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ACTA UNIVERSITATIS SZEGEDIENSIS

ACTA SCIENTIARUM MATHEMATICARUM

TOMUS 31

FASC. 1—2

SZEGED, 1970

NSTITUTUM BOLYAIANUM UNIVERSITATIS SZEGEDIENSIS

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ACTA SCIENTIARUM MATHEMATICARUM

31. KÖTET
1—2. FÜZET

SZEGED, 1970. JÚNIUS

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Characterization of classes of functions by polynomial approximation

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1. Introduction

Let $f(x)$ be a 2π-periodic, continuous function with the Fourier series

$$S(f) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \sum_{k=0}^{\infty} A_k(x),$$

and let $\tilde{f}(x)$ be the trigonometric conjugate of $f(x)$. Set

$$R_n(r;f) = \sum_{k=0}^{n} \left[1 - \left(\frac{k}{n+1}\right)^r\right] A_k.$$

The following theorems hold:

(A) $f \in \mathrm{Lip}\,1$ *if and only if* $\|\tilde{f} - R_n(1;\tilde{f})\|_C = O\left(\frac{1}{n}\right)$,

(B) $f' \in \mathrm{Lip}\,1$ *if and only if* $\|f - R_n(2;f)\|_C = O\left(\frac{1}{n^2}\right)$.

(A) was proved long ago by ALEXITS [1] and ZAMANSKY [5], (B) by ZAMANSKY [5].

More generally, *if $f(x)$ has $r-1$ continuous derivatives* ($r \geqq 1$), *we have*

(C) $f^{(r-1)} \in \mathrm{Lip}\,1$ *if and only if*

$$\|f - R_n(r;f)\|_C = O\left(\frac{1}{n^r}\right) \textit{ for } r \textit{ even}, \qquad \|\tilde{f} - R_n(r;\tilde{f})\|_C = O\left(\frac{1}{n^r}\right) \textit{ for } r \textit{ odd}.$$

This was also proved by ZAMANSKY [5], but the "if" part was given earlier by ZYGMUND [6].

The fact that for odd r we have to consider the approximation of the conjugate function $\tilde{f}(x)$ instead of $f(x)$, is inconvenient both from theoretical and practical point of view. Hence, it is natural to raise the following

Problem. Does there exist a sequence of linear operators $\{T_n\}$ such that $T_n f$ is a trigonometric polynomial or order n and, in the space $L^p (1 \leqq p \leqq \infty)$, the approximation property

(1) $$\|f - T_n f\|_p = O\left(\frac{1}{n^r}\right)$$

is necessary and sufficient for $f^{(r-1)} \in \text{Lip}\,(1, p)$? [1])

The answer is yes. This was proved by TRIGUB [8] who introduced, to this aim, a special summation process of the Fourier series, somewhat similar to the Rogosinski means. Recently, his result was generalized by ŽUK [9].

Now, we shall indicate a general method to form polynomial operators for solving similar problems. Our way has the advantage that the results can be localized easily; this fact seems to have some interest, because it offers access to the characterization by algebraic polynomials of non-periodical functions belonging to the class Lip 1 in the interior of an arbitrary interval (a, b).

2. Trigonometric operators which characterize the classes Lip $(1, p)$

We call T_n a trigonometric operator of order n if $T_n f$ is a trigonometric polynomial of order n. By Z_p we denote the L^p-Zygmund class, i.e. the class of functions satisfying the condition

$$\|f(x+h) + f(x-h) - 2f(x)\|_p = O(|h|).$$

We have the following general

Theorem 1. *Let $U_n f$ and $V_n f$ be trigonometric operators of order n such that*

(2) $$\|f - U_n f\|_p = O\left(\frac{1}{n^r}\right)$$

for $f^{(r-1)} \in Z_p$ and that the relation

(3) $$\|V_n^{(r)} f\|_p = O(1)$$

is equivalent to $f^{(r-1)} \in \text{Lip}\,(1, p)$, where $V_n^{(r)} f$ denotes the r th derivative of $V_n f$. Setting

(4) $$T_n f = U_n f + \frac{1}{n^r} V_n^{(r)} f,$$

condition (1) is both necessary and sufficient for $f^{(r-1)} \in \text{Lip}\,(1, p)$.

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[1]) Of course, we assume that Lip $(1, \infty)$ is the ordinary Lip 1 class and the ∞-norm is the C norm for continuous $f(x)$. Also, $f \in \text{Lip}\,(1, 1)$ is equivalent to $f \in BV$.

Proof. Suppose $f^{(r-1)} \in \mathrm{Lip}\,(1, p)$. Then (2) and (3) follow by assumption, and hence (1) is an immediate consequence of definition (4).

Inversely, suppose (1) satisfied. Then by the well-known Bernstein—Zygmund theorem (generalized to L^p) it follows that $f^{(r-1)} \in Z_p$; hence (2) holds good. So we get by (1), (2) and (4)

$$\frac{1}{n^r} \|V_n^{(r)} f\|_p = O\left(\frac{1}{n^r}\right);$$

therefore (3) is valid, too. But this is equivalent by assumption to $f^{(r-1)} \in \mathrm{Lip}\,(1, p)$, and so our theorem is proved.

Corollary. *Setting*

$$T_n f = 2\sigma_{2n-1}(f) - \sigma_{n-1}(f) + \frac{1}{n^r} \sigma_n^{(r)}(f),$$

where $\sigma_m(f)$ denotes the mth Fejér mean of the Fourier series $S(f)$, we get a trigonometric operator of order $2n-1$ for which (1) *is equivalent to* $f^{(r-1)} \in \mathrm{Lip}\,(1, p)$.

Indeed, $U_n f = 2\sigma_{2n-1}(f) - \sigma_{n-1}(f)$ is the nth de la Vallée Poussin mean which satisfies (2), furthermore, as well known (cf. ZYGMUND [7], chapter IV), condition (3) is equivalent for $V_n f = \sigma_n(f)$ to $f^{(r-1)} \in \mathrm{Lip}\,(1, p)$.

By this corollary our problem could be considered as essentially solved, the only lack is that the operator T_n defined in the corollary is not a trigonometric polinomial of order n, but of order $2n-1$. We can easily help there.

Theorem 2. *By choosing*

$$T_n f = R_n\left(r + \frac{1-(-1)^r}{2}; f\right) + \frac{1-(-1)^r}{2n^r} \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) k^r B_k, \tag{5}$$

where $B_k(x) = a_k \sin kx - b_k \cos kx$, condition (1) *is both necessary and sufficient for* $f^{(r-1)} \in \mathrm{Lip}\,(1, p)$, *where* $1 \leqq p \leqq \infty$.

If r is even, then $T_n f = R_n(r; f)$ and our statement is already proved by theorem (C). Hence we have to consider only the case of r odd. But then

$$T_n f = R_n(r+1; f) + \frac{1}{n^r} \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) k^r B_k. \tag{6}$$

Setting $U_n f = R_n(r+1; f)$, it is easily seen that $U_n f$ satisfies condition (2) if $f^{(r-1)} \in Z_p$ (cf. ZYGMUND [6] and ALJANČIĆ [2]). Now

$$V_n^{(r)} f = \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) k^r B_k$$

is, for odd r, the r times differentiated nth Fejér mean, therefore (cf. ZYGMUND [7], chapter IV), condition (3) is equivalent to $f^{(r-1)} \in \mathrm{Lip}\,(1, p)$. All the conditions of Theorem 1 being satisfied, our statement is a consequence of this theorem.

3. Localization and approximation by algebraic polynomials

We say that the function $f(x)$ satisfies the Lipschitz condition in the open interval (a, b), if $f \in \operatorname{Lip} 1$ in every closed interval $[a+\varepsilon, b-\varepsilon] \subset (a, b)$, the Lipschitz constant being dependent of the choice of $\varepsilon > 0$. We suppose $(a, b) \subset (-\pi, \pi)$ and set $f(x)=0$ for $x \in (-\pi, \pi)-(a, b)$. By $S(f)$ we understand the Fourier series of the function $f(x)$ extended in such a way to $(-\pi, \pi)$ and then periodically to the whole straight line.

For the study of local approximation we introduce the Riesz means of type r and order r of the series $S(f)$:

$$R_n^*(r; f) = \sum_{k=0}^{n} \left[1 - \left(\frac{k}{n+1}\right)^r\right]^r A_k.$$

Put

$$T_n f = R_n^*\left(r + \frac{1-(-1)^r}{2}; f\right) + \frac{1-(-1)^r}{2n^r} \sum_{k=0}^{n} \frac{A_{n-k}^r}{A_n^r} k^r B_k, \tag{5a}$$

wehre $A_m^l = \binom{m+l}{m}$.

Theorem 3. *By choosing T_n as in* (5a), *we have $f^{(r-1)} \in \operatorname{Lip} 1$ in (a, b) if and only if*

$$\|f - T_n f\|_{C(a+\varepsilon, b-\varepsilon)} = O\left(\frac{1}{n^r}\right) \tag{7}$$

for every $\varepsilon > 0$, where the constant in the O-sign may depend on ε.

Proof. Suppose first $f^{(r-1)} \in \operatorname{Lip} 1$ in (a, b). Then, for r even, $T_n f$ reduces to $R_n^*(r; f)$ and (7) follows by the train of thoughts of Sunouchi [4b]. If r is odd, $T_n f$ has a similar form as in (6). We get then

$$\|f - R_n^*(r+1; f)\|_{C(a+\varepsilon, b-\varepsilon)} = O\left(\frac{1}{n^r}\right); \tag{8}$$

further we also have (Sunouchi [4a])

$$\left\|\sum_{k=1}^{n} \frac{A_{n-k}^r}{A_n^r} k^r B_k\right\|_{C(a+\varepsilon, b-\varepsilon)} = O(1), \tag{9}$$

and then (7) follows from (6), (8) and (9).

Suppose now (7) is satisfied. $T_n f$ being a trigonometric polynomial of order n, it follows by the localized Bernstein—Zygmund theorem that $f^{(r-1)}(x)$ exists and belongs to the Zygmund class in $[a+2\varepsilon, b-2\varepsilon]$. Therefore

$$\|f - R_n^*(r+1; f)\|_{C(a+3\varepsilon, b-3\varepsilon)} = O\left(\frac{1}{n^r}\right),$$

and so we have also

$$\frac{1}{n^r}\left\|\sum_{k=1}^{n}\frac{A_{n-k}^r}{A_n^r}k^r B_k\right\|_{C(a+3\varepsilon,\, b-3\varepsilon)} = O\left(\frac{1}{n^r}\right).$$

From the last evaluation it follows by the above mentioned result of SUNOUCHI [4a] that $f^{(r-1)}\in \mathrm{Lip}\,1$ in $[a+3\varepsilon, b-3\varepsilon]$. Since $\varepsilon>0$ is arbitrary, this means $f^{(r-1)}\in\mathrm{Lip}\,1$ in the open interval (a, b), as we have stated.

We are now able also to characterize non-periodical functions satisfying the Lipschitz condition by their approximation with algebraic polynomials. Denote to this aim by $t_n(x)$ and $u_n(x)$ the nth normed Chebyshev polynomials of the first and second kind. Set

$$a_n = \int_{-1}^{1} f(x)t_n(x)\frac{dx}{\sqrt{1-x^2}},$$

where $f(x)=0$ outside some interval (a, b) with $-1<a<b<1$.

Theorem 4. *Set*

$$T_n f = \sum_{k=0}^{n}\left[1-\left(\frac{k}{n+1}\right)^2\right]^2 a_k t_k + \frac{1}{n}\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)k a_k u_k. \tag{10}$$

The function $f(x)$ belongs to Lip 1 *in (a, b) if and only if we have for every $\varepsilon>0$*

$$\|f-T_n f\|_{C(a+\varepsilon,\, b-\varepsilon)} = O\left(\frac{1}{n}\right). \tag{11}$$

Proof. Introducing the variable $\theta=\arccos x$, the function $f(x)$ is transformed in $f^*(\theta)=f(\cos\theta)$, the operator $T_n f$ in

$$T_n f^* = \sum_{k=0}^{n}\left[1-\left(\frac{k}{n+1}\right)^2\right]^2 a_k\cos k\theta + \frac{1}{n}\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)k a_k\frac{\sin k\theta}{\sin\theta},$$

and the open interval (a, b) in $(a', b')\subset(0, \pi)$. The first sum on the right hand side of (11) is $R_n^*(2; f^*)$, while the second equals $\frac{1}{n\sin\theta}\sigma_n'(f^*)$. For $\theta\in[a'+\delta, b'-\delta]$ with $\delta>0$ we have $0<\alpha\leqq\sin\theta\leqq1$, therefore the last expression and $n^{-1}\cdot\sigma_n'(f^*, \theta)$ are $=O(n^{-1})$ at the same time, i.e. (11) is equivalent to

$$\|f^*-T_n^*f^*\|_{C(a'+\delta,\, b'-\delta)} = O\left(\frac{1}{n}\right), \tag{12}$$

where $\delta>0$ is arbitrary and $T_n^*f^*$ is defined by

$$T_n^*f^* = R_n^*(2; f^*)+\frac{1}{n}\sigma_n'(f^*).$$

But, by Theorem 3, (12) is equivalent to the statement that $f^*(\theta)=f(\cos\theta)$ satisfies the Lipschitz condition in the open interval (a', b'). Returning to the variable $x=\cos\theta$ and taking into account that $f^*(\theta)$ in (a', b') and $f(x)$ in (a, b) satisfy the Lipschitz condition at the same time, our theorem is entirely proved.

Remarks. 1. It could be proved that, using the operator $T_n f$ defined in (5a), the condition

$$\|f-T_nf\|_{p(a,b)} = O\left(\frac{1}{n^r}\right)$$

is necessary and sufficient for $f^{(r-1)}\in \mathrm{Lip}\,(1, p)$ in the open interval $(a, b)\subset(-\pi, \pi)$. Hereby the L^p norm has to be taken over every interval $[a+\varepsilon, b-\varepsilon]\subset(a, b)$.

2. Let $\{p_n(x)\}$ be the system of orthonormal polynomials belonging to a weight function $w(x)\geqq 0$ having the property $0<m\leqq w(x)\leqq M$ for $x\in[a, b]\subset[-1, 1]$. Denoting by $\sigma_n(x)$ the nth Fejér mean of the expansion

$$f(x)\sim\sum_{k=0}^{\infty} c_k p_k(x),$$

one can prove by a theorem of G. Freud [3] that the Lebesgue constants of the de la Vallée Poussin delayed means $2\sigma_{2n-1}-\sigma_{n-1}$ are bounded in every subinterval $[a+\varepsilon, b-\varepsilon]$ of (a, b). Then, setting $T_nf=2\sigma_{2n-1}-\sigma_{n-1}+n^{-1}\cdot\sigma_n'$, one can prove easily that $f\in\mathrm{Lip}\,(1, p)$ in the open interval (a, b) if and only if

$$\|f-T_nf\|_{p(a,b)} = O\left(\frac{1}{n}\right) \quad (1\leqq p\leqq\infty),$$

where the L^p norm has to be taken over every closed subinterval of (a, b).

3. We were concerned with the case $\mathrm{Lip}\,(1, p)$, because only the case $\alpha=1$ of Lipschitz classes $\mathrm{Lip}\,(\alpha, p)$ is critical. But it can be seen easily that, in the theorems 2 and 3, we may substitute $\mathrm{Lip}\,(\alpha, p)$ for $\mathrm{Lip}\,(1, p)$, if we substitute in the statements $O(n^{-r+1-\alpha})$ for $O(n^{-r})$. Thus defined as in (5), then $f^{(r-1)}\in\mathrm{Lip}\,(\alpha, p)$ is equivalent to $\|f-T_nf\|_p=O(n^{-r+1-\alpha})$ for every positive $\alpha\leqq 1$ and $r=1, 2, \ldots$.

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(Received December 3, 1968)

On $|\overline{N}, p_n|$ summability factors

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1. Introduction

1. Introduction. Let Σa_n be an infinite series with the partial sums S_n. Let $\{p_n\}$ be a sequence of constants, real or complex, with partial sums $\{P_n\}$ and $P_{-1}=p_{-1}=0$. The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \quad (P_n \neq 0) \tag{1. 1}$$

defines the $(\overline{N}, p_n)$ means of the sequence $\{S_n\}$, or of the series Σa_n, generated by the sequence of constants $\{p_n\}$. If $\lim_{n\to\infty} t_n$ exists, we say that the series Σa_n is summable $(\overline{N}, p_n)$ [1]; and if the sequence $\{t_n\}$ is of bounded variation, that is

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty, \tag{1. 2}$$

then the series Σa_n is said to be absolutely summable $(\overline{N}, p_n)$, or simply summable $|\overline{N}, p_n|$.

The conditions of regularity of the method of summability $(\overline{N}, p_n)$ defined by (1. 1) are

$$\lim_{n\to\infty} |P_n| = \infty \tag{1. 3}$$

and

$$\sum_{v=0}^{n} |p_v| = O(|P_n|). \tag{1. 4}$$

If $\{p_n\}$ is real and non-negative, (1. 4) is automatically satisfied, and then (1. 3) is a necessary and sufficient condition for the regularity of the method. It is known that $|\overline{N}, p_n|$ is equivalent to $|R, P_n, 1|^*$, where $|R, P_n, 1|^*$ is a discrete Riesz mean of order one and type P_n.

In the particular case when $p_n = 1$, the $(\overline{N}, p_n)$ mean reduces to the familiar $(C, 1)$ mean. Also when $p_n = e^n$, $|\overline{N}, p_n|$ is equivalent to $|C, 0|$.

2. The sequence $\{\varepsilon_n\}$ is said to be a summability factor of the series Σa_n for a summability method Q, if $\Sigma a_n \varepsilon_n$ is summable by the method Q whereas, in general, Σa_n need not be summable. The summability factors for absolute Cesàro methods of summation were obtained by KOGBETLIANTZ [2]. He proved the following theorem.

Theorem. *If* Σa_n *is* $|C, \delta|$, *then* $\sum \frac{a_n}{n^{\delta-\gamma}}$ *is summable* $|C, \gamma|$ *for* $\gamma \leqq \delta$, $\gamma, \delta > 0$.

MOHANTY [3] has shown that whenever Σa_n is $|R, \log n, 1|$, the series $\sum \frac{a_n}{\log n}$ is $|C, 1|$ (see also TATCHELL [4]).

The object of this paper is to establish the following theorem, which is an analogue of the theorem of KOGBETLIANTZ and includes, amongst others, the above result of MOHANTY.

Theorem. *If* Σa_n *is* $|\bar{N}, p_n|$ *summable, then* $\sum \frac{a_n p_n Q_n}{q_n P_n}$ *is* $|\bar{N}, q_n|$ *summable provided* $\{p_n\}$ *and* $\{q_n\}$ *are positive sequences such that the sequences* $\left\{\Delta\left(\frac{Q_n}{q_n}\right)\right\}, \left\{\frac{Q_n p_n}{P_n q_n}\right\}$ *and* $\left\{\frac{Q_{n+1}}{q_{n+1}} \cdot \frac{\Delta p_n}{p_n}\right\}$ *are bounded.*

Proof. Let t_n denote the $(\bar{N}, p_n)$ mean of Σa_n. Then

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu = \frac{1}{P_n} \sum_{\nu=0}^{n} (P_n - P_{\nu-1}) a_\nu .$$

If τ_n denotes the $(\bar{N}, q_n)$ mean of $\sum \frac{a_n p_n Q_n}{q_n P_n}$, we similarly have

$$\tau_n = \frac{1}{Q_n} \sum_{\nu=0}^{n} (Q_n - Q_{\nu-1}) \frac{a_\nu p_\nu Q_\nu}{q_\nu P_\nu} = \sum_{\nu=0}^{n} \frac{a_\nu p_\nu Q_\nu}{P_\nu q_\nu} - \frac{1}{Q_n} \sum_{\nu=0}^{n} \frac{a_\nu p_\nu Q_{\nu-1} Q_\nu}{P_\nu q_\nu}$$

and

$$\tau_{n+1} = \sum_{\nu=0}^{n+1} \frac{a_\nu p_\nu Q_\nu}{P_\nu q_\nu} - \frac{1}{Q_{n+1}} \sum_{\nu=0}^{n+1} \frac{a_\nu p_\nu Q_{\nu-1} Q_\nu}{P_\nu q_\nu} .$$

Hence

$$\tau_{n+1} - \tau_n = \frac{a_{n+1} p_{n+1} Q_{n+1}}{P_{n+1} q_{n+1}} + \left(\frac{1}{Q_n} - \frac{1}{Q_{n+1}}\right) \sum_{\nu=0}^{n} \frac{a_\nu p_\nu Q_{\nu-1} Q_\nu}{P_\nu q_\nu} - \frac{a_{n+1} p_{n+1} Q_n Q_{n+1}}{P_{n+1} q_{n+1} Q_{n+1}} =$$

$$= \frac{a_{n+1} p_{n+1}}{P_{n+1}} + \left(\frac{q_{n+1}}{Q_n Q_{n+1}}\right) \sum_{\nu=0}^{n} \frac{a_\nu p_\nu Q_{\nu-1} Q_\nu}{P_\nu q_\nu} = \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{\nu=0}^{n+1} \frac{a_\nu p_\nu Q_{\nu-1} Q_\nu}{P_\nu q_\nu} =$$

$$= \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{\nu=0}^{n} \Delta\left(\frac{Q_{\nu-1} Q_\nu p_\nu}{P_\nu q_\nu}\right) s_\nu + \frac{q_{n+1}}{Q_n Q_{n+1}} \cdot \frac{Q_n Q_{n+1} p_{n+1} s_{n+1}}{P_{n+1} q_{n+1}} =$$

$$= -\frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{\nu=0}^{n} \frac{1}{p_\nu} \Delta\left(\frac{Q_{\nu-1} Q_\nu p_\nu}{P_\nu q_\nu}\right) \Delta P_{\nu-1} t_{\nu-1} - \frac{\Delta P_n t_n}{P_{n+1}} =$$

$$= -\frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{\nu=0}^{n} \frac{1}{p_\nu} \Delta\left(\frac{Q_{\nu-1} Q_\nu p_\nu}{P_\nu q_\nu}\right) \{t_{\nu-1} \Delta P_{\nu-1} + P_\nu \Delta t_{\nu-1}\} - \frac{t_n \Delta P_n + P_{n+1} \Delta t_n}{P_{n+1}} =$$

$$= \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{\nu=0}^{n} \Delta\left(\frac{Q_{\nu-1} Q_\nu p_\nu}{P_\nu q_\nu}\right) t_{\nu-1} - \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{\nu=0}^{n} \frac{P_\nu}{p_\nu} \Delta\left(\frac{Q_{\nu-1} Q_\nu p_\nu}{P_\nu q_\nu}\right) \Delta t_{\nu-1} + \frac{p_{n+1} t_n}{P_{n+1}} - \Delta t_n.$$

Further,

$$\sum_{\nu=0}^{n} \Delta\left(\frac{Q_{\nu-1} Q_\nu p_\nu}{P_\nu q_\nu}\right) t_{\nu-1} = -\sum_{\nu=0}^{n-1} \frac{Q_\nu Q_{\nu+1} p_{\nu+1}}{P_{\nu+1} q_{\nu+1}} \Delta t_{\nu-1} - \frac{Q_n Q_{n+1} p_{n+1} t_{n-1}}{P_{n+1} q_{n+1}}.$$

Consequently,

$$\tau_{n+1} - \tau_n = -\frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{\nu=0}^{n-1} \frac{Q_\nu Q_{\nu+1} p_{\nu+1}}{P_{\nu+1} q_{\nu+1}} \Delta t_{\nu-1} - \frac{p_{n+1} \Delta t_{n-1}}{P_{n+1}} - \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{\nu=0}^{n} \frac{P_\nu}{p_\nu} \Delta\left(\frac{Q_{\nu-1} Q_\nu p_\nu}{P_\nu q_\nu}\right) \Delta t_{\nu-1} - \Delta t_n.$$

Or,

$$\tau_n - \tau_{n+1} = \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{\nu=0}^{n} \frac{Q_\nu Q_{\nu+1} p_{\nu+1}}{P_{\nu+1} q_{\nu+1}} \Delta t_{\nu-1} + \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{\nu=0}^{n} \frac{P_\nu}{p_\nu} \Delta\left(\frac{Q_{\nu-1} Q_\nu p_\nu}{P_\nu q_\nu}\right) \Delta t_{\nu-1} + \Delta t_n.$$

Hence,

$$\sum_{n=0}^{\infty} |\tau_n - \tau_{n+1}| \leqq \sum_{n=0}^{\infty} \frac{q_{n+1}}{Q_n Q_{n+1}} \left|\sum_{\nu=0}^{n} \frac{Q_\nu Q_{\nu+1} p_{\nu+1}}{P_{\nu+1} q_{\nu+1}} \Delta t_{\nu-1}\right| + \sum_{n=0}^{\infty} \frac{q_{n+1}}{Q_n Q_{n+1}} \left|\sum_{\nu=0}^{n} \frac{P_\nu}{p_\nu} \Delta\left(\frac{Q_{\nu-1} Q_\nu p_\nu}{P_\nu q_\nu}\right) \Delta t_{\nu-1}\right| + \sum_{n=0}^{\infty} |\Delta t_n| = \Sigma_1 + \Sigma_2 + \Sigma_3 \quad \text{(say)}.$$

Now,

$$\Sigma_1 = \sum_{n=0}^{\infty} \frac{q_{n+1}}{Q_n Q_{n+1}} \left|\sum_{\nu=0}^{n} \frac{Q_\nu Q_{\nu+1} p_{\nu+1}}{P_{\nu+1} q_{\nu+1}} \Delta t_{\nu-1}\right| \leqq$$

$$\leqq \sum_{\nu=0}^{\infty} \frac{Q_\nu Q_{\nu+1} p_{\nu+1}}{P_{\nu+1} q_{\nu+1}} |\Delta t_{\nu-1}| \sum_{n=\nu}^{\infty} \left(\frac{1}{Q_n} - \frac{1}{Q_{n+1}}\right) = \sum_{\nu=0}^{\infty} \frac{Q_{\nu+1} p_{\nu+1}}{P_{\nu+1} q_{\nu+1}} |\Delta t_{\nu-1}| < \infty,$$

by hypothesis, and also $\Sigma_3 < \infty$ by hypothesis. Now,

$$\left|\sum_{\nu=0}^{n} \frac{P_\nu}{p_\nu} \Delta\left(\frac{Q_{\nu-1} Q_\nu p_\nu}{P_\nu q_\nu}\right) \Delta t_{\nu-1}\right| =$$

$$= \left|\sum_{\nu=0}^{n} \left\{\Delta\left(\frac{Q_{\nu-1} Q_\nu}{q_\nu}\right) + \frac{Q_{\nu+1} Q_\nu}{q_{\nu+1}} \left(1 - \frac{P_\nu p_{\nu+1}}{P_{\nu+1} p_\nu}\right)\right\} \Delta t_{\nu-1}\right| =$$

$$= \left| \sum_{\nu=0}^{n} \left\{ \Delta\left(\frac{Q_{\nu-1}Q_\nu}{q_\nu}\right) + \frac{Q_{\nu+1}Q_\nu}{q_{\nu+1}} \left(\frac{p_\nu P_{\nu+1} - P_\nu p_{\nu+1}}{P_{\nu+1}p_\nu}\right)\right\} \Delta t_{\nu-1} \right| =$$

$$= \left| \sum_{\nu=0}^{n} \left\{ \Delta\left(\frac{Q_{\nu-1}Q_\nu}{q_\nu}\right) + \frac{Q_\nu Q_{\nu+1}}{q_{\nu+1}} \left(\frac{p_\nu (P_{\nu+1} - P_\nu) + P_\nu \Delta p_\nu}{P_{\nu+1}p_\nu}\right)\right\} \Delta t_{\nu-1} \right| =$$

$$= \left| \sum_{\nu=0}^{n} \left\{ \Delta\left(\frac{Q_{\nu-1}Q_\nu}{q_\nu}\right) + \frac{Q_\nu Q_{\nu+1}p_{\nu+1}}{P_{\nu+1}q_{\nu+1}} + \frac{Q_\nu Q_{\nu+1}}{q_{\nu+1}} \cdot \frac{P_\nu}{P_{\nu+1}} \cdot \frac{\Delta p_\nu}{p_\nu}\right\} \Delta t_{\nu-1} \right| =$$

$$= \left| \sum_{\nu=0}^{n} -Q_\nu \Delta t_{\nu-1} + \sum_{\nu=0}^{n} Q_\nu \Delta\left(\frac{Q_\nu}{q_\nu}\right) \Delta t_{\nu-1} + \sum_{\nu=0}^{n} \frac{Q_\nu Q_{\nu+1} P_\nu \Delta p_\nu}{q_{\nu+1} P_{\nu+1} p_\nu} \Delta t_{\nu-1} + \right.$$

$$\left. + \sum_{\nu=0}^{n} \frac{Q_\nu Q_{\nu+1} p_{\nu+1}}{q_{\nu+1} P_{\nu+1}} \Delta t_{\nu-1} \right| \leqq \sum_{\nu=0}^{n} Q_\nu |\Delta t_{\nu-1}| + \sum_{\nu=0}^{n} Q_\nu \left| \Delta\left(\frac{Q_\nu}{q_\nu}\right)\right| |\Delta t_{\nu-1}| +$$

$$+ \sum_{\nu=0}^{n} \frac{Q_\nu Q_{\nu+1} p_{\nu+1}}{q_{\nu+1} P_{\nu+1}} |\Delta t_{\nu-1}| + \sum_{\nu=0}^{n} \left|\frac{\Delta p_\nu}{p_\nu}\right| |\Delta t_{\nu-1}| \frac{P_\nu Q_\nu Q_{\nu+1}}{P_{\nu+1} q_{\nu+1}}.$$

Hence,

$$\Sigma_2 = \sum_{n=0}^{\infty} \frac{q_{n+1}}{Q_n Q_{n+1}} \left| \sum_{\nu=0}^{n} \frac{P_\nu}{p_\nu} \Delta\left(\frac{Q_{\nu-1} Q_\nu p_\nu}{P_\nu q_\nu}\right) \Delta t_{\nu-1} \right| \leqq$$

$$\leqq \sum_{n=0}^{\infty} \frac{q_{n+1}}{Q_n Q_{n+1}} \left\{ \sum_{\nu=0}^{n} Q_\nu |\Delta t_{\nu-1}| + \sum_{\nu=0}^{n} Q_\nu \left|\Delta\left(\frac{Q_\nu}{q_\nu}\right)\right| |\Delta t_{\nu-1}| + \right.$$

$$\left. + \sum_{\nu=0}^{n} \frac{Q_\nu Q_{\nu+1} p_{\nu+1}}{P_{\nu+1} q_{\nu+1}} |\Delta t_{\nu-1}| + \sum_{\nu=0}^{n} \frac{Q_\nu Q_{\nu+1} P_\nu}{P_{\nu+1} q_{\nu+1}} \left|\frac{\Delta p_\nu}{p_\nu}\right| |\Delta t_{\nu-1}| \right\} =$$

$$= \sum_{\nu=0}^{\infty} |\Delta t_{\nu-1}| + \sum_{\nu=0}^{\infty} \left|\Delta\left(\frac{Q_\nu}{q_\nu}\right)\right| |\Delta t_{\nu-1}| + \sum_{\nu=0}^{\infty} \frac{Q_{\nu+1} p_{\nu+1}}{P_{\nu+1} q_{\nu+1}} |\Delta t_{\nu-1}| +$$

$$+ \sum_{\nu=0}^{\infty} \frac{Q_{\nu+1} P_\nu}{q_{\nu+1} P_{\nu+1}} \frac{|\Delta p_\nu|}{p_\nu} |\Delta t_{\nu-1}| = O(\Sigma |\Delta t_{\nu-1}|) = O(1).$$

Hence, $\Sigma |\tau_n - \tau_{n+1}| < \infty$. This establishes the theorem.

3. It is easy to see that, by taking $p_n = \frac{1}{n+1}$ and $q_n = 1$, the result of MOHANTY follows from our theorem.

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(Received May 2, 1968)

On embedding of classes H_p^ω

By L. LEINDLER in Szeged*)

Introduction

Let $\omega(\delta)$ be a nondecreasing continuous function on the interval (0, 1) having the properties:

$$\omega(0)=0, \quad \omega(\delta_1+\delta_2) \leqq \omega(\delta_1)+\omega(\delta_2) \quad \text{for} \quad 0 \leqq \delta_1 \leqq \delta_2 \leqq \delta_1+\delta_2 \leqq 1.$$

Such a function will be called a "modulus of continuity".

If $f(x)\in L^p(0, 1)$ $(1 \leqq p < \infty)$, the "modulus of continuity of $f(x)$ in $L^p(0, 1)$" is defined by

$$\omega_p(\delta, f) = \sup_{0\leqq h\leqq \delta} \left\{\int_0^{1-h} |f(x+h)-f(x)|^p\, dx\right\}^{1/p} \qquad (0\leqq\delta\leqq 1).$$

If given a number $p\geqq 1$ and a modulus of continuity $\omega(\delta)$, then $H_p^\omega \equiv H_p^{\omega(\delta)}$ will denote the collection of the functions $f(x)$ satisfying the condition $\omega_p(\delta, f)=O(\omega(\delta))$.

Let $\varphi(x)$ be a nonnegative nondecreasing function on $[0, \infty)$. The collection of the functions $f(x)$ having the property

$$\int_0^1 |f(x)|^r \varphi(|f(x)|)\, dx < \infty$$

will be denoted by $L^r\varphi(L)$.

Recently P. L. UL'JANOV has investigated in several papers (see for instance [4], [5] and [6]) the following problems:

1) Find a sufficient condition that

$$f\in L^r\varphi(L).$$

2) Find necessary and sufficient conditions in order that

$$H_p^{\omega(\delta)}\subset L^r\varphi(L).$$

*) This research was made while the author worked in the Steklov Institute Moscow as a visiting scientist.

Among others he proved the following theorems:

Theorem A. ([5], Theorem 1) *Suppose that* $f(x)\in L^p(0,1)$ *for some* $p\geqq 1$. *If* $p=1$ *then*

a) $$\sum_{n=1}^{\infty}[\varphi(n+1)-\varphi(n)]\,\omega_1\left(\frac{1}{n},f\right)<\infty \quad \textit{implies} \quad f\in L\varphi(L),$$

b) $$\sum_{n=1}^{\infty}\frac{1}{n^2}\,\varphi\left(36n\omega_1\left(\frac{1}{n},f\right)\right)<\infty \quad \textit{implies} \quad f\in\varphi(L),$$

while if $\nu\geqq p\geqq 1$ *then*

c) $$\sum_{n=1}^{\infty}n^{\frac{\nu}{p}-2}\,\omega_p^{\nu}\left(\frac{1}{n},f\right)<\infty \quad \textit{implies} \quad f\in L^{\nu}.$$

Theorem B. ([6], Theorem 3) *Suppose that* $1\leqq p<\nu<\infty$ *and* $0\leqq\beta<\infty$. *Then a necessary and sufficient condition for*

$$H_p^{\omega(\delta)}\subset L^{\nu}(\ln^+L)^{\beta}$$

is that

$$\sum_{n=1}^{\infty}n^{\frac{\nu}{p}-2}\,\omega^{\nu}\left(\frac{1}{n}\right)\ln^{\beta}(n+1)<\infty.$$

Theorem C. ([5], Theorem 2 and 4) *Let* $p=1$ *or* $p=2$. *Then a necessary and sufficient condition for*

$$H_p^{\omega(\delta)}\subset L^p\ln^+L$$

is that

$$\sum_{n=1}^{\infty}\frac{\omega^p(1/n)}{n}<\infty.$$

We remark that the method of proof of Theorem C for $p=1$ is different from what is given for $p=2$; furthermore, for $p=1$, UL'JANOV proved the following more general theorem:

Theorem D. ([5], Theorem 2) *If* $\varphi(t)$ *is an even and nonnegative function such that*

$$\varphi(t)\uparrow\infty \quad \textit{as} \quad t\uparrow \quad (0\leqq t<\infty), \qquad \varphi(n+1)-\varphi(n)\downarrow \quad \textit{as} \quad n\uparrow\infty,$$

and

$$\varphi(t^2)\leqq\varphi(t) \quad \textit{for any} \quad 0\leqq t<\infty,$$

then, for a given modulus of continuity $\omega(\delta)$,

$$H_1^{\varphi(\delta)}\subset L\varphi(L)$$

if and only if

$$\sum_{n=1}^{\infty}(\varphi(n+1)-\varphi(n))\,\omega\left(\frac{1}{n}\right)<\infty.$$

In connection with Theorem C, P. L. UL'JANOV raised in a conversation the problem: Is Theorem C valid for any $p \geqq 1$?

In the present paper we are going to give an affirmative answer to this question; that is, we prove that Theorem C can be extended to any $p \geqq 1$, moreover we generalize all the above mentioned theorems. We would like to point out that the most important parts of our theorems are the cases $p = \nu > 1$.

Our method of proof is partly similar to that of UL'JANOV's but in the case $p = \nu > 1$ it is quite different from his. In this case ($p = \nu > 1$) the kernel of our proof can be found in Lemma 7 and Lemma 8.

Theorem 1. *Let $\{\varphi_k\}$ be a nonnegative monotonic sequence of numbers, $\nu \geqq 1$ and $f(x) \in L^p(0, 1)$ for some $p \geqq 1$. Define*

$$\Phi(x) = \sum_{k=1}^{x} k^{\frac{\nu}{p}-2} \varphi_k.$$[1]

Then

(1) $$f \in L^{\nu - \frac{\nu}{p} + 1} \Phi(L)$$

follows from

(2) $$\sum_{k=1}^{\infty} k^{\frac{\nu}{p}-2} \varphi_k \omega_p^\nu \left(\frac{1}{n}, f\right) < \infty$$

in the following cases:

a) *if $\nu = p = 1$;*

b) *if $\nu = p > 1$ and*

(3) $$\sum_{k=m}^{\infty} \frac{\varphi_k}{k^{1+\varepsilon}} \leqq K \frac{\varphi_m}{m^\varepsilon}$$[2]

for certain $\varepsilon = \varepsilon(p) > 0$[3];

c) *if $\nu > p \geqq 1$ and*

(4) $$\varphi_k \leqq \varphi_{k+1}, \quad \sum_{k=m}^{\infty} \frac{\varphi_k}{k^2} \leqq K \frac{\varphi_m}{m}.$$

This theorem has the following corollary which was proved only for $p = 1$ till now (see Theorem A).

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[1]) Σ_a^b, where a and b are not integers, means a sum over all integers between a and b.

[2]) K and K_i denote either absolute constants or constants depending on certain functions and numbers which are not necessary to explain in detail, not necessarily the same at each occurrence.

[3]) See more on the meaning of $\varepsilon(p)$ in the proof of Lemma 8.

Corollary 1. *If* $p \geqq 1$, $\beta > -1$ *and*

$$\sum_{n=2}^{\infty} \frac{1}{n} (\ln n)^{\beta} \omega_p^p\left(\frac{1}{n}, f\right) < \infty,$$

then

$$f \in L^p (\ln^+ L)^{\beta+1};$$

if, furthermore, $\gamma > -1$ *and*

$$\sum_{n=8}^{\infty} \frac{(\ln \ln n)^{\gamma}}{n \ln n} \omega_p^p\left(\frac{1}{n}, f\right) < \infty,$$

then

$$f \in L^p (\ln^+ \ln L)^{\gamma+1}.$$

Let $\psi(x)$ be a nonnegative increasing function having the properties:

(5) $$\frac{\psi(x)}{x} \uparrow \quad \text{and} \quad \frac{\psi(x)}{x^{\eta}} \downarrow \quad \text{for some} \quad \eta > 1 \quad \text{as} \quad x \uparrow \infty.$$

Mulholland [3] investigated such functions and proved: *If* $\psi(x)$ *satisfies* (5), $\gamma > 1$ *and* $\lambda_n \geqq 0$, *then*

(6) $$\sum_{n=1}^{\infty} n^{-\gamma} \psi(\lambda_1 + \lambda_2 + \cdots + \lambda_n) \leqq K \sum_{n=1}^{\infty} n^{-\gamma} \psi(n\lambda_n)$$

holds, where $K = K(\psi, \gamma)$.

Using (6) and an estimate of Ul'janov (Lemma 4) we can prove

Theorem 2. *If* $\psi(x)$ *satisfies* (5), $p \geqq 1$, *and*

(7) $$\sum_{n=1}^{\infty} \frac{1}{n^2} \psi\left(n^{1/p} \omega_p\left(\frac{1}{n}, f\right)\right) < \infty,$$

then

$$f \in \psi(L).$$

This is an analogue of the part b) of Theorem A for any $p \geqq 1$.

Finally we prove the following

Theorem 3. *Let* $1 \leqq p \leqq \nu < \infty$. *Let* $\omega(\delta)$ *be a given modulus of continuity and* $\{\varphi_k\}$ *be a nonnegative monotonic sequence of numbers satisfying* $\varphi_{k^2} \leqq K\varphi_k$ *for any* k, *and if* $\nu > p$, *then moreover let* $\varphi_k \leqq \varphi_{k+1}$. *Then a necessary and sufficient condition that*

(8) $$H_p^{\omega(\delta)} \subset L^{\nu - \frac{\nu}{p} + 1} \Phi(L)$$

is that

(9) $$\sum_{n=1}^{\infty} n^{\frac{\nu}{p} - 2} \varphi_n \omega^{\nu}\left(\frac{1}{n}\right) < \infty,$$

where $\Phi(x)$ *means the same as in Theorem 1.*

We remark that in the case $\nu > p$ and $\varphi_k = (\ln k)^{\beta}$ $(\beta \geqq 0)$ Theorem 3 includes Theorem B.

Corollary 2. *Let $p \geqq 1$ and $\beta > -1$. Then a necessary and sufficient condition for*

$$H_p^{\omega(\delta)} \subset L^p (\ln^+ L)^{\beta+1}$$

is that

$$\sum_{n=2}^{\infty} \frac{(\ln n)^{\beta}}{n} \omega^p \left(\frac{1}{n}\right) < \infty.$$

This corollary is a generalization of Theorem C for any $p \geqq 1$ and gives an answer to the problem of UL'JANOV mentioned above. However, for $p = 1$ this corollary and also the following Corollary 3 were proved by UL'JANOV ([5], Corollary 4).

Corollary 3. *Let $p \geqq 1$ and $\gamma > -1$. Then a necessary and sufficient condition for*

$$H_p^{\omega(\delta)} \subset L^p (\ln^+ \ln^+ L)^{\gamma+1}$$

is that

$$\sum_{n=10}^{\infty} \frac{(\ln \ln n)^{\gamma}}{n \ln n} \omega^p \left(\frac{1}{n}\right) < \infty.$$

My grateful acknowledgement is due to Professor P. L. UL'JANOV for having called my attention to this problem.

§ 1. Lemmas

We require the following lemmas.

Lemma 1. ([2], Lemma 3) *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of nonnegative numbers such that*

$$\sum_{k=n}^{\infty} \alpha_k = \eta_n \alpha_n.$$

Then for any $\gamma \geqq 1$

$$\sum_{k=1}^{\infty} \alpha_k \left(\sum_{n=1}^{k} \beta_n\right)^{\gamma} \leqq K \sum_{k=1}^{\infty} \alpha_k (\eta_k \beta_k)^{\gamma}.$$

Lemma 2. ([5], Lemma 4) *If $f(x) \in L^p(0, 1)$, $1 \leqq p < \infty$ and*

$$\psi_n(x) = n \int_{k/n}^{(k+1)/n} f(t)\, dt \quad \text{for} \quad x \in \left[\frac{k}{n}, \frac{k+1}{n}\right) \qquad (k = 0, 1, \ldots, (n-1)),$$

then

$$\left\{\int_0^1 |f(x) - \psi_n(x)|^p\, dx\right\}^{1/p} \leqq 4\omega_p \left(\frac{1}{n}, f\right) \qquad (n = 1, 2, \ldots).$$

Lemma 3. ([5], Lemma 13) *Let $A(u)$ be a nonnegative nondecreasing function on* $[0, \infty)$ *such that* $A(u^2) \leqq KA(u)$ *for any* $u \in [0, \infty)$ *and let* $B(u)$ *be a nonnegative function on* $(0, 1]$. *Then*

$$\int_0^1 B(u) A(B(u))\, du < \infty \tag{1. 1}$$

implies

$$\int_0^1 B(u) A\left(\frac{1}{u}\right) du < \infty. \tag{1. 2}$$

Lemma 4. *Let* $p \geqq 1$ *and* $f(x) \in L^p(0, 1)$. *Let* $F(z)$ *be a nonnegative nonincreasing function such that*

$$\text{mes}\, \{x: x \in [0, 1], |f(x)| > y\} = \text{mes}\, \{z: z \in [0, 1], F(z) > y\}.$$

Then

$$\int_0^{1/n} F(z)\, dz \leqq K\omega_1\left(\frac{1}{n}, f\right) \tag{1. 3}$$

and for $p > 1$

$$F(2^{-n-1}) \leqq K\left(1 + \sum_{k=1}^{n} 2^{k/p} \omega_p(2^{-k}, f)\right) \qquad (n = 1, 2, \ldots) \tag{1. 4}$$

hold.

This lemma can be found in the proof of Theorem A ([5], Theorem 1) implicitly.

Lemma 5. ([5], Lemma 7) *If* $f(x) \in L(0, 1)$ *and* $0 \leqq \alpha \leqq 1$, *then*

$$\sup_{\substack{E \subset [0, 1] \\ |E| = \alpha}} \int_E |f(x)|\, dx = \int_0^\alpha F(z)\, dz,$$

where $F(z)$ *has the same meaning as in Lemma 4.*

Lemma 6. ([1], p. 78) *If* $\omega(\delta)$ *is a modulus of continuity, then there exists a concave function* $\omega_1(\delta)$ *such that*

$$\omega(\delta) \leqq \omega_1(\delta) \leqq 2\omega(\delta) \quad \text{if} \quad 0 \leqq \delta \leqq 1.$$

Lemma 7. *If* $p \geqq 1$ *and* $f(x) \in L^p(0, 1)$, *then*

$$\int_0^{1/2n} F(z)^p\, dz \leqq K(p)\left(\int_{1/2n}^{1/n} F(z)^p\, dz + \omega_p^p\left(\frac{1}{n}, f\right)\right) \tag{1. 5}$$

for any $n \geqq 1$, *where* $K(p)$ *depends only on* p, *and* $F(z)$ *means the same as in Lemma 4.*

Proof. Let us choose $M(>1)$ such that

$$q = \left(1+\frac{1}{M}\right)^p < 2. \tag{1.6}$$

Set

$$E_n^* = \left\{x\colon x\in[0,1],\ |f(x)| > F\left(\frac{1}{2n}\right)\right\} \quad\text{and}\quad E_n^{**} = \left\{x\colon x\in[0,1],\ |f(x)| = F\left(\frac{1}{2^n}\right)\right\}.$$

If mes $E_n^* < \frac{1}{2n}$, then let us define y such that

$$\text{mes}\,(E_n^{**}\cap(0,y)) = \frac{1}{2n} - \text{mes}\,E^*$$

and let

$$E_n = (E_n^{**}\cap(0,y))\cup E_n^*.$$

If mes $E_n^* = \frac{1}{2n}$, then we set $E_n = E_n^*$.

First we estimate $|f(x)|^p$ on E_n. Let $\psi_n(x)$ be the same function as in Lemma 2. Since the estimate

$$|f(x)|^p \leqq 2^p(|f(x)-\psi_n(x)|^p + |\psi_n(x)|^p)$$

in not fine enough for our aim we split the set E_n into the parts

$$E_n' = \{x\colon x\in E_n,\ M|f(x)-\psi_n(x) \leqq |\psi_n(x)|\}$$

and

$$E_n'' = \{x\colon x\in E_n,\ M|f(x)-\psi_n(x)| > |\psi_n(x)|\}.$$

Then we have by $|f(x)| \leqq |f(x)-\psi_n(x)| + |\psi_n(x)|$

$$\int_{E_n'} |f(x)|^p\,dx \leqq \int_{E_n'} \left(\frac{1}{M}+1\right)^p |\psi_n(x)|^p\,dx \leqq q \int_{E_n} |\psi_n(x)|^p\,dx$$

and

$$\int_{E_n''} |f(x)|^p\,dx \leqq \int_{E_n''} (M+1)^p |f(x)-\psi_n(x)|^p\,dx \leqq (M+1)^p \int_0^1 |f(x)-\psi_n(x)|^p\,dx.$$

Hence we get by Lemma 2 and Lemma 5

$$\int_0^{1/2n} F(z)^p\,dz = \int_{E_n} |f(x)|^p\,dx \leqq q \int_{E_n} |\psi_n(x)|^p\,dx + K_1(p)\,\omega_p^p\left(\frac{1}{n}, f\right) \leqq \tag{1.7}$$

$$\leqq q(\max|\psi_n(x)|)^p \frac{1}{2n} + K_1(p)\,\omega_p^p\left(\frac{1}{n}, f\right).$$

Since

$$\max|\psi_n(x)| \leqq n\int_0^{1/n} F(z)\,dz \leqq n\left\{\int_0^{1/n} F(z)^p\,dz\right\}^{1/p} n^{\frac{1-p}{p}} \leqq \left\{n\int_0^{1/n} F(z)^p\,dz\right\}^{1/p},$$

we obtain by (1. 7)

$$\int_0^{1/2n} F(z)^p\,dz \leqq \frac{q}{2}\int_0^{1/n} F(z)^p\,dz + K_1(p)\,\omega_p^p\left(\frac{1}{n}, f\right).$$

Hence, by (1. 6),

$$\left(1-\frac{q}{2}\right)\int_0^{1/2n} F(z)^p\,dz \leqq \frac{q}{2}\int_{1/2n}^{1/n} F(z)^p\,dz + K_1(p)\,\omega_p^p\left(\frac{1}{n}, f\right)$$

which gives the conclusion (1. 5).

Lemma 8. *If* $p \geqq 1$ *and* $f(x) \in L^p(0,1)$, *then there exist constants* $K(p)$ *and* $\varepsilon(p) > 0$ *depending only on p such that*

$$\int_0^{1/n} F(z)^p\,dz \leqq \frac{K(p)}{n^{\varepsilon(p)}}\left(\sum_{k=1}^{n} k^{\varepsilon(p)-1}\,\omega_p^p\left(\frac{1}{k}, f\right) + \int_0^1 F(z)^p\,dz\right) \tag{1. 8}$$

for any $n \geqq 1$.

Proof. By Lemma 7 there exists an integer $N = N(p)$ such that for any $n \geqq 1$

$$\int_0^{2^{-n}} F(z)^p\,dz \leqq N\left(\int_{2^{-n-1}}^{2^{-n}} F(z)^p\,dz + \omega_p^p(2^{-n}, f)\right). \tag{1. 9}$$

Let us define for every $n \geqq 1$

$$a_n = \int_{2^{-n-1}}^{2^{-n}} F(z)^p\,dz, \quad b_n = \omega_p^p(2^{-n}, f)$$

and

$$\alpha_n = \sum_{k=(n-1)N+1}^{nN} a_k, \quad \beta_n = \sum_{k=(n-1)N+1}^{nN} b_k.$$

Considering (1. 9) we have

$$\sum_{k=n}^{\infty} a_k \leqq N(a_n + b_n)$$

and hence an easy computation gives that for any $m \geqq 1$

$$\sum_{i=m+1}^{\infty} \alpha_i \leqq \alpha_m + \beta_m.$$

Indeed, we have for any nonnegative integer j

$$\sum_{i=m+1}^{\infty} \alpha_i = \sum_{k=mN+1}^{\infty} a_k \leqq \sum_{k=mN+1-j}^{\infty} a_k \leqq N(a_{mN+1-j} + b_{mN+1-j}).$$

Taking $j=1, 2, \ldots, N$ and summing we obtain

$$N \sum_{i=m+1}^{\infty} \alpha_i \leqq N \sum_{k=(m-1)N+1}^{mN} (a_k+b_k) = N(\alpha_m+\beta_m),$$

and hence we get the required inequality. Multiplying this inequality by max $(2^{m-2}, 1)$ for all m, $1 \leqq m \leqq n$, summing and cancelling we obtain

$$(1.10) \qquad 2^{n-1} \sum_{i=n+1}^{\infty} \alpha_i \leqq \alpha_1+\beta_1+\sum_{k=2}^{n} 2^{k-2}\beta_k.$$

Inserting the definition of α_i and β_i, (1. 10) implies

$$(1.11) \qquad \int_0^{2^{-(nN+1)}} F(z)^p\,dz \leqq 2^{4-n}\left\{\int_0^1 F(z)^p\,dz+\sum_{k=1}^{n} 2^k N\omega_p^p(2^{-(k-1)N},f)\right\} \leqq$$

$$\leqq \frac{K_1(p)}{2^n}\left\{\int_0^1 F(z)^p\,dz+\sum_{k=1}^{n} 2^k\omega_p^p(2^{-kN},f)\right\}.$$

If $2^{nN+1} \leqq m < 2^{(n+1)N+1}$, then from (1. 11) it follows with $\varepsilon = \frac{1}{N}$ that

$$(1.12) \qquad \int_0^{1/m} F(z)^p\,dz \leqq \frac{K_2(p)}{m^\varepsilon}\left\{\int_0^1 F(z)^p\,dz+\sum_{k=1}^{\overset{2}{\smile}\varepsilon\log m} 2^k\omega_p^p(2^{-kN},f)\right\}.$$

From this point on the proof is an easy computation. Indeed, we have

$$\sum_{k=1}^{\overset{2}{\smile}\varepsilon\log m} 2^k\omega_p^p(2^{-kN},f) \leqq 2^N \sum_{k=1}^{\overset{2}{\smile}\varepsilon\log m} \frac{2^k}{2^{kN}} \sum_{i=2^{(k-1)N}+1}^{2^{kN}} \omega_p^p\left(\frac{1}{i},f\right) \leqq$$

$$\leqq 4^N \sum_{k=1}^{\varepsilon\log m} \sum_{i=2^{(k-1)N}+1}^{2^{kN}} \frac{i^\varepsilon}{i}\,\omega_p^p\left(\frac{1}{i},f\right) \leqq 4^N \sum_{k=1}^{m} i^{\varepsilon-1}\omega_p^p\left(\frac{1}{i},f\right);$$

inserting this into (1. 12) we obtain (1. 8) in accordance with our statement.

The following two lemmas are slight improvements of Lemmas 11 and 12 of UL'JANOV [5].

Lemma 9. *Let $p>0$, $\alpha > 1-p$ and let $\omega(\delta)$ be a concave modulus of continuity. If the sequence $\{\varphi_k\}$ is monotonic,*

$$(1.13) \qquad \varphi_k \geqq 0, \quad \sum_{k=m}^{\infty} \frac{\varphi_k}{k^{\alpha+p}} \leqq K\frac{\varphi_m}{m^{\alpha+p-1}}$$

and

$$(1.14) \qquad \sum_{k=1}^{\infty} \varphi_k k^{-\alpha}\omega^p\left(\frac{1}{k}\right) = \infty,$$

then there exists a sequence of numbers $\{B_k\}$ *such that*

$$(1.15)\qquad B_k\downarrow 0,\quad B_k\leqq \omega\left(\frac{1}{k}\right),\quad \sum_{k=1}^{m} k^{p-1}B_k^p \leqq Km^p\omega^p\left(\frac{1}{m}\right)$$

and

$$(1.16)\qquad \sum_{k=1}^{\infty}\varphi_k k^{-\alpha}B_k^p=\infty.$$

Proof. In view of the conditions,

$$(1.17)\qquad n\omega\left(\frac{1}{n}\right)\uparrow\infty \quad\text{as}\quad n\to\infty.$$

By (1. 17) we can easily define a sequence of integers $\{n_k\}$ such that

$$(1.18)\qquad n_{k+1}\omega\left(\frac{1}{n_{k+1}}\right)>2n_k\omega\left(\frac{1}{n_k}\right)\quad\text{for}\quad k\geqq 1$$

and

$$(1.19)\qquad n\omega\left(\frac{1}{n}\right)\leqq 2n_k\omega\left(\frac{1}{n_k}\right)\quad\text{if}\quad n_k\leqq n<n_{k+1}.$$

Indeed, let $n_0=0$ and $n_1=1$. Suppose that $n_1<n_2<\cdots<n_k$ are defined so that they satisfy (1. 18) and (1. 19), and let n_{k+1} be the smallest integer μ having the property $\mu\omega\left(\frac{1}{\mu}\right)>2n_k\omega\left(\frac{1}{n_k}\right)$. Thus, by induction, we get the required sequence $\{n_k\}$.

Now we define the sequence $\{B_n\}$. Put

$$(1.20)\qquad B_n=\omega\left(\frac{1}{n_k}\right)\quad\text{for}\quad n_{k-1}<n\leqq n_k\qquad (k=1,2,\ldots).$$

It is clear that $B_n\downarrow 0$ and $B_n\leqq\omega\left(\frac{1}{n}\right)$. To prove the estimation (1. 15) we choose r such that $n_r<m\leqq n_{r+1}$. Then, by (1. 18) and (1. 20), we have

$$\sum_{n=1}^{m} n^{p-1}B_n^p\leqq\sum_{k=1}^{r}\sum_{n=n_{k-1}+1}^{n_k} n^{p-1}B_n^p+\sum_{n=n_r+1}^{m} n^{p-1}B_n^p\leqq$$

$$\leqq K(p)\left(\sum_{k=1}^{r}B_{n_k}^p n_k^p+m^pB_m^p\right)\leqq K_1(p)m^p\omega^p\left(\frac{1}{m}\right)$$

in accordance with our statement.

Finally we prove (1. 16). Since $n_{k+1}>2n_k$ and $\varphi_{2i}\leqq K_1\varphi_i$ follow from (1. 13), we have

$$\sum_{n=1}^{\infty} n^{-\alpha}\varphi_n B_n^p = \sum_{k=1}^{\infty}\sum_{n=n_{k-1}+1}^{n_k} n^{-\alpha}\varphi_n B_n^p \geqq \tag{1.21}$$

$$\geqq \sum_{k=1}^{\infty} B_{n_k}^p \cdot \sum_{n=\frac{n_k}{2}+1}^{n_k} \varphi_n n^{-\alpha} \geqq K_2 \sum_{k=1}^{\infty} B_{n_k}^p \varphi_{n_k} n_k^{1-\alpha}.$$

On the other hand by (1. 13) and (1. 19),

$$\sum_{n=n_k}^{n_{k+1}-1} \varphi_n n^{-\alpha}\omega^p\left(\frac{1}{n}\right) = \sum_{n=n_k}^{n_{k+1}-1} \varphi_n n^{-\alpha-p}\left(n\omega\left(\frac{1}{n}\right)\right)^p \leqq \tag{1.22}$$

$$\leqq 2^p\left(n_k\omega\left(\frac{1}{n_k}\right)\right)^p \sum_{n=n_k}^{\infty}\varphi_n n^{-\alpha-p} \leqq K_3\left(\omega\left(\frac{1}{n_k}\right)\right)^p \varphi_{n_k} n_k^{1-\alpha} = K_3 B_{n_k}^p \varphi_{n_k} n_k^{1-\alpha}.$$

In view of (1. 14), (1. 21) and (1. 22) we obtain (1. 16).

The proof is thus completed.

Lemma 10. *Let $\nu\geqq 1$, $1-\nu<\alpha<1$ and let $\omega(\delta)$ be a concave modulus of continuity. If the positive sequence $\{\varphi_k\}$ is increasing,*

$$\sum_{n=m}^{\infty} \varphi_n n^{-\alpha-\nu} \leqq K\varphi_m m^{1-\alpha-\nu} \tag{1.23}$$

and

$$\sum_{n=1}^{\infty} \varphi_n n^{-\alpha}\omega^\nu\left(\frac{1}{n}\right) = \infty, \tag{1.24}$$

then there exist a sequence of numbers $\{B_n\}$ and a sequence of integers $\{n_k\}$ such that

$$B_n\downarrow 0 \quad and \quad B_n\leqq \omega\left(\frac{1}{n}\right), \tag{1.25}$$

$$n_{k+1}>2n_k \quad and \quad B_{n_{k+1}}\leqq \frac{1}{2}B_{n_k} \qquad (k\geqq 1), \tag{1.26}$$

$$\sum_{n=1}^{m} n^{p-1}B_n^p \leqq K(p)m^p\omega^p\left(\frac{1}{m}\right) \quad for\ any \quad p>0, \tag{1.27}$$

$$\sum_{n=1}^{\infty}\varphi_n n^{-\alpha}B_n^\nu = \infty \quad and \quad \sum_{k=1}^{\infty}\varphi_{n_k} n_k^{1-\alpha}B_{n_k}^\nu = \infty \tag{1.28}$$

and

$$\sum_{n=1}^{\infty}\varphi_{2^n}2^{n(1-\alpha)}(B_{2^n}-B_{2^{n+1}})^\nu = \infty. \tag{1.29}$$

Proof. Since $\omega(\delta)$ is concave, by (1. 23) we have $n\omega\left(\frac{1}{n}\right)\uparrow\infty$. Using this we can define the sequence $\{n_k\}$. Let $n_0=0$ and $n_1=1$. Suppose that $n_1<n_2<\dots<n_k$ are defined so that $n_{i+1}>2n_i$ $(i=0, 1, \dots, k-1)$. Then denote by m_{k+1} the smallest integer μ having the property $\mu\omega\left(\frac{1}{\mu}\right)>2n_k\omega\left(\frac{1}{n_k}\right)$. By the definition of m_{k+1} we have

$$n\omega\left(\frac{1}{n}\right)\leqq 2n_k\omega\left(\frac{1}{n_k}\right) \quad \text{if} \quad n_k\leqq n<m_{k+1}, \tag{1.30}$$

$$m_{k+1}\omega\left(\frac{1}{m_{k+1}}\right)>2n_k\omega\left(\frac{1}{n_k}\right) \tag{1.31}$$

and

$$m_{k+1}>2n_k. \tag{1.32}$$

If

$$\omega\left(\frac{1}{m_{k+1}}\right)\leqq\frac{1}{2}\omega\left(\frac{1}{n_k}\right), \tag{1.33}$$

then set $n_{k+1}=m_{k+1}$. Conversely, if $\omega\left(\frac{1}{m_{k+1}}\right)>\frac{1}{2}\omega\left(\frac{1}{n_k}\right)$, then let n_{k+1} be the smallest integer μ satisfying $\omega\left(\frac{1}{\mu}\right)\leqq\frac{1}{2}\omega\left(\frac{1}{n_k}\right)$. In this case

$$\omega\left(\frac{1}{n}\right)>\frac{1}{2}\omega\left(\frac{1}{n_k}\right) \quad \text{for} \quad n_k\leqq n<n_{k+1} \tag{1.34}$$

and

$$\omega\left(\frac{1}{n_{k+1}}\right)\leqq\frac{1}{2}\omega\left(\frac{1}{n_k}\right). \tag{1.35}$$

Set

$$B_n=\omega\left(\frac{1}{n_k}\right) \quad \text{for} \quad n_{k-1}<n\leqq n_k \qquad (k=1, 2, \dots). \tag{1.36}$$

The statements (1. 25) and (1. 26) obviously follow from the definitions and (1. 27) can be proved as in Lemma 9.

Now we prove (1. 28). As in Lemma 9 we have

$$\sum_{n=1}^{\infty} n^{-\alpha}\varphi_n B_n^{\nu}\geqq K\sum_{k=1}^{\infty} B_{n_k}^{\nu}\varphi_{n_k}n_k^{1-\alpha}. \tag{1.37}$$

If $n_{k+1}=m_{k+1}$, then by (1. 23) and (1. 30) we get

$$\sigma_k\equiv\sum_{n=n_k}^{n_{k+1}-1}\varphi_n n^{-\alpha}\omega^{\nu}\left(\frac{1}{n}\right)=\sum_{n=n_k}^{n_{k+1}-1}\varphi_n n^{-\alpha-\nu}\left(n\omega\left(\frac{1}{n}\right)\right)^{\nu}\leqq \tag{1.38}$$

$$\leqq 2^{\nu}\sum_{n=n_k}^{\infty}\left(n_k\omega\left(\frac{1}{n_k}\right)\right)^{\nu}\varphi_n n^{-\alpha-\nu}\leqq K(\nu)\omega^{\nu}\left(\frac{1}{n_k}\right)\varphi_{n_k}n_k^{1-\alpha}.$$

If $n_{k+1} > m_{k+1}$, then by (1. 34) we obtain

$$\sigma_k \leqq \omega^\nu\left(\frac{1}{n_k}\right)\sum_{n=n_k}^{n_{k+1}-1}\varphi_n n^{-\alpha} \leqq K_1(\nu)\omega^\nu\left(\frac{1}{n_{k+1}-1}\right)\varphi_{n_{k+1}} n_{k+1}^{1-\alpha} \leqq \tag{1.39}$$

$$\leqq K_2(\nu)\omega^\nu\left(\frac{1}{n_{k+1}}\right)\varphi_{n_{k+1}} n_{k+1}^{1-\alpha}.$$

(1. 38) and (1. 39) imply, by (1. 36), that

$$\sigma_k \leqq K_3(\nu)(\varphi_{n_k} n_k^{1-\alpha} B_{n_k}^\nu + \varphi_{n_{k+1}} n_{k+1}^{1-\alpha} B_{n_{k+1}}^\nu).$$

Hence and from (1. 37), on account of (1. 24), the statements (1. 28) follow.

In order to prove (1. 29) we set $J_j = [2^j, 2^{j+1})$ $(j=0, 1, \ldots)$. If $n_k \in J_{i_k}$, then for any l, $l \neq k$, we have $n_l \notin J_{i_k}$, indeed $n_{k+1} > 2n_k$. Therefore, if $n_k \in J_{i_k}$, then

$$B_{2^{i_k}} - B_{2^{i_k+1}} = B_{n_k} - B_{n_{k+1}}.$$

Using this and $B_{n_{k+1}} \leqq \frac{1}{2} B_{n_k}$ we have

$$\sum_{n=0}^{\infty}\varphi_{2^n} 2^{n(1-\alpha)}(B_{2^n} - B_{2^{n+1}})^\nu \geqq K_1 \sum_{k=1}^{\infty}\varphi_{2^{i_k}} 2^{i_k(1-\alpha)}(B_{n_k} - B_{n_{k+1}})^\nu \geqq$$

$$\geqq K_2 \sum_{k=1}^{\infty}\varphi_{2^{i_k}} 2^{i_k(1-\alpha)} B_{n_k}^\nu \geqq K_3 \sum_{k=1}^{\infty}\varphi_{n_k} n_k^{1-\alpha} B_{n_k}^\nu.$$

Hence, by (1. 28), the statement (1. 29) follows, and this completes the proof.

§ 2. Proof of the theorems

Proof of Theorem 1. Let $F(x)$ be the same function as in Lemma 4. It is well known that for any nonnegative nondecreasing function $\chi(u)$ on $[0, \infty)$

$$\int_0^1 \chi(|f(x)|)\,dx = \int_0^1 \chi(F(x))\,dx$$

(see [7], p. 54). Therefore it is sufficient to prove our statements for $F(x)$.

Put

$$E_n = \{z\colon\ z \in [0, 1],\ n \leqq F(z) < n+1\} \qquad (n=0, 1, \ldots).$$

It is clear that $E_n E_m = 0$ $(n \neq m)$, $\sum_{n=0}^{\infty} E_n = [0, 1]$ and $E_n = \{\alpha_{n+1}, \alpha_n\}$, where $\alpha_n \downarrow 0$ as $n \to \infty$. Set

$$A_n = \int_0^{\alpha_n} F^p(z)\,dz.$$

Then $A_n \to 0$ as $n \to \infty$ and for $n \geqq n_0$

$$1 \geqq A_n = \int_0^{\alpha_n} F^p(z)\,dz \geqq n\alpha_n,$$

that is

(2. 1) $$\alpha_n \leqq 1/n \quad \text{for} \quad n \geqq n_0.$$

If $\nu = p$ then we have to prove that

$$I = \int_0^1 F^p(z)\Phi(F(z))\,dz < \infty.$$

Using Abel transformation we obtain

(2. 2) $$I \leqq \sum_{n=0}^{\infty} \int_{E_n} F^p(z)\,dz \sum_{k=1}^{n} \varphi_k k^{-1} \leqq \sum_{k=1}^{\infty} \varphi_k k^{-1} \sum_{n=k}^{\infty} \int_{E_n} F^p(z)\,dz \leqq \sum_{k=1}^{\infty} \varphi_k k^{-1} A_k.$$

Hence for $p = \nu = 1$, by (2), (1. 3) and (2. 1), the statement (1) obviously follows. If $\nu = p > 1$, using (2), (3), (2. 1), (2. 2) and Lemma 8, we get with $\varepsilon = \varepsilon(p)$ that

$$I \leqq K \sum_{k=1}^{\infty} \varphi_k k^{-1-\varepsilon} \left(\sum_{n=1}^{k} n^{\varepsilon-1} \omega_p^p\left(\frac{1}{n}, f\right) + 1 \right) \leqq$$

$$\leqq K_1 \left\{ \sum_{n=1}^{\infty} n^{\varepsilon-1} \omega_p^p\left(\frac{1}{n}, f\right) \sum_{k=n}^{\infty} \varphi_k k^{-1-\varepsilon} + 1 \right\} \leqq K_2 \left\{ \sum_{n=1}^{\infty} n^{-1} \varphi_n \omega_p^p\left(\frac{1}{n}, f\right) + 1 \right\} < \infty;$$

thus (1) is verified for $\nu = p > 1$.

If $\nu > p$, then

(2. 3) $$I_1 \equiv \int_0^1 F(z)^{\nu - \frac{\nu}{p} + 1} \Phi(F(z))\,dz = \sum_{n=1}^{\infty} \int_{2^{-n}}^{2^{-n+1}} F(z)^{\nu - \frac{\nu}{p} + 1} \Phi(F(z))\,dz \leqq$$

$$\leqq \sum_{n=1}^{\infty} 2^{-n} F(2^{-n})^{\nu - \frac{\nu}{p} + 1} \Phi(F(2^{-n})) \equiv \Sigma_1.$$

By (4)

$$\varphi_{N\cdot m} \leqq K(N)\varphi_m; \qquad m = 1, 2, \ldots,$$

and by (1. 4)

$$F(2^{-n}) \leqq K 2^n.$$

According to these we have

$$\Phi(F(2^{-n})) \leqq K_1 \varphi_{2^n} F(2^{-n})^{\frac{\nu}{p} - 1}.$$

Hence

$$\Sigma_1 \leqq K_1 \sum_{n=1}^{\infty} 2^{-n} \varphi_{2^n} F(2^{-n})^{\nu}.$$

Using this and (1. 4) we obtain

(2. 4)
$$\Sigma_1 \leqq K_2 \sum_{n=1}^{\infty} \varphi_{2^n} 2^{-n} \left\{1+\sum_{k=1}^{n} 2^{k/p}\omega_p(2^{-k}, f)\right\}^{\nu} \leqq$$
$$\leqq K_3+K_4 \sum_{n=1}^{\infty} \varphi_{2^n} 2^{-n} \left\{\sum_{k=1}^{n} 2^{k/p}\omega_p(2^{-k}, f)\right\}^{\nu} \leqq$$
$$\leqq K_5+K_6 \sum_{m=1}^{\infty} \varphi_m m^{-2} \left(\sum_{i=1}^{m} i^{\frac{1}{p}-1}\, \omega_p\left(\frac{1}{i}, f\right)\right)^{\nu} = K_5+\Sigma_2.$$

Using (2), (4) and Lemma 1 ($\alpha_k=\varphi_k k^{-2}$, $\eta_n \leqq Kn$, $\gamma=\nu$) we get

(2. 5)
$$\Sigma_2 \leqq K_7 \sum_{m=1}^{\infty} \varphi_m m^{-2} m^{\nu} \left(m^{\frac{1}{p}-1}\, \omega_p\left(\frac{1}{m}, f\right)\right)^{\nu} = K_7 \sum_{m=1}^{\infty} \varphi_m m^{\frac{\nu}{p}-2}\, \omega_p^{\nu}\left(\frac{1}{m}, f\right) < \infty.$$

Collecting the estimates (2. 3), (2. 4) and (2. 5), we obtain that $J_1<\infty$, that is, (1) is proved.

We have completed our proof.

Proof of Theorem 2. An application of Lemma 4, (5) and (6) now yield

$$\int_0^1 \psi(|f(x)|)\, dx = \int_0^1 \psi(F(x))\, dx = \sum_{n=0}^{\infty} \int_{2^{-n-1}}^{2^{-n}} \psi(F(x))\, dx \leqq$$
$$\leqq \sum_{n=0}^{\infty} 2^{-n-1} \psi(F(2^{-n-1})) \leqq \sum_{n=0}^{\infty} 2^{-n-1} \psi\left(K\left\{1+\sum_{k=1}^{n} 2^{k/p}\omega_p(2^{-k}, f)\right\}\right) \leqq$$
$$\leqq K_1 \sum_{n=0}^{\infty} 2^{-n} \psi\left(1+\sum_{i=1}^{2^n} i^{\frac{1}{p}-1}\, \omega_p\left(\frac{1}{i}, f\right)\right) \leqq$$
$$\leqq K_2 \sum_{m=1}^{\infty} (m+1)^{-2} \psi\left(1+\sum_{i=1}^{m} i^{\frac{1}{p}-1}\, \omega_p\left(\frac{1}{i}, f\right)\right) \leqq K_3 \sum_{m=1}^{\infty} m^{-2} \psi\left(m^{1/p}\omega_p\left(\frac{1}{m}, f\right)\right).$$

Hence, by (7), we obtain the statement of Theorem 2.

Proof of Theorem 3. The sufficiency of (9) has been proved by Theorem 1. The necessity of (9) will be proved indirectly. Suppose that

(2. 6)
$$\sum_{n=1}^{\infty} n^{\frac{\nu}{p}-2} \varphi_n \omega^{\nu}\left(\frac{1}{n}\right) = \infty,$$

yet, (8) holds. According to Lemma 6 we can assume that $\omega(\delta)$ is concave. Now we can construct a function $f_0(x)$ leading to contradiction.

If $\nu=p$, then all the requirements of Lemma 9 are satisfied, thus we have a sequence $\{\bar{B}_n\}$ such that it satisfies (1. 15) and (1. 16) with $\alpha=1$.

If $\nu > p$, using Lemma 10 with $\alpha = 2-\frac{\nu}{p}$, we have sequences $\{\hat{B}_n\}$ and $\{n_k\}$ satisfying (1. 25)—(1. 29) with $\alpha = 2-\frac{\nu}{p}$.

Now define $f_0(x)$ as follows:

$$f_0(x) = \begin{cases} \varrho_n, & \text{if} \quad x = 3\cdot 2^{-n-2}, \\ 0, & \text{if} \quad x = 0,\ x\in[1/2, 1],\ x = 2^{-n}, \\ \text{linear on} & [2^{-n-1}, 3\cdot 2^{-n-2}],\ [3\cdot 2^{-n-2}, 2^{-n}] \end{cases}$$

$(n = 1, 2, \ldots)$, where $\varrho_n = 2^{(n+1)\frac{1}{p}}(B_{2^n}^p - B_{2^{n+1}}^p)^{\frac{1}{p}}$ and

$$B_n = \begin{cases} \bar{B}_n, & \text{if} \quad p = \nu, \\ \hat{B}_n, & \text{if} \quad p < \nu. \end{cases}$$

First we show that $f_0(x)\in H_p^{\omega(\delta)}$.

Let

$$h\in(2^{-k-3}, 2^{-k-2}], \quad k \geqq 2. \tag{2. 7}$$

Then

$$\int_0^{1-h} |f_0(t+h)-f_0(t)|^p\,dt = \left(\int_0^{3h} + \int_{3h}^{1-h}\right)|f_0(t+h)-f_0(t)|^p\,dt = I_1+I_2.$$

By (1. 15) and (1. 25) we have

$$I_1 \leqq K(p)\int_0^{4h} |f_0(x)|^p\,dx \leqq K\int_0^{2^{-k}} |f_0(x)|^p\,dx \leqq K\sum_{n=k}^{\infty}\int_{2^{-n-1}}^{2^{-n}} |f_0(x)|^p\,dx \leqq$$

$$\leqq K_1\sum_{n=k}^{\infty}\varrho_n^p 2^{-n-1} = K_1\sum_{n=k}^{\infty}(B_{2^n}^p - B_{2^{n+1}}^p) \leqq K_1 B_{2^k}^p \leqq K_2\,\omega^p(h).$$

To estimate J_2 we use the inequalities

$$|f_0(t+h)-f_0(t)| \leqq h2^{n+2}(\varrho_n+\varrho_{n-1}), \quad \text{if} \quad 2^{-n-1} \leqq t \leqq 2^{-n} (1 \leqq n \leqq k-1)$$

and (1. 15). In fact

$$I_2 \leqq \int_{2^{-k}}^{2^{-1}} |f_0(t+h)-f_0(t)|^p\,dt = \sum_{n=1}^{k-1}\int_{2^{-n-1}}^{2^{-n}} |f_0(t+h)-f_0(t)|^p\,dt \leqq$$

$$\leqq K(p)h^p\sum_{n=0}^{k} 2^{n(p-1)}\varrho_n^p \leqq K_1 h^p\sum_{n=0}^{k} 2^{np}(B_{2^n}^p - B_{2^{n+1}}^p) \leqq K_1 h^p\sum_{n=0}^{k} 2^{np}B_{2^n}^p \leqq$$

$$\leqq K_2 h^p\sum_{i=1}^{2^k} i^{p-1}B_i^p \leqq K_3 h^p 2^{kp}\omega^p(2^{-k}) \leqq K_4\,\omega^p(h).$$

Summing up we get

(2. 8) $$f_0(x)\in H_p^\omega.$$

Next we prove that

(2. 9) $$f_0(x)\notin L^{\nu-\frac{\nu}{p}+1}\Phi(L).$$

First we demonstrate (2. 9) if $\nu=p$. In this case, by (2. 6) and (1. 16), we have

(2. 10) $$\sum_{n=1}^{N}\varphi_n n^{-1}B_n^p\to\infty \quad\text{as}\quad N\to\infty.$$

By (1. 13) there exists a number K_1 such that $\varphi_{2n}\leqq K_1\varphi_n$ for all n, and since $B_n\downarrow 0$ for any N there exists an integer N_1 such that $B_{N1}<\frac{1}{2}B_N$. Using these facts and (2. 10), an easy computation gives that

(2. 11) $$\sum_{n=1}^{\mu}\Phi(2^n)\varrho_n^p 2^{-n}\to\infty \quad\text{as}\quad \mu\to\infty.$$

Indeed, if $2^\mu>N_1$, we have

$$\sum_{k=1}^{N}\varphi_k k^{-1}B_k^p\leqq 2\sum_{k=1}^{N}\varphi_k k^{-1}(B_k^p-B_{N_1}^p)\leqq 2\sum_{k=1}^{2^\mu}\varphi_k k^{-1}(B_k^p-B_{2^\mu}^p)\leqq$$
$$\leqq 2\left(\sum_{n=1}^{\mu}\sum_{k=2^{n-1}+1}^{2^n}\varphi_k k^{-1}B_k-\Phi(2^\mu)B_{2^\mu}^p\right)+K_2\leqq$$
$$\leqq 2\left(\sum_{n=1}^{\mu}B_{2^{n-1}}^p\sum_{k=2^{n-1}+1}^{2^n}\varphi_k k^{-1}-\Phi(2^\mu)B_{2^\mu}^p\right)\leqq$$
$$\leqq K_3\left(\sum_{n=1}^{\mu}B_{2^{n-1}}^p\sum_{k=2^{n-2}+1}^{2^{n-1}}\varphi_k k^{-1}-\Phi(2^\mu)B_{2^\mu}^p\right)+K_2\leqq$$
$$\leqq K_3\left(\sum_{i=1}^{\mu-1}B_{2^i}^p(\Phi(2^i)-\Phi(2^{i-1}))-\Phi(2^\mu)B_{2^\mu}^p\right)+K_4\leqq$$
$$\leqq K_3\sum_{n=1}^{\mu-1}\Phi(2^n)(B_{2^n}^p-B_{2^{n+1}}^p)+K_4\leqq K_3\sum_{n=1}^{\mu}\Phi(2^n)\varrho_n^p 2^{-n-1}+K_4,$$

which proves (2. 11) by (2. 10).

It is clear that

$$\int_0^1|f_0(x)|^p\Phi\left(\frac{1}{x}\right)dx=\sum_{n=0}^{\infty}\int_{2^{-n-1}}^{2^{-n}}|f_0(x)|^p\Phi\left(\frac{1}{x}\right)dx\geqq$$
$$\geqq\sum_{n=0}^{\infty}\Phi(2^n)\int_{2^{-n-1}}^{2^{-n}}|f_0(x)|^p\,dx\geqq K_5\sum_{n=0}^{\infty}\Phi(2^n)\varrho_n^p 2^{-n},$$

and thus, by (2. 11), we have

(2. 12)
$$\int_0^1 |f_0(x)|^p \Phi\left(\frac{1}{x}\right) dx = \infty.$$

Since $\varphi_{k^2} \leqq K_1 \varphi_k$, we have
$$\Phi(u^2) \leqq K\Phi(u),$$
thus by (2. 12), applying Lemma 3, we obtain
$$\int_0^1 |f_0(x)|^p \Phi(|f_0(x)|)\, dx = \infty,$$
that is,

(2. 13)
$$f_0(x) \notin L^p\Phi(L).$$

Hereby (2. 9) is proved for $p = \nu$.

Next let us suppose that $\nu > p$. By $\varphi_{k^2} \leqq K_1 \varphi_k$ we have
$$\Phi(x) = \sum_{k=1}^{x} k^{\frac{\nu}{p}-2} \varphi_k \geqq \sum_{x/2}^{x} k^{\frac{\nu}{p}-2} \varphi_k \geqq K_1 \varphi_{[x]} x^{\frac{\nu}{p}-1}.$$
Using this we get

(2. 14)
$$\int_0^1 |f_0(x)|^{\nu-\frac{\nu}{p}+1} \Phi(|f_0(x)|)\, dx \geqq K_2 \int_0^1 |f_0(x)|^\nu \varphi(|f_0(x)|)\, dx,$$
where
$$\varphi(x) = \begin{cases} 0, & \text{if } x \in [0, 1), \\ \varphi_n, & \text{if } x = n, \\ \text{linear between } n \text{ and } n+1 \end{cases}$$
$(n = 1, 2, \ldots)$. Furthermore, an application of Lemma 10 with $\alpha = 2 - \dfrac{\nu}{p}$, especially (1. 29), gives that
$$\sum_{n=1}^{\infty} \varphi_{2^n} 2^{n\left(\frac{\nu}{p}-1\right)} (B_{2^n} - B_{2^{n+1}})^\nu = \infty$$
and this implies

(2. 15)
$$J \equiv \int_0^1 |f_0(x)|^\nu \varphi\left(\frac{1}{x}\right) dx = \infty.$$

Indeed, we have
$$J = \sum_{n=1}^{\infty} \int_{2^{-n}}^{2^{-n+1}} |f_0(x)|^\nu \varphi\left(\frac{1}{x}\right) dx \geqq K_1 \sum_{n=1}^{\infty} \varphi_{2^n} \varrho_n^\nu 2^{-n} \geqq$$
$$\geqq K_2 \sum_{n=1}^{\infty} \varphi_{2^n} 2^{n\left(\frac{\nu}{p}-1\right)} (B_{2^n}^p - B_{2^{n+1}}^p)^{\frac{\nu}{p}} \geqq K_2 \sum_{n=1}^{\infty} \varphi_{2^n} 2^{n\left(\frac{\nu}{p}-1\right)} (B_{2^n} - B_{2^{n+1}})^\nu.$$
Applying Lemma 3, by (2. 15) and (2. 14), we obtain (2. 9) also for $\nu > p$.

The statements (2. 8) and (2. 9), however, contradict (2. 6) and hereby the necessity of (9) is proved.

The proof of Theorem 3 is thus completed.

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(Received April 10, 1969)

On the convergence of Fourier series in every arrangement of the terms

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1. In our earlier paper [2] the following theorem was proved:

Theorem A. *If* $\{\varrho(n)\}$ *is any sequence of positive numbers for which*

$$\varrho(n) = o(\log\log n),\ ^{1})$$

then there exists a square integrable function on $(0, 2\pi)$ *whose Fourier series*

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

is such that

$$\sum_{n=1}^{\infty} (a_n^2 + b_n^2)\varrho(n) < \infty,$$

and which can be rearranged into an everywhere divergent series

$$\sum_{k=1}^{\infty} (a_{n(k)} \cos n(k)x + b_{n(k)} \sin n(k)x).$$

In this paper we are going to sharpen this result slightly by making use of an observation of TANDORI [1]. Our theorem reads as follows:

Theorem 1. *Let* $\eta\,(<1)$ *be a positive number. There exists a sequence of numbers* $a_1, b_1, \ldots, a_n, b_n, \ldots$ *such that*

$$\sum_{n} (a_n^2 + b_n^2) \log\log n\, (\log\log\log n)^{1-\eta} < \infty, \tag{2}$$

and the Fourier series (1) *can be rearranged into an almost everywhere divergent series.*

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[1]) In this paper log means logarithm with base 4 but, of course, this is not essential in our considerations.

As to the partial sums of rearranged Fourier series we have the following estimate:

Theorem 2. *If $\eta(<1)$ is a positive number, then there exists a square integrable function whose Fourier series*

$$\sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx)$$

can be rearranged in such a way that the partial sums $S_l(x)$ of the rearranged series

$$\sum_{k=1}^{\infty}(A_{n(k)} \cos n(k)x + B_{n(k)} \sin n(k)x)$$

satisfy the relation

$$\overline{\lim_{l\to\infty}} \frac{|S_l(x)|}{\sqrt{\log\log l(\log\log\log l)^{1-\eta}}} > 0$$

almost everywhere.

We remark that in the assertions of Theorem 1 and Theorem 2 the words "almost everywhere" can be replaced by "everywhere". This refinement needs the same technique as can be found in [2]. (See there Lemma 4.)

It seems to be very probable that the analogous results hold in the case of Walsh's orthogonal system, but we do not treat it here.

In the sequel we shall use the following notation as an abbreviation: $\lambda(n) = \log\log n(\log\log\log n)^{1-\eta}$ if $n \geqq 17$, and $\lambda(n)=1$ if $1 \leqq n \leqq 16$, where $\eta(<1)$ is a given positive number.

2. The proof of Theorem 1 is based on the same idea as was worked out in our cited paper [2] to prove Theorem A, except for that paper's Lemma 3, which will be improved in a simple way. Let us recall here this lemma:

Lemma 1.[2]) *Let α be a natural number, and let $\varepsilon(<\pi/2)$ be a positive number. Then there exist mutually disjoint*[3]) *trigonometric polynomials $R_k^{(i)}(x)$ and simple sets $E_k^{(i)}$ $(k=1, 2, \ldots, 3^i;\ i=1, 2, \ldots)$ with the following properties:*

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[2]) Here we give the original lemma with a little modification with respect to the frequencies occurring in $R_k^{(i)}(x)$ $(k=1, 2, \ldots, 3^i)$. Originally, we stated that they were contained between f_{i-1} and f_i $(i=1, 2, \ldots)$. Our new assertion concerning the occurring frequencies is essentially included in the proof of the lemma in question.

[3]) For a function $a_n \cos nx + b_n \sin nx$ $(\not\equiv 0)$ we call n its frequency. Two trigonometric polynomials are said to be disjoint if they have no terms of the same frequency.

(i) *the frequencies occurring in* $R_k^{(i)}(x)$ $(k=1, 2, \ldots, 3^i)$ *are contained between* $f_{\alpha+i-1}$ *and* $f_{\alpha+i}$ *where* $f_j=(C_1/\varepsilon)^j 4^{4^j}$ $(j=1, 2, \ldots)$; [4])

(ii) $$\int_{-\pi}^{\pi}\left(\sum_{k=1}^{3^i} R_k^{(i)}(x)\right)^2 dx \leqq C_2 \quad \textit{for} \quad i=1, 2, \ldots;$$

(iii) *the sets* $E_k^{(i)}$ $(k=1, 2, \ldots, 3^i)$ *corresponding to the same value of i are disjoint., the set*

$$F_i=\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]-\bigcup_{k=1}^{3^i} E_k^{(i)}$$

consists of at most $f_{\alpha+i}$ *disjoint intervals, the lengths of which are at least* $1/f_{\alpha+i}$, *and*

(4) $$\operatorname{mes}(F_i) \leqq \varepsilon\left(1-\frac{1}{2^i}\right); \text{[5]})$$

(iv) *for any natural number i, the trigonometric polynomials* $R_k^{(j)}(x)$ *with* $k=1, 2, \ldots, 3^j$; $j=1, 2, \ldots, i$ *can be arranged into a sequence*

$$U_1^{(i)}(x), U_2^{(i)}(x), \ldots, U_{J_i}^{(i)}(x), \quad \textit{where} \quad J_i=3+3^2+\cdots+3^i;$$

such that

(5) $$\sum_{l=1}^{\mu_k^{(i)}} U_l^{(i)}(x) \geqq \frac{i}{8} \quad \textit{for every} \quad x \in E_k^{(i)}$$

with $\mu_k^{(i)}(\leqq J_i)$ *not depending on the particular point x in* $E_k^{(i)}$ $(k=1, 2, \ldots, 3^i)$.

The only new observation — but an essential one — is the fact that the trigonometric polynomials $R_k^{(j)}(x)$ corresponding to the different values of j have to be considered with different "weights". More precisely, we give the following stronger form of Lemma 1:

Lemma 2. *Let* $i(\geqq 4)$ *and* α *be natural numbers,* $1<\alpha \leqq \sqrt{i}$, *and let* ε *be a real number,* $4^{-4^\alpha}<\varepsilon<\pi/2$. *Furthermore, let* $R_k^{(j)}(x)$ $(k=1, 2, \ldots, 3^j;\ j=1, 2, \ldots, i)$ *and* $E_k^{(i)}$ $(k=1, 2, \ldots, 3^i)$ *be, respectively, the disjoint trigonometric polynomials and the disjoint simple sets in the sense of Lemma 1; and let*

(6) $$D_j=\begin{cases}\dfrac{1}{\alpha \log \alpha} & \textit{if} \quad 1 \leqq j \leqq \alpha, \\[2ex] \dfrac{1}{j \log j} & \textit{if} \quad \alpha<j \leqq i.\end{cases}$$

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[4]) In the following $C_1, C_2, \ldots$ will denote positive constants.

[5]) mes (F) denotes the Lebesgue measure of the set F.

Then the coefficients $a_n = a_n^{(j)}$, $b_n = b_n^{(j)}$ defined by

$$\sum_{k=1}^{3^j} D_j R_k^{(j)}(x) = \sum_{n=f_{\alpha+j-1}+1}^{f_{\alpha+j}} (a_n \cos nx + b_n \sin nx) \qquad (j = 1, 2, \dots, i)$$

fulfil the inequality

$$\sum_{n=f_\alpha+1}^{f_{\alpha+i}} (a_n^2 + b_n^2)\lambda(n) \leqq \frac{C_3}{(\log \alpha)^\eta}. \tag{7}$$

Denoting by

$$V_1^{(i)}(x), V_2^{(i)}(x), \dots, V_{J_i}^{(i)}(x)$$

the arrangement of the trigonometric polynomials $D_j R_k^{(j)}(x)$ $(k = 1, 2, \dots, 3^j;$ $j = 1, 2, \dots, i)$ made in the same way as in Lemma 1, we have

$$\sum_{l=1}^{\mu_k^{(i)}} V_l^{(i)}(x) \geqq C_4 \log \frac{\log i}{\log \alpha} \tag{8}$$

for every $x \in E_k^{(i)}$ $(k = 1, 2, \dots, 3^i)$.

Lemma 2 can be proved by nearly the same argument as Lemma 1. For this reason we give only a sketch of the proof.

First of all, on the basis of the condition on ε, we find

$$\lambda(f_{\alpha+j}) \leqq \begin{cases} \lambda(f_{2\alpha}) \leqq C_5 \alpha (\log \alpha)^{1-\eta} & \text{if} \quad 1 \leqq j \leqq \alpha, \\ \lambda(f_{2j}) \leqq C_5\, j (\log j)^{1-\eta} & \text{if} \quad \alpha < j \leqq i. \end{cases}$$

From (ii) and (6), by a simple calculation, we obtain

$$\sum_{n=f_\alpha+1}^{f_{\alpha+i}} (a_n^2 + b_n^2)\lambda(n) \leqq \sum_{j=1}^{i} \lambda(f_{\alpha+j}) D_j^2 \int_{-\pi}^{\pi} \left(\sum_{k=1}^{3^j} R_k^{(j)}(x) \right)^2 dx \leqq$$

$$= C_2 \left\{ \sum_{j=1}^{\alpha} + \sum_{j=\alpha+1}^{i} \right\} \lambda(f_{\alpha+j}) D_j^2 \leqq C_2 C_5 \left\{ \frac{1}{(\log \alpha)^{1+\eta}} + \sum_{j=\alpha+1}^{i} \frac{1}{j (\log j)^{1+\eta}} \right\} \leqq \frac{C_3}{(\log \alpha)^\eta},$$ [6]

which is the assertion (7).

---

[6]) The last estimate follows from

$$\sum_{j=\alpha+1}^{i} \frac{1}{j (\log j)^{1+\eta}} \leqq C_6 \int_{\alpha}^{i} \frac{dx}{x (\log x)^{1+\eta}} \leqq \frac{C_6}{(\log \alpha)^\eta} \qquad (\eta > 0).$$

We shall also make use of another inequality that says

$$\sum_{j=\alpha+1}^{i} \frac{1}{j \log j} \geqq C_7 \int_{\alpha}^{i} \frac{dx}{x \log x} \geqq C_7 (\log \log i - \log \log \alpha).$$

As to (8), by applying (5) and (6), we get in the same way as in the proof of Lemma 1 that

$$\sum_{l=1}^{\mu_k^{(i)}} V_l(x) \geqq \frac{1}{8}\sum_{j=1}^{i} D_j = \frac{1}{8}\left\{\frac{1}{\log\alpha}+\sum_{j=\alpha+1}^{i}\frac{1}{j\log j}\right\} \geqq C_4 \log\frac{\log i}{\log\alpha} \quad (x\in E_k^{(i)}),$$

which concludes the proof of Lemma 2.

3. Proof of Theorem 1. Set $\varepsilon_m = 1/m$ $(m=1, 2, \ldots)$, and define the increasing sequences of natural numbers $\{\alpha_m\}$ and $\{i_m\}$ by recurrance with respect to m as follows: for $m=1$ put $\alpha_1=2$ and $i_1=4$, furthermore, the numbers α_{m+1} and i_{m+1} are chosen so large that the following conditions are satisfied for $m=1, 2, \ldots$:

(9)
$$f_{\alpha_m+i_m} = (C_1 m)^{\alpha_m+i_m} 4^{4^{\alpha_m+i_m}} < (C_1(m+1))^{\alpha_{m+1}} 4^{4^{\alpha_{m+1}}} = f_{\alpha_{m+1}},$$
$$\frac{1}{(\log\alpha_{m+1})^{\eta}} \leqq \frac{1}{(m+1)^2} \quad \text{and} \quad i_{m+1} = \alpha_{m+1}^2.$$

It is clear that these choices are possible.

Then apply subsequently Lemma 2 with $\varepsilon=\varepsilon_m$, $\alpha=\alpha_m$ and $i=i_m$ $(m=1, 2, \ldots)$. We obtain the trigonometric polynomials $V_l^{(i_m)}(x)$ $(l=1, 2, \ldots, J_{i_m};\ m=1, 2, \ldots)$ satisfying (7) and (8), and denote by $T_m(x)$ the sum of all $V_l^{(i_m)}\left(x-\frac{m\pi}{2}\right)$ corresponding to the same value of i_m. It is obvious that

$$T_m(x) = \sum_{n=f_{\alpha_m}+1}^{f_{\alpha_m+i_m}} (a_n\cos nx + b_n \sin nx) \qquad (m = 1, 2, \ldots).$$

Now consider the series

(10)
$$\sum_{m=1}^{\infty} T_m(x).$$

By virtue of (9) the trigonometric polynomials $T_m(x)$ and $T_{m'}(x)$ do not overlap for $m\neq m'$. Therefore, writing every $T_m(x)$ in (10) in extenso, we represent (10) in the form of trigonometric series

(11)
$$\sum_{n=1}^{\infty} (a_n\cos nx + b_n\sin nx),$$

where the coefficients a_n and b_n not occurring in any $T_m(x)$ equal 0.

Taking into consideration (7) and (9) we obtain that the inequality (2) holds.

Now write down the mutually disjoint trigonometric polynomials $V_l^{(i_m)}\left(x-\frac{m\pi}{2}\right)$ ($l=1, 2, \ldots, J_{i_m}$; $m=1, 2, \ldots$) in this order:

$$V_1^{(i_1)}\left(x-\frac{\pi}{2}\right),\ V_2^{(i_1)}\left(x-\frac{\pi}{2}\right), \ldots, V_{J_{i_1}}^{(i_1)}\left(x-\frac{\pi}{2}\right); \ldots$$

$$\ldots; V_1^{(i_m)}\left(x-\frac{m\pi}{2}\right),\ V_2^{(i_m)}\left(x-\frac{m\pi}{2}\right), \ldots, V_{J_{i_m}}^{(i_m)}\left(x-\frac{m\pi}{2}\right); \ldots$$

and label the occurring frequencies, in this order, by the subscript $n(k)$ ($k=1, 2, \ldots$). It is clear that the series

$$\sum_{k=1}^{\infty}\left(a_{n(k)}\cos n(k)x+b_{n(k)}\sin n(k)x\right) \tag{12}$$

is a well determined arrangement of the non-vanishing terms of (11).

On account of (8) and (9), for every $m(\geqq 1)$ we find that

$$\max_{f_{\alpha_m}<l\leqq f_{\alpha_m}+i_m}\left\{\sum_{k=f_{\alpha_m}+1}^{l}\left(a_{n(k)}\cos n(k)x+b_{n(k)}\sin n(k)\right\}\geqq C_4\log 2\right.$$

holds in $\left(\frac{m\pi}{2}-\frac{\pi}{4}, \frac{m\pi}{2}+\frac{\pi}{4}\right)$ except on a set F_m of measure less than $1/m$. We can easily see that almost every point of $(-\pi; \pi)$ is not contained in infinitely many F_m.[7]) Thus the series (12) is divergent almost everywhere, and the proof of Theorem 1 is complete.

4. Proof of Theorem 2. Starting with Theorem 1 it is possible to deduce Theorem 2. For this purpose we define the sequences $\{\varepsilon_m\}$, $\{\alpha_m\}$ and $\{i_m\}$ of numbers exactly as in the proof of Theorem 1, i.e. so that the conditions (9) are satisfied. We consider the trigonometric polynomials $D_j R_k^{(j)}(x)$ ($k=1, 2, \ldots, 3^j$; $j=1, 2, \ldots, i_m$) defined by Lemma 2 for $m=1, 2, \ldots$. Denote by

$$W_1^{(i_m)}(x),\ W_2^{(i_m)}(x), \ldots, W_{J_{i_m}}^{(i_m)}(x)$$

the arrangement of the trigonometric polynomials $\sqrt{\lambda(f_{\alpha_m+j})}\, D_j R_k^{(j)}(x)$ ($k=1, 2, \ldots, 3^j$; $j=1, 2, \ldots, i_m$) made in the same way as previously described in Lemma 1 concerning $R_k^{(j)}(x)$. It is clear that

$$U_m(x)=\sum_{l=1}^{J_{i_m}} W_l^{(i_m)}\left(x-\frac{m\pi}{2}\right)=\sum_{n=f_{\alpha_m}+1}^{f_{\alpha_m}+i_m}\left(A_n\cos nx+B_n\sin nx\right),$$

---

[7]) We consider the sets F on the whole line of real numbers modulo 2π.

where $A_n=\sqrt{\lambda(f_{\alpha_m+j})}\, a_n$ and $B_n=\sqrt{\lambda(f_{\alpha_m+j})}\, b_n$ if $f_{\alpha_m+j-1}<n\leqq f_{\alpha_m+j}$ $(j=1,2,\dots,i_m;$ $m=1,2,\dots)$, and the coefficients a_n, b_n are defined in Theorem 1.

Now we form the series

$$\sum_{m=1}^{\infty} U_m(x). \tag{13}$$

Since two polynomials $U_m(x)$ with different subscript are disjoint owing to (9), we obtain the trigonometric series

$$\sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx) \tag{14}$$

by writing out the terms of each trigonometric polynomial in (13). The series (14) is the Fourier series of a square integrable function. Indeed, by (2), we have

$$\sum_{n=1}^{\infty} (A_n^2+B_n^2) = \sum_{m=1}^{\infty} \int_{-\pi}^{\pi} U_m^2(x)\,dx =$$

$$= \sum_{m=1}^{\infty} \sum_{j=1}^{i_m} \lambda(f_{\alpha_m+j}) D_j^2 \int_{-\pi}^{\pi} \left(\sum_{k=1}^{3^j} R_k^{(j)}\left(x-\frac{m\pi}{2}\right)\right)^2 dx \leqq$$

$$\leqq C_8 \sum_{m=1}^{\infty} \sum_{n=f_{\alpha_m}+1}^{f_{\alpha_m+i_m}} (a_n^2+b_n^2)\lambda(n) < \infty.$$

Finally, let us rearrange the terms of the series (14) as we did in (11). In this way we obtain

$$\sum_{k=1}^{\infty} (A_{n(k)} \cos n(k)x + B_{n(k)} \sin n(k)x).$$

Denote by $S_l(x)$ the lth partial sum of this series. We are going to show that

$$\max_{f_{\alpha_m}<l\leqq f_{\alpha_m+i_m}} \frac{S_l(x)-S_{f_{\alpha_m}}(x)}{\sqrt{\lambda(f_{\alpha_m+i_m})}}$$

does not tend to zero almost everywhere (although $\alpha_m\to\infty$). To achieve this aim, let us consider the trigonometric polynomials $V_l^{(i_m)}(x)$ $(l=1,2,\dots,J_{i_m})$ and the simple sets $E_k^{(i_m)}$ $(k=1,2,\dots,3^{i_m})$ in the sense of Lemma 2. For every $x\in E_k^{(i_m)}$ $(k=1,2,\dots,3^{i_m})$ we have

$$\sum_{l=1}^{\mu_k^{(i_m)}} V_l^{(i_m)}(x) = \sum_{j=1}^{i_m} \Big(\sum_k{}^{*} D_j R_k^{(j)}(x)\Big) = \sum_{j=1}^{i_m} \frac{1}{\sqrt{\lambda(f_{\alpha_m+j})}} \Big(\sum_k{}^{*} D_j \sqrt{\lambda(f_{\alpha_m+j})}\, R_k^{(j)}(x)\Big),$$

where the sum $\sum_k^{*}$ is extended over every integer value of k $(1\leqq k\leqq 3^j)$ for which the trigonometric polynomial $D_j R_k^{(j)}(x)$ is equal to some $V_l^{(i_m)}(x)$ occurring on the left-hand side sum of this inequality ($j=1, 2, \ldots, i_m$). Performing an Abel transform we find that

$$\sum_{l=1}^{\mu_k^{(i_m)}} V_l^{(i_m)}(x) =$$

$$(15)\qquad = \sum_{l=1}^{i_m-1}\left(\frac{1}{\sqrt{\lambda(f_{\alpha_m+l})}}-\frac{1}{\sqrt{\lambda(f_{\alpha_m+l+1})}}\right)\sum_{j=1}^{l}\Big(\sum_k{}^* D_j\sqrt{\lambda(f_{\alpha_m+j})}\,R_k^{(j)}(x)\Big)+$$

$$+\frac{1}{\sqrt{\lambda(f_{\alpha_m+i_m})}}\sum_{j=1}^{i_m}\Big(\sum_k{}^* D_j\sqrt{\lambda(f_{\alpha_m+j})}\,R_k^{(j)}(x)\Big).$$

We notice that the last term on the right-hand side of (15) equals

$$\frac{1}{\sqrt{\lambda(f_{\alpha_m+i_m})}}\sum_{l=1}^{\mu_k^{(i_m)}} W_l^{(i_m)}(x).$$

We assert that the series on the right-hand side of (15) converges to zero almost everywhere (as $m\to\infty$). By a simple calculation we obtain

$$\sum_{l=1}^{i_m-1}\left(\frac{1}{\sqrt{\lambda(f_{\alpha_m+l})}}-\frac{1}{\sqrt{\lambda(f_{\alpha_m+l+1})}}\right)\int_{-\pi}^{\pi}\left|\sum_{j=1}^{l}\Big(\sum_k{}^* D_j\sqrt{\lambda(f_{\alpha_m+j})}\,R_k^{(j)}(x)\Big)\right|dx \leqq$$

$$\leqq \sqrt{2\pi}\sum_{l=1}^{i_m-1}\left(\frac{1}{\sqrt{\lambda(f_{\alpha_m+l})}}-\frac{1}{\sqrt{\lambda(f_{\alpha_m+l+1})}}\right)\left\{\sum_{j=1}^{l} D_j^2\lambda(f_{\alpha_m+j})\int_{-\pi}^{\pi}\Big(\sum_k{}^* R_k^{(j)}(x)\Big)^2 dx\right\}^{1/2}\leqq$$

$$\leqq C_8\sum_{l=\alpha_m+1}^{\alpha_m+i_m-1}\left(\frac{1}{\sqrt{\lambda(f_l)}}-\frac{1}{\sqrt{\lambda(f_{l+1})}}\right)\left\{\sum_{n=f_{\alpha_m}+1}^{f_{\alpha_m+i_m}}(a_n^2+b_n^2)\lambda(n)\right\}^{1/2},$$

where we took into consideration that the trigonometric polynomials $R_k^{(j)}(x)$ ($k=1, 2, \ldots, 3^j$; $j=1, 2, \ldots, i_m$) are mutually disjoint. On account of (2), by Beppo Levi's theorem this implies our above assertion. Denote by H the set of measure zero, on which the series on the right-hand side of (15) does not tend to zero.

By virtue of (8) and (9) we get that

$$\max_{f_{\alpha_m}<l\leqq f_{\alpha_m+i_m}}\frac{S_l(x)-S_{f_{\alpha_m}}(x)}{\sqrt{\lambda(f_{\alpha_m+i_m})}}\geqq\frac{1}{2\sqrt{\lambda(f_{\alpha_m+i_m})}}\sum_{l=1}^{\mu_k^{(i_m)}} W_l^{(i_m)}\left(x-\frac{m\pi}{2}\right)\geqq\frac{C_4\log 2}{2}$$

holds in $\left(\frac{m\pi}{2}-\frac{\pi}{4}, \frac{m\pi}{2}+\frac{\pi}{4}\right)$ provided $x\notin F_m\cup H$ and m is large enough, where

$$F_m=\bigcup_{k=1}^{3^{i_m}} E_k^{(i_m)},$$

and thus $\operatorname{mes}(F_m) < \varepsilon_m = 1/m$. It is obvious that almost every point of $(-\pi, \pi)$ is not contained in infinitely many $F_m \cup H$. Taking into account the elementary fact that

$$\max_{f_{\alpha_m} < l \leq f_{\alpha_m + i_m}} \frac{S_l(x) - S_{f_{\alpha_m}}(x)}{\sqrt{\lambda(f_{\alpha_m + i_m})}} \leq 2 \max_{f_{\alpha_m} \leq l \leq f_{\alpha_m + i_m}} \frac{|S_l(x)|}{\sqrt{\lambda(l)}},$$

we can see that

$$\overline{\lim_{l \to \infty}} \frac{|S_l(x)|}{\sqrt{\lambda(l)}} \geq \frac{C_4 \log 2}{4}$$

holds almost everywhere. This proves our statement (3), and finishes the proof of Theorem 2.

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(Received January 26, 1969)

The central limit theorem for multiplicative systems

By FERENC MÓRICZ in Szeged

1. ALEXITS introduced the following definitions (see [1], p. 186): The sequence of real measurable functions $\varphi_1(t), \varphi_2(t), \ldots$ defined in the interval [0, 1] is called a multiplicative system if all its finite products are Lebesgue integrable with

$$(1) \qquad \int_0^1 \varphi_{n_1}(t)\ldots\varphi_{n_k}(t)\,dt = 0 \qquad (n_1 < \cdots < n_k;\ k = 1, 2, \ldots).$$

The sequence $\{\varphi_n(t)\}$ is called an equinormed strongly multiplicative system (in abbreviation: ESMS) if the system $\{\varphi_{n_1}(t)\ldots\varphi_{n_k}(t)\}$ $(n_1 < \cdots < n_k;\ k = 1, 2, \ldots)$ is orthonormal, i.e.

$$(2) \qquad \int_0^1 \varphi_n(t)\,dt = 0, \quad \int_0^1 \varphi_n^2(t)\,dt = 1 \qquad (n = 1, 2, \ldots);$$

$$\int_0^1 \varphi_{n_1}^{r_1}(t)\ldots\varphi_{n_k}^{r_k}(t)\,dt = \int_0^1 \varphi_{n_1}^{r_1}(t)\,dt \ldots \int_0^1 \varphi_{n_k}^{r_k}(t)\,dt,$$

where $r_1, \ldots, r_k$ can be equal to 1 or 2.

2. These notions proved to be tractable and useful ones, because the behaviour of the series arising from the functions of an ESMS resembles, in many respects, that of series of independent functions. This is not suprising, as a sequence of independent functions with mean value 0 and dispersion 1 is an ESMS. Another example for ESMS, also having a lot of properties in common with the independent functions, is a strongly lacunary sequence of trigonometric functions, i.e. $\{\sqrt{2}\sin 2\pi m_k x\}$, where $m_{k+1}/m_k \geqq 3$ $(k = 1, 2, \ldots)$.

A number of authors have generalized the central limit theorem for lacunary trigonometric series. The most general result is due to SALEM and ZYGMUND [11], who state the following

Theorem A. *Let* $S_N(t)$ *denote the* Nth *partial sum of the lacunary trigonometric series* $\sum_{k=1}^{\infty} (a_k \cos m_k t + b_k \sin m_k t)$, $m_{k+1}/m_k \geqq q > 1$ $(k = 1, 2, \ldots)$, *and let*

$a_1, a_2, \ldots; b_1, b_2, \ldots$ be arbitrary sequences of real numbers for which

$$C_N = \left\{\frac{1}{2}\sum_{k=1}^{N}(a_k^2+b_k^2)\right\}^{1/2} \to \infty \quad \textit{and} \quad \{a_N^2+b_N^2\}^{1/2} = o(C_N).$$

Then for any set $E \subset [0, 2\pi]$ of positive measure the distribution functions

$$F_N(y; E) = \frac{\operatorname{mes}(\{t: \in E: S_N(t)/C_N < y\})}{\operatorname{mes}(E)} \text{ [1]}$$

tend pointwise to the Gaussian distribution with mean value 0 *and dispersion* 1.

The present author (see [6]) managed to generalize the above theorem almost word by word to ESMS. Before saying it in an explicit form, we introduc the notations

$$S_N(t) = \sum_{n=1}^{N} c_n \varphi_n(t), \quad C_N^2 = \sum_{n=1}^{N} c_n^2,$$

where $\{c_n\}$ is an arbitrary sequence of real numbers.

Theorem B. *Let $\{\varphi_n(t)\}$ be a uniformly bounded* ESMS. *If*

(3) $$\text{(i)}\quad C_N \to \infty \qquad \textit{and} \qquad \text{(ii)}\quad c_N = o(C_N),$$

then the distribution functions

$$F_N(y) = \operatorname{mes}\left(\left\{t \in [0, 1]: \frac{S_N(t)}{C_N} < y\right\}\right)$$

tend pointwise to the Gaussian distribution function

$$G(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{t^2}{2}}\,dt.$$

Another generalization of the central limit theorem is due to Révész [10], whose theorem reads as follows:

Theorem C. *Let $\{\varphi_n(t)\}$ be a uniformly bounded multiplicative system such that*

(4) $$\int_0^1 \varphi_m^2(t)\varphi_n^2(t)\,dt = \int_0^1 \varphi_n^2(t)\,dt = 1 \qquad (m \neq n).$$

If

$$C_N \to \infty \quad \textit{and} \quad \sum_{N=2}^{\infty} \frac{c_N^4}{C_N^4} \log^2 N < \infty,$$

then for every y, $F_N(y)$ tends to the Gaussian distribution function.

3. In this paper we propose to give a complete solution of the problem of the central limit theorem concerning uniformly bounded ESMS or only multiplicative

[1]) mes (E) denotes the Lebesgue measure of a set E.

systems satisfying (4). We shall prove an even more general result, namely, that in the case of ESMS the distribution function of $S_N(t)/C_N$ on every fixed set of positive measure tends to the Gaussian distribution function.

Theorem 1. *Let* $\{\varphi_n(t)\}$ *be a uniformly bounded* ESMS, *and let* E *be a subset of* $[0, 1]$ *with* $\text{mes}(E)>0$. *If* (3) *holds then*

$$F_N(y; E) = \frac{\text{mes}\{(t\in E: S_N(t)/C_N < y)\}}{\text{mes}(E)}$$

tends pointwise to the Gaussian distribution function.

We should like to point out that in case $E=[0, 1]$ the central limit theorem is valid under apparently weaker conditions than those of Theorem 1.

Theorem 2. *Let* $\{\varphi_n(t)\}$ *be a uniformly bounded multiplicative system satisfying* (4). *If* (3) *holds then* $F_N(y)$ *tends pointwise to the Gaussian distribution function.*

Evidently this theorem contains, as particular cases, both Theorem B and Theorem C.

The theorem which follows indicates that if $C_N\to\infty$ then the second condition (3) is indispensable for the validity of Theorem 1. We note that by a distribution function we mean any non-decreasing function $F(y)$, continuous to the left, with $\lim_{y\to-\infty} F(y)=0$ and $\lim_{y\to+\infty} F(y)=1$.

Theorem 3. *Let* $\{\varphi_n(t)\}$ *be a uniformly bounded* ESMS, *and let* $E\subset[0, 1]$ *be a set of positive measure. Suppose that* $C_N\to\infty$ *and that* $F_N(y; E)$ *tends to a distribution function* $F(y)$ *(at the points of continuity of the latter) such that either* $F(y)>0$ *or* $F(y)<1$ *for all finite* y. *Then* (3) (ii) *must hold.*

Theorem 1 and Theorem 3 generalize the results of SALEM and ZYGMUND [11] from lacunary trigonometric series to ESMS.

4. Suppose now that

$$\sum_{n=1}^{\infty} c_n\varphi_n(t) \tag{5}$$

is of the class L^2, i.e. $\sum_{n=1}^{\infty} c_n^2<\infty$. It is known that in this case the series (5) converges almost everywhere in [0, 1]. (See ALEXITS [2], or ALEXITS and TANDORI [3]). The remainder $\sum_{n=N}^{\infty} c_n\varphi_n(t)$ of (5) represents then a certain function $R_N(t)$. By $G_N(y; E)$ we shall mean the distribution function of $R_N(t)/D_N$ over the set $E\subset[0, 1]$ of positive measure, where $D_N^2=\sum_{n=N}^{\infty} c_n^2$. The proofs of the following two results are repetitions of those of Theorem 1 and Theorem 3.

Theorem 4. *Let* $\{\varphi_n(t)\}$ *be a uniformly bounded* ESMS, *and let* E *be a set with* $\operatorname{mes}(E)>0$. *Suppose that* (5) *is of the class* L^2, *and that* $c_N/D_N \to 0$. *Then* $G_N(y; E)$ *tends to the Gaussian distribution function.*

Theorem 5. *Let* $\{\varphi_n(t)\}$ *be a uniformly bounded* ESMS, *and let* E *be a set of positive measure. If* (5) *is of the class* L^2 *and if* $G_N(y; E)$ *tends to a distribution function* $F(y)$ *which is not constant outside a finite interval then* $c_N/D_N \to 0$.

Instead of the partial sums of (5) we may consider linear means of the partial sums. Analogous theorems are valid for ESMS just as much as for lacunary trigonometric series, and we refer again to the cited paper of Salem and Zygmund [11]. To make the analogy complete, let us mention the following fact, which plays an important role in the considerations of the above authors: If the series (5) is summable by a linear summation process, regular in the sense of Toeplitz, in a set of positive measure then (5) must converge almost everywhere. (See in [8] of the present author.)

5. Finally we remark that the notion of multiplicativity can be much more generally defined as follows (as for this terminology, see, in detail, Révész [9]): Let $\{\Omega, \mathcal{S}, \mathbf{P}\}$ be a probability space, and let $\xi_1, \xi_2, \ldots$ be a sequence of random variables on Ω with $E(\xi_n)=0$, $E(\xi_n^2)=1$ $(n=1, 2, \ldots)$, where $E(\xi)$ means the expectation of ξ.[2]) The sequence $\xi_1, \xi_2, \ldots$ is then called, for example, an ESMS if

$$E(\xi_{n_1}^{r_1} \ldots \xi_{n_k}^{r_k}) = E(\xi_{n_1}^{r_1}) \ldots E(\xi_{n_k}^{r_k}) \qquad (n_1 < \cdots < n_k;\ k = 2, 3, \ldots),$$

where $r_1, \ldots, r_k$ can be equal to 1 or 2.

In this paper we consider only the particular case, when $\Omega=[0, 1]$, $\mathcal{S}$ is the class of Lebesgue measurable subsets of [0, 1], and $\mathbf{P}$ is the common Lebesgue measure on it. Our results hold in the general case as well, without changing the proofs, but terms must be replaced by suitable ones in the probability theory.

6. To prove Theorem 1 and Theorem 2 our point of departure is the following lemma which is of interest in itself.

Lemma 1. *Let* $\{\varphi_n(t)\}$ *be a uniformly bounded system of real functions satisfying* (4). *If* (3) *holds then*

$$\frac{1}{C_N^2} \sum_{n=1}^{N} c_n^2 \varphi_n^2(t)$$

converges in measure to 1.

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[2]) I.e. $E(\xi) = \int_{\Omega} \xi(\omega)\, d\mathbf{P}$.

We note that this is an earlier result of the present author (see [7]), but there was given an unnecessarily complicated proof. That is why we give another proof, much simpler than the former one.

Proof of Lemma 1. We begin with

$$\frac{1}{C_N^2}\sum_{n=1}^{N} c_n^2\varphi_n^2(t) = 1+\frac{1}{C_N^2}\sum_{n=1}^{N} c_n^2(\varphi_n^2(t)-1) = 1+\xi_N(t).$$

We observe that owing to (4) the system $\{\varphi_n^2(t)-1\}$ is orthogonal[3]), on the other hand, it is not difficult to see that (3) is equivalent to

$$(\max_{1\leqq n\leqq N}|c_n|)/C_N \to 0. \tag{6}$$

Hence the measure of the set of points where $|\xi_N(t)|\geqq\varepsilon>0$ is less than

$$\frac{1}{\varepsilon^2}\int_0^1 \xi_N^2(t)\,dt = \frac{1}{\varepsilon^2}\frac{1}{C_N^4}\sum_{n=1}^{N} c_n^4\left\{\int_0^1 \varphi_n^4(t)\,dt-1\right\}$$

and so tends to 0 on account of (6) and the uniform boundedness of $\{\varphi_n(t)\}$. Thus the proof of Lemma 1 is complete.

7. Proofs of Theorem 1 and Theorem 2. The main idea used throughout the proof is due to SALEM and ZYGMUND (see [11]).

We make use of the classical method of characteristic functions. In view of Paul LÉVY's theorem it is enough to show that over any finite range of x the characteristic function of $F_N(y; E)$ tends to that of the Gaussian distribution with mean value 0 and dispersion 1, i.e. to $e^{-\frac{1}{2}x^2}$. The characteristic function of $F_N(y; E)$ is

$$\Phi_N(x) = \int_{-\infty}^{+\infty} e^{-ixy}\,dF_N(y; E).$$

From the definition of the Lebesgue integral we find that

$$\Phi_N(x) = \frac{1}{\operatorname{mes}(E)}\int_E \exp\left\{-\frac{ixS_N(t)}{C_N}\right\}dt = \frac{1}{\operatorname{mes}(E)}\int_E \prod_{n=1}^{N}\exp\left\{-\frac{ixc_n\varphi_n(t)}{C_N}\right\}dt. \tag{7}$$

Hence, using the fact that

$$\exp z = (1+z)\exp\{\tfrac{1}{2}z^2+O(|z|^3)\} = (1+z)\exp\{\tfrac{1}{2}z^2+o(|z|^2)\}$$

[3]) P. RÉVÉSZ called our attention to the fact that this can make proofs of theorems concerning ESMS simpler.

for $z \to 0$, we can write (7) in the form

$$\frac{1}{\operatorname{mes}(E)} \int_E e^{o(1)} \prod_{n=1}^{N} \left\{ \left(1 - \frac{ixc_n\varphi_n(t)}{C_N}\right) \exp\left(-\frac{x^2c_n^2\varphi_n^2(t)}{2C_N^2}\right) \right\} dt, \tag{8}$$

where the term $o(1)$ in $e^{o(1)}$ tends to 0 uniformly in t as $N \to \infty$, provided $x = O(1)$, which we assume from now on.

Observe now (since $1+u \leqq e^u$) that

$$\left| \prod_{n=1}^{N} \left(1 - \frac{ixc_n\varphi_n(t)}{C_N}\right) \right| \leqq \left\{ \prod_{n=1}^{N} \left(1 + \frac{x^2c_n^2K^2}{C_N^2}\right) \right\}^{1/2} \leqq e^{\frac{1}{2}x^2K^2},$$

where K denotes a common bound of the system $\{\varphi_n(t)\}$. Hence, by virtue of Lemma 1, it follows that, with an error tending uniformly to 0 as $N \to \infty$, the integral (8) is

$$\frac{1}{\operatorname{mes}(E)} e^{-\frac{1}{2}x^2} \int_E \prod_{n=1}^{N} \left(1 - \frac{ixc_n\varphi_n(t)}{C_N}\right) dt. \tag{9}$$

Denote the last integral by I_N. It is enough to show that I_N tends to $\operatorname{mes}(E)$.

By (1) this is immediate if $E = [0, 1]$ and thus the proof of Theorem 2 is complete.

Continuing the proof of Theorem 1, expand the integrand of (9) in the form

$$\prod_{n=1}^{N} \left(1 - \frac{ixc_n\varphi_n(t)}{C_N}\right) = 1 + \sum_{\nu \geqq 1} \alpha_\nu^{(N)} \psi_\nu(t),$$

where the numbers $\alpha_\nu^{(N)}$ depending also on x and the product system $\{\psi_\nu(t)\}$ are defined as follows: Let

$$\nu = 2^{n_1} + \cdots + 2^{n_k} \qquad (0 \leqq n_1 < \cdots < n_k;\quad k \geqq 1)$$

denote the dyadic representation of the index $\nu \geqq 1$. Set

$$\alpha_\nu^{(N)} = \begin{cases} \left(-\dfrac{ix}{C_N}\right)^k c_{n_1+1} \cdots c_{n_k+1} & \text{if } \ 1 \leqq \nu < 2^N, \\ 0 & \text{if } \qquad \nu \geqq 2^N; \end{cases}$$

and

$$\psi_\nu(t) = \varphi_{n_1+1}(t) \cdots \varphi_{n_k+1}(t).$$ [4]

By (2) it is obvious that the system $\{\psi_\nu(t)\}$ is orthonormal. In particular, we obtain

$$I_N = \operatorname{mes}(E) + \sum_{\nu \geqq 1} \gamma_\nu \alpha_\nu^{(N)},$$

[4]) $\{\psi_\nu(t)\}$ is called the W-system generated by $\{\varphi_n(t)\}$. (See in detail ALEXITS [1], p. 187.)

where the γ_ν are the Fourier coefficients of the characteristic function of E with respect to the system $\{\psi_\nu(t)\}$. In view of (6) each $\alpha_\nu^{(N)}$ tends to 0 as $N\to\infty$ ($\nu=1, 2, \ldots$). Hence, if ν_0 is fixed,

(10) $$\sum_{\nu\geqq 1}|\gamma_\nu\alpha_\nu^{(N)}| = \sum_{\nu\leqq\nu_0} + \sum_{\nu>\nu_0} \leqq o(1)+\Big(\sum_{\nu>\nu_0}\gamma_\nu^2\Big)^{1/2}\Big(\sum_{\nu>\nu_0}|\alpha_\nu^{(N)}|^2\Big)^{1/2}.$$

The first factor in the last product is arbitrarily small if ν_0 is large enough (since $\Sigma\gamma_\nu^2<\infty$), and we are going to show that the second factor is bounded. To estimate it we remove the restriction that $\nu>\nu_0$. This adds only non-negative terms to it, and the expression obtained in this way then collapses back into

$$\prod_{n=1}^{N}\left(1+\frac{x^2c_n^2}{C_N^2}\right).$$

Taking into account again that $1+u\leqq e^u$, we can see that this product is not greater than e^{x^2}. Collecting results we deduce that the second factor in the last product of (10) is bounded. Hence $I_N\to\text{mes}\,(E)$, and this concludes the proof of Theorem 1.

8. Proof of Theorem 3. We note that the proof does not require the full power of a uniformly bounded ESMS but only a much weacker assumption on $\{\varphi_n(t)\}$, namely, that it is a uniformly bounded system, say with K.

The following argument follows closely that of a similar theorem in the paper of Salem and Zygmund [11]. For the sake of completeness, we give the proof in detail here too.

Let us assume for example $F(y)<1$ for all finite y, and that (3) (ii) is false. Then there is an $\varepsilon>0$ such that $c_N/C_N>\varepsilon$ for infinitely many k; consider only such k. Let $E_N(y)$ denote the subset of E where $S_N(t)/C_N<y$. Obviously, $C_{N-1}/C_N<(1-\varepsilon^2)^{1/2}$ and the formula

$$\frac{S_N(t)}{C_N}=\frac{S_{N-1}(t)}{C_{N-1}}\cdot\frac{C_{N-1}}{C_N}+\frac{c_N\varphi_N(t)}{C_N}$$

shows that at every point t of $E_{N-1}(y)$ with $y>0$ we have

$$\frac{S_N(t)}{C_N}<y(1-\varepsilon^2)^{1/2}+K.$$

It follows that $E_{N-1}(y)$ is included in $E_N\big(y(1-\varepsilon^2)^{1/2}+K\big)$, and so

$$F_{N-1}(y;E)\leqq F_N\big(y(1-\varepsilon^2)^{1/2}+K;E\big).$$

Let y be a point of continuity of $F(y)$. Letting $N\to\infty$ gives

(11) $$F(y)\leqq F\big(y(1-\varepsilon^2)^{1/2}+K\big).$$

However, from the assumption that $F(y)$ is always less than 1 and from the fact that $F(y)\to 1$ as $y\to\infty$ it follows that there are points of continuity $y>0$ such that

$$F(y) > F\big(y(1-\varepsilon^2)^{1/2}+K\big).$$

This contradicts (11) and completes the proof of Theorem 3.

9. Finally, we produce an application which shows that the central limit theorems can be used to obtain exact estimates on the asymptotic behaviour of $S_N(t)$ in L^p norm ($p>0$). In [6] the present author has already proved that the mean of degree p of $S_N(t)$ and the C_N are equal to within a factor. To be more precise, our cited theorem reads as follows:

Theorem D. *Let $\{\varphi_n(t)\}$ be a uniformly bounded* ESMS. *Then, for every positive real number p, we have*

$$K_1^{(p)}C_N \leqq \left\{\int_0^1 |S_N(t)|^p\,dt\right\}^{1/p} \leqq K_2^{(p)}C_N, \tag{12}$$

where $K_1^{(p)}$ and $K_2^{(p)}$ are positive constants depending only on p.

Now we are able to improve these inequalities not only in case [0, 1] but for an arbitrary fixed set $E\subset[0,1]$ of positive measure when (3) is fulfilled. Namely, we show the following holds:

Theorem 6. *Let $\{\varphi_n(t)\}$ be a uniformly bounded* ESMS, *let $E\subset[0,1]$ be a set of positive measure, and let p be a positive number. If* (3) *holds then*

$$\frac{1}{C_N^p}\int_E |S_N(t)|^p\,dt \to \frac{\operatorname{mes}(E)}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}|x|^p e^{-\frac{x^2}{2}}\,dx. \tag{13}$$

This theorem makes the relation (12) much more exact in case (3). As to its proof we refer to a known result concerning the convergence of distribution functions and that of their moments.

Lemma 2. *Let p and δ be positive numbers, and let $F(y), F_1(y), F_2(y), \ldots$ be the distribution functions of the $L^{p+\delta}$ integrable functions $f(t), f_1(t), f_2(t), \ldots$ with*

$$\int_0^1 |f_n(t)|^{p+\delta}\,dt \leqq K \qquad (n=1,2,\ldots),$$

where K means a positive constant. If $F_n(y)$ converges to $F(y)$ at the points of continuity of the latter then

$$\int_0^1 |f_n(t)|^p\,dt \to \int_0^1 |f(t)|^p\,dt.$$

This lemma can be found, for example, in the book of LOÈVE ([4], pp. 178—185) or, in the special case $p=1$, it is proved in our paper [7]. The proof given in the latter can be extended to this more general case word for word, but Schwarz's inequality must be replaced by Hölder's inequality in it.

To prove Theorem 6, after having Lemma 2, it suffices to remark that

$$\int_E |f(t)|^p\,dt = \operatorname{mes}(E) \int_{-\infty}^{+\infty} |y|^p\,dF(y; E),$$

where $F(y; E)$ denotes the distribution function of $f(t)$ relative to the set E.

As to the integral on the right-hand side of (13), with the factor $1/\sqrt{2\pi}$, can be easily calculated in many cases, for example, it is equal to $1.3\ldots(p-1)$ if p is an even number.

It would be interesting to know whether Theorem 6 remains valid also in case the integrand on the left-hand side of (13), or only the integrand of (12), is substituted by $\max\limits_{1\leq n\leq N}|S_n(t)|$. The analogous result for independent functions is true (see [5]). Such a result, if valid, would considerably contribute to the investigation of the divergence features of the series arising from the functions of an ESMS.

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(Received August 19, 1969)

On Haar and Schauder series

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1. Consider the Haar functions [1]

$$\chi_0^{(0)}(x) \equiv 1 \quad (x \in [0, 1]);$$

$$\chi_m^{(k)}(x) = \begin{cases} \sqrt{2^m} & \left(x \in \left(\frac{k}{2^m}, \frac{2k+1}{2^{m+1}}\right)\right), \\ -\sqrt{2^m} & \left(x \in \left(\frac{2k+1}{2^{m+1}}, \frac{k+1}{2^m}\right)\right), \\ 0 & \left(x \in [0, 1] \setminus \left\{\left(\frac{k}{2^m}, \frac{2k+1}{2^{m+1}}\right) \cup \left(\frac{2k+1}{2^{m+1}}, \frac{k+1}{2^m}\right)\right\}\right), \end{cases}$$

($m = 0, 1, 2, \ldots$; $k = 0, 1, 2, \ldots, 2^m - 1$) and arrange them in a sequence $\{\chi_n\}_{n=0}^{\infty}$ by setting $\chi_0 = \chi_0^{(0)}$ and $\chi_n = \chi_m^{(k)}$ for $n = 2^m + k$.

The system of the functions

$$\varphi_n(x) = \int_0^x \chi_n(t)\, dt \qquad (0 \leqq x \leqq 1;\ n = 0, 1, 2, \ldots) \tag{1}$$

will be called the Schauder—Ciesielski system (cf. [2], [3]).

We associate with every function $f(x)$ on the interval [0, 1] the numerical sequence $\{c_n(f)\}_{n=0}^{\infty}$ defined by

$$c_0(f) = f(1) - f(0), \tag{2}$$

$$c_n(f) = -\int_0^1 f(x)\, d\chi_n(x) \left(= \sqrt{2^m}\left[2f\left(\frac{2k+1}{2^{m+1}}\right) - f\left(\frac{k}{2^m}\right) - f\left(\frac{k+1}{2^m}\right)\right]\right)$$

$$(n = 2^m + k;\ m = 0, 1, 2, \ldots;\ k = 0, 1, 2, \ldots, 2^m - 1)$$

and call

$$S(f) = f(0) + \sum_{n=0}^{\infty} c_n(f)\varphi_n(x) \tag{3}$$

the Schauder—Ciesielski series of the function $f(x)$.

If $f(x)$ is continuous on the interval [0, 1] then it is known that the series (3) converges uniformly to $f(x)$ ([2], [3]).

Theorem 1. *For an arbitrary function $f(x)$ on* [0, 1] *the series* (3) *converges to $f(x)$ at every dyadic rational point x, and at every point x where $f(x)$ is continuous.*

Proof. For the partial sums

$$S_n(x) = S_n(f; x) = f(0) + \sum_{l=0}^{n-1} c_l(f)\varphi_l(x) \tag{4}$$

we have evidently that

$$S_n(x) = S_{2^\nu}(x) \quad \text{or} \quad S_n(x) = S_{2^{\nu+1}}(x) \quad \text{if} \quad 2^\nu \leqq n < 2^{\nu+1}.$$

Thus it is enough to study the convergence behaviour of the partial sums $S_{2^\nu}(x)$.

Let x be an arbitrary but fixed point in the interval [0, 1]. For every ν we define the pair of dyadic rationals $\alpha_\nu(x)$, $\beta_\nu(x)$ by the inequalities:

$$\alpha_\nu(x) = \frac{l}{2^\nu} \leqq x < \frac{l+1}{2^\nu} = \beta_\nu(x) \qquad (0 \leqq l < 2^\nu - 1). \tag{5}$$

From (2) and (4) we deduce

$$S_{2^\nu}\left(f; \frac{l}{2^\nu}\right) = f\left(\frac{l}{2^\nu}\right) \qquad (\nu = 0, 1, 2, \dots;\ l = 0, 1, \dots, 2^\nu - 1). \tag{6}$$

Since $S_{2^\nu}(x)$ is linear on the interval $(\alpha_\nu(x), \beta_\nu(x))$, (2) and (4) imply that

$$S_{2^\nu}(f; x) = f(\alpha_\nu(x)) + 2^\nu(x - \alpha_\nu(x))\{f(\beta_\nu(x)) - f(\alpha_\nu(x))\}. \tag{7}$$

For dyadic irrational x our assertion follows from (7) if we take into account the inequalities $0 < (x - \alpha_\nu(x))2^\nu < 1$. For dyadic rational x the assertion is trivial.

2. A function $f(x)$ $(x \in [0, 1])$ will be called *D-differentiable* at x if

$$2^\nu[f(\beta_\nu(x)) - f(\alpha_\nu(x))]$$

tends to a finite limit $f'_D(x)$ as $\nu \to \infty$.

Theorem 2. *Let $f(x)$ be an arbitrary function defined on* [0, 1]. *Then the Haar series*

$$S'(x) = \sum_{n=0}^{\infty} c_n(f)\chi_n(x) \tag{8}$$

with the coefficients defined by (2) *is convergent at a dyadic irrational point x if and only if $f'_D(x)$ exists, and in this case its sum is equal to $f'_D(x)$.*

Proof. From (1) and (6) we deduce that

(9) $$s_{2^\nu}(x) = \sum_{l=0}^{2^\nu-1} c_l(f)\chi_l(x) = \sum_{l=0}^{2^\nu-1} c_l(f)\varphi_l'(x) = S'_{2^\nu}(x) =$$

$$= 2^\nu[S_{2^\nu}(f;\beta_\nu(x)) - S_{2^\nu}(f;\alpha_\nu(x))] = 2^\nu[f(\beta_\nu(x)) - f(\alpha_\nu(x))].$$

Since

(10) $$s_n(x) = s_{2^\nu}(x) \quad \text{or} \quad s_n(x) = s_{2^{\nu+1}}(x) \quad \text{for} \quad 2^\nu \leqq n < 2^{\nu+1},$$

our assertion is an immediate consequence of (9).

Theorem 3. *The series*

(11) $$\sum_{n=0}^{\infty} a_n \chi_n(x)$$

is the Haar—Fourier expansion of a Lebesgue-integrable function if and only if there exists an absolutely continuous function $F(x)$ for which $a_n = c_n(F)$, i.e.

(12) $$S'(F) = \sum_{n=0}^{\infty} a_n \chi_n(x).$$

Proof. Suppose (11) is the Haar—Fourier expansion of a function $f(x) \in L[0, 1]$, i.e. that

(13) $$a_n = \int_0^1 f(x)\chi_n(x)\,dx \qquad (n = 0, 1, 2, \ldots).$$

Let $F(x) = \int_0^x f(t)\,dt$. Integration by parts gives

(14) $$a_n = \int_0^1 F'(x)\chi_n(x)\,dx = -\int_0^1 F(x)\,d\chi_n(x) = c_n(F)$$

for $n = 1, 2, 3, \ldots$, while for $n = 0$ the equality $a_0 = c_0(F)$ is trivial.

Conversely, if $F(x)$ is absolutely continuous, then we can consider the series (11), where

$$a_n = -\int_0^1 F(x)\,d\chi_n(x) = \int_0^1 F'(x)\chi_n(x)\,dx,$$

i.e. $S'(F)$ is the Haar—Fourier expansion of the integrable function $F'(x)$.

Theorem 4. *All the partial sums of a Haar series* (11) *are non-negative everywhere if and only if there exists an increasing function $f(x)$ on* [0, 1] *for which* $S'(F) = \Sigma a_n \chi_n(x)$.

Proof. If $f(x)$ is an increasing function in [0, 1], then by the equation (9) the partial sums of the Haar series $S'(f)$ are non-negative everywhere.

Conversely, if for all $x \in [0, 1]$ and for every n the inequalities

$$s_n(x) = \sum_{l=0}^{n-1} a_l \chi_l(x) \geqq 0$$

hold, then the functions

(15)
$$S_n(x) = \int_0^x s_n(t)\,dt \qquad (n = 1, 2, \ldots)$$

are absolutely continuous and increasing for every n, and the equalities

$$S_n(0) = 0 \qquad S_n(1) = a_0$$

hold. Applying the theorem of Helly we can pick out a subsequence $\{S_{n_k}(x)\}_{k=1}^{\infty}$ of the sequence (15) such that the limit

(16)
$$\lim_{k\to\infty} S_{n_k}(x) = F(x)$$

exists for every x [4]. The function $F(x)$ is evidently increasing on [0, 1] and we have $F(0) = 0$ and $F(1) = a_0$.

Since the functions $S_{n_k}(x)$ satisfy the inequalities $0 \leqq S_{n_k}(x) \leqq a_0$ we have by the dominated convergence theorem of Lebesgue that

(17)
$$\lim_{k\to\infty} \int_0^1 S_{n_k}(x)\,d\chi_m(x) = \int_0^1 F(x)\,d\chi_m(x).$$

Since by the orthonormality of the Haar system the equalities

$$\int_0^1 S_{n_k}(x)\,d\chi_m(x) = -\int_0^1 \chi_m(x)\,dS_{n_k}(x) = -\int_0^1 \chi_m(x) s_{n_k}(x)\,dx = -a_m$$

are true for $n_k \geqq m$, it follows from (17) that

$$a_m = -\int_0^1 F\,d\chi_m, \quad \text{i.e.} \quad a_m = c_m(F) \qquad (m = 0, 1, 2, \ldots).$$

Q.E.D.

The following assertion can be proved in a quite similar way. (We have only to use the decomposition of a function $f(x)$ into its positive and negative parts.)

Theorem 5. *For the partial sums* $s_n(x) = \sum_{l=0}^{n-1} a_l \chi_l(x)$ *of a Haar series* (11) *we have*

$$I_n = \int_0^1 |s_n(x)|\,dx = O(1)$$

if and only if there exists a function $F(x)$ *of bounded variation on* [0, 1] *for which* $S'(F) = \sum_{l=0}^{\infty} a_l \chi_l(x)$.

3. In this section we mention some corollaries to the above theorems.

Theorem 6. *A Haar series with non-negative partial sums converges almost everywhere to* 0 *if and only if there exists an increasing singular function* $f(x)$ *for which the Haar series is identically equal to* $S'(f)$.

The theorem is an immediate consequence of Theorems 2 and 4.

Theorem 7. *There exists a Haar series with non-negative partial sums, which is not the Haar—Fourier expansion of some function* $f(x)\in L[0, 1]$.

This follows from Theorem 6 if we apply it to an increasing singular function. Thus the fact, conjectured by STEINHAUS for trigonometric series, and proved in [5], does not hold for Haar series.

Finally it is possible to give a class of continuous functions which are not differentiable almost everywhere; more precisely the following assertion is true:

Theorem 8. *If in a Schauder—Ciesielski series*

$$\sum_{n=0}^{\infty} a_n \varphi_n(x) \tag{18}$$

the coefficients a_n *tend monotonically to* 0 *then the series tends uniformly to a continuous function* $f(x)$ *and if at the same time*

$$\sum a_n^2 = \infty, \tag{19}$$

then $f(x)$ *is not differentiable almost everywhere.*

Proof. The series (18) is equiconvergent with the series

$$\sum_{m=0}^{\infty}\left(\sum_{k=0}^{2^m-1} a_{2^m+k}\varphi_{2^m+k}(x)\right) \equiv \sum_{m=0}^{\infty} \Phi_m(x),$$

and since $|\Phi_m(x)| \leqq a_0 2^{-\frac{m}{2}}$, the series (18) converges uniformly, and hence it is the Schauder—Ciesielski expansion of its continuous sum $f(x)$.

If the function $f(x)$ were differentiable on a set $E\subset[0, 1]$ of positive measure, then by Theorem 2 the Haar series

$$S'(f) = \sum_{n=0}^{\infty} a_n \chi_n(x)$$

would be convergent on E. But this is impossible, because, by [6], a Haar series is almost everywhere divergent if its coefficients are monotonic and satisfy condition (19).

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(Received August 9, 1968)

Über einseitige Approximation durch Polynome. II

Von G. FREUD und J. SZABADOS in Budapest

§ 1. Einführung

Vorliegende Arbeit setzt die Arbeit G. FREUD [1] fort und behandelt ebenfalls die einseitige L_1-Approximation bezüglich der reellen Zahlgeraden. In diesem Fall scheint die Funktion e^{-x^2} die natürliche Gewichtsfunktion zu sein.

Wir bezeichnen mit $V(r, s)$ (r, s natürliche Zahlen) die Klasse der Funktionen $F(x)$ in $(-\infty, +\infty)$, die die folgenden Bedingungen a), b) befriedigen:

a) $F(x)$ is $(r-1)$-mal stetig differenzierbar und $F^{(r-1)}(x)$ ist die Integralfunktion einer Funktion $F^{[r]}(x)$, die über jedes endliche Intervall eine beschränkte Schwankung hat und

(1) $$M_r(F) \equiv \int_{-\infty}^{+\infty} e^{-x^2} |dF^{[r]}(x)| < +\infty$$

genügt.

b) Es gilt mit passenden (von F abhängenden) nichtnegativen Zahlen A, B, s

(2) $$|F(x)| \leqq A + Bx^{2s} \qquad (-\infty < x < +\infty).$$

Satz. *Sei $F \in V(r, s)$. Zu jedem $n(=s, s+1, \ldots)$ gibt es Polynome höchtens $(2n-1)$-ten Grades $p_n(x)$ und $P_n(x)$, die die folgenden Bedingungen* (3), (4) *befriedigen:*

(3) $$p_n(x) \leqq F(x) \leqq P_n(x) \qquad (-\infty < x < +\infty)$$

und

(4) $$\int_{-\infty}^{+\infty} [P_n(x) - p_n(x)] e^{-x^2} dx = O\left(n^{-\frac{r+1}{2}}\right).$$

In § 2 beweisen wir den Satz für die speziellen Funktionen $\Gamma_\nu(x, \xi)$ und in § 3 betrachten wir einige Hilfssätze. Der Beweis für den allgemeinen Fall erfolgt in § 4.

§ 2. Einseitige Approximation von $\Gamma_\nu(x, \xi)$

Für jede ganze Zahl $\nu \geqq 0$ setzen wir

(5) $$\Gamma_\nu(x, \xi) = \begin{cases} 0 & \text{für } x \leqq \xi. \\ (x-\xi)^\nu & \text{für } x > \xi; \end{cases}$$

als Funktion von x gehört $\Gamma_\nu(x, \xi)$ offenbar zu $V\left(\nu, \left[\frac{\nu}{2}\right]+1\right)$.

Hilfssatz. *Für jede feste natürliche Zahl ν, jede ganze Zahl $n > \nu+1$, und jede reelle Zahl $\xi \in \left[-\frac{1}{2}\sqrt{n}, \frac{1}{2}\sqrt{n}\right]$, gibt es Polynome von x höchtens $(2n-2)$-ten Grades $p_{n,\nu}(x, \xi)$ und $P_{n,\nu}(x, \xi)$ mit den Eigenschaften*

(6) $$p_{n,\nu}(x, \xi) \leqq \Gamma_\nu(x, \xi) \leqq P_{n,\nu}(x, \xi) \qquad (-\infty < x < +\infty)$$

und

(7) $$\int_{-\infty}^{+\infty} [P_{n,\nu}(x, \xi) - p_{n,\nu}(x, \xi)] e^{-x^2} dx \leqq c_\nu n^{-\frac{\nu+1}{2}} e^{-\xi^2}.$$

Dabei hängen die Zahlen c_ν weder von n noch von ξ ab, ferner sind $p_{n,\nu}(x, \xi)$ und $P_{n,\nu}(x, \xi)$ für festes x stückweise stetige beschränkte Funktionen des Parameters ξ.

Beweis. Da wir mit Hilfe der Wahl eines größeren $c_\nu > 0$ die ersten Glieder der Folge $\{P_{n,\nu}(x, \xi)\}$ durch $(x-\xi)^{2[\nu/2]+2}$ und die ersten Glieder der Folge $\{p_{n,\nu}(x, \xi)\}$ durch 0 ersetzen können, reicht es den Satz für hinreichend große Werte von n zu beweisen.

Wir bezeichnen mit

$$x_{1,n} < x_{2,n} < \cdots < x_{n,n}$$

die Nullstellen des Hermiteschen Orthogonalpolynoms $H_n(x)$. Wegen $|x_{1,n}| = x_{n,n} \approx \sqrt{2n+1}$ (vgl. G. Szegő [2], S. 131) und $|\xi| \leqq \frac{1}{2}\sqrt{n}$ gibt es ein $\mu = \mu(n, \xi)$ derart, daß

(8) $$x_{\mu,n} \leqq \xi < x_{\mu+1,n}.$$

Es sei jetzt

(9) $$\eta = \left[\frac{\nu+1}{2}\right],$$

und es sei n_0 so groß, daß für jedes $n \geqq n_0$ und $|\xi| \leqq \frac{1}{2}\sqrt{n}$

(10) $$-\sqrt{n} < x_{\mu-\eta,n} < x_{\mu+\eta+1,n} < \sqrt{n}$$

gilt *), und im Weiteren sei $n \geqq n_0$.

*) Die Existenz eines solchen Wertes n_0 folgt z. B. aus (16).

Im Laufe der weiteren Konstruktion unterscheiden wir die Fälle von geraden und ungeraden Werten von ν.

Fall: ν eine gerade Zahl. Es seien $\varphi_n(x, \xi)$, bzw. $\Phi_n(x, \xi)$ die Polynome höchtens $2(n-\eta-1)$-ten Grades mit

$$\varphi_n(x_{1,n}, \xi) = \cdots = \varphi_n(x_{\mu+1,n}, \xi) = 0, \quad \varphi_n(x_{\mu+\eta+2,n}, \xi) = \cdots = \varphi_n(x_{n,n}, \xi) = 1,$$

$$\varphi_n'(x_{1,n}, \xi) = \cdots = \varphi_n'(x_{\mu,n}, \xi) = \varphi_n'(x_{\mu+\eta+2,n}, \xi) = \cdots = \varphi_n'(x_{n,n}, \xi) = 0;$$

bzw. mit

$$\Phi_n(x_{1,n}, \xi) = \cdots = \Phi_n(x_{\mu-\eta-1,n}, \xi) = 0, \quad \Phi_n(x_{\mu,n}, \xi) = \cdots = \Phi_n(x_{n,n}, \xi) = 1,$$

$$\Phi_n'(x_{1,n}, \xi) = \cdots = \Phi_n'(x_{\mu-\eta-1,n}, \xi) = \Phi_n'(x_{\mu+1,n}, \xi) = \cdots = \Phi_n'(x_{n,n}, \xi) = 0.$$

Aus der Diskussion des Verlaufes dieser Polynome ergibt sich (T. J. STIELTJES [3]; vgl. G. FREUD [4])

$$(11) \qquad \varphi_n(x, \xi) \leqq \Gamma_0(x, \xi) \leqq \Phi_n(x, \xi) \qquad (-\infty < x < +\infty)$$

und

$$(12) \qquad 0 \leqq \varphi_n(x, \xi) \leqq 1 \qquad (x_{\mu+1,n} \leqq x \leqq x_{\mu+\eta+2,n}),$$

$$0 \leqq \Phi_n(x, \xi) \leqq 1 \qquad (x_{\mu-\eta-1,n} \leqq x \leqq x_{\mu,n}).$$

Wir setzen

$$p_{n,\nu}(x, \xi) = (x-\xi)^\nu \varphi_n(x, \xi) \quad \text{und} \quad P_{n,\nu}(x, \xi) = (x-\xi)^\nu \Phi_n(x, \xi);$$

diese Polynome sind wegen (9) höchstens vom Grade $2n-2$ und aus (11) folgt das Erfülltsein von (6). Für jedes Polynom höchtens $2n-1$-ten Grades $\Pi_{2n-1}(x)$ gilt die Gauß—Jacobische Quadraturformel

$$(13) \qquad \int_{-\infty}^{+\infty} \Pi_{2n-1}(x) e^{-x^2}\, dx = \sum_{k=1}^{n} \Pi_{2n-1}(x_{k,n}) \lambda_{k,n},$$

wobei die „Christoffelschen Zahlen" $\lambda_{k,n} \geqq 0$ die Ungleichung

$$(14) \qquad \lambda_{k,n} \leqq a_1 n^{-1/2} e^{-x_{k,n}^2} \quad \left(|x_{k,n}| < \sqrt{n}\right)$$

mit einer absoluten Konstanten a_1 befriedigen (vgl. G. FREUD [5]).

Indem wir in (13) $\Pi_{2n-1}(x) = P_{n,\nu}(x, \xi) - p_{n,\nu}(x, \xi)$ setzen, erhalten wir unter Beachtung von (14)

$$(15) \qquad \int_{-\infty}^{+\infty} [P_{n,\nu}(x, \xi) - p_{n,\nu}(x, \xi)] e^{-x^2}\, dx =$$

$$= \sum_{k=\mu-\eta}^{\mu+\eta+1} (x_{k,n}-\xi)^\nu [\Phi_n(x_{k,n}, \xi) - \varphi_n(x_{k,n}, \xi)] \lambda_{k,n} \leqq$$

$$\leqq (x_{\mu+\eta+1,n} - x_{\mu-\eta,n})^\nu a_1 n^{-1/2} \sum_{k=\mu-\eta}^{\mu+\eta+1} e^{-x_{k,n}^2}.$$

Wir bedienen uns endlich der Ungleichung

$$x_{k+1,n}-x_{k,n}<\frac{a}{\sqrt{n}}\qquad(-\sqrt{n}<x_{k,n}<x_{k+1,n}<\sqrt{n}) \tag{16}$$

(vgl. etwa G. FREUD [5]). Für $\mu-\eta\leqq k\leqq\mu+\eta+1$ ist also unter Beachtung von $|\xi|\leqq\frac{1}{2}\sqrt{n}$

$$e^{-x_{k,n}^2}=e^{-(x_{k,n}-\xi)^2}e^{2\xi(\xi-x_{k,n})}e^{-\xi^2}<a_2(\eta)e^{\sqrt{n}\cdot a\frac{2\eta+2}{\sqrt{n}}}e^{-\xi^2}<a_3(\eta)e^{-\xi^2}, \tag{17}$$

wo $a_2(\eta), a_3(\eta), \dots$ nur von η abhängen. Aus (15), (16) und (17) folgt wegen (9) die Gültigkeit von (7). Es ist aus der Konstruktion selbst klar, daß $P_{n,\nu}(x,\xi)$ und $p_{n,\nu}(x,\xi)$ in jedem offenen Intervall $(x_{k,n}, x_{k+1,n})\subset\left[-\frac{1}{2}\sqrt{n}, \frac{1}{2}\sqrt{n}\right]$ Polynome von ξ sind, da $\varphi_n(x,\xi)$ und $\Phi_n(x,\xi)$ in einem solchen Intervall von ξ unabhängig sind. Dies zeigt die Gültigkeit des Hilfssatzes für gerade ν.

Fall: ν eine ungerade Zahl. Unter $\bar\varphi_n(x,\xi)$ bzw. $\bar\Phi_n(x,\xi)$ verstehen wir die Polynome höchtens $(2(n-\eta)-1)$-ten Grades (vgl. (9)) mit

$$\bar\varphi_n(x_{1,n},\xi)=\cdots=\bar\varphi_n(x_{\mu,n},\xi)=0,\quad \bar\varphi_n(x_{\mu+\eta+1,n},\xi)=\cdots=\bar\varphi_n(x_{n,n},\xi)=1,$$

$$\bar\varphi_n'(x_{1,n},\xi)=\cdots=\bar\varphi_n'(x_{\mu,n},\xi)=\bar\varphi_n'(x_{\mu+\eta+1,n},\xi)=\cdots=\bar\varphi_n'(x_{n,n},\xi)=0;$$

bzw.

$$\bar\Phi_n(x_{1,n},\xi)=\cdots=\bar\Phi_n(x_{\mu,n},\xi)=0,\quad \bar\Phi_n(x_{\mu+\eta,n},\xi)=\cdots=\bar\Phi_n(x_{n,n},\xi)=1,$$

$$\bar\Phi_n'(x_{1,n},\xi)=\cdots=\bar\Phi_n'(x_{\mu-1,n},\xi)=\bar\Phi_n'(x_{\mu+\eta+1,n},\xi)=\cdots=\bar\Phi_n'(x_{n,n},\xi)=0,$$

und es sei ferner für ungerade ν

$$p_{n,\nu}(x,\xi)=(x-\xi)^\nu\bar\varphi_n(x,\xi),\quad P_{n,\nu}^*(x,\xi)=(x-\xi)^\nu\bar\Phi_n(x,\xi).$$

Es gilt für $x\in[x_{\mu,n}, x_{\mu+\eta,n}]$

$$0\leqq\bar\varphi_n(x,\xi)\leqq 1,\quad 0\leqq\bar\Phi_n(x,\xi)\leqq 1 \tag{22}$$

(vgl. G. FREUD [4]) und weiter unter Beachtung von (16)

$$|p_{n,\nu}(x,\xi)|\leqq a_8(\eta)n^{-\nu/2},\quad |P_{n,\nu}^*(x,\xi)|\leqq a_9(\eta)n^{-\nu/2}\qquad(x_{\mu,n}\leqq x\leqq x_{\mu+\eta,n}). \tag{23}$$

Es gilt auch (vgl. G. FREUD [1])

$$\begin{aligned} &p_{n,\nu}(x,\xi)\leqq\Gamma_\nu(x,\xi)\quad(-\infty<x<+\infty),\\ &\Gamma_\nu(x,\xi)\leqq P_{n,\nu}^*(x,\xi)\quad(x\notin[x_{\mu,n}, x_{\mu+\eta,n}]). \end{aligned} \tag{24}$$

Es seien $l_{k,n}(x) = \dfrac{H_n(x)}{H_n'(x_{k,n})(x-x_{k,n})}$ die Grundpolynome der Lagrangeschen Interpolation über die Knotenpunkte $x_{k,n}$ $(k=1, 2, \ldots, n)$, und wir setzen

$$(25) \qquad \Omega_n(x) = \sum_{k=\mu}^{\mu+\eta-1}{}' \, [l_{k,n}(x)+l_{k+1,n}(x)]^2 \geqq 0.$$

Der Grad von $\Omega_n(x)$ ist höchtens gleich $2n-2$ und aus einer Bemerkung von P. ERDŐS—P. TURÁN [6] folgt

$$(26) \qquad \Omega_n(x) \geqq 1 \qquad (x_{\mu,n} \leqq x \leqq x_{\mu+\eta,n}).$$

Es sei

$$(27) \qquad P_{n,\nu}(x, \xi) = P_n^*(x, \xi) + B_\eta \Omega_n(x)$$

mit

$$(28) \qquad B_\eta = 2(x_{\mu+\eta,n} - x_{\mu,n})^\nu \leqq a_{10}(\eta) n^{-\nu/2}$$

(vgl. (16)). Dann ist infolge (5), (24), (23), (27) und (28)

$$(29) \qquad \Gamma_\nu(x, \xi) \leqq P_{n,\nu}(x, \xi) \qquad (-\infty < x < +\infty).$$

Aus der Quadraturformel (13) erhalten wir

$$\int_{-\infty}^{+\infty} [P_{n,\nu}(x, \xi) - p_{n,\nu}(x, \xi)] e^{-x^2} dx =$$

$$= \int_{-\infty}^{+\infty} [P_{n,\nu}^*(x, \xi) - p_{n,\nu}(x, \xi)] e^{-x^2} dx + B_\eta \int_{-\infty}^{+\infty} \Omega_n(x) e^{-x^2} dx \leqq$$

$$\leqq \sum_{k=\mu+1}^{\mu+\eta}{}' \, [P_{n,\nu}^*(x_{k,n}, \xi) - p_{n,\nu}(x_{k,n}, \xi)] \lambda_{k,n} + 2B_\eta \cdot \sum_{k=\mu}^{\mu+\eta} \lambda_{k,n}$$

und unter Beachtung von (23), (28), (14) und (17) schließen wir auf die Gültigkeit von (7). Die beiden letzten Behauptungen von Hilfssatz 1 zeigt man genau so, wie im Falle von geraden ν.

§ 3. Weitere Hilfssätze

Hilfssatz 2. *Für $x>0$ und $k=0, 1, 2, \ldots$ gilt*

$$(30) \qquad I_k(x) = \int_x^{+\infty} (\xi - x)^k e^{-\xi^2} d\xi \leqq \frac{k!}{(2x)^{k+1}} e^{-x^2}.$$

Beweis. Es ist

$$I_0(x) = \int_x^{+\infty} e^{-\xi^2} d\xi < \frac{1}{x} \int_x^{+\infty} \xi e^{-\xi^2} d\xi = \frac{1}{2x} e^{-x^2},$$

und für $k \geqq 1$

$$I_k(x) = k \int_x^{+\infty} (\xi - x)^{k-1} I_0(\xi)\, d\xi < k \int_x^{+\infty} (\xi - x)^{k-1} \frac{1}{2\xi} e^{-\xi^2} d\xi < \frac{k}{2x} I_{k-1}(x).$$

Hieraus erhalten wir durch Induktion die Beziehung (30).

Hilfssatz 3. *Aus den Bedingungen* b) *und* c) *für die Klasse* $V(r, s)$ *folgt, daß die Funktionen*

$$F^{(k)}(x) e^{-x^2} x^{r-k} \qquad (k = 0, 1, \dots, r-1) \tag{31}$$

und

$$F^{[r]}(x) e^{-x^2} \tag{32}$$

auf der ganzen reellen Achse beschränkt sind.

Beweis. Aus der Ungleichung

$$|e^{-x^2}[F^{[r]}(x) - F^{[r]}(0)]| \leqq$$

$$\leqq \left| \int_0^x e^{-\xi^2} |dF^{[r]}(\xi)| \right| \leqq \int_{-\infty}^{+\infty} e^{-\xi^2} |dF^{[r]}(\xi)| = M_r(F) < +\infty \qquad (-\infty < x < +\infty)$$

folgt die Beschränktheit des Ausdrucks (32). Man zeigt leicht, daß die k-te Derivierte der Funktion $\alpha(x) = x^{-r} e^{x^2}$ gleich

$$\alpha^{(k)}(x) = x^{-r-k} e^{x^2} \beta_{2k}(x)$$

ist, wo $\beta_{2k}(x)$ ein Polynom genau $2k$-ten Grades mit positivem Leitkoeffizienten bedeutet. Falls wir also $x_0 > 0$ hinreichend groß wählen, gilt

$$0 < \alpha^{(k)}(x) < a_{11}(r) x^{-r+k} e^{x^2} \qquad (x \geqq x_0;\ k = 0, 1, \dots, r), \tag{33}$$

und sogar

$$a_{12}(r) \leqq \alpha^{(r)}(x) e^{-x^2} \leqq a_{11}(r) \quad \text{und} \quad |F^{[r]}(x)| \leqq a_{13}(r) \alpha^{(r)}(x) \quad (x \geqq x_0). \tag{34}$$

Durch $r-k$-maliges Integrieren zwischen den Grenzen x_0 und x folgt hieraus

$$|F^{(k)}(x)| \leqq a_{13}(r) \alpha^{(k)}(x) + \Pi_{r-k}(x) \qquad (x \geqq x_0),$$

wo $\Pi_{r-k}(x)$ ein Polynom höchstens $r-k$-ten Grades ist, und weiter unter Beachtung von (33) gilt:

$$|F^{(k)}(x)| \leqq a_{14}(r) x^{-r+k} e^{x^2} \qquad (x \geqq x_0;\ k = 0, 1, \dots, r-1). \tag{35}$$

Die gleiche Abschätzung für $F(-x)$ anstelle von $F(x)$ ergibt, zusammen mit (35),

$$|F^{(k)}(x)| \leqq a_{14}(r) |x|^{-r+k} e^{x^2} \qquad (|x| \geqq x_0;\ k = 0, 1, \dots, r-1). \tag{36}$$

Durch eine Vergrößerung des Wertes von $a_{14}(r)$ (falls nötig) bleiben diese Ungleichungen auch für $|x| < x_0$ gültig, w. z. b. w.

§ 4. Beweis des Satzes

Ohne Einschränkung der Allgemeinheit kann man annehmen, daß $2s>r$. Es reicht auch offenbar Funktionen $F(x)$ zu betrachten, welche für $x<0$ verschwinden, da man eine beliebige Funktion $F(x)\in V(r,s)$ in der Form

$$F(x)=\sum_{\nu=0}^{r-1}\frac{F^{(\nu)}(0)}{\nu!}x^{\nu}+\frac{F^{[r]}(+0)+F^{[r]}(-0)}{2r!}x^{r}+F_1(x)+F_2(-x)$$

darstellen kann, mit für $x<0$ verschwindenden $F_1, F_2\in V(r,s)$. (Dabei sind in jedem Intervall, dessen Endpunkte von 0 verschieden sind die Schwankungen von $F_1^{[r]}(x)$ und $F_2^{[r]}(x)$ kleiner als die Schwankung von $F^{[r]}(x)$. Demzufolge gilt auch $M_r(F_1)\leqq M_r(F)$, $M_r(F_2)\leqq M_r(F)$.)

Es sei $\omega_n\in\left[\frac{1}{4}\sqrt{n},\ \frac{1}{2}\sqrt{n}\right]$ eine Stetigkeitsstelle von $F^{[r]}(x)$, und wir zerlegen $F(x)$ in

$$F(x)=\sum_{\nu=0}^{r-1}\frac{F^{(\nu)}(\omega_n)}{\nu!}(x-\omega_n)^{\nu}+\frac{F^{[r]}(\omega_n)}{r!}(x-\omega_n)^{r}+F^*(x)+F^{**}(x),\tag{37}$$

so daß F^* und F^{**} zu $V(r,s)$ gehören, $F^*(x)$ für $x\geqq\omega_n$ und $F^{**}(x)$ für $x<\omega_n$ verschwindet, und die Schwankungen von $F^{*[r]}(x)$ und $F^{**[r]}(x)$ in keinem Intervall größer als die Schwankung von $F^{[r]}(x)$ sind. Die Approximation von $F^*(x)$ und $F^{**}(x)$ behandeln wir einzeln.

a) *Einseitige Approximation von* $F^*(x)$.

Aus $F^{*(\nu)}(\omega_n)=0$ $(\nu=0,1,\dots,r-1)$ folgt

$$F^*(x)=\frac{1}{r!}\int_{\omega_n}^{x}(x-\xi)^r\,dF^{*[r]}(\xi)=\frac{(-1)^{r+1}}{r!}\int_0^{\omega_n}\Gamma_r(-x,-\xi)\,dF^{*[r]}(\xi)=\tag{38}$$

$$=\frac{(-1)^{r+1}}{r!}\int_0^{\omega_n}\Gamma_r(-x,-\xi)\,dF^{*[r]+}(\xi)+\frac{(-1)^r}{r!}\int_0^{\omega_n}\Gamma_r(-x,-\xi)\,dF^{*[r]-}(\xi),$$

wo $F^{*[r]+}(\xi)$ bzw. $F^{*[r]-}(\xi)$ die positive, bzw. die negative Schwankung von $F^{*[r]}(\xi)$ in $(0,\xi)$ bedeuten. Wir setzen

$$\Pi_{n,r}^{(1)}(x,\xi)=\begin{cases}P_{n,r}(x,\xi)\\ p_{n,r}(x,\xi)\end{cases},\quad \Pi_{n,r}^{(2)}(x,\xi)=\begin{cases}p_{n,r}(x,\xi) & \text{für gerade } r,\\ P_{n,r}(x,\xi) & \text{für ungerade } r;\end{cases}\tag{39}$$

und

$$p_n^*(x)=\frac{(-1)^{r+1}}{r!}\int_0^{\omega_n}\Pi_{n,r}^{(1)}(-x,-\xi)\,dF^{*[r]+}(\xi)+\frac{(-1)^r}{r!}\int_0^{\omega_n}\Pi_{n,r}^{(2)}(-x,-\xi)\,dF^{*[r]-}(\xi),$$

(40)

$$P_n^*(x)=\frac{(-1)^{r+1}}{r!}\int_0^{\omega_n}\Pi_{n,r}^{(2)}(-x,-\xi)\,dF^{*[r]+}(\xi)+\frac{(-1)^r}{r!}\int_0^{\omega_n}\Pi_{n,r}^{(1)}(-x,-\xi)\,dF^{*[r]-}(\xi).$$

Infolge des letzten Teiles von Hilfssatz 1 sind alle hier auftretenden Integrale sinnvoll und es sind $p_n^*(x)$ und $P_n^*(x)$ Polynome niedriger als $2n-1$-ten Grades. Aus (38) und (39) bzw. aus (38) und (40) schließen wir mit Hilfe von (6)

$$p_n^*(x) \leqq F^*(x) \leqq P_n^*(x) \qquad (-\infty < x < +\infty). \tag{41}$$

Aus (7), (39), (40) und (2) folgt weiter

$$\int_{-\infty}^{+\infty} [P_n^*(x)-p_n^*(x)]e^{-x^2}\,dx =$$

$$= \frac{1}{r!}\int_0^{\omega_n}\left\{\int_{-\infty}^{+\infty}[P_{n,r}(x,-\xi)-p_{n,r}(x,-\xi)]e^{-x^2}\,dx\right\}[dF^{*[r]+}(\xi)+dF^{*[r]-}(\xi)].$$

Unter Beachtung von (7) und (2) folgt

$$\int_{-\infty}^{+\infty} [P_n^*(x)-p_n^*(x)]e^{-x^2}\,dx \leqq \tag{42}$$

$$\leqq \frac{1}{r!}c_r n^{-\frac{r+1}{2}}\int_0^{\omega_n} e^{-\xi^2}|dF^{*[r]}(\xi)| \leqq \frac{c_r}{r!}n^{-\frac{r+1}{2}}M_r(F).$$

(41) und (42) zeigen die Gültigkeit unseres Satzes für den Summanden $F^*(x)$ von $F(x)$.

b) *Einseitige Approximation von* $F^{**}(x)$.

Aus (37) und (1) erhalten wir für $x \geqq \omega_n$

$$|F^{**}(x)| \leqq A+2^{2s-1}B\omega_n^{2s}+2^{2s-1}B(x-\omega_n)^{2s}+$$

$$+\sum_{\nu=0}^{r-1}\frac{|F^{(\nu)}(\omega_n)|}{\nu!}(x-\omega_n)^\nu+\frac{|F^{[r]}(\omega_n)|}{r!}(x-\omega_n)^r.$$

Da $F^{**}(x)$ für $x \leqq \omega_n$ verschwindet, folgt weiter

$$|F^{**}(x)| \leqq \left(A+\frac{B}{2}n^s\right)\Gamma_0(x,\omega_n)+2^{2s-1}B\cdot\Gamma_{2s}(x,\omega_n)+ \tag{43}$$

$$+\sum_{\nu=0}^{r-1}\frac{|F^{(\nu)}(\omega_n)|}{\nu!}\Gamma_\nu(x,\omega_n)+\frac{|F^{[r]}(\omega_n)|}{r!}\Gamma_r(x,\omega_n)\overset{\text{def}}{=}T_r(x,\omega_n) \qquad (-\infty < x < +\infty),$$

so daß für $n > 2s$ das Polynom höchtens $2n-1$-ten Grades

$$P_n^{**}(x) = \left(A+\frac{B}{2}n^s\right)P_{n,0}(x,\omega_n)+2^{2s-1}B\cdot P_{n,2s}(x,\omega_n)+ \tag{44}$$

$$+\sum_{\nu=0}^{r-1}\frac{|F^{(\nu)}(\omega_n)|}{\nu!}P_{n,\nu}(x,\omega_n)+\frac{|F^{[r]}(\omega_n)|}{r!}P_{n,r}(x,\omega_n)$$

die Beziehung

(45) $$-P_n^{**}(x) \leqq F^{**}(x) \leqq P_n^{**}(x) \quad (-\infty < x < +\infty)$$

befriedigt. Es gilt dann wegen (43), (44), (6) und (7)

$$\int_{-\infty}^{+\infty} P_n^{**}(x)e^{-x^2}\,dx = \int_{-\infty}^{+\infty} [P_n^{**}(x) - T_r(x, \omega_n)]e^{-x^2}\,dx + \int_{\omega_n}^{+\infty} T_r(x, \omega_n)e^{-x^2}\,dx \leqq$$

$$\leqq e^{-\omega_n^2}\left[\left(A + \frac{B}{2}n^s\right)n^{-1/2} + 2^{2s-1}B \cdot n^{-s-1/2} + \sum_{\nu=0}^{r-1} \frac{|F^{(\nu)}(\omega_n)|}{\nu!} n^{-\frac{\nu+1}{2}} + \right.$$

$$\left. + \frac{|F^{[r]}(\omega_n)|}{r!} n^{-\frac{r+1}{2}}\right] + \left(A + \frac{B}{2}n^s\right)\int_{\omega_n}^{+\infty} e^{-x^2}\,dx + 2^{2s-1}B \cdot \int_{\omega_n}^{+\infty} (x-\omega_n)^{2s}e^{-x^2}\,dx +$$

$$+ \sum_{\nu=0}^{r-1} \frac{|F^{(\nu)}(\omega_n)|}{\nu!} \int_{\omega_n}^{+\infty} (x-\omega_n)^\nu e^{-x^2}\,dx + \frac{|F^{[r]}(\omega_n)|}{r!} \int_{\omega_n}^{+\infty} (x-\omega_n)^r e^{-x^2}\,dx.$$

Aus den Hilfssätzen 2 und 3 erhalten wir endlich

$$\int_{-\infty}^{+\infty} P_n^{**}(x)e^{-x^2}\,dx = O\left(n^{-\frac{r+1}{2}}\right).$$

Dies zeigt die Gültigkeit des Satzes für den Teil $F^{**}(x)$ von $F(x)$, u. zw. mit der Wahl $p_n^{**}(x) = -P_n^{**}(x)$. Den Satz selbst erhalten wir endlich indem wir

$$p_n(x) = \sum_{\nu=0}^{r-1} \frac{F^{(\nu)}(\omega_n)}{\nu!}(x-\omega_n)^\nu + \frac{F^{[r]}(\omega_n)}{r!}(x-\omega_n)^r + p_n^*(x) + p_n^{**}(x)$$

und

$$P_n(x) = \sum_{\nu=0}^{r-1} \frac{F^{(\nu)}(\omega_n)}{\nu!}(x-\omega_n)^\nu + \frac{F^{[r]}(\omega_n)}{r!}(x-\omega_n)^r + P_n^*(x) + P_n^{**}(x)$$

setzen.

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(Eingegangen am 30. Januar 1969)

Operator conjugation with respect to symmetric and skew-symmetric forms

By J. BOGNÁR in Budapest

Let K be the real or complex number field, E the n-dimensional $(1 \leqq n < \infty)$ vector space over K, and B the set of all linear mappings of E into itself. The elements of B will be termed *operators*.

We recall that a *sesquilinear form* on E is a function $\varphi(x, y)$ of two variables $x, y \in E$ with values in K such that

$$\varphi(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 \varphi(x_1, y) + \alpha_2 \varphi(x_2, y)$$

and

$$\varphi(y, \alpha_1 x_1 + \alpha_2 x_2) = \bar{\alpha}_1 \varphi(y, x_1) + \bar{\alpha}_2 \varphi(y, x_2)$$

for every $\alpha_1, \alpha_2 \in K$; $x_1, x_2, y \in E$. The sesquilinear form φ is said to be *non-degenerate* if $\varphi(x, y) = 0$ for every y implies $x = 0$, and $\varphi(x, y) = 0$ for every x implies $y = 0$. The sesquilinear form φ is said to be *symmetric* if $\varphi(x, y) = \overline{\varphi(y, x)}$, and *skew-symmetric* if $\varphi(x, y) = -\overline{\varphi(y, x)}$ for every x, y in E.

Let $\varphi(x, y)$ be a symmetric or skew-symmetric non-degenerate sesquilinear form on E. It is well-known (see e.g. [1], section 99) that to every $T \in B$ there is a uniquely defined $T^* \in B$ such that

$$\varphi(Tx, y) = \varphi(x, T^* y) \qquad (x, y \in E;\ T \in B). \tag{1}$$

The operator T^* is called the *adjoint* of T with respect to φ. The mapping $T \to T^*$ $(T \in B)$ has the following properties:

$$(\alpha_1 T_1 + \alpha_2 T_2)^* = \bar{\alpha}_1 T_1^* + \bar{\alpha}_2 T_2^* \quad (\alpha_1, \alpha_2 \in K;\ T_1, T_2 \in B), \tag{2}$$

$$(T_1 T_2)^* = T_2^* T_1^* \quad (T_1, T_2 \in B), \tag{3}$$

$$T^{**} = T \quad (T \in B). \tag{4}$$

In the present note it will be shown that any mapping of B into itself satisfying the conditions (2)—(4) can be obtained in this way.

The idea was suggested us by E. FRIED's paper [2] where the case of a symmetric $\varphi(x, y)$ with *definite* $\varphi(x, x)$ (i.e. $\varphi(x, x) = 0$ only if $x = 0$) is treated.

The "existence" part of our result is implicitly contained in the concluding remarks of [2]. However, the question of uniqueness and the separate characterization of the symmetric and skew-symmetric case do not appear there. Our proof of existence seems to be not quite different from FRIED's one, but we need neither an *a priori* given inner product nor the characterization of operators commuting with every element of B.

Theorem. *Let $T \to T^*$ $(T \in B)$ be a mapping of B into itself with the properties* (2), (3), (4). *If $K=C$, the field of complex numbers, then there exist both a symmetric and a skew-symmetric non-degenerate sesquilinear form φ on E satisfying the relation* (1). *If $K=R$, the field of real numbers, then there exists either a symmetric or a skew-symmetric non-degenerate sesquilinear form φ on E satisfying* (1); *the form φ will be symmetric if and only if there is an operator $T_0 \in B$ such that*

$$\text{rank } T_0 = 1, \qquad T_0^* T_0 \neq 0. \tag{5}$$

In each of the above cases φ is, up to a real non-zero factor, uniquely determined.

Proof. If there is an operator T_0 with properties (5), we choose a basis $e_1, \dots, e_n$ of E such that

$$T_0 e_k = 0 \qquad (k=2, \dots, n). \tag{6}$$

In the opposite case let $e_1, \dots, e_n$ be any basis of E.

We define n^2 operators P_{jk} setting

$$P_{jk} e_r = \delta_{kr} e_j \qquad (j, k, r=1, \dots, n). \tag{7}$$

Turning our attention to the mapping $T \to T^*$ $(T \in B)$ we observe that

$$T^*=0 \quad \text{if and only if} \quad T=0. \tag{8}$$

Indeed, according to (2), $0^*=(2 \cdot 0)^*=2 \cdot 0^*$ i.e. $0^*=0$. Moreover, in view of (4), $T^*=0$ implies $T=T^{**}=(T^*)^*=0^*=0$.

Since by the definition (7)

$$P_{jk} \neq 0, \quad P_{jk} P_{rs} = \delta_{kr} P_{js} \quad (j, k, r, s = 1, \dots, n), \tag{9}$$

from (8) and (3) we infer

$$P_{jk}^* \neq 0, \quad P_{rs}^* P_{jk}^* = \delta_{kr} P_{js}^* \quad (j, k, r, s=1, \dots, n). \tag{10}$$

We will construct and study the required sesquilinear form φ by the aid of the basis $e_1, \dots, e_n$ and a new basis $f_1, \dots, f_n$. In order to define the latter, we need the operators P_{jk}^* and an operator S (resp. W) of rank 1 such that $S^*=S$ $(W^*=-W)$.

Let the relations (5) be satisfied by some $T_0 \in B$. We may assume that (6) holds, too. Setting $S=T_0^* T_0$ we have

$$S^*=S, \quad S \neq 0, \quad S e_k = 0 \qquad (k=2, \dots, n). \tag{11}$$

If (5) cannot be satisfied then in particular

$$P_{11}^*P_{11}=0, \quad \text{and} \quad P_{k1}^*P_{k1}=0, \quad (P_{11}+P_{k1})^*(P_{11}+P_{k1}) = 0 \qquad (k=2, \ldots, n).$$

Hence, in view of (2),

$$P_{11}^*P_{k1}+P_{k1}^*P_{11} = 0 \qquad (k=2, \ldots, n).$$

But for some $k\neq 1$ the operator $W=P_{11}^*P_{k1}$ is different from zero. In fact, otherwise it would follow that

$$P_{11}^*e_k=P_{11}^*P_{k1}e_1=0 \qquad (k=1, \ldots, n)$$

i.e. $P_{11}^*=0$. Thus we have

(12) $$W^* = -W, \quad W\neq 0, \quad We_k=0 \qquad (k=2, \ldots, n).$$

Let

(13) $$f_1 = \begin{cases} Se_1 & \text{if (5) can be fulfilled,} \\ We_1 & \text{otherwise,} \end{cases}$$

and let

(14) $$f_k=P_{1k}^*f_1 \qquad (k=2, \ldots, n).$$

The validity of (14) can be extended to $k=1$. Really, if $f_1=Se_1$, then $P_{11}^*f_1= =P_{11}^*Se_1=P_{11}^*S^*e_1=(SP_{11})^*e_1=S^*e_1=Se_1=f_1$; in the case $f_1=We_1$ a similar argument holds. Making use of this fact and of formula (10) we obtain:

(15) $$P_{jk}^*f_r=\delta_{jr}f_k \qquad (j, k, r=1, \ldots, n).$$

Assume that $\sum_{r=1}^{n} \alpha_r f_r=0$ for some $\alpha_1, \ldots, \alpha_n\in K$. Then $P_{j1}^* \sum_{r=1}^{n} \alpha_r f_r=0$, so that by (15) $\alpha_j f_1=0$ $(j=1, \ldots, n)$. On account of (11)—(13) $f_1\neq 0$. Hence $\alpha_1=\alpha_2= =\cdots=\alpha_n=0$. Consequently, $f_1, \ldots, f_n$ constitute a basis of E.

Now let $\varphi(x, y)$ be a non-degenerate sesquilinear form on E which satisfies the relation (1). Then necessarily

$$\varphi(e_j, f_k) = \varphi(P_{j1}e_1, f_k) = \varphi(e_1, P_{j1}^*f_k) = \delta_{jk}\varphi(e_1, f_1) \qquad (j, k = 1, \ldots, n),$$

and

$$\varphi\left(\sum_{j=1}^{n} \mu_j e_j, \sum_{k=1}^{n} \nu_k f_k\right) = \sum_{j,k=1}^{n} \mu_j \bar{\nu}_k \varphi(e_j, f_k) =$$

$$= \sum_{k=1}^{n} \mu_k \bar{\nu}_k \varphi(e_1, f_1) \qquad (\mu_k, \nu_k\in K;\ k = 1, \ldots, n),$$

where

$$\varphi(e_1, f_1)\neq 0.$$

Therefore if φ_1, φ_2 are two non-degenerate sesquilinear forms with the property

$$\varphi_k(Tx, y)=\varphi_k(x, T^*y) \qquad (x, y\in E;\ T\in B;\ k=1, 2),$$

then $\varphi_1=\lambda\varphi_2$, where

$$\lambda = \frac{\varphi_1(e_1, f_1)}{\varphi_2(e_1, f_1)} \tag{16}$$

is a non-zero (till now possibly complex) number.

Since a real multiple of a symmetric (skew-symmetric) form is symmetric (skew-symmetric), and a non-degenerate form cannot be symmetric and skew-symmetric at the same time, for $K=R$ we additionally find that the cases where φ can be chosen. symmetric or skew-symmetric, respectively, must be mutually disjoint.

If $K=C$ and both of the forms φ_1, φ_2 are required to be symmetric (resp. skew-symmetric), then the value of λ in (16) must be real. As a matter of fact, the relations

$$\varphi_k(e_1, f_1) = \varepsilon\overline{\varphi_k(f_1, e_1)} \qquad (\varepsilon = \pm 1;\ k = 1, 2)$$

imply $\lambda=\bar{\lambda}$.

Conversely, it is easy to see that for any fixed real non-zero number γ the formula

$$\varphi\left(\sum_{j=1}^{n}\mu_j e_j, \sum_{k=1}^{n}\nu_k f_k\right) = \gamma\sum_{k=1}^{n}\mu_k\bar{\nu}_k \qquad (\mu_k, \nu_k\in K;\ k = 1, \dots, n) \tag{17}$$

defines a non-degenerate sesquilinear form on E. Moreover, in view of (7) and (15) we have

$$\varphi(P_{jk}e_r, f_s) = \delta_{kr}\,\varphi(e_j, f_s) = \delta_{kr}\delta_{js}\gamma,$$

$$\varphi(e_r, P^*_{jk}f_s) = \delta_{js}\varphi(e_r, f_k) = \delta_{js}\delta_{kr}\gamma$$

i.e.

$$\varphi(P_{jk}e_r, f_s) = \varphi(e_r, P^*_{jk}f_s) \qquad (j, k, r, s = 1, \dots, n),$$

so that making use of the linearity of P_{jk} and the sequilinearity of φ we obtain

$$\varphi(P_{jk}x, y) = \varphi(x, P^*_{jk}y) \qquad (x, y\in E;\ j, k = 1, \dots, n).$$

Taking into account that any $T\in B$ is a linear combination of the operators P_{jk}, the relation (1) follows.

If $f_1=Se_1$ (cf. (13)), then the relations (1), (11), (17), (15), (4) and (7) yield:

$$\varphi(f_1, e_1) = \varphi(Se_1, e_1) = \varphi(e_1, Se_1) = \varphi(e_1, f_1) = \gamma,$$

$$\varphi(f_k, e_j) = \varphi(P^*_{1k}f_1, e_j) = \varphi(f_1, P_{1k}e_j) = \delta_{kj}\varphi(f_1, e_1) = \delta_{kj}\gamma,$$

$$\varphi\left(\sum_{k=1}^{n}\nu_k f_k, \sum_{j=1}^{n}\mu_j e_j\right) = \sum_{j,k=1}^{n}\nu_k\bar{\mu}_j\varphi(f_k, e_j) = \sum_{k=1}^{n}\nu_k\bar{\mu}_k\gamma.$$

Thus, in view of (17), φ is symmetric.

If $f_1 = We_1$ (cf. (13)) then, by virtue of (12), φ turns out to be skew-symmetric.

Finally, let $K=C$, and let φ be a symmetric (skew-symmetric) non-degenerate sesquilinear form satisfying the relation (1). Then $\varphi_1 = i\varphi$ is a skew-symmetric (resp. symmetric) non-degenerate sesquilinear form satisfying (1).

Added in proof. Professor KLAUS VALA kindly called my attention to the fact that FRIED's result referred to in the introduction is a special case of a theorem of Mackey and Kakutani (cf. C. E. RICKART, *General theory of Banach algebras*, Princeton—Toronto—London—New York, 1960; p. 265), where operators on a Banach space of arbitrary dimension are considered. The two proofs, however, seem to have nothing in common.

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(Received March 27, 1969)

J-unitary dilation of a general operator

By CHANDLER DAVIS in Toronto (Canada)[1])

According to a well-known theorem of SZ.-NAGY [13], [14, Thm. I. 4. 2], every contraction on Hilbert space has a unitary dilation on a larger space (and also has an extension on a larger space which is the adjoint of an isometry [14, Thm. I. 4. 1.). In this paper, by a modification of familiar methods, the corresponding result is obtained for an arbitrary closed densely-defined operator. The conclusion is different in that the dilation is only J-unitary (and the extension is only the adjoint of a J-isometry, or the J-adjoint of a J-isometry).

It is a pleasure to thank B. SZ.-NAGY and E. DURSZT for conversations which inspired this investigation, and C. FOIAŞ for suggestions which led to substantial improvements upon the first version.

1. Definitions

The subject will be a closed operator T whose domain $\boldsymbol{D}(T)$ is a dense linear set in a Hilbert space $\mathfrak{H}$. The inner product of $\mathfrak{H}$ will be denoted by $(\,,)$. I will construct later a Hilbert space $\mathfrak{K}$ of which $\mathfrak{H}$ is a linear subspace; the inner product of $\mathfrak{K}$ will be an extension of that of $\mathfrak{H}$, and will also be denoted by $(\,,)$. The orthoprojector on $\mathfrak{K}$ onto $\mathfrak{H}$ will be denoted by $P_{\mathfrak{H}}$. I will also construct an operator U, closed and densely defined in $\mathfrak{K}$, which is a "dilation" of T; this means that

$$(1.\,1)\qquad T^n = P_{\mathfrak{H}} U^n|_{\mathfrak{H}} \quad \text{and} \quad T^{*n} = P_{\mathfrak{H}} U^{-n}|_{\mathfrak{H}} \qquad (n = 1, 2, \ldots).$$

In addition $\mathfrak{K}$ will be a "J-space". This means [7] that the Hilbert space $\mathfrak{K}$ will have associated with it a canonical symmetry J, i.e., a fixed unitary hermitian operator J. In any J-space one considers along with J the complementary orthoprojectors J^+ and J^-. (I will use the notations A^+ and A^- for the positive part and the negative part of an arbitrary self-adjoint operator A [11, § 108].) It is often assumed that the ranges $\boldsymbol{R}(J^+)$ and $\boldsymbol{R}(J^-)$ are both non-zero, but here that is not

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[1]) This research was done while the author was in Szeged on a Senior Research Fellowship of the National Research Council of Canada.

assumed. In terms of J, a new continuous hermitian sesquilinear form is defined by

$$[k, k'] = (Jk, k') \qquad (k, k' \in \mathfrak{K}).$$

Unlike the inner product $(,)$, the "J-product" $[,]$ need not be definite; in particular, $[k, k] > 0$ for non-zero $k \in \boldsymbol{R}(J^+)$, while $[k, k] < 0$ for non-zero $k \in \boldsymbol{R}(J^-)$. For this reason J-spaces are also called Hilbert spaces with indefinite metric; but do not be misled. The norm is defined in terms of the definite inner product, not the J-product, and topological notions are defined in terms of the norm. The "J-adjoint" of any A is JA^*J.

A "J-isometric" operator U is a closed, densely defined operator which preserves the J-product:

(1. 2) $$[Uk, Uk'] = [k, k'] \qquad (k, k' \in \boldsymbol{D}(U)).$$

A J-isometry U is called "J-unitary" in case it has a densely defined inverse, which then is necessarily J-isometric as well. The terminology and notation of unbounded operators are used because the operator U which appears below really can be unbounded. This has obliged me to depart from the usual terminology [7], in which J-unitary operators are by definition bounded. (IOHVIDOV [5], [6] studies unbounded J-isometries, but in quite different context.)

2. The main lemma

Let $|T|$ denote $(T^*T)^{1/2}$, a self-adjoint operator with $\boldsymbol{D}(|T|) = \boldsymbol{D}(T)$, $\boldsymbol{R}(|T|) = \boldsymbol{R}(T^*)$; similarly $|T^*| = (TT^*)^{1/2}$. Let W denote the unique partial isometry such that $T = W|T| = |T^*|W$, $\boldsymbol{R}(W) = \overline{\boldsymbol{R}(T)}$, $\boldsymbol{R}(W^*) = \overline{\boldsymbol{R}(T^*)}$ [10].

It will be useful to have special notation and terminology for some operators and subspaces derived from these, which will figure prominently in the construction. Let $|T| = \int_0^\infty \lambda \, dE(\lambda)$ be the spectral resolution of $|T|$; $|T^*| = \int_0^\infty \lambda \, dF(\lambda)$, that of $|T^*|$. Define

$$J_T = \operatorname{sgn}(1 - T^*T) = \int_0^\infty \operatorname{sgn}(1 - \lambda^2) \, dE(\lambda),$$

$$Q_T = (|1 - T^*T|)^{1/2} = \int_0^\infty \sqrt{|1 - \lambda^2|} \, dE(\lambda),$$

$$D_T = ((1 - T^*T)^+)^{1/2} = \int_0^1 \sqrt{1 - \lambda^2} \, dE(\lambda),$$

$$X_T = ((1 - T^*T)^-)^{1/2} = \int_1^\infty \sqrt{\lambda^2 - 1} \, dE(\lambda),$$

which are self-adjoint operators. Clearly J_T and D_T are bounded and everywhere defined. As for the possibly unbounded operators $|T|$, Q_T, and X_T, they differ by operators which are bounded, and so they have the same domain (which is $\boldsymbol{D}(T)$).

In case T is a contraction, J_T^- and X_T are O, while Q_T is equal to D_T, the same operator denoted by that symbol in [14].

In spite of the choice of letter, J_T is not quite suitable for defining a J-product on $\mathfrak{H}$, since it can have a null-space. Its role in the eventual construction of the J-space $\mathfrak{K}$ will be less direct.

It is obvious that $J_T Q_T^2 = Q_T^2 J_T = 1 - T^* T$. It is less immediate, but worth noting, that

$$(2.1) \qquad (J_T Q_T h, Q_T l) = (h, l) - (Th, Tl) \qquad (h, l \in \boldsymbol{D}(T)).$$

To see this, we may introduce $|||\ |||$, the "graph norm for T", defined by

$$|||h|||^2 = \|h\|^2 + \|Th\|^2 \qquad (h \in \boldsymbol{D}(T)).$$

It makes $\boldsymbol{D}(T)$ into a (complete) Hilbert space, in which it is easy to prove that $\boldsymbol{D}(T^* T)$ is dense. But with respect to this norm, both sides of (2. 1) are continuous functions of h and l; and (2. 1) does hold for h and l belonging to $\boldsymbol{D}(T^* T)$; therefore it must hold in general.

Interchanging T with T^* and $E(\ )$ with $F(\ )$, we get operators J_{T^*}, Q_{T^*}, D_{T^*}, X_{T^*}, with properties corresponding.

Lemma. *Let the symbol* $\blacksquare$ *stand for either J, Q, D, or X. We have* $T\blacksquare_T = \blacksquare_{T^*} T$.

In the case $TD_T = D_{T^*} T$, this relation has been crucial in unitary dilation theory since its beginnings [5], and it continues its role here.

Proof. Each relation to be proved has the form

$$(2.2) \qquad W \int_0^\infty f(\lambda)\, dE(\lambda) = \left(\int_0^\infty f(\lambda)\, dF(\lambda) \right) W$$

for some piecewise-continuous function f vanishing at 0: for example, in the case of the equation $TX_T = X_{T^*} T$, take $f(\lambda) = \chi_{[1,\infty]}(\lambda) \cdot \lambda \sqrt{\lambda^2 - 1}$. Now the fact that $W|T| = |T^*| W$, with which we began, implies that $WE(\lambda) = F(\lambda) W$ for all λ if $E(\ )$ and $F(\ )$ are normalized in the same way. Using the criterion $\int_0^\infty f(\lambda)^2 \, d(E(\lambda)h, h) < \infty$ for a vector h to belong to the domain of $\int_0^\infty f(\lambda)\, dE(\lambda)$ and using the properties of W, we see easily that the two sides of (2. 2) have the same domain, and agree on the domain.

This proves the Lemma as stated, but it is worth noting the somewhat more delicate fact that (with the same notation)

$$(2.3) \qquad (\blacksquare_T h, T^* l) = (Th, \blacksquare_{T^*} l) \qquad (h \in \boldsymbol{D}(T), l \in \boldsymbol{D}(T^*)).$$

For J_T and D_T this does follow from the Lemma. For the other two cases, we may either use approximation in the graph norm as for (2. 1), or else use the fact that Q_T resp. X_T differs from $|T|$ by a bounded operator, which reduces (2. 3) rather quickly to the Lemma.

The operators J_T, J_{T^*}, Q_T, Q_{T^*} are all that are needed to prove the main theorem; the others will be used only for the discussion of the geometry of the dilation space, which will follow in § 4.

3. The dilation

Theorem. *Given any closed, densely defined operator T in $\mathfrak{H}$, there exists a Hilbert space $\mathfrak{K} \supseteq \mathfrak{H}$ and there exists a closed, densely defined operator U in $\mathfrak{K}$, with the following properties:*

(a) *$\mathfrak{K}$ is a J-space, with $\mathfrak{H} \subseteq J^+(\mathfrak{K})$* (i.e., $[h, l] = (h, l)$ for $h, l \in \mathfrak{H}$);

(b) *U is J-unitary, that is,* (1. 2) *holds and U^{-1} is densely defined;*

(c) *U is a dilation of T, that is,* (1. 1) *holds;*

(d) $\vee\{U^n \mathfrak{H}: n = 0, \pm 1, \pm 2, \ldots\} = \mathfrak{K}$.

(In stating (d), and occasionally below, I use an expression like $U\mathfrak{H}$ as short-hand for $U(\mathfrak{H} \cap \boldsymbol{D}(U))$.)

The construction follows quite closely that of Schäffer, as sharpened subsequently [14, I. 5]. I begin (as Schäffer did) with a space $\mathfrak{K}^0$, somewhat larger than desired but easy to describe: it is the direct sum of countably many isometric copies of $\mathfrak{H}$ (with the usual inner product). One of these copies I will identify at once with $\mathfrak{H}$. The unitary application of each copy onto the next one in order will be denoted by S; thus I write

$$(3.1) \qquad \mathfrak{K}^0 = \cdots \oplus S^{-2}\mathfrak{H} \oplus S^{-1}\mathfrak{H} \oplus \mathfrak{H} \oplus S\mathfrak{H} \oplus S^2\mathfrak{H} \oplus \cdots.$$

S will have a role as an operator (a bilateral shift of multiplicity dim $\mathfrak{H}$ acting on $\mathfrak{K}$) and as a device for indexing the component subspaces of $\mathfrak{K}^0$ in (3. 1). For any $k \in \mathfrak{K}^0$, k_i will mean the component of k in $S^i\mathfrak{H}$. (Thus, for instance, $(Sk)_i = Sk_{i-1}$.)

Now to define J, I take the diagonal operator-matrix

$$(3.2)\qquad \left[\begin{array}{ccc|c|ccc} \ddots & & & & & & \\ & J_{T^*} & & & & & \\ & & J_{T^*} & & & & \\ \hline & & & 1 & & & \\ \hline & & & & J_T & & \\ & & & & & J_T & \\ & & & & & & \ddots \end{array}\right].$$

That is, for an element $S^n h$ $(h \in \mathfrak{H})$ I define

$$J(S^n h) = \begin{cases} S^n(J_{T^*} h) & (n < 0), \\ h & (n = 0), \\ S^n(J_T h) & (n > 0). \end{cases}$$

This extends to a self-adjoint operator J on $\mathfrak{K}^0$ such that the space is spanned by its eigenspaces belonging to (at most) the three eigenvalues $0, \pm 1$. Define $\mathfrak{K} = \boldsymbol{R}(J) \subseteqq \mathfrak{K}^0$ and $[k, k'] = (Jk, k')$. Obviously conclusion (a) of the Theorem holds.

The operator U will be, essentially, an extension of

$$(3.3)\qquad V = \left[\begin{array}{cccc|c|cccc} \ddots & & & & & & & & \\ & O & & & & & & & \\ & 1 & O & & & & & & \\ & & 1 & O & & & & & \\ \hline & & & Q_{T^*} & T & & & & \\ \hline & & & -J_T T^* & Q_T & O & & & \\ & & & & & 1 & O & & \\ & & & & & & 1 & O & \\ & & & & & & & & \ddots \end{array}\right].$$

The simplest interpretation of (3.3) is that it defines an operator in $\mathfrak{K}^0$ for which

$$V(S^n h) = \begin{cases} S^{n+1} h & (n \neq -1, 0), \\ Q_{T^*} h \oplus - S J_T T^* h \in \mathfrak{H} \oplus S\mathfrak{H} & (n = -1;\ h \in \boldsymbol{D}(T^*)), \\ Th \oplus S Q_T h \in \mathfrak{H} \oplus S\mathfrak{H} & (n = 0;\ h \in \boldsymbol{D}(T)), \end{cases}$$

and which is then to be extended by linearity to the finite linear combinations of the vectors $S^n h$.

So defined, V takes $\boldsymbol{D}(V) \cap \mathfrak{K}$ into $\mathfrak{K}$. To verify this, we treat $(Vk)_1$, the only component about which there is any possible doubt. By definition, $(Vk)_1 = S Q_T k_0 - S J_T T^* S k_{-1}$. This is in $S(\boldsymbol{R}(J_T))$ as desired because Q_T and J_T, being by their definition self-adjoint operators with the same null-space, have $\overline{R(Q_T)} = \overline{R(J_T)}$ $(=R(J_T))$. Therefore we are free to treat (3. 3) as representing an operator with domain in $\mathfrak{K}$.

Next it must be proved that the operator V possesses a closure [10]. Suppose not; then there exists a sequence $(k^{(n)})$ in $D(V)$ such that $k^{(n)} \to 0$ but $Vk^{(n)}$ approaches a non-zero limit. It is easy to see that either $(Vk^{(n)})_0$ or $(Vk^{(n)})_1$ approaches a non-zero limit; the argument goes the same in either case, I will write it for the first alternative. We have $(Vk^{(n)})_0 = Tk_0^{(n)} + Q_{T*}Sk_{-1}^{(n)}$ by definition, where $(k_0^{(n)})$ is a sequence of vectors in $D(T)$, tending to 0 as $n\to\infty$, and $(h'^{(n)}) = (Sk_{-1}^{(n)})$ is a sequence of vectors in $D(T^*)$, tending to 0 as $n \to \infty$; but where the sequence $((Vk^{(n)})_0)$ approaches a non-zero limit. We can write $Tk_0^{(n)} = |T^*|h^{(n)}$, where $(h^{(n)}) = (Wk_0^{(n)})$ is a sequence of elements of $D(T^*)$ approaching 0 as $n\to\infty$. Now

$$(Vk^{(n)})_0 = |T^*|h^{(n)} + Q_{T*}h'^{(n)} = |T^*|(h^{(n)} + h'^{(n)}) + (Q_{T*} - |T^*|)h'^{(n)},$$

and because $Q_{T*} - |T^*|$ is bounded, the last term tends to 0 as $n\to\infty$. Therefore $|T^*|(h^{(n)} + h'^{(n)})$ tends to a non-zero limit. But this contradicts the fact that $|T^*|$ is a closed operator.

We may, therefore, extend V to a minimal closed linear operator U; its domain is clearly dense. It is also clear that $U(S^{-1}\mathfrak{H} \oplus \mathfrak{H}) \subseteqq \mathfrak{H} \oplus S\mathfrak{H}$.

As to the first equation in (1. 1), clearly T^n is an extension of $P_{\mathfrak{H}}U^n|\mathfrak{H}$. It is also easy to see that $D(U^n) \cap \mathfrak{H} = D(T^n)$.

The other half of (1. 1) is less apparent, and will be deferred.

The main formal idea in the construction comes out in verifying that U is J-isometric, that is, in verifying (1. 2).

In doing this, we may restrict attention to vectors in $D(V)$ — thus to k such that $k_0 \in D(T)$ and $k_{-1} \in S^{-1}(D(T^*))$. This is by an argument already invoked in § 2: namely, if $D(U)$ is considered in the graph norm for U, then $D(V)$ is dense in it and both sides of (1. 2) are continuous in k and k'. Also, by the usual polarization argument, it is enough to prove (1. 2) for $k = k'$. Define l by

$$k = k_{-1} \oplus k_0 \oplus l;$$

this decomposition is both orthogonal and J-orthogonal. It is obvious that the transformation $l \to Ul = Sl$ preserves both the inner product and the J-product. It is also obvious that $U(k_{-1} \oplus k) \in \mathfrak{H} \oplus S\mathfrak{H}$ is both orthogonal and J-orthogonal to Ul. Therefore it is enough to prove (1. 2) for $l = 0$. The right-hand member is then

$$[k, k] = (J(k_{-1} \oplus k_0), k_{-1} \oplus k_0) = (J_{T*}Sk_{-1}, Sk_{-1}) + (k_0, k_0).$$

The left-hand member of (1. 2) is

(3. 4) $$[Uk, Uk] = (J((Uk)_0 \oplus (Uk)_1), (Uk)_0 \oplus (Uk)_1)$$
$$= ((Uk)_0, (Uk)_0) + (J_T S^{-1}(Uk)_1, S^{-1}(Uk)_1)$$
$$= (Q_{T^*}Sk_{-1} + Tk_0, Q_{T^*}Sk_{-1} + Tk_0) +$$
$$+ (J_T(-J_T T^* Sk_{-1} + Q_T k_0), -J_T T^* Sk_{-1} + Q_T k_0).$$

The terms $(Tk_0, Tk_0) + (J_T Q_T k_0, Q_T k_0)$ add to (k_0, k_0) by (2. 1). The terms

$$2\,\mathrm{Re}\,(Q_{T^*}Sk_{-1}, Tk_0) + 2\,\mathrm{Re}\,(-J_T^2 T^* Sk_{-1}, Q_T k_0) =$$
$$= 2\,\mathrm{Re}\,\{(Q_{T^*}Sk_{-1}, Tk_0) - (T^* Sk_{-1}, Q_T k_0)\}$$

add to zero essentially by the Lemma — strictly, by (2. 3). The remaining terms in (3. 4) are equal to

$$(Q_{T^*}Sk_{-1}, Q_{T^*}Sk_{-1}) + (T^* Sk_{-1}, J_T T^* Sk_{-1}).$$

I apply (2. 1) (with T and T^* interchanged) to this expression, taking as the vectors h, l in (2. 1) the vectors $J_{T^*} Sk_{-1}$, $Sk_{-1} \in \boldsymbol{D}(T^*)$. It becomes

$$(Q_{T^*}J_{T^*}h, Q_{T^*}l) + (T^* J_{T^*}h, J_T T^* l) = (J_{T^*}Q_{T^*}h, Q_{T^*}l) + (J_T T^* h, J_T T^* l) = (h, l),$$

by use also of the Lemma. That is,

$$[Uk, Uk] = (k_0, k_0) + (h, l) = (k_0, k_0) + (J_{T^*} Sk_{-1}, Sk_{-1}) = [k, k].$$

(1. 2) is established.

It is also easy to prove the "forward half" of (d). Define

(3. 5) $$\mathfrak{K}_+ = \mathfrak{H} \oplus SJ_T(\mathfrak{H}) \oplus S^2 J_T(\mathfrak{H}) \oplus \cdots .$$

Since $U\mathfrak{H} \supseteqq SQ_T\mathfrak{H}$, which is dense in $SJ_T\mathfrak{H} = S\mathfrak{H} \cap \mathfrak{K}$, and since for $n > 1$ we have $U^n\mathfrak{H} = S^{n-1}U\mathfrak{H}$, it is clear (remembering (3.2)) that $\vee\{U^n\mathfrak{H}: n = 0, 1, 2, \dots\} = \mathfrak{K}_+$.

The remaining arguments concern the inverse of U. We know U is one-one by (1. 2); because if $Uk = 0$ then (1. 2) shows that Jk is orthogonal to the dense set $\boldsymbol{D}(U)$, and, J having zero null-space, this forces $k = 0$. Therefore U^{-1} exists; we have to consider $\boldsymbol{D}(U^{-1}) = \boldsymbol{R}(U)$.

Now it is evident that $\boldsymbol{R}(U) \supseteqq \oplus\{S^n\mathfrak{H}: n \neq 0,1\}$, and that when we consider restrictions to this subspace, U^{-1} agrees with S^{-1}. Take $k = k_0 \oplus k_1$, with $k_0 \in \boldsymbol{D}(T^*)$ and $k_1 \in S(\boldsymbol{D}(T))$. Define

(3. 6) $$l = l_{-1} \oplus l_0 \quad (Sl_{-1} = J_{T^*}Q_{T^*}k_0 - J_{T^*}TS^{-1}k_1, \quad l_0 = T^* k_0 + Q_T J_T S^{-1} k_1).$$

I will show that $Ul = k$. Circumspection is needed with this l because it need not be in $\boldsymbol{D}(V)$. As in earlier arguments, let us approximate by elements from the appropriate domains. Let $(k^{(n)})$ be a sequence of elements of $\mathfrak{K}$ such that (i) $k^{(n)} = k_0^{(n)} \oplus k_1^{(n)}$; (ii) $(k_0^{(n)})$ is a sequence approaching k_0 in $\boldsymbol{D}(T^*)$ considered with the graph norm for T^*, while each $k_0^{(n)}$ lies in the dense set $\boldsymbol{D}(TT^*)$; (iii) similarly, in $\boldsymbol{D}(T)$ with the graph norm for T, $S^{-1}k_1^{(n)} \to S^{-1}k_1$ and $S^{-1}k_1^{(n)} \in \boldsymbol{D}(T^*T)$. In particular,

$k^{(n)} \to k$ in $\mathfrak{K}^0$. Now define $l^{(n)}$ in terms of $k^{(n)}$ by putting superscripts on equations (3. 6). By the definition of the graph norm we have also $l^{(n)} \to l$ in $\mathfrak{K}^0$. Furthermore $l^{(n)} \in \mathfrak{K}$, $l \in \mathfrak{K}$ by the definition of $\mathfrak{K}$. But $Ul^{(n)} = Vl^{(n)}$ can be computed from (3. 3). One obtains, using the Lemma,

$$(Vl^{(n)})_0 = Q_{T^*}(J_{T^*}Q_{T^*}k_0^{(n)} - J_{T^*}TS^{-1}k_1^{(n)}) + T(T^*k_0^{(n)} + Q_T J_T S^{-1}k_1^{(n)}) = k_0^{(n)},$$

and similarly $(Vl^{(n)})_1 = k_1^{(n)}$. Now because U is a closed extension of V, $Ul = k$.

The most immediate consequence is that $\boldsymbol{R}(U)$ is dense; this was all that was lacking to complete the proof of (b).

But we have showed in addition that U^{-1} is an extension of the operator

$$(3.\ 7)\qquad \left[\begin{array}{c|c|c} \begin{matrix}\ddots & & \\ & O & 1 \\ & & O\end{matrix} & J_{T^*}Q_{T^*} & -J_{T^*}T \\ \hline & T^* & Q_T J_T \\ \hline & & \begin{matrix} O & 1 & \\ & O & \\ & & \ddots\end{matrix} \end{array}\right],$$

which is interpreted similarly to (3. 3). By the same reasoning used in connection with (3. 5), we see that $\bigvee\{U^{-n}\mathfrak{H}: n=0, 1, 2, \ldots\} = \cdots \oplus S^{-2}J_{T^*}\mathfrak{H} \oplus S^{-1}J_{T^*}\mathfrak{H} \oplus \mathfrak{H}$, and this completes the proof of (d). We also see at once that $P_{\mathfrak{H}}U^{-n}|\mathfrak{H}$ is an extension of T^{*n} $(n=1, 2, \ldots)$.

To complete the proof of (1. 1), it remains to show that $\boldsymbol{D}(U^{-n}) \cap \mathfrak{H} = \boldsymbol{D}(T^{*n})$. It suffices, just as before, to check that when $k_1 = 0$, we have $k \in \boldsymbol{D}(U^{-1})$ if and only if $k_0 \in \boldsymbol{D}(T^*)$. Let, then, $k \in \boldsymbol{R}(U)$, $k_1 = 0$. It is easy to see that we may assume $k \in \mathfrak{H}$ (i.e., $k = k_0$) without loss of generality. Because U is the minimal closed extension of V, we may take $k = Ul$, with sequences $(l^{(n)})$, $(k^{(n)})$ having the properties $k^{(n)} = Ul^{(n)}$, $l^{(n)} \to l$, $k^{(n)} \to k$, $Sl_{-1}^{(n)} \in \boldsymbol{D}(T^*)$, $l_0^{(n)} \in \boldsymbol{D}(T)$. Now $S^{-1}k_1^{(n)} = -J_T T^* Sl_{-1}^{(n)} + Q_T l_0^{(n)}$ approaches $S^{-1}k_1 = 0$. From this we want to prove that

$$k_0 = \lim k_0^{(n)} = \lim (Q_{T^*}Sl_{-1}^{(n)} + Tl_0^{(n)})$$

is in $\boldsymbol{D}(T^*)$. Take any $h \in \boldsymbol{D}(T)$; it will be sufficient to prove that $(k_0, Th) = (l_0, h)$, and for this it will be enough to prove that

$$(3.\ 8)\qquad (k_0^{(n)}, Th) - (l_0^{(n)}, h) \to 0.$$

Substituting the expression for $k_0^{(n)}$, and using (2. 1) and (2. 3),

$$\begin{aligned}(k_0^{(n)}, Th) &= (Q_{T^*}Sl_{-1}^{(n)}, Th) + (Tl_0^{(n)}, Th) \\ &= (Q_{T^*}Sl_{-1}^{(n)}, Th) - (J_T Q_T l_0^{(n)}, Q_T h) + (l_0^{(n)}, h) \\ &= (T^*Sl_{-1}^{(n)}, Q_T h) - (J_T Q_T l_0^{(n)}, Q_T h) + (l_0^{(n)}, h).\end{aligned}$$

Hence the left-hand member of (3. 8) is equal to

$$(T^* S l_{-1}^{(n)} - J_T Q_T l_0^{(n)}, Q_T h) = -(J_T S^{-1} k_1^{(n)}, Q_T h) \to -(J_T S^{-1} k_1, Q_T h) = 0,$$

as required. This completes the proof of the dilation property, and thereby that of the Theorem.

Corollary. *Under the hypotheses of the Theorem, there exists a Hilbert space* $\mathfrak{K}_+ \supseteqq \mathfrak{H}$ *and there exists a closed, densely defined operator* U_+ *in* $\mathfrak{K}_+$, *with the following properties:*

(a) $\mathfrak{K}_+$ *is a* J*-space, with* $\mathfrak{H} \subseteqq J^+(\mathfrak{K}_+)$;

(b) U_+ *is* J*-isometric, that is,* (1. 2) *holds*;

(c) T^* *is the restriction of* U_+^* *to* $\boldsymbol{D}(T^*)$;

(d) $\bigvee \{U_+^n \mathfrak{H}:\ n=0, 1, 2, \ldots\} = \mathfrak{K}_+$.

Namely, use the construction of the Theorem, and the space $\mathfrak{K}_+$ defined there (3. 5). It inherits the J-space structure of $\mathfrak{K}$, because $J|\mathfrak{K}_+$ is still a unitary hermitian operator; the same symbol may be used for this restriction. As U_+ we must of course take $U|\mathfrak{K}_+$. All the assertions of the Corollary follow at once from what has already been proved, except (c). But (c) follows from the definition of adjoint. Indeed, let $h \in \boldsymbol{D}(T^*)$ and $k \in \boldsymbol{D}(U_+)$ be given. As observed in the proof of (c) of the Theorem, $k_0 \in \boldsymbol{D}(T)$. Therefore

$$(U_+ k, h) = (T k_0, h) + (S Q_T k_0, h) + \big(S(k - k_0), h\big) = (T k_0, h) = (k_0, T^* h),$$

which is what is needed to prove that $T^* \subseteqq U_+^*$. The Corollary is established.

The Corollary does say, as promised in the Introduction, that every operator has an extension which is the adjoint (or the J-adjoint) of a J-isometry, but the operator about which it says so is T^*. Indeed, the extension in question is U_+^*; its ordinary adjoint U_+ is a J-isometry by (b); but then so is its J-adjoint JU_+J, because for all $k, k' \in \boldsymbol{D}(JU_+J)$ we have $Jk, Jk' \in \boldsymbol{D}(U_+)$ and

$$[JU_+Jk, JU_+Jk'] = [U_+Jk, U_+Jk'] = [Jk, Jk'] = [k, k'].$$

4. Geometry of the dilation space

The construction of § 3 carries over more than the algebraic manipulations from the contraction case. I will now exhibit the generalization of the geometric considerations related to the defect spaces [18, I. 3—4]. The geometry here must of course be richer, but it stays in close analogy.

Consider the following subspaces of $\mathfrak{H}$:

$$\mathfrak{D}_T = \overline{\boldsymbol{R}(D_T)} = \overline{\boldsymbol{R}\big((1-T^*T)^+\big)}, \text{ the "defect space" for } T;$$

$$\mathfrak{O}_T = \boldsymbol{N}(1-T^*T), \text{ the "isometric-like space" for } T; \tag{4. 1}$$

$$\mathfrak{X}_T = \overline{\boldsymbol{R}(X_T)} = \overline{\boldsymbol{R}\big((1-T^*T)^-\big)}, \text{ the "excess space" for } T.$$

Defect, isometric-like, and excess spaces for T^* are defined the same way. It is clear that

$$\mathfrak{H} = \mathfrak{D}_T \oplus \mathfrak{O}_T \oplus \mathfrak{X}_T = \mathfrak{D}_{T^*} \oplus \mathfrak{O}_{T^*} \oplus \mathfrak{X}_{T^*}, \tag{4.2}$$

and that the defect and isometric-like spaces have simple characterizations:

$$h \in \mathfrak{O}_T \Leftrightarrow h \in \boldsymbol{D}(T) \text{ and } |T|h = h; \quad h \in \mathfrak{D}_T \Leftrightarrow h \in \bigcap_n \boldsymbol{D}(|T|^n) \text{ and } |T|^n h \to 0. \tag{4.3}$$

It is also clear from definitions that $\mathfrak{D}_T = J_T^+(\mathfrak{H})$, $\mathfrak{X}_T = J_T^-(\mathfrak{H})$, and $\mathfrak{D}_T \oplus \mathfrak{X}_T = J_T(\mathfrak{H})$; and then from (3. 2) and (3. 3), we see that $\mathfrak{K}_+$ (defined by (3. 5)) has the following subspaces invariant for U:

$$\oplus\{S^n \mathfrak{D}_T \colon n = 0, 1, 2, \ldots\} \subseteqq J^+(\mathfrak{K}) \cap \boldsymbol{D}(U), \quad \oplus\{S^n \mathfrak{X}_T \colon n = 1, 2, \ldots\} \subseteqq J^-(\mathfrak{K}).$$

Symmetrically, we construct a positive and a negative subspace

$$\oplus\{S^{-n} \mathfrak{D}_{T^*} \colon n = 0, 1, 2, \ldots\}, \quad \oplus\{S^{-n} \mathfrak{X}_{T^*} \colon n = 1, 2, \ldots\}$$

invariant for U^{-1}.

The complications occur for $n = 0, \pm 1$. To describe the action of U there, note first that

$$\begin{aligned} W(\mathfrak{D}_T) &\subseteqq \mathfrak{D}_{T^*}, & W^*(\mathfrak{D}_{T^*}) &\subseteqq \mathfrak{D}_T, \\ W(\mathfrak{O}_T) &\subseteqq \mathfrak{O}_{T^*}, & W^*(\mathfrak{O}_{T^*}) &\subseteqq \mathfrak{O}_T, \\ W(\mathfrak{X}_T) &\subseteqq \mathfrak{X}_{T^*}, & W^*(\mathfrak{X}_{T^*}) &\subseteqq \mathfrak{X}_T. \end{aligned} \tag{4.4}$$

Indeed, the assertions regarding defect and isometric-like spaces are easily proved using the characterizations (4. 3); then the assertions regarding the excess spaces follow using (4. 2).

Now the action of U within $\mathfrak{H}$ — that is, the action of $T = W|T|$ — may be very complicated indeed, but one portion of the complication is brought into view by (4. 2) and (4. 4): We have two orthogonal decompositions of the space, and a pair of partial isometries relating the two.

The action of U "mixes" $J^+(\mathfrak{K})$ with $J^-(\mathfrak{K})$ only at two places: U takes $S^{-1}(\mathfrak{X}_{T^*}) \cap \boldsymbol{D}(U) \subseteqq J^-(\mathfrak{K})$ into $\mathfrak{X}_{T^*} \oplus S(\mathfrak{X}_T)$, although $\mathfrak{X}_{T^*} \subseteqq J^+(\mathfrak{K})$. Secondly, U takes $\mathfrak{X}_T \cap \boldsymbol{D}(U) \subseteqq J^+(\mathfrak{K})$ into $\mathfrak{X}_{T^*} \oplus S(\mathfrak{X}_T)$, although $S(\mathfrak{X}_T) \subseteqq J^-(\mathfrak{K})$.

One especially simple reducing subspace of T has been studied by APOSTOL [1] and DURSZT [2], extending [14, Thm. I. 3. 2]. In terms of the present paper, one may put the central idea as follows. Among subspaces $\mathfrak{U}$ of $\mathfrak{H}$ such that $T|\mathfrak{U}$ is unitary, there is a maximal one $\mathfrak{U}^\infty$, given by

$$\mathfrak{U}^\infty = \bigcap_{n=0}^{\infty} T^n(\mathfrak{O}_T \cap \mathfrak{O}_{T^*}) \cap \bigcap_{n=1}^{\infty} T^{*n}(\mathfrak{O}_T \cap \mathfrak{O}_{T^*}).$$

The part $T|\mathfrak{U}^\infty$ is called the "unitary part" of T; $T|\mathfrak{H} \ominus \mathfrak{U}^\infty$, the "completely non-unitary part".

This is easily proved using the Theorem of § 3, even if T is unbounded; along with the formula

$$\mathfrak{U}^{\infty} = \bigcap_{n=-\infty}^{\infty} U^n(\mathfrak{D}_T \cap \mathfrak{D}_{T^*}).$$

To be sure, the dilation is of interest only as regards the completely non-unitary part. None of the considerations of § 3 would have been affected if I had constructed $\mathfrak{K}^0$ from copies of $\mathfrak{H} \ominus \mathfrak{U}^{\infty}$ rather than copies of $\mathfrak{H}$.

5. Further remarks

1. In case T is a contraction ($\|T\| \leqq 1$), the construction given in § 3 leads to the same dilation as that of [14, I. 5], with J the identity operator and with U unitary. To see this, one need only compare the two step-by-step.

2. Unlike the case of Sz.-Nagy, the construction given here is not determined essentially uniquely by the conditions of the Theorem. Indeed, assuming that T is neither a contraction nor doubly-expansive ($\|Th\| \geqq \|h\|$ and $\|T^* h'\| \geqq \|h'\|$ for all h, h' in the respective domains), I will show how it can always be modified in a non-trivial way.

For we know, once those two cases are excluded, that either $\mathfrak{D}_T$ and $\mathfrak{X}_T$ are both non-zero, or $\mathfrak{D}_{T^*}$ and $\mathfrak{X}_{T^*}$ are both non-zero. This enables us to find operators $Z^{(n)}$ on $\mathfrak{H}$ such that

$$Z^{(0)} = 1; \quad Z^{(n)} J_T = J_T Z^{(n)} \quad (n = 1, 2, \ldots); \quad Z^{(n)} J_{T^*} = J_{T^*} Z^{(n)} \quad (n = -1, -2, \ldots);$$

$$\|Z^{(n)}\| \leqq M, \quad \|(Z^{(n)})^{-1}\| \leqq M \qquad (n = \pm 1, \pm 2, \ldots);$$

for some n,

$$Z^{(n)} \mathfrak{D}_T \neq \mathfrak{D}_T \text{ (if } n > 0), \quad \text{or} \quad Z^{(n)} \mathfrak{D}_{T^*} \neq \mathfrak{D}_{T^*} \text{ (if } n < 0).$$

Then define $Z = \sum_{-\infty}^{\infty} S^n Z^{(n)} S^{-n}$, a continuous, continuously invertible operator on $\mathfrak{K}^0$, and consider it restricted to $\mathfrak{K}$. Evidently all the properties asserted in the Theorem for U hold also for ZU, as does also the desirable property of having $\mathfrak{K}_+ \ominus \mathfrak{H}$ in its domain. Yet the geometry can be quite different. The conditions of the Theorem determine the structure of $\mathfrak{K}$ only with respect to [,], leaving a great deal of freedom as to the inner product (,).

In order to get a uniqueness assertion we must assume more.

Proposition. *Given $\mathfrak{H}$ and T, let $\mathfrak{K}$ and U be constructed as in the proof of the Theorem. Let $\mathfrak{K}'$ be a J-space (with canonical symmetry still denoted by J) and U' a J-unitary in it, which also satisfy conditions* (a)—(d) *of the Theorem. Assume* *further that* (for $n=1,2,\ldots$) $U'^n(\mathfrak{D}_T)\subseteqq J^+(\mathfrak{K}')$, $U'^n(\mathfrak{X}_T\cap D(T))\subseteqq J^-(\mathfrak{K}')$, $U'^{-n}(\mathfrak{D}_{T^*})\subseteqq J^+(\mathfrak{K}')$, $U'^{-n}(\mathfrak{X}_{T^*}\cap D(T^*))\subseteqq J^-(\mathfrak{K}')$. *Then there exists a unitary and J-unitary map Z' of $\mathfrak{K}$ onto $\mathfrak{K}'$ such that $U'=Z'UZ'^{-1}$.*

The proof exploits property (d) in the same way as in the case of contractions. There is no need to go into details.

3. Naturally the construction was motivated in large part by the hope of finding a geometric approach to characteristic functions for arbitrary operators. Just as Sz.-Nagy and Foiaş [14, VI] exhibit a natural geometric genesis of the characteristic function of a contraction, it is hoped to do the same for more general operators. The role of contractive analytic operator-valued function [14, V—VI] might be played by J-contractive ones [3] (further references in [12]). In fact, since the construction described here was found, this program has made some progress, leading to a geometric treatment of the characteristic functions studied earlier by Sahnovič [12] and Kužel' [9]. This will be described in future papers.

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(Received March 28, 1969)

On the unitary part of an operator on Hilbert space

By E. DURSZT in Szeged

Let T be a bounded (linear) operator on a Hilbert space H. If a subspace K of H reduces T and $T|K$ is a unitary operator, then $T|K$ is called a *unitary part* of T. A unitary part of T is *maximal* if it is an extension of every other unitary part of T.

A special case of Theorem 4 in [1] states the existence of the maximal unitary part $T|H_0$ for an arbitrary bounded operator T, and characterizes H_0 as the set of vectors φ of H satisfying

(1) $$T^*TA\varphi = A\varphi = TT^*A\varphi$$

for every finite product A of factors equal to T, T^*, and for $A=I$.

In case T is a contraction (i.e. $\|T\| \leqq 1$), H_0 can be characterized as the set of vectors φ in H for which

(2) $$\|T^n\varphi\| = \|\varphi\| = \|T^{*n}\varphi\| \quad (n = 1, 2, \ldots),$$

cf. [2], [3]. For a contraction T, conditions (2) are obviously equivalent to the following ones:

(2′) $$T^{*n}T^n\varphi = \varphi = T^nT^{*n}\varphi \quad (n = 1, 2, \ldots).$$

In this paper we give a new proof of the existence of H_0 for an arbitrary bounded T, and characterize H_0 in a way which is simpler than (1) and very similar to the characterization (2) in the case of contractions. Also, we give a characterization of the orthogonal complement $H \ominus H_0$. Finally, by giving a counterexample we show that the characterization of H_0 by (2) does not hold true in general if T is not a contraction, not even if T is power-bounded.

Consider an arbitrary bounded operator T on the Hilbert space H. We denote by H^0 the set of vectors φ of H for which:

(3) $$\|T^*T^n\varphi\| = \|T^n\varphi\| = \|\varphi\| = \|T^{*n}\varphi\| = \|TT^{*n}\varphi\| \quad (n=1, 2, \ldots)$$

and by H^1 the subspace spanned by the ranges of the operators

(4) $$T^n(I-TT^*) \quad \text{and} \quad T^{*n}(I-T^*T) \quad (n=0, 1, 2, \ldots).$$

Theorem.

(i) *H^0 is a subspace of H reducing T.*

(ii) *$T|H^0$ is the maximal unitary part of T.*

(iii) $H^1 = H \ominus H^0$.

Proof. Let $\varphi \in H^0$. Then

$$\|(I-T^*T)T^n\varphi\|^2 = \|T^n\varphi\|^2 - 2\|T^{n+1}\varphi\|^2 + \|T^*T^{n+1}\varphi\|^2 = 0 \qquad (n = 0, 1, \dots),$$

i.e.:

$$T^*T^{n+1}\varphi = T^n\varphi \qquad (n = 0, 1, 2, \dots).$$

Repeating this computation with T^* in place of T we get:

$$TT^{*n+1}\varphi = T^{*n}\varphi \qquad (n = 0, 1, 2, \dots).$$

Resuming:

(5) $$T^*T^{n+1}\varphi = T^n\varphi, \quad TT^{*n+1}\varphi = T^{*n}\varphi \qquad (n = 0, 1, 2, \dots).$$

So we have: (3) *implies* (5).

On the other hand, if (5) holds for a vector φ, then

$$\|T^*T^{n+1}\varphi\|^2 = \|T^n\varphi\|^2 = (T^{*n}T^n\varphi, \varphi) = (T^*T^n\varphi, T^{n-1}\varphi) =$$
$$= (T^{n-1}\varphi, T^{n-1}\varphi) = \|T^{n-1}\varphi\|^2 = \dots = \|\varphi\|^2,$$

and analogously

$$\|TT^{*n+1}\varphi\| = \|T^{*n}\varphi\| = \|\varphi\| \qquad (n = 0, 1, 2, \dots).$$

So we have: (3) *is equivalent to* (5).

From (5) it is obvious that H^0 is a subspace of H. In the special case $n=0$ we get from (5)

(6) $$T^*T\varphi = \varphi = TT^*\varphi \qquad (\varphi \in H^0).$$

This fact and (3) show that H^0 is invariant both for T and T^*, i.e., H^0 reduces T.

So (i) is proved.

Clearly (6) implies that $T|H^0$ is unitary on H^0. Suppose that $K(\subset H)$ reduces T and $T|K$ is unitary, and let $\varphi \in K$. In this case (3) holds for φ and consequently, $\varphi \in H^0$. Thus $K \subset H^0$, i.e. our statement (ii) is proved.

As regards (iii), (5) shows that the vector $\varphi \in H$ belongs to H^0 if and only if

$$H \perp (I-T^*T)T^n\varphi \quad \text{and} \quad H \perp (I-TT^*)T^{*n}\varphi \qquad (n = 0, 1, \dots),$$

or equivalently:

$$T^{*n}(I-T^*T)H \perp \varphi \quad \text{and} \quad T^n(I-TT^*)H \perp \varphi \qquad (n = 0, 1, \dots).$$

Thus we have: $\varphi \in H^0$ if φ is orthogonal to the ranges of the operators (4), i.e. if $\varphi \perp H^1$. This gives $H^1 = H \ominus H^0$.

So we finished the proof of the theorem.

As regards the counterexample, let $\{\psi_1, \psi_2\}$ be an orthonormal basis in a two-dimensional Hilbert space H and define T by the matrix

$$\begin{pmatrix} -1 & 0 \\ \sqrt{3} & 0 \end{pmatrix}.$$

We prove that there exists a non-zero vector satisfying (2) and such that the corresponding H_0 is $\{0\}$. Indeed, let $\varphi = \frac{1}{2}\psi_1 + \frac{\sqrt{3}}{2}\psi_2$. An easy computation shows that

$$\|\varphi\| = 1, \quad T\varphi = \psi_1 = T^*\varphi, \quad T^n = (-1)^{n-1}T \qquad (n = 1, 2, \dots),$$

and consequently

$$\|T^n\varphi\| = \|\varphi\| = \|T^{*n}\varphi\| \qquad (n = 1, 2, \dots),$$

i.e. (2) is fulfilled.

Next observe that T is not a unitary operator. In order to prove that $H_0 = \{0\}$ it suffices therefore to prove that T is not reduced by any non-trivial subspace. If H had a non-trivial subspace reducing T, then this subspace should be one-dimensional, i.e. spanned by an eigenvector of T. An easy computation shows that the two possible linearly independent eigenvectors of T are ψ_2 and $\frac{1}{2}\psi_1 - \frac{\sqrt{3}}{2}\psi_2$, but none of them spans an invariant subspace of T^*.

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(Received February 10, 1969)

Modèle de Jordan pour une classe d'opérateurs de l'espace de Hilbert

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Introduction

Cette Note fait suite de la Note [2]. Là on a étudié les opérateurs T d'un espace de Hilbert (séparable, complexe) qui appartiennent à une des classes $C_0(N)$[1]) et sont „sans multiplicité" dans un sens qu'on a précisé par une série de conditions équivalentes, l'une de ces conditions étant l'existence d'un vecteur cyclique pour T. On a démontré en particulier qu'un opérateur T de classe $C_0(N)$ admet un vecteur cyclique si T est quasi-similaire à un opérateur de classe $C_0(1)$ et dans ce cas seulement.

Or la classe $C_0(1)$ est constituée des opérateurs qui sont unitairement équivalents aux opérateurs $S(m)$ attachés à des fonctions intérieures non-constantes m (pour le disque unité); $S(m)$ est défini dans l'espace

$$\mathfrak{H}(m) = H^2 \ominus mH^2$$

par

$$S(m)u = P_{\mathfrak{H}(m)}(\lambda \cdot u(\lambda)), \quad \text{ou} \quad S(m)^* u = \frac{1}{\lambda}[u(\lambda) - u(0)],$$

$P_{\mathfrak{H}(m)}$ désignant la projection orthogonale de l'espace de Hardy H^2 dans son sous-espace $\mathfrak{H}(m)$.

Dans cette Note on envisagera des opérateurs appartenant à la classe plus générale C_0 et de multiplicité finie quelconque, au sens de la définition suivante:

Pour un opérateur borné quelconque T dans l'espace $\mathfrak{H}$, la *multiplicité* μ_T est le nombre cardinal minimum d'un sous-ensemble $\mathfrak{S}$ de $\mathfrak{H}$ tel que $\bigvee_{n=0}^{\infty} T^n \mathfrak{S} = \mathfrak{H}$.[2]) (Notons que $\mu_T = 1$ veut dire que T admet un vecteur cyclique.)

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[1]) Pour les définitions voir les Préliminaires.

[2]) Dans la Note [3] on a employé la notation $\dim_T \mathfrak{H}$ pour μ_T.

Le résultat principal dans la partie I de cette Note est que tout opérateur T de classe C_0 aux multiplicités μ_T et μ_{T^*} finies, est quasi-similaire à un opérateur de la forme

$$(*) \qquad S(m_1) \oplus S(m_2) \oplus \cdots \oplus S(m_K)$$

où $m_1, m_2, \ldots, m_K$ sont des fonctions intérieures non-constantes dont chacune est un diviseur de la précédente; de plus cet opérateur est déterminé par T d'une manière univoque et on a $K = \mu_T = \mu_{T^*}$ (théorème 2).

Tel opérateur $(*)$ sera appelé un *opérateur de Jordan* et désigné aussi par $S(m_1, m_2, \ldots, m_K)$.

Comme une application du théorème 2 on démontre dans la partie II que pour tout opérateur T de type envisagé le bicommutant $(T)''$ est constitué des fonctions de T (au sens du chap. IV de [1]). Ce résultat (théorème 3) généralise le fait bien connu dans l'algèbre linéaire que le bicommutant d'une matrice carrée est constitué des polynômes de cette matrice.

Notre théorème 2 présente certaines intersections avec un théorème de KISILEVSKY [5], mais il y a une grande différence dans les sujets et les méthodes des deux recherches. Là, il s'agit des opérateurs bornés A dont la partie imaginaire $(A - A^*)/(2i)$ est non-négative et de trace finie, et dont le spectre est constitué du seul ponit 0. Cela correspond par la transformation de Cayley à une contraction faible (au sens du chap. VIII de [1]) de classe C_0, dont le spectre est constitué du seul point 1. Par contre, notre recherche concerne les opérateurs de classe C_0 de multiplicité finie (en particulier les opérateurs des classes $C_0(N)$), mais sans aucune restriction pour le spectre. Il y a aussi la différence que notre méthode est fondée sur les notions de „quasi-affinité" et de „quasi-similitude", tandis que M. Kisilevsky utilise une décomposition en „somme approximative" des espaces en question. Nous discuterons les relations entre ces notions, dans le cas qui nous intéresse, dans le n° 7.

Préliminaires

Rappelons la définition de la classe C_0 d'opérateurs (*cf.* [1] chap. III). C_0 est constituée des contractions complètement non-unitaires T d'un espace de Hilbert $\mathfrak{H}$, pour lesquelles il existe une fonction $u \in H^\infty$ [3]) telle que $u \not\equiv 0$ et $u(T) = O$. Parmi ces fonctions il y a alors une fonction intérieure qui divise dans H^∞ toutes les autres; cette fonction intérieure est déterminée par T à un facteur constant de module 1 près et s'appelle la fonction minimum de T; on la désigne par m_T. Sauf pour l'opérateur O de l'espace banal $\{0\}$, la fonction m_T n'est pas constante. Inversement,

[3]) Les classes de Hardy envisagées sont celles pour le disque $|\lambda| < 1$.

pour toute fonction intérieure non-constante m il existe un opérateur T de classe C_0 telle que $m_T = m$; tel est en particulier l'opérateur $S(m)$ envisagé dans l'Introduction.

La classe C_0 est contenue dans la classe C_{00}, composée des contractions T telles $T^n \to O$ et $T^{*n} \to O$ pour $n \to \infty$. Inversement, tout opérateur $T \in C_{00}$ dont les indices de défaut

$$\mathfrak{d}_T = \dim \overline{(I - T^*T)^{\frac{1}{2}} \mathfrak{H}}, \quad \mathfrak{d}_{T^*} = \dim \overline{(I - TT^*)^{\frac{1}{2}} \mathfrak{H}}$$

sont égaux à un nombre fini N, appartient à la classe C_0; la sous-classe de C_0 formée par ces opérateurs s'appelle classe $C_0(N)$ (*cf.* [1], n° IX. 3).

La classe $C_0(0)$ comprend le seul opérateur O de l'espace banal $\{0\}$. La classe $C_0(1)$ est constituée des opérateurs qui sont unitairement équivalents aux opérateurs $S(m)$ attachés aux fonctions intérieures m non-constantes. Ce fait se généralise pour tout $N \geqq 1$: la classe $C_0(N)$ est constituée des opérateurs qui sont unitairement équivalents aux opérateurs $S(\Theta)$ attachés aux fonctions intérieures matricielles $\Theta(\lambda)$ d'ordre N, contractives pures[4]). L'opérateur $S(\Theta)$ est défini notamment dans l'espace

$$\mathfrak{H}(\Theta) = H^2(E^N) \ominus \Theta H^2(E^N) \quad ^{5})$$

par

$$S(\Theta)u = P_{\mathfrak{H}(\Theta)}(\lambda\, u(\lambda)), \quad \text{ou} \quad S(\Theta)^* u = \frac{1}{\lambda}(u(\lambda) - u(0)).$$

Pour l'opérateur $S(\Theta)$, la fonction minimum se calcule comme le quotient du déterminant de la matrice $\Theta(\lambda)$ par le plus grand diviseur intérieur commun des déterminants mineurs d'ordre $N-1$ de cette matrice. *Cf.* [1], chap. VI, en particulier le théorème VI. 5. 2.

Pour $T \in C_0$ on a $T^* \in C_0$ et $m_{T^*} = m_T^{\sim}$, et pour T unitairement équivalent à $S(\Theta) \in C_0(N)$, T^* est unitairement équivalent à $S(\Theta^{\sim}) \in C_0(N)$; [6]) *cf.* [1], chap. VI, théorème 3. 1 et formule (1. 6).[7]) Pour $T \in C_0(N)$ la restriction de T à un sous-espace invariant quelconque appartient à une classe $C_0(N')$ avec $N' \leqq N$; *cf.* le lemme IX. 3. 1 de [1].

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[4]) C'est-à-dire que la fonction $\Theta(\lambda)$ est définie et holomorphe dans le disque $|\lambda| < 1$, ses valeurs sont des contractions dans l'espace euclidien complexe E^N, la limite radiale $\Theta(e^{it})$ est un opérateur isométrique (donc aussi unitaire) dans E^N pour presque tous les points e^{it} du cercle unité, et de plus $\|\Theta(0)x\| < \|x\|$ pour tout vecteur non-nul x de E^N.

[5]) $H^2(E^N)$ est l'espace hilbertien de Hardy des fonctions $u = u(\lambda)$ à valeurs vecteurs dans E^N. Pour $N=1$ il s'agit donc de l'espace H^2 ordinaire.

[6]) Pour une fonction $\Theta(\lambda)$ à valeurs opérateurs on définit $\Theta^{\sim}(\lambda) = \Theta(\bar{\lambda})^*$; pour une fonction scalaire cette définition se réduit à $m^{\sim}(\lambda) = \overline{m(\bar{\lambda})}$.

[7]) Une transformation unitaire $u \to v = \Psi u$ de l'espace $\mathfrak{H}(\Theta)$ sur l'espace $\mathfrak{H}(\Theta^{\sim})$ par laquelle l'opérateur $S(\Theta)^*$ est transformé en l'opérateur $S(\Theta^{\sim})$, est définie par $v(z) = \bar{z} \cdot \Theta^{\sim}(z) u(\bar{z})$ $(z = e^{it})$; *cf.* [4], n° 4. On a notamment $\Psi S(\Theta)^* = S(\Theta^{\sim})\Psi$.

Pour tout opérateur $T \in C_{00}$ on a $\mu_T \leqq \mathfrak{d}_{T^*}$ et $\mu_{T^*} \leqq \mathfrak{d}_T$; *cf.* le n° 1 de [3]. En particulier, pour $T \in C_0(N)$ on a donc

$$(\ast) \qquad \mu_T,\ \mu_{T^*} \leqq N.$$

Proposition A. (Proposition 1 dans [2].) *Si l'opérateur $T \in C_0(N)$ admet un vecteur cyclique, il en est de même de T^* ainsi que de toute restriction de T à un sous-espace invariant.*

Proposition B. (Théorème 1 dans [2].) *Pour tout opérateur $T \in C_0(N)$ il existe un sous-espace invariant $\mathfrak{L}$ tel que la restriction $T' = T|\mathfrak{L}$ admette un vecteur cyclique dans $\mathfrak{L}$ et qu'on ait $m_{T'} = m_T$.*

Un opérateur borné X d'un espace $\mathfrak{H}_1$ à un espace $\mathfrak{H}_2$ est appelé une *quasi-affinité* s'il admet un inverse à domaine dense dans $\mathfrak{H}_2$. Un opérateur borné T_1 dans $\mathfrak{H}_1$ s'appelle une *transformée quasi-affine* d'un opérateur borné T_2 dans $\mathfrak{H}_2$, s'il existe une quasi-affinité X de $\mathfrak{H}_1$ à $\mathfrak{H}_2$ telle que $T_2X = XT_1$. Nous voulons indiquer cette relation aussi par la notation

$$T_1 \prec T_2 \quad \text{ou} \quad T_2 \succ T_1.$$

Cette relation d'ordre partiel est transitif et $T_1 \prec T_2$ entraîne $T_1^* \succ T_2^*$. Si de plus $T_1, T_2 \in C_0$, on a $m_{T_1} = m_{T_2}$.

Deux opérateurs dont chacun est une transformée quasi-affine de l'autre, sont dits *quasi-similaires.*

Proposition C. *La multiplicité μ_T est invariante par rapport à une quasi-similitude. De plus, $T_1 \prec T_2$ entraîne $\mu_{T_1} \geqq \mu_{T_2}$.*

En effet, si $\mathfrak{S}_1$ est un ensemble de vecteurs dans l'espace $\mathfrak{H}_1$, de cardinalité μ_T et tel que $\bigvee_{n=0}^{\infty} T_1^n \mathfrak{S}_1 = \mathfrak{H}_1$, et si X est une quasi-affinité de $\mathfrak{H}_1$ à $\mathfrak{H}_2$ telle que $T_2X = XT_1$, on a $T_2^n X = XT_1^n$ $(n = 0, 1, \ldots)$ et par conséquent

$$\bigvee_{n=0}^{\infty} T_2^n X \mathfrak{S}_1 = \bigvee_{n=0}^{\infty} XT_1^n \mathfrak{S}_1 = \overline{X \bigvee_{n=0}^{\infty} T_1^n \mathfrak{S}_1} = \overline{X\mathfrak{H}_1} = \mathfrak{H}_2,$$

d'où on déduit que μ_{T_2} est au plus égale à la cardinalité de l'ensemble $\mathfrak{S}_2 = X\mathfrak{S}_1$, donc à μ_{T_1}.

Proposition D. (Proposition 1 dans [3].) *Tout opérateur $T \in C_0$ admet comme transformée quasi-affine un opérateur $T_1 \in C_0$ dont les indices de défaut sont égaux à μ_T. Si $\mu_T < \infty$, on a donc $T_1 \in C_0(N)$ avec $N = \mu_T$.*

Terminons par remarquer que pour tout opérateur T dans l'espace E^N, de norme $\|T\| < 1$, on a $T \in C_0(N)$. Il s'ensuit que les résultats de cette Note s'appliquent en particulier aux matrices carrées finies, donc présentent des généralisations de certains faits appartenant à l'algèbre linéaire.

PARTIE I

1. Trois propositions et deux théorèmes

1. Il s'agit des propositions suivantes:

Proposition 1. *Pour tout opérateur T dans l'espace $\mathfrak{H}$, appartenant à une classe $C_0(N)$ $(N \geqq 1)$, il existe une décomposition de $\mathfrak{H}$ en somme $\mathfrak{L} \oplus \mathfrak{M}$ de deux sous-espaces orthogonaux dont $\mathfrak{M}$ est invariant pour T, telle que dans la matrice correspondante $T = \begin{bmatrix} T_{\mathfrak{L}} & O \\ * & T_{\mathfrak{M}} \end{bmatrix}$ l'opérateur $T_{\mathfrak{L}}$ (de $\mathfrak{L}$) admette un vecteur cyclique et ait la même fonction minimum que T: $m_{T_{\mathfrak{L}}} = m_T$. Dans ce cas on a de plus*

$$T \succ S(m_T) \oplus T_{\mathfrak{M}}.$$

Proposition 2. *Soient $m_1, \ldots, m_K$ des fonctions intérieures ayant pour diviseur commun une fonction intérieure non-constante m. Supposons que l'opérateur $S(m_1) \oplus \cdots \oplus S(m_K)$ est la transformée quasi-affine d'un opérateur T de classe $C_0(N)$. On a alors $K \leqq N$.*

Proposition 3. *Soient $S(m_1, \ldots, m_K)$ et $S(m'_1, \ldots, m'_{K'})$ deux opérateurs de Jordan dont le premier est une transformée quasi-affine du second. On a alors $K' = K$ et $m'_i = m_i$ $(i = 1, \ldots, K)$, donc les deux opérateurs coïncident.*

On démontrera ces propositions dans les n[os] 2—4.

2. Ici nous en déduisons d'abord le suivant

Théorème 1. *Pour tout opérateur $T \in C_0$, de multiplicité μ_T finie, il existe un opérateur de Jordan $S(m_1, \ldots, m_K)$ tel que*

$$T \succ S(m_1, \ldots, m_K). \tag{1.1}$$

Démonstration. En vertu de la proposition D il existe un opérateur $T_1 \in C_0(N_1)$ tel que $N_1 = \mu_T$ et

$$T \succ T_1. \tag{1.2}$$

D'après la proposition 1 il existe une restriction T_2 de T_1 à un sous-espace invariant $\mathfrak{M}_1$ telle qu'on ait

$$T_1 \succ S(m_1) \oplus T_2$$

où $m_1 = m_{T_1}$ $(= m_T)$. Etant une restriction de T_1, l'opérateur T_2 appartient à une classe $C_0(N_2)$ et m_{T_2} est un diviseur de m_1. Si $N_2 \geqq 1$, on peut appliquer le même raisonnement à T_2 au lieu de T_1, et on obtient alors que

$$T_1 \succ S(m_1) \oplus S(m_2) \oplus T_3$$

où $m_2 = m_{T_2}$, T_3 est une restriction de T_2 à un sous-espace invariant et par conséquent appartient à une classe $C_0(N_3)$, et m_{T_3} est un diviseur de m_2. Ce procédé ne peut être continué indéfiniment. En effet, si l'on a obtenu que

$$T_1 \succ S(m_1) \oplus \cdots \oplus S(m_k) \oplus T_{k+1}$$

pour certaines fonctions intérieures non-constantes $m_1, \ldots, m_K$ (dont chacune est un diviseur de la précédente) et pour un opérateur T_{k+1}, alors $S(m_1) \oplus \cdots \oplus S(m_k)$ est la transformée quasi-affine de la restriction de T_1 à un certain sous-espace invariant[8]; comme cette restriction appartient à une classe $C_0(N')$ avec $N' \leqq N_1$, il s'ensuit de la proposition 2 que $k \leqq N'$ ($\leqq N_1$). Or le procédé ne s'arrête à l'étape k-ième que si T_{k+1} est l'opérateur dans l'espace banal $\{0\}$. On conclut qu'il existe un opérateur de Jordan $S(m_1, \ldots, m_K)$ tel que

$$T_1 \succ S(m_1, \ldots, m_K). \tag{1. 3}$$

Les relations (1. 2) et (1. 3) entraînent la relation (1. 1).

Il manifeste que la fonction minimum d'un opérateur de Jordan $S(m_1, \ldots, m_K)$ est égale à m_1. Il s'ensuit que *pour tout opérateur de Jordan vérifiant la relation* (1. 1) *on a* $m_1 = m_T$.

Le résultat principal de cette partie est contenu dans le

Théorème 2. *Pour un opérateur T de classe C_0 et aux multiplicités μ_T, μ_{T^*} finies, il existe un opérateur de Jordan et un seul qui vérifie la relation* (1. 1). *Cet opérateur est même quasi-similaire à T et on a*

$$K = \mu_T = \mu_{T^*}; \tag{1. 4}$$

on l'appellera le modèle de Jordan de T.

Remarque. Grâce aux inégalités (⁑) dans les Préliminaires, le théorème 2 s'applique en particulier aux opérateurs des classes $C_0(N)$, $N \geqq 1$.

Démonstration. En vertu de la proposition D il existe un opérateur $T_{1*} \in C_0(N_{1*})$ tel que $N_{1*} = \mu_{T^*}$ et $T^* \succ T_{1*}$. On a alors aussi $T_{1*}^* \in C_0(N_{1*})$ et

$$T_{1*}^* \succ T. \tag{1. 5}$$

Envisageons deux opérateurs de Jordan, $S = (m_1, \ldots, m_K)$ et $S' = S(m_1', \ldots, m_{K'}')$, tels que $T \succ S$, $T^* \succ S'$ et d'ailleurs quelconques; l'existence de tels opérateurs est

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[8]) En effet, soient A et B deux opérateurs tels que $B \succ A$, et soit $\mathfrak{L}$ un sous-espace invariant pour A. Si X est une quasi-affinité telle que $BX = XA$, alors $\mathfrak{M} = \overline{X\mathfrak{L}}$ est un sous-espace invariant pour B, $X' = X|\mathfrak{L}$ est une quasi-affinité de $\mathfrak{L}$ à $\mathfrak{M}$, et on a $(B|\mathfrak{M})X' = X'(A|\mathfrak{L})$.

assuré par le théorème 1. On a alors

$$S'^* \succ T \succ S. \tag{1. 6}$$

Or S'^* est unitairement équivalent à l'opérateur de Jordan $S'^\sim = S(m_1'^\sim, \ldots, m_{K'}'^\sim)$, donc il dérive de (1. 6) par la proposition 3 que $S' = S$. Vu que S et $S'^\sim$ étaient choisis indépendamment l'un de l'autre, une des conséquences de cette égalité est que S est déterminé par T d'une manière univoque. En vertu de (1. 6), une autre conséquence est que T est quasi-similaire à S.

Reste à démontrer (1. 4). En vertu de la proposition C on a $\mu_T = \mu_S$. D'autre part on a $\mu_S \leqq K$ parce que $S \in C_0(K)$.[9]) Finalement, comme (1. 5) et (1. 6) entraînent $T_{1*}^* \succ S$ et que $T_{1*}^* \in C_0(\mu_{T^*})$, il s'ensuit par la proposition 2 que $K \leqq \mu_{T^*}$. Ainsi, nous avons

$$\mu_T \leqq K \leqq \mu_{T^*}.$$

Vu la symétrie des hypothèses faites en T et T^*, on doit avoir au côté de l'inégalité $\mu_T \leqq \mu_{T^*}$ aussi celle opposée $\mu_{T^*} \leqq \mu_T$. Donc $\mu_T = \mu_{T^*} = K$. Cela achève la démonstration.

En vertu de la proposition 3, le théorème 2 admet le suivant:

Corollaire 1. *Soient T et T' des opérateurs de type envisagé dans le théorème 2. Si $T \succ T'$, alors T et T' ont le même modèle de Jordan. Inversement, si T et T' ont le même modèle de Jordan, alors T et T' sont quasi-similaires.*

Remarque 1. Dans [2], n° 8, on a construit un opérateur de classe $C_0(2)$ et un opérateur de classe $C_0(1)$, qui sont quasi-similaires sans être similaires. Ainsi il n'est pas possible de remplacer, dans le corollaire ci-dessus, quasi-similitude par similitude.

Remarque 2. Nous savons que pour tout opérateur de Jordan vérifiant la relation (1. 1) on a $m_1 = m_T$. Pour T unitairement équivalent à un opérateur $S(\Theta) \in C_0(N)$, la fonction m_1 se calcule donc comme le déterminant de la matrice Θ divisé par le plus grand diviseur intérieur commun des déterminants mineurs d'ordre $N-1$. Il est fort probable que les autres fonctions dans la suite $m_1, \ldots, m_K$ peuvent être calculés d'une manière analogue.

Tout de même, il y a une relation utile que nous pouvons établir pour ce cas, notamment la suivante:

$$\text{dét}\, \Theta(\lambda) = m_1(\lambda) \cdots m_K(\lambda). \tag{1. 7}$$

[9]) En effet, on a $S = S(\Theta)$ pour la fonction matricielle $\Theta(\lambda)$ d'ordre K, de type diagonal, dont les éléments à la diagonale sont les fonctions scalaires $m_1, \ldots, m_K$. Cette fonction est évidemment intérieure et contractive pure.

Démonstration. Comme la fonction $\Theta(\lambda)$ coïncide avec la fonction caractéristique $\Theta_T(\lambda)$, son déterminant coïncide avec le déterminant $d_T(\lambda)$ de la fonction caractéristique de T. On fera l'induction sur K. Lorsque $K(=\mu_T)=1$, on a $d_T=m_T$ en vertu du théorème 2 de [2], ce qui prouve (1. 7) dans ce cas. Supposons que (1. 7) soit vérifiée pour $K=k-1$ et montrons qu'elle est alors vérifiée pour $K=k$ aussi. Soit donc $T\in C_0(N)$ tel que $K(=\mu_T)=k$. D'après la proposition 1 il existe une décomposition de l'espace de T en somme vectorielle de deux sous-espaces orthogonaux, soit $\mathfrak{L}$ et $\mathfrak{M}$, telle que $\mathfrak{M}$ soit invariant pour T et que, en posant $T_{\mathfrak{M}}=T|\mathfrak{M}$ et $T_{\mathfrak{L}}=(T^*|\mathfrak{L})^*$, $T_{\mathfrak{L}}$ ait un vecteur cyclique et vérifie la relation $m_{T_{\mathfrak{L}}}=m_T$. En vertu du lemme IX. 3. 1 de [1] on a alors pour les déterminants des fonctions caractéristiques correspondantes: $d_T=d_{T_{\mathfrak{L}}}d_{T_{\mathfrak{M}}}$. En vertu de la proposition 1 on a aussi: $T\succ S(m_T)\oplus T_{\mathfrak{M}}$. Soit $S(m_1',\dots,m_{K'}')$ le modèle de Jordan de $T_{\mathfrak{M}}$; $m_1'(=m_{T_{\mathfrak{M}}})$ est alors un diviseur de $m_1(=m_T)$. Il s'ensuit que $T\succ S(m_1,m_1',\dots,m_{K'}')$, d'où il dérive grâce à l'unicité du modèle que $K'=K-1=k-1$ et $m_i'=m_{i+1}$ $(i=1,\dots,k-1)$. Par l'hypothèse d'induction on a $d_{T_{\mathfrak{M}}}=m_1'\dots m_{k-1}'=m_2\dots m_k$; d'autre part on a $d_{T_{\mathfrak{L}}}=m_{T_{\mathfrak{L}}}$ $(=m_T=m_1)$ parce que $T_{\mathfrak{L}}$ admet un vecteur cyclique. On conclut que

$$d_T=d_{T_{\mathfrak{L}}}d_{T_{\mathfrak{M}}}=m_1m_2\cdots m_k$$

ce qui achève la démonstration.

Notons la conséquence suivante de la relation (1. 7):

Corollaire 2. *Un opérateur T d'une classe $C_0(N)$ $(N\geqq 1)$ et sa restriction T' à un sous-espace invariant propre ne peuvent avoir le même modèle de Jordan.*

Démonstration. Si $T=\begin{bmatrix}T' & *\\ O & T''\end{bmatrix}$ est la triangulation de T par rapport au sous-espace invariant en question et son complément orthogonal, on a $d_T=d_{T''}d_{T'}$ où la fonction $d_{T''}$ n'est pas constante (de module 1) parce que $d_{T''}(T'')=O$. Si T et T' avaient le même modèle de Jordan, on aurait $d_T=d_{T'}$ en vertu de (1. 7); contradiction.

2. Démonstration de la proposition 1

Soit T un opérateur dans l'espace $\mathfrak{H}$, de classe $C_0(N)$, $N\geqq 1$. Comme T^* est alors de même type, on obtient par la proposition B qu'il existe un sous-espace $\mathfrak{L}$ invariant pour T^* tel que $T^*|\mathfrak{L}$ admette un vecteur cyclique dans $\mathfrak{L}$ et qu'on ait

$$m_{T^*|\mathfrak{L}}=m_{T^*}. \tag{2. 1}$$

Comme $\mathfrak{M}=\mathfrak{H}\ominus\mathfrak{L}$ est alors invariant pour T, la décomposition

$$\mathfrak{H}=\mathfrak{L}\oplus\mathfrak{M}$$

engendre pour T et pour la projection orthogonale $P_{\mathfrak{L}}$ de $\mathfrak{H}$ à $\mathfrak{L}$ les formes matricielles

$$T = \begin{bmatrix} T_{\mathfrak{L}} & O \\ * & T_{\mathfrak{M}} \end{bmatrix} \quad \text{et} \quad P_{\mathfrak{L}} = \begin{bmatrix} I_{\mathfrak{L}} & O \\ O & O \end{bmatrix},$$

d'où l'on obtient pour $n=0, 1, 2, \ldots$

$$\text{(2. 2)} \qquad P_{\mathfrak{L}} T^n = \begin{bmatrix} I_{\mathfrak{L}} & O \\ O & O \end{bmatrix} \begin{bmatrix} T_{\mathfrak{L}}^n & O \\ * & T_{\mathfrak{M}}^n \end{bmatrix} = \begin{bmatrix} T_{\mathfrak{L}}^n & O \\ O & O \end{bmatrix} = T_{\mathfrak{L}}^n P_{\mathfrak{L}}.$$

Comme on a $T_{\mathfrak{L}} = (T^*|\mathfrak{L})^*$, il s'ensuit de la proposition A que $T_{\mathfrak{L}}$ admet aussi un vecteur cyclique, soit f, donc on a

$$\mathfrak{L} = \bigvee_{n=0}^{\infty} T_{\mathfrak{L}}^n f.$$

Posons

$$\mathfrak{L}_1 = \bigvee_{n=0}^{\infty} T^n f;$$

de (2. 2) il résulte aussitôt que

$$\text{(2. 3)} \qquad \mathfrak{L} = \overline{P_{\mathfrak{L}} \mathfrak{L}_1}.$$

De plus, comme $f \in \mathfrak{L} \cap \mathfrak{L}_1$, on a

$$u(T_{\mathfrak{L}})f = P_{\mathfrak{L}} u(T) f = P_{\mathfrak{L}} u(T|\mathfrak{L}_1) f$$

pour toute fonction $u \in H^\infty$. On a donc en particulier

$$m_{T|\mathfrak{L}_1}(T_{\mathfrak{L}}) f = 0.$$

Puisque f est cyclique pour $T_{\mathfrak{L}}$, cela entraîne

$$\text{(2. 4)} \qquad m_{T|\mathfrak{L}_1}(T_{\mathfrak{L}}) = O.$$

Or (2. 1) entraîne

$$\text{(2. 5)} \qquad m_{T_{\mathfrak{L}}} = m_{(T^*|\mathfrak{L})^*} = (m_{T^*|\mathfrak{L}})^\sim = (m_{T^*})^\sim = m_T.$$

Ainsi on déduit de (2. 4) que m_T doit être un diviseur de $m_{T|\mathfrak{L}_1}$. Comme $T|\mathfrak{L}_1$ est la restriction de T à un sous-espace invariant, on conclut que

$$\text{(2. 6)} \qquad m_{T|\mathfrak{L}_1} = m_T.$$

Comme $T|\mathfrak{L}_1$ appartient à une classe $C_0(N_1)$, $N_1 \leqq N$, et admet le vecteur cyclique f, on déduit de la proposition D et de (2. 6) que $T|\mathfrak{L}_1 \succ S(m_T)$; donc il existe une quasi-affinité

$$X\colon \mathfrak{H}(m_T) \to \mathfrak{L}_1$$

telle que

$$\text{(2. 7)} \qquad X S(m_T) = (T|\mathfrak{L}_1) X = TX.$$

D'autre part, (2. 5) entraîne

$$m_{T^*_{\mathfrak{L}}} = (m_{T_{\mathfrak{L}}})^{\sim} = (m_T)^{\sim} = (m_{S(m_T)})^{\sim} = m_{S(m_T)^*} = m_{S(m_{\tilde{T}})},$$

et il s'ensuit de nouveau de la proposition D que $T^*_{\mathfrak{L}} \succ S(m_{\tilde{T}})\,(\succ S(m_T)^*)$ et que par conséquent $T_{\mathfrak{L}} \prec S(m_T)$. Donc il existe une quasi-affinité

$$Y\colon \mathfrak{L} \to \mathfrak{H}(m_T)$$

telle que

(2. 8) $$S(m_T)Y = YT_{\mathfrak{L}}.$$

Ecrivons, pour simplifier, m au lieu de m_T et définissons

$$\mathfrak{H}' = \mathfrak{H}(m) \oplus \mathfrak{M} \quad \text{et} \quad T' = S(m) \oplus T_{\mathfrak{M}};$$

T' est une contraction dans $\mathfrak{H}'$.

Nous voulons montrer que $T' \prec T$. A cet effet, envisageons l'opérateur

$$X'\colon \mathfrak{H}' \to \mathfrak{H},$$

défini par la formule

(2. 9) $$X'(g \oplus h) = Xg + h \quad \text{pour} \quad g \in \mathfrak{H}(m) \quad \text{et} \quad h \in \mathfrak{M}.$$

X' est évidemment linéaire et borné, et (2. 7) entraîne que

$$TX'(g \oplus h) = T(Xg + h) = XS(m)g + T_{\mathfrak{M}}h =$$
$$= X'\big(S(m)g \oplus T_{\mathfrak{M}}h\big) = X'T'(g \oplus h),$$

d'où

(2. 10) $$TX' = X'T'.$$

Reste à montrer que X' est une quasi-affinité de $\mathfrak{H}'$ à $\mathfrak{H}$, c'est-à-dire que

(2. 11) $$\overline{X'\mathfrak{H}'} = \mathfrak{H}$$

et que

(2. 12) $$X'(g \oplus h) = 0 \quad \text{entraîne} \quad g \oplus h = 0.$$

Or, la relation (2. 11) dérive d'une manière simple des relations (2. 9) et (2. 3) et de ce que X est une quasi-affinité de $\mathfrak{H}(m)$ à $\mathfrak{L}_1$. On a notamment

$$\overline{X'\mathfrak{H}'} = \overline{X\mathfrak{H}(m) + \mathfrak{M}} = \overline{\mathfrak{L}_1 + \mathfrak{M}} = \overline{P_{\mathfrak{L}}\mathfrak{L}_1 + \mathfrak{M}} = \overline{\mathfrak{L} + \mathfrak{M}} = \mathfrak{L} \oplus \mathfrak{M} = \mathfrak{H}.$$

Quant à (2. 12), cela veut dire que $Xg = -h$ pour un $g \in \mathfrak{H}(m)$ et un $h \in \mathfrak{M}$ entraîne $g = 0$, c'est-à-dire que l'opérateur $P_{\mathfrak{L}}X$ est inversible, ou, ce qui revient au même, que l'opérateur

$$W = YP_{\mathfrak{L}}X\colon \mathfrak{H}(m) \to \mathfrak{H}(m)$$

est inversible.

Observons d'abord que par (2. 3) et puisque X et Y sont des quasi-affinités, on a

$$\overline{W\mathfrak{H}(m)} = \overline{YP_{\mathfrak{L}}X\mathfrak{H}(m)} = \overline{YP_{\mathfrak{L}}\mathfrak{L}_1} = \overline{Y\mathfrak{L}} = \mathfrak{H}(m). \tag{2. 13}$$

D'autre part, on déduit de (2. 2), (2. 7) et (2. 8) que

$$S(m)\,W = S(m)Y\,P_{\mathfrak{L}}\,X = YT_{\mathfrak{L}}P_{\mathfrak{L}}X = YP_{\mathfrak{L}}TX = YP_{\mathfrak{L}}XS(m) = W\,S(m).$$

Faisons usage d'un théorème de SARASON [6] suivant lequel tout opérateur borné permutable à $S(m)$ est de la forme $\varphi(S(m))$ avec $\varphi \in H^\infty$. Donc on a

$$W = w(S(m)) \quad \text{où} \quad w \in H^\infty. \tag{2. 14}$$

Puisque (2. 13) entraîne $W \neq O$, on a certainement $w \not\equiv 0$.

Soit $g \in \mathfrak{H}(m)$ tel que $Wg = 0$; par (2. 14) cela veut dire que

$$P_{\mathfrak{H}(m)}wg = 0, \quad \text{donc} \quad wg \in mH^2, \quad wg = mu,$$

avec un $u \in H^2$. Supposons que $g \not\equiv 0$. On a alors aussi $u \not\equiv 0$ et par conséquent nous pouvons prendre les factorisations canoniques

$$w = w_i w_e, \quad g = g_i g_e, \quad u = u_i u_e$$

des fonctions $w \in H^\infty$ et $g, u \in H^2$, en produit de leurs facteurs intérieurs et extérieurs. La relation $wg = mu$ entraîne $w_i g_i = mu_i$ et par conséquent

$$g_i w = mw' \quad \text{avec} \quad w' = u_i w_e \in H^2,$$

d'où

$$g_i(S(m))\,w(S(m)) = m(S(m))\,w'(S(m)) = O$$

parce que $m(S(m)) = O$. Grâce à la relation (2. 13) on en déduit que $g_i(S(m)) = O$, ce qui veut dire que

$$P_{\mathfrak{H}(m)}(g_i v) = 0, \quad \text{donc} \quad g_i v \in mH^2$$

pour toute fonction $v \in \mathfrak{H}(m)$. Choisissons en particulier $v = 1 - \overline{m(0)}m$: nous obtenons ainsi que $g_i \in mH^2$, donc $g_i = mh$, avec un $h \in H^2$. Comme g_i et m sont des fonctions intérieures, il en est de même de h. On a donc $g = mhg_e \in mH^2$. Comme $g \in \mathfrak{H}(m)$, on conclut que $g = 0$: contradiction. Donc $Wg = 0$ entraîne $g = 0$: W est inversible, et la démonstration est terminée.

3. Démonstration de la proposition 2

1. Nous commençons par un lemme qui nous sera utile aussi dans le n°5.

Lemme 1. *Soient m et m' deux fonctions intérieures non-constantes, dont m est un diviseur de m', donc*

$$m' = mq \tag{3. 1}$$

avec une fonction q intérieure. Envisageons les espaces

$$\mathfrak{H}=\mathfrak{H}(m) \quad (= H^2 \ominus mH^2), \qquad \mathfrak{H}'=\mathfrak{H}(m') \quad (= H^2 \ominus m'H^2)$$

et leurs opérateurs $S=S(m)$ et $S'=S(m')$. Dans ces conditions,

$$\mathfrak{H}^0 = qH^2 \ominus m'H$$

est un sous-espace de $\mathfrak{H}'$ invariant pour S' et l'opérateur $S^0 = S'|\mathfrak{H}^0$ est unitairement équivalent à S. Notamment, R: $u \to qu$ $(u \in \mathfrak{H})$ est une transformation unitaire de $\mathfrak{H}$ à $\mathfrak{H}^0$ et on a

(3. 2) $$S^0 R = RS.$$

D'autre part, l'opérateur $q(S')$ applique $\mathfrak{H}'$ sur $\mathfrak{H}^0$ et l'opérateur

(3. 3) $$Q = R^{-1} q(S')$$

applique $\mathfrak{H}'$ sur $\mathfrak{H}$:

(3. 4) $$Q\mathfrak{H}' = \mathfrak{H}.$$

De plus on a

(3. 5) $$SQ = QS'.$$

Démonstration. Le fait que R est unitaire, de $\mathfrak{H}$ à $\mathfrak{H}^0$, découle immédiatement de (3. 1) et de ce que q est une fonction intérieure.

Observons ensuite que pour $v \in \mathfrak{H}$ et $u=qv$ $(\in \mathfrak{H}^0)$ on a

$$q \cdot Sv = q \cdot P_{\mathfrak{H}}(\lambda v) = q(\lambda v + mv_1) = \lambda u + m' w_1$$

et

$$S'u = P_{\mathfrak{H}'}(\lambda u) = \lambda u + m' w_2$$

où $w_1, w_2 \in H^2$. Il s'ensuit que

$$S'u - q \cdot Sv \in m'H^2.$$

D'autre part, $S'u \in \mathfrak{H}'$ et $q \cdot Sv \in q\mathfrak{H} = \mathfrak{H}^0 \subset \mathfrak{H}'$, donc $S'u - q \cdot Sv \in \mathfrak{H}'$. Ainsi on a nécessairement $S'u - q \cdot Sv = 0$, donc

$$S'(qv) = q \cdot Sv \qquad (v \in \mathfrak{H}).$$

Cela montre que $\mathfrak{H}^0$ $(=q\mathfrak{H})$ est invariant pour S' et que $S^0 = S'|\mathfrak{H}^0$ vérifie la relation (3. 2).

Observons ensuite que

$$q(S')\mathfrak{H}' = P_{\mathfrak{H}'}(q\mathfrak{H}') = P_{\mathfrak{H}'} q(H^2 \ominus m'H^2) =$$

$$= P_{\mathfrak{H}'}[q(H^2 \ominus mH^2) \oplus m'(H^2 \ominus qH^2)] = P_{\mathfrak{H}'}(q\mathfrak{H}) = P_{\mathfrak{H}'}\mathfrak{H}^0 = \mathfrak{H}^0$$

parce que $\mathfrak{H}^0 \subset \mathfrak{H}'$. En appliquant R^{-1} on en déduit la relation (3. 4). Finalement, (3. 1) et (3. 2) entraînent

$$SQ = SR^{-1}q(S') = R^{-1}S^0 q(S') = R^{-1}S'q(S') = R^{-1}q(S')S' = QS',$$

c'est-à-dire la relation (3. 5). Le lemme est démontré.

2. Pour démontrer la proposition 2, envisageons des fonctions intérieures $m_1, \ldots, m_K$ ayant pour diviseur intérieur commun une fonction intérieure non-constante m, et supposons que l'opérateur $S(m_1)\oplus\cdots\oplus S(m_K)$ est la transformée quasi-affine d'un opérateur T de classe $C_0(N)$. D'après le lemme 1 l'opérateur $S(m)$ est, pour $k=1, \ldots, K$, unitairement équivalent à la restriction de $S(m_k)$ à un sous-espace invariant; par conséquent l'opérateur

$$S^{(K)}(m) = S(m)\oplus\cdots\oplus S(m) \qquad (K \text{ termes})$$

est unitairement équivalent à la restriction de $S(m_1)\oplus\cdots\oplus S(m_K)$ à un sous-espace invariant. Mais alors $S^{(K)}(m)$ est la transformée quasi-affine d'une restriction T' de T à un sous-espace invariant (voir [8])), donc d'un opérateur de classe $C_0(N')$ avec $N'\leqq N$. Si l'on montre que cela entraîne $K\leqq N'$, on aura démontré à fortiori $K\leqq N$.

Il est manifeste que T' peut être remplacé dans nos considérations par son modèle fonctionnel, c'est-à-dire par un opérateur $S(\Theta)$ où Θ est une fonction matricielle d'ordre N', contractive pure et intérieure. D'autre part, $S^{(K)}(m)$ est évidemment égal à $S(m\, I_K)$ où I_K désigne l'opérateur unité dans E^K.

Par l'hypothèse faite, il existe une quasi-affinité

$$X\colon\ \mathfrak{H}(m\, I_K) \to \mathfrak{H}(\Theta)$$

pour laquelle

$$S(\Theta)X = X\, S(mI_K). \tag{3.6}$$

Cela entraîne, en vertu de la généralisation du théorème de Sarason donnée par les auteurs dans [4], qu'il existe une fonction analytique bornée $\{E^K, E^{N'}, A(\lambda)\}$ telle que

$$X = P_{\mathfrak{H}(\Theta)} A|\mathfrak{H}(m\, I_K). \tag{3.7}$$

D'autre part, on déduit de (3. 6) que l'opérateur $S(\Theta)$ a la même fonction minimum que l'opérateur $S(m\, I_K)$, donc m. Par conséquent il existe une fonction matricielle analytique bornée $\Omega(\lambda)$, d'ordre N', telle que

$$\Omega(\lambda)\Theta(\lambda) = \Theta(\lambda)\Omega(\lambda) = m(\lambda)\, I_{N'}.\text{[10])} \tag{3.8}$$

Envisageons alors la fonction analytique bornée $\{E^K, E^{N'}, B(\lambda)\}$ définie par

$$B(\lambda) = \Omega(\lambda)A(\lambda)$$

et soit

$$u\in H^2(E^K) \quad \text{tel que} \quad Bu\in m H^2(E^{N'}).$$

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[10]) $\Omega(\lambda)$ est l'adjointe algébrique de la matrice $\Theta(\lambda)$, divisée par le plus grand diviseur intérieur de ses éléments (fonctions dans H^∞).

On déduit de (3. 8) que

$$m\,Au = \Theta\Omega Au = \Theta Bu \subset \Theta\, m H^2(E^{N'}) = m\Theta H^2(E^{N'}),$$

d'où

$$Au \in \Theta H^2(E^{N'}).$$

Posons $v = P_{\mathfrak{H}(mI_K)}u$. On a alors $u - v \in mH^2(E^K)$, d'où, par (3. 8),

$$A(u-v) \in AmH^2(E^K) = mAH^2(E^K) = \Theta\Omega AH^2(E^K) \subset \Theta H^2(E^{N'}).$$

Ainsi on a

$$Av = Au - A(u-v) \in \Theta H^2(E^N)$$

et par conséquent $Xv = P_{\mathfrak{H}(\Theta)}Av = 0$. Puisque X est inversible, cela entraîne $v=0$, donc $u \perp \mathfrak{H}(m\,I_K)$, $u \in mH^2(E^K)$. Résumons:

(3. 9) $\qquad u \in H^2(E^K)$ et $Bu \in mH^2(E^{N'})$ entraînent $u \in mH^2(E^K)$.

Soient $b_{ij}(\lambda)$ ($i=1, \ldots, N'$; $j=1, \ldots, K$) les éléments de la matrice $B(\lambda)$; les fonctions $b_{ij}(\lambda)$ appartiennent à H^∞. Il est impossible que toutes ces fonctions soient divisibles par $m(\lambda)$ parce que, autrement, (3. 9) entraînerait $u \in mH^2(E^K)$ pour *toute fonction* $u \in H^2(E^K)$ et par conséquent $m(\lambda)$ serait constante: contradiction.

Il existe donc un mineur de la matrice $[b_{ij}(\lambda)]$ dont le déterminant $\Delta(\lambda)$ ($\in H^\infty$) n'est pas divisible par $m(\lambda)$ et dont l'ordre est maximal. Il ne restreint pas la généralité de supposer que c'est le mineur

$$[b_{ij}(\lambda)] \qquad (i=1, \ldots, r;\ j=1, \ldots, r).$$

On a évidemment $1 \leqq r \leqq \min\{K, N'\}$:

Supposons que $K > N'$ et montrons que cela nous conduit à une contradiction. Envisageons à cet effet le déterminant

$$\begin{vmatrix} b_{11} & \ldots & b_{1r} & b_{1,r+1} \\ \vdots & & & \vdots \\ b_{r1} & \ldots & b_{rr} & b_{r,r+1} \\ x_1 & \ldots & x_r & x_{r+1} \end{vmatrix} = \sum_{j=1}^{r+1} x_j u_j$$

que nous avons développé suivant sa dernière ligne. On a alors

$$\sum_{j=1}^{r+1} b_{ij}u_j = \begin{cases} 0 \text{ pour } i=1, \ldots, r; \\ \text{le déterminant d'un mineur d'ordre } r+1 \text{ de } B(\lambda) \text{ pour } i = r+1, \ldots, N'. \end{cases}$$

Par conséquent, en posant

$$u = [u_1, \ldots, u_{r+1}, 0, \ldots, 0] \qquad (K \text{ composantes})$$

on aura $u \in H^2(E^K)$ tel que $(Bu)_i$ est divisible par m pour $i=1, \ldots, N'$, donc

$Bu \in mH^2(E^{N'})$. En vertu de (3.9) cela entraîne que $u \in mH^2(E^K)$, ce qui est impossible puisque u_{r+1} $(=\Delta)$ n'est pas divisible par m.

On a donc nécessairement $K \leqq N'$, ce qui achève la démonstration de la proposition 2.

4. Démonstration de la proposition 3

1. Posons $S = S(m_1, \ldots, m_K)$ et $S' = S(m'_1, \ldots, m'_{K'})$. Il suffit de démontrer que $K \leqq K'$ et que m_j est un diviseur de m'_j pour $j = 1, \ldots, K$, car les relations de sens opposé s'ensuivent de celles-ci en les appliquant au couple $\{S'^*, S^*\}$ au lieu du couple $\{S, S'\}$.

L'inégalité $K \leqq K'$ découle immédiatement de la proposition 2 parce qu'il est évident que $S' \in C_0(K')$.

Par hypothèse il existe une quasi-affinité

$$X\colon \mathfrak{H} = \mathfrak{H}(m_1) \oplus \cdots \oplus \mathfrak{H}(m_K) \to \mathfrak{H}(m'_1) \oplus \cdots \oplus \mathfrak{H}(m'_{K'}) = \mathfrak{H}'$$

telle que $S'X = XS$.

Fixons un k, $1 \leqq k \leqq K$, et posons

$$\mathfrak{M}_k = \overline{m'_k(S)\mathfrak{H}} \quad \text{et} \quad \mathfrak{M}'_k = \overline{m'_k(S')\mathfrak{H}'}.$$

Puisque

$$\overline{X\mathfrak{M}_k} = \overline{Xm'_k(S)\mathfrak{H}} = \overline{m'_k(S')X\mathfrak{H}} = \overline{m'_k(S')\mathfrak{H}'} = \mathfrak{M}'_k,$$

on voit que $X_k = X|\mathfrak{M}_k$ est une quasi-affinité de $\mathfrak{M}_k$ à $\mathfrak{M}'_k$. Il est manifeste que $\mathfrak{M}_k$ est invariant pour S, et $\mathfrak{M}'_k$ pour S', et que de plus les restrictions

$$Z_k = S|\mathfrak{M}_k, \quad Z'_k = S'|\mathfrak{M}'_k$$

vérifient la relation

$$Z'_k X_k = X_k Z_k;$$

Z_k est donc une transformée quasi-affine de Z'_k.

Introduisons les notations suivantes:

$$\mathfrak{H}_i = \mathfrak{H}(m_i), \quad S_i = S(m_i), \quad \mathfrak{H}_{ik} = \overline{m'_k(S_i)\mathfrak{H}_i}, \quad S_{ik} = S_i|\mathfrak{H}_{ik} \qquad (i = 1, \ldots, K),$$

$$\mathfrak{H}'_j = \mathfrak{H}(m'_j), \quad S'_j = S(m'_j), \quad \mathfrak{H}'_{jk} = \overline{m'_k(S'_j)\mathfrak{H}'_j}, \quad S'_{jk} = S'_j|\mathfrak{H}'_{jk} \qquad (j = 1, \ldots, K');$$

il est évident que

$$Z_k = S_{1k} \oplus \cdots \oplus S_{Kk} \quad \text{et} \quad Z'_k = S'_{1k} \oplus \cdots \oplus S'_{K'k}.$$

Chacun des termes aux seconds membres est un opérateur de classe $C_0(N)$ avec $N = 1$ ou 0, et d'après le lemme 2 qu'on va établir tout de suite, les fonctions minimum correspondantes sont

$$(4.1) \quad m_{S_{ik}} = \frac{m_i}{m_i \wedge m'_k} \quad (i = 1, \ldots, K) \quad \text{et} \quad m_{S'_{jk}} = \frac{m'_j}{m'_j \wedge m'_k} \quad (j = 1, \ldots, K'),$$

où le signe $\wedge$ indique le plus grand diviseur intérieur des fonctions intérieures en question.

Désignons les fonctions (4. 1) plus simplement par m_{ik} et m'_{jk}, selon les cas. S_{ik} est unitairement équivalent à $S(m_{ik})$ et S'_{jk} est unitairement équivalent à $S(m'_{jk})$, donc leurs sommes orthogonales Z_k et Z'_k sont unitairement équivalentes à

$$U_k = S(m_{1k}) \oplus \cdots \oplus S(m_{Kk}) \quad \text{et} \quad U'_k = S(m'_{1k}) \oplus \cdots \oplus S(m'_{K'k}),$$

selon les cas. Il s'ensuit que U_k est une transformée quasi-affine de U'_k. De plus on a $U_k \in C_0(N_k)$ et $U'_k \in C_0(N'_k)$ où N_k est le nombre des fonctions m_{ik} $(i=1, \ldots, K)$ non-constantes, et de même pour N'_k. Or, en vertu du lemme 2, $m_{i+1,k}$ est un diviseur de m_{ik}, et $m'_{j+1,k}$ est un diviseur de m'_{jk}. Puisque $m'_{kk}=1$, on a donc $N'_k \leqq k-1$.

Supposons que m_k n'est pas un diviseur de m'_k. Dans ce cas $m_k \wedge m'_k$ ne coïncide pas avec m_k, donc m_{kk} n'est pas constante et par conséquent $N_k \geqq k$. Comme on peut écarter de la somme orthogonale U_k les termes égaux à O (donc correspondant aux fonctions m_{ik} constantes), on obtient en appliquant la proposition 2 que

$$k \leqq N_k \leqq N'_k \leqq k-1.$$

Cette contradiction prouve que m_k est un diviseur de m'_k.

Comme k était arbitraire, la démonstration de la proposition est complète.

2. Nous avons fait usage du suivant

Lemme 2. a) *Soit T une contraction de classe C_0 dans l'espace $\mathfrak{H}$ et soit p une fonction scalaire intérieure quelconque. Le sous-espace $\mathfrak{H}_p = \overline{p(T)\mathfrak{H}}$ est invariant pour T et la restriction $T_p = T|\mathfrak{H}_p$ a sa fonction minimum m_{T_p} égale à $m_T/(m_T \wedge p)$.*

b) *Si m, m', p sont des fonctions scalaires intérieures et m est un diviseur de m', alors $m/(m \wedge p)$ est un diviseur de $m'/(m' \wedge p)$.*

Démonstration. a) L'invariance de $\mathfrak{H}_p$ pour T est évidente. Pour une fonction scalaire intérieure quelconque u on a

$$\overline{u(T_p)\mathfrak{H}_p} = \overline{u(T)\overline{p(T)\mathfrak{H}}} = \overline{u(T)p(T)\mathfrak{H}}.$$

Pour qu'on ait $u(T_p)=O$ il faut donc et il suffit que $u(T)p(T)=O$, c'est-à-dire que up soit un multiple de m_T. Or, si $up=m_T v$ pour une fonction intérieure v, on a

$$(4.2) \qquad up' = m'v \quad \text{pour} \quad p' = \frac{p}{m_T \wedge p} \quad \text{et} \quad m' = \frac{m_T}{m_T \wedge p};$$

comme $p' \wedge m' = 1$, on déduit de (4. 2) que u est divisible par m'. D'autre part, $u=m'$ vérifie (4. 2) avec $v=p'$. On conclut que $m_{T_p}=m'$.

b) Une partie du raisonnement ci-dessus fournit que pour des fonctions intérieures m, p quelconques on a

$$\frac{m}{m \wedge p} = \bigwedge_{u \in A(m,p)} u$$

où $A(m, p)$ désigne l'ensemble des fonctions intérieures u pour lesquelles up est divisible par m. Or, si m est un diviseur de m', on a $A(m, p) \supset A(m', p)$ et par conséquent $\bigwedge_{u \in A(m,p)} u$ est un diviseur de $\bigwedge_{u \in A(m',p)} u$.

PARTIE II

5. Le théorème sur le bicommutant et sa démonstration

Pour une contraction complètement non-unitaire T dans l'espace $\mathfrak{H}$ on désigne par N_T la classe des fonctions $\varphi(\lambda) = \dfrac{u(\lambda)}{v(\lambda)}$ telles que $u, v \in H^\infty$ et que $v(T)$ a un inverse à domaine dense dans $\mathfrak{H}$. Pour telle fonction φ on définit $\varphi(T) = v(T)^{-1}u(T)$; $\varphi(T)$ est un opérateur non nécessairement borné, mais fermé et de domaine dense dans $\mathfrak{H}$. (*Cf.* le chap. IV de l'édition anglaise de [1].)

Tout opérateur borné B dans $\mathfrak{H}$, qui permute à T, permute aussi à $\varphi(T)$:

$$\varphi(T)B \supset B\varphi(T).$$

Nous allons montrer que cette propriété est caractéristique des fonctions $\varphi(T)$, du moins pour T de type envisagé dans le théorème 2 et pour $\varphi(T)$ borné. Notamment, on a le suivant

Théorème 3. *Pour un opérateur T dans $\mathfrak{H}$, de type envisagé dans le théorème* 2, *tout opérateur $A \in (T)''$ est de la forme $A = \varphi(T)$ où $\varphi \in N_T$.*

Démonstration. D'après le théorème 2, T est quasi-similaire à un opérateur de Jordan

$$S = S(m_1) \oplus \cdots \oplus S(m_K) \quad \text{dans} \quad \mathfrak{G} = \mathfrak{H}(m_1) \oplus \cdots \oplus \mathfrak{H}(m_K).$$

Soient

$$X: \mathfrak{G} \to \mathfrak{H} \quad \text{et} \quad Y: \mathfrak{H} \to \mathfrak{G}$$

des quasi-affinités pour lesquelles

$$(5.1) \qquad TX = XS \quad \text{et} \quad SY = YT.$$

D'après le lemme 1 du n°3 il existe, pour $k = 1, \ldots, K$, un sous-espace $\mathfrak{H}_k^0$ de $\mathfrak{H}(m_1)$, invariant pour $S(m_1)$ et tel que

$$S_k^0 = S(m_1)|\mathfrak{H}_k^0$$

soit unitairement équivalent à $S(m_k)$:

$$(5.2) \qquad S_k^0 R_k = R_k S(m_k) \quad \text{avec} \quad R_k: \mathfrak{H}(m_k) \to \mathfrak{H}_k^0 \quad \text{unitaire}.$$

Notons que $\mathfrak{H}_1^0=\mathfrak{H}(m_1)$ et $S_1^0=S(m_1)$. De plus il existe un opérateur borné Q_k de $\mathfrak{H}(m_1)$ à $\mathfrak{H}(m_k)$ tel que

$$S(m_k)Q_k=Q_k S(m_1) \quad \text{et} \quad Q_k\mathfrak{H}(m_1)=\mathfrak{H}(m_k). \tag{5. 3}$$

Auprès de l'opérateur S dans $\mathfrak{G}$, envisageons aussi les opérateurs

$$S^0 = S_1^0\oplus\cdots\oplus S_K^0 \quad \text{dans} \quad \mathfrak{G}^0 = \mathfrak{H}_1^0\oplus\cdots\oplus\mathfrak{H}_K^0$$

et

$$\hat{S} = S(m_1)\oplus\cdots\oplus S(m_1) \quad \text{dans} \quad \hat{\mathfrak{G}} = \mathfrak{H}(m_1)\oplus\cdots\oplus\mathfrak{H}(m_1) \qquad (K \text{ termes}).$$

Posons

$$R = R_1\oplus\cdots\oplus R_K \quad \text{et} \quad Q = Q_1\oplus\cdots\oplus Q_K;$$

il s'ensuit de (5. 2) et (5. 3) que R est unitaire,

$$R\mathfrak{G} = \mathfrak{G}^0, \quad S^0R = RS, \tag{5. 4}$$

et

$$Q\hat{\mathfrak{G}} = \mathfrak{G}, \quad SQ = Q\hat{S}. \tag{5. 5}$$

De (5. 1) et (5. 4) on obtient $S^0RY=RYT$; comme S^0 est évidemment la restriction de S à $\mathfrak{G}^0$, cette relation peut s'écrire aussi sous la forme

$$\hat{S}RY=RYT. \tag{5. 6}$$

Soit W un opérateur borné quelconque dans $\hat{\mathfrak{G}}$ permutant à $\hat{S}$. Grâce à (5. 1), (5. 5) et (5. 6) on a

$$TXQWRY=XSQWRY=XQ\hat{S}WRY=XQW\hat{S}RY=XQWRSY=XQWRYT,$$

donc $XQWRY$ permute à T. Il permute alors à tout opérateur A dans $(T)''$, donc on a

$$AXQ\cdot W\cdot RY = XQ\cdot W\cdot RYA.$$

En posant

$$B=RYAXQ \quad \text{et} \quad C=RYXQ$$

il en dérive la relation

$$BWC=CWB. \tag{5. 7}$$

Observons que B et C permutent à $\hat{S}$; en effet on a

$$\hat{S}B = \hat{S}RYAXQ \overset{(5.6)}{=\!=} RYTAXQ = RYATXQ \overset{(5.1)}{=\!=} RYAXSQ \overset{(5.5)}{=\!=} RYAXQ\hat{S} = B\hat{S}$$

et de même pour C (cas $A=I$).

Par leur définition, les opérateurs B et C appliquent l'espace $\hat{\mathfrak{G}}$ dans l'espace $\mathfrak{G}^0$. Comme $\mathfrak{G}^0\subset\hat{\mathfrak{G}}$, on peut les considérer aussi comme des opérateurs dans $\hat{\mathfrak{G}}$. Soient

$$W=[W_{ij}], \quad B=[B_{ij}], \quad C=[C_{ij}] \qquad (i,j=1,\dots,K)$$

les matrices de ces opérateurs correspondant à la décomposition $\hat{\mathfrak{G}} = \mathfrak{H}(m_1) \oplus \cdots \cdots \oplus \mathfrak{H}(m_1)$. Les éléments de ces matrices sont des opérateurs bornés dans $\mathfrak{H}(m_1)$ permutant à $S(m_1)$.

Comme W peut être une matrice quelconque $[W_{ij}]$ dont les éléments permutent à $S(m_1)$, on peut choisir en particulier $W_{ij} = I$ pour $(i, j) = (k, 1)$ et $W_{ij} = O$ pour $(i, j) \neq (k, 1)$. On obtient alors de (5. 7)

(5. 8) $$B_{ik} C_{1j} = C_{ik} B_{1j} \qquad (i, k, j = 1, \ldots, K).$$

Observons que $C\ (= RYXQ)$ applique $\hat{\mathfrak{G}}$ sur une variété dense dans $\mathfrak{G}^0$. Cela s'ensuit de (5. 4), (5. 5) et de ce que X et Y sont des quasi-affinités. Les composantes de rang 1 des vecteurs dans $C\hat{\mathfrak{G}}$ font alors une variété dense dans la composante de rang 1 de $\mathfrak{G}^0$, c'est-à-dire dans $\mathfrak{H}(m_1)$. Cela veut dire que les éléments de la forme

$$\sum_{j=1}^{K} C_{1j} g_j \quad (g_j \in \mathfrak{H}(m_1))$$

sont denses dans $\mathfrak{H}(m_1)$, donc pour tout $g \in \mathfrak{H}(m_1)$ il existe des $g_j^{(n)} \in \mathfrak{H}(m_1)$ $(j = 1, \ldots, K;\ n = 1, 2, \ldots)$ tels que

$$g = \lim_{n \to \infty} \sum_{j=1}^{K} C_{1j} g_j^{(n)},$$

d'où en vertu de (5. 8) on déduit que

(5. 9) $$B_{ik} g = \lim_{n \to \infty} C_{ik} \sum_{j=1}^{K} B_{1j} g_j^{(n)} \qquad (i, k = 1, \ldots, K).$$

Comme B_{ik} et C_{ik} permutent à $S(m_1)$, il s'ensuit par le théorème déjà cité de SARASON [6] qu'il existe des fonctions $b_{ik}, c_{ik} \in H^\infty$ pour lesquelles

$$B_{ik} = b_{ik}(S(m_1)) \quad \text{et} \quad C_{ik} = c_{ik}(S(m_1)).$$

Les relations (5. 8) impliquent

(5. 10) $$b_{ik} c_{1j} - c_{ik} b_{1j} \in m_1 H^2 \qquad (i, k, j = 1, \ldots, K)$$

et (5. 9) entraîne, en l'appliquant au cas $g = 1 - \overline{m_1(0)}\, m_1\ (\in \mathfrak{H}(m_1))$, que

(5. 11) $$b_{ik} = \lim_{n \to \infty} [c_{ik} h_{ik}^{(n)} + m_1 l_{ik}^{(n)}]$$

où $h_{ik}^{(n)}, l_{ik}^{(n)} \in H^2$ et la convergence est dans la métrique hilbertienne de H^2. De (5. 11) on déduit que tout diviseur intérieur commun de c_{ik} et m_1 est un diviseur de b_{ik} aussi.

Envisageons en particulier le plus grand diviseur intérieur commun de c_{1j} et m_1, que nous désignons par p_j (lorsque $c_{1j}=0$ on a $p_j=m_1$). Posons

$$b_j' = b_{1j}/p_j, \quad c_j' = c_{1j}/p_j \quad \text{et} \quad m_j' = m_1/p_j \qquad (j=1, \ldots, K);$$

en vertu de (5. 10) on a

$$b_{ik}c_j' - c_{ik}b_j' = m_j'u_{ikj}, \quad u_{ikj} \in H^2. \tag{5. 12}$$

Notons que b_j', c_j' et m_j' appartiennent à H^∞, m_j' est même une fonction intérieure, et c_j' et m_j' n'ont pas de diviseur intérieur non-constant.

Puisque chaque m_j' est un diviseur de m_1, $M=m_1' \vee \ldots \vee m_K'$ (le plus petit multiple intérieur commun de $m_1', \ldots, m_K'$) est aussi un diviseur de m_1. Observons aussi que

$$q=m_1/M$$

est un diviseur de $m_1/m_j'=p_j$ pour $j=1, \ldots, K$. Par conséquent q est un diviseur de c_{1j}, donc $c_{1j}=qd_j$ où $d_j \in H^\infty$ $(j=1, \ldots, K)$. Comme on a alors

$$\sum_{j=1}^{K} C_{1j}g_j = \sum_{j=1}^{K} c_{1j}(S(m_1))g_j = q(S(m_1)) \sum_{j=1}^{K} d_j(S(m_1))g_j$$

pour $g_j \in \mathfrak{H}(m_1)$ et que ces éléments sont denses dans $\mathfrak{H}(m_1)$, on conclut que, à fortiori,

$$\overline{q(S(m_1))\mathfrak{H}(m_1)} = \mathfrak{H}(m_1).$$

D'après le lemme 2 (n°4) cela entraîne que $m_1/(q \wedge m_1)=m_1$, $q \wedge m_1 = 1$, donc (q étant un diviseur de m_1) $q=1$. Ainsi on a

$$m_1 = m_1' \vee \ldots \vee m_K'.$$

En appliquant un lemme sur l'arithmétique des fonctions intérieures, démontré dans [2], on conclut qu'il existe des fonctions intérieures m_k'' $(k=1, \ldots, K)$ telles que

a) m_k'' est un diviseur de m_k',

b) $m_k'' \wedge m_h''=1$ pour $k \neq h$,

c) $m_1=m_1'' \vee \ldots \vee m_K''$ $(=m_1'' \cdots m_K'')$.

Nous déduisons de (5. 12) que

$$b_{ik}v - c_{ik}w = m_1 z \tag{5. 13}$$

où

$$v = \sum_{j=1}^{K} m_1'' \ldots m_{j-1}'' c_j' m_{j+1}'' \ldots m_K'', \quad w = \sum_{j=1}^{K} m_1'' \ldots m_{j-1}'' b_j' m_{j+1}'' \ldots m_K'' \in H^\infty$$

et

$$z = \frac{1}{m_1} \sum_{j=1}^{K} m_1'' \ldots m_{j-1}'' m_j' u_{ikj} m_{j+1}'' \ldots m_K'' = \sum_{j=1}^{K} \frac{m_j'}{m_j''} u_{ikj} \in H^2.$$

Soit r un diviseur intérieur commun de v et m_1 (dans H^∞). Puisque la factorisation $m_1 = m_1'' \cdots m_K''$ est en facteurs premiers deux-à-deux, on a pour r une factorisation correspondante $r = r_1 \cdots r_K$ où chaque r_k est un diviseur de m_k''. Comme m_k'' figure dans tous les termes de la somme définissant v sauf dans celui de rang $j=k$, r_k est un diviseur de tous ces termes. Comme d'autre part r_k est un diviseur de v, il doit être un diviseur aussi du terme de rang $j=k$. Or on a $m_k'' \wedge m_i'' = 1$ et à fortiori $r_k \wedge m_i'' = 1$ pour $i \neq k$, d'où il s'ensuit que r_k doit être un diviseur de c_k'. Mais c_k' et m_k' n'ont pas de diviseur commun non-constant, donc $r_k = 1$. Cela étant valable pour $k=1, \ldots, K$, on a aussi $r=1$.

Ainsi, nous venons de démontrer que les fonctions v et m_1 n'ont pas de diviseur intérieur commun non-constant. On en fera usage tout à l'heure.

De (5. 13) il dérive que

$$B_{ik} v(S(m_1)) - C_{ik} w(S(m_1)) = O \qquad (i, k = 1, \ldots, K),$$

d'où

$$Bv(\hat{S}) - Cw(\hat{S}) = O, \quad RY[AXQv(\hat{S}) - XQw(\hat{S})] = O$$

et par conséquent

$$AXQv(\hat{S}) - XQw(\hat{S}) = O. \tag{5.14}$$

Or (5. 5) entraîne $Q v(\hat{S}) = v(S)Q$ et (5. 1) entraîne $Xv(S) = v(T)X$, donc on a $XQv(\hat{S}) = v(T)XQ$ et la même chose pour w; par (5.14) il en résulte que

$$(Av(T) - w(T))XQ = O.$$

Puisque $\overline{XQ\hat{\mathfrak{G}}} = \overline{X\mathfrak{G}} = \mathfrak{H}$, *cf.* (5. 5), cela entraîne

$$Av(T) - w(T) = O.$$

Finalement, le fait que v et $m_1 (= m_T)$ n'ont pas de diviseur intérieur commun non-constant, entraîne que la fonction v appartient à la classe K_T, c'est-à-dire que $v(T)$ admet un inverse à domaine dense dans $\mathfrak{H}$.[11]) Vu aussi que, évidemment, A permute à $v(T)$, on conclut que $\varphi = w/v$ appartient à la classe N_T et qu'on a $A = \varphi(T)$.

[11]) Comme m_T n'est pas constante, le fait que v et m_T n'ont pas de diviseur commun intérieur non-constant exclue la possibilité $v=0$. Soit $v = v_e\, v_i$ la factorisation canonique de v en ses facteurs extérieur v_e et intérieur v_i. On a $v(T) = v_e(T) v_i(T)$, et $v_e(T)$ est inversible (*cf.* n° III. 3 de [1]). Puisque $m_T(T)h = 0$ pour tout $h \in \mathfrak{H}$, il s'ensuit du lemme III.4.5 de [1] que $v_i(T)h = 0$ pour un h entraîne $(v_i \wedge m_T)(T)h = 0$. Comme dans notre cas $v_i \wedge m_T = 1$, cela veut dire que $h=0$; donc $v_i(T)$ est aussi inversible. Ainsi $v(T)$ est inversible. — En appliquant le même raisonnement à $v^\sim$, $m_1^\sim$ et T^*, on obtient que $v(T)^*$ $(= v^\sim(T^*))$ est aussi inversible. Cela prouve que $v(T)^{-1}$ a domaine dense dans $\mathfrak{H}$.

Cela achève la démonstration du théorème 3.

Corollaire. *Soit $T\in C_0(N)$. Pour que T soit sans multiplicité* (dans le sens de [2]) *il faut et il suffit que l'algèbre $(T)'$, formée par les opérateurs bornés permutables à T, soit commutative.*

En effet, l'une des propriétés caractérisant les opérateurs sans multiplicité T de classe $C_0(N)$ est que tout $B\in(T)'$ est de la forme $B=\varphi(T)$ avec $\varphi\in N_T$. Donc, si T est sans multiplicité, $(T)'$ est commutatif. Inversement, si $(T)'$ est commutatif, on a $(T)'=(T)''$, et par conséquent, en vertu du théorème 3, tout opérateur dans $(T)'$ est de la forme $\varphi(T)$.

6. Le rôle de la classe N_T des fonctions

Les opérateurs A qu'on a envisagés dans le théorème 3 étaient bornés, mais les fonctions $\varphi(\lambda)$ par lesquelles on les a représentés sous la forme $A=\varphi(T)$ étaient non nécessairement bornées dans le disque unité $|\lambda|<1$. La question se pose s'il existe même une représentation $A=w(T)$ par une fonction bornée dans ce disque, c'est-à-dire par $w\in H^\infty$.

Nous allons montrer par un contre-exemple que cela n'est pas le cas. En effet, *nous construisons un opérateur $T\in C_0(2)$ et un opérateur borné B tels que B peut être représenté sous la forme $B=\varphi(T)$ avec $\varphi\in N_T$, mais ne peut pas être représenté sous la forme $B=w(T)$ avec $w\in H^\infty$.*

Pour commencer nous choisissons deux fonctions intérieures scalaires non-constantes $u(\lambda)$ et $v(\lambda)$ telles que

$$u\wedge v=1; \tag{6.1}$$

on les précisera plus tard. On définit alors la fonction $\{E^2, E^2, \Theta(\lambda)\}$ par la matrice

$$\Theta(\lambda)=\frac{1}{\sqrt{2}}\begin{bmatrix} u(\lambda) & u(\lambda)\\ v(\lambda) & -v(\lambda)\end{bmatrix}; \tag{6.2}$$

c'est évidemment une fonction intérieure, contractive pure, donc l'opérateur $T=S(\Theta)$ qu'elle engendre dans l'espace $\mathfrak{H}(\Theta)$ est de classe $C_0(2)$. La condition (6.1) entraîne que la fonction minimum de T est égale au déterminant de la matrice $\Theta(\lambda)$. En vertu du théorème 2 de [2], T est alors sans multiplicité et par suite tout opérateur $B\in(T)'$ est de la forme $B=\varphi(T)$ avec $\varphi\in N_T$.

Envisageons en particulier l'opérateur B défini dans $\mathfrak{H}(\Theta)$ par

$$Bh=P_{\mathfrak{H}(\Theta)}Jh \quad \text{où } J=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}.$$

B est évidemment borné, et comme on a

$$(6.3) \qquad J\Theta = \Theta J' \quad \text{où} \quad J' = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

B permute à T (*cf.* théorème 2 de [4]). Ainsi on a $B=\varphi(T)$ pour une fonction $\varphi \in N_T$. Supposons qu'on a aussi

$$(6.4) \qquad B = w(T)$$

pour une fonction $w \in H^\infty$. Or, pour notre $T = S(\Theta)$, l'opérateur $w(T)$ s'obtient par la formule

$$w(T)h = P_{\mathfrak{H}(\Theta)}(wh) \qquad (h \in \mathfrak{H}(\Theta)),$$

ainsi (6. 4) veut dire que $P_{\mathfrak{H}(\Theta)}(wh - Jh) = 0$, donc

$$(6.5) \qquad wh - Jh \in \Theta H^2(E^2)$$

et cela pour tout $h \in \mathfrak{H}(\Theta)$. Comme tout élément de la forme $h = (I_2 - \Theta(\lambda)\Theta(0)^*)x$, où $x \in E^2$, appartient à $\mathfrak{H}(\Theta)$, on obtient de (6. 5), en faisant usage aussi de (6. 3), que pour tout $x \in E^2$ il existe un $y \in H^2(E^2)$ tel que

$$(6.6) \qquad (wI_2 - J)x = \Theta y;$$

vu que $\Theta(e^{it})$ est unitaire p. p. sur le cercle unité et que $w \in H^\infty$, il s'ensuit même que $y \in H^\infty(E^2)$. Soient $y_1 = [y_{11}, y_{21}]$ et $y_2 = [y_{12}, y_{22}]$ les éléments de $H^\infty(E^2)$ qui correspondent de cette façon aux vecteurs $x_1 = [1, 0]$ et $x_2 = [0, 1]$ de E^2; on déduit de (6. 2) et (6. 6) que

$$w+1 = \frac{1}{\sqrt{2}} u(y_{11} + y_{21}), \qquad w-1 = \frac{1}{\sqrt{2}} v(y_{12} - y_{22}),$$

d'où

$$(6.7) \quad 1 = ua + vb, \quad \text{avec} \quad a = \frac{1}{2\sqrt{2}}(y_{11} + y_{21}) \in H^\infty, \quad b = \frac{1}{2\sqrt{2}}(-y_{12} + y_{22}) \in H^\infty.$$

Or il est possible de choisir u et v de façon que telle équation soit impossible; c'est le cas par exemple si $u(\lambda) = \exp\left(\frac{\lambda+1}{\lambda-1}\right)$ et $v(\lambda)$ est un produit de Blaschke dont les zéros sont réels et tendent vers 1 (*cf.* le n°8.2 de [2]). Tel choix de u et v rend donc l'équation (6. 4) impossible pour $w \in H^\infty$.

7. Décomposition approximative pour un opérateur de classe $C_0(N)$

D'après une définition due à KISILEVSKY [5] (*cf.* aussi [7]) l'espace de Hilbert $\mathfrak{K}$ est appelé *somme approximative* de ses sous-espaces $\mathfrak{K}_j$ $(j\in\Gamma)$ si l'on a

$$\mathfrak{K} = \bigvee_{j\in\Gamma} \mathfrak{K}_j \tag{7.1}$$

et

$$\Big(\bigvee_{j\in\Gamma'} \mathfrak{K}_j\Big)\cap\Big(\bigvee_{j\in\Gamma''} \mathfrak{K}_j\Big) = \{0\} \tag{7.2}$$

pour toute partition de l'ensemble des indices Γ en parties disjointes non vides Γ' et Γ''.

Théorème 4. *Pour un opérateur T dans l'espace $\mathfrak{K}$, de classe $C_0(N)$ $(N\geqq 1)$, les propriétés suivantes sont équivalentes:*

(i) *T est quasi-similaire à l'opérateur de Jordan $S=S(m_1, \ldots, m_K)$ défini dans l'espace $\mathfrak{H} = \mathfrak{H}(m_1)\oplus\cdots\oplus\mathfrak{H}(m_K)$;*

(ii) *Il existe une décomposition de $\mathfrak{K}$ en somme approximative de sous-espaces $\mathfrak{K}_1, \ldots, \mathfrak{K}_K$ invariants pour T, telle que chaque $T_j=T|\mathfrak{K}_j$ est sans multiplicité et m_j $(=m_{T_j})$ est un diviseur de m_{j-1} (pour $j>1$).*

Démonstration. (ii)→(i). Comme T_j est sans multiplicité, il existe une quasi-affinité X_j: $\mathfrak{H}(m_j)\to\mathfrak{K}_j$ telle que $TX_j=X_jS(m_j)$; *cf.* [2]. En définissant

$$X(h_1\oplus\cdots\oplus h_K) = \sum_{j=1}^{K} X_j h_j$$

pour $h_j\in\mathfrak{H}(m_j)$ $(j=1, \ldots, K)$, on obtient un opérateur X: $\mathfrak{H}\to\mathfrak{K}$, évidemment borné et tel que $XS(m_1, \ldots, m_K)=TX$. De (7. 1) il dérive que $\overline{X\mathfrak{H}}=\mathfrak{K}$, et de (7. 2) il dérive que $Xh=0$ entraîne $h=0$; donc X est une quasi-affinité. En vertu du théorème 2, $S(m_1, \ldots, m_K)$ est alors quasi-similaire à T.

(i)→(ii). Soient X: $\mathfrak{H}\to\mathfrak{K}$ et Y: $\mathfrak{K}\to\mathfrak{H}$ des quasi-affinités vérifiant les relations $TX=XS$ et $SY=YT$. Posons

$$\mathfrak{H}_j=\mathfrak{H}(m_j) \quad \text{et} \quad \mathfrak{K}_j=\overline{X\mathfrak{H}_j} \qquad (j=1, \ldots, K);$$

$\mathfrak{K}_j$ est invariant pour T, $X_j=X|\mathfrak{H}_j$ est une quasi-affinité $\mathfrak{H}_j\to\mathfrak{K}_j$, et pour $T_j=T|\mathfrak{K}_j$ on a $T_jX_j=X_jS(m_j)$, d'où il s'ensuit que T_j est sans multiplicité et que $m_{T_j}=m_j$. La relation (7. 1) est manifeste (pour $\Gamma=\{1, \ldots, K\}$). Il reste à démontrer (7. 2).

Posons à cet effet $\mathfrak{H}' = \bigoplus_{j\in\Gamma'} \mathfrak{H}_j$, $\mathfrak{H}'' = \bigoplus_{j\in\Gamma''} \mathfrak{H}_j$, $\mathfrak{K}' = \bigvee_{j\in\Gamma'} \mathfrak{K}_j$ et $\mathfrak{K}''= \bigvee_{j\in\Gamma''} \mathfrak{K}_j$; il est manifeste que $\mathfrak{K}'=\overline{X\mathfrak{H}'}$ et $\mathfrak{K}''=\overline{X\mathfrak{H}''}$. Envisageons alors la somme directe

$$\mathfrak{K}_\circ = \mathfrak{K}'\oplus\mathfrak{K}''$$

et son opérateur

$$T_\circ = (T|\mathfrak{K}')\oplus(T|\mathfrak{K}''),$$

qui est évidemment aussi d'une classe $C_0(N_o)$, $N_o \geqq 1$. En posant

$$X_o(h'+h'') = Xh' \oplus Xh'' \quad \text{pour} \quad h' \in \mathfrak{H}', \quad h'' \in \mathfrak{H}'',$$

on définit une quasi-affinité X_o: $\mathfrak{H} \to \mathfrak{K}_o$ telle que

(7. 3) $$T_o X_o = X_o S.$$

D'autre part, en posant

$$Y_o(k' \oplus k'') = Y(k'+k'') \quad \text{pour} \quad k' \in \mathfrak{K}', \quad k'' \in \mathfrak{K}'',$$

on définit un opérateur borné Y_o: $\mathfrak{K}_o \to \mathfrak{H}$, avec $Y_o \mathfrak{K}_o$ dense dans $\mathfrak{H}$ et tel que

(7. 4) $$SY_o = Y_o T_o.$$

L'opérateur $Z_o = Y_o^*$: $\mathfrak{H} \to \mathfrak{K}_o$ est alors inversible et tel que

(7. 5) $$Z_o S^* = T_o^* Z_o;$$

de (7. 5) on déduit que $\overline{Z_o \mathfrak{H}}$ est invariant pour T_o^*. En désignant par $[T_o^*]$ la restriction de T_o^* à ce sous-espace, on aura

(7. 6) $$[Z_o] S^* = [T_o^*][Z_o]$$

où $[Z_o]$ désigne la quasi-affinité $\mathfrak{H} \to \overline{Z_o \mathfrak{H}}$ induite par Z_o. En vertu de (7. 6) on a donc $[T_o^*] \succ S^*$; par conséquent $[T_o^*]$ a le modèle de Jordan $S^* = S(\tilde{m_1}, \ldots, \tilde{m_K})$. D'autre part, (7. 3) entraîne que T_o^* a aussi le modèle de Jordan égal à S^*. En vertu du corollaire 2 du théorème 2 cela n'est possible que si $\overline{Z_o \mathfrak{H}} = \mathfrak{K}_o$. On conclut que Z_o — et alors Y_o aussi — sont des quasi-affinités.

Cela entraîne (7. 2). En effet, pour $k \in \mathfrak{K}' \cap \mathfrak{K}''$ on a $k_o = k \oplus (-k) \in \mathfrak{K}_o$ et $Y_o k_o = Y(k-k) = 0$, d'où, Y_o étant inversible, on déduit que $k_o = 0$ et par conséquent $k = 0$.

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(Reçu le 20. février 1969)

l_p-faktorisierbare Operatoren in Banachräumen

Von ALBRECHT PIETSCH in Jena (DDR)

Ein beschränkter linearer Operator T, der einen Banachraum E in einem Banachraum F abbildet, heißt *l_p-faktorisierbar* $(1 \leqq p \leqq \infty)$, wenn er sich in ein Produkt

$$T: \ E \xrightarrow{A} l_p \xrightarrow{Y} F$$

von zwei beschränkten linearen Operatoren $A \in \mathbf{L}(E, l_p)$ und $Y \in \mathbf{L}(l_p, F)$ aufspalten läßt. Setzt man

$$\varphi_p(T) = \inf \|A\| \, \|Y\|,$$

wobei das Infimum über alle möglichen Faktorisierungen von T gebildet wird, so ist die Klasse $\mathbf{F}_p$ aller l_p-faktorisierbaren Operatoren zwischen beliebigen Banachräumen ein vollständiges Operatorenideal mit der Norm φ_p.

Bemerkenswert ist die Tatsache, daß das Ideal $\mathbf{F}_p(H, H)$ für jeden Hilbertraum H und $1 < p < \infty$ sowie $p \neq 2$ gerade aus allen kompakten linearen Operatoren besteht, während man in den Grenzfällen $p = 1$ und $p = \infty$ die Hilbert—Schmidt-Operatoren erhält. Für $p = 2$ ergibt sich das Ideal aller beschränkten linearen Operatoren mit separablem Bildraum.

Als Anwendung der gewonnenen Ergebnisse erhalten wir ein interessantes Nuklearitätskriterium für lokalkonvexe Räume.

1. Die vollständigen Normideale $[\mathbf{F}_p, \varphi_p]$

Mit $\mathbf{L}$ bezeichnen wir die Klasse aller beschränkten linearen Operatoren zwischen beliebigen Banachräumen, und $\mathbf{L}(E, F)$ ist die Menge derjenigen Operatoren $T \in \mathbf{L}$, die den Banachraum E in den Banachraum F abbilden.

Eine Klasse $\mathbf{A}$ von beschränkten linearen Operatoren heißt *Ideal* (Vgl. [9]), wenn für die Mengen

$$\mathbf{A}(E, F) = \mathbf{L}(E, F) \cap \mathbf{A}$$

die folgenden Aussagen gelten:

(A) *Aus* $S, T \in \mathbf{A}(E, F)$ *folgt* $S + T \in \mathbf{A}(E, F)$.

(I_1) *Aus* $T \in \mathbf{L}(E, F)$ *und* $S \in \mathbf{A}(F, G)$ *folgt* $ST \in \mathbf{A}(E, G)$.

(I_2) *Aus* $T \in \mathbf{A}(E, F)$ *und* $S \in \mathbf{L}(F, G)$ *folgt* $ST \in \mathbf{A}(E, G)$.

Eine Abbildung α, die jedem Operator $T \in \mathbf{A}$ eine nicht negative Zahl $\alpha(T)$ zuordnet, nennt man *Idealnorm*, wenn die folgenden Bedingungen erfüllt sind:

(O) *Aus* $\alpha(T) = 0$ *folgt* $T = O$.

(NA) *Für* $S, T \in \mathbf{A}(E, F)$ *gilt* $\alpha(S+T) \leqq \alpha(S) + \alpha(T)$.

(NI_1) *Für* $T \in \mathbf{L}(E, F)$ *und* $S \in \mathbf{A}(F, G)$ *gilt* $\alpha(ST) \leqq \alpha(S)\|T\|$.

(NI_2) *Für* $T \in \mathbf{A}(E, F)$ *und* $S \in \mathbf{L}(F, G)$ *gilt* $\alpha(ST) \leqq \|S\|\alpha(T)$.

Ein Operatorenideal $\mathbf{A}$, auf dem eine Idealnorm α gegeben ist, wird als *Normideal* $[\mathbf{A}, \alpha]$ bezeichnet. Ein Normideal $[\mathbf{A}, \alpha]$ heißt *vollständig*, wenn die einzelnen Komponenten $\mathbf{A}(E, F)$ vollständig sind.

Wir formulieren nun das Hauptergebnis dieses Abschnittes.

Satz 1. *Die Klasse* $[\mathbf{F}_p, \varphi_p]$ *ist ein vollständiges Normideal.*

Wir verzichten auf den trivialen Nachweis der Eigenschaften (I_1), (I_2), (NI_1), (NI_2) und (O). Die Gültigkeit von (A) und (NA) ergibt sich zusammen mit der Vollständigkeit aus

Hilfssatz 1. *Für jede Folge von Operatoren* $T_n \in \mathbf{F}_p(E, F)$ *mit*

$$\sum \varphi_p(T_n) < +\infty$$

wird durch den Ansatz

$$T = \sum T_n$$

ein Operator $T \in \mathbf{F}_p(E, F)$ *definiert, und es gilt*

$$\varphi_p(T) \leqq \sum \varphi_p(T_n).$$

Beweis. Zu einer vorgegebenen Zahl $\varepsilon > 0$ bestimmen wir Faktorisierungen

$$T_n:\ E \xrightarrow{A_n} l_p \xrightarrow{Y_n} F$$

mit

$$\|A_n\| \leqq [\varphi_p(T_n) + 2^{-n}\varepsilon]^{1/p} \quad \text{und} \quad \|Y_n\| \leqq [\varphi_p(T_n) + 2^{-n}\varepsilon]^{1/p'}.$$

Außerdem betrachten wir neben dem Folgenraum l_p den entsprechenden Doppelfolgenraum $\boldsymbol{l}_p$ und setzen (n wird festgehalten)

$$I_n:\ \{\xi_i\} \to \{\xi_i \delta_{kn}\} \quad \text{und} \quad P_n:\ \{\xi_{ik}\} \to \{\xi_{in}\}.$$

Dann liefern die Operatoren

$$A = \sum I_n A_n \quad \text{und} \quad Y = \sum Y_n P_n$$

die gewünschte Faktorisierung

$$T\colon\ E \xrightarrow{A} l_p \xrightarrow{Y} F.$$

Dabei gelten die Ungleichungen

$$\|A\| \leqq \{\sum \|A_n\|^p\}^{1/p} \quad \text{und} \quad \|Y\| \leqq \{\sum \|Y_n\|^{p'}\}^{1/p'}.$$

Damit ist unsere Behauptung bewiesen, denn die Banachräume l_p und $\boldsymbol{l}_p$ sind isomorph, und es besteht die Abschätzung

$$\varphi_p(T) \leqq \|A\|\,\|Y\| \leqq \sum \varphi_p(T_n) + \varepsilon.$$

2. Ein Darstellungssatz für l_p-faktorisierbare Operatoren

Wenn man die bekannten Darstellungssätze (vgl. [6]) für Operatoren $A \in \mathbf{L}(E, l_p)$ und $Y \in \mathbf{L}(l_p, F)$ ausnutzt, ergibt sich

Satz 2. *Ein Operator $T \in \mathbf{L}(E, F)$ ist genau dann l_p-faktorisierbar $(1 < p < \infty)$, wenn er sich in der Form*

$$Tx = \sum \langle x, a_n \rangle y_n$$

darstellen läßt, so daß die Ungleichungen

$$\sum |\langle x, a_n \rangle|^p < +\infty \quad \text{und} \quad \sum |\langle y_n, b \rangle|^{p'} < +\infty$$

für alle $x \in E$ bzw. alle $b \in F'$ bestehen. Setzt man

$$\varepsilon_p[a_n] = \sup_{\|x\| \leqq 1} \{\sum |\langle x, a_n \rangle|^p\}^{1/p} \quad \text{und} \quad \varepsilon_{p'}[y_n] = \sup_{\|b\| \leqq 1} \{\sum |\langle y_n, b \rangle|^{p'}\}^{1/p'},$$

so gilt die Identität

$$\varphi_p(T) = \inf \{\varepsilon_p[a_n]\, \varepsilon_{p'}[y_n]\},$$

falls das Infimum über alle möglichen Darstellungen von T gebildet wird.

3. l_p-faktorisierbare Operatoren in Hilberträumen

Im folgenden characterisieren wir die l_p-faktorisierbaren Operatoren in einem beliebigen Hilbertraum H. Dazu benötigen wir

Hilfssatz 2. *Für die identische Abbildung*

$$I_n\colon\ l_2^n \to l_2^n$$

und $1 < p < \infty$ gilt mit einer von $n = 1, 2, \ldots$ unabhängigen Konstanten c_p die Ungleichung

$$\varphi_p(I_n) \leqq c_p.$$

Beweis. Durchläuft $e=\{\varepsilon_i\}$ die 2^n-elementige Menge aller n-tupel mit $\varepsilon_i=\pm 1$, so besteht für alle $x=\{\xi_i\}$ mit einer Konstanten a_p die Littlewood-Chintchinsche Ungleichung (vgl. [3], [5])

$$\{\sum_e |\sum_i \varepsilon_i \xi_i|^p\}^{1/p} \leqq 2^{n/p} a_p \{\sum |\xi_i|^2\}^{1/2}.$$

Deshalb erhält man durch den Ansatz

$$A_n\colon \{\xi_i\} \to \{\eta_e = \sum_i \varepsilon_i \xi_i\}$$

einen Operator mit

$$\|A_n\colon l_2^n \to l_p^{2^n}\| \leqq 2^{n/p} a_p,$$

und für den (dualen) Operator

$$A_n'\colon \{\eta_e\} \to \{\zeta_k = \sum_e \varepsilon_k \eta_e\}$$

gilt

$$\|A_n'\colon l_p^{2^n} \to l_2^n\| \leqq 2^{n/p'} a_{p'}.$$

Aus der Identität $I_n = 2^{-n} A_n' A_n$ ergibt sich abschließend die behauptete Ungleichung

$$\varphi_p(I_n) \leqq a_p a_{p'} = c_p.$$

Nach diesen Vorbereitungen erhalten wir

Satz 3. *Für $1<p<\infty$ und $p\neq 2$ besteht das Ideal $\mathbf{F}_p(H, H)$ aus allen kompakten linearen Operatoren.*

Beweis.

(1) Wir betrachten zuerst einen ausgearteten Operator T. Weil der Bildraum $B(T)$ zu dem Hilbertraum l_2^n mit $n=\dim B(T)$ isomorph ist, hat man

$$(\Delta) \qquad \varphi_p(T) \leqq \varphi_p(I_n)\|T\| \leqq c_p\|T\|.$$

Da jeder kompakte lineare Operator T durch ausgeartete Operatoren approximiert werden kann, überträgt sich die Ungleichung (Δ), und T gehört zu $\mathbf{F}_p(H, H)$.

(2) Weil man jeden l_p-faktorisierbaren Operator T in der folgenden Weise zerlegen kann,

$$T\colon H \to l_2 \to l_p \to l_2 \to H,$$

ergibt sich die behauptete Kompaktheit aus der Tatsache, daß für $r>s$ alle Operatoren aus $\mathbf{L}(l_r, l_s)$ kompakt sind (vgl. [10]).

Den Beweis der folgenden Behauptung findet man indirekt bei A. Grothendieck [2] oder J. Lindenstrauss—A. Pełczyński [4].

Satz 4. *Die Ideale* $\mathbf{F}_1(H, H)$ *und* $\mathbf{F}_\infty(H, H)$ *bestehen aus allen Hilbert—Schmidt-Operatoren.*

Ohne Beweis formulieren wir abschließend den trivialen

Satz 5. *Das Ideal* $\mathbf{F}_2(H, H)$ *besteht aus allen beschränkten linearen Operatoren mit separablem Bildraum.*

4. $\mathfrak{M}$-faktorisierbare Operatoren

Der Begriff des l_p-faktorisierbaren Operators kann folgendermaßen verallgemeinert werden. Wir betrachten eine beliebige Klasse $\mathfrak{M}$ von Banachräumen und bezeichnen einen Operator $T \in \mathbf{L}(E, F)$ als *$\mathfrak{M}$-faktorisierbar*, wenn es Banachräume $M_n \in \mathfrak{M}$, Operatoren $A_n \in \mathbf{L}(E, M_n)$ und $Y_n \in \mathbf{L}(M_n, F)$ mit

$$\sum \|A_n\| \, \|Y_n\| < +\infty$$

gibt, so daß

$$T = \sum Y_n A_n$$

gilt. Setzt man

$$\varphi_{\mathfrak{M}}(T) = \inf \sum \|A_n\| \, \|Y_n\|,$$

wobei das Infimum über alle möglichen Darstellungen gebildet wird, so ist die Klasse $[\mathbf{F}_{\mathfrak{M}}, \varphi_{\mathfrak{M}}]$ aller M-faktorisierbaren Operatoren das kleinste vollständige Normideal $[\mathbf{A}, \alpha]$, das alle identischen Abbildungen

$$I_M \colon M \to M, \quad M \in \mathfrak{M},$$

enthält.

(1) Die von uns betrachteten l_p-faktorisierbaren Operatoren erhält man, wenn $\mathfrak{M}$ nur aus dem Banachraum l_p besteht.

(2) Enthält $\mathfrak{M}$ lediglich den eindimensionalen Banachraum, so ergeben sich die nuklearen Operatoren.

(3) Die Klasse aller Hilberträume liefert das interessante Normideal der sogenannten Hilbert-Operatoren, die bereits von A. Grothendieck [2] und J. Lindenstrauss—A. Pełczyński [4] untersucht wurden.

5. Projektive Spektren von Banachräumen

Eine Folge von Banachräumen E_n, zwischen denen beschränkte lineare Operatoren

$$T_n \colon E_{n+1} \to E_n$$

definiert sind, heißt *projektives Spektrum* (vgl. [1]). Für jedes projektive Spektrum wird die Menge E aller Folgen

$$x = \{x_n\} \quad \text{mit} \quad T_n x_{n+1} = x_n$$

zu einem (F)-Raum, wenn man die Halbnormen

$$p_n(x) = \|x_n\|$$

einführt. Umgekehrt läßt sich jeder (F)-Raum auf diese Weise aus einem projektiven Spektrum von Banachräumen erzeugen.

Aus der Theorie der nuklearen lokalkonvexen Räume (vgl. [7]) ist bekannt, daß man jeden nuklearen (F)-Raum sogar aus einem projektiven Spektrum von Hilberträumen gewinnen kann, in dem die Operatoren T_n nuklear sind. Als unmittelbare Folgerung aus dieser Feststellung ergibt sich

Satz 6. (Vgl. [11], S. 101) *Jeder nukleare (F)-Raum kann mit beliebigen Zahlen $p \neq q$ aus einem projektiven Spektrum*

$$(*) \qquad \to l_p \xrightarrow{T_{n+1}} l_q \xrightarrow{T_n} l_p \to$$

erzeugt werden.

Es erhebt sich nun die umgekehrte Frage, ob jedes projektive Spektrum $(*)$ einen nuklearen (F)-Raum liefert.

Satz 7. *Für $p=1$ und $2 \leqq q \leqq \infty$ bzw. $1 \leqq p \leqq 2$ und $q = \infty$ wird durch jedes projektive Spektrum $(*)$ ein nuklearer (F)-Raum erzeugt.*

Beweis. Unsere Behauptung ergibt sich unmittelbar aus der Tatsache, daß jeder Operator aus $\mathbf{L}(l_\infty, l_q)$ mit $1 \leqq q \leqq 2$ absolut-2-summierend ist (vgl. [4], [8]).

Satz 8. *Für $1 < p, q < \infty$ wird durch ein projektives Spektrum $(*)$ nicht immer ein nuklearer (F)-Raum erzeugt.*

Beweis. Wir betrachten eine Nullfolge von reellen Zahlen λ_i mit

$$\sum |\lambda_i|^r = +\infty \quad \text{für} \quad r \geqq 1$$

und definieren den Operator T durch die Zuordnung

$$T\colon \{\xi_i\} \to \{\lambda_i \xi_i\}.$$

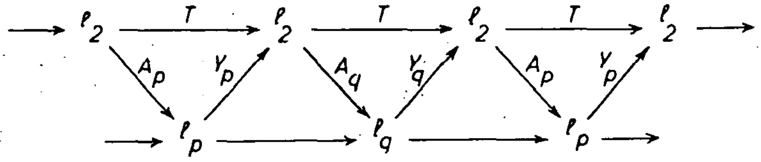
Dann gilt

$$T \in \mathbf{F}_p(l_2, l_2) \quad \text{für} \quad 1 < p < \infty,$$

und es gibt Faktorisierungen

$$T\colon l_2 \xrightarrow{A_p} l_p \xrightarrow{Y_p} l_2 \quad \text{und} \quad T\colon l_2 \xrightarrow{A_q} l_q \xrightarrow{Y_q} l_2.$$

Folglich liefert das projektive Spektrum



keinen nuklearen (F)-Raum.

Problem. *Erzeugt jedes projektive Spektrum* $(*)$ *mit* $p=1$ *und* $1<q<2$ *bzw.* $2<p<\infty$ *und* $q=\infty$ *stets einem nuklearen* (F)*-Raum?*

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(Eingegangen am 18. März, 1969)

Some remarks on expectations

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In [3] the notion of a $\mathcal{G}$-finite (see below) von Neumann algebra $\mathfrak{A}$ is developed and gives rise to an expectation which is a generalization of the concept of the center trace [2, III, § 5]. In § 2 we discuss ultraweakly closed ideals invariant under a group of automorphisms in a $\mathcal{G}$-finite algebra and certain normal state which serve to replace characters [2, p. 275]. We then remove a restriction from one of STØRMER's result [5] on expectations and examine the effect of this expectation on characterizing certain ideals.

1. In this paragraph we discuss consequences of the expectation $A \to A^{\mathcal{G}}$ as given in [3].

Definition. If $\mathfrak{A}$ is a von Neumann algebra and $\{\alpha_g | g \in \mathcal{G}\}$ is a group of automorphisms acting on $\mathfrak{A}$, then $\mathfrak{A}$ is said to be $\mathcal{G}$-finite if whenever $A \in \mathfrak{A}^+$, $A \neq 0$, there exists an invariant, normal state ϱ such that $\varrho(A) \neq 0$.

In [3] it is shown that if $\mathfrak{A}$ is $\mathcal{G}$-finite and $\mathcal{K}(T, \mathcal{G})$ equals the strong closure of $co\{\alpha_g(T) | g \in \mathcal{G}\}$ then $\mathcal{K}(T, \mathcal{G}) \cap \mathfrak{A}^{\mathcal{G}}$ contains a unique point, where $\mathfrak{A}^{\mathcal{G}}$ is the von Neumann algebra of elements fixed by all α_g. This is then used to define the faithful normal map $T \to T^{\mathcal{G}}$ [3, p. 240].

Proposition 1. *Let $\mathfrak{A}$ be a $\mathcal{G}$-finite and suppose $\mathfrak{m}$ is an ultraweakly closed ideal in $\mathfrak{A}$ invariant under the $\{\alpha_g\}$. Then $\mathfrak{m} \cap \mathfrak{A}^{\mathcal{G}} = \{T^{\mathcal{G}} | \, T \in \mathfrak{m}\}$. If $\mathfrak{A}^{\mathcal{G}} \supseteqq \mathfrak{Z}$, every two-sided ultraweakly closed ideal is invariant.*

Proof. Let $\mathfrak{m}$ be u.w. closed and suppose $\mathfrak{m}$ is a left ideal. $\mathfrak{m} = \mathfrak{A}E$ [2, p. 45] with E a unique projection in $\mathfrak{A}$. Since $\alpha_g(\mathfrak{m}) = \mathfrak{m}$ we have $\alpha_g(E) = E$ by uniqueness. Let $T \in \mathfrak{m} \cap \mathfrak{A}^{\mathcal{G}}$ then $T = T^{\mathcal{G}}$ [3, p. 241]. Conversely suppose $T \in \mathfrak{m}$ then $T = SE$ so $T^{\mathcal{G}} = (SE)^{\mathcal{G}} = S^{\mathcal{G}} E \in \mathfrak{m}$. Moreover $(T^{\mathcal{G}})^{\mathcal{G}} = T^{\mathcal{G}}$ so $T^{\mathcal{G}} \in \mathfrak{m} \cap \mathfrak{A}^{\mathcal{G}}$ [3, p. 240]. The last statement follows from the previous remarks and the fact that any such ideal looks like $\mathfrak{A}z$ with $z \in \mathfrak{Z}$ [2, p. 45].

Remark 1. If we suppose $\mathfrak{A}^{\mathcal{G}} \supseteqq \mathfrak{Z}$ then if $\mathfrak{m} = \mathfrak{A}z$ is an ultraweakly closed two-sided ideal in $\mathfrak{A}$ then clearly $\mathfrak{A}^{\mathcal{G}} z$ $(= \mathfrak{m} \cap \mathfrak{A}^{\mathcal{G}})$ is one in $\mathfrak{A}^{\mathcal{G}}$. If $\mathfrak{n}$ is an ultraweakly closed two-sided ideal in $\mathfrak{A}^{\mathscr{G}}$ there is (if at all) at most one ultraweakly closed two-sided ideal in $\mathfrak{A}$ giving rise to $\mathfrak{n}$ in this manner. This is the case for if $\mathfrak{A}^{\mathscr{G}} z_1 = \mathfrak{A}^{\mathscr{G}} z_2$ then since $I \in \mathfrak{A}^{\mathscr{G}}$, $z_1 \leqq z_2$ and $z_2 \geqq z_1$, so $z_1 = z_2$. If we make the additional hypothesis that $\mathfrak{Z}$ is the center of $\mathfrak{A}^{\mathscr{G}}$, then the correspondence becomes complete and clearly preserves maximality.

The appropriate replacement for characters seems to be $\mathscr{G}$-clustering states, where

Definition. Let ϱ be an invariant state. ϱ is said to be $\mathscr{G}$-clustering [5, p. 18] if $\varrho(AB^{\mathscr{G}}) = \varrho(A)\varrho(B)$.

Proposition 2. *Let $\mathfrak{A}$ be $\mathscr{G}$-finite and suppose ϱ is a normal $\mathscr{G}$-clustering state. Then the support* [2, p. 61] *of ϱ is a minimal projection in $\mathfrak{A}^{\mathscr{G}}$, lying in the center of $\mathfrak{A}^{\mathscr{G}}$. Conversely to every minimal projection lying in the center of $\mathfrak{A}^{\mathscr{G}}$ there corresponds a unique normal $\mathscr{G}$-clustering state on $\mathfrak{A}$.*

Proof. Since ϱ is invariant we have that E_ϱ, the support of ϱ, belongs to $\mathfrak{A}^{\mathscr{G}}$ (this is noted in [3]). The map $A \to A^{\mathscr{G}}$ takes $\mathfrak{A}$ onto $\mathfrak{A}^{\mathscr{G}}$ thus ϱ restricted to $\mathfrak{A}^{\mathscr{G}}$ is a normal multiplicative state (for all normal invariant states ψ we have $\psi(A^{\mathscr{G}}) = \psi(A)$ [3, p. 240]). It is now clear that E_ϱ is also the support of ϱ restricted to $\mathfrak{A}^{\mathscr{G}}$ and thus by a result of PLYMEN [4] is minimal in $\mathfrak{A}^{\mathscr{G}}$ and lies in the center of $\mathfrak{A}^{\mathscr{G}}$.

Conversely suppose E belongs to the center of $\mathfrak{A}^{\mathscr{G}}$ and is minimal in $\mathfrak{A}^{\mathscr{G}}$. Then [4] there exists a unique, normal, multiplicative state $\bar{\varrho}$ on $\mathfrak{A}^{\mathscr{G}}$ whose support is E. We then define $\varrho(A) = \bar{\varrho}(A^{\mathscr{G}})$. ϱ is normal by the normality of $A \to A^{\mathscr{G}}$. Further $\varrho(\alpha_g(A)) = \bar{\varrho}([\alpha_g(A)]^{\mathscr{G}}) = \bar{\varrho}(A^{\mathscr{G}}) = \varrho(A)$ [3, p. 240], thus ϱ is invariant. For $A, B \in \mathfrak{A}$ we have $\varrho(AB^{\mathscr{G}}) = \bar{\varrho}((AB^{\mathscr{G}})^{\mathscr{G}}) = \bar{\varrho}(A^{\mathscr{G}} B^{\mathscr{G}}) = \bar{\varrho}(A^{\mathscr{G}})\bar{\varrho}(B^{\mathscr{G}}) = \varrho(A)\varrho(B)$ i.e. ϱ is $\mathscr{G}$-clustering on $\mathfrak{A}$. The uniqueness follows from the fact that the state $\bar{\varrho}$ is uniquely determined by E and the fact that a normal invariant state is uniquely determined by its values on $\mathfrak{A}^{\mathscr{G}}$ [3, p. 242].

Under appropriate conditions we obtain the analogue of [Proposition 5. 2, p. 277].

Proposition 3. *Suppose $\mathfrak{A}$ is $\mathscr{G}$-finite and $\mathfrak{Z}$ is the center of $\mathfrak{A}^{\mathscr{G}}$. Then there exists a one-to-one correspondence between maximal two-sided ultraweakly closed ideals in $\mathfrak{A}$ and normal $\mathscr{G}$-clustering states on $\mathfrak{A}$.*

Proof. By Remark 1 it suffices to exhibit a correspondence with ideals in $\mathfrak{A}^{\mathscr{G}}$.

We consider the kernel of $\varrho|\mathfrak{A}^{\mathscr{G}}$. By the $\mathscr{G}$-clustering and a result of PLYMEN, this equals $\mathfrak{A}^{\mathscr{G}}(I - E_\varrho)$ which is a two sided ultraweakly closed ideal in $\mathfrak{A}^{\mathscr{G}}$. Suppose there exists $\mathfrak{m}$ with $\mathfrak{A}^{\mathscr{G}} \supset \mathfrak{m} \supset \mathfrak{A}^{\mathscr{G}}(I - E_\varrho)$, $\mathfrak{m}$ two sided u.w. closed ideal in $\mathfrak{A}^{\mathscr{G}}$. Since $\mathfrak{Z}$ is the center of $\mathfrak{A}^{\mathscr{G}}$, $\mathfrak{m} = \mathfrak{A}^{\mathscr{G}} z$ with $z \in \mathfrak{Z}$. Thus $I > z > I - E_\varrho$ and $O < I - z < E_\varrho$ which contradicts the minimality of E_ϱ. Thus $\mathfrak{A}^{\mathscr{G}}(I - E_\varrho)$ is maximal.

2. We now discuss a result of STØRMER [5] and obtain a more explicit ideal correspondence.

Definition. Let $\mathfrak{A}$ be a von Neumann algebra and $\mathfrak{B}$ a von Neumann subalgebra of $\mathfrak{A}$. Then a positive linear map Φ of $\mathfrak{A}$ onto $\mathfrak{B}$ is called an expectation if $\Phi(I)=I$ and $\Phi(BA)=B\Phi(A)$ for $B\in\mathfrak{B}$ and $A\in\mathfrak{A}$.

In [5] STØRMER constructs an expectation onto a subalgebra of $\mathfrak{Z}$ under the condition that the algebra is acted upon by a large group of automorphisms given by unitaries.

Definition. Let $\mathcal{U}$ be a group of unitaries giving rise to automorphisms of $\mathfrak{A}$, a C^*-algebra. Then $\mathcal{U}$ is said to be a large group of automorphisms if $co(UAU^{-1}: U\in\mathcal{U})\cap\mathfrak{A}' \neq \emptyset$ for $A\in\mathfrak{A}$ (the closure is in the strong topology).

We show that one can obtain a normal invariant expectation onto the same subalgebra of $\mathfrak{Z}$ without this assumption. We do not however obtain the full strength of STØRMER's results.

Theorem 1. *Let $\mathfrak{A}$ be a von Neumann algebra acted upon by a group of automorphisms $\{\alpha_g\}$. Set $\mathfrak{B}=\mathfrak{A}^g\cap\mathfrak{Z}$ and suppose there exists a normal state, ϱ, invariant under the $\{\alpha_g\}$ which is faithful on $\mathfrak{B}$. Then there exists an expectation, Φ taking $\mathfrak{A}$ onto $\mathfrak{B}$ such that*

(i) $\varrho(B\Phi(X))=\varrho(BX)$ $B\in\mathfrak{B}$ *and* $X\in\mathfrak{A}$,

(ii) $\Phi(\alpha_g(A))=\Phi(A)$,

(iii) Φ *is normal,*

(iv) *if* $\mathfrak{m}$ *is an ultraweakly closed two-sided invariant ideal and* $X\in\mathfrak{m}$, *then* $\Phi(X)\in\mathfrak{m}$.

Proof. The existence of an expectation with property (i) is a special case of a result of DE KORVIN [1]. One first realizes $\mathfrak{B}$ as a Hilbert algebra with inner product $(A, B)=\varrho(B^*A)$. Then one defines $\sigma(B)=\varrho(BX)$ for $X\in\mathfrak{A}^+$, $B\in\mathfrak{B}$. Riesz' lemma and a standard Hilbert algebra argument yield the desired result.

From (i)

$$\varrho\, B\Phi(\alpha_g(X))=\varrho(B\alpha_g(X))=\varrho(\alpha_g(BX))=\varrho(BX)=\varrho(B\Phi(X)) \qquad B\in\mathfrak{B}.$$

Thus $\Phi(\alpha_g(X))=\Phi(X)$ since ϱ is faithful on $\mathfrak{B}$.

Normality follows as in [5, p. 10] since the map Φ is positive.

Now let $\mathfrak{m}$ be an ultraweakly closed two-sided ideal in $\mathfrak{A}$. Then $\mathfrak{m}=\mathfrak{A}z$. If $\mathfrak{m}$ is invariant then $\alpha_g(\mathfrak{m})=\mathfrak{m}$. By the uniqueness of z, $\alpha_g(z)=z$ for all g and $z\in\mathfrak{B}$. We must show that if $X\in\mathfrak{m}$ then $\Phi(X)\in\mathfrak{m}$ or equivalently $z\Phi(X)=\Phi(X)$. But for $X\in\mathfrak{m}$

$$\varrho(Bz\Phi(X))=\varrho(BzX)=\varrho(BX)=\varrho(B\Phi(X)).$$

An appropriate choice of B gives $z\Phi(X)=\Phi(X)$.

The expectation Φ' that STØRMER constructs has the nice property that it preserves normal invariant states in that if ψ is any such $\psi\circ\Phi' = \psi$. While this is not necessarily true for the above expectation, nevertheless we have

Corollary. *Let $\mathfrak{A}$ be as in the theorem. If ψ is an invariant, multiplicative, normal state on $\mathfrak{A}$ then $\psi\circ\Phi = \psi$.*

Proof. Since ψ is invariant so is ker ψ (the kernel of ψ), which PLYMEN has shown [4] is an ultraweakly closed two-sided ideal. By (iv) of the theorem $\Phi(\ker\psi)\subseteq$ $\subseteq \ker\psi$ so $\ker\psi\circ\Phi \supseteq \ker\psi$. Thus $\psi\circ\Phi = \lambda\psi$. However $\Phi(I)=I$ so $\lambda=1$ and $\psi\circ\Phi = \psi$.

In this case we can, following [2, p. 273], obtain a characterization of the ideal $\mathfrak{m}$ corresponding to an ideal $\mathfrak{n}$ in $\mathfrak{B}$.

Proposition 4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be as in Theorem 1. Let $\mathfrak{n}$ be a two-sided ultraweakly closed ideal in $\mathfrak{B}$. Let $\mathfrak{m}=\{T\in\mathfrak{A}\,|\,\Phi(T_1TT_2)\in\mathfrak{n}$ for $T_1, T_2\in\mathfrak{A}\}$. Then $\mathfrak{m}$ is the largest two-sided ideal of $\mathfrak{A}$ that $\mathfrak{m}\cap\mathfrak{B}\subseteq\mathfrak{n}$, $\mathfrak{m}$ is invariant and ultraweakly closed.* $\mathfrak{m}\cap\mathfrak{B}=\mathfrak{n}$.

Proof. Linearity and ultraweak continuity [2, p. 56] of Φ imply that $\mathfrak{m}$ is a two-sided ultraweakly closed ideal. Suppose now that $T\in\mathfrak{m}\cap\mathfrak{B}$. Then $\Phi(T)=T\in\mathfrak{n}$ so $\mathfrak{m}\cap\mathfrak{B}\subseteq\mathfrak{n}$. If $T\in\mathfrak{n}$ then for $T_1, T_2\in\mathfrak{A}$ we have $\Phi(T_1TT_2)=\Phi(TT_1T_2)=$ $=T\Phi(T_1T_2)\in\mathfrak{n}$, i.e. $T\in\mathfrak{m}\cap\mathfrak{B}$. — $\mathfrak{m}$ is invariant for if $T\in\mathfrak{m}$ then by (ii) of Theorem 1

$$\Phi(T_1\alpha_g(T)T_2)=\Phi(\alpha_g(T_1'TT_2'))=\Phi(T_1'TT_2')\in\mathfrak{n}.$$

Suppose $\mathfrak{m}'$ is another ultraweakly closed two-sided invariant ideal in $\mathfrak{A}$ with $\mathfrak{m}'\cap\mathfrak{B}\subseteq\mathfrak{n}$. By (iv) of Theorem 1 we have $\mathfrak{m}'\cap\mathfrak{B}=\{\Phi(T)|\,T\in\mathfrak{m}'\}$, i.e. $T\in\mathfrak{m}'$ gives $\Phi(T)\in\mathfrak{n}$. Since $\mathfrak{m}'$ is an ideal $\Phi(T_1TT_2)\in\mathfrak{n}$, i.e. $\mathfrak{m}'\subseteq\mathfrak{m}$.

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(Received December 12, 1968)

Über die kovariante Ableitung der Vektoren in verallgemeinerten Linienelementräumen

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§ 1. Einleitung

In [3] begründeten wir eine Übertragungstheorie der Vektoren und der verallgemeinerten Vektoren in einem $\mathfrak{M}_n$-Raum, d. h. in einem Raum, in dem die Grundelemente (x^i, v^i) dem Transformationsgesetz:

$$(1.1)\qquad \begin{cases} \hat{x}^i = \hat{x}^i(x^1, x^2, \ldots, x^n), \\ \hat{v}^i = \hat{v}^i(\bar{v}^1, \bar{v}^2, \ldots, \bar{v}^n), \quad \bar{v}^j = \dfrac{\partial \hat{x}^j}{\partial x^r} v^r, \\ \text{Det}\left(\dfrac{\partial \hat{x}^i}{\partial x^j}\right) \neq 0, \quad \text{Det}\left(\dfrac{\partial \hat{v}^i}{\partial \bar{v}^j}\right) \neq 0 \end{cases}$$

genügen. Die Funktionen $\hat{v}^i(\bar{v})$ sollen in den $\bar{v}^i$ immer homogen von erster Ordnung sein. In [4] bestimmten wir verschiedene mögliche Type der kovarianten Ableitungen der verallgemeinerten Vektoren, d. h. die bezüglich (1. 1) dem Transformationsgesetzt

$$(1.2a)\qquad \hat{X}^i = \frac{\partial \hat{v}^i}{\partial v^r} X^r \qquad \text{bzw.} \qquad (1.2b)\qquad \hat{Y}_i = \frac{\partial v^t}{\partial \hat{v}^i} Y_t$$

genügen.

Im folgenden wollen wir die möglichen Formen der kovarianten Ableitungen der gewöhnlichen Vektoren im $\mathfrak{M}_n$-Raum bestimmen, d. h. die kovarianten Ableitungen der Vektoren, die bezüglich (1. 1) dem Transformationsgesetz

$$(1.3a)\qquad \hat{X}^i = \frac{\partial \hat{x}^i}{\partial x^r} X^r \qquad \text{bzw.} \qquad (1.3b)\qquad \hat{Y}_i = \frac{\partial x^t}{\partial \hat{x}^i} Y_t$$

genügen. Bezüglich des Begriffs der kovarianten Ableitung verweisen wir auf die fundamentale Arbeit [1] der Theorie der geometrischen Objekte (vgl. insb. Kapitel IV. 1). Wir bemerken, daß die Theorie der allgemeinen zweiten kovarianten Ableitung $_{(2)}\nabla_k$ nicht so vollständig ist, wie die der ersten kovarianten Ableitung; die fundamentalen Funktionalgleichungen aber, von denen die zweiten kovarianten Ableitungen bestimmt werden können, werden wir in § 4 angeben.

§ 2. Fundamentalgrößen des $\mathfrak{M}_n$-Raumes

Wir werden die folgenden Bezeichnungen benützen:

$$(2.\,1)\qquad \hat{p}^i_r \overset{\text{def}}{=} \frac{\partial \hat{v}^i}{\partial \bar{v}^r},\quad p^j_s \overset{\text{def}}{=} \frac{\partial \bar{v}^j}{\partial \hat{v}^s},\quad \hat{q}^i_r \overset{\text{def}}{=} \frac{\partial \hat{x}^i}{\partial x^r} \equiv \frac{\partial \bar{v}^i}{\partial v^r},\quad q^j_s \overset{\text{def}}{=} \frac{\partial x^j}{\partial \hat{x}^s} \equiv \frac{\partial v^j}{\partial \bar{v}^s};$$

diese stimmen mit den in [4] benützten Bezeichnungen überein.

Die Grundgrößen eines $\mathfrak{M}_n$-Raumes, in dem eine Übertragungstheorie der Vektoren definiert ist (vgl. [3] § 3—§6) sind die folgenden:

I. Der Pseudotensor $a^i_j(x, v)$ mit dem Transformationsgesetz[1]):

$$(2.\,2)\qquad \hat{a}^i_j(\hat{x}, \hat{v}) = \frac{\partial \hat{v}^i}{\partial v^r}\frac{\partial x^s}{\partial \hat{x}^j}\, a^r_s(x, v) \equiv \hat{p}^i_t \hat{q}^t_r q^s_j a^r_s(x, v),$$

da nach (2. 1) offenbar

$$(2.\,2a)\qquad \frac{\partial \hat{v}^i}{\partial v^r} \equiv \frac{\partial \hat{v}^i}{\partial \bar{v}^t}\frac{\partial \bar{v}^t}{\partial v^r} \equiv \hat{p}^i_t \hat{q}^t_r$$

besteht.

Es soll immer

$$(2.\,2b)\qquad \operatorname{Det}(a^i_j) \neq 0$$

gelten, woraus folgt, daß a^i_j einen eindeutig bestimmten inversen Pseudotensor b^i_r definiert, d. h. die Relationen

$$(2.\,3)\qquad b^i_r a^r_j = \delta^i_j \quad \text{bzw.} \quad b^r_i a^j_r = \delta^j_i$$

bezüglich b^i_j eindeutig lösbar sind, und es gilt:

$$(2.\,4)\qquad \hat{b}^i_j(\hat{x}, \hat{v}) = \frac{\partial v^t}{\partial \hat{v}^j}\frac{\partial \hat{x}^i}{\partial x^r}\, b^r_t(x, v) \equiv q^t_s p^s_j \hat{q}^i_r b^r_t(x, v)$$

da nach (2. 1) offenbar

$$(2.\,4a)\qquad \frac{\partial v^t}{\partial \hat{v}^j} \equiv \frac{\partial v^t}{\partial \bar{v}^s}\frac{\partial \bar{v}^s}{\partial \hat{v}^j} \equiv q^t_s p^s_j$$

besteht.

B e m e r k u n g. Die Relationen (2. 2*a*) und (2. 4*a*) werden wir im folgenden öfters ohne einen direkten Hinweis auf diese Gleichungen anwenden. Die beiden Relationen von (2. 3) sind nach einem wohlbekannten Satz der Tensoralgebra gleichwertig.

---

[1]) Bezüglich des Begriffs der verallgemeinerten Tensoren und Pseudotensoren vgl. [3] § 2. Die verallgemeinerten Tensoren könnten bezüglich der Grundelementtransformation (1. 1) auch als gewöhnliche Tensoren betrachtet werden, doch wollen wir die Terminologie von unserer Arbeit [3] behalten.

II. Die in i, k symmetrischen Übertragungsparameter $M_i{}^j{}_k$ mit dem Transformationsgesetz:

$$\hat{M}_i{}^j{}_k = M_b{}^b{}_c p_i^t q_t^a \hat{p}_r^j \hat{q}_b^r p_k^s q_s^c - \hat{p}_{st}^j p_i^s p_k^t, \tag{2. 5}$$

wo

$$\hat{p}_{st}^j \overset{\text{def}}{=} \frac{\partial^2 \hat{v}^j}{\partial \bar{v}^s \partial \bar{v}^t}$$

bedeutet (vgl. z. B. [4], Formel (1. 6)).

III. Die Übertragungsparameter $L_j^{*i}{}_k$ mit dem Transformationsgesetz:

$$\hat{L}_j^{*i}{}_k = L_a^{*b}{}_c p_j^r q_r^a \hat{p}_s^i \hat{q}_b^s q_k^c + \hat{p}_{bc}^i p_j^b \hat{q}_t^c q_k^r L_o^{*t}{}_r + \hat{p}_{bc}^i \hat{q}_r^c p_j^b q_{tk}^r \bar{v}^t + \hat{p}_r^i \hat{q}_t^r q_{sk}^t p_j^s, \tag{2. 6}$$

wo der Index „o" die Überschiebung mit v^j, und

$$q_{sk}^t \overset{\text{def}}{=} \frac{\partial^2 x^t}{\partial \hat{x}^s \partial \hat{x}^k}$$

bedeuten (vgl. [4], Formel (1. 6)).

Die Größen $M_j{}^i{}_k$ und $L_j^{*i}{}_k$ sind für die Festlegung der kovarianten Ableitung der verallgemeinerten Vektoren nötig (vgl. [4]), während die Pseudotensoren a_j^i bzw. b_j^i bei der Definition der kovarianten Ableitungen der gewöhnlichen Vektoren im $\mathfrak{M}_n$-Raum benützt werden.

§ 3. Die allgemeine erste kovariante Ableitung

Nach diesen Vorbereitungen gehen wir zur Definition der allgemeinen ersten kovarianten Ableitung ${}_{(1)}\nabla_k$ über. Diese allgemeine erste kovariante Ableitung soll so definiert werden, daß sie eine Verallgemeinerung der in [3] durch $\overset{*}{\nabla}_k$ bezeichneten Ableitung sei. Aus § 5 und aus den Gleichungen (4. 12) und (4. 14) von [3] kann leicht berechnet werden, daß für einen gewöhnlichen Vektor:

$$\overset{*}{\nabla}_k X^i \overset{\text{def}}{=} \frac{\partial}{\partial v^k}(a_r^i X^r) + M_j{}^i{}_k a_r^j X^r, \quad \overset{*}{\nabla}_k Y_i \overset{\text{def}}{=} \frac{\partial}{\partial v^k}(b_i^t Y_t) - M_i{}^j{}_k b_j^t Y_t.$$

Definition 1. Die allgemeine erste kovariante Ableitung ${}_{(1)}\nabla_k$ eines kontra- bzw. kovarianten Vektors $\vec{Z}$ vom Transformationsgesetz (1. 3) ist ein verallgemeinerter Tensor zweiter Stufe, der von $\vec{Z}$, $\partial_{v^i}\vec{Z}$, a_j^i, $\partial_{v^k} a_j^i$ und vom Hilfsobjekt $M_j{}^i{}_k$ abhängig ist, und dessen kovariante Stufenzahl um eins größer, als die von $\vec{Z}$ ist (vgl. [3] § 2).

Die einzelnen Funktionen der verschiedenen kovarianten Ableitungen sollen jetzt und im folgenden immer in allen ihren Veränderlichen stetig sein. Wir beweisen für die Form der ersten kovarianten Ableitung eines kontravarianten Vektors den folgenden

Satz 1. *Die allgemeine erste kovariante Ableitung eines kontravarianten Vektors X^i im $\mathfrak{M}_n$-Raum ist eine Funktion von $a^a_b X^b$, $a^a_b \partial_{vj} X^b$ und $b^s_j \overset{*}{\nabla}_c a^a_s$, wo*

$$\overset{*}{\nabla}_k a^i_j \overset{\text{def}}{=} \frac{\partial a^i_j}{\partial v^k} + M_t{}^i{}_k a^t_j. \tag{3. 1}$$

Vor dem Beweis des Satzes 1 wollen wir zeigen, daß die durch (3. 1) bestimmte Größe selbst ein Pseudotensor ist. Auf Grund von (2. 1) ist nämlich

$$\frac{\partial \hat{a}^i_j}{\partial \hat{v}^k} = \frac{\partial \hat{v}^i}{\partial v^r} \frac{\partial v^t}{\partial \hat{v}^k} \frac{\partial x^s}{\partial \hat{x}^j} \frac{\partial a^r_s}{\partial v^t} + \frac{\partial^2 \hat{v}^i}{\partial v^r \partial v^t} \frac{\partial v^t}{\partial \hat{v}^k} \frac{\partial x^s}{\partial \hat{x}^j} a^r_s,$$

woraus in Hinsicht auf (2. 5) und (2. 1), ferner wegen der Relation

$$\frac{\partial^2 \hat{v}^i}{\partial v^r \partial v^t} \frac{\partial v^t}{\partial \hat{v}^k} = -\frac{\partial \hat{v}^i}{\partial v^t} \frac{\partial^2 v^t}{\partial \hat{v}^s \partial \hat{v}^k} \frac{\partial \hat{v}^s}{\partial v^r}$$

— die offenbar besteht, da $\dfrac{\partial \hat{v}^i}{\partial v^t} \dfrac{\partial v^t}{\partial \hat{v}^k} = \delta^i_k$ ist — leicht folgt:

$$\overset{*}{\nabla}_k \hat{a}^i_j = \frac{\partial \hat{v}^i}{\partial v^r} \frac{\partial v^s}{\partial \hat{v}^k} \frac{\partial x^t}{\partial \hat{x}^j} \overset{*}{\nabla}_s a^r_t, \tag{3. 2}$$

und das beweist unsere Behauptung bezüglich des pseudotensoriellen Charakters von (3. 1).

Es ist bemerkenswert, daß in (3. 1) nur ein $M_j{}^i{}_k$ enthaltendes Glied nötig ist, obwohl a^i_j zwei Indizes hat. Das folgt daraus, daß in der Transformationsformel (2. 2) von a^i_j nur bezüglich des Indexen „i" ein $\dfrac{\partial \hat{v}^i}{\partial v^r}$-Faktor vorkommt, während $\dfrac{\partial x^s}{\partial \hat{x}^j}$ bezüglich der Ableitung $\dfrac{\partial}{\partial v^k}$ eine Konstante ist.

Beweis des Satzes 1. Auf Grund der Definition 1 ist:

$$_{(1)}\nabla_s X^r = F^r_s\left(X^i, \frac{\partial X^i}{\partial v^j}, M_j{}^i{}_k, a^i_j, \frac{\partial a^i_j}{\partial v^k}\right). \tag{3. 3}$$

Da $_{(1)}\nabla_s X^r$ nach der Definition 1 ein verallgemeinerter Tensor sein muß, gilt nach einer Transformation (1. 1) die Transformationsformel

$$_{(1)}\nabla_k \hat{X}^i = \hat{p}^i_a \hat{q}^a_r q^s_b p^b_k \, {}_{(1)}\nabla_s X^r, \tag{3. 4}$$

wo die Größen $\hat{p}^i_a$, $\hat{q}^a_r$, q^s_b und p^b_k durch (2. 1) festgelegt sind. In Hinsicht auf (3. 3),

(2. 2), (2. 5) und (2. 6) bekommt man für die erste kovariante Ableitung F_r^t das folgende Funktionalsgleichungssystem

$$F_k^i\left(\hat{q}_b^a X^b, \hat{q}_b^a q_t^s p_j^t \frac{\partial X^b}{\partial v^s}, p_a^t q_t^e \hat{p}_r^b \hat{q}_l^r p_c^s q_s^j M_e{}^l{}_j - \right.$$
$$\left. - \hat{p}_{rs}^b p_a^r p_c^s, \hat{p}_t^a \hat{q}_r^t q_b^s a_s^r, \hat{p}_t^a \hat{q}_r^t q_b^s q_l^m p_c^l \frac{\partial a_s^r}{\partial v^m} + \hat{p}_{tm}^a p_c^m \hat{q}_r^t q_b^s a_s^r\right) = \tag{3. 5}$$
$$= \hat{p}_m^i \hat{q}_r^m q_t^s p_k^t F_s^r\left(X^a, \frac{\partial X^a}{\partial v^b}, M_a{}^b{}_c, a_b^a, \frac{\partial a_b^a}{\partial v^c}\right).$$

Da die Relationen (3. 5) für die $\hat{p}_s^a$, $\hat{q}_b^a$, $\hat{p}_{rs}^b$ eine Identität bilden, bekommt man für

$$\hat{p}_j^i = \hat{q}_j^i = p_j^i = q_j^i = \delta_j^i, \quad \hat{p}_{ac}^b = M_a{}^b{}_c \text{ [2])}$$

die Relation:

$$F_k^i\left(X, \frac{\partial X}{\partial v}, 0, a, \overset{*}{\nabla} a\right) = F_k^i\left(X, \frac{\partial X}{\partial v}, M, a, \frac{\partial a}{\partial v}\right),$$

wo wir wegen leichterer Übersicht die Indizes bei den Argumenten von F_k^i wegließen. Unsere letzte Formel zeigt nun, daß die erste kovariante Ableitung von X^i von $M_a{}^b{}_c$ nicht explizit abhängig ist, d. h. daß sie die Form:

$$_{(1)}\nabla_k X^i = \Phi_k^i\left(X^a, \frac{\partial X^a}{\partial v^b}, a_b^a, \overset{*}{\nabla}_c a_b^a\right) \tag{3. 6}$$

hat.

Die Transformationsformel (3. 4) gibt somit nach (3. 3) für Φ_k^i das Funktionalgleichungssystem:

$$\Phi_k^i\left(\hat{q}_b^a X^b, \hat{q}_b^a q_l^m p_j^l \frac{\partial X^b}{\partial v^m}, \hat{p}_t^a \hat{q}_r^t q_b^s a_s^r, \hat{p}_t^a \hat{q}_r^t q_b^s q_l^m p_c^l \overset{*}{\nabla}_m a_s^r\right) = \tag{3. 7}$$
$$= \hat{p}_t^i \hat{q}_r^t q_h^s p_k^h \Phi_s^r\left(X^a, \frac{\partial X^a}{\partial v^j}, a_b^a, \overset{*}{\nabla}_c a_b^a\right).$$

Wählen wir jetzt $\hat{q}_b^a = a_b^a$, $\hat{p}_b^a = b_b^a$, so werden wegen $\hat{p}_r^i p_j^r = \delta_j^i$, $\hat{q}_r^i q_j^r = \delta_j^i$ die Relationen $p_j^r = a_j^r$, $q_j^r = b_j^r$ bestehen und aus (3. 7) bekommt man für die Funktionen Φ_k^i die Funktionalgleichung:

$$\Phi_k^i\left(a_b^a X^b, a_b^a \frac{\partial X^b}{\partial v^j}, \delta_b^a, b_c^s \overset{*}{\nabla}_c a_s^a\right) = \Phi_k^i\left(X^a, \frac{\partial X^b}{\partial v^j}, a_b^a, \overset{*}{\nabla}_c a_b^a\right),$$

und diese Formel beweist wegen (3. 6) eben den Satz 1.

---

[2]) Diese Substitution ist wegen der Symmetrie von $M_a{}^b{}_c$ in a, c möglich. Offenbar folgt auch aus $\hat{p}_j^i = \hat{q}_j^i = \delta_j^i$ die Relation $p_j^i = q_j^i = \delta_j^i$.

Die allgemeine erste kovariante Ableitung eines kontravarianten Vektors hat also die Form:

$$(3.8)\qquad {}_{(1)}\nabla_k X^i = \varphi^i_k\left(a^a_b X^b,\, a^a_b \frac{\partial X^b}{\partial v^j},\, b^s_b \overset{*}{\nabla}_c a^a_s\right).$$

Wir wollen nun zeigen, daß die durch $\overset{*}{\nabla}_k$ bezeichnete kovariante Ableitung der gewöhnlichen Vektoren mit dem Transformationsgesetz (1. 3*a*), in unserem Aufsatz [3] auch die Form von (3. 8) hat. Die kovariante Ableitung und invariantes Differential eines gewöhnlichen Vektors bildet man nach § 5 der Arbeit [3] in der Weise, daß man dem gewöhnlichen Vektor X^i durch $a^i_r X^r$ einen verallgemeinerten Vektor von dem Transformationsgesetz (1. 2*a*) zuordnet und dann die kovariante Ableitung bzw. das invariante Differential dieses verallgemeinerten Vektors bildet. Es ist somit in Hinsicht auf (3. 1):

$$\overset{*}{\nabla}_k X^i = \frac{\partial}{\partial v^k}(a^i_r X^r) + M_j{}^i{}_k a^j_r X^r \equiv (b^r_s \overset{*}{\nabla}_k a^i_r) a^s_m X^m + a^i_r \frac{\partial X^r}{\partial v^k},$$

und das zeigt, daß $\overset{*}{\nabla}_k X^i$ tatsächlich die Form (3. 8) hat, wie behauptet wurde.

Bezüglich der allgemeinen ersten kovarianten Ableitung eines kovarianten Vektors Y_i gilt der

Satz 2. *Die allgemeine erste kovariante Ableitung eines kovarianten Vektors Y_i im $\mathfrak{M}_n$-Raum ist eine Funktion von $b^t_a Y_t$, $b^t_a \frac{\partial Y_t}{\partial v_j}$, $a^j_m \overset{*}{\nabla}_d b^m_a$, wo*

$$(3.9)\qquad \overset{*}{\nabla}_d b^m_a \overset{\text{def}}{=} \frac{\partial b^m_a}{\partial v^d} - M_a{}^s{}_d b^m_s.$$

Ebenso wie bei der Formel (3. 1) kann leicht gezeigt werden, daß die durch (3. 9) bestimmte Größe ein Pseudotensor ist. Nach (2. 4) und (2. 5) kann leicht verifiziert werden, daß die folgende Transformationsformel besteht:

$$(3.10)\qquad \overset{*}{\nabla}_d \hat{b}^c_a = \frac{\partial v^r}{\partial \hat{v}^a}\frac{\partial \hat{x}^c}{\partial x^t}\frac{\partial v^m}{\partial \hat{v}^d}\overset{*}{\nabla}_m b^t_r.$$

Beweis des Satzes 2. Auf Grund der Definition 1 ist

$$(3.11)\qquad {}_{(1)}\nabla_k Y_i = F_{ik}\left(Y_a,\, \frac{\partial Y_a}{\partial v^b},\, M_a{}^b{}_c,\, b^a_b,\, \frac{\partial b^a_b}{\partial v^c}\right),$$

wo wir statt der a^a_b die inversen Größen b^a_b gesetzt haben. Bilden wir nun — wie im kontravarianten Fall — das Transformationsgesetz von ${}_{(1)}\nabla_k Y_i$, das nach der

Definition ein verallgemeinerter rein kovarianter Tensor ist, so erhält man für F_{ik} in Hinsicht auf (2. 2), (2. 4), (2. 5) und (2. 6) das Funktionalgleichungssystem:

$$
(3.\,12)\qquad F_{ik}\left(q_a^r Y_r,\, q_a^r q_i^s p_b^l \frac{\partial Y_r}{\partial v^s},\, p_a^t q_t^m \hat{p}_r^b \hat{q}_l^r p_c^s q_s^j M_m{}^l{}_j - \right.
$$

$$
\left. -\hat{p}_{rs}^b p_a^r p_c^s,\, q_s^r p_a^s \hat{q}_t^c b_r^t,\, q_s^r p_a^s \hat{q}_t^c q_h^m p_d^h \frac{\partial b_r^t}{\partial v^m} + p_{ad}^s q_s^r \hat{q}_t^c b_r^t \right) =
$$

$$
= q_m^r p_k^m q_i^s p_i^t F_{rs}\left(Y_a,\, \frac{\partial Y_a}{\partial v^b},\, M_a{}^b{}_c,\, b_a^c,\, \frac{\partial b_a^c}{\partial v^d}\right),
$$

wo

$$
p_{ad}^s \overset{\text{def}}{=} \frac{\partial^2 \bar{v}^s}{\partial \hat{v}^a\, \partial \hat{v}^d}
$$

bedeutet.

Da nach (2. 1) $\hat{p}_r^b p_a^r = \delta_a^b$ besteht, bekommt man durch partielle Ableitung nach $\hat{v}^d$

$$
(3.\,13)\qquad \hat{p}_{rt}^b p_a^r p_d^t = -\hat{p}_r^b p_{ad}^r .
$$

Setzen wir jetzt in (3. 12)

$$
p_b^a = \hat{p}_b^a = q_b^a = \hat{q}_b^a = \delta_b^a, \quad p_{ac}^b = -M_a{}^b{}_c,
$$

so zeigt sich nach (3. 13), daß F_{ik} von den $M_a{}^b{}_c$ nicht explizit, sondern nur durch $\overset{*}{\nabla}_d b_a^m$ abhängt. Die allgemeine kovariante Ableitung von Y_i wird somit die Form

$$
(3.\,14)\qquad {}_{(1)}\nabla_k Y_i = \Phi_{ik}\left(Y_a,\, \frac{\partial Y_a}{\partial v^b},\, b_c^a,\, \overset{*}{\nabla}_d b_c^a\right) \equiv F_{ik}\left(Y_a,\, \frac{\partial Y_a}{\partial v^b},\, 0,\, b_a^c,\, \frac{\partial b_a^c}{\partial v^b}\right)
$$

haben. Statt (3. 12) erhält man für die Funktionen Φ_{ik} das Funktionalgleichungssystem:

$$
(3.\,15)\qquad \Phi_{ik}\left(q_a^r Y_r,\, q_a^r q_e^s p_b^e \frac{\partial Y_r}{\partial v^s},\, q_s^r p_a^s \hat{q}_t^c b_r^t,\, q_e^r p_a^e \hat{q}_t^c q_s^m p_d^s \overset{*}{\nabla}_m b_r^t\right) =
$$

$$
= q_m^r p_k^m q_i^s p_i^t \Phi_{rs}\left(Y_a,\, \frac{\partial Y_a}{\partial v^b},\, b_a^c,\, \overset{*}{\nabla}_d b_a^c\right).
$$

Wählen wir jetzt $\hat{q}_b^a = a_b^a$, $\hat{p}_b^a = b_b^a$, so gilt für die inversen Größen $q_b^a = b_b^a$, $p_b^a = a_b^a$, da nach (2. 3) a_b^a und b_b^a zu einander inverse Größen sind. Aus (3. 15) wird dann

$$
\Phi_{ik}\left(Y_a,\, \frac{\partial Y_a}{\partial v^b},\, b_a^c,\, \overset{*}{\nabla}_d b_a^c\right) = \Phi_{ik}\left(b_a^r Y_r,\, b_a^r \frac{\partial Y_r}{\partial v^b},\, \delta_a^c,\, a_t^c \overset{*}{\nabla}_d b_a^t\right),
$$

und das drückt nach (3. 14) eben die Behauptung des Satzes 2 aus.

Zum Schluß dieses Paragraphen zeigen wir noch, daß die in unserem Aufsatz [3] durch $\overset{*}{\nabla}_k$ bezeichnete kovariante Ableitung des kovarianten Vektors Y_i von den im Satz 2 angegebenen Größen abhängig ist. Es ist nämlich in Hinsicht auf (3. 9)

$$\overset{*}{\nabla}_k Y_i = \frac{\partial}{\partial v^k}(b^t_i Y_t) - M_i{}^r{}_k b^t_r Y_t \equiv (a^r_t \overset{*}{\nabla}_k b^t_i) b^s_r Y_s + b^t_i \frac{\partial Y_t}{\partial v^k}$$

und das beweist unsere Behauptung.

Wir wollen noch bemerken, daß die in den Sätzen 1 und 2 angegebenen Größen verallgemeinerte Vektoren bzw. Tensoren sind, wie das aus den entsprechenden Transformationsformeln leicht verfiziert werden kann.

§ 4. Die allgemeine zweite kovariante Ableitung

Die allgemeine zweite kovariante Ableitung ${}_{(2)}\nabla_k$ der gewöhnlichen Vektoren mit dem Transformationsgesetz (1. 3), soll die in unserer Arbeit [3] durch ∇_k bezeichnete kovariante Ableitung verallgemeinern. Aus § 5 und ferner aus den Gleichungen (4. 11) und (4. 13) von [3] folgt, daß

$$\tilde{\nabla}_k X^i = \frac{\partial}{\partial x^k}(a^i_t X^t) - \left\{\frac{\partial}{\partial v^s}(a^i_t X^t)\right\} L^{*s}_{o}{}_k + L^{*i}_s{}_k a^s_t X^t,$$

$$\tilde{\nabla}_k Y_i = \frac{\partial}{\partial x^k}(b^t_i Y_t) - \left\{\frac{\partial}{\partial v^s}(b^t_i Y_t)\right\} L^{*s}_{o}{}_k - L^{*s}_i{}_k b^t_s Y_t$$

ist. $\tilde{\nabla}_k$ bezeichnet die kovariante Ableitung für gewöhnliche Vektoren.

D e f i n i t i o n 2. Die allgemeine zweite kovariante Ableitung ${}_{(2)}\nabla_k$ eines kontra- bzw. kovarianten Vektors $\vec{Z}$ vom Transformationsgesetz (1. 3) ist ein Pseudotensor zweiter Stufe, der von $\vec{Z}$, $\partial_{x^i}\vec{Z}$, $\partial_{v^i}\vec{Z}$, a^i_j, $\partial_{x^k} a^i_j$, $\partial_{v^k} a^i_j$ und vom Hilfsobjekt $L^*_j{}^j{}_k$ abhängig ist. Die Transformationsformeln seien die folgenden:

$$(4.\ 1)\qquad {}_{(2)}\nabla_k \hat{X}^i = \frac{\partial \hat{v}^i}{\partial v^s}\frac{\partial x^r}{\partial \hat{x}^k}\,{}_{(2)}\nabla_r X^s,$$

$$(4.\ 2)\qquad {}_{(2)}\nabla_k \hat{Y}_i = \frac{\partial v^s}{\partial \hat{v}^i}\frac{\partial x^r}{\partial \hat{x}^k}\,{}_{(2)}\nabla_r Y_s.$$

Wir beginnen mit der Untersuchung des kontravarianten Falles. Nach Definition 2 ist

$$(4.\ 3)\qquad {}_{(2)}\nabla_k X^i = f^i_k\left(X^a, \frac{\partial X^a}{\partial x^b}, \frac{\partial X^a}{\partial v^b}, a^a_b, \frac{\partial a^a_b}{\partial x^c}, \frac{\partial a^a_b}{\partial v^c}, L^{*b}_a{}_c\right),$$

wo die Form der Funktionen f^i_k bestimmt werden soll. Für f^i_k werden wir das charakteristische Funktionalgleichungssystem bestimmen, die explizite Lösung ist aber noch ein ungelöstes Problem.

Für die Bestimmung dieses Funktionalgleichungssystem müssen wir die transformierten Komponenten der Größen $\frac{\partial X^a}{\partial x^b}, \frac{\partial X^a}{\partial v^b}, \frac{\partial a^a_b}{\partial x^c}, \frac{\partial a^a_b}{\partial v^c}$ berechnen. Beachten wir nun, daß für eine von (x^i, v^i) abhängige Größe die Operatoren $\frac{\partial}{\partial \hat{x}^i}, \frac{\partial}{\partial \hat{v}^i}$ die Form

$$\frac{\partial}{\partial \hat{x}^k} = q^s_k \frac{\partial}{\partial x^s} + q^s_{rk} \bar{v}^r(\hat{v}) \frac{\partial}{\partial v^s}, \quad \frac{\partial}{\partial \hat{v}^k} = q^r_s p^s_k \frac{\partial}{\partial v^r}$$

haben, ferner die Identitäten

$$\frac{\partial}{\partial \hat{x}^b} \hat{q}^a_r = \frac{\partial^2 \hat{x}^a}{\partial x^r \partial x^s} \frac{\partial x^s}{\partial \hat{x}^b} \equiv \hat{q}^a_{rs} q^s_b \equiv -\hat{q}^a_s q^s_{tb} \hat{q}^t_r,$$

$$\frac{\partial}{\partial \hat{v}^k} \hat{p}^a_r \hat{q}^r_s = \frac{\partial^2 \hat{v}^a}{\partial v^m \partial v^s} \frac{\partial v^m}{\partial \hat{v}^k} \equiv \hat{p}^a_{mt} p^t_k \hat{q}^m_s$$

bestehen, so wird:

$$\frac{\partial \hat{X}^a}{\partial \hat{x}^b} = \hat{q}^a_r q^s_b \frac{\partial X^r}{\partial x^s} + \hat{q}^a_r q^s_{eb} \bar{v}^e(\hat{v}) \frac{\partial X^r}{\partial v^s} + \hat{q}^a_{rs} q^s_b X^r, \tag{4. 4}$$

$$\frac{\partial \hat{X}^a}{\partial \hat{v}^b} = \hat{q}^a_r q^s_e p^e_b \frac{\partial X^r}{\partial v^s}, \tag{4. 5}$$

$$\frac{\partial \hat{a}^a_b}{\partial \hat{x}^c} = \hat{p}^a_e \hat{q}^e_r q^s_b \left(\frac{\partial a^r_s}{\partial x_t} q^t_c + \frac{\partial a^r_s}{\partial v^t} q^t_{mc} \bar{v}^m(\hat{v}) \right) + \hat{p}^a_e (\hat{q}^e_{rt} q^t_c q^s_b + \hat{q}^e_r q^s_{bc}) a^r_s, \tag{4. 6}$$

$$\frac{\partial \hat{a}^a_b}{\partial \hat{v}^c} = \hat{p}^a_e \hat{q}^e_r q^s_b q^t_h p^h_c \frac{\partial a^r_s}{\partial v^t} + a^r_s \hat{p}^a_{et} p^t_c \hat{q}^e_r q^s_b. \tag{4. 7}$$

Aus (4. 1) bekommen wir somit für die Funktionen f^i_k — die die zweite kovariante Ableitung der gewöhnlichen kontravarianten Vektoren bestimmen — das charakteristische Funktionalgleichungssystem:

$$f^i_k \left(\hat{X}^a, \frac{\partial \hat{X}^a}{\partial \hat{x}^b}, \frac{\partial \hat{X}^a}{\partial \hat{v}^b}, \hat{a}^a_b, \frac{\partial \hat{a}^a_b}{\partial \hat{x}^c}, \frac{\partial \hat{a}^a_b}{\partial \hat{v}^c}, \hat{L}^{*b}_{a\ c} \right) = \tag{4. 8}$$

$$= \hat{p}^i_t \hat{q}^t_s q^r_k f^s_r \left(X^a, \frac{\partial X^a}{\partial x^b}, \frac{\partial X^a}{\partial v^b}, a^a_b, \frac{\partial a^a_b}{\partial x^c}, \frac{\partial a^a_b}{\partial v^c}, L^{*b}_{a\ c} \right),$$

wo selbstverständlich $\hat{X}^a = \hat{q}^a_r X^r$, ferner für $\frac{\partial \hat{X}^a}{\partial \hat{x}^b}, \frac{\partial \hat{X}^a}{\partial \hat{v}^b}, \ldots$ die entsprechenden Werte aus (4. 4)—(4. 7), (2. 2) und aus (2. 6) gesetzt werden sollen.

Im kovarianten Fall muß die allgemeine zweite kovariante Ableitung eines kovarianten Vektors Y_i die Form:

(4. 9) $$_{(2)}\nabla_k Y_i = f_{ik}\left(Y_a, \frac{\partial Y_a}{\partial x^b}, \frac{\partial Y_a}{\partial v^b}, b^a_b, \frac{\partial b^a_b}{\partial x^c}, \frac{\partial b^a_b}{\partial v^c}, L^{*b}_{a\ c}\right)$$

haben[3]). Wir verfahren auch jetzt ähnlich dem vorigen Falle. Auf Grund von

(4. 10) $$\hat{Y}_a = q^r_a Y_r$$

und (2. 4) wird:

(4. 11) $$\frac{\partial \hat{Y}_a}{\partial \hat{x}^b} = q^t_a q^s_b \frac{\partial Y_t}{\partial x^s} + q^t_a q^s_{eb} \bar{v}^e \frac{\partial Y_t}{\partial v^s} + q^t_{ab} Y_t,$$

(4 12) $$\frac{\partial \hat{Y}_a}{\partial \hat{x}^b} = q^t_a q^s_e p^e_b \frac{\partial Y_t}{\partial v^s},$$

(4. 13) $$\frac{\partial \hat{b}^a_b}{\partial \hat{x}^c} = q^t_e p^e_b \hat{q}^a_s \left(\frac{\partial b^s_t}{\partial x^r} q^r_c + \frac{\partial b^s_t}{\partial v^r} q^r_{ch} \bar{v}^h\right) + p^e_b (q^t_{ec} \hat{q}^a_s + q^t_e \hat{q}^a_{sm} q^m_c) b^s_t,$$

(4. 14) $$\frac{\partial \hat{b}^a_b}{\partial \hat{v}^c} = q^t_e p^e_b \hat{q}^a_s q^r_f p^f_c \frac{\partial b^s_t}{\partial v^r} + q^t_e p^e_{bc} \hat{q}^a_s b^s_t.$$

Auf Grund von (4. 2) und (4. 9) bekommt man das Funktionalgleichungssystem:

(4. 15) $$f_{ik}\left(\hat{Y}_a, \frac{\partial \hat{Y}_a}{\partial \hat{x}^b}, \frac{\partial \hat{Y}_a}{\partial \hat{v}^c}, \hat{b}^a_b, \frac{\partial \hat{b}^a_b}{\partial \hat{x}^c}, \frac{\partial \hat{b}^a_b}{\partial \hat{v}^c}, \hat{L}^{*b}_{a\ c}\right) =$$

$$= q^s_i p^t_i q^r_k f_{sr}\left(Y_a, \frac{\partial Y_a}{\partial x^b}, \frac{\partial Y_a}{\partial v^b}, b^a_b, \frac{\partial b^a_b}{\partial x^c}, \frac{\partial b^a_b}{\partial v^c}, L^{*b}_{a\ c}\right),$$

wo $\hat{Y}_a, \frac{\partial \hat{Y}_a}{\partial \hat{x}^b}, \ldots$ aus den Formeln (4. 10)—(4. 14), (2. 4) und (2. 6) substituiert werden sollen. (4. 8) bzw. (4. 15) muß somit in den $\hat{p}^a_b$, $\hat{q}^a_b$, p^a_b, q^a_b, $\hat{p}^a_{bc}$, $\hat{q}^a_{bc}$, p^a_{bc}, q^a_{bc} eine Identität sein; diese Größen sind aber voneinander nicht unabhängig. Wenn $\hat{p}^a_b$, $\hat{q}^a_b$, $\hat{p}^a_{bc}$ und $\hat{q}^a_{bc}$ angegeben sind, so sind die übrigen schon eindeutig bestimmt, wie wir das in § 3 gezeigt haben.

Die Bestimmung der Lösung der Funktionalgleichungssysteme (4. 8) und (4. 15) ist noch ein ungelöstes Problem. Man kann nicht $L^{*b}_{a\ c}$ eliminieren, wie in den gewöhnlichen Linienelementräumen (vgl. [2]) da in den $\mathfrak{M}_n$-Räumen $L^{*b}_{a\ c}$ in a, c nicht symmetrisch ist. Mit der in [2] verwandten Methode wäre nur der symmetrische

[3]) Die b^a_b sind nach (2. 2b) und (2. 3) eindeutige Funktionen der a^a_b.

Teil $L^{*\,b}_{(a\ c)}$ eliminierbar, der zurückbleibende schiefsymmetrische Teil $L^{*\,b}_{[a\ c]}$ hat aber in den allgemeinen $\mathfrak{M}_n$-Räumen keinen tensoriellen Charakter (vgl. die Transformationsformel (2. 6) von $L^{*b}_{a\ c}$ und der Satz 2 des Aufsatzes [3]).

§ 5. Vergleichung der kovarianten Ableitungen mit denen in Linienelementräumen

Wir untersuchten den Fall der gewöhnlichen Linienelementräumen $\mathfrak{L}_n$ in unserem Aufsatz [2]. Dieser Fall ist unter den Grundtransformationen (1. 1) dadurch gekennzeichnet, daß in (1. 1) $\hat{v}^i \equiv \bar{v}^i$ gesetzt werden soll. Daraus folgt, daß $\frac{\partial}{\partial v^i}$ in den $\mathfrak{L}_n$-Räumen eine tensorielle Operation ist; somit wird das Hilfsobjekt $M_i{}^j{}_k$ überflüssig und es kann ${}_{(1)}\nabla_k X^i = f^i_k \left(\frac{\partial X^a}{\partial v^b}\right)$ gesetzt werden. $\frac{\partial}{\partial v^k}$ ist die einfachste erste fundamentale kovariante Ableitung des $\mathfrak{L}_n$-Raumes.

Bei der ${}_{(2)}\nabla_k$-Ableitung ist im $\mathfrak{L}_n$-Raum $a^i_k = b^i_k = \delta^i_k$, $\nabla_k a^i_j = \nabla_k b^i_j = 0$, und aus der Formel (4. 10) wäre noch auch die explizite Abhängigkeit von X^a eliminierbar (vgl. [2] Satz 3.). Unsere Formel (4. 16) geht im wesentlichen — abgesehen von der Abhängigkeit von v^i — in (3. 21) von [2] über; in den $\mathfrak{M}_n$-Räumen haben wir aber eine explizite Abhängigkeit von den v^i nicht vorausgesetzt, da jetzt v^i wegen $\hat{v}^i \equiv \frac{\partial \hat{v}^i}{\partial v^s} v^s$ kein gewöhnlicher Vektor ist.

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(Eingegangen am 6, Dezember 1968)

On immersion of locally bounded curvature

By J. SZENTHE in Szeged

According to results of J. NASH ([11]) and N. H. KUIPER ([10]) C^1-immersions are too general to admit a reasonable generalization of the curvature theory of C^2-immersions. The idea to extend the curvature theory to a restricted class is therefore justified and in fact this has been done at first by J. HJELMSLEV ([8]) and G. BOULIGAND ([3]) in case of C^1-surfaces in 3-dimensional euclidean space. They were mainly interested in the curvature theory of curves on C^1-surfaces, i.e. in generalizations of the theorems of Euler and Meusnier. Later on various related results have been obtained by others[1]). In the first part of this paper a class of C^1-immersions of k-dimensional manifolds into n-dimensional euclidean space is introduced, which will be called *immersions of locally bounded curvature*, and it is shown that in their case the second fundamental tensor can be defined in a way which resembles very much the standard one. In the case $n=3$, $k=2$ similar results have been achieved by H. BUSEMANN and W. FELLER ([5]) and A. V. POGORELOV ([14]) for considerably wider classes with more refined methods. In the second part of the paper the case $k = n-1$, i.e. hypersurfaces of locally bounded curvature are considered. It is shown that the theorem on the uniqueness of C^2-hypersurfaces with given first and second fundamental forms generalizes to them which gives another point of considering immersions of locally bounded curvature.

1. Preliminaries

Some prerequisites of technical nature are provided in this section.

Let E^n be the n-dimensional euclidean space, and V_k^n $(k = 1, ..., n-1)$ the euclidean vector space formed by its k-vectors. Oriented k-dimensional subspaces of V_1^n will be identified, as usual, with simple unit k-vectors, consequently the set $S_k^n(\subset V_k^n)$ of simple unit k-vectors will stand for the set of oriented k-dimensional subspaces of V_1^n as well. Let further B^n be the set of complete orthonormal systems

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[1]) An account of related results can be found in BUSEMANN [4].

in V_1^n, $Q(n, k)$ the group of isometric isomorphisms of V_k^n, and $O(n)$ that of orthogonal $n \times n$ matrices with real entries. The facts which follow are well known. $O(n)$ is a simply transitive right transformation group of B^n with the definition: $b\alpha = \left(\sum_{i=1}^{n} \alpha_{ij} a_i\right)_{j=1,\dots,n}$ for $(a_i)_{i=1,\dots,n} = b \in B^n$, $\|\alpha_{ij}\| = \alpha \in O(n)$. Distance on B^n and $O(n)$ is defined by $\sigma(b', b'') = \left[\sum_{i=1}^{n} (a_i' - a_i'')^2\right]^{\frac{1}{2}}$ and $\mu(\alpha', \alpha'') = \left[\sum_{i,j=1}^{n} (\alpha_{ij}' - \alpha_{ij}'')^2\right]^{\frac{1}{2}}$, respectively. If $b \in B^n$ is fixed then $\Phi_b = b\alpha$ defines a distance preserving map $\Phi_{,n}: (O(n), \mu) \to (B^n, \sigma)$. With the above definition, $O(n)$ is a distance preserving transformation group of B^n, and μ is left and right invariant. If $(a_i)_{i=1,\dots,n} = b \in B^n$ is fixed and $\sum_{i=1}^{n} x^i a_i = x \in V_1^n$ then with the definition $\alpha x = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} x^j a_i$ the group $O(n)$ is a left transformation group of V_1^n and since each of these transformations is an isometric isomorphism of V_1^n an isomorphism Ψ_b: $O(n) \to Q(n, 1)$ is obtained. But there is the standard homomorphism $\Sigma^k: Q(n, 1) \to Q(n, k)$, hence for any fixed $b \in B^n$ a homomorphism $\Lambda_b^k = \Sigma^k \circ \Psi_b: O(n) \to Q(n, k)$ is defined. If $X_0^k \in S_k^n$ and $H(X_0^k) \subset Q(n, k)$ is the subgroup of elements which leave X_0^k fixed then the inverse image of $H(X_0^k)$ under Λ_b^k is a subgroup $H_b(X_0^k)$ of $O(n)$ and there is a one to one correspondence between the left coset space $O(n)/H_b(X_0^k)$ and S_k^n, where the left coset corresponding to $X^k \in S_k^n$ is formed by those elements α of $O(n)$ for which $\Lambda_b^k(\alpha)$ sends X_0^k into X^k.

A distance preserving map $\theta: (O(n), \mu) \to E^{n^2}$ is defined by $\theta(\alpha) = (x^1, \dots, x^{n^2})$, where $x^l = \alpha_{ij}$ for $l = (i-1)n + j$. If $O(n)$ is considered as a Lie group then θ is a C^∞-embedding, therefore there is a Riemannian metric on $O(n)$ for which θ is isometric. Let ϱ_e be the distance function of this Riemannian metric ([1], 124); $\varrho_e(\alpha', \alpha'')$ is equal to the infimum of the length of curves joining α', α'' if their length is calculated with respect to the distance function μ ([5]). Therefore ϱ_e and the above Riemannian metric are left and right invariant ([9], 169—172). If $b \in B^n$, $X_0^k \in S_k^n$ are fixed then a distance function $\bar{\varrho}_e$ is defined on the left coset space $O(n)/H_b(X_0^k)$ by $\bar{\varrho}_e(\alpha H_b(X_0^k), \beta H_b(X_0^k)) = \inf \{\varrho_e(\xi, \eta) | \xi \in \alpha H_b(X_0^k), \eta \in \beta H_b(X_0^k)\}$. The one to one correspondence $O(n)/H_b(X_0^k) \leftrightarrow S_k^n$ yields a distance function d^k on S_k^n for which this correspondence will be distance preserving. The distance function d^k does not depend on the particular choice of b and X_0^k and it will be called the *auxiliary metrization of* S_k^n.

Lemma 1. 1. *Let G be a compact Lie group, H a subgroup and ϱ the distance function of a Riemannian metric on G which is left and right invariant. Let the distance function $\bar{\varrho}$ on the left coset space G/H be defined by*

$$\bar{\varrho}(\alpha H, \beta H) = \inf \{\varrho(\xi, \eta) | \xi \in \alpha H, \eta \in \beta H\}.$$

Then there exist $\delta>0$ *and* $A>0$ *such that in case* $\bar{\varrho}(H, \alpha H)<\delta$ *a unique* $\xi\in\alpha H$ *with* $\varrho(\varepsilon, \xi)=\bar{\varrho}(H, \alpha H)$ *exists,* ε *being the identity in* G*; further in case* $\bar{\varrho}(H, \alpha_l H)<\delta$ $(l=1,2)$ *the inequality* $\varrho(\xi_1, \xi_2)\leqq A\bar{\varrho}(\alpha_1 H, \alpha_2 H)$ *holds for the corresponding* $\xi_l\in\alpha_l H$. *There is a Riemannian metric on* G/H *with distance function* $\tilde{\varrho}\geqq\bar{\varrho}$.

Proof. The existence of a $\xi\in\alpha H$ with $\varrho(\varepsilon, \xi)=\bar{\varrho}(H, \alpha H)$ for any left coset αH is obvious. Since the given Riemannian metric is left and right invariant, H and its left cosets are totally geodesic submanifolds ([1], 136—137). There is such a $\delta'>0$ that the spherical neighborhood $U(\vartheta)$ of of ε with radius ϑ is convex if $0<\vartheta\leqq\delta'$ ([1], 246——150). Assume that for some αH with $\bar{\varrho}(H,\alpha H)\leqq\delta'$ there are $\xi', \xi''\in\alpha H$ with $\xi'\neq\xi''$, $\varrho(\varepsilon, \xi')=\varrho(\varepsilon, \xi'')=\bar{\varrho}(H, \alpha H)=\vartheta$. Since $\xi', \xi''\in U(\vartheta)$, there is a unique minimizing geodesic arc joining them, which with the exception of its endpoints lies in the interior of $U(\vartheta)$. But αH is totally geodesic, therefore this geodesic arc is in αH. By the invariance of the Riemannian metric $\bar{\varrho}(H, \alpha H)=\inf\{\varrho(\varepsilon, \zeta)|\zeta\in\alpha H\}$, therefore $\bar{\varrho}(H, \alpha H)<\vartheta$, which is a contradiction. There is a canonical coordinate system of the first kind $\varphi\colon W\to E^m$ and one of the second kind $\psi\colon W\to E^m$, both defined on the neighborhood W of ε, such that 1) if g_{ij} $(i, j=1, \dots, m)$ are the components of the fundamental tensor of the given Riemannian metric in the coordinate system φ, then $g_{ij}(\varphi(\varepsilon))=\delta_{ij}$; 2) if $\psi(\zeta)=(z^1, \dots, z^m)$ for $\zeta\in W$, then $\psi(\zeta)=(0, \dots, 0, z^{s+1}, \dots, z^m)$ for $\zeta\in H\cap W$; moreover, if $\alpha H\cap W$ is not empty then there is a unique $\bar{\alpha}\in\alpha H\cap W$ such that $\zeta=\bar{\alpha}\cdot\bar{\xi}$ with $\bar{\xi}\in H\cap W$ holds for any $\zeta\in\alpha H\cap W$ and

$$\psi(\bar{\alpha})=(\bar{a}^1, \dots, \bar{a}^s, 0, \dots, 0), \quad \psi(\bar{\xi})=(0, \dots, 0, \bar{z}^{s+1}, \dots, \bar{z}^m),$$

$$\psi(\zeta)=(\bar{a}^1, \dots, \bar{a}^s, \bar{z}^{s+1}, \dots, \bar{z}^m);$$

3) if $\varphi(\zeta)=(y^1, \dots, y^m)$ for $\zeta\in W$ and $y^i=\chi^i(z^1, \dots, z^m)$ $(i=1, \dots, m)$ are the transition functions from ψ to φ, then $\chi^i(\underset{1}{0}, \dots, \underset{j-1}{0}, \underset{j}{t}, \underset{j+1}{0}, \dots, \underset{m}{0})=\delta_{ij}t$ for $j=s+1, \dots, m$, if $(\underset{1}{0}, \dots, \underset{j-1}{0}, \underset{j}{t}, \underset{j+1}{0}, \dots, \underset{m}{0})\in\psi(W)$ ([9], II. 62—86). Let $\gamma\colon[0, 1]\to G$ be the unique minimizing geodesic arc joining ε and $\zeta\in W$; then γ is given by $\gamma^i(t)=y^i\cdot t$ $(0\leqq t\leqq 1)$ in the coordinate system φ. Hence

$$\varrho(\varepsilon, \zeta)=\int_0^1\left[\sum_{i,j=1}^m g_{ij}(\varphi\circ\gamma(t))y^iy^j\right]^{\frac{1}{2}}dt=\int_0^1\left[\sum_{i,j=1}^m g_{ij}(\varphi(\varepsilon))y^iy^j\right]^{\frac{1}{2}}dt=\left[\sum_{i=1}^m (y^i)^2\right]^{\frac{1}{2}}.$$

Therefore if αH is such a coset that the corresponding $\xi\in\alpha H\cap W$ and $\psi(\xi)= =(\bar{a}^1, \dots, \bar{a}^s, \bar{x}^{s+1}, \dots, \bar{x}^m)$, then

$$F_j(\bar{a}^1, \dots, \bar{a}^s, \bar{x}^{s+1}, \dots, \bar{x}^m)=\sum_{i=1}^m \chi^i(\bar{a}^1, \dots, \bar{a}^s, \bar{x}^{s+1}, \dots, \bar{x}^m)\cdot\frac{\partial\chi^i}{\partial z^j}\bigg|_{\psi(\xi)}=0$$

for $j = s+1, \dots, m$. But

$$\frac{\partial F_j}{\partial z^l} = \sum_{i=1}^{m} \left(\frac{\partial \chi^i}{\partial z^l} \frac{\partial \chi^i}{\partial z^j} + \chi^i \frac{\partial^2 \chi^i}{\partial z^l \partial z^j} \right) \quad \text{and hence} \quad \frac{\partial F_j}{\partial z^l}\Big|_{\psi(\varepsilon)} = \delta_{jl}$$

for $j, l = s+1, \dots, m$. Consequently there is a neighborhood $U \subset W$ of ε with $\frac{\partial (F_{s+1}, \dots, F_m)}{\partial (z^{s+1}, \dots, z^m)}\Big|_{\psi(\zeta)} \neq 0$ for $\zeta \in U$. Therefore by the implicit function theorem there are analytic functions $\omega^{s+1}(\bar a^1, \dots, \bar a^s), \dots, \omega^m(\bar a^1, \dots, \bar a^s)$ defined on a neighborhood V of the origin in E^s such that

$$F_j(\bar a^1, \dots, \bar a^s, \omega^{s+1}(\bar a^1, \dots, \bar a^s), \dots, \omega^m(\bar a^1, \dots, \bar a^s)) = 0 \qquad (j = s+1, \dots, m)$$

for $(\bar a^1, \dots, \bar a^s) \in V$ and there are no other solutions of $F_j(\bar a^1, \dots, \bar a^m, z^{s+1}, \dots, z^m) = 0$ $(j = s+1, \dots, m)$ in V. Let αH be such a coset that $\xi \in \alpha H \cap W$ and $(\bar a^1, \dots, \bar a^s) \in V$. Then $x^j = \omega^j(\bar a^1, \dots, \bar a^s)$ $(j = s+1, \dots, m)$. Let $\delta'' > 0$ be such that $\xi \in \alpha H \cap W$, $(\bar a^1, \dots, \bar a^s) \in V$ if $\bar\varrho(H, \alpha H) \leqq \delta''$. Put $\delta = \min(\frac{1}{4}\delta', \delta'')$ and assume that $\bar\varrho(H, \alpha_1 H) = \vartheta_1 \leqq \bar\varrho(H, \alpha_2 H) = \vartheta_2 \leqq \delta$. Since $\varrho(\xi_1, \xi_2) \leqq \frac{1}{2}\delta'$, there is a unique $\xi_2' \in \alpha_2 H$ with $\bar\varrho(\alpha_1 H, \alpha_2 H) = \varrho(\xi_1, \xi_2')$. Further there are such bounds $A' \leqq A''$ that

$$A' \left[\sum_{i=1}^{m} (z_1^i - z_2^i)^2 \right]^{\frac{1}{2}} \leqq \varrho(\zeta_1, \zeta_2) \leqq A'' \left[\sum_{i=1}^{m} (z_1^i - z_2^i)^2 \right]^{\frac{1}{2}}$$

for $\zeta_1, \zeta_2 \in V$, $\psi(\zeta_l) = (z_l^1, \dots, z_l^m)$ $(l = 1, 2)$ ([5]). Therefore

$$\frac{\varrho(\xi_1, \xi_2)}{\bar\varrho(\alpha_1 H, \alpha_2 H)} = \frac{\varrho(\xi_1, \xi_2)}{\varrho(\xi_1, \xi_2')} \leqq \frac{A'' \left[\sum_{i=1}^{s} (\bar a_1^i - \bar a_2^i)^2 + \sum_{j=s+1}^{m} (x_1^j - x_2^j)^2 \right]^{\frac{1}{2}}}{A' \left[\sum_{i=1}^{s} (\bar a_1^i - \bar a_2^i)^2 + \sum_{j=s+1}^{m} (x_1^j - x_2'^j)^2 \right]^{\frac{1}{2}}} \leqq$$

$$\leqq \frac{A''}{A'} \left[1 + \frac{\sum_{j=s+1}^{m} (x_1^j - x_2^j)^2}{\sum_{i=1}^{s} (\bar a_1^i - \bar a_2^i)^2} \right]^{\frac{1}{2}} \leqq$$

$$\leqq \frac{A''}{A'} \left(1 + \left[\frac{\sum_{j=s+1}^{m} (\omega^j(\bar a_1^1, \dots, \bar a_1^s) - \omega^j(\bar a_2^1, \dots, \bar a_2^s))^2}{\sum_{i=1}^{s} (\bar a_1^i - \bar a_2^i)^2} \right]^{\frac{1}{2}} \right).$$

But

$$\frac{|\omega^j(\bar a_1^1, \dots, \bar a_1^s) - \omega^j(\bar a_2^1, \dots, \bar a_2^s)|}{\left[\sum_{i=1}^{s} (\bar a_1^i - \bar a_2^i)^2 \right]^{\frac{1}{2}}} \leqq \sum_{i=1}^{s} \left\{ \left| \frac{\partial \omega^j}{\partial \bar a^i} (\bar a_1^1, \dots, \bar a_1^s) \right| + \right.$$

$$\left. + \int_0^1 \left| \frac{\partial \omega^j}{\partial \bar a^i} (\bar a_1^1 + t(\bar a_2^1 - \bar a_1^1), \dots, \bar a_1^s + t(\bar a_2^s - \bar a_1^s)) - \frac{\partial \omega^j}{\partial \bar a^i} (\bar a_1^1, \dots, \bar a_1^s) \right| dt \right\}.$$

Hence the quantity under square root is bounded and the existence of a bound A with $\varrho(\xi_1, \xi_2) \leqq A\bar{\varrho}(\alpha_1 H, \alpha_2 H)$ for $\bar{\varrho}(H, \alpha_l H) \leqq \delta$ $(l=1,2)$ follows. A Riemannian metric with distance function $\tilde{\varrho} \geqq \bar{\varrho}$ can be evidently given by the standard construction of a homogeneous Riemannian metric on G/H based on the Haar measure of G([1], 136).

Lemma 1.2. *There is such a bound B that $d^k(X_1^k, X_2^k) \leqq B \cdot \|X_1^k - X_2^k\|$ for any $X_1^k, X_2^k \in S_k^n$, where the norm is taken in the euclidean vector space V_k^n.*

Proof. Since S_k^n is a C^∞-submanifold of V_k^n this embedding defines a Riemannian metric on S_k^n which has a distance function ϱ', and admits such a bound B' that $\varrho'(X_1^k, X_2^k) \leqq B' \cdot \|X_1^k - X_2^k\|$ for $X_1^k, X_2^k \in S_k^n$ ([5]). Let $\bar{\varrho}$ be the distance function provided by Lemma 1.1, then there is such a B'' that $\tilde{\varrho} \leqq B'' \cdot \varrho'$. Consequently $B = B' \cdot B''$ is the bound required.

2. Immersions of locally bounded curvature

Immersions of locally bounded curvature are introduced in this section and the basic concepts of the curvature theory of C^2-immersions are generalized for them.

Let f: $M^k \to E^n$ be a C^1-immersion of the C^1-manifold M^k and for $p \in M^k$ let U be an oriented neighborhood of p in M^k. Then the tangent space $T_q M^k$ for $q \in U$ is mapped by the induced map of the tangent bundles f_*: $TM^k \to TE^n$ onto an oriented k-dimensional subspace of $T_{f(q)}E^n$ which in turn is mapped by $\exp_{f(q)}$: $T_{f(q)}E^n \to E^n$ onto an oriented k-plane L_q^k of E^n which defines a simple unit k-vector $X_q^k \in S_k^n$. The immersion f defines a Riemannian metric on M^k; let d be its distance function. If there is such a K_p that $\limsup_{q \to p} \dfrac{\|X_q^k - X_p^k\|}{d(p,q)} \leqq K_p$, then f is said to be *of bounded curvature at p with the bound K_p*. If p has a neighborhood V such that f is of bounded curvature at every $q \in V$ with the same bound K_V, then f is said to be *of locally bounded curvature at p with the bound K_V*. If f is of locally bounded curvature at every point of M then it is called an *immersion of locally bounded curvature.*

Lemma 2.1. *Let the C^1-immersion f: $M^k \to E^n$ of the C^1-manifold M^k be of locally bounded curvature at $p \in M^k$. Then there is a coordinate system α: $U \to E^k$ of the C^1-manifold M^k defined on a neighborhood U of p such that the second derivatives of the vector valued function $x_\alpha = f \circ \alpha^{-1}$: $\alpha(U) \to E^n$ exist are measurable, and independent of the order derivations almost everywhere on $\alpha(U)$.*

Proof. Let π_p: $E^n \to L_p^k$ be the orthogonal projection on L_p^k. There is a neighborhood U' of p in M^k such that $\alpha = \pi_p \circ f$: $U' \to L_p^k$ yields a coordinate system of the C^1-manifold M^k. Let V be the neighborhood of p on which f is of locally bounded curvature with the bound K_V according to the assumption of the lemma. Choose $\delta'>0$ such that $U(2\delta')$, the spherical neighborhood of p with radius $2\delta'$ taken according to the distance function d, is contained in V. Then $\frac{\|X^k_{q_1}-X^k_{q_2}\|}{d(q_1, q_2)} \leqq K_V$ $d(q_l, p)\leqq\delta'$ $(l=1, 2)$. In fact, by assuming the contrary and considering successive bisections of a minimizing geodesic arc joining q_1, q_2, one would arrive at a point of V where K_V cannot be a bound for f. Put $\delta=\min\left(\delta', \frac{1}{K_V}\right)$, then $U=U(\delta)$, the spherical neighborhood of p with radius δ, is contained in U'. To verify the last assertion it suffices to see that there is no $q\in U$ with $\langle X^k_q, X^k_p\rangle=0$ ([13], 117—119); but this is obvious since $q\in U$ and $\langle X^k_q, X^k_p\rangle=0$ would imply that $d(p, q)\geqq\frac{\sqrt{2}}{K_V}$. If orthonormal coordinate systems are suitably chosen in E^n and L^k_p, then

$$x_\alpha(u^1, \dots, u^k) = \left(u^1, \dots, u^k, x_\alpha^{k+1}(u^1, \dots, u^k), \dots, x_\alpha^n(u^1, \dots, u^k)\right)$$

with $(u^1, \dots, u^k) = \alpha(q)$ for $q\in U'$. Put

$$Y^k(u^1, \dots, u^k) = \frac{\partial x_\alpha}{\partial u^1}\Big|_{\alpha(q)} \wedge \dots \wedge \frac{\partial x_\alpha}{\partial u^k}\Big|_{\alpha(q)} \quad \text{and} \quad N(u^1, \dots, u^k) = \|Y^k(u^1, \dots, u^k)\|$$

for $q\in U'$. Let Q be the solid k-dimensional cube spanned by the basic vectors of the coordinate system of L^k_p; then its inverse image in L^k_q under π_p is a solid k-dimensional parallelepiped Q_q, which is spanned by the vectors $\frac{\partial x_\alpha}{\partial u^1}\Big|_{\alpha(q)}, \dots, \frac{\partial x_\alpha}{\partial u^k}\Big|_{\alpha(q)}$, for $q\in U'$. But $N(u^1, \dots, u^k)$ is equal to the k-dimensional volume of Q_q and

$$\langle Y^k(u^1, \dots, u^k), X^k_p\rangle = N(u^1, \dots, u^k)\langle X^k_q, X^k_p\rangle = 1 \quad ([10], 56-57).$$

Therefore

$$|N(u^1_1, \dots, u^k_1)-N(u^1_2, \dots, u^k_2)| = \left|\frac{\langle X^k_{q_2}-X^k_{q_1}, X^k_p\rangle}{\langle X^k_{q_1}, X^k_p\rangle\cdot\langle X^k_{q_2}, X^k_p\rangle}\right| \leqq$$

$$\leqq \frac{\|X^k_{q_2}-X^k_{q_1}\|}{|1-\frac{1}{2}\|X^k_{q_1}-X^k_p\|^2|\cdot|1-\frac{1}{2}\|X^k_{q_2}-X^k_p\|^2|} \leqq 4\cdot\|X^k_{q_2}-X^k_{q_1}\|$$

if $q_1, q_2\in U$. There is a $C>0$ such that $d(q_1, q_2)^2 \leqq C^2\cdot\sum_{i=1}^k (u^i_1-u^i_2)^2$ for $q_1, q_2\in U'$. Consequently

$$|N(u^1_1, \dots, u^k_1)-N(u^1_2, \dots, u^k_2)| \leqq 4\cdot K_V\cdot d(q_1, q_2) \leqq 4\cdot K_V\cdot C\cdot\left[\sum_{i=1}^k (u^i_1-u^i_2)^2\right]^{\frac{1}{2}}$$

for $q_1, q_2 \in U$. Therefore

$$\|Y^k(u_1^1, \dots, u_1^k) - Y^k(u_2^1, \dots, u_2^k)\| \leqq N(u_1^1, \dots, u_1^k) \cdot \|X_{q_1}^k - X_{q_2}^k\| +$$

$$+ |N(u_1^1, \dots, u_1^k) - N(u_2^1, \dots, u_2^k)| \leqq 6 \cdot K_V \cdot d(q_1, q_2) \leqq 6 \cdot K_V \cdot C \cdot \left[\sum_{i=1}^{k} (u_1^i - u_2^i)^2\right]^{\frac{1}{2}}$$

if $q_1, q_2 \in U$. This means in other words that $Y^k: \alpha(U) \to V_k^n$ is a Lipschitz map. Since

$$\left\| \frac{\partial x_\alpha}{\partial u^i}\bigg|_{\alpha(q_1)} - \frac{\partial x_\alpha}{\partial u^i}\bigg|_{\alpha(q_2)} \right\| \leqq \|Y^k(u_1^1, \dots, u_k^1) - Y^k(u_2^1, \dots, u_2^k)\|$$

it follows that $\frac{\partial x_\alpha}{\partial u^i}: \alpha(U) \to V_1^n$ is a Lipschitz map as well. Therefore by Rademacher's theorem ([13], 271—272), $\frac{\partial^2 x_\alpha}{\partial u^j \partial u^i}$ $(i, j = 1, \dots, k)$ exist almost everywhere on $\alpha(U)$ and are measurable. The fact that $\frac{\partial^2 x_\alpha}{\partial u^j \partial u^i} = \frac{\partial^2 x_\alpha}{\partial u^i \partial u^j}$ almost everywhere on $\alpha(U)$ follows by an obvious application of Fubini's theorem.

If the C^1-immersion $f: M^k \to E^n$ is of locally bounded curvature at p and $\alpha: U \to E^k$ is a coordinate system of the C^1-manifold M^k on the neighborhood U of p constructed according to the proof of the preceding lemma then $\alpha: U \to E^k$ will be called a *distinguished coordinate system.*

Let $f: M^k \to E^n$ be a C^1-immersion, NM^k its normal bundle, $\pi: NM^k \to M^k$ the projection in the normal bundle and $\nu: NM^k \to E^n$ the normal map of the immersion; $\pi^{-1}(p) = N_p M^k$ is a euclidean vector space and the restriction of ν to it is an isometric vector space isomorphism. Assume that f is of locally bounded curvature on the neighborhood V of p with the bound K_V. Let $(a_1, \dots, a_n) = b \in B^n$ be such a base that $a_1 \wedge \dots \wedge a_k = X_p^k$ and $\Lambda_b^k: O(n) \to Q(n, k)$ the corresponding homomorphism. Then $H_b(X_p^k) \subset O(n)$ is the subgroup of elements which leave X_p^k fixed, and by the correspondence $O(n)/H_b(X_p^k) \leftrightarrow S_k^n$ for any $q \in M^k$ there is a left coset $\alpha_q H_b(X_p^k)$ consisting of those elements which send X_p^k to X_q^k. Let $\delta > 0$ be the number provided by Lemma 1. 1 for the case $G = O(n)$, $H = H_b(X_p^k)$, $\varrho = \varrho_e$ and B the bound given by Lemma 1. 2. There is a neighborhood $U(\delta')$ of radius $\delta' > 0$ of p such that $U(2\delta') \subset V$ and $\delta' \leqq \frac{\delta}{K_V \cdot B}$. Consequently if $q \in U(\delta')$, then

$$\bar{\varrho}_e(H_b(X_p^k), \alpha_q H_b(X_p^k)) = d^k(X_p^k, X_q^k) \leqq B \cdot \|X_p^k - X_q^k\| \leqq \delta.$$

Hence by Lemma 1. 1 for any $q \in U(\delta')$ there exists a unique $\xi_q \in \alpha_q H_b(X_p^k)$ with

$\varrho_e(\varepsilon, \xi_q)=\bar{\varrho}_e(H_b(X_p^k), \alpha_q H_b(X_p^k))$. The field of bases $\beta\colon U \to B^n$ defined on $U=U(\delta')$ by $\beta(q) = b\cdot\xi_q = (\bar{w}_1(q), \dots, \bar{w}_n(q))$ $(q\in U)$ will be called a *distinguished field of bases*. Assume that a distinguished coordinate system $\alpha\colon U \to E^n$ is given as well. Then $(w_1(u^1, \dots, u^k), \dots, w_n(u^1, \dots, u^k))$, $(u^1, \dots, u^k)\in\alpha(U)$ with $w_i(u^1, \dots, u^k)=w_i(q)$ $(i=1, \dots, n)$ for $(u^1, \dots, u^k)=\alpha(q)$ $(q\in U)$ is called a *coordinate representation of the distinguished field of bases*. If $w\in\pi^{-1}(U)$ then

$$v(w) = \sum_{j=k+1}^{n} t^j\bar{w}_j(q) = \sum_{j=k+1}^{n} t^j w_j(u^1, \dots, u^k) \text{ and } \zeta_{\alpha\beta}(w) = (u^1, \dots, u^k, t^{k+1}, \dots, t^n)$$

defines a coordinate system $\zeta_{\alpha\beta}\colon \pi^{-1}(U) \to \alpha(U)\times E^{n-k}$ for the normal bundle; this will be called a *distinguished coordinate system of the normal bundle*. The map $z_{\alpha\beta}\colon \alpha(U)\times E^{n-k} \to E^n$ defined by $z_{\alpha\beta}(u^1, \dots, u^k, t^{k+1}, \dots, t^n)= \sum_{j=k+1}^{n} t^j w_j(u^1, \dots, u^k)$ is called a *distinguished coordinate representation of the normal map*.

Lemma 2. 2. *Let the C^1-immersion $f\colon M^k \to E^n$ be of locally bounded curvature at $p\in M^k$, let $\alpha\colon U\to E^k$, $\beta\colon U \to B^n$ be a distinguished coordinate system and a distinguished field of bases on the neighborhood U of p. Then the corresponding coordinate representation $z_{\alpha\beta}\colon\alpha(U)\times B_\vartheta^{n-k} \to E^n$ of the normal map is a Lipschitz map, where B_ϑ^{n-k} is the solid ball of radius $\vartheta>0$ at the origin in E^{n-k}.*

Proof. Since

$$\frac{\|z_{\alpha\beta}(u_1^1, \dots, u_1^k, t_1^{k+1}, \dots, t_1^n)-z_{\alpha\beta}(u_2^1, \dots, u_2^k, t_2^{k+1}, \dots, t_2^n)\|}{\left[\sum_{i=1}^{k}(u_1^i-u_2^i)^2+\sum_{j=k+1}^{n}(t_1^j-t_2^j)^2\right]^{\frac{1}{2}}} \leqq$$

$$\leqq \frac{\|x_\alpha(u_1^1, \dots, u_1^k)-x_\alpha(u_2^1, \dots, u_2^k)\|}{\left[\sum_{i=1}^{k}(u_1^i-u_2^i)^2\right]^{\frac{1}{2}}} + \frac{\left[\sum_{j=k+1}^{n}(t_1^j-t_2^j)^2\right]^{\frac{1}{2}}}{\left[\sum_{j=k+1}^{n}(t_1^j-t_2^j)^2\right]^{\frac{1}{2}}}+$$

$$+\frac{\left\|\sum_{j=k+1}^{n} t_2^j\big(w_j(u_1^1, \dots, u_1^k)-w_j(u_2^1, \dots, u_2^k)\big)\right\|}{\left[\sum_{i=1}^{k}(u_1^i-u_2^i)^2\right]^{\frac{1}{2}}}$$

it sufficies to find bounds for the first and the last term. With the notations of and according to the proof of the preceding lemma

$$\frac{\|x_\alpha(u_1^1,\dots,u_1^k)-x_\alpha(u_2^1,\dots,u_2^k)\|}{\left[\sum_{i=1}^{k}(u_1^i-u_2^i)^2\right]^{\frac{1}{2}}}=$$

$$=\left\{1+\sum_{j=k+1}^{n}\left(\sum_{i=1}^{n}\frac{\partial x_\alpha^j}{\partial u^i}\Big|_{\alpha(q_1)}\frac{u_2^i-u_1^i}{\left[\sum_{s=1}^{k}(u_1^s-u_2^s)^2\right]^{\frac{1}{2}}}+\frac{R_{ji}}{\left[\sum_{s=1}^{k}(u_1^s-u_2^s)^2\right]^{\frac{1}{2}}}\right)^2\right\}^{\frac{1}{2}}\leqq$$

$$\leqq\left\{1+\sum_{j=k+1}^{n}\left(\sum_{i=1}^{k}\left|\frac{\partial x_\alpha^j}{\partial u^i}\right|\Big|_{\alpha(q_1)}+\frac{|R_{ji}|}{\left[\sum_{s=1}^{k}(u_1^s-u_2^s)^2\right]^{\frac{1}{2}}}\right)^2\right\}^{\frac{1}{2}}\leqq$$

$$\leqq 1+\sum_{j=k+1}^{n}\sum_{i=1}^{k}\left(\left|\frac{\partial x_\alpha^j}{\partial u^i}\right|\Big|_{\alpha(q_1)}+\frac{|R_{ji}|}{\left[\sum_{s=1}^{k}(u_1^i-u_2^i)^2\right]^{\frac{1}{2}}}\right)\leqq$$

$$\leqq 1+\sqrt{2}\,\|Y^k(u_1^1,\dots,u_1^k)\|+\frac{1}{\left[\sum_{s=1}^{k}(u_1^s-u_2^s)^2\right]^{\frac{1}{2}}}\sum_{j=k+1}^{n}\sum_{i=1}^{k}|R_{ji}|.$$

But

$$\|Y^k(u_1^1,\dots,u_1^k)\|=N(u_1^1,\dots,u_1^k)=\frac{1}{\langle X_{q_1}^k,X_p^k\rangle}=\frac{1}{1-\frac{1}{2}\|X_{q_1}^k-X_p^k\|^2}\leqq 2$$

and

$$R_{ji}=\int_0^1\left(\frac{\partial x_\alpha^j}{\partial u^i}\Big|_{\alpha(q(t))}-\frac{\partial x_\alpha^j}{\partial u^i}\Big|_{\alpha(q_1)}\right)(u_2^i-u_1^i)\,dt,$$

where

$$\alpha(q(t))=(u_1^1+t(u_2^1-u_1^1),\dots,u_1^k+t(u_2^k-u_1^k)),\quad (0\leqq t\leqq 1).$$

Therefore

$$\sum_{j=k+1}^{n}\sum_{i=1}^{k}|R_{ji}|\leqq\int_0^1\sum_{j=k+1}^{n}\sum_{i=1}^{k}\left|\left(\frac{\partial x_\alpha^j}{\partial u^i}\Big|_{\alpha(q(t))}-\frac{\partial x_\alpha^j}{\partial u^i}\Big|_{\alpha(q_1)}\right)(u_2^i-u_1^i)\right|dt\leqq$$

$$\leqq\left[\sum_{s=1}^{k}(u_1^i-u_2^i)^2\right]^{\frac{1}{2}}\cdot\int_0^1\left[2\sum_{j=k+1}^{n}\sum_{i=1}^{k}\left(\frac{\partial x_\alpha^j}{\partial u^i}\Big|_{\alpha(q(t))}-\frac{\partial x_\alpha^j}{\partial u^i}\Big|_{\alpha(q_1)}\right)^2\right]^{\frac{1}{2}}dt\leqq$$

$$\leqq\left[2\sum_{i=1}^{k}(u_1^i-u_2^i)^2\right]^{\frac{1}{2}}\int_0^1\left\|Y^k(\alpha(q(t)))-Y^k(\alpha(q_1))\right\|dt\leqq$$

$$\leqq\left[2\sum_{i=1}^{k}(u_1^i-u_2^i)^2\right]^{\frac{1}{2}}\cdot 6\cdot K_V\cdot d(q_1,q_2).$$

Consequently

$$\frac{\|x_\alpha(u_1^1, \ldots, u_1^k) - x_\alpha(u_2^1, \ldots, u_2^k)\|}{\left[\sum_{i=1}^{k} (u_1^i - u_2^i)^2\right]^{\frac{1}{2}}} \leqq 1 + 8\sqrt{2}.$$

By Lemmas 1.1 and 1.2

$$\left\| \sum_{j=k+1}^{n} t_j\big(w_j(u_1^1, \ldots, u_1^k) - w_j(u_2^1, \ldots, u_2^k)\big) \right\| \leqq$$

$$\leqq \vartheta \left[\sum_{j=k+1}^{n} \big(w_j(u_1^1, \ldots, u_1^k) - w_j(u_2^1, \ldots, u_2^k)\big)^2 \right]^{\frac{1}{2}} \leqq \vartheta \cdot \sigma(b\xi_{q_1}, b\xi_{q_2}) =$$

$$= \vartheta \cdot \mu(\xi_{q_1}, \xi_{q_2}) \leqq \vartheta \cdot \varrho_e(\xi_{q_1}, \xi_{q_2}) \leqq \vartheta \cdot A \cdot \bar{\varrho}_e\big(\alpha_1 H_b(X_p^k), \alpha_2 H_b(X_p^k)\big) =$$

$$= \vartheta \cdot A \cdot d^k(X_{q_1}^k, X_{q_2}^k) \leqq \vartheta \cdot A \cdot B \cdot \|X_{q_1}^k - X_{q_2}^k\| \leqq \vartheta \cdot A \cdot B \cdot K_V \cdot d(q_1, q_2) \leqq$$

$$\leqq \vartheta \cdot A \cdot B \cdot K_V \cdot C \left[\sum_{i=1}^{k} (u_1^i - u_2^i)^2 \right]^{\frac{1}{2}}.$$

Theorem 2.1. *Let $f\colon M^k \to E^n$ be an immersion of locally bounded curvature. Then its second fundamental tensor and second fundamental forms exist almost everywhere on M^k.*

Proof. Let $\alpha\colon U \to E^k$ be a distinguished coordinate system, $\beta\colon U \to B^n$ a distinguished field of bases. By the preceding lemma $w_i\colon \alpha(U) \to V_1^n$ $(i=1, \ldots, n)$ is a Lipschitz map and therefore, according to Rademacher's theorem, $\frac{\partial w_i}{\partial u^s}$ $(s=1, \ldots, k)$ exist almost everywhere on $\alpha(U)$ and are measurable. As a consequence the usual definition of the second fundamental tensor applies almost everywhere on U as follows. Let $w(u^1, \ldots, u^k) = \sum_{j=k+1}^{n} \beta^j(u^1, \ldots, u^k) w_j(u^1, \ldots, u^k)$ be the coordinate representation of a normal field, where β^j are differentiable functions on $\alpha(U)$, and $v = \sum_{i=1}^{k} \alpha^i \frac{\partial x_\alpha}{\partial u^i}\Big|_{\alpha(q)}$, then $D(v, w)$, the derivative of w in the direction of v, exists if $\frac{\partial w_j}{\partial u^i}\Big|_{\alpha(q)}$ $(i=1, \ldots, k;\ j=k+1, \ldots, n)$ all exist. Let $S_q(v, w)$ be the orthogonal projection of $D(v, w)$ on X_q^k, then

$$S_q(v, w) = \sum_{i=1}^{k} \sum_{r=1}^{k} \sum_{s=1}^{k} \sum_{j=k+1}^{n} g^{ir}(u^1, \ldots, u^k) \alpha^s \beta^j(u^1, \ldots, u^k) \left\langle \frac{\partial x_\alpha}{\partial u^r}, \frac{\partial w_j}{\partial u^s} \right\rangle\Big|_{\alpha(q)} \frac{\partial x_\alpha}{\partial u^i}\Big|_{\alpha(q)}$$

where g^{ir} are the contravariant components of the first fundamental tensor. Therefore

$S_q(v, w)$ defines a tensor $S_q\colon T_qM^k \times N_qM^k \to T_qM^k$. Consequently a tensor field S is obtained which is defined for almost every $q \in M^k$ and is called the *second fundamental tensor of the immersion.* If $\|w\|=1$ and w is fixed then $l_w(v, v') = \langle S_q(v, w), v'\rangle = \sum_{r=1}^{k}\sum_{s=1}^{k}\sum_{j=k+1}^{n} \alpha'^r \alpha^s \beta^j \left\langle \frac{\partial x_\alpha}{\partial u^r}, \frac{\partial w_j}{\partial u^s}\right\rangle$ is a bilinear form

$$l_w\colon T_qM^k \times T_qM^k \to E^1$$

in $v = \sum_{i=1}^{k} \alpha^i \frac{\partial x_\alpha}{\partial u^i}\Big|_{\alpha(q)}$ and $v' = \sum_{i=1}^{k} \alpha'^i \frac{\partial x_\alpha}{\partial u^i}\Big|_{\alpha(q)}$. This is called the *second fundamental form of the immersion at q in the normal direction* $w \in N_qM^k$.

In case of a hypersurface of locally bounded curvature $f\colon M^{n-1} \to E^n$ there is essentially one choice for w, therefore the hypersurface has one second fundamental form l. The above theorem and Lemma 2. 1 give the following

Corollary. *Let $f\colon M^{n-1} \to E^n$ be a hypersurface of locally bounded curvature and l its second fundamental form, then $l_q\colon T_qM^{n-1} \times T_qM^{n-1} \to E^1$ is symmetric for almost every $q \in M^{n-1}$. The coefficients of l are measurable in any distinguished coordinate system of the hypersurface.*

3. The uniqueness of hypersurfaces of locally bounded curvature with given first and second fundamental forms

The uniqueness theorem for C^∞-hypersurfaces is the following: Let the C^∞-hypersurfaces $f_1, f_2\colon M^{n-1} \to E^n$ $(n \geqq 3)$ have common first and second fundamental forms where the second fundamental forms are taken with respect to C^∞-unit normal fields $\overline{w}_n^1, \overline{w}_n^2\colon M^{n-1} \to V_1^n$ of the hypersurfaces f_1, f_2. Then there exists a distance preserving transformation $\Phi\colon E^n \to E^n$ with $f_2 = \Phi \circ f_1$ ([7], 79—81).

This theorem will be generalized to hypersurfaces of locally bounded curvature in this section.

Lemma 3. 1. *Let the C^1-hypersurface $f\colon M^{n-1} \to E^n$ be of locally bounded curvature at $p \in M^{n-1}$ with the bound K_V, let $\alpha\colon U \to E^{n-1}$ be a distinguished coordinate system, $\beta\colon U \to B^n$ a distinguished field of bases on the neighborhood U of p and $z_{\alpha\beta}\colon \alpha(U) \times E^1 \to E^n$ the corresponding coordinate representation of the normal map. Then there exist a neighborhood $\tilde{U} \subset U$ of p and numbers $\delta, \vartheta > 0$ such that*

$$\|z_{\alpha\beta}(u_1^1, \dots, u_1^{n-1}, t_1^n) - z_{\alpha\beta}(u_2^1, \dots, u_2^{n-1}, t_2^n)\| \geqq \delta \left[\sum_{i=1}^{n-1} (u_1^i - u_2^i)^2 + (t_1^n - t_2^n)^2\right]^{\frac{1}{2}}$$

if

$$(u_l^1, \dots, u_l^{n-1}, t_l^n) \in \alpha(\tilde{U}) \times B_\vartheta^1 \qquad (l = 1, 2).$$

Proof. If $u_1^i = u_2^i$ $(i = 1, \ldots, n-1)$, the inequality holds with $\delta = 1$ for any $\vartheta > 0$, therefore it will be assumed that $(u_l^1, \ldots, u_l^{n-1})$ $(l = 1, 2)$ correspond to different $q_1, q_2 \in U$. Assume further that $t_1^n \geqq t_2^n$ and put $t = t_2^n$, $\Delta t = t_1^n - t_2^n$,

$$\Delta u = \left[\sum_{i=1}^{n-1} (u_2^i - u_1^i)^2\right]^{\frac{1}{2}}, \quad x = x_\alpha(u_1^1, \ldots, u_1^{n-1}) - x_\alpha(u_2^1, \ldots, u_2^{n-1}),$$

$$\tilde{w}_l = \overline{w}_n(q_l), \quad y = x + t(\tilde{w}_1 - \tilde{w}_2), \quad z = x + t_1^n \tilde{w}_1 - t_2^n \tilde{w}_2, \quad d = d(q_1, q_2).$$

Let ε_l be the angle of the vector x and the subspace $X_{q_l}^{n-1}$ and put $\gamma_l = \sphericalangle(x, \tilde{w}_l)$, $\gamma' = \sphericalangle(y, \tilde{w}_1)$, $\gamma'' = \sphericalangle(x, \tilde{w}_1 - \tilde{w}_2)$, $2\gamma = \sphericalangle(\tilde{w}_1, \tilde{w}_2)$ where $\gamma'' = 0$ if $\tilde{w}_1 = \tilde{w}_2$. According to the proof of Lemma 2. 2, $1 \leqq \frac{\|x\|}{\Delta u} \leqq 1 + 8\sqrt{2}$ if $q_1, q_2 \in U$. Let $\alpha' : U' \to E^{n-1}$ be a distinguished coordinate system at $q_1 \in U$ and let $q_2 \in U' \cap U$. Then for a coordinate system suitably chosen in E^n we have $|\operatorname{tg} \varepsilon_1| = \left|\frac{\Delta x_{\alpha'}^n}{\Delta u'}\right|$ where, like below, primes refer to analogous quentities in case of the coordinate system α'. Therefore

$$|\operatorname{tg} \varepsilon_1| = \left|\left(\sum_{i=1}^{n-1} \frac{\partial x_{\alpha'}^n}{\partial u'^i}\bigg|_{\alpha'(q_2)} \frac{u_1'^i - u_2'^i}{\Delta u'} + \frac{R'_{ni}}{\Delta u'}\right)\right| \leqq \sum_{i=1}^{n-1} \left(\left|\frac{\partial x_{\alpha'}^n}{\partial u'^i}\right|\bigg|_{\alpha'(q_2)} + \frac{|R'_{ni}|}{\Delta u'}\right) \leqq$$

$$\leqq \sqrt{2}\, \|Y^{n-1}(u_2'^1, \ldots, u_2'^{n-1})\| + \frac{1}{\Delta u'} \sum_{i=1}^{n-1} |R'_{ni}| \leqq$$

$$\leqq \sqrt{2}\, \|Y^{n-1}(u_1'^1, \ldots, u_1'^{n-1}) - Y^{n-1}(u_2'^1, \ldots, u_2'^{n-1})\| +$$

$$+ \frac{1}{\Delta u'} \sum_{i=1}^{n-1} |R'_{ni}| \leqq \sqrt{2} \cdot 6 \cdot K_V \cdot C' \cdot \Delta u' + \sqrt{2} \cdot 6 \cdot K_V \cdot C' \cdot \Delta u' = 12 \cdot \sqrt{2} \cdot K_V \cdot C' \cdot \Delta u',$$

with respect to the proof of Lemmas 2. 1. and 2. 2. Hence $\frac{|\operatorname{tg} \varepsilon_1|}{\|x\|} \leqq \frac{|\operatorname{tg} \varepsilon_1|}{\Delta u'} \leqq$ $\leqq 12 \cdot \sqrt{2} \cdot K_V \cdot C'$ if $q_1 \in U$, $q_2 \in U' \cap U$. But the value of C' which may depend on q_1 is bounded on a neighborhood of p, since f is a C^1-hypersurface. Therefore a neighborhood $\tilde{U} \subset U$ of p and a bound D exist with $\frac{|\operatorname{tg} \varepsilon_l|}{\|x\|} \leqq D$ for $q_1, q_2 \in \tilde{U}$ $(l = 1, 2)$. Obviously

$$\|z\|^2 = (\Delta t)^2 + \|x\|^2 + t^2 \|\tilde{w}_1 - \tilde{w}_2\|^2 -$$

$$- 2\Delta t \left[\|x\|^2 + t^2 \|\tilde{w}_1 - \tilde{w}_2\|^2 - 2t \|x\| \cdot \|\tilde{w}_1 - \tilde{w}_2\| \cdot \cos \gamma''\right]^{\frac{1}{2}} \cdot \cos \gamma' -$$

$$- 2t \|x\| \cdot \|\tilde{w}_1 - \tilde{w}_2\| \cdot \cos \gamma'.$$

Hence

$$\|z\|^2 \geqq (\Delta t)^2 + \|x\|^2 - t^2 \cdot K_V^2 \cdot d^2 -$$

$$-2\Delta t\left[\|x\|^2 + t^2 \cdot K_V^2 \cdot d^2 + 2t \cdot \|x\| \cdot K_V \cdot d \cdot \cos\gamma'\right]^{\frac{1}{2}} - 2t \cdot \|x\| \cdot K_V \cdot d = (\Delta t)^2 +$$

$$+(\Delta u)^2\left[\frac{\|x\|^2}{(\Delta u)^2} - t^2 \cdot K_V^2 \cdot \frac{d^2}{(\Delta u)^2} - 2\Delta t \frac{\|x\|}{\Delta u}\left(\frac{\|x\|}{\Delta u} + t \cdot K_V \cdot \frac{d}{\Delta u}\right)\frac{\cos\gamma'}{\|x\|} - \right.$$

$$\left. - 2t \cdot K_V \cdot \frac{\|x\|}{\Delta u} \cdot \frac{d}{\Delta u}\right].$$

But

$$\frac{\cos\gamma'}{\|x\|} = \frac{\langle \tilde{w}_1, x + t(\tilde{w}_1 - \tilde{w}_2)\rangle}{\|x\| \cdot \|x + t(\tilde{w}_1 - \tilde{w}_2)\|} =$$

$$= \left[\frac{\cos\gamma_1}{\|x\|} + 2t\frac{\sin^2\gamma}{\|x\|}\right] \cdot \left[1 + t^2 \frac{\|\tilde{w}_1 - \tilde{w}_2\|^2}{\|x\|^2} + 2t\left(\frac{\cos\gamma_1}{\|x\|} - \frac{\cos\gamma_2}{\|x\|}\right)\right]^{-\frac{1}{2}}$$

and as obvious calculations show $|\cos\gamma_l| \leqq |\sin\varepsilon_l| \leqq |\text{tg}\,\varepsilon_l|$, $\dfrac{\|\tilde{w}_1 - \tilde{w}_2\|}{\|x\|} \leqq K_V \cdot C$, $\dfrac{\sin\gamma}{\|x\|} = K_V \cdot C$. Therefore $\dfrac{\cos\gamma'}{\|x\|}$ is bounded if $q_1, q_2 \in \tilde{U}$ and t is sufficiently small. The above estimates yield the existence of such $\delta, \vartheta > 0$ that $\|z\| \geqq \delta[(\Delta t)^2 + (\Delta u)^2]^{\frac{1}{2}}$ if $q_1, q_2 \in \tilde{U}$ and $|t_1^n|, |t_2^n| \leqq \vartheta$.

Theorem 3. 1. *Let f_1, f_2: $M^{n-1} \to E^n$ be hypersurfaces of locally bounded curvature, g^1, g^2 their first fundamental forms, and l^1, l^2 their second fundamental forms which are calculated with respect to continuous unit normal fields $\tilde{w}_n^1, \tilde{w}_n^2$: $M^{n-1} \to V_1^n$ of the hypersurfaces. Assume that $g^1 = g^2$ and $l_q^1 = l_q^2$ for almost every $q \in M^{n-1}$. Then there exists a distance preserving transformation Φ: $E^n \to E^n$ such that $f_2 = \Phi \circ f_1$.*

Proof. Let $N^l M^{n-1}$ be the normal bundle and γ_l: $N^l M^{n-1} \to E^n$ be the normal map of the hypersurface f_l $(l = 1, 2)$. In consequence of the preceding lemma for any $p \in M^{n-1}$ there is a distinguished coordinate system α_l: $U \to E^{n-1}$ and a distinguished field of bases β_l: $U \to B^n$ of f_l $(l = 1, 2)$ on a neighborhood U of p such that if z_l: $\alpha_l(U) \times E^1 \to E^n$ is the corresponding representation of the normal map, then $\delta, \vartheta > 0$ and $\tilde{U} \subset U$ exist for which the assertion of the lemma holds. Here β_1, β_2 are chosen so as to give $\overline{w}_n^l(q) = \tilde{w}_n^l(q)$ for $q \in U$. Let π_l: $N^l M^{n-1}$ be the projection in the bundle and $V^l = \{w \mid w \in \pi_l^{-1}(\tilde{U}),\ \|w\| < \vartheta\}$; then the restriction of v^l to V^l is one-to-one. Consequently there is such a distance function ϱ^l that v_l: $(V^l, \varrho^l) \to E^n$ is distance preserving. By convexification of the distance function ϱ_l an intrinsic distance function $\bar{\varrho}^l$ is obtained on V^l ([2], [4], 77). The map γ_l: $(V^l, \bar{\varrho}^l) \to E^n$ is locally distance preserving. If $w \in V^1$ then there is a unique $w' \in V^2$ such that $\pi_1(w) = \pi_2(w') = q$ and if $\gamma_1(w) = f_1(q) + t^n \overline{w}_n^1(q)$ then $v_2(w') = f_2(q) + t^n \overline{w}_n^2(q)$. Let

$\Psi_{\tilde{U}}\colon V^1 \to V^2$ be defined by $\Psi_{\tilde{U}}(w)=w'$, then $\Psi_{\tilde{U}}$ is one-to-one and maps V^1 onto V^2. The following argument will show that $\Psi_{\tilde{U}}\colon (V^1, \bar{\varrho}^1) \to (V^2, \bar{\varrho}^2)$ is distance preserving. Let $\bar{g}^1_{ab}(u^1, \dots, u^{n-1}, t^n)$ be defined for $a, b=1, \dots, n$, and $(u^1, \dots, u^{n-1}, t^n) \in \alpha_1(\tilde{U}) \times B^1_\vartheta$ by

$$\bar{g}^1_{ab} = \begin{cases} \left\langle \dfrac{\partial z_1}{\partial u^a}, \dfrac{\partial z_1}{\partial u^b} \right\rangle & \text{if } a, b = 1, \dots, n-1 \\ \left\langle \dfrac{\partial z_1}{\partial u^a}, \dfrac{\partial z_1}{\partial t^n} \right\rangle & \text{if } a = 1, \dots, n-1;\ b = n, \\ \left\langle \dfrac{\partial z_1}{\partial t^n}, \dfrac{\partial z_1}{\partial u^b} \right\rangle & \text{if } a = n;\ b = 1, \dots, n-1, \\ \left\langle \dfrac{\partial z_1}{\partial t^n}, \dfrac{\partial z_1}{\partial t^n} \right\rangle & \text{if } a = b = n. \end{cases}$$

Then

$$\bar{g}^1_{ab} = g^1_{ab} + t^n(l^1_{ab} + l^1_{ba}) + (t^n)^2 \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \sum_{r=1}^{n-1} \sum_{s=1}^{n-1} g^1_{ij} g_1^{ri} g_1^{sj} l^1_{ar} l^1_{bs}$$

if $a\, b = 1, \dots, n-1$, where g^1_{ab}, g_1^{ab} are the covariant and contravariant components of the first fundamental tensor and the l^1_{ab} coefficients of the second fundamental form of f_1 in the coordinate system α_1; $\bar{g}^1_{ab}=0$ if $a=1, \dots, n-1$; $b=n$ or $a=n$; $b=1, \dots, n-1$; $\bar{g}^1_{ab}=1$ if $a=b=n$. The functions $\bar{g}^1_{ab}$ are measurable on $\alpha_1(\tilde{U}) \times B^1_\vartheta$. Let $\hat{x}(u^1, \dots, u^{n-1})$ $\hat{w}_n(u^1, \dots, u^{n-1})$ be defined by $\hat{x}(u^1, \dots, u^{n-1}) = f_2(g)$, $\hat{w}_n(u^1, \dots, u^{n-1}) = \bar{w}^2_n(g)$, $(u^1, \dots, u^{n-1}) = \alpha_1(g)$, $(g \in \tilde{U})$ and put $\hat{z}(u^1, \dots, u^{n-1}, t^n) = \hat{x}(u^1, \dots, u^{n-1}) + t^n \hat{w}_n(u^1, \dots, u^{n-1})$ for $(u^1, \dots, u^{n-1}, t^n) \in \alpha_1(\tilde{U}) \times B^1_\vartheta$. Let the $\bar{\hat{g}}^2_{ab}(u^1, \dots, u^{n-1}, t^n)$ be defined for $a, b=1, \dots, n$ and $(u^1, \dots, u^{n-1}, t^n) \in \alpha_1(\tilde{U}) \times B^1_\vartheta$ by

$$\bar{\hat{g}}^2_{ab} = \begin{cases} \left\langle \dfrac{\partial \hat{z}}{\partial u^a}, \dfrac{\partial \hat{z}}{\partial u^b} \right\rangle & \text{if } a, b = 1, \dots, n-1, \\ \left\langle \dfrac{\partial \hat{z}}{\partial u^a}, \dfrac{\partial \hat{z}}{\partial t^n} \right\rangle & \text{if } a = 1, \dots, n-1;\ b = n, \\ \left\langle \dfrac{\partial \hat{z}}{\partial t^n}, \dfrac{\partial \hat{z}}{\partial u^b} \right\rangle & \text{if } a = n;\ b = 1, \dots, n-1, \\ \left\langle \dfrac{\partial \hat{z}}{\partial t^n}, \dfrac{\partial \hat{z}}{\partial t^n} \right\rangle & \text{if } a = b = n. \end{cases}$$

Then

$$\bar{\hat{g}}^2_{ab} = \hat{g}^2_{ab} + t^n(\hat{l}^2_{ab} + \hat{l}^2_{ba}) + (t^n)^2 \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \sum_{r=1}^{n-1} \sum_{s=1}^{n-1} \hat{g}^2_{ij} \hat{g}_2^{ri} \hat{g}_2^{sj} \hat{l}^2_{ar} \hat{l}^2_{bs}$$

if $a, b = 1, \ldots, n-1$, where $\hat{g}^2_{ab}, \hat{g}^{ab}_2, \hat{l}^2_{ab}$ are the corresponding quantities of f_2 in the coordinate system α_1; $\bar{g}^2_{ab}=0$ if $a = 1, \ldots, n-1$; $b=n$ or $a=n$; $b = 1, \ldots, n-1$; $\bar{g}^2_{ab}=1$ if $a=b=n$. The measurability of the functions $\bar{g}^2_{ab}$ follows again by the Corollary to Theorem 2. 1 but a coordinate transformation have to be considered as well. According to assumptions of the theorem $\bar{g}^1_{ab}=\bar{g}^2_{ab}$ $(a, b=1, \ldots, n)$ almost everywhere on $\alpha_1(\tilde{U})\times B^q_1$. If $w_1, w_2 \in V^1$ are sufficiently near then there is an n-dimensional parallelepiped P in $v_1(V^1)$ formed by a set S of straight line segments parallel and congruent to the one joining $v_1(w_1), v_1(w_2)$ and such that their endpoints fill two $(n-1)$-dimensional cubes the centres of which are $v_1(w_1)$ and $v_1(w_2)$. Let the arc φ: $[0, 1] \to V^1$ be such that $v_1 \circ \varphi:[0, 1] \to E^n$ is a linear representation of a segment in S, and let the functions $\varphi^h(t)$ $(h=1, \ldots, n;\ 0 \leqq t \leqq 1)$ be defined by $z_1(\varphi^1(t), \ldots, \varphi^n(t)) = v_1 \circ \varphi(t)$ $(0 \leqq t \leqq 1)$. In consequence of Lemma 3. 1 $\varphi^h(t)$ are Lipschitz functions; therefore $\dot{\varphi}(t)$ exist almost everywhere on $[0, 1]$ and are measurable in case of any segment in the set S. Obvious applications of Fubini's theorem to the set P yield that there are segments in S arbitrary near to the one joining $v_1(w_1)$, $v_1(w_2)$ such that 1) $\bar{g}^1_{ab}(\varphi^1(t), \ldots, \varphi^n(t))$, $\bar{g}^2_{ab}(\varphi^1(t), \ldots, \varphi^n(t))$ $(a, b=1, \ldots, n)$ are measurable on $[0, 1]$; 2) $\bar{g}^1_{ab}(\varphi^1(t), \ldots, \varphi^n(t)) = \bar{g}^2_{ab}(\varphi^1(t), \ldots, \varphi^n(t))$ $(a, b=1, \ldots, n)$ almost everywhere on $[0, 1]$. The distance of the points $v_1 \circ \varphi(0)$, $v_1 \circ \varphi(1)$ in case of such segments is

$$\int_0^1 \left[\sum_{a,b=1}^{n} \bar{g}^1_{ab}(\varphi^1(t), \ldots, \varphi^n(t))\dot{\varphi}^a(t)\dot{\varphi}^b(t)\right]^{\frac{1}{2}} dt =$$

$$= \int_0^1 \left[\sum_{a,b=1}^{n} \bar{g}^2_{ab}(\varphi^1(t), \ldots, \varphi^n(t))\dot{\varphi}^a(t)\dot{\varphi}^b(t)\right]^{\frac{1}{2}} dt.$$

But by Tonelli's theorem the last integral is equal to the length of the curve $\hat{z}(\varphi^1(t), \ldots, \varphi^n(t))$ $(0 \leqq t \leqq 1)$ since it is of bounded variation in consequence of Lemma 2. 2. Furthermore we have $\hat{z}(\varphi^1(0), \ldots, \varphi^n(0)) = v_2 \circ \Psi_{\tilde{U}} \circ \varphi(0)$ and $\hat{z}(\varphi^1(1), \ldots, \varphi^n(1)) = v_2 \circ \Psi_{\tilde{U}} \circ \varphi(1)$, and therefore

$$\varrho^1(\varphi(0), \varphi(1)) \geqq \varrho^2(\Psi_{\tilde{U}} \circ \varphi(0), \Psi_{\tilde{U}} \circ \varphi(1)).$$

Consequently $\varrho^1(w_1, w_2) \geqq \varrho^2(\Psi_{\tilde{U}}(w_1), \Psi_{\tilde{U}}(w_2))$, and changing the role of f_1, f_2 in the above argument gives $\varrho^1(w_1, w_2) \leqq \varrho^2(\Psi_{\tilde{U}}(w_1), \Psi_{\tilde{U}}(w_2))$.These imply that $\Psi_{\tilde{U}}$: $(V^1, \varrho^1) \to (V^2, \varrho^2)$ is locally distance preserving and $\Psi_{\tilde{U}}$: $(V^1, \bar{\varrho}^1) \to (V^2, \bar{\varrho}^2)$ is distance preserving. Proceeding in like manner a sequence of neighborhoods $\{\tilde{U}_m\}_{m=1,2,\ldots}$ covering M^{n-1} can be obtained with the corresponding sequences $\{(V^l_m, \bar{\varrho}^l_m)\}_{m=1,2,\ldots}$ of metrized neighborhoods in $N^l M^{n-1}$ and the distance preserving maps Ψ_m: $(V^1_m, \bar{\varrho}^1_m)$. The set $V_l = \bigcup_{m=1}^{\infty} V^l_m$ is a neighborhood of the zero section in $N^l M^{n-1}$ and since $\bar{\varrho}^l_{m'}, \bar{\varrho}^l_{m''}$ are equal on $V^l_{m'} \cap V^l_{m''}$ there is an intrinsic distance function $\bar{\varrho}^l$ on V_l which is equal to $\bar{\varrho}^l_m$ on V^l_m $(m=1, 2, ..., ; l=1, 2)$. Further $\Psi_{m'}$ and $\Psi_{m''}$ coincide on $V^1_{m'} \cap V^1_{m''}$, therefore there is a distance preserving map $\Psi: (V_1, \bar{\varrho}_1) \to (V_2, \bar{\varrho}_2)$ which coincides with Ψ_m on V^1_m for $m=1, 2, \dots$. Hence $v_l: (V_l, \bar{\varrho}_l) \to E^n$ is locally distance preserving. There is an open subset of V^1_1 with compact closure A such that $v_1: (A, \bar{\varrho}_1) \to E^n$ and $v_2 \circ \Psi: (A, \bar{\varrho}_1) \to E^n$ are distance preserving, therefore there is a distance preserving transformation $\Phi: E^n \to E^n$ with $v_2 \circ \Psi = \Phi \circ v_1$ on A. Assume that the last equality does not hold on V_1. Then there is a $w^* \in V_1$ nearest to A with $v_2 \circ \Psi(w^*) \neq \Phi \circ v_1(w^*)$. But w^* has a neighborhood V^* on which $v_1, v_2 \circ \Psi$ are distance preserving, consequently there is a distance preserving transformation $\Phi^*: E^n \to E^n$ with $v_2 \circ \Psi = \Phi^* \circ v_1$ on V^*. Since $\bar{\varrho}_1$ is intrinsic and locally euclidean there is an open subset $V' \subset V^*$ such that $v_2 \circ \Psi = \Phi \circ v_1$ on V'. Consequently $v_2 \circ \Psi = \Phi \circ v_1$ on V^* in contradiction with the above assumption. Therefore $v_2 \circ \Psi = \Phi \circ v_1$ on V_1. The restriction of this equality to the zero section in $N^1 M^{n-1}$ yields the assertion of the theorem.

Remark. The assumption of Theorem 3. 1 that there are continuous unit normal fields on the whole manifold M^{n-1} for both hypersurfaces does not mean an essential restriction since if M^{n-1} is not orientable its orientable covering manifold can be considered instead.

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(Received May 15, 1969)

Translations of regular algebras

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By an *algebra* we shall mean any non-empty set together with a non-empty set of finitary operations. Our further terminology is essentially that of [1].

After MALCEV [4], an algebra is called *regular* if no two congruences of this algebra have a congruence-class in common. We call a mapping φ of the algebra A with set of operations Ω into itself an *elementary translation* of A, if there exists an n-ary operation $\omega \in \Omega$, a natural number i $(1 \leqq i \leqq n)$ and elements $a_j \in A$ $(j = 1, \ldots, i-1, i+1, \ldots, n)$ such that for any $x \in A$ $x\varphi = a_1 \cdots a_{i-1} x a_{i+1} \cdots a_n \omega$ holds. If we say "derived operation" instead of "operation", then we obtain the definition of a *derived translation*. By *translation* we mean an arbitrary product of finitely many elementary translations. Thus, the identical mapping of A is not necessarily a translation, but it is always a derived translation. The set $T(A)$ of all translations of a given algebra A forms a semigroup of transformations of A, which is a subsemigroup of the semigroup $D(A)$ of all derived translations.

It is known that if any two congruences of an arbitrary algebra in the variety $\mathfrak{A}$ commute, then for any $A \in \mathfrak{A}$ $D(A)$ is a transitive semigroup of transformations. The same also holds in those varieties in which the lattice of congruences of every algebra is distributive [2]. Among other investigations concerning regular algebras, MALCEV has formulated a proposition as follows: in order that an algebra A be regular, it is necessary that $D(A)$ be transitive [4]. (We remark that in MALCEV's text the term "translation" is used for "derived translation".) THURSTON in [3] asserts without proof — with reference to [4] — that if a variety $\mathfrak{A}$ is *regular* (that is, all of its algebras are regular), then $T(A)$ is transitive for all $A \in \mathfrak{A}$.

We are going to show that MALCEV's proposition is not valid in general, and we give some description of regular algebras with intransitive semigroups of derived translations. Then we prove that THURSTON's assertion holds even in a more general form. Finally we characterize regular varieties by a certain property of translations for all algebras in such a variety. We begin with

Theorem 1. *On any set having at least two elements there may be defined a regular algebra with intransitive semigroup of derived translations.*

Proof. Let A be an arbitrary set having at least two elements and let $a \in A$. For every pair of different elements $x, y \in A$ $(x \neq a)$ define on A a unary operation ε_{xy} in the following way: for any $z \in A$ let

$$z\varepsilon_{xy} = \begin{cases} y & \text{if } z = x, \\ a & \text{otherwise.} \end{cases} \tag{1}$$

We shall prove that A, with these operations, will be an algebra with the properties desired. A mapping of A into itself shall be a derived translation if and only if it is identical or it is a product of finitely many operations. By definition a is invariant under any translation of A. Verify that A is regular; for this purpose it is sufficient to prove that A is simple.

Let Φ be a congruence on A having a class C with at least two elements, say c, d. Then $c \not\equiv a$ (Φ) implies the existence of ε_{cd} for which we have $c\varepsilon_{cd} = d$, $d\varepsilon_{cd} = a$, according to (1). Clearly, $c\varepsilon_{cd} \equiv d\varepsilon_{cd}(\Phi)$, and thus $d \equiv a(\Phi)$, whence $c \equiv a(\Phi)$. This contradiction shows that $a \in C$. Set now e.g. $c \neq a$, and let b be a further element in A. Then $c\varepsilon_{cb} = b$ and $a\varepsilon_{cb} = a$. Since $c\varepsilon_{cb} \equiv a\varepsilon_{cb}(\Phi)$, we have $b \equiv a(\Phi)$, that is, $b \in C$. Hence $C = A$, and the simplicity of A is verified.

In the following we write $a \to b$ $[a \Rightarrow b]$ for $a, b \in A$ if there exists a $\tau \in T(A)$ $[\delta \in D(A)]$ such that $a\tau = b$ $[a\delta = b]$. It is clear that $a \to b$ implies $a \Rightarrow b$. The corresponding complementary relations will be denoted by $a \not\to b$ and $a \not\Rightarrow b$, respectively.

Our next theorem gives some information about the structure of any regular algebra A with intransitive $D(A)$. To arrange the proof conveniently we first formulate several lemmas.

Lemma 1. *In any algebra, the relation $\to$ is transitive.*

Lemma 2. *In any algebra, the relation $\Rightarrow$ is the reflexive closure of $\to$.*

These lemmas are obvious from definitions.

Lemma 3. *For any regular algebra A having at least three elements and for arbitrary $a, b, c \in A$ with $a \neq b$, there exists $\tau \in T(A)$ such that either $a\tau = c$, $b\tau \neq c$, or $a\tau \neq c$, $b\tau = c$.*

Proof. We shall call a triplet (x, y, z) *proper* if x, y, z are pairwise different. In the case when (a, b, c) is proper, this lemma is implicit in Malcev [4].

Let now (a, b, c) be not proper. Assume $b = c$ and let $d \in A$, (a, b, d) proper. Then there exists a $\tau_1 \in T(A)$ for which either

$$a\tau_1 = d, \quad b\tau_1 = d_1 \neq d \tag{2}$$

or

$$a\tau_1 = d_1 \neq d, \quad b\tau_1 = d. \tag{3}$$

If (d, d_1, c) is not proper, then $d_1=c$ and thus exactly one of $a\tau_1$ and $b\tau_1$ equals c. If (d, d_1, c) is proper there exists a $\tau_2\in T(A)$ for which either

(4) $$d\tau_2=c,\quad d_1\tau_2\neq c,$$

or

(5) $$d\tau_2\neq c,\quad d_1\tau_2=c.$$

Take $\tau=\tau_1\tau_2$. Now (2) and (4) as well as (3) and (5) imply $a\tau=c$, $b\tau\neq c$; (2) and (5) as well as (3) and (4) imply $a\tau\neq c$, $b\tau=c$.

Lemma 4. *Let A be a regular algebra having at least three elements $a, b, c\in A$ with $a\neq b$. If $a\nrightarrow c$, then $b\rightarrow c$.*

Proof. Obvious from Lemma 3.

Theorem 2. *For any algebra A having at least two elements the following statements are equivalent:*

I. *A is regular and $D(A)$ is intransitive.*

II. *A is regular and $T(A)$ is intransitive.*

III. *A is simple and there exists an $a\in A$ such that, for any different $x, y\in A$, $x\nrightarrow y$ holds if and only if $x=a$, $y\neq a$.*

IV. *A is simple and there exists an $a\in A$ such that, for any different $x, y\in A$, $x\not\Rightarrow y$ holds if and only if $x=a$, $y\neq a$.*

Proof. Clearly, I implies II, and IV implies I. Further, III implies IV by Lemma 2. The only non-trivial part of theorem is that III is a consequence of II.

This is immediate in the case when A has exactly two elements. In the other case, let $a, b\in A$, $a\nrightarrow b$. By Lemma 4, $c\rightarrow b$ for any $c\in A$ $(c\neq a)$, whence $a\nrightarrow c$ by Lemma 1. Thus $a\rightarrow d$ $(d\in A)$ implies $d=a$. Hence by Lemma 4 for each pair $x, y\in A$ $(x\neq a)$ we have $x\rightarrow y$.

Let now Φ be an arbitrary congruence on A which has a class consisting of at least two elements. Denote by C the class of including a. It follows from the regularity of A that C contains at least one element e different from a. Choose an element x from A arbitrarily. Then $e\rightarrow x$, that is, there exists a translation τ on A such that $e\tau=x$. $e\equiv a(\Phi)$ implies $e\tau\equiv a\tau(\Phi)$, and, because of $a\tau=a$, $x\equiv a(\Phi)$ is valid, whence $x\in C$. Thus $C=A$, and this fact shows that A is simple.

Theorem 2 shows that the example in the proof of Theorem 1 is a typical one.

Theorem 3. *Let A an arbitrary algebra. If $A\times A$ is regular, then $T(A)$ is transitive.*

Proof. We may assume that A has at least two elements. Let $a\nrightarrow b$ for $a, b\in A$, and let $c\in A$, for which $a\rightarrow c$ holds. Consider the following relations θ, Φ on $A\times A$:

for any $(x, y), (u, v) \in A \times A$ let $(x, y) \equiv (u, v)(\theta)$ if and only if $x = u$; furthermore, let $(x, y) \equiv (u, v)\ (\Phi)$ if and only if $x = u$ and $a \to x$. It is easy to see, that θ and Φ are congruences on $A \times A$, and $\theta \neq \Phi$; e.g., $(b, b) \equiv (b, c)(\theta)$, but $(b, b) \not\equiv (b, c)(\Phi)$. Nevertheless, θ and Φ have a congruence-class in common, e.g. such a class is formed by the set of all elements of $A \times A$ with c as first component. Thus we see that $A \times A$ is not regular, qu.e.d.

Corollary. (THURSTON [3]) *Let $\mathfrak{A}$ be a regular variety and $A \in \mathfrak{A}$. Then $T(A)$ is transitive.*

Proof. $A \times A \in \mathfrak{A}$, thus $A \times A$ is regular, and hence we may apply Theorem 3.

We remark that the corollary holds not only for varieties, but for classes containing with any algebra its direct product with itself, too.

Finally we show that the condition of Lemma 3 is able to characterize regular varieties.

Theorem 4. *A variety $\mathfrak{A}$ is regular if and only if*

(i) *for any $A \in \mathfrak{A}$ and for all $a, b, c \in A$ $(a \neq b)$ there exists a $\tau \in T(A)$ such that either $a\tau = c$, $b\tau \neq c$, or $a\tau \neq c$, $b\tau = c$.*

Proof. Suppose that $\mathfrak{A}$ regular. If $A \in \mathfrak{A}$ has at least three elements, then (i) follows immediately from Lemma 3. Let A have exactly two elements. We must show that there exists a $\tau \in T(A)$ the range of which contains more than one element. Indeed, in the contrary case the range of any translation of $A \times A$ consists of one element, in contradiction with Lemma 3.

Now suppose that $\mathfrak{A}$ is not regular. This implies, as THURSTON has proved in [3], the existence of an algebra $A \in \mathfrak{A}$ such that the identical congruence and some non-identical congruence Φ have a common class on A. Let c be the unique element of this class, and let $a \equiv b(\Phi)$ $(a \neq b)$. If $a\tau = c$ for $\tau \in T(A)$, then $b\tau \equiv a\tau(\Phi)$, and hence $b\tau = c$ holds. Similarly, $b\tau = c$ implies $a\tau = c$. Thus, (i) is false on $\mathfrak{A}$.

We can observe that Theorem 4 remains valid if we replace the assumption that $\mathfrak{A}$ is a variety with a weaker requirement, namely that $\mathfrak{A}$ contains with any algebra all its homomorphic images and its direct product with itself.

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(Received February 12, 1969)

A semiring whose Green's relations do not commute

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We call Green's relations on a semiring R the equivalence relations defined thus: $a\mathcal{L}b$ ($a\mathcal{R}b$) if and only if a and b generate the same principal left (right) ideal, $\mathcal{D}=\mathcal{L}\vee\mathcal{R}$ and $\mathcal{H}=\mathcal{L}\cap\mathcal{R}$. The relations $\mathcal{L}$ and $\mathcal{R}$ commute for a large class of semirings which includes semirings with a commutative addition or with a globally idempotent (or weakly reductive) multiplication. In these cases, they have properties analogous to the properties of Green's relations in semigroup theory and apply to the study of ideals of a semiring (cf. [2]). However $\mathcal{L}$ and $\mathcal{R}$ do not commute in general and the purpose of this paper is to give a counterexample, which we have been unable to obtain by more elementary methods.

Our example is the semiring R generated by the two elements set $\{b, d\}$ and subject to the relations:

$$b = d^2b+2db+b = b+2bd+bd^2.$$

Since these relations can also be written under the form: $b = d(db+b)+db+b = b+bd+(b+bd)d$, it is clear that in R the relation: $db+b\ \mathcal{L}b\mathcal{R}\ b+bd$ holds. We shall prove that there exists no element e of R such that $db+b\ \mathcal{R}e\mathcal{L}\ b+bd$.

1. It is possible to construct R as the quotient of the free semiring F on the set $\{b, d\}$ by the smallest congruence on F containing the binary relation $\mathcal{G}$ consisting of the two pairs:

$$(b, d^2b+2db+b), \qquad (b, b+2bd+bd^2). \tag{1}$$

However, to obtain a suitable description of R, we need to refine this construction by using the construction of F itself which we first recall briefly (cf. [3]).

Let S be the free multiplicative semigroup on $\{b, d\}$, i.e. the set of all monomials $x_1x_2\cdots x_n$ ($n>0$, $x_i\in\{b, d\}$. Then consider the free additive semigroup W on S which is the set of all sums $w_1+w_2+\cdots+w_n$ where $n>0$ and $w_i\in S$ with addition defined by juxtaposition. The multiplication in S can be extended to a associative multiplication of W in the following way:

$$(w_1+\cdots+w_n)(w'_1+\cdots+w'_p) = \sum_{i=1}^{n}\left(\sum_{j=1}^{p} w_iw'_j\right),$$

for all $w_i, w'_j \in S$. We shall denote by S^1 (W^0) the set resulting from the adjunction of a formal identity to the multiplicative (additive) semigroup $S(W)$.

Finally let $\mathcal{D}$ be the transitive closure of the binary relation $\mathcal{F}$ defined as the set of all pairs having either the form (w, w) with $w \in W$ or (x, y) or (y, x) with

$$x = u + \sum_{i=1}^{m} \left(\sum_{j=1}^{n} w_i w'_j \right) + v, \quad y = u + \sum_{j=1}^{n} \left(\sum_{i=1}^{m} w_i w'_j \right) + v,$$

where $m, n > 0$, $w_i, w'_j \in S$ for all i, j and $u, v \in W^0$. Then $F = W/\mathcal{D}$ is the free semiring on $\{b, d\}$.

We shall now describe R as the quotient of W by a suitable congruence. First let $(w, w') \in \mathcal{G}^*$ if and only if $(w, w') \in \mathcal{G}$ or $(w', w) \in \mathcal{G}$. Observe that the binary relation $\mathcal{G}^*$ consists of four pairs and is symmetric. Then let $\mathcal{B}$ be the binary relation on W defined by:

$$\mathcal{B} = \{(u + swt + v,\ u + sw't + v);\ \ u, v \in W^0,\ \ s, t \in S^1,\ \ (w, w') \in \mathcal{G}^*\}.$$

Also let $P = \mathcal{B} \cup \mathcal{F}$. Then we have the following:

Lemma 1. *The smallest congruence $\mathcal{C}$ on W containing both $\mathcal{G}$ and $\mathcal{D}$ is the transitive closure of $\mathcal{P}$. Furthermore $R = W/\mathcal{C}$.*

Proof. Clearly any congruence on W which contains both $\mathcal{G}$ and $\mathcal{D}$ must also contain $\mathcal{G}^*$, $\mathcal{B}$, $\mathcal{P}$ and therefore the transitive closure $\mathcal{C}'$ of $\mathcal{P}$. Thus $\mathcal{C}' \subseteq \mathcal{C}$. To show the inverse inclusion, we shall successively prove that $\mathcal{C}'$ contains both $\mathcal{G}$ and $\mathcal{D}$, and that $\mathcal{C}'$ is a congruence on W.

Trivially $\mathcal{C}'$ contains $\mathcal{G}$; also, since $\mathcal{F} \subseteq \mathcal{P}$, certainly the transitive closure $\mathcal{C}^*$ of $\mathcal{P}$ contains the transitive closure $\mathcal{D}$ of $\mathcal{F}$.

Since both $\mathcal{B}$ and $\mathcal{F}$ are symmetric and $\mathcal{F}$ is reflexive, $\mathcal{C}'$ is an equivalence relation. Also both $\mathcal{B}$ and $\mathcal{F}$ admit the addition of W, so does $\mathcal{P}$ whence $\mathcal{C}'$ is an additive congruence on W. It is left to show that $\mathcal{C}'$ is also a multiplicative congruence. To this end it suffices to prove that $\mathcal{P}$ has the following property: if $z \in W$ and $(x, y) \in \mathcal{P}$, then $(zx, zy) \in \mathcal{C}'$ and $(xz, yz) \in \mathcal{C}'$.

Observe first that, since $\mathcal{D}$ is a congruence on W containing $\mathcal{F}$, for all $(x, y) \in \mathcal{F}$ and $z \in W$, we have: $(zx, zy) \in \mathcal{D}$ and $(xz, yz) \in \mathcal{D}$. Thus, since $\mathcal{D} \subseteq \mathcal{C}'$, $(zx, zy) \in \mathcal{C}'$ and $(xz, yz) \in \mathcal{C}'$ for all $(x, y) \in \mathcal{F}$, $z \in W$.

On the other hand, let $x = u + swt + v$ and $y = u + sw't + v$ be such that $u, v \in W^0$, $s, t \in S^1$ and $(w, w') \in \mathcal{G}^*$. If first $z \in S$, then

$$(zx,\ zu + zswt + zv) \in \mathcal{D}, \quad (zy,\ zu + zsw't + zv) \in \mathcal{D}$$

by distributivity modulo $\mathcal{D}$. Also $(zu + zswt + zv,\ zu + zsw't + zv) \in \mathcal{B}$. Since $\mathcal{C}'$ is an additive congruence containing both $\mathcal{B}$ and $\mathcal{D}$, it follows that $(zx, zy) \in \mathcal{C}'$; similarly $(xz, yz) \in \mathcal{C}'$.

If now $z\in W$ so that $z = z_1+z_2+\dots+z_r$ for some $r>0$ and $z_i\in S$. By the above, $(z_ix, z_iy)\in\mathscr{C}'$ for all i. Since

$$zx = z_1x+z_2x+\dots+z_rx \quad\text{and}\quad zy = z_1y+z_2y+\dots+z_ry,$$

and $\mathscr{C}'$ is an additive congruence, we obtain $(zx, zy)\in\mathscr{C}'$. Similarly $(xz, yz)\in\mathscr{C}'$.

It is routine to check that $R=W/\mathscr{C}$, which completes the proof.

2. Let K be the set of all elements $x = x_1+x_2+\dots+x_r$ $(r>0,\ x_i\in S)$ of W having the following two properties:

(A) for all i, $x_i=d^{p_i}bd^{q_i}$ for some $p_i, q_i\geqq 0$;

(B) there exists k such that $1\leqq k\leqq r$, $p_k=q_k=0$, $p_i>0$ if $i<k$ and $q_i>0$ if $i>k$.

It is convenient to write the elements of K under the form $x = x_\lambda+b+x_\varrho$, where x_λ, x_ϱ have the obvious meaning. Observe that all monomials of x_λ are divisible on the left by d and all monomials of x_ϱ are divisible on the right by d.

Lemma 2. *Let $x\in K$ and $x'\in W$ satisfy $(x, x')\in\mathscr{C}$. Then $x'\in K$. Furthermore $(x_\lambda+b, x'_\lambda+b)\in\mathscr{C}$ and $(b+x_\varrho, b+x'_\varrho)\in\mathscr{C}$.*

Proof. Clearly, it is enough to show that $\mathscr{P}$ has these properties. We consider successively the two cases $(x, x')\in\mathscr{F}$ and $(x, x')\in\mathscr{B}$. We may also assume $x\neq x'$.

1) Let $(x, x')\in\mathscr{F}$ and $x\in K$. Then we can write for instance:

$$x = u+\sum_{i=1}^{m}\left(\sum_{j=1}^{n} w_iw'_j\right)+v, \quad x' = u+\sum_{j=1}^{n}\left(\sum_{i=1}^{m} w_iw'_j\right)+v,$$

for some $u, v\in W^0$, $w_i, w'_j\in S$. We shall study only the case when $u, v\in W$, the cases when $u=0$ or $v=0$ being simpler. Clearly, since x satisfy (A), and x, x' have same monomials up to the order, then x' satisfy (A) too.

Write $u = u_1+u_2+\dots+u_{m'}$, $v = v_1+v_2+\dots+v_{n'}$ with $m', n'>0$, $u_i, v_j\in S$. Since $w_i, w'_j\in W^2$, but $b\notin W^2$, certainly $w_iw'_j\neq b$, so that $x\in K$ implies that either $b=u_k$ for some $k=1, 2, \dots, m'$ or $b=v_{k'}$ for some $k'=1, 2, \dots, n'$.

Assume that $b=u_k$ for some k; then $x_\lambda = u_1+\dots+u_{k-1}$ and

$$x_\varrho = u_{k+1}+\dots+u_{m'}+\sum_{i=1}^{m}\left(\sum_{j=1}^{n} w_iw'_j\right)+v.$$

Then set $x'_\lambda=x_\lambda$ and

$$x'_\varrho = u_{k+1}+\dots+u_{m'}+\sum_{j=1}^{n}\left(\sum_{i=1}^{m} w_iw'_j\right)+v.$$

Clearly, since all monomials of x_λ are divisible on the left by d, so are all monomials of x'_λ; also $(x_\lambda+b, x'_\lambda+b)\in\mathscr{F}$ trivially. Now, looking at the above expressions of x_ϱ and x'_ϱ, we see that $(b+x_\varrho, b+x'_\varrho)\in\mathscr{F}$; in particular x_ϱ and x'_ϱ have the same monomials up to the order which implies that all monomials of x'_ϱ are divisible on the right by d, since the monomials of x_ϱ have this property. We therefore obtain that $x' = x_\lambda + b + x'_\varrho$ satisfies (B) whence $x' \in K$.

The case when $b = v_{k'}$ for some k' is treated similarly.

2) Let now $(x, x') \in \mathscr{B}$ and $x \in K$; then $x = u + swt + v$, $x' = u + sw't + v$ for some $u, v \in W^0$, $s, t \in S^1$, $(w, w') \in \mathscr{G}^*$. Two subcases are to be considered:

a) If $s \neq 1$ or $t \neq 1$, then $swt \in W^2$ so that no monomial of swt can be equal to b. Thus $x \in K$ implies $b = u_k$ for some monomial u_k of u (and hence $u \neq 0$) or that $b = v_{k'}$ for some monomial $v_{k'}$ of v (and hence $v \neq 0$).

If first $u = u_1 + \dots + u_{m'}$ $(m' > 0,\ u_i \in S)$ and $b = u_k$ for some $1 \leqq k \leqq m'$, then set

$$x_\lambda = u_1 + u_2 + \dots + u_{k-1} = x'_\lambda,$$

$$x_\varrho = u_{k+1} + \dots + u_{m'} + swt + v,$$

$$x'_\varrho = u_{k+1} + \dots + u_{m'} + sw't + v.$$

All possible forms of $w \in W$ such that $(w, w') \in \mathscr{G}^*$ for some $w' \in W$ are: $b, b + 2bd + bd^2, d^2b + 2db + b$; hence sbt figures as a monomial of swt in all cases; since $x \in K$, and since swt is a term of a sum equal to x_ϱ, $s = d^p$, $t = d^q$ for some $p \geqq 0$ and $q > 0$. Then it is obvious to check on all possible forms of swt and $sw't$ that all their monomials are of the form $d^{p'}bd^{q'}$ with $p' \geqq 0$ and $q' > 0$. Since the set of monomials of x_ϱ and x'_ϱ can differ only by monomials of swt and $sw't$, $x \in K$ implies that $x' \in K$. Obviously $(x_\lambda + b, x'_\lambda + b) \in \mathscr{P}$ since $x_\lambda = x'_\lambda$; also $(b + x_\varrho, b + x'_\varrho) \in \mathscr{B}$ is obvious on the form of x_ϱ and x'_ϱ.

The case when $b = v_{k'}$ for some monomial $v_{k'}$ of v is treated in a similar way.

b) If finally $s = t = 1$, then the different possible forms of $x \in K$ are: $u + b + v$, $u + b + v$, $u + d^2b + 2db + b + v$, $u + b + 2bd + bd^2 + v$; they correspond respectively to the following forms of x': $u + d^2b + 2db + b + v$, $u + b + 2bd + bd^2$, $u + b + v$, $u + b + v$. It is then easy to check on these forms that the result holds also in this case.

The following lemma will be needed later on:

Lemma 3. *Let $x = x_1 + x_2 + \dots + x_r$, with $r > 0$ and $x_i \in S$ for all i, be such that $(db + b, x) \in \mathscr{C}$. Then the two sets $A_x = \{i;\ x_i = bd\}$ and $B_x = \{j;\ x_j = dbd\}$ have an even cardinality.*

Proof. The result holds certainly for $x = db + b$, for then $A_x = B_x = \emptyset$. Clearly, it is enough to show that if $(x, x') \in \mathscr{P}$ and $|A_x|$ and $|B_x|$ are even, so are $|A_{x'}|$ and $|B_{x'}|$.

This is clear if $(x, x') \in \mathscr{F}$, for then x and x' have the same monomials up to the order.

Let now $(x, x') \in \mathscr{B}$ so that $x = u + swt + v$, $x' = u + sw't + v$ for some $u, v \in W^0$,

$s=d^p$, $t=d^q$ with $p, q \geqq 0$ (for $x, x' \in K$) and $(w, w') \in \mathcal{G}^*$. Then we have: $|A_x| = |A_u|+|A_{swt}|+|A_v|$, $|B_x| = |B_u|+|B_{swt}|+|B_v|$; also $|A_{x'}| = |A_u|+|A_{sw't}|+|A_v|$, $|B_{x'}| = |B_u|+|B_{sw't}|+|B_v|$. Thus it is enough to show that $|A_{swt}|$ and $|A_{sw't}|$ ($|B_{swt}|$ and $|B_{sw't}|$) differ only by an even number. This is done by direct inspection of the pairs $(swt, sw't)$. Observe that it is enough to consider pairs $(w, w') \in \mathcal{G}$ (since the result is symmetric in x and x') and integers $p, q \leqq 1$. The cases which are left to study are given by the following table:

	swt	$sw't$
$s=t=1$	b	$b+2bd+bd^2$
	b	$d^2b+2db+b$
$s=1,\ t=d$	bd	$bd+2bd^2+bd^3$
	bd	$d^2bd+2dbd+bd$
$s=d,\ t=1$	db	$db+2dbd+dbd^2$
	db	d^3b+2d^2b+db
$s=t=d$	dbd	$dbd+2dbd^2+dbd^3$
	dbd	$d^3bd+2d^2bd+dbd$

It is clear by the table that $|A_{swt}|$ and $|A_{sw't}|$($|B_{swt}|$ and $|B_{sw't}|$) differ only by an even number which completes the proof.

3. Let $a = db+b$ and $c = b+bd$. Also denote by π the canonical projection of W to $R=W/\mathcal{C}$. We wish to show that there exists no element y of W such that the relation $\pi(a)\ \mathcal{R}\pi(y)\ \mathcal{L}\pi(c)$ holds in R. Assume that on the contrary such an element y exists.

Since $\pi(a)$ and $\pi(y)$ generate the same principal right ideal of R, there exist $y_1, y_2, \ldots, y_r, a_1, a_2, \ldots, a_{r'} \in W^1$, (where W^1 is obtained by adjunction of a formal identity to W) such that:

$$(2)\qquad \left(y, \sum_{i=1}^{r} ay_i\right) \in \mathcal{C} \quad \text{and} \quad \left(a, \sum_{j=1}^{r'} ya_j\right) \in \mathcal{C}.$$

Using the distributivity modulo $\mathcal{C}$, we may first assume that $y_i, a_j \in S^1$ for all i, j; also it follows from (2) that

$$(3)\qquad \left(a, \sum_{j=1}^{r'} \left(\sum_{i=1}^{r} ay_i a_j\right)\right) \in \mathcal{C}.$$

Lemma 4. *With the above notation, $y_1=a_1=1$; furthermore, for all $i, j>1$, $y_i=d^{q_i}$ and $a_j=d^{q'_j}$ for some $q_i, q'_j>0$. Finally, $\sum_{i=1}^{r} ay_i \in K$.*

Proof. Let $x = \sum_{j=1}^{r'} \left(\sum_{i=1}^{r} (d b y_i a_j + b y_i a_j) \right)$. Since $a = db + b$, by distributivity modulo $\mathscr{C}$ and in view of (3), we see that $(a, x) \in \mathscr{C}$. Thus $x \in K$ by Lemma 2. In particular, $by_{i_0} a_{j_0} = b$ for some i_0, j_0. Then we must have $i_0 = j_0 = 1$: otherwise there would exist some $i_1 \leqq i_0$, $j_1 \leqq j_0$ such that $i_1 \neq i_0$ or $j_1 \neq j_0$; then $by_1 a_1$ would be a monomial of x_λ which is not divisible on the left by d; this is impossible for $x \in K$. Therefore $y_1 = a_1 = 1$.

Furthermore, for all $i, j > 1$, $by_1 a_j = ba_j$ and $by_i a_1 = by_i$ are monomials of x_ϱ, whence $a_j = d^{q'_j}$ and $y_i = d^{q_i}$ for some $q_i, q'_j > 0$. The last statement follows.

Lemma 5. *For any $y \in W$ such that $\pi(a) \mathscr{R} \pi(y) \mathscr{L} \pi(c)$, $(y, db + b + bd) \in \mathscr{C}$.*

Proof. With the above notation, Lemma 4 implies that $\left(\sum_{i=1}^{r} a y_i \right)_\lambda = db$. Since $\left(y, \sum_{i=1}^{r} a y_i \right) \in \mathscr{C}$ holds by formula (2) and $\sum_{i=1}^{r} a y_i \in K$ by Lemma 4, applying Lemma 2, we obtain $(y_\lambda + b, db + b) \in \mathscr{C}$. A similar reasoning using $\pi(y) \mathscr{L} \pi(c)$ would imply $(b + y_\varrho, b + bd) \in \mathscr{C}$. Since $\mathscr{C}$ is an additive congruence, it follows that

$$(y, db + b + bd) \in \mathscr{C}$$

which completes the proof.

Finally consider $z = \sum_{j=1}^{r'} (dba_j + ba_j + bda_j)$. First Lemma 5 and formula (2) imply that $(a, z) \in \mathscr{C}$. Also, using Lemma 4, it is easy to check that $A_z = \{j; j > 1, q'_j = 1\} \cup \{1\}$ and $B_z = \{j; j > 1, q'_j = 1\}$. Thus $|A_z|$ and $|B_z|$ are of different parity which contradicts $(a, z) \in \mathscr{C}$ in view of Lemma 3. Therefore there exists no $y \in W$ such that $\pi(a) \mathscr{R} \pi(y) \mathscr{L} \pi(c)$ and we have proved:

Theorem 6. *The Green's relations of the semiring R generated by the set $\{b, d\}$ and subject to the relations $d^2 b + 2db + b = b = b + 2bd + bd^2$ do not commute.*

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(Received May 28, 1969)

Reduktion eines Problems bezüglich der Brown—McCoyschen Radikalringe

Von F. SZÁSZ in Budapest

In der Ringtheorie, Gruppentheorie und Kategorientheorie spielen die Radikale und Halbeinfachkeiten bekanntlich eine sehr wichtige Rolle. Eine wohlbestimmte Äquivalenz des Radikals und Halbeinfachkeit in gewissen, natürliche Bedingungen erfüllenden Kategorien wurde von F. SZÁSZ und R. WIEGANDT [16] untersucht. Die genannten Bedingungen sind übrigens sowohl von der Kategorie aller assoziativen Ringe, als auch von der Kategorie aller Gruppen erfüllt. Eine ähnliche kategorientheoretische Untersuchung findet man bei RJABUHIN [13]. Dagegen scheinen Radikal und Halbeinfachkeit für beliebige Kategorien (und zwar für die Kategorie der nichtassoziativen Ringe) nicht äquivalent zu sein. Nach Resultaten von KUROSCH [8] läßt sich übrigens jede Klasse von assoziativen Ringen sowohl in eine Klasse von Radikalringen als auch in eine Klasse von halbeinfachen Ringen für geeignete allgemeine Radikale einbetten.

Jedoch ist wesentlich weniger über Radikalringe als über halbeinfache Ringe für die nützlichen konkreten Radikale (wie z.B. das untere Nilradikal, das obere Nilradikal [2], das Levitzkische lokal nilpotente Radikal [10], das Jacobsonsche Radikal [6], das Brown—McCoysche Radikal [3] usw.) bekannt. Deshalb ist zweckmäßig die Radikalringe als in gewissem Sinne „schlechte" und mühsamer untersuchbare Ringe für konkrete, nützbare Radikale zu diskutieren.

Bezüglich der nötigen Begriffe verweisen wir auf die Bücher [5], [6], [7], [9] und [12].

In dieser Arbeit werden wir ein Problem über die Brown—McCoyschen Radikalringe untersuchen. Bekanntlich ist jeder Nilring und jeder Jacobsonsche Radikalring auch ein Brown—McCoyscher Radikalring. Ein Ring ist genau dann ein Brown—McCoyscher Radikalring, wenn er auf keinen von Null verschiedenen einfachen Ringen mit Einselement homomorph abgebildet werden kann.

In der Arbeit [14] des Verfassers sind gewisse Klassen von sogenannten E_i-Ringen (für $i=1, 2, 3, 4, 5$) untersucht. Dementsprechend bedeutet ein E_2-Ring einen solchen Ring A, für den der maximale triviale A-Rechtsuntermodul M_0 von jedem A-Rechtsmodul M ein direkter Summand von M ist. Hierbei versteht man unter einem trivialen A-Modul einen Modul N mit $NA=0$. In anderer Abfassung stammt der Begriff eines E_2-Ringes von KERTÉSZ [7], wobei auch gefragt ist, ob jeder E_2-Ring ein zweiseitiges Einselement haben soll. Jeder Ring mit Einselement ist nämlich ein E_2-Ring.

In der Arbeit [14] des Verfassers sind einige Kriterien für die Existenz des Einselementes in einem Ring mit der Hilfe von E_2-Ringen bzw. E_5-Ringen angegeben, und ebendort wurde vom Verfasser auch folgendes Problem aufgeworfen:

(P) Gibt es einen von Null verschiedenen Brown—McCoyschen Radikalring, der gleichzeitig auch ein E_2-Ring ist?

In der vorliegenden Arbeit werden wir einige Eigenschaften, der im obigen Problem (P) erwähnten Ringe bestätigen, ohne ihre Existenz zu beweisen. In der Arbeit [14] hat Verfasser bewiesen, daß es keinen von Null verschiedenen Jacobsonschen Radikalring gibt, der auch ein E_2-Ring ist (Satz 2. 3. 2 von [14]), und daß $a \in aA + AaA$ für jedes Element a eines E_2-Ringes (sogar allgemeiner schon eines E_5-Ringes) gilt. Dr. G. MICHLER hat mir in einem Brief geschrieben, daß genau mit den Methoden des Beweises für den Satz 2. 3. 2 in [14] auch die schärfere Relation $a \in aA$ für jedes Element a von jedem E_2-Ring A bewiesen werden kann. Die Ringe A, für die $a \in aA$ für jedes Element $a \in A$ gilt, wurden in meiner Arbeit [14] E_3-Ringe genannt. Also ergibt sich:

(∗) *Jeder E_2-Ring ist ein E_3-Ring.*

Gilt nämlich $a \notin aA$ für ein Element $a \in A$ eines E_2-Ringes A, so gibt es nach dem Zornschen Lemma ein maximales solches Rechtsideal R von A, für das $a \notin R$ und $R \supseteqq aA$ bestehen. Dann ist der A-Rechtsmodul A/R subdirekt irreduzibel, und wegen $aA \subseteqq R$ liegt der minimale A-Untermodul M_1/R von A/R im maximalen trivialen A-Untermodul M_0/R von A/R. Da A ein E_2-Ring ist, erhält man $M_0 = A$, $A^2 \subseteqq R$, was dem in [14] bewiesenen Resultat $A^2 = A$ widerspricht. Hiernach ist die Behauptung (∗) von MICHLER mit den Methoden meines Beweises für den Satz 2. 3. 2 [14] wirklich bewiesen.

Der Ring aller linearen Transformationen von endlichem Rang eines Vektorraumes von unendlichem Rang über einem Schiefkörper ist ein solcher von Null verschiedene Brown—McCoysche Radikalring, der ein von Neumannscher regulärer Ring, folglich auch ein E_3-Ring ist, obwohl kein von Null verschiedener Jacobsonscher Radikalring nach meinem Satz 2. 3. 2 bzw. nach der Michlerschen Behauptung (∗) ein E_2-Ring ist.

Jetzt bestätigen wir einige Eigenschaften solcher von Null verschiedenen Brown—McCoyschen Radikalringe, die auch E_2-Ringe sind. Die Existenz solcher Ringe ist nicht bewiesen. Es gilt aber der

Satz. *Existiert ein von Null verschiedener Brown—McCoyscher Radikalring A, der gleichzeitig auch ein E_2-Ring ist, so hat A folgende Eigenschaften:*

1) *A läßt sich auch als ein linksprimitiver oder ein rechtsprimitiver Ring (folglich als ein Primring) wählen;*

2) *jedes Element a von A ist ein Linksnullteiler in A; wenn dabei A auch ein Primring ist, so gilt schon* $axa = 0$ *mit* $xa \neq 0$ *für jedes* $a \neq 0$ $(a \in A)$; *und somit hat dann jedes Linksideal ein von Null verschiedenes nilpotentes Element xa;*

3) *es gilt* $A(1-a)A = A$ *für jedes Element* $a \in A$;

4) *für jedes von Null verschiedene Element* $a \in A$ *existiert ein* $b \in A$ *mit* $ab \notin Aa$, *und somit hat A kein nichttriviales Zentrum, weiterhin gilt* $Aa \neq A$ *für jedes* $a \in A$;

5) *es gilt* $a \in (a+n)A + A(a+n)A$ *für jedes* $a \in A$ *und für jede ganze rationale Zahl n;*

6) *in A existieren maximale Linksideale, und es gilt* $LA = A$ *für jedes maximale Linksideal L von A;*

7) *die Maximalbedingung gilt nicht für die Linksideale der Gestalt*

$$L_a = \{x:\ x \in A,\ xa = x\}$$

für jedes $a \in A$, *und somit ist A kein linksnoetherscher Ring.*

Bemerkung 1. Weitere Eigenschaften der im Satz erwähnten Ringe können nach den Resultaten der §§ 2 und 3 meiner Arbeit [14] abgefaßt werden.

Bemerkung 2. Die im Satz erwähnte Eigenschaft 2) besagt, daß das Linksideal Aa für jedes $a \in A$ hinreichend groß im Sinne ist, daß Aa Rechtsnullteiler $b = b_a$ für jedes $a \in A$ enthält, für die $ab = 0$ gilt, dagegen drückt die Eigenschaft 4) aus, daß das Linksideal Aa für jedes $a \in A$ auch hinreichend klein im anderen Sinne ist, daß Aa nicht alle Produkte ab enthält. Diese „groß" und „klein" Begriffe sind freilich von den üblichen modultheoretischen Begriffen verschieden (s. z. B. Lambek [9]). Es ist ebenfalls interessant, daß nach der Eigenschaft 6) maximale Linksideale in A so existieren, daß A nach 7) kein linksnoetherscher Ring ist.

Beweis des Satzes. Wir verifizieren die im Satz erwähnten Eigenschaften des Ringes A nacheinander.

1) Jedes homomorphe Bild eines Brown—McCoyschen Radikalringes ist ein Radikalring von demselben Typ. Ebenfalls ist jedes homomorphe Bild von jedem E_2-Ring nach [14] auch ein E_2-Ring. Weiterhin stimmt ein im Satz betrachtete E_2-Ring A nach [14] nicht mit seinem Jacobsonschen Radikal $J(A)$ überein, und $A/J(A)$ ist dann sowohl eine subdirekte Summe von rechtsprimitiven Ringen als auch eine subdirekte Summe von linksprimitiven Ringen. Dabei sind diese subdirekten Summanden S_α homomorphe Bilder von A, also Brown—McCoysche Radikalringe und E_2-Ringe, die jetzt auch Primringe sind. Deshalb kann A durch S_α ersetzt werden.

2) Ist ein Element $a \in A$ kein Linksnullteiler von A, so kann ein Widerspruch folgendermaßen abgeleitet werden. Nach der Behauptung $(*)$ gibt es ein $b \in A$ mit $a = ab$, folglich mit $a(1-b)A = 0$. Da a jetzt kein Linksnullteiler ist, erhält man $(1-b)A = 0$, und somit ist b ein Linkseinselement von A. Nach dem Zornschen Lemma gibt es dann ein maximales solches Ideal M, daß $b \notin M$ besteht, denn es gilt offenbar $b \notin (1-b)A = 0$. Dann ist A/M wegen $(A(1-b))^2 = 0$ ein einfacher Ring mit Einselement, was der Definition der Brown—McCoyschen Radikalringe widerspricht. Damit ist der erste Teil von 2 bewiesen.

Ist nun A insbesondere ein Primring, so erhält man $cAd \neq 0$ für jedes von Null verschiedene c und d, folglich $cA \cap Ad \neq 0$. Es sei jetzt a ein beliebiges von Null verschiedenes Element von A, und $b \neq 0$ ein Element mit $ab = 0$, welches nach dem vorigen existiert. Dann gibt es ein $x \in A$ und ein $y \in A$ mit $0 \neq xa = by \in bA \cap Aa$, woraus man $axa = aby = 0$ erhält. Offenbar ist xa ein von Null verschiedenes nilpotentes Element im beliebigen Hauptlinksideal $(a)_l$.

3) Wegen Satz 2. 1. 3 meiner Arbeit [14] ergibt sich $A(1-a) \subseteq A(1-a)A$ für jedes Element a von A, und somit ist $A/A(1-a)A$ ein Ring mit dem Rechtseinselement $a + A(1-a)A$. Gilt $a \notin A(1-a)A$ und ist M ein maximales solches (nach dem Zornschen Lemma existierendes) Ideal, für das $a \notin M$ und $M \supseteq A(1-a)A$ bestehen, so ist A/M ein einfacher Ring bekanntlich mit dem zweiseitigen Einselement $a + M$. Dieser Widerspruch zur Definition der Brown—McCoyschen Radikalringe beweist $a \in A(1-a)A$ für jedes $a \in A$. Ist nämlich $a + M$ kein Linkseinselement von A/M, so ist $(1-a)A + M/M$ ein nilpotentes von Null verschiedenes Rechtsideal von A/M, was unmöglich ist. Daraus folgt aber wegen $A(1-a) \subseteq A(1-a)A$ auch $A(1-a)A = A$ für jedes $a \in A$, w. z. b. w.

4) Existiert ein Element $a \in A$ mit $ab \in Aa$ für jedes $b \in A$, so ergibt sich $aA \subseteq Aa$. Es sei weiterhin $c \in A$ so gewählt, daß $a = ac$ gilt, was nach der Behauptung $(*)$ für E_2-Ringe immer möglich ist. Nach der Eigenschaft 3) erhält man $A(1-c)A = A$ mit diesem c, folglich ergibt sich wegen $aA \subseteq Aa$ auch $aA = aA(1-c)A \subseteq$ $\subseteq Aa(1-c)A = A.0.A = 0$. Daraus folgt aber $a = 0$, denn jeder E_2-Ring ist nach [14] linksannihilatorfrei. Es gilt also $aA \nsubseteq Aa$ für jedes $a \in A$. Hiernach besteht auch das Zentrum von A nur aus 0, und es gilt wegen $aA \nsubseteq Aa$ insbesondere auch $Aa \neq A$ für jedes Element $a \in A$.

5) Gibt es ein von Null verschiedenes Element a von A und eine ganze rationale Zahl n mit

$$a \notin (a+n)A + A(a+n)A = B,$$

so ist $a + B$ im Faktorring A/B ein Linksmultiplikator von A/B. Da auch A/B gleichzeitig mit A ein E_2-Ring ist, erhält man nach Satz 3. 4. 1 aus [14], daß A/B ein zweiseitiges Einselement hat. Dann läßt sich aber A/B auch auf einen einfachen Ring mit Einselement homomorph abbilden. Diese Tatsache widerspricht der Voraussetzung, daß A ein Brown—McCoyscher Radikalring ist, und somit ist 5) bewiesen.

6) Da jedes homomorphe Bild eines E_2-Ringes nach Satz 2. 3. 1 bzw. 2. 2. 2 von [14] bzw. Behauptung $(*)$ keinen von Null verschiedenen Linksannihilator besitzt, ist A auch ein E_5-Ring im Sinne [14]. Es gilt folglich $L_1 \subseteqq L_1 A$ für jedes Linksideal L_1 von A, und A besitzt nach Satz 2. 3. 3 [14] maximale Linksideale L. Es gilt also entweder $LA = L$ oder $LA = A$. Im Falle $LA = L$ ist L auch ein zweiseitiges Ideal von A, und somit ist A/L, da nur triviale Linksideale hat, entweder ein Zeroring oder ein Schiefkörper. Der erste Unterfall widerspricht aber der Tatsache $A^2 = A$ (vgl. Satz 2. 2. 1 von [14]), und der zweite Unterfall widerspricht der Definition der Brown—McCoyschen Radikalringe. Es gilt also $LA = A$ für jedes maximale Linksideal L von A.

7) Aus dem vorigen folgt leicht, daß A kein Rechtseinselement hat, d.h, $A(1-b) \neq 0$ für jedes $b \in A$ ist, denn nach der Eigenschaft 3) stets $A(1-b)A = A$ gilt. Das im Satz definierte Linksideal

$$L_b = \{x\colon\ x \in A,\ xb = x\}$$

ist genau ein Linksideal L, für das $L(1-b) = 0$ besteht. Wegen $A(1-b) \neq 0$ ergibt sich $L_b \neq A$ für jedes b, und nach der Behauptung $(*)$ folgt $b \in bA$, also $b = bc$ mit einem $c \in A$, und somit $b \in L_c$ mit einem $c = c_b \in A$ für jedes $b \in A$. Es sei weiterhin $x \circ y = x + y - xy$ für jedes $x, y \in A$. Man erhält offenbar $L_x \subseteqq L_{x \circ y}$ für beliebiges x und y. Es sei jetzt b ein beliebiges Element und a ein solches Element von A, für das $a(1-b) \neq 0$, also $a \notin L_b$ gilt. Dann existiert nach Behauptung $(*)$ ein $c \in A$ mit $a(1-b) = a(1-b)c$, folglich mit $a(1-b \circ c) = 0$ also $a \in L_{b \circ c}$. Daher ist $L_{b \circ c}$ echt größer als L_b, und somit gilt in A nicht die Maximalbedingung für die Linksideale der Gestalt L_x. Insbesondere ist A deshalb auch kein linksnoetherscher Ring, w. z. b. w.

Damit haben wir auch den Satz bewiesen.

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(Eingegangen am 4. Dezember, 1968)

Über dreifaktorisierbare Gruppen. I

Von J. SZÉP in Budapest und G. ZAPPA in Firenze

Es sei G eine Gruppe und H eine echte Untergruppe von G. Es seien $H_1, H_2, \dots$ sämtliche Konjugierte von H in G. Bekanntlich ist das Produkt $\overline{H}=H_1 H_2 \dots$ eine Gruppe [1]. Es gilt außerdem der Satz:

Ist G eine nichtnilpotente endliche Gruppe, so hat G eine nilpotente Untergruppe H mit der Eigenschaft $\overline{H}=G$ [1].

Im Fall $\overline{H}=G$ benötigt man aber nicht notwendigerweise alle Konjugierte von H um mit Hilfe des Produkts die ganze Gruppe G darzustellen. ORE zeigte, daß das Produkt von zwei Konjugierten einer $H \subset G$ die Gruppe G nicht darstellt, also ist die minimale Anzahl der Konjugierten von H drei, um die ganze Gruppe zu erhalten.

Es erhebt sich die Aufgabe, die Struktur der „dreifaktorisierbaren" Gruppen $G=H_1 H_2 H_3$ zu untersuchen, wobei H_1, H_2, H_3 konjugierte (echte) Untergruppen von G sind.

Eine Zusammenfassung der Ergebnisse dieser Arbeit wurde in [4] veröffentlicht.

I

Definition. Man nennt die Gruppe G *dreifaktorisierbar*, wenn in G eine echte Untergruppe H existiert, für die

(1) $$G=aHa^{-1}HbHb^{-1} \qquad (a, b \in G)$$

gilt. Die Faktorisation (1) heißt eine *Dreifaktorisation*.

Satz 1. *Eine Gruppe G ist dreifaktorisierbar (mit H) dann und nur dann, wenn es ein Element $a \in G$ gibt mit*

$$G=Ha^{-1}HaH.$$

Beweis. Es ist ausreichend, den Fall „nur dann" zu beweisen. Wir nehmen an, daß (1) gilt. Dann ist

$$G=a^{-1}Gb=a^{-1}aHa^{-1}HbHb^{-1}b=Ha^{-1}HbH.$$

Es existieren also drei Elemente h_1, h_2, h_3 von H mit $1=h_1 a^{-1} h_2 b h_3$. Daraus folgt $b=h_2^{-1} a h_1^{-1} h_3^{-1}$, und so erhalten wir

$$G=Ha^{-1}HbH=Ha^{-1}Hh_2^{-1}ah_1^{-1}h_3^{-1}H=Ha^{-1}HaH.$$

Bemerkung. Ist G eine n-faktorisierbare Gruppe, d.h. gilt

$$G=H_1 \dots H_n,$$

wobei $H_1, \dots, H_n$ konjugierte Untergruppen sind, so kann man erreichen, daß im Produkt der erste und der letzte Faktor identisch werden. Der Beweis erfolgt ähnlich wie im Fall des Satzes 1.

II

Jetzt werden wir einige Kriterien für die Existenz eines Normalkomplements von H herleiten.

Satz 2. *Es sei G eine endliche Gruppe von der Form $G=Ha^{-1}HaH$, wobei H eine Hall-Untergruppe und a ein Element von G sind. Dann gilt $N_G(H)=H$.*

Beweis. Es sei Π die Menge der verschiedenen Primzahlen in der Ordnung von H und x ein Element von $N_G(H)$. Es gilt

$$x=h_1 a^{-1} h_2 a h_3 \qquad (h_1, h_2, h_3 \in H).$$

Es ergibt sich $a^{-1}h_2 a=h_1^{-1}xh_3^{-1}\in N_G(H)$. Das Element $a^{-1}h_2 a$ ist ein Π-Element und mit H vertauschbar. Die Gruppe $\{H, a^{-1}h_2 a\}$ ist eine Π-Untergruppe von G. H ist eine Hall-Untergruppe von G, also gilt

$$\{H, a^{-1}h_2 a\}=H, \quad \text{d.h.} \quad a^{-1}h_2 a\in H.$$

Daraus folgt $x\in H$.

Satz 3. *Es sei G eine endliche Gruppe und H eine Hall-Untergruppe von G, die im Zentrum ihres Normalisators enthalten ist. Dann existiert ein Normalteiler N von G mit $G=HN$, $H\cap N=1$.*

Beweis. Es sei Π die Menge der verschiedenen Primzahlen $p_1, p_2, \dots, p_r$ in der Ordnung von H. Es sei P_i die p_i-Sylowgruppe von H $(i=1, \dots, r)$ und $x\in N_G(P_i)$. Die Gruppe H ist abelsch und außerdem ist $H\subseteqq C(P_i)$, also gilt

$$x^{-1}Hx\subseteqq C(x^{-1}P_i x)=C(P_i).$$

Daraus folgt, daß H und $x^{-1}Hx$ abelsche (nilpotente) Hall-Untergruppen von $C(P_i)$ sind. Nach einem Satz von Wielandt [3] sind die Untergruppen H und $x^{-1}Hx$ in $C(P_i)$ konjugiert. Dann existiert ein $y\in C(P_i)$ mit $y^{-1}(x^{-1}Hx)\,y=H$, d.h. es gilt $(xy)^{-1}Hxy=H$. Daraus folgt $xy\in N_G(H)$. H ist eine abelsche Gruppe, und es gilt $xy \in C(P_i)$, d.h. $x \in C(P_i)$. Das Element x ist ein beliebiges Element von $N_G(P_i)$. also gilt $N_G(P_i) = C(P_i)$ und $P_i \subseteqq Z(N_G(P_i))$. Nach BURNSIDE enthält G genau ein p_i-Normalkomplement und so ist G eine p_i-nilpotente Gruppe. Der Durchschnitt der p_i-Normalkomplemente ($i - 1, 2, \ldots, r$) ist ein Normalkomplement N von H in G, d.h. $G = NH$, $N \cap H = 1$.

Aus Satz 2 und 3 folgt der

Satz 4. *Es sei G eine dreifaktorisierbare endliche Gruppe $G = Ha^{-1}HaH$, wobei H eine abelsche Hall-Untergruppe von $G (a \in G)$ ist. Dann gibt es ein Normalteiler N von G derart, daß $G = HN$, $H \cap N = 1$.*

Im Satz 4 ist die Gruppe H eine abelsche Gruppe. Diese Annahme kann man nicht durch „nilpotent" substituieren. Dazu betrachten wir die Gruppe S_4. S_4 hat eine Untergruppe H von der Ordnung 8 (also ist H nilpotent), außerdem hat S_4 die Dreifaktorisation $S_4 = Ha^{-1}HaH (a \in S_4)$. Doch hat S_4 keinen Normalteiler mit dem Index 3.

Die Substitution der abelschen Untergruppe durch eine nilpotente Gruppe ist auch in dem Fall nicht möglich, wenn die Ordnung von G ungerade ist. Betrachten wir dazu die Gruppe G mit der Ordnung $3^7 \cdot 7$, die folgendermaßen definiert ist:

$$a_i^3 = 1 \ (i = 1, \ldots, 7), \quad b^7 = 1, \quad c^3 = 1, \quad a_1 a_2 \ldots a_7 = 1,$$

$$a_i a_j = a_j a_i \ (i, j = 1, \ldots, 7), \quad b^{-1} a_i b = a_{i+1} \ (i = 1, \ldots, 6),$$

$$b^{-1} a_7 b = a_1, \quad c^{-1} a_1 c = a_2, \quad c^{-1} a_2 c = a_4, \quad c^{-1} a_3 c = a_6, \quad c^{-1} a_4 c = a_1,$$

$$c^{-1} a_5 c = a_3, \quad c^{-1} a_6 c = a_5, \quad c^{-1} a_7 c = a_7, \quad c^{-1} bc = b^2.$$

Die Elemente $a_1, \ldots, a_7, c$ erzeugen eine Untergruppe H von der Ordnung 3^7 (also ist H eine nilpotente Hall-Gruppe), außerdem gilt $G = Hb^{-1}HbH$. Doch G hat keinen Normalteiler von der Ordnung 7.

Trotzdem gibt es eine Verallgemeinerung des Satzes 4 für den Fall, daß H ungerade ist. Im Beweis der Verallgemeinerung brauchen wir einen Satz von THOMPSON [2]. Es bezeichne $J(P)$ die im Satz von THOMPSON auftretende Gruppe wobei P eine Sylow-Gruppe bedeutet. ($J(P)$ ist die mit den abelschen Untergruppen erzeugte Untergruppe von P, für die die Anzahl der Basisgeneratoren maximal ist.)

Satz 5. *Es sei G eine endliche Gruppe und H eine nilpotente Hall-Gruppe von G. Es seien $p_1, \ldots, p_s$ die in der Ordnung von H auftretenden verschiedenen Primzahlen und P_i die p_i-Sylow-Gruppe von H. G habe die folgenden Eigenschaften:*

$$\text{a)} \ N_G(H) = H, \qquad \text{b)} \ J(P_i) = P_i, \qquad \text{c)} \ P_i' \subseteqq Z(P_i)$$

($i = 1, 2, \ldots, s$). *Ist H von ungerader Ordnung, so gibt es einen Normalteiler N von G, derart, daß $G = HN$, $H \cap N = 1$.*

Beweis 1. Ist $T \supset H$ eine Untergruppe von G, so ist $\{H^T\} = T$. Im entgegengesetzten Fall gäbe es ein $x \in T$ mit $x \notin \{H^T\}$. Es gilt aber $x^{-1}\{H^T\}x = \{H^T\}$, und nach Wielandt gibt es ein $y \in \{H^T\}$ sodaß $y^{-1}x^{-1}Hxy = H$, d.h. $xy \in N_G(H) = H \subseteqq \{H^T\}$. Daraus folgt $x \in \{H^T\}$, was ein Widerspruch ist.

2. Ist P_i ein Normalteiler von G, dann ist G p_i-nilpotent. Es gilt $G = P_i S$ $(|P_i|, |S|) = 1$. Es sei $H_i = P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_s$. Dann ist $\{H_i^G\} \subseteqq S$. Es gilt aber $\{H_i^G\} = S$, weil im entgegengesetzten Fall $P_i\{H_i^G\} = \{H^G\} \subset G$ gelten würde, was wegen 1 unmöglich ist.

3 $C_G(Z(P_i))$ ist p_i-nilpotent. Der Beweis erfolgt durch Induktion. Wegen 2 kann man annehmen, daß $Z(P_i) \subset P_i$ ist. Gilt $C_G(Z(P_i)) \subset G$, dann gelten unsere Bedingungen für $C_G(Z(P_i))$, also ist $C_G(Z(P_i))$ nach Induktion p_i-nilpotent. Gilt $C_G(Z(P_i)) = G$, so betrachten wir die Gruppe $G/Z(P_i)$. Für $G/Z(P_i)$ gelten unsere Bedingungen $(P_i/Z(P_i)$ ist eine abelsche Gruppe, d.h. $J(P_i)/Z(P_i) = P_i/Z(P_i))$. Nach Induktion ist $G/Z(P_i)$ p_i-nilpotent, also existiert eine Untergruppe V mit $V = Z(P_i)S = Z(P_i) \times S$, die ein Normalteiler von G ist. Wegen $p_i \nmid |S|$ ist S das p_i-Komplement in G.

4. $N_G(J(P_i))$ ist p_i-nilpotent. Wegen $J(P_i) = P_i$ und 2 gilt die Behauptung.

Bezüglich 3 und 4 ergibt sich nach dem Satz von Thompson daß G p_i-nilpotent ist. Dies gilt für $i = 1, \ldots, s$ also erhalten wir den Satz.

Nach Satz 2 und 5 bekommt man den

Satz 6. *Es sei G eine dreifaktorisierbare Gruppe $G = Ha^{-1}HaH$, wobei H eine nilpotente Hall-Gruppe von G mit ungerader Ordnung ist. Es seien $p_1, \ldots, p_s$ die in der Ordnung von H vorkommenden verschiedenen Primzahlen. Es sei P_i die p_i-Sylow-Gruppe von H. Nehmen wir an, daß $J(P_i) = P_i$ $(i = 1, \ldots, s)$ und $P_i' \subseteqq Z(P)$ gilt. Dann gibt es einen Normalteiler N von G derart, daß $G = HN$, $H \cap N = 1$.*

Bemerkungen. Für den Fall der Gruppe von der Ordnung $3^7 \cdot 7$ gilt in unserem Gegenbeispiel $J(H) = J(P) = \{a_1, \ldots, a_7\}$ d.h. $J(P) \neq P$.

Wir bemerken noch, daß zwischen den dreifaktorisierbaren Gruppen auch einfache Gruppen existieren. Dazu betrachten wir die alternierende Gruppe A_5 (über 5 Ziffern); in dieser bilden die Permutationen, die eine Ziffer unverändert lassen, eine Untergruppe H von der Ordung 12, und es gilt $G = Ha^{-1}HaH$, wobei $a \in G$ ein geeignetes Element ist.

III

Wir beschäftigen uns jetzt mit auflösbaren dreifaktorisierbaren endlichen Gruppen und stellen sämtliche maximale Dreifaktorisationen von solchen Gruppen vor.

Definition. Es sei G eine dreifaktorisierbare Gruppe $G=Ha^{-1}HaH$ $(a\in G)$. Man nennt die Dreifaktorisation von G maximal, wenn es keine Gruppe $H_1\subset G$ gibt, für die $G=H_1b^{-1}H_1bH_1$ $(b\in G)$ mit $H\subset H_1$ gilt.

Der Kern der Untergruppe H von G ist $\bigcap H_i$, wobei H_i sämtliche Konjugierte von H in G durchläuft.

Definition. Man sagt, die Untergruppe $H\subset G$ in G sei antinormal, wenn $\bigcap H_i=1$ gilt, d.h. wenn H keinen echten Normalteiler von G enthält.

Ist M der Kern von H, so sieht man leicht, daß H/M antinormal in G/M ist. Es ergibt sich leicht der

Satz 7. *Es sei $G=Ha^{-1}HaH$ eine endliche dreifaktorisierbare Gruppe und M der Kern von H. Dann gilt*

$$G/M=(H/M)(aM)^{-1}(H/M)(aM)(H/M).$$

Die Gruppe H/M ist antinormal in G/M, also reduziert sich die Untersuchung der maximalen Dreifaktorisationen der G auf den Fall, in dem die Untergruppe H antinormal ist.

Wir beweisen den

Satz 8. *Es sei G eine endliche auflösbare Gruppe. G besitzt eine maximale Dreifaktorisation $G=Ha^{-1}HaH$ ($a\in G$, H ist antinormal in G) dann und nur dann, wenn die folgenden Bedingungen erfüllt sind:*

a) *$G=HN$, wobei N ein minimaler Normalteiler von G ist mit $H\cap N=1$ und $C_G(N)=N$.*

b) *Es gibt ein $b\in N$ derart, daß jedes Element von N mit den Elementen der Menge L, die aus den Konjugierten von b in H besteht, wenigstens in einer Weise in der Form $l_1^{-1}l_2$ $(l_1,l_2\in L)$ darstellbar ist.*

Beweis. Zuerst beweisen wir den Fall „dann". Wir nehmen an, daß $G=Ha^{-1}HaH$ $(H\subset G)$ eine endliche auflösbare Gruppe mit antinormaler H ist; ferner sei $Ha^{-1}HaH$ eine maximale Dreifaktorisation. Es sei N ein minimaler Normalteiler von G. Man kann annehmen, daß $N\neq G$. Es gilt $G=HN$ $(H\cap N=1)$, weil man im Fall $HN\subset G$ die Dreifaktorisation $G=(HN)(a^{-1}HNa)HN$ erhalten würde, was der Maximalität der Dreifaktorisation widerspricht; außerdem widerspricht der Fall $H\cap N>1$ der antinormalen Eigenschaft von H.

Die Untergruppe $C_G(N)$ ist ein Normalteiler von G, also ist $N_G(H\cap C_G(N))\supseteq H$ und $N_G(H\cap C_G(N))\supseteq N$. Daraus folgt $N_G(H\cap C_G(N))=G$. H ist antinormal in G, also ist $H\cap C_G(N)=1$ oder $H\cap C_G(N)=H$. Im letzten Fall wäre H normal in G, was unmöglich ist. So bleibt der Fall $H\cap C_G(N)=1$, d.h. der Index von $C_G(N)$ in G stimmt mit $|H|$ überein. So folgt $C_G(N)=N$ und damit ist a) bewiesen.

Das Element a kann man in der Form $a=hb$ $(h\in H,\ b\in N)$ schreiben, also gilt $G=Ha^{-1}HaH=Hb^{-1}HbH$. Es sei n ein beliebiges Element von N. Es gilt $n==h_1b^{-1}h_2bh_3$ $(h_1, h_2, h_3\in H)$. N ist ein Normalteiler von G, also folgt aus $n=(h_1b^{-1}h_1^{-1})(h_1h_2bh_2^{-1}h_1^{-1})(h_1h_2h_3)$, daß $h_1h_2h_3\in N$ gilt. Wegen $h_1h_2h_3\in H$ folgt $h_1h_2h_3=1$. Es seien $h_1^{-1}=h_4$, $(h_1h_2)^{-1}=h_5$; so bekommt man

$$n=h_4^{-1}b^{-1}h_4h_5^{-1}bh_5=(h_4^{-1}bh_4)^{-1}h_5^{-1}bh_5,$$

d.h. $n=l_1^{-1}l_2$ mit $l_1=h_4^{-1}b\ h_4\in L$, $l_2=h_5^{-1}b\ h_5\in L$. Damit haben wir b) bewiesen.

Wir beweisen nun den Fall „nur dann" des Satzes. Wir nehmen an, daß a) und b) gelten, a ein Element von L ist, und für beliebiges $n\in N$ gilt: $n=l_1^{-1}l_2$, mit $l_1, l_2\in L$. Also folgt $n=(h_1^{-1}ah_1)^{-1}(h_2^{-1}ah_2)$ für geeignete h_1, h_2 von H. Deshalb gilt $n=h_1^{-1}a^{-1}h_1h_2^{-1}\ ah_2\in Ha^{-1}HaH$, und so ergibt sich

$$N\subseteqq Ha^{-1}HaH,\quad G=HN\subseteqq HHa^{-1}HaH=Ha^{-1}HaH,$$

d.h. G hat eine Dreifaktorisation.

Ist diese Dreifaktorisation nicht maximal, so gibt es eine Untergruppe $X\subset G$, $H\subset X$ mit $G=Xb^{-1}XbX$ $(b\in G)$. In diesem Fall ist $D=X\cap N\neq 1$ ein Normalteiler von X und auch von N, d.h. $X\cap N$ ist ein Normalteiler von G. Es ist klar, daß $D\neq 1$, $D\neq N$ und daß N kein minimaler Normalteiler ist, was der früheren Annahme widerspricht, daß N ein minimaler Normalteiler in N ist. Also ist die Dreifaktorisation $G=Ha^{-1}HaH$ maximal. Somit ist der Satz bewiesen.

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(Eingegangen am 14. Dezember, 1968)

On (m, n)-ideals in regular duo semigroups

By SÁNDOR LAJOS in Budapest

Let S be a semigroup. Following the notation and terminology of A. H. CLIFFORD and G. B. PRESTON [1] we shall say that S is regular if $a \in aSa$ for each element a in S. A semigroup S is called duo semigroup if every one-sided ideal of S is a two-sided ideal. We shall prove that in a regular duo semigroup every bi-ideal is a two-sided ideal. More generally we establish that every (m, n)-ideal of a regular duo semigroup is also a two-sided ideal, if m, n are non-negative integers such that $m+n > 0$. For the definition and fundamental properties of (m, n)-ideals we refer to the author's paper [2].

First we need the following result.

Theorem 1. *A non-empty subset A of a regular semigroup S is a bi-ideal of S if and only if there exists a left ideal L and a right ideal R of S such that $A = RL$.*

Proof. Let A be a bi-ideal of a regular semigroup S. We show that it is the product of the smallest right ideal of S containing A and the smallest left ideal of S containing A:

$$A = (A \cup AS)(A \cup SA). \tag{1}$$

It is easy to see that the inclusion

$$A \subseteq (A \cup AS)(A \cup SA) = A^2 \cup ASA \tag{2}$$

holds because $a = axa \in ASA$ for each element a in A. The converse inclusion

$$(A \cup AS)(A \cup SA) \subseteq A \tag{3}$$

also holds since A is a bi-ideal of S.

Conversely we prove that if S is an arbitrary semigroup, L is a left ideal and R is a right ideal of S then the product RL is a bi-ideal of S. It is evident that

$$(RL)(RL) \subseteq RL, \tag{4}$$

that is the product RL is a subsemigroup of S. On the other hand

$$(RL)S(RL) \subseteq RSL \subseteq RL, \tag{5}$$

i.e. RL is a bi-ideal of S. This completes the proof of Theorem 1.

Theorem 2. *Every bi-ideal of a regular duo semigroup is a two-sided ideal.*

Proof. The statement of Theorem 2 is an easy consequence of Theorem 1 because the product of two-sided ideals is also a two-sided ideal.

Corollary. *Every quasi-ideal of a regular duo semigroup is a two-sided ideal.*

This follows from Theorem 2 because every quasi-ideal is a bi-ideal.

Theorem 3. *Suppose that S is a regular duo semigroup and m, n are non-negative integers so that $m+n>0$. Then every (m, n)-ideal of S is a two-sided ideal of S.*

Proof. The statement of Theorem 3 follows from Theorem 2 and from Theorem 1.5 in the author's paper [2] by mathematical induction.

Let S be a semigroup which is a semilattice of groups. It is known that S is a regular duo semigroup (see [1] or [3]). Applying Theorems 2 and 3 we obtain the following results.

Theorem 4. *Let S be a semigroup which is a semilattice of groups. Then every bi-ideal of S is a two-sided ideal.*

Theorem 5. *Suppose that S is a semigroup which is a semilattice of groups and m, n are non-negative integers such that $m+n>0$. Then every (m, n)-ideal of S is a two-sided ideal of S.*

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(Received March 25, 1969)

Bibliographie

H. Federer, Geometric Measure Theory (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band 153), XIV + 676 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1969.

This is the first systematic and detailed exposition of the subject, from the foundations up to the most recent results, including many which were not previously published. The book can serve both as an excellent reference book and as a textbook; the reader is merely assumed to be familiar with the very elements of set theory, topology, linear algebra, and commutative ring theory.

The abundance and variety of the material peresented in the book makes an exhausting description in a short review quite impossible. So we can only comment on the general plan of the book, and mention some samples of the most characteristic results contained.

Chapter I contains a systematic account of the methods of multilinear algebra which are used throughout the book. From among these basic tools of geometric measure theory the author uses exterior and alternating algebras to discuss oriented m-dimensional vector subspaces of Euclidean n-space, and uses symmetric algebras to treat higher differentials of mappings, for example in WHITNEY's theorem or in the theory of strongly elliptic systems of second order partial differential equations.

The techniques of the general theory of integration are developed in Chapter II, the material of which can be warmly recommended to students in a higher course on real analysis. The author's exposition of general measure theory features equally the set theoretic approach of CARATHÉODORY and the function lattice approach of P. J. DANIELL and F. RIESZ. It includes not only the fundamental facts on Lebesgue integration, but also numerous additional topics like the theory of Suslin sets, the theory of covariant measures over homogeneous spaces of locally compact groups, the main properties of Hausdorff-type measures, etc. Connections between integration and linear operations, including the Radon-Nikodým theorem, are discussed in great detail first for an arbitrary lattice of real valued functions, and later for continuous functions on a locally compact space. It then proceeds to give an account of the theory of covering and derivation, which comprises several modern results about derivates based on generalizations of the covering theorems of VITALI and BESICOVITCH. Finally, we find an exhaustive study of CARATHÉODORY's construction of measure, by which he achieved the significant extension of measure theory to lower dimensional subsets of the space, obtaining for example a reasonable notion of area for a two-dimensional surface in Euclidean 3-space.

Chapter III contains the basic facts concerning integration with respect to m-dimensional measures over subsets of Euclidean n-space. It centers about tangential and rectifiability properties of sets and the transformation formulae corresponding to Lipschitzian maps. Some generalizations of differentiation are defined here; tangent spaces of arbitrary subsets of R^n, differentiable submanifolds, etc. are discussed in detail. Then the author exhaustively elaborates the theory of area and coarea of Lipschitzian maps, with a fine application of the coarea formula in integral geometry, and a generalization of the calculus of area and coarea, etc. The rest of the chapter deals with the characterization of rectifiable sets by their projection properties within the framework of structure theory, and with some effect of high order smoothness of functions on the Hausdorff measures of certain associated sets.

In Chapter IV the author employs distributions in the sense of L. SCHWARTZ, and currents as introduced by G. DE RHAM for use in the theory of harmonic forms. The principal objects of the investigation are the normal, rectifiable and integral currents and the integral flat chains. The author follows the lead of H. WHITNEY, *Geometric integration theory* (Princeton, 1957), in using Lipschitzian maps and the notion of mass, but the two books have very different aims, WHITNEY'S book being directed to cohomology with general cochains, whereas this one to homology with general chains. This chapter contains a number of topics about integration and differential forms over oriented-sets including the formulae of Gauss, Green and Stokes. The reader will find several versions of these classical results; research on the problem of finding the most natural and general forms of them has greatly contributed to the development of geometric measure theory. Another fundamental result of this chapter is the deformation theorem, which yields basic isoperimetric estimates and links the theory of integral currents to the classical integral homology theory.

Recently the methods of geometric measure theory have led to a very considerable progress in the study of general elliptic variation problems, including the multidimensional problem of least area. Chapter V is entirely devoted to applications of the theory of integral currents to the calculus of variations. The theory is based on the concept of the integral of a positive parametric integrand Φ of degree m over an m-dimensional rectifiable current T. The problem of regularity and smoothness of minimizing currents has not yet been completely solved, but some very significant partial results have been obtained, and these are systematically collected here. The study of singularities of minimizing currents is likely to remain a fruitful field of research in years to come. The treatment of strongly elliptic systems of second order partial differential equations is not intended to be as comprehensive as in C. B. MORREY, *Multiple integrals in the calculus of variations* (New York, 1966), but it contains a complete exposition of the results needed for geometric applications. We note that Fourier transformation is not used in the book.

The enumeration of the contents could hardly give a right impression of the richness of the monograph. The presentation is concise, but always clear and well-readable. The last three chapters are in particular useful for those interested in further research work. It is perhaps not exaggerated to assert that this book is of basic importance for everybody who wants to keep pace with up-to-date developments in analysis.

Ferenc Móricz (Szeged)

Ju. M. Berezanskiĭ, Expansions in eigenfunctions of selfadjoint operators (Translations of Mathematical Monographs, vol. 17) VI+809 pages, American Mathematical Society, Providence, 1968.

Developments in the application of the general theory of selfadjoint operators to spectral problems for differential and difference equations have been very rapid in recent years. Nevertheless, at the time of preparation of the Russian original of the present book, there were no books in existence in which relevant questions were discussed in any complete way. The book of M. A. NAĬMARK (*Linear differential operators;* Moscow, 1954) discusses the spectral theory of selfadjoint ordinary differential and difference equations. Analogous questions for partial differential equations are dealt with in only one chapter of the book of GEL'FAND and ŠILOV (*Generalized functions.* Part III; Moscow, 1968) and in one chapter of the book by DUNFORD and SCHWARTZ (*Linear operators.* Part II; New York, 1963). The author has tried to fill this gap by undertaking the job of giving a comprehensive account of the subject in the form of a monograph.

The reviewer is aware of the fact that a rough chapter-by-chapter description does not do justice to the extremely rich content of the book. The limited space here, however, does not allow to do more.

The nature of the subject makes it necessary to introduce the relevant notions of modern functional analysis. This background is given in a very self-contained manner as an Introduction, which is a chapter by itself. A great deal of discussions in the book are based on generalized functions of finite order. The theory of these functions is given in the first chapter and is formulated in a conveniently abstract form. The shorter Chapter II introduces and discusses the general concepts of the theory of boundary problems. First boundary value problems for linear partial differential equations are considered, then formal schemes for the application of functional methods to the analysis of these problems are presented. Chapter III is devoted to the study of boundary value problems for elliptic equations. Here the chief attention is turned to questions on when each generalized solution of an elliptic equation is sufficiently smooth both in the interior of the domain and up to and including its boundary. These questions are of central importance when we construct expansions with respect to eigenfunctions. Chapter IV contains a number of examples of problems connected with non-elliptic equations, which should be considered as illustrations of the method developed in Chapter II. The discussion of the theory announced in the title of the book actually commences in Chapter V. The general theory of expansions in eigenfunctions (generalized or ordinary) is given here in detail. In this exposition, the main ideas are represented by two methods. The first (due to the author) relies essentially on the use of Radon—Nikodym-type derivatives and the second on the notion of von Neumann's direct integral. The results obtained in the course of studying the general situation are then interpreted for concrete cases such as arbitrary selfadjoint operators in $L^2(G)$, $G\subset R^n$, and Carleman's operators. Chapter VI contains a thorough analysis of the results of the preceding chapter for operators in L^2 connected with elliptic equations. A brief summary of the corresponding results for ordinary differential problems is also given here. Chapter VII is devoted to the study of the spectral theory of selfadjoint difference operators in l^2. Finally, in Chapter VIII the theory of selfadjoint (differential and difference) operators acting in a space with scalar product generated by a positive definite kernel is constructed. M. G. KREIN's results on integral representations for positive definite functions play here a role of central importance.

Each chapter is illuminated with a great many examples, both classical and recent. The book concludes with very instructive bibliographical and historical comments and a very rich bibliography. In this presentation the author has overcome the difficulties inherent in the material treated. Thus the book, in spite of its intricate subject, is quite readable. One of methods for achieving this consists in the author's repeating important concepts and definitions rather than using cross references. The reviewer is convinced that this excellent monograph is to become classic in this field.

The American Mathematical Society has performed a very valuable service in translating the original Russian edition into English. This English edition makes it possible for this very interesting book to appeal to a wider class of interested mathematicians.

I. Kovács (Szeged)

J. L. Bell and **A. B. Slomson, Models and ultraproducts: an introduction,** ix+322 pages, North-Holland Publishing Company, Amsterdam—London, 1969.

This monograph on ultraproducts appeals to a wide circle of readers. The practising mathematician may be attracted by the accounts giving insight into the major trends of recent development in the field as far as this is possible without an unfavourable increase of the size of the book; the undergraduate will appreciate the fact that everything is presented in a way available also to him; finally it serves everyone's comfort that the author is successful in avoiding the overpredantry that endangers many of the writers on logics in a way that this does not amount to carelessness or looseness in any disturbing quantity. Indeed, the expert reader may supply the subtle points not considered in the text; and, on the other hand, it would not really be to the point to overload the undergraduate reader with refined distinctions where these do not play a prominent role.

The importance of the subject of the book can hardly be overstressed. It is indeed the concept of ultrafilters that directly or indirectly penetrated into widely different fields of mathematics and which — via non-standard analysis — proved that even in disciplines seemingly remote from logics the methods of the latter may turn out very valuable. Though the present book is concerned only with the model-theoretic applications of ultrafilters, it is really hard to imagine this otherwise. These applications form the bases of all others; this is true even for topology — as is shown by non-standard methods — where the use of ultrafilters to replace the convergent sequences of analysis seems in no way to be connected with ultraproducts.

The book begins with introductory chapters on propositional and predicate calculus, model theory with the proof of the Löwenheim-Skolem theorem, compactness theorem, completeness theorem, etc. Another proof, using the maximal ideal theorem instead of the full strength of the axiom of choice, of the compactness theorem is also included. As is known, the existence of such a proof has practical importance in that it makes possible to establish many classical results in analysis as a consequence of a principle weaker than the axiom of choice by using non-standard analysis.

After these basic questions the authors touch upon more specific problems concerning the cardinality of ultraproducts, connections between semantical and algebraic properties of structures, characterization of elementary equivalence with the aid of iterated ultrapowers (ultralimits), completeness and model completeness, algebraically homogeneous and universal structures and saturated models, an extremely useful tool in up-to-date research in model theory; various applications of ultraproducts, among them one to the construction of non-standard models of arithmetic; after considerations of the effect of extending the language so as to include generalized quantifiers the book is concluded with an account of infinitary languages — here a celebrated result of W. Hanf of inconpactness of cardinals is considered, though not in its strongest form, the importance on which is underlined by the fact that it decided (in the negative direction) a long open problem of seemingly pure set-theoretical nature: whether or not it is possible to introduce a countably additive non-atomic measure on the set of all subsets of the first inaccessible cardinal.

The material encompassed in this gap-filling work is still in a boiling stage. Though this means that the final word cannot yet be proclaimed on the subject, it certainly increases the actuality value of this book.

A. Máté (Szeged)

Theory of Finite Groups, A Symposium, Edited by **Richard Brauer** and **Chih-Han Sah,** XIII + 263 pages, W. A. Benjamin, Inc., New York—Amsterdam, 1969.

This book contains abstracts of lectures held at a symposium on finite groups at Harvard University in 1968. The principal subject of this symposium was one of the most exciting recent problems of finite group theory: the description of simple groups.

After half a century of stagnation, research in the field of finite simple groups got a new impulse in the fifties, owing above all to the new ideas raised by R. Brauer and C. Chevalley to study centralizers of involutions and the Sylow 2-subgroup as well as to apply the algebraic-geometrical method. These ideas have considerably deepened our general knowledge on simple groups, and also led to the discovery of some new types of simple groups. The greatest part of the book is attached to these three ideas. The authors of the 36 abstracts are well-known specialists of finite group theory, among them R. Brauer, W. Feit, D. Gorenstein, Z. Janko, M. Suzuki, J. Tits. The articles contain not only results but also sketch their proofs in a more or less detailed form, so they give abundant and valuable information for the competent reader on the present state of research.

May we mention some samples from the content. M. SUZUKI describes a (new) simple permutation group of order $2^{13}\cdot 3^7\cdot 5^2\cdot 7\cdot 11\cdot 13$ on $2\cdot 3^4\cdot 11$ letters. G. HIGMAN and J. MCKAY prove the existence of a simple group of order $2^7\cdot 3^5\cdot 5\cdot 17\cdot 19$ studied by Z. JANKO. J. L. ALPERIN and R. BRAUER give a description of the possible orders and the possible centralizers of involutions of a simple group with quasi-dihedral Sylow 2-subgroup. CHIH-HAN SAH attacks Schreier's conjecture on the solvability of automorphism groups of simple groups.

In view of the rapid progress in this domain, we can praise the publisher, who, by applying a high-quality multiplying technique, published this precious book very shortly after the symposium.

Béla Csákány (Szeged)

Daniel Ponasse, Logique mathématique, 164 pages, Office Central de Librairie, Paris, 1967.

The book, an introductory course for post-graduate students, is extremely successful in saying much in a limited space while avoiding any notable adverse effect of its compactness. It presents some fundamental topics in mathematical logics; in particular, it developes the propositional calculus and first-order predicate calculus first from syntactic and then from semantic aspect along parallel lines; gives an insight into the notions of deducibility, deductive systems, completeness, and consistency; proves several equivalent versions of the completeness theorem; considers Boolean rings and algebras with regard to their applications to logical calculi; studies the completeness problem in first-order predicate calculus, the Löwenheim-Skolem theorem and the finiteness principle (compactness theorem) and concludes with an account of first order calculus with equality.

A. Máté (Szeged)

K. Chandrasekharan, Introduction to analytic number theory (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 148), VIII+140 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1968.

These introductory lectures are prepared with the intention to raise the interest of students and non-specialists in how the analytical means seemingly remote from the field are in fact closely connected with deep problems of number theory. The large amount of literature on the subject understandably makes it difficult to say anything new about the classical results of the subject; through careful selection and grouping of the material the author carried out the undertaken task eminently. After elementary discussions on factorization and congruences the book deals with approximation of irrationals, quadratic residues, arithmetical functions, Chebyshev's estimate for the distribution of primes, uniform distribution modulo 1, Minkowski's theorem on lattice points in convex sets, Dirichlet's theorem on primes in arithmetical progressions and the prime number theorem. The work is complemented with a short historical survey.

A. Máté (Szeged)

C. G. Lekkerkerker, Geometry of numbers (Series Bibliotheca Mathematica, Vol. 8), 510 pages, Wolters-Noordhoff Publishing, Groningen, North-Holland Publishing Company, Amsterdam—London, 1969.

The geometry of numbers, a subject the basic question of which is when a body in the n-dimensional Euclidean space contains an integral point different from the origin, was initiated by a relatively simple theorem of H. MINKOWSKI and its surpising arithmetical applications. The subject has in the last decades aquired a considerable level of development. This monograph intends to satisfy the long-felt need for a comprehensive up-to-date account of the subject. It is natural that such a deep study as this is mainly addressed to practising mathematicians or more advanced students; nevertheless, since only a minimal amount of preliminary knowledge is required; everyone interested in the subject can make use of this book.

The material is so arranged that orientation on the present state of knowledge is very quick; this is achieved by laying stress on problems and not on the methods used in the solutions of these. Six major topics are considered in six chapters ensuing upon the first one that sums up the preliminaries.

The second chapter starts with the fundamental theorem of Minkowski and its various extensions and generalizations, among them Siegel's interesting observation that Minkowski's theorem is a special case of Parseval's formula for multiple Fourier series, and finally homogeneous and inhomogeneous minima of convex bodies with respect to a lattice is studied. In the third chapter families of lattices are considered, Mahler's selection theorem is proved, the role of the critical determinant is discussed, and questions of packings and converings of the plane are studied. Questions concerning the inhomogeneous determinant of a set are also touched upon.

The next two chapters deal with continuity properties connected with, and methods for reduction of, star bodies; methods for estimation from below of the critical and the inhomogeneous determinant are discussed. The last two chapters give a systematic study of arithmetical problems of the field. The absolute homogeneous minima of various forms are considered and applications to the theory of Diophantine approximation are exhibited; finally, questions related to the inhomogeneous minima of forms are investigated.

At the end of the book there is a vast bibliography that is complete for the period 1935—65.

A. Máté (Szeged)

F. Klein, Elementarmathematik vom höheren Standpunkte aus. Band 1: XII+309 Seiten, Band 2: XII+302 Seiten, Band 3: X+238 Seiten (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 14—16), 4. Auflage, Berlin, Verlag von Julius Springer, Nachdruck 1968.

Klein's weltberühmte Elementarmathematik hat eine epochemachende Bedeutung im Mathematiklehren. Der Inhalt der Bände ist der folgende. Band 1: Arithmetik, Algebra, Analysis; Band 2: Geometrie; Band 3: Präzisions- und Approximationsmathematik. Der gegenwertige Ausgabe ist ein Nachdruck von Fr. Seyfart's Umarbeitung (1933).

Ich zitiere die Worte des Buches über Bildungsreformen: „Die unabweisbare Notwendigkeit solcher Reformen liegt darin begründet, daß die diejenigen mathematischen Begriffsbildungen betreffen, die heutzutage die Anwendungen der Mathematik auf alle möglichen Gebiete durchaus beherrschen und ohne die alle Studien an der Hochschule, schon die einfachsten Studien über Experimentalphysik, gänzlich in der Luft schweben."

J. Berkes (Szeged)

INDEX — TARTALOM

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70-70 — Szegedi Nyomda

Felelős szerkesztő és kiadó: Szőkefalvi-Nagy Béla
A kézirat nyomdába érkezett: 1970. január hó
Megjelenés: 1970. június hó

Példányszám: 1400. Terjedelem: 16 1/4 (A/5) ív
Készült monószedéssel, íves magasnyomással, az MSZ 5601-24 és az MSZ 5602-55 szabvány szerint