. It follows that

$$\sup_{x+y=2t} f(x)g(y) = \max_{x+y=2t} f(x)g(y) = \lim_{i\to\infty} \max_{x+y=2t} f_i(x)g_i(y),$$

whence we conclude

$$\int_0^1 \sup_{x+y=2t} f(x)g(y)\,dt = \lim_{i\to\infty} \int_0^1 \max_{x+y=2t} f_i(x)g_i(y)\,dt \geqq$$

$$\geqq \lim_{i\to\infty} \left(\int_0^1 f_i^2(x)\,dx\right)^{\frac{1}{2}} \left(\int_0^1 g_i^2(y)\,dy\right)^{\frac{1}{2}} = \left(\int_0^1 f^2(x)\,dx\right)^{\frac{1}{2}} \left(\int_0^1 g^2(y)\,dy\right)^{\frac{1}{2}}.$$

As the next and final step in the proof we show that every Lebesgue measurable and finitely valued function defined in [0, 1] is the limit in measure of a sequence of functions $f_i \in F$ $(i=1, 2, \dots)$, where

$$(2.11)\qquad f_i(x) \leqq f(x) \qquad (0 \leqq x \leqq 1;\ i=1, 2, \dots).$$

To prove this we denote by $d_1, \dots, d_n$ $(d_1 < \dots < d_n)$ the values of the function f and by $D_1, \dots, D_n$ those measurable sets where f takes on these values. Let us cover $\bar{D}_j = [0, 1] \setminus D_j$ by a sequence of intervals

$$I_{i1}^{(j)}, \quad I_{i2}^{(j)}, \dots \qquad (i=1, 2, \dots;\ j=1, \dots, n),$$

where the sum of the lengths of these intervals tends to the Lebesgue measure of $\bar{D}_j$ as $i\to\infty$. Let us define f_i in the following manner

$$(2.12)\quad f_i(x) = d_j \quad \text{if} \quad x \notin \bigcup_{k=1}^{\infty} I_{ik}^{(j)} \qquad (j=1, \dots, n) \quad \text{and} \quad f_i(x) = 0 \text{ otherwise.}$$

For every $i=1, 2, \dots$ we have $f_i \in F$, (2. 11) is fulfilled, and the sequence (2. 12) converges to f in measure.

If the sequence g_i $(i=1, 2, \ldots)$ is defined in a similar way in connection with g then we conclude

$$\int_0^1 \sup_{x+y=2t} f(x)g(y)\,dt \geqq \int_0^1 \sup_{x+y=2t} f_i(x)g_i(y)\,dt \geqq$$

$$\geqq \Big(\int_0^1 f_i^2(x)\,dx\Big)^{\frac{1}{2}} \Big(\int_0^1 g_i^2(y)\,dy\Big)^{\frac{1}{2}} \to \Big(\int_0^1 f^2(x)\,dx\Big)^{\frac{1}{2}} \Big(\int_0^1 g^2(y)\,dy\Big)^{\frac{1}{2}}.$$

This completes the proof of Theorem 1.

3. The main theorems. The main result of this paper is the following

Theorem 2. *Let $Q(\mathbf{x})$ be a convex function defined on the entire n-dimensional space R^n. Suppose that $Q(\mathbf{x}) \geqq a$, where a is some real number. Let $\psi(z)$ be a function defined on the infinite interval $[a, \infty)$. Suppose that $\psi(z)$ is non-negative, non-increasing, differentiable, and $-\psi'(z)$ is logarithmic concave*[3]*. Consider the function $f(\mathbf{x}) = \psi(Q(\mathbf{x}))$ $(\mathbf{x} \in R^n)$ and suppose that it is a probability density*[4]*, i.e.*

$$\int_{R^n} f(\mathbf{x})\,d\mathbf{x} = 1. \tag{3.1}$$

Denote by $P\{C\}$ the integral of $f(\mathbf{x})$ over the measurable subset C of R^n. If A and B are any two convex sets in R^n, then the following inequality holds:

$$P\{\lambda A + (1-\lambda)B\} \geqq (P\{A\})^{\lambda}(P\{B\})^{1-\lambda} \qquad (0<\lambda<1). \tag{3.2}$$

Remark 1. Condition (3. 1) implies that $\psi(z) \to 0$ as $z \to \infty$. Otherwise $f(\mathbf{x})$ would have a positive lower bound contradicting the finiteness of the integral (3. 1).

Remark 2. We supposed that $Q(\mathbf{x})$ is bounded from below. Dropping this assumption and allowing z to vary on the entire real line, where we suppose that $\psi(z)$ satisfies the same conditions as before, we can deduce from the other assumptions of Theorem 2 that $Q(\mathbf{x})$ is bounded from below.

For if $Q(\mathbf{x})$ were unbounded from below then for every real number b the set

$$\{\mathbf{x} \mid Q(\mathbf{x}) \leqq b\} \tag{3.3}$$

would be unbounded and convex. Consequently the Lebesgue measure of (3. 3) would equal infinity. Now the function $\psi(z)$ cannot vanish everywhere because of

[3]) A function $h(\mathbf{x})$ defined on a convex set K is said to be logarithmic concave if for any $\mathbf{x}, \mathbf{y} \in K$ and $0<\lambda<1$ we have $h(\lambda\mathbf{x}+(1-\lambda)\mathbf{y}) \geqq [h(\mathbf{x})]^{\lambda}[h(\mathbf{y})]^{1-\lambda}$.

[4]) It would be enough to suppose that the integral of $f(\mathbf{x})$ is finite on the entire space R^n.

(3. 1). Thus if $Q(\mathbf{x})$ is unbounded from below then $f(\mathbf{x})$ is greater than or equal to a positive number on a set of infinite Lebesgue measure. This contradicts (3. 1).

Remark 3. We may allow $Q(\mathbf{x})$ to take on the value ∞. In this case we require that $\psi(\infty)=0$.

Proof of Theorem 2. Consider the one parameter family of sets

$$E(v)=\{\mathbf{x} \mid f(\mathbf{x})\geqq v\}=\{\mathbf{x} \mid Q(\mathbf{x})\leqq \psi^{-1}(v)\} \qquad (v>0), \tag{3. 4}$$

and the corresponding Lebesgue measures $F(v)=\mu\{E(v)\}$ $(v>0)$. As the integral of $f(\mathbf{x})$ is finite over the entire space R^n it follows that the measures $F(v)$ are finite for every v. Furthermore all non-empty sets $E(v)$ $(v>0)$ are convex, thus they must be bounded as well. Finally, the sets (3. 4) are closed because $Q(\mathbf{x})$ is continuous. The integral of $f(\mathbf{x})$ on R^n can be expressed in the form

$$\int_{R^n} f(\mathbf{x})\,d\mathbf{x} = -\int_0^\infty v\,dF(v) = \int_0^\infty F(v)\,dv, \tag{3. 5}$$

where we have used partial integration and the following formulas

$$F(v)=0\big(v>\psi(a)\big), \quad \lim_{v\to 0} vF(v)=0.$$

The first relation is trivial, the proof of the second relation is given below. For any $\varepsilon>0$ we have

$$-\int_0^\infty v\,dF(v) \geqq -\int_\varepsilon^\infty v\,dF(v) = \varepsilon F(\varepsilon)+\int_\varepsilon^\infty F(v)\,dv \geqq \int_\varepsilon^\infty F(v)\,dv.$$

Thus the integral on the right hand side of (3. 5) is finite. Taking this into account we see from the line above that $\lim_{\varepsilon\to 0}\varepsilon F(\varepsilon)$ exists. This limit cannot be positive as $\int_0^\varepsilon F(v)\,dv$ is finite.

Let us introduce the notations

$$K(v)=\{\mathbf{x} \mid Q(\mathbf{x})\leqq v\}, \quad L(v)=\mu\{K(v)\} \qquad (-\infty<v<\infty),$$

where μ is again the symbol of Lebesgue measure. Then, for every $v>0$, $E(v)=K(\psi^{-1}(v))$ and $F(v)=L(\psi^{-1}(v))$. Using this notation we can rewrite (3. 5) in the form

$$\int_{R^n} f(\mathbf{x})\,d\mathbf{x} = \int_0^\infty F(v)\,dv = \int_0^{\psi(a)} L\big(\psi^{-1}(v)\big)\,dv.$$

Applying the transformation $z=\psi^{-1}(v)$ and observing that $\psi^{-1}(0)=\infty$, we obtain that

$$\int_{R^n} f(\mathbf{x})\,dx = \int_a^\infty L(z)[-\psi'(z)]\,dz.$$

The above reasoning can be applied for an arbitrary measurable subset C of R^n with the difference that instead of $E(v)$, $K(v)$ we have to work with the intersections $E(v)\cap C$ and $K(v)\cap C$. Introducing the notation $L(C, v) = \mu\{K(v)\cap C\}$, we can write

$$\int_C f(\mathbf{x})\,d\mathbf{x} = \int_a^\infty L(C, z)[-\psi'(z)]dz. \tag{3.6}$$

By the convexity of the function $Q(\mathbf{x})$ we have for any $v_1 \geqq a$, $v_2 \geqq a$ and $0<\lambda<1$,

$$K(\lambda v_1 + (1-\lambda)v_2) \supset \lambda K(v_1) + (1-\lambda)K(v_2). \tag{3.7}$$

Let A and B be any convex sets in R^n. Considering the Minkowski sum $\lambda A+(1-\lambda)B$ with the same λ as in (3. 7), it is easy to see that

$$K(\lambda v_1 + (1-\lambda)v_2)\cap[\lambda A+(1-\lambda)B] \supset \lambda[K(v_1)\cap A] + (1-\lambda)[K(v_2)\cap B].$$

By the Theorem of Brunn,

$$[L(\lambda A+(1-\lambda)B,\ \lambda v_1+(1-\lambda)v_2]^{\frac{1}{n}} \geqq \lambda[L(A, v_1)]^{\frac{1}{n}} + (1-\lambda)[L(B, v_2)]^{\frac{1}{n}}. \tag{3.8}$$

We shall use the following consequence of (3. 8):

$$L(\lambda A+(1-\lambda)B,\ \lambda v_1+(1-\lambda)v_2) \geqq [L(A, v_1)]^{\lambda}[L(B, v_2]^{1-\lambda}. \tag{3.9}$$

The function $-\psi'(z)$ is logarithmic concave in the interval $z\geqq a$; hence for any $v_1\geqq a$, $v_2 \geqq a$ we have

$$-\psi'(\tfrac{1}{2}(v_1+v_2)) \geqq [-\psi'(v_1)]^{\frac{1}{2}}\,[-\psi'(v_2)]^{\frac{1}{2}}. \tag{3.10}$$

Putting $\lambda = \frac{1}{2}$ in (3. 9) and multiplying the inequalities (3. 9), (3. 10) we obtain

$$L(\tfrac{1}{2}A+\tfrac{1}{2}B, \tfrac{1}{2}v_1+\tfrac{1}{2}v_2)[-\psi'(\tfrac{1}{2}v_1+\tfrac{1}{2}v_2)] \geqq$$
$$\geqq \{L(A, v_1)[-\psi'(v_1)]\}^{\frac{1}{2}}\{L(B, v_2)[-\psi'(v_2)]\}^{\frac{1}{2}}.$$

It follows from this that

$$L(\tfrac{1}{2}A+\tfrac{1}{2}B, z)[-\psi'(z)] \geqq \sup_{v_1+v_2=2z} \{L(A, v_1)[-\psi(v_1)]\}^{\frac{1}{2}}\{L(B, v_2)[-\psi'(v_2)]\}^{\frac{1}{2}}. \tag{3.11}$$

Now we apply Theorem I for the functions on the right hand side of (3. 11). First taking into account (3. 11) we conclude the following result

$$\int_a^\infty L(\tfrac{1}{2}A+\tfrac{1}{2}B, z)[-\psi'(z)]\,dz \geqq$$

$$\geqq \int_a^\infty \sup_{v_1+v_2=2z} \{L(A, v_1)[-\psi'(v_1)]\}^{\frac{1}{2}} \{L(B, v_2)[-\psi'(v_2)]\}^{\frac{1}{2}}\,dz \geqq$$

$$\geqq \left\{\int_a^\infty L(A, v_1)[-\psi'(v_1)]\,dv_1\right\}^{\frac{1}{2}} \left\{\int_a^\infty L(B, v_2)[-\psi'(v_2)]\,dv_2\right\}^{\frac{1}{2}}.$$

In view of (3. 6) this means

$$P\{\tfrac{1}{2}A+\tfrac{1}{2}B\} = \int_{\frac{1}{2}A+\frac{1}{2}B} f(\mathbf{x})\,d\mathbf{x} \geqq \Big[\int_A f(\mathbf{x})\,d\mathbf{x}\Big]^{\frac{1}{2}} \Big[\int_B f(\mathbf{x})\,d\mathbf{x}\Big]^{\frac{1}{2}} = [P\{A\}]^{\frac{1}{2}}[P\{B\}]^{\frac{1}{2}}. \tag{3.12}$$

Thus inequality (3. 2) is proved for $\lambda = \frac{1}{2}$.

The assertion for the case of an arbitrary λ can be deduced from here by a continuity argument. First we remark that if A_1, A_2, A_3, A_4 are arbitrary convex sets in R^n then (3. 12) implies

$$P\{\tfrac{1}{4}A_1+\tfrac{1}{4}A_2+\tfrac{1}{4}A_3+\tfrac{1}{4}A_4\} = P\{\tfrac{1}{2}(\tfrac{1}{2}A_1+\tfrac{1}{2}A_2)+\tfrac{1}{2}(\tfrac{1}{2}A_3+\tfrac{1}{2}A_4)\} \geqq$$

$$\geqq [P\{\tfrac{1}{2}A_1+\tfrac{1}{2}A_2\}]^{\frac{1}{2}} [P\{\tfrac{1}{2}A_3+\tfrac{1}{2}A_4\}]^{\frac{1}{2}} \geqq [P\{A_1\}]^{\frac{1}{4}} [P\{A_2\}]^{\frac{1}{4}} [P\{A_3\}]^{\frac{1}{4}} [P\{A_4\}]^{\frac{1}{4}}.$$

A similar inequality holds for any convex sets C_i $(i=1, \dots, 2^N)$, where N is a positive integer. Define the sets

$$A_i = A \quad (i=1, \dots, j), \qquad B_i = B \quad (i=1, \dots, k),$$

where we suppose that $j+k$ is a power of 2, furthermore

$$\lim_{j,k\to\infty} \frac{j}{j+k} = \lambda. \tag{3.13}$$

Let $j+k = 2^N$. It follows that

$$P\left\{\frac{A_1+\cdots+A_j+B_1+\cdots+B_k}{2^N}\right\} = P\left\{\frac{j}{2^N}\frac{A_1+\cdots+A_j}{j}+\frac{k}{2^N}\frac{B_1+\cdots+B_k}{k}\right\} =$$

$$= P\left\{\frac{j}{2^N}A+\frac{k}{2^N}B\right\} \tag{3.14}$$

because A and B are convex sets. On the other hand we have

$$P\{2^{-N}(A_1+\cdots+A_j+B_1+\cdots+B_k)\} \geqq \left(\prod_{i=1}^{j} P\{A_i\}\right)^{2^{-N}} \left(\prod_{i=1}^{k} P\{B_i\}\right)^{2^{-N}} = \tag{3.15}$$
$$= (P\{A\})^{j2^{-N}}(P\{B\})^{k2^{-N}}.$$

Comparing (3. 14) and (3. 15) we conclude

$$P\left\{\frac{j}{2^N} A+\frac{k}{2^N} B\right\} \geqq (P\{A\})^{j2^{-N}}(P\{B\})^{k2^{-N}}. \tag{3.16}$$

Taking into account (3. 16) and the continuity in λ of the function $P\{\lambda A+(1-\lambda)B\}$, we see that (3. 2) holds for arbitrary $0<\lambda<1$. Thus the proof of Theorem 2 is complete.

Theorem 3. *Let $f(\mathbf{x})=\psi(Q(\mathbf{x}))$ be a probability density in R^n satisfying the conditions of Theorem 2 and $A\subset R^n$ a convex set. Then the function*

$$h(\mathbf{t}) = P\{A+\mathbf{t}\} \quad (\mathbf{t}\in R^n) \tag{3.17}$$

is logarithmic concave in R^n.

Proof. Let $\mathbf{t}_1$, $\mathbf{t}_2$ be arbitrary vectors in R^n and let $0<\lambda<1$. Then we have

$$\lambda(A+\mathbf{t}_1)+(1-\lambda)(A+\mathbf{t}_2) = A+[\lambda\mathbf{t}_1+(1-\lambda)\mathbf{t}_2].$$

In fact if $\mathbf{x}\in A$, $\mathbf{y}\in A$ then

$$\lambda(\mathbf{x}+\mathbf{t}_1)+(1-\lambda)(\mathbf{y}+\mathbf{t}_2) = [\lambda\mathbf{x}+(1-\lambda)\mathbf{y}]+[\lambda\mathbf{t}_1+(1-\lambda)\mathbf{t}_2]$$

and we supposed that A is convex. Thus by Theorem 2

$$P\{A+[\lambda\mathbf{t}_1+(1-\lambda)\mathbf{t}_2]\} = P\{\lambda(A+\mathbf{t}_1)+(1-\lambda)(A+\mathbf{t}_2)\} \geqq$$
$$\geqq (P\{A+\mathbf{t}_1\})^{\lambda}(P\{A+\mathbf{t}_2\})^{1-\lambda},$$

which means that

$$h(\lambda\mathbf{t}_1+(1-\lambda)\mathbf{t}_2) \geqq [h(\mathbf{t}_1)]^{\lambda}[h(\mathbf{t}_2)]^{1-\lambda}.$$

Theorem 4. *Let $F(\mathbf{x})$ be a continuous multivariate probability distribution function the probability density of which is of the form $f(\mathbf{x})=\psi(Q(\mathbf{x}))$ and satisfies the conditions of Theorem 2. Then $F(\mathbf{x})$ is a logarithmic concave function in R^n.*

Proof. Apply Theorem 3 to the set $A=\{\mathbf{z}|\mathbf{z}\leqq\mathbf{0}\}$ and take into account that $F(\mathbf{x}) = P\{A+\mathbf{x}\}$ for $\mathbf{x}\in R^n$.

4. Examples of probability measures satisfying the conditions of Theorem 1. The most important multivariate probability distribution is the normal distribution. Its density is given by

$$(4.1)\qquad f(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}|C|^{\frac{1}{2}}} e^{-\frac{1}{2}(\mathbf{x}-\mathbf{m})'C^{-1}(\mathbf{x}-\mathbf{m})} \quad (\mathbf{x}\in R^n),$$

where $\mathbf{m}\in R^n$ is an arbitrary vector and C is a positive definite matrix the determinant of which is denoted by $|C|$. Vectors are considered as column matrices as well and the prime means transpose. This function satisfies the conditions of Theorem 2. In fact $f(\mathbf{x})$ can be written as

$$f(\mathbf{x}) = \psi(Q(\mathbf{x})) \quad (\mathbf{x}\in R^n)$$

with

$$(4.2)\qquad \psi(z) = Ke^{-z^{\alpha}} \quad (z \geqq 0) \quad \text{and} \quad Q(\mathbf{x}) = \left[\frac{1}{2}(\mathbf{x}-\mathbf{m})'C^{-1}(\mathbf{x}-\mathbf{m})\right]^{1/\alpha},$$

where α is any fixed number satisfying $1\leqq\alpha\leqq 2$ further K is the constant standing on the right hand side in (4. 1). That $\psi(z)$ has the required property, is trivial. Only $Q(\mathbf{x})$ needs a remark. It is well known that a function

$$(\mathbf{x}'D\mathbf{x})^{\frac{1}{2}} \qquad (\mathbf{x}\in R^n)$$

is convex in the entire space provided D is positive semidefinite. This implies the convexity of $Q(\mathbf{x})$ in (4. 2).

Three further probability distributions will be discussed. In all cases we shall show that the probability densities are logarithmic concave in the entire space R^n.

The probability density $f(X)$ of the Wishart distribution is defined by

$$f(X) = \frac{|X|^{\frac{N-p-2}{2}} e^{-\frac{1}{2}\mathrm{Sp}\, C^{-1}X}}{2^{\frac{N-1}{2}p}\, \pi^{\frac{p(p-1)}{4}}\, |C|^{\frac{N-1}{2}} \prod_{i=1}^{p} \Gamma\left(\frac{N-i}{2}\right)}$$

if X is positive definite, and $f(X)=0$ otherwise. Here C and X are $p\times p$ matrices, C is fixed and positive definite while X contains the variables. In view of the symmetry of the matrix, the number of independent variables is $n = \frac{1}{2}\, p(p+1)$. We suppose that $N \geqq p+2$. It is well known that the set of positive definite[5] $p\times p$ matrices is convex in the $n = \frac{1}{2}p(p+1)$-dimensional space.

[5]) Any positive definite (or semi-definite) matrix is supposed to be symmetrical in this paper.

We show that $f(X)$ is logarithmic concave on this set[6]). To this it is enough to remark that for any $0<\lambda<1$ and any pair X_1, X_2 of positive definite matrices the inequality

$$|\lambda X_1+(1-\lambda)X_2| \geqq |X_1|^\lambda |X_2|^{1-\lambda} \tag{4. 3}$$

holds, where we have a strict inequality if $X_1 \neq X_2$ (see [1]).

The multivariate beta distribution has the probability density $f(X)$ defined by

$$f(X) = \frac{c(n_1, p)c(n_2, p)}{c(n_1+n_2, p)} |X|^{\frac{1}{2}(n_1-p-1)}|I-X|^{\frac{1}{2}(n_2-p-1)},$$

if X and $I-X$ are positive definite, and $f(X)=0$ otherwise (see [7]), where

$$\frac{1}{c(k, p)} = 2^{\frac{pk}{2}} \pi^{\frac{p(p-1)}{4}} \prod_{i=1}^{p} \Gamma\left(\frac{k-i+1}{2}\right),$$

I is the unit matrix, I and X are of order $p \times p$, p is a positive integer. We suppose that $n_1 \geqq p+1$, $n_2 \geqq p+1$. The number of independent variables of the function $f(X)$ is equal to $n = \frac{1}{2}p(p+1)$.

It is clear that the set of positive definite matrices X for which $I-X$ is also positive definite, is convex. The function $f(X)$ is zero outside this set and is logarithmic concave on this set which can be seen very easily on the basis of (4. 3).

Finally we consider the Dirichlet distribution (see e.g. [11]) the probability density of which is given by the formula

$$f(\mathbf{x}) = K x_1^{p_1-1} \dots x_n^{p_n-1}(1-x_1 \cdots -x_n)^{p_{n+1}-1}$$

if $x_i>0$ $(i=1, \dots, n)$, $x_1+\dots+x_n < 1$, and $f(\mathbf{x})=0$ otherwise. Here we have set $K = \frac{\Gamma(p_1+\dots+p_{n+1})}{\Gamma(p_1)\cdots\Gamma(p_{n+1})}$. The logarithm of this function in the positivity domain is

$$\log f(\mathbf{x}) = \log K+\sum_{i=1}^{n}(p_i-1)\log x_i+(p_{n+1}-1)\log(1-x_1-\dots-x_n). \tag{4. 4}$$

We suppose that $p_i \geqq 1$ $(i=1, \dots, n+1)$. This implies that the function (4. 4) is concave. In fact the second term is trivially concave while $\log(1-x_1-\dots-x_n)$ is an increasing and concave function of a linear function. Hence the assertion.

5. Application to stochastic programming. Let us now return to Problem (1.1) and consider the x-function in the first constraint which is given separately in (1. 2). We show if the random variables $\beta_1, \dots, \beta_m$ have a continuous joint distribution

[6]) If a function is logarithmic concave on a convex set and equal to zero elsewhere then the function is logarithmic concave in the entire space.

satisfying the conditions of Theorem 2, then the function $h(\mathbf{x})$ is logarithmic concave in the entire space R^n. We recall that the functions $g_1(\mathbf{x}), \ldots, g_m(\mathbf{x})$ are supposed to be concave in R^n.

Let $\mathbf{x}, \mathbf{y} \in R^n$ and $0 < \lambda < 1$. In view of the concavity of the functions $g_1(\mathbf{x}), \ldots, g_m(\mathbf{x})$ we have

$$g_i(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \geqq \lambda g_i(\mathbf{x}) + (1-\lambda) g_i(\mathbf{y}) \qquad (i = 1, \ldots, m).$$

The function $P\{\beta_1 \leqq z_1, \ldots, \beta_m \leqq z_m\}$ of the variables $z_1, \ldots, z_m$ is logarithmic concave by Theorem 4, and also a probability distribution function; hence it is monotonic non-decreasing in all variables. Taking these into account we conclude

$$h(\lambda\mathbf{x}+(1-\lambda)\mathbf{y}) = P\{g_1(\lambda\mathbf{x}+(1-\lambda)\mathbf{y}) \geqq \beta_1, \ldots, g_m(\lambda\mathbf{x}+(1-\lambda)\mathbf{y}) \geqq \beta_m\} \geqq$$
$$\geqq P\{\lambda g_1(\mathbf{x})+(1-\lambda)g_1(\mathbf{y}) \geqq \beta_1, \ldots, \lambda g_m(\mathbf{x})+(1-\lambda)g_m(\mathbf{y}) \geqq \beta_m\} \geqq$$
$$\geqq [P\{g_1(\mathbf{x}) \geqq \beta_1, \ldots, g_m(\mathbf{x}) \geqq \beta_m\}]^{\lambda}[P\{g_1(\mathbf{y}) \geqq \beta_1, \ldots, g_m(\mathbf{y}) \geqq \beta_m\}]^{1-\lambda} =$$
$$= [h(\mathbf{x})]^{\lambda}[h(\mathbf{y})]^{1-\lambda};$$

what was to be proved.

Considering Problem (1. 1), we may take the logarithm of both sides of the first constraint. Then we obtain a convex programming problem. For some reason we may leave it in the original form (the computational solution may prefer this form), then we have a quasi-convex programming problem because a logarithmic concave function is always quasi-concave. Any of these versions can be solved by non-linear programming methods (see e. g. [4], [8], [12]). We emphasize again that this short remark concerning the application of the theory presented in this paper is just for illustration and to disclose the origin of the problem.

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Über das Maximum der Summen orthogonaler Funktionen

Von KÁROLY TANDORI in Szeged

1. Im folgenden betrachten wir orthonormierte Systeme $\{\varphi_n(x)\}_1^\infty$ im Grundintervall (0, 1). Für eine reelle Zahlenfolge $\{a_n\}_1^\infty$ setzen wir

$$\|\{a_n\};\infty\| = \sup\left\{\int_0^1 \left(\sup_{1\le i\le j} |a_i\varphi_i(x)+\cdots+a_j\varphi_j(x)|\right)^2 dx\right\}^{\frac{1}{2}}$$

und

$$\|\{a_n\};K\| = \sup_{|\varphi_n|\le K}\left\{\int_0^1 \left(\sup_{1\le i\le j} |a_i\varphi_i(x)+\cdots+a_j\varphi_j(x)|\right)^2 dx\right\}^{\frac{1}{2}},$$

wobei das Supremum über alle orthonormierten Systeme $\{\varphi_n(x)\}_1^\infty$, bzw. über alle orthonormierten Systeme mit

(1) $$|\varphi_n(x)|\le K \qquad (0\le x\le 1;\ n=1,2,\dots)$$

zu bilden ist ($K\ge 1$). Offensichtlich ist $\|\{a_n\};K\|\le\|\{a_n\};\infty\|$ ($K\ge 1$). Es ist eine Frage, ob auch eine Ungleichung

$$\|\{a_n\};\infty\| \le C(K)\|\{a_n\};K\| \qquad (K>1)$$

gilt, mit einer nur von K abhängigen Konstanten $C(K)$. (Im folgenden bezeichnen $C_1(K), C_2(K), \dots$ gewisse nur von K abhängige positive Konstanten, $C_1, C_2, \dots$ sind aber positive absolute Konstanten.) Dieses Problem ist noch ungelöst; nur sind gewisse Teilresultate bekannt. Der Wert $\|\{a_n\};K\|$, bzw. $\|\{a_n\};\infty\|$ hängt nähmlich von der Anordnung der Folge $\{a_n\}$ ab, und für gewisse Anordnungen ist eine solche Ungleichung gültig. In einer vorigen Arbeit [2] haben wir bewiesen, daß

$$\sup_P\|\{a_n\};\infty\| \le C_1(K)\sup_P\|\{a_n\};K\| \qquad (K>1),$$

wobei $\sup_P$ das Supremum für jede Anordnung der Folge $\{a_n\}$ bedeutet. In dieser Note werden wir Folgendes beweisen:

Satz. *Für jede Folge* $\{a_n\}_1^\infty$ *gilt*

$$\inf_P \|\{a_n\}; \infty\| \leqq C_2(K) \inf_P \|\{a_n\}; K\| \qquad (K>1),$$

wobei $\inf_P$ *bedeutet, daß das Infimum für jede Anordnung der Folge* $\{a_n\}$ *gebildet wird.*

2. Unsere Behauptung folgt aus dem folgenden Hilfssatz.

Hilfssatz I. *Für jede Folge* $\{a_n\}_1^\infty$ *von* 0 *verschiedenen Zahlen und für jedes* N *gilt*

$$\sum_{n=1}^{\nu(N)} a_n^2 \log_+^2 \frac{a_1^2+\cdots+a_{\nu(N)}^2}{a_n^2} \leqq C_3(K) \|\{a_n\}_1^{\nu(N)}; K\|^2 \qquad (K>1), \tag{2}$$

wobei $\{a_n\}_1^{\nu(N)}$ *die Folge* $\{a_1, \ldots, a_{\nu(N)}, 0, \ldots\}$ *bezeichnet,* $\nu(N) = 1+32+\cdots+32^N$ *ist, weiterhin die Funktion* $\log_+ x$ *folgenderweise definiert wird:*

$$\log_+ x = \begin{cases} \log x, & x \geqq 2, \\ 1 & \textit{sonst}. \end{cases}$$

(Man bezeichnet mit log den Logarithmus mit der Basis 2.)

Wegen der offensichtlich gültigen Ungleichung

$$\left(\sum_{n=1}^{\infty} a_n^2\right)^{\frac{1}{2}} \leqq \|\{a_n\}; K\|$$

können wir ohne Beschränkung der Allgemeinheit $\{a_n\} \in l^2$ annehmen. Da der Wert $\|\{a_n\}; \infty\|$, bzw. $\|\{a_n\}; K\|$ nur von den Koeffizienten $a_n \neq 0$ abhängt, können wir $a_n \neq 0$ $(n=1, 2, \ldots)$ voraussetzen. Dann folgt aus (2) offensichtlich

$$\sum_{n=1}^{\infty} a_n^2 \log_+^2 \frac{a_1^2+\cdots+a_m^2+\cdots}{a_n^2} \leqq C_3(K) \|\{a_n\}_1^\infty; K\| \qquad (K>1),$$

und hieraus:

$$\sum_{n=1}^{\infty} a_n^2 \log_+^2 \frac{a_1^2+\cdots+a_m^2+\cdots}{a_n^2} \leqq C_3(K) \inf_P \|\{a_n\}_1^\infty; K\| \qquad (K>1). \tag{3}$$

(Eine ähnliche Abschätzung haben wir für $\|\{a_n\}; \infty\|$ schon in der Arbeit [3] bewiesen.)

Es sei $\{a_{n_k}\}$ eine Anordnung von $\{a_n\}$, für die $|a_{n_1}| \geqq \cdots \geqq |a_{n_k}| \geqq \cdots$ besteht. Dann ist

$$\begin{aligned} \sum_{n=1}^{\infty} a_n^2 \log_+^2 \frac{a_1^2+\cdots+a_m^2+\cdots}{a_n^2} &= \sum_{k=1}^{\infty} a_{n_k}^2 \log_+^2 \frac{a_{n_1}^2+\cdots+a_{n_l}^2+\cdots}{a_{n_k}^2} \geqq \\ &\geqq C_1\left(a_{n_1}^2 + \sum_{k=2}^{\infty} a_{n_k}^2 \log^2 k\right). \end{aligned} \tag{4}$$

Weiterhin, in der Arbeit [4] haben wir bewiesen, daß

$$\|\{\gamma_n\};\infty\| \leqq C_2\left(\gamma_1^2+\sum_{n=2}^{\infty}\gamma_n^2\log^2 n\right)^{\frac{1}{2}}$$

für jede Folge $\{\gamma_n\}$ besteht. So ist

(5) $$\inf_P \|\{a_n\};\infty\| \leqq \|\{a_{n_k}\};\infty\| \leqq C_2\left(a_{n_1}^2+\sum_{k=2}^{\infty}a_{n_k}^2\log^2 k\right)^{\frac{1}{2}}.$$

Aus (3), (4) und (5) erhalten wir die Behauptung unseres Satzes.

3. Es soll nun der Hilfssatz I bewiesen werden, Ohne Beschränkung der Allgemeinheit können wir $a_n>0$ $(n=1,2,\dots,\nu(N))$ annehmen. $\{a_{n_k}\}$ $(k=1,\dots,\nu(N))$ bezeichne eine Anordnung der Folge $\{a_n\}_1^{\nu(N)}$, für die $a_{n_1}\geqq\cdots\geqq a_{n_{\nu(N)}}$ besteht. Es sei Z_k $(k=1,\dots,N)$ die Menge der Indizes n_l mit $\nu(k-1)<l\leqq\nu(k)$, und $Z_0=\{n_1\}$. Die Elemente von Z_k bezeichnen wir in natürlicher Anordnung mit $m_1(k),\dots,m_{32^k}(k)$ $(m_1(k)<\cdots<m_{32^k}(k))$. Wir setzen

$$b_n=\min_{m\in Z_k}a_m \qquad (n\in Z_k;\ k=0,\dots,N).$$

Dann ist $\{b_n\}_1^{\nu(N)}$ eine Folge von positiven Zahlen mit $b_n\leqq a_n$ $(n=1,\dots,\nu(N))$. Es sei weiterhin

$$\beta_n=\min_{m\in Z_{k-1}}a_m \quad (n\in Z_k;\ k=1,\dots,N),\quad \beta_{n_1}=a_{n_1}.$$

Dann ist $a_n\leqq\beta_n$ $(n=1,\dots,\nu(N))$.

Nach dem Hilfssatz I der Arbeit [3] gilt also

(6) $$\sum_{n=1}^{\nu(N)}a_n^2\log_+^2\frac{a_1^2+\cdots+a_{\nu(N)}^2}{a_n^2}\leqq C_3\sum_{n=1}^{\nu(N)}\beta_n^2\log_+^2\frac{\beta_1^2+\cdots+\beta_{\nu(N)}^2}{\beta_n^2}.$$

Da

$$\sum_{n\in Z_k}\beta_n^2=32\sum_{n\in Z_{k-1}}b_n^2 \qquad (k=1,\dots,N)$$

ist, erhalten wir durch eine einfache Rechnung

$$\sum_{n=1}^{\nu(N)}\beta_n^2\log_+^2\frac{\beta_1^2+\cdots+\beta_{\nu(N)}^2}{\beta_n^2}=\sum_{k=0}^{N}\sum_{n\in Z_k}\beta_n^2\log_+^2\frac{\beta_1^2+\cdots+\beta_{\nu(N)}^2}{\beta_n^2}\leqq$$

(7) $$\leqq b_{n_1}^2\log_+^2\frac{b_{n_1}^2+32\sum_{k=0}^{N-1}\left(\sum_{n\in Z_k}b_n^2\right)}{b_{n_1}^2}+32\sum_{k=1}^{N-1}\left(\sum_{n\in Z_k}b_n^2\log_+^2\frac{b_{n_1}^2+32\sum_{k=0}^{N-1}\left(\sum_{n\in Z_k}b_n^2\right)}{b_n^2}\right)\leqq$$

$$\leqq C_4\sum_{n=1}^{\nu(N)}b_n^2\log_+^2\frac{b_1^2+\cdots+b_{\nu(N)}^2}{b_n^2}.$$

Aus (6) und (7) folgt

(8) $$\sum_{n=1}^{\nu(N)}a_n^2\log_+^2\frac{a_1^2+\cdots+a_{\nu(N)}^2}{a_n^2}\leqq C_5\sum_{n=1}^{\nu(N)}b_n^2\log_+^2\frac{b_1^2+\cdots+b_{\nu(N)}^2}{b_n^2}.$$

Nach dem Hilfssatz II der Arbeit [5] gilt

$$\|\{c_n\}; K\| \leqq C_4(K)\|\{d_n\}; K\| \quad (|c_n| \leqq |d_n|;\ n=1, 2, \ldots; K>1),$$

und so ist

$$\|\{b_n\}_1^{\nu(N)}; K\| \leqq C_4(K)\|\{a_n\}_1^{\nu(N)}; K\| \qquad (K>1). \tag{9}$$

Wir werden nun die Abschätzung

$$\sum_{n=1}^{\nu(N)} b_n^2 \log_+^2 \frac{b_1^2+\cdots+b_{\nu(N)}^2}{b_n^2} \leqq C_5(K)\|\{b_n\}_1^{\nu(N)}; K\| \qquad (K>1) \tag{10}$$

beweisen. Aus (8), (9) und (10) erhalten wir die Behauptung des Hilfssatzes I.

4. Zum Beweis von (10) können wir ohne Beschränkung der Allgemeinheit

$$\sum_{n=1}^{\nu(N)} b_n^2 = 1 \tag{11}$$

annehmen.

Wir werden erstens den folgenden bekannten, in wesentlichen von MENCHOFF [1] stammenden Hilfssatz (siehe [5], Hilfssatz VI) anwenden.

Hilfssatz II. *Es sei $K>1$, $p(\geqq 2)$ eine natürliche Zahl und $1\leqq c \leqq p/4$. Dann gibt es ein in* $(0, 1)$ *orthonormiertes System von Treppenfunktionen $h_l(c, p; x)$* $(l=1, \ldots, p^2)$ *mit folgenden Eigenschaften: es gilt $|h_l(c, p; x)| \leqq K$* $(0\leqq x\leqq 1;\ l=1, \ldots, p^2)$; *es gibt ein Intervall $E(\subseteqq (0, 1))$ mit* $\operatorname{mes}(E) \geqq C_6(K)c^{-1}$ *derart, daß für $x\in E$ ein Index $m(x)(<p^2)$ mit $h_l(c, p; x)\geqq 0$* $(l=1, \ldots, m(x))$ *und*

$$\sum_{l=1}^{m(x)} h_l(c, p; x) \geqq C_7(K)\sqrt{c}\, p \log p$$

existiert.

Es sei $I_0=(-1, 0)$, $I_k=\left(\frac{1}{2^{k+1}}, \frac{1}{2^k}\right)$ $(k=1, \ldots, N)$ und $J=\left(\frac{1}{2}, 2\right)$. Wir werden ein in $(-1, 2)$ orthonormiertes System von Treppenfunktionen $\psi_n(x)$ $(n=1, \ldots, \nu(N))$ mit folgenden Eigenschaften definieren:

a) es gilt $|\psi_n(x)|\leqq K$ $\left(0\leqq x\leqq 1;\ n\in \bigcup_{l=1}^{k} Z_l\right)$,

b) im Falle $x\in I_{l_0}(0\leqq l_0\leqq k)$ ist $\psi_n(x)=0$ $\left(n\in \bigcup_{l=0}^{k} Z_l \setminus Z_{l_0}\right)$

c) weiterhin besteht $\psi_{n_1}(x)=1$ $(x\in(-1, 0))$, und für jedes $l(1\leqq l\leqq k)$ gibt es eine meßbare Menge $E_l(\subseteqq I)$ mit

$$\operatorname{mes}(E_l)\geqq C_6(K)\frac{1}{2^{l+1}}$$

derart, daß für $x \in E_l$ mit geeignetem Indizes $\nu(x)$ $(1 \leqq \nu(x) \leqq 32^l)$

$$\sum_{i=1}^{\nu(x)} b_{m_i(l)} \psi_{m_i(l)}(x) \geqq \sqrt{2}\, C_7(K)\, l \sqrt{2^l \sum_{i=1}^{32^l} b^2_{m_i(l)}}$$

besteht $(k = 1, \dots, N)$.

Wir setzen

$$\psi_{n_1}(x) = \begin{cases} 1 & (x \in (-1, 0)), \\ 0 & \text{sonst.} \end{cases}$$

Es sei k_0 $(0 \leqq k_0 < N)$ eine ganze Zahl. Wir nehmen an, daß die Treppenfunktionen $\psi_n(x)$ $\left(n \in \bigcup_{l=1}^{k_0} Z_l\right)$ derart definiert sind, daß sie in $(-1, 2)$ ein orthonormiertes System bilden, und a), b), c) für $k = k_0$ erfüllt sind.

Wir wenden den Hilfssatz II im Falle $c = 1, p = 4^{k_0+1}$ an. Die so erhaltenen Funktionen, bzw. die so erhaltene Menge bezeichnen wir mit $\chi_m(x)$ $(m = 1, \dots, 16^{k_0+1})$, bzw. mit E. Es sei $f(x)$ eine in $(0, 1)$ definierte Funktion und H eine Untermenge von $(0, 1)$. Ist $I = (a, b)$ ein endliches Intervall, dann setzen wir

$$f(I; x) = \begin{cases} f\left(\dfrac{x-a}{b-a}\right) & (a < x < b), \\ 0 & \text{sonst,} \end{cases}$$

weiterhin bezeichne $H(I)$ diejenige Menge, die aus H durch die Transformation $y = (b-a)x + a$ entsteht.

Wir setzen

$$\tilde{\psi}_n(x) = \chi_i(I_{k_0+1}; x) \quad (n = m_{(i-1)2^{k_0+1}+s}(k_0+1),\ s = 1, \dots, 2^{k_0+1};\ i = 1, \dots, 16^{k_0+1}).$$

Nach dem obigen gelten offensichtlich

$$\int_{-1}^{1} \tilde{\psi}_n^2(x)\, dx = \frac{1}{2^{k_0+2}} \qquad (n \in Z_{k_0+1}), \tag{12}$$

$$a_{n,m} = \int_{-1}^{1} \tilde{\psi}_n(x) \tilde{\psi}_m(x)\, dx \leqq \frac{1}{2^{k_0+2}} \qquad (n, m \in Z_{k_0+1}), \tag{13}$$

$$\int_{-1}^{1} \tilde{\psi}_n(x) \tilde{\psi}_m(x)\, dx = 0 \qquad (n, m \in Z_{k_0+1}, |n-m| > 2^{k_0+1}). \tag{14}$$

Der folgende Hilfssatz ist bekannt. (S. MENCHOFF [1].)

Hilfssatz III. *Es seien d und q positive ganze Zahlen, $0 < d < q$. Zu jedem Indexpaar (i, j) mit $1 \leqq i \leqq q$, $1 \leqq j \leqq q$ und $|i-j| = d$ soll eine von Null verschiedene*

Zahl $\alpha_{i,j}$ zugeordnet werden; wir bezeichnen mit B_d das Maximum der absoluten Beträge der Zahlen $\alpha_{i,j}$. In jedem Intervall (u, v) mit

$$v-u > 2B_d$$

können dann Treppenfunktionen $\bar{\varphi}_l(x)$ $(l=1, \dots, q)$ mit folgenden Eigenschaften definiert werden:

$$|\bar{\varphi}_l(x)|=1 \qquad (u<x<v;\ l=1, \dots, q),$$

$$\int_u^v \bar{\varphi}_i(x)\bar{\varphi}_j(x)\,dx = -\alpha_{i,j} \qquad (|i-j| = d,\ 1 \leqq i \leqq q,\ 1 \leqq j \leqq q),$$

$$\int_u^v \bar{\varphi}_i(x)\bar{\varphi}_j(x)\,dx = 0 \qquad (i \neq j,\ |i-j| \neq d,\ 1 \leqq i \leqq q,\ 1 \leqq j \leqq q).$$

Auf Grund von (12), (13) und (14), durch Anwendung dieses Hilfssatzes können wir Treppenfunktionen $\bar{\psi}_{m_i(k_0+1)}(x)$ $(i=1, \dots, 32^{k_0+1})$ mit folgenden Eigenschaften angeben:

(15) $$|\bar{\psi}_{m_i(k_0+1)}(x)| = \begin{cases} 1 & (x\in(1, 2)) \\ 0 & \text{sonst} \end{cases} \qquad (i=1, \dots, 32^{k_0+1}),$$

(16) $$\int_1^2 \bar{\psi}_n(x)\bar{\psi}_m(x)\,dx = -\alpha_{n,m} \qquad (n\neq m;\ n, m\in Z_{k_0+1}).$$

Da die Funktionen $\psi_n(x)$ in $(-1, 2)$ Treppenfunktionen sind, können wir eine Einteilung des Intervalls J auf paarweise disjunkte Intervalle J_r $(r=1, \dots, \varrho)$ derart angeben, daß jede Funktion $\psi_n(x)$ in jedem Intervall J_r $(1 \leqq r \leqq \varrho)$ konstant ist; die zwei Hälften von J_r bezeichnen wir mit J'_r, bzw. mit J''_r $(r=1, \dots, \varrho)$. Dann setzen wir

(17) $$\psi_n(x) = \frac{1}{D}\left(\tilde{\psi}_n(x)+\sum_{r=1}^{\varrho} \bar{\psi}_n(J'_r; x)-\sum_{r=1}^{\varrho} \bar{\psi}_n(J''_r; x)\right) \qquad (n\in Z_{k_0+1}),$$

wobei

(18) $$D^2 = \int_{I_{k_0+1}} \tilde{\psi}_n^2(x)\,dx+\int_J \bar{\psi}_n^2(x)\,dx$$

ist.

$\psi_n(x)$ $(n\in Z_{k_0+1})$ sind offensichtlich Treppenfunktionen. Wegen (16), (17) und (18) bilden die Funktionen $\psi_n(x)$ $\left(n\in \bigcup_{l=0}^{k_0+1} Z_l\right)$ ein orthonormiertes System in $(-1, 2)$. Auf Grund von (12) und (15) gilt

(19) $$1 \leqq D^2 \leqq 2,$$

und so besteht

(20) $$|\psi_n(x)| \leqq K \quad \text{für} \quad -1 \leqq x \leqq 2;\ n\in Z_{k_0+1}.$$

Auf Grund der Definition von $\psi_n(x)$ gilt auch

(21) $$\psi_n(x)=0 \quad \text{für} \quad n\in Z_{k_0+1}; \qquad x\in(-1,1),\ x\notin I_{k_0+1}.$$

Auf Grund des Hilfssatzes II, weiterhin der Definition von b_n und $\psi_n(x)$, aus (19) folgt mit $E_{k_0+1}=E(I_{k_0+1})(\subseteq I_{k_0+1})$

(22) $$\sum_{i=1}^{\nu(x)} b_{m_i(k_0+1)}\psi_{m_i(k_0+1)}(x) \geqq \frac{1}{\sqrt{2}} C_7(K) \min_{m\in Z_{k_0+1}} a_m 4^{k_0+1}\cdot\log 4^{k_0+1}\cdot 2^{k_0+1} =$$

$$= \sqrt{2}\, C_7(K) \sqrt{2^{k_0+1}\sum_{i=1}^{32^{k_0+1}} b^2_{m_i(k_0+1)}(k_0+1)}, \quad \text{für} \quad x\in E_{k_0+1}$$

mit geeigneten Indizes $\nu(x)$ $(1\leqq\nu(x)\leqq 32^{k_0+1})$, und mit

(23) $$\operatorname{mes}(E_{k_0+1}) \geqq C_6(K)\frac{1}{2^{k_0+1}}.$$

Aus (20), (21), (22) und (23) ergibt sich, daß a), b) und c) im Falle $k=k_0+1$ für das in $(-1,2)$ orthonormierte System $\{\psi_n(x)\}$ $\left(n\in\bigcup_{l=0}^{k_0+1} Z_l\right)$ von Treppenfunktionen erfüllt werden. Das Funktionensystem $\{\psi_n(x)\}$ $\left(n\in\bigcup_{l=0}^{N} Z_l\right)$ mit den erwähnten Eigenschaften erhalten wir also durch Induktion.

Aus b) und c) bekommen wir leicht

(24) $$\int_{-1}^{2}\Big(\max_{1\leqq i\leqq j\leqq \nu(N)} |b_i\psi_i(x)+\cdots+b_j\psi_j(x)|\Big)^2 dx \geqq$$
$$\geqq b^2_{n_1}+C_7^2(K)C_6(K)\sum_{k=1}^{N}\Big(\sum_{n\in Z_k} b_n^2\Big)k^2 \geqq C_8(K)\left[b^2_{n_1}+\sum_{k=1}^{N}\Big(\sum_{n\in Z_k} b_n^2\Big)k^2\right].$$

Aus (11) folgt

(25) $$\frac{1}{b_n^2} \geqq 32^k \qquad (n\in Z_k).$$

Wir bezeichnen mit M die Menge derjenigen Indizes k $(1\leqq k\leqq N)$, für die

$$\frac{1}{b_n^2} \geqq 32^{4k} \qquad (n\in Z_k)$$

besteht, und M' sei die Menge der übrigen Indizes k $(1\leqq k\leqq N)$. Dann setzen wir

(26) $$\sum_{n=1}^{\nu(N)} b_n^2 \log_+^2\frac{1}{b^2_{n_1}} = b^2_{n_1}\log_+^2\frac{1}{b_n^2}+\sum_{k\in M}\left(\sum_{n\in Z_k} b_n^2\log_+^2\frac{1}{b^2_{n_1}}\right)+\sum_{k\in M'}\left(\sum_{n\in Z_k} b_n^2\log_+^2\frac{1}{b_n^2}\right).$$

Da $x\log\dfrac{1}{x}$ für $0<x<\dfrac{1}{2}$ eine monoton wachsende Funktion ist, auf Grund von (25), weiterhin aus der Definition von M und M' erhalten wir

$$\sum_{k\in M}\left(\sum_{n\in Z_k} b_n^2\log_+^2\frac{1}{b_n^2}\right)+\sum_{k\in M'}\left(\sum_{n\in Z_k} b_n^2\log_+^2\frac{1}{b_n^2}\right)=$$

$$(27)\qquad \begin{aligned} &=\sum_{k\in M}\left(\sum_{n\in Z_k} b_n^2\log^2\frac{1}{b_n^2}\right)+\sum_{k\in M'}\left(\sum_{n\in Z_k} b_n^2\log^2\frac{1}{b_n^2}\right)\leqq\\ &\leqq C_6\sum_{k\in M}\left(\sum_{n\in Z_k} b_n\right)+20\sum_{k\in M'}\left(\sum_{n\in Z_k} b_n^2\right)k^2\leqq\\ &\leqq C_6\sum_{k\in M}\frac{1}{32^k}+20\sum_{k\in M}\left(\sum_{n\in Z_k} b_n^2\right)k^2\leqq C_7\left(1+\sum_{k=1}^{N}\Big(\sum_{n\in Z_k} b_n^2\Big)k^2\right).\end{aligned}$$

Ist $b_{n_1}^2\geqq\frac{1}{2}$, dann gilt wegen (11)

$$b_{n_1}^2\log_+^2\frac{1}{b_{n_1}^2}\leqq b_{n_1}^2\leqq 1,$$

ist aber $b_{n_1}^2<\frac{1}{2}$, dann gilt

$$b_{n_1}^2\log_+^2\frac{1}{b_{n_1}^2}\leqq C_8 b_{n_1}<C_8.$$

Also ist

$$(28)\qquad b_{n_1}^2\log_+^2\frac{1}{b_{n_1}^2}\leqq C_9.$$

Weiterhin folgt aus (11)

$$(29)\qquad b_{n_1}^2+\sum_{k=1}^{N}\Big(\sum_{n\in Z_k}{}' b_n^2\Big)k^2\geqq 1.$$

Aus (26), (27), (28) und (29) erhalten wir

$$(30)\qquad \sum_{n=1}^{\nu(N)} b_n^2\log_+^2\frac{1}{b_n^2}\leqq C_{10}\left(b_{n_1}^2+\sum_{k=1}^{N}\Big(\sum_{n\in Z_k} b_n^2\Big)k^2\right).$$

Aus (24) und (30) ergibt sich

$$(31)\qquad \int_{-1}^{2}\Big(\max_{1\leqq i\leqq j\leqq\nu(N)}|b_i\psi_i(x)+\cdots+b_j\psi_j(x)|\Big)^2dx\geqq C_9(K)\left(\sum_{n=1}^{\nu(N)} b_n^2\log_+^2\frac{1}{b_n^2}\right).$$

Wegen $K>1$ gibt es eine Konstante $C_{10}(K)$ $(0<C_{10}(K)<1)$, für die

$$\frac{C_{10}(K)}{3}+(1-C_{10}(K))K^2=1$$

erfüllt ist. Wir setzen

$$\varphi_n(x) = \begin{cases} \psi_n\left(\frac{3x}{C_{10}(K)}-1\right) & (x\in(0, C_{10}(K))), \\ Kr_n((C_{10}(K), 1); x) & (x\in(C_{10}(K), 1)), \\ 0 & \text{sonst} \end{cases}$$

$(n=1, \ldots, \nu(N))$, wobei $r_n(x)=\operatorname{sign}\sin 2^n\pi x$ die n-te Rademachersche Funktion bezeichnet. Die Funktionen $\varphi_n(x)$ bilden in (0, 1) ein durch K beschränktes orthonormiertes System. Weiterhin folgt aus (31)

$$\int_0^1 \Big(\max_{1\leqq i\leqq j\leqq \nu(N)} |b_i\varphi_i(x)+\cdots+b_j\varphi_j(x)|\Big)^2\,dx \geqq C_{11}(K)\sum_{n=1}^{\nu(N)} b_n^2\log_+^2\frac{1}{b_n^2}.$$

Daraus erhalten wir endlich wegen

$$\|\{b_n\}_1^{\nu(N)}; K\|^2 = \sup_{|\varphi_n|\leqq K}\int_0^1 \Big(\max_{1\leqq i\leqq j\leqq \nu(N)} |b_i\varphi_i(x)+\cdots+b_j\varphi_j(x)|\Big)^2\,dx$$

die Behauptung des Hilfssatzes I.

5. Bemerkung. Aus (3), auf Grund eines Satzes der Arbeit [5] folgt die folgende Behauptung.

Es sei $K>1$. Ist

$$\sum_{n=1}^{\infty} a_n^2\log_+^2\frac{a_1^2+\cdots+a_m^2+\cdots}{a_n^2}=\infty,$$

dann gibt es ein orthonormiertes System $\{\varphi_n(x)\}$ mit (1), für welches die Reihe

$$\Sigma a_n\varphi_n(x)$$

in (0, 1) fast überall divergiert.

Für nicht notwendigerweise beschränkte Systeme wurde dies in der Arbeit [3] bewiesen.

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Über die regulären duo-Elemente in Gruppoid-Verbänden

Von OTTO STEINFELD in Budapest

Ein assoziativer Ring (eine Halbgruppe) A heißt *regulär*, wenn für jedes Element a von A gilt:

$$a \in aAa.$$

Die folgende Charakterisierung stammt von L. KOVÁCS [2]: Ein assoziativer Ring (eine Halbgruppe) A ist dann und nur dann regulär, wenn für jedes Linksideal L und Rechtsideal R von A

$$RL = R \cap L$$

gilt.

Unter einem *duo-Ring* (einer *duo-Halbgruppe*) verstehen wir einen assoziativen Ring (eine Halbgruppe), dessen (deren) alle einseitigen Ideale zweiseitige Ideale sind[1]).

In den Arbeiten [3], [4], hat S. LAJOS die regulären duo-Ringe (-Halbgruppen) folgenderweise charakterisiert: Für einen assoziativen Ring (eine Halbgruppe) A sind die folgenden Bedingungen äquivalent:

(α) A ist regulär und duo;

(β) der Durchschnitt und das Produkt irgendwelcher Linksideale L_1 und L_2 von A stimmen überein, und dasselbe gilt für irgendwelche Rechtsideale R_1 und R_2 von A;

(γ) für alle Linksideale L und Rechtsideale R von A besteht $L \cap R = LR$.

Wir werden diese Charakterisierungen für die Elemente gewisser teilweise geordneten Gruppoide verallgemeinern.

Ein teilweise geordnetes Gruppoid $\langle L; \leqq \rangle$ nennen wir einen *Gruppoid-Verband*, wenn L bezüglich seiner teilweise Ordnung $\leqq$ einen vollständigen Verband $\langle L; \wedge, \vee \rangle$ bildet, in dem die Bedingungen

(1) $$a^2 \leqq a \qquad (\text{für jedes } a \in L)$$

[1]) Diese Begriffe sind englisch "duo ring" bzw. "duo semigroup" genannt. Siehe E. H. FELLER [1] und S. LAJOS [4].

und

(2) $$0 \cdot e = e \cdot 0 = 0$$

erfüllt sind, wobei 0 und e das kleinste bzw. das größte Element von L bezeichnen. Mit L bezeichnen wir stets einen Gruppoid-Verband.

Wir sagen, daß das Element b von L ein *Absorbent* des Elementes a von L ist, wenn

(3) $$b \leqq a$$

und

(4) $$ab \leqq b, \quad ba \leqq b$$

bestehen. Das Element b heißt ein *Linksabsorbent (Rechtsabsorbent)* von a, wenn (3) und (4_1) [(3) und (4_2)] gelten.

Ein Element k von L heißt ein *Quasiabsorbent* von $a(\in L)$, wenn

(5) $$k \leqq a \quad \text{und} \quad ka \wedge ak \leqq k$$

bestehen.

Diese Begriffe wurden in unserer Arbeit [5] definiert.

Behauptung 1. *Der Durchschnitt $r \wedge l$ eines Rechtsabsorbenten r und eines Linksabsorbenten l des Elementes a von L ist ein Quasiabsorbent von a.*

Beispiele 1. Definiert man das Produkt $B \cdot C$ der Unterringe B, C eines assoziativen Ringes A als denjenigen Unterring von A, der durch alle Elemente bc ($b \in B$; $c \in C$) erzeugt ist, so bildet die Menge aller Unterringe von A einen Gruppoid-Verband L_1 bezüglich dieser Multiplikation und des mengentheoretischen Enthaltenseins. Der aus dem Nullelement bestehende Unterring von A ist das kleinste Element des Gruppoid-Verbandes L_1, und A ist sein größtes Element. Die Links-, Rechts- und Quasiabsorbenten des Elementes A von L sind die Links-, Rechts- und Quasiidealen des Ringes A.

2. Es sei H_0 eine Halbgruppe mit Nullelement 0. Ähnlich zu dem vorigen Beispiel bildet die Menge aller Unterhalbgruppen mit 0 von H_0 einen Gruppoid-Verband L_2. Die Links-, Rechtsund Quasiideale von H_0 werden in L_2 die Links-, Rechtsund Quasiabsorbenten des Elementes H_0 von L_2.

Ein Element a des Gruppoid-Verbandes L heißt *duo-Element*, wenn alle Linksabsorbenten und alle Rechtsabsorbenten von a Absorbenten von a sind.

Von jetzt an schreiben wir je eine bedingte Assoziativitäts- bzw. Distributivitätsregel vor, die zu unseren Untersuchungen nötig sind.

Voraussetzung (A). Sind k_1, k_2 und k_3 Quasiabsorbenten des Elementes a von L, so sei

$$(k_1 k_2) k_3 = k_1 (k_2 k_3).$$

Voraussetzung (D_v). Für das Element a von L seien die Distributivitätsregeln

$$k_1(k_2 \vee k_2 a) = k_1 k_2 \vee k_1(k_2 a) \quad \text{und} \quad (k_2 \vee k_2 a)k_1 = k_2 k_1 \vee (k_2 a)k_1,$$

$$k_1(k_2 \vee ak_2) = k_1 k_2 \vee k_1(ak_2) \quad \text{und} \quad (k_2 \vee ak_2)k_1 = k_2 k_1 \vee (ak_2)k_1$$

für alle Quasiabsorbenten k_1 und k_2 von a erfüllt.

Es ist nicht schwer zu zeigen, daß die Voraussetzungen (A) und (D_v) in den Gruppoid-Verbänden L_1 und L_2 erfüllt sind.

Voraussetzung (K). Für jeden Rechtsabsorbenten r und Linksabsorbenten l des Elementes a von L gelte $rl = r \wedge l$.

In der Arbeit [5] haben wir ein Element a von L *regulär* genannt, falls a die Voraussetzungen (A), (D_v) und (K) erfüllt.

Der folgende Satz verallgemeinert und ergänzt die erwähnten Ergebnisse von S. Lajos [3], [4].

Satz. *Sind die Voraussetzungen* (A) *und* (D_v) *für das Element a des Gruppoid-Verbandes L erfüllt, so sind die folgenden Bedingungen äquivalent:*

(i) *a ist regulär und duo;*

(ii) *für jede Quasiabsorbenten k_1, k_2 von a gilt $k_1 \wedge k_2 = k_1 k_2$;*

(iii) *für jede Linksabsorbenten l_1, l_2 und Rechtsabsorbenten r_1, r_2 von a bestehen $l_1 \wedge l_2 = l_1 l_2$, und $r_1 \wedge r_2 = r_1 r_2$;*

(iv) *für jeden Quasiabsorbenten k von a gelten*

$$(k \vee ka)^2 = k \vee ak \quad \textit{und} \quad (k \vee ak)^2 = k \vee ka;$$

(v) *für jeden Linksabsorbenten l und Rechtsabsorbenten r gilt $l \wedge r = lr$.*

Zum Beweis des Satzes benützen wir die folgende Umkehrung der Behauptung 1.

Behauptung 2. *Jeder Quasiabsorbent k des regulären Elementes a von L ist in der Form*

$$k = r \wedge l = rl$$

darstellbar, wo r und l einen geeigneten Rechtsabsorbenten bzw. Linksabsorbenten von a bezeichnen.

Beweis. Infolge der Voraussetzungen (A) und (D_v) bezeichnen die Elemente $l = k \vee ak$ und $r = k \vee ka$ den durch den Quasiabsorbenten k erzeugten Linksabsorbenten bzw. Rechtsabsorbenten von a. (Siehe die Arbeit [5].) Andererseits besteht wegen der Voraussetzungen (K), (A), (D_v) und wegen (5)

$$k \leqq (k\vee ka)\wedge(k\vee ak) = r\wedge l = rl = (k\vee ka)(k\vee ak) =$$

$$= k^2\vee(ka)k\vee k(ak)\vee(ka)(ak) \leqq ka\wedge ak \leqq k,$$

woraus Behauptung 2 folgt.

Beweis des Satzes. (i)⇒(ii). Nach Behauptung 2 bestehen $k_1 = r_1\wedge l_1$ und $k_2 = r_2\wedge l_2$ mit geeigneten Rechtsabsorbenten r_1, r_2 und Linksabsorbenten l_1, l_2 von a. Da das Element a duo ist, sind die Quasiabsorbenten $k_1 = r_1\wedge l_1$ und $k_2 = r_2\wedge l_2$ von a Absorbenten von a. Dieses und die Voraussetzung (K) implizieren (ii).

Die Implikationen (ii)⇒(iii) und (ii)⇒(v) gelten trivialerweise.

(ii)⇒(iv). Infolge (ii) gilt $ka = ak = a\wedge k = k$ für jeden Quasiabsorbenten k von a, woraus wieder wegen (ii)

$$(k\vee ka)^2 = k\vee ka = k = k\vee ak \quad \text{und} \quad (k\vee ak)^2 = k\vee ak = k = k\vee ka$$

folgen.

Wir haben noch die Implikationen (iii)⇒(i), (iv)⇒(i) und (v)⇒(i) zu zeigen.

(iii)⇒(i). Im Falle $l_2 = a$ folgt $l_1 a = l_1\wedge a = l_1$ aus (iii), d.h. jeder Linksabsorbent l_1 von a ist ein Rechtsabsorbent von a. Ähnlich sieht man ein, daß jeder Rechtsabsorbent r_2 von a ein Linksabsorbent von a ist. So ist a ein duo-Element von L. Dieses und Bedingung (iii) sichern die Regularität des Elementes a.

Ganz ähnlich kann man die Implikation (v)⇒(i) einsehen.

(iv)⇒(i). Ist l ein Linksabsorbent von a, so bekommt man aus (iv)

$$la \leqq l\vee la = (l\vee al)^2 \leqq l^2 \leqq l.$$

Dieses bedeutet, daß l ein Rechtsabsorbent von a ist.

Dualerweise sieht man ein, daß jeder Rechtsabsorbent von a auch ein Linksabsorbent von a ist.

Um die Regularität von a zu zeigen, betrachten wir einen Rechtsabsorbenten r und einen Linksabsorbenten l von a. Da das Element a duo ist, sind die Elemente r, l und $r\wedge l$ Absorbenten von a. So bekommt man

$$r\wedge l = (r\wedge l)\vee(r\wedge l)a = (r\wedge l)\vee a(r\wedge l).$$

Dieses und Bedingung (iv) implizieren $r\wedge l = (r\wedge l)^2 \leqq rl$, womit die Regularität von a bewiesen ist.

Damit ist der Beweis beendet.

Bemerkung. Spezialisiert man den Satz für die regulären duo-Ringe und duo-Halbgruppen, so liefern die Bedingungen (ii) und (iv) nach unserem Wissen neue Charakterisierungen dieser Strukturklassen. Wir möchten hier nur das folgende Korollar erwähnen:

Ein assoziativer Ring (eine Halbgruppe) A ist dann und nur dann regulär und duo, wenn jede Quasiideale K_1, K_2 *von A die Bedingung* $K_1 \cap K_2 = K_1 K_2$ *erfüllen.*

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On minimal biideals of rings

By FERENC A. SZÁSZ in Budapest

In this paper by a ring we always mean an associative ring (cf. N. JACOBSON [4]). For arbitrary subsets C and D of a ring A the product CD will mean the subgroup generated by all products $c \cdot d$ with $c \in C$ and $d \in D$. By a biideal B of a ring A we understand a subring B of A satisfying the condition $BAB \subseteqq B$.

Obviously, every one-sided ideal is a biideal. The biideals for semigroups are special cases of the (m, n)-ideals, introduced by S. LAJOS [5]. The concept of biideal for semigroups was introduced by R. A. GOOD and D. R. HUGHES [3] (in addition A. H. CLIFFORD—G. B. PRESTON [2]). For biideals of rings we refer the reader to [7], whose Proposition 3 asserts that for any biideal B and any subset T of a ring A the products BT and TB are again biideals of A. Biideals of rings occurred earlier also in the author's papers [9] and [10]. Obviously, any biideal B of a two-sided regular ring A is, by $B \subseteqq BA = AB = BA \cap AB = BA$, $AB = BAB \subseteqq B$, a two-sided ideal of A (cf. S. LAJOS and the author [6]). Important particular cases of biideals are the quasiideals which were studied for rings by O. STEINFELD [8]. The quasiideal Q of a ring A is in fact a submodule satisfying $QA \cap AQ \subseteqq Q$.

J. CALAIS [1] gave an example of a biideal, which is a product of two quasiideals, but which itself is not a quasiideal. But it is still an open problem whether there exists a ring A having a minimal biideal B such that $B^2 = 0$ and B is not a quasiideal of A.

In this paper we are interested in minimal biideals of rings.

Theorem 1. *If the biideal B of a ring A is a division ring, then B is a minimal biideal of A.*

Proof. Assume that C is an arbitrary biideal of A satisfying $C \subseteqq B$. Then $CAC \subseteqq C$ implies $CBC \subseteqq C$ and thus C is also a biideal of B. Consequently, by Theorem 1 of [7] C is a left ideal of a right ideal of B. But the division ring B has only trivial left ideals and right ideals, and therefore either $C = 0$ or $C = B$. Consequently B is a minimal biideal of A.

Conversely, we have also the following

Theorem 2. *For any minimal biideal B of a ring A the following holds: either $B^2=0$, or B is a division ring.*

Proof. By Proposition 3 of [7], B^2 is a biideal of A, and by the assumed minimality of B, we have either $B^2=0$ or $B^2=B$. We assume $B^2=B$ which implies $B^3=B$.

First we prove the existence of the two-sided unity element of the subring B and then we show that B is a division ring.

We note that the definition of the biideal B of a ring A implies that the set S of all elements xby (b runs over B, x and y are fixed elements of B) coincides with the subring generated by the set S.

Since $B^3=B$, there exist elements $b_1, b_2 \in B$ with $b_1Bb_2 \neq 0$. B being a subring, by Proposition 3 of [7] we have $b_1Bb_2=B$ and, by $0\neq B=B^2=b_1Bb_2b_1Bb_2$, obviously $b_2b_1\neq 0$, too. Since $b_1Bb_2=B$, there exist two elements $b_3, b_4 \in B$ satisfying $b_1=b_1b_3b_2$ and $b_2=b_1b_4b_2$, whence

$$0\neq b_2b_1=b_1b_4b_2b_1b_3b_2=b_1b_4(b_2b_1)=(b_2b_1)b_3b_2\in Bb_2b_1\cap b_2b_1B$$

follows. $Bb_2b_1\subseteq B$, $b_2b_1B\subseteq B$ being true, Propositions 1 and 3 of [7] imply that $Bb_2b_1\cap b_2b_1B$ is also a biideal of A which is contained in B. Thus the fact $b_2b_1\neq 0$ and the minimality of B give $B = Bb_2b_1\cap b_2b_1B$. Consequently, there exist four further elements b_5, b_6, b_7 and b_8 of B satisfying

$$b_1=b_5b_2b_1=b_2b_1b_6\neq 0 \quad \text{and} \quad b_2=b_7b_2b_1=b_2b_1b_8\neq 0.$$

As for the element $e=b_5b_2b_1b_8$, since $b_1\neq 0$ and $b_2\neq 0$, we first observe that

$$0\neq e=b_5b_2b_1b_8=b_1b_8=b_5b_2$$

and

$$e^2=(b_5b_2)(b_1b_8)=e\in B$$

hold. Furthermore $e=e^3\in eBe\subseteq B$, thus Proposition 3 of [7] and the minimality of B imply $B=eBe$.

Therefore e is the two-sided unity element of the subring B.

Let ebe be any nonzero element of eBe. Then $B'=eBe$, ebe is contained in eBe. Furthermore, since $e^3\cdot ebe\neq 0$, by virtue of Proposition 3 of [7], and the minimality of B, B' is a nonzero biideal of A, consequently $B'=B$. Thus there exists an element $eb'e\in B$ satisfying $eb'e\cdot ebe = e$.

Therefore B is a division ring indeed, which completes the proof.

Theorem 3. *If a minimal biideal B of a ring A contains an element b such that b is neither a left divisor of zero, nor a right divisor of zero in A, then A must have a two-sided unity element.*

Proof. Evidently $b^3 \neq 0$. Then, since $b^3 \in bAb \subseteqq B$, in virtue of Proposition 3 of [7] and the minimality of B we have $bAb = B$. Hence there exists an element $a \in A$ such that $b = bab$ holds. Then for any $x \in A$ and $y \in A$, by making use of the two-sided cancelling rule concerning b, we obtain from $xb = xbab$ and $by = baby$ that $x = xba$ and $y = aby$. Consequently $e = ba$ is a right unity element and $f = ab$ a left unity element of A, therefore $e = fe = f$ is the two-sided unity element of the ring.

Theorem 4. *If R is a minimal right ideal and L a minimal left ideal of a ring A, then either $RL = 0$ or RL is a minimal biideal of A.*

Proof. Assume $RL \neq 0$. If B' is a biideal of A satisfying $0 \neq B' \subset B = RL$, then from $B' \subset RL \subseteqq R$ we conclude that $B'A \subseteqq R$. The minimality of R also implies $B'A = R$, because in the case $B'A = 0$ the biideal B' is also a nontrivial right ideal of A which is contained in R. Similarly one also has $L = AB'$ and thus the contradiction

$$B = RL = B'A \cdot AB' \subseteqq B'AB' \subseteqq B' \subset B$$

completes the proof of Theorem 4.

In some special cases the converse statement to Theorem 4 also holds. In fact we have

Theorem 5. *Any minimal biideal B of a ring A without nonzero nilpotent ideals can be represented in the form $B = RL$, where R is a minimal right ideal and L is a minimal left ideal of A.*

Proof. By virtue of $BAB \subseteqq B$ and Proposition 3 of [7] we have $BAB = B$. In fact, in case $BAB = 0$, the right ideal BA is nilpotent, consequently $BA = 0$, $B^2 = 0$, $B = 0$, which is impossible. Therefore $B = BABAB$, which, by virtue of $BABAB \subseteqq \subseteqq BA^2B \subseteqq BAB$, implies $B = BA \cdot AB$.

We shall prove that $R = BA$ is a minimal right ideal, and $L = AB$ is a minimal left ideal of A.

If R' is a right ideal of A satisfying $0 \subset R' \subset R$, then by Proposition 3 of [7] $B' = R'AB$ is a biideal of A such that $B' \subseteqq BAAB \subseteqq B$ holds. By the minimality of B we have either $B' = 0$ or $B' = B$. But $B' = 0$ implies $R'AR' \subseteqq R'ABA = B'A = 0$, $(R'A)^2 = 0$, $R'A = 0$, $(R')^2 = 0$, $R' = 0$, which is impossible. Therefore $B' = B$, consequently $B = R'AB \subseteqq R'$, $BA \subseteqq R'A \subseteqq R' \subset BA$. This is a contradiction, and thus the verification of the minimality of the right ideal $R = BA$ is complete. For $L = AB$ the proof is similar.

Theorem 6. *Any ring A without nonzero nilpotent ideals and with minimum condition on principal right ideals is a sum of minimal biideals of A.*

Proof. By [9] we have $A=\sum_{\alpha} R_\alpha=\sum_{\beta} L_\beta$, where R_α are minimal right ideals and L_β minimal left ideals of A. Then $A=A^2=\sum_{\alpha,\beta} R_\alpha L_\beta$, and Theorem 4 implies Theorem 6.

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Axiomatic characterization of $\sum$-semirings

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In memoriam A. Rényi

§ 1. Introduction

J. PŁONKA [4] introduced the concept of a sum of a join-direct system of algebras and showed that if we form a sum of a non-trivial join-direct system of algebras in an equational class the new algebra satisfies only those regular equations which are satisfied in all algebras of the direct system.

Now if we take the equational class $\mathfrak{R}$ of all associative rings and form all possible sums of join-direct systems over it, we obtain an equational class $\mathfrak{R}_\Sigma$ of additively commutative semirings. By a *semiring* $\langle R, +, \circ \rangle$ we mean a universal algebra with two associative operations $+$ and $\circ$, such that $\circ$ is distributive with respect to $+$. It is *additively commutative* if $\langle R, + \rangle$ is a commutative semigroup.

It is not true that all the additively commutative semirings can be obtained by sums of joint-direct systems over associative rings.

In this note we give a simple axiomatic characterization of those semirings R which are in $\mathfrak{R}_\Sigma$, and we call them Σ-semirings. Every Σ-semiring has a unique way of representation as a sum of join-direct system of rings.

§ 2. Basic concepts and lemmas

Let $\langle I, \leqq \rangle$ be a join-semilattice, with join denoted by $\vee$.

A system $\mathfrak{U} = \langle \langle I, \leqq \rangle, \{R_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leqq j} \rangle$ is called a join-direct system of associative rings if it is a direct system of associative rings whose underlying index set is a join-semilattice and

(i) for each $i \in I$, $\langle R_i, +_i, \circ_i \rangle$ is an associative ring and $R_i \cap R_j = \emptyset$ for $i \neq j$.

(ii) If $i \leqq j$ in I, then $\varphi_{ij}: R_i \to R_j$ is a ring homomorphism, subject to the conditions: (a) $\varphi_{ii}(x) = x$ for all x in R_i, (b) $i \leqq j \leqq k$ in I, then $\varphi_{jk}\varphi_{ij} = \varphi_{ik}$.

Any join-direct system $\mathfrak{U}$ of associative rings gives us an additively commutative semiring R as follows:

Set $R=\bigcup_{i\in I} R_i$ and define $+$ and $\circ$ on R by

$x+y = \varphi_{ik}(x)+_k \varphi_{jk}(y)$ and $x\circ y=\varphi_{ik}(x)\circ_k \varphi_{jk}(y)$ if $x\in R_i$, $y\in R_j$, and $k=i\vee j$.

Then $\langle R, +, \circ\rangle$ is an additively commutative semiring. We shall call it the *sum* of $\mathfrak{U}$ and denote it by $R=S(\mathfrak{U})$.

Now let us define a unary operation $*$ on R by setting $*x = -x$ if $x\in R_i$, where $-x$ is the additive inverse of x in R_i.

It can be seen that $*$ has the following properties, for all x and y in R:

(1) $*(*x)=x$, (2) $x+(*x)+x = x$, (3) $*(x+y) = (*x)+(*y)$,

(4) $x\circ(*y) = (*x)\circ y = *(x\circ y)$, (5) $(x+(*x))\circ y = x+(*x)+y+(*y)$.

Now we can state our main theorem.

Theorem 1. *A semiring* $\langle R, +, \circ\rangle$ *is a* Σ*-semiring if and only if:*

(A) $\langle R, +, \circ\rangle$ *is additively commutative and*

(B) *a unary operation* $*\colon R\to R$ *can be defined satisfying the above conditions* (1)—(5).

To demonstrate it, we shall need the following

Lemma 1. *Let R be a semiring satisfying conditions* (A) *and* (B) *of the above theorem. Then we have, for all x and y in R,*

(a) $x\circ(y+(*y)) = (y+(*y))\circ x = (x+(*x))\circ y = y\circ(x+(*x))$,

(b) *if* $x+(*x)+y+(*y) = y+(*y)$, *then* $x\circ(y+(*y)) = y+(*y)$,

(c) *if* $x+(*x) = y+(*y)$, *then* $x\circ(y+(*y)) = y+(*y)$.

Proof. (a) follows immediately by interchanging the variables x and y in (5), using commutativity of $+$ and distributivity of $\circ$ with respect to $+$. (b) is trivial and (c) follows from (2) and (b).

Lemma 2. *Let R be a semiring satisfying the conditions* (A) *and* (B) *of the theorem. Let* $E(R) = \{x+(*x)\mid x\in R\}$. *Then $E(R)$ is the set of all additive idempotents of R, and all elements of $E(R)$ are multiplicative idempotents. Furthermore, if we define $\leqq$ on $E(R)$ by setting $a\leqq b$ if and only if $a+b = b$ for $a, b\in E(R)$, then* $\langle E(R); \leqq\rangle$ *is a join-semilattice.*

Proof. Let $x \in R$, then

$$(x+(^*x))+(x+(^*x)) = (x+(^*x)+x)+(^*x) = x+(^*x) \quad \text{by (2)},$$

therefore $x+(^*x)$ is an additive idempotent.

Conversely, suppose e is an additive idempotent in R. Then by (2), $e = e+e+ +(^*e) = e+(^*e)$ is in $E(R)$.

Observe $^*(x+(^*x)) = (^*x)+(^*(^*x)) = (^*x)+x = x+(^*x)$, and by Lemma 1 (b) we have $x \circ (x+(^*x)) = x+(^*x)$, $(^*x)\circ(x+(^*x)) = x+(^*x)$. Therefore $x\circ(x+(^*x))+(^*x)\circ(x+(^*x)) = x+(^*x)+x+(^*x)$ and then $[x+(^*x)]\circ \circ[x+(^*x)] = x+(^*x)$. Therefore $x+(^*x)$ is a multiplicative idempotent. Clearly under the relation $\leqq$, $E(R)$ becomes a partially ordered set. Let $e, f \in E(R)$. We claim that $e+f = e \vee f$. Since

$$e+(e+f) = (e+e)+f = e+f$$

we have $e \leqq e+f$. Similarly, $f \leqq e+f$. Suppose $e, f \leqq g$ in $E(R)$. Then $e+g = g$, $f+g = g$. Thus $(e+g)+(f+g) = g+g$ so $(e+f)+g = g$. Therefore, $e+f \leqq g$. This shows $e \vee f = e+f$. Hence $\langle E(R); \leqq \rangle$ is a join-semilattice.

§ 3. Proof of the theorem

The necessity of the conditions (A) and (B) was proved in § 2.

Now suppose we have a semiring R which satisfies the conditions of the theorem. Define a relation $\equiv$ on R as follows: $x \equiv y$ if and only if $x+(^*x) = y+(^*y)$.

Clearly $\equiv$ is an equivalence and therefore partitions R into disjoint classes. It is clear that each class contains one and only one element of $E(R)$. Therefore, we denote the class containing an element a of $E(R)$ by R_a. Define $+_a$ and $\circ_a$ on R_a by restricting the operations $+$ and $\circ$ of R to R_a.

We want to show that $\langle R_a, +_a, \circ_a \rangle$ is an associative ring with a as its zero.

First we show that R_a is closed under $+_a$ and $\circ_a$. Let $x, y \in R_a$, then $x+(^*x) = y+(^*y) = a$. Thus $(x+y)+(^*(x+y)) = x+(^*x)+y+(^*y) = a+a = a$, $(x\circ y)+(^*(x\circ y)) = (x\circ y)+(^*x)\circ y = (x+(^*x))\circ y = x+(^*x)$ by Lemma 1(c). Therefore $x+y$, $x\circ y \in R_a$. Moreover, it is clear that $^*x \in R_a$.

To see that $\langle R_a, +_a \rangle$ is an abelian group with zero a, let $x \in R_a$. Then

$$x+a = x+(x+(^*x)) = x \quad \text{and} \quad x+(^*x) = a.$$

Hence $\langle R_a, +_a, \circ_a \rangle$ is an associative ring.

Now for each $a \leqq b$ in $E(R)$, define a map $\varphi_{ab}\colon R_a \to R_b$ by $\varphi_{ab}(x) = x+b$ for all x in R_a. Then

I) φ_{ab} is a ring homomorphism. Let $x, y \in R_a$. Then

$$\varphi_{ab}(x+y) = x+y+b = (x+b)+(y+b) = \varphi_{ab}(x) +_b \varphi_{ab}(y)$$

and

$$\begin{aligned}\varphi_{ab}(x) \circ_b \varphi_{ab}(y) &= (x+b)\circ(y+b) = x\circ y + b\circ y + x\circ b + b\circ b \\ &= x\circ y + b\circ y + b\circ x + b\circ b && \text{(by Lemma 1(a))} \\ &= x\circ y + b\circ(x+y) + b && \text{(by Lemma 2)} \\ &= x\circ y + b + b && \text{(by Lemma 1(b))} \\ &= x\circ y + b = \varphi_{ab}(x\circ y).\end{aligned}$$

II) $\varphi_{aa}(x) = x+a = x+x+(^*x) = x$ for all x in R_a.

III) If $a \leqq b \leqq c$ in $E(R)$, then $\varphi_{bc}\varphi_{ab} = \varphi_{ac}$ because

$\varphi_{bc}(\varphi_{ab}(x)) = \varphi_{bc}(x+b) = (x+b)+c = x+c = \varphi_{ac}(x)$ for all x in R_a.

The proof will be complete if we show that

$$R = S(\langle\langle E(R); \leqq\rangle, \{R_a\}_{a\in E(R)}, \{\varphi_{ab}\}_{a\leqq b}\rangle).$$

Clearly $R = \bigcup_{a\in E(R)} R_a$. Define operations $\oplus$ and $\odot$ on R as follows: for x, y in R

$x \oplus y = \varphi_{ac}(x) +_c \varphi_{bc}(y)$ and $x \odot y = \varphi_{ac}(x) \circ_c \varphi_{bc}(y)$ if $x \in R_a$, $y \in R_b$, $c = a+b$.

We want to show that $\oplus = +$ and $\odot = \circ$.

Let $x, y \in R$, $x \in R_a$, $y \in R_b$, $c = a+b$. We have

$$\begin{aligned}x\oplus y &= \varphi_{ac}(x) +_c \varphi_{bc}(y) = (x+c)+(y+c) = x+y+c = \\ &= x+y+a+b = (x+a)+(y+b) = \varphi_{aa}(x)+\varphi_{bb}(y) = x+y.\end{aligned}$$

Also

$$\begin{aligned}x\odot y &= \varphi_{ac} \circ_c \varphi_{bc}(y) = (x+c)\circ(y+c) = x\circ y + c\circ y + x\circ c + c\circ c = \\ &= x\circ y + c + c + c \text{ (by Lemma1 (b))} = x\circ y + c.\end{aligned}$$

Now $x\circ y \in R_c$ for

$$\begin{aligned}x\circ y + (^*(x\circ y)) &= x\circ y + (^*x)\circ y = (x+(^*x))\circ y \\ &= (x+(^*x)) + (y+(^*y)) = a+b = c && \text{(Condition (5)).}\end{aligned}$$

Therefore $x \odot y = \varphi_{cc}(x\circ y) = x\circ y$. Hence

$$R = S(\langle\langle E(R), \leqq\rangle, \{R_a\}_{a\in E(R)}, \{\varphi_{ab}\}_{a\leqq b}\rangle).$$

Corollary. *The class of all Σ-semirings form an equational class of semirings and it includes the class of all associative rings as an equational subclass.*

§ 4. Some remarks on Σ-semirings

Remark 1. It is clear that every Σ-semiring is an additively regular semiring, i.e., a semiring such that the equation $a+x+a=a$ always has a solution (cf. [1]). However, not all additively commutative and additively regular semirings are Σ-semirings.

Consider the 3-element additively commutative and additively regular semiring R with the following tables:

$+$	a	b	c
a	a	c	c
b	c	b	c
c	c	c	c

$\circ$	a	b	c
a	b	b	b
b	b	b	b
c	b	b	b

The only possible unary operation $*: R\to R$ which can be defined that satisfies condition (B) (1)—(4) is:

$$^*a=a, \quad ^*b=b, \quad ^*c=c.$$

However $(a+(^*a))\circ b \neq a+(^*a)+b+(^*b)$.

Remark 2. Additively regular semirings arise naturally if we consider the endomorphism semiring of a Σ-semimodule over a ring R.

By a Σ-semimodule we mean a system $\langle M, +, \{f_a\}_{a\in R}, ^*\rangle$ where:

(1) $\langle M, +\rangle$ is a commutative semigroup,

(2) for each $a\in R$, $f_a: M\to M$ satisfies:

$$f_a(x+y)=f_a(x)+f_a(y), \quad f_{a+b}(x)=f_a(x)+f_b(x), \quad f_{a\circ b}(x)=f_a(f_b(x)),$$

(3) $*: M\to M$ satisfies:

$$^*(^*x)=x, \quad f_r(^*x)={}^*(f_r(x)), \quad ^*(x+y)=(^*x)+(^*y),$$

$$x+{}^*x+x=x, \quad f_r(x+(^*x))=x+(^*x).$$

The concept of Σ-semimodule is the generalization of the usual left R-module. In [3], it was shown that every Σ-semimodule M is a sum of join-direct system of R-modules, i.e. $M=S(\langle\langle E(M);\leqq\rangle, \{M_a\}_{a\in E(M)}, \{\psi_{ab}\}_{a\leqq b}\rangle)$, where $E(M)$ is the set of all idempotents of M and M_a is R-module for each $a\in E(M)$. $\psi_{ab}: M_a\to M_b$ is a module homomorphism which takes x to $x+b$ for all x in M_b.

A mapping $\varphi: M\to M$ is called an R-endomorphism of M if for $x, y\in M$ and $a\in R$ we have

$$\varphi(x+y)=\varphi(x)+\varphi(y), \quad \varphi(f_a(x))=f_a(\varphi(x)), \quad \varphi(^*x)={}^*(\varphi(x)).$$

Let $\mathrm{End}_R(M)$ denote the set of all R-endomorphisms of M.
For $\varphi, \psi \in \mathrm{End}_R(M)$ we define:

$$(\varphi+\psi)(x) = \varphi(x)+\psi(x), \quad (^*\varphi)(x)={}^*(\varphi(x)), \quad (\varphi\circ\psi)(x) = \varphi(\psi(x)).$$

Then $\langle \mathrm{End}_R(M), +, \circ\rangle$ is an additively commutative semiring and * satisfies conditions (1)—(4) of Theorem 1.

Theorem 2. *Let* $A=\langle \mathrm{End}_R(M), +, \circ\rangle$ *be the endomorphism semiring of a* Σ*-semimodule* M. A *is a* Σ*-semiring if and only if* M *is an* R*-module and in this case* A *is a ring.*

Proof. The if part is straightforward. Suppose A is a Σ-semiring then for each $\varphi, \psi \in A$ we have $(\varphi+(^*\varphi))\circ\psi = \varphi+(^*\varphi)+\psi+(^*\psi)$.

Now let $x\in M$ then we have

$$\begin{aligned}((\varphi+(^*\varphi))\circ\psi)(x) &= (\varphi+(^*\varphi))(\psi(x)) = \varphi(\psi(x))+(^*\varphi)(\psi(x)) = \\ &= (\varphi\circ\psi)(x)+(\varphi\circ\psi)(^*x) = (\varphi\circ\psi)(x+{}^*x), \\ (\varphi+(^*\varphi)+\psi+(^*\psi))(x) &= \varphi(x)+(^*\varphi)(x)+\psi(x)+(^*\psi)(x) = \\ &= (\varphi+\psi)(x)+(\varphi+\psi)(^*x) = (\varphi+\psi)(x+{}^*x).\end{aligned}$$

This implies the restrictions $\varphi+\psi|_{E(M)}$ and $\varphi\circ\psi|_{E(M)}$ are equal.

Now if $E(M)$ has more than 2 elements, say $a\leqq b$, consider the following two R-endomorphisms of M

$$\varphi_1(x)=a, \quad \varphi_2(x)=b \quad \text{for every} \quad x\in M.$$

Then $(\varphi_1+\varphi_2)(x) = a+b = b$ and $(\varphi_1\circ\varphi_2)(x) = \varphi_1(\varphi_2(x)) = \varphi_1(b) = a$ for every $x\in M$. Therefore $\varphi_1+\varphi_2 \neq \varphi_1\circ\varphi_2$ on $E(M)$: a contradiction. Thus $|E(M)|=1$ which implies that M is an R-module.

Remark 3. S. M. Yusuf [6] called an additively commutative semiring whose additive semigroup is an inverse semigroup an *additively inversive hemiring*.

If we take away (4) and (5) in condition (B) of Theorem 1, we obtain an axiomatic characterization of additively inversive hemirings. This implies immediately that the class of all additively inversive hemirings is an equational class which contains the class of Σ-semirings as an equational subclass.

Since (4) always holds in additively inversive hemiring, if we consider Σ-semirings as algebras of type $\langle 2, 2, 1\rangle$, they can be defined by the following independent axioms: 1) $\langle R, +, \circ\rangle$ is an additively commutative semiring, 2) $^*(^*x)=x$, 3) $^*(x+y) = {}^*x+{}^*y$, 4) $x+(^*x)+x = x$, 5) $x\circ(y+(^*y)) = x+(^*x)+y+(^*y)$.

Remark 4. Let R be a $\bar{\Sigma}$-semiring. If we define a map $f: R^2 \to R$ by setting $f(x, y) = x+y+(^*y)$, then it can be checked that f is a *partition function* of R (for the terminology see [4]). By Theorem 2 of [4], it induces a sum-representation of R. This representation is essentially the same as the one we obtained in the proof of Theorem 1, and by Theorem 1 of [5] this is the only possible sum-representation of R by rings.

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Added in proof. Consider Σ-semirings as algebras of $\langle 2, 2, 1\rangle$, one can show that the lattice of equational subclasses of Σ-semirings is isomorphic to the direct product of the lattice of equational subclasses of associative rings and the two element chain. The following problem is still unsolved: what is the lattice of equational subclasses of additively inverse hemirings?

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Remarks on endomorphism rings of torsion-free abelian groups

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1. The commutativity of the endomorphism ring

In this paper we study endomorphism rings of torsion-free abelian groups. In [2], Problem 46(a) FUCHS asks to determine all abelian groups with commutative endomorphism ring. Later FUCHS has shown the following [3]. Call a family of groups $G_\alpha\,(\alpha\in I)$ a *rigid* system if Hom $(G_\alpha, G_\beta)=0$ or a subgroup of the rationals according as $\alpha\neq\beta$ or $\alpha=\beta$. To every cardinal m, less than the first inaccessible aleph, there exists a rigid system consisting of 2^m torsion-free groups of cardinality m. .

The groups in a rigid system are obviously always indecomposable and they have commutative endomorphism rings. So the question arises: if the endomorphism ring of a torsion-free abelian group G is commutative, is G then indecomposable? It is easy to construct a counter-example. Let p_1, p_2 be different primes. G_{p_1} is the group of the rationals whose denominators are powers of p_1; G_{p_2} is similar with respect to p_2. Then $\{G_{p_1}, G_{p_2}\}$ is a rigid system and $E(G)\cong E(G_{p_1})+E(G_{p_2})$ (ring-direct sum), since G_{p_i} is a fully invariant subgroup of $G=G_{p_1}+G_{p_2}$ (direct sum) $(i=1, 2)$. Hence $E(G)$ is commutative, but $G=G_{p_1}+G_{p_2}$ is decomposable.

Conversely, assume that G is an indecomposable group. Is $E(G)$ then a commutative ring? For well-known indecomposable groups, such as the group Z of integers, the group Q of rationals, the group $Z(p)$ of p-adic integers, any pure subgroup G of $Z(p)$, this is true. However, one can construct a counter-example as follows:

Let R be the ring of integer quaternions i.e. elements of the form $a_0+a_1 i+$ $+a_2 j+a_3 k$ with $a_i\in Z$ $(i=0, 1, 2, 3)$ and $i^2=j^2=k^2=-1$, $ij=k=-ji$, $ik=$ $=-j=-ki, jk=i=-kj$ with obvious addition and multiplication. R is a reduced, torsion-free ring of rank 4. By a theorem of CORNER [1] every reduced torsion-free ring A of finite rank n is isomorphic to the endomorphism ring $E(G)$ of some reduced, torsion-free group G of rank $2n$. Hence R is isomorphic to the endomorphism ring $E(G)$ of some reduced, torsion-free group G of rank 8.

Since R has no zero-divisors, the same is true for $E(G)$. Hence 0 and 1 are the only idempotents in $E(G)$. But this implies that G is indecomposable, for if $G = G_1 + G_2$ for subgroups G_1, G_2, then the projections $\pi_i: G \to G_i$, $i = 1, 2$, are orthogonal idempotents of $E(G)$ whose sum $\pi_1 + \pi_2 = 1$. So we get either $\pi_1 = 1$, $\pi_2 = 0$ or $\pi_1 = 0$, $\pi_2 = 1$ which means either $G_2 = 0$ or $G_1 = 0$. Hence G is indecomposable, but $E(G) \cong R$ is not commutative. Thus we have to impose stronger conditions on the group G in order that its ring of endomorphisms be commutative. We recall from [4]:

Definition 1. (cf. [4], definition 2. 1) For groups G and H, we say that

(i) G is quasi-contained in H ($G \dot{\subseteq} H$) if $nG \subseteq H$ for some non-zero integer n;
(ii) G is quasi-equal to H ($G \doteq H$) if $G \dot{\subseteq} H$ and $H \dot{\subseteq} G$;
(iii) G is quasi-decomposable if there exist non-zero independent groups A and B such that $G \doteq A + B$;
(iv) G is strongly indecomposable if G is not quasi-decomposable.

Now suppose that G is a torsion-free group of rank 2. Then G is strongly indecomposable or $G = G_1 + G_2$, $G_1 \cong G_2$, or $G \doteq G_1 + G_2$, G_i of incomparable types, or $G \doteq S + B$, type $B <$ type S.

Let $E(G)$ be the ring of endomorphisms of G. Then $E(G)$ is a torsion-free ring and $QE(G)$ is the minimal Q-algebra containing $E(G)$. $QE(G)$ can be characterized as the set of linear transformation Φ of QG (minimal Q-algebra containing G) such that $n\Phi(G) \subseteq G$ for some $n \neq 0$ in Z.

The algebra $QE(G)$ is the *ring of quasi-endomorphisms* of G and will be denoted by $\mathrm{E}(G)$. Now if G is strongly indecomposable then $\mathrm{E}(G)$ is a quadratic number field, Q, or the ring of 2×2 triangular matrices $\left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| a, b \in Q\right\}$ with equal diagonal elements. In all cases $\mathrm{E}(G)$ is commutative, hence $E(G)$, which is a subring of $\mathrm{E}(G)$, is commutative. Hence:

If G is a strongly indecomposable group of rank 2, then $E(G)$ is commutative.

Although the condition of strong indecomposability of G is sufficient for the commutativity of $E(G)$ it is not necessary, as may be seen from $G = G_1 + G_2$, G_i of incomparable types (cf. first counter-example). We can extend this result to torsion-free groups of prime rank, in case G is irreducible.

Definition 2. A group G is irreducible if it has no proper non-trivial pure fully invariant subgroups (cf. [4], definition 5. 1).

Now let G be a strongly indecomposable group of prime rank. If G is irreducible, then $E(G)$ is commutative. By Corollary 5. 6 [4], $\mathrm{E}(G) = \Gamma$ is a division ring and by Theorem 5. 5, $[\Gamma : Q] = \text{rank } G = p$ (p a prime).

Now let F be the center of Γ, then $[\Gamma:Q]=[\Gamma:F][F:Q]=p$; but $[\Gamma:F]=n^2$, so $n^2|p$ which implies $n=1$, hence $\Gamma=F$ or $\mathrm{E}(G)=\Gamma$ is commutative. Then $E(G)$, as a subring of $\mathrm{E}(G)$, is commutative. For irreducible groups G of prime rank, REID [4] has shown that G is either strongly indecomposable or equal to a direct sum of isomorphic rank one groups. Hence for these groups indecomposability implies strongly indecomposability. Hence:

Theorem 1. *Let G be an irreducible, indecomposable torsion-free group of prime rank. Then $E(G)$ is commutative.*

One might ask whether strong indecomposability is always sufficient for commutativity of the endomorphism ring. The answer is no and the counter-example is again the ring R of integer quaternions. As we have seen, $R\cong E(G)$, where G is a reduced torsion-free group of rank 8. Now the ring $\mathrm{E}(G)$ of quasi-endomorphisms of G is the quaternion field F with basis 1, i, j, k over Q.

Since F is a field it is a *local ring*, that is, a ring R with identity such that $R/J(R)$ is a division ring, where $J(R)$ is the Jacobson radical of R.

By Corollary 4. 3 [4], a torsion-free group G of finite rank is strongly indecomposable if and only if $\mathrm{E}(G)$ is a local ring. Since $F=\mathrm{E}(G)$ is such a ring, it follows that G is strongly indecomposable. However, $E(G)\cong R$ is not commutative.

For the class of irreducible groups of prime rank we have seen that they are either strongly indecomposable or equal to a direct sum of isomorphic rank one groups. Now assume that G is such a group and $E(G)$ is commutative. Then the number of direct summands in a direct sum representation of G cannot be greater than one.

Hence G is strongly indecomposable or G is a rank one group. A rank one group is clearly strongly indecomposable. Hence, if we use Theorem 1, we get:

Theorem 2. *Let G be an irreducible group of prime rank. Then $E(G)$ is commutative if and only if G is strongly indecomposable.*

If we omit the condition that the rank of G should be prime, we have the following result:

Theorem 3. *Let G be an irreducible group of finite rank k, such that k is square free. Then $E(G)$ is commutative if and only if G is strongly indecomposable.*

Proof. Assume $E(G)$ is commutative, then $\mathrm{E}(G)$ is commutative. Since G is irreducible, $\mathrm{E}(G)=\Gamma_m$ where Γ is a division algebra, m is the number of strongly indecomposable summands in a quasi-decomposition of G and $m[\Gamma:Q]=\operatorname{rank} G$ [4]. Since Γ_m is commutative, it follows that $m=1$, $\mathrm{E}(G)=\Gamma$ and G is strongly indecomposable. Conversely, assume that G is strongly indecomposable. Since G is irreducible, G has a quasi-decomposition $G \doteq \sum_{i=1}^{m} G_i$ with each G_i strongly indecomposable [4]. It follows that $m=1$ and $\mathrm{E}(G)=\Gamma$ is a division ring. Moreover $[\Gamma:Q]=\text{rank } G=k$. Since the dimension of Γ over its center must be a square dividing k, this dimension is 1 and $\mathrm{E}(G)=\Gamma$ is commutative. Hence $E(G)$ is commutative. Note that Theorem 2 is a special case of Theorem 3.

From [4] we use the

Definition 3. Let G be a torsion-free group of finite rank. Let S be the pure subgroup of G generated by the collection of non-zero minimal pure fully invariant subgroups of G. We call S the pseudo-socle of G.

Reid [4] has shown that $G=S$ if and only if $\mathrm{E}(G)$ is semi-simple. So we investigate the commutativity of $E(G)$ under the condition that the radical of $\mathrm{E}(G)$ is zero. First we remark that the quasidecomposition of a torsion-free group of finite rank is essentially unique i.e. if G has finite rank then any quasi-decomposition of G has only finitely many summands and if

$$\sum_{i=1}^{s} H_i \doteq G \doteq \sum_{j=1}^{t} K_j$$

with the H_i and K_j strongly indecomposable ($i=1, \dots, s$; $j=1, \dots, t$), then $s=t$ and for some permutation π of $\{1, 2, \dots, t\}$ we have K_j is quasi-isomorphic to $H_{\pi(j)}$ ($j=1, \dots, t$) [4].

Theorem 4. *Let G be a torsion-free group of finite rank with $\mathrm{E}(G)$ semi-simple. but not simple. Then $E(G)$ is commutative if and only if in any quasi-decomposition of G the summands have commutative endomorphism rings.*

Proof. Assume $E(G)$ is commutative, then $\mathrm{E}(G)$ is commutative. Since $\mathrm{E}(G)$ has D.C.C. on right ideals and is semi-simple, we get $\mathrm{E}(G) \cong \Delta_1 + \dots + \Delta_m$ (direct sum), where Δ_i is a field ($i=1, \dots, m$). Identify $\mathrm{E}(G)$ with this direct sum and write $\mathrm{E}(G) = \sum_{i=1}^{m} f_i \mathrm{E}(G)$, where $\Delta_i = f_i \mathrm{E}(G)$ ($i=1, \dots, m$) and f_i induces the projection of $\mathrm{E}(G)$ onto Δ_i. To this decomposition of $\mathrm{E}(G)$ there corresponds a quasi-decomposition of $G \doteq \sum_{i=1}^{m} Gf_i$ with $\mathrm{E}(Gf_i) \cong f_i \mathrm{E}(G) f_i = \Delta_i$, so that $\mathrm{E}(Gf_i)$ is a field. Hence Gf_i is strongly indecomposable ($i=1, \dots, m$) ([4], Corollary 4. 3). Hence any quasi-decomposition of G has m strongly indecomposable summands and each of these summands has a commutative quasi-endomorphism ring and therefore a commutative endomorphism ring.

Conversely, assume that the condition for G with respect to quasi-decomposability is satisfied. Since $\mathrm{E}(G)$ has D.C.C. on right ideals and is semi-simple, it may be identified with a finite direct sum of matrix rings over division rings: $\mathrm{E}(G)=$

$= \Delta_1 + \cdots + \Delta_n$ (Wedderburn). This implies there is a set $\{e_1, \ldots, e_n\}$ of non-zero mutually orthogonal idempotents of $\mathrm{E}(G)$ whose sum is the identity in $\mathrm{E}(G): 1 = e_1 + e_2 + \cdots + e_n$. Then there is a quasi-decomposition $G \doteq \sum_{i=1}^{n}{}' Ge_i$ of G, which corresponds to the direct decomposition of $\mathrm{E}(G)$ ([4], Theorem 3. 1). Now $\mathrm{E}(Ge_i) \cong e_i \mathrm{E}(G) e_i = \Delta_i e_i = \Delta_i$, since e_i is the unit element for Δ_i, so that Δ_i must be commutative. Hence $\mathrm{E}(G)$ is commutative and therefore $E(G)$ is commutative. This completes the proof of the theorem.

From the semi-simplicity of $\mathrm{E}(G)$ one easily derives that the components Ge_i in a quasi-decomposition of G have a semi-simple quasi-endomorphism ring $\mathrm{E}(Ge_i)$, since the radical of $e_i \mathrm{E}(G) e_i$ $(\cong \mathrm{E}(Ge_i))$ is $e_i N e_i$, where N is the radical of $\mathrm{E}(G)$. Hence Theorem 4 reduces the case of groups G of finite rank with $\mathrm{E}(G)$ semi-simple but not simple to the case of strongly indecomposable groups G of finite rank with $\mathrm{E}(G)$ semisimple but not simple.

Next assume that G is a strongly indecomposable group with semi-simple $\mathrm{E}(G)$. Then $\mathrm{E}(G)$ is a division algebra ([4], Corollary 4. 3). Now we have the following sufficient condition in order that $E(G)$ be commutative: *G has a commutative $E(G)$ if G has a non-zero minimal pure fully invariant subgroup P, whose rank k is square-free.*

(Note that the case $G = P$ or G is irreducible is contained in Theorem 3.)

Indeed, if the condition is satisfied, then rank $P = [\mathrm{E}(G): Q] = k$, k square-free. Since the dimension of $\mathrm{E}(G)$ over its center must be a square dividing k, $\mathrm{E}(G)$ is commutative and an algebraic number field. Hence $E(G)$ is commutative.

The condition is satisfied if the rank of G is 2 or 3. If G is irreducible, $G = P$ and the rank of G is square-free. If G is not irreducible, there exists a minimal non-zero pure fully invariant subgroup P in G, distinct from G, and the rank of P is 1 or 2. Hence the condition is satisfied.

2. The Jacobson radical

All the groups G considered here are torsion-free groups of finite rank. So $\mathrm{E}(G)$ always satisfies the D.C.C. for right ideals. It is well known that under this condition G is strongly indecomposable if and only if $\mathrm{E}(G)|N$ is a division ring, where N is the Jacobson radical of $\mathrm{E}(G)$ (Corollary 4. 3, [4]), i.e. $\mathrm{E}(G)$ is a local ring.

We prove now

Theorem 5. *Let G be a torsion-free group such that* $\mathrm{E}(G)$ *satisfies the D.C.C. on right ideals. Then the Jacobson radical of $E(G) (= J(E(G)))$ is zero implies that the Jacobson radical of* $\mathrm{E}(G) (= J(\mathrm{E}(G)))$ *is zero i.e.* $\mathrm{E}(G)$ *is semi-simple.*

Proof. Since $\mathrm{E}(G)$ satisfies D.C.C. for right ideals, $J(\mathrm{E}(G))$ coincides with the union of all left nilpotent ideals in $\mathrm{E}(G)$ and $J(\mathrm{E}(G))$ is nil. Hence $J(\mathrm{E}(G))$ is a pure ideal in $\mathrm{E}(G)$, since the nil radical of a torsion-free ring is a pure ideal ([2], p. 271). It follows that nil radical of $E(G) = E(G) \cap$ nil radical of $\mathrm{E}(G)$, according to the correspondence between pure ideals in $E(G)$ and $\mathrm{E}(G)$. So we get nil radical of $E(G) = E(G) \cap J(\mathrm{E}(G))$ and then $E(G) \cap J(\mathrm{E}(G)) \subseteqq J(E(G))$.

Now suppose $J(E(G)) = 0$ and let $\varphi \in J(\mathrm{E}(G))$. Then $\varphi \in \mathrm{E}(G)$, so $\exists n \neq 0 \in Z$ such that $n\varphi \in E(G)$. Also $n\varphi \in J(\mathrm{E}(G))$, hence $n\varphi \in J(\mathrm{E}(G)) \cap E(G) \subseteqq J(E(G)) = 0$, so $n\varphi = 0$, which implies $\varphi = 0$, since $\mathrm{E}(G)$ is torsion-free. Hence $J(\mathrm{E}(G)) = 0$. This completes the proof of Theorem 5.

Since $\mathrm{E}(G)$ is semi-simple if and only if $G = S$, it follows immediately:

Corollary. *Let G be a torsion-free group of finite rank. If the Jacobson radical $J(E(G))$ of the endomorphism ring $E(G)$ is zero, then $G = S$.*

One may ask whether $J(E(G)) = 0$ is a necessary condition in order that $J(\mathrm{E}(G)) = 0$. This is not the case as may be seen from the following example. Let $G = Z(p)$ be the group of p-adic integers. Then $E(G) = Z(p)$ and $\mathrm{E}(G) = K(p)$, the p-adic number field. Hence $J(\mathrm{E}(G)) = 0$, but $J(E(G)) = pZ(p)$, so $J(E(G)) \neq 0$. Of course, if $E(G)$ satisfies D.C.C. on right ideals, then nil radical of $E(G) = J(E(G)) = E(G) \cap J(\mathrm{E}(G))$. Hence $J(E(G)) = 0$ if and only if $J(\mathrm{E}(G)) = 0$ in this case.

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S-objects in an abelian category

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1. Introduction

An abelian group G is an *S-group* if whenever K is a direct summand of G, then $G \cong G \oplus K$ [1]. G is an ID-*group* if G has an isomorphic proper direct summand [2]. In this paper we extend these concepts to an arbitrary abelian category with the emphasis on S-objects. Section 2 contains a few general properties of S-objects. In section 3 we investigate the relation of S-objects to ID-objects. We show that an ID-object in a C_3-category (i.e., satisfies the Grothendieck axiom A. B. 5) contains a non-zero S-object and we give a condition such that an S-object A in a complete C_3-category is isomorphic to an interdirect sum of countably many copies of A. In the last section we restrict our discussion to the category of abelian groups. We show several cases of a cancellation property for S-groups and conclude with the result that an abelian group whose torsion subgroup is an ID-group has a non-zero direct summand which is an S-group.

Throughout this paper **A** will denote an abelian category and A an arbitrary object in **A**. The word group will mean abelian group. Most of the notation is based on MITCHELL [6] with some taken from FUCHS [4] and the two main resource papers [1] and [2].

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2. S-objects

(2. 1) Definition. An object $A \in \mathbf{A}$ is an *S-object* if whenever B is a direct summand of A, then $A \cong A \oplus B$.

Theorem 2. 3, based on a similar result for direct sums of groups [1, Th. 3, p. 74] gives two large classes of S-objects.

(2. 2) Lemma. *Let* **A** *be complete (cocomplete). If* $A = \underset{i<\omega}{\times} A_i \left(\underset{i<\omega}{\oplus} A_i\right)$, *where* $A_i \cong A$ *for each i, then A is an* S-*object.*

Proof. Suppose $A = B \oplus L$. Then $A_i = B_i \oplus L_i$, where $B_i \cong B$ and $L_i \cong L$ for all i. Hence $A = \underset{i<\omega}{\times} A_i = \underset{i<\omega}{\times} (B_i \oplus L_i) \cong (\underset{i<\omega}{\times} B_i) \oplus (\underset{i<\omega}{\times} L_i) = B_0 \oplus (\underset{i<\omega}{\times} B_{i+1}) \oplus$ $\oplus (\underset{i<\omega}{\times} L_i) \cong B_0 \oplus (\underset{i<\omega}{\times} (B_{i+1} \oplus L_i)) \cong B \oplus \underset{i<\omega}{\times} A_i = B \oplus A$. Therefore, A is an S-object.

Dually, A is an S-object if $A = \underset{i<\omega}{\oplus} A_i$.

(2. 3) Theorem. *Let* **A** *be complete (cocomplete). If* $A = \underset{\lambda \in \Lambda}{\times} B_\lambda \left(\underset{\lambda \in \Lambda}{\oplus} B_\lambda\right)$, *where* $|\Lambda| \geqq \aleph_0$ *and* $B_\lambda \cong B$ *for each* λ, *then A is an* S-*object.*

Proof. Partition the index set Λ into $\aleph_0$ disjoint subsets Λ_i such that $|\Lambda_i| = |\Lambda|$ for all i. Then $A = \underset{\lambda \in \Lambda}{\times} B_\lambda \cong \underset{i \in \omega}{\times} \left(\underset{i \in \Lambda_i}{\times} B_\lambda\right) = \underset{i<\omega}{\times} A_i$ where $A_i = \underset{i \in \Lambda_i}{\times} B_\lambda \cong A$ for each i. Therefore A is an S-object by Lemma 2. 2.

Dually, A is an S-object if $A = \underset{\lambda \in \Lambda}{\oplus} B_\lambda$.

Kaplansky [5, p. 12] raises three questions which he notes might be appropriate to consider for any specific structure of groups. It follows directly from the definition that test problems I and II are satisfied by S-objects in an arbitrary **A**.

(2. 4) Proposition. (Kaplansky's test problems I and II.) *Let A and B be* S-*objects in* **A** *then:* I. *A isomorphic to a direct summand of B and B isomorphic to a direct summand of A implies* $A \cong B$, *and* II. $A \oplus A \cong B \oplus B$ *implies* $A \cong B$.

For an S-object A, it is obvious that $A \cong \underset{n}{\oplus} A$ for any $n < \omega$ since $A \cong A \oplus A$. However, $A \not\cong \underset{\aleph_0}{\oplus} A$ in general as the following example shows.

(2. 5) Example. Let $P = \underset{\aleph_0}{\times} Z$ where Z is the additive group of the integers. Then P is an S-group by Theorem 2. 3 and $\underset{\aleph_0}{\oplus} P \cong \underset{\aleph_0}{\oplus} (Z \oplus P) \cong (\underset{\aleph_0}{\oplus} Z) \oplus (\underset{\aleph_0}{\oplus} P)$. Nunke [7, Th. 5, p. 69] shows that every direct summand of a product of copies of Z is a product of copies of Z. Thus $P \not\cong \underset{\aleph_0}{\oplus} P$.

3. ID-objects

Many of the results in this section are extensions and applications to S-objects of the results and techniques in [2].

(3. 1) Definition. An object $A \in$ **A** is called an ID-*object* if A has an isomorphic proper direct summand.

(3. 2) Lemma. *If $A \neq 0$ is an* S-*object, then A is an* ID-*object.*

(3. 3) Lemma. *An object $A \in \mathbf{A}$ is an* ID-*object if and only if there exist $\varphi, \psi \in [A, A]$ such that $\psi\varphi = 1_A$ and $\varphi\psi \neq 1_A$.* ($[A, A]$ is the set of all morphisms from A to A in $\mathbf{A}$.)

Proof. Let A be an ID-object, then $A = B \oplus L$, $L \neq 0$, and there is an isomorphism $\varphi_1 : A \rightarrowtail\!\!\twoheadrightarrow B$. Let $\varphi = u_B \varphi_1$ where u_B is the injection of B into the coproduct. Let $\psi : B \oplus L \to A$ be the unique map defined by the definition of coproduct such that $\psi u_B = \varphi_1^{-1}$ and $\psi u_L = 0$. Then $\psi\varphi = \psi u_B \varphi_1 = \varphi_1^{-1} \varphi_1 = 1_A$ and $\varphi\psi(A) = B$ so $\varphi\psi \neq 1_A$.

Conversely, if $\psi\varphi = 1_A$, then φ is a monomorphism and the exact sequence $0 \to A \underset{\psi}{\overset{\varphi}{\rightleftarrows}} A \to A/\varphi(A) \to 0$ splits so that $A = \varphi(A) \oplus A/\varphi(A) = \varphi(A) \oplus \operatorname{Ker} \psi$ [6, Prop. 19. 1*, p. 32]. But $\varphi\psi \neq 1_A$ implies $\operatorname{Ker} \psi \neq 0$. Therefore, A is isomorphic to a proper direct summand $\varphi(A)$.

Thus, ID-objects can be studied by means of the following definition.

(3. 4) Definition. An ID-*system* is a triple $\langle A; \varphi, \psi \rangle$ where $A \in \mathbf{A}$ and $\varphi, \psi \in [A, A]$ such that $\psi\varphi = 1_A$.

Since any S-object A is an ID-object it determines an ID-system. An S-object actually determines a set of distinct ID-systems. This is shown in the following characterization of S-objects.

(3. 5) Proposition. *Let $\mathbf{B}$ be a representative set of non-isomorphic direct summands of A. A is an* S-*object if and only if there exists a set $\{(\varphi_B, \psi_B) : B \in \mathbf{B}\} \subset$ $\subset [A, A] \times [A, A]$ such that $\psi_B \varphi_B = 1_A$ and $\operatorname{Ker} \psi_B \cong B$ for all $B \in \mathbf{B}$.*

Proof. We need to show first that $\mathbf{B}$ is a set. If B is a direct summand of A, then the projection onto B followed by the injection of B into A is a morphism $\gamma_B \in [A, A]$ such that $\gamma_B(A) \cong B$. Thus if $C \not\cong B$ as subobjects, $\gamma_B(A) \not\cong \gamma_C(A)$ so $\gamma_B \neq \gamma_C$. Therefore, $\mathbf{B}$ is in one-to-one correspondence with a class of distinct morphisms in $[A, A]$. Since $[A, A]$ is a set, $\mathbf{B}$ is a set.

If A is an S-object and $A = B \oplus M$, $B \in \mathbf{B}$, then there is an isomorphism $\alpha : A \oplus B \rightarrowtail\!\!\twoheadrightarrow A$. Let $u : A \to A \oplus B$ be the injection of A into the coproduct and p the projection onto A. Define $\varphi_B = \alpha u$ and $\psi_B = p\alpha^{-1}$. Then $\psi_B \varphi_B = p\alpha^{-1}\alpha u = pu = 1_A$ and $\operatorname{Ker} \psi_B \cong \operatorname{Ker} p = B$.

Conversely, let $A = B' \oplus M$ and $B \in \mathbf{B}$ such that $B \cong B'$ as subobjects of A. $\psi_B \varphi_B = 1_A$, so φ_B is monic and $A = \varphi_B(A) \oplus \operatorname{Ker} \psi_B$ as in Lemma 3. 3. But $\varphi_B(A) \cong A$ and $\operatorname{Ker} \psi_B \cong B \cong B'$ so $A \cong A \oplus B'$. Therefore, A is an S-object.

(3. 6) Theorem. *Let $\mathbf{A}$ be C_3, A an* ID-*object in $\mathbf{A}$, then A contains a non-zero* S-*object.*

Proof. Since A is an ID-object, there is an ID-system $\langle A; \varphi, \psi\rangle$ for A such that $\operatorname{Ker}\psi \neq 0$. Let $H = \operatorname{Ker}\psi$. Then by repeatedly applying φ to A, A splits as $A = H \oplus \varphi(A) = H \oplus \varphi(H) \oplus \varphi^2(A) = \cdots$, where $\varphi^n(A) \cong A$ and $\varphi^n(H) \cong H$ for all $n < \omega$ ($\varphi^0(H) = H$). Then $\{\varphi^n(H) : n < \omega\}$ is a set of subobjects of A such that $\bigoplus_{n=0}^{m} \varphi^n(H)$ is a direct summand of A for every $m < \omega$. Clearly $\left\{\bigoplus_{n=0}^{m} \varphi^n(H) : m < \omega\right\}$ is a direct system and $\varinjlim_{m<\omega}\left(\bigoplus_{n=0}^{m} \varphi^n(H)\right) = \bigoplus_{n<\omega} \varphi^n(H)$ (see [6, p. 48, Example 1]). But $\varinjlim_{m<\omega}\left(\bigoplus_{n=0}^{m} \varphi^n(H)\right) = \bigcup_{n<\omega} \varphi^n(H) \subset A$ by [6, Prop. 1. 2, p. 82] since **A** is C_3. Therefore, $\bigoplus_{n<\omega} \varphi^n(H)$ is a subobject of A and by Theorem 2. 3 it is an S-object.

(3. 7) Corollary. *Let* **A** *be* C_3, *A an S-object. Then A contains an S-object isomorphic to* $\bigoplus_{\aleph_0} A$.

Proof. Since $A \cong A \oplus A$, let $\langle A; \varphi, \psi\rangle$ be an ID-system for A such that $H = \operatorname{Ker}\psi \cong A$. Then $\varphi^n(H) \cong H \cong A$ and $\bigoplus_{n<\omega} \varphi^n(H) \cong \bigoplus_{\aleph_0} A$ so the results follows from Theorem 3. 6 and its proof.

By imposing additional hypotheses we are able to extend the conclusion in 3. 7 such that an S-object is isomorphic to an interdirect sum of countably many copies of itself. In Theorem 3. 9 we let $\varphi^\omega A = \bigcap_{n<\omega} \varphi^n(A)$. This intersection exists since we assume A to be complete.

(3. 8) Definition. Let **A** be complete C_3 with $\{A_i : i \in I\} \subset \mathbf{A}$. An object $A \in \mathbf{A}$ is called an *interdirect sum* of the A_i if

$$\bigoplus_{i\in I} A_i \subset A \subset \bigtimes_{i\in I} A_i$$

(3. 9) Theorem. *Let* **A** *be complete C_3. If A is an S-object with ID-system $\langle A; \varphi, \psi\rangle$ where $\operatorname{Ker}\psi \cong A$ and if $\varphi^\omega A$ is a direct summand of A, then A is isomorphic to an interdirect sum of countably many copies of A.*

Proof. Let $H = \operatorname{Ker}\psi$ and $K = \varphi^\omega A$. From the proof of Theorem 3. 6 we have $\bigoplus_{n<\omega} \varphi^n(H) \subset A$ and

$$(*) \qquad A = H \oplus \varphi(H) \oplus \cdots \oplus \varphi^n(H) \oplus \varphi^{n+1}(A).$$

Thus let $\alpha_n : A \to \varphi^n(H)$ be the projection defined by $(*)$ and let $p_n : \bigtimes_{n<\omega} \varphi^n(H) \to \varphi^n(H)$ be the projection from the product. Then by the definition of product there exists a unique $\alpha : A \to \bigtimes_{n<\omega} \varphi^n(H)$ such that $p_n\alpha = \alpha_n$ for all $n < \omega$. Let $L = \operatorname{Im}\alpha$.

Now from $(*)$ we see that $\operatorname{Ker}\alpha_n = H \oplus \cdots \oplus \varphi^{n-1}(H) \oplus \varphi^{n+1}(A)$.

Thus $\bigcap_{n=0}^{m} \operatorname{Ker}\alpha_n = \varphi^{m+1}(A)$, and $\bigcap_{n<\omega} \operatorname{Ker}\alpha_n = \bigcap_{n<\omega} \varphi^n(A) = \varphi^\omega A$.

Thus, by an exercise in MITCHELL [6, Ex. 8, p. 37] $\operatorname{Ker}\alpha=\varphi^{\omega}A=K$. Since $\varphi^{\omega}A$ is a direct summand by hypothesis, $A = K\oplus L$.

Claim $A\cong L$. $A = H\oplus\varphi(A)$ by (*) and $K\subset\varphi(A)$ by definition. Thus, by the modular law [3, p. 103, Exercise A] $\varphi(A) = A\cap\varphi(A) = K\oplus[L\cap\varphi(A)]$ implies $A = H\oplus K\oplus[L\cap\varphi(A)]$ so that $L \cong A/K \cong H\oplus[L\cap\varphi(A)]$. Now $H\cong A$ and A an S-object implies $H\oplus K \cong A\oplus K \cong A \cong H$. Hence $A = K\oplus H\oplus[L\cap\varphi(A)] \cong$ $\cong H\oplus[L\cap\varphi(A)] \cong L$.

Finally, we need to show that $\bigoplus_{n<\omega}\varphi^n(H) \subset L$. Let β_n and γ_n be the injection of $\varphi^n(H)$ into A and $\bigtimes_{n<\omega}\varphi^n(H)$ respectively. Then $\alpha\beta_n=\gamma_n$ since $p_j\alpha\beta_n=\alpha_j\beta_n$ is the identity on $\varphi^n(H)$ if $j=n$ and is 0 if $j\neq n$ and similary for $p_j\gamma_n$. Thus α restricted to $\bigoplus_{n<\omega}\varphi^n(H)$ is the natural map $\delta\colon \bigoplus_{n<\omega}\varphi^n(H)\to \bigtimes_{n<\omega}\varphi^n(H)$. By hypothesis and [6, Cor. 1. 3, p. 83], **A** is C_2, thus δ is a monomorphism. Since α factors through L we have $\bigoplus_{n<\omega}\varphi^n(H)\subset L\subset \bigtimes_{n<\omega}\varphi^n(H)$.

Since $\varphi^n(H)\cong A$ for all $n<\omega$, L is isomorphic to an interdirect sum of countably many copies of A. Since $A\cong L$, the proof is complete.

4. Applications to abelian groups

In this section we restrict our attention to the category of abelian groups. We start with a cancellation property for S-groups. This follows the standard pattern of considering the reduced and divisible cases separately.

(4. 1) Proposition. *Suppose G is an S-group, $G = K\oplus L$, K finitely generated, then $G\cong L$.*

Proof. G an S-group implies $G \cong K\oplus G$ so that $K\oplus G \cong K\oplus L$. Thus $G\cong L$ by [8, Cor. 8, p. 900].

(4. 2) Theorem. *Let G be a reduced p-group, G an S-group, and $G = K\oplus L$ where K contains no non-zero S-group, then $G\cong L$.*

Proof. Suppose K is infinite and let B be a basic subgroup for K. Then K infinite implies $|B|=m\geqq\aleph_0$ so that $B[p] \cong \bigoplus_m C(p)$ is an S-group. Thus K is finite and therefore finitely generated. By Proposition 4. 1, $G\cong L$.

(4. 3) Corollary. *Let T be a reduced torsion group, T an S-group, and $T = K\oplus L$ where K contains no non-zero S-group, then $T\cong L$.*

Proof. $T = \bigoplus_{p\in\pi} T_p$ and each T_p is an S-group [1, Cor. 2, p. 72]. Also $T = K\oplus L$ implies $T_p = K_p\oplus L_p$ and K_p contains no non-zero S-group since K contains no non-zero S-group. Thus Theorem 4. 2 implies $T_p\cong L_p$ and so $T\cong L$.

The conditions in Theorem 4. 2 are not sufficient to guarantee that G is an S-group. That is, the following is an example of a group G such that if $G = K \oplus L$ and K contains no non-zero S-groups, then $G \cong L$, however, G is not an S-group.

(4. 4) Example. By ZIPPIN [9, p. 98—99], there is a reduced countable p-group G such that $f(G, n) = \aleph_0$, $n < \omega$, and $f(G, \omega) = 1$ where $f(G, n)$ is the n^{th} Ulm invariant of G. If $G = K \oplus L$ where K contains no non-zero S-group, then, as in 4. 2, K is finite so that $f(K, n)$ is finite for $n < \omega$ and $f(K, \omega) = 0$. By the properties of Ulm invariants and by ULM's theorem [5, p. 27], it follows that $G \cong L$. However, G is not an S-group since $f(G, \omega) = 1$ [1, Th. 2, p. 73].

(4. 5) Theorem. *Let D be a divisible group, D an* S-*group, and $D = K \oplus L$ where K contains no non-zero* S-*groups, then $D \cong L$.*

Proof. By [1, Th. 2, p. 73] the torsion free rank of D is zero or infinite and the p-rank of D is zero or infinite for each $p \in \pi$. Now K is also divisible and if its torsion free rank were infinite or if its p-rank were infinite for any p, K would contain an S-group by Theorem 2. 3 (or [1, Th. 3, p. 74]). Thus, the torsion free rank of L and the p-rank of L for each p must be the same as the corresponding rank of D. Therefore, $D \cong L$.

We can now prove the general torsion case by splitting the group into its divisible and reduced components and applying 4. 3 and 4. 5. We also need the fact that a group is an S-group if and only if its reduced and divisible components are both S-groups [1, Cor. 1, p. 72].

(4. 6) Theorem. *Let T be a torsion group, T an* S-*group, and $T = K \oplus L$ where K contains no non-zero* S-*groups, then $T \cong L$.*

We next note that for groups Theorem 3. 9 has a special interpretation [see 2].

(4. 7) Proposition. *If G is an* S-*group with* ID-*system $\langle G; \varphi, \psi \rangle$ where* Ker $\psi \cong G$ *and if $\varphi^{\omega} G$ is a direct summand of G, then G is isomorphic to a total shift invariant subgroup of $\underset{\aleph_0}{\times} G$.*

The following gives a more involved example than Theorem 2. 3 of an S-group and demonstrates a simple application of Proposition 4. 7 (and thus of Theorem 3. 9).

(4. 8) Example. Let $P = \underset{\aleph_0}{\times} Z$ and $F = \underset{\aleph_0}{\oplus} Z$ where Z is the additive group of the integers. P and F are both S-groups by Theorem 2. 3. We will show that $P \oplus F$ is also an S-group.

Suppose $P \oplus F = A \oplus B$. Let φ be the projection of $P \oplus F$ onto F. Letting φ_A be the restriction of φ to A we get the exact sequence $0 \to \text{Ker } \varphi_A \to A \xrightarrow{\varphi_A} F$ where $\varphi_A(A)$ is free since it is a subgroup of a free group. Since $A/\text{Ker } \varphi_A \cong \varphi_A(A)$, we have $A = \text{Ker } \varphi_A \oplus L$, where L is free [4, Th. 9. 2, p. 38]. Clearly L has countable rank. Now, Ker $\varphi_4 = A\cap P$ and $A\cap P$ is a direct summand of $P\oplus F$ since $P\oplus F = A\oplus B = A\cap P\oplus L\oplus B$. Thus $A\cap P\subset P$ implies $P = A\cap P\oplus P\cap(L\oplus B)$ by the modular law. So $A = A\cap P\oplus L$ where $A\cap P$ is a direct summand of P and hence a product of copies of Z [7, Th. 5, p. 69]. Therefore, $A\cong\bigoplus_n Z$, $A\cong P$, $A\cong F$, or $A\cong P\oplus F$. In any case, $P\oplus F\oplus A\cong P\oplus F$ so $P\oplus F$ is an S-group.

Let $G = (\bigoplus_{i<\omega} Z_i)\oplus(\times_{j<\omega} Z'_j)$ where $Z_i\cong Z\cong Z'_j$ for all i and j. Then $G\cong P\oplus F$ and so is an S-group. It is obvious that G is an interdirect sum of countably many copies of Z. With 4. 7 we can also show that G is isomorphic to a total shift invariant subgroup of $\times_{\aleph_0} G$. Define $\varphi\colon G\to G$ by $\varphi(Z_i)=Z_{2i}$ and $\varphi(Z'_j)=Z'_{2j}$, then

$$G = [(\bigoplus_{i<\omega} Z_{2i}(\oplus)\times_{j<\omega} Z'_{2j})]\oplus[(\bigoplus_{i<\omega} Z_{2i-j})\oplus(\times_{j<\omega} Z'_{2j-1})] = \varphi(G)\oplus H$$

and $\varphi^n(G) = (\bigoplus_{i<\omega} Z_{2^n i})\oplus(\times_{j<\omega} Z'_{2^n j})$ so $\varphi^\omega G=0$ and is thus a direct summand of G and Proposition 4. 7 applies.

Clearly if an object A has a direct summand which is an S-object, A is an ID-object. We conclude with the converse for torsion groups. 4. 10 may also be considered a special case of Theorem 3. 6.

(4. 9) Lemma. *If a reduced p-group G is an* ID-*group, then G has a non-zero direct summand which is a bounded* S-*group.*

Proof. By [2, Th. 2. 9, p. 23], G an ID-group implies $f(G, n)$ is infinite for some integer n. Thus $B_n = \bigoplus_{f(G,n)} C(p^n)$ is an S-group (B_n is the n^{th} component of a basic subgroup for G). But B_n is bounded and is a direct summand of G.

(4. 10) Theorem. *If G_T is an* ID-*group, then G has a non-zero direct summand which is an* S-*group.* (G_T is the maximum torsion subgroup of G.)

Proof. Since G_T is an ID-group, by [2, Th. 2. 6, p. 23] $(G_T)_p$ is an ID-group for some p. Let $(G_T)_p = D_p\oplus R_p$ where D_p is divisible and R_p is reduced. Then, by [2, Th. 2.8, p. 23], D_p or R_p is an ID-group. If D_p is an ID-group, D_p has infinite p-rank and is thus an S-group by [1, Th. 3, p. 74]. D_p is also a direct summand of G since it is divisible. If R_p is an ID-group, then by Lemma 4. 9, R_p has a non-zero direct summand K which is a bounded S-group. Since R_p is pure in G, K is also pure in G and is a direct summand of G by [5, Th. 7, p. 18].

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Bibliographie

Zelling S. Harris, Mathematical structures of language (Interscience Tracts in Pure and Applied Mathematics, No. 21), IX+230 pages, New York—London—Sydney—Toronto, Interscience Publishers, John Wiley and Sons, 1968. — 112 s.

The book is an expansion of a lecture given by the author at the Courant Institute of Mathematical Sciences. It does not attempt to present a unified treatment of what is called mathematical linguistics; such a treatment at today's stage of development of the subject could probably not be given. Rather, the book is a report on the author's own work.

The pursued aim is not to build an elegant mathematical theory which has some relevance to linguistics; rather, to define a mathematical structure that comes as close to describing natural languages as seems possible at the present time. The main result of the book is, as the author claims, the definition of such a structure. The structure finally arrived at is rather complicated: it has a family of primitive arguments and five finite families of operators acting on primitive arguments or operators. This is, however, not unexpected if we consider how complicated a natural language really is.

Once such a structure has been defined, one can prove theorems about it. The extent how far these theorems are interpretable as true properties of natural languages may be a good check of how close the given structure comes to describing natural languages. Also, the study of related mathematical structures should be inspiring for linguistics.

The book is written in a lucid style, with many illustrating examples from the English language. Its chapter headings are: 1. Introduction. 2. Properties of language relevant to a mathematical formulation. 3. Sentence forms. 4. Sentence transformations. 5. Structures defined by transformations. 6. Regularization beyond language. 7. The abstract system. 8. The interpretation. The book ends with an Index. At the end of the Introduction a list of works is given that contain more detailed information about parts of the material.

Attila Máté (Szeged)

P. Rosenstiehl and J. Mothes, Mathematics in management: the language of sets, statistics and variables, translated from French, xvi+392 pages, Amsterdam, North—Holland, 1968.

The traditional approach to teaching mathematics in high-school is to provide a basis for those wanting to continue their studies in engineering or science. As a result, up to quite recently, most other people gladly severed all ties with mathematics as something irrelevant to their lives at the age of eighteen. Yet, it was proven quite some time ago, that the applications of mathematics are not restricted to engineering and science. In particular, efficient business management cannot live without them.

Of course, there are specialists in applications of mathematics to business and industry, and business administrators and managers need not have such a specialized knowledge. What they need is an overall picture of where and how to apply mathematics to problems in management. This will enable them to judge when they should invoke the help of specialists; and, in fact, without such a knowledge, they may have extreme difficulty in communicating their problems to the mathematician.

This was kept in mind when, in the beginning of the 1960s, l'Ecole des Hautes Etudes Commerciales started to include a new course in mathematics in its programme. The French original of this book (Mathématiques de l'action, Dunod, Paris, 1968) is based on the experience gathered from this course during several years. This should by itself be a guarantee of the quality of the book.

As it should seem clear from what has been said so far, this book is intended for people whose main interest is not science. This does not mean that low standards of mathematical precision are applied. In fact, the material is presented in a clear and rigorous way. A great number of illustrating examples and excercises are given; many of these help one to grasp the relevancy of the discussed material to problems encountered in management.

The contents of the book can be best illustrated by the chapter and section headings: I. Subsets and partitions of a finite set (1. Elements and sets. 2. The set $\mathcal{P}(E)$ of the subsets of a finite set E. 3. Boolean algebra. 4. Partitions of a finite set). II. Organisation, classification and enumeration (1. General remarks on the statistics of a set. 2. The genealogy of simplexes. 3. Compartments and objects. 4. Morphisms). III. Events and probability (1. The language of events. 2. Probability: a measure of events. 3. Numerical estimation of probabilities). IV. Random variables (1. Discrete random variables. 2. Continuous random variables. 3. Two-dimensional random rariables). V. Common probabilistic models (1. Discrete models. 2. Continuous models. 3. Confrontation of the observations and the model).

Each chapter ends with a summary, practical excercises with solutions, and the description of one of more fields of applications. The book ends with a few tables useful in statistics and a subject index.

As the above description of the contents shows, the special considerations in the preparations of this book do not make its scope so limited as it would seem natural. The book should be useful to everyone who directly or indirectly may be confronted with applications of mathematics, including those interested in various branches of science. A further volume is planned on programming.

Attila Máté (Szeged)

S. A. Naimpally—B. D. Warrack, Proximity spaces (Cambridge Tracts in Mathematics and Mathematical Physics, No. 59), X+128 pages, Cambridge University Press, 1970.

The idea of using the relation of "nearness" of two subsets of a space as a basic tool of introducing a structure goes back to a congress talk of F. Riesz in 1908. However, this idea was not systematically developed earlier than the work of V. A. Efremovič (1952), who defined proximity spaces axiomatically. Since that time, the theory of these spaces produced interesting and deep results and found important applications, so that a monograph on this subject fills a serious gap in the literature of general topology.

The authors divided the book into four chapters preceded by a short account on historical background. The first of them presents basic definitions and facts, the second gives the theory of Smirnov compactification (with the help of the method of clusters). In the third chapter we find the most important interrelationships between proximity and uniformity, including some generalized concepts of uniformity (Alfsen—Njåstad uniformities, contiguities). The main subject of the final chapter is a survey of various kinds of generalized proximities and further generalizations of uniformities (syntopogenous spaces of the referee, generalized topological spaces of D. Doičinov, sequential proximitcs of S. G. Mrówka, generalized proximities of S. Leader, M. W. Lodato, W. J. Pervin, generalized uniformities of C. J. Mozzochi, etc.); a similar subject (local proximity) was previously presented in Chapter 2. Each chapter is followed by a series of references to the fairly complete bibliography standing at the end of the volume.

This monograph is very useful for a reader interested in modern developments of general topology. Although nothing else is postulated than basic knowledge from the theory of topological spaces, it is advantageous to be familiar with the theory of uniform structures because some concepts (e. g. that of a uniformly continuous mapping) are used without a definition. The referee succeeded in finding only a very small number of misprints and errors.

Á. Császár (Budapest)

Josef Stoer—Christoph Witzgall, Convexity and optimization in finite dimensions. I (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 163), IX+293 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1970. — DM 54, —

This book provides an excellent summary of mathematical results which are basic for the linear and nonlinear continuous variable programming in finite dimensions. The results of the various authors are discussed in the frame of a unified theory in a clear, elegant manner. The book consists of six chapters. Chapter 1 is devoted to the algorithmic solution of linear inequalities originated by Fourier. Farkas' theorem, the main transposition theorems, the duality theorem of linear programming and the complementary slackness theorems are deduced from the theory obtainable from the elimination procedure. Chapter 2 contains the basic theory of convex polyhedra. Beyond the classical results of Minkowski, Farkas, Carathéodory, Motzkin, Weyl, attention is paid to the important later results, among which we mention the combinatorial type Gale diagram characterizing the face structure of convex polyhedra: Chapter 3 deals with convex sets, their topological, combinatorial, extremal properties, supporting sets, separation and fixed point theorems. Chapter 4 deals with the properties of convex functions, the conjugate function theory of Fenchel and various generalizations of convexity. Chapters 5 and 6 are devoted respectively to the strongly related duality theory and saddle point theorems. Fenchel's duality theorem was generalized by Rockafellar and this is again generalized in Chapter 5 and then the previous theorems (proved by Gale, Kuhn, Tucker for linear programs and Dennis, Dorn, Eisenberg and Cottle for nonlinear programs) are shown to be special cases of this one. A similar line is followed in Chapter 6. The classical theorems of von Neumann and Kakutani were generalized by Sion while this generalization is extended in this book to the noncompact case. This contains as a special case the Kuhn—Tucker saddle point theorem. A direct approach to the Kuhn—Tucker theory and explanation of its connection with classical calculus is also given. In the foreword the authors promise to treat the algorithms of convex optimization in a subsequent volume.

A. Prékopa (Budapest)

H. Störmer, Semi-Markoff-Prozesse mit endlich vielen Zuständen (Lecture Notes in Operations Research and Mathematical Systems, Vol. 34), VII+126 Seiten, Berlin—Heidelberg—New York, Springer Verlag, 1970. — DM 12, —

Der Begriff der Semi-Markoff-Prozesse wurde von R. Pyke eingeführt. Diese Prozesse sind durch endlich viele oder abzählbar unendlich viele Zustände, durch die Übergangswahrscheinlichkeiten und durch die Verteilungsfunktionen für die Zustandsdauern angegeben und enthalten als Spezialfälle die Klasse der Erneuerungsprozesse, die Klasse der Markoff-Ketten und der Markoff-

Prozesse mit stetigem Zeitparameter. Die Semi-Markoff-Prozesse sind besonders geeignet zur Beschreibung einer großen Anzahl von Zufallsvorgängen in Natur, Wirtschaft und Technik; man kann z. B. sie für die Betrachtung der Wachstumspozesse, der Lagerhaltungs- und Warteschlangenprobleme oder der Probleme der Zuverlässigkeit von Systemen anwenden.

Im ersten Teil des Buches wird ein Abriß der Erneuerungstheorie angegeben; die Ergebnisse der Erneuerungstheorie liefern nämlich die wesentlichen mathematischen Hilfsmittel für die Behandlung der Semi-Markoff-Prozesse. Im zweiten Teil werden die für die verschiedenen Anwendungen wichtigsten Resultate der Theorie der Semi-Markoff-Prozesse hergeleitet. Nur die Semi-Markoff-Prozesse mit endlich vielen Zuständen werden diskutiert; diese haben nämlich für die Anwendungen besondere Bedeutung, und diese kann man mit den einfachen Mitteln des Matrizenkalkuls behandeln. Die Betrachtungsweise ist sehr klar und das Buch eignet sich vorzüglich dafür, daß man daraus eine Übersicht über diese für die Anwendungen wichtige Theorie gewinnt.

K. Tandori (Szeged)

F. Ferschl, Markovketten (Lecture Notes in Operations Research and Mathematical Systems, Vol. 35), VI+168 Seiten, Berlin—Heidelberg—New York, Springer-Verlag, 1970. — DM 14,—

Dieses Buch ist die Ausarbeitung einer Vorlesung, die in den Jahren 1969—70 für Studenten der Volkswirtschaft gehalten wurde. Die wichtigsten Grundbegriffe und Ergebnisse der Theorie von Markovketten mit abzählbar unendlich vielen Zuständen und ihre wichtigsten Anwendungen in der Volkswirtschaft (Theorie der Warteschlangen, Erneuerungstheorie, Ruinprobleme) werden kurz, aber klar zusammengefaßt. Die Titel der einzelnen Kapitel sind die folgenden: Die Definition stochastischer Prozesse; Die Definition von Markovketten; Übergangswahrscheinlichkeiten; Die graphentheoretische Analyse von Markovketten; Das Rückkerverhalten von Markovketten; Stationäre- und Gleichgewichtsverteilungen; Transienz- und Rekurrenzkriterien; Algebraische Methoden zur Berechnung der Übergangswahrscheinlichkeiten. Am Anfang der einzelnen Kapitel — wo es notwendig ist — werden die entsprechenden Hilfsmittel (z. B. Hilfsmittel aus der Wahrscheinlichkeitstheorie, aus der Graphentheorie, aus der Reihentheorie) betrachtet. Es ist erwähnenswert, daß der Verfasser zur Einführung der verschiedenen Zustände der Markovketten die Begriffe von gerichteten Graphen und von der Theorie der Relationen anwendet. Mit dieser Betrachtungsmethode wird es möglich, die verschiedenen Begriffe klar einzuführen; diese abstrakte Betrachtungsmethode ist aber nur für Mathematiker interessant. Das Buch betrachtet ausführlicher auch die Methoden der Matrizenrechnung, und so gibt es praktisch handhabbare Rechenmethode für die Bestimmung der Potenzen von stochastischen Matrizen. Am Ende des Buches gibt es ein Literaturverzeichnis, in welchem die wichtigsten Werke über Markovketten kurz rezensiert werden.

K. Tandori (Szeged)

F. Bartholomes and G. Hotz, Homomorphismen und Reduktionen linearer Sprachen (Lecture Notes in Operations Research and Mathematical Systems, Vol. 32), XII+143 Seiten, Berlin—Heidelberg—New York, Springer-Verlag, 1970.

Es ist bekannt, daß man die linearen Chomsky-Sprachen als direkte Verallgemeinerungen der endlichen Automaten betrachten kann, im Sinne, daß die durch endliche Automaten darstellbaren Mengen genau mit den Satzmengen der einseitig linearen Sprachen zusammenfallen. Auf Grund dieser Tatsache darf man erwarten, daß sich eine ganze Reihe von Ergebnissen der Theorie von endlichen Automaten auf lineare Sprachen übertragen läßt. Das Hauptziel dieser Monographie ist die Realisation dieses Programms dadurch, daß man zu jeder linearen Sprache eine endlich erzeugte

Kategorie zuordnet, deren Objekte und Morphismen Wortmengen mit einer ganz speziellen Struktur bzw. die Klassen gewisser Ableitungen dieser Sprache sind. Die systematische Anwendung der Methoden der Theorie von Kategorien gestattet einen Überblick darüber, welche Sätze über lineare Sprachen *rein algebraischer und welche spezifisch sprachentheoretischer Natur sind.*

In § 1 wird es gezeigt, daß jede durch endliche Automaten darstellbare Menge als Satzmenge einer linkslinearen Sprache auftritt. § 2 enthält gewisse spezielle kategorien-theoretische Vorbereitungen. Hier wird es sich zeigen, daß eine umkehrbar eindeutige Beziehung zwischen den aus der Automatentheorie bekannten normalen Standardereignissen und den endlich erzeugten freien Kategorien existiert. In §§ 3 und 4 werden der Homomorphiesatz und der Begriff des Reduktionsverbandes der endlichen Automaten auf die linearen Sprachen übertragen und nachher für endlich erzeugte freie Kategorien formuliert. § 5 enthält Untersuchungen über die Homomorphismen und Reduktionen der linearen Sprachen. Die Reduktionen sind im wesentlichen surjektive Funktoren zwischen den den linearen Sprachen zugeordneten freien Kategorien. In § 6 findet man einige Bemerkungen über die lokal eindeutigen und eindeutigen linearen Sprachen.

I. Peák (Szeged)

Paul F. Byrd—Morris D. Friedman, Handbook of Elliptic Integrals for Engineers and Scientists (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 67), XVI+358 Seiten, zweite, verbesserte Auflage, Berlin—Heidelberg—New York, Springer-Verlag, 1971.

Die erste Auflage dieses Buches erschien 1971. Der vorliegenden zweiten, verbesserten Auflage ist eine ergänzende Bibliographie hinzugefügt, die mehrere Hinweise auf die numerischen Näherungsmethoden und auf die entsprechenden Algorithmen für Rechenapparaten enthält. Das Buch umfaßt umgefähr 3000 verschiedene Formeln und im Appendix mehrere Werttabellen, die die Auswertung von elliptischen Integralen erleichtern. Die entsprechenden Beweise sind nicht diskutiert, nur die notwendigen Begriffe und die Formeln sind mitgeteilt. So its dieses Buch in erster Reihe für diejenigen Fachleute brauchbar, die in ihrer Tätigkeit nicht-elementare Integrale auswerten sollen.

Károly Tandori (Szeged)

D. S. Mitrinović, in cooperation with **P. M. Vasić**, **Analytic inequalities** (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 165), XII+400 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1970.

From the author's *introduction*: "If it is true that 'all analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove', we may expect that 'Analytic inequalities' will be of some help to them."

The aim of the present monograph is mainly to collect inequalities not dealt with in the classical work "Inequalities" by Hardy, Littlewood, and Pólya, and the book "Inequalities" by Beckenbach and Bellman. Some overlap was of course inevitable. However, as is claimed in the preface, even in the presentation of classical inequalities new facts have been added.

The collection is very rich, although it was impossible to strive for completeness. Where proofs or details could not be included for lack of space, references are given to original works. The first part, entitled "Introduction", concentrates on convex functions. The author considers the second part, entitled "General inequalities", the main part of the book. It is subdivided into twenty seven sections, some of which are further subdivided.

Studied here are, among many other topics: Young's and Hölder's inequality, the inverse of Hölder's inequality due to Dias, Goldman, and Metcalf, inequalities involving means, the λ-method of Mitrinović and Vasić, which may be used to connect various, seemingly unrelated inequalities, Steffenson's and Turán's inequalities, integral inequalities involving derivatives, and inequalities for vector norms. The third part, entitled "Particular inequalities", collects over 450 special results.

This collection should be very useful as a reference book for any research mathematician in analysis, but it may be useful to other people, like engineers, physicists, statisticians, etc., who might encounter inequalities in their works, and students may also benefit from parts of the book.

Attila Máté (Szeged)

LIVRES REÇUS PAR LA REDACTION

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