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Nilpotent groups and automorphisms*)

JOHN DAUNS and KARL HEINRICH HOFMANN[1]) in New Orleans (Louisiana, U.S.A.)

I. First an arbitrary endomorphism $A \times V \to A_1 \times V_1$ of semi-direct products $A \times V$, $A_1 \times V_1$ of arbitrary groups A, V, A_1, V_1 is described by four functions $f_1 : A \to A_1$, $f_2 : V \to A_1$, $c_1 : A \to V_1$, and $c_2 : V \to V_1$. Under additional hypotheses, automorphisms of $A \times V$ leaving the subgroup $1 \times V \lhd A \times V$ invariant are studied.

II. If K is any field, set $V = K^n$. Let A be the group of all upper triangular matrices $\alpha = \|a_{ij}\|$ $(0 \leqq i, j \leqq n;\ a_{ij} \in K;\ a_{ij} = 0$ for $i > j;\ a_{ii} \neq 0)$. Form the semi-direct product $A \times V$:

$$(\beta, w)(\alpha, v) = (\beta\alpha, w\alpha + v) \qquad (\alpha, \beta \in A;\ v, w \in K^n);$$

$$w\alpha = (w_1, \ldots, w_n)\|a_{ij}\|, \qquad (\alpha = \|a_{ij}\| \in A;\ w = (w_1, \ldots, w_n) \in K^n).$$

Secondly, the general methods of I are used to compute the automorphism group Aut $A \times V$. Modulo all the inner automorphisms, there is exactly one non-inner automorphism $\sigma: A \times V \to A \times V$ with $\sigma(1 \times V) \neq 1 \times V$; σ is found explicitly.

III. The quotients of the descending central series of the commutator subgroup $N = [A \times V, A \times V]$ are K-vector spaces. Lastly, all normal subgroups $W \lhd N$ whose image in each quotient of the descending central series is a one dimensional vector space are determined.

The automorphism group Aut $A \times V$ of the holomorph $A \times V$ of a group V has received considerable attention (see [6], [7], [11] and [12]). In all of the above papers, those automorphisms of $A \times V$ which leave the normal subgroup $1 \times V \lhd A \times V$ invariant, play a significant role. Suppose V is any abelian o-group which is divisible by 2 and A^+ the group of all order preserving transformations of V. It has been shown (see [Harvey; Theorem 2. 1, p. 24]), that if A^+ can be ordered in any manner whatever so that it becomes an o-group, then:

(i) $1 \times V$ is o-characteristic in the o-holomorph $A^+ \times V$;

(ii) every o-automorphism of $A^+ \times V$ is inner.

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Let A and V be as in II of the introduction with K an ordered field. Let $A^+ \lhd A$ be all $\|a_{ij}\| \in A$ with $a_{ii} > 0$ for $i = 1, \ldots, n$. If $V = K^n$ is ordered lexicographically, then A^+ is a group of order preserving transformations of V. Now take K to be the rationals. Then A^+ is precisely all order preserving transformations of V. Since clearly A^+ can be naturally ordered so that it becomes an o-group, any automorphism σ of $A^+ \times V$ which does not leave $1 \times V$ invariant satisfies:

(i) σ does not preserve the order of V;

(ii) σ is not inner.

If $N \lhd A^+ \times V$ is the commutator subgroup of $A^+ \times V$, then the quotients of the descending central series of N are vector spaces. Since the image of $1 \times V$ in each quotient is a one dimensional vector space, if σ is any automorphism of $A^+ \times V$, $\sigma(1 \times V)$ should have the same property. These considerations were the motivation for classifying all normal subgroups $W \lhd N$ with one dimensional images in each quotient. If in the above example K is taken as the reals, then the group of all order preserving transformations of V consists of matrices having zeroes below the diagonal, strictly positive entries on the diagonal, and rational-linear (in general discontinuous) linear maps of K into K as the entries above the diagonal. Due to our inability to handle such groups, this note deals with groups of the above general kind, where the entries of the matrices are in an arbitrary field K (sometimes assumed to be not of characteristic 2).

1. Automorphisms of semi-direct products

The main objective of this first section is to determine all those automorphisms F of a semi-direct product $A \times V$ of two groups A and V having the property that $F[1 \times V] = 1 \times V$. Most of the propositions are established in greater generality than later needed. In fact, for the most part it is not even necessary to assume that V is abelian — let alone a vector space, or even a finite dimensional one. However, it has to be assumed that A is a group of automorphisms of V and that the inner automorphisms by elements of V belong to A.

1. 1. Notation. If A and V are any groups, then an *action* of A on V is a map $A \times V \to V$, $(v, \alpha) \to v\alpha$, with the properties

$$(v + w, \alpha) = v\alpha + w\alpha \quad \text{and} \quad (v, \alpha\beta) = (v\alpha)\beta, \quad (\alpha, \beta \in A;\ v, w \in V).$$

Although written additively, V is not assumed to be abelian. With respect to any fixed action of A on V, the semi-direct product $A \times V$ will be written as follows:

$$(\alpha, v)(\beta, w) = (\alpha\beta, v\beta + w) \qquad (\alpha, \beta \in A;\ v, w \in V).$$

The identity elements of V and A will be denoted by 0 and 1. Inner automorphisms and commutators in any group will always be written as $\beta^{\alpha}=\alpha^{-1}\beta\alpha$ and $[\alpha, \beta]=\alpha^{-1}\beta^{-1}\alpha\beta$. If A_1, V_1 are two other such groups, then the functions f_i, c_i $(i=1, 2)$ are defined by

$$F[(\alpha, 0)]=(f_1(\alpha), c_1(\alpha)),\ F[(1, v)]=(f_2(v), c_2(v)) \qquad (\alpha\in A, v\in V).$$

If $f\colon B\to A$ is any homomorphism of any group B into A, then a map $\tau\colon B\to V$ is a *crossed homomorphism with respect to f provided* $(\alpha\beta)\tau=(\alpha\tau)f(\beta)+\beta\tau$ holds for all $\alpha, \beta\in B$. It is *inner* if there is a $y\in V$ for which $\alpha\tau = -yf(\alpha)+y$ for all $\alpha\in B$.

For any arbitrary group V, Aut V will denote the group of all automorphisms of V. The centralizer and normalizer of a subgroup A in Aut V will be denoted by $C(A<\text{Aut } V)$ and $N(A<\text{Aut } V)=\{T\in\text{Aut } V|T^{-1}AT=TAT^{-1}=A\}$. Every element $v\in V$ gives rise to an inner automorphism $\bar{v}\in\text{Aut } V$.

Remark. If A acts on an abelian group V, the crossed homomorphisms form a group $Z^1(A, V)$ under pointwise addition. The inner automorphisms form a subgroup $B^1(A, V)$. The factor group $Z^1(A, V)/B^1(A, V)$ is the first cohomology group of A with respect to the given action of A on V.

In the next proposition c_1 and c_2 are crossed homomorphisms with respect to f_1 and f_2. Note that equation (iv) implies that A leaves the kernel of f_2 invariant.

Proposition 1.2. *Let* $F\colon A\times V\to A_1\times V_1$ *and* f_i, c_i, $i=1, 2$, *be as above arbitrary semi-direct products. Then the following hold for all* $\alpha, \beta\in A$ *and* $v, w\in V$:

(i) $F[(\alpha, v)]=(f_1(\alpha)f_2(v), c_1(\alpha)f_2(v)+c_2(v))$;

(ii) f_1 *and* f_2 *are homomorphisms*;

(iii) $c_1(\alpha\beta)=c_1(\alpha)f_1(\beta)+c_1(\beta)$; $c_2(v+w)=c_2(v)f_2(w)+c_2(w)$;

(iv) $f_2(w\beta)=f_1(\beta)^{-1}f_2(w)f_1(\beta)$;

(v) $c_2(w\beta) = -c_1(\beta)f_2(w\beta)+c_2(w)f_1(\beta)+c_1(\beta)$.

Conversely, if f_1, f_2, c_1, *and* c_2 *are any functions satisfying* (ii)—(v), *then* F *defined by equation* (i) *is an endomorphism.*

Proof. As an illustration, (iv) and (v) will be proved. The proofs of (i)—(iii) are similar and even simpler. Computing $F[(\beta, 0)]F[(1, w\beta)]$ and $F[(1, w)]F[(\beta, 0)]$ by (i) and then equating the first and second components, we obtain

(iv) $f_2(w\beta)=f_2(w)^{f_1(\beta)}$,

(v) $c_1(\beta)f_2(w\beta)+c_2(w\beta) = c_2(w)f_1(\beta)+c_1(\beta)$.

Since it is not clear that (ii)—(v) are all the relations that interrelate the functions f_i, c_i, the proof of the converse will be indicated. Equation (i) shows that

$$F[(\alpha\beta, v\beta+w)]=(f_1(\alpha\beta)f_2(v\beta+w), c_1(\alpha\beta)f_2(v\beta+w)+c_2(v\beta+w)),$$

$$F[(\alpha, v)]\ F[(\beta, w)] = (f_1(\alpha)f_2(v)f_1(\beta)f_2(w), c_1(\alpha)f_2(v)f_1(\beta)f_2(w)+ + c_2(v)f_1(\beta)f_2(w)+c_1(\beta)f_2(w)+c_2(w)).$$

By (iv) and (ii) the first components of the above two equations are equal. Use of (iv) and (iii) gives that

$$c_1(\alpha)f_2(v)f_1(\beta)f_2(w) = c_1(\alpha\beta)f_2(v\beta+w) - c_1(\beta)f_2(v\beta+w).$$

Thus it only remains to show that

$$c_2(v\beta+w) = -c_1(\beta)f_2(v\beta+w) + c_2(v)f_1(\beta)f_2(w) + c_1(\beta)f_2(w) + c_2(w).$$

But this follows from (iii) and (v), since

$$c_2(v\beta+w) = [-c_1(\beta)f_2(v\beta) + c_2(v)f_1(\beta) + c_1(\beta)]f_2(w) + c_2(w).$$

From now on, three simplifications will be assumed throughout. First $V=V_1$, $A=A_1$; secondly A will be taken in $A \subseteqq \operatorname{Aut} V$; and thirdly, only automorphisms F of $A \times V$ will be considered.

The proof given in [Harvey; p. 7] of the next corollary for the case when V is abelian generalizes to non-abelian V.

Corollary 1.3. *If $\tau: A \to V$ is a crossed homomorphism with $\tau(A) \subseteqq$ center V, then*

$$F: A \times V \to A \times V, \quad F[(\alpha, v)] = (\alpha, \alpha\tau + v) \qquad ((\alpha, v) \in A \times V)$$

is an automorphism of $A \times V$ leaving $1 \times V$ elementwise fixed. Conversely, every automorphism of $A \times V$ leaving $1 \times V$ elementwise fixed is necessarily of the above form. Furthermore, suppose $\tau(\alpha) = -y\alpha + y$, for all $\alpha \in A$ and some $y \in$ center V. Then $F[(\alpha, v)] = (\alpha, \alpha\tau + v) = (0, y)^{-1}(\alpha, v)(0, y)$ for $(\alpha, v) \in A \times V$.

Note that the converse of (i) of the next corollary is also true, i.e., $f_2 \equiv 1$ if and only if $F[1 \times V] \subseteqq 1 \times V$.

Corollary 1.4. *In Proposition 1.2 assume $V=V_1$, $A=A_1$, and that F is an automorphism of $A \times V$ onto itself. Suppose $F[1 \times V] \subseteqq 1 \times V$. Then*

(i) $f_2 \equiv 1$,
(ii) $f_1(A) = A$,
(iii) *c_2 is an injective homomorphism.*

Corollary 1.5. *Let $T: V \to V$ be defined by $vT = c_2(v)$ for all $v \in V$. Assume that:*
(a) *$F: A \times V \to A \times V$ is an automorphism,*
(b) $F[1 \times V] = 1 \times V$,
(c) $\{\bar{v} | v \in V\} \subseteqq A$.
Then:
(i) $f_2 \equiv 1$;
(ii) *f_1 is an isomorphism of A onto A; $F^{-1}[(\alpha, 0)] = (f_1^{-1}(\alpha), 0)$ for all $\alpha \in A$;*

(iii) $F^{-1}[(1, v)] = (1, vT^{-1})$ *for all* $v \in V$;

(iv) $f_1(\beta) = T^{-1}\beta T\overline{c_1(\beta)}^{-1}$;

(v) $T^{-1}AT = TAT^{-1} = A$; $T \in N(A < \text{Aut } V)$.

Proof. Conclusions (i)—(iv) are immediate consequences of Proposition 1. 2. It follows from (iii), (iv), and (*c*) that $T^{-1}AT \subseteq A$ and $TAT^{-1} \subseteq A$ and hence $T^{-1}AT = TAT^{-1} = A$.

The next lemma shows how to construct automorphisms of $A \times V$ which leave $1 \times V$ invariant.

Lemma 1. 6. *Consider any group* V *(not assumed to be abelian) and any subgroup* $A \subseteq \text{Aut } V$.

(i) *For any* $S \in N(A < \text{Aut } V)$ *and any* $y \in V$, *the map* $F\colon A \times V \to A \times V$ *defined by*

$$F[(\beta, w)] = (\beta^s, -y\beta^s + wS + y) \qquad (\beta \in A,\ w \in V)$$

is an automorphism.

(ii) *For* S *and* y *as in* (i),

$$(S, y) \in \text{Aut } V \times V \quad \text{and} \quad F[(\beta, w)] = (S, y)^{-1}(\beta, w)(S, y).$$

Thus the automorphism in (i) *is inner if and only if* $S \in A$.

(iii) *In addition assume that* $\{\bar{v} | v \in V\} \subseteq A$ *and let* $T \in \text{Aut } V$. *Then* T *extends to an automorphism* $F\colon A \times V \to A \times V$ *if and only if* $T \in N(A < \text{Aut } V)$.

Proof. (i) and (ii). Conclusion (ii) proves (i). (iii) If T is obtained from an automorphism F of $A \times V$ by $F[(1, v)] = (1, vT)$ for $v \in V$, then by Corollary 1. 5 (v) $T \in N(A < \text{Aut } V)$. Conversely, if $T \in N(A < \text{Aut } V)$, then the map $F[(\beta, w)] = (\beta^T, wT)$ for $(\beta, w) \in A \times V$ is an automorphism by (i) of this lemma.

From now on the group V will be abelian, later a vector space, and, finally, a finite dimensional one. The following lemma is well known (see [Harvey; p. 11)]; its proof is omitted.

Lemma 1. 7. *Let* V *be any abelian group,* $A \subseteq \text{Aut } V$ *any subgroup, and* $f_1\colon A \to A$ *a homomorphism of* A *into* A. *Assume that:*

(a) *The map* $\bar{2}\colon V \to V$, $v \to 2v$, *is an isomorphism of* V *onto* V;

(b) $\bar{2} \in A$;

(c) $f_1(\bar{2}) = \bar{2}$.

Then every crossed homomorphism τ *with respect to the action* f_1, *is inner. In fact,* $\alpha\tau = -yf_1(\alpha) + y$, *where* $y = -\bar{2}\tau$.

Definition 1. 8. Consider a vector space V over a field K and a field automorphism $\mu: K \to K$. Any K-basis $\{v(\lambda)|\lambda \in \Lambda\}$, where Λ is an indexing set, defines a map $\tilde{\mu}: V \to V$ by:

$$\text{if} \quad v = \Sigma\{k(\lambda)v(\lambda)|k(\lambda) \in K, \lambda \in \Lambda\}, \quad \text{define} \quad v\tilde{\mu} = \Sigma k(\lambda)\mu v(\lambda).$$

Suppose $A \subseteqq \operatorname{Aut} V$ is any subgroup having the property that

$$A = \{\tilde{\mu}^{-1}\alpha\tilde{\mu}|\alpha \in A\} = \{\tilde{\mu}\alpha\tilde{\mu}^{-1}|\alpha \in A\}.$$

Then an automorphism $\bar{\mu}: A \times V \to A \times V$ may be defined by

$$(\beta, w)\bar{\mu} = (\tilde{\mu}^{-1}\beta\tilde{\mu}, w\tilde{\mu}) \qquad (\beta \in A, w \in V).$$

The subgroup of Aut V consisting of all K-linear automorphisms of V will be denoted by $\operatorname{Aut}_K V$.

Remarks 1. If in the above definition $\alpha \in A$ is K-linear, then so is $\tilde{\mu}^{-1}\alpha\tilde{\mu}$. The matrix with respect to the basis $\{v(\lambda)|\lambda \in \Lambda\}$ of $\tilde{\mu}^{-1}\alpha\tilde{\mu}$ is obtained by applying μ to each entry of the matrix of α.

2. The automorphism $\bar{\mu}$ depends upon the choice of basis; whether $\tilde{\mu} \in N(A < \operatorname{Aut} V)$ may also depend upon the choice of the basis.

Lemma 1. 9. *Consider a vector space V over a field K and a subgroup $A \subseteqq \operatorname{Aut} V$. Let E: $V \to V$ be the identity map. Assume that*

(a) *F: $A \times V \to A \times V$ is an automorphism;*
(b) $F[1 \times V] = 1 \times V$;
(c) $C(A < \operatorname{Aut} V) = \{kE|k \in K \setminus \{0\}\}$.

Then:

(i) *There is a (bijective) field automorphism μ: $K \to K$ such that*

$$(cv)T = (c\mu^{-1})(vT) \qquad (v \in V, c \in K).$$

(ii) *In addition assume that for some choice of basis in V, $\tilde{\mu} \in N(A < \operatorname{Aut} V)$. Then the automorphism $F \circ \bar{\mu}$: $A \times V \to A \times V$ satisfies:*

$$F \circ \bar{\mu}[(1, v)] = (1, v\tilde{\mu}T), \quad (cv)(\tilde{\mu}T) = c(v\tilde{\mu}T) \qquad (v \in V, c \in K).$$

Proof (i) It follows from equation (v) of Proposition 1. 2, that for any $\beta \in A$, $w \in V$, and $c \in K$, we have

$$w\beta(cE)T = wTf_1(\beta)f_1(cE), \quad w(cE)\beta T = wTf_1(cE)f_1(\beta).$$

Since by Corollary 1. 5 (ii) the map T is surjective, it follows that $f_1(\beta)f_1(cE) = f_1(cE)f_1(\beta)$ and $f_1(cE) \in C(A < \operatorname{Aut} V)$. Thus there is a map ν: $K \to K$ such that

$f_1(cE)=(cv)E$. It is easily seen that v is an injective homomorphism. There is a similar map $\mu: K \to K$ associated with the automorphism F^{-1} which satisfies $v \circ \mu = \mu \circ v =$identity. Thus v and μ are epimorphic and $v=\mu^{-1}$. Conclusion (ii) follows immediately.

2. Automorphisms of linear groups

In this section the general facts about automorphisms of semi-direct products developed in section 1 are used to find the automorphism groups of a certain class of groups. The next definition gives this class of groups as well as various subgroups which will be of major interest throughout the rest of the discussion.

Definition 2. 1. Let K be an arbitrary field and G the group of all $(n+1)\times\times(n+1)$ upper triangular matrices P with entries from K of the form

$$P=\|a_{ij}\| \ (0\leqq i, j\leqq n); \quad a_{ij}=0 \quad \text{if} \quad i>j; \ a_{ii}\neq 0$$

for all i.

Two normal subgroups $N\subset G^1\subset G$ of G are defined by:

$$N=\{P\in G|a_{ii}=1,\ i=0, \ldots, n\}; \quad G^1=\{P\in G|a_{00}=1\}.$$

The normal subgroups Γ and Γ^1 of G are defined by:

$\Gamma=\{P\in N|a_{ij}=0$ for $i<j$, except in the first row and the last column$\}$,

$\Gamma^1=\{P\in N|a_{ij}=0$ for $i<j$, except in the first two rows and the last column$\}$.

Let $\alpha\in K$ be any scalar. Two normal subgroups B^1 and $B^1(\alpha)$ of N are defined as follows:

$$B^1=\{P\in N|a_{ij}=0 \quad \text{for } i<j \quad \text{unless} \quad i=0\},$$

$$B^1(\alpha)=\{P\in N|a_{ij}=0 \text{ for } i<j \text{ unless } i=0, \text{ or } (i,j)=(1,n) \text{ and } a_{1n}=\alpha a_{01}\}.$$

The groups B^1 and N are normal in G. Note that $B^1(0)=B^1$. For $\alpha\neq 0$, $B^1(\alpha)$ is normal in N but not in G^1. By taking transposes of all elements of $B^1, B^1(\alpha), \Gamma^1$ around the second diagonal, we obtain three other groups B_1, $B_1(\alpha)$, Γ_1. The subgroup of G consisting of all diagonal matrices is denoted by D. The element of D whose diagonal entries are $\lambda_0, \ldots, \lambda_n$ will be denoted by diag $(\lambda_0, \ldots, \lambda_n)$. Set $D^1=G^1\cap D$.

If $N_0=N=[G, G]$ and $N_j=[N_{j-1}, N], j=1, \ldots, n-1$; then N is a group of nilpotency class n having the same descending and ascending central series

$$[G, G]=N=N_0\supset N_1\supset \ldots \supset N_{n-1}\supset N_n=1.$$

The group N_j consists precisely of all matrices having j strings of zeroes parallel to the main diagonal. The entries of a matrix in the $(j+1)$-st string parallel to the main diagonal will be referred to as the $(j+1)$-st *layer*. The center N_{n-1} of N will be denoted by Z for simplicity. The usual matrix unit with all zeroes except for a one in the p-th row and q-th column will be denoted by E_{pq}, $0\leqq p, q\leqq n$. For

$j=0, \ldots, n$, there is an isomorphism α_j of K-vector spaces

$$\alpha_j : K^{n-j} \to N_j/N_{j+1}, \qquad \alpha_j[(x_1, \ldots, x_{n-j})] = I + \sum_{i=1}^{n-j} x_i E_{i-1,\, j+i}.$$

For the reader's convenience we include schematic diagrams showing the forms of the elements in the various groups where the $a_{ij} \in K$ are arbitrary

$$B^1 : \begin{vmatrix} 1 & a_{01} & a_{02} & & a_{0,n-1} & a_{0n} \\ & & & & & \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & & 0 & 1 \end{vmatrix} \qquad B^1(\alpha) : \begin{vmatrix} 1 & a_{01} & a_{02} & & a_{0,n-1} & a_{0n} \\ 0 & 1 & 0 & & 0 & \alpha a_{01} \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & & 0 & 1 \end{vmatrix}$$

$$\Gamma : \begin{vmatrix} 1 & a_{01} & a_{02} & & & a_{0,n} \\ 0 & 1 & 0 & & & a_{1,n} \\ & & & & & \\ 0 & 0 & 0 & & 1 & a_{n-1,n} \\ 0 & 0 & 0 & & 0 & 1 \end{vmatrix} \qquad \Gamma^1 : \begin{vmatrix} 1 & a_{01} & a_{02} & & a_{0,n-1} & a_{0n} \\ 0 & 1 & a_{12} & & a_{1,n-1} & a_{1n} \\ 0 & 0 & 1 & & 0 & a_{2n} \\ 0 & 0 & 0 & & 1 & a_{n-1,n} \\ 0 & 0 & 0 & & 0 & 1 \end{vmatrix}$$

$$\alpha[(x_1, \ldots, x_{n-j})] = \begin{vmatrix} 1 & \overbrace{0 \quad 0 \quad\quad 0}^{j} & x_1 & 0 & & 0 & 0 \\ 0 & 1 \quad 0 \quad\quad 0 & 0 & x_2 & & 0 & 0 \\ & & & & & & \\ 0 & 0 \quad 0 & & & & 0 & x_{n-j} \\ 0 & 0 \quad 0 & & & & 0 & 1 \end{vmatrix} N_{j+1} \in N_j/N_{j+1} (j=0, \ldots, n-1).$$

Figure 1.

Let n be any integer; set $V=K^n$. Let A be the group of all $n \times n$ matrices α with zeroes below the diagonal, arbitrary elements above the diagonal and non-zero elements on the diagonal. Elements of V are viewed as row vectors and in $A \times V$, A acts on these by right multiplication. The group $A \times V$ can be identified as a subgroup of the general linear group $Gl(n+1, K)$ as follows:

$$\alpha = \begin{vmatrix} a_{11} & a_{12} & & a_{1n} \\ 0 & a_{22} & & a_{2n} \\ & & & \\ 0 & 0 & & a_{nn} \end{vmatrix} \qquad v = (a_{01}, a_{02}, \ldots, a_{0n}) \qquad (\alpha \in A,\ v \in V);$$

$$(\alpha, v) = \begin{vmatrix} 1 & v \\ 0 & \alpha \end{vmatrix} = \begin{vmatrix} a_{00} & a_{01} & & a_{0n} \\ 0 & a_{11} & & a_{1n} \\ & & & \\ 0 & 0 & & a_{nn} \end{vmatrix} \in Gl(n+1, K);$$

$$(\alpha, v)(\alpha', v') = (\alpha\alpha', v\alpha' + v'), \quad \begin{vmatrix} 1 & v \\ 0 & \alpha \end{vmatrix} \begin{vmatrix} 1 & v' \\ 0 & \alpha' \end{vmatrix} = \begin{vmatrix} 1 & v\alpha' + v' \\ 0 & \alpha\alpha' \end{vmatrix} \qquad (\alpha' \in A,\ v' \in V).$$

Sometimes $G^1 = A \times V$ will be viewed as a matrix subgroup of $Gl(n+1, K)$ and denoted by G^1, whereas at other times, when in our considerations its semi-direct product structure plays an important role, it will be written as a semi-direct product $A \times V$. In case K is an ordered field, the normal subgroup of G consisting of all matrices with strictly positive entries on the diagonal will be denoted by G^+; set $G^{+1} = G^1 \cap G^+$, $D^1 = D \cap G^1$, and $D^{+1} = D^1 \cap G^+$. Similarly A^+ will consist of all α having strictly positive diagonal entries. Thus just as $A \times V$ can be identified with G^1, so $A^+ \times V$ can be identified with G^{+1}.

Next some automorphisms of G^1 and G^{+1} are defined. If $\mu: K \to K$ is any field automorphism, then $\tilde{\mu}: V \to V$ will always be defined with respect to the natural basis by

$$v\tilde{\mu} = (v_1\mu, \dots, v_n\mu) \qquad (v_1, \dots, v_n) \in K^n.$$

Let $\tilde{V}$ denote the subgroup of Aut $A \times V$ consisting of all automorphisms $F: A \times V \to A \times V$ such that the restriction $F|1 \times V \in \mathrm{Aut}_K V$, where $\mathrm{Aut}_K V$ was the group of all K-linear isomorphisms of V.

For the remainder of this definition suppose now that K is an ordered field. The subgroup of all $\tilde{\mu}$ obtained from order preserving automorphisms will be denoted by U. The element $F_i = I - 2E_{ii} \in G$ $(i = 0, 1, \dots, n)$ defines an automorphism $\bar{F}_i: G^{+1} \to G^{+1}$ by

$$\bar{F}_i(g) = F_i^{-1} g F_i \qquad (g \in G^{+1}; \quad i = 0, 1, \dots, n).$$

Note that $F_i \in G^1$ for $i = 1, \dots, n$, that $F_0 \notin G^1$. However, $F_0 = -F_1 \dots F_n$ and hence $\bar{F}_0 = \bar{F}_1 \dots \bar{F}_n$. The $\bar{F}_1, \dots, \bar{F}_n$ generate a subgroup $\mathscr{F}$ of Aut $A^+ \times V$. The group of inner automorphisms of Aut $A^+ \times V$ will be denoted by J.

The objective is to find all automorphisms of G^{+1}.

Proposition 2.2. *Consider the group $G^{+1} = A^+ \times V$ of Definition 2.1 and any automorphism $F: A^+ \times V \to A^+ \times V$ such that $F[1 \times V] = 1 \times V$. Let $c_2: V \to V$ be defined by $F[(1, v)] = (1, c_2(v))$ for $v \in V$. Then:*

(i) *There is an order preserving field automorphism $\mu: K \to K$ such that*

$$F[(1, cv)] = (1, (c\mu^{-1})c_2(v)) \qquad (c \in K, v \in V).$$

(ii) *If $T: V \to V$ is defined by $F \circ \bar{\mu}[(1, v)] = (1, vT)$ $(v \in V)$, then $T \in N(A^+ < \mathrm{Aut}\, V) = A$.*

(iii) *There is a $y \in V$ such that for any $(\alpha, v) \in A^+ \times V$,*

$$F \circ \bar{\mu}[(\alpha, v)] = (\alpha^T, -y\alpha^T + vT + y) = \begin{vmatrix} 1 & y \\ 0 & T \end{vmatrix}^{-1} \begin{vmatrix} 1 & v \\ 0 & \alpha \end{vmatrix} \begin{vmatrix} 1 & y \\ 0 & T \end{vmatrix}.$$

(iv) $\tilde{V} = \mathscr{F} J.$

Proof. (i) First, it is easy to see that $C(A^+ < \mathrm{Aut}\, V)$ are the scalar operators. The automorphisms μ and μ^{-1} given by Lemma 1. 9 (i) clearly preserve the order of K. (ii) It is well known and easy to prove that $N(A^+ < \mathrm{Aut}_K\, V) = A$. In order to show that $T \in \mathrm{Aut}_K\, V$, take $c \in K$ and $(v_1, \ldots, v_n) \in V$. Then

$$(1, (cv)T) = F[(1, (cv)\bar{\mu}] = F[1, (c\mu)(v\bar{\mu})] = (1, cc_2(v\bar{\mu})) = (1, c(vT)).$$

Thus $T \in \mathrm{Aut}_K\, V$ and it follows from Corollary 1. 5 (v) that $T \in N(A^+ < \mathrm{Aut}_K\, V)$.

(iii) By Proposition 1. 2 there are functions $f_1, f_2 \equiv 1, c_1, c_2 \equiv T$ corresponding to the automorphism $F \circ \bar{\mu}$ such that

$$F \circ \bar{\mu}[(\alpha, v)] = (f_1(\alpha), c_1(\alpha) + vT) \qquad (\alpha, v) \in A \times V.$$

By Corollary 1. 5 (iv), $f_1(\alpha) = T^{-1}\alpha T$ for $\alpha \in A$. By Lemma 1. 7, c_1 is of the form $c_1(\alpha) = -y\alpha^T + y$ for some $y \in V$. Then the above equation becomes

$$F \circ \bar{\mu}[(\alpha, v)] = (\alpha^T, -y\alpha^T + v^T + y).$$

(iv) The automorphism F can be realized as an inner automorphism by the elements $(1, y) \in A^+ \times V$ and $(T, 0) \in A \times V$ as follows:

$$F \circ \bar{\mu}[(\alpha, v)] = (T, y)^{-1}(\alpha, v)(T, y) = (1, y)^{-1}(T, 0)^{-1}(\alpha, v)(T, 0)(1, y)$$

$$((\alpha, v) \in A \times V).$$

However, $(T, 0)$ equals a product of some of $F_1, \ldots, F_n$ times an element of $A^+ \times V$. Thus $\tilde{V} = \mathscr{F}J$.

Remark. The last Proposition 2. 2 remains valid verbatim if A^+ is replaced by A throughout.

The second step in determining the automorphism group of G^{+1} is to show that any automorphism $F\colon G^1 \to G^1$ maps either $F(B^1) = B^1$ or $F(B^1) = B_1$. Since the previous Proposition 2. 2 completely determines all automorphisms $F\colon G^1 \to G^1$ satisfying $F(B^1) = B^1$, the final step will be the construction of an automorphism $\sigma\colon G^1 \to G^1$ satisfying $\sigma(B^1) = B_1$. The next definition and sequence of lemmas is needed in order to accomplish the second step.

Definition 2. 3. For $0 \leqq p < \varkappa \leqq n$, let $A_{p\varkappa}$ be the set of all $\|a_{\alpha\beta}\| \in N$ with $a_{\alpha\beta} = 0$ for $\alpha \geqq p+1$ and for $\beta \leqq \varkappa - 1$.

Remarks. 1. The group $A_{p\varkappa}$ is abelian and $A_{p\varkappa} \lhd G^1$.

2. It can be shown that $A_{p,\,p+1}$ $(0 \leqq p \leqq n-1)$ is a maximal normal abelian subgroup of N. It is our conjecture that there are no others.

3. Note that $A_{01} = B^1$, $A_{n-1,n} = B_1$; also $A_{0j} \subseteqq B^1$ and $A_{in} \subseteqq B_1$ for all j and i.

4. If $j = \varkappa - p - 1$, then $N_{j-1} \subset A_{p\varkappa} \subseteqq N_j$; $A_{p\varkappa} = N_j$ if and only if $A_{0n} = N_{n-1} = Z$.

Suppose $\|a_{\alpha\beta}\| \in N_j$ with $a_{p\varkappa} \neq 0$ where $\varkappa - p - 1 = j$, $0 \leqq j \leqq n-1$. Suppose i and j satisfy $i \leqq p$ and $\varkappa \leqq j$. The next sequence of lemmas will show that by inner automorphisms from G^1, $\|a_{\alpha\beta}\|$ can be transformed into $I + cE_{ij}$, where $c \in K$. In the diagram below, $\|a_{\alpha\beta}\|$ has non-trivial entries in the triangular region in the upper right hand corner. The group $A_{p\varkappa}$ consists of all elements having non-trivial entries in the rectangle region inside the triangular region. The j-th layer is represented by the line connecting $(0, \varkappa - p)$ and $(n - \varkappa + p, n)$ entries.

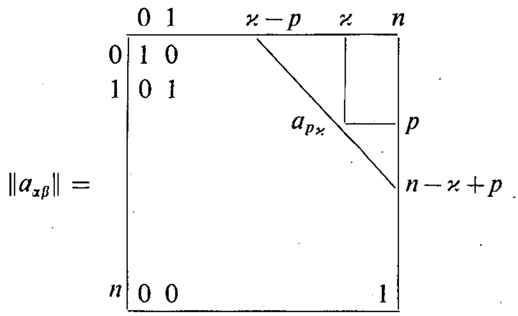

Figure 2.

The next lemma describes the types of elements that can be constructed by application of inner automorphisms.

Lemma 2.4. *Let $\|\beta_{ij}\|$, $\beta_{ij} \in K$, $0 \leqq i, j \leqq n$, be any matrix.*

(i) *If $\|\beta'_{ij}\|$ is defined by*

$$\|\beta'_{ij}\| = (I - cE_{pq})\|\beta_{ij}\|(I + cE_{pq}) \qquad (0 \leqq p,\ q \leqq n),$$

then $\|\beta'_{ij}\|$ is obtained from $\|\beta_{ij}\|$ by subtracting c times the q-th row from the p-th row and adding c-times the p-th column to the q-th column.

(ii) *If $S = \operatorname{diag}(\lambda_0, \ldots, \lambda_n)$, where $\lambda_0, \ldots, \lambda_n \in K \setminus \{0\}$, and if $S^{-1}\|\beta_{ij}\|S = \|\beta'_{ij}\|$, then $\beta'_{ij} = \lambda_i^{-1}\beta_{ij}\lambda_j$ $(0 \leqq i, j \leqq n)$.*

(iii) *If $S = \operatorname{diag}(1, \lambda, \lambda^2, \ldots, \lambda^n)$, and if $S^{-1}\|\beta_{ij}\|S = \|\beta'_{ij}\|$, then $\beta'_{\varkappa, j+\varkappa} = \beta_{\varkappa, j+\varkappa}\lambda^j$ $(\varkappa = 0, \ldots, n-j)$; i.e. $S^{-1}\|\beta_{ij}\|S$ is obtained from $\|\beta_{ij}\|$ by multiplying the j-th layer of $\|\beta_{ij}\|$ by λ^j.*

Remark. The inner automorphism by the diagonal element S in (iii) of the above Lemma 2.4 induces scalar multiplication by λ^{j+1} on $N_j/N_{j+1} = K^{n-j}$. If the ground field K does not contain all the j-th roots of its elements for $2 \leqq j \leqq n$, then t may be impossible to obtain all scalar multiplications on N_j/N_{j+1} from inner automorphisms. The following fact will not be later used. If $d = \operatorname{diag}(1, \lambda, \lambda^2, \ldots, \ldots, \lambda^{j-1}, 1, \lambda, \lambda^2, \ldots, \lambda^{n-j})$ and if $(\beta_{0, j+1}, \ldots, \beta_{n-j-1, n})$ is the $(j+1)$-st layer of $\|\beta_{ij}\|$, then $(\lambda\beta_{0,j+1},\ldots,\lambda\beta_{j-1,2j},\lambda^{j+1}\beta_{j,2j+1},\ldots,\lambda^{j+1}\beta_{n-j-1,n})$ is the $(j+1)$-st layer of $d^{-1}\|\beta_{ij}\|d$.

Lemma 2. 5. *Let $P=\|a_{ij}\|$ $(0\leqq i,j\leqq n)$ be any matrix of the form $P=\begin{vmatrix} B & C \\ 0 & D\end{vmatrix}$ where B is $(j+1)\times(j+1)$, C is $(j+1)\times(n-j)$, D is $(n-j)\times(n-j)$, and where B and D have inverses B^{-1}, D^{-1}. For $\lambda\in K$ set $d(j,\lambda)=E_{00}+\ldots+E_{jj}+ +\lambda(E_{j+1,j+1}+\ldots+E_{nn})$. Then:*

(i) $P^{-1}=\begin{vmatrix} B^{-1} & -B^{-1}CD^{-1} \\ 0 & D^{-1}\end{vmatrix}$,

(ii) $d(j,\lambda)^{-1}Pd(j,\lambda)=\begin{vmatrix} B & \lambda C \\ 0 & D\end{vmatrix}$,

(iii) $[P,d(j,\lambda)]=P^{-1}d(j,\lambda)^{-1}Pd(j,\lambda)=\begin{vmatrix} I & (\lambda-1)B^{-1}C \\ 0 & I\end{vmatrix}$.

(iv) *If B has ones on the diagonal, zeroes below and if the last t rows of B are those of the identity matrix, then the last t rows of $B^{-1}C$ are those of C.*

Proof. Conclusions (i), (ii), and (iii) are immediate, while (iv) is a consequence of (i) with D an $t\times t$ matrix.

Lemma 2. 6. *For any subgroup $W\lhd G^1$, if for some j, $N_j\cap W$ contains an element $\|a_{\alpha\beta}\|$ with $a_{p\varkappa}\neq 0$ for p and $\varkappa$ satisfying $\varkappa-p-1=j$, then there is an element $\|a''_{ij}\|\in \in A_{p\varkappa}\cap W$ with $a''_{p\varkappa}\neq 0$.*

Proof. Applying the previous Lemma 2. 5 with $j=\varkappa-1$, we get a matrix $P=\|a'_{\alpha\beta}\|$ as follows

$$P_1=[P,d(\varkappa-1,\lambda)]=\begin{vmatrix} I & (\lambda-1)B^{-1}C \\ 0 & I\end{vmatrix},\qquad a'_{p\varkappa}=(\lambda-1)\,a_{p\varkappa}.$$

Due to the fact that from the p-th row on (and including the p-th row) the entries of B are those of the identity matrix, also the matrices $B^{-1}C$ and C agree in the p-th and all subsequent rows. Lemma 2. 5 will be applied a second time to P_1 with $j=p$; in the decomposition

$$P_1=\begin{vmatrix} B_1 & C_1 \\ 0 & D_1\end{vmatrix},\qquad C_1 \text{ is } (p+1)\times(n-p);$$

$B_1=I$ and columns $p+1$ to $\varkappa-1$ inclusive of C_1 are zero. For $\lambda\in K$ let $P_2=\|a''_{ij}\|\in A_{p\varkappa}$ be defined by

$$P_2=[P_1,d(p,\lambda_1)]=\begin{vmatrix} I & (\lambda_1-1)C_1 \\ 0 & I\end{vmatrix},\qquad a''_{p\varkappa}=(\lambda_1-1)(\lambda-1)\,a_{p\varkappa}.$$

The p-th row of P_2 from the $\varkappa$-th column on inclusive is that of P multiplied by $(\lambda_1-1)(\lambda-1)$. (In fact, the same is true for the rows $p, \ldots, \varkappa-1$.)

The proof of the next lemma follows from Lemma 2. 4. It should be observed that the next two lemmas require the use of inner automorphisms from N but not from G^1.

Lemma 2. 7. *If $c \in K$ and $P=\|a_{\alpha\beta}\| \in A_{p\varkappa}$ are arbitrary then:*

(i) *The inner automorphism by $I+cE_{ip}$ $(0 \leqq i \leqq p-1)$ subtracts c times the p-th row from the i-th (with the exception that the (i, p) entry remains unchanged).*

(ii) *The inner automorphism by $I+cE_{\varkappa j}$, $\varkappa+1 \leqq j \leqq n$, adds c-times the $\varkappa$-th column to the j-th (except for the $(\varkappa, j)$ entry which remains unchanged).*

(iii) *Consequently, if $a_{p\varkappa} \neq 0$, and if $a_1, a_2 \in K$ are any non-zero scalars, then $\|a_{\alpha\beta}\|$ can be transformed by inner automorphisms from N into elements Q, Q_1, and Q_2 of the form*

$$Q = I+a_{p\varkappa}E_{p\varkappa}+T, \quad T = \Sigma\{b_{ij}E_{ij} | i \leqq p-1;\ \varkappa+1 \leqq j\},$$

$$Q_1 = I+a_{p\varkappa}E_{p\varkappa}+a_1E_{p,\varkappa+1}+T, \quad Q_2 = I+a_{p\varkappa}E_{p\varkappa}+a_2E_{p-1,\varkappa}+T.$$

Lemma 2. 8. *If $\|a_{\alpha\beta}\| \in A_{p\varkappa}$ is an element with $a_{p\varkappa} \neq 0$, then the normal subgroup of N generated by $\|a_{\alpha\beta}\|$ is precisely $A_{p\varkappa}$.*

Proof. It suffices to show that the subgroup of N generated by $\|a_{\alpha\beta}\|$ contains all elements of the form $I+cE_{ij}$, where $0 \neq c \in K$ and $i \leqq \varkappa, p \leqq j$. By application of inner automorphisms from N, $\|a_{\alpha\beta}\|$ can be transformed into elements Q, Q_1, and Q_2 as in the last Lemma 2. 7. But then

$$Q^{-1}Q_1 = I+a_1E_{p,\varkappa+1}, \quad Q^{-1}Q_2 = I+a_2E_{p-1,\varkappa} \qquad (a_1, a_2 \in K;\ a_1 \neq 0,\ a_2 \neq 0).$$

It is now clear that by a finite number of applications of the above process, the element $I+cE_{ij}$ can be obtained.

The previous lemmas imply the next proposition. It is false if the hypothesis that $W \lhd G^1$ is weakened to $W \lhd N$. (See Figure 2.)

Proposition 2. 9. *For a subgroup $W \lhd G^1$, if for some $j=0, \ldots, n-1$, the group $N_j \cap W$ contains an element $\|a_{\alpha\beta}\|$ with $a_{p\varkappa} \neq 0$ for p and $\varkappa$ satisfying $\varkappa-p-1=j$, then $A_{p\varkappa} \subseteqq W$. In particular, if $\|a_{\alpha\beta}\| \in W$ and if for some i and j, $a_{ij} \neq 0$, then $I+cE_{ij} \in W$ for all $c \in K$.*

Remark. The previous Proposition 2. 9 has the following interesting consequence. Suppose $W \subseteqq N$, $W \lhd G^1$, and $\|a_{\alpha\beta}\| \in W$. If $\|b_{\alpha\beta}\|$ is obtained from $\|a_{\alpha\beta}\|$ by replacing all $a_{\alpha\beta} \neq 0$, $\alpha < \beta$, with arbitrary scalars $b_{\alpha\beta}$, then also $\|b_{\alpha\beta}\| \in W$.

Corollary 2. 10. *If $F\colon G^1 \to G^1$ is any automorphism, then either $F(B^1)=B^1$ or $F(B^1)=B_1$.*

Proof. If $F(B^1)\nsubseteq \Gamma$, let j be the smallest integer such that $N_j \cap F(B^1)$ contains an element $\|a_{\alpha\beta}\|$ with $a_{p\varkappa}\neq 0$, $\varkappa-p-1=j$, and with $p\neq 0$, $\varkappa\neq n$. Then the second center of N is $N_{n-2}\subseteq A_{p\varkappa}\subseteq F(B^1)$. Thus the nilpotency class of $G^1/F(B^1)\leqq \leqq n-2$, whereas class of G^1/B^1 is $n-1$. This is a contradiction, since $G^1/B^1\cong \cong G^1/F(B^1)$. Thus $F(B^1)\subseteq\Gamma$. Since F induces an automorphism on N_0/N_1, there is an $\|a_{\alpha\beta}\|\in F(B^1)$ with either $a_{01}\neq 0$ or $a_{n-1,n}\neq 0$. Thus either $B^1\subseteq F(B^1)$ or $B_1\subseteq F(B^1)$. If $B^1\subseteq F(B^1)$, but $B^1\neq F(B^1)$, then by the last Proposition 2.9, $F(B^1)$ would have to contain a group of the form A_{pn} with $1\leqq p$. However, then $F(B^1)$ would not be abelian. A similar argument applies if $B_1\subseteq F(B^1)$ and $B_1\neq F(B^1)$.

The last step in determining the group of automorphisms of G^1 is to construct an automorphism $\sigma\colon G^1\to G^1$ such that $\sigma(B^1)=B_1$.

2.11. Homomorphisms of semi-direct products. The group B_1 is embedded in a semi-direct product $K\times B_1 \lhd G^1$ consisting of all (λ, b) of the form:

$$(\lambda, b)=\begin{vmatrix} I & b \\ 0 & \lambda \end{vmatrix}, \quad b=\begin{vmatrix} b_0 \\ \vdots \\ b_{n-1} \end{vmatrix}\in K^n$$

$$\left(0\neq\lambda\in K;\quad (\lambda,b)(\lambda',b')=(\lambda\lambda', b\lambda'+b');\quad (\lambda',b')\in K\times B_1\right).$$

Similarly B^1 is embedded in another semi-direct product $K\times B^1\lhd G^1$ consisting of all $[\lambda, a]$

$$[\lambda, a]=\operatorname{diag}(1,\lambda,\dots,\lambda)\begin{vmatrix} 1 & a \\ 0 & I \end{vmatrix}$$

$$\left(a=(a_1,\dots,a_n)\in K^n;\quad 0\neq\lambda\in K;\quad [\lambda,a][\lambda',a']=[\lambda\lambda', a'+\lambda' a],\qquad [\lambda',a']\in K\times B^1\right).$$

The map $K\times B_1\to K\times B^1$, $(\lambda, b)\to[\lambda, b]$ is an isomorphism.

The group G is a direct product $G=K\oplus G^1$. Define a map $p\colon G\to G^1$ by:

$$g=\|a_{ij}\|\in G,\ p(g)=\|a_{ij}a_{00}^{-1}\|=a_{00}^{-1}I\|a_{ij}\|.$$

Note that both G^1 and G are semi-direct products $G=D\times N$ and $G^1=D^1\times N$.

Definition 2.12. An anti-automorphism $\tau'\colon G\to G$ is defined by transposing around the second diagonal, i.e. by

$$g=\|a_{ij}\|\in G,\ \tau'(g)=\|b_{ij}\|;\ b_{ij}=a_{n-i,n-j}$$

$(0\leqq i, j\leqq n)$. An automorphism $\tau\colon G\to G$ is defined by $\tau(g)=\tau'(g^{-1})$ for $g\in G$. Thirdly, by use of the map p of 2.11, a map $\sigma\colon G\to G^1$ is defined by $\sigma(g)=p[\sigma(g)]$, $g\in G$.

The matrix with zeroes everywhere but ones on the second diagonal will be denoted by P. A superscript t denotes the transpose of a matrix; $-t$ denotes the inverse of the transpose.

Remarks. 1. It is asserted that $\tau'(g)=Pg^tP$, $\tau(g)=Pg^{-t}P$, $\tau'(g^{-1})=(\tau'(g))^{-1}$ $(g\in G)$. For any matrix g, the matrix Pg is obtained simply by rotating g hundred eighty degrees about its horizontal axis of symmetry. Similarly, gP is obtained by rotating g hundred eighty degrees about its vertical axis of symmetry. Thus PgP is obtained by transposing g about both of its diagonals, the order being immaterial. Thus $\tau'(g)=Pg^tP$. The other two equations follow from the fact that $P^2=I$.

2. Since $G=D\times N$ and $G^1=D^1\times N$ are semi-direct products, for $g=dn$; $n\in N$, $d=\text{diag}\,(\lambda_0, \ldots, \lambda_n)$, $\lambda_i\in K$; $\sigma(g)=p[\tau(d)]\tau(n)$, where $p[\tau(d)]=\text{diag}\,(1, \lambda_n\lambda_{n-1}^{-1}\ldots, \lambda_n\lambda_0^{-1})$. Thus all elements of the form diag $(1, \lambda, \lambda^2, \ldots, \lambda^n)$ $(\lambda\in K)$ are left invariant by σ.

The main properties of the map σ are given by the next proposition.

Proposition 2. 13. *Let the notation be as in* 2. 11 *and* 2. 12. *Then:*

(i) *For any* $g=\|a_{ij}\|\in G$, $\sigma(g)=(a_{nn}I)Pg^{-t}P$.

(ii) *The restriction* $\sigma|G^1\colon G^1\to G^1$ *is an automorphism.*

(iii) $\sigma(K\times B^1)=K\times B_1$.

(iv) *There does not exist an* $M\in Gl(n+1; K)$ *such that for all* $h\in G^1$,

$$\sigma(h)=M^{-1}hM \quad or \quad \sigma(h)=M^{-1}h^{-t}M.$$

Proof. Conclusions (i)—(iii) are easily verified using the formula for σ given in Remark 2 above. For $g=\text{diag}\,(1, \lambda_1, \ldots, \lambda_n)n\in G^1$, with $n\in N$, determinant $g=$ $=(\lambda_1\ldots\lambda_{n-1})^{-1}\lambda_n^n$. Thus (iv) follows.

Finally, we are in a position to combine Propositions 2. 2, 2. 9, and 2. 13 to find all automorphisms of G^{+1}.

Theorem I. *Let K be any ordered field and* $G^{+1}=A^+\times V$; $\bar F_i$, $i=0, \ldots, n$; $\mathscr{F}$, $\bar\mu$, U, *and J as in Definition* 2. 1 *and as in* 2. 12. *Then:*

(i) $\sigma^2=1$; $\mathscr{F}$ *is abelian*; $\bar F_0=\bar F_1\ldots\bar F_n$; $\bar F_i^2=1$; $\sigma^{-1}\bar F_i\sigma=\bar F_{n-i}$ $(i=0, \ldots, n)$.

(ii) *The following subgroups of* Aut $A^+\times V$ *are semi-direct products:*

$$\{\sigma\}\times\mathscr{F},\ \{\sigma\}\times J,\ \mathscr{F}\times J,\ \{\sigma\}\times\mathscr{F}J;$$

$$\text{Aut}\,A^+\times V=U\oplus\{\sigma\}\times[\mathscr{F}\times J].$$

In particular,

$$\frac{\text{Aut}\,A^+\times V}{J}\cong U\oplus\{\sigma\}\times\mathscr{F}.$$

Proof. Conclusion (i) is clear. (i) Since $F_iF_j=F_jF_i$, $0\leqq i, j\leqq n$, and since $F_0=-F_1\ldots F_n$, it follows that $\mathscr{F}$ is abelian and that $\bar F_0=\bar F_1\ldots\bar F_n$. The geometric characterization of PF_i and F_iP shows that $PF_i=F_{n-i}P$, $i=0, \ldots, n$. For $g=\|a_{ij}\|\in G^{+1}_{1}$,

$$\bar F_i[\sigma(g)]=F_i(a_{nn}I)(Pg^{-t}P)F_i=(a_{nn}I)P(F_{n-i}gF_{n-i})^{-t}P=\sigma[\bar F_{n-i}(g)].$$

Thus $\sigma^{-1}\bar F_i\sigma=\bar F_{n-i}$, $i=0, \ldots, n-i$. Thus (i) follows; it immediately implies (ii).

The situation becomes somewhat simpler if the group A^+ of the last theorem is replaced by A as is done in the next corollary. Its proof is an immediate consequence of the remark following Proposition 2. 2.

Corollary 2. 14. *For any field K, let $G^1 = A \times V$, $\bar{\mu}$, and σ be as in* 2. 1 *and* 2. 12. *Let $\bar{U} \subseteqq \text{Aut}\, A \times V$ be the subgroup generated by all $\bar{\mu}$; $\bar{J}$ denotes the group of inner automorphisms of $A \times V$. Then:*

(i) $\sigma^2 = 1$,

(ii) $\{\sigma\} \times \bar{J}$ *is a semi-direct product*; $\text{Aut}\, A \times V = \bar{u} \oplus \{\sigma\} \times \bar{J}$.

3. Subgroups whose images in the quotients of a central series are one dimensional

In the notation of the previous section

$$N = N_0 \supset N_1 \supset \dots \supset N_{n-1} \supset N_n = 1$$

is both the descending and ascending central series of N; $N_j/N_{j-1} \cong K^{n-j}$, $j = 0, \dots, n-1$. The objective of this section is to characterize all normal subgroups $W \subseteqq N$ such that each $(W \cap N_j)N_{j+1}/N_{j+1}$ is a one dimensional K-vector space.

Definition 3. 1. A subgroup $W \subseteqq N$ is called *K-linear*, if it has the property that for any $k \in K$ and $P \in W$, if Pk is the matrix obtained by multiplying every non-diagonal entry of P by k, then $Pk \in W$. For any subgroup $W \subseteqq N$, (which is not assumed to be K-linear) the *rank* of $(W \cap N_j)N_{j+1}/N_{j+1}, j = 0, \dots, n-1$, is defined as the number of linearly independent elements over K that it contains.

Example. Let $n = 4$ and let $a, b, c, d, e \in K$ be arbitrary constants. The most general abelian subgroup M of N consists of all (x, y, z) where for any $x, y, z \in K$,

$$(x, y, z) = \begin{vmatrix} 1 & ax & ay+dx & z \\ 0 & 1 & cx & by+ex \\ 0 & 0 & 1 & bx \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

Since two elements (x, y, z) and $(x', y', z') \in M$ multiply according to the rule

$$(x, y, z)(x', y', z') = (x+x', y+y'+cxx', z''),$$

(where $z'' = z + z' + ab(xy' + x'y) + (ae + db)xx'$), it follows that for $0 \neq k \in K$, the map $M \to M$, $P \to Pk$ is not a homomorphism. However, this map is in general for any n a homomorphism for all the normal abelian subgroups of N considered here.

Lemma 3. 2. (i) *If $W \subseteqq N$ is a subgroup such that $(W \cap N_j)N_{j+1}/N_{j+1}$ is a K-vector space for all $j = 0, \dots, n-1$, then W is K-linear.*

(ii) *Let n be the nilpotency class of G^1 and suppose that K contains the j-th roots of all of its elements for $2 \leqq j \leqq n$. Then if W is a normal subgroup of G^1 and $W \subset N$, then W is K-linear.*

Proof. Conclusion (i) is easily proved; (ii) follows from (i) and Lemma 2. 4 (iii). The subgroup W in the next lemma need not be K-linear.

Lemma 3. 3. *Suppose $W \subseteqq N$ is any subgroup such that:*

(a) *W is invariant under inner automorphisms from $\Gamma^1\Gamma_1$;*

(b) *$(W \cap N_j)N_{j+1}/N_{j+1}$ is of rank at most one for all $j=0, \ldots, n-1$.*
Then $W \subseteqq \Gamma$.

Proof. For $n=1$, $N=\Gamma$. Assuming the lemma to be true for $1, \ldots, n-1$, it will be proved for groups $W \subset N$ of $(n+1)\times(n+1)$ matrices. Replacing N by N/B_1 and W by WB_1/B_1 and using induction, we obtain that $WB_1/B_1 \subseteqq \Gamma/B_1$. Thus $W \subseteqq \Gamma_1$. Similarly, $W \subseteqq \Gamma^1$. Thus $W \subseteqq \Gamma_1 \cap \Gamma^1$. Suppose there is a $g \in W$ of the form

$$g = \begin{vmatrix} 1 & a & a_{n-1} & z \\ & & s & b_1 \\ & & & b \\ & & & 1 \end{vmatrix} \qquad a=(a_1, \ldots, a_{n-2}), \quad b = \begin{vmatrix} b_2 \\ \vdots \\ b_{n-1} \end{vmatrix} \qquad (a_j, b_j \in K; \quad 0 \neq s \in K).$$

For arbitrary $c, k \in K$ we have

$$(I-kE_{n-1,n})(I-cE_{01})g(I+cE_{01})(I+kE_{n-1,n}) = \begin{vmatrix} 1 & a & a_{n-1}-cs & z_0 \\ & & & b_1+ks \\ & & & b \\ & & & 1 \end{vmatrix}$$

(where $z_0 = z - cb_1 + ka_{n-1}$). Since c and k are arbitrary, there are elements $h, f \in W$ of the form

$$h = \begin{vmatrix} 1 & a & 0 & z_1 \\ & & s & b_1 \\ & & & b \\ & & & 1 \end{vmatrix} \qquad f = \begin{vmatrix} 1 & a & a_{n-1} & z_2 \\ & & & 0 \\ & & & b \\ & & & 1 \end{vmatrix}$$

$$(b_1, a_{n-1}, z_1, z_2 \in K; \quad b_1 \neq 0, a_{n-1} \neq 0).$$

Then

$$g^{-1} = \begin{vmatrix} 1 & -a & (a_1 s - a_{n-1}) & z_3 \\ & & -s & (s\, b_{n-1} - b_1) \\ & & & -b \\ & & & 1 \end{vmatrix}$$

(where $z_3 = -z-(a_1b_1+\dots+a_{n-1}b_{n-1})-a_1sb_{n-1}$), and

$$g^{-1}h = \begin{vmatrix} 1 & 0 & 0 & z_4 \\ & & 0 & -b_1 \\ & & & 0 \\ & & & 1 \end{vmatrix} \qquad g^{-1}f = \begin{vmatrix} 1 & 0 & -a_{n-1} & z_5 \\ & & & 0 \\ & & & 0 \\ & & & 1 \end{vmatrix}$$

for some $z_4, z_5 \in K$. Thus $s=0$ and $W \subseteqq \Gamma$.

Notation 3. 4. Consider a subgroup W with $Z \subseteqq W \subseteqq \Gamma$. Its elements will be written as $[a; z; b]$, where

$$[a; z; b] = \begin{vmatrix} 1 & a & z \\ 0 & I & b \\ 0 & 0 & 1 \end{vmatrix} \qquad \left(a = (a_1, \dots, a_{n-1}); \quad b = \begin{vmatrix} b_1 \\ \vdots \\ b_{n-1} \end{vmatrix}; \quad a_j, b_i \in K \right).$$

Then the elements of W/Z and B^1/Z are canonically of the form $[a; 0; b]$ and $[a; 0; 0]$. There is a homomorphism $\pi^1\colon W/Z \to B^1/Z$ defined by $\pi^1([a; 0; b]) = [a; 0; 0]$. Similar remarks apply to π_1 and B_1/Z. Note that

$$[a; z; b]^{-1} = [-a; a \cdot b - z; -b], \qquad a \cdot b = a_1b_1 + \dots + a_{n-1}b_{n-1},$$

$$[a; 0; b]^{-1} = [-a; 0; -b] \quad \text{in} \quad B^1/Z.$$

Note that hypothesis (b) of the next lemma implies that $Z \subseteqq W$.

Lemma 3. 5. *Suppose the subgroup $W \subseteqq N$ satisfies:*

(a) *W is invariant under inner automorphisms from $\Gamma^1\Gamma_1$;*

(b) *$(W \cap N_j)N_{j+1}/N_{j+1}$ is a one dimensional K-vector space, $j=0, \dots, n-1$.*

(c) *There exists $[a; z: b] \in W$ with $a_1 \neq 0$.*

Then:

(i) *π^1 is a bijective isomorphism.*

(ii) *There exist linear functionals $f_i\colon K^{n-1} \to K$, $i=1, \dots, n-1$, such that every element of W is of the form*

$$[t; z; f(t)] = \begin{vmatrix} 1 & t & z \\ 0 & I & f(t) \\ 0 & 0 & 1 \end{vmatrix}$$

$$\left(t = (t_1, \dots, t_{n-1}) \in K^{n-1}; \quad f(t) = \begin{vmatrix} f_1(t) \\ f_{n-1}(t) \end{vmatrix} \in K^{n-1}; \quad z \in K \right).$$

Proof. (i) In order to show that π^1 is surjective, it suffices to show that for any q in $2 \leqq q \leqq n-1$, there is an element $w \in W$ of the form

$$w = [(0, \dots, 0, c_q, 0, \dots, 0); z; b] \qquad b \in K^{n-1}, z \in K$$

for some $0 \neq c_q \in K$. For any $c \in K$, the element $(I+ca_1^{-1}E_{1q})^{-1}[a;z;b](I+c_qa_1E_{1q})$ has the same entries as $[a; z; b]$, except in positions $(0, q)$ and $(1, n)$, where $(0, q)$ entry $=a_q+c$ $(q=2, ..., n-1)$ and $(1, n)$ entry $=b_1-ca_1^{-1}b_q$. Thus there are $u, u^{-1}, r, ru^{-1} \in W$ of the form

$$u=[(a_1, 0, ..., 0); 0; (b_1', b_2, ..., b_{n-1})],$$
$$u^{-1}=[(-a_1, 0, ..., 0); a_1b_1'; (-b_1, -b_2, ..., -b_{n-1})],$$
$$r=[(a_1, 0, ... 0, c_q, 0, ..., 0); 0; (b_1'', b_2, ..., b_{n-1})],$$
$$ru^{-1}=[(0, ... 0, c_q, 0, ..., 0); z; (\bar{b}_1, 0, ..., 0)],$$

where $0 \neq c_q \in K$ and where, in fact, b_1', b_1'', $\bar{b}_1$, and z are

$$b_1'=b_1+a_1^{-1}(a_1b_1+...+a_nb_n), \qquad b_1''=b_1'-c_qa_1^{-1}b_q,$$
$$\bar{b}_1=-b_1''+b_1=-c_qa_1^{-1}b_q, \; z=-c_qb_q.$$

Since kernel $\pi^1=\{[0; 0; b] | b \in K^{n-1}\}$, the hypothesis (b) with $\pi^1(W)=B^1/Z$ imply that π^1 is a bijective isomorphism.

(ii) Since an arbitrary element $[t; 0; b] \in W/Z$ with $t, b \in K^{n-1}$ is uniquely determined by its first component t, the functions $f_1, ..., f_{n-1}$: $K^{n-1} \to K$ are uniquely defined by setting $(f_1(t), ..., f_{n-1}(t))=b$. Let f: $K^{n-1} \to K^{n-1}$ be the map $f(t)= =(f_1(t), ..., f_{n-1}(t))$. Since for any $t, t' \in K^{n-1}$, $[t; 0; f(t)][t'; 0; f(t')] = =[t+t'; 0; f(t)+f(t')] = [t+t'; 0; f(t+t')]$, we have $f(t+t')=f(t)+f(t')$. Since by assumption (b) and Lemma 3. 2 (i) the group W is K-linear, it follows that for any $c \in K$ and any $[t; 0; f(t)] \in W/Z$, $[ct; 0; cf(t)] \in W/Z$. But $[ct; 0; cf(t)] = =[ct; 0; f(ct)]$; thus $f(ct) = cf(t)$ and the f_i are K-linear functionals f_i: $K^{n-1} \to K$.

Remark. The assumption (b) of the last Lemma 3. 5 in conjunction with Lemma 3. 3 implies that $W \subseteqq \Gamma$. Assumption (b) of the last Lemma 3. 5 guarantees that there is an element $[a; z; b] \in W$ with either $a_1 \neq 0$ or $b_1 \neq 0$. Thus hypothesis (c) is no real restriction but merely a notational convenience.

Lemma 3. 6. *Assume that the subgroup* $W \subseteqq N$ *satisfies* (b) *and* (c) *of the previous Lemma* 3. 5 *and that in addition* $W \triangleleft N$. *Then:*

(i) *There are* $\alpha, \beta \in K$ *such that every element of* W *is of the form*
$[t; z; (\alpha t_1+\beta t_2, -\beta t_1, 0, ..., 0)]$ $(t_1, ..., t_{n-1}) \in K^{n-1}$ $(z \in K)$;

(ii) *If characteristic* $K=2$, W *is abelian.*

(iii) *If characteristic* $K \neq 2$, *and if in addition* W *is abelian, then* $\beta=0$ *and* $W=B^1(\alpha)$.

Proof. (i) Let $[t; z; f(t)] \in W$ be an arbitrary element with $t \in K^{n-1}$ and $z \in K$ For $c \in K$ and any indices i and r satisfying $0 \leqq i-r<i \leqq n-1$, let $t \in K^{n-1}$, $z' \in K$ be defined by

(1) $$[t'; z'; f(t')] = (I-cE_{i-r,i})[t; z; f(t)](I+cE_{i-r,i}).$$

The inner automorphism has only changed the $(0, i)$, $(i-r, n)$, and $(0, n)$ entries in the manner indicated in the following diagram:

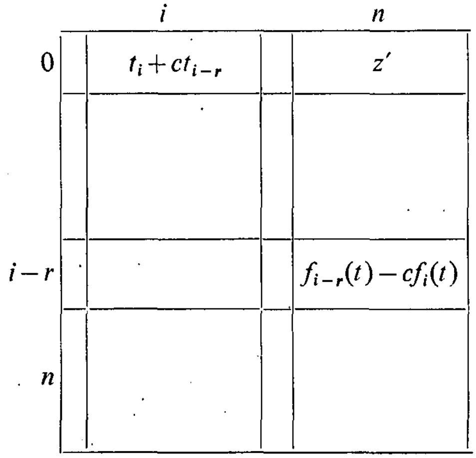

Figure 3.

Thus $t'_j=t_j$ for $j\neq i$ and $t_i=t_i+ct_{i-r}$. Then equation (1) shows that

$$f_{i-r}(t')=f_{i-r}(t)-cf_i(t)$$

which in turn implies that

$$f_i(t) = -f_{i-r}(0, \ldots, 0, t_{i-r}, 0, \ldots, 0) \qquad (0\leqq i-r<i\leqq n),$$

where t_{i-r} is in the i-th position. Suppose $i\geqq 3$; for $r=1, 2$ the above becomes

$$(2) \qquad r = 1\colon f_i(t) = -f_{i-1}(0, \ldots, 0, t_{i-1}, 0, \ldots, 0),$$

$$r = 2\colon f_i(t) = -f_{i-2}(0, \ldots, 0, t_{i-2}, 0, \ldots, 0).$$

where t_{i-1} and t_{i-2} are in the i-th position. Since for arbitrary $t_{i-1}, t_{i-2}\in K$, $f_{i-1}(0, \ldots, 0, t_{i-1}, 0, \ldots, 0)=f_{i-2}(0, \ldots, 0, t_{i-2}, 0, \ldots, 0)$, it follows that both of these are identically zero for all choices of $t_{i-1}, t_{i-2}\in K$; consequently $f_i\equiv 0$ for $i\geqq 3$. The equation $f_k(t')=f_k(t)$, $k\neq i-r$ implies that

$$f_k(0, \ldots, 0, t_{i-r}, 0, \ldots, 0)=0 \qquad (k\neq i-r,\ 1\leqq k\leqq n),$$

where the t_{i-r} is in the i-th position. Take a fixed k, $1\leqq k\leqq n$ and $r=1$; then the above equation holds for all i except $i=k+1$ and $i=1$. Since f_k is linear and $t_{i-r}\in K$ is arbitrary, this implies that

$$f_k(t)=f_k(t_1, \ldots, t_{n-1})=f_k(t_1, 0, \ldots, 0, t_{k+1}, 0, \ldots, 0),$$

where the t_{k+1} is in the $(k+1)$-st position. In particular, for $k=1$ there are $\alpha, \beta\in K$ such that $f_1(t) = \alpha t_1+\beta t_2$. But now equation (2) with $i=2$ becomes $f_2(t) = -f_1(0, t_1, 0, \dots, 0) = -\beta t_1$. Thus (i) has been proved.

(ii) and (iii) If $[t; z; f(t)]$ and $[t'; z'; f(t')]$ are arbitrary elements of W, then a necessary and sufficient condition that they commute is that $2\beta(t_1t_2'-t_1't_2)=0$.

Corollary 3. 7. *If $K_0, K_1, \dots, K_n$ are any multiplicative subgroups of $K\setminus\{0\}$, let $G(K_0, K_1, \dots, K_n)$ denote the subgroup of G consisting of all matrices $\|a_{ij}\|$ with $a_{ii}\in K_i$ $(i=0, 1, \dots, n)$. Suppose the subgroup $W\lhd N$ satisfies the hypotheses of the last Lemma* 3. 6. *Then $\alpha=\beta=0$ and $W=B^1$ if either one of the following two conditions hold:*

(i) $K_0=K_1=\dots=K_{n-1}=\{1\}$, $K_n\neq\{1\}$;

(ii) $K_0=K_2=\dots=K_n=\{1\}$; $K_1\neq\{1\}, \{1, -1\}$.

In particular, if $W\lhd G^1$, then $W=B^1$.

Proof. Let $d(\varkappa)=\operatorname{diag}(1, \dots, 1, \lambda, 1, \dots, 1)$ where $0\neq\lambda\in K$ is located in the $\varkappa$-th row and column. Let $w=[t; z; (\alpha t_1+\beta t_2, -\beta t_1, 0, \dots, 0)]\in W$, where $t\in K^{n-1}$, $z\in K$. Then

$$d(n)^{-1}wd(n) = [t; \lambda z; (\lambda\alpha t_1+\lambda\beta t_2, -\beta t_1, 0, \dots, 0)],$$

$$d(1)^{-1}wd(1) = [(\lambda t_1, t_2, \dots, t_n); z(\lambda^{-1}\alpha t_1+\lambda^{-1}\beta t_2, -\beta t_1, 0, \dots, 0)].$$

Thus $\alpha t_1+\beta t_2=\lambda(\alpha t_1+\beta t_2)$, $\lambda\neq 1$ for all $t_1; t_2\in K$ implies that $\alpha=\beta=0$. Similarly, $\lambda(\alpha t_1+\beta t_2)=\lambda^{-1}(\alpha t_1+\beta t_2)$ and $\lambda^2-1\neq 0$ also implies that $\alpha=\beta=0$.

The next lemma is proved by tedious but straightforward computations; its proof is omitted.

Lemma 3. 8. *For any constant $\alpha\in K$, $B^1(\alpha)$ and $B_1(\alpha)$ are maximal normal abelian subgroups of N; B^1 and B_1 are maximal normal abelian subgroups of G^1.*

The results of this section are summarized in the next theorem.

Theorem II. *Let the notation be as in* 2. 1 *and* 3. 4. *Suppose $W\subseteqq N$ is a subgroup satisfying:*

(a) $W\lhd N$, (b) $(W\cap N_j)N_{j+1}/N_{j+1}$ *is a one dimensional K-vector space for each $j=0, \dots, n-1$. Then $W\subseteqq\Gamma$ and consequently there exists an element $[a; z; b]\in W$ with either $a_1\neq 0$ or $b_1\neq 0$. Assume:*

(c) $[a; z; b]\in W$, $a_1\neq 0$.

Then:

(i) *There exist $\alpha, \beta\in K$ such that W consists of all elements of the form*

$$[t; z; (\alpha t_1+\beta t_2, -\beta t_1, 0, \dots, 0)] \qquad (t\in K^{n-1}, z\in K).$$

(ii) *If characteristic $K=2$, W is abelian.*

(iii) *If characteristic* $K \neq 2$, *and if* W *is abelian, then necessarily* $\beta = 0$ *and* $W = B^1(\alpha)$.

(iv) *If, in addition to* (a), (b), *and* (c), W *also satisfies* (d) $W \lhd G^1$, *then* $W = B^1$. *In particular,* W *is abelian. In the other case when* $b_1 \neq 0$, *the obvious analogues of* (i)—(iv) *hold.*

(v) *For any* $\alpha \in K$ ($\alpha = 0$ *is not excluded),* $B^1(\alpha)$ *and* $B_1(\alpha)$ *are maximal normal abelian groups of* N; B^1 *and* B_1 *are maximal normal abelian subgroups of* G^1.

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Square extensions of finite rings

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Let R and S be rings. We say that a ring T is an *extension of S by R* if S is an ideal in T and T/S is isomorphic to R. Let us call an extension T of S by R a *square extension*, if $S=T^2$, where T^2 is the ideal in T generated by all products of elements of T. Now T/T^2 is a zero-ring, so in order that there exist a square extension of S by R, R must be a zero-ring. Henceforth we assume that R is a zero-ring and moreover that R is a finite ring. On the other hand, if S^2 is the ideal in S generated by all products of elements in S, then S/S^2 is a zero-ring. We assume that S/S^2 is also finite. Our problem is to find necessary and sufficient conditions for the existence of a square extension of S by R. We shall reduce this problem to the case in which the additive group of S is a finite abelian elementary p-group and S is a zero-ring. In Theorem 4 we get the result that there does not exist a split square extension of S by R. Next we get a partial result on the existence of non-split square extensions of S by R (Theorem 5). Finally we determine all rings of order 8, which may occur either as a square extension of a ring of order 4 or as a square extension of a ring of order 2.

First we note that the ideal S^2 of S is an ideal not only in S, but also in every extension of S, since S^2 is a characteristic subring of S.

Theorem 1. *T is a square extension of S by R if and only if T/S^2 is a square extension of S/S^2 by R.*

Proof. From the isomorphism $T/S\cong T/S^2/S/S^2$ it follows that T is an extension of S by R if and only if T/S^2 is an extension of S/S^2 by R. Now suppose $T^2=S$, then $(T/S^2)^2=T^2/S^2=S/S^2$. Conversely, if $S/S^2=(T/S^2)^2$, then $S/S^2=T^2/S^2$ and hence $S=T^2$. This theorem reduces the problem to the case in which S is a finite zero-ring.

If $S=(0)$, then every extension T of S by R is a square extension because R is a zero-ring. Therefore, we assume that S is a non-trivial finite zero-ring. At this

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point we want to summarize the theory of extensions of S by R, where R and S are finite zero-rings. Let T be an extension of S by R, so that $T/S \cong R$. Let φ: $T \to R$ be the epimorphism whose kernel is S. An element $\bar{u}$ of T is called a *representative* of $u \in R$ if $\varphi(\bar{u}) = u$. Let $(z_1, \dots, z_l)$ be a basis of the additive group R^+ of R and let m_i be the order of z_i. An l-tuple $(\bar{z}_1, \dots, \bar{z}_l)$ is called a *representative set* of the basis if each $\bar{z}_i$ is a representative of z_i. As the products $\bar{z}_i a$, $a\bar{z}_i$ $(a \in S)$ are all in S, the mappings $a \to \bar{z}_i a$, $a \to a\bar{z}_i$ are endomorphisms of S^+, which will be denoted by $\eta_l(z_i)$ and $\eta_r(z_i)$ resp. Thus $\eta_l(z_i)a = \bar{z}_i a$ and $a\eta_r(z_i) = a\bar{z}_i$.

It is clear that if we choose another representative of $z_i \in R$, for instance $\bar{z}'_i$, then $\bar{z}'_i a = \bar{z}_i a$ and $a\bar{z}'_i = a\bar{z}_i$, as $\bar{z}'_i = \bar{z}_i$ (mod S) and S is a zero-ring. Hence the induced endomorphisms are completely determined by the element $z_i \in R$. So we get a set of $2l$ endomorphisms of S^+ and we divide them into pairs: $(\eta_l(z_1), \eta_r(z_1))$, $(\eta_l(z_2), \eta_r(z_2))$, $\dots$, $(\eta_l(z_l), \eta_r(z_l))$. Each of these pairs is a double homothetism of S, since S is a zero-ring and the endomorphisms $\eta_l(z_i)$ and $\eta_r(z_i)$ are commuting. As T is an associative ring these double homothetisms are pairwise related (cf. [2]). Now we consider the mapping: $z_i \to \eta(z_i) = (\eta_l(z_i), \eta_r(z_i))$, which associates with each $z_i \in R$ the corresponding double homothetism of S and we extend η by linearity. We claim that η is a homomorphism of R into a maximal ring D of related double homothetisms of S. First we remark that if $\bar{z}_i$ and $\bar{z}_j$ are arbitrary representatives in T then $\bar{z}_i\bar{z}_j \in S$, as $\varphi(\bar{z}_i\bar{z}_j) = \varphi(\bar{z}_i)\varphi(\bar{z}_j) = z_i z_j = 0$. Hence $\bar{z}_i(\bar{z}_j a) = \eta_l(z_i)(\eta_l(z_j)a) = 0$ for all $a \in S$. This implies $\eta_l(z_i)\eta_l(z_j) =$ zero-endomorphism for all $z_i, z_j \in R$. In the same way it can be shown that $\eta_r(z_i)\eta_r(z_j) =$ zero-endomorphism for all $z_i, z_j \in R$. As the product of the double homothetisms $(\eta_l(z_i), \eta_r(z_i))(\eta_l(z_j), \eta_r(z_j)) = (\eta_1(z_i)\eta_l(z_j), \eta_r(z_i)\eta_r(z_j)) = (0, 0)$ in D, it follows that the mapping η maps R homomorphically into a ring D; the homomorphic image $\eta(R)$ is a zero-subring of a maximal ring of related double homothetisms of S. As we saw earlier each product $\bar{z}_i\bar{z}_j \in S$; we define $\bar{z}_i\bar{z}_j = \{z_i, z_j\}$ for all i, j with $1 \leq i \leq l$, $1 \leq j \leq l$; the elements $\{z_i, z_j\}$ are called a *multiplicative factor set*. Finally we know that $m_i\bar{z}_i \in S$, as $\varphi(m_i\bar{z}_i) = m_i z_i = 0$. So we get another set of elements $m_i\bar{z}_i = b_i$ in S.

It is easy to check that the homomorphism η, the multiplicative factor set $\{z_i, z_j\}$ and the set $\{b_i\}$ have the following properties:

(1) $\{z_i, 0\} = \{0, z_j\} = 0$, if 0 is a representative of $0 \in R$.

(2) $\eta_l(z_i)\{z_j, z_k\} = \{z_i, z_j\}\eta_r(z_k)$,

(3) $(b_i)\eta_r(z_j) = m_i\{z_i, z_j\}$,

(4) $\eta_l(z_j)(b_i) = m_i\{z_j, z_i\}$, for all $z_i, z_j, z_k \in R$, $b_i \in S$, m_i as integers.

Hence given an extension T of S by R, T determines with the representative set $(\bar{z}_1, \dots, \bar{z}_l)$ a homomorphism η of R into a maximal ring of related double homothetisms of S, a multiplicative factor set $\{z_i, z_k\}$ and a set $\{b_i\}$ $(b_i \in S)$, such that the properties (1)—(4) are satisfied.

Conversely, assume that R and S are given finite zero-rings and that $\eta: R \to D$ is a given homomorphism of R into a maximal ring D of related double homothetisms of S. Let the functions $\{z_i, z_j\}$ of $R \times R$ into S and the set $\{b_i\}$ $(b_i \in S)$ be given for all i, j with $1 \leqq i \leqq l$, $1 \leqq j \leqq l$, such that (1)—(4) hold. Consider the set of all symbols $\sum_{i=1}^{l} n_i \bar{z}_i + s$, $0 \leqq n_i < m_i$, $s \in S$. Define equality by: $\sum_{i=1}^{l} n_i \bar{z}_i + s = \sum_{i=1}^{l} u_i \bar{z}_i + v$ if and only if $n_i = u_i$ for all i and $s = v$. Define addition by: $\left(\sum_{i=1}^{l} n_i \bar{z}_i + s\right) + \left(\sum_{i=1}^{l} u_i \bar{z}_i + v\right) = \sum_{i=1}^{i} (n_i + u_i)\bar{z}_i + s + v$, where $m_i \bar{z}_i = b_i$ and the sum is reduced mod $m_i \bar{z}_i$. Define multiplication by:

$$\left(\sum_{i=1}^{l} n_i \bar{z}_i + s\right)\left(\sum_{i=1}^{l} u_i \bar{z}_i + v\right) = \sum_{i=1}^{l} \sum_{j=1}^{l} n_i u_j \{z_i, z_j\} + \sum_{i=1}^{l} n_i \big(\eta_l(z_i) v\big) + \sum_{j=1}^{l} u_j \big(s \eta_r(z_j)\big).$$

It is easy to check that the set T of all symbols $\sum_{i=1}^{l} n_i \bar{z}_i + s$ with the addition and multiplication just defined is a ring. Now $T^2 \subseteqq S$, hence S is an ideal in T and $T/S \cong R$ under $\sum_{i=1}^{l} n_i \bar{z}_i + S \to \sum_{i=1}^{l} n_i z_i$. Further $\bar{z}_i v = \eta_l(z_i) v \in S$, $v \bar{z}_i = v \eta_r(z_i) \in S$ for all $v \in S$, hence the double homothetisms $\eta(z_i) = \big(\eta_l(z_i), \eta_r(z_i)\big)$ of S are induced by inner double homothetisms $(\bar{z}_{i_l}, \bar{z}_{i_r})$ of T. So T is an extension of S by R which, with the representative set $\bar{z}_i$, induces the given homomorphism η. Since $\bar{z}_i \bar{z}_j = \{z_i, z_j\}$ for all i, j and $m_i \bar{z}_i = b_i$ for all i, T has, with the same representative set $\bar{z}_i$, the multiplicative factor set $\{z_i, z_j\}$ and the additive set $\{b_i\}$.

We call an extension T of S by R combined with the homomorphism $\eta: R \to D$, where D is some maximal ring of related double homothetisms of S, an η-extension of S by R.

Let T be any η-extension of S by R which has, for the representative set $\bar{z}_i$, the multiplicative factor set $\{z_i, z_j\}$ and the additive set $\{b_i\}$. Another representative set of T/S may be: $\bar{z}_1', \bar{z}_2', \ldots, \bar{z}_l'$, where $\bar{z}_i' = \bar{z}_i + \psi_{z_i}$, $\psi_{z_i} \in S$ for $i = 1, \ldots, l$. Then $\bar{z}_i' \bar{z}_j' = (\bar{z}_i + \psi_{z_i})(\bar{z}_j + \psi_{z_j}) = \{z_i, z_j\} + \eta_l(z_i)(\psi_{z_j}) + (\psi_{z_i})\eta_r(z_j)$ and $m_i \bar{z}_i' = m_i(\bar{z}_i + \psi_{z_i}) = b_i + m_i \psi_{z_i}$. Hence the new factor sets are

(5) $$\{z_i, z_j\}' = \{z_i, z_j\} + \eta_l(z_i)(\psi_{z_j}) + (\psi_{z_i})\eta_r(z_j)$$

and

(6) $$b_i' = b_i + m_i \psi_{z_i}.$$

We shall call two factor sets $\{z_i, z_j\}$, $\{b_i\}$ and $\{z_i, z_j\}'$, $\{b_i\}'$ *equivalent* if there exists a mapping $\psi: R \to S(\psi_0 = 0)$ such that (5) and (6) hold. Hence any two factor sets corresponding to the same η-extension of S by R are equivalent.

On the other hand, we shall call two η-extensions T and T' of S by R *equivalent* (and write $T \sim T'$) if there exists an isomorphism α: $T \to T'$ such that α is the identity on S and $\varphi = \alpha\varphi'$, where φ: $T \to R$ and φ': $T' \to R$ are the epimorphisms whose kernels are S. With these definitions we get the result: Let T_1 and T_2 be two η-extensions of S by R. Then $T_1 \sim T_2$ if and only if, for some choice of representative sets in T_1 resp. T_2, the corresponding factor sets $\{z_i, z_j\}_1$, $\{b_i\}_1$, resp. $\{z_i, z_j\}_2$. $\{b_i\}_2$ are equivalent. More explicitly, if T_k, with representative set $\{\bar{z}_i\}_k$, has the factor set $\{z_i, z_j\}_k$, $\{b_i\}_k$ $(k=1, 2)$, then the isomorphism α: $T_1 \to T_2$ is given by $\left(\sum_{i=1}^{l} n_i(\bar{z}_i)_1 + s\right)\alpha = \sum_{i=1}^{l} n_i(\bar{z}_i)_2 + s + \sum_{i=1}^{l} n_i\psi_{z_i}$, where ψ: $R \to S$ $(\psi_0 = 0)$ is a mapping such that (5) and (6) hold for ψ and the factor sets. The proof is straightforward.

An η-extension T of S by R is said to be a *splitting extension* over S if and only if, for some choice of representative set, all $\{z_i, z_j\}$ are 0 and all b_i are 0. Also, $T = S \oplus R$ (ringtheoretical direct sum) if and only if T is a 0-extension of S by R $(\eta = 0)$ and, for some choice of representative set, all $\{z_i, z_j\}$ are 0 and all b_i are 0. The direct sum extension is a zero-ring, since R and S are supposed to be zero-rings.

Let T be an η-extension of S by R. A subring K of S is an ideal in T if and only if K is invariant under the double homothetisms of S, which occur as images in η: $R \to D$. Now the $\eta(z_i) = (\eta_l(z_i), \eta_r(z_i))$ are double homothetisms of K and T/K is an η^*-extension of S/K by R. If η: $R \to D$ is such that $\eta(z_i) = (\eta_l(z_i), \eta_r(z_i))$ then η^*: $R \to D^*$, where D^* is a maximal ring of related double homothetisms of S/K, is defined by $\eta^*(z_i) = (\eta_l^*(z_i), \eta_r^*(z_i))$, where $\eta_l^*(z_i)(s+K) = \eta_l(z_i)s + K$ and $(s+K)\eta_r^*(z_i) = s\eta_r(z_i) + K$. Since K is invariant in $\eta(R)$, this definition does not depend on the particular choice of a representative s in $s+K$. It is easy to show that $\eta^*(z_i) = (\eta_l^*(z_i), \eta_r^*(z_i))$ is a double homothetism of S/K and that any two of such double homothetisms are related. It can be shown also that η^* is a homomorphic mapping. Hence η^*: $R \to D^*$ is a homomorphism of R into a maximal ring of related double homothetisms of S/K. If T has the representative set $\bar{z}_i$, $i = 1, \ldots, l$, then a representative set of T/K is the set $\bar{z}_i + K$, $i = 1, \ldots l$. We have $(\bar{z}_i + K)(\bar{z}_j + K) = \{z_i, z_j\} + K$ and $m_i(\bar{z}_i + K) = b_i + K$, hence the corresponding factor sets are $\{z_i, z_j\} + K$ and $b_i + K$ for all i, j with $1 \leqq i, j \leqq l$ Moreover $(\bar{z}_i + K)(s+K) = \eta_l(z_i)s + K = \eta_l^*(z_i)(s+K)$ and $(s+K)(\bar{z}_i + K) = s\eta_r(z_i) + K = (s+K)\eta_r^*(z_i)$, hence η^* is induced by inner double homothetisms of T/K.

The following lemma is obvious now; in fact the proof is similar to that of Theorem 1.

Lemma 1. *If T is an η-square extension of S by R then, for each subring K of S invariant under the double homothetisms in $\eta(R)$, T/K is an η^*-square extension of S/K by R.*

Lemma 2. *Suppose that $S=S_1\oplus S_2$ (direct sum) and the orders q_1, and q_2 of S_1 resp. S_2 are relatively prime. If there exist η' resp. η''-square extensions of S_1 resp. S_2 by R, then there exists an $(\eta'+\eta'')$-square extension of S by R.*

Proof. Let $\{z_i, z_j\}'$, b_i' resp. $\{z_i, z_i\}''$, b_i'' be factor sets is S_1 resp. S_2 for an η'-resp. η''-extension of S_1 resp. S_2 by R. Here $\eta'\colon R\to D_1$ is a homomorphism of R into a maximal ring of related double homothetisms of S_1 and $\eta''\colon R\to D_2$ is a homomorphism of R into a maximal ring of related double homothetisms of S_2. Extend the double homothetisms $\eta'(z_i)=(\eta_l'(z_i), \eta_r'(z_i))$ of S_1 by letting them act trivially on S_2. Then define $\eta_l'(z_i)(s_1+s_2)=\eta_l'(z_i)s_1$ and $(s_1+s_2)\eta_r'(z_i)=s_1\eta_r'(z_i)$ for all $(\eta_l'(z_i), \eta_r'(z_i))$ in $\eta'(R)$ and all $s_1\in S_1$ and all $s_2\in S_2$. Similarly, extend the double homothetisms $\eta''(z_i)=(\eta_l''(z_i), \eta_r''(z_i))$ of S_2 by letting them act trivially on S_1. Then define $\eta_l''(z_i)(s_1+s_2)=\eta_l''(z_i)s_2$ and $(s_1+s_2)\eta_r''(z_i)=s_2\eta_r''(z_i)$ for all $(\eta_l''(z_i), \eta_r''(z_i))$ in $\eta''(R)$ and all $s_1\in S_1$ and all $s_2\in S_2$. It is easy to show now that both the extended $\eta'(z_i)$ and the extended $\eta''(z_i)$ are double homothetisms of S. Moreover the double homothetisms $\eta'(z_i)$ and $\eta''(z_i)$ of S are related double homothetisms. It follows that the sum $\eta'(z_i)+\eta''(z_i)$ is again a double homothetism of S, ([1]). We define now: $\eta'+\eta''(z_i)=\eta'(z_i)+\eta''(z_i)$ for all $z_i\in R$ and extend $\eta'+\eta''$ by linearity. Thus $\eta'+\eta''(z_i)$ is that double homothetism of S which is the sum of $\eta'(z_i)$ and $\eta''(z_i)$. More explicitly: $\eta'+\eta''(z_i)=(\eta_l'(z_i)+\eta_l''(z_i), \eta_r'(z_i)+\eta_r''(z_i))$, where $(\eta_l'(z_i)+\eta_l''(z_i))(s_1+s_2) = \eta_l'(z_i)(s_1+s_2)+\eta_l''(z_i)(s_1+s_2) = \eta_l'(z_i)s_1+\eta_l''(z_i)s_2$ for all $s_1\in S_1$ and all $s_2\in S_2$ and a similar formula holds for $\eta_r'(z_i)+\eta_r''(z_i)$. Then $\eta'+\eta''\colon R\to D$ is a homomorphic mapping of R into a maximal ring D of related double homothetisms of S, as the extended η' and η'' are homomorphisms of R into D. Here we may take $D=D_1\oplus D_2$. In order to construct an $\eta'+\eta''$-square extension of S by R, we use the sets $\{z_i, z_j\}'+\{z_i, z_j\}''$, $b_i'+b_i''$ in S as factor sets. As $\{z_i, z_j\}'$, b_i' with η' and $\{z_i, z_j\}''$, b_i'' with η'' both satisfy the conditions (1)—(4), it follows that $\{z_i, z_j\}'+\{z_i, z_j\}''$, $b_i'+b_i''$ together with $\eta'+\eta''$ satisfy the conditions (1)—(4). Hence we have obtained an $\eta'+\eta''$-extension T of $S=S_1\oplus S_2$ by R. Now we have to prove that $T^2=S$. First we remark that S_2 is mapped into itself under $\eta'+\eta''$. As T is an $\eta'+\eta''$-extension of S by R it follows that T/S_2 is an η^*-extension of S/S_2 by R, (Lemma 1). The corresponding factor set is $\{z_i, z_j\}'+S_2$, $b_i'+S_2$. Now since $\{z_i, z_j\}'$, b_i' corresponds to an η'-square extension of S_1 by R, it follows that T/S_2 is a square extension of S/S_2 by R. So $(T/S_2)^2=S/S_2$ and in the same way $(T/S_1)^2=S/S_1$. As T is an $\eta'+\eta''$-extension of S by R with the factor set $\{z_i, z_j\}'+\{z_i, z_j\}''$, $b_i'+b_i''$, it is clear that $T^2\subseteqq S$. So we have to prove $S\subseteqq T^2$. From $(T/S_2)^2=S/S_2=S_1$ it follows that, if s_1 is a given element of S_1, there exists an element $a\in T^2$ such that $s_1\equiv a \pmod{S_2}$. From $(T/S_1)^2=S/S_1=S_2$ it follows that, if s_2 is a given element of S_2, there exists an element $b\in T^2$ such that $s_2\equiv b \pmod{S_1}$. Then $q_2s_1\equiv q_2a \pmod{q_2S_2=0}$, so $q_2s_1\in T^2$.

As the order of s_1 is relatively prime to q_2 it follows that $s_1 \in T^2$. Similarly $q_1 s_2 \equiv \equiv q_1 b \pmod{q_1 S_1 = 0}$, so $q_1 s_2 \in T^2$. As the order of s_2 is relatively prime to q_1 it follows that $s_2 \in T^2$. From $s_1 \in T^2, s_2 \in T^2$ for all $s_1 \in S_1, s_2 \in S_2$ it follows that $S_1 + S_2 = S \subseteqq T^2$. Then $T^2 = S$ and T is an $\eta' + \eta''$-square extension of S by R. We apply the lemmas 1 and 2 in the following theorem:

Theorem 2. *Let R and S be finite zero-rings. There exists a square extension T of S by R if and only if, for each p_i-Sylow subgroup A_i of S (p_i a prime), there exists a square extension of A_i by R.*

Proof. Let $S = A_1 \oplus \ldots \oplus A_k$, where the p_i-Sylow subgroup A_i has the order $p_i^{\alpha_i}$ $i = 1, \ldots, k$. Now the orders $p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_k^{\alpha_k}$ are relatively prime. Thus, if there exist square extensions of $A_1, A_2, \ldots, A_k$ by R, then there exists a square extension T of S by R by the preceding Lemma 2.

Conversely let us suppose that T is a square extension of S by R. Now the A_i are characteristic subrings of S, i.e. they are invariant under all double homothetisms of S. Hence the direct sum $A_1 \oplus \ldots \oplus A_{i-1} \oplus A_{i+1} \oplus \ldots \oplus A_k$ is a characteristic subring of S. Therefore $T/A_1 \oplus \ldots \oplus A_{i-1} \oplus A_{i+1} \oplus \ldots \oplus A_k$ is a square extension of $S/A_1 \oplus \ldots \oplus A_{i-1} \oplus A_{i+1} \oplus \ldots \oplus A_k = A_i$ by R (Lemma 1). This theorem reduces the problem to the case in which S^+ is a finite abelian p-group.

Theorem 3. *Let S^+ be a finite abelian p-group, and S a zero-ring. Let R be a finite zero-ring. T is a square extension of S by R if and only if T/pS is a square extension of S/pS by R.*

Proof. First we remark that pS is a characteristic subring of S, for if $\alpha = (\alpha_1, \alpha_2)$ is an arbitrary double homothetism of S, then $\alpha_1(ps) = p\alpha_1(s)$ and $(ps)\alpha_2 = p(s)\alpha_2$ for all $s \in S$. Hence T is a square extension of S by R implies T/pS is a square extension of S/pS by R (Lemma 1). Conversely, suppose T/pS is a square extension of S/pS by R. Then $T/pS/S/pS \cong T/S \cong R$ and T is an extension of S by R. From $(T/pS)^2 = S/pS$ it follows that, if b is a given element in T^2, there exists an element $s \in S$ such that $b \equiv s \pmod{pS}$. Thus $T^2 \subseteqq S$. Conversely, if s is a given element in S, there exists an element $a \in T^2$ such that $s \equiv a \pmod{pS}$. Then $ps = a_0 + p^2 s_1$, where $a_0 = pa \in T^2$ and $s_1 \in S$, $p^2 s_1 = a_1 + p^3 s_2$, where $a_1 \in T^2$, $s_2 \in S$, $\ldots, p^{k-1} s_{k-2} = = a_{k-2} + p^k s_{k-1} = a_{k-2} \in T^2$, if we assume that $p^k S = 0$. Tracing back we find $ps \in T^2$ and as s is an arbitrary element in S we have $pS \subseteqq T^2$. But this implies $(T/pS)^2 = T^2/pS = S/pS$, hence $S = T^2$. T is a square extension of S by R. We note that $S^+/(pS)^+$ is an elementary abelian p-group and therefore we have reduced the problem to the case where S^+ is an elementary abelian p-group of finite rank.

Let $\eta\colon R \to D$ be a fixed homomorphism of R into a maximal ring of related double homothetisms of S. We consider the set $S_{\eta(r)}$ of all elements of the form $\eta_l(r)s, s'\eta_r(r)$, where $\eta(r) = (\eta_l(r), \eta_r(r))$ is a fixed element of $\eta(R)$ and s is a variable element in S, s' is a variable element in S, independent of s. Then let S^* be the subring of S, generated by all $S_{\eta(r)}$ for $r \in R$, which is denoted by $S^* = \langle S_{\eta(r)} \rangle$. Finally, if T is an extension of S by R, then M will denote the multiplicative factor set for some choice of representative set in T, i.e. $M = (\{z_i, z_j\}, 1 \leqq i \leqq l, 1 \leqq j \leqq l)$. Now we can prove:

Lemma 3. *T is a square η-extension of S by R if and only if S is generated by M and S^*: $S = \langle M, S^* \rangle$.*

Proof. It is sufficient to show that, given an η-extension T of S by R, $\langle M, S^* \rangle = T^2$. Let us assume that this has been proved. Then if T is a square η-extension of S by R we get $T^2 = S = \langle M, S^* \rangle$. Conversely, if $S = \langle M, S^* \rangle$ for some η-extension T of S by R, then, as $\langle M, S^* \rangle = T^2$, we get $T^2 = S$ and T is a square η-extension of S by R. Now we are going to prove that $T^2 = \langle M, S^* \rangle$ for a given η-extension T of S by R. For the multiplication in T we have:

$$\left(\sum_{i=1}^{l} n_i \bar{z}_i + s \right) \cdot \left(\sum_{j=1}^{l} u_j \bar{z}_j + v \right) = \sum_{i=1}^{l} \sum_{j=1}^{l} n_i u_j \{z_i, z_j\} + \sum_{i=1}^{l} n_i (\eta_l(z_i) v) + \sum_{j=1}^{l} u_j (s \eta_r(z_j))$$

where $(\bar{z}_1, \dots, \bar{z}_l)$ is a representative set of the basis $(z_1, \dots, z_l)$ in R, $s, v \in S$ and n_i, u_j are integers for $1 \leqq i \leqq l, 1 \leqq j \leqq l$. Thus $T^2 \subseteqq \langle M, S^* \rangle$. Now the generators of $\langle M, S^* \rangle$ are the elements $\{z_i, z_j\}$ of M and all elements of the form $\eta_l(z_i)v$, $s\eta_r(z_j)$ where $z_i, z_j \in (z_1, \dots, z_l)$ in R and $v, s \in S$. As $\{z_i, z_j\} = \bar{z}_i \bar{z}_j$, $\eta_l(z_i)v = \bar{z}_i v$ and $s\eta_r(z_j) = s\bar{z}_j$ it follows that all generators of $\langle M, S^* \rangle$ belong to T^2, hence $\langle M, S^* \rangle \subseteqq T^2$. Then $\langle M, S^* \rangle = T^2$.

Next we investigate the η-extensions of S by R which are splitting extensions over S. First we consider the case where S^+ is an elementary abelian p-group of rank 1. We prove:

Lemma 4. *Let $S^+ = (0, a, \dots, (p-1)a)$ be an elementary abelian p-group of rank 1. S is a zero-ring, i.e. $a^2 = 0$. Let R^+ be the direct sum of l cyclic groups (z_i) of order m_i, $i = 1, \dots, l$. R is a zero-ring, i.e. $z_i z_j = 0$ for all i, j with $1 \leqq i \leqq l$, $1 \leqq j \leqq l$. Then there does not exist a splitting square η-extension T of S by R, whatever η may be.*

Proof. Let T be an η-extension of S by R with representative set $(\bar{z}_1, \dots, \bar{z}_l)$. Addition and multiplication in T are performed according to: $\left(\sum_{i=1}^{l} n_i \bar{z}_i + sa \right) + \left(\sum_{i=1}^{l} u_i \bar{z}_i + va \right) = \sum_{i=1}^{l} (n_i + u_i) \bar{z}_i + (s+v)a$, with $n_i + u_i$ reduced mod $m_i (i = 1, \dots, l)$ and $s + v$ reduced mod p; $\left(\sum_{i=1}^{l} n_i \bar{z}_i + sa \right) \left(\sum_{i=1}^{l} u_i \bar{z}_i + va \right) = \sum_{i=1}^{l} n_i v (\eta_l(z_i) a) + \sum_{j=1}^{l} u_j s (a \eta_r(z_j))$, if we assume that T is a splitting extension over S. But then

$0=a(\bar{z}_1^2)=(a\bar{z}_1)\bar{z}_1=(a\eta_r(z_1))\eta_r(z_1)$, which implies that $a\eta_r(z_1)=0$. So we get $a\eta_r(z_i)=0$ for all z_i with $i=1, \ldots, l$. Similarly $\eta_l(z_i)a=0$ for all z_i with $i=1, \ldots, l$. Hence T is a zero-ring and $T^2\neq S$, as $S\neq(0)$. In this case there exists no splitting square η-extension T of S by R.

Theorem 4. *Let R and S be finite zero-rings. Let $\eta\colon R\to D$ be an arbitrary homomorphism of R into a maximal ring of related double homothetism of S. Then there does not exist an η-square extension T of S by R, such that T splits over S.*

Proof. It is sufficient to show that there does not exist an η-square extension T of S by R, such that T splits over S for the case that S^+ is an elementary abelian p-group of finite rank. Let us assume that this has been proved. First let S^+ a finite abelian p-group, not elementary, S a zero-ring and R a finite zero-ring. Then pS ($\neq 0$, $\neq S$) is a characteristic subring of S. Suppose T is an η-square extension of S by R which splits over S. Then, by Lemma 1, T/pS is an η^*-square extension of S/pS by R and from the results preceding Lemma 1, it is easy to see that T/pS splits over S/pS. But S/pS is an elementary abelian p-group, hence by assumption there does not exist an η^*-square extension of S/pS which splits over S/pS. So we get that there does not exist an η-square extension T of S by R which splits over S in case S^+ is a finite abelian p-group and R and S are finite zero-rings. Next let S^+ be an arbitrary finite abelian group and S a zero-ring. Let $S^+=A_1\oplus\ldots\oplus A_k$, where the p_i-Sylow subgroup A_i has the order $p_i^{\alpha_i}$, $i=1, \ldots\ k$, and the p_i are primes. Suppose T is an η-square extension of S by R, which splits over S. Then, again by Lemma 1, $T/A_1\oplus\ldots\oplus A_{i-1}\oplus A_{i+1}\oplus\ldots\oplus A_k$ is an η^*-square extension of $S/A_1\oplus\ldots\oplus\ldots\oplus A_{i-1}\oplus A_{i+1}\oplus\ldots\oplus A_k=A_i$ by R, which splits over A_i, $1\leqq i\leqq k$. But A_i^+ is a finite abelian p_i-group, hence there does not exist an η^*-square extension T of A_i by R which splits over A_i. This contradiction implies that there does not exist an η-square extension T of S by R which splits over S, if R and S are finite zero-rings.

Now S^+ is supposed to be an elementary abelian p-group of finite rank and we are going to prove that there does not exist an η-square extension T of S by R which splits over S whatever η may be. For a split extension, for some choice of representative set, $\{z_i, z_j\}=0$ and $b_i=0$ for all i and j, $1\leqq i\leqq l$, $1\leqq j\leqq l$. Hence T is an η-square extension of S by R which splits over S if and only if $S=S^*=\langle S_{\eta(r)}|r\in R\rangle$ (Lemma 3). Now suppose that T is an η-square extension of S by R which splits over S. Since $S=S^*\neq 0$, $\eta(R)\neq 0$, where $\eta(R)$ is the image of R in the homomorphical mapping $\eta\colon R\to D$. Since R is generated by the z_i, $1\leqq i\leqq l$, it is clear that $\eta(R)$ is generated by the pairs (A_i, B_i), $1\leqq i\leqq l$, where $A_i=\eta_l(z_i)$, $B_i=\eta_r(z_i)$, such that $\eta(z_i)=(\eta_l(z_i), \eta_r(z_i))$ is the double homothetism of S corresponding to $z_i\in R$. The $2l$ endomorphisms A_i, B_j have the properties:

(i) $A_i A_k = 0$, $B_j B_t = 0$ for all i, j, k, t with $i \leqq i, k, j, t \leqq l$;
(ii) $A_i B_j = B_j A_i$ for all i, j with $1 \leqq i, j \leqq l$.

In particular both the A_i and B_j are nilpotent endomorphisms such that $A_i^2 = 0$ and $B_j^2 = 0$ for all i, j with $1 \leqq i, j \leqq 1$. Since $\eta(R) \neq 0$, at least one of these endomorphisms is $\neq 0$, say $A_1 \neq 0$. Now consider the set $A_1 S = \{A_1 s | s \in S\}$. Then $A_1 S$ is a subring of S, as $A_1 s_1 + A_1 s_2 = A_1(s_1 + s_2)$ and $(A_1 s_1)(A_1 s_2) = 0$. Moreover $A_1 S$ is invariant under $A_1, \ldots, A_l; B_1, \ldots, B_l$, as $A_i(A_1 S) = 0$ (i), $B_j(A_1 S) = = A_1(B_j S) \subseteqq A_1 S$ for all A_i, B_j (ii) with $1 \leqq i, j \leqq l$. This means $A_1 S$ is a subring of S invariant under the double homothetisms of $\eta(R)$. Further $A_1 S \neq 0$, as $A_1 \neq 0$ and $A_1 S \neq S$. If $A_1 S = S$ then $A_1 s = A_1$ $(A_1 s') = 0$ for every $s \in S$ (i) and this would imply $A_1 = 0$ which is a contradiction. By Lemma 1, as T is an η-square extension of by R, T/A_1 S is an η^*-square extension of $S/A_1 S$ by R, where η^* is induced by η. In fact, $\eta^*\colon R \to D^*$, D^* a maximal ring of related double homothetisms of $S/A_1 S$, is such that $\eta^*(z_i) = (\eta_l^*(z_i), \eta_r^*(z_i))$, where, by definition, $\eta_l^*(z_i)(s + A_1 S) = = \eta_l(z_i)s + A_1 S$ and $(s + A_1 S)\eta_r^*(z_i) = s\eta_r(z_i) + A_1 S$. Since $S = S^* = \langle S_{\eta(r)} | r \in R \rangle$, it follows from the definition of $\eta_l^*(z_i)$, that $S/A_1 S = (S/A_1 S)^* = \langle S/A_1 S_{\eta^*(r)} | r \in R \rangle$. Hence $T/A_1 S$ is an η^*-square extension of $S/A_1 S$ by R, which splits over $S/A_1 S$. As $A_1 S \neq 0$, and $A_1 S \neq S$, the dimension of $S/A_1 S$ is less than r and greater than 0, if we consider S^+ as an r-dimensional vector space over the prime Galois field $F = GF(p)$. By Lemma 4, there does not exist an η-square extension T of S by R, which splits over S, in case S^+ has dimension 1. So, by induction on the dimension of S, it follows that there does not exist an η-square extension T of S by R which splits over S whatever η may be. This completes the proof of Theorem 4.

Next we investigate the existence of 0-square extensions of S by R i.e. extensions where the homomorphism $\eta\colon R \to D$ is the zero-homomorphism. Here we get the result:

Theorem 5. *Let S be a zero-ring and S^+ an elementary abelian p-group of finite rank r. Let R be a finite zero-ring, where $R^+ = \sum_{i=1}^{l} \oplus z_i$, $O(z_i) = m_i$, $1 \leqq i \leqq l$. Then there exists a 0-square extension T of S by R if and only if the following conditions are satisfied:* (i) $l^2 \geqq r$; (ii) *if $(n-1)^2 < r \leqq n^2$ for some n with $1 \leqq n \leqq l$, then $p | m_i$ for at least n integers $m_i (1 \leqq i \leqq l)$.*

Proof. Let T be a 0-square extension of S by R. Then $T^2 = S$ and S is generated by M, for some choice of representative set (Lemma 3). As S has rank r, the number of generators of S in M is greater than or equal to r. Since $O(M) = l^2$, it follows that $l^2 \geqq r$. As $\eta(R) = 0$ we must have $m_i\{z_i, z_j\} = 0$ and $m_i\{z_j, z_i\} = 0$ for a fixed z_i and all z_j, $1 \leqq j \leqq l$ ((3) and (4)). But if $\{z_i, z_j\} \neq 0$ then it has order p, hence $p | m_i$ if $\{z_j, z_i\} \neq 0$ for any z_j. Likewise if $\{z_j, z_i\} \neq 0$ for any z_j then $p | m_i$. The question is now: how many different elements $z_i (\in R)$ have the property that either $\{z_i, z_j\} \neq 0$

or $\{z_j, z_i\} \neq 0$ for at least one $z_j (\in R)$? Now let $(n-1)^2 < r \leqq n^2$ for some n with $1 \leqq n \leqq l$, and let B be a basis of S in M. Since $(n-1)^2 < r = O(B)$, there are more than $(n-1)^2$ elements $\{z_i, z_j\}$ in M $(1 \leqq i \leqq l, 1 \leqq j \leqq l)$ which are not equal to 0, i.e. the elements of the basis B. It is clear now that the minimal number of *different* $z_i (\in R)$ which occur either as a first or as a second component in at least one element in B is n. Hence $p | m_i$ for at least n integers m_i $(1 \leqq i \leqq l)$.

Conversely suppose the conditions (i) and (ii) are satisfied. We define functions $\{z_i, z_j\}$ of $R \times R$ into S for the basic elements of R in the following way. First let $\{z_i, 0\} = \{0, z_j\} = 0$ for all z_i, z_j with $1 \leqq i \leqq l$ and $1 \leqq j \leqq l$. We know $r \leqq l^2$, hence we may suppose that $(n-1)^2 < r \leqq n^2$ for some n with $1 \leqq n \leqq l$. We denote $r = (n-1)^2 + v$, where $1 \leqq v \leqq 2n-1$. Now S has rank r and let $(s_1, \ldots, s_r)$ be a basis of S. From (ii) we infer that there are n integers, say $m_1, \ldots, m_n$, such that $p | m_i$ for all i with $1 \leqq i \leqq n$. Then set $\{z_1, z_1\} = s_1, \{z_1, z_2\} = s_2, \ldots, \{z_1, z_{n-1}\} = = s_{n-1}, \{z_2, z_1\} = s_n, \ldots, \{z_2, z_{n-1}\} = s_{2n-2}, \ldots, \{z_{n-1}, z_1\} = s_{n^2-3n+3}, \ldots, \{z_{n-1}, z_{n-1}\} = = s_{(n-1)^2}$ and set $\{z_i, z_n\}$ and/or $\{z_n, z_i\}$ equal to $s_{(n-1)^2+1}, \ldots, s_r$ for v functions $\{z_i, z_n\}$ and/or $\{z_n, z_i\}$ with $1 \leqq i \leqq n$. Then set all other $\{z_i, z_j\} = 0$. It is clear now that S is generated by the set of all $\{z_i, z_j\}$ with $1 \leqq i \leqq n$ and $1 \leqq j \leqq n$. If we put $\eta(R) = 0$ then the conditions (1)—(4) are satisfied for the functions $\{z_i, z_j\}$ $(1 \leqq i \leqq l, 1 \leqq j \leqq l)$ and an arbitrary set $b_i \in S$ $(1 \leqq i \leqq l)$. Hence T is an 0-extension of S by R, if we define T as the set of all symbols $\sum_{i=1}^{l} n_i \bar{z}_i + s$ $(s \in S, n_i$ integers) with the addition and multiplication: $\left(\sum_{i=1}^{l} n_i \bar{z}_i + s\right) + \left(\sum_{i=1}^{l} u_i \bar{z}_i + v\right) = = \sum_{i=1}^{l} (n_i + z_i) \bar{z}_i + s + v$, where $m_i \bar{z}_i = b_i (\in S)$ for $1 \leqq i \leqq l$, $\left(\sum_{i=1}^{l} n_i \bar{z}_i + s\right)\left(\sum_{i=1}^{l} u_i \bar{z}_i + v\right) = = \sum_{i=1}^{l} \sum_{j=1}^{l} n_i u_j \{z_i, z_j\}$. As $S = \langle M \rangle$, it follows that T is a 0-square extension of S by R, which completes the proof of Theorem 5.

Now we determine the rings T which may occur as a square extension of a ring S of order 2 by a ring R of order 4. Both S and R are supposed to be zero-rings. Let $S^+ = (0, a)$ with $2a = 0$ and $a^2 = 0$. Let $R^+ = (z_1) \oplus (z_2)$ be the direct sum of two cyclic groups (z_1) and (z_2) both of order 2 and $z_1^2 = z_1 z_2 = z_2 z_1 = z_2^2 = 0$. Now the endomorphism ring of S^+ consists of the zero-endomorphism and the identity mapping. Hence in this case we must have $\eta(R) = 0$, so that there are only 0-square extensions of S by R possible. As the conditions of Theorem 5 are satisfied there exist 0-square extensions of S by R. There are 2 cases: (i) $2\bar{z}_1 = 2\bar{z}_2 = 0$, which means $b_1 = b_2 = 0$ in S. (ii) at least one of b_1 and $b_2 \neq 0$.

(i) In this case the elements a, $\bar{z}_1$ and $\bar{z}_2$ all have order 2 and we get $T^+ = = (a) \oplus (\bar{z}_1) \oplus (\bar{z}_2)$ is of typus (2, 2, 2). As $\eta(R) = 0$, $a\bar{z}_1 = a\bar{z}_2 = \bar{z}_1 a = \bar{z}_2 a = 0$. If $\{z_1, z_1\}, \{z_1, z_2\}, \{z_2, z_1\}$ and $\{z_2, z_2\}$ are 0, then $T^2 = (0)$ which contradicts that $T^2=S$. Hence we must have at least one of the four elements $\{z_1, z_1\}$, $\{z_1, z_2\}$ $\{z_2, z_1\}$ and $\{z_2, z_2\}$ equal to a. We get 15 different rings T with multiplications:

	a	$\bar{z}_1$	$\bar{z}_2$
a	0	0	0
$\bar{z}_1$	0	0	0
$\bar{z}_2$	0	0	a

	a	$\bar{z}_1$	$\bar{z}_2$
a	0	0	0
$\bar{z}_1$	0	a	0
$\bar{z}_2$	0	0	0

	a	$\bar{z}_1$	$\bar{z}_2$
a	0	0	0
$\bar{z}_1$	0	a	a
$\bar{z}_2$	0	a	a

	a	$\bar{z}_1$	$\bar{z}_2$
a	0	0	0
$\bar{z}_1$	0	0	a
$\bar{z}_2$	0	a	a

	a	$\bar{z}_1$	$\bar{z}_2$
a	0	0	0
$\bar{z}_1$	0	a	0
$\bar{z}_2$	0	0	a

	a	$\bar{z}_1$	$\bar{z}_2$
a	0	0	0
$\bar{z}_1$	0	a	a
$\bar{z}_2$	0	a	0

	a	$\bar{z}_1$	$\bar{z}_2$
a	0	0	0
$\bar{z}_1$	0	a	0
$\bar{z}_2$	0	a	a

	a	$\bar{z}_1$	$\bar{z}_2$
a	0	0	0
$\bar{z}_1$	0	a	a
$\bar{z}_2$	0	0	a

	a	$\bar{z}_1$	$\bar{z}_2$
a	0	0	0
$\bar{z}_1$	0	0	0
$\bar{z}_2$	0	a	0

	a	$\bar{z}_1$	$\bar{z}_2$
a	0	0	0
$\bar{z}_1$	0	0	0
$\bar{z}_2$	0	a	a

	a	$\bar{z}_1$	$\bar{z}_2$
a	0	0	0
$\bar{z}_1$	0	0	a
$\bar{z}_2$	0	0	0

	a	$\bar{z}_1$	$\bar{z}_2$
a	0	0	0
$\bar{z}_1$	0	0	a
$\bar{z}_2$	0	0	a

	a	$\bar{z}_1$	$\bar{z}_2$
a	0	0	0
$\bar{z}_1$	0	a	0
$\bar{z}_2$	0	a	0

	a	$\bar{z}_1$	$\bar{z}_1$
a	0	0	0
$\bar{z}_1$	0	a	a
$\bar{z}_2$	0	0	0

	a	$\bar{z}_1$	$\bar{z}_2$
a	0	0	0
$\bar{z}_1$	0	0	a
$\bar{z}_2$	0	a	0

Thus we get 15 non-equivalent 0-square extensions T of S by R.

(ii) In this case at least one of the elements $\bar{z}_1$ and $\bar{z}_2$ is of order 4, and T^+ is of typus (2, 4), say $T^+=(\bar{z}_1)\oplus(\bar{z}_2)$ where $O(\bar{z}_1)=2$ and $O(\bar{z}_2)=4$. For the multiplication in T one has again: $\bar{z}_1^2=k_1 a, \bar{z}_1\bar{z}_2=k_2 a, \bar{z}_2\bar{z}_1=k_3 a, \bar{z}_2^2=k_4$ a where

$0 \leqq k_i \leqq 1$, $i=1, 2, 3, 4$. Hence we get the same multiplication tables as in case (i), if we omit the first row and the first column. Thus we find 15 non-equivalent 0-square extensions T of S by R. Next we suppose S to be a zero-ring of order 2 as above and $R^+=(z)$ a cyclic group of order 4. R is a zero-ring i.e. $z^2=0$. Again $\eta(R)=0$ so there are only 0-square extensions of S by R possible and by Theorem 5 there are such extensions. As $\bar{z}^2=0$ or a, we get $\{z, z\}=0$ or a. But if $\{z, z\}=0$ then $T^2=(0)$, contradiction. So we must have $\{z, z\}=a$. We have two possibilities for the addition according to $4\bar{z}=0$ or a, which means $b=0$ or a. If $b=0$, then $T^+=(a)\oplus(\bar{z})$ is of typus (2, 4), if $b=a$, then $T^+=(\bar{z})$ is a cyclic group of order 8. Thus we get 2 non-equivalent 0-square extensions T of S by R. Finally we want to discuss the rings T which may occur as a square extension of a ring S of order 4 by a ring R of order 2. Both R and S are supposed to be zero-rings. Let $S^+=(a_1)\oplus\oplus(a_2)$ be the direct sum of two cyclic groups (a_1) and (a_2) each of order 2 and $a_1^2=a_1a_2=a_2a_1=a_2^2=0$. Let $R^+=(0, z)$ with $2z=0$ and $z^2=0$. As the condition (i) of Theorem 5 is not satisfied in this case ($l=1, r=2$), there do not exist 0-square extensions of S by R now. The nilpotent endomorphisms in the endomorphismring of S^+ are: $s_1: a_1\to 0, a_2\to 0$; $s_2: a_1\to 0, a_2\to a_1$; $s_3: a_1\to a_2, a_2\to 0$; $s_4: a_1\to a_1+a_2$; $a_2\to a_1+a_2$. So the possible double homothetisms are (s_1, s_1), (s_1, s_2), (s_1, s_3), (s_1, s_4), (s_2, s_1), (s_2, s_2), (s_3, s_1), (s_3, s_3), $(s_4, s_1)(s_4, s_4)$, which may occur as the element $(\eta_l(z), \eta_r(z))$ in $\eta(R)$. For $\bar{z}^2=\{z, z\}$ as well as for $2\bar{z}=b$ we may choose $0, a_1, a_2$ or a_1+a_2. But as $2\{z, z\}=0$ we must have $(b)\eta_r(z)=\eta_1(z)(b)=0$, ((3) and (4)). Then we distinguish the following cases:

(i) Let $b=a_1$. Then $(\eta_l(z), \eta_r(z))=(s_2, s_2)$ for a square extension of S by R. As $S^*=\langle S_{\eta(r)}\rangle=(0, a_1)$ we must have $\{z, z\}=a_2$ or a_1+a_2 for a square extension of S by R (Lemma 3). Since $\eta_l(z)=\eta_r(z)=s_2$ the condition (2) is satisfied. The additive group T^+ of a square extension T of S by R has the form: $T^+=(\bar{z})\oplus(a_2)$ where $(\bar{z})$ has order 4 and a_2 has order 2. So T^+ is of typus (2, 4). For the multiplication in T one has: $a_2^2=0$, $\bar{z}a_2=s_2a_2=a_1$; $a_2\bar{z}=a_2s_2=a_1$ and $\bar{z}^2=a_2$ or a_1+a_2. Hence one gets 2 non-equivalent η-square extensions T of S by R.

(ii) Let $b=a_2$. Then we must take $(\eta_l(z), \eta_r(z))=(s_3, s_3)$ for a square extension of S by R. As $S^*=\langle S_{\eta(r)}\rangle=(0, a_2)$ we must have $\{z, z\}=a_1$ or a_1+a_2 (Lemma 3). Since $\eta_l(z)=\eta_r(z)=s_3$ the condition (2) is satisfied. The additive group T^+ of a square extension T of S by R has the form: $T^+=(\bar{z})\oplus(a_1)$ where $(\bar{z})$ has order 4 and a_1 has order 2. So T^+ is of typus (2, 4). For the multiplication in T one has: $a_1^2=0$, $\bar{z}a_1=s_3a_1=a_2$, $a_1\bar{z}=a_1s_3=a_2$ and $\bar{z}^2=a_1$ or a_1+a_2. Hence one gets 2 non-equivalent η-square extensions T of S by R.

(iii) Let $b=a_1+a_2$. Now we must have $(\eta_l(z), \eta_r(z))=(s_4, s_4)$ for a square extension of S by R. As $S^*=\langle S_{\eta(r)}\rangle=(0, a_1+a_2)$ we must have $\{z, z\}=a_1$ or a_2, (Lemma 3). Since $\eta_l(z)=\eta_r(z)=s_4$ the condition (2) is satisfied. The additive group T^+ of a square extension T of S by R has the form: $T^+=(\bar{z})\oplus(a_1)$, where $(\bar{z})$

has order 4 and a_1 has order 2. So T^+ is of typus (2, 4). For the multiplication in T one has: $a_1^2=0$, $\bar{z}a_1=s_4a_1=a_1+a_2$, $a_1\bar{z}=a_1s_4=a_1+a_2$ and $\bar{z}^2=a_1$ or a_2. Hence one gets 2 non-equivalent η-square extensions T of S by R.

(iv) Let $b=0$. Then the conditions (3) and (4) are satisfied. For a square extension T of S by R we need only satisfy condition (2): $\eta_l(z)\{z, z\}=\{z, z\}\eta_r(z)$. We have again different cases:

(iv. a) Let $\{z, z\}=a_1$. Now we must have $(\eta_l(z), \eta_r(z))=(s_3, s_3)$ or (s_4, s_4). In both cases the condition (2) is satisfied. So we get 2 rings T each of which has an additive group $T^+=(a_1)\oplus(a_2)\oplus(\bar{z})$ of typus (2, 2, 2). Hence there are 2 square extensions T of S by R, an η'-square extension where $\eta'(z)=(s_3, s_3)$ and an η''-square extension where $\eta''(z)=(s_4, s_4)$.

(iv. b) Let $\{z, z\}=a_2$. Then we must have $(\eta_l(z), \eta_r(z))=(s_2, s_2)$ or (s_4, s_4). In both cases the condition (2) is satisfied. Thus we get 2 rings T each of which has an additive group $T^+=(a_1)\oplus(a_2)\oplus(\bar{z})$ of typus (2, 2, 2). So there are 2 square extensions T of S by R, an η'-square extension for $\eta'(z)=(s_2, s_2)$ and an η''-square extension for $\eta''(z)=(s_4, s_4)$.

(iv. c) Let $\{z, z\}=a_1+a_2$. Here we must have $(\eta_l(z), \eta_r(z))=(s_2, s_2)$ or (s_3, s_3). In both cases the condition (2) is satisfied. Again we get 2 rings T each of which has as an additive group $T^+=(a_1)\oplus(a_2)\oplus(\bar{z})$ of typus (2, 2, 2). Therefore we get 2 square extensions T of S by R, an η-square extension where $\eta(z)=(s_2, s_2)$ and an η'-square extension where $\eta'(z)=(s_3, s_3)$.

(iv. d) Let $\{z, z\}=0$. Now we would get a square extension T of S by R which splits over S which is impossible by Theorem 4. Hence there do not exist square extensions in this case.

There is a second class of rings T which may occur as a square extension of a ring S of order 4 by a ring R of order 2. Now we put $S^+=(a)$ is a cyclic group of order 4 and $a^2=0$ (S is a zero-ring). Again $R^+=(0, z)$ with $2z=0$ and $z^2=0$. The nilpotent endomorphism in the endomorphismring of S^+ are: $s_1: a\to 0$, and $s_2: a\to 2a$. So the pairs (s_1, s_1), (s_1, s_2), (s_2, s_1) and (s_2, s_2) may occur as the element $(\eta_l(z), \eta_r(z))$ in $\eta(R)$. The elements $\bar{z}^2=\{z, z\}$ and $2\bar{z}=b$ in an extension T of S by R must satisfy the conditions (3) and (4), i.e. $(b)\eta_r(z)=2\{z, z\}$ and $\eta_l(z)(b)=$ $=2\{z, z\}$, $(b\in S, \{z, z\}\in S)$. This implies that if $b=0$ or $b=2a$, then $\{z, z\}=0$ or $\{z, z\}=2a$. In either case $T^2=(0)$ or $T^2=(0, 2a)$ and $T\neq S$, so T is not a square extension of S by R. Hence we must have $b=a$ or $b=3a$. By the conditions (3) and (4) we get square extensions if we take $(\eta_l(z), \eta_r(z))=(s_2, s_2)$ and $\{z, z\}=a$ or $3a$, (cf. also Lemma 3). The condition (2) is satisfied.

(i) Let $\{z, z\}=a$ and $b=a$ resp. $b=3a$. Let T_1 be an η-extension of S by R with factor set $\{z, z\}=a$, $b=a$ and let T_2 be an η-extension of S by R with factor set $\{z, z\}'=a$, $b'=3a$. Then $T_1\sim T_2$ as the conditions (5) and (6) are satisfied for $\psi_z=a$. Here $(\eta_l(z), \eta_r(z))=(s_2, s_2)$ and T_1 and T_2 have the same additive group

$T^+ = (\bar{z})$ which is a cyclic group of order 8. As $S = \langle M, S^* \rangle$ both for T_1 and T_2, we get 2 equivalent η-square extensions of S by R (Lemma 3).

(ii) Let $\{z, z\} = 3a$ and $b = a$ resp. $b = 3a$. In the same way as in case (i) we get 2 equivalent η-square extensions T_1 and T_2 of S by R, where T_1 resp. T_2 has the factor set $(3a, a)$ resp. $(3a, 3a)$. Both T_1 and T_2 have again the additive group $T^+ = (\bar{z})$ (cyclic of order 8).

Remark. Our results obtained in Theorems 1, 2 and 3 and Lemmas 1, 2 and 3 are quite analogous to the corresponding Theorems and Lemmas in the paper: H. ONISHI, Commutator extensions of finite groups *Mich. Math. J.*, **13** (1966), 119—126, if one replaces "commutator extension" by "square extension". In fact, the results of ONISHI for finite groups led us to consider the situation for finite rings.

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Über die Struktur der Hauptidealhalbgruppen. I

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§ 1. Einleitung. Vorbereitungen

Unter einer Hauptidealhalbgruppe verstehen wir eine Halbgruppe H, deren sämtliche Rechts- und Linksideale durch ein Element erzeugt sind. Die Untersuchung solcher Halbgruppen wird in drei Stufen durchgeführt: zuerst beschäftigen wir uns mit der Struktur „im Grossen", d.h., wir zeigen, daß die Green'sche Relation $\mathscr{I}$ (bei GREEN $\mathscr{F}$, s. [3]) in einer Hauptidealhalbgruppe eine Kongruenz ist, und beschreiben dann die Faktorhalbgruppe $H/\mathscr{I}$; im zweiten Schritt wird dasselbe bezüglich der Relation $\mathscr{Q}$ (bei GREEN $\mathscr{H}$) gemacht; endlich wird die Operation zwischen Elementen der $\mathscr{Q}$-Klassen untersucht. In diesem ersten Teil werden die ersten zwei Fragen erledigt.

Die Terminologie stimmt meistens mit der von [2] überein. Eine Abweichung findet nur im Falle der Green'schen $\mathscr{H}$-Klassen statt, die wir $\mathscr{Q}$-Klassen nennen wollen, da sie in derselben Beziehung mit den Quasiidealen stehen, wie die $\mathscr{L}$-, $\mathscr{R}$- und $\mathscr{I}$-Klassen mit den Links-, Rechts-, bzw. zweiseitigen Idealen. Die Green'schen Klassen werden wir durch L, R, I, Q usw. bezeichnen, die ein- und zweiseitigen Ideale durch $\mathbf{L}, \mathbf{R}, \mathbf{I}$ usw. In einer beliebigen Halbgruppe H wird für $a \in H$ die a enthaltende $\mathscr{I}$-Klasse durch $I(a)$, das von a erzeugte Ideal durch $\mathbf{I}(a)$ bezeichnet. $I(a) > I(b)$ bedeutet $\mathbf{I}(a) \supset \mathbf{I}(b)$. Für die verwandten Begriffe werden analoge Bezeichnungen gebraucht.

Die ein- und zweiseitigen Idealverbände einer Hauptidealhalbgruppe ist sehr einfach zu beschreiben:

1. 1. (LYAPIN [4]). *Dann und nur dann sind sämtliche Ideale (Linksideale, Rechtsideale) einer Halbgruppe durch ein Element erzeugt, wenn die Relation* $\mathbf{I} \subset \mathbf{I}'$ ($\mathbf{L} \subset \mathbf{L}'$, $\mathbf{R} \subset \mathbf{R}'$) *eine duale Wohlordnung ist.*

In [4] ist diese Tatsache nur für zweiseitige Ideale formuliert, der Beweis ist aber natürlich allgemeingültig.

Aus 1. 1 folgt, daß man die Ideale und auch die Rechts- und Linksideale mit Ordnungszahlen so numerieren kann, daß $\mathbf{I}_\alpha \supset \mathbf{I}_\beta$ ($\mathbf{L}_\alpha \supset \mathbf{L}_\beta$, $\mathbf{R}_\alpha \supset \mathbf{R}_\beta$) äquivalent mit $\alpha<\beta$ ist. Die zugehörigen Green'schen Klassen werden ebenso numeriert, also $I_\alpha>I_\beta\Leftrightarrow\alpha<\beta$ usw. Dabei laufen die Indizes der Ideale, der Links- und der Rechtsideale im allgemeinen verschiedene Mengen durch, nämlich die Menge der Ordnungszahlen, die kleiner als γ, γ_l bzw. γ_r sind. Es ist klar, daß $\gamma\leqq\gamma_l$, $\gamma\leqq\gamma_r$.

Da aus $\mathbf{L}(b)\supseteqq\mathbf{L}(a)$ in jeder Hauptidealhalbgruppe $\mathbf{I}(b)\supseteqq\mathbf{I}(a)$ folgt, so gilt in Hauptidealhalbgruppen wegen 1. 1 auch

1. 2. *In einer Hauptidealhalbgruppe folgt aus* $\mathbf{I}(a)\supset\mathbf{I}(b)$ *stets auch* $\mathbf{L}(a)\supset\mathbf{L}(b)$.

Die duale Aussage besteht natürlich auch. Von jetzt an werden wir aus dualen Aussagen immer nur eine formulieren.

§ 2. Die Faktorhalbgruppe $H/\mathcal{I}$

Unser Zweck ist die Existenz der im Titel des Paragraphen genannten Halbgruppe zu beweisen. Dazu brauchen wir aber noch weitere Vorbereitungen.

Eine $\mathcal{I}$-Klasse I der Hauptidealhalbgruppe H nennen wir *geschichtet*, falls I aus mehr als einer $\mathcal{L}$-Klasse besteht. Später werden wir sehen, daß dieser Begriff selbstdual ist. Zunächst möchten wir zeigen:

Satz 1. *Eine geschichtete $\mathcal{I}$-Klasse der Hauptidealhalbgruppe H ist eine Unterhalbgruppe der letzteren.*

Dem Beweis schicken wir einige Hilfssätze voraus.

2. 1. *In einer beliebigen Halbgruppe H ist die Abbildung* $\mathbf{L}\to\mathbf{L}a$ $(a\in H)$ *des Linksidealverbandes in sich monoton und aus*

$$\mathbf{L}\subset\mathbf{L}',\quad \mathbf{L}a\subset\bar{\mathbf{L}}\subset\mathbf{L}'a \tag{1}$$

folgt die Existenz eines $\mathbf{L}''$ *mit*

$$\mathbf{L}\subset\mathbf{L}''\subset\mathbf{L}',\quad \mathbf{L}''a=\bar{\mathbf{L}}.$$

(Diese Tatsache ist ein Spezialfall einer viel allgemeineren mengentheoretischen Tatsache.) Die erste Behauptung ist trivial und wohlbekannt. Gilt ferner (1), so bezeichnen wir den Rechtsquotienten $\bar{\mathbf{L}}\cdot a=\{x|xa\in\bar{\mathbf{L}}\}$ durch $\mathbf{L}_0$ und es sei $\mathbf{L}''=\mathbf{L}_0\cap\mathbf{L}'$. Offenbar gilt $\mathbf{L}''a\subseteqq\bar{\mathbf{L}}$. Andererseits gibt es wegen $\bar{\mathbf{L}}\subset\mathbf{L}'a$ für jedes $\bar{b}\in\bar{\mathbf{L}}$ ein $b\in\mathbf{L}'$ mit $ba=\bar{b}$; da somit auch $b\in\mathbf{L}_0$ gilt, ist $b\in\mathbf{L}''$, also $\bar{\mathbf{L}}\subseteqq\mathbf{L}''a$, was zu beweisen war.

Für unseren Fall ergibt 2. 1 den Spezialfall

2. 2. *Gelten in der Hauptidealhalbgruppe H die Gleichungen* $\mathbf{L}_\mu a=\mathbf{L}_\alpha$, $\mathbf{L}_{\mu+\nu}a=\mathbf{L}_{\alpha+\beta}$, *so ist* $\beta\leqq\nu$. *Aus* $\mathbf{L}_\alpha b=\mathbf{L}_\alpha$, $\mathbf{L}_\alpha\supseteqq\mathbf{L}_\mu$ *folgt im speziellen* $\mathbf{L}_\mu b\supseteqq\mathbf{L}_\mu$.

In der Tat, nach 2. 1 wird die Menge $\{\mathbf{L}_{\mu+\xi}\}_{0\leqq\xi\leqq\nu}$ auf die Menge $\{\mathbf{L}_{\alpha+\eta}\}_{0\leqq\eta\leqq\beta}$ abgebildet.

2. 3. *Gibt es unter den $\mathscr{L}$-Klassen, die in der $\mathscr{I}$-Klasse I einer Halbgruppe H enthalten sind, eine maximale, sei denn L_0, so gibt es für jedes $L \subseteqq I$ ein $a \in H^1$ mit $La \subseteqq L_0$ (d.h. $\mathbf{L}a = \mathbf{I}$).*

Ist nämlich $L \subseteqq I$, $x \in L$, $y \in L_0$, so gibt es Elemente $b, a \in H^1$ derart, daß $y = bxa$. Offensichtlich ist $L(xa) \geqq L(bxa) = L_0$, aber wegen $xa \in I$ und der Maximalität von L_0 muß hier die Gleichheit statthaben. Es gilt also $xa \in L_0$, dann aber auch $La \subseteqq L_0$.

Im Falle einer Hauptidealhalbgruppe H bedeutet dies, daß für eine geschichtete $\mathscr{I}$-Klasse I_σ mit $\mathbf{I}_\sigma = \mathbf{L}_\alpha$ und für $L_\mu \subset I_\sigma$ stets ein $a \in H$ existiert, so daß $L_\mu a \subseteqq L_\alpha$ (d.h. $\mathbf{L}_\mu a = \mathbf{I}_\sigma$). Ein solches Element werden wir ein *Vergrößerungselement für* L_μ oder *für* $\mathbf{L}_\mu$ nennen.

2. 4. *Sei I_σ eine $\mathscr{I}$-Klasse der Hauptidealhalbgruppe H. Gilt $xy \in \mathbf{I}_{\sigma+1}$ für ein $x \in I_\sigma$ und ein $y \in H$, so gilt auch $\mathbf{I}_\sigma y \subseteqq \mathbf{I}_{\sigma+1}$.*

Wäre nämlich $\mathbf{I}_\sigma y \supset \mathbf{I}_{\sigma+1}$, dann hätten wir $\mathbf{I}_\sigma \subseteqq \mathbf{I}_{\sigma+1} \cdot{}^{\cdot} y$, folglich $\mathbf{I}_\sigma \supset \mathbf{I}_{\sigma+1} \cdot{}^{\cdot} y \supset \supset \mathbf{I}_{\sigma+1}$, d.h.

$$\text{(2)} \qquad \mathbf{I}_\sigma = \mathbf{L}_\alpha, \quad \mathbf{I}_{\sigma+1} \cdot{}^{\cdot} y = \mathbf{L}_{\alpha+\nu}, \quad \mathbf{I}_{\sigma+1} = \mathbf{L}_{\alpha+\xi}, \qquad 0 < \nu < \xi$$

(hier ist $\mathbf{I}_{\sigma+1} \cdot{}^{\cdot} y = \{x | xy \in \mathbf{I}_{\sigma+1}, x \in H\}$). Dann könnte man aber nach 2. 3 ein $b \in H$ derart finden, daß $\mathbf{I}_\sigma yb = \mathbf{L}_\alpha yb = \mathbf{L}_\alpha$ gilt. Andererseits wäre

$$\mathbf{L}_{\alpha+\nu} yb \subseteqq \mathbf{I}_{\sigma+1} b \subseteqq \mathbf{I}_{\sigma+1} = \mathbf{L}_{\alpha+\xi},$$

was aber wegen 2. 2 und (2) unmöglich ist. Somit muß tatsächlich $\mathbf{I}_\sigma y \subseteqq \mathbf{I}_{\sigma+1}$ bestehen.

Es gilt endlich

2. 5. *Ist in der Hauptidealhalbgruppe H*

$$\mathbf{I}_\sigma = \mathbf{L}_\alpha, \quad \mathbf{I}_{\sigma+1} = \mathbf{L}_{\alpha+\xi}, \qquad \xi \neq 1,$$

so ist ξ ein Limeszahl.

Dies folgt aus dem folgenden bekannten Satz:

2. 6 (Lyapin [4], S. 233—234.). *Sei $\mathbf{I}$ das durch die $\mathscr{I}$-Klasse I erzeugte Ideal. Ist $L \cup (\mathbf{I} \setminus I)$ ein Linksideal für eine $\mathscr{L}$-Klasse $L \subseteqq I$, so ist es ein Linksideal für alle $\mathscr{L}$-Klassen mit $L \subseteqq I$.*

Wäre nun $\xi = \eta + 1$, so wäre

$$\mathbf{L}_{\alpha+\eta} = L_{\alpha+\eta} \cup \mathbf{I}_{\sigma+1} = L_{\alpha+\eta} \cup (\mathbf{I}_\sigma \setminus I_\sigma)$$

ein Linksideal, also z.B. auch $\mathbf{L} = L_\alpha \cup \mathbf{I}_{\sigma+1}$ wäre ein solches; dann hätte aber das Linksideal $\mathbf{L} \cup \mathbf{L}_{\alpha+\eta}$ kein erzeugendes Element.

Jetzt können wir zum Beweis des Satzes übergehen.

Nehmen wir an, daß die geschichtete $\mathcal{I}$-Klasse I_σ keine Halbgruppe ist. Dann gibt es Elemente $x, y \in I_\sigma$ derart, daß $xy \in \mathbf{I}_{\sigma+1}$ ist. Nach 2. 4 ist dann $\mathbf{I}_\sigma y \subseteqq \mathbf{I}_{\sigma+1}$ und jetzt nach dem Dualen desselben Hilfssatzes $z\mathbf{I}_\sigma \subseteqq \mathbf{I}_{\sigma+1}$ für jedes $z \in \mathbf{I}_\sigma$, d.h.

(3) $$\mathbf{I}_\sigma^2 \subseteqq \mathbf{I}_{\sigma+1}.$$

Betrachten wir das Linksideal

$$\mathbf{L} = \{x \mid \mathbf{I}_\sigma x \subset \mathbf{I}_\sigma\},$$

(es ist sogar ein Ideal) und es sei $\mathbf{L} = \mathbf{L}(a)$. Wir zeigen zunächst, daß $\mathbf{I}_\sigma a \subseteqq \mathbf{I}_{\sigma+1}$ ist (also auch $\mathbf{I}_\sigma \mathbf{L} \subseteqq \mathbf{I}_{\sigma+1}$). Im Falle $\mathbf{L} = \mathbf{I}_\sigma$ ist das trivial. Ist dagegen $\mathbf{L} \supset \mathbf{I}_\sigma$, so gibt es für jedes $y \in \mathbf{I}_\sigma$ ein $x \in H$ mit $xa = y$, d.h.

(4) $$(\mathbf{I}_\sigma . \cdot a) a = \mathbf{I}_\sigma.$$

Wegen der Definition von a besteht auch

(5) $$\mathbf{I}_\sigma a \subset \mathbf{I}_\sigma,$$

also für $\mathbf{L}_{\varrho_1} = \mathbf{I}_\sigma . \cdot a$ muß wegen (4) und (5) $\mathbf{L}_{\varrho_1} \supset \mathbf{I}_\sigma$ gelten. Ist noch $\mathbf{L}(a) \supset \mathbf{L}_{\varrho_1}$, so gilt wieder $(\mathbf{L}_{\varrho_1} . \cdot a) a = \mathbf{L}_{\varrho_1}$ und $\mathbf{L}_{\varrho_2} = \mathbf{L}_{\varrho_1} . \cdot a \supset \mathbf{L}_{\varrho_1}$. Setzt man diesen Prozess fort, so erhält man eine Linksidealkette

$$\mathbf{L}_{\varrho_1} \subset \mathbf{L}_{\varrho_2} \subset \ldots,$$

die in endlich vielen Schritten abbrechen muß. Dies geschieht, sobald $\mathbf{L}_{\varrho_n} \supseteqq \mathbf{L}(a)$ ist. Dann haben wir

$$\mathbf{I}_\sigma = \mathbf{L}_{\varrho_1} a = \mathbf{L}_{\varrho_2} a^2 = \ldots = \mathbf{L}_{\varrho_n} a^n,$$

also wegen $a \in \mathbf{L}_{\varrho_n}$ auch $a^{n+1} \in \mathbf{I}_\sigma$. Folglich gilt nach (3) $\mathbf{I}_\sigma a^{n+1} \subseteqq \mathbf{I}_{\sigma+1}$ und das ergibt, daß es eine kleinste k $(0 \leqq k \leqq n)$ existiert, so daß $\mathbf{I}_\sigma a^{k+1} \subseteqq \mathbf{I}_{\sigma+1}$. Wäre dabei $k > 0$, so hätten wir

$$\mathbf{I}_\sigma \supset \mathbf{I}_\sigma a^k \supset \mathbf{I}_{\sigma+1}, \quad \mathbf{I}_\sigma a^{k+1} \subseteqq \mathbf{I}_{\sigma+1}.$$

Dann wäre aber $xa \in \mathbf{I}_{\sigma+1}$ für $x \in \mathbf{I}_\sigma a^k \setminus \mathbf{I}_{\sigma+1}$, und $xa \notin \mathbf{I}_{\sigma+1}$ für $x \in \mathbf{I}_\sigma \setminus \mathbf{I}_\sigma a^k$, im Widerspruch mit 2. 4. Somit muß $\mathbf{I}_\sigma a \subseteqq \mathbf{I}_{\sigma+1}$ gelten.

Nehmen wir nun ein beliebiges Element $b \notin \mathbf{L}$ und es sei wieder $\mathbf{I}_\sigma = \mathbf{L}_\alpha$. Wegen $\mathbf{I}_\sigma b = \mathbf{I}_\sigma$ und 2. 2 müßen für die Linksideale

$$\mathbf{L}_{\alpha_0} = \mathbf{L}_\alpha b, \quad \mathbf{L}_{\alpha_1} = \mathbf{L}_{\alpha+1} b, \ldots, \quad \mathbf{L}_{\alpha_m} = \mathbf{L}_{\alpha+m} b, \ldots$$

die Ungleichungen $\alpha_i \leqq \alpha + i$ gelten. Da $\mathbf{L}_\xi x = \mathbf{L}_\eta$ offensichtlich gleichbedeutend mit $L_\xi x \subseteqq L_\eta$ ist, folgt aus dem Gesagten, daß $\left(\bigcup_{i=0}^m L_i\right) b \subseteqq \bigcup_{i=0}^m L_i$ für $m = 0, 1, 2, \ldots$.

Dann sind aber die Mengen $M_m = \bigcup_{i=0}^m L_i \cup \mathbf{I}_{\sigma+1}$ Rechtsideale, da

$$M_m b = \left(\bigcup_{i=0}^m L_i\right) b \cup \mathbf{I}_{\sigma+1} b \subseteqq \bigcup_{i=0}^m L_i \cup \mathbf{I}_{\sigma+1} = M_m$$

für $b \notin \mathbf{L}$ und $M_m b \subseteqq \mathbf{I}_{\sigma+1}$ für $b \in \mathbf{L}$ besteht. Dabei ist $M_0 \subset M_1 \subset M_2 \subset \ldots$ wegen 2. 5 eine unendliche wachsende Rechtsidealkette, im Widerspruch mit 1. 1. Dies vollendet den Beweis von Satz 1.

Nehmen wir zwei Elemente $a, b \notin \mathbf{I}_{\sigma+1}$. Ist $ab \in \mathbf{I}_{\sigma+1}$ d.h. $\mathbf{L}(a)\mathbf{R}(b) \subseteqq \mathbf{I}_{\sigma+1}$, so gibt es wegen $I_\sigma \cap \mathbf{L}(a) \neq \emptyset$, $I_\sigma \cap \mathbf{R}(b) \neq \emptyset$ auch Elemente $a', b' \in I_\sigma$ mit $a'b' \in \mathbf{I}_{\sigma+1}$. Dies ergibt

2. 7. *Ist I_σ geschichtet, so ist $H \setminus \mathbf{I}_{\sigma+1}$ eine Halbgruppe* (d.h. $\mathbf{I}_{\sigma+1}$ ein Primideal).

Wir brauchen noch weitere Hilfssätze.

2. 8. *Für jede $\mathscr{L}$-Klasse L_μ einer geschichteten $\mathscr{I}$-Klasse I_σ gibt es ein Vergrößerungselement in I_σ.*

Nach Satz 1 ist für ein beliebiges $x \in I_\sigma$ noch $\mathbf{L}_\mu x \supset \mathbf{I}_{\sigma+1}$. Nach 2. 3 existiert dann ein $a \in H^1$ mit $\mathbf{L}_\mu xa = \mathbf{I}_\sigma$. Somit ist xa ein Vergrösserungselement und $xa \in I_\sigma$.

2. 9. *Für jedes $a \in H$ gibt es in I_σ ein L_μ, so daß $\mathbf{L}_\mu a \neq \mathbf{I}_\sigma$.*

Im entgegengesetzten Falle wäre nämlich $\mathbf{L}_\mu a = \mathbf{I}_\sigma$ für $\mathbf{L}_\mu \supset \mathbf{I}_{\sigma+1}$, $\mathbf{I}_{\sigma+1} a \subseteqq \mathbf{I}_{\sigma+1}$ im Widerspruch mit 2. 2.

2. 10. *Ist*

$$\mathbf{I}_\sigma = \mathbf{L}_\alpha, \quad \mathbf{L}_{\alpha+1} \supset \mathbf{I}_{\sigma+1},$$

so sind $H \setminus \mathbf{L}_{\alpha+1}$ und L_α Teilhalbgruppen.

Es genügt das erste zu zeigen, da $L_\alpha = \mathbf{I}_\sigma \cap (H \setminus \mathbf{L}_{\alpha+1})$ ist. Sind aber $a, a' \in H \setminus \mathbf{L}_{\alpha+1}$, $aa' \in \mathbf{L}_{\alpha+1}$, so hat man

$$\mathbf{I}_\sigma a' \subseteqq \mathbf{L}(a)\, a' \subset \mathbf{L}_{\alpha+1}. \tag{6}$$

Andererseits gibt es nach 2. 8 in I_σ ein b mit $\mathbf{I}_\sigma b = \mathbf{I}_\sigma$. Da dann $b \in L(a')$, d.h. $b = xa'$ ist, so besteht

$$\mathbf{I}_\sigma = \mathbf{I}_\sigma b = \mathbf{I}_\sigma xa' \subseteqq \mathbf{I}_\sigma a',$$

im Widerspruch mit (6).

2. 11. $\mathbf{I}_\sigma a = \mathbf{I}_\sigma$ *für jedes* $a \in H \setminus \mathbf{L}_{\alpha+1}$.

Für $b \in \mathbf{I}_\alpha$ hat man nämlich $ba \in L_{\bar{\alpha}}$, also $\mathbf{I}_\sigma a = \mathbf{L}(b)a = \mathbf{L}(ba) = \mathbf{I}_\sigma$.

Satz 2. *Eine geschichtete $\mathscr{I}$-Klasse I_σ kann keine $\mathscr{R}$-Klasse sein.*

Es sei wieder $\mathbf{I}_\sigma = \mathbf{L}_\alpha$ und nehmen wir ein $a \in L_\alpha$. Nach 2. 9 gibt es ein $L_\mu \subset I_\sigma$, so daß $\mathbf{L}_\mu a \neq \mathbf{I}_\sigma$ und nach 2. 8 gibt es ein $b \in I_\sigma$ mit $\mathbf{L}_\mu b = \mathbf{I}_\sigma$. Wäre nun $R(a) = R(b)$, so gälte $a = bx$, also $L(x) \geqq L_\alpha$, d.h. $x \in H \setminus \mathbf{L}_{\alpha+1}$. Hieraus folgt aber wegen 2. 11

$$\mathbf{L}_\mu a = \mathbf{L}_\mu bx = \mathbf{I}_\sigma x = \mathbf{I}_\sigma,$$

im Widerspruch mit dem Wahl von $\mathbf{L}_\mu$.

Dieser Satz besagt, daß der Begriff der geschichteten $\mathscr{I}$-Klasse selbstdual ist.

Somit sind die dualen Aussagen von 2. 5, 2. 8, 2. 9, 2. 10 und 2. 11 auch gültig. Die duale Definition der geschichteten $\mathcal{J}$-Klassen werden wir im folgenden als gleichberechtigte mit der früher angegebenen anschauen.

Jetzt ist schon alles bereit für den Beweis des ersten Hauptresultats:

Satz 3. *In einer Hauptidealhalbgruppe H ist die Green'sche Relation $\mathcal{J}$ eine Kongruenz.*

Wir haben $I_\varrho I_\sigma \subseteqq I_\tau$ für beliebige ϱ, σ und passendes τ zu zeigen. Wir unterscheiden zwei Fälle.

a) Weder I_ϱ, noch I_σ sind geschichtet. Dann ist $I_\varrho = L_\mu$, $I_\sigma = R_\nu$, also nach [5], Lemma 1 ist $I_\varrho I_\sigma$ sogar in einer einzigen $\mathcal{D}$-Klasse enthalten.

b) I_σ ist geschichtet. Wir behaupten, daß dann immer

(7) $$I_\varrho I_\sigma \subseteqq \min(I_\varrho, I_\sigma)$$

besteht. Sei nämlich I_ϱ daß größte unter den $\mathcal{J}$-Klassen, für welche (7) nicht erfüllt ist. Es kann nicht $I_\varrho \geqq I_\sigma$ sein, da dann $I_\varrho I_\sigma \subseteqq I_\sigma$ besteht und (7) aus 2. 7 folgt. Es ist also $I_\varrho < I_\sigma$. Dabei ist I_ϱ nicht geschichtet, sonst konnte man ϱ und σ vertauschen und das vorige Argument träte wieder in Kraft. Wir haben also $I_\varrho = L_\alpha$, und wegen $I_\varrho a \subseteqq\!\!\!\!/\; I_\varrho$ für $a \in I_\sigma$

$$\mathbf{I}_\varrho a \subseteqq \mathbf{L}_{\alpha+1} = \mathbf{I}_{\varrho+1},$$

d. h.

(8) $$I_\varrho a \cap I_\varrho = \emptyset.$$

Andererseits gibt es wegen $L(a) > L_\alpha$ für jedes $b \in I_\varrho$ ein $x \in H$ mit $xa = b$. Dabei kann x wegen (8) nicht in I_ϱ liegen, aber auch $x \in I_\xi$, $I_\sigma > I_\xi > I_\varrho$ ist unmöglich, da für diese I_ξ nach der Annahme

$$I_\xi a \subseteqq I_\xi I_\sigma \subseteqq I_\xi,$$

gilt und trivialerweise auch $I_\xi \geqq I_\sigma$ ausgeschloßen ist. Dieser Widerspruch vollendet den Beweis, da der Fall, wo I_ϱ geschichtet ist, durch Dualisieren des betrachteten Falles entsteht.

Es ist klar, daß ein homomorphes Bild einer Hauptidealhalbgruppe wieder eine solche ist. $H/\mathcal{J}$ ist also eine Hauptidealhalbgruppe, in welcher jede $\mathcal{J}$-Klasse aus einem Element besteht. Die so beschaffene Halbgruppen sind leicht zu beschreiben:

Nehmen wir eine transfinite Folge $\mathfrak{f} = (n_0, \ldots, n_\sigma, \ldots)_{\sigma<\tau}$, deren Glieder natürliche Zahlen und Symbole ∞ sind und $\tau \geqq 1$ ist. Jedem $\sigma < \tau$ ordnen wir eine zyklische Halbgruppe H_σ der Ordnung n_σ zu, welche im Falle eines endlichen n_σ auch den Index n_σ hat, und es sei $H_\varrho \cap H_\sigma = \emptyset$ für $\varrho \neq \sigma$. Für Elemente verschiedener H_σ definieren wir die Multiplikation durch

(9) $$(h_\sigma h_\varrho =)\, h_\varrho h_\sigma = h_{\max(\varrho,\sigma)} \qquad (h_\varrho \in H_\varrho,\, h_\sigma \in H_\sigma).$$

Dann ist $H_\mathfrak{f} = \bigcup_{\sigma<\tau} H_\sigma$ offensichtlich eine kommutative Halbgruppe.

Satz 4. *$H_{\mathfrak{f}}$ ist eine Hauptidealhalbgruppe, in der jede $\mathscr{I}$-Klasse ein Element hat, und jede Hauptidealhalbgruppe mit dieser Eigenschaft ist einer $H_{\mathfrak{f}}$ isomorph.*

In der Tat, betrachten wir ein Ideal $\mathbf{I}$ von $H_{\mathfrak{f}}$ und sei σ der kleinste Index, für welchen $\mathbf{I} \cap H_\sigma \neq \emptyset$, a das erzeugende Element von $H_\sigma(=\langle a \rangle)$, und k der kleinste Exponent für welchen $a^k \in \mathbf{I}$. Da die höheren Potenzen von a Mehrfachen von a^k sind und nach (9) dasselbe für sämtliche Elemente der H_ϱ mit $\varrho > \sigma$ gilt, so ist

$$\mathbf{I} = \mathbf{I}(a^k) = \Big(\bigcup_{\sigma < \varrho < \tau} H_\varrho\Big) \cup H_\sigma a^k \cup a^k.$$

Umgekehrt, sei H eine Hauptidealhalbgruppe, in welcher jede $\mathscr{I}$-Klasse aus einem Element besteht. Der Ordnungstypus der Idealkette von H soll γ sein und $h_\sigma(\sigma < \gamma)$ soll das einzige Element der $\mathscr{I}$-Klasse I_σ bedeuten. Ferner, definiere man die folgenden Ordnungszahlmengen: Λ sei die Menge der Ordnungszahlen $\lambda(<\gamma)$ vom zweiten Typus und

$$\Gamma = \{\sigma \mid \sigma = \xi + 1, h_\xi^2 = h_\xi\}, \quad \Delta = \Lambda \cup \Gamma.$$

Endlich, für beliebiges $\sigma < \gamma$ bezeichne σ' das Maximum von Δ unter σ:

$$\sigma' \leqq \sigma, \quad \sigma' \in \Delta,$$

$$\sigma' < \tau \in \Delta \Rightarrow \sigma < \tau.$$

Ein solches σ' gibt es offensichtlich. Auch ist es klar, daß dann σ in der Form

$$\sigma = \sigma' + k - 1$$

darstellbar ist, wo k eine natürliche Zahl bedeutet. Wir zeigen, daß dann

$$h_\sigma = h_{\sigma'}^k,$$

gilt und auch (9) für $\varrho < \sigma'$ besteht.

Nehmen wir an, daß dies für jedes $\tau < \sigma$ schon bewiesen ist. Ist dabei $\sigma \notin \Delta$ (also $\sigma = \xi + 1$ und $k > 1$), so gibt es ein $h_\eta \in H$ mit $h_\eta h_\xi = h_\sigma$, η minimal. Wegen der Induktionsannahme gilt (9) für h_ξ statt h_σ und für $\varrho < \sigma'$, also muß $\eta \geqq \sigma'$ sein. Wäre aber $\eta > \sigma'$, so hätten wir $h_{\sigma'}^k = h_{\sigma'} h_\xi = h_\xi = h_{\sigma'}^{k+1}$, also auch $h_\xi^2 = h_\xi$, d.h. $\sigma = \xi + 1 \in \Delta$, entgegen unserer Annahme. Somit ist $\eta = \sigma'$ und $h_\sigma = h_{\sigma'} h_\xi = h_{\sigma'}^k$. Ist ferner $\varrho < \sigma'$ so gilt

$$h_\varrho h_\sigma = h_\varrho h_{\sigma'} h_\xi = h_{\sigma'} h_\xi = h_\sigma$$

und ebenso die duale Gleichung.

Es sei nun $\sigma \in \Delta$, $\varrho < \sigma$. Wieder gibt es ein h_η mit $h_\varrho h_\eta = h_\sigma$. Wir wollen zeigen, daß $\eta = \sigma$ ist. Dazu genügt es einzusehen, daß $h_\varrho h_\tau \neq h_\sigma$ für $\varrho, \tau < \sigma$. Ist dies falsch und z.B. $\varrho \leqq \tau$, so ist jedenfalls $\tau' \leqq \varrho$, da sonst $h_\varrho h_\tau = h_\tau$ besteht. Dann ist aber $h_{\varrho'} = h_{\tau'}$ und

$$h_\varrho h_\tau = h_{\tau'}^l = h_{\tau' + l - 1}$$

für ein $l < \omega$. Ist dabei $\sigma \in \Lambda$, so ist wegen $\tau' \leqq \tau < \sigma$ auch $\tau' + l - 1 < \sigma$. Ist dagegen

$\sigma \in \Gamma$, $\sigma = \xi + 1$, dann ist $h_\xi^2 = h_\xi$ und $h_\xi h_\sigma = h_\xi \cdot h_\xi x = h_\xi x = h_\sigma$ und auch

$$I(h_\sigma) \geqq I(h_\varrho h_\sigma) \geqq I(h_\xi h_\sigma) = I(h_\sigma),$$

also $h_\varrho h_\sigma = h_\sigma$. Somit ist unsere Behauptung bewiesen.

Sind nun $\sigma_0(=0), \sigma_1, \ldots, \sigma_\alpha, \ldots$ die Elemente der Menge Δ, der Größe nach geordnet, so besteht die zyklische Halbgruppe $H_\alpha = \langle h_{\sigma_\alpha} \rangle$ nach den Obigen aus den Elementen h_σ mit $\sigma_\alpha \leqq \sigma < \sigma_{\alpha+1}$. Dabei besteht nach dem Gezeigten auch (9). Ist also $n_\alpha = o(H_\alpha)$ endlich, so ist $h_{\sigma_\alpha}^{n_\alpha+1} \neq h_{\sigma_\alpha}^l$ mit $l < n_\alpha$, da sonst die verschiedenen Elemente $h_{\sigma_\alpha}^l$ und $h_{\sigma_\alpha}^{n_\alpha}$ $\mathcal{I}$-äquivalent wären, im Widerspruch mit der Voraussetzung über H. Der Index von H_α ist hiermit n_α. Dies bedeutet aber $H \cong H_{\mathfrak{f}}$, wo $\mathfrak{f} = (n_0, \ldots, n_\alpha, \ldots)_{\sigma_\alpha \in \Delta}$ ist.

§ 3. Die Kongruenzrelation $\mathcal{Q}$

Wir erinnern den Leser, daß die Relation $\mathcal{Q}$ durch $\mathcal{Q} = \mathcal{R} \cap \mathcal{L}$ definiert ist. Wir zeigen sogleich:

Satz 5. *In einer Hauptidealhalbgruppe H ist $\mathcal{Q}$ eine Kongruenz.*

Es sei $a\mathcal{Q}a'$, $b\mathcal{Q}b'$; wir wollen $ab\mathcal{Q}a'b'$ zeigen. Betrachten wir das Element ab'. Trivialerweise gilt

(10) $$ab\mathcal{R}ab'$$

Ferner gibt es Elemente x, y mit $b' = bx$, $b = b'y$. Wegen $\mathbf{L}(b) = \mathbf{L}(b')$ ist $\mathbf{L}(b)x = \mathbf{L}(b')$. Für $L_\alpha \leqq L(b)$ gilt also $L_\alpha x \subseteqq L_{\alpha'}$ mit $L_{\alpha'} \leqq L(b)$. Andererseits haben wir $cxy = c$ für $c \in \mathbf{L}(b)$, also $L_\alpha xy = L_\alpha$ und somit $L_{\alpha'} y \cap L_\alpha \neq \emptyset$, also $L_{\alpha'} y \subseteqq L_\alpha$ (eigentlich sogar $L_{\alpha'} y = L_\alpha$). Dies bedeutet aber, daß die Abbildung $L_\alpha \to L_{\alpha'}$ eine ein-eindeutige Abbildung der Menge der $\mathcal{L}$-Klassen von $\mathbf{L}(b)$ in sich ist. Da dabei diese Abbildung nach 2. 1 monoton und „stetig" (im Sinne von (1)) ist, ferner überführt sie $L(b)$ (die maximale $\mathcal{L}$-Klasse von $\mathbf{L}(b)$) in sich, muß sie die identische Abbildung sein. U.a. gilt $L(ab)x \subseteqq L(ab)$, also $ab'\mathcal{L}ab$. Mit (10) zusammen ergibt dies $ab\mathcal{Q}ab'$.

Ein dualer Gedankengang führt zu $ab'\mathcal{Q}a'b'$, also

$$ab\,\mathcal{Q}\,ab'\,\mathcal{Q}\,a'b',$$

was zu beweisen war.

Im § 2 haben wir erhalten, daß jede geschichtete $\mathcal{I}$-Klasse I_σ der Hauptidealhalbgruppe H selber eine Halbgruppe ist. Nach einem Satz von Clifford [1], den man auf die Rees'sche Faktorhalbgruppe $H/\mathbf{I}_{\sigma+1}$ anwendet, ist I_σ sogar einfach. Es kann aber vorkommen, daß sie schon keine Hauptidealhalbgruppe ist. Nimmt man doch statt H die Halbgruppe $H/\mathcal{Q}$, d.h. eine Hauptidealhalbgruppe, in welcher die $\mathcal{Q}$-Klassen je ein Element haben, so sind in dieser sie geschichteten $\mathcal{I}$-Klassen schon einfache Hauptidealhalbgruppen. Die folgenden Überlegungen dienen zur Begründung dieser Behauptung.

Im folgenden, wie vorher, sei I_σ eine geschichtete $\mathcal{J}$-Klasse und $\mathbf{I}_\sigma = \mathbf{R}_\beta = \mathbf{L}_\alpha$. Wir bemerken zuerst, daß es nach dem Dualen von 2. 11 für $a \in R_\beta$ ein $e \in \mathbf{I}_\sigma$ (also $e \in I_\sigma$) mit $ae = a$ gibt. Dann gilt natürlich $xe = x$ für alle $x \in \mathbf{L}(a)$. Ein solches e nennen wir ein *äußeres Rechtseinselement* für $\mathbf{L}(a)$. Bezeichnen wir durch $\mathbf{L}_\varepsilon$ das maximale Linksideal, für welches ein äußeres Rechtseinselement e in $\mathbf{I}_\sigma$ gibt; nach dem gesagten gilt $\mathbf{I}_\sigma \supseteqq \mathbf{L}_\varepsilon \supset \mathbf{I}_{\sigma+1}$. Wegen $He \supseteqq \mathbf{L}_\varepsilon$ ist $e \in H \setminus \mathbf{L}_{\varepsilon+1}$ klar.

3. 1. *Hat die $\mathcal{R}$-Klasse R_ϱ einen nichtleeren Durchschnitt mit $\mathbf{I}_\sigma \setminus \mathbf{L}_\varepsilon$, so besteht R_ϱ aus einer einzigen $\mathcal{Q}$-Klasse.*

In der Tat, es sei $a, b \in I_\sigma$, $a\mathcal{R}b$, aber nicht $a\mathcal{L}b$. Bestimmtheitshalber sei $L(a) > L(b)$; dann genügt es $a \in \mathbf{L}_\varepsilon$ zu zeigen. Nach $a\mathcal{R}b$ gibt es Elemente x, y mit $ax = b$, $by = a$, d.h. $axy = a$. Dabei muß $x \in \mathbf{I}_\sigma$ sein, da sonst nach 2. 11 $\mathbf{I}_\sigma x = \mathbf{L}_\alpha x = \mathbf{L}_\alpha$ und nach 2. 2 $L(ax) \geqq L(a)$ wäre, was der Annahme widerspricht. Dann ist aber $xy \in \mathbf{I}_\sigma$ ein äußeres Rechtseinselement für $\mathbf{L}(a)$, also $\mathbf{L}(a) \subseteqq \mathbf{L}_\varepsilon$, was zu beweisen war.

Aus 3. 1 folgt unmittelbar

3. 2. *Für $\alpha \leqq \mu < \varepsilon$ ist L_μ die Vereinigungsmenge von $\mathcal{R}$-Klassen.*

Jetzt können wir beweisen

3. 3. *H sei eine Hauptidealhalbgruppe, I_σ eine geschichtete $\mathcal{J}$-Klasse in H. Für $a \in I_\sigma$, $b \notin \mathbf{I}_\sigma$ gilt $ab\mathcal{Q}a$.*

Zuerst zeigen wir $ab\mathcal{R}a$, oder, was dasselbe ist, $R_\varrho b \subseteqq R_\varrho$ für $R_\varrho \subset I_\sigma$. Für $\varrho = \beta$ (d.h. $\mathbf{R}_\varrho = \mathbf{I}_\sigma$) folgt dies aus dem Dualen von 2. 10. Nehmen wir an, daß R_ϱ die maximale $\mathcal{R}$-Klasse ist, für welche $R_\varrho b \subseteqq\!\!\!\!/\; R_\varrho$ ist. Da dann

$$\mathbf{I}_\varrho b = \mathbf{I}_\sigma, \quad \bigcup_{\xi<\varrho} R_\xi b \subseteqq \bigcup_{\xi<\varrho} R_\xi, \quad \mathbf{R}_{\varrho+1} b \subseteqq \mathbf{R}_{\varrho+1}$$

ist, muß $R_\varrho b \supset R_\varrho$ gelten, also es gibt ein $a \in R_\varrho$ mit $ab \in R_\varrho$. Wäre nun $R_\varrho \subseteqq\!\!\!\!/\; \mathbf{L}_\varepsilon$, so wäre R_ϱ nach 3. 1 eine $\mathcal{Q}$-Klasse und dann folgt aus $a \in R_\varrho$, $ab \in R_\varrho$ wegen des bekannten Lemma von GREEN (S. [3]), daß $R_\varrho b = R_\varrho$ ist. Deshalb muß $R_\varrho \subseteqq L_\varepsilon$ sein. Dann ist aber $ae = a$ für ein beliebiges $a \in R_\varrho$ und wegen $e \in R(b)$ gibt es ein x mit $bx = e$, $abx = a$, also $ab\mathcal{R}a$, und damit $R_\varrho b \subseteqq R_\varrho$.

Um $ab\mathcal{L}a$ zu zeigen, bemerken wir, daß dies für $L(a) > L_\varepsilon$ schon aus dem gezeigten folgt, da für solche a nach 3. 2 $ab\mathcal{R}a \Rightarrow ab\mathcal{L}a$. Für $a \in L_\varepsilon$ haben wir wie oben $a = ae = abx$. Es sei $L_\alpha b \subseteqq L_{\alpha'}$ für $\alpha \geqq \varepsilon$. Wie beim Beweis von Satz 5, erhalten wir dann, daß $L_\alpha \to L_{\alpha'}$ die identische Abbildung für die $L_\alpha \leqq L_\varepsilon$ ist, also auch $ab \in L(a)$ besteht. Dies vollendet den Beweis.

Aus dem bewiesenen folgt unmittelbar

Satz 6. *Besteht in der geschichteten $\mathcal{J}$-Klasse I_σ jede $\mathcal{Q}$-Klasse aus einem Element, so ist I_σ eine Hauptidealhalbgruppe.*

In der Tat, nach 2. 10 und Satz 1 ist $H \setminus \mathbf{I}_{\sigma+1}$ eine Halbgruppe. Es ist leicht zu sehen, daß es sogar eine Hauptidealhalbgruppe ist und ihre einseitigen Ideale genau die Untermengen der Form $\mathbf{L} \setminus \mathbf{L}_{\sigma+1}$ bzw. $\mathbf{R} \setminus \mathbf{I}_{\sigma+1}$ sind, wo $\mathbf{L}$ ($\mathbf{R}$) ein beliebiges Links- (Rechts-) ideal mit $\mathbf{L} \supset \mathbf{I}_{\sigma+1}$ ($\mathbf{R} \supset \mathbf{I}_{\sigma+1}$) ist. Nach 3. 3 müßen die durch $a \in I_\sigma$ erzeugten rechtsseitigen Hauptideale in $H \setminus \mathbf{I}_{\sigma+1}$ und in I_σ zusammenfallen, da die von a verschiedenen rechtsseitigen Vielfachen von a nur durch Multiplikation mit einem $x \in I_\sigma$ entstehen können. Da aber jedes Rechtsideal die Vereinigung rechtsseitiger Hauptideale ist, sind die Rechtsideale in $H \setminus \mathbf{I}_{\sigma+1}$ und I_σ die gleiche, d.h. nur die Hauptideale. Das Duale folgt aus dem Dualen von 3. 3.

Betrachten wir jetzt eine Hauptidealhalbgruppe H, in der jede $\mathcal{Q}$-Klasse aus einem Element besteht und nehmen wir ein $b \in H$, welches in einer nichtgeschichteten $\mathcal{I}$-Klasse enthalten ist (und somit das einzige Element der letzteren ist). Liegt eine Potenz b^n von b in einer geschichteten $\mathcal{I}$-Klasse I_σ, so gilt nach 3. 3 $b^n a = ab^n = a$ für jedes $a \in I_\sigma$, also b^n ist das Einselement von I_σ.

Nehmen wir eine einfache Hauptidealhalbgruppe E mit Einselement, deren jede $\mathcal{Q}$-Klasse aus einem Element besteht, und eine zyklische Halbgruppe Z, die eine Periode der Länge 1 hat, d.h., für deren erzeugendes Element z die Gleichung $z^{n+1} = z^n$ $(n = o(Z))$ gilt. Eine Idealerweiterung von Z durch E nennen wir eine Halbgruppe vom Typ $Z \circ E$, falls z^n gleich dem Einselement von E ist und folglich $zx = xz = x$ für $x \in E$ gilt. Aus den oben gesagten erhielt man leicht:

Satz 7. *Jede Hauptidealhalbgruppe H, in der sämtliche $\mathcal{Q}$-Klassen aus je einem Element bestehen, läßt sich als die Vereinigung $\bigcup_{\alpha<\tau} H_\alpha$ einer wohlgeordneten Menge ihrer Unterhalbgruppen H_α darstellen, wo jede H_α zu einem der folgenden Typen gehört:*

a) *unendliche zyklische Halbgruppen,*

b) *endliche zyklische Halbgruppen mit einer Periode der Länge* 1,

c) *einfache Hauptidealhalbgruppen, in denen die $\mathcal{Q}$-Klassen aus einem Element bestehen,*

d) *Halbgruppen vom Typ $Z \circ E$,*

und das Produkt von Elementen aus verschiedenen H_α durch (9) *definiert ist. Umgekehrt, jede so beschaffene Halbgruppe ist eine Hauptidealhalbgruppe mit $\mathcal{Q}$-Klassen aus je einem Element.*

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On oscillation of the number of primes in an arithmetical progression

By I. KÁTAI in Budapest

1. J. E. LITTLEWOOD [1] proved — in the contrary to an assertion of RIEMANN — that, for a suitable sequence $x_1' < x_2' < \dots$ of integers, the inequality

$$\pi(x_\nu') > \operatorname{li} x_\nu'$$

holds. SKEWES [2] has obtained an upper bound for the first x for which the difference $\pi(x) - \operatorname{li} x$ is positive, namely exp exp exp exp (7,705). Later S. KNAPOWSKI [3] — using the ideas of P. TURÁN — gave another proof of this fact. In the last year, S. LEHMAN [4] gave a better upper bound, namely $1{,}65 \cdot 10^{1165}$.

Recently S. KNAPOWSKI and P. TURÁN gave an explicit, localized $\Omega_\pm$ estimation for the difference $\pi(x, 4, 1) - \frac{1}{2} \operatorname{li} x$, where, in general, $\pi(x, k, l)$ denotes the number of primes in the arithmetical progression $\equiv l \pmod k$ not exceeding x.

The investigation of the oscillation behavior of $\pi(x, 4{,}3) - \frac{1}{2} \operatorname{li} x$ is a simpler case. However, for this we need another method.

In the following, $c, c_0, c_1, \dots, \delta$ will denote explicitly calculable numerical constants (e. c. n. c.), not the same in every place. $e_1(x)$ means e^x and $e_\nu(x) = e_1(e_{\nu-1}(x))$, further $\log_1 x = \log x$ and $\log_\nu x = \log(\log_{\nu-1} x)$. Throughout the paper the letter p is preserved for primes.

We shall prove the following

Theorem 1. *For every $T > c_0$ we have*

$$\max_{T \leqq x \leqq T^\varkappa} \frac{\pi(x, 4, 3) - \frac{1}{2} \operatorname{li} x}{\sqrt{x}/\log x} > \delta, \qquad \min_{T \leqq x \leqq T^\varkappa} \frac{\pi(x, 4, 3) - \frac{1}{2} \operatorname{li} x}{\sqrt{x}/\log x} < -\delta,$$

where δ and c_0 are e. c. n. positive constants and $\varkappa = (2 + \sqrt{3})^2$.

In their papers [5], [6] KNAPOWSKI and TURÁN dealt with the oscillation behavior of the functions

$$a(x) = \sum_{p \equiv l_1 \pmod 8} \log p \cdot e^{-px} - \sum_{p \equiv l_2 \pmod 8} \log p \cdot e^{-px},$$

$$b(x) = \sum_{p \equiv l_1 \pmod 8} e^{-px} - \sum_{p \equiv l_2 \pmod 8} e^{-px}.$$

In [5] they proved that if $0<\delta<c_1$, then for $l_1\not\equiv l_2\not\equiv 1 \pmod 8$ we have

$$\max_{\delta\leqq x\leqq\delta^{1/3}} a(x) > \frac{1}{\sqrt{\delta}}\, e_1\left(-22\frac{\log(1/\delta)\cdot\log_3(1/\delta)}{\log_2(1/\delta)}\right),$$

where c_1 is an e. c. n. c. In [6] they proved that for $l_1, l_2=3, 5, 7$ $(l_1\neq l_2)$ we have

$$\max_{\delta\leqq x\leqq\delta^{1/3}} |b(x)| \geqq \frac{1}{\sqrt{\delta}}\, e_1\left(-23\frac{\log(1/\delta)\cdot\log_3(1/\delta)}{\log_2(1/\delta)}\right).$$

The authors remarked: "To the more difficult problem of one-sided theorems (for $b(x)$) we hope to return." This problem seems still to be open.

From our Theorem 5 it follows that, for $0<y<c$ and for all $l_1\not\equiv l_2\not\equiv 1 \pmod 8$, the inequality

$$\max_{y^{\varkappa}\leqq x\leqq y} b(x)\sqrt{x}\log(1/x) > \delta$$

holds, where $\varkappa=(2+\sqrt{3})^2$.

We formulate now some theorems the proofs of which are similar to the proof of Theorem 1.

Let $N_k(l)$ denote the number of solutions of the congruence $x^2\equiv l \pmod k$. For the moduli k in

(A) 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 19, 24,

the position of the zeros of $L(s,\chi)$ for all $\chi \pmod k$ is known in the neighbourhood of the real line. Especially, it was proved by Haselgrove, that the $L(s,\chi)$ are non-vanishing on the real line in the critical strip.

Theorem 2. *For k in* (A) *and for all of those pairs l_1, l_2 for which $N_k(l_1)=N_k(l_2)$, $l_1\not\equiv l_2 \pmod k$, we have*

$$\max_{T\leqq x\leqq T^{\varkappa}} \frac{\pi(x,k,l_1)-\pi(x,k,l_2)}{\sqrt{x}/\log x} > \delta, \qquad \min_{T\leqq x\leqq T^{\varkappa}} \frac{\pi(x,k,l_1)-\pi(x,k,l_2)}{\sqrt{x}/\log x} < -\delta,$$

if $T>c$, where $\varkappa=(2+\sqrt{3})^2$, c and δ are e. c. n. positive constants.

Theorem 3. *For all k in* (A) *and for all l for which $N_k(l)=0$, we have the inequalities*

$$\max_{T\leqq x\leqq T^{\varkappa}} \frac{\pi(x,k,l)-\dfrac{\mathrm{li}\,x}{\varphi(k)}}{\sqrt{x}/\log x} > \delta, \qquad \min_{T\leqq x\leqq T^{\varkappa}} \frac{\pi(x,k,l)-\dfrac{\mathrm{li}\,x}{\varphi(k)}}{\sqrt{x}/\log x} < -\delta,$$

whenever $T>c$, where δ and c are e. c. positive numerical constants.

Let

$$\sigma(x,k,l)=\sum_{p\equiv l\,(\mathrm{mod}\,k)} e^{-p/x}, \quad s(x)=\sum_{n=2}^{\infty}\frac{1}{\log n}e^{-n/x}.$$

The following assertions hold.

Theorem 4. *For every k in* (A) *and for all l for which $N_k(l)=0$, we have*

$$\max_{T\leqq x\leqq T^{\varkappa}}\frac{\sigma(x,k,l)-\dfrac{s(x)}{\varphi(k)}}{\sqrt{x}/\log x}>\delta, \quad \min_{T\leqq x\leqq T^{\varkappa}}\frac{\sigma(x,k,l)-\dfrac{s(x)}{\varphi(k)}}{\sqrt{x}/\log x}<-\delta$$

if $T>c$, where $\varkappa=(2+\sqrt{3})^2$, further c and δ are e. c. positive numerical constants.

Theorem 5. *For every k in* (A) *and all l_1, l_2 for which $N_k(l_1)=N_k(l_2)$, $l_1\not\equiv l_2$* (mod k), *we have*

$$\max_{T\leqq x\leqq T^{\varkappa}}\frac{\sigma(x,k,l_1)-\sigma(x,k,l_2)}{\sqrt{x}/\log x}>\delta,$$

if $T>c$, where $\varkappa=(2+\sqrt{3})^2$, further c and δ are positive e. c. n. c.

The method of the proofs of our Theorems is the same as was elaborated for the omega-estimation of $M(x)=\sum_{n\leqq x}\mu(n)$ in my dissertation [7] and in the paper [8]. However, we use here an idea of Rodosskу in a deeper form [9].

2. Some lemmas.

Lemma 1. *If*

$$F(w)=\sum_{n=1}^{\infty}\frac{a_n}{n^w}$$

is absolutely convergent for $\sigma\geqq\sigma_0$, then

$$\sum_{n=1}^{\infty}a_n e_1\left(-\frac{\log^2 n}{4u}\right)=\frac{i\sqrt{u}}{\sqrt{\pi}}\int_{(\sigma_0)}F(w)\,e_1(w^2u)\,dw. \tag{2.1}$$

For the proof see [9].

Lemma 2. [9] *For $0<\alpha\leqq 1$ and $u\to\infty$,*

$$\frac{1}{2u}\int_1^{\infty}x^{\alpha-1}\log x\cdot e_1\left(-\frac{\log^2 x}{4u}\right)dx=2\sqrt{\pi u}\,e_1(\alpha^2 u)+O(1). \tag{2.2}$$

Lemma 3. [9] *Let $u \geqq 1$ and y, z be defined by*

$$\log y = 2u\left(1-\frac{\sqrt{3}}{2}\right), \quad \log z = 2u\left(1+\frac{\sqrt{3}}{2}\right). \tag{2.3)-(2.4}$$

The following inequalities hold:

$$\frac{1}{2u}\int_1^y x^{-1/2}\log x\cdot e_1\left(-\frac{\log^2 x}{4u}\right)dx < ce_1\left(\frac{u}{4}\right), \tag{2.5}$$

$$\frac{1}{2u}\int_z^\infty x^{-1/2}\log x\cdot e_1\left(-\frac{\log^2 x}{4u}\right)dx < ce_1\left(\frac{u}{4}\right). \tag{2.6}$$

Lemma 4. *Let*

$$R(u) = \frac{i\sqrt{u}}{\sqrt{\pi}}\int_{(2)}\log\left(w-\frac{1}{2}\right)e_1(w^2u)\,dw. \tag{2.7}$$

Then

$$|R(u)| > \frac{c}{\sqrt{u}}e_1(u/4), \qquad \text{if } u > c_1.$$

Proof. Using the well-known formula

$$\log s = \int_0^\infty \frac{e^{-v}-e^{-sv}}{v}\,dv \quad (\operatorname{Re} s > 0)$$

due to EULER, we obtain that

$$R(u) = \frac{i\sqrt{u}}{\sqrt{\pi}}\int_{(2)}\int_0^\infty \frac{e_1(-v)-e_1\left(-(w-\frac{1}{2})v\right)}{v}\,dv\,e_1(w^2u)\,dw =$$

$$= \pi\int_0^\infty\left[-e_1(-v)+e_1\left(-\frac{v^2}{4u}+\frac{v}{2}\right)\right]\frac{dv}{v}.$$

Since

$$\left|\int_0^1\left[e_1\left(-\frac{v}{4u}+\frac{v}{2}\right)-e_1(-v)\right]\frac{dv}{v}\right| < c \quad\text{and}\quad \int_1^\infty e_1(-v)\frac{dv}{v} < c,$$

the inequality

$$R_1(u) \overset{\text{def}}{=} \int_1^\infty e_1\left(-\frac{v^2}{4u}+\frac{v}{2}\right)\frac{dv}{v} = R(u)+O(1)$$

holds. Substituting $e_1(v)=x$ we obtain

$$R_1(u) = \int_e^\infty e_1\left(-\frac{\log^2 x}{4u}\right)\frac{x^{-1/2}}{\log x}\,dx,$$

thus

$$R_1(u) \geqq 2u(\log z)^{-2}\frac{1}{2u}\int_y^z e_1\left(-\frac{\log^2 x}{4u}\right)\log x\cdot x^{-1/2}\,dx \geqq$$

$$\geqq cu(\log z)^{-2}\sqrt{u}\,e_1(u/4) \geqq ce_1(u/4)\cdot u^{-1/2}, \quad c>0.$$

(See Lemmas 2, 3 and (2. 4).) Hence the assertion follows.

From Lemma 4 one can deduce the following

Lemma 5. *Let*

(2.8) $$J(u) = \frac{i\sqrt{u}}{\sqrt{\pi}}\int_{(2)}\log(w-\tfrac{1}{2})\,\Gamma(w)\,e_1(w^2 u)\,dw.$$

Then

$$|J(u)| > ce_2(u/4)u^{-1/2}, \quad c>0.$$

Proof. Let L denote the broken line with vertices $1-i\infty$, $1-i\cdot 2$, $1/4-i\cdot 2$, $1/4+i\cdot 2$, $1+i\cdot 2$, $1+i\cdot\infty$. Let $\Gamma(\omega)=\Gamma(\tfrac{1}{2})+\varphi(\omega)$. So the inequalities

(2. 9) $$|\varphi(\omega)|\leqq c|w-1/2|, \qquad |\log(w-\tfrac{1}{2})\varphi(w)|<c|w-\tfrac{1}{2}|^{3/4}$$

hold on the line L. Let now

$$J(u) = \frac{i\sqrt{u}\,\Gamma(\frac{1}{2})}{\sqrt{\pi}}\int_{(2)}\log(w-\tfrac{1}{2})\,e_1(w^2u)\,dw + \frac{i\sqrt{u}}{\sqrt{\pi}}\int_L \varphi(w)\,e_1(w^2u)\log(w-\tfrac{1}{2})\,dw.$$

From (2. 9) it follows that the absolute value of second integral is majorized by $ce_1(u|4)u^{-1}$. For the first integral we use Lemma 4, and we obtain the assertion stated in Lemma 5.

3. Let us now introduce the following notations:

(3.1)-(3.2) $$f(s) = \sum_{p\equiv 3\,(\mathrm{mod}\,4)} p^{-s}; \quad g(s) = \frac{1}{2}\sum_{n=2}^{\infty}\frac{(\log n)^{-1}}{n^s};$$

(3.3) $$F(s) = f(s)-g(s) = \sum_{n=2}^{\infty}\frac{a_n}{n^s},$$

where the coefficients a_n of $F(s)$ are defined by

(3.4) $$a_n = \begin{cases} 1-\frac{1}{2}(\log n)^{-1}, & \text{if } n=p\equiv -1\,(\mathrm{mod}\,4),\ p \text{ prime},\\ -\frac{1}{2}(\log n)^{-1} & \text{otherwise.}\end{cases}$$

Let $\zeta(s)$ be the Riemann zeta-function and let

$$L(s,\chi)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}.$$

We have evidently that

$$f(s)=\frac{1}{2}\log\frac{\zeta(s)}{L(s,\chi)}+h(s), \tag{3.5}$$

where $h(s)$ is a function represented by an absolutely convergent Dirichlet series in the halfplane $\operatorname{Re} s>1/3$ and hence regular.

Further we have $\frac{dg(s)}{ds}=-\zeta(s)$ and so $\frac{d}{ds}(g(s)+\log(s-1))=-\zeta(s)+\frac{1}{s-1}$. Since the right hand side is an integral function, so is $g(s)+\log(s-1)$ regular on the whole plane. Hence it follows that $F(s)$ is regular at the point $s=1$. Further it is known that in the domain $0<\sigma<1, 0\leqq t\leqq 10,24$ the function $L(s,\chi)$ has a unique simple zero, namely at the point

$$\varrho=\tfrac{1}{2}+i\cdot 6{,}02\ldots=\tfrac{1}{2}+i\cdot\gamma. \tag{3.6}$$

In this domain $\zeta(s)$ is non-vanishing.

Let now

$$I(\tau)=\frac{i\sqrt{u}}{\sqrt{\pi}}\int_{(2)}\Gamma(w+i\tau)\,e_1(w^2u)\,dw, \tag{3.7}$$

where τ is a real number.

We shall now give an upper estimation for (3. 7) in the special cases $\tau=0$ and $\tau=\gamma$.

Let Γ denote the broken line with vertices $1{,}5-i\cdot\infty$; $1{,}5-4i$; $0{,}4-4i$; $0{,}4+4i$; $1{,}5+4i$; $1{,}5+i\cdot\infty$. For the estimation of $I(0)$ we transform the integration line in (3. 7) to Γ and we obtain

$$|I(0)|<ce_1(0{,}16\cdot u). \tag{3.8}$$

Choose now $\tau=\gamma$. Then the function $F(w+i\gamma)$ has a logarithmic singularity at the point $w=1/2$ and

$$F(w+i\gamma)=-\log(w-\tfrac{1}{2})+F_1(w),$$

where $F_1(w)$ is a regular function on the broken line Γ and on the right hand side of Γ. So we have

$$I(\gamma)=\frac{i\sqrt{u}}{\sqrt{\pi}}\int_{\Gamma}F_1(w)\,e_1(w^2u)\,dw-\frac{i\sqrt{u}}{\sqrt{\pi}}\int_{(2)}\log\left(w-\frac{1}{2}\right)e_1(w^2u)\,dw=P(u)-R(u).$$

For $P(u)$ we have the estimation $|P(u)| < ce_1(0, 16u)$. From Lemma 4

$$|R(u)| > \frac{ce_1(u/4)}{\sqrt{u}}$$

follows. So we have

(3.9) $$|I(\gamma)| > \frac{ce_1(u/4)}{\sqrt{u}}.$$

4. Let now

(4.1) $$A(x) = \sum_{n \leqq x} a_n,$$

where the a_n are defined by (3. 4). It is evident, that

(4.2) $$\pi(x, 4, 3) - \tfrac{1}{2}\,\mathrm{li}\, x = A(x) + O(1).$$

From Lemma 1 it follows that the $I(\tau)$ in (3. 7) can be represented as

$$I(\tau) = \sum_{n=2}^{\infty} a_n e_1\left(-\frac{\log^2 n}{4u} - i\tau \log n\right).$$

By partial integration follows:

(4.3) $$I(\tau) = \int_1^{\infty} A(x)\, x^{-1}\left(\frac{\log x}{2u} + i\tau\right) e_1\left(-\frac{\log^2 x}{4u} - i\tau \log x\right) dx.$$

Let further $I(\tau, 1, y)$, $I(\tau, y, z)$, $I(\tau, z, \infty)$ denote the integral on the right hand side extended for the intervals $[1, y]$, $[y, z]$, $[z, \infty]$, respectively. Let the values y, z be choosen as in (2. 3), (2. 4). Using the trivial estimation $|A(x)| < cx(\log x)^{-1}$ we have

$$|I(\tau, 1, y)| < c \int_2^{y} (\log x)^{-1}\left[\frac{\log x}{2u} + |\tau|\right] e_1\left(-\frac{\log^2 x}{4u}\right) dx \leqq$$

$$\leqq c(1 + |\tau|) \int_2^{y} \frac{1}{\log x}\, e_1\left(-\frac{\log^2 x}{4u}\right) dx = c(1 + |\tau|) \int_{\log 2}^{\log y} t^{-1} e_1\left(t - \frac{t^2}{4u}\right) dt$$

and by partial integration,

$$\int_2^{y} \frac{1}{\log x}\, e_1\left(-\frac{\log^2 x}{4u}\right) dx < cu^{-1} e_1(u/4).$$

Hence

(4.3) $$|I(\tau, 1, y)| < c(1 + |\tau|)\, u^{-1} e_1(u/4)$$

follows. Using similar computations we obtain

(4.4) $$|I(\tau, z, \infty)| < c(1+|\tau|)\, u^{-1}\, e_1(u/4).$$

Let now assume that for a fixed positive δ one of the inequalities

(4.5) $$\max_{y \leqq x \leqq z} \left(A(x) - \delta \frac{x^{1/2}}{\log x} \right) \leqq 0,$$

(4.6) $$\min_{y \leqq x \leqq z} \left(A(x) + \delta \frac{x^{1/2}}{\log x} \right) \geqq 0$$

holds. Using this assumption we obtain such an inequality for $I(\gamma)$ and $I(0)$ which contradicts (3. 8), (3. 9).

Indeed, we have

$$|I(\tau, y, z)| \leqq \left| \int_y^z \frac{A(x) \pm \delta \dfrac{x^{1/2}}{\log x}}{x} \left| \frac{\log x}{2u} + i\tau \right| e_1\left(-\frac{\log^2 x}{2u} \right) dx \right| +$$

$$+ \delta \int_y^z \frac{x^{-1/2}}{\log x} \left| \frac{\log x}{2u} + i\tau \right| e_1\left(-\frac{\log^2 x}{4u} \right) dx.$$

Using the inequality

$$\left| \frac{\log x}{2u} + i\tau \right| < c(1+|\tau|) \frac{\log x}{2u}$$

and our assumption, i.e. that one of the functions

$$A(x) \pm \delta \frac{x^{1/2}}{\log x}$$

has constant sign on the interval $[y, z]$, we have

$$|I(\tau, y, z)| \leqq c(1+|\tau|) I(0, y, z) + c\delta (1+|\tau|) \int_y^z \frac{x^{-1/2}}{\log x} e_1\left(-\frac{\log^2 x}{4u} \right) dx.$$

For the integral on the right hand side we have

$$\int_y^z \frac{x^{-1/2}}{\log x} e_1\left(-\frac{\log^2 x}{4u} \right) dx < \frac{c}{u} \int_1^\infty x^{-1/2} e_1\left(-\frac{\log^2 x}{4u} \right) dx < \frac{c}{\sqrt{u}} e_1(u/4)$$

by Lemma 2. Hence

$$|I(\tau, y, z)| < c(1+|\tau|)\, |I(0, y, z)| + c\delta (1+|\tau|)\, e_1(u/4) u^{-1}$$

and by (4. 3), (4. 4)

$$|I(\tau)| < c(1+|\tau|)\left\{|I(0)| + \delta\frac{e_1(u/4)}{\sqrt{u}} + \frac{e_1(u/4)}{u}\right\}. \tag{4.7}$$

Let now $\tau=\gamma$. Taking into account the inequalities (3. 8), (3. 9) we get

$$c_1 u^{-1/2} e_1(u/4) < c_2 e_1(0,16u) + \delta c_2 u^{-1/2} e_1(u/4) + c_3 u^{-1} e_1(u/4),$$

where $c_1>0$. This is impossible if $\delta<c_1/c_2$ and u is sufficiently large. Hence it follows that the inequalities cannot hold, i.e. we have

$$\max_{y\leq x\leq z}\frac{A(x)\log x}{\sqrt{x}} > \delta, \quad \min_{y\leq x\leq z}\frac{A(x)\log x}{\sqrt{x}} < -\delta,$$

if $u>c$.

Taking into account that

$$A(x) = \pi(x,4,3) - \tfrac{1}{2}\operatorname{li} x + O(1),$$

and that $z=y^{\varkappa}$ Theorem 1 follows.

5. In this section we give a sketch of Theorem 5 in the special case $k=8$ We shall use the following generalization of Lemma 1.

Lemma 6. *Let*

$$h(s) = \int_1^{\infty} x^{-s}\, dA(x) \tag{5.1}$$

absolutely and uniformly convergent in the halfplane $\sigma>\sigma_1(>0)$. Then

$$\int_1^{\infty} e_1\left(-\frac{\log^2 x}{4u}\right) dA(x) = \frac{i\sqrt{u}}{\sqrt{\pi}}\int_{(\sigma)} h(w)\, e_1(w^2 u)\, dw. \tag{5.2}$$

The proof of this Lemma is very similar to that of Lemma 1 and so can be omitted.

Let l_1, l_2 be two different among the numbers 3, 5, 7, further let ε_p be defined by the relation

$$\varepsilon_p = \begin{cases} 1, & \text{if } p\equiv l_1 \pmod 8, \\ -1, & \text{if } p\equiv l_2 \pmod 8, \\ 0 & \text{otherwise,} \end{cases} \tag{5.3}$$

and let

(5.4)–(5.5) $\quad g(s) = \sum_p \frac{\varepsilon_p}{p^s}, \quad s(x) = \sigma(x,8,l_1) - \sigma(x,8,l_2) = \sum_p \varepsilon_p e^{-p/x}.$

Using a well-known relation we have

$$\Gamma(s)g(s)=\int_0^\infty y^{s-1}\sum_p \varepsilon_p e^{-py}\,dy=\int_0^\infty \frac{s(x)}{x^{s+1}}\,dx=\int_0^1+\int_1^\infty=l(s)+h(s). \tag{5.6}$$

Here the function $l(s)$ is regular in the halfplane $\operatorname{Re} s=\sigma>0$ and $|l(s)|<c$ if $\sigma\geqq 1/10$, because $|s(x)|<c$ in the interval $0\leqq x\leqq 1$. Using now Lemma 6 with

$$dA(x)=\frac{s(x)}{x^{1+i\tau}}\,dx,\quad h(s)=\int_1^\infty \frac{s(x)}{x^{s+1}}\,dx, \tag{5.7}$$

we obtain

$$\int_1^\infty e_1\left(-\frac{\log^2 x}{4u}-i\tau\log x\right)s(x)\frac{dx}{x}=\frac{i\sqrt{u}}{\sqrt{\pi}}\int_{(2)} h(w+i\tau)\,e_1(w^2u)\,dw. \tag{5.8}$$

Let us now introduce the following notations:

$$I(\tau,a,b)=\int_a^b e_1\left(-\frac{\log^2 x}{4u}-i\tau\log x\right)s(x)\frac{dx}{x}, \tag{5.9}$$

$$K(\tau)=\frac{i\sqrt{u}}{\sqrt{\pi}}\int_{(2)} h(w+i\tau)\,e_1(w^2u)\,dw. \tag{5.10}$$

In the proof an essential role is played by some numerical data due to P. C. Haselgrove (see S. Knapowski and P. Turán [5], p. 254). Let

$$L(s,\chi_1)\overset{\text{def}}{=}\sum_{n=0}^{\infty}\left\{\frac{1}{(8n+1)^s}+\frac{1}{(8n+3)^s}-\frac{1}{(8n+5)^s}-\frac{1}{(8n+7)^s}\right\},$$

$$L(s,\chi_2)\overset{\text{def}}{=}\sum_{n=0}^{\infty}\left\{\frac{1}{(4n+1)^s}-\frac{1}{(4n+3)^s}\right\},$$

$$L(s,\chi_3)\overset{\text{def}}{=}\sum_{n=0}^{\infty}\left\{\frac{1}{(8n+1)^s}-\frac{1}{(8n+3)^s}-\frac{1}{(8n+5)^s}+\frac{1}{(8n+7)^s}\right\}.$$

Then in the domain

$$0<\sigma<1,\ |t|\leqq 12$$

the zeros of $L(s,\chi_1)$ are

$$\tfrac12\pm i\cdot 4{,}899\ldots,\quad \tfrac12\pm i\cdot 7{,}628\ldots,\quad \tfrac12\pm i\cdot 10{,}806\ldots$$

those of $L(s,\chi_2)$

$$\tfrac12\pm 2\cdot 6{,}020\ldots,\quad \tfrac12\pm 2\cdot 10{,}243\ldots,$$

and those of $L(s, \chi_3)$

$$\tfrac{1}{2} \pm i \cdot 3{,}576\ldots, \quad \tfrac{1}{2} \pm i \cdot 7{,}434\ldots, \quad \tfrac{1}{2} \pm i \cdot 9{,}503\ldots .$$

In particular, they are simple and different from each other.

We shall use that for the function $g(s)$ in (5. 4)

$$g(s) = \frac{-1}{4} \sum_{\chi \,(\mathrm{mod}\, 8)} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \log L(s, \chi) + u(s), \tag{5.11}$$

where the function $u(s)$ has an absolutely convergent Dirichlet series representation in the halfplane $\sigma > \frac{1}{3}$, because 3, 5, 7 are quadratic non-residues mod 8. So we have

$$h(s) = -\frac{\Gamma(s)}{4} \sum_{\chi \,(\mathrm{mod}\, 8)} (\bar{\chi}(l_1) - \bar{\chi}(l_2)) \log L(s, \chi) + v(s), \tag{5.12}$$

where $v(s)$ is a regular and bounded function in the strip $\frac{1}{3} < \sigma < 10$. Transforming the integration line in (5.10) to the broken line Γ (see (3.8)) we have

$$|K(0)| < c e_1(0{,}16 u). \tag{5.13}$$

Choose $\tau = \gamma$ where $\frac{1}{2} + i\gamma$ is the first singularity of $g(s)$ in the upper halfplane ($\mathrm{Im}\, s > 0$). Using Lemma 5 instead of Lemma 4 we have

$$|K(\gamma)| > c u^{-1/2} e_1(u/4), \qquad c > 0. \tag{5.14}$$

Let now y, z be chosen as in (2. 3), (2. 4) and assume that one of the inequalities

$$\max_{y \leqq x \leqq z} \left(s(x) - \delta \frac{\sqrt{x}}{\log x} \right) \leqq 0, \qquad \min_{y \leqq x \leqq z} \left(s(x) + \delta \frac{\sqrt{x}}{\log x} \right) \geqq 0 \tag{5.15)-(5.16}$$

be satisfied with a positive δ. Using a similar argument as in the section 4, we can deduce from this assumption the inequality

$$|I(\tau, 1, \infty)| < c(1 + |\tau|)\{|I(0, 1, \infty)| + \delta u^{-1/2} e_1(u/4) + u^{-1} e_1(u/4)\}. \tag{5.17}$$

Taking into account that $I(\tau, 1, \infty) = K(\tau)$ and choosing $\tau = \gamma$, the inequality (5.17) contradicts the inequalities (5.13), (5.14) for a sufficiently small positive δ and for $u > c$. So the inequalities (5.15)—(5.16) for this δ cannot hold and hence the assertion follows.

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Quasitriangular operators

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Every square matrix with complex entries is unitarily equivalent to a triangular one. In other words, if A is an operator on a finite-dimensional Hilbert space H, then there exists an increasing sequence $\{M_n\}$ of subspaces such that $\dim M_n = n$ ($n=0, \ldots, \dim H$), and such that each M_n is invariant under A. On a Hilbert space of dimension $\aleph_0$ the appropriate definition is this: A is *triangular* if there exists an increasing sequence $\{M_n\}$ of finite-dimensional subspaces whose union spans H such that each M_n is invariant under A. It is easy, but not obviously desirable, to fill in the dimension gaps, and hence to justify the added assumption that $\dim M_n = n$ ($n=0, 1, 2, \ldots$).

In many considerations of invariant subspaces ($AM \subset M$) it is convenient to treat their projections instead ($AE = EAE$). In terms of projections a necessary and sufficient condition that an operator A on a separable Hilbert space H be triangular is that

(Δ) *there exists an increasing sequence $\{E_n\}$ of projections of finite rank such that $E_n \to 1$ (strong topology) and such that $AE_n - E_nAE_n = 0$ for all n.*

This formulation suggests an asymptotic generalization of itself. An operator A is *quasitriangular* if

(Δ_1) *there exists an increasing sequence $\{E_n\}$ of projections of finite rank such that $E_n \to 1$ (strong topology) and such that $\|AE_n - E_nAE_n\| \to 0$.*

(Informally: E_n is approximately invariant under A.) The concept (but not the name) has been seen before; it plays a central role in the proofs of the Aronszajn—Smith theorem [1] on the existence of invariant subspaces for compact operators, and in the proofs of its various known generalizations [2], [3], [5].

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It is interesting and useful to examine a variant of the condition (Δ_1); the variant requires that

(Δ_2) *there exists a sequence* $\{E_n\}$ *of projections of finite rank such that* $E_n \to 1$ *(strong topology) and such that* $\|AE_n - E_nAE_n\| \to 0$.

The only difference between Δ_1 and Δ_2 is that the latter does not require the sequence $\{E_n\}$ to be increasing.

There is still another pertinent condition. The set of all projections of finite rank, ordered by range inclusion, is a directed set. Since $E \to \|AE - EAE\|$ is a net on that directed set, it makes sense to say that

$$(\Delta_0) \qquad \liminf_{E \to 1} \|AE - EAE\| = 0.$$

What it means is that for every positive number ε and for every projection E_0 of finite rank there exists a projection E of finite rank such that $E_0 \leqq E$ and $\|AE - EAE\| < \varepsilon$.

The purpose of this paper is to initiate a study of quasitriangular operators. The study begins with the observation that approximately invariant projections that are large (in the sense of having large ranks) always exist (Section 1). The main result is the characterization of quasitriangular operators; it asserts (for separable spaces) that the conditions Δ_0, Δ_1, and Δ_2 are mutually equivalent (Section 2). This characterization is applied to show that there exist operators that are *not* quasitriangular. On the other hand the set of quasitriangular operators is quite rich (Section 3); it is closed under the formation of polynomials, it is closed in the norm topology of operators, it is closed under the formation of countable direct sums, and it contains, for example, all operators of the form $N + K$ where N is normal and K is compact. The paper concludes with a few questions (Section 4). Sample: is it true for every operator A that either A or A^* is quasitriangular?

Section 1

Sequences of approximately invariant projections that are not required to be "large" always exist. A precise statement is this: for each operator A there exists a sequence $\{E_n\}$ of non-zero projections of finite rank such that $\|AE_n - E_nAE_n\| \to 0$; in fact the E_n's can be chosen to have rank 1. The proof is immediate from the existence of approximate eigenvalues and eigenvectors. Let λ be a scalar and $\{e_n\}$ a sequence of unit vectors such that $\|Ae_n - \lambda e_n\| \to 0$. If the projections E_n are defined by $E_n f = (f, e_n)e_n$, then

$$(AE_n - E_nAE_n)f = (f, e_n)(Ae_n - (Ae_n, e_n))e_n.$$

Since $(Ae_n, e_n) \to \lambda$, it follows that $\|AE_n - E_nAE_n\| \to 0$.

Since every operator has approximately invariant projections of rank 1, it is tempting to conclude, via the formation of finite spans, that every operator on an infinite-dimensional Hilbert space has approximately invariant projections of arbitrarily large finite ranks. The theory of approximate invariance turns out, however, to be surprisingly delicate. It is, for instance, not true that the span of two approximate eigenvectors is approximately invariant. More precisely, there exists a 3×3 matrix A and there exist two projections F and G of rank 1 such that F is invariant under A, G is nearly invariant under A, but if $E=F\vee G$, then $\|AE-EAE\|=1$. In detail: put

$$A=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix},$$

let F be the projection onto $\langle 0,1,0\rangle$, and let G be the projection onto $\langle a,b,0\rangle$, where $|a|^2+|b|^2=1$ and a is "small" (but not 0). It is easy to verify that

$$F=\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix},\quad G=\begin{pmatrix}|a|^2 & ab^* & 0\\ a^*b & |b|^2 & 0\\ 0 & 0 & 0\end{pmatrix},\quad E=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$\|AF-FAF\|=0,\quad \|AG-GAG\|=|a|,\quad \text{and}\quad \|AE-EAE\|=1.$$

This example, informal in its interpretation of "nearly invariant", can be used to construct an example of two sequences of approximately invariant projections, in the precise technical sense, such that the sequence of their spans is not approximately invariant, as follows. Let H be the direct sum $H_1\oplus H_2\oplus\ldots$ of 3-dimensional spaces such as played a role in the preceding paragraph, and let the operator A on H be the direct sum $A_1\oplus A_2\oplus\ldots$ of the corresponding operators. Let F_n be the direct sum projection whose summand with index n is the previous F and whose other summands are 0; let G_n be the direct sum projection whose summand with index n is the previous G with $a=\frac{1}{n}$ and whose other summands are 0. It follows that $\|AF_n-F_nAF_n\|=0$ for all n, $\|AG_n-G_nAG_n\|\to 0$, and, if $E_n=F_n\vee G_n$, then $\|AE_n-E_nAE_n\|=1$.

It is slightly surprising that, despite the evidence of the preceding example, approximately invariant projections of arbitrarily large ranks always exist.

Theorem 1. *If A is an operator on an infinite-dimensional Hilbert space, ε is a positive number, and n is a positive integer, then there exists a projection E of rank n such that* $\|AE-EAE\|<\varepsilon$.

Proof. For $n=1$, the result was derived from the existence of approximate eigenvectors. The idea of the inductive proof that follows is that although near invariance is not preserved by the formation of spans, it is preserved by the formation of orthogonal spans. Given ε and n, assume the result for n, and let F be a projection of rank n such that $\|AF-FAF\|<\varepsilon/2$. Since the compression of A to $\operatorname{ran}(1-F)$ (i.e., the restriction of $(1-F)A(1-F)$ to $\operatorname{ran}(1-F)$) has approximately invariant projections of rank 1, it follows that there exists a projection G of rank 1 such that $G\perp F$ and

$$\|(1-F)A(1-F)G-G(1-F)A(1-F)G\|<\varepsilon/2.$$

(Find G on $\operatorname{ran}(1-F)$ first and then extend it by definining it to be 0 on $\operatorname{ran} F$.) Since $G(1-F)=(1-F)G=G$, the last inequality is equivalent to

$$\|(1-F)(1-G)AG\|<\varepsilon/2.$$

If $E=F+G$, then E is a projection of rank $n+1$ and

$$\|AE-EAE\| = \|(1-E)AE\| = \|(1-F)(1-G)A(F+G)\| =$$
$$= \|(1-G)(1-F)AF+(1-F)(1-G)AG\| \leqq \|(1-F)AF\|+\|(1-F)(1-G)AG\|<\varepsilon.$$

Section 2

It is trivial that the definition of quasitriangularity (Δ_1) implies the weakened form (Δ_2) (obtained from (Δ_1) by omitting the word "increasing"). It is also quite easy to prove that if, on a separable Hilbert space, $\liminf_{E\to 1}\|AE-EAE\|=0$ (Δ_0), then A is quasitriangular (Δ_1). Indeed, let $\{e_1, e_2, \ldots\}$ be an orthonormal basis for the space. By (Δ_0) there exists a projection E_1 of finite rank such that $e_1\in\operatorname{ran} E_1$ and $\|AE_1-E_1AE_1\|<1$. Again, by (Δ_0), there exists a projection E_2 of finite rank such that $E_1\leqq E_2$, $e_2\in\operatorname{ran} E_2$, and $\|AE_2-E_2AE_2\|<\frac{1}{2}$. In general, inductively, use (Δ_0) to get a projection E_{n+1} of finite rank such that $E_n\leqq E_{n+1}$, $e_{n+1}\in\operatorname{ran} E_{n+1}$, and $\|AE_{n+1}-E_{n+1}AE_{n+1}\|<\dfrac{1}{n+1}$. Conclusion: $\{E_n\}$ is an increasing sequence of projections of finite rank such that $E_n\to 1$ and such that $\|AE_n-E_nAE_n\|\to 0$; in other words A is quasitriangular, as promised.

The non-trivial implication along these lines is the one from (Δ_2) to (Δ_0). The proof depends on a lemma according to which if two projections have the same finite rank and are near, then there is a "small" unitary operator that transforms one onto the other. (For unitary operators "small" means "near to 1".) A possible quantitative formulation goes as follows.

Lemma 1. *If E and F are projections of the same finite rank such that $\|E-F\|=\varepsilon<1$, then the infimum of $\|1-W\|$, extended over all unitary operators W such that $W^*EW=F$, is not more than $2\varepsilon^{\frac{1}{2}}$.*

The lemma can be improved, but the improvement takes considerably more work and for present purposes it is not needed. A trivial improvement is to drop the assumption that E and F have the same rank and recapture it from the known result [7, p. 58] that the inequality $\|E-F\|<1$ implies rank $E=$ rank F. Another qualitative improvement is to drop the assumption that the ranks are finite and pay for it by introducing partial isometries instead of unitary operators. The best kind of improvement is quantitative; the estimate $2\varepsilon^{\frac{1}{4}}$ can be sharpened to $2^{\frac{1}{2}}[1-(1-\varepsilon^2)^{\frac{1}{2}}]^{\frac{1}{2}}$. For a discussion of such results and references to related earlier work see [4]. Conjecturally the sharpened estimate is best possible, but the proof of that does not seem to be in the literature.

Proof. The equality of rank E and rank F implies the existence of a unitary operator W_0 such that $W_0^*EW_0=F$. Write E, F, and W_0 as operator matrices, according to the decomposition $1=E+(1-E)$, so that, for instance, $E=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$. If $W_0=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$, then $W_0^*=\begin{pmatrix}A^* & C^*\\ B^* & D^*\end{pmatrix}$, and therefore

$$W_0^*W_0=\begin{pmatrix}A^*A+C^*C & A^*B+C^*D\\ B^*A+D^*C & B^*B+D^*D\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

The matrix of $F\,(=W_0^*EW_0)$ can now be computed; it turns out to be $\begin{pmatrix}A^*A & A^*B\\ B^*A & B^*B\end{pmatrix}$. Since the norm of each entry of a matrix is dominated by the norm of the matrix, it follows that

$$\|C^*C\|=\|1-A^*A\|\leqq\varepsilon \quad\text{and}\quad \|1-D^*D\|=\|B^*B\|\leqq\varepsilon.$$

Observe next that if U and V are unitary operators on ran E and ran $(1-E)$ respectively, and if $W_1=\begin{pmatrix}U & 0\\ 0 & V\end{pmatrix}$, then W_1 commutes with E, and, therefore, W_1W_0 transforms E onto F (just as W_0 does). The purpose of the rest of the proof is to choose U and V so as to make $\|1-W_1W_0\|$ small. Since

$$1-W_1W_0=\begin{pmatrix}1-UA & -UB\\ -VC & 1-VD\end{pmatrix},$$

and since the norm of a matrix is dominated by the square root of the sum of the squares of the norms of its entries, it is sufficient to prove that by appropriate choices of U and V the entries of the last written matrix can be made to have small norms. The off-diagonal entries of $1-W_1W_0$ are easy to estimate:

$$\|-VC\|^2=\|C\|^2=\|C^*C\|\leqq\varepsilon \quad\text{and}\quad \|-UB\|^2=\|B\|^2=\|B^*B\|\leqq\varepsilon.$$

In these estimates U and V are arbitrary unitary operators; it is only in the next step that they have to be chosen so as to make something small.

Observe that since the ranks of E and F are finite, the lemma loses no generality if it is stated for finite-dimensional spaces only; the infinite-dimensional case is recaptured by applying the finite-dimensional lemma to E and F restricted to ran $E \vee$ ran F and extending the resulting unitary operator by defining it to be the identity on the orthogonal complement of ran $E \vee$ ran F. In the finite-dimensional case A is the product of a unitary operator and $(A^*A)^{\frac{1}{2}}$ (polar decomposition); let U be the inverse of the unitary factor. With this choice $1-UA$ becomes $1-P$, where $P=(A^*A)^{\frac{1}{2}}$. Since $\|A\| \leqq 1$, so that $0 \leqq P^2 \leqq P \leqq 1$, it follows that $0 \leqq 1-P \leqq \leqq 1-P^2$, and hence that

$$\|1-UA\| = \|1-P\| \leqq \|1-P^2\| = \|1-A^*A\|.$$

A similar argument for D produces a unitary V such that $\|1-VD\| \leqq \|1-D^*D\|$. Conclusion:

$$\|1-W_1W_0\|^2 \leqq 2(\varepsilon+\varepsilon^2) \leqq 4\varepsilon,$$

and the proof of the lemma is complete.

The ground is now prepared for the proof of the principal result.

Theorem 2. *If $\{E_n\}$ is a sequence of projections of finite rank such that $E_n \to 1$ (strong topology) and such that $\|AE_n - E_nAE_n\| \to 0$, then* $\liminf_{E\to 1} \|AE-EAE\| = 0.$

Proof. It is to be proved that if $\varepsilon > 0$ and if E_0 is a projection with rank $E_0 = n_0 < \infty$, then there exists a projection E of finite rank such that $E_0 \leqq E$ and $\|AE-EAE\| < \varepsilon$.

Let δ be a temporarily indeterminate positive number; it will be specified, in terms of ε, later. Suppose that $\{e_1, \ldots, e_{n_0}\}$ is an orthonormal basis for ran E_0. The two limiting assumptions imply the existence of a positive integer n such that $\|e_j - E_ne_j\| < \delta/\sqrt{n_0}$ $(j=1, \ldots, n_0)$ and $\|AE_n - E_nAE_n\| < \delta$. The first of these inequalities implies that if δ is sufficiently small, then the set $\{E_ne_1, \ldots, E_ne_{n_0}\}$ is linearly independent. (The proof is easy and is omitted here; it is explicitly carried out in [6].) Let F_0 be the projection (of rank n_0) onto their span; note that $F_0 \leqq E_n$. (The n here used will remain fixed from now on.)

If $f \in \operatorname{ran} E_0$, so that $f = \sum_{j=1}^{n_0} \xi_j e_j$, then

$$\|f-F_0f\|^2 = \|\sum_j \xi_j(e_j - E_ne_j)\|^2 \leqq (\sum_j |\xi_j| \cdot \|e_j - E_ne_j\|)^2 \leqq$$

$$\leqq \sum_j |\xi_j|^2 \cdot \sum_j \|e_j - E_ne_j\|^2 \leqq \|f\|^2 \cdot n_0 \left(\delta/\sqrt{n_0}\right)^2,$$

and therefore

$$\|E_0 - F_0E_0\| \leqq \delta.$$

This shows that E_0 is approximately dominated by F_0; what is needed for the rest of the proof is the stronger assertion that E_0 is approximately equal to F_0.

By definition, ran F_0 is spanned by the vectors $F_0 e_j (=E_n e_j)$, $j=1, \ldots, n_0$; it follows that ran $F_0 =$ ran $F_0 E_0$. In other words, the restriction of F_0 to ran E_0 maps ran E_0 onto ran F_0. Call that restriction T; then T is a linear transformation from a space of dimension n_0 onto a space of dimension n_0, and, consequently, T is invertible. Since the spaces involved are finite-dimensional, the transformation T^{-1} is bounded, but that is not enough information; what is needed is an effective estimate of $\|T^{-1}\|$. That turns out to be easy to get. If $f \in$ ran E_0, then

$$\|F_0 f\| \geqq \|f\| - \|f - F_0 f\| \geqq \|f\| - \delta\|f\| = (1-\delta)\|f\|,$$

and therefore $\|T^{-1}\| \leqq \dfrac{1}{1-\delta}$.

The inequality $\|E_0 - F_0 E_0\| \leqq \delta$ shows that F_0 is near to E_0 on ran E_0; the next step is to show that F_0 is near to E_0 on $\text{ran}^\perp E_0$. Suppose therefore that $f \perp$ ran E_0, i.e., that $E_0 f = 0$, and write $g = T^{-1} F_0 f$. Since $g \in$ ran E_0, it follows that $F_0 g = Tg = F_0 f$, or $F_0 E_0 g = F_0 f$; note that $\|g\| \leqq \dfrac{1}{1-\delta}\|f\|$. Since $\|F_0 f - E_0 g\| \leqq$ $\leqq \|F_0 f - F_0 E_0 g\| + \|F_0 E_0 g - E_0 g\| \leqq \delta\|g\| \leqq \dfrac{\delta}{1-\delta}\|f\|$, it follows that

$$\|F_0 f\|^2 = (F_0 f, f) \leqq |F_0 f - E_0 g, f)| + |(E_0 g, f)| \leqq \frac{\delta}{1-\delta}\|f\|^2$$

($(E_0 g, f) = 0$ because $E_0 f = 0$), and hence that

$$\|F_0(1-E_0)\| \leqq \left(\frac{\delta}{1-\delta}\right)^{1/2}.$$

This inequality together with $\|E_0 - F_0 E_0\| \leqq \delta$ yields

$$\|E_0 - F_0\| \leqq \delta + \left(\frac{\delta}{1-\delta}\right)^{1/2} = \gamma.$$

Lemma 1 is now applicable. Choose δ small enough to make sure that $\gamma < 1$ and conclude that there exists a unitary operator W such that $W^* E_0 W = F_0$ and $\|1 - W\| \leqq 2\sqrt{\gamma}$. Write $E = W E_n W^*$. Since $F_0 \leqq E_n$, it follows that $E_0 \leqq E$; all that remains is to verify that E can be forced to be within ε of being invariant under A. That is easy; since

$$\|AE - EAE\| = \|A(WE_n W^*) - (WE_n W^*)A(WE_n W^*)\|,$$

and since the right hand term depends continuously on W, it follows that if W is chosen sufficiently near to 1 (i.e., if δ is chosen sufficiently small), then the right hand term can be made arbitrarily near to $\|AE_n - E_nAE_n\|$, within $\varepsilon/2$ of it, say. Since $\|AE_n - E_nAE_n\| < \delta$, it might now be necessary to make δ a little smaller still, so as to guarantee $\delta < \varepsilon/2$; after this modification it will follow that, indeed, $\|AE - EAE\| < \varepsilon$.

The first definition of quasitriangularity (Δ_1) is quite hard ever to disprove; how does one show that there does not exist a sequence with the required properties? Theorem 2 makes the job easier. For an example suppose that $\{e_0, e_1, e_2, ...\}$ is an orthonormal basis and let U be the corresponding unilateral shift. The properties of U that will be needed are that it is an isometry ($U^*U = 1$) whose adjoint has a non-trivial kernel ($U^*e_0 = 0$).

Theorem 3. *The unilateral shift is not quasitriangular.*

Proof. Let E_0 be the projection (of rank 1) onto e_0. The proof will show that if E is a projection of finite rank such that $E_0 \leqq E$ (i.e., $e_0 \in \operatorname{ran} E$), then $\|UE - EUE\| = 1$.

Put $D = UE - EUE = (1-E)UE$. Clearly $\|D\| \leqq 1$; the problem is to prove the reverse inequality. Observe that $D^*D = EU^*(1-E)\cdot(1-E)UE = EU^*UE - EU^*EUE = E - (EUE)^*(EUE)$. The finite-dimensional space $\operatorname{ran} E$ reduces both E and EUE, and on its orthogonal complement both those operators vanish. It follows that if T is the restriction of EUE to $\operatorname{ran} E$, then $\|D^*D\| = \|1 - T^*T\|$; the symbol "1" here refers, of course, to the identity operator on $\operatorname{ran} E$.

Now use the assumption that $e_0 \in \operatorname{ran} E$ and observe that $T^*e_0 = EU^*Ee_0 = EU^*e_0 = 0$. Since T^* is an operator on a finite-dimensional space and has a non-trivial kernel, the same is true of T^*T. (The falsity of this implication on infinite-dimensional spaces is shown by U itself.) If f is a unit vector in $\ker T^*T$, then $\|(1 - T^*T)f\| = 1$, and therefore $\|1 - T^*T\| \geqq 1$; the proof of the theorem is complete.

Section 3

It is not difficult to see that a polynomial in a quasitriangular operator is quasitriangular. Suppose indeed that $\{E_n\}$ is a sequence of projections such that $\|AE_n - E_nAE_n\| \to 0$, and let p be a polynomial. Since $AE_n - E_nAE_n$ is linear in A, it is sufficient to prove the assertion for monomials, $p(z) = z^k$, and that can be done by induction. The case $k=1$ is covered by the hypothesis. (Note icidentally that constant terms can come and go with impunity: $(A+\lambda)E_n - E_n(A+\lambda)E_n = AE_n - E_nAE_n$.) The induction step from k to $k+1$ is implied by the identity:

$$(1-E_n)A^{k+1}E_n = \big((1-E_n)A^{k+1}E_n - (1-E_n)AE_nA^kE_n\big) + (1-E_n)AE_nA^kE_n =$$
$$= (1-E_n)A\big((1-E_n)A^kE_n\big) + \big((1-E_n)AE_n\big)A^kE_n.$$

W. B. ARVESON has proved that an operator similar to a quasitriangular one is also quasitriangular. The result of the preceding paragraph and ARVESON's result are closure properties of the set of all quasitriangular operators. The next two results are of the same kind.

Theorem 4. *A countable direct sum of quasitriangular operators is quasitriangular.*

Proof. Suppose that for each $j(=1, 2, 3, \ldots)$ $A^{(j)}$ is an operator and $\{E_n^{(j)}\}$ is a sequence of projections of finite rank such that $\|A^{(j)}E_n^{(j)} - E_n^{(j)}A^{(j)}E_n^{(j)}\| \to 0$ as $n \to \infty$. Write $A = A^{(1)} \oplus A^{(2)} \oplus \ldots$. For each fixed k, find n_k so that $\|A^{(j)}E_n^{(j)} - E_n^{(j)}A^{(j)}E_n^{(j)}\| < \frac{1}{k}$ when $1 \leqq j \leqq k$ and $n \geqq n_k$; write $E_k = E_{n_k}^{(1)} \oplus \ldots \oplus E_{n_k}^{(k)} \oplus 0 \oplus 0 \oplus \ldots$. The E_k's are projections of finite rank. Since $E_{n_k}^{(j)} \to 1$ as $k \to \infty$ (strong topology) for each j, it follows that

$$E_k\langle f^{(1)}, f^{(2)}, f^{(3)}, \ldots\rangle \to \langle f^{(1)}, f^{(2)}, f^{(3)}, \ldots\rangle$$

whenever the vector $\langle f^{(1)}, f^{(2)}, f^{(3)}, \ldots\rangle$ is finitely non-zero. The boundedness of the sequence $\{E_k\}$ implies that $E_k \to 1$ (strong topology). Since $\|AE_k - E_kAE_k\| = \max\{\|A^{(j)}E_{n_k}^{(j)} - E_{n_k}^{(j)}A^{(j)}E_{n_k}^{(j)}\|: j = 1, \ldots, k\} < \frac{1}{k}$, the proof is complete.

Theorem 5. *The set of quasitriangular operators is closed in the norm topology.*

Proof. Suppose that A_n is quasitriangular and $\|A_n - A\| \to 0$. Given a positive number ε and a projection E_0 of finite rank, find n_0 so that $\|A - A_{n_0}\| < \varepsilon/3$, and then find a projection E of finite rank such that $E_0 \leqq E$ and $\|A_{n_0}E - EA_{n_0}E\| < \varepsilon/3$. It follows that $\|AE - EAE\| \leqq \|AE - A_{n_0}E\| + \|A_{n_0}E - EA_{n_0}E\| + \|EA_{n_0}E - EAE\| < \varepsilon$.

Theorem 4 implies (and it is obvious anyway) that (on a separable Hilbert space, as always) every diagonal operator is quasitriangular. Since every normal operator is in the closure of the set of diagonal operators, Theorem 5 implies that every normal operator is quasitriangular.

A similar application of Theorem 5 shows that every compact operator is quasitriangular; what is needed is the easy observation that every operator of finite rank is quasitriangular. For compact operators, however, more is true; not only does there exist a well behaved sequence of projections, but in fact all "large" sequences are well behaved. That is: if A is compact and if $\{E_n\}$ is a sequence of projections such that $E_n \to 1$ (strong topology), then $\|AE_n - E_nAE_n\| \to 0$. The following formulation in terms of the directed set of projections of finite rank is more elegant; the assertion is that lim inf can be replaced by lim.

Lemma 2. *If A is compact, then* $\lim\limits_{E \to 1} \|AE - EAE\| = 0$.

Proof. Given a positive number ε, find an operator F of finite rank such that $\|A-F\|<\varepsilon/2$, and then find a projection E_0 of finite rank such that $FE_0=E_0F=F$. If E is a projection of finite rank such that $E_0 \leqq E$, then

$$\|AE-EAE\| \leqq \|AE-FE\| + \|FE-EFE\| + \|EFE-EAE\| < \varepsilon.$$

Lemma 2 implies that an operator of the form $A+K$, where A is quasitriangular and K is compact, is quasitriangular; in particular so is every operator of the form $N+K$, where N is normal and K is compact.

Still other quasitriangular operators of interest have arisen in the various generalizations of the Aronszajn—Smith theorem on invariant subspaces of compact operators. Thus, for instance, a crucial step in the treatment of polynomially compact operators [5] is the proof that every polynomially compact operator with a cyclic vector is quasitriangular. In their generalization of the invariant subspace theorem for polynomially compact operators, ARVESON and FELDMAN [2] need and prove the statement that every quasinilpotent operator with a cyclic vector is quasitriangular.

Section 4

Quasitriangular operators first arose in connection with the invariant subspace problem, but their status in that connection is still not settled.

Question 1. *Does every quasitriangular operator have a non-trivial invariant subspace?*

Experience with compact and polynomially compact operators suggests that the answer to Question 1 is yes. On the other hand, if the answer is yes, then it follows that every quasinilpotent operator has a non-trivial invariant subspace. Since it is a not unreasonable guess that the general invariant subspace question is equivalent to the one for quasinilpotent operators, and since the answer to the general invariant subspace question is more likely no than yes, the compact and polynomially compact experience comes under suspicion.

PETER ROSENTHAL suggested a more concrete way of connecting Question 1 with quasinilpotent operators. It is quite a reasonable conjecture that the spectrum of every unicellular operator is a singleton. (An operator is unicellular if its lattice of invariant subspaces is a chain.) Every transitive operator is obviously unicellular. (An operator is transitive if it has no non-trivial invariant subspaces.) The truth of the conjecture would imply therefore that, except for an additive scalar, every transitive operator is quasinilpotent, and hence, once again, an affirmative answer to Question 1 would imply an affirmative answer to the general invariant subspace question.

Question 2. *If the direct sum of two operators is quasitriangular, are both summands quasitriangular?*

This question is due to CARL PEARCY. He has proved that if $A\oplus 0$ is quasitriangular, then A must be, but the general case is open. An interesting related question concerns the unilateral shift U: is $U\oplus U^*$ quasitriangular? If the answer to Question 2 is yes, then the answer to this question about U must be no. What is known, as a special case of PEARCY's result, is that $U\oplus 0$ is not quasitriangular.

Question 3. *Is it true for every operator that either it or its adjoint is quasitriangular?*

The only example presented above of an operator that is not quasitriangular is the unilateral shift U; a glance at the matrix of U proves that U^* is quasitriangular. If the answer to Question 3 is yes, then Question 1 is equivalent to the general invariant subspace question. Since $U\oplus U^*$ is unitarily equivalent to its own adjoint, it follows that an affirmative answer to Question 3 would imply that $U\oplus U^*$ is quasitriangular, and, therefore, that the answer to Question 2 is no. There are other interesting and unknown special cases of Question 3. Thus, for instance, by an improvement of the argument that proved that U is not quasitriangular, PEARCY has obtained a large class of operators that are not quasitriangular; one of them is $3U+U^*$. It is not known whether the adjoint of that operator is quasitriangular.

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A characterization of thin operators

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Let $\mathfrak{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathscr{L}(\mathfrak{H})$ denote the algebra of all bounded, linear operators on $\mathfrak{H}$. In [2], HALMOS initiated the study of the class of *quasitriangular* operators on $\mathfrak{H}$. These operators may be defined as follows. Let $\mathscr{P}$ denote the directed set consisting of all finite dimensional (orthogonal) projections in $\mathscr{L}(\mathfrak{H})$ under the usual ordering ($P \leqq Q$ if and only if $(Px, x) \leqq (Qx, x)$ for all $x \in \mathfrak{H}$.) For a fixed $A \in \mathscr{L}(\mathfrak{H})$, the map $P \to \|PAP - AP\|$ is a net on $\mathscr{P}$, and A is quasitriangular provided

$$\liminf_{P \in \mathscr{P}} \|PAP - AP\| = 0.$$

(The definition of quasitriangularity given in [2] is actually somewhat different. That the above is an equivalent definition is [2, Theorem 2].) Among the quasitriangular operators are the operators of the form $\lambda + C$ where λ is a scalar and C is a compact operator. In this note we call such operators $\lambda + C$ *thin* operators. Among the quasitriangular operators are also the operators A that satisfy

$$\text{(H)} \qquad \lim_{P \in \mathscr{P}} \|PAP - AP\| = 0.$$

HALMOS has conjectured that an operator has property (H) if and only if it is thin. The purpose of this note to prove that conjecture.

To accomplish this, we first obtain an interesting characterization of the η-function of BROWN and PEARCY [1] in terms of the nets $P \to \|PAP - AP\|$. Recall that the η-function is defined on $\mathscr{L}(\mathfrak{H})$ by the equation

$$\eta(A) = \lim_{P \in \mathscr{P}} \Big[\sup_{\substack{x \in (1-P)\mathfrak{H} \\ \|x\| = 1}} \|Ax - (Ax, x)x\| \Big].$$

Theorem 1. *For every* $A \in \mathscr{L}(\mathfrak{H})$,

$$\eta(A^*) = \limsup_{P \in \mathscr{P}} \|PAP - AP\|.$$

Proof. It clearly suffices to prove that

$$\eta(A) = \limsup_{P \in \mathscr{P}} \|PA(1-P)\|$$

for every $A \in \mathscr{L}(\mathfrak{H})$, since then

$$\eta(A^*) = \limsup_{P \in \mathscr{P}} \|PA^*(1-P)\| = \limsup_{P \in \mathscr{P}} \|(1-P)AP\|.$$

Thus let $A \in \mathscr{L}(\mathfrak{H})$ be fixed, and let

$$\limsup_{P \in \mathscr{P}} \|PA(1-P)\| = \alpha.$$

Let also $\varepsilon > 0$ and $P_0 \in \mathscr{P}$ be given. Then, by definition, there exists $P_1 \in \mathscr{P}$ such that $P_1 \geqq P_0$ and $\|P_1 A(1-P_1)\| > \alpha - \varepsilon$. It follows that there is a unit vector $y \in (1-P_1)\mathfrak{H}$ such that $\|P_1 A(1-P_1)y\| > \alpha - \varepsilon$. Since Ay can be written as $Ay = [Ay - (Ay, y)y] + (Ay, y)y$ and $P_1 y = 0$, we have

$$\|Ay - (Ay, y)y\| \geqq \|P_1 Ay\| = \|P_1 A(1-P_1)y\| > \alpha - \varepsilon.$$

Since $\eta(A)$ can be written as

$$\eta(A) = \limsup_{P \in \mathscr{P}} [\sup_{\substack{x \in (1-P)\mathfrak{H} \\ \|x\| = 1}} \|Ax - (Ax, x)x\|],$$

we have shown that $\eta(A) \geqq \alpha$.

To complete the proof, we show that $\alpha \geqq \eta(A)$. Let $\delta > 0$ and a finite dimensional projection P_2 be given. It suffices to exhibit a finite dimensional projection $Q \geqq P_2$ and a unit vector z in the range of $1-Q$ such that $\|QA(1-Q)z\| > \eta(A) - \delta$. To find such a projection Q and such a vector z, we proceed as follows. The definition of $\eta(A)$ guarantees that there exists a projection $P_3 \in \mathscr{P}$ such that for every finite dimensional projection $P \geqq P_3$, there exists a unit vector x_p in the range of $(1-P)$ such that $\|Ax_p - (Ax_p, x_p)x_p\| > \eta(A) - \delta$. Choose $P_4 \geqq P_2, P_3$, and let $z (= x_{p_4})$ be a unit vector in the range of $(1-P_4)$ such that $\|Az - (Az, z)z\| > \eta(A) - \delta$. Finally, let Q be the finite dimensional projection that is the supremum of P_4 and the one dimensional projection whose range is $Az - (Az, z)z$. Since z is perpendicular to the range of P_4 and also to the vector $Az - (Az, z)z$, z is perpendicular to the range of Q. In other words, z is a unit vector in the range of $1-Q$, and the inequality

$$\|QA(1-Q)z\| = \|QAz\| = \|Q[Az - (Az, z)z] + Q(Az, z)z\| = \|Az - (Az, z)z\| > \eta(A) - \delta$$

completes the proof.

Theorem 2. *An operator $A \in \mathscr{L}(\mathfrak{H})$ has property (H) if and only if A is thin.*

Proof. Clearly A is thin if and only if A^* is thin, and according to [1, Theorem 1], A^* is thin if and only if $\eta(A^*) = 0$. Finally, from Theorem 1 we see that $\eta(A^*) = 0$ if and only if

$$\limsup_{P \in \mathscr{P}} \|(1-P)AP\| = 0,$$

or, what is the same thing, if and only if A has property (H).

We conclude this note by observing that the problem treated above makes sense in any von Neumann algebra. To be specific, let $\mathscr{A}$ be any von Neumann algebra, let $\mathscr{I}$ be any uniformly closed ideal in $\mathscr{A}$, and let $\mathscr{P}$ denote the directed set of projections in $\mathscr{I}$. It is not hard to see that every operator of the form $A=\lambda+J$, where $J\in\mathscr{I}$, satisfies

$$\lim_{P\in\mathscr{P}}\|PAP-AP\|=0.$$

Is the converse true?

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On the power-bounded operators of Sz.-Nagy and Foiaş

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1. In [6] Sz.-Nagy and Foiaş considered, for each $\varrho > 0$, the class C_ϱ of operators T on a given complex Hilbert space $\mathfrak{H}$ having the following property: for some Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ as a subspace and some *unitary* operator U on $\mathfrak{K}$, $T^n = \varrho P_{\mathfrak{H}} U^n$ $(n = 1, 2, 3, \ldots)$, where $P_{\mathfrak{H}}$ denotes the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$. It had been shown previously that $C_1 = \{T: \|T\| \leqq 1\}$ (see Sz.-Nagy [5]) and that $C_2 = \{T: w(T) \leqq 1\}$ (see Berger [1]), where $w(T)$ denotes the "numerical radius" of T, namely $\sup\{|(Th, h)|: h \in \mathfrak{H} \text{ and } \|h\| \leqq 1\}$. It seemed natural to us to introduce the functions w_ϱ defined on the space $\mathscr{L}(\mathfrak{H})$ of operators on $\mathfrak{H}$ in such a way that (a) w_ϱ is homogeneous $(w_\varrho(zT) = |z| w_\varrho(T))$, and (b) $w_\varrho(T) \leqq 1 \Leftrightarrow T \in C_\varrho$. In this way we obtain a family of "operator radii" which includes the familiar norms $\|\cdot\| (= w_1(\cdot))$ and $w(\cdot) (= w_2(\cdot))$ and which has a number of interesting properties. Recently we received from J. P. Williams a preprint of [8] where he, too, introduces the functions w_ϱ, stressing properties different from those which concern us here.

One can, of course, show that $w_\varrho(T^n) \leqq (w_\varrho(T))^n$ for all $\varrho > 0$ and all $n \geqq 1$ (recall the "power inequality" $w(T^n) \leqq (w(T))^n$ of Berger); here however we shall deal with somewhat different kinds of multiplicative behavior in the operator radii $w_\varrho(\cdot)$ (see § 4 and § 6 below). A basic result of this nature is the inequality $w_{\varrho\sigma}(TS) \leqq$ $\leqq w_\varrho(T) w_\sigma(S)$, holding whenever T and S double commute.

We shall also show that another well-known "operator radius", namely the spectral radius $v(\cdot)$ may be adjoined in a natural way to our family $\{w_\varrho(\cdot)\}_{\varrho > 0}$; in fact, if we let $w_\infty(T) = \lim_{\varrho \to 0} w_\varrho(T)$, we find that $w_\infty(T) = v(T)$. This result, and others concerning the relationship between $v(T)$ and $w_\varrho(T)$ are discussed in § 5.

These techniques may be applied to yield information about the classes C_ϱ themselves. We shall see, for example, that although Sz.-Nagy and Foiaş have shown that $\bigcup_{\varrho > 0} C_\varrho$ does not contain every "power-bounded" operator (see [6], § 4), nevertheless $\bigcup_{\varrho > 0} C_\varrho$ is dense in the class of all power-bounded operators.

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2. We shall use the following two characterizations of the classes C_ϱ. Both of these theorems are immediate consequences of the Theorem of [6] (or of its proof)

For $T\in\mathscr{L}(\mathfrak{H})$ and $\varrho>0$ define $T_\varrho(n)$ as follows:

$$T_\varrho(n)=\frac{1}{\varrho}T^n \text{ if } n=1,2,\dots;\quad T_\varrho(0)=I;\quad T_\varrho(n)=\frac{1}{\varrho}(T^*)^{-n} \text{ if } n=-1,-2,\dots.$$

Theorem 2. 1. *Given $\varrho>0$ and $T\in\mathscr{L}(\mathfrak{H})$ we have $T\in C_\varrho$ if, and only if, $\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}T_\varrho(n)\geqq 0$ for every θ and r such that $0\leqq r<1$. It is understood that the series converges absolutely, i. e., $\sum_{n=-\infty}^{\infty} r^{|n|}\|T_\varrho(n)\|<\infty$, whenever $0\leqq r<1$.*

Theorem 2. 2. *Given $\varrho>0$ and $T\in\mathscr{L}(\mathfrak{H})$, we have $T\in C_\varrho$ if, and only if, $v(T)\leqq 1$, and for each $h\in\mathfrak{H}$ and each complex z such that $|z|<1$,*

$$(*)\qquad \operatorname{Re}\big((I-zT)h,h\big)\geqq\left(1-\frac{\varrho}{2}\right)\|(I-zT)h\|^2.$$

If $\varrho\leqq 2$, the condition on the spectral radius is redundant [1]).

As Sz.-Nagy and Foiaş point out in [6], it is a simple matter to use Theorem 2. 2 to derive the earlier results of Sz.-Nagy and Berger that $C_1=\{T: \|T\|\leqq 1\}$ and $C_2=\{T: w(T)\leqq 1\}$.

3. For each $p>0$, we define the function w_ϱ on $\mathscr{L}(\mathfrak{H})$ as follows:

$$w_\varrho(T)=\inf\left\{u: u>0, \frac{1}{u}T\in C_\varrho\right\}.$$

Theorem 3.1. *$w_\varrho(\cdot)$ has the following properties:*

(1) $w_\varrho(T)<\infty$;

(2) $w_\varrho(T)>0$ *unless* $T=0$; *in fact,* $w_\varrho(T)\geqq\frac{1}{\varrho}\|T\|$;

(3) $w_\varrho(zT)=|z|w_\varrho(T)$;

(4) $w_\varrho(T)\leqq 1\Leftrightarrow T\in C_\varrho$.

Proof. To prove (1) we need only show that, for some $v>0$, $vT\in C_\varrho$. However, if $0\leqq r<1$ and $z=re^{i\theta}$,

$$\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}(vT)_\varrho(n)=I-\frac{2}{\varrho}\operatorname{Re}\sum_{n=1}^{\infty}(zvT)^n\geqq\left(1-\frac{2}{\varrho}\sum_{n=1}^{\infty}(v\|T\|)^n\right)I\geqq O$$

provided $v\|T\|$ is sufficiently small. For such v, then, by Theorem 2. 1, $vT\in C_\varrho$.

[1]) By a recent result, it is actually redundant for *any* ϱ; *cf.* Ch. Davis, The shell of a Hilbert-space operator, *Acta Sci. Math.*, **29** (1968), 69—86 (Prop. 8. 3).

(2) follows once we observe that, if $0<u<\frac{1}{\varrho}\|T\|$, we have $\left\|\frac{1}{u}T\right\|>\varrho$ so that we cannot have $\frac{1}{u}T=\varrho P_{\mathfrak{H}}U$ for any unitary operator U.

For the proofs of (3) and (4) we shall need the following result: $T\in C_\varrho$ and $|z|\leqq 1 \Rightarrow zT\in C_\varrho$. To see this note if that $T\in C_\varrho$ we have a unitary operator U on $\mathfrak{K}\supset\mathfrak{H}$ such that $T^n=\varrho P_{\mathfrak{H}}U^n$ $(n=1, 2, 3, \ldots)$; thus $(zT)^n=\varrho P_{\mathfrak{H}}(zU)^n$. But, if $|z|\leqq 1$, then $\|zU\|\leqq 1$ so that $zU\in C_1$ for the new space $\mathfrak{K}$; letting V be a unitary operator on $\mathfrak{K}_1\supset\mathfrak{K}$ such that $(zU)^n=P_{\mathfrak{K}}V^n$, we see that (with the obvious interpretation) $(zT)^n=\varrho P_{\mathfrak{H}}V^n$, so that, indeed, $zT\in C_\varrho$.

Recalling Theorem 2.1, it is clear that $O\in C_\varrho$ for every $\varrho>0$, and it follows easily that $w_\varrho(O)=0$. Thus (3) certainly holds when $|z|=0$. Turning to the case where $|z|>0$, write $z=re^{i\theta}$ and observe that, by the result of the last paragraph, we can assert that, for every $S\in\mathscr{L}(\mathfrak{H})$, $e^{i\theta}S\in C_\varrho \Leftrightarrow S\in C_\varrho$. We may thus perform the following calculation:

$$|z|\,w_\varrho(T)=r\left(\inf\left\{u: u>0, \frac{1}{u}T\in C_\varrho\right\}\right)=$$

$$=\inf\left\{ru: u>0, \frac{1}{ru}rT\in C_\varrho\right\}=\inf\left\{ru: u>0, \frac{1}{ru}re^{i\theta}T\in C_\varrho\right\}=$$

$$=\inf\left\{u: u>0, \frac{1}{u}zT\in C_\varrho\right\}=w_\varrho(zT).$$

The implication $(\Leftarrow)$ in (4) is immediate from the definition of w_ϱ. To prove $(\Rightarrow)$ assume that $w_\varrho(T)\neq 0$ and observe that we always have $u_n>0$ such that $\frac{1}{u_n}T\in C_\varrho$ and $u_n\downarrow w_\varrho(T)$; it follows easily, using Theorem 2. 2, that $\left(\lim\frac{1}{u_n}\right)T\in C_\varrho$, i.e., that $\frac{T}{w_\varrho(T)}\in C_\varrho$. If $w_\varrho(T)(=|w_\varrho(T)|)\leqq 1$, we conclude that $T=w_\varrho(T)\left(\frac{T}{w_\varrho(T)}\right)\in C_\varrho$. Finally, if $w_\varrho(T)=0$, then $T=O$ by (2), and, as noted earlier, we always have $O\in C_\varrho$. Q.e.d.

For $\varrho=1$ and $\varrho=2$, of course, $w_\varrho(\cdot)$ is actually a norm; more generally we have the following result.

Theorem 3. 2. *The function w_ϱ is a norm on $\mathscr{L}(\mathfrak{H})$ whenever $0<\varrho\leqq 2$.*

Proof. Equivalently, we must show that C_ϱ is a convex body in $\mathscr{L}(\mathfrak{H})$ whenever $\varrho\leqq 2$. Suppose, then, that $T, S\in C_\varrho$; by Theorem 2. 2 we have, for every $h\in\mathfrak{H}$ and complex z such that $|z|<1$,

$$\mathrm{Re}\,((I-zT)h, h)\geqq\left(1-\frac{\varrho}{2}\right)\|(I-zT)h\|^2$$

and an analogous inequality for S. It follows that, if $\lambda, \mu \geqq 0$ and $\lambda+\mu=1$, we have

$$\operatorname{Re}\left((I-z(\lambda T+\mu S))h, h\right) \geqq \left(1-\frac{\varrho}{2}\right)[\lambda\|(I-zT)h\|^2+\mu\|(I-zS)h\|^2].$$

For any $x, y \in \mathfrak{H}$ we have $\lambda\|x\|^2+\mu\|y\|^2 \geqq \|\lambda x+\mu y\|^2$, as the following calculation shows: $\lambda\|x\|^2+\mu\|y\|^2-\|\lambda x+\mu y\|^2 = (\lambda-\lambda^2)\|x\|^2+(\mu-\mu^2)\|y\|^2-2\lambda\mu \operatorname{Re}(x, y) \geqq$ $\geqq \lambda(1-\lambda)\|x\|^2+(1-\mu)\mu\|y\|^2-2\lambda\mu\|x\|\cdot\|y\| = \lambda\mu(\|x\|-\|y\|)^2 \geqq 0$. Since $\varrho \leqq 2$, we have $\left(1-\frac{\varrho}{2}\right) \geqq 0$; thus

$$\operatorname{Re}\left((I-z(\lambda T+\mu S))h, h\right) \geqq \left(1-\frac{\varrho}{2}\right)\|\lambda(I-zT)h+\mu(I-zS)h\|^2 =$$

$$= \left(1-\frac{\varrho}{2}\right)\|(I-z(\lambda T+\mu S))h\|^2.$$

Using Theorem 2. 2 again, we conclude that $\lambda T+\mu S \in C_\varrho$. Q.e.d.

As a by-product of the results of § 6, we shall see that $w_\varrho(\cdot)$ fails to be a norm whenever $\varrho > 2$.

4. In this section we discusss some of the basic inequalities governing the operator radii $w_\varrho(\cdot)$.

The following theorem comes as no surprise; it is simply a generalization of BERGER's proof of the "power inequality" $w(T^n) \leqq (w(T))^n$ (a conjecture of HALMOS).

Theorem 4.1. *For each $\varrho > 0$ and $T \in \mathscr{L}(\mathfrak{H})$ we have $w_\varrho(T^k) \leqq (w_\varrho(T))^k$ $(k=1, 2, 3, \ldots)$.*

Proof. By Theorem 3.1, $w_\varrho(\cdot)$ is homogeneous so that we need only show that $w_\varrho(T) \leqq 1 \Rightarrow w_\varrho(T^k) \leqq 1$, or equivalently that $T \in C_\varrho \Rightarrow T^k \in C_\varrho$. But if U is a unitary operator on $\mathfrak{K} \supset \mathfrak{H}$ such that $T^n = \varrho P_\mathfrak{H} U^n$, then $(T^k)^n = \varrho P_\mathfrak{H}(U^k)^n$ and U^k is unitary. Q.e.d.

In the next theorem we derive a different sort of inequality concerning the behavior of the w_ϱ with respect to operator multiplication.

Theorem 4. 2. *If $\varrho, \sigma > 0$ and $T, S \in \mathscr{L}(\mathfrak{H})$, we have $w_{\varrho\sigma}(TS) \leqq w_\varrho(T) \cdot w_\sigma(S)$ provided T and S double commute (i.e., $TS=ST$ and $TS^*=S^*T$).*

Proof. Again it is clear, using Theorem 3.1 ((3) and (4)), that we need only show that $TS \in C_{\varrho\sigma}$ whenever $T \in C_\varrho$ and $S \in C_\sigma$ and T, S double commute.

By Theorem 2.1 we have $\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}T_\varrho(n) \geqq O$ and $\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}S_\sigma(n) \geqq O$ in the sense described in that theorem. Now it is not hard to prove (see [4], Theorem 3.3) that if, in the appropriate sense, $\sum_{-\infty}^{\infty} r^{|n|}e^{in\theta}A_n \geqq O$ and $\sum_{-\infty}^{\infty} r^{|n|}e^{in\theta}B_n \geqq O$, then we also have $\sum_{-\infty}^{\infty} r^{|n|} e^{in\theta} A_n B_n \geqq O$, provided $A_n B_m = B_m A_n$ for all choices of n and m. Since T and S double commute we may apply this result to conclude that $\sum_{-\infty}^{\infty} r^{|n|} e^{in\theta} T_\varrho(n) S_\sigma(n) \geqq O$.

Finally, we note that, for every n, $T_\varrho(n) S_\sigma(n) = (TS)_{\varrho\sigma}(n)$ so that, using Theorem 2.1 once more, we indeed have $TS \in C_{\varrho\sigma}$. Q.e.d.

In connection with the essential fact of the last theorem — namely that $T \in C_\varrho$, $S \in C_\sigma$ and T, S double commute imply $TS \in C_{\varrho\sigma}$ — we wish to mention another proof of this result, sent to us recently by Professor SZ.-NAGY, see [5*]. In that proof the "unitary $\varrho\sigma$-dilation" for TS is given explicitly in the form UV where U and V are commuting unitary ϱ-and σ-dilations of T and S respectively, constructed simultaneously on a space $\mathfrak{K} \supset \mathfrak{H}$.

If $\varrho = 2$, $\sigma = 1$ in the theorem just proved we obtain the inequality $w(TS) \leqq \leqq w(T) \cdot \|S\|$ (if T, S double commute). This result occurs in [4], where a number of proofs of the inequality are discussed.

At this point it is important to determine the value of $w_\varrho(I)$ for each $\varrho > 0$.

Theorem 4.3. *For* $\varrho \geqq 1$, $w_\varrho(I) = 1$; *for* $0 < \varrho < 1$, $w_\varrho(I) = \dfrac{2}{\varrho} - 1$.

Proof. We must determine for which values $u > 0$ we have $\frac{1}{u} I \in C_\varrho$. Using Theorem 2.2 we see that it is necessary and sufficient that

$$(*) \qquad \operatorname{Re}\left(1 - \frac{z}{u}\right) \geqq \left(1 - \frac{\varrho}{2}\right)\left|1 - \frac{z}{u}\right|^2$$

whenever $|z| < 1$ and that $v\left(\frac{1}{u} I\right) \leqq 1$. The last condition implies that, in any case, $u \geqq 1$.

Rewriting (*) in the form $\left(1 - \frac{\varrho}{2}\right) \leqq \operatorname{Re}\left(1 - \frac{z}{u}\right)^{-1}$, we see that we must consider the values of $\operatorname{Re} w^{-1}$ where w lies inside the circle c_1 of radius $\frac{1}{u}$ centered at 1. Since $\frac{1}{u} \leqq 1$ it is clear that, inverting c_1 in the unit circle, we obtain a circle (or half-plane) c_2 having $\left(1 + \frac{1}{u}\right)^{-1}$ as its most westerly point. Thus, the additional condition imposed on u by (*) is $\left(1 - \frac{\varrho}{2}\right) \leqq \left(1 + \frac{1}{u}\right)^{-1}$; this holds automatically if $\left(1 - \frac{\varrho}{2}\right) \leqq 0$ and otherwise reduces to $u \geqq \frac{2}{\varrho} - 1$.

Thus $\frac{1}{u} I \in C_\varrho \Leftrightarrow u \geqq \max\left(1, \frac{2}{\varrho}-1\right)$ so that, indeed, $w_\varrho(I) = \max\left(1, \frac{2}{\varrho}-1\right)$. Q.e.d.

It should be pointed out that the theorem above is included in a result of DURSZT (see [3, Theorem 1]) which, upon introducing the functions $w_\varrho(\cdot)$, amounts to the evaluation of $w_\varrho(T)$ for any normal T. In § 5, on the other hand, we shall see that Theorem 4. 3 combined with some general inequalities yields the theorem of DURSZT in a somewhat extended form.

We can now prove some preliminary results concerning the behavior of $w_\varrho(T)$ for fixed T as ϱ varies.

Theorem 4. 4. *Suppose $T \in \mathscr{L}(\mathfrak{H})$ and $0 < \varrho < \varrho'$. Then $w_{\varrho'}(T) \leqq w_\varrho(T)$ and $w_\varrho(T) \leqq \left(\frac{2\varrho'}{\varrho} - 1\right) w_{\varrho'}(T)$. Thus $w_\varrho(T)$ is continuous and non-increasing as ϱ increases.*

Proof. Simply combine Theorems 4. 2 and 4. 3 as follows:

$$w_{\varrho'}(T) = w_{\left(\frac{\varrho'}{\varrho}\right)\varrho}(IT) \leqq w_{\frac{\varrho'}{\varrho}}(I) \cdot w_\varrho(T) = 1 \cdot w_\varrho(T);$$

$$w_\varrho(T) = w_{\left(\frac{\varrho}{\varrho'}\right)\varrho'}(IT) \leqq w_{\frac{\varrho}{\varrho'}}(I) \cdot w_{\varrho'}(T) = \left(\frac{2\varrho'}{\varrho} - 1\right) \cdot w_{\varrho'}(T). \quad \text{Q. e. d.}$$

In view of Theorem 3.1, the fact that $w_\varrho(T)$ is non-increasing as ϱ increases implies that $C_{\varrho'} \supset C_\varrho$ whenever $\varrho' > \varrho$. In [6] (§ 3) SZ.-NAGY and FOIAŞ discuss the problem of determining when these inclusions are strict. In essence, they consider the operator A defined by the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (relative to an orthonormal basis) on a 2-dimensional subspace of $\mathfrak{H}$ (and vanishing on the orthogonal complement) and show that $\varrho A \in C_{\varrho+1} \setminus C_{\varrho-\varepsilon}$ whenever $\varepsilon > 0$ and $\varrho \geqq 1$, and that $\frac{\varrho}{2-\varrho} A \in C_\varrho \setminus C_{\left(\frac{\varrho}{2-\varrho} - \varepsilon\right)}$ whenever $\varepsilon > 0$ and $\varrho < 1$. Actually, as DURSZT was the first to point out (see [3, Theorem 2]), we can show that $\varrho A \in C_\varrho \setminus C_{\varrho-\varepsilon}$ for every $\varrho > 0$ and $\varepsilon > 0$, so that the classes C_ϱ form a *strictly* increasing scale (as ϱ increases). By Theorem 3.1, it is sufficient to show that $w_\varrho(\varrho A) = 1$ and $w_{\varrho-\varepsilon}(\varrho A) > 1$; but it is clear that $w_\varrho(A) = \frac{1}{\varrho}$, for every $\varrho > 0$, by means of the following observation, which we shall have occasion to use several times again.

Theorem 4. 5. *Suppose $T \in \mathscr{L}(\mathfrak{H})$, $\|T\| = 1$, and $T^2 = O$. Then, for every $\varrho > 0$, $w_\varrho(T) = \frac{1}{\varrho}$.*

Proof. As $w_1(T) = \|T\| = 1$ we have $T \in C_1$, i.e., for some unitary operator U on $\mathfrak{K} \supset \mathfrak{H}$ we have $T^n = P_{\mathfrak{H}} U^n$ $(n = 1, 2, 3, \ldots)$. Since $T^2 = O$, $(\varrho T)^n = \varrho T^n$ $(n = 1, 2,$

3, ...), so that we have $(\varrho T)^n = \varrho P_{\mathfrak{H}} U^n$ $(n=1, 2, 3, \ldots)$, i.e., $\varrho T \in C_\varrho$. Thus $w_\varrho(\varrho T) \leqq 1$ and $w_\varrho(T) \leqq \frac{1}{\varrho}$. But, by Theorem 3.1 (2), $w_\varrho(T) \geqq \frac{1}{\varrho}\|T\| = \frac{1}{\varrho}$. Q.e.d.

As we have noted above, we have as an immediate consequence the following fact.

Corollary 4.6 (Durszt). *Provided $\mathfrak{H}$ is at least 2-dimensional, we have $C_{\varrho'} \supset C_\varrho$ strictly whenever $\varrho' > \varrho$ (>0).*

5. In this section we discuss the relationship between the spectral radius $v(T)$ and the operator radii $w_\varrho(T)$.

Since $w_\varrho(T)$ decreases with increasing ϱ and is always non-negative, we may define, for each $T \in \mathscr{L}(\mathfrak{H})$, $w_\infty(T) = \lim_{\varrho \to \infty} w_\varrho(T)$.

Theorem 5.1. *For every $T \in \mathscr{L}(\mathfrak{H})$, $w_\infty(T) = v(T)$.*

Proof. We have $\frac{T}{w_\varrho(T)} \in C_\varrho$ so that, by Theorem 2. 2, $v\left(\frac{T}{w_\varrho(T)}\right) \leqq 1$; thus $v(T) \leqq w_\varrho(T)$ for every ϱ.

On the other hand, suppose that $v(T) < 1$. For some $s > 1$ we also have $v(sT) < 1$ and since, by the spectral radius formula, $\|(sT)^n\|^{\frac{1}{n}} \to v(sT)$, we see that for some $B < \infty$ we have $s^n \|T^n\| \leqq B$ $(n=1, 2, 3, \ldots)$. Thus, if $|z| < 1$, $\left\|\sum_{n=1}^{\infty} (zT)^n\right\| \leqq \sum_{n=1}^{\infty} \|T^n\| \leqq$ $\leqq \sum_{n=1}^{\infty} \frac{B}{s^n} = M(<\infty)$. It follows that if $0 \leqq r < 1$ we have, setting $z = re^{i\theta}$,

$$\sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} T_\varrho(n) = I + \frac{2}{\varrho} \operatorname{Re}\left(\sum_{n=1}^{\infty} (zT)^n\right) \geqq \left(1 - \frac{2}{\varrho}\left\|\sum_{n=1}^{\infty} (zT)^n\right\|\right) I \geqq \left(1 - \frac{2}{\varrho} M\right) I \geqq o$$

as soon as $\varrho \geqq 2M$.

Using Theorem 2.1, it is clear that, whenever $v(T) < 1$, there is some ϱ such that $T \in C_\varrho$, i.e., $w_\varrho(T) \leqq 1$. Now if $v(T) \neq 0$, and $\varepsilon > 0$ we have $v\left(\frac{T}{(1+\varepsilon)v(T)}\right) = \frac{1}{1+\varepsilon} < 1$ so that, for some ϱ, $w_\varrho\left(\frac{T}{(1+\varepsilon)v(T)}\right) \leqq 1$, i.e., $(1+\varepsilon)v(T) \geqq w_\varrho(T) (\geqq v(T))$. Clearly, then, $w_\infty(T) = v(T)$ in this case. If $v(T) = 0$, then for any n $v(nT) = 0 < 1$ so that for some ϱ $w_\varrho(nT) \leqq 1$, i.e., $w_\varrho(T) \leqq \frac{1}{n}$. Thus $w_\infty(T) = 0\ (= v(T))$. Q.e.d.

An operator T in any one of the operator classes C_ϱ is "power bounded", i.e., the sequence $\{\|T^n\|\}_1^\infty$ is bounded; in fact, $\|T^n\| = \|\varrho P_{\mathfrak{H}} U^n\| \leqq \varrho$. Sz.-Nagy and Foiaş show, however, by constructing an example (see [6], §4), that there are power-bounded operators not lying in any of the classes C_ϱ. Nevertheless, we have the following result.

Theorem 5. 2. *The family of power-bounded operators* $\bigcup_{\varrho>0} C_\varrho$ *is dense (with respect to the ordinary operator norm) in the class of all power-bounded operators.*

Proof. If T is power-bounded the $\nu(T)=\lim \|T^n\|^{\frac{1}{n}}\leqq 1$. Thus, for any r such that $0\leqq r<1$, we have $\nu(rT)<1$ and hence, by Theorem 5.1, there is some ϱ such that $w_\varrho(rT)\leqq 1$, i.e., $rT\in C_\varrho$, hence the assertion follows.

If $T\in\mathscr{L}(\mathfrak{H})$ and $w(T)=\|T\|$, then we actually have $\nu(T)=w(T)=\|T\|$, i.e. $w_1(T)=w_2(T)\Rightarrow \nu(T)=w_1(T)$. We may even replace 1 and 2 in the above statement by *any* distinct values of ϱ. Indeed, we have the following:

Theorem 5. 3. *If* $T\in\mathscr{L}(\mathfrak{H})$ *is such that* $w_{\varrho_0}(T)>\nu(T)$, *then* $w_\varrho(T)$ *is strictly decreasing at* ϱ_0, *i.e.,* $\varrho>\varrho_0\Rightarrow w_\varrho(T)<w_{\varrho_0}(T)$.

Proof. We may assume that $w_{\varrho_0}(T)=1$ and $\nu(T)<1$, and prove that, if $\varrho>\varrho_0$, $w_\varrho(T)<1$. By Theorems 3.1 and 2. 2 we have $T\in C_{\varrho_0}$ and hence, for each $h\in\mathfrak{H}$ and complex z such that $|z|<1$,

$$(*)\qquad \mathrm{Re}((I-zT)h,h)\geqq \left(1-\frac{\varrho_0}{2}\right)\|(I-zT)h\|^2.$$

Now $\alpha=\inf\left(\|(I-zT)h\|^2\colon |z|<1,\ h\in\mathfrak{H},\ \|h\|=1\right)>0$, since we would otherwise have $h_n\in\mathfrak{H}$ and complex z_n such that $\|h_n\|=1$, $|z_n|<1$, and $\|(I-z_nT)h_n\|\to 0$; by passing to a subsequence we could assume that $z_n\to z_0$, and it is easy to see that $\|(I-z_0T)h_n\|\to 0$ in this case: thus we would have $1/z_0$ in the spectrum of T, contradicting the assumption that $\nu(T)<1$.

If we choose $b>1$ such that, whenever $|z|<1$ and $\|h\|=1$, we have

$$|\mathrm{Re}((I-zbT)h,h)-\mathrm{Re}((I-zT)h,h)|<\frac{\varrho-\varrho_0}{2}\cdot\frac{\alpha}{2}$$

and

$$\left|\left(1-\frac{\varrho}{2}\right)\|(I-zbT)h\|^2-\left(1-\frac{\varrho}{2}\right)\|(I-zT)h\|^2\right|<\frac{\varrho-\varrho_0}{2}\cdot\frac{\alpha}{2},$$

it is easy to see that $(*)$ implies

$$\mathrm{Re}((I-zbT)h,h)\geqq\left(1-\frac{\varrho}{2}\right)\|(I-zbT)h\|^2$$

for all such z and h. But this inequality is independent of the value of $\|h\|$, so that, by Theorem 2. 2, we have $bT\in C_\varrho$ provided we have chosen $b(>1)$ small enough so that, in addition, $\nu(bT)\leqq 1$. In this case $w_\varrho(bT)\leqq 1$, i.e., $w_\varrho(T)\leqq\frac{1}{b}<1$. Q.e.d.

The following theorem finds its natural place in this section.

Theorem 5. 4. *For any* $T\in\mathscr{L}(\mathfrak{H})$ *and* $\varrho>0$ *we have* $w_\varrho(T)\geqq w_\varrho(I)\,\nu(T)$.

This result follows upon recalling Theorem 2. 2 and the fact that T has an approximate eigenvalue λ such that $|\lambda| = v(T)$.

By Theorem 4. 2 we have $w_\varrho(T) \leqq w_\varrho(I) w_1(T)$ and this combined with the last theorem and our evaluation of $w_\varrho(I)$ (i.e., Theorem 4. 3) yields the following extension of a theorem of DURSZT (see [3, Theorem 1]). The extension is implicit in DURSZT's work, and has also been pointed out by BERGER and STAMPFLI (see [2, Theorem 6]).

Theorem 5. 5. *For any $T \in \mathcal{L}(\mathfrak{H})$ such that $v(T) = \|T\|$ (such T have been called "normaloid" operators, and include, of course, the normal operators) we have*

$$w_\varrho(T) = \|T\| \, w_\varrho(I) = \begin{cases} \|T\| \left(\dfrac{2}{\varrho} - 1 \right), & \text{if } \; 0 < \varrho < 1, \\ \|T\|, & \text{if } \; \varrho \geqq 1. \end{cases}$$

6. Upon considering the "power inequality" of Theorem 4.1 one naturally asks to what extent the operator radii $w_\varrho(\cdot)$ are multiplicative, i.e., under what conditions do we have an inequality of the following type: $w_\varrho(TS) \leqq w_\varrho(T) \cdot w_\varrho(S)$. Although it does not seem possible, except in very special cases, to derive the power inequality from a more general inequality involving a pair of operators we shall describe here some results along these lines.

Let us first observe that in the case where T and S may be quite unrelated, and in the case where they are assumed to double commute, the problem may be settled in a fairly satisfactory way.

Theorem 6.1. *For any $T, S \in \mathcal{L}(\mathfrak{H})$ and $\varrho \geqq 1$ we have $w_\varrho(TS) \leqq \varrho^2 w_\varrho(T) \cdot w_\varrho(S)$; this result is best possible, provided $\mathfrak{H}$ is at least 2-dimensional.*

Proof. Using Theorems 4. 4 and 3.1 (2) we have at once $w_\varrho(TS) \leqq w_1(TS) \leqq$ $\leqq w_1(T) w_1(S) \leqq (\varrho w_\varrho(T))(\varrho w_\varrho(S))$.

On the other hand, if $\dim(\mathfrak{H}) \geqq 2$ we may define operators A and B on some 2-dimensional subspace by the matrices (relative to an orthonormal basis) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ respectively, and require that A and B vanish on the orthogonal complement. By Theorem 4. 5, $w_\varrho(A) = w_\varrho(B) = \dfrac{1}{\varrho}$. Now AB corresponds to the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ so that $w_1(AB) = v(AB) = 1$, and hence $w_\varrho(AB) = 1$ whenever $\varrho \geqq 1$. This example shows that the inequality of the theorem cannot be improved. Q.e.d.

Theorem 6. 2. *If $T, S \in \mathcal{L}(\mathfrak{H})$ and T and S double commute, then $w_\varrho(TS) \leqq$ $\leqq \varrho w_\varrho(T) w_\varrho(S)$ for all $\varrho > 0$. This result is best possible, at least if* $\dim(\mathfrak{H}) \geqq 4$.

Proof. Using Theorems 4. 2 and 3.1 (2) we have $w_\varrho(TS)\leqq w_1(T)w_\varrho(S)\leqq$ $\leqq(\varrho w_\varrho(T))w_\varrho(S)$.

On the other hand, if $\dim(\mathfrak{H})\geqq 4$ we may define operators C and D on some 4-dimensional subspace by the matrices (relative to an orthonormal basis) $\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$ and $\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$ respectively, and require that C and D vanish on the orthogonal complement. It is easy to verify that C and D double commute and that CD corresponds to the matrix $\begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$. Applying Theorem 4. 5, we see that $w_\varrho(C)=w_\varrho(D)=w_\varrho(CD)=\frac{1}{\varrho}$, for every $\varrho>0$. It follows that our inequality cannot be improved. Q.e.d.

When we simply assume that T and S commute, the situation is much less clear. Since $w_1(\cdot)=\|\cdot\|$ and $w_\infty(\cdot)=v(\cdot)$, we have $w_1(TS)\leqq w_1(T)\cdot w_1(S)$ and, provided T and S commute, $w_\infty(TS)\leqq w_\infty(T)\cdot w_\infty(S)$. The case of $w_2(\cdot)\,(=w(\cdot))$ is settled by the following theorem, which also shows that the constant in Theorem 6.1 can be improved if we assume T and S commute, at least when $\sqrt{2}<\varrho\leqq 2$.

Theorem 6. 3. *If $T, S\in\mathcal{L}(\mathfrak{H})$, T and S commute, and $w_\varrho(\cdot)$ is a norm (and hence, by Theorem 3. 2, whenever $\varrho\leqq 2$), then $w_\varrho(TS)\leqq 2w_\varrho(T)w_\varrho(S)$. This result is best possible for $\varrho=2$, at least if* $\dim(\mathfrak{H})\geqq 4$.

Proof. We may assume that $w_\varrho(T)=w_\varrho(S)=1$ and prove that $w_\varrho(TS)\leqq 2$. In the following calculation we use both the assumption that $w_\varrho(\cdot)$ is a norm and the "power inequality" of Theorem 4.1:

$$w_\varrho(TS)=w_\varrho\left(\tfrac14[(T+S)^2-(T-S)^2]\right)\leqq$$

$$\leqq\tfrac14\left[w_\varrho((T+S)^2)+w_\varrho((T-S)^2)\right]\leqq\tfrac14\left[(w_\varrho(T+S))^2+(w_\varrho(T-S))^2\right]\leqq$$

$$\leqq\tfrac14\left[(w_\varrho(T)+w_\varrho(S))^2+(w_\varrho(T)+w_\varrho(S))^2\right]=2.$$

To see that the inequality $w_2(TS)\leqq 2w_2(T)\cdot w_2(S)$ cannot be improved (if $\dim(\mathfrak{H})\geqq 4$), recall that, by Theorem 6. 2, the inequality is best possible even under the assumption that T and S *double* commute. Q.e.d.

Corollary 6. 4. *For $\varrho>2$, $w_\varrho(\cdot)$ fails to be a norm on $\mathcal{L}(\mathfrak{H})$.*

Proof. Compare Theorems 6. 3 and 6. 2. Q.e.d.

The following theorem shows that Theorem 6. 3 can be much improved if one of the operators is normal.

Theorem 6. 5. *Suppose T and S are commuting operators in $\mathscr{L}(\mathfrak{H})$ and that T is normal. Then, for all $\varrho > 0$, $w_\varrho(TS) \leqq w_\varrho(T) w_\varrho(S)$.*

Proof. Since S commutes with the normal operator T, FUGLEDE's theorem (see ROSENBLUM [7] for a slick proof) tells us that S and T double commute. Hence, by Theorem 4. 2, $w_\varrho(TS) \leqq w_1(T) w_\varrho(S)$. But, as T is normal, $v(T) = \|T\| \; (= w_1(T))$, so that for all $\varrho > 0$ $w_1(T) \leqq w_\varrho(T)$. Thus $w_\varrho(TS) \leqq w_\varrho(T) w_\varrho(S)$. Q.e.d.

While it does not seem clear whether or not the inequalities of Theorems 6. 2 and 4. 2 can be extended to the case where the operators merely commute, it is usually possible to say something more in this case than in the case where the operators are quite arbitrary. Our final theorem is a rather curious example of a result of this nature. Note that for arbitrary $T, S \in \mathscr{L}(\mathfrak{H})$ we have, for $\varrho \geqq 1$, $w_\varrho(TS) \leqq \|TS\| \leqq \leqq \|T\| \cdot \|S\| \leqq \varrho w_\varrho(T) \cdot \|S\|$ (we have used Theorems 4. 4 and 3. 1(2)); furthermore we can actually have equality under these conditions (consider the operators A and B introduced in the proof of Theorem 6. 1). Of course, if T and S *double* commute, Theorem 4. 2 tells that $w_\varrho(TS) \leqq w_\varrho(T) \cdot \|S\|$. Whether or not we can say the same if T and S merely commute, we *do* have the following improvement over the case where T and S may be completely unrelated.

Theorem 6. 6. *Suppose $\varrho > 1$, and T and S are commuting operators in $\mathscr{L}(\mathfrak{H})$. Then provided $T \neq O$ and $S \neq O$, $w_\varrho(TS) < \varrho w_\varrho(T) \cdot \|S\|$.*

Proof. Since, as we have noted above, we have $w_\varrho(TS) \leqq \|TS\| \leqq \|T\| \cdot \|S\| \leqq \leqq \varrho w_\varrho(T) \cdot \|S\|$, the theorem could fail only if we had $w_\varrho(TS) = \|TS\|$. In this case, by Theorem 5. 3, $w_\varrho(TS) = v(TS)$; but this is impossible because, since T and S commute, we would have $w_\varrho(TS) = v(TS) \leqq v(T) \cdot v(S) \leqq w_\varrho(T) \cdot \|S\|$, as well as $w_\varrho(TS) = \varrho w_\varrho(T) \cdot \|S\|$. Q.e.d.

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A remark on a class of power-bounded operators in Hilbert space

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The notion of unitary dilation was generalized by SZ.-NAGY and C. FOIAŞ [1], [2] by considering the classes C_ϱ of operators T in Hilbert spaces H whose powers admit a representation

$$T^n = \varrho \cdot \text{pr } U^n \qquad (n=1, 2, 3, \dots)$$

where U is a unitary operator in some Hilbert space $H_1 \supset H$.

H. LANGER has proposed the following further generalization: if A is a positive self-adjoint operator, $mI \leqq A \leqq MI$, where $m > 0$, consider the class C_A of operators whose powers admit a representation

$$QT^nQ = \text{pr } U^n \qquad (n=1, 2, 3, \dots)$$

where $Q = A^{-\frac{1}{2}}$ and U is a unitary operator in some Hilbert space $H_1 \supset H$; see [2], p. 54.

The aim of this note is to prove the following

Theorem. *C_A is a increasing function of A in the sense that $A_1 \leqq A_2$ implies $C_{A_1} \subseteqq C_{A_2}$.*

Proof. We use the following characterization of C_A indicated by H. LANGER (see [2], p. 54): $T \in C_A$ if and only if

1° the spectrum of T lies in the closed unit disc,

2° $(Ah, h) - \text{Re}\,(z(A-I)Th, h) + |z|^2((A-2I)Th, Th) \geqq 0$ for $|z| \leqq 1$ and $h \in H$.

The relation 2° can be written in the form:

$$((A-I)h, h) + (h, h) - 2\text{Re}\,(z(A-I)Th, h) - |z|^2 \|Th\|^2 + |z|^2((A-I)Th, Th) \geqq 0$$

or, equivalently,

$$\|h\|^2 - |z|^2 \|Th\|^2 + ((A-I)(I-2T)h, (I-zT)h) \geqq 0.$$

Since the left-hand side is an increasing function of the self-adjoint operator A, the theorem is proved.

Corollary 1. *If* $T \in C_A$ *then* $T \in C_{\|A\|}$.
This follows from the fact that $A \leqq \|A\| \cdot I$.

Corollary 2. *Every operator* T *in* C_A *is similar to a contraction.*
This follows from Corollary 1 and the theorem of [3].

Corollary 3. *There exist power-bounded operators which do not belong to any class* C_A.

Indeed in [1] there is given a power-bounded operator which belongs to none of the classes C_ϱ, thus it belongs to none of the classes C_A, either.

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A problem on lacunary series

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The limiting distributions of lacunary trigonometric sums

$$S_N(t) = (\tfrac{1}{2}N)^{-\frac{1}{2}} \sum_{k=1}^{N} \cos(y_k t) \tag{1}$$

where $1 \leqq y_1$, $qy_k \leqq y_{k+1}$ $(1 \leqq k < \infty)$, were considered by SALEM and ZYGMUND [3], who showed that, over any fixed set of positive Lebesgue measure, S_N tends in law to the normal, or Gaussian, distribution of mean 0 and variance 1. HELSON and KAHANE [1] showed that certain consequences of lacunarity persist if the Lebesgue measure is replaced by a Baire probability measure whose Fourier—Stieltjes transform meets the condition $\hat{\mu}(u) = O(|u|^{-\alpha})$ for some $\alpha > 0$. The nearest metric analogue of this is

$$\mu([a, a+h]) \leqq Mh^{\beta} \quad \text{for all } a, \text{ and } h > 0, \tag{2}$$

where β and $M = M(\beta)$ are positive constants. In this case nothing like the Salem—Zygmund result is necessarily valid, even if (2) holds for every $\beta < 1$. However, if we treat the coefficients y_k as functions of a variable x, we can obtain a similar result, at least in a special case.

Theorem. *For $x > 1$, let $y_k = x^k$, that is*

$$S_N(t) = (\tfrac{1}{2}N)^{-\frac{1}{2}} \sum_{k=1}^{N} \cos(x^k t).$$

Then for almost all $x > 2$

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} e^{-i\lambda S_N(t)} \mu(dt) = e^{-\frac{1}{2}\lambda^2} \qquad (-\infty < \lambda < \infty).$$

1. Lemma. *There exist a number $\delta > 0$, depending only on β, and a number M' depending only on M and β* (cf. (2)) *with the property: For any real function $f(x)$, $0 \leqq x \leqq 1$, of class C^2, with $f'' \geqq r > 0$,*

$$I = \int_0^1 \left| \int_{-\infty}^{\infty} e^{itf(x)} \mu(dt) \right|^2 dx \leqq M' r^{-\delta}.$$

Proof. The inner integral can be written

$$\int_{-\infty} \int_{-\infty} e^{i(t-s)f(x)} \mu(dt)\, \mu(ds)$$

so

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{1} e^{i(t-s)f(x)}\, dx\, \mu(dt)\, \mu(ds) \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t-s)\, \mu(dt)\, \mu(ds).$$

By VAN DER CORPUT's lemma [2]

$$|h(t-s)| \leqq C|r(t-s)|^{-\frac{1}{2}} \qquad (-\infty < s,\ t < \infty),$$

for an absolute constant C; clearly also $|h| \leqq 1$. Let $\eta > 0$ so that

$$\iint_{|t-s| \leqq \eta} \mu(dt)\, \mu(s) = \int_{-\infty}^{\infty} \int_{s-\eta}^{s+\eta} \mu(dt)\, \mu(ds) \leqq 2M\eta^{\beta}.$$

For every choice of $\eta > 0$ we find

$$I \leqq 2M\eta^{\beta} + Cr^{-\frac{1}{2}}\eta^{-\frac{1}{2}}.$$

Choosing $\eta = r^{-1/1+2\beta}$ we obtain

$$I \leqq (2M + C)\, r^{-\beta/1+2\beta}.$$

2. If $c_1, \ldots, c_N$ are real numbers, then

$$\left| \exp\left[-i \sum_{k=1}^{N} c_k \right] - \sum_{k=1}^{N} (1 - ic_k) \sum_{k=1}^{N} \left(1 - \frac{1}{2} c_k^2 \right) \right| \leqq 2 \sum_{k=1}^{N} |c_k|^3 \sum_{k=1}^{N} (1 + c_k^2)^{3/2}.$$

Hence

$$\exp(-i\lambda S_N(t)) = O(N^{-\frac{1}{2}}) + \sum_{k=1}^{N} \left(1 - i\lambda(\tfrac{1}{2}N)^{-1} \cos(x^k t)\right) \sum_{k=1}^{N} \left(1 - \lambda^2 N^{-1} \cos^2(x^k t)\right).$$

(The symbols O and o always refer to a bound, uniform for any interval $-B \leqq \lambda \leqq B$). Using the formula $\cos 2v = 2\cos^2 v - 1$, we see that the second factor converges in μ-measure to $e^{-\frac{1}{2}\lambda^2}$, provided

$$\sum_{1 \leqq j < k \leqq N} \sum \left| \int_{-\infty}^{\infty} \cos(2x^j t) \cos(2x^k t)\, \mu(dt) \right| = o(N^2),$$

or

$$\sum_{1 \leqq j < k \leqq N} \sum \left| \int_{-\infty}^{\infty} \cos(2x^j t \pm 2x^k t)\, \mu(dt) \right| = o(N^2).$$

But by the lemma and the Schwarz inequality

$$\sum_{1 \leqq j < k} \sum \int_{q}^{q+1} \left| \int_{-\infty}^{\infty} \cos(2x^j t \pm 2x^k t)\, \mu(dt) \right| dx < \infty,$$

for any $q>1$. Here we applied the lemma to the functions $2x^k \pm 2x^j$ $(1 \leqq j<k)$ whose derivatives are easily estimated. It follows that the convergence of the second factor takes place for almost all $x>1$.

3. Set

$$J_N(x) = \int_{-\infty}^{\infty} \prod_{k=1}^{N} [1 - i\lambda(\tfrac{1}{2}N)^{-\frac{1}{2}} \cos(x^k t)]\mu(dt).$$

First, we estimate the sum of the terms involving a single cosine, or a product of two cosines. The first kind give a sum

$$\prod_{k=1}^{N} O(N^{-\frac{1}{2}}) \int_{-\infty}^{\infty} \cos(x^k t)\,\mu(dt)$$

and the second

$$\sum_{1\leqq j<k\leqq N}\sum O(N^{-1}) \int_{-\infty}^{\infty} \cos(x^k t \pm x^j t)\,\mu(dt).$$

We already dealt with integrals similar to these, and showed that the sums converge to 0 for almost all $x>1$.

From now on we assume

$$q+1 \geqq x \geqq q > 2$$

so that if $0<k_1<\ldots<k_r$ are integers $(2\leqq r)$,

$$\frac{d^2}{dx^2}(x^{k_r} \pm x^{k_{r-1}} \pm \ldots \pm x^{k_1}) \geqq Aq^{k_r}$$

for some $A=A(q)>0$.

Consider now the part of $J_N(x)$ involving products of exactly m cosines, $3\leqq m\leqq N$. We divide this further according to the largest power of x involved, and obtain

$$\sum_{1\leqq k_1<\ldots<k_m\leqq N}\ldots\sum \left(\frac{\sqrt{2}}{2}\right)^m O(B^m N^{-\frac{1}{2}m}) \int_{-\infty}^{\infty} \cos(\pm x^{k_m} t \pm \ldots \pm x^{k_1} t)\,\mu(dt).$$

(The number B is chosen so that $|\lambda| \leqq B$.)

By the lemma and the inequality on the second derivative, there is an $r\in(0,1)$ such that

$$\int_q^{q+1} \left| \int_{-\infty}^{\infty} \cos(tx^{k_m} \pm \ldots \pm tx^{k_1})\,\mu(dt)\right| dx = O(r^{k_m}).$$

Thus the integral of the modulus of that part of $J_N(x)$ not already disposed of, is of the order of magnitude

$$\varphi(N) = \sum_{m=3}^{N} N^{-\frac{1}{2}m}(2B)^m \sum_{k=m}^{N} \binom{k-1}{m-1} r^k \leqq \sum_{k=3}^{N}\sum_{m=3}^{k} r^k N^{-\frac{1}{2}m}(2B)^m \binom{k}{m}.$$

By TAYLOR's formula

$$\sum_{m=3}^{k}(2BN^{-\frac{1}{2}})^m \binom{k}{m} \leqq \frac{1}{6}(2BN^{-\frac{1}{2}})^3 \sum_{m=3}^{k}(2BN^{-\frac{1}{2}})^{m-3}\binom{k}{m} m(m-1)(m-2) \leqq$$
$$\leqq k^3(2BN^{-\frac{1}{2}})^3(1+2BN^{-\frac{1}{2}})^k.$$

Thus $\varphi(N) \ll \sum_{k=3}^{N} N^{-\frac{3}{2}} r^k k^3 (1+2BN^{-\frac{1}{2}})^k$. If N is so large that $(1+2BN^{-\frac{1}{2}})\,r < r^{\frac{1}{2}}$, $\varphi(N) \ll N^{-\frac{3}{2}}$. Since $\sum N^{-\frac{3}{2}} < \infty$, $J_N(x) \to 0$ for almost all $x \in [q, q+1]$, and the proof is complete.

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On general multiplication of infinite series

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1

The most general definition of the product of two infinite series can be obtained as follows:

Definition 1. Let us denote by N the set of all pairs (k, l) of positive integers, and let π be a sequence $N_1, N_2, \dots, N_m, \dots$ of finite, mutually disjoint subsets of N such that $N = \bigcup_{m=1}^{\infty} N_m$. Given two infinite series [1])

$$(1.1) \qquad A = \sum_{k=1}^{\infty} a_k \quad \text{and} \quad B = \sum_{l=1}^{\infty} b_l$$

we call the series

$$(1.2) \qquad C = \sum_{m=1}^{\infty} c_m = \sum_{m=1}^{\infty} \Big(\sum_{(k,l) \in N_m} a_k b_l \Big)$$

the product of the series (1.1) *obtained by the method corresponding to the sequence* π, or simply *by the method* (π), and we denote it by $\pi(\Sigma a_k, \Sigma b_l)$ or shortly by $\pi(A, B)$.

Definition 2. The method (π) will be called *perfect* if for any two convergent series (1. 1), the product series $\pi(A, B)$ also converges and its sum is equal to the product of the sums of the factor series.

Definition 3. The method (π) will be said to have property M_1 (resp. M_2) if for any series A and B the convergence of A (resp. B) and the absolute convergence of B (resp. A) implies the convergence of $\pi(A, B)$, its sum being equal to the product of the sums of the factor series.

Definition 4. If a method (π) has both properties M_1 and M_2 we will say that it has the *Mertens property.*

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[1]) In the sequel the series $\Sigma a_k, \Sigma b_l, \Sigma c_m, \dots$ will be denoted by the corresponding capital letters $A, B, C, \dots$ independently of their convergence or divergence.

Definition 5. The method (π) will be said to have the *Abel property*, if for any convergent series A and B the convergence of $\pi(A, B)$ implies that the sum of $\pi(A, B)$ is equal to the product of the sums of A and B.

R. Rado [1] has given necessary and sufficient conditions for a method (π) to be perfect. Rado considers merely methods with sets N_m consisting of one element. For the general case of definition 1 the perfectness and the Mertens property were characterized by A. Alexiewicz [2] in necessary and sufficient form.

Although the theorems of Alexiewicz solve the convergence problem of a method (π) — apart from the Abel property — in the classical sense, there arises the following question:

If we pass over the classical view of convergence, i. e. if we agree that the "convergence" of a series Σc_n means that the series $\Sigma \pm c_n$ is convergent with the probability 1 taking at random the signs of its terms, then how can we modify the theorems of Alexiewicz, and how we stand with the problem of the Abel property?

We can formulate our problem — due to Rademacher [3], Kolmogoroff and Khintchine [4] — analytically in the following manner:

If $\{r_n(t)\}_{n=1}^{\infty}$ denotes the system of the Rademacher functions i.e. if

$$r_n(t) = \text{sign}\,(\sin 2^n \pi t) \qquad (n = 1, 2, 3, \dots) \tag{1.3}$$

in the interval $0 \leqq t \leqq 1$, then for a given method (π) what can we say about the convergence of the series

$$\pi\big(A(x), B(y)\big) = \sum_{m=1}^{\infty} \Big(\sum_{(k,l) \in N_m} a_k b_l r_k(x)\, r_l(y) \Big) \tag{1.4}$$

at the points (x, y) of the unit square $Q = \{(x, y);\, 0 \leqq x \leqq 1, 0 \leqq y \leqq 1\}$, assuming that the factor series

$$A(x) = \sum_{k=1}^{\infty} a_k r_k(x) \quad \text{and} \quad B(y) = \sum_{l=1}^{\infty} b_l r_l(y) \tag{1.5}$$

are convergent almost everywhere in [0, 1], i.e. assuming that the conditions $\Sigma a_k^2 < \infty$ and $\Sigma b_l^2 < \infty$ are fulfilled?

In section 2 we shall prove that every method (π) possesses the Mertens property in the above sense.

In section 3 we shall show that every method (π) becomes perfect if we put the terms of the product series into brackets in suitable form, and at the same time we mention a conjecture in the theory of Walsh series, which is essentially equivalent to the perfectness of every method (π).

Finally in section 4 we prove that every method (π) has the Abel property.

2

Theorem 1. *If the conditions*

$$\sum_{k=1}^{\infty} |a_k| < \infty \quad \text{and} \quad \sum_{l=1}^{\infty} b_l^2 < \infty \tag{2.1}$$

hold, then the product series (1. 4) *of the series* (1. 5) *— generated by an arbitrary given method* (π) *— converges almost everywhere on the unit square* Q.

Proof. First of all we cite a lemma — discovered by ZYGMUND and MARCINKIEWICZ [5] — which will be used in the sequel.

Lemma 1. *If the functions of an orthonormal system* $\{\varphi_n(x)\}_{n=1}^{\infty}$ *in* $L^2(0, 1)$ *are stochastically independent*[1]) *with the integral mean* 0, *then for any finite coefficient system* $\{c_k\}_{k=1}^{n}$ *the following inequality is true*:

$$\int_0^1 \Big[\operatorname*{Max}_{\substack{m \\ (1 \leq m \leq n)}} \Big| \sum_{k=1}^{m} c_k \varphi_k(x) \Big| \Big]^2 dx \leq 8 \int_0^1 \Big[\sum_{k=1}^{n} c_k \varphi_k(x) \Big]^2 dx = 8 \sum_{k=1}^{n} c_k^2. \tag{2.2}$$

In order to prove the convergence of the series (1. 4) it is enough to show that under the conditions (2.1) the series

$$\pi^*(A(x), B(y)) = \sum_{(k,\, l) \in N^*} a_k b_l r_k(x) r_l(y), \tag{2.3}$$

arising from (1. 4) by omitting brackets [2]), converges almost everywhere on Q, too.

Let us consider for each index n the subseries

$$S_n(x, y) = \sum_{m=1}^{\infty} a_n b_{\nu(n,\, m)} r_n(x) r_{\nu(n,\, m)}(y) \tag{2.4}$$

---

[1]) A system $\{f_k(x)\}_{k=1}^{n}$ of measurable functions defined on the interval [0, 1] will be called stochastically independent if for an arbitrary given system of intervals $I_k = (\alpha_k, \beta_k)$ $(k = 1, \ldots, n)$ the equality

$$m\Big(\bigcap_{k=1}^{n} E\{f_k \in I_k\}\Big) = \prod_{k=1}^{n} m(E\{f_k \in I_k\})$$

is valid, where $E\{f_k \in I_k\}$ means the set of all $x \in [0, 1]$ for which the inequalities $\alpha_k < f_k(x) < \beta_k$ hold and $m(H)$ denotes the Lebesgue measure of the set H.

A sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ $(x \in [0, 1])$ is stochastically independent, if any finite subsequence of it is stochastically independent in the above sense.

From these definitions follows that any rearrangement $\{f_{n_k}\}_{k=1}^{\infty}$ of a stochastically independent system $\{f_n(x)\}_{n=1}^{\infty}$ remains stochastically independent.

[2]) N^* means the sequence of all elements of $N = \{(k, l)\}$ generated by the decomposition $\bigcup_{m=1}^{\infty} N_m = N$ of the method (π).

which contains — *in unaltered order* — all the terms of (2. 3) having the factor $a_n r_n(x)$.

Since the sequence $\{b_{\nu(n,m)}\}_{m=1}^{\infty}$ is a permutation of the original sequence $\{b_l\}_{l=1}^{\infty}$ (generated by the method (π)), therefore we get from the second condition of (2.1) that

$$\sum_{m=1}^{\infty} a_n^2 b_{\nu(n,m)}^2 = a_n^2 \sum_{m=1}^{\infty} b_{\nu(n,m)}^2 = a_n^2 \sum_{l=1}^{\infty} b_l^2 < \infty$$

is valid for each index n.

This inequality guarantees for each n the existence of such a sequence

$$1 < m_1^{(n)} < m_2^{(n)} < \ldots < m_j^{(n)} < m_{j+1}^{(n)} < \ldots, \tag{2.5}$$

for which the inequalities

$$\sum_{m=m_j^{(n)}}^{\infty} b_{\nu(n,m)}^2 < \frac{1}{4^n} \cdot \frac{1}{4^j} \qquad (j = 1, 2, \ldots)$$

are true, and therefore, using for each n the notation $m_0^{(n)} = 1$ we get from (2.1) the following estimate:

$$\Omega = \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \sqrt{\sum_{m=m_j^{(n)}}^{m_{j+1}^{(n)}-1} a_n^2 b_{\nu(n,m)}^2} \equiv \Omega_1 + \Omega_2 \equiv$$

$$\equiv \sum_{n=1}^{\infty} \left(\sum_{m=1}^{m_1^{(n)}-1} a_n^2 b_{\nu(n,m)}^2 \right)^{\frac{1}{2}} + \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \left(\sum_{m=m_j^{(n)}}^{m_{j+1}^{(n)}-1} a_n^2 b_{\nu(n,m)}^2 \right)^{\frac{1}{2}} \leqq \tag{2.6}$$

$$\leqq \sum_{n=1}^{\infty} |a_n| \left(\sum_{m=1}^{m_1^{(n)}-1} b_{\nu(n,m)}^2 \right)^{\frac{1}{2}} + \sum_{n=1}^{\infty} |a_n| \frac{1}{2^n} \sum_{j=1}^{\infty} \frac{1}{2^j} \leqq \sqrt{B'} \sum_{n=1}^{\infty} |a_n| + \sqrt{A'} \sqrt{C'} < \infty,$$

where A', B' and C' mean the sums of the convergent series Σa_k^2, Σb_l^2, and $\Sigma 4^{-n}$, respectively [3]).

Secondly we construct from the series (2. 4) by the help of the sequences (2. 5) the following series

$$S_n^*(x, y) = \sum_{j=0}^{\infty} \left(\sum_{m=m_j^{(n)}}^{m_{j+1}^{(n)}-1} a_n b_{\nu(n,m)} r_n(x) r_{\nu(n,m)}(y) \right) \equiv \sum_{j=0}^{\infty} F_j^{(n)}(x, y),$$

and let us denote for each pair (j, n) of indices a general segment of $F_j^{(n)}(x, y)$ by

$$\{F_j^{(n)}(x,y)\}_{\nu}^{\mu} = \sum_{m=\nu}^{\mu} a_n b_{\nu(n,m)} r_n(x) r_{\nu(n,m)}(y) \quad (m_j^{(n)} \leqq \nu \leqq \mu \leqq m_{j+1}^{(n)} - 1).$$

[3]) (2. 6) shows that the condition $\Sigma |a_n| < \infty$ was only used for the estimate $\Omega_1 < \infty$, and that the weaker condition $\Sigma a_n^2 < \infty$ is enough to ensure the validity of $\Omega_2 < \infty$.

Taking into account that for each n and for each $x \in [0, 1]$ the inequalities $|r_n(x)| \leqq 1$ hold, we get for each quartet (j, n, ν, μ) of indices and for each point $(x, y) \in Q$ the following inequality:

$$|\{F_j^{(n)}(x, y\}_\nu^\mu| = |r_n(x)| \left| \sum_{m=\nu}^{\mu} a_n b_{\nu(n,m)} r_{\nu(n,m)}(y) \right| \leqq \tag{2.7}$$
$$\leqq \underset{(m_j^{(n)} \leqq \varrho \leqq m_{j+1}^{(n)}-1)}{2\,\underset{\varrho}{\mathrm{Max}}} \left| \sum_{m=m_j^{(n)}}^{\varrho} a_n b_{\nu(n,m)} r_{\nu(n,m)}(y) \right| = 2\delta_j^{(n)}(y).$$

Since the Rademacher functions $\{r_{\nu(n,m)}(y)\}_{m=1}^{\infty}$ evidently satisfy the conditions of Lemma 1, so the functions $\delta_j^{(n)}(y)$ satisfy, according to (2. 2), the following integral inequalities:

$$\int_0^1 [\delta_j^{(n)}(y)]^2\, dy \leqq 8 \sum_{m=m_j^{(n)}}^{m_{j+1}^{(n)}-1} a_n^2 b_{\nu(n,m)}^2. \tag{2.8}$$

Introducing the non-negative functions

$$\Delta_j^{(n)}(x, y) = 2\delta_j^{(n)}(y)$$

defined for each pair (j, n) of indices in the unit square Q, we can deduce from (2. 7 and (2. 8) the following two properties:

$$|\{F_j^{(n)}(x, y)\}_\nu^\mu| \leqq \Delta_j^{(n)}(x, y) \text{ for each pair } (j, n) \text{ of indices,} \tag{2.9a}$$

$$\iint_Q [\Delta_j^{(n)}(x, y)]^2\, dx\, dy = 4 \int_0^1 [\delta_j^{(n)}(y)]^2\, dy \leqq 32 \sum_{m=m_j^{(n)}}^{m_{j+1}^{(n)}-1} a_n^2 b_{\nu(n,m)}^2. \tag{2.9b}$$

Using the Schwarz inequality and the rearrangement theorem of series with positive terms [6] we get from (2. 6) and (2. 9b) that

$$\sum_{\lambda=1}^{\infty} \left(\sum_{j+n=\lambda} \iint_Q \Delta_j^{(n)}(x, y)\, dx\, dy \right) \leqq \sum_{\lambda=1}^{\infty} \left(\sum_{j+n=\lambda} \left[\iint_Q \{\Delta_j^{(n)}(x, y)\}^2\, dx\, dy \right]^{\frac{1}{2}} \right) =$$
$$= \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \left(\iint_Q [\Delta_j^{(n)}(x, y)]^2\, dx\, dy \right)^{\frac{1}{2}} \leqq \sqrt{32} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \left(\sum_{m=m_j^{(n)}}^{m_{j+1}^{(n)}-1} a_n^2 b_{\nu(n,m)}^2 \right) = \sqrt{32}\, \Omega < \infty,$$

and so, in consequence of the Beppo Levi theorem, the series

$$\sum_{\lambda=1}^{\infty} \sum_{j+n=\lambda} \Delta_j^{(n)}(x, y) \tag{2.10}$$

converges almost everywhere on Q.

Finally writing consecutive indices in the series (2. 3) we get that each segment

$$\sum_{\tau=p}^{q} \varphi_\tau(x, y) \tag{2.11}$$

of the series

$$\sum_{\tau=1}^{\infty} \varphi_\tau(x, y) \equiv \sum_{(k,l)\in N^*} a_k b_l r_k(x) r_l(y) \equiv \pi^*(A(x), B(y)) \tag{2.12}$$

is a sum of finitely many $\{F_j^{(n)}(x, y)\}_\nu^\mu$, because we preserved the order of the terms of (2. 12) when forming the series (2. 4). Choosing therefore the lower index p in (2. 11) so large that among the terms of the sum

$$\sum_{\tau=1}^{p-1} \varphi_\tau(x, y)$$

every term $a_k b_l r_k(x) r_l(y)$ of the finite sum

$$\sum_{\lambda=1}^{M} \sum_{j+n=\lambda} F_j^{(n)}(x, y)$$

occurs, then we get by (2. 9a) from the convergence of the series (2. 10):

$$\left|\sum_{\tau=p}^{q} \varphi_\tau(x, y)\right| \leqq \sum_{\lambda=M+1}^{\infty} \left(\sum_{j+n=\lambda} \Delta_j^{(n)}(x, y)\right) \to 0$$

when p and $q \to \infty$, which proves the convergence of the series (2. 12), q.e.d.

3.

Let be given two sequences $\{a_k\}_{k=1}^\infty$ and $\{b_l\}_{l=1}^\infty$ satisfying the conditions

$$\sum_{k=1}^{\infty} a_k^2 < \infty \quad \text{and} \quad \sum_{l=1}^{\infty} b_l^2 < \infty, \tag{3.1}$$

respectively, and let us consider a given method (π) defined in Definition 1.

Since the product series

$$\pi(\sum a_k^2, \sum b_l^2) = \sum_{m=1}^{\infty} \left(\sum_{(k,l)\in N_m} a_k^2 b_l^2\right) = (\sum a_k^2)(\sum b_l^2)$$

converges [6], there exists an increasing sequence $\{m_\nu\}_{\nu=1}^\infty$ of indices, such that

$$\sum_{\nu=0}^{\infty} \left(\sum_{m=m_\nu}^{m_{\nu+1}-1} \sum_{(k,l)\in N_m} a_k^2 b_l^2\right)^{\frac{1}{2}} = \Lambda < \infty \qquad (m_0 = 1) \tag{3.2}$$

holds.

In the light of this fact let us put into brackets the terms of the product series

$$\pi(A(x), B(y)) = \sum_{m=1}^{\infty}{}' \left(\sum_{(k,l)\in N_m} a_k b_l r_k(x) r_l(y) \right)$$

of the almost everywhere convergent Rademacher series $\Sigma\, a_k r_k(x)$ and $\Sigma\, b_l r_l(y)$ by the help of the sequence $\{m_\nu\}_{\nu=0}^{\infty}$, i.e. we consider the series

$$\pi^*(A(x), B(y)) = \sum_{\nu=0}^{\infty} \left[\sum_{m=m_\nu}^{m_{\nu+1}-1} \sum_{(k,l)\in N_m} a_k b_l r_k(x) r_l(y) \right] \equiv \sum_{\nu=0}^{\infty} \Phi_\nu(x, y).$$

We assert that last series converges absolutely almost everywhere on the unit square Q. For our purpose it is enough to observe that the functions

(3. 3) $$R_{k,l}(x, y) = r_k(x) r_l(y) \qquad (k=1, 2, \ldots;\ l=1, 2, \ldots)$$

are orthonormal on Q and so by Schwarz inequality we get from (3. 2) that

$$\sum_{\nu=0}^{\infty} \iint_Q |\Phi_\nu(x, y)|\, dx\, dy \leqq \sum_{\nu=0}^{\infty} \left(\iint_Q \Phi_\nu^2(x, y)\, dx\, dy \right)^{\frac{1}{2}} = \Lambda,$$

which proves our assertion and the following

Theorem 2. *If the conditions* (3. 1) *are fulfilled, then for every method* (π) *we can choose an increasing sequence of indices such that the associated product series*

$$\pi^*(\Sigma \pm a_k, \Sigma \pm b_l) = \sum_{\nu=0}^{\infty} \left[\sum_{m=m_\nu}^{m_{\nu+1}-1} \sum_{(k,l)\in N_m} (\pm a_k)(\pm b_l) \right]$$

converges absolutely for almost all signings of the factor series.

Note. If the functions (3. 3) had been stochastically independent on Q, then applying the two-dimensional form of Lemma 1 we should have proved from (3. 2) the perfectness of every method (π) in the strict sense.

In the sequel we indicate a rather interesting problem in the theory of Walsh series, which is essentially equivalent to the problem of the perfectness of general methods (π).

To this end we introduce a convenient form of a famous transformation due to F. Riesz [7], [8].

Before all we co-ordinate the unit interval $I = I_0 = \{t;\ 0 \leqq t \leqq 1\}$ with the unit square $Q = Q_0 = \{(x, y);\ 0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1\}$, in sign:

0°) $$I_0 \leftrightarrow Q_0.$$

In the first step we decompose the interval I_0 by the points $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ to four closed intervals $I_{1,1}, I_{1,2}, I_{1,3}, I_{1,4}$, and similarly we divide the square Q_0 by the help of the straight lines $x = \frac{1}{2}$ and $y = \frac{1}{2}$ into four congruent closed subsquares $Q_{1,1}, Q_{1,2}, Q_{1,3}, Q_{1,4}$.

In the case of the intervals $\{I_{1,k}\}_{k=1}^{4}$ the increasing order of the second index k corresponds to the increasing direction of the variable $t\in I_0$ and in the case of the subsquares $\{Q_{1,k}\}_{k=1}^{4}$ the increasing order of the second index is indicated by the scheme in figure 1, i.e. the subsquares $\{Q_{1,k}\}_{k=1}^{4}$ are represented by figure 2, and we order the elements of the systems $\{I_{1,k}\}_{k=1}^{4}$ and $\{Q_{1,k}\}_{k=1}^{4}$ mutually to each other, in sign

1°) $$I_{1,k}\leftrightarrow Q_{1,k} \qquad (k=1,2,3,4).$$

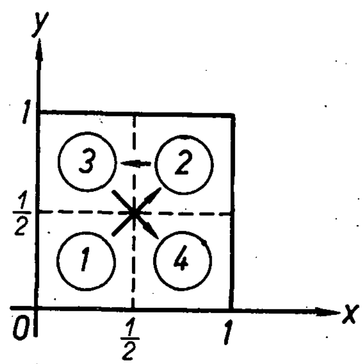

Figure 1

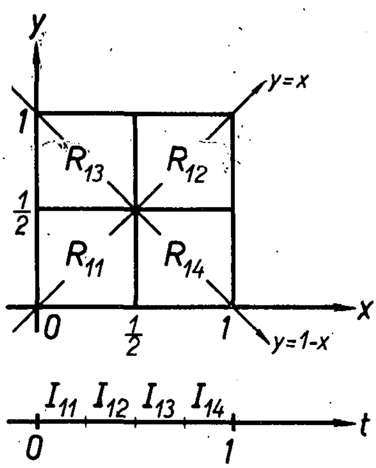

Figure 2

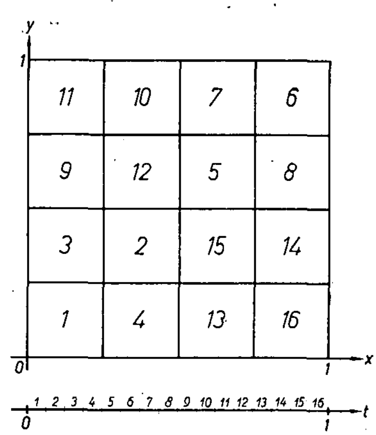

Figure 3

Next we decompose each interval $I_{1,k}$, resp. each quarter $Q_{1,k}$, into four congruent and closed subintervals, resp. subsquares, and we denote the so created systems by

(3.4) $$I_{2,1}, I_{2,2}, \ldots, I_{2,16}; \quad Q_{2,1}, Q_{2,2}, \ldots, Q_{2,16}.$$

In (3. 4) the increasing order of the second indices of the intervals $\{I_{2,k}\}_{k=1}^{16}$ corresponds to the increasing direction of the variable t, and the second indices of the squares $\{Q_{2,k}\}_{k=1}^{16}$ are given so, that the one-to-one mapping

2°) $$I_{2,k} \leftrightarrow Q_{2,k} \qquad (k = 1, 2, \ldots, 16)$$

has the following two properties:

α) if $I_{2,l} \subset I_{1,k}$ then $Q_{2,l} \subset Q_{1,k}$,

β) if the squares $Q_{2,n}, Q_{2,m}, Q_{2,p}, Q_{2,r}$ are subsquares of a square $Q_{1,k}$, then the increasing order of the second indices is as indicated by the direction scheme in Figure 1.

The mapping 2°) is illustrated by figure 3, where only the second indices are written out in the corresponding subintervals and subsquares.

Iterating the steps 0°), 1°), 2°) periodically we get a sequence $\mathscr{I}$ of intervals

(3.5) $$\mathscr{I} = \{I_0; I_{1,1}, I_{1,2}, I_{1,3}, I_{1,4}; \ldots, I_{n,1}, I_{n,2}, I_{n,4^n}; \ldots\}$$

and a sequence $\mathscr{Q}$ of squares

(3.6) $$\mathscr{Q} = \{Q_0; Q_{1,1}, Q_{1,2}, Q_{1,3}, Q_{1,4}; \ldots, Q_{n,1}, Q_{n,2}, \ldots, Q_{n,4^n}; \ldots\}$$

for which the one-to-one mapping of the elements

(3.7) $$I_{n,k} \leftrightarrow Q_{n,k} \qquad (n = 0, 1, 2, \ldots, \quad k = 1, 2, \ldots, 4^n)$$

has the following three properties:

(I) for each index $n(=1, 2, \ldots)$, $\bigcup_{k=1}^{4^n} I_{n,k} = I_0$ and $\bigcup_{k=1}^{4^n} Q_{n,k} = Q_0$;

(II) for each pair (n, k) of indices there exists such an index l, for which (3.8) $I_{n,k} \subset I_{n-1,l}$ and in this case $Q_{n,k} \subset Q_{n-1,l}$ is valid, and conversely if $Q_{n,k} \subset Q_{n-1,l}$ then $I_{nk} \subset I_{n-1,l}$;

(III) $m(I_{n,k}) = m(Q_{n,k})$, i.e. (3. 7) preserves the measure.

The mapping (3. 7) of the sequences (3. 5) and (3. 6) generates a correspondence between the points of the unit interval I and the unit square Q in the following way:

Definition 6. To each value $t \in I$ let correspond the point (or points) $T(t) = (x, y) \in Q$ for which

(3.9) $$T(t) = (x, y) = \bigcap_{n=1}^{\infty} Q_{n,\nu(n)}, \quad \text{if} \quad t = \bigcap_{n=1}^{\infty} I_{n,\nu(n)},$$ [4)]

[4)] If t is not dyadic rational, then the subsequence $\{I_{n,\nu(n)}\}_{n=1}^{\infty}$ of (3. 5) is uniquely determined by t, and so $T(t) = (x, y) \in Q$ is also uniquely determined.

and conversely to each point $(x,y)\in Q$ let correspond the point (or points) $V(x,y)=t\in I$ for which

$$V(x,y)=t=\bigcap_{n=1}^{\infty} I_{n,\mu(n)}, \quad \text{if} \quad (x,y)=\bigcap_{n=1}^{\infty} Q_{n,\mu(n)}. \tag{3.10}$$

Considering the properties (I)—(III) of the mapping (3. 7) it is easy to see the validity of

Theorem 3. *The transformations* (3. 9) *and* (3. 10) *are inverse of one another apart from sets of measure zero, and both of them are measure preserving transformations.*

Definition 7. The functions $f(t)\,(t\in I)$ and $g(x,y)$ $((x,y)\in Q)$ will be called equivalent, in sign

$$f(t)\simeq g(x,y), \tag{3.11}$$

if for almost all pairs of corresponding points $(t=V(x,y),\ (x,y)=T(t))$ the equality $f(t)=g(x,y)$ holds.

Theorem 3 has the following two corollaries:

Theorem 4. *The function $f(t)$ is measurable resp. integrable on I if and only if the equivalent function $g(x,y)$ is measurable resp. integrable on Q, and in the latter case*

$$\int_0^1 f(t)\,dt=\iint_Q g(x,y)\,dx\,dy.$$

Theorem 5. *If the elements of the sequences $\{g_k(x,y)\}_{k=1}^{\infty}$ $((x,y)\in Q)$ and $\{f_k(t)\}_{k=1}^{\infty}$ $(t\in I)$ are term by term equivalent in the sense of definition 7, then the series $\sum_{k=1}^{\infty} g_k(x,y)$ converges almost everywhere on Q if and only if the series $\sum_{k=1}^{\infty} f_k(t)$ is convergent almost everywhere on I.*

Finally considering our direction scheme in figure 1, it is easy to see by induction the following.

Theorem 6. *If $\{r_n(t)\}_{n=1}^{\infty}$ denotes the system of the Rademacher functions, then for the functions of two variables*

$$\varrho_k(x,y)\equiv r_k(x);\quad \sigma_l(x,y)\equiv r_l(y)\qquad (k=1,2,\dots\ l=1,2,\dots)$$

defined on Q, the following relations are true:

$$\varrho_k(x,y)\simeq r_{2k}(t)\quad \text{and}\quad \sigma_l(x,y)\simeq r_{2l}(t)r_{2l-1}(t)\qquad (k=1,2,\dots\ l=1,2,\dots)$$

in the sense of Definition 7.

By means of Theorems 5 and 6 we can join the theory of multiplication of infinite series with the theory of Walsh series [9].

In defining the functions of the Walsh system it is convenient to follow PALEY's modification [10]:

Definition 8. If $\{r_n(t)\}_{n=1}^{\infty}$ denotes the system of Rademacher functions defined in (1. 3) then the Walsh functions $\{w_n(t)\}_{n=0}^{\infty}$ are given in the following form:

$$w_0(t) \equiv 1, \tag{3.12}$$
$$w_n(t) = r_{\nu_1+1}(t)\, r_{\nu_2+1}(t) \dots r_{\nu_k+1}(t)$$

for $n = 2^{\nu_1} + 2^{\nu_2} + \dots + 2^{\nu_k}$, where the integers ν_i are uniquely determined by $\nu_{i+1} < \nu_i$.

Definition 9. Let $\{w_{n_i}(t)\}_{i=1}^{\infty}$ be such a subsequence of the sequence $\{w_n(t)\}_{n=0}^{\infty}$ of the Walsh functions, whose elements have at most three factors in the Paley representation (3. 12), i.e. whose elements can be written in the form

$$w_{n_i}(t) = r_{\nu_1+1}(t)\, r_{\nu_2+1}(t)\, r_{\nu_3+1}(t).$$

Conjecture. If $\sum c_i^2 < \infty$ then the lacunary Walsh series

$$\sum_{i=1}^{\infty} c_i w_{n_i}(t) \tag{3.13}$$

converges unconditionally [5]) almost ewerywhere.

Theorem 7. *Let the sequences* $\{a_k\}_{k=1}^{\infty}$ *and* $\{b_l\}_{l=1}^{\infty}$ *satisfy the conditions* (3. 1) *and let us consider an arbitrary method* (π) *defined in Definition 1. If the above conjecture is true then the product series*

$$\pi\left(\sum a_k r_k(x), \sum b_l r_l(y)\right) = \sum_{m=1}^{\infty} \left(\sum_{(k,l)\in N_m} a_k b_l r_k(x) r_l(y) \right) \tag{3.14}$$

converges almost everywhere on the unit square Q.

Proof. From the conditions (3. 1) it follows that the sum

$$\sum_{m=1}^{\infty} \left(\sum_{(k,l)\in N_m} a_k^2 b_l^2 \right)$$

converges, and according to the theorems 5 and 6 the series (3. 14) is equiconvergent with the Walsh series

$$\sum_{m=1}^{\infty} \left(\sum_{(k,l)\in N_m} a_k b_l r_{2k}(t) r_{2l}(t) r_{2l-1}(t) \right),$$

which is a rearranged and associated series of type (3. 13).

---

[5]) i. e. the series (3.13) converges by any rearrangement of its terms apart from a set of measure zero (which set may depend of course on the rearrangement in question).

Note. The conjecture is true in that special case, when in (3. 13) the indices n_i are in an increasing order; more generally the following theorem is true [11]:

Theorem 8. *If* $\{w_{n_i}(x)\}_{i=1}^{\infty}$ *denotes such a subsequence of the sequence* $\{w_n(x)\}_{n=0}^{\infty}$ *whose elements have at most N factors in Paley's representation* (3. 12), *where N is an arbitrary but fixed natural number, then the convergence of the series* Σc_i^2 *involves the convergence almost everywhere of the lacunary Walsh series*

$$\sum c_i w_{n_i}(x). \tag{3.15}$$

Proof. In order to prove the theorem it is enough to show that the following assertion is true:

If for a given Walsh series $\sum_{n=0}^{\infty} c_n w_n(x)$ the coefficients satisfy the condition $\sum c_n^2 < \infty$ and if the elements of the monotonically increasing sequence n_k have a dyadic expression of the form

$$n_k = 2^{\nu_1} + 2^{\nu_2} + \ldots + 2^{\nu_l}, \tag{3.16}$$

l being limited by an arbitrary chosen but fixed natural number N, then the partial sums

$$s_{n_k}(x) = \sum_{\lambda=0}^{n_k} c_\lambda w_\lambda(x)$$

converge almost everywhere.

We can get the proof of the last assertion from a lemma of L. Leindler [12] applying it to the Walsh system whose Lebesgue constants are known [13].

Lemma 2. *If* $\{n_k\}$ *is a positive non-decreasing sequence of indices and* $\{\varphi_n(x)\}_{n=0}^{\infty}$ *is such an orthonormal system on the interval* $[a, b]$ *whose Lebesgue functions*

$$L_{n_k}(x) = \int_a^b \left| \sum_{i=0}^{n_k} \varphi_i(x)\varphi_i(u) \right| du$$

are uniformly bounded on a set $E \subset [a, b]$, *then the condition* $\sum c_i^2 < \infty$ *implies that the subsequence*

$$s_{n_k}(x) = \sum_{i=0}^{n_k} c_i \varphi_i(x)$$

of the partial sums of the orthogonal series $\sum c_i \varphi_i(x)$ *is almost everywhere convergent on the set E.*

If the orthonormal system $\{\varphi_n(x)\}$ is the Walsh system $\{w_n(x)\}_{n=0}^{\infty}$, then we can apply the above lemma in a very convenient form to the case of the index sequence (3. 16). In fact, N. J. FINE [13] showed that the Lebesgue functions of the Walsh system do not depend on x and have the following explicit form

$$L_n(x) \equiv L_n = l - \sum_{1 \leqq p < r \leqq l} 2^{(\nu_r - \nu_p)}$$

if $n = 2^{\nu_1} + 2^{\nu_2} + \ldots + 2^{\nu_l}$. Consequently, for the indices (3. 16) we have $L_{n_k} \leqq l \leqq N$ and therefore our theorem is proved.

4

Theorem 9. *If the conditions*

$$\sum a_k^2 < \infty \quad \text{and} \quad \sum b_l^2 < \infty$$

hold, and if the general product series

$$\pi(A(x), B(y)) = \sum_{m=1}^{\infty} \left(\sum_{(k,l) \in N_m} a_k b_l r_k(x) r_l(y) \right) \tag{4.1}$$

of the almost everywhere convergent series

$$A(x) = \sum_{k=1}^{\infty} a_k r_k(x) \quad \text{and} \quad B(y) = \sum_{l=1}^{\infty} b_l r_l(y) \tag{4.2}$$

converges almost everywhere on the unit square Q to the sum $S(x, y)$, then $S(x, y) = A(x)B(y)$ almost everywhere on Q.

Proof. Since the rectangular product

$$\sum_{m=1}^{\infty} \left(\sum_{l=1}^{m} a_m b_l r_m(x) r_l(y) + \sum_{k=1}^{m-1} a_k b_m r_k(x) r_m(y) \right)$$

of the series (4. 2) converges almost everywhere on Q to the sum $A(x)B(y)$, and since the product series (4. 1) is, according to (3. 3), such an orthogonal series on Q for which the square sum of its coefficients is finite, so the theorem is an immediate consequence of the Riesz—Fischer theorem.

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On a problem of summability of orthogonal series

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Introduction

Let $\{\varphi_k(x)\}$ $(k=0, 1, \dots)$ be an orthonormal system on the finite interval (a, b). We shall denote by $s_n(x)$ the n-th partial sum of the orthogonal series

$$\sum_{k=0}^{\infty} a_k \varphi_k(x). \tag{1}$$

Let $T=(\alpha_{ik})$ $(i, k=0, 1, \dots)$ be a double infinite matrix of numbers. The sum

$$t_i(x) = \sum_{k=0}^{\infty} \alpha_{ik} s_k(x) \qquad (i = 0, 1, \dots)$$

is called the i-th *T-mean* of the series (1), provided that the series on the right-hand side converges. We say that the series (1) is *T-summable* to the sum s at the point $x_0(\in(a, b))$ if $t_i(x_0)$ exists for all i (perhaps except finitely many of them), and $\lim_{i\to\infty} t_i(x_0)=s$. A T-summation process is said to be *permanent* if $\lim_{n\to\infty} s_n=s$ implies $\lim_{i\to\infty} t_i=s$. The necessary and sufficient conditions for the permanence of a summation process are known. (See ALEXITS [1], p. 65.)

For any given orthonormal system $\{\varphi_k(x)\}$ and for any summation matrix T we shall consider the following functions

$$L_i(T; \{\varphi_k\}; x) = \int_a^b \left| \sum_{k=0}^{\infty} \alpha_{ik} \left(\sum_{l=0}^{k} \varphi_l(x)\, \varphi_l(t) \right) \right| dt = \int_a^b \left| \sum_{l=0}^{\infty} \left(\sum_{k=l}^{\infty} \alpha_{ik}\, \varphi_k(x)\, \varphi_k(t) \right) \right| dt,$$

provided they exist. The function $L_i(T; \{\varphi_k\}; x)$ is called the i-th *Lebesgue function* of the orthonormal system $\{\varphi_k(x)\}$ concerning the T-summation process. The order of magnitude of the Lebesgue functions may, in many cases, be decisive for the convergence problems.

In particular, taking

$$\alpha_{ik} = \frac{1}{i+1} \quad (k = 0, 1, \dots, i), \qquad \alpha_{ik} = 0 \quad (k = i+1, i+2, \dots) \quad (i = 0, 1, \dots),$$

we obtain the classical $(C, 1)$-summation process. Now, we have

$$L_i\big((C,1);\{\varphi_k\};x\big)=\int_a^b\left|\sum_{k=0}^{i}\left(1-\frac{k}{i+1}\right)\varphi_k(x)\,\varphi_k(t)\right|dt.$$

In this case KACZMARZ [3] has proved the following theorem:

Let $\{\varphi_k(x)\}$ be an arbitrary orthonormal system in (a, b). If $\{\mu_k\}$ is a positive, non-decreasing number sequence for which the relation

$$\sup_{\nu(x)}\int_a^b\frac{L_{\nu(x)}\big((C,1);\{\varphi_k\};x\big)}{\mu_{\nu(x)}}\,dx<\infty \tag{2}$$

holds, where the sup is taken over all the measurable functions $\nu(x)$ assuming only integer values, then

$$\sum_{k=0}^{\infty}a_k^2\,\mu_k<\infty$$

implies the $(C, 1)$-summability of the orthogonal series (1) *almost everywhere.*

It is obvious that the condition (2) is equivalent to the following one:

$$\sup_i\frac{L_i\big((C,1);\{\varphi_k\};x\big)}{\mu_i}\in L(a,b).$$

KACZMARZ formulated this theorem under the condition requiring somewhat more than (2), namely

$$L_i\big((C,1);\{\varphi_k\};x\big)=O(\mu_i)\qquad(a\leqq x\leqq b),$$

however, the above sharper assertion can also be obtained from his proof.

KACZMARZ [3] has generalized the above theorem also for the $(C,\beta>0)$-summation. (In this case, we have

$$\alpha_{ik}=\frac{A_{i-k}^{(\beta-1)}}{A_i^{(\beta)}}\quad(k=0,1,\ldots,i),\quad\alpha_{ik}=0\,(k=i+1,i+2,\ldots)\quad(i=0,1,\ldots),$$

where $A_i^{(\beta)}=\binom{i+\beta}{i}$.) (See also TANDORI [8].)

SUNOUCHI [7] and LEINDLER [4] have transferred these results to the Riesz summation of orthogonal series. (In this case

$$\alpha_{ik}=\frac{\lambda_{k+1}-\lambda_k}{\lambda_{i+1}}\quad(k=0,1,\ldots,i),\quad\alpha_{ik}=0\quad(k=i+1,i+2,\ldots)\,(i=0,1,\ldots),$$

where $\{\lambda_i\}$ is a positive, strictly increasing sequence of numbers with $\lambda_0=0$ and $\lambda_n \to \infty$.)

To our knowledge, no analogous theorem for other summation processes is yet proved. The following problem can be quite naturally raised: if for any T-summation process the condition

$$\sup_{\nu(x)} \int_a^b \frac{L_{\nu(x)}(T;\{\varphi_k\};x)}{\mu_{\nu(x)}}\,dx < \infty$$

or the stronger one

(3) $$L_i(T;\{\varphi_k\};x) = O(\mu_i) \qquad (a \leqq x \leqq b)$$

is fulfilled, is then the orthogonal series (1) under the condition $\sum_{k=0}^{\infty} a_k^2 \mu_k < \infty$ with the concerning process summable almost everywhere?

EFIMOV [2] has essentially showed that, under the condition (3), $\sum_{k=0}^{\infty} a_k^2 \mu_k < \infty$ with $\mu_k \to \infty$ does not imply the almost everywhere T-summability of the orthogonal series $\sum_{k=0}^{\infty} a_k \varphi_k(x)$ for every permanent T-summation process. In his proof, however, the condition $\mu_k \to \infty$ is very important one.

In this paper we give a construction in which the condition $\mu_k \to \infty$ is not essential. We are going to deal only with the important special case $\mu_k = 1$ ($k=0, 1, \ldots$). Our theorem reads as follows:

Theorem. *There exist an orthonormal system* $\{\varphi_k(x)\}$ *in* (0, 1), *a coefficient sequence* $\{c_k\}$, *and a permanent T-summation process such that* $\sum_{k=0}^{\infty} c_k^2 < \infty$ *and the relation*

(4) $$\sup_{\nu(x)} \int_a^b L_{\nu(x)}(T;\{\varphi_k\};x)\,dx < \infty$$

holds, where the sup is taken over all the measurable functions $\nu(x)$ *assuming only integer values, but the orthogonal series*

(5) $$\sum_{k=0}^{\infty} c_k \varphi_k(x)$$

is not T-summable almost everywhere in (0, 1).

The proof of our theorem will be accomplished by a direct construction. The T-summation process occurring in the theorem can be chosen as it was found by MENCHOFF [6] and applied to clarify another question.

It is an open question to prove this theorem under the following stronger condition instead of (4):

$$L_i(T; \{\varphi_k\}; x) = O(1) \quad (0 \leqq x \leqq 1).$$

This problem seems to be difficult.

§ 1. Lemmas

We require two lemmas to prove our theorem. In the following $C_1, C_2, \ldots$ will denote positive absolute constants.

Lemma 1. *Let n be a natural number. Then there exist an orthonormal system $\{\omega_l(x)\}$ $(l=0, 1, \ldots, 2^{2^n}-1)$ of step-functions in $(0, 1)$, a coefficient sequence $\{b_l\}$ $(l=0, 1, \ldots, 2^{2^n}-1)$, and a simple set $E\,(\subseteqq(0, 1))$* [1] *with the following properties: the integral of each function $\omega_l(x)$ extended over $(0, 1)$ vanishes,*

$$\sum_{l=0}^{2^{2^n}-1} b_l^2 \leqq 1, \tag{6}$$

$$L_{2^{2^n}-1}(\{\omega_l\}; x) = \int_0^1 \left| \sum_{l=0}^{2^{2^n}-1} \omega_l(x)\,\omega_l(t) \right| dt \leqq 1 \quad (0 \leqq x \leqq 1), \tag{7}$$

$$|E| = \frac{1}{8},\text{[2]} \tag{8}$$

and

$$\max_{0 \leqq s \leqq 2^{2^n}-1} \left| \sum_{l=0}^{s} b_l \omega_l(x) \right| \geqq C_1 \sqrt{n} \quad \textit{if} \quad x \in E. \tag{9}$$

Proof. This lemma have been essentially proved in an earlier paper of Tandori [9]. For the sake of completeness, we give its proof in detail here also.

Let $r_n(x) = \operatorname{sign} \sin 2^n \pi x$ be the n-th Rademacher function $(n=0, 1, \ldots)$. Let be $w_0(x) \equiv 1$ in $(0, 1)$; if $k \geqq 1$ and $2^{\nu_1} + 2^{\nu_2} + \ldots + 2^{\nu_p}$ $(\nu_1 < \nu_2 < \ldots < \nu_p)$ is the dyadic representation of k, then let us put $w_k(x) = r_{\nu_1+1}(x) r_{\nu_2+1}(x) \ldots r_{\nu_p+1}(x)$. The Walsh functions $w_k(x)$ $(k=0, 1, \ldots)$ defined in this manner are step-functions, orthogonal and obviously normed. It is known (see e.g. Alexits [1], p. 188) that for all natural numbers N

$$L_{2^N-1}(\{w_k\}; x) = \int_0^1 \left| \sum_{k=0}^{2^N-1} w_k(x)\, w_k(t) \right| dt \leqq 1. \tag{10}$$

[1] A set E will be said simple if it is the union of finitely many, non-overlapping intervals.
[2] $|E|$ denotes the Lebesgue measure of the set E.

Let a be a natural number and let us consider the functions

$$\varphi_a\left(\frac{l}{2^a};x\right) = \frac{1}{2^{a-1}}\prod_{k=2}^{a}\left(1+r_k\left(\frac{l}{2^a}+\frac{l}{2^{a+1}}\right)r_k(x)\right) \qquad (l=0,1,\dots,2^a-1).$$

It is obvious that the functions $\varphi_a(l/2^a;x)$ are linear combinations of the Walsh functions $w_0(x), w_2(x), \dots, w_{2^a-2}(x)$ and that the following equalities are true:

$$\varphi_a\left(\frac{l}{2^a};x\right) = \begin{cases} 1 & \text{if } \dfrac{l}{2^a}<x<\dfrac{l+1}{2^a}, \quad \text{or} \\ & \dfrac{1}{2}+\dfrac{l}{2^a}<x<\dfrac{1}{2}+\dfrac{l+1}{2^a}, \quad \text{or} \quad \dfrac{l}{2^a}-\dfrac{1}{2}<x<\dfrac{l+1}{2^a}-\dfrac{1}{2}, \\ 0 & \text{elsewhere;} \end{cases}$$

and

$$\int_0^1 \varphi_a^2\left(\frac{l}{2^a};x\right)dx = \frac{1}{2^{a-1}}.$$

Now let us consider the following functions:

$$\Phi_1(0;x) = \varphi_2(0;x);$$

$$\Phi_1(1;x) = r_3(x)\varphi_2(0;x), \quad \Phi_2(1;x) = -r_3(x)r_1(x)\varphi_2(0;x);$$

$$\Phi_1(2;x) = r_4(x)\varphi_3(0;x), \quad \Phi_2(2;x) = -r_1(x)\Phi_1(2;x),$$

$$\Phi_3(2;x) = r_5(x)\varphi_3\left(\frac{1}{2^3};x\right), \quad \Phi_4(2;x) = -r_1(x)\Phi_3(2;x);$$

generally,

$$\Phi_{2j+1}(k;x) = r_{2+2^{k-1}+j}(x)\sum_l \varphi_{2+2^{k-2}+[j/2]}(x_l;x) \quad (j=0,1,\dots,2^{k-1}-1), \text{[3]}$$

where the points x_l denote the left-hand side endpoints of the subintervals of $(0,\frac{1}{2})$, in which the function $\Phi_{j+1}(k-1;x)$ is positive, and finally

$$\Phi_{2j}(k;x) = -r_1(x)\Phi_{2j-1}(k;x) \qquad (j=1,2,\dots,2^{k-1}).$$

It is clear that for an arbitrary natural number $n(\geqq 2)$ the functions $\Phi_r(k;x)$ $(k=0,1,\dots,n-1;\ r=1,2,\dots,2^k)$ possess the following properties: these functions are also linear combinations of the Walsh functions, namely

(11) $$\Phi_r(k;x) = \sum_i b_i(r,k)\,w_{n(i,r,k)}(x) \qquad (n(1,r,k)<n(2,r,k)<\dots);$$

[3]) $[\alpha]$ denotes the integer part of α.

the different functions $\Phi_r(k; x)$ have no common Walsh function in their representation (11); in this representation of the function $\Phi_r(k; x)$ $(k=0, 1, \dots, n-1;$ $r=1, 2, \dots, 2^k)$ only some of the Walsh functions $w_0(x), w_1(x), \dots, w_{2^{2^{n-1}}+2^{2^{n-2}+1}-1}(x)$ occur; furthermore, the inequality

$$(12)\qquad \sum_{r=1}^{2^k} \int_0^1 \Phi_r^2(k; x)\,dx \leqq 1 \qquad (k=0, 1, \dots, n-1)$$

is satisfied.

Now, let us consider the following sum:

$$S_n(x) = \Phi_1(0; x) + \sum_{k=1}^{n-1} \sum_{j=0}^{2^{k-1}-1} \left(\Phi_{2j+1}(k; x) + 2\Phi_{2(j+1)}(k; x)\right) =$$

$$= \sum_{l=0}^{2^{2^{n-1}}+2^{2^{n-2}+1}-1} b_l(n)\, w_l(x).$$

On account of (12) we get

$$(13)\qquad \int_0^1 S_n^2(x)\,dx = \sum_{l=0}^{2^{2^{n-1}}+2^{2^{n-2}+1}-1} b_l^2(n) \leqq 5n.$$

Finally, set us arrange the terms $\Phi_j(k; x)$ of the sum $S_n(x)$ by recurrence with respect to k: let

$$s_1(S_n; x) = \Phi_1(0; x) + \Phi_1(1; x) + 2\Phi_2(1; x),$$

$$s_2(S_n; x) = \Phi_1(0; x) + \Phi_1(1; x) + \Phi_1(2; x) + 2\Phi_2(2; x) +$$

$$+ 2\Phi_2(1; x) + \Phi_3(2; x) + 2\Phi_4(2; x),$$

and so on. In general, from $s_\mu(S_n; x)$ we obtain $s_{\mu+1}(S_n; x)$ in such a manner that for every term $\Phi_{2j+1}(\mu; x)$ and $\Phi_{2(j+1)}(\mu; x)$ $(j=0, 1, \dots, 2^{\mu-1}-1)$ we look for the place where they occur in $s_\mu(S_n; x)$, and then immediately after them we insert the sums $\Phi_{2^2j+1}(\mu+1; x) + 2\Phi_{2^2j+2}(\mu+1; x)$ and $\Phi_{2^2j+3}(\mu+1; x) + 2\Phi_{2^2j+4}(\mu+1; x)$, respectively. Now, let us choose the set $\bar{E}$ that is the set of the points of the interval $(0, \frac{1}{4})$ at which $w_l(x) \neq 0$ $(l=0, 1, \dots, 2^{2^{n-1}}+2^{2^{n-2}+1}-1)$ (i.e. apart from a finite number of the dyadically rational points). It is clear that this $\bar{E}$ is a simple set and $|\bar{E}| = \frac{1}{4}$. From the definition of $\Phi_r(k; x)$ we get that the maximum of the partial sums of the prescribed rearrangement of the sum $S_n(x)$ will equal n in the points of $\bar{E}$.

If we substitute the representations (11) in the above rearrangement of $S_n(x)$

and label the occurring Walsh functions, in this order, by the subscript n_i ($i=0, 1, \ldots$ $\ldots, 2^{2^{n-1}}+2^{2^{n-2}+1}-1$) then we have

$$S_n(x) = \sum_{i=0}^{2^{2^{n-1}}+2^{2^{n-2}+1}-1} b_{n_i}(n)\, w_{n_i}(x).$$

Then the above assertion may be written as follows:

$$\max_{1\leqq s\leqq 2^{2^{n-1}}+2^{2^{n-2}+1}-1} \left| \sum_{i=0}^{s} b_{n_i}(n)\, w_{n_i}(x) \right| = n \qquad (x\in \bar{E}). \tag{14}$$

Now we put

$$\bar{\omega}_i(x) = w_{n_i}(x) \qquad (i = 0, 1, \ldots, 2^{2^{n-1}}+2^{2^{n-2}+1}-1),$$

$$\bar{\omega}_i(x) = w_i(x) \qquad (i = 2^{2^{n-1}}+2^{2^{n-2}+1}, \ldots, 2^{2^n}-1);$$

$$b_i = \frac{b_{n_i}(n)}{\sqrt{5n}} \qquad (i = 0, 1, \ldots, 2^{2^{n-1}}+2^{2^{n-2}+1}-1),$$

$$b_i = 0 \qquad (i = 2^{2^{n-1}}+2^{2^{n-2}+1}, \ldots, 2^{2^n}-1).$$

This is possible as $2^{2^{n-1}}+2^{2^{n-2}+1}-1 \leqq 2^{2^n}-1$. Finally, we set

$$\omega_i(x) = \begin{cases} \bar{\omega}_i(2x) & \text{if } x\in \left(0, \frac{1}{2}\right), \\ -\bar{\omega}_i(2x-1) & \text{if } x\in \left(\frac{1}{2}, 1\right), \\ 0 & \text{elsewhere,} \end{cases}$$

($i=0, 1, \ldots, 2^{2^n}-1$). Furthermore, let E be the set arising from $\bar{E}$ as the result of the linear transformation of the interval (0, 1) into the subinterval (0, $\frac{1}{2}$).

It is obvious that E is a simple set and the assertion (8) is satisfied. We can easily see that the function system $\{\bar{\omega}_i(x)\}$ ($i=0, 1, \ldots, 2^{2^n}-1$) is a rearrangement of the Walsh functions $\{w_i(x)\}$ ($i=0, 1, \ldots, 2^{2^n}-1$). From (10) we have

$$\int_0^1 \left| \sum_{l=0}^{2^{2^n}-1} \bar{\omega}_l(x)\, \bar{\omega}_l(t) \right| dt \leqq 1 \qquad (0\leqq x\leqq 1).$$

Hence, by a simple calculation we get that assertion (7) is also satisfied. Furthermore, by virtue of (13) and (14), the inequalities (6) and (9) hold. Finally, taking into account the construction of $\omega_l(x)$ it is obvious that

$$\int_0^1 \omega_l(x)\, dx = 0 \qquad (l = 0, 1, \ldots, 2^{2^n}-1).$$

The proof is thus completed.

Lemma 2. *Let n be a natural number, λ real number such that $0<\lambda<1$, furthermore, let I_1, I_2, I_3 be arbitrary, mutually disjoint subintervals of the interval* $(0, 1)$ *for which $|I_2| \leqq |I_1|$ and $|I_3| \leqq |I_1|$ are satisfied. Then there exist an orthonormal system $\{\psi_k(x)\}$ $(k=1, 2, \ldots, 2.2^{2^n})$ of step-functions in $(0, 1)$, a coefficient sequence $\{d_k\}$ $(k=1, 2, \ldots, 2.2^{2^n})$, and a simple set $F(\subseteqq I_1)$ having the following properties: the integral of each function $\psi_k(x)$ extended over $(0, 1)$ vanishes,*

$$\sum_{k=1}^{2.2^{2^n}} d_k^2 \leqq 1 \qquad (d_k = 0 \quad \text{if} \quad k = 2^{2^n}+1, \ldots, 2.2^{2^n}), \tag{15}$$

$$|F| = \frac{|I_1|}{8}, \tag{16}$$

$$\max_{1\leqq s<2^{2^n}} \left| \sum_{k=1}^{s} d_k \psi_k(x) \right| \geqq C_2 \lambda \sqrt{n} \quad \text{if} \quad x \in F; \tag{17}$$

for the Lebesgue functions of this system the following upper estimates hold:

$$L_{2^{2^n}}(\{\psi_k\}; x) = \int_0^1 \left| \sum_{k=1}^{2^{2^n}} \psi_k(x)\psi_k(t) \right| dt \leqq \begin{cases} C_3 \lambda & (x \in I_1), \\ C_4/\sqrt{|I_2|} & (x \in I_2), \\ C_5/\sqrt{|I_3|} & (x \in I_3), \\ 0 & \text{elsewhere}; \end{cases} \tag{18}$$

$$L_{2.2^{2^n}}(\{\psi_k\}; x) \leqq \begin{cases} C_6 \lambda & (x \in I_1), \\ C_7/\sqrt{|I_2|} & (x \in I_2), \\ 1 & (x \in I_3), \\ 0 & \text{elsewhere}; \end{cases} \tag{19}$$

furthermore, for the function

$$R_i(x) = \int_0^1 \left| \sum_{k=1}^{i} \psi_k(x)\psi_k(t) + \sum_{k=2^{2^n}+1}^{2.2^{2^n}-i} \psi_k(x)\psi_k(t) \right| dt \qquad (1 \leqq i < 2^{2^n})$$

we have also the following upper estimate:

$$R_i(x) \leqq \begin{cases} C_8 \lambda + L_n \sqrt{|I_3|}/\sqrt{|I_1|} & (x \in I_1), \\ C_9 L_n/\sqrt{|I_2|} & (x \in I_2), \\ C_{10} L_n/\sqrt{|I_3|} & (x \in I_3), \\ 0 & \text{elsewhere} \end{cases} \tag{20}$$

with

$$L_n = \max_{\substack{0\leqq i \leqq 2^{2^n}-1, \\ 0 \leqq x \leqq 1}} L_i(\{\omega_l\}; x);$$

the functions $\omega_l(x)$ occurring here are defined by Lemma 1. (As the functions $\omega_l(x)$ are uniformly bounded, L_n is a finite number for every n.)

Proof. Let $f(x)$ be an arbitrary function defined in the interval $(0, 1)$, furthermore, let $I=(a, b)$ be an arbitrary subinterval of $(0, 1)$ and H an arbitrary subset of $(0, 1)$. Now, we proceed from the interval $(0, 1)$ to the interval I by means of the linear tarnsformation $y=(x-a)/(b-a)$ $(a \leqq x \leqq b,\ 0 \leqq y \leqq 1)$, and put

$$f(I;x)=\begin{cases} f\left(\dfrac{x-a}{b-a}\right) & (a \leqq x \leqq b), \\ 0 & \text{elsewhere}; \end{cases}$$

let $H(I)$ be the set into which H is carried over by this linear transformation.

Let $\{\omega_l(x)\}$ $(l=0, 1, \ldots, 2^{2^n}-1)$, $\{b_l\}$ $(l=0, 1, \ldots, 2^{2^n}-1)$ and E denote the corresponding orthonormal system, the coefficient sequence, and the simple set occurring in Lemma 1, respectively.

Let us put

$$d_l=\begin{cases} b_{l-1} & \text{for} \quad 1 \leqq l \leqq 2^{2^n}, \\ 0 & \text{for} \quad 2^{2^n}+1 \leqq l \leqq 2.2^{2^n}; \end{cases}$$

furthermore, $F=E(I_1)$. It then follows from (6) and (8) that (15) and (16) are fulfilled. The functions $\psi_k(x)$ are defined as follows: for $k=1, 2, \ldots, 2^{2^n}$ let us set

$$\psi_k(x)=\frac{\lambda}{\sqrt{2|I_1|}}\omega_{k-1}(I_1;x)+\frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}}\omega_{k-1}(I_2;x)+\frac{1}{\sqrt{2|I_3|}}\omega_{k-1}(I_3;x),$$

and for $k=2^{2^n}+1, \ldots, 2.2^{2^n}$

$$\psi_k(x)=\frac{\lambda}{\sqrt{2|I_1|}}\omega_{2.2^{2^n}-k}(I_1;x)+\frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}}\omega_{2.2^{2^n}-k}(I_2;x)-\frac{1}{\sqrt{2|I_3|}}\omega_{2.2^{2^n}-k}(I_3;x).$$

By a simple calculation we get from these definitions that the functions $\psi_k(x)$ form an orthonormal system in $(0, 1)$. If $x \in F$ then there exists $y \in E$ such that

$$\psi_k(x)=\frac{\lambda}{\sqrt{2|I_1|}}\omega_{k-1}(y) \qquad (k=1, 2, \ldots, 2^{2^n}),$$

thus the correctness of (17) follows from (9). On account of Lemma 1, it is clear that

$$\int_0^1 \psi_k(x)\,dx=0 \qquad (k=1, 2, \ldots, 2.2^{2^n}).$$

It remains to be proved that the inequalities (18), (19) and (20) are also satisfied.

First of all, we remark that the functions $\psi_k(x)$ vanish outside the set $I_1 \cup I_2 \cup I_3$. According to the definition of the functions $\psi_k(x)$, by calculating the integrals on the right-hand side, we obtain for $x \in I_1$

(21)
$$L_{2^{2^n}}(\{\psi_k\}; x) = \frac{\lambda}{\sqrt{2|I_1|}} \left(\int_{I_1} + \int_{I_2} + \int_{I_3} \right) \left| \sum_{k=1}^{2^{2^n}} \omega_{k-1}(I_1; x) \psi_k(t) \right| dt =$$
$$= \frac{\lambda}{\sqrt{2|I_1|}} \left(\frac{\lambda}{\sqrt{2|I|_1}} |I_1| + \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}} |I_2| + \frac{|I_3|}{\sqrt{2|I_3|}} \right) \int_0^1 \left| \sum_{l=0}^{2^{2^n}-1} \omega_l(y) \omega_l(t) \right| dt;^{4)}$$

for $x \in I_2$

(22)
$$L_{2^{2^n}}(\{\psi_k\}; x) = \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}} \left(\int_{I_1} + \int_{I_2} + \int_{I_3} \right) \left| \sum_{k=1}^{2^{2^n}} \omega_{k-1}(I_2; x) \psi_k(t) \right| dt =$$
$$= \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}} \left(\frac{\lambda}{\sqrt{2|I_1|}} |I_1| + \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}} |I_2| + \frac{|I_3|}{\sqrt{2|I_3|}} \right) \int_0^1 \left| \sum_{l=0}^{2^{2^n}-1} \omega_l(y') \omega_l(t) \right| dt;$$

and for $x \in I_3$

(23)
$$L_{2^{2^n}}(\{\psi_k\}; x) = \frac{1}{\sqrt{2|I_3|}} \left(\int_{I_1} + \int_{I_2} + \int_{I_3} \right) \left| \sum_{k=1}^{2^{2^n}} \omega_{k-1}(I_3; x) \psi_k(t) \right| dt =$$
$$= \frac{1}{\sqrt{2|I_3|}} \left(\frac{\lambda}{\sqrt{2|I_1|}} |I_1| + \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}} |I_2| + \frac{|I_3|}{\sqrt{2|I_3|}} \right) \int_0^1 \left| \sum_{l=0}^{2^{2^n}-1} \omega_l(y'') \omega_l(t) \right| dt.$$

By paying attention to (7), from (21), (22) and (23) we obtain the estimate (18)

Now we treat the Lebesgue function $L_{2 \cdot 2^{2^n}}(\{\psi_k\}; x)$. We also distinguish three subcases as above. If $x \in I_1$, we get

(24)
$$L_{2.2^{2^n}}(\{\psi_k\}; x) = \left(\int_{I_1} + \int_{I_2} + \int_{I_3} \right) \left| \sum_{k=1}^{2.2^{2^n}} \psi_k(x) \psi_k(t) \right| dt =$$
$$= \frac{\lambda}{\sqrt{2|I_1|}} \left(\frac{2\lambda}{\sqrt{2|I_1|}} |I_1| + \frac{2\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}} |I_2| \right) \int_0^1 \left| \sum_{l=0}^{2^{2^n}-1} \omega_l(y) \omega_l(t) \right| dt;$$

4) Let y, y' and y'' denote the image points into which the points $x \in I_1$, $x \in I_2$ and $x \in I_3$ are carried over by the corresponding linear transformations transferring the intervals I_1, I_2 and I_3 into the interval (0, 1), respectively.

if $x \in I_2$ then

$$(25)\quad \begin{aligned} L_{2.2^{2^n}}(\{\psi_k\}; x) &= \left(\int_{I_1} + \int_{I_2} + \int_{I_3}\right) \left| \sum_{k=1}^{2.2^{2^n}} \psi_k(x)\psi_k(t) \right| dt = \\ &= \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}} \left(\frac{2\lambda}{\sqrt{2|I_1|}} |I_1| + \frac{2\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}} |I_2| \right) \int_0^1 \left| \sum_{l=0}^{2^{2^n}-1} \omega_l(y')\omega_l(t) \right| dt; \end{aligned}$$

and if $x \in I_3$ then

$$(26)\quad \begin{aligned} L_{2.2^{2^n}}(\{\psi_k\}; x) &= \left(\int_{I_1} + \int_{I_2} + \int_{I_3}\right) \left| \sum_{k=1}^{2.2^{2^n}} \psi_k(x)\psi_k(t) \right| dt = \\ &= \frac{1}{2|I_3|} 2|I_3| \int_0^1 \left| \sum_{l=0}^{2^{2^n}-1} \omega_l(y'')\omega_l(t) \right| dt. \end{aligned}$$

By virtue of (7), (24), (25) and (26) we have also the estimate (19).

The validity of (20) follows in a similar way as before. According to the definition of the function $R_i(x)$, we have for $x \in I_1$

$$(27)\quad \begin{aligned} R_i(x) &= \left(\int_{I_1} + \int_{I_2} + \int_{I_3}\right) \left| \sum_{k=1}^{i} \psi_k(x)\psi_k(t) + \sum_{k=2^{2^n}+1}^{2.2^{2^n}-i} \psi_k(x)\psi_k(t) \right| dt = \\ &= \frac{\lambda}{\sqrt{2|I_1|}} \left\{ \left(\frac{\lambda}{\sqrt{2|I_1|}} |I_1| + \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}} |I_2| \right) \int_0^1 \left| \sum_{l=0}^{2^{2^n}-1} \omega_l(y)\omega_l(t) \right| dt + \right. \\ &\quad \left. + \frac{1}{\sqrt{2|I_3|}} |I_3| \int_0^1 \left| \sum_{l=0}^{i-1} \omega_l(y)\omega_l(t) - \sum_{l=i}^{2^{2^n}-1} \omega_l(y)\omega_l(t) \right| dt \right\}; \end{aligned}$$

for $x \in I_2$

$$(28)\quad \begin{aligned} R_i(x) &= \left(\int_{I_1} + \int_{I_2} + \int_{I_3}\right) \left| \sum_{k=1}^{i} \psi_k(x)\psi_k(t) + \sum_{k=2^{2^n}+1}^{2.2^{2^n}-i} \psi_k(x)\psi_k(t) \right| dt = \\ &= \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}} \left\{ \left(\frac{\lambda}{\sqrt{2|I_1|}} |I_1| + \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}} |I_2| \right) \int_0^1 \left| \sum_{l=0}^{2^{2^n}-1} \omega_l(y')\omega_l(t) \right| dt + \right. \\ &\quad \left. + \frac{1}{\sqrt{2|I_3|}} |I_3| \int_0^1 \left| \sum_{l=0}^{i-1} \omega_l(y')\omega_l(t) - \sum_{l=i}^{2^{2^n}-1} \omega_l(y')\omega_l(t) \right| dt \right\}; \end{aligned}$$

and finally for $x \in I_3$

$$R_i(x) = \left(\int_{I_1} + \int_{I_2} + \int_{I_3}\right) \left| \sum_{k=1}^{i} \psi_k(x)\psi_k(t) + \sum_{k=2^{2^n}+1}^{2.2^{2^n}-i} \psi_k(x)\psi_k(t) \right| dt =$$

$$\text{(29)} \quad \frac{1}{\sqrt{2|I_3|}} \left\{ \left(\frac{\lambda}{\sqrt{2|I_1|}} |I_1| + \frac{\sqrt{1-\lambda^2}}{\sqrt{2|I_2|}} |I_2| \right) \int_0^1 \left| \sum_{l=0}^{i-1} \omega_l(y'')\omega_l(t) - \sum_{l=i}^{2^{2^n}-1} \omega_l(y'')\omega_l(t) \right| dt + \right.$$

$$\left. + \frac{|I_3|}{\sqrt{2|I_3|}} \int_0^1 \left| \sum_{l=0}^{2^{2^n}-1} \omega_l(y'')\omega_l(t) \right| dt \right\}.$$

Taking into consideration that $|I_2| < 1$, $|I_3| < 1$ and $L_n \geqq 1$, from (27), (28) and (29) we obtain the estimate (20). This completes the proof of Lemma 2.

§ 2. Proof of the theorem

Let $\{\nu_n\}$ and $\{N_n\}$ ($n = 2, 3, \ldots$) be the following sequences of natural numbers:

$$\nu_n = 2^{2^{n^6}} \qquad (n = 2, 3, \ldots),$$

$$N_2 = 0, \quad N_n = \sum_{i=2}^{n-1} 2\nu_i \qquad (n = 3, 4, \ldots).$$

Define the matrix $T = \{\alpha_{ik}\}$ ($i, k = 0, 1, 2, \ldots$) occurring in our theorem as follows:

$$\alpha_{00} = 1, \quad \alpha_{0k} = 0 \qquad (k = 1, 2, \ldots),$$

and in general, for an arbitrary natural number $n (\geqq 2)$ we distinguish three subcases: if $N_n < i < N_n + \nu_n$ then we put

$$\alpha_{ii} = \tfrac{1}{2}, \quad \alpha_{i,\, N_{n+1}-(i-N_n)} = \tfrac{1}{2}, \quad \alpha_{ik} = 0 \qquad \text{otherwise;}$$

if $i = N_n + \nu_n$ then

$$\alpha_{i,\, N_n+\nu_n} = 1, \quad \alpha_{ik} = 0 \qquad \text{otherwise;}$$

and finally if $N_n + \nu_n < i \leqq N_{n+1}$ then

$$\alpha_{i,\, N_{n+1}} = 1, \quad \alpha_{ik} = 0 \qquad \text{otherwise.}$$

From the definition of the matrix T it immediately follows that the conditions

$$\alpha_{ik} \geqq 0 \quad (i, k = 0, 1, 2, \ldots); \quad \lim_{i \to \infty} \alpha_{ik} = 0 \quad (k = 0, 1, 2, \ldots);$$

$$\sum_{k=0}^{\infty} \alpha_{ik} = 1 \quad (i = 0, 1, 2, \ldots)$$

are satisfied. Therefore, on account of a theorem (see e.g. ALEXITS [1], p. 65) we infer the permanence of the T-summation process.

To define the orthonormal system $\{\varphi_k(x)\}$ $(k=0, 1, 2, \ldots)$ and the coefficient sequence $\{c_k\}$ $(k=0, 1, 2, \ldots)$ occurring in our theorem we apply induction. The construction is similar to that of TANDORI [10].

Let $\lambda_n=1/n$ $(n=2, 3, \ldots)$ be. First of all, let us consider three sequences of subintervals $\{I_1(n)\}$, $\{I_2(n)\}$ and $\{I_3(n)\}$ of the interval (0, 1) so that the conditions

(30) $$I_i(n)\cap I_j(n)=O \qquad (i\neq j;\quad n=2,3,\ldots);$$

(31) $$\begin{aligned} &I_i(n')\cap I_i(n'')=O \qquad (i=2,3;\quad n'\neq n'';\quad n',n''=2,3,\ldots);\\ &I_2(n')\cap I_3(n'')=O \qquad (n',n''=2,3,\ldots);\end{aligned}$$

(32) $$I_1(n)=\left[\frac{2^{m+1}-n}{2^m},\frac{2^{m+1}-n+1}{2^m}\right) \qquad (2^m<n\leqq 2^{m+1};\quad m=0,1,2,\ldots);$$

(33) $$\sum_{n=2}^{\infty} L_{n^6}\left(\sqrt{|I_2(n)|}+\sqrt{|I_3(n)|}\right)<\infty,$$

where L_n is defined in Lemma 2, and

(34) $$\frac{L_{n^6}\sqrt{|I_3(n)|}}{\sqrt{|I_1(n)|}}\leqq\lambda_n \qquad (n=2,3,\ldots)$$

should be satisfied. It is obvious that both intervals $I_2(n)$ and $I_3(n)$ can be chosen in accordance with these requirements.

From (31) we can easily see that every point x of (0, 1) belongs to at most one of all the subintervals $I_2(n)$ and $I_3(n)$. Furthermore, by (32) it follows that every point $x\in(0,1)$ lies in $I_1(n)$ for infinitely many values of n, and for every non-negative integer m there exists a uniquely determined natural number $n_m(x)$ for which $2^m<n_m(x)\leqq 2^{m+1}$ and $x\in I_1(n_m(x))$. By the definition of $\{\lambda_n\}$ we get immediately that

(35) $$\sum_{m=0}^{\infty}\lambda_{n_m(x)}\leqq\sum_{m=0}^{\infty}\frac{1}{2^m}=2.$$

Now we are going to construct a system $\{\varphi_k(x)\}$ $(k=0, 1, 2, \ldots)$ of orthonormal step-functions in (0, 1), a coefficient sequence $\{c_k\}$ $(k=0, 1, 2, \ldots)$, and a sequence of simple subsets $G_n(\subseteq I_1(n))$ $(n=2, 3, \ldots)$ in (0, 1) so that the following relations should be satisfied:

(36) $$\sum_{k=N_n+1}^{N_n+\nu_n} c_k^2\leqq\frac{1}{n^2} \quad\text{and}\quad c_k=0 \quad\text{for}\quad k=N_n+\nu_n+1,\ldots,N_{n+1} \qquad (n=2,3,\ldots);$$

(37) $$|G_n|=\frac{|I_1(n)|}{8};$$

(38) $$\max_{N_n<i\leqq N_n+\nu_n}\left|\sum_{k=N_n+1}^{i} c_k\varphi_k(x)\right|\geqq C_2 n \quad\text{if}\quad x\in G_n \qquad (n=2,3,\ldots);$$

furthermore,

$$(39)\qquad \int_0^1 \left|\sum_{k=N_n+1}^{N_n+\nu_n} \varphi_k(x)\varphi_k(t)\right| dt \leqq \begin{cases} C_3\lambda_n & (x\in I_1(n)),\\ C_4/\sqrt{|I_2(n)|} & (x\in I_2(n)),\\ C_5/\sqrt{|I_3(n)|} & (x\in I_3(n)),\\ 0 & \text{elsewhere}; \end{cases} \quad (n=2,3,\ldots);$$

$$(40)\qquad \int_0^1 \left|\sum_{k=N_n+1}^{N_{n+1}} \varphi_k(x)\varphi_k(t)\right| dt \leqq \begin{cases} C_6\lambda_n & (x\in I_1(n)),\\ C_7/\sqrt{|I_2(n)|} & (x\in I_2(n)),\\ 1 & (x\in I_3(n)),\\ 0 & \text{elsewhere}; \end{cases} \quad (n=2,3,\ldots);$$

$$(41)\qquad S_i(n;x) = \int_0^1 \left|\sum_{k=N_n+1}^{N_n+i} \varphi_k(x)\varphi_k(t) + \sum_{k=N_n+\nu_n+1}^{N_{n+1}-i} \varphi_k(x)\varphi_k(t)\right| dt \leqq$$

$$\leqq \begin{cases} C_8\lambda_n + L_{n^6}\sqrt{|I_3(n)|}/\sqrt{|I_1(n)|} & (x\in I_1(n)),\\ C_9 L_{n^6}/\sqrt{|I_2(n)|} & (x\in I_2(n)),\\ C_{10} L_{n^6}/\sqrt{|I_3(n)|} & (x\in I_3(n)),\\ 0 & \text{elsewhere} \end{cases}$$

$$(N_n < i < N_n+\nu_n;\quad n=2,3,\ldots).$$

We notice that, on account of (34) and (41), the estimate

$$(42)\qquad S_i(n;x) \leqq \begin{cases} C_{11}\lambda_n & (x\in I_1(n)),\\ C_9 L_{n^6}/\sqrt{|I_2(n)|} & (x\in I_2(n)),\\ C_{10} L_{n^6}/\sqrt{|I_3(n)|} & (x\in I_3(n)),\\ 0 & \text{elsewhere} \end{cases}$$

$$(N_n < i < N_n+\nu_n;\quad n=2,3,\ldots)$$

also follows.

Let $\varphi_0(x)\equiv 1$ and $c_0=0$ be. We apply Lemma 2 with $n=2^6$, $\lambda=\lambda_2$ and $I_i=I_i(2)$ $(i=1,2,3)$ (on account of (30) it is permissible). We get the orthonormal system $\{\psi_k(x)\}$ $(k=1,2,\ldots,2\nu_2)$, the coefficient sequence $\{d_k\}$ $(k=1,2,\ldots,2\nu_2)$, and the simple set F satisfying (15)—(20). Now we write

$$\varphi_k(x)=\psi_k(x),\quad c_k=\frac{d_k}{2}\quad (k=1,2,\ldots,N_3),\ \text{ and }\quad G_2=F.$$

According to Lemma 2 the step-functions $\varphi_k(x)$ $(k=0,1,\ldots,N_3)$ are orthonormal, and the relations (36)—(41) hold for $n=2$.

Now, $n_0(\geqq 2)$ being arbitrary, we assume that the step-functions $\varphi_k(x)$ $(k=0,1,\ldots,N_{n_0+1})$, the coefficients c_k $(k=0,1,\ldots,N_{n_0+1})$, and the simple sets $G_n\,(\subseteqq I_1(n))$ $(n=2,3,\ldots,n_0)$ are already determined such that these functions are orthogonal and normed in (0, 1) and that the requirements (36)—(41) are satisfied for each integer $n \leqq n_0$. We are going to construct the functions, coefficients, and simple set corresponding to n_0+1 so that these also satisfy (36)—(41).

We can divide the intervals $I_1(n_0+1)$, $I_2(n_0+1)$ and $I_3(n_0+1)$ into a finite number of mutually disjoint subintervals

$$I_1(n_0+1) = \bigcup_{i=1}^{q_1} J_i(1), \quad I_2(n_0+1) = \bigcup_{i=1}^{q_2} J_i(2), \quad I_3(n_0+1) = \bigcup_{i=1}^{q_3} J_i(3)$$

on which every function $\varphi_k(x)$ $(k=0, 1, \ldots, N_{n_0+1})$ remains constant, and every set $G_n \cap I_1(n_0+1)$ $(n=2, 3, \ldots, n_0)$ can be represented as the union of some intervals $J_i(1)$.

We begin with applying Lemma 1 with $n=(n_0+1)^6$. We get the functions $\omega_l(x)$ $(l=0, 1, \ldots, 2^{2(n_0+1)^6}-1)$. Next applying Lemma 2 with $n=(n_0+1)^6$, $\lambda=\lambda_{n_0+1}$ and $I_i=I_i(n_0+1)$ $(i=1, 2, 3)$, we obtain the functions $\psi_k(x)$ $(k=1, 2, \ldots, 2\nu_{n_0+1})$, the coefficients d_k $(k=1, 2, \ldots, 2\nu_{n_0+1})$, and the simple set F_{n_0+1}. Let us put

$$\varphi_{N_{n_0+1}+l}(x) = \frac{\lambda_{n_0+1}}{\sqrt{2|I_1(n_0+1)|}} \sum_{i=1}^{q_1} \omega_{l-1}(J_i(1); x) + \frac{\sqrt{1-\lambda_{n_0+1}^2}}{\sqrt{2|I_2(n_0+1)|}} \sum_{i=1}^{q_2} \omega_{l-1}(J_i(2); x) +$$

$$+\frac{1}{\sqrt{2|I_3(n_0+1)|}} \sum_{i=1}^{q_3} \omega_{l-1}(J_i(3); x) \qquad (l=1, 2, \ldots, \nu_{n_0+1}),$$

$$\varphi_{N_{n_0+1}+\nu_{n_0+1}+l}(x) = \frac{\lambda_{n_0+1}}{\sqrt{2|I_1(n_0+1)|}} \sum_{i=1}^{q_1} \omega_{\nu_{n_0+1}-l}(J_i(1); x) +$$

$$+\frac{\sqrt{1-\lambda_{n_0+1}^2}}{\sqrt{2|I_2(n_0+1)|}} \sum_{i=1}^{q_2} \omega_{\nu_{n_0+1}-l}(J_i(2); x) - \frac{1}{\sqrt{2|I_3(n_0+1)|}} \sum_{i=1}^{q_3} \omega_{\nu_{n_0+1}-l}(J_i(3); x)$$

$$(l=1, 2, \ldots, \nu_{n_0+1}).$$

It is clear that the functions $\varphi_k(x)$ $(k=N_{n_0+1}+1, \ldots, N_{n_0+2})$ are also step-functions. By virtue of Lemma 1 and the definition, we can easily prove that the functions $\varphi_k(x)$ $(k=0, 1, \ldots, N_{n_0+2})$ are orthonormal in (0, 1).

Let us put

$$c_{N_{n_0+1}+k} = \frac{d_k}{n_0+1} \qquad (k=1, 2, \ldots, 2\nu_{n_0+1}).$$

From (16) it follows that (36) is satisfied for $n=n_0+1$. Finally, we set

$$G_{n_0+1} = \bigcup_{i=1}^{q_1} E(J_i(1)).$$

It is obvious that G_{n_0+1} is a simple set, and on account of Lemma 1, (37) holds for $n=n_0+1$.

If $x \in G_{n_0+1}$ then there exists a point $y \in F_{n_0+1}$ such that

$$\varphi_{N_{n_0+1}+k}(x) = \psi_k(y) \qquad (k = 1, 2, \dots, 2\nu_{n_0+1}).$$

Taking into consideration of the definition of the coefficients c_k and (17), we obtain (38) for $n = n_0 + 1$.

According to the definition of the functions $\varphi_k(x)$ $(N_{n_0+1} < k \leqq N_{n_0+2})$ and the proof of Lemma 2, if $x \in (0, 1)$ then for an appropriately chosen y we have

$$\int_0^1 \left| \sum_{k=N_{n_0+1}+1}^{N_{n_0+1}+\nu_{n_0+1}} \varphi_k(x)\varphi_k(t) \right| dt = \int_0^1 \left| \sum_{l=1}^{\nu_{n_0+1}} \psi_l(y)\psi_l(t) \right| dt.$$

To show this, let $x \in I_1(n_0+1) \cup I_2(n_0+1) \cup I_3(n_0+1)$ be fixed. Then by simple integral transformations we get that the left-hand side equals

$$\frac{\lambda_{n_0+1}}{\sqrt{2|I_1(n_0+1)|}} \sum_{i=1}^{q_1} \int_{J_i(1)} \left| \sum_{l=1}^{\nu_{n_0+1}} \psi_l(y)\,\omega_{l-1}(J_i(1); t) \right| dt +$$

$$+ \frac{\sqrt{1-\lambda_{n_0+1}^2}}{\sqrt{2|I_2(n_0+1)|}} \sum_{i=1}^{q_2} \int_{J_i(2)} \left| \sum_{l=1}^{\nu_{n_0+1}} \psi_l(y)\,\omega_{l-1}(J_i(2); t) \right| dt +$$

$$+ \frac{1}{\sqrt{2|I_3(n_0+1)|}} \sum_{i=1}^{q_3} \int_{J_i(3)} \left| \sum_{l=1}^{\nu_{n_0+1}} \psi_l(y)\,\omega_{l-1}(J_i(3); t) \right| dt =$$

$$= \frac{\lambda_{n_0+1}}{\sqrt{2|I_1(n_0+1)|}} \int_0^1 \left| \sum_{l=1}^{\nu_{n_0+1}} \psi_l(y)\,\omega_{l-1}(t) \right| dt \sum_{i=1}^{q_1} |J_i(1)| +$$

$$+ \frac{\sqrt{1-\lambda_{n_0+1}^2}}{\sqrt{2|I_2(n_0+1)|}} \int_0^1 \left| \sum_{l=1}^{\nu_{n_0+1}} \psi_l(y)\,\omega_{l-1}(t) \right| dt \sum_{i=1}^{q_2} |J_i(2)| +$$

$$+ \frac{1}{\sqrt{2|I_3(n_0+1)|}} \int_0^1 \left| \sum_{l=1}^{\nu_{n_0+1}} \psi_l(y)\,\omega_{l-1}(t) \right| dt \sum_{i=1}^{q_3} |J_i(3)| =$$

$$= \left(\int_{J_1(n_0+1)} + \int_{J_2(n_0+1)} + \int_{J_3(n_0+1)} \right) \left| \sum_{l=1}^{\nu_{n_0+1}} \psi_l(y)\psi_l(t) \right| dt = \int_0^1 \left| \sum_{l=1}^{\nu_{n_0+1}} \psi_l(y)\psi_l(t) \right| dt.$$

Here we took into consideration that

$$\sum_{i=1}^{q_1} |J_i(1)| = |I_1(n_0+1)|, \quad \sum_{i=1}^{q_2} |J_i(2)| = |I_2(n_0+1)|, \quad \sum_{i=1}^{q_3} |J_i(3)| = |I_3(n_0+1)|.$$

Similarly, we have also the following equations:

$$\int_0^1 \left| \sum_{k=N_{n_0+1}+1}^{N_{n_0+2}} \varphi_k(x)\,\varphi_k(t) \right| dt = \int_0^1 \left| \sum_{l=1}^{2\nu_{n_0+1}} \psi_l(x)\,\psi_l(t) \right| dt,$$

and

$$\int_0^1 \left| \sum_{k=N_{n_0+1}+1}^{N_{n_0+1}+i} \varphi_k(x)\,\varphi_k(t) + \sum_{k=N_{n_0+1}+\nu_{n_0+1}+1}^{N_{n_0+2}-i} \varphi_k(x)\,\varphi_k(t) \right| dt =$$

$$= \int_0^1 \left| \sum_{l=1}^{i} \psi_l(y)\,\psi_l(t) + \sum_{l=\nu_{n_0+1}+1}^{2\nu_{n_0+1}-i} \psi_l(y)\,\psi_l(t) \right| dt \qquad (i = 1, 2, \ldots, \nu_{n_0+1}-1);$$

here $y \in I_1(n_0+1)$, $y \in I_2(n_0+1)$, $y \in I_3(n_0+1)$ and $y \notin \bigcup_{i=1}^{3} I_i(n_0+1)$ according to $x \in I_1(n_0+1)$, $x \in I_2(n_0+1)$, $x \in I_3(n_0+1)$ and $x \notin \bigcup_{i=1}^{3} I_i(n_0+1)$, respectively. By (18), (19) and (20) we get (39), (40) and (41) also for $n = n_0+1$.

Thus we obtained the orthonormal system $\{\varphi_k(x)\}$, the coefficient sequence $\{c_k\}$, and the sequence of simple sets $\{G_n\}$ by induction, which fulfil the requirements (36)—(41).

Let us consider the sets

$$H_m = \bigcup_{n=2^m+1}^{2^{m+1}} G_n \qquad (m = 1, 2, \ldots).$$

By virtue of the definition of the intervals $I_1(n)$ and (36), we have

(43) $$|H_m| = \frac{1}{8} \qquad (m = 1, 2, \ldots).$$

According to the definition of the sets G_n, it can easily be seen that the sets H_m are stochastically independent. Applying the Borel—Cantelli lemma we get

$$|\overline{\lim_{m\to\infty}}\, H_m| = 1.$$

If $x \in \overline{\lim}_{m\to\infty} H_m$ then the inequality (38) is satisfied for infinitely many values of m and hence

(44) $$\overline{\lim_{n\to\infty}} \left(\max_{N_n < i \leq N_n+\nu_n} \left| \sum_{k=N_n+1}^{i} c_k\,\varphi_k(x) \right| \right) = \infty$$

holds almost everywhere.

As to the Lebesgue functions

$$L_i(\{\varphi_k\}; x) = \int_0^1 \left| \sum_{k=0}^{i} \varphi_k(x)\,\varphi_k(t) \right| dt$$

of the system $\{\varphi_k(x)\}$ with $i=N_n$ and $i=N_n+\nu_n$, we have

$$L_{N_n}(\{\varphi_k\};x) \leqq 1+\sum_{r=1}^{n}\int_0^1\left|\sum_{k=N_{r-1}+1}^{N_r}\varphi_k(x)\varphi_k(t)\right|dt,$$

as $\varphi_0(x)\equiv 1$. From the definition of the intervals $I_i(n)$ $(i=1,2,3;\, n=2,3,\dots)$, by (35) and (40), it follows

$$L_{N_n}(\{\varphi_k\};x) \leqq \begin{cases} C_{12} & \left(x\notin \bigcup_{l=2}^{\infty}(I_2(l)\cup I_3(l))\right), \\ C_{13}/\sqrt{|I_2(p)|} & (x\in I_2(p)), \\ C_{14} & (x\in I_3(q)) \quad (n=2,3,\dots). \end{cases} \tag{45}$$

It follows exactly in the same way as before that

$$L_{N_n+\nu_n}(\{\varphi_k\};x) \leqq 1+\sum_{r=1}^{n}\int_0^1\left|\sum_{k=N_{r-1}+1}^{N_r}\varphi_k(x)\varphi_k(t)\right|dt+\int_0^1\left|\sum_{k=N_n+1}^{N_n+\nu_n}\varphi_k(x)\varphi_k(t)\right|dt,$$

and taking into consideration (35) and (39), we get the estimate

$$L_{N_n+\nu_n}(\{\varphi_k\};x) \leqq \begin{cases} C_{15} & \left(x\notin \bigcup_{l=2}^{\infty}(I_2(l)\cup I_3(l))\right), \\ C_{16}/\sqrt{|I_2(p)|} & (x\in I_2(p)), \\ C_{17}/\sqrt{|I_3(q)|} & (x\in I_3(q)) \quad (n=2,3,\dots). \end{cases} \tag{46}$$

Hence and by (45) and (46), in virtue of (33), we obtain that

$$\int_0^1 (\sup_n L_{N_n}(\{\varphi_k\};x))\,dx<\infty, \qquad \int_0^1 (\sup_n L_{N_n+\nu_n}(\{\varphi_k\};x))\,dx<\infty.$$

Furthermore, (36) implies $\sum_{k=0}^{\infty} c_k^2<\infty$. Denote by $s_i(x)$ the i-th partial sum of the series (5). On account of a theorem of LEINDLER [5] it follows that $\{s_{N_n}(x)\}$ and $\{s_{N_n+\nu_n}(x)\}$ converge almost everywhere.

The above mentioned theorem of LEINDLER reads as follows:

Let $\{\varphi_k(x)\}$ $(k=0,1,\dots)$ *be an arbitrary orthonormal system in* (a,b). *If for a monotone increasing sequence* $\{n_r\}$ *of indices the inequality*

$$L_{n_r}(\{\varphi_k\};x)=O(1) \qquad (a\leqq x\leqq b)$$

holds, then under the condition $\sum_{k=0}^{\infty} a_k^2<\infty$ *the* n_r*-th partial sums of the orthogonal series* (1) *converge almost everywhere.*

A more detailed analysis of LEINDLER's proof shows that the assertion remains valid under the weaker condition:

$$\sup_r L_{n_r}(\{\varphi_k\}; x) \in L(a, b).$$

Let us denote by $t_i(x)$ the i-th T-mean of the orthogonal series (5). If $N_n < i < N_n + \nu_n$ then on account of the definition of the matrix T and the sequence $\{c_k\}$, we have

$$t_i(x) = \tfrac{1}{2} s_i(x) + \tfrac{1}{2} s_{N_{n+1}-i}(x) = \tfrac{1}{2} s_{N_n}(x) + \tfrac{1}{2} \sum_{k=N_n+1}^{i} c_k \varphi_k(x) + \tfrac{1}{2} s_{N_n+\nu_n}(x).$$

Hence, if we pay attention to (44), it follows from the convergence of $\{s_{N_n}(x)\}$ and $\{s_{N_n+\nu_n}(x)\}$ that

$$\overline{\lim_{i\to\infty}} |t_i(x)| = \infty$$

almost everywhere. Thus the orthogonal series (5) is not T-summable almost everywhere in (0, 1).

To accomplish the proof of our theorem, we have to show that for the Lebesgue functions concerning the T-summation the relation (4) is satisfied.

If $N_n + \nu_n \leqq i \leqq N_{n+1}$ then

$$L_i(T; \{\varphi_k\}; x) = L_{N_{n+1}}(\{\varphi_k\}; x) \quad \text{and} \quad L_i(T; \{\varphi_k\}; x) = L_{N_n+\nu_n}(\{\varphi_k\}; x),$$

respectively, thus in virtue of (45) and (46) the following estimate

$$L_i(T; \{\varphi_k\}; x) \leqq \begin{cases} C_{18} & \left(x \notin \bigcup_{l=2}^{\infty} (I_2(l) \cup I_3(l))\right), \\ C_{19}/\sqrt{|I_2(p)|} & (x \in I_2(p)), \\ C_{20}/\sqrt{|I_3(q)|} & (x \in I_3(q)) \end{cases} \tag{47}$$

$$(N_n + \nu_n \leqq i \leqq N_{n+1}; \quad n = 2, 3, \dots)$$

is true.

Finally, let $N_n < i < N_n + \nu_n$ be, i.e. $i = N_n + j$ $(1 \leqq j < \nu_n)$. Then

$$L_i(T; \{\varphi_k\}; x) = \tfrac{1}{2} \int_0^1 \left| \sum_{k=0}^{N_n+j} \varphi_k(x)\varphi_k(t) + \sum_{k=0}^{N_{n+1}-j} \varphi_k(x)\varphi_k(t) \right| dt.$$

A simple calculation shows

$$L_i(T; \{\varphi_k\}; x) \leqq \tfrac{1}{2} \int_0^1 \left| \sum_{k=0}^{N_n} \varphi_k(x)\varphi_k(t) \right| dt + \tfrac{1}{2} \int_0^1 \left| \sum_{k=0}^{N_n+\nu_n} \varphi_k(x)\varphi_k(t) \right| dt +$$

$$+ \tfrac{1}{2} \int_0^1 \left| \sum_{k=N_n+1}^{N_n+j} \varphi_k(x)\varphi_k(t) + \sum_{k=N_n+\nu_n+1}^{N_{n+1}-j} \varphi_k(x)\varphi_k(t) \right| dt = \tag{48}$$

$$= \tfrac{1}{2}\left(L_{N_n}(\{\varphi_k\}; x) + L_{N_n+\nu_n}(\{\varphi_k\}; x) + S_j(n; x)\right).$$

By virtue of (42) we get

$$S_j(n;x) \leqq \begin{cases} C_{11} & \left(x \notin \bigcup_{l=2}^{\infty} (I_2(l) \cup I_3(l))\right), \\ C_9 L_{p^6}/\sqrt{|I_2(p)|} & (x \in I_2(p)), \\ C_{10} L_{q^6}/\sqrt{|I_3(q)|} & (x \in I_3(q)) \end{cases} \tag{49}$$

$$(1 \leqq j < \nu_n; \quad n = 2, 3, \dots).$$

From the inequalities (45), (46), (48) and (49) it follows

$$L_i(T; \{\varphi_k\}; x) \leqq \begin{cases} C_{21} & \left(x \notin \bigcup_{l=2}^{\infty} (I_2(l) \cup I_3(l))\right), \\ C_{22} L_{p^6}/\sqrt{|I_2(p)|} & (x \in I_2(p)), \\ C_{23} L_{q^6}/\sqrt{|I_3(q)|} & (x \in I_3(q)) \end{cases} \tag{50}$$

$$(N_n < i < N_n + \nu_n; \quad n = 2, 3, \dots).$$

(Here we again took into consideration that $L_n \geqq 1$ for every n.) From (47) and (50) we infer that

$$\int_0^1 \sup_i L_i(T; \{\varphi_k\}; x)\, dx \leqq C_{24} \left(1 + \sum_{n=2}^{\infty} L_{n^6} \left(\sqrt{|I_2(n)|} + \sqrt{|I_3(n)|}\right)\right)$$

holds. Hence on account of (33) we obtain that (4) is fulfilled.

We have thus completed the proof of our theorem.

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Berichtigung zur Arbeit „Über die starke Summation von Fourierreihen" *)

Von KÁROLY TANDORI in Szeged

Der Beweis des Satzes I dieser Arbeit ist falsch. Mit der dort angewandten Methode kann man nur die folgende, ziemlich komplizierte Behauptung beweisen:

Ist $f(t)$ nach 1 periodisch und in $[0, 1]$ *Lebesgue-integrierbar, so gibt es für fast alle Punkte* $x \in [0, 1]$ *eine positive Intervallfunktion* $\Phi_x(I)$ *mit* $\sum_{n=0}^{\infty} \Phi_x\left(\left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]\right) < \infty$ *derart, daß für* $0 < k < \infty$ *und* $0 < h \to 0$ *gilt:*

$$\text{(1)} \qquad \int_{h}^{2h} |f(x+u)-f(x)|\, du \int_{u-k}^{u+k} |f(x+v)-f(x)|\, dv = o(h^2 \Phi((h, 2h])) + o(hk),$$

undzwar gleichmäßig in Bezug auf k.

Ähnlicherweise, wie in der erwähnten Arbeit, kann bewiesen werden, daß aus (1) die H_2-Summierbarkeit der Fourierreihe von $f(t)$ in dem Punkt x folgt.

(Eingegangen am 28. März 1968)

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*) *Acta Sci. Math.*, **16** (1955), 65—73.

Bibliographie

C. A. Rogers, Packing and covering, VIII+111 pages, Cambridge University Press, 1964.

The study of packing and covering problems represents the most developed branch of discrete geometry whose results find direct applications in the theory of numbers, in the analysis, in the theory of information, and in other branches of mathematics.

The first monograph dealing with packing and covering is FEJES TÓTH's well-known discrete geometry book (*Lagerungen in der Ebene auf der Kugel und im Raum*, Berlin—Göttingen—Heidelberg, 1953) in which the author confined his attention mainly to packing and covering in two and three-dimensional space of constant curvature. Some results on this subject are treated also in his recent book *Regular Figures* (Oxford—London—New York—Paris, 1964).

The vaste literature of packing and covering, to which many well-known mathematicians (including COXETER, DELAUNAY, DIRICHLET, FEJES TÓTH, GAUSS, HLAWKA, LAGRANGE, MINKOWSKI, ROGERS, SEGRE, THUE, VORONOI) have contributed, has reached the point where a systematization into a theory has become possible, indeed necessary. We can be grateful to Professor ROGERS, who has himself enriched the theory with many beautiful results, that he has undertaken the task to write this monograph.

The book treats mainly problems in n-dimensional Euclidean space, where n is larger than 3, and the packing and covering system is formed by a finite or countably infinite system of translates of a single set which will usually be convex. In the introduction the author gives an excellent historical outline of the subject. In the succeeding chapters he gives a systematic (but by no means exhaustive) account of the general results of the subject and their derivations. For a closer idea about the content of this part let us mention the titles of the chapters: 1. Packing and covering densities; 2. The existence of reasonably dense packings; 3. The existence of reasonably economical coverings; 4. The existence of reasonably dense lattice packings; 5. The existence of reasonably economical lattice coverings; 6. Packings of simplices cannot be very dense; 7. Packings of spheres cannot be very dense; 8. Coverings with spheres cannot be very economical.

The author uses in his work an analytical method and only one diagram illustrates the book, in contrary to FEJES TÓTH who gives preference to the syntetical method, richly illustrating his works.

ROGERS's monograph is thus a useful complement to FEJES TÓTH's discrete geometry monograph. It is written with great care, is easy to read, and its economical and logical structure is quite excellent.

J. Molnár (Bamako—Budapest)

L. Rédei, The theory of finitely generated commutative semigroups, XIII+350 pages, Budapest, Akadémiai Kiadó, 1965.

This book is a translation of the original German text edited in 1963 by Teubner Verlag and Akadémiai Kiadó. The theory contained in this monograph has become a well-known and separate part of the theory of algebraic semigroups. We can omit to give here a detailed report on the work as there are such thorough references at disposal as those of E. A. BEHRENS (MR, **28** (1964), 5130) or Št. SCHWARZ (*Acta Sci. Math.*, **25** (1964), 175—176). One can hope that the English edition rendering this original and important topic available for a wider class of researchmen will start new investigations in this field, concerning especially some possible generalizations, links with other parts of semigroup theory and the application of RÉDEI's results to special classes of semigroups.

G. Pollák (Szeged)

A. F. Timan, Theory of approximation of functions of a real variable, XII+631 pages, Hindustan Publishing Corporation (India), Delhi-7, 1966.

The present book is a translation from the Russian original, published in 1960 by "Fizmatgiz", Moscow. It presents a detailed account of the new results of the theory of approximation. As the author points out in the foreword, this monograph systematically investigates the relationship between the various structural properties of real functions and the character of their possible approximation by polynomials or other simple functions. The investigations carried out in this book are based on the classical approximation theorem of WEIERSTRASS, the concept of TSCHEBYSCHEFF of the best approximation and the converse theorem of BERNSTEIN on the existence of a function with a given sequence of best approximation. The chapter headings give a more detailed outline of the presentation: I. WEIERSTRASS's theorem. — II. The best approximation. — III. Certain compact classes of functions and their structural characteristics. This chapter includes properties of various moduli of continuity, properties of various classes of analytic functions, quasi-analytic classes of functions, and properties of conjugate functions. — IV. Certain properties of algebraic polynomials and transcendental integral functions of exponential type. Here we find interpolation formulae, WIENER—PALEY's theorem, some extremal properties of polynomials and transcendental integral functions. — V. Direct theorems of the constructive theory of functions. — VI. Converse theorems. Constructive characteristics of certain classes of functions. — VII. Additional theorems on the connection between the best approximations of functions and their structural properties. — VIII. Linear processes of approximation of functions by polynomials. Certain estimates connected with them. — IX. Certain deductions from the theory of functions and functional analysis. This includes basic theorems without proof.

At the end of each chapter there is a section which contains various problems and theorems supplementing the material of the main text.

There is a useful Bibliography (containing some 350 items) and a detailed Index.

The book is well-organized and the presentation is clear.

L. Leindler (Szeged)

Ralph Abraham and **Joel Robbin, Transversal Mappings and Flows,** X+161 pages, W. A. Benjamin, Inc., New York—Amsterdam, 1967.

Since the initiative work of G. REEB, S. SMALE and R. THOM the qualitative theory of ordinary differential equations has merged into a rapidly developing new theory. This process has been marked by the application of differential topology. It has commenced by considering a system of first order ordinary differential equations as a vector field on a differentiable manifold and the solutions of the system as the flows of the latter. The successes of this process, however, were mainly due to the fact that, roughly speaking, the vector fields of a compact differentiable manifold form an (infinite-dimensional) differentiable manifold, which fact proves to be very useful when dealing with questions of the qualitative theory. An exposition of the fundamental ideas and results of the new theory, originating from this process, is the goal of this book.

The first chapter is a review of differential theory. Definitions and basic theorems are given, while in some cases for the proof the reader is referred to DIEUDONNÉ's *Foundations of Modern Analysis* or to LANG's *Introduction to Differentiable Manifolds.* It contains an original result, the converse of TAYLOR's theorem, and the exposition is remarkable in various respects.

The second chapter deals with the topologies of spaces of vector bundle sections and contains a proof of the smoothness of the evaluation map. The latter theorem prepares for the application of transversality technique which has been introduced by R. THOM and is applied here systematically.

The third chapter contains a proof of the SMALE's Density Theorem concerning the regular values of the so-called FREDHOLM mappings. It is obtained as the last one in a sequence of fundamental theorems, each implied by the preceding one. These theorems are: the Rough Composition Theorem of KNESER and GLAESER, and the Density Theorems of MÉTIVIER, SARD and SMALE.

The basic facts of transversality theory are given in the fourth chapter. It contains the Openness and Isotopy Theorems of THOM and the Density Theorem of ABRAHAM.

The fundamental concepts and theorems of the theory of vector fields and their flows are given in the fifth chapter. These are formulated both traditionally and in terms of transversality by which the adequacy of this new technique becomes evident.

Pseudocharts for closed orbits, the FLOQUET normal form and a proof of the Stable Manifold Theorem of SMALE, are the topics discussed in the sixth chapter.

A new proof of the theorem of S. SMALE and I. KUPKA on generic properties of flows is given in the seventh chapter. By the application of transversality technique it is easier than the original one. Important tools of this proof are a theorem of A. TARSKI and A. SEIDENBERG on the structure of semialgebraic sets and the perturbation theory of vector fields.

From the Appendices the first two provide some perequisites, the third, however, is an original research article by AL KELLEY on stable, center-stable, center-unstable, and unstable manifolds.

The above material up to now has been accessible mainly in research papers. The authors' self-contained exposition is worked out carefully, with the intent to attain a maximum of clarity. It serves not only to arouse interest in, but also to yield a very readable introduction into this developing modern subject.

J. Szenthe (Szeged)

H. Halberstam—K. F. Roth, Sequences, Vol. 1, XX+291 pages, Oxford, Clarendon Press, 1966.

The arithmetical properties of special integer sequences (e. g. the distribution of prime numbers in arithmetical progressions, the additive properties of the sequence of squares) are extensively studied in number theory. Many of these properties hold for all or for wide classes of integer sequences. The main theme of this book is the study of those general arithmetical properties which are satisfied for extensive classes of such sequences.

In Chapters I, II, and III general laws related to the addition of sequences are established. Chapter I deals with the density relationship. This chapter contains among others the theorems of MANN, DYSON, VAN der CORPUT, BESICOVICH, ERDŐS on the Schnirelmann density and their asymptotic and p-adic analogues, the theorem of LINNIK on the essential components. Chapter II contains the theorem of ERDŐS and FUCHS. Concerning the number of representations of integers as the sum of two summands taken from a given set, this theorem investigates the discrepancy of the asymptotic of the mean value of this number. Chapter III is devoted to the treatment of the probability methods which serve to prove existence theorems for integer sequences with a given growth of the representation function.

Chapter IV gives a very good account of the sieve methods of V. BRUN and A. SELBERG, and of the large sieve.

Chapter V deals with some interesting properties of integer sequences which depend on the multiplicative structure of the integers. Results of ERDŐS, ERDŐS—DAVENPORT, BESICOVITCH, and others, concerning primitiv sequences and sets of multiples are treated.

The style is clear, the authors are masters of their subject.

Imre Kátai (Budapest)

The Theory of Groups, Proceedings of the International Conference held at the Australian National University Canberra, 10—20 August, 1965. Edited by **L. G. Kovács** and **B. H. Neumann,** XVII+397 pages, Gordon and Breach Science Publishers, New York—London—Paris, 1967.

This volume contains more than half a hundred articles which represent all the essential branches of current research in group theory. To make more perceptible the tendencies of development, we list the authors and indicate the results of the more important papers, in the following order: 1) Simple groups, 2) Varieties of groups, 3) Some purely group-theoretical problems, 4) Connections between groups and other algebraic systems.

1. W. FEIT deals with groups having a cyclic Sylow subgroup. Z. JANKO proves that a nontrivial simple group with abelian 2-Sylow subgroups having no doubly transitive permutation representation coincides with the Janko new simple group. R. REE in his article deals with classification of involutions in some Chevalley groups and computes the centralizers of these elements.

2. W. BRISLEY investigates varieties generated by all the proper factors of a critical group. N. D. GUPTA proves some theorems on metabelian groups contained in certain varieties. G. HIGMAN's paper deals with the form of functions, describing orders of relatively free groups. His other article applies the theory of the representation of the general linear groups to varieties of p-groups. L. G. KOVÁCS and M. F. NEWMAN study varieties in which every proper subvariety is a Cross variety. J. D. MACDONALD shows that if critical p-groups generate the same variety then this holds

for the sets of their proper factors too. HANNA NEUMANN summarizes some developments in the field of varieties of groups. In SHEILA OATES' paper we find some investigations on the number of generators of a simple group. SOPHIE PICCARD gives an analysis of the notion of group, free modulo n. P. M. WEICHSEL studies finite critical p-groups which generate join-irreducible varieties.

3. CHRISTINE W. AYOUB studies the minimum number of conjugate classes which a finite p-group can possess. REINHOLD BAER takes part in the volume with three articles. He analyzes the interrelations between the properties characterizing nilpotent groups in the finite case. He characterizes also the polycyclic groups and the noetherian groups possessing a polycyclic subgroup of finite index. In his third paper he discovers some parallelism between theories of artinian and noetherian groups. G. BAUMSLAG gives a review on the present status of the theory of finitely presented groups. H. S. M. COXETER describes some geometric aspects of the isomorphism between the Lorentz group and the group of homographies. J. D. DIXON proves a theorem of Schur—Zassenhaus type. T. HAWKES introduces and investigates the notion of f-Prefrattini subgroup. K. A. HIRSCH presents some results of his student B. WEHRFRITZ, e. g., the proof of the conjugacy of Sylow subgroups in any periodic linear group. N. ITO proves that a nonsolvable transitive permutation group of degree p — where p is a Fermat prime — containing an odd permutation coincides with the symmetric group of degree p. O. H. KEGEL gives a characterization of finite supergroups. R. STEINBERG treats the Galois cohomology of linear algebraic groups. G. SZEKERES determines all finite metabelian groups with two generators. O. TAMASCHKE presents a generalized character theory of finite groups. G. E. WALL constructs a counter-example for a conjecture of D. R. HUGHES. H. WIELANDT gives a survey on subnormal and relatively maximal subgroups and states some problems. In his other article he deals with automorphisms of doubly transitive permutation groups and as application he obtains a special case of SCHREIER'S conjecture on the automorphism group of a finite simple group. G. ZAPPA introduces and studies the notion of the Hall S-partition of a group.

4. L. W. ANDERSON and R. P. HUNTER give conditions for the minimal two-sided ideal of a compact connected semigroup to be a group. Their other paper deals also with certain groups connected with the semigroup theory. A. L. S. CORNER gives a characterization of endomorphism rings of countable reduced torsion-free abelian groups as topological rings. L. FUCHS lists the most useful properties of orderable groups and indicates some unsolved problems. M. HALL Jr. presents some applications of block designs to group theory. F. LOONSTRA investigates the ordered set of abelian extensions of an abelian group. P. J. LORIMER generalizes the notion of characteristic for any finite projective plane in a manner eliminating the disadvantages of the earlier generalization. K. W. WESTON presents an interesting connection between group and ring theory. H. SCHWERDTFEGER investigates groups which may be considered as a slightly modified projective plane.

From this list it is clear that this valuable book will be useful for all algebraists interested in group theory.

B. Csákány (Szeged)

Hans Hermes, Einführung in die Verbandstheorie (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 73), XII+209 Seiten, zweite, erweiterte Auflage, Springer-Verlag, Berlin—Heidelberg—New York, 1967.

Die vorliegende zweite Auflage hat den Charakter und den Aufbau der ersten behalten, sie wurde aber mit einigen neuen Paragraphen ergänzt. (Vgl. die Besprechung der ersten Auflage in diesen *Acta*, **16** (1955), 275.) So werden Erzeugungs- und Entscheidungsverfahren für die in verschiedenen, durch Gleichungssysteme definierbaren Verbandsklassen gültigen Termgleichungen angegeben (§ 26, 27); es werden die klassischen und die intuitionistischen Aussagenstrukturen, sowie ihre Beziehungen zu den dualen Primidealen der Booleschen bzw. pseudo-Booleschen (d. h., nach unten beschränkten relativ pseudokomplementären) Verbände behandelt (§ 28, 29; „duales Primideal" wird hier einfach „Primideal" genannt); zur Vorbereitung der letzten dient ein Paragraph über die pseudo-Booleschen Verbände, in dem — unter anderem — gezeigt wird, daß diese Verbandsklasse durch ein Gleichungssystem definierbar ist; ferner werden die Kongruenzrelationen in Verbänden eingehender behandelt (§ 31, 32; hier findet man z. B. den Funayama — Nakayamaschen Satz, hinreichende Bedingungen für die Komplementarität des Kongruenzverbandes und den Satz von HASHIMOTO über die Beziehung zwischen den Kongruenzrelationen und den Idealen). Übrigens wurde der Text der ersten Auflage nur stellenweise abändert; Beispiel 8.4 und Abbildung 19.1 B wurden berichtigt.

G. Szász (Nyiregyháza)

Proceedings of Symposia in Pure Mathematics, vol. X. **Singular integrals,** VI+375 pages, American Mathematical Society, Providence, Rhode Island, 1967.

This volume, edited by A. P. CALDERÓN, contains the material of the Symposium on Singular Integrals and their Applications held at the University of Chicago, in Chicago, April 20—22, 1966. The rich and deep material is dedicated to Professor ANTONI ZYGMUND in celebration of his sixty-fifth anniversary and in recognition of his decisive contribution to the field of singular integrals.

The volume consists of twenty papers written by outstanding authors, furthermore, of an author and subject index. The papers, in general, deal with integral transformations of the form

$$\tilde{f}(x) = \text{P. V.} \int f(x-y)K(y)dy = \lim_{\varepsilon\to 0} \int_{|y|\geqq\varepsilon} f(x-y)K(y)dy$$

(P. V. means "principal value") or of similar types and with their various applications, where $x=(x_1, x_2, ..., x_n)$, $y=(y_1, y_2, ..., y_n)$ are points of the real n-dimensional Euclidean space, $|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$, and the kernel $K(y)$ fulfils certain conditions of homogeneity and integrability.

The purpose of the paper of B. BAJSANSKI and R. COIFMAN is to prove the boundedness of the maximal operator associated with some singular integrals considered by A. P. CALDERÓN. The paper of A. P. CALDERÓN presents the development of the algebraic formalism of singular integral operators as sketched in his earlier papers. In their joint paper A. P. CALDERÓN, MARY WEISS, and A. ZYGMUND investigate the existence of singular integrals and show that, under certain conditions, the singular integrals are operators of weak type (1, 1). M. V. CORDES treats some properties of the mapping $f\to\tilde{f}$ in L^2. E. B. FABES and H. JODEIT, JR. introduce so-called parabolic singular integral operators and apply such operators to boundary value problems for parabolic equations. E. B. FABES and N. M. RIVIERE, continuing their earlier investigations, extend some results of CALDERÓN and ZYGMUND to the case of kernels with mixed homogeneity. K. O. FRIEDRICHS and P. D. LAX deal with symmetrizable differential operators, LARS HÖRMANDER with pseudo-differential operators and hypoelliptic equations, M. JODEIT, JR. with symbols of parabolic singular integrals, B. FRANK JONES, JR. with applications of singular integrals to the solution of boundary value problems for the heat equation. Then the further papers follow: PAUL KRÉE: A Class of Singular Integrals. Pseudo-differential Operators on Non-quasi-analytic Function Spaces, P. D. LAX and L. NIRENBERG: A Sharp Inequality for Pseudo-differential and Difference Operators, J. E. LEWIS: Mixed Estimates for Singular Integrals and an Application to Initial Value Problems in Parabolic Differential Equations, UMBERTO NERI: Singular Integral Operators on Manifolds, JOHN C. POLKING: Boundary Value Problems for Parabolic Systems of Partial Differential Equations, CORA SADOSKY and MISCHA COTLAR: On quasi-homogeneous Bessel Potential Operators, R. T. SEELEY: Complex Powers of an Elliptic Operator, Elliptic Singular Integral Equations, E. M. STEIN: Singular Integrals, Harmonic Functions, and Differentiability Properties of Functions of Several Variables, RICHARD L. WHEEDEN: Hypersingular Integrals and Summability of Fourier Integrals and Series.

The above enumeration shows that this volume, which is very rich in its material, gives a comprehensive view of the modern, developing and important theory of singular integrals.

F. Móricz—K. Tandori (Szeged)

Olivier Dimon Kellogg, Foundations of Potential Theory (Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band XXXI) IX+384 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1967.

This book is the reprint from the first edition of 1929. The first edition was reviewed by F. RIESZ in these *Acta*, **5** (1930—32), 137—138.

D. E. Men'šov, Limits of indeterminacy in measure of trigonometric and orthogonal series (Proceedings of the Steklov Institute of Mathematics, number 99 (1967)), 67 pages, American Mathematical Society, Providence, Rhode Island, 1968.

This book is the translation by R. P. BOAS of the Russian original. It contains the proof of three, very general and complicated theorems concerning the lower and upper limits in measure, or in other words, the limits of indeterminacy in measure, of measurable functions. The first two theorems — somewhat in a less general form — have been published by the author earlier. (On limits of indeterminacy with respect to measure and limit functions of trigonometric and orthogonal series, *Dokl. Akad. Nauk SSSR*, **160** (1965), 1254—1256.) These results are based on the author's earlier investigations related to convergence in measure of trigonometric series (On convergence in measure of trigonometric series, *Trudy Mat. Inst. Steklov*, **32** (1950), 1—99; On the limit functions of trigonometric series, *Trudy Moskov. Mat. Obšč.*, **7** (1958), 291—334) as well as on the results of A. A. TALALJAN (Limit functions of series in bases of the space L_p, *Mat. Sbornik* **56** (98) (1962), 353—374), and at the same time they are improvements of those. The theorems in this book essentially assert that the limits of indeterminacy in measure of the sequences of partial sums of series with respect to a complete orthogonal system, or more generally, to a normalized basis in $L_p(p>1)$, and of series of arbitrary measurable terms, have many properties in common.

F. Móricz—K. Tandori (Szeged)

Pál Révész, The Laws of Large Numbers, 176 pages, Akadémiai Kiadó, Budapest, 1967.

The laws of large numbers have always occupied an important position in the history of the calculus of probability, referring to both theoretical and practical applications. Although there exists already a huge literature on this subject, there appeared no monograph which would systematically elaborate all known laws. The present work fills this gap, giving a general survey of the results and the most important methods of proof in this field. Occasionally, when the proof of a theorem requires very special methods, it is omitted. Several open questions are also mentioned.

To define exactly the field of the laws of large numbers seems to be very difficult. We can say, in an attempt to obtain a definition, that a law of large numbers asserts the convergence, in a certain sense, of the average

$$\eta_n = \frac{\xi_1+\xi_2+\dots+\xi_n}{n}$$

of the random variables $\xi_1, \xi_2, \dots$ to a random variable η (p. 8). By making use of different modes of convergence, different types of the laws of large numbers can be obtained. Three kinds of convergence are considered in the book: stochastic convergence (or convergence in measure), convergence with probability 1 (or convergence almost everywhere), and mean convergence. According to this, mainly, the weak laws, the strong laws, and the mean laws of large numbers are studied.

In connection with the laws of large numbers the author investigates the rate of convergence. Hence, the laws of the iterated logarithm are also treated in the book.

Theorems on the convergence of a series of the form $\Sigma c_k \xi_k$, where $\{\xi_k\}$ is a sequence of random variables and $\{c_k\}$ is a sequence of real numbers, cannot be considered as a law of large numbers. However, this class of theorems is occasionally also studied because the convergence of a series of the above form immediately implies a law of large numbers by using the Kronecker lemma.

The book contains eleven chapters with a complete bibliography and author index.

In Chapter 0 the author has collected the most important definitions and theorems without proof which are applied in the book. The reader should be familiar with the most fundamental results and concepts of probability, stochastic processes, measure theory, ergodic theory, functional analysis, etc.

Chapter 1 deals with the special concepts and general theorems of the laws of large numbers.

Chapter 2 is devoted to the laws of large numbers of independent random variables. The author presents some fundamental results of KOLMOGOROV such as the Kolmogorov inequality, the so-called three series theorem, another theorem that gives a necessary and sufficient condition for the weak law of large numbers, etc. One virtue of the book is that there are occasionally given more essentially different proofs of the same theorem, e. g. the gap method and the method of high moments for the proof of the strong laws of large numbers. The end of this chapter studies the asymptotical properties of weighted averages, and the case of convergence to $+\infty$.

Chapter 3 contains the author's own results that are analogues of some theorems of the previous chapter for the case in which the random variables are not independent but only strongly multiplicative. These investigations are based on an inequality, also due to him, which can be considered as a generalization of the Rademacher—Menšov inequality, well-known in the theory of orthogonal series.

Chapter 4 discusses the laws of large numbers for stationary sequences by making use of the results of ergodic theory. The difficulties from the special point of view of probability lie in the investigation of the condition of ergodicity. This investigation seems to be very difficult and not quite solved.

The main problem of Chapter 5 is the following one: under what conditions can we find a subsequence of an arbitrary sequence of random variables obeying a law of large numbers? Among others, the author investigates this question in the case of Walsh functions $\{w_n(x)\}$ (and, presents the analogous results related to the sequence $\{\sin nx\}$), and he proves that if we consider a subsequence $w_{n_k}(x)$ (resp. $\sin n_k x$) for which $n_{k+1}/n_k \geqq q > 1$ then we can obtain practically the same results as for independent sequences. The main result of this chapter, due also to the author, asserts that from any sequence of uniformly bounded random variables we can choose a subsequence which has properties similar to those of an independent system.

After presenting the fundamental theorem of symmetrically dependent random variables, the results of Chapter 4 concerning the strong laws of large numbers are sharpened in Chapter 6. At the end of this chapter the author discusses the connection between the equinormed strongly multiplicative systems and the quasi-multiplicative systems.

Chapter 7 deals with the laws of large numbers of Markov chains. The laws of large numbers as well as the limit theorems for non-homogeneous Markov chains are based on the different kinds of measures of ergodicity. The author introduces some of them, and several theorems are mentioned.

Chapter 8 contains some general laws of large numbers which are not related to any concrete class of stochastic processes. In this chapter there are no restrictions on the kind of dependence, only on the strength of it. After introducing the notion of mixing for a sequence of random variables, general theorems are treated.

Up to the Chapter 9 random variables taking values on the real line occur. In general, similar results can be obtained for random variables taking values in a finite dimensional Banach space. The situation is not much more complicated if the values of the random variables are in a Hilbert space. The real difficulty is in the treatment of the random variables taking values in a Banach space. The author follows the treatment of Beck.

In general, a law of large numbers states that the average of the first n terms of a sequence of random variables is practically constant if n is large enough. In many practical applications the number of the experiments (i. e. the integer n) depends on chance. Chapter 10 deals with the questions of the sum of a random number of independent random variables.

The results of the previous chapters are applied in Chapter 11 to number theory, to statistics, and to information theory. To begin with, two classes of expansions of the numbers $x \in [0, 1]$ are studied: the first is the so-called Cantor series, the second is a very general expansion introduced by Rényi. In connection with this second expansion only the case of the continued fractions is treated in detail. As regards applications in statistics, the book investigates the estimation of the distribution and of the density functions.

The book is well-readable and, in spite of its relatively short extent, the most important results of the laws of large numbers are presented in it with complete proofs.

F. Móricz (Szeged)

Pál Révész, Die Gesetze der grossen Zahlen (Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, Mathematische Reihe, Band 35), 176 Seiten, Birkhäuser-Verlag, Basel und Akadémiai Kiadó, Budapest, 1968.

The German edition is the translation of the English original, reviewed above.

András Ádám, Truth functions and the problem of their realization by two-terminal graphs, 206 pages, 34 figures and 11 tables, Akadémiai Kiadó, Budapest, 1968.

The concept of truth functions appeared firstly in mathematical logic, mainly by studying the simplest functions such as negation, disjunction, conjunction, and equivalence. The functions in question are called frequently Boolean functions too, for the logician Boole investigated them firstly systematically. The discovery of the applicability of these functions to engineering and cybernetics has given rise to the development of a self-reliant theory.

The author gives a systematic survey on the main directions of the theory of truth functions from a mathematician's point of view. According to this, the author mentions only in short remarks the technical origine or significance of the matter presented.

The book is divided into two parts; it contains ten chapters, an appendix, bibliography, and indices of names and subjects. For the convenience of the reader, the symbols which are used in the book are also summarized. The first part treats the theory of truth functions considered from the point of view of "discrete analysis", the second one deals with the problem of their realization by two-terminal graphs, mainly by using graph-theoretical methods.

The content of the book is almost self-contained; the presupposed knowledge of the reader does not exceed some fundamental notions and results of mathematics (lying mainly in the theories of sets, numbers and groups).

The majority of the results contained in the book has appeared solely in the original articles of several authors; only a small part of them was already elaborated in books. Such results also occur which have never been published before. A virtue of the book is the fact that the proofs presented are always given in a complete form, and this makes the subject more readable than in the original publications.

In Chapter 1 the fundamental concepts of this topics are considered. § 5 contains an unpublished theorem of T. BAKOS that is useful for the design of logical machines, more precisely, for the planning of technically advantageous enumeration of the places of the definition of a truth function.

Chapter 2 is devoted to the presentation of the fundamental researches of QUINE on prime implicants and disjunctive normal forms of minimal length. These results have an important applicability to the problems of simplification of electrical networks. The author presents two algoritms of QUINE to determine all the prime implicants of a truth function, and a further method to find all the representation of a truth function by irredundant disjunctive normal forms. The end of this chapter contains some results of the author on relations between repetition-free superpositions of truth functions and prime implicants.

Chapter 3 treats interrelations between conjunctive and disjunctive normal forms in order to reach all prime implicants of a truth function in a more economical way than in the preceding chapter. From the theoretical point of view the theorem of NELSON, from the view-point of practical applications the method of VOISHVILLO solve completely the problem of finding all the prime implicants starting with an arbitrary conjunctive normal form of a truth function.

Chapter 4 deals with the characterization of the systems of truth functions which are functionally complete concerning superposition. In §§ 14—15 the functionally complete systems of truth functions, in §§ 16—17 the complete systems of certain special automata consisting of a function and a non-negative integer number expressing time delay are considered.

Chapter 5 is devoted to the questions of uniqueness of the "deepest" decomposition by repetition-free superpositions. The main result is due to KUZNETSOV whose theorem asserts that if a truth function is given, then its deepest repetition-free superpositional decompositions are necessarily "almost conciding" with each other.

Chapter 6 deals with numerical questions, particularly, with the number of certain sets consisting of truth functions. The author presents three permutation groups in connection with the set of the truth functions of n variables, and discusses the following problem: What is the number of classes of the essentially different truth functions? The "essentially different" is meant in three distinct senses, namely, two truth functions belong to the same class if one of them is originated from the other either by applying a permutation of its variables or by substituting some variables by their negatives or by applying simultaneously both previous processes. The treatment of this subject is based chiefly upon the results of G. PÓLYA.

Chapter 7 presents notions and results concerning linearly separable functions, i. e. treats the possibility of assigning real numbers to the variables of a truth function in such a manner that the function value is true exactly if the sum of the numbers, assigned to the true variables, exceeds a given threshold.

The first part of Chapter 8 presents general preliminaries of graph theory. The author deals in detail with the questions of series-parallel decomposition of 2-graphs (2-graphs always means a strongly connected two-terminal graph), of canonical decomposition of indecomposable 2-graphs, etc. The results of §§ 38—39 are due to the author. Introducing the notion of completable and separating pair of edges and that of quasi-series decomposition, he gives a new method of building up 2-graphs from simpler ones.

In Chapter 9 the author introduces several concepts of realization of truth functions: (i) by functional elements, (ii) by 2-graphs that is attributed to SHESTAKOV and SHANNON, (iii) by threeterminal graphs that was proposed by L. KALMÁR. In the rest the repetition-free realization of truth functions by 2-graphs is studied. § 45 contains some own results of the author concerning the problem of existence, after introducing the notion of the quasi-series decomposition of truth functions. The chapter ends with the result of TRAHTENBROT on the solution of the problem of unicity.

Chapter 10 mentions some aspects of the problem of optimal realization containing two theorems of LUNTS.

In the Appendix the author mentions some possibilities of the future development by presenting the notion of stochastic truth functions introduced by JOHN VON NEUMANN.

The book will be useful both for the theoretical-minded mathematicians who either want to make research in the theory of truth functions or to be thoroughly acquainted with the more essential results of this topic, and expectably also for scientists, well-educated in mathematics, who may apply the theory of truth functions in their work.

F. Móricz (Szeged)

K. Reidemeister, Vorlesungen über Grundlagen der Geometrie (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 32.), berichtigter Nachdruck, X+147 Seiten, Berlin—Heidelberg—New York, Springer-Verlag, 1968.

Seit Erscheinung der Originalausgabe (für ihre Besprechung siehe: diese *Acta*, **5** (1930—32), S. 250) hat sich die Grundidee des Buches, bei der Begründung der ebenen affinen Geometrie dem Begriff der Gewebe eine Hauptrolle zu erteilen, bekanntlich als fruchtbar erwiesen. Die für die Grundlagen der Geometrie Interessierten werden also sicher diesen Nachdruck begrüssen. Durch Hinweise auf die nach 1930 erschienene Literatur werden die inzwischen erreichten Ergebnisse und die Anhaltspunkte zu aktuellen Forschungsfragen angegeben.

J. Szenthe (Szeged)

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