ACTA UNIVERSITATIS SZEGEDIENSIS

ACTA SCIENTIARUM MATHEMATICARUM

ADIUVANTIBUS

L. KALMÁR, L. RÉDEI ET K. TANDORI

REDIGIT

B. SZ.-NAGY

TOMUS XXVIII

FASC. 1—2

SZEGED, 1967

INSTITUTUM BOLYAIANUM UNIVERSITATIS SZEGEDIENSIS

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28. KÖTET
1—2. FÜZET

SZEGED, 1967. JÚNIUS

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SZEGED, 1967. DECEMBER

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On rootless operators and operators without logarithms

By DON DECKARD and CARL PEARCY in Miami (Florida, U. S. A.)

1. Introduction

Let $\mathfrak{H}$ be an infinite dimensional, separable, complex Hilbert space, and denote by $\mathfrak{L}(\mathfrak{H})$ the algebra of all bounded linear operators on $\mathfrak{H}$. Topological properties of subsets of $\mathfrak{L}(\mathfrak{H})$ under discussion always refer to the uniform operator topology. An operator $A \in \mathfrak{L}(\mathfrak{H})$ is said to be *rootless* if for every positive integer $n \geqq 2$, A fails to have any n-th roots. Throughout this note, the set of *invertible, rootless* operators on $\mathfrak{H}$ is denoted by $\mathfrak{R}$. That $\mathfrak{R}$ is non-empty was first proved by HALMOS, LUMER, and SCHÄFFER in [2], and that $\mathfrak{R}$ has non-void interior was established in [3], and again proved in [5] and [6]. In fact, all previously known examples of operators in $\mathfrak{R}$ are interior points of $\mathfrak{R}$, and this caused LUMER [5] to ask if $\mathfrak{R}$ is open. In this note we generalize the methods of [1], which led to the construction of a certain class of operators without square roots, and thereby prove the following

Theorem 1. *The set $\mathfrak{R}$ is not open and is not closed relative to the invertible operators.*

Closely related to the question of whether an invertible operator A has roots is the question of whether A has a logarithm; i.e., whether there is some $B \in \mathfrak{L}(\mathfrak{H})$ satisfying $\exp(B) = A$. We denote the set of all invertible operators on $\mathfrak{H}$ that *fail* to have a logarithm by $\mathfrak{L}$. (A necessary and sufficient condition that an operator belong to $\mathfrak{L}$ is known [4, page 285], but it has not yielded specific examples of operators in $\mathfrak{L}$.) It is clear that $\mathfrak{R} \subset \mathfrak{L}$, and the existence of invertible operators with square roots but no fourth roots [1, 6] implies that the above inclusion is proper. That $\mathfrak{L}$ has non-void interior follows from the fact that $\mathfrak{R}$ does. However, the questions as to whether $\mathfrak{L}$ is open, or closed relative to the invertible operators, seem to have gone unanswered, and we furnish answers as follows:

Theorem 2. *The set $\mathfrak{L}$ is not open and is not closed relative to the invertible operators.*

2. Preliminaries

Before discussing the idea used in the proofs of Theorems 1 and 2, we introduce the following terminology.

Let N denote the set of integers greater than 2, and let $N_2, N_3, \ldots$ be infinite disjoint subsets of N whose union is N. The sets N and N_p, $p = 2, 3, \ldots,$ will remain fixed throughout the paper, and will frequently be regarded as increasing sequences without further apology. Notation such as $\lim_{n\in N_p}\{a_n\}$ will be used for simplicity, and should be given the obvious interpretation.

For each $n\in N$, let $\mathfrak{H}_n$ be n-dimensional complex Hilbert space, and let $\mathfrak{L}(\mathfrak{H}_n)$ be the algebra of all (linear) operators on $\mathfrak{H}_n$. Denote by $\mathfrak{H}$ the Hilbert space $\sum_{n\in N}\oplus\mathfrak{H}_n$, and let the algebra $\mathfrak{I}\subset\mathfrak{L}(\mathfrak{H})$ be the C^*-sum $\mathfrak{I}=\sum_{n\in N}\oplus\mathfrak{L}(\mathfrak{H}_n)$. Then, of course, $\mathfrak{I}$ consists of all operators $A=\sum_{n\in N}\oplus A_n$ where $A_n\in\mathfrak{L}(\mathfrak{H}_n)$ and the sequence $\{\|A_n\|\}_{n\in N}$ is bounded.

Note that to prove that neither $\mathfrak{R}$ nor $\mathfrak{L}$ is open it clearly suffices to prove the following proposition.

I) *There exists an operator $A\in\mathfrak{I}\cap\mathfrak{R}$ and a sequence $\{G^{(k)}\}$ of operators in $\mathfrak{I}$ such that each $G^{(k)}$ has a logarithm and such that $\|G^{(k)}-A\|\to 0$.*

On the other hand, to show that neither $\mathfrak{R}$ nor $\mathfrak{L}$ is closed relative to the class of invertible operators, it is enough to prove following proposition.

II) *There exists an operator $C\in\mathfrak{I}$ having a logarithm and a sequence $\{B^{(k)}\}$ of operators in $\mathfrak{I}\cap\mathfrak{R}$ such that $\|B^{(k)}-C\|\to 0$.*

Our task of proving Theorems 1 and 2 will be accomplished by proving I) and II). To do this, we show that it suffices to prove the following

Theorem 3. *There exists a sequence $\{A_n\}_{n\in N}$, $A_n\in\mathfrak{L}(\mathfrak{H}_n)$, satisfying:*

a) $\sum_{n\in N}\oplus A_n\in\mathfrak{R}\cap\mathfrak{I}$,

b) *every operator $B=\sum\oplus B_n$ in $\mathfrak{I}$ such that* spectrum $B_n=$ spectrum A_n *for $n\in N$ and $B_n=A_n$ for all sufficiently large n, satisfies $B\in\mathfrak{R}$.*

Furthermore, there exists a sequence $\{C_n\}_{n\in N}$, $C_n\in\mathfrak{L}(\mathfrak{H}_n)$, satisfying:

c) spectrum $C_n=$ spectrum A_n *for* $n\in N$,

d) $\|C_n-A_n\|\to 0$,

e) $\sum_{n\in N}\oplus C_n$ *has a logarithm in $\mathfrak{I}$ (and thus is itself an invertible operator in $\mathfrak{I}$).*

Proof of I) and II) using Theorem 3. With the notation as above, define:

$$A=\sum_{n\in N}\oplus A_n,\qquad C=\sum_{n\in N}\oplus C_n,$$

and for each $k\in N$,

$$B^{(k)}=C_3\oplus\ldots\oplus C_k\oplus A_{k+1}\oplus A_{k+2}\oplus\ldots,$$

$$G^{(k)}=A_3\oplus\ldots\oplus A_k\oplus C_{k+1}\oplus C_{k+2}\oplus\ldots.$$

Since C has a logarithm in $\mathfrak{I}$, and every invertible operator on a finite dimensional space has a logarithm, each $G^{(k)}$ has a logarithm. Since obviously $\|G^{(k)}-A\|\to 0$, I) is proved. Since each $B^{(k)}\in\mathfrak{R}\cap\mathfrak{I}$ and obviously $\|B^{(k)}-C\|\to 0$, II) is proved.

Thus to prove Theorems 1 and 2, it suffices to prove Theorem 3, and the remainder of the paper is devoted to this task.

3. A construction

To begin the proof of Theorem 3, we wish to produce for each $p>1$ an operator of the form $\sum_{n\in N_p} \oplus A_n$ that has no p-th root. Thus we must suitably modify and generalize the lemmas of [1, § 2] to make them applicable to the problem of p-th roots. Throughout the paper we denote by ω_n^r the n-th root of unity

$$\omega_n^r = e^{2\pi i r/n} \qquad (r = 0, \pm 1, \pm 2, \dots).$$

Lemma 3.1. *Let $p\geqq 2$ and $n\geqq 3$ be integers, and let $\{\lambda_n^r\}_{r=1}^n$ be complex numbers such that $(\lambda_n^r)^p=\omega_n^r$ for $r=1, 2, \dots, n$. Then*

$$\lambda_n^r = \omega_{pn}^r\,\omega_p^{k(r)} \quad \textit{for} \quad r = 1, 2, \dots, n,$$

where each $k(r)$ is some integer satisfying $1\leqq k(r)\leqq p$, and either

(α) *for some r satisfying $1\leqq r\leqq n-1$,*

$$\left|\frac{(\lambda_n^r)^p-(\lambda_n^{r+1})^p}{\lambda_n^r-\lambda_n^{r+1}}\right| \leqq \frac{2\pi}{n|1-\omega_{3p}^2|}, \qquad \textit{or}$$

(β) $k(1)=k(2)=\dots=k(n)$, *in which case,*

$$\left|\frac{(\lambda_n^1)^p-(\lambda_n^n)^p}{\lambda_n^1-\lambda_n^n}\right| \leqq \frac{2\pi}{n|1-\omega_{3p}^2|}.$$

Proof. Suppose there is an r satisfying $1\leqq r\leqq n-1$ such that $k(r)\neq k(r+1)$, and note that

$$\left|\frac{(\lambda_n^r)^p-(\lambda_n^{r+1})^p}{\lambda_n^r-\lambda_n^{r+1}}\right| = \left|\frac{(\omega_{pn}^r)^p-(\omega_{pn}^{r+1})^p}{\omega_{pn}^r\,\omega_p^{k(r)}-\omega_{pn}^{r+1}\,\omega_p^{k(r+1)}}\right| =$$

$$= \left|\frac{\omega_n^r-\omega_n^{r+1}}{\omega_{pn}^r[\omega_p^{k(r)}-\omega_{pn}^1\,\omega_p^{k(r+1)}]}\right| = \left|\frac{1-\omega_n^1}{1-\omega_{pn}^1\,\omega_p^{[k(r+1)-k(r)]}}\right|.$$

Now $k=k(r+1)-k(r)$ is a non-zero integer satisfying $-(p-1)\leqq k\leqq p-1$, and it is clear that the distance from 1 to $\omega_{pn}^1\omega_p^{-1}$ along the unit circle is less than or equal to the distance from 1 to the point $\omega_{pn}^1\omega_p^k$ along the unit circle. Thus

$$|1-\omega_{pn}^1\,\omega_p^k| \geqq |1-\omega_{pn}^1\,\omega_p^{-1}|.$$

Furthermore, $|1-\omega_{pn}^1\omega_p^{-1}|$ attains its minimum as a function of n at $n=3$, so

$$|1-\omega_{pn}^1\,\omega_p^k| \geqq |1-\omega_{3p}^1\,\omega_p^{-1}| = |1-\omega_{3p}^2|$$

and thus

$$\left|\frac{1-\omega_n^1}{1-\omega_{pn}^1\,\omega_p^k}\right| \leqq \left|\frac{1-\omega_n^1}{1-\omega_{3p}^2}\right| \leqq \frac{2\pi}{n|1-\omega_{3p}^2|}.$$

On the other hand, if $k(1)=k(2)=\dots=k(n)$, then

$$\left|\frac{(\lambda_n^1)^p-(\lambda_n^n)^p}{\lambda_n^1-\lambda_n^n}\right| = \left|\frac{1-\omega_n^1}{1-\omega_{pn}^1\,\omega_p^{-1}}\right| \leqq \frac{2\pi}{n|1-\omega_{3p}^2|},$$

as before.

Lemma 3.2. *Suppose that $K \in \mathfrak{L}(\mathfrak{H})$ has the distinct eigenvalues $\{\lambda_i\}_{i\in I}$, and suppose that for $i \in I$, the eigenspace corresponding to λ_i is spanned by the vector x_i. If J is any operator on $\mathfrak{H}$ that commutes with K, and $\mathfrak{K}$ is any subspace of $\mathfrak{H}$ spanned by some subset of the $\{x_i\}_{i\in I}$, then $\mathfrak{K}$ is an invariant subspace for J.*

Proof. It suffices to show that for $i \in I$, $Jx_i = \alpha_i x_i$ for some scalar α_i. If $Jx_i = y_i$, then $Ky_i = KJx_i = JKx_i = \lambda_i y_i$, so that by hypothesis $y_i = \alpha_i x_i$.

The following corollaries are immediate.

Corollary 3.3. *If p and n are positive integers, T is an $n \times n$ complex matrix in upper triangular form having n distinct eigenvalues, and $R^p = T$, then R is also in upper triangular form.*

Corollary 3.4. *Suppose $K \in \mathfrak{J}$ and satisfies the hypotheses of Lemma 3.2 and the additional hypothesis that the vectors $\{x_i\}_{i\in I}$ span $\mathfrak{H}$. Suppose also that $J \in \mathfrak{L}(\mathfrak{H})$ and satisfies $J^p = K$ for some positive integer p. Then $J \in \mathfrak{J}$.*

The following easy computation is designated as a lemma for convenience in referring to it later.

Lemma 3.5. *Let T be the $n \times n$ complex matrix*

$$\left(\begin{array}{c|c} B & z \\ \hline 0 & \lambda \end{array}\right)$$

where B is an $(n-1)\times(n-1)$ matrix, z is a $(n-1)$-vector, and λ is a scalar not in the spectrum of B. If p is any positive integer, then T^p is the matrix

$$\left(\begin{array}{c|c} B^p & x \\ \hline 0 & \lambda^p \end{array}\right),$$

where x is the $(n-1)$-vector $x = (B-\lambda)^{-1}(B^p - \lambda^p)z$.

(We call an $n \times n$ matrix *upper triangular* if the elements below its diagonal are all equal to 0.)

The following lemma is obtained from Lemma 3.5 by induction on the size of the matrix.

Lemma 3.6. *Let p and n be positive integers larger than 1. Let T be an upper triangular $n \times n$ matrix whose diagonal elements are $\mu_1, \ldots, \mu_n$, where the μ_i are distinct complex numbers. For $i = 1, 2, \ldots, n$, let λ_i be such that $(\lambda_i)^p = \mu_i$. Then there exists exactly one upper triangular $n \times n$ matrix R whose diagonal elements are $\lambda_1, \ldots, \lambda_n$, and for which $R^p = T$.*

With these preparatory lemmas out of the way, we proceed with some additional definitions needed to prove Theorem 3. The sequences of operators we shall consider can most easily be described matricially, so we assume given for each $n \in N$ an orthonormal basis X_n for $\mathfrak{H}_n$, and the matrices exhibited hereafter are to be regarded as the corresponding operators. For each pair (p, n) of integers with $p > 1$ and

$n \in N_p$, we define Q_n to be the unique operator on $\mathfrak{H}_n$ whose matrix is upper triangular and has the diagonal elements $(1+1/n)^{1/p}\,\omega_{pn}^k$ $(k=1, \ldots, n)$, satisfying

$$(Q_n)^p = (1+1/n)\begin{pmatrix} \omega_n^1 & n^{-\frac{1}{2}} & & & \\ & \omega_n^2 & n^{-\frac{1}{2}} & & \\ & & \ddots & \ddots & \\ & & & \omega_n^{n-1} & n^{-\frac{1}{2}} \\ & & & & \omega_n^n \end{pmatrix}$$

Next define (for each $p>1$) the sequence $\{c_n\}_{n\in N_p}$ of complex numbers by

$$c_n = \left(\frac{\omega_n^1 - 1}{\omega_{pn}^1 - \omega_p^1}\right) q_{1n}^{(n)}$$

Finally, (for each $p>1$) define the sequence $\{f_n\}_{n\in N_p}$ as follows:

γ) if the sequence $\{c_n\}_{n\in N_p}$ does not converge to zero, set $f_n=0$ for $n\in N_p$;

δ) if the sequence $\{c_n\}_{n\in N_p}$ converges to zero, set $f_n=0$ for all $n\in N_p$ such that $|c_n|>1$, and set $f_n=1-c_n$ for those $n\in N_p$ such that $|c_n|\leqq 1$.

Lemma 3. 7. *For each pair (p, n) with $p>1$ and $n\in N_p$, let A_n be the operator*

$$A_n = (1+1/n)\begin{pmatrix} \omega_n^1 & n^{-\frac{1}{2}} & & & f_n \\ & \omega_n^2 & n^{-\frac{1}{2}} & & \\ & & \ddots & \ddots & \\ & & & \omega_n^{n-1} & n^{-\frac{1}{2}} \\ & & & & \omega_n^n \end{pmatrix}$$

and let T_n be the unique operator of the form

$$T_n = (1+1/n)^{1/p}\begin{pmatrix} \omega_{pn}^1 & & & \\ & \omega_{pn}^2 & t_{ij}^{(n)} & \\ & & \ddots & \\ & & & \omega_{pn}^n \end{pmatrix}$$

satisfying $(T_n)^p = A_n$. Then $t_{1n}^{(n)} = (c_n + f_n)\left(\dfrac{\omega_{pn}^1 - \omega_p^1}{\omega_n^1 - 1}\right)$.

The proof of this lemma is an easy calculation using Lemmas 3. 5 and 3. 6 and is omitted.

4. The proof of Theorem 3

Note that if for each integer $p>1$ a sequence $\{t_n\}_{n\in N_p}$ has been defined, then these sequences give rise in an obvious way to a sequence $\{t_n\}_{n\in N}$. The following lemma proves the first half of Theorem 3.

Lemma 4. 1. *For each pair of integers* (p, n) *with* $p>1$ *and* $n\in N_p$, *let* $A_n\in\mathfrak{L}(\mathfrak{H}_n)$ *be as in Lemma* 3. 7. *Let* $n_0\geqq 3$ *be a fixed integer, and let* $B=\sum_{n\in N}\oplus B_n$ *belong to* $\mathfrak{I}$ *and satisfy:*

(ϱ) *for* $3\leqq n\leqq n_0$, *the eigenvalues of* B_n *are identical with those of* A_n, *and*

(τ) *for* $n>n_0$, $B_n=A_n$.

Then $B\in\mathfrak{R}$; *i.e., B is an invertible rootless operator.*

Proof. The inverse on $\mathfrak{H}_n$ of each A_n can be computed directly, and an easy calculation shows that

$$\|A_n^{-1}\|\leqq(1-1/\sqrt{n})^{-1}+|f_n|.$$

Since the sequence $\{f_n\}_{n\in N}$ is bounded by construction, $B^{-1}=\sum_{n\in N}\oplus B_n^{-1}\in\mathfrak{I}$. Now suppose that for some $p>1$ there is an operator $S\in\mathfrak{L}(\mathfrak{H})$ satisfying $S^p=B$. Since B satisfies the hypotheses of Corollary 3.4, $S\in\mathfrak{I}$, and we write $S=\sum_{n\in N}\oplus S_n$. By Corollary 3. 3, for $n>n_0$, S_n is in upper triangular form. Thus for each $n\in N_p$ satisfying $n>n_0$, let

$$S_n=(1+1/n)^{1/p}\begin{pmatrix}\lambda_n^1 & & & s_{ij}^{(n)}\\ & \lambda_n^2 & & \\ & & \ddots & \\ & & & \lambda_n^n\end{pmatrix}.$$

Direct computation shows that, for each n,

$$s_{i,i+1}=\left(\frac{\lambda_n^i-\lambda_n^{i+1}}{(\lambda_n^i)^p-(\lambda_n^{i+1})^p}\right)(1/\sqrt{n}),\qquad i=1,2,\ldots,n-1,$$

and we note that the $\{\lambda_n^r\}_{r=1}^n$ satisfy the hypotheses of Lemma 3. 1. Thus by Lemmas 3. 1 and 3. 6, for each $n\in N_p$ satisfying $n>n_0$

$$|s_{i,i+1}^{(n)}|\geqq\frac{\sqrt{n}\,|1-\omega_{3p}^2|}{2\pi}$$

for some i satisfying $1\leqq i\leqq n-1$ or $S_n=\omega_p^{k(n)}T_n$, where T_n is as defined in Lemma 3. 7 and $k(n)$ is some integer. Since $\|S_n\|\geqq|s_{i,i+1}^{(n)}|$ and S is assumed to be bounded, there must exist an integer $n_1\geqq n_0$ such that $S_n=\omega_p^{k(n)}T_n$ for all $n\in N_p$ satisfying $n>n_1$. But then by Lemma 3. 7,

$$s_{1n}^{(n)}=\omega_p^{k(n)}\frac{(c_n+f_n)(\omega_{pn}^1-\omega_p^1)}{(\omega_n^1-1)}\quad\text{for}\quad n\in N_p,\quad n>n_1,$$

and applying Lemma 3. 1 [case (β)],

$$|s_{1n}^{(n)}|\geqq\frac{|c_n+f_n|\cdot|1-\omega_{3p}^2|\,n}{2\pi}.$$

By construction, the sequence $\{c_n + f_n\}_{n \in N_p}$ does not converge to zero, so the sequence $\{s_{1n}^{(n)}\}_{n \in N_p}$ is unbounded, contradicting $S \in \mathfrak{L}(\mathfrak{H})$. Thus the lemma is proved.

The following lemma completes the proof of Theorem 3.

Lemma 4. 2. *For each $n \in N$ let $C_n \in \mathfrak{L}(\mathfrak{H}_n)$ be the operator*

$$C_n = (1 + 1/n) \begin{pmatrix} \omega_n^1 & & & & f_n \\ & \omega_n^2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \omega_n^n \end{pmatrix}$$

where f_n is as previously defined, and let $D_n \in \mathfrak{L}(\mathfrak{H}_n)$ be the operator

$$D_n = \log(1 + 1/n) I + \begin{pmatrix} 2\pi i/n & & & & & d_n \\ & 4\pi i/n & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & 2(n-1)\pi i/n & \\ & & & & & 0 \end{pmatrix},$$

where $d_n = \dfrac{2 f_n \pi i}{n(\omega_n^1 - 1)}$. *Then for $n \in N$,* $\exp(D_n) = C_n$ *and* $\|D_n\| \leqq 11$, *so that* $D = \sum_{n \in N} \oplus D_n \in \mathfrak{I}$, *and* $\exp(D) = C$.

Proof. Compute, using the fact that $|f_n| \leqq 2$.

Question. Is it possible for an invertible operator to have roots of all orders and yet fail to have a logarithm?

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(Received February 10, 1966)

Complete sets of unitary invariants for compact and trace-class operators

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I. Introduction

A complete set of unitary invariants for operators in a family $\mathscr{F}$ of operators on a (complex) Hilbert space is an indexed collection $\{O_\gamma\}_{\gamma\in\Gamma}$ of objects attached to each operator in $\mathscr{F}$ such that if $A, B\in\mathscr{F}$, then A is unitarily equivalent to B if and only if $O_\gamma(A)=O_\gamma(B)$ for each $\gamma\in\Gamma$.

For several families of operators complete sets of unitary invariants are known. For example, probably the best known family is the family of normal operators, where the theory of spectral multiplicity provides such a complete set of unitary invariants (see [2]). However, no complete set of unitary invariants has been found for arbitrary operators. The object of this paper is to solve the problem for compact operators. RADJAVI [5] has recently given a completely different characterization of unitary equivalence for compact operators.

The first problem which one encounters in trying to obtain a complete set of unitary invariants for compact operators on a Hilbert space $\mathfrak{H}$ is that of obtaining a complete set of unitary invariants for $n\times n$ matrices, that is, of solving the problem in the special case that $\mathfrak{H}$ is finite dimensional. Such a set of invariants was provided by SPECHT [7], who obtained the following result: Let Ω denote the free multiplicative semigroup in the free variables x and y. Two $n\times n$ matrices A and B are unitarily equivalent if and only if $t[\omega(A, A^*)]=t[\omega(B, B^*)]$ for each $\omega(x, y)\in\Omega$, where $t(A)$ denotes the trace of A in the usual sense.

PEARCY has shown in [4] that for each n there is a finite subset Ω_n of Ω (containing at most 4^{n^2} members) such that two $n\times n$ matrices A and B are unitarily equivalent if and only if $t[\omega(A, A^*)]=t[\omega(B, B^*)]$ for each $\omega(x, y)\in\Omega_n$. We shall refer to the above two sets of invariants as the Specht and Specht—Pearcy invariants, respectively.

Throughout this paper we shall denote the null space of an operator A by $\mathfrak{N}(A)$, the closure of the range of A by $\mathfrak{R}(A)$, and the operator $(A^*A)^{\frac{1}{2}}$ by $[A]$.

Since compact operators on a Hilbert space can be uniformly approximated by operators of finite rank, which are essentially operators on finite dimensional

*) This paper is essentially a thesis presented to the Rice University in partial fulfillment of the requirements for the degree of Doctor of Philosophy. The author wishes to thank his thesis director, Professor ARLEN BROWN, and also Professor CARL PEARCY for their helpful suggestions and constructive criticism.

spaces, it is reasonable to expect the above sets of invariants to provide some sort of complete sets of unitary invariants for compact operators. This is, indeed, the case. In § III we show that if the appropriate approximates of two compact operators A and B are unitarily equivalent and if $\dim [\mathfrak{N}(A) \cap \mathfrak{N}(A^*)] = \dim [\mathfrak{N}(B) \cap \mathfrak{N}(B^*)]$, then A and B are unitarily equivalent. This, together with the choice of approximate canonical approximating sequences, yields complete sets of unitary invariants for compact operators.

In § IV we make use of a class of compact operators on Hilbert space having well defined numerical traces. This class, called the trace class, has been studied extensively by SCHATTEN (see [6]). We show that if f is a strictly increasing continuous real valued function on the non-negative reals such that $f(0)=0$, then

$$\{t[f([A])\omega(A, A^*)] : \omega(x, y) \in \Omega\} \quad \text{and} \quad \dim [\mathfrak{N}(A) \cap \mathfrak{N}(A^*)]$$

form a complete set of unitary invariants for operators A such that $f([A])$ is a member of the trace class. With each compact operator A we associate a function f_A such that $f_A([A])$ is in the trace class and such that $f_A = f_B$ if A and B are unitarily equivalent; this then extends SPECHT's theorem to compact operators.

SPECHT's theorem extends more directly to the trace class. For this class

$$\{t[\omega(A, A^*)] : \omega(x, y) \in \Omega\} \quad \text{and} \quad \dim [\mathfrak{N}(A) \cap \mathfrak{N}(A^*)]$$

form a complete set of unitary invariants. The same result holds for the Schmidt-class (the class of Hilbert—Schmidt operators), except that the words x and y must be omitted.

II. Preliminaries

We say, as usual, that two operators A and B on a Hilbert space $\mathfrak{H}$ are unitarily equivalent if there is a unitary operator U on $\mathfrak{H}$ such that $UAU^* = B$.

We denote by $\mathfrak{S}(A)$ the subspace $\mathfrak{H} \ominus [\mathfrak{N}(A) \cap \mathfrak{N}(A^*)]$; the subspace $\mathfrak{N}(A) \cap \mathfrak{N}(A^*)$ is the largest subspace which reduces A and on which A is the zero operator.

Definition. Two operators A and B are *isometrically equivalent* if there is a partial isometry U with initial space $\mathfrak{S}(A)$ and final space $\mathfrak{S}(B)$ such that $UAU^* = B$ (or, equivalently, $UA = BU$).

If A and B are unitarily equivalent, say via a unitary operator U, then U maps $\mathfrak{N}(A) \cap \mathfrak{N}(A^*)$ isometrically onto $\mathfrak{N}(B) \cap \mathfrak{N}(B^*)$ and $\mathfrak{S}(A)$ isometrically onto $\mathfrak{S}(B)$, so that A and B are also isometrically equivalent and $\dim [\mathfrak{N}(A) \cap \mathfrak{N}(A^*)] = \dim [\mathfrak{N}(B) \cap \mathfrak{N}(B^*)]$. Conversely, if A and B are isometrically equivalent and if $\dim [\mathfrak{N}(A) \cap \mathfrak{N}(A^*)] = \dim [\mathfrak{N}(B) \cap \mathfrak{N}(B^*)]$, then it is obvious that A and B are unitarily equivalent.

An operator A on $\mathfrak{H}$ is said to be of finite rank if $\dim \mathfrak{R}(A) < \infty$. If $\{\varphi_i\}$ is an orthonormal basis for $\mathfrak{H}$, we define the trace $t(A)$ of an operator A of finite rank to be $\Sigma_i (A\varphi_i, \varphi_i)$. The sum is finite and is independent of the basis chosen (§ IV). If $\mathfrak{H}_1$ is an m-dimensional subspace of $\mathfrak{H}$ containing $\mathfrak{S}(A)$, we can choose $\{\varphi_i\}$ such that $\varphi_1, \ldots, \varphi_m$ is a basis for $\mathfrak{H}_1$. Then the trace of A is the trace of the restriction of A to $\mathfrak{H}_1$ as calculated for operators on finite dimensional spaces.

Let A and B be of finite rank and suppose that $t[\omega(A, A^*)] = t[\omega(B, B^*)]$ for each $\omega(x, y) \in \Omega$. Then, by SPECHT's theorem, there is a unitary operator U_1 defined on the subspace $\mathfrak{H}_1$ spanned by $\mathfrak{S}(A)$ and $\mathfrak{S}(B)$ which implements the unitary equivalence of the restrictions of A and B to $\mathfrak{H}_1$. The operator U which equals U_1 on $\mathfrak{H}_1$ and which equals the identity operator on $\mathfrak{H} \ominus \mathfrak{H}_1$ then implements the unitary equivalence of A and B.

If A and B are of finite rank and if $\dim \mathfrak{S}(A) = \dim \mathfrak{S}(B) = n$, there is a unitary operator V which maps $\mathfrak{S}(A)$ isometrically onto $\mathfrak{S}(B)$. If, in addition, the n-dimensional Specht—Pearcy invariants of A and B are equal, the restrictions of VAV^* and B to $\mathfrak{S}(B)$ are unitarily equivalent as operators on $\mathfrak{S}(B)$. Thus, as above, A is unitarily equivalent to B.

We summarize these results in the following

Lemma 2. 1. *Each of the following is a complete set of unitary invariants for operators A of finite rank:*

(1) $\{t\{\omega[A, A^*)]: \omega(x, y) \in \Omega\}$

(2) $\dim \mathfrak{S}(A)$ and $\{t[\omega(A, A^*)]: \omega(x, y) \in \Omega_{\dim \mathfrak{S}(A)}\}$.

III. Unitary equivalence of compact operators

In this section we establish a sort of "continuity" property for isometric equivalence and then use this result to obtain complete sets of unitary invariants for compact operators.

Lemma 3. 1. *Suppose that P and Q are projections of finite rank and that $\{P_n\}$ and $\{Q_n\}$ are sequences of projections converging uniformly to P and Q, respectively. Suppose also that for each n there is a partial isometry U_n whose initial space contains $\mathfrak{R}(P_n)$ and whose final space contains $\mathfrak{R}(Q_n)$ such that $U_n P_n = Q_n U_n$. Then there is a subsequence $\{U_{n_k}\}$ of $\{U_n\}$ such that the sequence of the restrictions of the U_{n_k}'s to $\mathfrak{R}(P)$ converges to a linear map sending $\mathfrak{R}(P)$ isometrically onto $\mathfrak{R}(Q)$.*

Proof. Let $x_1, \ldots, x_p$ be an orthonormal basis of $\mathfrak{R}(P)$. It suffices to find a subsequence $\{U_{n_k}\}$ of $\{U_n\}$ such that $U_{n_k} x_i \to y_i$ strongly, $i = 1, \ldots, p$, where $y_1, \ldots, y_p$ is some orthonormal basis of $\mathfrak{R}(Q)$. Since $QU_n x_i \in \mathfrak{R}(Q)$, which is finite dimensional, and since

$$1 \geqq \|QU_n x_i\| = \|U_n P_n x_i - U_n P_n x_i + QU_n x_i\| = \|U_n P_n x_i - Q_n U_n x_i + QU_n x_i\| \geqq$$
$$\geqq \|U_n P_n x_i\| - \|Q_n U_n x_i - QU_n x_i\| = \|P_n x_i\| - \|(Q_n - Q)U_n x_i\| =$$
$$= \|Px_i + P_n x_i - Px_i\| - \|(Q_n - Q)U_n x_i\| \geqq 1 - (\|P_n - P\| + \|Q_n - Q\|) \to 1,$$

there is a subsequence $\{U_{n_k}\}$ of $\{U_n\}$ such that $QU_{n_k} x_i \to y_i$, $i = 1, \ldots, p$, and $\|y_i\| = 1$. Moreover,

$$0 \leqq \|U_n x_i - QU_n x_i\| = \|U_n P x_i - U_n P_n x_i + Q_n U_n x_i - QU_n x_i\| \leqq$$
$$\leqq \|U_n(P - P_n)x_i\| + \|(Q_n - Q)U_n x_i\| \to 0,$$

so

$$U_{n_k} x_i \to y_i, \quad i = 1, \ldots, p.$$

Also,

$$\| U_n P_n x_i - U_n x_i \| = \| U_n (P_n - P) x_i \| \to 0,$$

so

$$U_{n_k} P_{n_k} x_i \to y_i .$$

If $i \neq j$,

$$|(U_n P_n x_i, U_n P_n x_j)| = |(P_n x_i, P_n x_j)| =$$

$$= |(P_n x_i, x_j) - (x_i, x_j)| = |([P_n - P] x_i, x_j)| \leqq \| P_n - P \| \to 0,$$

and hence

$$(y_i, y_j) = 0.$$

Since, from [1], p. 73, if $\|P_n - P\| < 1$ and $\|Q_n - Q\| < 1$, then

$$\dim \mathfrak{R}(P) = \dim \mathfrak{R}(P_n) = \dim \mathfrak{R}(Q_n) = \dim \mathfrak{R}(Q),$$

it follows that $y_1, \ldots, y_p$ is a basis of $\mathfrak{R}(Q)$. This completes the proof of the lemma.

Lemma 3. 2. *Suppose that $\{P_k\}$ and $\{Q_k\}$ are sequences of projections of finite rank and that, for each k, $\{P_{k,n}\}$ and $\{Q_{k,n}\}$ are sequences of projections converging in the uniform topology to P_k and Q_k, respectively. Suppose also that, for each n, there is a partial isometry U_n whose initial space contains $\mathfrak{R}(P_{k,n})$ and whose final space contains $\mathfrak{R}(Q_{k,n})$ such that, for each k, $U_n P_{k,n} = Q_{k,n} U_n$. Then there is a partial isometry U such that for each k the initial space of U contains $\mathfrak{R}(P_k)$ and the final space of U contains $\mathfrak{R}(Q_k)$ and such that $UP_k = Q_k U$.*

Proof. We first choose subsequences $\{U_n^{(r)}\}$ of $\{U_n\}$ inductively. Let $\{U_n^{(0)}\} = \{U_n\}$, and suppose that $\{U_n^{(0)}\}, \ldots, \{U_n^{(r)}\}$ have been chosen. By lemma 3. 1, we may choose $\{U_n^{(r+1)}\}$ to be a subsequence of $\{U_n^{(r)}\}$ converging uniformly on $\mathfrak{R}(P_{r+1})$. The diagonal sequence $\{U_n^{(n)}\}$ converges on $\mathfrak{R}(P_k)$ to a map sending $\mathfrak{R}(P_k)$ isometrically onto $\mathfrak{R}(Q_k)$ for each k. Let $\mathfrak{M}$ be the submanifold spanned by $\{\mathfrak{R}(P_k)\}_{k=1}^{\infty}$ and let $x \in \mathfrak{M}$, say $x = x_1 + \ldots + x_r$, where $x_k \in \mathfrak{R}(P_k)$, $k = 1, \ldots, r$. Since the sequence of vectors $\{U_n^{(n)} x_k\}_{n=1}^{\infty}$ converges strongly for each $k = 1, \ldots, r$, and since $U_n^{(n)} x = U_n^{(n)} x_1 + \ldots + U_n^{(n)} x_r$, the sequence of operators $\{U_n^{(n)}\}$ converges strongly on $\mathfrak{M}$ to an operator U_0 (defined on $\mathfrak{M}$) such that $U_0 P_k = Q_k U_0$, $k = 1, 2, \ldots$. Also, setting

$$\varepsilon_n = \|P_1 - P_{1,n}\| \, \|x_1\| + \ldots + \|P_r - P_{r,n}\| \, \|x_r\|,$$

we have

$$\|x\| \geqq \| U_n x \| = \| U_n x_1 + \ldots + U_n x_r \| =$$

$$= \| U_n [P_{1,n} x_1 + \ldots + P_{r,n} x_r] + U_n [(P_1 - P_{1,n}) x_1 + \ldots + (P_r - P_{r,n}) x_r] \| \geqq$$

$$\geqq \| U_n [P_{1,n} x_1 + \ldots + P_{r,n} x_r] \| - \varepsilon_n = \| P_{1,n} x_1 + \ldots + P_{r,n} x_r \| - \varepsilon_n =$$

$$= \| x_1 + \ldots + x_r + (P_{1,n} - P_1) x_1 + \ldots + (P_{r,n} - P_r) x_r \| - \varepsilon_n \geqq$$

$$\geqq \| x_1 + \ldots + x_r \| - 2\varepsilon_n \to \| x \|,$$

so $\| U_0 x \| = \| x \|$. The extension U of U_0 defined by continuity on the closure of $\mathfrak{M}$ and defined to be zero on $\mathfrak{H} \ominus \mathfrak{M}$ has the desired properties.

Theorem 1. *Let A and B be compact operators on a Hilbert space. If there exist sequences $\{A_n\}$ and $\{B_n\}$ of (not necessarily compact) operators converging uniformly to A and B, respectively, such that, for each n, A_n is isometrically equivalent to B_n, then A is isometrically equivalent to B.*

Proof. We denote by Σ_A the spectrum of A, by $\operatorname{Re} A$ the operator $(A+A^*)/2$, and by $\operatorname{Im} A$ the operator $(A-A^*)/2i$. If Δ is a Borel subset of the line, we denote by $E_n(\Delta)$, $E(\Delta)$, $F_n(\Delta)$, $F(\Delta)$, $G_n(\Delta)$, $G(\Delta)$, $H_n(\Delta)$ and $H(\Delta)$ the spectral projections of $\operatorname{Re} A_n$, $\operatorname{Re} A$, $\operatorname{Im} A_n$, $\operatorname{Im} A$, $\operatorname{Re} B_n$, $\operatorname{Re} B$, $\operatorname{Im} B_n$, and $\operatorname{Im} B$, respectively, associated with Δ. Since A_n is isometrically equivalent to B_n, there is a partial isometry U_n with initial space $\mathfrak{S}(A_n)$ and final space $\mathfrak{S}(B_n)$ such that $U_nA_n=B_nU_n$. If Δ is any Borel subset of the line not containing zero, $\mathfrak{R}[E_n(\Delta)]$ and $\mathfrak{R}[F_n(\Delta)]$ are contained in $\mathfrak{S}(A_n)$, and $\mathfrak{R}[G_n(\Delta)]$ and $\mathfrak{R}[H_n(\Delta)]$ are contained in $\mathfrak{S}(B_n)$. As in the case of unitary equivalence, $U_nE_n(\Delta)=G_n(\Delta)U_n$ and $U_nF_n(\Delta)=H_n(\Delta)U_n$. In order to show that A is isometrically equivalent to B, it suffices to show that there is a partial isometry U with initial space $\mathfrak{S}(A)$ and final space $\mathfrak{S}(B)$ such that $UE(\Delta)=G(\Delta)U$ and $UF(\Delta)=H(\Delta)U$ for all Borel subsets of the line not containing zero. In fact, since each non-zero member of $\Sigma_{\operatorname{Re} A}$ or $\Sigma_{\operatorname{Im} A}$ is isolated, it suffices to show that $\Sigma_{\operatorname{Re} A}=\Sigma_{\operatorname{Re} B}$, $\Sigma_{\operatorname{Im} A}=\Sigma_{\operatorname{Im} B}$, and that if $\lambda\neq 0$ then $UE[(\lambda-\varepsilon,\lambda+\varepsilon)]=G[(\lambda-\varepsilon,\lambda+\varepsilon)]U$ and $UF[(\lambda-\varepsilon,\lambda+\varepsilon)]=H[(\lambda-\varepsilon,\lambda+\varepsilon)]U$ for all sufficiently small $\varepsilon>0$.

We first show that if $\lambda\neq 0$, then $\lambda\in\Sigma_{\operatorname{Re} A}$ if and only if for each $\varepsilon>0$, $E_n[(\lambda-\varepsilon,\lambda+\varepsilon)]\neq 0$ for $n>n_0(\varepsilon)$. This and the analogous results for $\Sigma_{\operatorname{Im} A}$, $\Sigma_{\operatorname{Re} B}$ and $\Sigma_{\operatorname{Im} B}$ guarantee that $\Sigma_{\operatorname{Re} A}=\Sigma_{\operatorname{Re} B}$ and $\Sigma_{\operatorname{Im} A}=\Sigma_{\operatorname{Im} B}$.

If $\lambda\notin\Sigma_{\operatorname{Re} A}$, let ε be less than the distance d from λ to $\Sigma_{\operatorname{Re} A}$. Then $\|(\zeta-\operatorname{Re} A)^{-1}\|$ is bounded for $|\zeta-\lambda|<\varepsilon$, say by M. One can easily see by power series expansions that if $\|\operatorname{Re} A_n-\operatorname{Re} A\|<1/M$, then $(\zeta-\operatorname{Re} A_n)$ is invertible, so that the interval $(\lambda-\varepsilon,\lambda+\varepsilon)$ contains no points of $\Sigma_{\operatorname{Re} A_n}$.

If $\lambda\in\Sigma_{\operatorname{Re} A}$, $\lambda\neq 0$, let d be the distance from λ to $\Sigma_{\operatorname{Re} A}-\{\lambda\}$; d is positive since A is compact. We shall show that $E_n[(\lambda-\varepsilon,\lambda+\varepsilon)]\to E[(\lambda-\varepsilon,\lambda+\varepsilon)]$ uniformly, at least for $0<\varepsilon<d/3$. As above, the intervals $(\lambda-2d/3,\lambda-\varepsilon)$ and $(\lambda+\varepsilon,\lambda+2d/3)$ contain no points of $\Sigma_{\operatorname{Re} A_n}$ for n sufficiently large. Thus

$$E_n[(\lambda-\varepsilon,\lambda+\varepsilon)] = (1/2\pi i)\oint_C (\zeta-\operatorname{Re} A_n)^{-1}\,d\zeta$$

and

$$E[(\lambda-\varepsilon,\lambda+\varepsilon)] = (1/2\pi i)\oint_C (\zeta-\operatorname{Re} A)^{-1}\,d\zeta,$$

where C is the circle $|\zeta-\lambda|=d/2$. Since inversion is a continuous operation where it is defined,

$$\|E_n[(\lambda-\varepsilon,\lambda+\varepsilon)]-E[(\lambda-\varepsilon,\lambda+\varepsilon)]\| =$$
$$= (1/2\pi)\left\|\oint_C [(\zeta-\operatorname{Re} A_n)^{-1}-(\zeta-\operatorname{Re} A)^{-1}]\,d\zeta\right\| \leqq$$
$$\leqq (1/2\pi)\oint_C \|(\zeta-\operatorname{Re} A_n)^{-1}-(\zeta-\operatorname{Re} A)^{-1}\|\,|d\zeta|\to 0 \quad\text{as}\quad n\to\infty.$$

Now, let $\alpha_1,\alpha_2,\ldots$ and $\beta_1,\beta_2,\ldots$ be the distinct non-zero members of $\Sigma_{\operatorname{Re} A}$ and $\Sigma_{\operatorname{Im} A}$, respectively, $\Delta_k=(\alpha_k-d/3,\alpha_k+d/3)$ where d is the distance from

α_k to $\Sigma_{\mathrm{Re}\,A}-\{\alpha_k\}$, and $\Delta_k'=(\beta_k-d/3,\ \beta_k+d/3)$, where d is the distance from β_k to $\Sigma_{\mathrm{Im}\,A}-\{\beta_k\}$. Set

$$P_{2k-1}=E(\Delta_k), \qquad P_{2k-1,\,n}=E_n(\Delta_k),$$
$$P_{2k}=F(\Delta_k'), \qquad P_{2k,\,n}=F_n(\Delta_k'),$$
$$Q_{2k-1}=G(\Delta_k), \qquad Q_{2k-1,\,n}=G_n(\Delta_k),$$
$$Q_{2k}=H(\Delta_k'), \qquad Q_{2k,\,n}=H_n(\Delta_k').$$

An application of lemma 3.2 completes the proof.

We now apply the preceding results to the problem of obtaining complete sets of unitary invariants for compact operators on Hilbert space. For this purpose, let A and B be any two compact operators on a Hilbert space $\mathfrak{H}$. We order the distinct non-zero eigenvalues of Re A, Im A, Re B and Im B and denote these sequences by $\{\alpha_k\}$, $\{\beta_k\}$, $\{\gamma_k\}$ and $\{\delta_k\}$, respectively. We require of the orderings that $|\alpha_k|\geqq|\alpha_{k+1}|$, that $|\alpha_k|=|\alpha_{k+1}|$ implies that $\alpha_k>0$ and $\alpha_{k+1}<0$, and analogously for $\{\beta_k\}$, $\{\gamma_k\}$ and $\{\delta_k\}$. (This guarantees that if $\Sigma_{\mathrm{Re}\,A}=\Sigma_{\mathrm{Re}\,B}$, then the sequences $\{\alpha_k\}$ and $\{\gamma_k\}$ are identical, and similarly for $\Sigma_{\mathrm{Im}\,A}$ and $\Sigma_{\mathrm{Im}\,B}$.) If E_k, F_k, G_k, and H_k are the spectral projections of Re A, Im A, Re B and Im B corresponding to α_k, β_k, γ_k and δ_k, respectively, then A and B can be written $A=\sum_k\alpha_kE_k+i\sum_k\beta_kF_k$ and $B=\sum_k\gamma_kG_k+i\sum_k\delta_kH_k$. We write $A_n=\sum_{k=1}^n\alpha_kE_k+i\sum_{k=1}^n\beta_kF_k$ and $B_n=\sum_{k=1}^n\gamma_kG_k+i\sum_{k=1}^n\delta_kH_k$, with obvious modifications if any of the sequences are finite. Then $\{A_n\}$ and $\{B_n\}$ converge uniformly to A and B, respectively.

Now suppose that A is isometrically equivalent to B, say $UAU^*=B$. Then $B=\sum_k\alpha_kUE_kU^*+i\sum_k\beta_kUF_kU^*$. Thus, since the spectral representation of an operator is unique, $\alpha_k=\gamma_k$, $\beta_k=\delta_k$, $UE_kU^*=G_k$, and $UF_kU^*=H_k$ for each k. It follows that $UA_nU^*=B_n$, so for each n, A_n is unitarily equivalent to B_n. On the other hand, if, for each n, A_n is unitarily equivalent to B_n, then A is isometrically equivalent to B by theorem 1. We have thus proved

Theorem 2. *Let A be compact, let the sequence $\{A_n\}$ be obtained from the Cartesian decomposition of A as described above, and let I be either of the complete sets of unitary invariants for operators of finite rank described in lemma 2. 1. Then $\{I(A_n)\}_{n=1}^\infty$ is a complete set of isometric invariants for A. The addition of* dim $[\mathfrak{N}(A)\cap\mathfrak{N}(A^*)]$ *to the above collection of isometric invariants yields a complete set of unitary invariants for A.*

A different complete set of unitary invariants can be obtained by using the polar decomposition of a compact operator to obtain a canonical set of approximating operators of finite rank. Let $\{\mu_k\}$ be the non-zero eigenvalues of $[A]$, $\mu_1>\mu_2>\dots$, and let E_k be the (finite dimensional) spectral projection of $[A]$ associated with μ_k. Then the series $\sum_k\mu_kE_k$ converges to $[A]$ in the uniform topology. Let $A=W[A]$ be the polar decomposition of A, and denote by W_k the partial isometry of finite rank WE_k. The series $\sum_k\mu_kW_k=\sum_k\mu_kWE_k=W\sum_k\mu_kE_k$ converges to A in the uniform topology. Let $B=\sum_k \nu_k V_k$ in a similar fashion, and suppose U implements the isometric equivalence of A and B, $UAU^*=B$. Let T_k be the partial isometry UW_kU^* and let F_k be the projection on the initial space of T_k. The series $\sum_k T_k$ converges in the strong operator topology to a partial isometry T and

$$B = U(\sum_k \mu_k W_k)U^* = \sum_k \mu_k UW_k U^* = \sum_k \mu_k T_k = \sum_k \mu_k T_k F_k = T\sum_k \mu_k F_k.$$

The operator $\sum_k \mu_k F_k$ is positive, so, by the unicity of the polar decomposition of an operator, $\nu_k=\mu_k$ and $V_k=T_k=UW_kU^*$. Thus, if $A_n=\sum_{k=1}^{n}\mu_k W_k$ and $B_n=\sum_{k=1}^{n}\nu_k V_k$, we have $UA_nU^*=B_n$. Conversely, $\{A_n\}$ and $\{B_n\}$ converge uniformly to A and B, respectively, so, by theorem 1, we have

Theorem 3. *Let A be compact, let $A_n=\sum_{k=1}^{n}\mu_k W_k$ be obtained from the polar decomposition of A as described above, and let I be either of the complete sets of unitary invariants for operators of finite rank described in lemma 2. 1. Then $\{I(A_n)\}_{n=1}^{\infty}$ is a complete set of isometric invariants for A. The addition of* $\dim[\mathfrak{N}(A)\cap\mathfrak{N}(A^*)]$ *to the above collection of isometric invariants yields a complete set of unitary invariants for A.*

IV. Unitary invariants involving traces

Before discussing the Schmidt- and trace-classes of operators we prove a lemma which will be useful in the proof of theorem 4.

Lemma 4. 1. *Suppose that $\{a_k\}$ and $\{b_k\}$ are sequences of complex numbers, that $\{\mu_k\}$ and $\{\nu_k\}$ are strictly decreasing sequences of real numbers converging to zero, and that $\sum_k |a_k|\mu_k^2<\infty$ and $\sum_k |b_k|\nu_k^2<\infty$. Suppose also that, for each positive integer p, $\sum_k a_k\mu_k^{2p}=\sum_k b_k\nu_k^{2p}$. Then:*

(1) *If, for each k, $a_k, b_k\neq 0$, then $a_k=b_k$ and $\mu_k=\nu_k$ for each k.*

(2) *If $\mu_{k_1}=\nu_{k_1}$, then $a_{k_1}=b_{k_1}$.*

Proof. The series $\sum_k a_k\mu_k^2/(z^2-\mu_k^2)$ converges uniformly in any domain in which z is uniformly bounded away from $\{\pm\mu_k\}$ to a function which we shall denote $f(z)$, and similarly for $g(z)=\sum_k b_k\nu_k^2/(z^2-\nu_k^2)$. $f(z)$ has a pole of order one and residue $\pm\frac{1}{2}a_k\mu_k$ at $z=\pm\mu_k$ for each k such that $a_k\neq 0$, a limit point of poles at $z=0$, and is holomorphic elsewhere; $g(z)$ has a pole of order one and residue $\pm\frac{1}{2}b_k\nu_k$ at $z=\pm\nu_k$ for each k such that $b_k\neq 0$, a limit point of poles at $z=0$, and is holomorphic elsewhere. For z in the domain $\{z:|z|>\mu_1\}$ we can expand $\mu_k^2/(z^2-\mu_k^2)$ about $z=\infty$ to obtain

$$f(z)=\sum_k a_k\sum_p(\mu_k/z)^{2p}.$$

In order to change the order of summation, we note that

$$\sum_k |a_k| \sum_p (\mu_k/|z|)^{2p} = \sum_k |a_k| \mu_k^2/(|z|^2-\mu_k^2) \leqq (\sum_k |a_k| \mu_k^2)/(|z|^2-\mu_1^2) < \infty;$$

thus

$$f(z) = \sum_p (\sum_k a_k \mu_k^{2p}) 1/z^{2p}.$$

Similarly

$$g(z) = \sum_p (\sum_k b_k v_k^{2p}) 1/z^{2p}$$

for $|z| > v_1$. Thus, by hypothesis, $f(z)=g(z)$ for $|z|>\max(\mu_1, v_1)$. By analytic continuation, $f(z)$ and $g(z)$ are identical, and the conclusion of the lemma follows.

The reader is referred to [6] for the proofs of the following and other interesting facts about the trace- and Schmidt-classes.

Let A be an operator on a Hilbert space $\mathfrak{H}$ and let $\{\varphi_i\}$ be an orthonormal basis of $\mathfrak{H}$. A is in the *Schmidt-class* (σc) if $\sum_i \|A\varphi_i\|^2 < \infty$; the sum is independent of the basis chosen. The Schmidt-class is a proper subset of the set of compact operators. If $\mathfrak{H}$ is L_2 of the unit interval, (σc) consists of all operators of the form

$$(Af)(x) = \int K(x, y) f(y)\, dy$$

where $K(x, y)$ is in L_2 of the unit square.

An operator A is in the *trace-class* (τc) if A is the product of two members of the Schmidt-class. The following are equivalent:

(1) $A \in (\tau c)$,

(2) $[A] \in (\tau c)$,

(3) $[A]^{\frac{1}{2}} \in (\sigma c)$,

(4) $\sum_i ([A]\varphi_i, \varphi_i) < \infty$ for some, and thus every, orthonormal basis $\{\varphi_i\}$ of $\mathfrak{H}$.

If A is in the trace class and $\{\varphi_i\}$ is an orthonormal basis of $\mathfrak{H}$, then $\sum |(A\varphi_i, \varphi_i)| < \infty$. The trace $t(A) = \sum (A\varphi_i, \varphi_i)$ of A is independent of the basis with respect to which it is computed. If $A, B \in (\tau c)$, X is any bounded operator, and c is a complex number, then

(1) $t(A^*) = \overline{t(A)}$,

(2) $t(cA) = ct(A)$,

(3) $(A+B) \in (\tau c)$ and $t(A+B) = t(A) + t(B)$,

(4) $AX, XA \in (\tau c)$ and $t(AX) = t(XA)$ (the traces of commutators are zero).

Definition. Let f be any continuous strictly increasing real valued function on the non-negative real numbers such that $f(0)=0$. The class $(\tau c)_f$ is the set of all operators A such that $f([A]) \in (\tau c)$.

It is easy to see that an operator A is compact if and only if $f([A])$ is compact; thus $(\tau c)_f$ is a subset of the compacts. If A is compact, $A = \sum_k \mu_k W_k$ as in § III, then $[A] = \sum_k \mu_k E_k$, where E_k is the projection $W_k^* W_k$. We denote by f_A the convex support (see [3]) of the set of points $(\mu_k, 1/(k^2 \dim [\mathfrak{R}(E_k)]))$. If $\{\varphi_i\}$ is an orthonormal basis of $\mathfrak{H}$ consisting of eigenvectors of $[A]$, then

$$\sum_i (f([A])\varphi_i, \varphi_i) = \sum_k \dim [\mathfrak{R}(E_k)] f(\mu_k) \leqq \sum_k \dim [\mathfrak{R}(E_k)]/(k^2 \dim [\mathfrak{R}(E_k)]) < \infty,$$

so $A \in (\tau c)_{f_A}$. If A and B are compact and unitarily equivalent, then so are $[A]$ and $[B]$, so $f_A = f_B$. Thus, if I_f is a complete set of unitary invariants for $(\tau c)_f$, f_A and $I_{f_A}(A)$ form a complete set of unitary invariants for all compact operators.

Although we shall not need to make use of this fact, we note that an easy application of lemma 4. 1 shows that $\{t[(f(A))^n]\}_{n=1}^{\infty}$ is a complete set of isometric invariants for the positive members of $(\tau c)_f$.

Theorem 4. *Let Ω denote the free multiplicative semigroup of all words $\omega(x, y)$ in the free variables x and y. A complete set of isometric invariants for operators A in $(\tau c)_f$ is*

$$\{t[f([A])\omega(A, A^*)]: \omega(x, y) \in \Omega\}.$$

The addition of $\dim [\mathfrak{N}(A) \cap \mathfrak{N}(A^*)]$ *to the above set of isometric invariants yields a complete set of unitary invariants for $(\tau c)_f$.*

Proof. Since traces are independent of the bases with respect to which they are computed and since $t[f([A])\omega(A, A^*)]$ is not affected by the dimension of $\mathfrak{N}(A) \cap \cap \mathfrak{N}(A^*)$, $t[f([A])\omega(A, A^*)]$ is preserved under isometric equivalence.

Now suppose that A and B are in $(\tau c)_f$ and that $t[f([A])\omega(A, A^*)] = t[f([B]) \cdot \omega(B, B^*)]$ for each $\omega(x, y) \in \Omega$. Let $A = \sum_k \mu_k W_k$, $A_n = \sum_{k=1}^{n} \mu_k W_k$, $B = \sum_k \nu_k V_k$, and $B_n = \sum_{k=1}^{n} \nu_k V_k$ as in § III. By theorem 3, it suffices to show that $t[\omega(A_n, A_n^*)] = t[\omega(B_n, B_n^*)]$ for each $\omega(x, y) \in \Omega$ and each n.

We first show that $\mu_k = \nu_k$ for each k. Choose an orthonormal set of vectors $\{\varphi_i\}$ such that $\varphi_{i_k}, \ldots, \varphi_{i_{k+1}-1}$ is a basis of the initial space of W_k. Since $f([A])(A^*A)^p = \sum_k f(\mu_k)\mu_k^{2p} W_k^* W_k$ is in (τc), we have, for each positive integer p,

$$t[f([A])(A^*A)^p] = \sum_i (f([A])(A^*A)^p \varphi_i, \varphi_i) =$$
$$= \sum_k \sum_{i=i_k}^{i_{k+1}-1} (f([A])(A^*A)^p \varphi_i, \varphi_i) = \sum_k f(\mu_k) t(W_k^* W_k) \mu_k^{2p}.$$

Similarly

$$t[f([B])(B^*B)^p] = \sum_k f(\nu_k) t(V_k^* V_k) \nu_k^{2p}.$$

Setting $a_k = f(\mu_k) t(W_k^* W_k) \neq 0$ and $b_k = f(\nu_k) t(V_k^* V_k) \neq 0$, we conclude from lemma 4. 1 that $\mu_k = \nu_k$ for each k.

For each $\omega(x, y) \in \Omega$ we write $\omega(x, y) = \prod_{j=1}^{r} z_j$, where $z_j = x$ or $z_j = y$. Since the traces of commutators are zero and the trace of the adjoint of an operator is the complex conjugate of the trace of the operator, it suffices to show that $t[\omega(A_n, A_n^*)] = t[\omega(B_n, B_n^*)]$ for each ω such that $z_1 = y$.

In an induction argument later in the proof we shall consider products involving not only A and A^* but also the partial isometries $W_1, W_1^*, W_2, W_2^*, \ldots$, and the corresponding products involving $B, B^*, V_1, V_1^*, V_2, V_2^*, \ldots$. For this purpose we introduce the free semigroup $\hat{\Omega}$ of words $\hat{\omega}(x, y, x_1, y_1, x_2, y_2, \ldots) = \prod_{j=1}^{m} \zeta_j$ where $\zeta_j \in \{x, y, x_1, y_1, x_2, y_2, \ldots\}$. Denote by $\lambda(\hat{\omega})$ the number of j's, $1 \leqq j \leqq m$, such that $\zeta_j \in \{x_1, y_1, x_2, y_2, \ldots\}$. (Thus if no ζ_j is equal to x or y, $\lambda(\hat{\omega})$ is the length m of $\hat{\omega}$.) For simplicity of notation we write

$$\hat{\omega}(A) = \hat{\omega}(A, A^*, W_1, W_1^*, W_2, W_2^*, \ldots\}$$

and

$$\hat{\omega}(B) = \hat{\omega}(B, B^*, V_1, V_1^*, V_2, V_2^*, \ldots\}.$$

With each $\omega(x, y) = \prod_{j=1}^{r} z_j \in \Omega$ and each r-tuple $k_1, \ldots, k_r$ of positive integers we associate the member $\hat{\omega}_{\omega, k_1, \ldots, k_r}(x, y, x_1, y_1, \ldots) = \prod_{j=1}^{r} \zeta_j$ of $\hat{\Omega}$ such that $\zeta_j = x_{k_j}$ if $z_j = x$ and $\zeta_j = y_{k_j}$ if $z_j = y$. Then

$$\omega(A_n, A_n^*) = \sum_{k_1, \ldots, k_r = 1}^{n} \mu_{k_1} \ldots \mu_{k_r} \hat{\omega}_{\omega, k_1, \ldots, k_r}(A)$$

and

$$\omega(B_n, B_n^*) = \sum_{k_1, \ldots, k_r = 1}^{n} \nu_{k_1} \ldots \nu_{k_r} \hat{\omega}_{\omega, k_1, \ldots, k_r}(B).$$

We now give an example to illustrate the notation introduced above. If $\omega(x, y) = y^2 x$, the word $\hat{\omega}_{y^2 x, k_1, k_2, k_3}(x, y, x_1, y_1, \ldots)$ is then $y_{k_1} y_{k_2} x_{k_3}$. We have

$$\omega(A_n, A_n^*) = \left(\sum_{k_1=1}^{n} \mu_{k_1} A_{k_1}^* \right) \left(\sum_{k_2=1}^{n} \mu_{k_2} A_{k_2}^* \right) \left(\sum_{k_3=1}^{n} \mu_{k_3} A_{k_3} \right) =$$

$$= \sum_{k_1, k_2, k_3 = 1}^{n} \mu_{k_1} \mu_{k_2} \mu_{k_3} A_{k_1}^* A_{k_2}^* A_{k_3} = \sum_{k_1, k_2, k_3 = 1}^{n} \mu_{k_1} \mu_{k_2} \mu_{k_3} \hat{\omega}_{y^2 x, k_1, k_2, k_3}(A).$$

Since we already know that $\mu_k = \nu_k$ for all k, it suffices to show that $t[\hat{\omega}_{\omega, k_1, \ldots, k_r}(A)] = t[\hat{\omega}_{\omega, k_1, \ldots, k_r}(B)]$ for all $\omega(x, y) \in \Omega$ such that $z_1 = y$ and for all $k_1, \ldots, k_r$; that is, that $t[\hat{\omega}(A)] = t[\hat{\omega}(B)]$ for all $\hat{\omega} = \prod_{j=1}^{r} \zeta_j$ such that $\zeta_j \in \{x_1, y_1, x_2, y_2, \ldots\}$ and $\zeta_1 = y_{k_1}$. We note that for such an $\hat{\omega}$, since $W_k^* W_k W_{k_1}^* = \delta_{k, k_1} W_{k_1}$,

$$f([A]) \hat{\omega}(A) = \sum_k f(\mu_k) W_k^* W_k \hat{\omega}(A) = f(\mu_{k_1}) \hat{\omega}(A);$$

similarly,

$$f([B]) \hat{\omega}(B) = f(\nu_{k_1}) \hat{\omega}(B) = f(\mu_{k_1}) \hat{\omega}(B).$$

Thus, for such an $\hat{\omega}$, if $t[f([A])\hat{\omega}(A)] = t[f([B])\hat{\omega}(B)]$, then $t[\hat{\omega}(A)] = t[\hat{\omega}(B)]$. We conclude the proof by proving the following by induction on $\lambda(\hat{\omega})$:

($*$) If $\hat{\omega} \in \hat{\Omega}$, then $t[f([A])\hat{\omega}(A)] = t[f([B])\hat{\omega}(B)]$.

Note that, since then traces of commutators are zero, if ($*$) holds for all $\hat{\omega} \in \hat{\Omega}$ such that $\lambda(\hat{\omega}) = q$, then $t[\hat{\omega}(A) f([A])] = t[\hat{\omega}(B) f([B])]$ if $\lambda(\hat{\omega}) = q$, and $t[\hat{\omega}_1(A) \cdot f([A]) \hat{\omega}_2(A)] = t[\hat{\omega}_1(B) f([B]) \hat{\omega}_2(B)]$ if $\lambda(\hat{\omega}_1) + \lambda(\hat{\omega}_2) = q$.

If $\lambda(\hat{\omega})=0$, then there is an $\omega(x, y)\in\Omega$ such that $\hat{\omega}(A)=\omega(A, A^*)$ and $\hat{\omega}(B)=\omega(B, B^*)$, so $(\ast)$ is true by hypothesis.

We now suppose that $(\ast)$ holds for $\lambda=q$, that $\lambda(\hat{\omega})=q+1$, and prove that $t[f([A])\hat{\omega}(A)]=t[f([B])\hat{\omega}(B)]$. By taking adjoints if necessary and using the fact that the traces of commutators are zero, it suffices to show that $t[CW_k]=t[DV_k]$ in the three cases

(i) $C=f([A])\hat{\omega}_0(A),\quad D=f([B])\hat{\omega}_0(B),\ \lambda(\hat{\omega}_0)=q,$

(ii) $C=\hat{\omega}_0(A)f([A]),\quad D=\hat{\omega}_0(B)f([B]),\ \lambda(\hat{\omega}_0)=q,$

and

(iii) $C=\hat{\omega}_1(A)f([A])\hat{\omega}_2(A),\quad D=\hat{\omega}_1(B)f([B])\hat{\omega}_2(B),\quad \lambda(\hat{\omega}_1)+\lambda(\hat{\omega}_2)=q.$

In each of the three cases, the induction hypothesis guarantees that

$$t[CA(A^*A)^p]=t[DB(B^*B)^p]$$

for each positive integer p. As above, we choose an orthonormal set of vectors $\{\varphi_i\}$ such that $\varphi_{i_k}, \ldots, \varphi_{i_{k+1}-1}$ is a basis of the initial space of W_k. Then

$$\mu_k^{2p+1}\, t[CW_k] = t[\mu_k^{2p+1}\, CW_k] = \sum_{i=i_k}^{i_{k+1}-1} (CA(A^*A)^p\varphi_i, \varphi_i),$$

so

$$t[CA(A^*A)^p] = \sum_i (CA(A^*A)^p\varphi_i, \varphi_i) =$$

$$= \sum_k \sum_{i=i_k}^{i_{k+1}-1} (CA(A^*A)^p\varphi_i, \varphi_i) = \sum_k \mu_k^{2p+1}\, t[CW_k].$$

Now, since $CA(A^*A)^p$ is in the trace class,

$$\sum_k \mu_k^{2p+1}\, |t[CW_k]| = \sum_k \left| \sum_{i=i_k}^{i_{k+1}-1} (CA(A^*A)^p\varphi_i, \varphi_i)\right| \leqq \sum_i |(CA(A^*A)^p\varphi_i, \varphi_i)| < \infty.$$

Similarly

$$t[DB(B^*B)^p] = \sum_k v_k^{2p+1}\, t[DV_k] = \sum_k \mu_k^{2p+1}\, t[DV_k]$$

and

$$\sum_k v_k^{2p+1}\, |t[DV_k]| < \infty.$$

Setting $a_k=\mu_k t[CW_k]$ and $b_k=\mu_k t[DV_k]$, we can conclude from lemma 4.1 that $t[CW_k]=t[DV_k]$ for all k, which completes the proof of theorem.

Corollary 4.2. *Let Ω denote the free multiplicative semigroup in the free variables x and y. Complete sets of isometric invariants for operators A in the trace- and Schmidt-classes are $\{t[\omega(A, A^*)]:\omega(x, y)\in\Omega\}$ and $\{t[(A^*A)\omega(A, A^*)]:\omega(x, y)\in\Omega\}$, respectively. The addition of* $\dim[\mathfrak{N}(A)\cap\mathfrak{N}(A^*)]$ *to the above sets of isometric invariants yields complete sets of unitary invariants.*

Proof. The Schmidt-class is the class $(\tau c)_f$ where $f(x)=x^2$, so the result for the Schmidt-class is a special case of theorem 4. The result for the trace-class follows from the fact that the trace-class is a subset of the Schmidt-class.

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RICE UNIVERSITY AND HUMBLE OIL & REFINING COMPANY

(Received November 27, 1965)

On the endomorphism ring of direct sums of groups

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§ 1

In this paper we investigate the commutativity of endomorphism rings $E(G)$ of groups G and apply the results on the rings R, which can be defined on G. A ring R is said to be defined on G, in case the additive group of R, denoted by R^+, is G. In the special case that G is a discrete direct sum of groups we obtain conditions for the uniqueness of the holomorphs of rings R, defined on G.

In [5] SZELE—SZENDREI have completely solved the problem of the commutativity of $E(G)$, in case G is a torsion group. For the case of mixed groups they have got some partial results. We consider a group G, which is a discrete direct sum of groups G_λ and obtain necessary and sufficient conditions that $E(G)$ be commutative (Theorem 2 and 2a). As a special case we have the torsion-free completely decomposable groups $G = \sum_\lambda A_\lambda$, where the A_λ are torsion-free groups of rank 1, i. e. subgroups of the additive group of all rationals $\mathfrak{R}$ (Theorem 3, Corollaries 3 and 4). Then we apply our result to torsion groups and obtain Theorem 4, which occurs as Theorem 1 in [5]. We also investigate the finite and finitely generated groups. A finite or a finitely generated group G has a commutative $E(G)$ if and only if G is a cyclic group (Corollaries 5 and 6). For mixed groups we have Theorem 5, due to SZELE—SZENDREI [5], and, in a special case, Corollary 7.

As to the holomorphs of a ring, we first prove a theorem for rings R, which are the ring-theoretic discrete direct sum of rings R_λ ($\lambda \in \Lambda$). In Theorem 1 we give a necessary and sufficient condition that such a ring R have one holomorph. For the definition of holomorph we refer to our paper [3]. From Theorem 1 a result of WEINERT—EILHAUER is easily obtained [6] (Corollary 1) and likewise our Theorem 1 in [3], (Corollary 2). In Theorem 6 we consider a ring R which is defined on a group $G = \sum_\lambda G_\lambda$ (discrete direct sum), where the G_λ are fully invariant subgroups of G. The ring R is the direct sum of its ideals G_λ (as rings). Now the uniqueness of the holomorph of R depends only on the same property for the direct summands G_λ of R. In the special case that the G_λ are rational groups, each G_λ (as a ring) has one holomorph $P(G_\lambda)$, which is isomorphic to the direct sum $G_\lambda \oplus G_\lambda$ (G_λ as a ring) (Theorem 7).

The groups, used in this paper, are all abelian groups, the rings are associative rings. For the definition of group-theoretic notions such as type of an element of a torsion-free group, divisible group, etc. we refer to the book of L. FUCHS [2].

§ 2

Theorem 1. *A ring $R=\sum_{\lambda\in\Lambda} R_\lambda$ (ring-theoretic discrete direct sum) has one holomorph if and only if each R_λ $(\lambda\in\Lambda)$ has one holomorph and each R_λ is invariant for the components of double homothetisms of R.*

Proof. First suppose that R has one holomorph. Consider the projection $\eta_\lambda: R\to R_\lambda$ of R $(r\to r_\lambda)$. It is easily seen that $(\eta_\lambda, \eta_\lambda)$ is a double homothetism of R. Now suppose that (α_1, α_2) is an arbitrary double homothetism of R. As $(\alpha_1,\alpha_2)\sim$ $\sim(\eta_\lambda,\eta_\lambda)$ (R has one holomorph) we have $\alpha_1\eta_\lambda=\eta_\lambda\alpha_1$ or $\alpha_1\eta_\lambda(r)=\eta_\lambda\alpha_1(r)$ or $\alpha_1(r_\lambda)=$ $=\eta_\lambda\{\alpha_1(r)\}\in R_\lambda$ for every $r_\lambda\in R_\lambda$. This shows that R_λ is invariant for the components of double homothetisms of R. Then take two arbitrary double homothetisms (α_1^*, α_2^*) and (β_1^*, β_2^*) of R_λ. Then we define $\alpha_1(r)=\alpha_1^*(r_\lambda)$ and $\alpha_2(r)=\alpha_2^*(r_\lambda)$, $\beta_1(r)=\beta_1^*(r_\lambda)$ and $\beta_2(r)=\beta_2^*(r_\lambda)$, for $r\in R$ and r_λ is the projection of r (λ is fixed). Now one proves easily, that (α_1,α_2) and (β_1,β_2) are double homothetisms of R. As R has one holomorph, $\alpha_1\beta_2(r)=\beta_2\alpha_1(r)$ and $\alpha_2\beta_1(r)=\beta_1\alpha_2(r)$ for all $r\in R$. Or $\alpha_1\beta_2^*(r_\lambda)=\beta_2\alpha_1^*(r_\lambda)$ and $\alpha_2\beta_1^*(r_\lambda)=\beta_1\alpha_2^*(r_\lambda)$ or $\alpha_1^*\beta_2^*(r_\lambda)=\beta_2^*\alpha_1^*(r_\lambda)$ and $\alpha_2^*\beta_1^*(r_\lambda)=\beta_1^*\alpha_2^*(r_\lambda)$. This proves $(\alpha_1^*,\alpha_2^*)\sim(\beta_1^*,\beta_2^*)$ and R_λ has one holomorph.

Conversely, let us suppose that each R_λ $(\lambda\in\Lambda)$ has one holomorph and is invariant for the components of double homothetisms of R. We take two arbitrary double homothetisms (α_1,α_2) and (β_1,β_2) of R. Then $\alpha_1(\sum_\lambda r_\lambda)=\sum_\lambda \alpha_1 r_\lambda$ and $\alpha_1 r_\lambda\in R_\lambda$ for each λ, $\beta_2(\sum_\lambda r_\lambda)=\sum_\lambda\beta_2 r_\lambda$ and $\beta_2 r_\lambda\in R_\lambda$ for each λ. And $(\alpha_1\beta_2-\beta_2\alpha_1)(\sum_\lambda r_\lambda)=$ $=\sum_\lambda(\alpha_1\beta_2-\beta_2\alpha_1)r_\lambda$, where $(\alpha_1\beta_2-\beta_2\alpha_1)r_\lambda\in R_\lambda$ for each λ. Consider a fixed direct summand R_λ of R and define $\alpha_1^*(r_\lambda)=\alpha_1(r_\lambda)$ and $\alpha_2^*(r_\lambda)=\alpha_2(r_\lambda)$ for each $r_\lambda\in R_\lambda$. Then (α_1^*,α_2^*) is a double homothetism of R_λ. Likewise (β_1^*,β_2^*) is a double homothetism of R_λ, if we define $\beta_1^*(r_\lambda)=\beta_1(r_\lambda)$, $\beta_2^*(r_\lambda)=\beta_2(r_\lambda)$ for each $r_\lambda\in R_\lambda$. As R_λ has one holomorph, one gets $(\alpha_1^*,\alpha_2^*)\sim(\beta_1^*,\beta_2^*)$, which means $\alpha_1^*\beta_2^*=\beta_2^*\alpha_1^*$. Therefore $(\alpha_1^*\beta_2^*-\beta_2^*\alpha_1^*)$ (r_λ) $=$ $(\alpha_1\beta_2-\beta_2\alpha_1)$ $(r_\lambda)=0$ for each r_λ in R_λ. As this is the case for each R_λ, we obtain that $(\alpha_1\beta_2-\beta_2\alpha_1)$ $(\sum_\lambda r_\lambda)=0$. Likewise $(\alpha_2\beta_1-\beta_1\alpha_2)(\sum_\lambda r_\lambda)=0$. Therefore $(\alpha_1,\alpha_2)\sim(\beta_1,\beta_2)$, i. e. R has one holomorph.

Corollary 1. *If R = $R^2\oplus n_R$ (direct sum of the ideal generated by all products in R and the annullator in R), then R has one holomorph if and only if the endomorphism ring of n_R^+ is commutative* (see Weinert—Eilhauer [6], Theorem 4).

It is clear that both R^2 and n_R are invariant for components of double homothetisms of R. From R = $R^2\oplus n_R$ and n_R has one holomorph it follows that R^2 has one holomorph. Therefore n_R has one holomorph is a necessary and sufficient condition for the uniqueness of the holomorph of R. As n_R is a zero-ring this is the case if and only if $E(n_R^+)$ is commutative (see Rédei [4]).

In the special case that $R=\sum_{\lambda\in\Lambda} R_\lambda$ and $\mathrm{Hom}\,(R_{\lambda_i}^+, R_{\lambda_j}^+)=0$ for $i\neq j$ we have that $E(R^+)$ = $\sum_{\lambda\in\Lambda} E(R_\lambda^+)$ (direct sum) and each R_λ^+ is a fully invariant subgroup of R^+. Particularly, the R_λ are invariant for the components of double homothetisms of R. So we get:

Corollary 2. $R=\sum_{\lambda} R_{\lambda}$ *with* Hom $(R_{\lambda_i}^{+}, R_{\lambda_j}^{+})=0$ *for* $i\neq j$ *has one holomorph if and only if each of the* R_{λ} *has one holomorph.*

Moreover the holomorph of R is the direct sum of the holomorphs of the R_{λ}, (cf. Theorem 6). Again specializing we have that a finite ring R is the direct sum of its p-components R_p and the holomorph of R is the direct sum of the holomorphs of the R_p (cf. Theorem 1, [3]), if each of the R_p has one holomorph.

§ 3

In order to get further information about the holomorphs of direct sums of rings, we have to investigate the commutativity of the endomorphism rings of direct sums of groups.

Theorem 2. *The endomorphism ring of a discrete direct sum* $G=\sum_{\lambda} G_{\lambda}$ *of groups* G_{λ} *is commutative if and only if each of the summands* G_{λ} *has a commutative* $E(G_{\lambda})$ *and none of* G_{λ} *can be mapped homomorphically onto a non-zero subgroup of another* G_{λ}.

Proof. *Necessity.* As $E(G)$ is commutative, it follows that every endomorphic image of G is fully invariant (Lemma 1, [5]). As every direct summand is an endomorphic image, it follows that the G_{λ} are fully invariant subgroups of G $(\lambda\in\Lambda)$. Suppose now that G_{λ_i} is mapped homomorphically onto a subgroup $(\neq 0)$ of G_{λ_j} by the homomorphism $\vartheta\in$ Hom $(G_{\lambda_i}, G_{\lambda_j})$ $(\lambda_i\neq\lambda_j)$. We define the mapping ϑ' of G into itself by: $\vartheta' g_{\lambda}=0$ if $g_{\lambda}\in G_{\lambda}$ with $\lambda\neq\lambda_i$; $\vartheta' g_{\lambda_i}=\vartheta g_{\lambda_i}$ if $g_{\lambda_i}\in G_{\lambda_i}$. Then ϑ' is an endomorphism of G or $\vartheta'\in E(G)$. But $\vartheta' G_{\lambda_i}\nsubseteq G_{\lambda_i}$, since ϑ' coincides with ϑ on G_{λ_i}. Therefore G_{λ_i} is not fully invariant, which is a contradiction. We conclude that none of G_{λ} can be mapped homomorphically onto a non-zero subgroup of another G_{λ}. Now let σ_{λ}, ϱ_{λ} be two arbitrary endomorphisms of G_{λ} (λ is fixed). G_{λ} is an endomorphic image of G and let η_{λ} be the projection of G onto G_{λ}. Then we can extend the endomorphisms σ_{λ} resp. ϱ_{λ} of G_{λ} to endomorphisms σ resp. ϱ of G defining $\sigma(\sum g_{\mu})=\sum_{\mu}\sigma g_{\mu}$ and $\sigma g_{\mu}=0$ if $g_{\mu}\in G_{\mu}$ with $\mu\neq\lambda$, $\sigma g_{\lambda}=\sigma_{\lambda} g_{\lambda}$ if $g_{\lambda}\in G_{\lambda}$ and likewise for ϱ with respect to ϱ_{λ}. Then $\sigma\varrho(\eta_{\lambda} g)=\varrho\sigma(\eta_{\lambda} g)$ $(g\in G)$, as $E(G)$ is commutative, or $\sigma\varrho_{\lambda}(g_{\lambda})=\varrho\sigma_{\lambda}(g_{\lambda})$, $g_{\lambda}\in G_{\lambda}$, or $\sigma_{\lambda}\varrho_{\lambda}(g_{\lambda})=\varrho_{\lambda}\sigma_{\lambda}(g_{\lambda})$ for every $g_{\lambda}\in G_{\lambda}$. This means $\sigma_{\lambda}\varrho_{\lambda}=\varrho_{\lambda}\sigma_{\lambda}$ or $E(G_{\lambda})$ is commutative.

Sufficiency. Let α be an arbitrary endomorphism of G. Then $\alpha(\sum_{\lambda} g_{\lambda})=\sum_{\lambda}\alpha g_{\lambda}$. Take a fixed G_{λ}. Now $\alpha g_{\lambda}=\sum_{\mu} g_{\lambda\mu}$ $(g_{\lambda\mu}\in G_{\mu})$ is a finite sum and if we put $\alpha_{\lambda\mu} g_{\lambda}=g_{\lambda\mu}$, then $\alpha_{\lambda\mu}$ clearly belongs to Hom (G_{λ}, G_{μ}). Therefore $\alpha_{\lambda\mu}=0$ for $\lambda\neq\mu$, and $g_{\lambda\mu}=0$ for $\lambda\neq\mu$. Then $\alpha g_{\lambda}=\sum_{\mu} g_{\lambda\mu}=g_{\lambda\lambda}\in G_{\lambda}$, which means that G_{λ} is a fully invariant subgroup of G. $E(G)=\sum_{\lambda\in\Lambda} E(G_{\lambda})$ (direct sum) and as each G_{λ} has a commutative $E(G_{\lambda})$, it follows that $E(G)$ is commutative.

From the proof above we see that Theorem 2 also may be read as:

Theorem 2a. *Let* $G=\sum_{\lambda} G_{\lambda}$ *be a discrete direct sum of groups* G_{λ}. *Then* $E(G)$

is commutative if and only if each G_λ has a commutative $E(G_\lambda)$ and is a fully invariant subgroup of G.

Theorem 3. *A completely decomposable torsion-free group $G=\sum_\lambda A_\lambda$, where the A_λ are torsion-free groups of rank 1 and G is their direct sum, has a commutative $E(G)$ if and only if the types of the components A_λ are pairwise incomparable.*

Proof. First we remark that, if A_{λ_i} and A_{λ_j} are two torsion-free groups of rank 1, of type $\mathfrak{a}$ and $\mathfrak{b}$ respectively, then A_{λ_i} is isomorphic to a subgroup of A_{λ_j} if and only if $\mathfrak{a}\leqq\mathfrak{b}$. Now suppose that the conditions of the theorem are satisfied. Then we show, that none of the groups A_λ can be mapped homomorphically onto a non-zero subgroup of another A_λ. For let $A_{\lambda_i}, A_{\lambda_j}$ be torsion-free groups of rank 1 $(\lambda_i\neq\lambda_j)$ and let φ be a homomorphism of A_{λ_i} onto a subgroup $(\neq 0)$ of A_{λ_j}. Then it is easy to see, that Ker $(\varphi)=0$ or φ is a monomorphism (isomorphism into). This means A_{λ_i} is isomorphic to a subgroup of A_{λ_j} i. e. $\varphi(A_{\lambda_i})$, but this is impossible by the remark above as the types of A_{λ_i} and A_{λ_j} are incomparable. As the A_λ are rational groups, they have commutative endomorphism rings, and $E(G)$ is commutative by Theorem 2.

Conversely, if $E(G)$ is commutative, then again none of the A_λ is isomorphic to a subgroup of another A_λ by theorem 2. This means, the types of the components A_λ are pairwise incomparable.

The class of completely decomposable groups comprises all groups of rank 1, all free abelian groups as well as all divisible torsion-free abelian groups. Thus we have the corollaries:

Corollary 3. *A free abelian group G has a commutative $E(G)$ if and only if $G\cong C(\infty)$ (infinite cyclic group).*

Corollary 4. *A divisible torsion-free abelian G has a commutative $E(G)$ if and only if $G\cong\mathfrak{R}$, where $\mathfrak{R}$ is the additive group of all rational numbers.*

§ 4

a) *Torsion groups.* Every torsion group may be represented as a direct sum of p-groups G_p belonging to different primes p. The G_p, uniquely determined by G, are called the *p-components* of G. They are fully invariant subgroups of G. Therefore by Theorem 2a, $G=\sum_p G_p$ has a commutative $E(G)$ if and only if each G_p has a commutative $E(G_p)$. Then we have to characterize the p-groups with commutative endomorphism ring. Now let p be a fixed prime and consider the p-component G_p of G. The center of $E(G_p)$ is the ring $\mathfrak{P}$ of p-adic integers or the residue class ring $I/(p^k)$ of the integers mod p^k, where I is the ring of rational integers ([2], Theorem 56. 3). Therefore, $E(G_p)$ is commutative if and only if $E(G_p)$ is either the ring $\mathfrak{P}$ of p-adic integers or the ring $I/(p^k)$ of integers mod p^k. We now use: if A is a group $C(p^k)$ $(k=1, 2, \ldots, \infty)$, and B is a p-group such that $E(B)\cong E(A)$, then $B\cong A$, (see [2], p. 215). Incase $E(G_p)\cong\mathfrak{P}=E(C(p^\infty))$, we have $G_p\cong C(p^\infty)$. In case $E(G_p)\cong \cong I/(p^k)=E(C(p^k))$, we have $G_p\cong C(p^k)$. Thus a p-component G_p of G has a commutative $E(G_p)$ if and only if G_p is either $C(p^\infty)$ or $C(p^k)$. Then $G=\sum_p G_p$ has a commutative $E(G)$ if and only if G is a direct sum of groups $C(p^k)$ $(k=1, 2, \ldots, \infty)$ for different primes p.

Theorem 4. *An abelian torsion group G has a commutative $E(G)$ if and only if G is a subgroup of C, where C is the additive group of rational numbers* mod 1 (cf. [5], § 4, Theorem 1).

If G is a finite abelian group, then components $C(p^\infty)$ do not occur in a direct decomposition of $G = \sum_p G_p$ in p-components. But then G is a direct sum of a finite number of cyclic groups $C(p^k)$ for different primes p, that means, G is cyclic. So we get:

Corollary 5. *A finite abelian group G has a commutative $E(G)$ if and only if G is a cyclic group.*

More generally, a *finitely generated* group G is a direct sum of a finite number of cyclic groups of infinite and/or prime power order, say $G = \sum_m C(\infty) + \sum_p C(p^k)$. Let G have a commutative $E(G)$. If G is torsion-free, then $G = C(\infty)$ (Corollary 3). If G is a torsion group, then $G = \sum_p C(p^k)$ for different primes p, or G is a cyclic group (Corollary 5). If G is a mixed group, then the torsion-free component of G is $C(\infty)$, as none of the direct summands can be mapped homomorphically onto another one. The maximal torsion subgroup of G is $\sum_p C(p^k)$ and as $E(G)$ is commutative, $\sum_p C(p^k)$ has a commutative endomorphism ring (Theorem 2). Then $\sum_p C(p^k)$ is a subgroup of C (Theorem 4); in this case, as G is finitely generated, $\sum_p C(p^k)$ is a cyclic group $C(n)$ (Corollary 5). Now $G = C(\infty) + C(n)$ is impossible, as $\mathrm{Hom}(C(\infty), C(n)) \cong C(n)$ and this contradicts the commutativity of $E(G)$. Therefore a mixed group G, which is finitely generated and has commutative $E(G)$, is impossible. We have proved:

Corollary a) 6. *A finitely generated abelian group G has a commutative $E(G)$ if and only if G is a cyclic (infinite or finite) group.*

Remark. a) For a *torsion group G*, SZELE—SZENDREI [5] have proved that G has a commutative $E(G)$ if and only if G has this property *locally*, i. e. every finitely generated subgroup of G has a commutative $E(G)$. By Corollary 6, this means, every finitely generated subgroup of G is cyclic or G is locally cyclic. Now a torsion group G is locally cyclic if and only if it is a subgroup of C, which is again Theorem 4.

b) For a *torsion-free group G* it is clear that if every finitely generated subgroup F of G has a commutative $E(F)$, then G has a commutative $E(G)$. For, according to Corollary 6, this means that every finitely generated subgroup is $C(\infty)$, or G is locally cyclic. But a locally cyclic torsion-free group G is a rational group or a subgroup of $\mathfrak{R}$, the additive group of all rationals. Therefore G has a commutative $E(G)$. The converse does not hold. A counter-example is: let $p_1, p_2, \ldots$ be an infinite sequence of different prime numbers and let R_{p_n} be the additive group of those rationals, whose denominator is relatively prime to p_n. Then the complete direct sum $G = \sum_{p_n}^{*} R_{p_n}$ has a commutative $E(G)$ (SZELE—SZENDREI [5]), but G is not locally cyclic.

c) *Mixed groups.* Let G be an arbitrary (mixed) group and p be an arbitrary prime number. If the group G contains an element of order p, then p is called *relevant* for G. Let $G = T + J$ be a *splitting* mixed group, i. e. G decomposes into a direct sum of a torsion group T and a torsion-free group J. Here we have the following theorem, due to SZELE—SZENDREI [5]:

Theorem 5. *Let $G = T + J$ be a splitting mixed group, where T is the torsion subgroup of G. Then $E(G)$ is commutative if and only if T is a locally cyclic group containing no subgroup of type $C(p^\infty)$ and J has a commutative $E(J)$ and $pJ = J$ holds for all primes p relevant for G.*

Remark. As a special case of Theorem 5 we consider the mixed groups G with bounded maximal torsion subgroup. Let G be a mixed group with bounded maximal torsion subgroup T ($nT = 0$). Then G is a splitting mixed group: $G = T + J$ ([2], Corollary 50. 4). Now suppose that G has a commutative $E(G)$. By Theorem 5, T is a locally cyclic group containing no subgroup of type $C(p^\infty)$. From $nT = 0$ we infer that only those cyclic components $C(p^k)$ can occur in T, for which $p|n$. As n has only a finite number of prime divisors, it follows that T has a finite number of direct summands, i. e. T is a cyclic group and a subgroup of $C(n)$. We may assume, without loss of generality, that n is the least positive integer such that $nT = 0$. Then we get $T = C(n)$. Evidently we also have $T = G[n]$, where $G[n]$ is the set of all $g \in G$ with $ng = 0$. Now it is clear that $J \cong G/T = G/G[n] \cong nG$, i. e. the set of all ng with $g \in G$, hence $E(nG)$ is commutative by Theorem 5. As $T = C(n)$, the prime divisors p_i of n are relevant for G. From Theorem 5 it follows that $p_i J = J$ for all $p_i | n$. Hence $nJ = J$ or $nG = J$, as $nG = nJ$. Conversely, if G is a mixed group with bounded maximal torsion subgroup $T = C(n)$, then again $T = G[n]$. If nG is the torsion-free component of G, then we have the direct decomposition $G = G[n] + nG$. Both $G[n]$ and nG are fully invariant subgroups of G. Moreover T as a cyclic group has a commutative $E(T)$. By Theorem 2a the commutativity of $E(nG)$ is sufficient now for the commutativity of $E(G)$. Thus we get:

Corollary 7. *Let G be a mixed group with bounded maximal torsion subgroup T such that $nT = 0$ and n is the least positive integer with this property. Then $E(G)$ is commutative if and only if $T = C(n)$ and nG is the torsion-free component of G and has a commutative $E(nG)$.*

Now we want to apply these results to the investigation of rings which can be defined on direct sums of groups. Let G be an arbitrary (abelian) group. An (associative) ring R *on* G is a ring R, such that $R^+ = G$. Such a ring R has one holomorph if the endomorphism ring $E(R^+) = E(G)$ is commutative [6]. If G is a discrete direct sum of groups, and every direct summand is a fully invariant subgroup of G, the structure of the holomorph of a ring R on G can be described.

Theorem 6. *Let $G = \sum_{\lambda \in \Lambda} G_\lambda$ be a discrete direct sum of groups G_λ, such that each G_λ is a fully invariant subgroup of G. Then in each ring R on G the G_λ are ideals and R is their direct sum in ring-theoretic sense. A ring R on G has one holomorph* *if and only if each of the G_λ (as a ring) has one holomorph. If R has one holomorph $P(R)$, then $P(R)$ is an interdirect sum of the holomorphs $P(G_\lambda)$ $(\lambda \in \Lambda)$.*

Proof. Let g be a fixed element of G. Then multiplication of the elements of G from the left by g in a ring R on G induces an endomorphism of G. As G_λ is fully invariant in G, we get $gg_\lambda \in G_\lambda$ for each $g_\lambda \in G_\lambda$. Likewise we find that g, operating on the right side on the elements of G, induces an endomorphism of G and therefore $g_\lambda g \in G_\lambda$ for each $g_\lambda \in G_\lambda$. G_λ is a two-sided ideal in G. Moreover $g_\lambda g_\mu \in G_\lambda \cap G_\mu = (0)$ for $\lambda \neq \mu$ or $G_\lambda G_\mu = (0)$. As G is a direct sum of groups G_λ, we infer that R is a direct sum of its ideals G_λ in ring-theoretic sense. Then, each G_λ is fully invariant in G implies in particular that each G_λ is invariant for the components of double homothetisms of R. By Theorem 1, R has one holomorph if and only if each of the G_λ (as a ring) has one holomorph. Finally we have to prove that the holomorph $P(R)$ of R is an interdirect sum of the holomorps $P(G_\lambda)$, $(\lambda \in \Lambda)$. Let D resp. D_λ be the maximal ring of related double homothetisms of R resp. G_λ $(\lambda \in \Lambda)$. The elements of the holomorph $P(R)$ are the pairs (α, a), $\alpha \in D$, $a \in R$ and sum and product are obtained as follows: $(\alpha, a)+(\beta, b) = (\alpha+\beta, a+b)$, $(\alpha, a)(\beta, b) = (\alpha\beta, \beta_2 a+\alpha_1 b+ab)$ with $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$. As $G = \sum_\lambda G_\lambda$ is the discrete direct sum of its fully invariant subgroups G_λ, it is clear that $E(G)$ is the complete direct sum of the groups $E(G_\lambda)$. Likewise D is the complete direct sum of the rings D_λ. Any $\alpha \in D$ induces a well-defined double homothetism α_λ of D_λ for every λ. If $\alpha = (\alpha_1, \alpha_2) \in D$ and α_1 induces $\alpha_{1\lambda}$ in D_λ, α_2 induces $\alpha_{2\lambda}$ in D_λ, then $\alpha_\lambda = (\alpha_{1\lambda}, \alpha_{2\lambda})$ is a double homothetism of D_λ. Every double homothetism $\alpha_\lambda \in D_\lambda$ (λ fixed) may be obtained as the "λ^{th} component" of a double homothetism $\alpha \in D$. The mapping $(\alpha, a) \to \langle \dots, (\alpha_\lambda, a_\lambda), \dots \rangle$ is a homomorphism of $P(R) = D \circ R$ into the complete direct sum of the $P(G_\lambda) = D_\lambda \circ G_\lambda$. Moreover, this homomorphism is an isomorphism, because if $(\alpha_\lambda, a_\lambda) = (0, 0)$ holds for all $\lambda \in \Lambda$, then $(\alpha, a) = (0, 0)$. Then $P(R)$ is isomorphic to a subring of the complete direct sum of the rings $P(G_\lambda) = D_\lambda \circ G_\lambda$ i. e. an interdirect sum of the rings $P(G_\lambda)$ $(\lambda \in \Lambda)$. This completes the proof of Theorem 6.

Now we will give examples of groups, which satisfy the requirements of Theorem 6. In the *torsion* case, we have that every torsion group G may be represented as a direct sum of its p-components G_p. These p-components G_p are fully invariant subgroups of G. Therefore Theorem 6 may be applied to torsion groups. If G is a finite group, say of order n, then, if $n = p_1^{k_1} \dots p_r^{k_r}$, G is the direct sum of r subgroups G_i of order $p_i^{k_i}$ $(i = 1, \dots, r)$. Every ring R on G is a finite ring and the ring-theoretic direct sum of finite p_i-rings R_{p_i}, which are rings on G_i $(i = 1, \dots, r)$ and annihilate each other for different primes p_i. The ring R has one holomorph if and only if each of the R_{p_i} has one holomorph. Moreover $P(R)$ is the direct sum of the $P(R_{p_i})$. This establishes Theorem 1 of my paper [3], (cf. also Corollary 2 of this paper).

In the *torsion-free* case, we consider a torsion-free group G which is the direct sum of homogeneous groups such that the types of the components G are pairwise incomparable. By a *homogeneous* group we mean a torsion-free group all of whose elements $\neq 0$ are of one and the same type $\mathfrak{a}$. We denote by $G(\mathfrak{a})$ the set of all elements a in G for which $T(a) \geqq \mathfrak{a}$. Now let G_λ be a fixed homogeneous component of G of type $\mathfrak{a}_\lambda$. As the types of the components G_λ are pairwise incomparable, we get $G(\mathfrak{a}_\lambda) = G_\lambda$. Now the subgroups $G(\mathfrak{a})$ are, for any type $\mathfrak{a}$, fully invariant in G. Therefore G_λ is a fully invariant subgroup of G for every λ. We do not know, however, whether a homogeneous group G_λ has a commutative $E(G_\lambda)$. If the homogeneous components G_λ are torsion-free groups of rank 1 or rational groups the group $G=\sum_\lambda G_\lambda$ is completely decomposable. If now the types of the rational groups are pairwise incomparable, then the G_λ are fully invariant in G. A ring R on G is the direct sum of its ideals G_λ. In this case, any ring R on G has one holomorph, as each of the G_λ (as a ring) has one holomorph. The last result is due to the fact, that each of the G_λ (as a rational group) has a commutative $E(G_\lambda)$ and this is a sufficient condition for the uniqueness of the holomorph $P(G_\lambda)$. The uniqueness of the holomorph of R is also an easy consequence of Theorem 3, as the ring $E(R^+)=E(G)$ is commutative. By Theorem 6, $P(R)$ is an interdirect sum of the holomorphs $P(G_\lambda)$ $(\lambda\in\Lambda)$.

If G_λ is a rational group, then any ring R_λ on G_λ is a subring of the rational number field or a zero-ring [1]. Now we have the theorem:

Theorem 7. *Let G_λ denote a subgroup of the additive group $\mathfrak{R}$ of all rationals and assume that $1\in G_\lambda$. Let R_λ be a non-zero ring on G_λ and let $1\times 1=1$ in R_λ. Then the holomorph $P(R_\lambda)$ of R_λ is isomorphic to $R_\lambda\oplus R_\lambda$ (ring-theoretic direct sum).*

Proof. Any $\eta\in E(R_\lambda^+)=E(G_\lambda)$ maps 1 upon a rational r and this r characterizes η. A double endomorphism (α_1,α_2) of R_λ^+, $\alpha_1\in E(R_\lambda^+)$, $\alpha_2\in E^\circ(R_\lambda^+)$ is a double homothetism of R_λ if the following conditions are satisfied: $\alpha_1(ab)=(\alpha_1 a)b$, $\alpha_2(ab)=a(\alpha_2 b)$, $(\alpha_2 a)b=a(\alpha_1 b)$ and $\alpha_2(\alpha_1 a)=\alpha_1(\alpha_2 a)$ for all $a,b\in R_\lambda$. As $1\times 1=1$ in R_λ, the multiplication in R_λ is the usual one of rational numbers. Now, if $\alpha_1 1=r_1$ and $\alpha_2 1=r_2$ $(r_1,r_2\in R_\lambda)$, it is clear that $\alpha_1 a=r_1 a$, $\alpha_2 a=r_2 a$ for all $a\in R_\lambda$. This means, that $\alpha_1(ab)=(\alpha_1 a)b$, $\alpha_2(ab)=a(\alpha_2 b)$ and $\alpha_2(\alpha_1 a)=\alpha_1(\alpha_2 a)$ for all $a,b\in R_\lambda$. From $(\alpha_2 a)b=a(\alpha_1 b)$ it follows that $r_2(ab)=r_1(ab)$ for all $a,b\in R_\lambda$. As R_λ has no zero-divisors (R_λ is a subring of the rational number field), we get $r_1=r_2$ or $\alpha_1=\alpha_2$. The double homothetisms of R_λ have the form (α,α), where $\alpha\in E(R_\lambda^+)$. Now R_λ has one maximal ring D_λ of related double homothetisms, as all double homothetisms are pairwise related. The mapping $(\eta,\eta)\to\eta$ provides an isomorphism of D_λ onto $E(R_\lambda^+)$. Now every double homothetism $(\eta,\eta)\in D_\lambda$ is an inner one, i. e. every (η,η) is induced by a rational number $r\in R$ such that $\eta a=ra$ for all $a\in R_\lambda$. Therefore $D_\lambda=D_{o_\lambda}=$ ring of all inner double homothetisms of R_λ. It is known, that $R_\lambda/n_{R_\lambda}\cong D_{o_\lambda}$, where n_{R_λ} is the annullator of R_λ (Rédei [4]). But $n_{R_\lambda}=(0)$, therefore $R_\lambda\cong D_{o_\lambda}=D_\lambda$. The elements of $P(R_\lambda)$ are pairs (η,a), $\eta=(\eta,\eta)\in D_\lambda$, $a\in R_\lambda$. We write these elements as (a,b), $a,b\in R_\lambda$, as $R_\lambda\cong D_\lambda$. Addition and multiplication are defined by

$$(a,b)+(c,d)=(a+c,b+d),\qquad (a,b)(c,d)=(ac,bc+ad+bd).$$

In this case $P(R_\lambda)=D_\lambda\circ R_\lambda$ is a direct sum. For let $(a,b)\to\pi(a,b)=(a,a+b)$ be a permutation of the elements of R_λ. Then we define: $(a,b)\dot{+}(c,d)=\pi(\pi^{-1}(a,b)+\pi^{-1}(c,d))$ and $(a,b)\dot{\times}(c,d)=\pi(\pi^{-1}(a,b)\pi^{-1}(c,d))$, and it turns out that $(a,b)\dot{+}(c,d)=(a+c,b+d)$ and $(a,b)\dot{\times}(c,d)=(ac,bd)$. Then $P(R_\lambda)=D_\lambda\circ R_\lambda\cong D_\lambda\oplus R_\lambda\cong R_\lambda\oplus R_\lambda$. Finally, let $G=T+J$ be a splitting *mixed* group, where T is the torsion subgroup of G and both T and J satisfy the conditions of Theorem 5. T is the maximal torsion subgroup of G and therefore T is a fully invariant subgroup of G. As $pJ=J$ for all primes relevant for G, it is clear that the equation $p^n x=a$ $(a\in J)$ is solvable in J for every natural number n and every prime p relevant for G. Then J is a fully invariant subgroup of G. Thus $G=T+J$ is the direct sum of its fully invariant subgroups T and J and we may apply Theorem 6.

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(Received March 19, 1966)

Einige Kriterien für die Existenz des Einselementes in einem Ring

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§ 1. Einleitung

Unter einem Ring werden wir in dieser Arbeit stets einen assoziativen Ring verstehen. Da die Ringe mit einem Einselement bzw. mit einem einseitigen Einselement bekanntlich eine wichtige Rolle in der Theorie der Ringe und der Operatorenmoduln spielen, ist es zweckmäßig Kriterien für die Existenz des Einselementes in einem Ring zu untersuchen. Bezüglich der Untersuchungen dieser Art verweisen wir vor allem auf die grundlegende Arbeit [1] von R. BAER.

Die Ringe mit Einselement bzw. mit Rechtseinselement sind auch vom Gesichtspunkt ringtheoretischer Zerlegungssätze aus wichtig (s. die Arbeiten [5], [7], [11], [14]).

Wir werden in dieser Arbeit weitere Kriterien für die Existenz des Einselementes in einem Ring angeben. Dafür werden wir im § 2 zuerst einige Ringklassen, und zwar hauptsächlich zwei Ringklassen betrachten. Diese zweien Ringklassen bestehen aus den Ringen mit laut linksannullatorfreien homomorphen Bildern, bzw. aus den Ringen A, für die der maximale triviale (d. h. von A annullierte) Untermodul M_0 von jedem A-Rechtsmodul M ein direkter Summand von M ist. Es werden auch einige offene Fragen bezüglich der vorkommenden Ringklassen aufgeworfen werden.

§ 2. Über einige Ringklassen

Um kurz zu sprechen, benützen wir die folgenden Bezeichnungen bzw. Benennungen.

E_0-Ringe, und E_1-Ringe bezeichnen Ringe mit einem Einselement bzw. mit einem Rechtseinselement.

E_2-Ring bedeutet einen solchen Ring A, für den der maximale triviale Untermodul M_0 von jedem A-Rechtsmodul M ein direkter Summand von M ist.

Ein Ring A, für den $a \in aA$ für jedes $a \in A$ gilt, heißt E_3-Ring.

Einen Ring, dessen sämtliche homomorphen Bilder keinen von Null verschiedenen Linksannullator und Rechtsannullator besitzen, nennen wir E_4-Ring.

Zum Schluß wird ein Ring mit laut linksannullatorfreien homomorphen Bildern E_5-Ring genannt.

Mit der Hilfe einer Peirceschen Zerlegung kann bewiesen werden, daß jeder E_0-Ring ein E_2-Ring ist. Jeder E_0-Ring ist ein E_i-Ring für $i=1, 2, 3, 4$ oder 5. Weiterhin ist ein E_1-Ring sowohl ein E_3-Ring, als auch ein E_5-Ring. Sowohl die

E_3-Ringe, als auch die E_4-Ringe sind E_5-Ringe. Die durch die Matrizen

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{und} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

über einem Primkörper erzeugte Algebra A zeigt die Existenz eines solchen E_3-Ringes, der kein E_4-Ring ist, denn A ist ein E_1-Ring mit von Null verschiedenen Rechtsannullatoren, und somit ist A kein E_0-Ring. Weiterhin zeigt jeder einfache Radikalring A mit $A^2 = A \neq 0$ (vgl. SĄSIADA [13]) die Existenz solcher Ringe, die sowohl E_5-Ringe, als auch E_4-Ringe, aber weder E_2-Ringe noch E_3-Ringe sind. Ein Radikalring im Sinne von JACOBSON ist nämlich offenbar kein E_3-Ring, und es wird in dieser Arbeit bewiesen werden, daß kein E_2-Ring ein von Null verschiedener Radikalring ist. Im Buch [10] von A. KERTÉSZ ist folgende Behauptung (mit anderer Abfassung) bewiesen: Ein Ring A ist dann und nur dann ein E_0-Ring, wenn es $M = M_0 \oplus MA$ für jeden A-Rechtsmodul M mit dem maximalen trivialen Untermodul M_0 gilt, wobei $\oplus$ eine moduldirekte Summe bezeichnet. Hiernach bedeutet die Klasse der E_2-Ringe eine formale Verallgemeinerung der E_0-Ringe.

Wir bestätigen jetzt einige Eigenschaften der E_5-Ringe bzw. der E_2-Ringe. Es gilt der folgende

Satz 2. 1. (1) *Jedes homomorphe Bild A' eines E_5-Ringes A ist ebenfalls ein E_5-Ring.* (2) *Für den Kern R' von jedem A'-Endomorphismus φ' von jedem homomorphen Bild A' eines E_5-Ringes A gilt $R' = \{x'; x' \in A', x'A' \subseteq R'\}$.* (3) *Es gilt $L \subseteq LA$ für jedes Linksideal L eines E_5-Ringes A.*

Beweis. (1) Da jedes homomorphe Bild A'' von A' auch ein homomorphes Bild des E_5-Ringes A ist, hat A'' keinen von Null verschiedenen Linksannullator von A'', und somit ist A' ein E_5-Ring. — (2) Bezeichnen wir die Menge $\{x'; x' \in A', x'A' \subseteq R'\}$ mit S', so ist S' ein Rechtsideal von A', für das $S'A' \subseteq R'$ gilt. Dann ergibt sich $(S'\varphi')A' \subseteq R'\varphi'$ und somit $S'\varphi' = 0$, denn $R'\varphi'$ ist Null und A' ist linksannullatorfrei. Also gilt $S' \subseteq R'$ und wegen $R' \subseteq S'$ auch $S' = R'$. — (3) Das Produkt LA ist für jedes Linksideal L ein Ideal von A, und da $l + LA$ für jedes $l \in L$ ein Linksannullator des E_5-Ringes A/LA ist, ergibt sich $l \in LA$ für jedes $l \in L$ und somit $L \subseteq LA$, w. z. b. w.

Satz 2. 2. (1) *Es gilt $A^2 = A$ für jeden E_5-Ring A.* (2) *Für jeden E_5-Ring A und für jedes $a \in A$ besteht $a \in aA + AaA$.* (3) *Das Jacobsonsche Radikal J von jedem E_5-Ring A stimmt mit dem Durchschnitt Φ_l aller maximalen Linksideale von A überein, bzw. es gilt $J = A$, wenn A kein maximales Linksideal besitzt.*

Beweis. (1) Ist $L = A$, so erhält man nach dem Satz 2. 1. 3. $A \subseteq A^2$ und somit $A^2 = A$. (2) Ist L das durch a in A erzeugte Hauptlinksideal $(a)_l$, so folgt aus dem Satz 2. 1. 3 $a \in L \subseteq LA = aA + AaA$. (3) Ist $L = J$, so ergibt sich nach dem Satz 2. 1. 3 $J \subseteq JA$. Da aber J der Durchschnitt aller modularen maximalen Linksideale von A ist, gilt $\Phi_l \subseteq J$, und nach einem Satz von HILLE [6] (Seite 486, Theorem 22. 15. 3) bzw. von KERTÉSZ [9] erhält man $JA \subseteq \Phi_l$, woraus wegen $\Phi_l \subseteq J \subseteq JA \subseteq \Phi_l$ notwendig $J = \Phi_l$ folgt.

Satz 2. 3. (1) *Jeder E_2-Ring A ist ein E_5-Ring, und jedes homomorphe Bild A' eines E_2-Ringes ist ebenfalls ein E_2-Ring.* (2) *Es gibt keinen von Null verschiedenen E_2-Ring A, der ein Radikalring im Sinne von* JACOBSON *ist.* (3) *Jeder E_2-Ring A besitzt maximale Linksideale.* (4) *Ist B eine beliebige Everettsche Ringerweiterung eines beliebigen E_2-Ringes A, so gibt es ein Ideal C von B und eine additive Untergruppe D von B^+ mit $B^+ = C \oplus D$, $CA=0$, $A \subseteqq D$, $D^2A \supseteqq DA$ und $dA \neq 0$ für jedes von Null verschiedene Element $d \in A$.*

Beweis. (1) Zuerst beweisen wir, daß jedes homomorphe Bild A' eines E_2-Ringes A ebenfalls ein E_2-Ring ist. Ist nämlich M' ein A'-Rechtsmodul mit dem maximalen trivialen Untermodul $(M')_0$, so wird M' mit dem Vorschrift $am = a'm'$ $(m=m')$ ein A-Rechtsmodul M, wenn a ein beliebiges Urbild von a' bei einem festen Homomorphismus φ von A auf A' ist. Dann ist $M_0 = (M')_0$ der maximale triviale Untermodul des A-Rechtsmoduls M mit $M = M_0 \oplus M_1$, wobei M_1 sowohl ein A-Untermodul, als auch ein A'-Untermodul von M ist. Also ist A' mit A ebenfalls ein E_2-Ring. Wir zeigen jetzt, daß jeder E_2-Ring ein E_5-Ring ist. Dafür genügt es nach dem vorigen zu bestätigen, daß jeder E_2-Ring A linksannullatorfrei ist. Es sei nämlich A_1 die kanonische (Dorrohsche) Ringerweiterung des E_2-Ringes A mit einem Einselement. A_1 besteht bekanntlich aus sämtlichen Paaren (a, m), wobei $a \in A$ und m eine ganze rationale Zahl ist. Definiert man die Gleichheit und Subtraktion der Paare komponentenweise und die Multiplikation mit den Regeln $(a, m)\cdot(b, n) = (ab+mb+na, mn)$ bzw. $(a, m)b = (ab+mb, 0)$, so wird der Ring A_1 ein A-Rechtsmodul. Da A ein E_2-Ring ist, ergibt sich für A_1 eine modultheoretische Zerlegung $A_1 = A_0 \oplus A_2$, wobei A_0 der maximale triviale A-Untermodul des A-Rechtsmoduls A_1 ist. Es gilt nun $(0, 1) = (a, m)+(-a, 1-m)$ mit $(a, m) \in A_0$, $(-a, 1-m) \in A_2$. Da jetzt es $ab+mb = 0$ für jedes $b \in A$ gilt, erhält man $(b, 0) = (0, 1)b = (-a, 1-m)b \in A_2$ für jedes $b \in A$. Deshalb ergibt sich $(b, 0)A = (bA, 0) \neq 0$ für jedes von Null verschiedene Element b von A, folglich $bA \neq 0$ für jedes von Null verschiedene Element $b \in A$. Also ist A linksannullatorfrei, und somit ist jeder E_2-Ring auch ein E_5-Ring. (2) Nehmen wir an, daß ein von Null verschiedener Radikalring A ein E_2-Ring ist. Ist a ein beliebiges von Null verschiedenes Element, so sei R maximal in der Menge derjenigen Rechtsideale S von A, für die $a \notin S$ gilt. Nach dem Lemma von Zorn gibt es gewiß ein solches Rechtsideal R. Dann ist aber der A-Rechtsmodul A/R subdirekt unzerlegbar, denn jeder von Null verschiedene A-untermodul des A-Rechtsmoduls A/R enthält $a+R$. Da A ein Radikalring ist, gilt $MA \subseteqq R$ für den minimalen A-Untermodul M/R von A/R. Da A auch ein E_2-Ring ist, liegt M/R im maximalen trivialen A-Untermodul A_0/R von A/R, für den $A/R = A_0/R \oplus A_0'/R$ mit einem Rechtsideal A_0' von A gilt. Da aber $M \neq R$ gilt und A/R subdirekt unzerlegbar ist, ergibt sich $A_0/R = A/R$, und somit $A^2 \subseteqq R$. Nach den Sätzen 2. 3. 1 und 2. 2. 1 gilt aber $A^2 = A$, und somit wegen $A \subseteqq R$ auch $A = R$, was den Bedingungen $a \notin R$ und $a \in A$ widerspricht. Dieser Widerspruch beweist, daß kein von Null verschiedener E_2-Ring ein Radikalring im Sinne von JACOBSON ist. (3) Besitzt nun ein E_2-Ring $A(\neq 0)$ kein maximales Linksideal, so gilt nach den Sätzen 2. 3. 1 und 2. 2. 3 $J = A$, was nach dem vorigen unmöglich ist. Also hat jeder von Null verschiedene E_2-Ring A maximale Linksideale. (4) Es seien B eine beliebige Everettsche Ringerweiterung des E_2-Ringes A, C der maximale triviale A-Untermodul des A-Rechtsmoduls B und D ein A-Untermodul von B mit $B = C \oplus D$. Da A ein Ideal von B ist, ist auch C ein Ideal in B mit $CA = 0$. Offenbar gilt $dA \neq 0$ für jedes $d \in D$ mit $d \neq 0$. Weiterhin erhält man nach den Sätzen 2. 3. 1 und 2. 2. 1 $A = A^2 \subseteqq BA \subseteqq CA + DA = DA \subseteqq D$ und wegen $DB \subseteqq DC + D^2$, $DC \subseteqq C$, $CA = 0$ und $A = A^2 \subseteqq BA \subseteqq A$ ergibt sich $DA \subseteqq D^2 A$, w. z. b. w.

§ 3. Kriterien für die Existenz des Einselementes in einem Ring

Ein von Null verschiedenes Element a eines Ringes A heißt Rechtsmultiplikator von A, wenn es eine ganze rationale Zahl n gibt, derart, daß $xa = nx$ für jedes $x \in A$ gilt. Ähnlich lassen sich auch die Linksmultiplikatoren definieren.

Es gilt der folgende

Satz 3. 1. (1) *Ein beliebiger Ring A hat dann und nur dann ein Rechtseinselement, wenn A ein E_5-Ring ist, der einen solchen Rechtsmultiplikator besitzt, der in A kein Rechtsnullteiler ist.* (2) *Jeder torsionsfreie E_4-Ring mit einem Rechtsmultiplikator hat ein Einselement.* (3) *Ein beliebiger Ring A hat dann und nur dann ein Einselement, wenn A ein E_5-Ring ist, der einen solchen Linksmultiplikator besitzt, der in A kein Linksnullteiler ist.* (4) *Jeder torsionsfreie E_5-Ring mit einem Linksmultiplikator besitzt ein Einselement.*

Beweis. Da die Bedingungen im Satz 3. 1. 1 für ein Rechtseinselement von A notwendig erfüllt sind, genügt es nur das Hinreichen der Bedingungen zu beweisen. Es sei also A ein E_5-Ring, der einen solchen Rechtsmultiplikator a hat, der in A kein Rechtsnullteiler ist. Dann gibt es eine ganze rationale Zahl n mit $xa = nx \neq 0$ für jedes von Null verschiedene $x \in A$, woraus folgt, daß A kein Element x mit $x \neq 0$ und $nx = 0$ besitzt. Weiterhin existieren wegen $Aa = nA$ und nach den Sätzen 2. 2. 1 und 2. 2. 2 Elemente b und $c \in A$ mit $a = ab + nb$, woraus $nx = xa = xab + nxc = nxb + nxc = nx(b + c)$ und somit $n(x(b + c) - x) = 0$ folgt. Da A kein Element y mit $y \neq 0$ und $ny = 0$ hat, gilt $x(b + c) = x$ für jedes $x \in A$, und somit ist $b + c$ ein Rechtseinselement von A. (2) Ist A ein torsionsfreier E_4-Ring mit einem Rechtsmultiplikator a, so ist a wegen der Torsionsfreiheit und da A rechtsannullatorfrei ist, kein Rechtsnullteiler, und somit läßt sich der Satz 3. 1. 1 für den Beweis der Existenz eines Rechtseinselementes anwenden. Da A ein E_4-Ring ist, ist jedes Rechtseinselement auch ein Einselement von A. (3) Es sei A ein E_5-Ring, der einen solchen Linksmultiplikator a besitzt, der in A kein Linksnullteiler ist. Dann gilt $ax = nx \neq 0$ für eine feste ganze rationale Zahl n und für jedes $x \in A$ mit $x \neq 0$. Da $aA = nA$, $AaA = AnA = nA^2 = nA$ und $a \in aA + AaA$ bestehen, gibt es ein b mit $a = nb$ $(b \in A)$, woraus $nbx = nx \neq 0$ für jedes $x \in A$ mit $x \neq 0$ und somit $n(bx - x) = 0$ und $bx = x$ für jedes $x \in A$ folgt. Also ist b ein Linkseinselement von A. Da jedes Element von $A(1-b) = \{x - xb; x \in A\}$ ein Linksannullator des E_5-Ringes A ist, gilt $A(1-b) = 0$ und somit auch $xb = x$ für jedes $x \in A$. Also ist b ein zweiseitiges Einselement von A. (4) Ist A ein torsionsfreier E_5-Ring mit einem Linksmultiplikator a, so gilt $ax = nx$ für jedes $x \in A$ mit festem $n \neq 0$, denn A hat keinen von Null verschiedenen Linksannullator. Wegen der Torsionsfreiheit von A^+ ist a kein Linksnullteiler von A, und somit läßt sich der Satz 3. 1. 3 für den Beweis der Existenz des Einheitselementes von A anwenden, w. z. b. w.

Satz 3. 2. (1) *Ein beliebiger Ring A hat dann und nur dann ein Einselement, wenn A ein E_5-Ring ist, der ein solches Element a mit $Aa \subseteqq aA$ besitzt, das in A kein*

Linksnullteiler ist. (2) *Jeder E_5-Ring A mit einem von Null verschiedenen Zentrum Z besitzt ein homomorphes Bild A' mit einem Einselement e'.*

Beweis. (1) Offenbar auch jetzt genügt es nur das Hinreichen der Bedingung im Satz 3. 2. 1 zu beweisen. Ist A ein E_5-Ring, der ein solches Element a mit $Aa \subseteqq aA$ besitzt, das in A kein Linksnullteiler ist, so gilt $AaA \subseteqq aA$ und somit nach dem Satz 2. 2. 2 auch $a \in aA$. Also gibt es ein Element $b \in A$ mit $a = ab$ und $a(1-b)A$. Da a in A kein Linksnullteiler ist, ergibt sich $(1-b)A = 0$ und somit ist b ein Linkseinselement von A. Da jedes Element von $A(1-b)$ ein Linksannullator des E_5-Ringes A ist, erhält man auch $A(1-b) = 0$, und somit ist b ein zweiseitiges Einselement von A. (2) Ist z ein von Null verschiedenes Element des Zentrums Z eines E_5-Ringes A, so gibt es nach dem Satz 2. 2. 2 ein Element $b \in A$ mit $z = zb = bz$. Ist das Ideal A_z von A der Annullator von z, so gilt $A_z \neq A$, denn $z \neq 0$ ist kein Annullator des E_5-Ringes A. Weiterhin ergibt sich für das Ideal $K = (1-b)A + A(1-b) + A(1-b)A$ gewiß $K \subseteqq A_z$. Dann ist A/K ein von Null verschiedenes homomorphes Bild von A mit zweiseitigem Einselement $e' = b + K$, w. z. b. w.

Satz 3. 3. (1) *Ein beliebiger Ring A hat dann und nur dann ein Einselement, wenn A ein E_5-Ring ist, der ein solches Zentrumelement $z \neq 0$ besitzt, das in A kein Nullteiler ist.* (2) *Ein beliebiger Ring A hat dann und nur dann ein Einselement, wenn A ein E_5-Ring ist, der ein solches Element a mit $aA = A$ besitzt, das mit jedem Rechtsideal der Gestalt $(1-b)A$ von A (für jedes $b \in A$) vertauschbar ist, d. h. es gilt $a(1-b)A = (1-b)Aa$.*

Beweis. Es genügt nur das Hinreichen der Bedingung zu beweisen. (1) Ist $z \neq 0$ ein Zentrumelement des E_5-Ringes A, das in A kein Nullteiler ist, so gilt $Az \subseteqq zA$, und somit läßt sich der Satz 3. 2. 1 für den Beweis der Existenz des Einselementes in A anwenden. (2) Es seien A ein E_5-Ring, a ein solches Element von A mit $aA = A$, für das $a(1-b)A = (1-b)A \cdot a$ für jedes $b \in A$ gilt. Es gibt wegen $aA = A$ ein Element $c \in A$ mit $ac = a$, woraus wegen $(1-c)Aa = a(1-c)A = (a(1-c))A = 0.A = 0$ und wegen $aA = A$ bzw. wegen $A^2 = A$ folgt gewiß $(1-c)AaA = (1-c)A = 0$. Also ist c ein Linkseinselement von A, und da jedes Element von $A(1-c)$ ein Linksannullator des E_5-Ringes A ist, ergibt sich $A(1-c) = 0$, und somit ist c ein zweiseitiges Einselement von A, w. z. b. w.

Satz 3. 4. (1) *Ein beliebiger Ring A hat dann und nur dann ein Einselement, wenn A ein E_2-Ring mit einem Linksmultiplikator ist.* (2) *Ein beliebiger Ring A hat dann und nur dann ein Einselement, wenn A ein solcher E_2-Ring mit einem Rechtsmultiplikator ist, der keinen von Null verschiedenen Rechtsannullator besitzt.*

Beweis. Für die Notwendigkeit der Bedingungen in beiden Behauptungen bemerken wir, daß jeder E_0-Ring ein E_2-Ring ist. Ist nämlich e das Einselement eines E_0-Ringes A und ist M ein beliebiger A-Rechtsmodul, so ist Me wegen $MeA = MA \subseteqq M$ und $Me \supseteqq MeAe = MeA$ ein A-Rechtsmodul und $M(1-e)$ ist der maximale triviale A-untermodul von $M = M(1-e) \oplus Me$. Also folgt der Beweis des Hinreichens beider Bedingungen. (1) Es sei A ein E_2-Ring mit einem Linksmultiplikator $b \neq 0$. Dann gibt es eine ganze rationale Zahl n mit $(b+n)A = 0$. Hiernach gehört das Paar (b, n) aus der kanonischen Ringerweiterung A_1 von A mit einem Einselement zum maximalen trivialen Untermodul A_0 des A-Rechtsmoduls A_1 (vgl. mit dem Beweis des Satzes 2. 3. 1). Da A nach dem Satz 2. 3. 1 linksannullatorfrei ist, existiert zu jedem n höchstens ein (eventuell kein) $b\in A$ mit $(b, n)\in A_0$. Es sei nun $(a, m)\in A_0$ ein solches Paar, für das der absolute Wert $|m|$ minimal in der Menge aller $|n|$ mit $(b, n)\in A_0$ ist. Es kann offenbar $m>0$ vorausgesetzt werden, denn A ist ein E_2-Ring und somit linksannullatorfrei. Gilt nun $n = mq+r$ mit $0\leqq r<m$ für ein beliebiges Paar (b, n) von A_0, so erhält man wegen $(b-qa)x = = bx-qax = -nx+qmx = -rx$ für jedes $x\in A$ gewiß $(b-qa, r)\in A_0$, was im Falle $r\neq 0$ der Minimalität von $|m|$ mit $(a, m)\in A_0$ widerspricht. Also $r=0$ ist notwendig, und somit auch $(b-qa)x = 0$ für jedes $x\in A$. Da aber A keinen von Null verschiedenen Linksannullator besitzt (vgl. Satz 2. 3. 1), ergibt sich $b=qa$ und somit $(b, n)=q(a, m)$. Also ist die Abelsche Gruppe A_0^+ zyklisch und offenbar unendlich. Da A ein E_2-Ring ist, erhält man $A_1 = A_0\oplus A_2$ mit einem A-Untermodul A_2 von A_1, und es gilt

$$(0, 1) = k(a, m)+(-ka, 1-km)$$

mit $k(a, m)\in A_0$, $(-ka, 1-km)\in A_0$ und mit einer ganzen rationalen Zahl k. Wegen $A_0\neq 0$ ergibt sich $k\neq 0$. Da

$$(x, 0) = (0, 1)x = (-ka, 1-km)a\in A_2$$

für jedes $x\in A$ gilt, erhält man wegen $(-ka, 0)\in A_2$ auch $(0, 1-km)\in A_2$, woraus wegen $m(0, 1-km)\in A_2$ gewiß

$$(1-km)(a, m) = ((1-km)a, 0)+(0, m(1-km))\in A_2\cap A_0$$

und somit $1-km = 0$ folgt, denn (a, m) erzeugt eine unendliche zyklische additive Gruppe. Wegen $m>0$ und $km=1$ ergibt sich $m=1$, und somit ist $e = -a$ wegen $ax+x = 0$ für jedes $x\in A$ ein Linkseinselement von A. Da jedes Element von $A(1-e)$ ein Linksannullator des E_2-Ringes A ist, gilt nach dem Satz 2. 3. 1 $A(1-e) = 0$, und somit ist e das Einselement von A. (2) Zuerst zeigen wir, daß jeder Rechtsmultiplikator eines rechtsannullatorfreien Ringes A auch ein Linksmultiplikator ist, und somit läßt sich der Satz 3. 4. 1 zum Beweis des Satzes 3. 4. 2 anwenden. Es sei also A ein rechtsannulatorfreier Ring mit einem Rechtsmultiplikator a. Dann gibt es eine ganze rationale Zahl n mit $xa=nx$ für jedes $x\in A$, woraus man wegen $(xy)a=n(xy)$ auch

$$x(ya)=(xy)a=n(xy)=(nx)y=(xa)y=x(ay)$$

für jedes $x, y\in A$ erhält. Also gilt $x(ya-ay) = 0$ für jedes $x, y\in A$ und somit $A(ya-ay) = 0$ für jedes $y\in A$. Da aber A keinen von Null verschiedenen Rechtsannullator besitzt, ergibt sich $ya=ay$ für jedes $y\in A$ und somit $ay=ny$ für jedes $y\in A$. Also ist a auch ein Linksmultiplikator von A. Ist nun A insbesondere ein E_2-Ring, so hat A nach dem vorigen ein Einselement, w. z. b. w.

Zum Schluß möchte ich einige offene Fragen erwähnen.

Problem 1. Was ist eine notwendige und hinreichende Bedingung dafür, daß die im Satz 2. 3. 4 vorkommende additive Untergruppe D ein Rechtsideal bzw. ein Ideal von B ist?

Problem 2. Gibt es einen von Null verschiedenen E_2-Ring, der ein Radikalring im Sinne von BROWN—McCOY [2] ist?

Problem 3. (A. KERTÉSZ [10]) Hat jeder E_2-Ring ein Einselement?

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(Eingegangen am 23. April 1966)

О семействах автоматных отображений

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Как известно, между множеством всех автоматных отображений $\varphi: F(X) \to F(X)$, обладающих фиксированным алфавитом X ($\overline{\overline{X}} < \aleph_0$) и множеством (инициальных) автоматов $\mathbf{A}$, их осуществляющих, можно установить однозначное соответствие $\mathbf{A}_i \to \varphi_i$ так, что произведение $\varphi_1 \dots \varphi_k$ отображеный $\varphi_1, \dots, \varphi_k$ осуществляется суперпозицией $\mathbf{A}_1 \dots \mathbf{A}_k$ автоматов $\mathbf{A}_1, \dots, \mathbf{A}_k$. Таким образом, если мы интересуемся отображениями, осуществляемыми суперпозициями некоторых автоматов $\mathbf{A}_i$ ($i=1, \dots, k$), то достаточно ограничиваться исследованием подполугруппы с образующими $\varphi_1, \dots, \varphi_k$ полугруппы H_X всех автоматных отображений φ: $F(X) \to F(X)$.

Б. Чакань предложил поставить в соответствие классу всех конечных автоматов некоторую алгебраическую систему так, что — аналогично вышесказанным — при рассмотрении отображений, осуществляемых композициями (произведениями) некоторого типа данных конечных автоматов можно было ограничиваться рассмотрением подсистемы, порожденной соответствующими элементами этой алгебраической системы. В настоящей статье дается решение этой проблемы относительно R-произведений [1]. При этом доказывается, что с точки зрения представления отображений квази-суперпозиция автоматов, введенная в [2] равносильна R-произведению тех же автоматов.

§ 1

X и Y будут обозначать произвольные конечные множества. Пусть $\varphi: F(X) \to F(Y)$ — произвольное автоматное отображение, а $\varphi_p: F(X) \to F(Y)$ $\big(p \in F(X)\big)$ — отображение, переводящее слово $q\big(\in F(X)\big)$ в слово r, удовлетворяющее равенству $\varphi(pq) = \varphi(p)r$ [1]). Следуя Рэни [3], мы будем называть φ_p *состоянием* отображения φ. Мы говорим, что φ — отображение *веса* n (обозначение: $s(\varphi) = n$), если φ обладает n различными состояниями. φ называется отображением *конечного веса*, если $n < \aleph_0$. Класс всех автоматных отображений конечного веса будем обозначать через L. Для начального

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[1]) $F(X)$ — свободная полугруппа с системой образующих X, e — единичный элемент полугруппы $F(X)$, удовлетворяющий равенствам $\varphi(e) = e$ и $\varphi_e = \varphi$.

отрезка $p'=x_1...x_i$ $(0\leqq i\leqq n)$ слова $p=x_1...x_n$ будем пользоваться обозначением $p(i+1)$. В частности, $p(1)=e$ и $p(n+1)=p$.

Пусть $\varphi\colon F(X)\to F(Y)$ — любое отображение конечного веса, V и Z — произвольные конечные множества, а ϑ и ϑ' — однозначные отображения V в X и $\langle\varphi_p|p\in F(X)\rangle\times V$ в Z, соответственно.

Отображения ϑ и ϑ' мы продолжим и на $F(V)$: если $q=v_1...v_n\in F(V)$, то

$$\vartheta(v_1...v_n)=\vartheta(v_1)...\vartheta(v_n)$$

и

$$\vartheta'(\varphi_p, q)=\vartheta'(\varphi_{p(\vartheta(q)(1))}, v_1)...\vartheta'(\varphi_{p(\vartheta(q)(n))}, v_n).$$

Отображение $\psi\colon F(V)\to F(Z)$, для которого выполняется $\psi(q)=\vartheta'(\varphi, q)$ $(q\in F(V))$, будем называть *производным отображением* от φ и обозначать через $(\varphi, \vartheta, \vartheta')$, если только необходимо знать, о каких отображениях (ϑ, ϑ') идет речь.[2]) Отношение между отображением и его производным выражается символом $\prec$ (например $\psi\prec\varphi$).

Подкласс L_i класса L назовем *семейством отображений*, если L_i содержит вместе с каждым отображением $\varphi\in L_i$ и все производные от φ. Подмножество $\langle\varphi_1, ..., \varphi_k\rangle$ семейства L_i $(L_i\subseteqq L)$ назовем *базисом* L_i (обозначение: $L_i=$ $=[\varphi_1, ..., \varphi_k]$), если любое $\varphi\in L_i$ является производным хотя бы от одного φ_j $(j=1, ..., k)$. Естественным путем определяются семейства с конечным базисом. Такие семества в дальнейшем будут отличаться — вместо оговорки — жирным шрифтом.

Пусть $\varphi\colon F(X)\to F(Y)$ и $\psi\colon F(V)\to F(Z)$ — произвольные отображения конечного веса. Определим отображение ξ множества $F(X\times V)$ в множество $F(\langle\varphi_p|p\in F(X)\rangle\times\langle\psi_q|q\in F(V)\rangle)$ следующим образом:

$$\xi((p, q))=(\varphi, \psi)_{(p,q)(1)}...(\varphi, \psi)_{(p,q)(n)}=(\varphi_{p(1)}, \psi_{q(1)})...(\varphi_{p(n)}, \psi_{q(n)}),$$

где $n=d((p, q))=d(p)=d(q)$ ($d(p)$ обозначает длину слова p). Нетрудно показать, что $\xi\in L$ и вес ξ не больше произведения весов φ и ψ. ξ называется *прямой суммой* отображений φ и ψ. Мы пишем: $\xi=\varphi+\psi$.

Берем совокупность $\mathbf{L}$ всех семейств отображений с конечным базисом. В $\mathbf{L}$ мы вводим две двуместных операции: *сложение* и *произведение*, которые

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[2]) Пусть $\mathbf{A}=\mathbf{A}(X, A, a_0, Y, \delta, \lambda)$ — минимальный автомат, индуцирующий отображение φ. Как известно, в качестве множества A можно брать $\langle\varphi_p|p\in F(X)\rangle$. В этом случае

$$\delta(\varphi_p, x)=\varphi_{px},$$

$$\lambda(\varphi_p, x)=\varphi_p(x).$$

Рассмотрим произвольные конечные множества V и Z, далее, однозначные отображения $\vartheta\colon V\to X$ и $\vartheta'\colon\langle\varphi_p|p\in F(X)\rangle\times V\to Z$. Обозначим через $\mathbf{B}=\mathbf{B}(V, A, a_0, Z, \delta', \lambda')$ автомат, для которого выполняется

$$\delta'(\varphi_p, v)=\delta(\varphi_p, \vartheta(v)),$$

$$\lambda'(\varphi_p, v)=\vartheta'(\varphi_p, v)\qquad(\varphi_p\in A,\quad v\in V).$$

Нетрудно показать, что $(\varphi, \vartheta, \vartheta')$ не что иное, как отображение, индуцируемое автоматом $\mathbf{B}$.

Заметим еще, что отношение $\prec$ — транзитивно.

обозначим символами $\oplus$ и $\odot$, соответственно. А именно, для произвольных $\mathbf{L}_i, \mathbf{L}_j \in \mathbf{L}$ пусть

$$\mathbf{L}_i \oplus \mathbf{L}_j = \bigcup_{\varphi_i, \psi_j} [\varphi_i + \psi_j] (\varphi_i \in \mathbf{L}_i, \psi_j \in \mathbf{L}_j);$$

и

$$\mathbf{L}_i \odot \mathbf{L}_j = \bigcup_{\varphi_i, \psi_j} [\varphi_i \psi_j] \big(\varphi_i(: F(X) \to F(Y)) \in \mathbf{L}_i, \ \psi_j(: F(Y) \to F(Z)) \in \mathbf{L}_j\big).$$

Автоматное отображение $\psi: F(U) \to F(V)$ назовем *гомоморфным образом* отображения $\varphi: F(X) \to F(Y)$, если найдутся такие однозначные отображения X на U и Y на V: ξ и ξ' соответственно, что для любого $p \in F(X)$ имеет место соотношение $\xi'(\varphi(p)) = \psi(\xi(p))$. Здесь ξ и ξ' продолжены на $F(X)$ и $F(Y)$ естественным (гомоморфным в обычном смысле слова) образом. Если ξ и ξ' являются взаимно однозначными, то мы говорим, что φ и ψ *изоморфны* и пишем $\varphi \cong \psi$.

Отметим, что одновременное выполнение $\varphi \cong \psi$ и $\varphi \in \mathbf{L}_i$ влечет за собой $\psi \in \mathbf{L}_i$. Далее, из соотношений $\varphi \cong \psi$ и $\varphi' \cong \psi'$ не обязательно вытекает $\varphi\varphi' \cong \psi\psi'$.

Следует также отметить, что результаты этой статьи могли бы быть получены и без употребления термина „класс", если вместо любого множества из n элементов было бы рассмотрено фиксированное множество $\langle 1, \ldots, n \rangle$. Это, однако, привело бы к громоздким вычислениям, не дающим нам ничего существенно нового.

§ 2

В этой части производится несколько более подробное исследование понятий, введенных в § 1. Относительно производных автоматных отображений имеет место

Лемма 1. *Если $\varphi \in L$, то и любое производное ψ от φ содержится в L; далее, $s(\psi) \leqq s(\varphi)$.*

Сперва покажем, что если имеет место $\psi \prec \varphi$, то и ψ является автоматным отображением, т. е. ψ сохраняет длину слов и начальный отрезок слов отображает в начальный отрезок образа. Пусть $\psi(: F(V) \to F(Z)) = \big(\varphi(: F(X) \to F(Y)), \gamma, \gamma'\big)$. По определению производного отображения очевидно, что ψ сохранит длин слов.

Пусть $q = v_1 \ldots v_n, r = v_{n+1} \ldots v_m$ $(q, r \in F(V))$ — произвольные слова. Тогда выполняется равенство

$$\psi(qr) = \psi(v_1 \ldots v_n v_{n+1} \ldots v_m) = \gamma'(\varphi_{\gamma(q)(1)}, v_1) \ldots \gamma'(\varphi_{\gamma(q)(n)}, v_n) \cdot \gamma'(\varphi_{\gamma(qr)(n+1)}, v_{n+1}) \ldots$$
$$\ldots \gamma'(\varphi_{\gamma(qr)(m)}, v_m) = \psi(q)s \qquad (s \in F(Z)),$$

т. е. ψ отображает начальный отрезок слов в начальный отрезок образа.

Надо еще доказать, что выполняется и $s(\psi) \leqq s(\varphi)$. Для этой цели достаточно показать, что из равенств $\gamma(q) = p$, $\gamma(q') = p'$ и $\varphi_p = \varphi_{p'}$ $(q, q' \in F(V); p, p' \in F(X))$ следует равенство $\psi_q = \psi_{q'}$. Последнее утверждение верно, так как для произвольного $r \in F(V)$ имеет место равенство $\psi_q(r) = \gamma'(\varphi_{\gamma(q)}, r) = \gamma'(\varphi_{\gamma(q')}, r) = \psi_{q'}(r)$.

Лемма 2. *Если* $\mathbf{L}_i, \mathbf{L}_j \in \mathbf{L}$, *то* $\mathbf{L}_i \oplus \mathbf{L}_j \in \mathbf{L}$ *и* $\mathbf{L}_i \odot \mathbf{L}_j \in \mathbf{L}$.

Ввиду транзитивности отношения $\prec$, по определению суммы и произведения очевидно, что сумма и произведение двух семейств отображений является семейством отображений. Далее, так как вес $\varphi+\psi$, $\varphi\psi$ и производных отображений от них не больше чем $s(\varphi)s(\psi)$, то существует такое натуральное число n, что для всех $\xi \in \mathbf{L}_i \oplus \mathbf{L}_j$ и $\xi \in \mathbf{L}_i \odot \mathbf{L}_j$ справедливы неравенства $s(\xi) \leqq n$. Поэтому, если мы покажем, что каждое такое семейство $L'(\subseteqq L)$ отображений, для которого при фиксированном натуральном n выполняется $s(\varphi) \leqq n$ $(\varphi \in L')$, обладает конечным базисом, то и получается доказательство леммы 2.

Итак, пусть L' — такое семейство отображений, для которого существует натуральное n так, что $s(\varphi) \leqq n$ справедливо для всех $\varphi \in L'$. Пусть $\varphi\colon F(X) \to F(Y)$ $(\varphi \in L')$ — некоторое отображение, где $\overline{\overline{X}} > n^n$. В этом случае найдутся два символа $x, x' \in X$ таких, что для всех $p \in F(X)$ имеем $\varphi_{px} = \varphi_{px'}$. Действительно, так как число всех однозначных отображений в себя множества из n элементов равно n^n, то наше утверждение справедливо.[3]) Пусть $\overline{\varphi}\colon F(X^*) \to F(\langle \varphi_p | p \in F(X) \rangle)$ $(X^* = X \setminus x')$ — отображение, переводящее слово $x_1 \ldots x_l \in F(X^*)$ в слово $\varphi\,\varphi_{x_1} \ldots \varphi_{x_1 \ldots x_{l-1}}$. Очевидно, что $\overline{\varphi}$ представяется в виде

$$\overline{\varphi} = (\varphi, \vartheta, \vartheta') \quad (\vartheta\colon X^* \to X,\ \vartheta'\colon \langle \varphi_p | p \in F(X) \rangle \times X^* \to \langle \varphi_{\overline{p}} | \overline{p} \in F(X^*) \rangle),$$

где $\vartheta(\overline{x}) = \overline{x}$ и $\vartheta'(\varphi_{\overline{p}}, \overline{x}) = \varphi_{\overline{p}}$ $(\overline{x} \in X^*, \overline{p} \in F(X^*))$, т. е. $\overline{\varphi} \prec \varphi$. Покажем, что выполняется и отношение $\varphi \prec \overline{\varphi}$, иными словами, имеет место равенство $\varphi = (\overline{\varphi}, \gamma, \gamma')$, где γ, γ' — подходящие отображения. Определим $\gamma\colon X \to X^*$ и $\gamma'\colon \langle \varphi_p | p \in F(X^*) \rangle \times X \to Y$ следующим образом:

$$\gamma(x_i) = \begin{cases} x_i, & \text{если } x_i \neq x' \\ x, & \text{если } x_i = x' \end{cases} \quad \text{и} \quad \gamma'(\overline{\varphi}_p, x_i) = \varphi_p(x_i)\,(p \in F(X^*)).$$

Нетрудно доказать, что для этой пары (γ, γ') справедливо равенство $\varphi = (\overline{\varphi}, \gamma, \gamma')$. Но $\overline{\overline{X^*}} < \overline{\overline{X}}$, и таким образом, продолжая этот процесс, принимая во внимание транзитивность отношения $\prec$ и конечность[4]) числа отображений $\varphi\colon F(X) \to F(\langle \varphi_p | p \in F(X) \rangle)$, при $\overline{\overline{X}} \leqq n^n$ и $s(\varphi) \leqq n$ получается, что L' обладает конечным базисом. Этим лемма 2 доказана.

Относительно произведения семейств отображений справедлива следующая

Лемма 3. *Для любых элементов* $\mathbf{L}_i, \mathbf{L}_j \in \mathbf{L}$ *выполняется равенство*

$$\mathbf{L}_i \odot \mathbf{L}_j = \langle \varphi_i \psi_j |\ \varphi_i(\colon F(X) \to F(Y)) \in \mathbf{L}_i,\ \ \psi_j(\colon F(Y) \to F(Z)) \in \mathbf{L}_j \rangle.$$

Пусть ε имеет вид $\varepsilon = (\varphi\psi, \gamma, \gamma') \in \mathbf{L}_i \odot \mathbf{L}_j$, где $\varepsilon\colon F(U) \to F(V)$, $\varphi\colon F(X) \to F(Y)$ и $\psi\colon F(Y) \to F(Z)$ $(\varphi \in \mathbf{L}_i, \psi \in \mathbf{L}_j)$. Покажем существование таких $\varphi' \prec \varphi$ и $\psi' \prec \psi$,

[3]) Здесь речь идет об отображениях $x\colon \varphi_p \to \varphi_{px}$, где φ и $x \in X$ — фиксированы; $p \in F(X)$ и $\varphi \in L'$, т. е. $\overline{\overline{\langle \varphi_p | p \in F(X) \rangle}} \leqq n$.

[4]) Здесь изоморфные отображения считаются тождественными.

что $\varepsilon = \varphi' \psi'$. Рассмотрим отображения

$$\varphi'(:F(U) \to F(U \times \langle \varphi_p | p \in F(X) \rangle \times Y)) = (\varphi, \varrho, \varrho')$$
$$(\varrho' : \langle \varphi_p | p \in F(X) \rangle \times U \to U \times \langle \varphi_p | p \in F(X) \rangle \times Y)$$

и

$$\psi'(:F(U \times \langle \varphi_p | p \in F(X) \rangle \times Y) \to F(V)) = (\psi, \vartheta, \vartheta'),$$

где

$$\varrho(u) = \gamma(u),\ \varrho'(\varphi_p, u) = (u, \varphi_p, \varphi_p(\varrho(u)))\,(p \in F(X),\ u \in U)$$

и

$$\vartheta(u, \varphi_p, y) = y,\ \vartheta'(\psi_q, (u, \varphi_p, y)) = \gamma'((\varphi_p, \psi_q), u)\,(q \in F(Y)).$$

Пусть $r = u_1 \dots u_n \in F(U)$ — произвольное слово. Тогда

$$\varepsilon(r) = \gamma'((\varphi\psi), u_1) \dots \gamma'((\varphi\psi)_{\gamma(r)(n)}, u_n) = \gamma'((\varphi, \psi), u_1) \dots$$
$$\dots \gamma'((\varphi_{\gamma(r)(n)}, \psi_{\varphi(\gamma(r)(n))}), u_n)$$

и

$$\varphi'(r) = \varrho'(\varphi, u_1) \dots \varrho'(\varphi_{\varrho(r)(n)}, u_n) = (u_1, \varphi, \varphi(\varrho(u_1))) \dots (u_n, \varphi_{\varrho(r)(n)}, \varphi_{\varrho(r)(n)}(\varrho(u_n))) =$$
$$= (u_1, \varphi, \varphi(\gamma(u_1))) \dots (u_n, \varphi_{\gamma(r)(n)}, \varphi_{\gamma(r)(n)}(\gamma(u_n)));$$
$$(\varphi'\psi')(r) = \psi'(\varphi'(r)) = \psi'((u_1, \varphi, \varphi(\gamma(u_1))) \dots (u_n, \varphi_{\gamma(r)(n)}, \varphi_{\gamma(r)(n)}(\gamma(u_n)))) =$$
$$= \vartheta'(\psi, (u_1, \varphi, \varphi(\gamma(u_1)))) \dots \vartheta'(\psi_{\varphi(\gamma(r)(n))}, (u_n, \varphi_{\gamma(r)(n)}(\gamma(u_n)))) =$$
$$= \gamma'((\varphi, \psi), u_1) \dots \gamma'((\varphi_{\gamma(r)(n)}, \psi_{\varphi(\gamma(r)(n))}), u_n);$$
$$\varepsilon(r) = (\varphi'\psi')(r).$$

Лемма 3 доказана.

Предложение. *Множество* $\mathbf{L}$ *является структурно упорядоченной алгебраической системой относительно сложения, умножения и теоретико-множественного включения.*

Для того, чтобы $\mathbf{L}$ при упорядочении $\subseteqq$ образовало структуру относительно операций пересечения и объединения, достаточно показать, что пересечение $\mathbf{L}_i \cap \mathbf{L}_j$ произвольных $\mathbf{L}_i$, $\mathbf{L}_j \in \mathbf{L}$ обладает конечным базисом; аналогичное утверждение справедливо, очевидно, для объединений. $\mathbf{L}_i \cap \mathbf{L}_j \neq \emptyset$, так как $[\varphi] \subseteqq \mathbf{L}_i$ выполняется для всех $\mathbf{L}_i$, где $s(\varphi) = 1$. Далее, мы видим, что $\mathbf{L}_i \cap \mathbf{L}_j$ является семейством отображений. Так как весы отображений, содержащихся в $\mathbf{L}_i$ или $\mathbf{L}_j$ имеют общую верхнюю границу, то и весы отображений из $\mathbf{L}_i \cap \mathbf{L}_j$ обладают общей верхней границей. Отсюда, в силу леммы 2, получается существование конечного базиса для $\mathbf{L}_i \cap \mathbf{L}_j$.

Теперь покажем ассоциативность сложения и умножения, т. е. докажем справедливость равенств $(\mathbf{L}_i \oplus \mathbf{L}_j) \oplus \mathbf{L}_k = \mathbf{L}_i \oplus (\mathbf{L}_j \oplus \mathbf{L}_k)$ и $(\mathbf{L}_i \odot \mathbf{L}_j) \odot \mathbf{L}_k = \mathbf{L}_i \odot (\mathbf{L}_j \odot \mathbf{L}_k)$ для всех $\mathbf{L}_i, \mathbf{L}_j, \mathbf{L}_k \in \mathbf{L}$.

Ассоциативность произведения, в силу леммы 3, тривиально, так как произведение отображений является ассоциативным. Ассоциативность сложения вытекает из того, что элементы из $(\mathbf{L}_i \oplus \mathbf{L}_j) \oplus \mathbf{L}_k$ и только они представляются в виде $((\varphi + \psi + \tau), \gamma, \gamma')$ $(\varphi \in \mathbf{L}_i, \psi \in \mathbf{L}_j, \tau \in \mathbf{L}_k)$; то же самое верно для элементов из $\mathbf{L}_i \oplus (\mathbf{L}_j \oplus \mathbf{L}_k)$.

Для завершения доказательства нашего предложения надо еще показать (см. [4] стр. 191), что имеют место следующие соотношения:

$$\mathbf{L}_i \oplus (\mathbf{L}_j \cup \mathbf{L}_k) = (\mathbf{L}_i \oplus \mathbf{L}_j) \cup (\mathbf{L}_i \oplus \mathbf{L}_k),$$

$$(\mathbf{L}_j \cup \mathbf{L}_k) \oplus \mathbf{L}_i = (\mathbf{L}_j \oplus \mathbf{L}_i) \cup (\mathbf{L}_k \oplus \mathbf{L}_i)$$

$$\mathbf{L}_i \odot (\mathbf{L}_j \cup \mathbf{L}_k) = (\mathbf{L}_i \odot \mathbf{L}_j) \cup (\mathbf{L}_i \odot \mathbf{L}_k),$$

и

$$(\mathbf{L}_j \cup \mathbf{L}_k) \odot \mathbf{L}_i = (\mathbf{L}_j \odot \mathbf{L}_i) \cup (\mathbf{L}_k \odot \mathbf{L}_i) \quad (\mathbf{L}_i, \mathbf{L}_j, \mathbf{L}_k \in \mathbf{L}).$$

Это, однако, вытекает из самого определения сложения и умножения. Отсюда получается и монотонность этих операций.

Заметим, что семейство $\mathbf{L}_i$ ($\in \mathbf{L}$) отображений, для отображения φ которого выполняется $s(\varphi)=1$, является единичным элементом для умножения и сложения. Поэтому мы его обозначим через $\mathbf{L}_e$.

§ 3

Операции и структурное упорядочение, введенные в предыдущих параграфах, связаны между собой, кроме монотонности, и следующим законом.

Теорема 1. *Для любых* $\mathbf{L}_i, \mathbf{L}_j \in \mathbf{L}$ *имеет место* $\mathbf{L}_i \oplus \mathbf{L}_j \subseteqq \mathbf{L}_i \odot \mathbf{L}_j$.

Для доказательства теоремы 1, ввиду определения сложения и транзитивности отношения $\prec$, достаточно показать, что для произвольных φ, ψ ($\varphi \in \mathbf{L}_i, \psi \in \mathbf{L}_j$) выполняется $\varphi + \psi \in \mathbf{L}_i \odot \mathbf{L}_j$. Пусть φ и ψ имеют вид $\varphi: F(X) \to F(Y)$ и $\Psi: F(V) \to F(Z)$. Рассмотрим отображения φ' и ψ', определенные следующим образом: $\varphi' = (\varphi, \gamma, \gamma')$ и $\psi' = (\psi, \vartheta, \vartheta')$, где

$$\gamma\big((x, v)\big) = x, \quad \gamma'\big(\varphi_p, (x, v)\big) = (x, v, \varphi_p),$$

$$\vartheta\big((x, v, \varphi_p)\big) = v, \quad \vartheta'\big(\psi_q, (x, v, \varphi_p)\big) = (\varphi_p, \psi_q)$$

$$\big((x, v) \in X \times V, \quad p \in F(X), \quad (x, v, \varphi_p) \in X \times V \times \langle \varphi_p | p \in F(X) \rangle, \quad q \in F(V)\big).$$

Покажем, что $\varphi + \psi = \varphi'\psi'$. Отсюда, ввиду $\varphi' \prec \varphi$ и $\psi' \prec \psi$, получим доказательство теоремы 1.

Пусть $r = (x_1, v_1) \dots (x_n, v_n) \in F(X \times V)$ — произвольное слово. Тогда

$$(\varphi'\psi')(r) = \psi'(\varphi'(r)) = \psi'\big((x_1, v_1, \varphi_{r(1)_1})\big) \dots \big(x_n, v_n, \varphi_{r(n)_1})\big) = (\varphi_{r(1)_1}, \psi_{r(1)_2}) \dots$$

$$\dots (\varphi_{r(n)_1}, \psi_{r(n)_2}) = (\varphi, \psi)_{r(1)} \dots (\varphi, \psi)_{r(n)} = (\varphi + \psi)(r) \text{ [5]}.$$

Этим теорема 1 доказана.

Следствие. *Если* $\mathbf{L}_k \in \mathbf{L}$ *получается из элементов* $\mathbf{L}_1, \dots, \mathbf{L}_m$ *путем* l*-кратного применения сложения и умножения, то существует такое* $\mathbf{L}'_k \in \mathbf{L}$, $\mathbf{L}'_k \supseteqq \mathbf{L}_k$, *которое получается из* $\mathbf{L}_1, \dots, \mathbf{L}_m$ *путем* l*-кратного применения единственного умножения.*

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[5]) Здесь $(x_1, y_1) \dots (x_n, y_n)_i$ $(i = 1, 2)$ означает слово $x_1 \dots x_n$, если $i = 1$, и слово $y_1 \dots y_n$, если $i = 2$.

В самом деле, такое $\mathbf{L}'_k$ получается из $\mathbf{L}_k$, заменяя в выражении $\mathbf{L}_k$ через $\mathbf{L}_1, \ldots, \mathbf{L}_m$ знак $\oplus$ везде знаком $\odot$.

Автомат $\mathbf{A}=\mathbf{A}(X, A, a_0, A, \delta, \lambda)$ называется автоматом Медведева, если для всех $a \in A$ и $x \in X$ имеет место $\lambda(a, x)=a$.

Рассмотрим конечное или бесконечное множество $\mathfrak{A}$ конечных автоматов Медведева и пусть $R=\langle 1, \ldots, k\rangle$ — некоторое частично упорядоченное множество. Каждому элементу $i \in R$ однозначно сопоставим элемент из $\mathfrak{A}$. Автомат, сопоставляемый элементу i, мы будем обозначать через $\mathbf{A}_i$. Пусть X, Y — любые конечные множества, а ξ и χ — отображения, отображающие $A_1 \times \ldots \times A_k \times X$ в $X_1 \times \ldots \times X_k$ и в Y, соответственно. Автомат $\mathbf{A} = \mathbf{A}(X, A, a_0, Y, \delta, \lambda)$ мы будем называть R-произведением автоматов $\mathbf{A}_1, \ldots, \mathbf{A}_k$ относительно множеств X, Y и отображений ξ, χ, если выполняются условия:

(1) $$A = A_1 \times \ldots \times A_k,$$

(2) $$a_0 = (a_{10}, \ldots, a_{k0}),$$

(3) $$\delta((a_1, \ldots, a_k), x) = (\delta_1(a_1, x_1), \ldots, \delta_k(a_k, x_k)),$$

где

$$(x_1, \ldots, x_k) = \xi(a_1, \ldots, a_k, x) = (\xi_1(a_1, \ldots, a_k, x), \ldots, \xi_k(a_1, \ldots, a_k, x))$$

и ξ_i $(1 \leqq i \leqq k)$ не зависит от элементов из A_j, если j не больше, чем i относительно частичного упорядочения R,

(4) $$\lambda((a_1, \ldots, a_k), x) = \chi(a_1, \ldots a_k, x).$$

Пусть $\mathbf{A}=\mathbf{A}(X', A, a_0, \delta')$ означает произвольный автомат Медведева; X, Y — любые множества, а $\gamma\colon X \to X'$, $\lambda\colon A \times X \to Y$ — произвольные отображения. Через $\mathbf{A}^{(\gamma, \lambda)}=\mathbf{A}^{(\gamma, \lambda)}(X, A, a_0, Y, \delta, \lambda)$ мы обозначим автомат, выходной функцией которого является λ, а для функции перехода выполняется $\delta(a, x)= =\delta'(a, \gamma(x))$ $(a \in A, x \in X)$.

Суперпозицию $\mathbf{A}_1^{(\gamma_1, \lambda_1)} \ldots \mathbf{A}_k^{(\gamma_k, \lambda_k)}$ (если существует) мы назовем квази-суперпозицией относительно $\langle X_i, Y_i, \delta_i, \lambda_i | i=1, \ldots, k\rangle$ автоматов $\mathbf{A}_i=\mathbf{A}_i(X'_i, A_i, a_{i_0}, \delta'_i)$ $(i=1, \ldots, k)$.

Понятно, что квази-суперпозиция является частным случаем R-произведения.

В этом параграфе φ^i означает отображение, индуцируемое автоматом $\mathbf{A}_i$. Обратно, если задано автоматное отображение φ^i, то $\mathbf{A}_i$ служит для обозначения некоторого из автоматов, индуцирующих φ^i.

Подсистему с образующими $\mathbf{L}_1, \ldots, \mathbf{L}_k$ алгебраической системы $\mathbf{L}$ мы обозначим через $\{\mathbf{L}_1, \ldots, \mathbf{L}_k\}$.

Алгебраическая система $\mathbf{L}$ и R-произведения автоматов тесно связаны между собой, как показывает

Теорема 2. *Автоматное отображение φ представляется некоторым R-произведением автоматов Медведева $\mathbf{A}_1, \ldots, \mathbf{A}_k$ тогда и только тогда, если существует такое $\mathbf{L}^*\{\in[\varphi^1], \ldots, [\varphi^k]\}$, что выполняется $\varphi \in \mathbf{L}^*$*

Предположим, что φ представляется R-произведением $\mathbf{A}$ автоматов $\mathbf{A}_1, \ldots, \mathbf{A}_k$, относительно множеств X, Y и отображений ξ, χ. Доказательство проведем индукцией по числу l сомножителей R-произведения. Впервые пусть — $i=1$, т. е. $\mathbf{A}$ обладает единственным сомножителем $\mathbf{A}_i=\mathbf{A}_i(X_i, A_i, a_{i_0}, \delta_i)$. Покажем, что φ представляется в виде $(\varphi^i, \gamma, \gamma')$, где γ, γ' — подходящие отображения. Пусть $\gamma(x)=\xi(x)$ и $\gamma'(\varphi^i_{\gamma(p)}, x)=\chi(a_{i_0}\cdot\gamma(p), x)$. Такой выбор отображения γ' является однозначным, так как для любого отображения φ^i, индуцируемого автоматом Медведева, из $\varphi^i_{pi}=\varphi^i_{qi}$ вытекает $a_{i0}p_i=$ $=a_{i_0}q_i(p_i, q_i\in F(X_i))$. Далее, пусть $p=x_1...x_n\in F(X)$ — произвольное слово. Тогда, выполняется равенство $\gamma'(\varphi^i, p)=\gamma'(\varphi^i, x_1)\cdot\gamma'(\varphi^i_{\gamma(p)(2)}, x_2)...\gamma'(\varphi^i_{\gamma(p)(n)}, x_n)=$ $=\chi(a_{i_0}, x_1)\cdot\chi(a_{i_0}(\gamma(p)(2)), x_2)...\chi(a_{i_0}(\gamma(p)(n)), x_n) = \varphi(p)$. Этим случай $l=1$ исчерпан. Пусть наше утверждение справедливо для чисел, меньших чем $l(\geqq 2)$ и пусть φ представляется R-произведением $\mathbf{A}$ автоматов $\mathbf{A}_j\in\langle\mathbf{A}_i|i=1, ...$ $..., k\rangle$ $(j=1, ..., l)$ относительно множеств X, Y и отображений ξ, χ. Пусть $\mathbf{A}_l$ — один из минимальных элементов относительно частичного упорядочения R. Теперь рассмотрим R_1-произведение $\mathbf{A}'=\mathbf{A}'(X', A', a_0', Y', \delta', \lambda')$ автоматов $\mathbf{A}_1, ..., \mathbf{A}_{l-1}$ относительно $X'=X$, $Y'=A_1\times...\times A_{l-1}\times X$, $\xi': A_1\times...$ $...\times A_{l-1}\times X\to X_1\times...\times X_{l-1}$, $\chi': A_1\times...\times A_{l-1}\times X\to A_1\times...\times A_{l-1}\times X$ и тривиальное R_2-произведение $\mathbf{A}''=\mathbf{A}''(X'', A'', a_0'', Y'', \delta'', \lambda'')$ автомата $\mathbf{A}_l$ относительно $X''=A_1\times...\times A_{l-1}\times X$, $Y''=Y$, $\xi'': A_1\times...\times A_{l-1}\times X\to X_l$, $\chi'': A_1\times...$ $...\times A_{l-1}\times X\to Y$, где R_1 получается ограничением R на множество $\langle 1, ..., l-1\rangle$. Более точно, пусть выполняются

$$\xi'(a_1, ..., a_{l-1}, x)=(\xi_1(a_1, ..., a_{l-1}, a_l, x)...\xi_{l-1}(a_1, ..., a_{l-1}, a_l, x)),$$
$$\chi'(a_1, ..., a_{l-1}, x) = (a_1, ..., a_{l-1}, x),$$
$$\xi''(a_l, (a_1, ... a_{l-1}, (x) = \xi_l(a_1, ..., a_{l-1}, a_l, x),$$
$$\chi''(a_l, (a_1, ..., a_{l-1}, x)) = \chi(a_1, ..., a_{l-1}, a_l, x).$$

(В определении ξ' $a_l\in A_l$ — произвольннй.) Определение ξ' не может содержать противоречие, так как ξ_i $(i=1, ..., l)$ не зависит от $a_l(\in A_l)$. Обозначим через φ' и φ'' отображение, индуцируемое автоматом $\mathbf{A}'$ и $\mathbf{A}''$, соответственно. Докажем, что $\varphi=\varphi'\varphi''$. Для доказательства этого достаточно показать, что можно установить инициальный $\mathfrak{A}$-изоморфизм (см. [5]) между $\mathbf{A}$ и суперпозицией $\mathbf{A}^* = \mathbf{A}^*(X, A^*, a_0^*, Y, \delta^*, \lambda^*)=\mathbf{A}'\mathbf{A}''$. Рассмотрим тождественные отображения множеств $X, A_1\times...\times A_l$ и Y на себя. Они и дают искомый изоморфизм. Пусть $x\in X$, $(a_1, ..., a_l)\in A_1\times...\times A_l$ — произвольные. Тогда

$$\delta((a_1, ..., a_l), x) = (\delta_1(a_1, \xi_1(a_1, ..., a_l, x)), ..., \delta_l(a_l, \xi_l(a_1, ..., a_l, x)))$$

и

$$\delta^*((a_1, ..., a_l), x) = (\delta'((a_1, ..., a_{l-1}), x), \delta''(a_l, \lambda'((a_1, ..., a_{l-1}), x))) =$$
$$= (\delta_1(a_1, \xi_1(a_1, ..., a_l, x)), ..., \delta_l(a_l, \xi_l(a_1, ..., a_l, x))),$$

т. е.

$$\delta((a_1, ..., a_l), x) = \delta^*((a_1, ..., a_l), x).$$

Аналогично,

$$\lambda((a_1, ..., a_l), x) = \psi(a_1, ..., a_l, x)$$

и

$$\lambda^*((a_1, ..., a_l), x) = \lambda''(a_l, \lambda'((a_1, ..., a_{l-1}), x)) =$$
$$= \lambda''(a_l, (a_1, ..., a_{l-1}, x)) = \psi(a_1, ..., a_l, x),$$

т. е.

$$\lambda((a_1, \ldots, a_l), x) = \lambda^*((a_1, \ldots, a_l), x).$$

Этим показано, что $\varphi=\varphi'\varphi''$. Но в силу индуктивного предположения для φ' и φ'' выполняется $\varphi'\in\mathbf{L}'\in\{[\varphi^1], \ldots, [\varphi^k]\}$ и $\varphi''\in\mathbf{L}''\in\{[\varphi^1], \ldots, [\varphi^k]\}$, соответственно. Необходимость теоремы 2 этим доказана.

Обратно, предположим, что $\varphi\in\mathbf{L}^*\in\{[\varphi^1], \ldots, [\varphi^k]\}$. Путем индукции мы покажем, что φ представляется некоторой *квази-суперпозицией* автоматов из $\langle \mathbf{A}_i | i=1, \ldots, k\rangle$. Впервые, пусть выполняется $\varphi\in[\varphi^i]$ и $\varphi(: F(X)\to F(Y)) = (\varphi^i(: F(X_i)\to F(Y_i)), \gamma, \gamma')$. Тогда φ индуцируется автоматом $\mathbf{A}_i^{(\gamma, \lambda)}$, где $\lambda=\gamma'$. Действительно, так как $\mathbf{A}_i$ является автоматом Медведева, то $\varphi^i_{p_i} = \varphi^i_{q_i}$ $(p_i, q_i\in F(X_i))$ тогда и только тогда, если $a_{i_0}p_i=a_{i_0}q_i$. Поэтому, принимая во внимание конструкцию автомата $\mathbf{A}_i^{(\gamma, \lambda)}$ вытекает справедливость нашего утверждения.

Теперь предположим, сто φ принадлежит к такому $\mathbf{L}^*$, которое получается из $[\varphi^1], \ldots, [\varphi^k]$ путем $l(\geqq 2)$-кратного применения сложения и умножения. Тогда, по следствию теоремы 1, отображение φ содержится в элементе $\mathbf{L}^{**}$ совокупности $\mathbf{L}$, получающемся из $[\varphi^1], \ldots, [\varphi^k]$ путем l-кратного применения единственного умножения, т. е. $\varphi=\psi_1\ldots\psi_l$ $(\psi_i\in\langle[\varphi^1]\cup\ldots\cup[\varphi^k]\rangle$, $i=1, \ldots, l)$. В силу индуктивного предположения $\psi_1\ldots\psi_{l-1}$ представляется некоторой *квази-суперпозицией* $\mathbf{A}_1^{(\gamma_1, \lambda_1)}\ldots\mathbf{A}_{l-1}^{(\gamma_{l-1}, \lambda_{l-1})}$ автоматов $\mathbf{A}_j(\in\langle\mathbf{A}_1, \ldots, \mathbf{A}_k\rangle)$ $(j=1, \ldots, l-1)$, а φ_l — как выше указано — индуцируется некоторым автоматом $\mathbf{A}_i^{(\gamma_i, \lambda_i)}$ $(\mathbf{A}_i\in\langle\mathbf{A}_1, \ldots, \mathbf{A}_k\rangle)$. Таким образом $\varphi=\psi_1\ldots\psi_{l-1}\psi_l$ представляется суперпозицией $\mathbf{A}_1^{(\gamma_1, \lambda_1)}\ldots\mathbf{A}_{l-1}^{(\gamma_{l-1}, \lambda_{l-1})}\cdot\mathbf{A}_i^{(\gamma_i, \lambda_i)}$, т. е. одной из *квази-суперпозиций* автоматов $\mathbf{A}_1, \ldots, \mathbf{A}_k$. Этим теорема 2 полностью доказана.

Из теоремы 2, принимая во внимание и доказательство достаточности этой теоремы, вытекает

Теорема 3. *Если φ индуцируется некоторым R-произведением с l сомножителями автоматов из $\langle\mathbf{A}_i|i=1, \ldots, k\rangle$, то φ представляется и некоторой квази-суперпозицией с l сомножительями автоматов из $\langle\mathbf{A}_i|i=1, \ldots, k\rangle$.*

§ 4

Элемент $\mathbf{L}_i\in\mathbf{L}$ $(\mathbf{L}_i\neq\mathbf{L}_e)$ будем называть *минимальным*, если для любого $\mathbf{L}_j\in\mathbf{L}$ из $\mathbf{L}_j\subseteqq\mathbf{L}_i$ вытекает $\mathbf{L}_j=\mathbf{L}_e$, или $\mathbf{L}_j=\mathbf{L}_i$. Относительно минимальных элементов алгебраической системы $\mathbf{L}$ имеет место

Теорема 4. *Элемент $\mathbf{L}_i(\neq\mathbf{L}_e)$ является минимальным тогда и только тогда, если найдется такое $\varphi(: F(X)\to F(Y))\in\mathbf{L}_i$ и простое число n, что*

(1) $$\mathbf{L}_i = [\varphi],$$

(2a) $$\varphi_p = \varphi_q \ (p, q\in F(X))\Leftrightarrow d(p) \equiv d(q)(\mathrm{mod}\, n),$$

или

(1) $$\mathbf{L}_i = [\varphi],$$

(2b) $$\varphi_p = \varphi_q \ (p, q\in F(X))\Leftrightarrow d(p), d(q)>0.$$

Для доказательства достаточности сперва предположим, что выполняется (1) и (2а). В этом случае автомат $\mathbf{A}=\mathbf{A}(X, A, a_0, Y, \delta, \lambda)$ с множеством состояний и с функцией переходов

$$A = \langle a_0, \ldots, a_{n-1}\rangle,$$

$$\delta(a_i, x_j) = \begin{cases} a_{i+1}, & \text{если} \quad i<n-1, \\ a_0, & \text{если} \quad i = n-1 \end{cases}$$

является минимальным автоматом, индуцирующим отображение φ, которое служит базисом семейства $\mathbf{L}_i$.

Если покажем, что для любого отображения ψ в случае $\psi \prec \varphi$ справедливо хотя бы одно из $s(\psi)=1$ и $\varphi \prec \psi$, то будет доказано, что семейство $\mathbf{L}_i$ — минимально. В самом деле, для любого отображения ψ, содержащегося в семействе $\mathbf{L}_j$ ($\subseteqq \mathbf{L}_i$) в случае $s(\psi)=1$ имеем $\mathbf{L}_j=\mathbf{L}_i$, а если существует ψ из $\mathbf{L}_j$ такое, что $\varphi \prec \psi$, то $\mathbf{L}_j=[\psi]=[\varphi]=\mathbf{L}_i$.

Рассмотрим произвольное отображение ψ: $F(U)\to F(V)$ из сежейства $\mathbf{L}_i$. Так как $\psi \prec \varphi$, то $\psi=(\varphi, \gamma, \gamma')$, где γ, γ' — подходящие отображения. Принимая во внимание транзитивность отношения $\prec$, это значит, что автомат $\mathbf{B}=\mathbf{B}(U, B, b_0, V, \delta', \lambda')$, где

$$B = A\,; b_0 = a_0,$$

$$\delta'(a_i, u) = \delta(a_i, \gamma(u)) = \begin{cases} a_{i+1}, & \text{если} \quad i<n-1, \\ a_0, & \text{если} \quad i = n-1, \end{cases}$$

$$\lambda'(a_i, u) = \gamma'(a_i, u) \qquad (a_i \in A, \quad u \in U)$$

индуцирует отображение ψ.

Нетрудно показать (см. [5]), что вес отображения ψ равен числу попарно неэквивалентных состояний автомата $\mathbf{B}$. Докажем, что их число равно 1, или же n. Если разбиваем B на такие классы, что в одном и том же классе содержатся только эквивалентные элементы, то при таком разбиении π для любых состояний a, b и входного сигнала u из $a \equiv b(\pi)$ вытекает $\delta'(a, u) \equiv \delta'(b, u)$ (π). Разбиения с этим последним свойством в дальнейшем будут называться *допустимыми разбиениями* (см. [1]). Поэтому для нашей цели достаточно показать, что $\mathbf{B}$ имеет только тривиальные допустимые разбиения.

Предположим, что π — некоторое допустимое разбиение автомата $\mathbf{B}$, m — максимум мощности классов по π, а $\pi(a)$ — произвольный класс мощности m. Так как $a_i q = a_j q$ тогда и только тогда, если $a_i = a_j$ $(q \in F(U))$, класс $\pi(a_i q)$ также содержит в точности m элементов. Но произвольный класс по π представляется в виде $\pi(aq)$, где q — подходящее слово из $F(U)$. Поэтому, ввиду простоты n, из отношения $m|n$ имеем $m=1$ или $m=n$. Это значит, что $s(\psi)=1$ или $s(\psi)=n$.

В дальнейшем, как уже отметили в начале доказательства, достаточно рассмотреть случай $s(\psi)=n$. Но в этом случае автомат $\mathbf{B}$ является минимальным автоматом, индуцирующим ψ. Рассмотрим пару отображений (ϑ, ϑ'), где ϑ — произвольное отображение множества X в U и для любых $a \in A$ и $x \in X$ $\vartheta'(a, x)=\lambda(a, x)$. Пусть далее $\mathbf{C}=\mathbf{C}(X, C, c_0, Y, \delta_C, \lambda_C)$, где $C=B=A$, $c_0=b_0=a_0$, $\delta_C(a, x)=\delta'(a, \vartheta(x))$ и $\lambda_C(a, x)=\lambda(a, x)$ $(a \in C, x \in X)$. Заметим, что

С индуцирует производное отображение от отображения ψ относительно пары (ϑ, ϑ'). Таким образом $\varphi=(\psi, \vartheta, \vartheta')$, т. е. $\varphi \prec \psi$.

Пусть выполняется теперь (1) и (2b). Покажем опять же, что для любого отображения $\psi(\prec\varphi)$ имеет место $s(\psi)=1$ или же $\varphi\prec\psi$. Достаточно рассмотреть случай $s(\psi)>1$. В этом случае, ввиду леммы 1, $s(\psi)=2$.

Для множества состояний и функции переходов минимального автомата $\mathbf{A}=\mathbf{A}(X, A, a_0, Y, \delta, \lambda)$, индуцирующего φ, выполняются

$$A = \langle a_0, a_1\rangle,$$
$$\delta(a_0, x) = \delta(a_1, x) = a_1 \qquad (x\in X).$$

Если отображение ψ представляется в виде $\psi=(\varphi, \gamma, \gamma')$, то ψ индуцируется автоматом $\mathbf{B}=\mathbf{B}(U, B, b_0, V, \delta', \lambda')$, где

$$B = A;\, b_0 = a_0,$$
$$\delta'(a_i, u) = \delta(a_i, \gamma(u)) = a_1 \qquad (a_i\in A, \quad u\in U),$$
$$\lambda'(a_i, u) = \gamma'(a_i, u).$$

Так как вес ψ равен 2, то φ представляется в виде $\varphi=(\psi, \vartheta, \vartheta')$, где ϑ — произвольное отображение множества X в U и для любых $a\in\mathbf{A}$ и $x\in X$ выполняется $\vartheta'(a, x)=\lambda(a, x)$. Этим достаточность условий теоремы 4 доказана.

С обратной стороны предположим, что либо не выполняется (1), либо же не выполняется ни одно из условий (2a) и (2b). Если не имеет место (1), то $\mathbf{L}_i$ обладает таким элементом φ, что $s(\varphi)>1$. В этом случае $[\varphi]\subseteq\mathbf{L}_i, [\varphi]\neq\mathbf{L}_e$, $[\varphi]\neq\mathbf{L}_i$, т. е. $\mathbf{L}_i$ не минимален.

Итак, предположим, что не выполняется ни одно из (2a) и (2b), но выполняется (1), т. е. $\mathbf{L}_i$ представляется в виде $[\varphi]$ $(\varphi\colon F(X)\to F(Y))$. Обозначим через H_φ множество таких производных отображений $\psi_i=(\varphi, \gamma, \gamma')$ от отображения φ, которые подчинены условиям

$$\psi_i : F(\langle x_i\rangle) \to F(\langle \varphi_p | p\in F(\langle x_i\rangle)\rangle), \quad \text{где } x_i \text{— любой элемент из } X,$$
$$\gamma(x_i) = x_i,$$
$$\gamma'(\varphi_p, x_i) = \varphi_p \qquad (p\in F(\langle x_i\rangle)),$$
$$s(\psi_i) \geqq 2.$$

Множество H_φ непусто, ибо в противном случае φ имело бы вес 1.

Обозначим через $\mathbf{A}=\mathbf{A}(X, A, a_0, Y, \delta, \lambda)$ минимальный автомат, индуцирующий отображение φ. Тогда, ввиду определения отображений ψ_i минимальные автоматы $\mathbf{B}_i=\mathbf{B}_i(\langle x_i\rangle, B_i, b_{i_0}, B_i, \delta_i, \lambda_i)$, индуцирующие отображения ψ_i — следующие:

$$B_i = \langle a_0 p | p\in F(\langle x_i\rangle)\rangle; \quad b_{i_0} = a_{i_0},$$
$$\delta_i(b, x_i) = \delta(b, x_i) \qquad (b\in B_i),$$
$$\lambda_i(b, x_i) = b.$$

Из этого непосредственно вытекает, что для любого $\psi_i \in H_\varphi$ существуют натуральные числа m_i и n_i так, что

$$b_{i_0}p \neq b_{i_0}q, \quad \text{если} \quad d(p), d(q) < m_i \quad \text{и} \quad p \neq q,$$

$$b_{i_0}p = b_{i_0}q \Leftrightarrow p = q, \quad \text{или} \quad d(p), d(q) \geqq m_i \quad \text{и} \quad d(p) \equiv d(q) \pmod{n_i}.$$

Предположим теперь, что $s(\varphi) > 2$. Будем различать следующие случаи:

(I) существует $\psi_i \in H_\varphi$ так, что $m_i + n_i < s(\varphi)$,
(II) не выполняется (I) и найдется такое $\psi_i \in H_\varphi$, что $m_i \neq 0$,
(III) не выполняются (I) и (II), а n — составное число,
(IV) не выполняются (I) и (II), а n — простое число.

В случае (I), так как вес ψ_i равен $m_i + n_i < s(\varphi)$ и из $\varphi_i \prec \varphi_j$ вытекает $s(\varphi_i) \leqq \leqq s(\varphi_j)$ (см. лемму 1), то φ не будет производным от ψ_i, т. е. $[\psi_i] \subset [\varphi]$, $[\psi_i] \neq \neq [\varphi]$ и $[\psi_i] \neq \mathbf{L}_e$.

Обозначим через ψ_i отображение, удовлетворящее условию (II). В этом случае существует производное $\xi_i\colon F(\langle x_i \rangle) \to F(\langle z_1, z_2 \rangle)$ от φ так, что $s(\xi_i) = 2$ (здесь z_1, z_2 — произвольные символы). Действительно, в качестве ξ_i можно брать отображение, индуцируемое автоматом $\mathbf{C}_i = \mathbf{C}_i(\langle x_i \rangle, C_i, c_{i0}, \langle z_1, z_2 \rangle, \delta_i', \lambda_i')$, определенным следующим образом

$$C_i = B_i;\ c_{i_0} = b_{i_0},$$

$$\delta_i'(b_{ij}, x_i) = \delta_i(b_{ij}, x_i),$$

$$\lambda_i'(b_{ij}, x_i) = \begin{cases} z_1, & \text{если} \quad j = 0, \\ z_2, & \text{в противном случае.} \end{cases}$$

Очевидно, что отображение ξ_i, индуцируемое автоматом $\mathbf{C}$, представляется в виде $\xi_i = (\psi_i, \gamma, \gamma')$, где

$$\gamma(x_i) = x_i,$$

$$\gamma'(\varphi_p, x_i) = \begin{cases} z_1, & \text{если} \quad p = e, \\ z_2, & \text{если} \quad p \neq e \end{cases} \qquad (p \in F(\langle x_i \rangle)).$$

Ввиду транзитивности отношения $\prec$ имеем $\xi_i \prec \varphi$ и так как $s(\xi_i) = 2 < s(\varphi)$, то $[\xi_i] \subset [\varphi]$.

Рассмотрим теперь случай (III), и пусть ψ_i — любое отображение из H_φ, удовлетворящее условию (III). Далее, пусть k — такое натуральное число, что $k|n$ и $1 < k < n$. Тогда нетрудно показать, что вес отображения ξ_i, индуцируемое следующим автоматом $\mathbf{C}_i = \mathbf{C}_i(\langle x_i \rangle, C_i, c_{i_0}, Z_i, \delta_i', \lambda_i')$, равен k:

$$C_i = B_i, \quad c_{i_0} = b_{i_0}, \quad Z_i = \langle z_{i_0}, \dots, z_{i_{k-1}} \rangle,$$

$$\delta_i'(b_{ij}, x_i) = \delta_i(b_{ij}, x_i),$$

$$\lambda_i'(b_{i_0}p, x_i) = z_{i_l} \Leftrightarrow d(p) \equiv l \pmod{k} \qquad (p \in F(\langle x_i \rangle),\ \ 0 \leqq l \leqq k-1).$$

Но ξ_i представимо в виде $\xi_i = (\psi_i, \gamma, \gamma')$, где

$$\gamma(x_i) = x_i,$$

$$\gamma'(\varphi_p, x_i) = z_{i_l} \Leftrightarrow d(p) \equiv l \pmod{k}.$$

Таким образом $\xi_i(\prec \psi_i) \prec \varphi$ и $1 < s(\xi_i) < s(\varphi)$, значит, $[\xi_i] \subset [\varphi]$.

Наконец, рассмотрим случай (IV). Напомним, что минимальный автомат, индуцирующий φ, мы обозначали через $\mathbf{A}=\mathbf{A}(X, A, a_0, Y, \delta, \lambda)$. Если $\varphi \prec \psi_i$, то автомат Медведева $\mathbf{A}(X, A, a_0, \delta)$ является гомоморфным образом автомата $\mathbf{C}_i=\mathbf{C}_i(X, C_i, c_{i_0}, \delta_i)$, где

$$c_i = \langle c_{i_0}, \ldots, c_{i_{n-1}} \rangle,$$

$$\delta_i(c_{ij}, x_i) = \begin{cases} c_{ij+1}, & \text{если} \quad j<n-1, \\ c_{i_0}, & \text{если} \quad j = n-1 \quad (x_i \in X). \end{cases}$$

Так как $s(\varphi)=n$, т. е. $\overline{\overline{A}}=n$, то автомат $\mathbf{A}(X, A, a_0, \delta)$ и $\mathbf{C}_i$ — изоморфны. Поэтому для отображения φ выполняется условие (2a). Получилось противоречие. Таким образом нами установлено, что не имеет место $\varphi \prec \psi_i$. Отсюда $[\psi_i] \subset [\varphi]$. Этим и случай (IV) исследован до конца.

Для окончания доказательства необходимости рассмотрим случай $s(\varphi)=2$. $\mathbf{A}=\mathbf{A}(X, A, a_0, Y, \delta, \lambda)$ $(\overline{\overline{A}}=2)$ означает по-прежнему автомат, индуцирующий φ. В этом случае произвольный элемент ψ_i из H_φ индуцируется одним из следующих автоматов $\mathbf{B}_i=\mathbf{B}_i(\langle x_i \rangle, B_i, b_{i_0}, \delta_i, \lambda_i)$ и $\mathbf{B}'_i=\mathbf{B}'_i(\langle x_i \rangle, B_i, b_{i_0}, \delta'_i, \lambda'_i)$, где

$$B_i = \langle b_{i_0}, b_{i_1} \rangle,$$

$$\delta_i(b_{ij}, x_i) = \begin{cases} b_{i_1}, & \text{если} \quad j=0, \\ b_{i_0}, & \text{если} \quad j=1, \end{cases}$$

$$\delta'_i(b_{ij}, x_i) = b_{i_1} \qquad (j=0, 1),$$

$$\lambda_i(b_{ij}, x_i) = \lambda'_i(b_{ij}, x_i) = b_{ij} \qquad (j=0, 1).$$

Если $\varphi \prec \psi_i$, то автомат Медведева $\mathbf{A}(X, A, a_0, \delta)$ является гомоморфным образом одного из автоматов $\mathbf{C}_i=\mathbf{C}_i(X, C_i, c_{i_0}, \delta_{i_c})$ и $\mathbf{C}'_i=\mathbf{C}'_i(X, C_i, c_{i_0}, \delta'_{i_c})$, где

$$C_i = B_i; \qquad c_{i_0} = b_{i_0},$$

$$\delta_{i_c}(b_{ij}, x) = \begin{cases} b_{i_1}, & \text{если} \quad j=0, \\ b_{i_0}, & \text{если} \quad j=1, \end{cases}$$

$$\delta'_{i_c}(b_{ij}, x) = b_{i_1} \qquad (j=0, 1; \quad x \in X).$$

Таким образом, ввиду равенства $s(\varphi)=2$, автомат $\mathbf{A}(X, A, a_0, \delta)$ изоморфен одному из автоматов $\mathbf{C}_i$ и $\mathbf{C}'_i$. А в этом случае для φ выполняется (2a) или (2b), что противоречит нашим предположениям. Мы получили, что для произвольного $\psi_i (\in H_\varphi)$ имеет место $[\psi_i] \subset [\varphi]$. Этим теорема 4 полностью доказана.

Применением теоремы 4 получается

Теорема 5. *Алгебраическая система* $\mathbf{L}$ *не обладает конечной совокупностью образующих элементов.*

Рассмотрим конечное число произвольно выбранных из $\mathbf{L}$ элементов $\mathbf{L}_1, \ldots, \mathbf{L}_k$. Покажем, что существует семейство $\mathbf{L}^* \in \mathbf{L}$ не принадлежащее к подсистеме, порожденной множеством $\langle \mathbf{L}_1, \ldots, \mathbf{L}_k \rangle$. Пусть m означает максимум весов отображений из $\mathbf{L}_1 \cup \ldots \cup \mathbf{L}_k$. Если $n(>m)$ — простое число и $\varphi: F(X) \to F(Y)$ такое отображение, что $\varphi_p=\varphi_q \Leftrightarrow d(p) \equiv d(q) \pmod{n}$ $(p, q \in F(X))$, то $[\varphi]\notin\{L_1, ..., L_k\}$. Действительно, так как для любого $\mathbf{L}_i\in\mathbf{L}$ справедливо $\mathbf{L}_i\geqq\mathbf{L}_e$, далее, $\mathbf{L}_e$ — единичный элемент относительно сложения и умножения, а умножение и сложение — монотонные операции, то $\mathbf{L}_i\oplus\mathbf{L}_j\supseteqq\mathbf{L}_i,\mathbf{L}_i\oplus\mathbf{L}_j\supseteqq\mathbf{L}_j$, и $\mathbf{L}_i\odot\mathbf{L}_j\supseteqq\mathbf{L}_i,\mathbf{L}_j$ $(\mathbf{L}_i, \mathbf{L}_j\in\mathbf{L})$. По теореме 4 неравенство $\mathbf{L}_i\leqq[\varphi]$ справедливо для единственного $\mathbf{L}_e$, т. е. элемент $[\varphi]$ не получается из отличных от него элементов путем применения сложения и умножения. Кроме того, $[\varphi]$ не содержится в рассмотренном множестве $\langle\mathbf{L}_1, ..., \mathbf{L}_k\rangle$. Теорема 5 доказана.

Совокупность $N=\langle\mathbf{L}_i|i=1, 2, ...\rangle$ образующих алгебраической системы $\mathbf{L}$ мы называем *минимальной*, если для произвольного $\mathbf{L}_i\in N$ множеством $N\backslash\mathbf{L}_i$ $\mathbf{L}$ уже не порождается.

Теорема 6. *Алгебраическая система* $\mathbf{L}$ *обладает минимальной совокупностью образующих, и произвольная ее совокупность образующих содержит минимальную совокупность образующих элементов.*

Заметим тот простой факт, что существует лишь конечное число таких семейств, весы отображений которых не провосходят данное натуральное число m.

Обозначим через G^m множество всех таких семейств, вес каждого отображения которых не больше, чем m. Так как G^m — конечно, то существует подмножество $H^m\subseteqq G^m$ так, что справедливы $\{H^m\}\supseteqq G^m$ и $\{H^m\backslash\mathbf{L}_m\}\not\supseteqq G^m$ для любого $\mathbf{L}_m\in H^m$. Присоединим H^m к множеству всех таких отображений из $G^{m+1}\backslash G^m$, которые не содержатся в $\{H^m\}$. Из полученного множества можно выбрать подмножество H^{m+1} так, что $\{H^{m+1}\}\supseteqq G^{m+1}$ и $\{H^{m+1}\backslash\mathbf{L}_{m+1}\}\not\supseteqq G^{m+1}$ $(\mathbf{L}_{m+1}\in H^{m+1})$. Выполняется и отношение $H^m\subseteqq H^{m+1}$, так как ни один $\mathbf{L}_m$ из H^m не содержится в $\{H^{m+1}\backslash\mathbf{L}_m\}$.

Объединение $H=H^m\cup H^{m+1}\cup...$ полученных множеств H^i $(i=m, m+1, ...)$ является минимальной совокупностью образующих. Действительно, ни один $\mathbf{L}_m\in H$ не содержится в $\{H\backslash\mathbf{L}_m\}$, так как выполняются $H^m\subseteqq H^{m+1}\subseteqq...$ и $\mathbf{L}_m\in H^i$ для некоторого $i\geqq m$.

Пусть S_1 — некоторая совокупность образующих алгебраической системы $\mathbf{L}$ и $S_1^m=G^m\cap S_1$. Из-за монотонности операций очевидно, что справедливо $G^m\subseteqq\{S_1^m\}$. Обозначим через S^m такое подмножество множества S_1^m, которое обладает свойствами, потребованными в предыдущих рассуждениях от H^m. Так как $S_2^{m+1}=G^{m+1}\cap S_1$ содержит S_1^m, то справедливо и включение $S^m\subseteqq S_2^{m+1}$. Пусть $S_1^{m+1}=S^m\cup S_2^{m+1}\backslash\{S^m\}$, а S^{m+1} — такое подмножество множества S_1^{m+1}, которое обладает свойствами, потребованными от H^{m+1}. Ясно, что $S=S^m\cup S^{m+1}\cup...$ является минимальной совокупностью образующих и $S\subseteqq S_1$. Этим теорема 6 доказана.

§ 5

Пусть φ: $F(X)\to F(Y)$ — произвольное автоматное отображение. Обозначим через φ': $F(X)\to F(\langle\varphi_p|p\in F(X)\rangle)$ отображение, отображающее любое слово $p\in F(X)$ $(d(p)=n)$ в слово $\varphi_{p(1)}...\varphi_{p(n)}$. Разобьем множество X на такие классы, что x и x' $(x, x'\in X)$ содержятся в одном и том же классов, тогда и только тогда, если $\varphi_{px}=\varphi_{px'}$ для всех $p\in F(X)$, а полученное разбиение обозначим через π. Выбирая по одному элементу из каждого такого класса, полученное так множество обозначим через $\bar{X}$. Пусть $\varphi^*\colon F(\bar{X})\to F(\langle\varphi_p|p\in F(X)\rangle)$ — отображение, для которого выполняется $\varphi^*(q)=\varphi'(q)$ $(q\in F(\bar{X}))$ при каждом $q\in F(\bar{X})$. Нетрудно показать, что $\varphi^*\prec\varphi'\prec\varphi$ и $\varphi\prec\varphi'\prec\varphi^*$. [6])

Рассмотрим произвольный базис $\langle\varphi^1, \ldots, \varphi^k\rangle$ семейства $\mathbf{L}_i$. Базис $\langle\varphi^1, \ldots, \varphi^k\rangle$ назовем *минимальным*, если $[\langle\varphi^1, \ldots, \varphi^k\rangle\backslash\varphi^i]\neq\mathbf{L}_i$ для любого $\varphi^i\in\langle\varphi^1, \ldots, \varphi^k\rangle$. Легко убедиться в том, что *если* $\langle\varphi^1, \ldots, \varphi^k\rangle$ *является базисом* $\mathbf{L}_i$, *то* и $\langle\varphi^{1*}, \ldots, \varphi^{k*}\rangle$ *является базисом* $\mathbf{L}_i$. Далее, *если* $\langle\varphi^1, \ldots, \varphi^k\rangle$ — *минимальный базис* $\mathbf{L}_i$, то и $\langle\varphi^{1*}, \ldots, \varphi^{k*}\rangle$ — *минимальный базис* $\mathbf{L}_i$.

По определению отображения φ^* видно, что минимальный базис $\langle\varphi^{1*}, \ldots, \varphi^{k*}\rangle$ будет „самым простым" в том смысле, что оно не обладает „лишными сигналами".

Минимальные базисы $\langle\varphi^{1*}, \ldots, \varphi^{k*}\rangle$ семейства $\mathbf{L}_i$ отображений однозначно определены, как показывает следующая

Теорема 7. *Если* $\langle\varphi^1, \ldots, \varphi^k\rangle$ *и* $\langle\psi^1, \ldots, \psi^l\rangle$ — *минимальные базисы одного и того же семейств отображений, то* $k=l$; *далее, можно установить между множествами* $\langle\varphi^{1*}, \ldots, \varphi^{k*}\rangle$ *и* $\langle\psi^{1*}, \ldots, \psi^{k*}\rangle$ *взаимно однозначное соответствие так, что соответствующие отображения изоморфны.*

Пусть $\langle\varphi^1, \ldots, \varphi^k\rangle$, $\langle\psi^1, \ldots, \psi^l\rangle$ — минимальные базисы семейства $\mathbf{L}_i$ отображений. Тогда для любого ψ^i $(i=1, \ldots, l)$ найдется φ^j $(1\leqq j\leqq k)$ так, что $\psi^i\prec\varphi^j$. С другой стороны справедливо и отношение $\varphi^j\prec\psi^i$, так как в противном случае, из транзитивности отношения $\prec$ и выполнения $\varphi^j\prec\psi^u$ $(u\neq i, 1\leqq u\leqq l)$ получалось бы $\psi^i(\prec\varphi^j)\prec\psi^u$, что противоречит минимальности базиса $\langle\psi^1, \ldots, \psi^l\rangle$. Поэтому для любого ψ^i $(i=1, \ldots, l)$ существует φ^j $(1\leqq j\leqq k)$ так, что $[\psi^i]=[\varphi^j]$. Таким же образом можно показать, что для произвольного φ^j $(j=1, \ldots, k)$ найдется ψ^i $(1\leqq i\leqq l)$ так, что $[\varphi^j]=[\psi^i]$. Так мы установили, что $k=l$ и существует взаимно однозначное соответствие между $\langle\varphi^1, \ldots, \varphi^k\rangle$ и $\langle\psi^1, \ldots, \psi^k\rangle$ так, что для соответствующих друг другу φ^j, ψ^i имеем $[\varphi^j]=[\psi^i]$ $(i, j=1, \ldots, k)$. Остается показать, что из $[\varphi]=[\psi]$ $(\varphi, \psi\in L)$ вытекает существование изоморфизма между φ^* и ψ^*. Пусть φ^* и ψ^* имеют вид $\varphi^*\colon F(X)\to F(Y)$ и $\psi^*\colon F(U)\to F(V)$. Так как $[\varphi^*]=[\psi^*]$, то выполняется равенство $\varphi^*=(\psi^*, \gamma, \gamma')$, где γ, γ' — подходящие отображения.

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[6]) Легко видеть, что пары отображений, для которых $\varphi^*=(\varphi', \vartheta, \vartheta')$ и $\varphi'=(\varphi, \gamma, \gamma')$ — следующие:

$$\vartheta(\bar{x}) = \bar{x} \qquad (\bar{x}\in\bar{X}),$$
$$\vartheta'(\varphi'_p, \bar{x}) = \varphi'_p(\bar{x}) \qquad (p\in F(X),\quad \bar{x}\in\bar{X}),$$
$$\gamma(x) = x \qquad (x\in X),$$
$$\gamma'(\varphi_p, x) = \varphi_p \qquad (p\in F(X),\quad x\in X).$$

А пары отображений, обезпечивающие равенства $\varphi=(\varphi', \varrho, \varrho')$ и $\varphi'=(\varphi^*, \tau, \tau')$ — следующие:

$$\varrho(x) = x \qquad (x\in X),$$
$$\varrho'(\varphi'_p, x) = (\varphi'_p(x))(x) \qquad (p\in F(X), x\in X),$$
$$\tau(x) = \bar{x}(x\in X, \bar{x}\in\bar{X} \text{ и } x\equiv\bar{x}(\pi)),$$
$$\tau'(\varphi^*_p, x) = \varphi^*_p(\bar{x}) \qquad (p\in F(\bar{X}),\quad x\in X,\quad \bar{x}\in\bar{X} \text{ и } x\equiv\bar{x}(\pi)).$$

Отображение $\gamma: X \to V$ является взаимно однозначным. Действительно, если существуют $x, x'(\in X; x \neq x')$, для которых $\gamma(x)=\gamma(x')$, то выполняется $\varphi^*_{px}=\varphi^*_{px'}$ при любом $p \in F(X)$. Но это противоречит определению отображения φ^*. Далее, так как для произвольного $p=x_1 \dots x_n \in F(X)$

$$(*) \qquad \gamma'(\psi^*, p) = \gamma'(\psi^*, x_1) \cdot \gamma'(\psi^*_{\gamma(x_1)}, x_2) \dots \gamma'(\psi^*_{\gamma(x_1 \dots x_{n-1})}, x_n) =$$

$$= \varphi^* \varphi^*_{x_1} \dots \varphi^*_{x_1 \dots x_{n-1}}$$

и $\gamma: F(X) \to F(U)$ — взаимно однозначное отображение, то $\varrho(\varphi^*_p)=\psi^*_{\gamma(p)}$ является взаимно однозначным отображением множества $\langle \varphi^*_p | p \in F(X) \rangle$ в $\langle \psi^*_{\gamma(p)} | p \in F(X) \rangle$. Из равенства $(*)$ вытекает, что (γ, ϱ) изоморфно отображает φ^* в ψ^*.

Обратно, так как ψ^* представляется в виде $\psi^*=(\varphi^*, \vartheta, \vartheta')$, то можно доказать аналогичным путем, что ψ^* изоморфно отображается в φ^*. Значит, φ^* и ψ^* изоморфны. Теорема 7 доказана.

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(Поступило 20/IV/1966 г.)

О многотактных автоматах

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Л. Кальмар в работе [1] ввел алгебраическую систему, которую можно рассматривать, как алгебраическую модель автоматической вычислительной машины. В одной из своих неопубликованных лекций им же был предложен упрощенный вариант этой модели, названный многотактным автоматом. В настоящей работе вводится понятие $\varkappa$-автомата, эквивалентное понятию многотактного автомата в отношении осуществимых ими отображений, допускающее, однако, более простой алгебраический подход. С помощью понятия $\varkappa$-автомата для многотактных автоматов Кальмара доказывается аналог некоторых результатов, полученных раньше для автоматов Мили.

§ 1

Система $\mathbf{A}=\mathbf{A}(X, A, a_0, Y, \delta, \lambda)$ называется *$\varkappa$-автоматом*, если X, A, Y — множества, $a_0 \in A$, δ — отображение множества $A \times \langle X \cup e \rangle$ в A, λ — отображение множества $A \times \langle X \cup e \rangle$ в свободную полугруппу $F(Y)$ с единичным элементом e, для которых выполняются

$$\delta(a, e)=a \quad \text{и} \quad \lambda(a, e)=e \qquad (a \in A). \tag{1}$$

(Множество X, A и Y называется множеством входных сигналов, состояний и выходных сигналов, соответственно, а a_0 — начальным состоянием. Функцию δ и λ назовем функцией переходов и выходов, соответственно.)

Символ e называется *пустым словом*. Его появление на входе или на выходе $\varkappa$-автомата означает отсутствие входного или выходного сигнала. Симбол e не входит ни в множество X, ни в множество Y, его использование служит только для упрощения наших рассуждений.

$\varkappa$-автомат $\mathbf{A}$ называется *конечным*, если каждое из множеств X, A и Y является конечным.

Пусть $a \in A$ — любое состояние некоторого $\varkappa$-автомата $\mathbf{A}$. Мы говорим, что отображение $\varphi_a: F(X) \to F(Y)$ *индуцируется состоянием a $\varkappa$-автомата* $\mathbf{A}$,

*) F. GÉCSEG (Szeged).

если φ_a переводит произвольное слово $p=x_1...x_n \in F(X)$ в произведение $q_1...q_n$ слов $q_1, ..., q_n \in F(Y)$, определенных условиями

(2) $$\delta(a, x_1) = a_1, \delta(a_1, x_2) = a_2, ..., \delta(a_{n-1}, x_n) = a_n$$

и

(3) $$\lambda(a, x_1) = q_1, \lambda(a_1, x_2) = q_2, ..., \lambda(a_{n-1}, x_n) = q_n.$$

Отображением, *индуцируемым* $\varkappa$*-автоматом* **A** называется φ_{a_0}, индуцируемое начальным состоянием a_0 $\varkappa$-автомата **A**.

Система $\mathbf{A}=\mathbf{A}(X, A', A'', a_0', Y, \delta', \delta'', \lambda', \lambda'')$ называется *многотактным автоматом Кальмара*, где

(I) A' — множество состояний покоя, $a_0' \in A'$,
(II) A'' — множество состояний работы,
(III) $A=A' \cup A''$ — множество состояний,
(IV) X — множество входных сигналов,
(V) Y — множество выходных сигналов,
(VI) $\delta': A' \times X \to A$ — функция переходов при покое,
(VII) $\delta'': A'' \to A$ — функция переходов при автоматическом режиме,
(VIII) $\lambda': A' \times X \to Y$ — (частичная) функция выходов при покое,
(IX) $\lambda'': A'' \to Y$ — (частичная) функция выходов при автоматическом режиме.

В согласии с нашим подходом предпологается, что функции λ' и λ'' вполне определенные (т. е. нечастичные), но в некоторых местах их значения равны пустому слову e.

Автомат **A** Кальмара называется *конечным*, если множества X, A и Y конечные, далее, существует натуральное число $k(\mathbf{A})$ так, что автомат **A** из любого состояния работы $a'' \in A''$ (автоматически) переходит в некоторое состояние покая в числе тактов, не превосходящем $k(\mathbf{A})$.

Отображения, индуцируемые автоматами Кальмара определяются аналогично отображениям, индуцируемым $\varkappa$-автоматами.

В дальнейшем мы занимаемся лишь *конечными инициальными* $\varkappa$-автоматами и *конечными инициальными* автоматами Кальмара.

Два автомата (любой из которых может быть либо $\varkappa$-автомат либо автомат Кальмара) с общими множествами входов и выходов называются *эквивалентными*, если они индуцируют одно и то же отображение.

Теперь покажем, что понятие $\varkappa$-автомата эквивалентно понятию автомата Кальмара в указанном выше смысле. Именно, имеет место следующее

Предложение 1. *Для произвольного автомата Кальмара существует эквивалентный ему* $\varkappa$*-автомат, с другой стороны, любой* $\varkappa$*-автомат эквивалентен некоторому автомату Кальмара.*

Рассмотрим произвольный многотактный автомат **A**. Состояние покоя $a' \in A'$ мы назовем *x-конечным состоянием*, если $\delta'(a', x)$ содержится в A'', т. е. $\delta'(a', x)$ — состояние работы. Пусть $\overline{\delta'(a', x)}$ обозначает состояние из A', в которое **A** автоматически переходит из состояния $\delta'(a', x)$ и обозначим через $\overline{\lambda'(a', x)}$ слово из $F(Y)$, которое автомат **A** выдает в течение этого автоматического перехода. Тогда, как легко усмотреть, $\varkappa$-автомат $\mathbf{B}=\mathbf{B}(X, B, b_0, Y, \delta, \lambda)$, для которого

(4) $$B = A'; b_0 = a'_0,$$

(5) $$\delta(a', x) = \begin{cases} \delta'(a', x), & \text{если } a' \text{ — не } x\text{-конечное состояние;} \\ \overline{\delta'(a', x)}, & \text{если } a' \text{ — } x\text{-конечное состояние,} \end{cases}$$

(6) $$\lambda(a', x) = \begin{cases} \lambda'(a', x), & \text{если } a' \text{ — не } x\text{-конечное состояние,} \\ \overline{\lambda'(a', x)}, & \text{если } a' \text{ — } x\text{-конечное состояние,} \end{cases}$$

эквивалентен автомату Кальмара $\mathbf{A}$.

Обратно, пусть $\mathbf{B}=\mathbf{B}(X, B, b_0, Y, \delta, \lambda)$ — произвольный $\varkappa$-автомат. Предположим, что для некоторого $x \in X$ и $b \in B$ справедливо следующее соотношение

(7) $$\lambda(b, x) = y_1 \ldots y_k \qquad (y_i \in Y; \quad i = 1, \ldots, k),$$

где $k > 1$. Тогда, обозначим через (b, x) множество $\langle b_1^x, \ldots, b_{k-1}^x \rangle$, где ни одного из символов b_i^x $(i=1, \ldots, k-1)$ не содержится в B. Пусть B' — объединение множеств (b, x) для всех $b \in B$ и $x \in X$, удовлетворяющих условию (7). Нетрудно показать, что автомат Кальмара $\mathbf{A}=\mathbf{A}(X, A', A'', a'_0, Y, \delta', \delta'', \lambda', \lambda'')$, описанный в следующем, эквивалентен $\mathbf{B}$:

$$A' = B; \quad a'_0 = b_0,$$

$$A'' = B',$$

$$\delta'(b, x) = \begin{cases} \delta(b, x), & \text{если } b \text{ и } x \text{ не удовлетворяют (7),} \\ b_1^x, & \text{если } b \text{ и } x \text{ удовлетворяют (7),} \end{cases}$$

$$\delta''(b_i^x) = \begin{cases} b_{i+1}^x, & \text{если} \quad i < k-1, \\ \delta(b, x), & \text{если} \quad i = k-1, \end{cases}$$

$$\lambda'(b, x) = \begin{cases} \lambda(b, x), & \text{если } b \text{ и } x \text{ не удовлетворяют (7),} \\ y_1, & \text{если } b \text{ и } x \text{ удовлетворяют (7),} \end{cases}$$

$$\lambda''(b_i^x) = y_{i+1} \qquad (i = 1, \ldots, k-1).$$

Этим Предложение 1 полностью доказано.

§ 2

Пусть X — произвольное конечное множество. Тогда каждый $\varkappa$-автомат и автомат Кальмара с общим множеством X входных и выходных сигналов индуцирует отображение свободной полугруппы $F(X)$ с единичным элементом в себя. При данном X обозначим через K_X множество всех отображений, индуцирующихся автоматами Кальмара с общим множеством X входов и выходов. Всилу Предложения 1, множество K_X совпадает с множеством всех отображений, индуцируемых $\varkappa$-автоматами, обладающими с общим множеством X входных и выходных сигналов.

Покажем, что справедливо следущее

Предложение 2. *Множество K_X образует полугруппу относительно обычного произведения отображений.*

Пусть φ и ψ — отображения из K_X, индуцируемые $\varkappa$-автоматами $\mathbf{A}=\mathbf{A}(X, A, a_0, X, \delta, \lambda)$ и $\mathbf{B}=\mathbf{B}(X, B, b_0, X, \delta', \lambda')$, соответственно. Так как произведение отображений ассоциативно, то для доказательства Предложения 2 достаточно показать, что $\varphi\psi$ содержится в K_X. Это, однако, верно, потому что автомат $\mathbf{C}=\mathbf{C}(X, C, c_0, X, \delta'', \lambda'')$ для которого выполняются

$$C = A \times B; \quad c_0 = (a_0, b_0),$$
$$\delta''((a, b), x) = (\delta(a, x)), u[\delta'(b, \lambda(a, x))],\ ^1)$$
$$\lambda''((a, b), x) = \lambda'(b, \lambda(a, x)),$$

является $\varkappa$-автоматом и $\mathbf{C}$ индуцирует $\varphi\psi$. Этим Предложение 2 доказано.

Автомат $\mathbf{C}$ называется *суперпозицией* $\varkappa$-автомата $\mathbf{A}$ с $\varkappa$-автоматом $\mathbf{B}$. Понятие суперпозиции естественным образом обобщается для произвольного вполне упорядоченного конечного множества $\varkappa$-автоматов.

Докажем, что K_X $(\overline{\overline{X}}\geqq 2)$ не обладает конечной системой образующих элементов. Прежде всего докажем лемму, из которой наше утверждение вытекает.

Пусть $\mathbf{A}=\mathbf{A}(X, A, a_0, X, \delta, \lambda)$ — произвольный $\varkappa$-автомат. Для всех троек $(\mathbf{A}, a, x)$, где состояние a и входный сигнал x произвольны из $\mathbf{A}$, можно однозначно сконструировать $\varkappa$-автомат $\mathbf{A}^{(a, x)}$ (ср. [2]) следующим образом: x — единственный входный сигнал, a — начальное состояние, $A^{(a, x)}$ — множество состояний, которое содержит, кроме a, все состояния вида $ax...x$ автомата $\mathbf{A}$ (и только их); множество X — выходный алфавит, а функции перехода и выхода получаются из функций δ и λ путем ограничения их области определения парами $(\overline{a}, x)$, где $\overline{a}\in A^{(a, x)}$.

Покажем, что имеет место следующая

Лемма. *Если $\varkappa$-автомат $\mathbf{A}=\mathbf{A}(X, A, a_0, X, \delta, \lambda)$ представляется в виде суперпозиции $\varkappa$-автоматов $\mathbf{A}_i=\mathbf{A}_i(X, A_i, a_{i_0}, X, \delta_i, \lambda_i)$ $(i=1, ..., k)$, то для произвольного $a\in A$ и произвольного $x\in X$ автомат $\mathbf{A}^{(a, x)}$ имеет такое множество допустимых разбиений $\pi_0^*, ..., \pi_j^*$ $(\pi_l^* > \pi_{l+1}^*;\ l=0, ..., j-1)$, что для $0\leqq l\leqq j-1$ число классов разбиения π_{l+1}^*, входящих в произвольный класс разбиения π_l^*, не превосходит s, где $s=\max\limits_{1\leqq i\leqq k} \overline{\overline{A}}_i$.*[2])

(Разбиение π множества состояний A $\varkappa$-автомата $\mathbf{A}=\mathbf{A}(X, A, a_0, Y, \delta, \lambda)$ называется *допустимым*, если для произвольных $x\in X$ и $a, a'\in A$ из $a\equiv a'(\pi)$ следует $\delta(a, x)\equiv\delta(a', x)$ (π); ср. [2].)

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[1]) $u(p)$ означает последную букву слова p.

[2]) Здесь π_0^* означает тривиальное разбиение, содержащее единственный класс, а π_j^* — разбиение, имеющее только одноэлементные классы.

Определим разбиения π_i ($1 \leqq i \leqq k$) $\varkappa$-автомата **A** следующим образом: $(a_1, \ldots, a_i, \ldots, a_k) = a$ и $(a'_1, \ldots, a'_i, \ldots, a'_k) = a'$ принадлежат к одному и тому же классу по π_i, если $a_t = a'_t$ (при $t = 1, \ldots, i$). Покажем, что разбиения π_i — допустимы. Пусть $x \in X$ — произвольный входной сигнал. Тогда

$$\delta(a, x) = (a_1, q_1, \ldots, a_i, q_i, \ldots, a_k, q_k),$$

$$\delta(a', x) = (a'_1, q'_1, \ldots, a'_i, q'_i, \ldots, a'_k, q'_k).$$

Но, по определению суперпозиции, q_t ($1 \leqq t \leqq i$) зависит только от x и $a_1, \ldots$ $\ldots, a_{t-1}$. Поэтому, в силу равенств $a_1 = a'_1, \ldots, a_i = a'_i$ справедливы равенства $q_1 = q'_1, \ldots, q_i = q'_i$ и так имеют места $a_1, q_1 = a'_1, q'_1, \ldots, a_i, q_i = a'_i, q'_i$, т. е. $\delta(a, x) \equiv \delta(a', x)\,(\pi_i)$. Значит, разбиения π_i ($i = 1, \ldots, k$) — допустимы. Обозначим через π_0 — тривиальное разбиение, содержащее единственный класс. Тогда $\pi_0 > \pi_1 > \ldots > \pi_k$, и легко убедиться, что число классов разбиения π_{m+1}, содержащихся в произвольном классе разбиения $\pi_m (0 \leqq m \leqq k-1)$ равно $\overline{\overline{A}}_{m+1}$, причем $\overline{\overline{A}}_{m+1} \leqq s$.

Пусть теперь разбиение π_i^* ($i = 0, \ldots, k$) $\varkappa$-автомата $\mathbf{A}^{(a, x)}$ — следующее: $a \equiv a'(\pi_i^*)$ ($a, a \in A^{(a, x)}$) тогда и только тогда, если $a \equiv a'(\pi_i)$. Классы по π_i^* представляются в виде пересечений классов по π_i и множества $A^{(a, x)}$. Поэтому очевидно, что если множество разбиений $\langle \pi_i^* \rangle$ $\varkappa$-автомата $\mathbf{A}^{(a, x)}$ разделить на классы совпадающих разбиений и сохранить по одному разбиению из каждого такого класса, то для получаемого множества разбиений π_l^* ($l = 0, \ldots, j$) утверждение Леммы выполняется.

Так как произвольный автомат Мили является $\varkappa$-автоматом, то из этой Леммы дословным повторением доказательства Теоремы работы [2] вытекает следующее

Предложение 3. *Если* $\overline{\overline{X}} \geqq 2$, *то полугруппа* K_X *не обладает конечной системой образующих элементов.*

Отметим, что А. Г. Курош предложил рассматривать отображения свободных полугрупп в себя, переводящие слова с одиноковыми начальными отрезками в такие же слова, но не сохраняющие объязательно длин слов. Исследованные автоматы являются в известном смысле решениями этой проблемы.

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(Поступило 27/VI/1966. г.)

On interpolation of L_p spaces with weight functions

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0. Introduction

According to a classical theorem of BERNSTEIN (1914) a sufficient condition for a function f on the real line $(-\infty, \infty)$ to be given by an absolutely convergent Fourier integral,

$$f(x) = \frac{1}{2\pi}\int e^{ix\xi}\hat{f}(\xi)\,d\xi, \quad \hat{f}\in L_1,$$

is that $f\in L_2$ and satisfies a Lipschitz condition of exponent $>\frac{1}{2}$ in L_2, i. e.

$$\|f(x+t)-f(x)\|_{L_2} = O(t^{\frac{1}{2}+\varepsilon}) \text{ for some } \varepsilon>0 \text{ and as } t\to 0.$$

A more precise condition in this sense reads:

$$\text{(0. 1)} \qquad \int_0^\infty \|f(x+t)-f(x)\|_{L_2}\frac{dt}{t^{3/2}}<\infty.$$

Now the requirement $f\in L_2$ is inessential. (Usually this theorem is given for Fourier series but the change to integrals is immediate; see [9], vol. 1, pp. 240—241; see also [8].) More recently BEURLING (see [2]) has shown that condition (0. 1) on f is equivalent to the following one on $\hat{f}$:

$$\text{(0. 2)} \qquad \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\omega(|\xi|)\,d\xi<\infty \quad \text{for some } \textit{non-decreasing } \omega \text{ with } \int_0^\infty \frac{d\xi}{\omega(|\xi|)}<\infty.$$

This is of importance in some questions of Spectral Synthesis (see [1], [2]).

The purpose of this note is to show how this as well as some other results of [2] can be interpreted from the point of view of the theory of interpolation spaces. The plan is as follows. We first (Section 1) briefly summarize some general notions on interpolation spaces $(A_0, A_1)_{\theta,q}$ which will be needed in what follows. Then (Section 2) we specialize to the case of spaces $L_p(w)$, L_p space with weight function w. In particular we characterize $(L_p(w_0), L_p(w_1))_{\theta,q}$ when $q=1$ or $q=\infty$. This should be contrasted to known results when $q=p$ (see [7]). The applications to Fourier integrals finally are given Section 3.

1. General notions on interpolation spaces

Let A_0 and A_1 be two Banach spaces both continuously imbedded in the same topological vector space $\mathfrak{A}$.

If $0<t<\infty$, $f\in A_0+A_1$, set

$$K(t,f)=K(t,f;A_0,A_1)=\inf_{f=f_0+f_1}(\|f_0\|_{A_0}+t\,\|f_1\|_{A_1})$$

and, if $0<t<\infty$, $f\in A_0\cap A_1$,

$$J(t,f)=J(t,f;A_0,A_1)\cdot\max\left(\|f\|_{A_0},\,t\,\|f\|_{A_1}\right).$$

There are basically two ways of obtaining interpolation spaces:

1° We impose a "growth condition" on $K(t,f)$ of the form

$$\Phi[K(t,f)]<\infty.$$

where Φ is a suitable functional.

2° Upon representing f in the form

$$f=\int_0^\infty f(t)\frac{dt}{t}\quad\text{(non unique!)}$$

we impose a "growth condition" on $J(t,f(t))$ of the form

$$\Phi\left[J(t,f(t))\right]<\infty.$$

The most important special case is when

$$\Phi[\varphi]=\Phi_{\theta,q}[\varphi]=\left[\int_0^\infty (t^{-\theta}\varphi(t))^q\frac{dt}{t}\right]^{1/q},\ 0<\theta<1,\quad 1\leqq q\leqq\infty.$$

In this case the two constructions 1° and 2° lead to the same spaces (up to an equivalence of norm) which we shall denote by $(A_0,A_1)_{\theta,q}$.

For more details about these spaces, see e. g. [6], see also [3], [4], [5].

2. The case of L_p spaces with weight functions

Let Z be a locally compact space provided with a positive measure μ. Let w be a positive μ-measurable function (weight function). We denote by $L_p(w)$, $1<p<\infty$, the space of μ-measurable function f such that $|wf|^p$ is μ-integrable and endow it with the norm

$$\|f\|_{L_p(w)}=\|wf\|_{L_p}=\left(\int_Z|wf|^p\,d\mu\right)^{1/p}.$$

Let now w_0 and w_1 be any two fixed such functions and take

$$A_0 = L_p(w_0), \quad A_1 = L_p(w_1), \qquad 1 < p < \infty.$$

Thus in what follows

$$K(t,f) = K(t,f;\, L_p(w_0),\, L_p(w_1)), \quad J(t,f) = J(t,f;\, L_p(w_0),\, L_p(w_1)).$$

By [7] we have

(2. 1) $$K(t,f) \sim \|f\|_{L_p(\min(w_0,\, tw_1))},$$

(2. 2) $$J(t,f) \sim \|f\|_{L_p(\max(w_0,\, tw_1))}.$$

Put further

$$K_1(t,f) = \|f\|_{L_p(w)}, \quad w = w^{(1)} \begin{cases} = w_0 & (w_0 < tw_1), \\ = 0 & (w_0 \geqq tw_1), \end{cases}$$

$$K_2(t,f) = \|f\|_{L_p(w)}, \quad w = w^{(2)} \begin{cases} = 0 & (w_0 < tw_1), \\ = tw_1 & (w_0 \geqq tw_1), \end{cases}$$

$$K_3(t,f) = \|f\|_{L_p(w)}, \quad w = w^{(3)} \begin{cases} \sim w_0 \sim tw_1 & (w_0 < tw_1 < 2w_0), \\ = 0 & (\text{elsewhere}). \end{cases}$$

Then holds

Theorem 2. 1. *The following inequalities are valid:*

(2. 3) $$C^{-1} K_i(t,f) \leqq K(t,f) \leqq C \left\{ \int_0^\infty \left[\varphi_i\left(\frac{1}{\sigma}\right) K_i(t\sigma, f) \right]^p \frac{d\sigma}{\sigma} \right\}^{1/p} \qquad (i = 1, 2, 3)$$

where

$$\varphi_1(\sigma) \begin{cases} = \sigma & (\sigma < 1), \\ = 0 & (\sigma \geqq 1), \end{cases} \quad \varphi_2(\sigma) \begin{cases} = 0 & (\sigma < 1), \\ = 1 & (\sigma \geqq 1), \end{cases}$$

$$\varphi_3(\sigma) = \min(1, \sigma) \begin{cases} = \sigma & (\sigma < 1), \\ = 1 & (\sigma \geqq 1). \end{cases}$$

It follows that

$$f \in (L^p(w_0),\, L^p(w_1))_{\theta,q} \qquad (q \geqq p)$$

if and only if

$$\left[\int_0^\infty [t^{-\theta} K_i(t,f)]^q \frac{dt}{t} \right]^{\frac{1}{q}} < \infty \qquad (i = 1, 2, 3).$$

Proof. For simplicity we consider the case $i = 1$ only; the other two cases can be treated in a similar fashion. The inequality to the left in (2. 3)

$$K_1(t,f) \leqq CK(t,f)$$

is trivial, by (2. 1), since clearly $w^{(1)} \leqq \min(w_0, tw_1)$. Remains the inequality to the right. But we get, again by (2. 1),

$$(K(t,f))^p \leqq C^p \int_Z |\min(w_0, tw_1)f|^p \, d\mu \leqq$$

$$\leqq C^p \Big[\int_{w_0 < 2tw_1} |w_0 f|^p \, d\mu + 2^{-p} \int_{2tw_1 \leqq w_0 < 2^2 tw_1} |w_0 f|^p \, d\mu +$$

$$+ 2^{-2p} \int_{2^2 tw_1 \leqq w_0 < 2^3 tw_1} |w_0 f|^p \, d\mu + \ldots \Big] \leqq$$

$$\leqq C^p [[K_1(2t,f)]^p + 2^{-p}[K_1(2^2 t,f)]^p + 2^{-2p}[K_1(2^3 t,f)]^p + \ldots] \leqq$$

$$\leqq C^p \int_1^\infty \left[\frac{1}{\sigma} K_1(t\sigma, f) \right]^p \frac{d\sigma}{\sigma}$$

and this inequality too follows.

If $f \in (L_p(w_0), L_p(w_1))_{\theta,q}$ the trivial half of (2. 3) shows that

$$\left[\int_0^\infty [t^{-\theta} K_1(t,f)]^q \frac{dt}{t} \right]^{1/p} < \infty .$$

Conversely, if this condition holds and moreover $q \geqq p$, upon writing the other half of (2. 3) as

$$[t^{-\theta} K(t,f)]^p \leqq C^p \int_1^\infty \sigma^{(\theta-1)p} [(\sigma t)^{-\theta} K_1(\sigma t, f)]^p \frac{d\sigma}{\sigma}$$

we get by MINKOWSKI's inequality

$$\left[\int_0^\infty [t^{-\theta} K(t,f)]^q \frac{dt}{t} \right]^{p/q} \leqq C^p \int_1^\infty \sigma^{(\theta-1)p} \frac{d\sigma}{\sigma} \left[\int_0^\infty [t^{-\theta} K_1(t,f)]^q \frac{dt}{t} \right]^{1/q} < \infty$$

which shows $f \in (L_p(w_0), L_p(w_1))_{\theta,q}$.

Next we prove — this is our main new contribution —

Theorem 2. 2 *We have*

(2. 4) $$f \in (L_p(w_0), L_p(w_1))_{\theta,1}$$

if and only if

(2. 5) $$f \in L_p\left(w_0 \varphi\left(\frac{w_1}{w_0} \right) \right)$$

for some non-decreasing φ *such that*

$$\int_0^\infty \left(\frac{t^\theta}{\varphi(t)} \right)^{p'} \frac{dt}{t} < \infty \qquad \left(\frac{1}{p} + \frac{1}{p'} = 1 \right).$$

Proof i) Assume that (2. 5) holds true. Put

$$f_k = \begin{cases} f & (w_0 < 2^k w_1 \leqq 2w_0), \\ 0 & (\text{elsewhere}). \end{cases}$$

Then, by (2. 2),

$$J(2^k, f_k) \leqq C\Big[\int_{w_0<2^k w_1\leqq 2w_0} |w_0 f|^p\, d\mu\Big]^{1/p} \leqq$$

$$\leqq C\frac{1}{\varphi(2^{-k})}\Big[\int_{w_0<2^k w_1\leqq 2w_0} \Big|w_0\varphi\Big(\frac{w_1}{w_0}\Big)f\Big|^p d\mu\Big]^{1/p},$$

the last estimate because φ is non-decreasing. It follows by HÖLDER's inequality that

$$\sum_{-\infty}^{\infty} 2^{-k\theta} J(2^k, f_k) \leqq C\|f\|_{L_p\left(w_0\varphi\left(\frac{w_1}{w_0}\right)\right)}\Big[\sum_{-\infty}^{\infty}\Big(\frac{2^{-k\theta}}{\varphi(2^{-k})}\Big)^{p'}\Big]^{1/p'} \leqq$$

$$\leqq C\|f\|_{L_p\varphi\left(w_0\varphi\left(\frac{w_1}{w_0}\right)\right)}\Big[\int_0^\infty\Big(\frac{t^\theta}{\varphi(t)}\Big)^{p'}\frac{dt}{t}\Big]^{1/p'} < \infty.$$

If we put

$$f(t) = (\log 2)^{-1} f_k \qquad (2^k \leqq t < 2^{k+1})$$

this gives

$$\int_0^\infty \frac{J(t, f(t))}{t^\theta}\frac{dt}{t} < \infty$$

and, since also $f = \int_0^\infty f(t)\frac{dt}{t}$, we have established (2. 4).

ii) Conversely assume (2. 4) holds true. Then

$$\int_0^\infty \frac{K(t,f)}{t^\theta}\frac{dt}{t} < \infty$$

or, in view of (2. 1),

$$\int_Z \Big|w_0\varphi\Big(\frac{w_1}{w_0}\Big)f\Big|^p d\mu < \infty$$

with

$$\varphi(\lambda) = \Big[\int_0^\infty \frac{[\min(1, t\lambda)]^p}{t^\theta[K(t,f)]^{p-1}}\frac{dt}{t}\Big]^{1/p}$$

which is obviously non-decreasing. But

$$K(t,f) \leqq \max(1, t\lambda) K\Big(\frac{1}{\lambda}, f\Big).$$

Hence

$$(\varphi(\lambda))^p \geqq \frac{\lambda^\theta}{\left[K\left(\frac{1}{\lambda}, f\right)\right]^{p-1}} \int_0^\infty \frac{[\min(1, t\lambda)]^p}{(\lambda t)^\theta [\max(1, t\lambda)]^p} \frac{dt}{t} = C \frac{\lambda^\theta}{\left[K\left(\frac{1}{\lambda}, f\right)\right]^{p-1}}$$

or

$$\left[\frac{\lambda^\theta}{\varphi(\lambda)}\right]^{p'} \leqq C\lambda^\theta K\left(\frac{1}{\lambda}, f\right)$$

and it follows at once the crucial condition:

$$\int_0^\infty \left(\frac{\lambda^\theta}{\varphi(\lambda)}\right)^{p'} \frac{d\lambda}{\lambda} < \infty.$$

Thus we have shown that f satisfies (2. 5).

We conclude by mentioning a sort of dual result.

Theorem 2. 3. *We have*

(2. 6) $$f \in (L_p(w_0), L_p(w_1))_{\theta, \infty}$$

if and only if

(2. 7) $f \in L_p\left(w_0 \varphi\left(\frac{w_1}{w_0}\right)\right)$ *for all non-decreasing* φ *such that*

$$\int_0^\infty \left(\frac{\varphi(t)}{t^\theta}\right)^p \frac{dt}{t} < \infty.$$

Prof. i) Assume that (2. 7) holds true. We claim that then

(2.8) $$\sup \|f\|_{L_p\left(w_0 \varphi\left(\frac{w_1}{w_0}\right)\right)} \Big/ \left[\int_0^\infty \left(\frac{\varphi(t)}{t^\theta}\right)^p \frac{dt}{t}\right]^{\frac{1}{p}} < \infty.$$

Indeed if this is not the case there is a sequence of non-decreasing functions φ_ν such that

$$\int_Z \left| w_0 \varphi_\nu\left(\frac{w_1}{w_0}\right) f\right|^p d\mu = 1, \quad \int_0^\infty \left(\frac{\varphi_\nu(t)}{t^\theta}\right)^p \frac{dt}{t} \leqq \frac{1}{\nu^2}.$$

If we set

$$(\varphi(t))^p = \sum_\nu (\varphi_\nu(t))^p$$

then φ is obviously non-decreasing too and

$$\int_0^\infty \left(\frac{\varphi(t)}{t^\theta}\right)^p \frac{dt}{t} \leqq \sum_\nu \frac{1}{\nu^2} < \infty$$

but

$$\int_Z \left| w_0 \varphi\left(\frac{w_1}{w_0}\right) f \right|^p d\mu = \sum_\nu 1 = \infty.$$

Thus we get a contradiction[1]). Upon taking $\varphi(t) = \min(1, t\, t_0)$ in (2. 8) we get

$$\int_Z |\min(w_0, t_0 w_1) f|^p \, d\mu \leqq C^p t_0^{\theta p},$$

so by (2. 1) we get (2. 6).

ii) Conversely asume (2. 6) holds true. Then by (2. 1)

$$\int_Z |\min(w_0, t w_1) f|^p \, d\mu \leqq C^p t^{\theta p}.$$

Multiply by $\left(\varphi\left(\frac{1}{t}\right)\right)^p$ and integrate! We get

$$\int_Z \left[\int_0^\infty \left[\varphi\left(\frac{1}{t}\right) \min(w_0, t w_1)\right]^p \frac{dt}{t}\right] |f|^p \, d\mu \leqq C^p \int_0^\infty t^{\theta p} \left[\varphi\left(\frac{1}{t}\right)\right]^p \frac{dt}{t} < \infty.$$

But, since φ in non-decreasing,

$$\int_0^\infty \left[\varphi\left(\frac{1}{t}\right) \min(w_0, t w_1)\right]^p \frac{dt}{t} \geqq \int_{\frac{w_0}{2w_1}}^{\frac{w_0}{w_1}} \left[\varphi\left(\frac{w_1}{w_0}\right) t w_1\right]^p \frac{dt}{t} \geqq C^p \left(\varphi\left(\frac{w_1}{w_0}\right)\right)^p w_0^p.$$

Thus we have established (2. 7).

Remark 2. 1. Theorem 2. 2 and 2. 3 should be compared to the known result ([see [7])

$$(L_p(w_0), L_p(w_1))_{\theta, p} = L_p(w_0^{1-\theta} w_1^\theta).$$

It would be inteiesting to have a characterization of $(L_p(w_0), L_p(w_1))_{\theta, q}$ which covers the remaining case $1 < q < \infty$, $q \neq p$ too.

3. Applications to Fourier integrals

We consider the space $\dot{W}_p^{s,q}$ of functions f on $(-\infty, \infty)$ such that

$$\left[\int_0^\infty \left\| f(x+Nt) - \binom{N}{1} f(x+(N-1)t) + \ldots + (-1)^N f(x) \right\|_{L_p}^q \frac{dt}{t^{sq+1}}\right]^{1/q} < \infty$$

[1]) We owe the above argument to Nils-Olof Wallin.

where $1<p<\infty$, $1 \leqq q \leqq \infty$, $0<s<N$. This definition is essentially independent of N. In view of [3], [5], [6] we have

$$\dot{W}_p^{s,q} = (L_p, \dot{H}_p^N)_{\frac{s}{N}, q}$$

where $\dot{H}_p^N$ is the space of functions whose Nth order derivative is in L_p (with respect to Haar measure).

Now take $p=2$. By PARSEVAL's formula we then get

$$(3.1) \qquad f \in \dot{W}_2^{s,q} \quad \text{if and only if} \quad \hat{f} \in [L_2(1), L_2(|\xi|^N)]_{\frac{s}{N}, q}$$

Thus we may apply the result of Section 2.

Let us write

$$L_p^{s,q} = [L_p(1), L_p(|\xi|^N)]_{\frac{s}{N}, q}.$$

On applying theorem 2.1 ($q=\infty$, $i=1$) we get: $\hat{f} \in L_p^{s,\infty}$ if and only if

$$(3.2) \qquad \left[\int_{|\xi| \geqq 1/t} |\hat{f}(\xi)|^p \, d\xi \right]^{1/p} \leqq Ct^s.$$

In view of the duality theorem of LIONS (see [4] or [5], chap. III) this is clearly related to [2], theorem II.

On applying theorem 2.2 we get: $\hat{f} \in L_p^{s,1}$ if and only if

$$(3.3) \qquad \int |\hat{f}(\xi)\varphi(|\xi|)|^p \, d\xi < \infty$$

for some *non-decreasing* φ such that

$$\int_0^\infty \left(\frac{t^s}{\varphi(t)} \right)^{p'} \frac{dt}{t} < \infty.$$

In particular ($p=2$) we get by (3.1): $f \in \dot{W}_2^{s,1}$ if and only if

$$(3.4) \qquad \int |\hat{f}(\xi)\varphi(|\xi|)|^2 \, d\xi < \infty$$

for some *non-decreasing* φ such that

$$\int_0^\infty \left(\frac{t^s}{\varphi(t)} \right)^2 \frac{dt}{t} < \infty.$$

If $s=\frac{1}{2}$ the condition on φ reads simply $\int_0^\infty \frac{dt}{(\varphi(t))^2} < \infty$ and setting $\omega(t)=(\varphi(t))^2$ we get (0.2). This is essentially [2], theorem III.

As in [2] similar results hold if $|\xi|^N$ is replaced by $(1+|\xi|)^N$.

It is also clear that we in the above fashion can treat the case of Fourier integrals in any number of variables.

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(Received March 5, 1966)

On multiplicative characters

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Let $f(n)$ be a multiplicative number-theoretical function, i. e.

$$f(mn)=f(m)f(n), \qquad (m, n)=1,$$

satisfying

(1) $$|f(n)| \leqq 1 \qquad (n=1, 2, \ldots).$$

H. DELANGE [1] proved that for the fulfilment of the relation

(2) $$M(x) \stackrel{\text{def}}{=} \sum_{n \leqq x} f(n) = o(x)$$

a sufficient condition is given by (1) and

(3) $$\sum_p \frac{1-\operatorname{Re} f(p)}{p} = +\infty.$$

It is natural to ask for a condition, which turns out to be besides (1) sufficient for the fulfilment of the relation

(4) $$M(x; k, l) \stackrel{\text{def}}{=} \sum_{\substack{n \leqq x \\ n \equiv l \pmod k}} f(n) = o(x).$$

It is clear that (2) alone is a too weak condition for the fulfilment of (4). If for example $f(n)=\chi(n)$, where $\chi(n)$ stands for an arbitrary but fixed non-principal character mod k, then

$$M(x) = \sum_{n \leqq x} \chi(n) = O(x),$$

and on the other-hand for every l with $(l, k)=1$, $1 \leqq l \leqq k$,

$$M(x; k, l) = \sum_{\substack{n \leqq x \\ n \equiv l \pmod k}} \chi(n) = \chi(l) \left\{ \left[\frac{x-l}{k} \right] + 1 \right\} \neq o(x).$$

We will show in the sequel that the characters are exceptional in a certain sense.

Theorem 1. *Let $f(n)$ be an arbitrary but fixed multiplicative function satisfying (1). Let k and l be given natural numbers with $(k, l)=1$. Suppose that*

$$f(p)=\alpha, \quad \text{if} \quad p \equiv l \pmod k, \ p \text{ prime},$$

and suppose that the value of α *is different from all the values* $\chi(l)$ *taken by any* $\chi \bmod k$. *Then for every* m *satisfying* $(m, k)=1$

$$\sum_{\substack{n\leqq x\\ n\equiv m(\operatorname{mod} k)}} f(n) = o(x) \tag{5}$$

holds.

Theorem 1 is a consequence of the following

Theorem 2. *Let* $g(n)$ *be an arbitrary but fixed multiplicative function satisfying* (1). *Suppose that for a pair* k, l *of coprime natural numbers*

$$g(p)=\beta, \quad \text{if} \quad p\equiv l \ (\operatorname{mod} k), \ p \text{ prime}$$

holds, where $\beta\neq 1$. *Then*

$$\sum_{n\leqq x} g(n) = o(x).$$

Deduction of Theorem 1 from Theorem 2. Let $g_\chi(n)=\chi(n)f(n)$, where $\chi(n)$ stands for an arbitrary character mod k. Then Theorem 2 applies by trivial arguments and gives the relation

$$\sum_{n\leqq x} \chi(n)f(n) = o(x).$$

From this, using

$$\sum_{\substack{n\leqq x\\ n\equiv m\,(\operatorname{mod} k)}} f(n) = \frac{1}{\varphi(k)} \sum_{\chi} \bar{\chi}(m) \sum_{n\leqq x} \chi(n)f(n),$$

the statement of Theorem 1 follows.

For the proof of Theorem 2 we need three lemmas. Before formulating the first of them we quote the following preliminaries.

Let $\mathcal{P}$ be an arbitrary infinite subset of the rational primes. Let the function $V_{\mathcal{P}}(n)$ be defined by

$$V_{\mathcal{P}}(n) \overset{\text{def}}{=} \sum_{\substack{p\mid n\\ p\in\mathcal{P}}} 1. \tag{6}$$

$V_{\mathcal{P}}(n)$ is an additive function. Let the number A_x be defined by

$$A_x \overset{\text{def}}{=} \sum_{\substack{p\leqq x\\ p\in\mathcal{P}}} \frac{1}{p}. \tag{7}$$

Then the following result of P. TURÁN [2] yields, which we state as

Lemma 1.

$$\sum_{n\leqq x} |V_{\mathcal{P}}(n)-A_x| = O(xA_x^{1/2}),$$

$$\sum_{n\leqq x} (V_{\mathcal{P}}(n)-A_x)^2 = O(xA_x).$$

Lemma 2. *For a coprime pair k, l of natural numbers*

$$\sum_{\substack{p\leqq x\\ p\equiv l(\mathrm{mod}\ k)}} \frac{1}{p} = \frac{1}{\varphi(k)} \log\log x + o(1).$$

This is an easy consequence of the prime-number theorem for arithmetical progressions.

Lemma 3. *Let $f(n)$ be an arbitrary multiplicative function satisfying* (1). *Then in the notation of* (2) *and* (7)

$$M(x)A_x = \sum_{\substack{p\leqq x\\ p\in\mathcal{P}}} f(p)M\left(\frac{x}{p}\right) + O(xA_x^{1/2}). \tag{8}$$

For the proof, see [3].

Proof of Theorem 2. Let us define $\mathcal{P}$ as the set of rational primes satisfying

$$p\equiv l \ (\mathrm{mod}\ k). \tag{9}$$

For the sake of brevity we take

$$h \overset{\mathrm{def}}{=} \frac{1}{\varphi(k)} \log\log x. \tag{10}$$

Then using lemma 3 and the condition of Theorem 2 we have

$$M(x)h = \beta \sum_{\substack{p\leqq x\\ p\equiv l(\mathrm{mod}\ k)}} M\left(\frac{x}{p}\right) + O(xh^{1/2}). \tag{11}$$

We have to deal with two cases: 1) $|\beta|<1$ and 2) $|\beta|=1$. In the first case the statement of Theorem 2 can be deduced from (11) in a very simple way.

Take

$$\overline{\lim_{x\to\infty}} \frac{|M(x)|}{x} = \tau. \tag{12}$$

Then by (11)

$$|M(x)|h \leqq |\beta|\,(\tau+\varepsilon)x \sum_{\substack{p\leqq x\\ p\equiv l(\mathrm{mod}\ k)}} \frac{1}{p} + o(xh)$$

holds for every $\varepsilon>0$ when $x\geqq x_0(\varepsilon)$. From this one can deduce by lemma 2 the inequality $|\tau|\leqq|\beta|\tau$.

In the case $|\beta|<1$ this implies $\tau=0$ i. e. the statement of the theorem is then true.

So it only remains to deal with the case $|\beta|=1$. Let δ be an arbitrary but fixed number with $0<\delta<\frac{1}{2}$. Using

$$\sum_{x^\delta\leqq p\leqq x} \left|M\left(\frac{x}{p}\right)\right| = O\left(x \sum_{x^\delta\leqq p\leqq x} \frac{1}{p}\right) = O(x)$$

and applying (11) with x/q instead of x, where $q \leqq x^\delta$ stands for a prime $\equiv l \pmod k$, we deduce by adding the results that

$$(13)\qquad h \sum_{\substack{q \leqq x^\delta \\ q \equiv l \pmod k}}' M\left(\frac{x}{q}\right) = \beta \sum_{\substack{p \leqq x^\delta,\, q \leqq x^\delta \\ p \equiv q \equiv l \pmod k}} M\left(\frac{x}{pq}\right) + O(xh^{3/2}).$$

Remark that by the deduction of (13) we have used the relation

$$\log\log\frac{x}{q} = \log\log x + O(1), \qquad q \leqq x^\delta.$$

From (13), by the modified form of (11), it follows that

$$(14)\qquad M(x)h^2 = \beta^2 \sum_{\substack{p \leqq x^\delta \\ q \leqq x^\delta \\ p \equiv q \equiv l \pmod k}} M\left(\frac{x}{pq}\right) + O(xh^{3/2}).$$

The first term on the right-hand side of (14) can be described as

$$2\beta^2 \sum_{\substack{pq \leqq x^\delta \\ p \leqq q \\ p \equiv q \equiv l \pmod k}} M\left(\frac{x}{pq}\right) + o(xh).$$

From (11), multiplying by h, and from (14) we obtain finally

$$(15)\qquad h^2\left(\frac{1}{\beta} - \frac{1}{2\beta^2}\right) M(x) = h \sum_{\substack{p \leqq x^\delta \\ p \equiv l \pmod k}} M\left(\frac{x}{p}\right) - \sum_{\substack{pq \leqq x^\delta \\ p \leqq q \\ p \equiv q \equiv l \pmod k}} M\left(\frac{x}{pq}\right) + O(xh^{3/2}).$$

Let now ε be an arbitrary but fixed positive number, and let us define the numbers n_ν by $n_\nu = (1+\varepsilon)^\nu$ $(\nu = 1, 2, \ldots)$. Then

$$\left| M\left(\frac{x}{n_\nu}\right) - M\left(\frac{x}{m}\right) \right| \leqq \left| \frac{x}{n_\nu} - \frac{x}{m} \right| + 1 \leqq \frac{\varepsilon x}{m} + 1$$

holds for $m \in [n_\nu, n_{\nu+1}]$. So by (15) we have

$$(16)\qquad h^2\left(\frac{1}{\beta} - \frac{1}{2\beta^2}\right) M(x) = \sum_{\nu \leqq \frac{\delta \log x}{\log(1+\varepsilon)}} a_\nu M\left(\frac{x}{n_\nu}\right) + O(\varepsilon x h^2),$$

where a_ν is given by

$$(17)\qquad a_\nu = h \sum_{\substack{p \in [n_\nu, n_{\nu+1}] \\ p \equiv l \pmod k}} 1 - \sum_{\substack{pq \in [n_\nu, n_{\nu+1}] \\ p \leqq q \\ p \equiv q \equiv l \pmod k}} 1.$$

From the prime-number theorem for arithmetical progressions it follows that

$$h \sum_{\substack{p \in [n_\nu, n_{\nu+1}] \\ p \equiv l \pmod k}} = \frac{h\varepsilon n_\nu}{\varphi(k) \log n_\nu}(1+o(1)),$$

$$\sum_{\substack{pq \in [n_\nu, n_{\nu+1}] \\ p \leqq q \\ p \equiv q \equiv l \pmod k}} = \frac{1}{\varphi^2(k)} \cdot \frac{\varepsilon n_\nu}{\log n_\nu} \log\log n_\nu (1+o(1)).$$

So for $n_\nu \leqq x^{\delta/2}$ we have

$$a_\nu = \frac{1}{\varphi^2(k)} \frac{\varepsilon n_\nu}{\log n_\nu} \log \frac{\log x}{\log n_\nu}(1+o(1)). \tag{18}$$

Now as in the case $|\beta|<1$ we have to show that $\tau=0$. From (16) it follows by (18) that

$$h^2 \left| \frac{1}{\beta} - \frac{1}{2\beta^2} \right| |M(x)| \leqq \frac{x(\tau+\varepsilon)}{\varphi^2(k)} \varepsilon \sum_{n_\nu \leqq x^\delta} \frac{\log\log x - \log\log n_\nu}{\log n_\nu} + o(\varepsilon x h^2). \tag{19}$$

From (19) one has the inequality

$$h^2 \left| \frac{1}{\beta} - \frac{1}{2\beta^2} \right| |M(x)| \leqq \frac{1}{2} x(\tau+\varepsilon) \frac{\varepsilon}{\log(1+\varepsilon)} h^2 + o(x\varepsilon h^2). \tag{20}$$

Letting first $x \to \infty$ then $\varepsilon \to 0$ it results

$$\left| \frac{1}{\beta} - \frac{1}{2\beta^2} \right| \tau \leqq \frac{\tau}{2}.$$

Hence $\tau=0$ or $\left| \frac{1}{\beta} - \frac{1}{2\beta^2} \right| \leqq \frac{1}{2}$. The second possibility can not occur in our case. For it implies $|\beta| \geqq 1$ and it holds in the case $|\beta|=1$ for $\beta=1$ only. But $\beta=1$ is impossible by the assumption of the theorem. Thus $\tau=0$ and Theorem 2 is proved.

A similar argument leads to the following

Theorem 3. *Let $f(n)$ be an arbitrary multiplicative function satisfying* (1). *Suppose that $f(p)=\alpha$ if $p \equiv l \pmod k$, p prime, holds for some k, l with $(k, l)=1$, where $|\alpha|<1$ or $\frac{\arg \alpha}{2\pi}$ is an irrational number. Then for every pair k^*, l^* of natural numbers*

$$\sum_{\substack{n \leqq x \\ n \equiv l^* \pmod{k^*}}} f(n) = o(x).$$

We note for the proof, that it is sufficient to deal with the case $(l^*, k^*)=1$ only. In this case it is enough to show the fulfilment of

$$\sum_{n \leqq x} g_\chi(n) = o(x),$$

where $g_\chi(n) = f(n)\chi(n)$ and $\chi(n)$ stands for an arbitrary but fixed character mod k^*.

The proof of Theorem 2 applies in this case too, the only difference is in the choice of the set $\mathcal{P}$ of Theorem 2, which requires but obvious modifications. We omit the details.

As an immediate consequence of Theorem 2 we mention the following

Theorem 4. *Let $f(n)$ be an arbitrary multiplicative function satisfying* (1). *Suppose that for a given natural number k and for every l coprime to k*

$$f(p)=\alpha_l \quad \text{if} \quad p\equiv l \ (\text{mod}\ k), \ p \text{ prime.}$$

Suppose further that there exists for every character $\chi(n)$ mod k at least one prime p with

$$\chi(p)\neq f(p).$$

Then for all l with $1\leqq l\leqq k$, $(k, l)=1$, we have

$$\sum_{\substack{n\leqq x \\ n\equiv l\,(\text{mod}\,k)}} f(n) = o(x).$$

Observe that Theorem 3 has some consequences for the distribution of the fractional parts of the values taken by an additive function. Let $h(n)$ be an additive function i. e. $h(mn)=h(m)+h(n)$ for $(m, n)=1$. Suppose that

$$h(p)=\alpha \quad \text{if} \quad p\equiv l \ (\text{mod}\ k), \ p \text{ prime}$$

holds for a given coprime pair k, l of natural numbers, where α denotes an arbitrary irrational number.

Under these conditions we have the following

Theorem 5. *The values of $\{h(n)\}$ are uniformly distributed in every arithmetical progression.* (Here { } stands for the fractional part.)

The proof of Theorem 5 can be obtained by applying Theorem 3 to the functions

$$f_t(n) = e^{2\pi i t h(n)}$$

and using WEYL's theorem [4] concerning uniform distribution.

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(Received July 9, 1966)

An example in the theory of Fourier series

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A. PLESSNER proved in [1] that if a trigonometric series converges on a set E with $mE>0$, then its conjugate series converges almost everywhere on E. This fact was proved independently by J. MARCINKIEWICZ and A. ZYGMUND [2] too.

In the present note we are going to prove the following

Theorem. *There exists a sequence $\{\alpha_n\}$ of non-negative numbers with the following properties. The series*

$$\frac{\alpha_0}{2}+\sum_{n=1}^{\infty}\alpha_n\cos nx \tag{1}$$

is the Fourier series of an $f(x)\geqq 0, f(x)\in L^q[-\pi,\pi]$ for every $q>0$. The series (1) diverges unboundedly on an everywhere dense set of second category in $[-\pi,\pi]$. There exists an infinite sequence of natural numbers with

$$s_{n_k}(x)=\frac{\alpha_0}{2}+\sum_{\nu=1}^{n_k}\alpha_\nu\cos\nu x\geqq 0,\qquad (x\in[-\pi,\pi];\quad k=1,2,\ldots). \tag{2}$$

Finally the series

$$\sum_{n=1}^{\infty}\alpha_n\sin nx \tag{3}$$

converges uniformly in $[-\pi,\pi]$, and so proves to be the Fourier series of a continuous function.

Remark. By the quoted result of PLESSNER the series (1) converges almost everywhere on $[-\pi,\pi]$.

For the proof of the Theorem we need three lemmas.

Lemma 1. *Suppose that all partial sums of the series*

$$\frac{b_0}{2}+\sum_{n=1}^{\infty}b_n\cos nx \tag{4}$$

are non-negative for every $x\in[-\pi,\pi]$. Then the same is true for the series

$$\frac{b_0^2}{2}+\sum_{n=1}^{\infty}b_n^2\cos nx. \tag{5}$$

Proof. Let us denote the partial sums of (4) by

$$u_n(x) = \frac{b_0}{2} + \sum_{\nu=1}^{n} b_\nu \cos \nu x, \qquad (n=0, 1, 2, \ldots).$$

So we have

$$\frac{b_0^2}{2} + \sum_{\nu=1}^{n} b_\nu^2 \cos \nu x = \frac{1}{\pi} \int_{-\pi}^{\pi} u_n(t) u_n(x-t)\, dt$$

which, by the supposed non-negativity of $u_n(x)$, proves the statement of the lemma.

Lemma 2. *Let $\{c_n\}$ be a decreasing sequence of positive numbers satisfying $c_n = 0\left(\frac{1}{n}\right)$. Then*

$$\left| \sum_{\nu=1}^{n} c_\nu \sin \nu x \right| < K \qquad (n=1, 2, \ldots).$$

This lemma represents a well-known result (see [3], Vol. I. pp. 182—183). The third thing we need is a theorem of P. TURÁN, which we formulate as

Lemma 3. *All partial sums of the series*

$$(6) \qquad 1 + \sum_{n=1}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \cos nx$$

are non-negative for every $x \in [-\pi, \pi]$.

Before going to prove the Theorem, we mention that the coefficients of (6) form a decreasing sequence satisfying

$$(7) \qquad c_1 n^{-\frac{1}{2}} < (-1)^n \binom{-\frac{1}{2}}{n} < c_2 n^{-\frac{1}{2}}$$

with suitable $c_1, c_2 > 0$.

Let now $\{\lambda_r\}$ be an arbitrary but fixed sequence of positive numbers with

$$\sum_{r=1}^{\infty} \frac{1}{\lambda_r} < +\infty .$$

We introduce the notation

$$a_0 = 4, \quad a_n = \binom{-\frac{1}{2}}{n}^2 \qquad (n = 1, 2, \ldots).$$

Then for every natural number u we have by (7)

$$(8) \qquad \frac{a_0}{2} + \sum_{v=1}^{V} a_{uv} > B$$

for any $B > 0$ if $V = V(B)$ is large enough.

As the next step we define two sequences $\{l_r\}$ and $\{m_r\}$ of natural numbers by induction on r.

Let $l_1 = 1$ and let m_1 be the least natural number satisfying the condition

$$\frac{a_0}{2} + \sum_{\nu=1}^{m_1} a_\nu > \frac{1}{\lambda_1}.$$

The choice of m_1 is always possible by (8).

Suppose now that the numbers l_u and m_u are already defined for $1 < u < r-1$. Determine l_r and m_r by the conditions

1. l_r is the least natural number which is a multiple of l_{r-1} and satisfying

$$l_r > l_{r-1} m_{r-1}.$$

2. If l_r is chosen as mentioned, m_r is the least natural number satisfying

$$\frac{a_0}{2} + \sum_{\nu=1}^{m_r} a_{\nu l_r} > \frac{1}{\lambda_r}.$$

The choice of m_r is always possible by (8). Thus the sequences $\{l_r\}$ and $\{m_r\}$ are defined for every value of $r \geqq 1$.

Let now the trigonometric polynomials $U_r(x)$ and $V_r(x)$ be defined by

$$\text{(9)} \qquad U_r(x) = \frac{a_0}{2} + \sum_{\nu=1}^{m_r} a_{\nu l_r} \cos \nu l_r x, \quad V_r(x) = \sum_{\nu=1}^{m_r} a_{\nu l_r} \sin \nu l_r x \qquad (r = 1, 2, \ldots).$$

We shall show that

$$\text{(10)} \qquad \sum_{r=1}^{\infty} \frac{1}{\lambda_r} U_r(x)$$

and

$$\text{(11)} \qquad \sum_{r=1}^{\infty} \frac{1}{\lambda_r} V_r(x)$$

are conjugate trigonometric series possessing the properties required in the Theorem.

First of all, observe that the series

$$\text{(12)} \qquad \sum_{n=1}^{\infty} a_n \sin nx$$

has monotonically decreasing coefficients satisfying $a_n = O\left(\frac{1}{n}\right)$ by (7) and (8).

Thus, using Lemma 2, we get that

$$\text{(13)} \qquad \left| \sum_{\nu=1}^{n} a_\nu \sin \nu x \right| < K \qquad (n = 1, 2, \ldots)$$

holds for every $x \in [-\pi, \pi]$. On the other hand, we have

$$\text{(14)} \qquad V_r(x) = \frac{1}{l_r} \sum_{t=0}^{l_r - 1} \left\{ \sum_{\nu=1}^{m_r l_r} a_\nu \sin \nu \left(x + \frac{2\pi t}{l_r} \right) \right\}.$$

So one can deduce from (14), using (13), that

$$\text{(15)} \qquad |V_r(x)| < K \qquad (x \in [-\pi, \pi];\ r = 1, 2, \ldots).$$

holds. (15) means that the series (11) converges uniformly in $[-\pi, \pi]$. Thus

$$\tilde{f}(x) \overset{\text{def}}{=} \sum_{r=1}^{\infty} \frac{1}{\lambda_r} V_r(x) \tag{16}$$

is a continuous function. By the choice of the sequences $\{l_r\}$ and $\{m_r\}$, $V_r(x)$ and $V_s(x)$ do not contain common sines if $r \neq s$.

By the uniform convergence of (16) and the remark done before we have

$$\alpha_n = \begin{cases} \frac{1}{\lambda_r} a_{\nu l_r} & \text{if} \quad n = \nu l_r, \quad 1 \leqq \nu \leqq m_r \\ 0 & \text{otherwise,} \end{cases} \tag{17}$$

where α_n denotes the nth Fourier sine coefficient of $\tilde{f}(x)$. By (17), using the definition of the sequences $\{a_n\}$ and $\{\lambda_r\}$ we get the inequality

$$\alpha_n \geqq 0 \qquad (n = 1, 2, \ldots). \tag{18}$$

Now (18) means by a theorem of PALEY [5], that the Fourier series of $\tilde{f}(x)$ converges uniformly.

A representation similar to (14) shows that

$$U_r(x) \geqq 0 \qquad (x \in [-\pi, \pi];\ r = 1, 2, \ldots)$$

holds. Indeed, by Lemma 3 and Lemma 1 all the partial sums of the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{19}$$

are non-negative for every $x \in [-\pi, \pi]$. Using this fact we get

$$\sum_{r=1}^{\infty} \frac{1}{\lambda_r} \int_{-\pi}^{\pi} U_r(x)\, dx < +\infty \tag{20}$$

which means by the theorem of Beppo Levi, that the series (10) converges a. e. on $[-\pi, \pi]$ to an $f(x) \geqq 0$, $f(x) \in L[-\pi, \pi]$.

A similar argument shows that

$$f(x) \sim \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos nx. \tag{21}$$

$f(x)$ and $\tilde{f}(x)$ being given by conjugate Fourier-series and since $\tilde{f}(x) \in L^q[-\pi, \pi]$ for every $q > 0$, it follows by known arguments that $f(x) \in L^q[-\pi, \pi]$ holds for every $q > 0$.

Now we note that the series (21) diverges unboundedly on an everywhere dense set of second category in $[-\pi, \pi]$. In fact

$$\frac{1}{\lambda_r} U_r\left(2\pi \frac{l}{l_s}\right) \geqq 1, \qquad 0 \leqq l < l_s - 1, \quad r \geqq s$$

holds by $l_s | l_r$ if $r \geqq s$ and by the choice of the numbers $\{l_r\}$ and $\{m_r\}$. From this, noting that the numbers $2\pi \frac{l}{l_s}$ $(s=1, 2, \dots)$ lie everywhere dense in $[0, 2\pi]$, the assertion follows. We quote the result that if a series of continuous functions diverges unboundedly on an everywhere dense set in $[-\pi, \pi]$, then the set of points where the series diverges unboundedly is of the second category in $[-\pi, \pi]$. For the proof see for example [6].

We conclude by remarking that for the choice $n_k = l_k m_k$ the non-negativity of the partial sums required in the theorem follows without difficulty.

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(Received July 9, 1966)

A remark on the theory of multiplicative functions

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In [2] E. M. WRIGHT presents a proof of the prime-number theorem, which uses elementary methods and depends on ideas introduced by A. SELBERG in the theory of numbers. His method used there enables one to prove more general results of the same character concerning multiplicative functions. We call in the sequel a function $f(n)$, defined on the domain of the natural numbers, *multiplicative* if for coprime integers m and n the relation

$$f(mn) = f(m)f(n)$$

holds. In this note we prove a theorem concerning multiplicative functions of that kind. It can be stated as follows.

Theorem. *Let $f(n)$ be a multiplicative function, which takes the three values* 0, 1, -1 *only. Let k be a positive integer. Suppose that there exists a natural number l, for which $l \leqq k$ and $(l, k) = 1$, and the relation*

$$\sum_{\substack{n \leqq x \\ n \equiv l \pmod{k}}} \frac{f(n)}{n} = O(1)$$

is satisfied. Then for all u with $1 \leqq u \leqq k$, $(u, k) = 1$, the relation

$$F_u(x) \overset{\text{def}}{=} \sum_{\substack{n \leqq x \\ n \equiv u \pmod{k}}} f(n) = o(x)$$

holds.

We mention, that in the case $k = 1$, $l = 1$, when

$$f(n) \equiv \mu(n),$$

where $\mu(n)$ stands for MOEBIUS's function, our result presents the prime number theorem. For the detailed elementary deduction of the prime number theorem from $\sum_{n \leqq x} u(n) = o(x)$, and for an elementary proof of the mentioned relation (and thus of the prime number theorem) see [3].

The proof of the theorem consists of two different parts. In the first part we prove the inequality

$$(1) \qquad |F_u(x)|\log^2 x \leqq \frac{2}{\varphi(k)} \sum_{\substack{v\leqq k\\(v,k)=1}} \int_1^x \left|F_v\left(\frac{x}{t}\right)\right| \log t\, dt + O(x\log x),$$

where $\varphi(k)$ denotes EULER's function, and in the second one we deduce from (1) the statement of the theorem.

For the sake of completeness we give the proof in full details. The method used by us shows a great formal similarity to that of [2]. The difference between the two methods lies primarily in the fact, that we make use, besides the original formulas of SELBERG, of some further formulas closely related to them. In the paper we make free use of the terminology applied in [2]. Before concluding these preliminary remarks, we observe that the result of the paper does not seem to be obtainable by analytical methods, thus in this case elementary methods seem to go further than those of the theory of functions.

Proof of the first part of the theorem

We shall need the following lemmas:

Lemma 1. *Let $c_1, c_2, \ldots$ be a sequence of numbers,*

$$C(t) = \sum_{n\leqq t} c_n \qquad (1\leqq t<\infty),$$

and $f(t)$ a function of t. Then

$$\sum_{n\leqq x} c_n f(n) = \sum_{n\leqq x-1} C(n)\{f(n)-f(n+1)\} + C(x)f([x]).$$

If, in addition, $c_j=0$ for $j<n_1$ and $f(t)$ has a continuous derivative for $t\geqq n_1$, then

$$\sum_{n\leqq x} c_n f(n) = C(x)f(x) - \int_{n_1}^x C(t)f'(t)\,dt.$$

For the proof of the lemma see [1], theorem 421, p. 346.

Lemma 2. (SELBERG's formula.) *Let k be a positive integer. Then if $1\leqq u\leqq k$, $(u,k)=1$, we have*

$$(2) \qquad \sum_{\substack{n\leqq x\\ n\equiv u \pmod k}} \Lambda(n)\log n + \sum_{\substack{mn\leqq x\\ mn\equiv u \pmod k}} \Lambda(m)\Lambda(n) = \frac{2}{\varphi(k)} x\log x + O(x).$$

Here $\varphi(k)$ stands for EULER's function. For the proof see [4].

Lemma 3. *Let $f(n)$ be any multiplicative function satisfying the condition*

$$|f(n)|\leqq 1$$

for all values of the positive integer n. Then

$$F_u(x)\log x-\sum_{n\le x} f(n)\Lambda(n)F_{un^{-1}}\left(\frac{x}{n}\right)=O(x), \tag{3}$$

where $\Lambda(n)$ denotes VON MANGOLDT'*s function, and n^{-1} stands for the number m determined by the conditions*

$$mn\equiv 1 \pmod k), \qquad 1\le m\le k.$$

Proof. We start with the obvious formula

$$\sum_{\substack{n\le x\\ n\equiv u \pmod k}} f(n)\log\frac{x}{n}=O(x).$$

Hence

$$F_u(x)\log x-\sum_{\substack{n\le x\\ n\equiv u \pmod k}} f(n)\log n=F_n(x)\log x-\sum_{p\le x} f(p)\log p\, F_{up^{-1}}\left(\frac{x}{p}\right)+O(x)=O(x).$$

Now considering that

$$\Lambda(p)=\log p \quad\text{and}\quad \sum_{n\le x,\, n\ne p} f(n)\Lambda(n)F_{un^{-1}}\left(\frac{x}{n}\right)=O(x),$$

we get at once the statement of the lemma.

After these preliminaries we perform the first part of the proof, i. e. the proof of the inequality (1).

If we replace n by m and x by x/n in (3), we have

$$F_u\left(\frac{x}{n}\right)\log\frac{x}{n}-\sum_{m\le\frac{x}{n}} f(m)\Lambda(m)F_{um^{-1}}\left(\frac{x}{mn}\right)=O(x).$$

Hence

$$\left\{F_u(x)\log x-\sum_{n\le x} f(n)\Lambda(n)F_{un^{-1}}\left(\frac{x}{n}\right)\right\}\log x+$$

$$+\sum_{n\le x} f(n)\Lambda(n)\left\{F_{un^{-1}}\left(\frac{x}{n}\right)\log\frac{x}{n}-\sum_{m\le\frac{x}{n}} f(m)\Lambda(m)F_{um^{-1}n^{-1}}\left(\frac{x}{mn}\right)\right\}=$$

$$=O(x\log x)+O\left(x\sum_{n\le x}\frac{\Lambda(n)}{n}\right)=O(x\log x),$$

that is

$$F_u(x)\log^2 x=\sum_{n\le x} f(n)\,\Lambda(n)\log n F_{un^{-1}}\left(\frac{x}{n}\right)+$$

$$+\sum_{mn\le x} f(m)f(n)\Lambda(m)\Lambda(n)F_{um^{-1}n^{-1}}\left(\frac{x}{mn}\right)+O(x\log x),$$

whence

$$(4)\qquad |F_u(x)|\log^2 x \leqq \sum_{\substack{1\leqq v\leqq k\\ (v,k)=1}} \sum_{n\leqq x} a_n^{(v)} \left|F_{uv^{-1}}\left(\frac{x}{n}\right)\right| + O(x\log x),$$

where

$$a_n^{(v)} = \begin{cases} \Lambda(n)\log n + \sum\limits_{hk=n} \Lambda(h)\Lambda(k), & \text{if} \quad n\equiv v (\mathrm{mod}\ k),\\ 0 \quad \text{otherwise}, \end{cases}$$

and

$$\sum_{n\leqq x} a_n^{(v)} = \frac{2}{\varphi(k)} x\log x + O(x)$$

by (2).

We now replace the inner sum on the right-hand side of (4) by an integral. To do so, we shall prove that

$$(5)\qquad \sum_{n\leqq x} a_n^{(v)} \left|F_{uv^{-1}}\left(\frac{x}{n}\right)\right| = \frac{2}{\varphi(k)}\int_1^x \left|F_{uv^{-1}}\left(\frac{x}{t}\right)\right| \log t\, dt + O(x\log x).$$

Noticing the fact, that if v runs through a restricted system of residues mod k, then the same is true of uv^{-1} for an arbitrary u with $(u, k)=1$, which means that (4) and (5) together will conclude the proof of (1).

We remark that if $t>t'\geqq 0$

$$||F_{uv^{-1}}(t)| - |F_{uv^{-1}}(t')|| \leqq |F_{uv^{-1}}(t) - F_{uv^{-1}}(t')| =$$
$$= |(F_{uv^{-1}}(t) + G_{uv^{-1}}(t)) - (F_{uv^{-1}}(t') + G_{uv^{-1}}(t')) - G_{uv^{-1}}(t) + G_{uv^{-1}}(t')| \leqq$$
$$\leqq H_{uv^{-1}}(t) - H_{uv^{-1}}(t'),$$

where

$$H_{uv^{-1}}(t) \overset{\text{def}}{=} F_{uv^{-1}}(t) + 2G_{uv^{-1}}(t), \qquad G_{uv^{-1}}(t) \overset{\text{def}}{=} \sum_{\substack{n\leqq t\\ n\equiv uv^{-1}}} 1,$$

and that $H_{uv^{-1}}(t)$ is a steadily increasing function of t, $H_{uv^{-1}}(t)=O(t)$. Using lemma 1, we obtain that

$$(6)\qquad \begin{aligned} \sum_{n\leqq x-1} n\left\{H_{uv^{-1}}\left(\frac{x}{n}\right) - H_{uv^{-1}}\left(\frac{x}{n+1}\right)\right\} &= \sum_{n\leqq x} H_{uv^{-1}}\left(\frac{x}{n}\right) - [x]\, H_{uv^{-1}}\left(\frac{x}{[x]}\right) = \\ &= O\left(x\sum_{n\leqq x}\frac{1}{n}\right) = O(x\log x). \end{aligned}$$

Now we prove (5) in two steps. First if we put

$$c_1 = 0, \quad c_n = a_n^{(v)} - \frac{2}{\varphi(k)}\int_{n-1}^{n} \log t\, dt, \qquad f(n) = \left|F_{uv^{-1}}\left(\frac{x}{n}\right)\right|$$

in the lemma 1, we have

$$C(x) = \sum_{n \leqq x} a_n^{(v)} - \frac{2}{\varphi(k)} \int_1^{[x]} \log t\, dt = O(x)$$

and

$$\sum_{n \leqq x} a_n^{(v)} \left| F_{uv-1}\left(\frac{x}{n}\right)\right| - \frac{2}{\varphi(k)} \sum_{2 \leqq n \leqq x} \left| F_{uv-1}\left(\frac{x}{n}\right)\right| \int_{n-1}^{n} \log t\, dt =$$

(7) $$= \sum_{n \leqq x-1} C(n) \left\{ \left| F_{uv-1}\left(\frac{x}{n}\right)\right| - \left| F_{uv-1}\left(\frac{x}{n+1}\right)\right| \right\} + C(x) F_{uv-1}\left(\frac{x}{[x]}\right) =$$

$$= O\left(\sum_{n \leqq x-1} n \left\{ H_{uv-1}\left(\frac{x}{n}\right) - H_{uv-1}\left(\frac{x}{n+1}\right)\right\}\right) + O(x) = O(x \lg x)$$

by (6).

Next

$$\left| \left| F_{uv-1}\left(\frac{x}{n}\right)\right| \int_{n-1}^{n} \log t\, dt - \int_{n-1}^{n} \left| F_{uv-1}\left(\frac{x}{t}\right)\right| \log t\, dt \right| \leqq$$

$$\leqq \int_{n-1}^{n} \left| \left| F_{uv-1}\left(\frac{x}{n}\right)\right| - \left| F_{uv-1}\left(\frac{x}{t}\right)\right| \right| \log t\, dt \leqq$$

$$\leqq \int_{n-1}^{n} \left\{ H_{uv-1}\left(\frac{x}{t}\right) - H_{uv-1}\left(\frac{x}{n}\right)\right\} \log t\, dt \leqq (n-1)\left\{ H_{uv-1}\left(\frac{x}{n-1}\right) - H_{uv-1}\left(\frac{x}{n}\right)\right\}.$$

Hence

$$\sum_{2 \leqq n \leqq x} \left| F_{uv-1}\left(\frac{x}{n}\right)\right| \int_{n-1}^{n} \log t\, dt - \int_{1}^{x} \left| F_{uv-1}\left(\frac{x}{t}\right)\right| \log t\, dt =$$

(8) $$= O\left(\sum_{n \leqq x-1} n \left\{ H_{uv-1}\left(\frac{x}{n}\right) - H_{uv-1}\left(\frac{x}{n+1}\right)\right\}\right) + O(x \log x) = O(x \log x).$$

Combining (7) and (8), we get (5).

Proof of the second part of the theorem

First we give our inequality (1) another form. We introduce the functions

$$V_v(\xi)=e^{-\xi}F_v(e^{\xi}), \qquad 1\leqq v\leqq k,\ (v,k)=1.$$

If we write $x=e^{\xi}$, $t=xe^{-\eta}$, we have

$$\int_1^x \left|F_v\left(\frac{x}{t}\right)\right|\log t\,dt = x\int_0^{\xi}|V_v(\eta)|\,(\xi-\eta)\,d\eta = x\int_0^{\xi}|V_v(\eta)|\int_{\eta}^{\xi}d\zeta\,d\eta = x\int_0^{\xi}\int_0^{\zeta}|V_v(\eta)|\,d\eta\,d\zeta$$

by interchanging the order of integration. Then our inequality (1) becomes

$$(9)\qquad \xi^2|V_v(\xi)|\leqq\frac{2}{\varphi(k)}\sum_{\substack{v\leqq k\\(v,k)=1}}\int_0^{\xi}\int_0^{\zeta}|V_v(\eta)|\,d\eta\,d\zeta+O(\xi).$$

The functions $V_v(\xi)$ are bounded as $\xi\to\infty$. Hence we may write

$$\alpha_v=\overline{\lim_{\xi\to\infty}}|V_v(\xi)|,\quad \beta_v=\overline{\lim_{\xi\to\infty}}\frac{1}{\xi}\int_0^{\xi}|V_v(\eta)|\,d\eta,$$

since both these upper limits exist. Clearly

$$(10)\qquad |V_v(\xi)|\leqq\alpha_v+o(1),$$

and

$$\int_0^{\xi}|V_v(\eta)|\,d\eta\leqq\beta_v\xi+o(\xi).$$

Using this in (9), we get

$$\xi^2|V_u(\xi)|\leqq\frac{2}{\varphi(k)}\sum_{\substack{v\leqq k\\(v,k)=1}}\int_0^{\xi}\{\beta_v\zeta+o(\xi)\}\,d\zeta+O(\xi)=\xi^2\frac{1}{\varphi(k)}\sum_{\substack{v\leqq k\\(v,k)=1}}\beta_v+o(\xi^2),$$

and from this

$$|V_u(\xi)|\leqq\frac{1}{\varphi(k)}\sum_{\substack{v\leqq k\\(v,k)=1}}\beta_v+o(1).$$

Hence

$$(11)\qquad \alpha_u\leqq\frac{1}{\varphi(k)}\sum_{\substack{v\leqq k\\(v,k)=1}}\beta_v.$$

In the sequel, let

$$(12)\qquad \alpha\overset{\text{def}}{=}\max_{\substack{1\leqq u\leqq k\\(u,k)=1}}\alpha_u.$$

To the completion of the proof it is enough to show that $\alpha=0$. We suppose that $\alpha>0$ and prove that this leads to a contradiction. For all v in question

$$\beta_v \leqq \alpha_v$$

holds by trivial arguments. So (11) gives that

$$\alpha \leqq \frac{1}{\varphi(k)} \sum_{\substack{v \leqq k \\ (v,k)=1}} \beta_v \leqq \frac{1}{\varphi(k)} \sum_{\substack{v \leqq k \\ (v,k)=1}} \alpha_v \leqq \alpha \tag{13}$$

and this can only hold if for all values of v

$$\beta_v = \alpha_v = \alpha$$

is fulfilled. Now from the assumption $\alpha>0$ we shall derive that

$$\beta_l < \alpha_l = \alpha$$

for the index l occurring in the theorem, and this will give a contradiction in the inequality (13).

For the proof we require two further lemmas.

Lemma 4. *Let the meaning of l be that of the theorem. Then there is a fixed positive number A, such that for every positive ξ_1, $\xi_2 \geqq \xi_1$ we have*

$$\left| \int_{\xi_1}^{\xi_2} V_l(\eta)\, d\eta \right| < A.$$

Proof. If we put $x=e^{\xi}$, $t=e^{\eta}$, we have

$$\int_0^{\xi} V_l(\eta)\, d\eta = \int_1^x \frac{F_l(t)}{t^2}\, dt = \sum_{\substack{n \leqq x \\ n \equiv l \pmod{k}}} \frac{f(n)}{n} - \frac{F_l(x)}{x} = O(1)+O(1) = O(1)$$

using the condition for $f(n)$ required in the theorem, and applying lemma 1 with

$$c_n = \begin{cases} f(n), & \text{if } \ n \equiv l \pmod{k}, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f(t) = \frac{1}{t}.$$

Hence we get for the numbers ξ_1 and ξ_2 that

$$\left| \int_{\xi_1}^{\xi_2} V_l(\eta)\, d\eta \right| = \left| \int_0^{\xi_2} V_l(\eta)\, d\eta - \int_0^{\xi_1} V_l(\eta)\, d\eta \right| \leqq$$

$$\leqq \left| \int_0^{\xi_2} V_l(\eta)\, d\eta \right| + \left| \int_0^{\xi_1} V_l(\eta)\, d\eta \right| = O(1)+O(1) = O(1),$$

and this gives the statement of the lemma.

Lemma 5. *If* $\eta_0>0$, *and* $V_l(\eta_0)=0$, *then*

$$\int_0^{\alpha} |V_l(\eta_0+\tau)|\, d\tau \leqq \frac{2}{k}\alpha^2+O\left(\frac{1}{\eta_0}\right),$$

where the meaning of α *is the same as in* (12).

Proof. We start with a simple remark. If we put $T_u(x)$ for

$$\sum_{\substack{n\leqq x\\ n\equiv u \,(\mathrm{mod}\, k)}} 1$$

then as one can see it without difficulty these functions satisfy the relation

$$T_l(x)\log x+\sum_{n\leqq x}\Lambda(n)T_{ln^{-1}}\left(\frac{x}{n}\right)=\frac{2}{k}x\log x+O(x).$$

Combining this with (3) we have

$$\{T_l(x)+F_l(x)\}\log x+\sum_{n\leqq x}\Lambda(n)Q_{ln^{-1}}\left(\frac{x}{n}\right)=\frac{2}{k}x\log x+O(x), \tag{14}$$

where $Q_{ln^{-1}}(y)$ stands for

$$\sum_{\substack{m\leqq y\\ m\equiv ln^{-1}(\mathrm{mod}\, k)}} \{1-f(n)f(m)\}.$$

If we take into account that the function $T_l(x)+F_l(x)$ steadily increases, and that for $Q_{ln^{-1}}(y)$

$$Q_{ln^{-1}}(y)\geqq 0$$

holds and this function has the same monotonity property as the function mentioned before, we get that for any positive x_0 and $x\geqq x_0$

$$0\leqq\{T_l(l)+F_l(x)\}\log x-\{T_l(x_0)+F_l(x_0)\}\log x_0\leqq\frac{2}{k}\{x\log x-x_0\log x_0\}+O(x).$$

From this we deduce that

$$|F_l(x)\log x-F(x_0)\log x_0|\leqq\frac{1}{k}\{x\log x-x_0\log x_0\}+O(x) \tag{15}$$

by virtue of the trivial relation

$$T_l(x)=\frac{x}{k}+O(1).$$

We put $x=e^{\eta_0+\tau}$, $x_0=e^{\eta_0}$, so that $F_l(x_0)=0$. We have, since $0\leqq\tau\leqq\alpha$

$$|V_l(\eta_0+\tau)|\leqq\frac{1}{k}\left(1-\left(\frac{\eta_0}{\eta_0+\tau}\right)e^{-\tau}\right)+O\left(\frac{1}{\eta_0}\right)=$$

$$=\frac{1}{k}(1-e^{-\tau})+O\left(\frac{1}{\eta_0}\right)\leqq\frac{1}{k}\tau+O\left(\frac{1}{\eta_0}\right)$$

by (15), and so

$$\int_0^\alpha |V_l(\eta_0+\tau)|\,d\tau \leqq \frac{1}{k}\int_0^\alpha \tau\,d\tau + O\left(\frac{1}{\eta_0}\right) = \frac{1}{2k}\alpha^2 + O\left(\frac{1}{\eta_0}\right),$$

which gives the statement of the lemma.

We now write

$$\delta = \frac{2Ak+(4k-1)\alpha^2}{2k\alpha} > \alpha,$$

take any positive number ζ and consider the behaviour of $V_l(\eta)$ in the interval $\zeta \leqq \eta \leqq \zeta+\delta-\alpha$. By the definition of $V_l(\eta)$, this function can change sign in the interval mentioned above only in the case, if there is an η_0 lying in $(\zeta, \zeta+\delta-\alpha)$ for which $V_l(\eta_0)=0$, owing to the fact that $f(n)$ takes the three values $0, 1, -1$ only. Hence in our interval, either $V_l(\eta_0)=0$ for some η_0 or $V_l(\eta)$ does not change sign at all. In the first case, we use (10) and lemma 5, and have

$$\int_\zeta^{\zeta+\delta} |V_l(\eta)|\,d\eta = \int_\zeta^{\eta_0} + \int_{\eta_0}^{\eta_0+\alpha} + \int_{\eta_0+\alpha}^{\zeta+\delta} |V_l(\eta)|\,d\eta \leqq$$

$$\leqq \alpha(\eta_0-\zeta) + \frac{1}{2k}\alpha^2 + \alpha(\zeta+\delta-\eta_0-\alpha) + o(1) =$$

$$= \alpha\left(\delta - \left(1-\frac{1}{2k}\right)\alpha\right) + o(1) = \alpha'\delta + o(1)$$

for large ζ, where $\alpha' = \alpha\left(1-\left(1-\frac{1}{2k}\right)\frac{\alpha}{\delta}\right) < \alpha$. In the second one we have

$$\int_\zeta^{\zeta+\delta-\alpha} |V_l(\eta)|\,d\eta = \left|\int_\zeta^{\zeta+\delta-\alpha} V_l(\eta)\,d\eta\right| < A$$

by lemma 4. Hence

$$\int_\zeta^{\zeta+\delta} |V_l(\eta)|\,d\eta = \int_\zeta^{\zeta+\delta-\alpha} + \int_{\zeta+\delta-\alpha}^{\zeta+\delta} |V_l(\eta)|\,d\eta < A + \alpha^2 + o(1) = \alpha''\delta + o(1),$$

where

$$\alpha'' = \frac{A+\alpha^2}{\delta} = \alpha\left(\frac{2kA+2k\alpha^2}{2kA+(4k-1)\alpha^2}\right) = \alpha\left(1-\left(1-\frac{1}{2k}\right)\frac{\alpha}{\delta}\right) = \alpha'.$$

Thus we have always

$$\int_\zeta^{\zeta+\delta} |V_l(\eta)|\,d\eta \leqq \alpha'\delta + o(1)$$

where $o(1)\to\infty$ as $\zeta\to\infty$. If $M=\left[\frac{\xi}{\delta}\right]$,

$$\int_0^\xi |V_l(\eta)|\,d\eta = \sum_{m=0}^{M-1}\int_{m\delta}^{(m+1)\delta}|V_l(\eta)|\,d\eta+\int_{M\delta}^{\xi}|V_l(\eta)|\,d\eta \leqq$$

$$\leqq \alpha' M\delta + o(M) + O(1) = \alpha'\xi + o(\xi).$$

Hence

$$\beta_l = \overline{\lim_{\xi\to\infty}}\,\frac{1}{\xi}\int_0^\xi |V_l(\eta)|\,d\eta \leqq \alpha' < \alpha,$$

and this inequality gives the contradiction as desired. So our theorem is proved.

Before finishing the paper, we mention that the condition $(l, k)=1$ is essential in the theorem, as the following example shows. Let $f(n)$ be the function defined by

$$f(n)=\begin{cases}\chi(n), & \text{if } n\equiv 1 \pmod 2,\\ \chi(m), & \text{if } n=2^\alpha m, \quad m\equiv 1 \pmod 2,\end{cases}$$

where $\chi(n)$ stands for the non-principal character mod 4. It is easy to see that the series

$$\sum_{\substack{n=1\\ n\equiv 2 \pmod 4}}^{\infty}\frac{f(n)}{n},$$

being an alternating series of Leibniz type, converges and so for its partial sums

$$\sum_{\substack{n\leqq x\\ n\equiv 2 \pmod 4}}\frac{f(n)}{n}=O(1)$$

holds. On the other hand it follows from the construction that

$$\sum_{\substack{n\leqq x\\ n\equiv 1 \pmod 4}} f(n) = \sum_{\substack{n\leqq x\\ n\equiv 1 \pmod 4}}\chi(n) = \sum_{\substack{n\leqq x\\ n\equiv 1 \pmod 4}} 1 = \frac{x}{4}+O(1)\neq o(x).$$

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(Received August 15, 1966)

Minimal spectral sets of compact operators

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1. Introduction

In his paper [9], VON NEUMANN introduced the notion of a spectral set for an operator T on a Hilbert space. He proved that each spectral set is a superset of a *minimal* spectral set, but aside from the trivial case in which the spectrum of T is spectral, there are no other known minimal spectral sets. In the present paper we obtain a necessary and sufficient condition for the minimality of certain spectral sets of finite-dimensional (or compact) operators. A corollary is that the disk $|z| \leqq |T|$ is a minimal spectral set if T is compact and completely non-normal. An example shows that the result is not true without the adjective "compact". A few other results, based on an interpretation of VON NEUMANN's work as an extension of the Schwarz lemma, are also included.

2. Preliminaries

We begin with a summary of the relevant results of [9].

If T is a (bounded linear) operator on a complex Hilbert space, a closed set X is a *spectral set* of T if X contains the spectrum $\sigma(T)$ of T and if

$$\|u(T)\| \leqq \|u\|_X = \sup \{|u(z)| : z \in X\}$$

for each rational function $u(z)$ with poles off X. Any closed superset of a spectral set is again spectral, and, less trivially, any spectral set contains a *minimal* spectral set, i. e., a spectral set no proper closed subset of which is spectral. For example, if T is normal, then $\sigma(T)$ is a minimal spectral set of T. (A result of HALMOS implies the same conclusion if T is merely subnormal [4].) There is exactly one spectral set of T, or, equivalently, $\sigma(T)$ is spectral for T, if and only if the intersection of any two spectral sets of T is spectral. In general, the spectrum of T is not big enough to be spectral for T. Thus if X is spectral for T and X is "thin" in the sense that rational functions with poles off X are uniformly dense in the continuous functions on X, then T must be normal.

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[1]) The results of this paper constitute a portion of the author's thesis written under the supervision of Professor ARLEN BROWN at The University of Michigan.

The disk $|z| \leqq \|T\|$ is always a spectral set of T. This result is equivalent to the assertion that a half-plane is a spectral set of any T whose *numerical range* $W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$ is contained in that half-plane.

Finally, let us remark that as a consequence of the identical relation $\|(T-\alpha I)x\|^2 - \|(I-\bar{\alpha}T)x\|^2 = (1-|\alpha|^2)[\|T\alpha\|^2 - \|x\|^2]$, we have for $\varphi_\alpha(z) = (z-\alpha)(1-\bar{\alpha}z)^{-1}$ ($|\alpha| < 1$):

$$\|\varphi_\alpha(T)\| \leqq 1 \quad \text{if} \quad \|T\| \leqq 1, \quad \text{and} \quad \|\varphi_\alpha(T)\| = 1 \quad \text{if} \quad \|T\| = 1.$$

3. Functions of a contraction

The classical Schwarz lemma is concerned with functions analytic in the open unit disk D. If we identify each $z \in D$ with the operator $z \cdot I$ we obtain a natural embedding of D into the set $\mathscr{D}$ of proper contractions on the Hilbert space H. It is reasonable to expect that the conclusion of the lemma is valid for all $T \in \mathscr{D}$. This is essentially what VON NEUMANN proved.

Theorem 1. (Schwarz lemma.) *Let H be a Hilbert space, $\mathscr{D}$ the set of proper contractions on H. If f is analytic in D, $f(0) = 0$ and $\|f\|_D \leqq 1$, then $\|f(T)\| \leqq \|T\|$ for each $T \in \mathscr{D}$. Moreover, equality can hold for some $T_0 \in \mathscr{D}$ only if $f(T) \equiv \gamma \cdot T$ for some constant γ of modulus* 1.

Proof. Note that if $T \in \mathscr{D}$, then f is analytic in a neighborhood of $\sigma(T)$ so that there is no difficulty in defining $f(T)$. Now if $T \in \mathscr{D}$, then by VON NEUMANN's theorem we have

$$\|f(T)\| \leqq \sup \{|f(z)| : |z| \leqq \|T\|\}$$

and since $f(0) = 0$, the usual version of the Schwarz lemma implies that the right member of this inequality is $\leqq \|T\|$. Moreover, we can have $\|f(T)\| = \|T\|$ for some $T \in \mathscr{D}$ only if there is a z_0 with $|z_0| = \|T\| < 1$ and $|f(z_0)| = \|T\|$. This occurs only if $f(z) \equiv \gamma \cdot z$ for some constant γ of modulus 1.

Corollary 1. *Let T be an operator, X a closed set containing $\sigma(T)$, and let μ be an interior point of X. If $\|u(T)\| \leqq \|u\|_X$ for each rational function which vanishes at μ, then X is spectral for T.*

Proof. Let v be a rational function with $\|v\|_X = 1$. We claim that $\|v(T)\| \leqq 1$. If $v(\mu) = 0$, this is true by hypothesis, otherwise $v(\mu) = \alpha$ has modulus less than 1 by the maximum principle, so that $\varphi_\alpha(z)$ is a conformal map of the disk D onto itself. Then $u(z) = \varphi_\alpha(v(z))$ is rational, vanishes at μ, and has bound 1 on X. Hence $u(T)$ is a contraction. But then, so is $v(T) = \varphi_\alpha^{-1}(u(T))$.

Corollary 2. *Consider the "two-dimensional shift" A_2 whose matrix relative to an orthonormal basis is*

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $\|\alpha + \beta A_2\| = \frac{1}{2}\{|\beta| + \sqrt{4|\alpha|^2 + |\beta|^2}\}$.

Proof. We have $\|\varphi_\alpha(A_2)\|=1$ for all $|\alpha|<1$. Since

$$\varphi_\alpha(A_2) = (A_2-\alpha)(1-\bar\alpha A_2)^{-1} = (A_2-\alpha)(1+\bar\alpha A_2) = -\alpha+(1-|\alpha|^2)A_2,$$

this gives

$$\|A_2-\alpha(1-|\alpha|^2)^{-1}\| = (1-|\alpha|^2)^{-1}.$$

Put $\lambda=\alpha(1-|\alpha|^2)^{-1}$ and compute

$$(1-|\alpha|^2)^{-1} = \frac{1}{2}\{1+\sqrt{4|\lambda|^2+1}\}$$

to arrive at

$$\|A_2-\lambda\| = \frac{1}{2}\{1+\sqrt{4|\lambda|^2+1}\}.$$

This yields the proposition for $\beta\neq0$. If $\beta=0$, the result is trivial.

According to the Sz.-Nagy—Foiaş theory of contractions, a contraction T is *completely non-unitary* if T has no reducing subspace restricted to which T is unitary. A compact contraction is completely non-unitary if and only of its spectrum lies in D. (This follows from the fact that if $|\lambda|=\|A\|$, then $Ax=\lambda x$ is equivalent to $\langle Ax, x\rangle=\langle\lambda x, x\rangle$ and therefore to $A^*x=\bar\lambda x$.) The (unique, strong, minimal) unitary dilation $U=\int\lambda\, dE_\lambda$ of T has spectrum equal to the unit circle ∂D, and for each $x\neq0$ in the domain of T, Lebesgue measure on the circle is equivalent to the measure $\langle E(\cdot)x, x\rangle$ [6].

Theorem 2. *Let T be a compact completely non-unitary contraction. If f is analytic in $|z|<1$ and bounded by 1 there, then $\|f(T)\|=1$ only if f is an inner function.*

Proof. First of all, since T is completely non-unitary, the operator $f(T)$ is well-defined for each bounded analytic function and for x, y in H,

$$\langle f(T)x, y\rangle=\langle f(U)x, y\rangle$$

where U is the unitary dilation of T [7]. Secondly, it is elementary that any compact operator attains its bound.

Now suppose that $f(T)$ has norm 1, where f is analytic and of bound 1 in $|z|<1$. Replacing $f(z)$, if necessary, by $\varphi_\alpha(f(z))$, where $\alpha=f(0)$, we may suppose that $f(0)=0$. Then $f(T)$ is compact and so we can choose a unit vector x in H so that $f(T)x$ has norm 1. Then

$$1 = \|f(T)x\|^2 \leqq \|f(U)x\|^2 = \int |f(\lambda)|^2 d\langle E_\lambda x, x\rangle.$$

Since $|f(\lambda)|\leqq1$ on ∂D, this implies that $|f(\lambda)|$ has the value 1 on ∂D almost everywhere with respect to $\langle E(\cdot)x, x\rangle$. By the result referred to above, this in turn implies that f has modulus 1 on ∂D almost everywhere with respect to Lebesgue measure. This however is exactly the requirement the f be an inner function.

Remark. Using a result of Havinson [3, Theorem 2. 5] a stronger version of Theorem 2 can be obtained. Under the same hypothesis on f and T one can show that f is a Blaschke product with only finitely many zeros.

4. Completely non-normal operators

We call an operator T *completely non-normal* if T has no reducing subspace restricted to which T is normal. Any operator T is the direct sum of a normal operator N and a completely non-normal operator T_0 (*cf.* e. g. [10]). It follows that a closed set X is spectral for T if and only if X is spectral for T_0 and X contains $\sigma(N)$, so the theory of spectral sets reduces to the study of completely non-normal operators.

If T is completely non-normal and $\psi(z)=(az+b)(cz+d)^{-1}$ is analytic on $\sigma(T)$, then $\psi(T)$ is also completely non-normal. Hence the class of completely non-normals is closed with respect to translation, inversion, adjunction, and scalar multiplication. It is not however closed with respect to products or sums.

If $\dim H \geqq 3$, the notions of completely non-normal and completely non-unitary are distinct. If H is two-dimensional there are essentially only two completely non-normal operators as the following theorem shows.

Theorem 3. *Let T be a completely non-normal operator of norm 1 on a two-dimensional space. Either $T=\alpha+\beta A_2$ with $|\beta|=1-|\alpha|^2$, or there exist unit vectors x_1, x_2 and scalars λ_1, λ_2 such that $Tx_i=\lambda_i x_i$. In the second case the following relations are valid:*

(i) $0<|\langle x_1, x_2\rangle|<1$,

(ii) $\sqrt{1-|\langle x_1, x_2\rangle|^2} = |(\lambda_2-\lambda_1)(1-\bar{\lambda}_1\lambda_2)^{-1}|$.

Observe that if T is the two-dimensional completely non-normal operator with two eigenvalues and norm 1, then for any function f analytic in D, the above theorem implies that $f(T)$ has norm 1 if and only if

$$|(f(\lambda_2)-f(\lambda_1))(1-\overline{f(\lambda_1)}f(\lambda_2))^{-1}| = |(\lambda_2-\lambda_1)(1-\bar{\lambda}_1\lambda_2)^{-1}|.$$

Consequently, if f also has bound 1 on D, then the Schwarz lemma implies that f is a conformal map of D onto itself. It is easy to see that the same conclusion is also valid for the other two-dimensional completely non-normal operator. We record these facts as a

Corollary. *Let T be a completely non-normal operator of norm 1 on a two-dimensional space. Then the only non-constant functions analytic in D which satisfy*

$$\|f(T)\| = \|f\|_D = 1$$

are the conformal maps of D onto itself.

We conclude this section with a decomposition theorem which, although it does not appear in the literature, is probably known to specialists. Consider a compact set X containing the spectrum of an operator T, and suppose that X is the union of two non-empty disjoint sets X_1 and X_2. Put

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{d\lambda}{\lambda - z}$$

where γ is a rectifiable path surrounding X_1 and containing X_2 in its exterior. Then f is analytic on X, identically 1 on X_1 and identically zero on X_2. Put $E=f(T)$.

Theorem 4. *If X is spectral for T, then E is a self-adjoint projection, the range of E reduces T, and X_1 is spectral for the restriction of T to $E(H)$.*

Proof. The operator E is idempotent and commutes with T [2] so that it suffices to show that E is self-adjoint and that X_1 is spectral for $T|E(H)$.

The first assertion is almost trivial. Thus by approximating the integral defining f, one sees that f is the uniform limit on X of functions which are rational and bounded on X, hence

$$\|E\| = \|f(T)\| \leqq \|f\|_X = 1.$$

It remains only to observe that an idempotent of norm 1 is necessarily self-adjoint; this is straightforward and we omit the details.

It now follows that $H_1 = E(H)$ reduces T. To prove that X_1 is spectral for $T_1 = T|H_1$, let u be a rational function with poles off X_1. Set

$$v(z) = \frac{1}{2\pi i} \int_\gamma \frac{u(\lambda)}{\lambda - z} d\lambda$$

where γ is the path used to define E, and note that $v(z)$ is a uniform limit of functions which are rational and bounded on X. Also

$$v(z) = \begin{cases} u(z) & z \in X_1, \\ 0 & z \in X_2, \end{cases}$$

and so

$$\|v(T)\| \leqq \|v\|_X = \|u\|_{X_1}.$$

Now use the fact that $u(T_1) = v(T)|H_1$ (see [2, p. 574] for example) to conclude that $\|u(T_1)\| \leqq \|u\|_{X_1}$. Thus X_1 is spectral for T_1.

Corollary 1. *If the spectrum of T consists of a single point, in particular, if T is quasi-nilpotent, then each minimal spectral set of T is connected. The same conclusion is valid for any completely non-normal operator if* dim $H \leqq 3$, *but is otherwise false.*

Proof. The first assertion is clear, and the second follows from the fact that if E is a self-adjoint projection with one-dimensional range, and if E commutes with T, then T is normal on the range of E.

To complete the proof observe that the set

$$X = \{z: |z| \leqq 1\} \cup \{z: |z-3| \leqq 1\}$$

is not connected, and is spectral for the completely non-normal operator $T = A_2 \oplus \oplus (A_2 + 3)$. We will prove later that X is in fact a minimal spectral set for T.

Corollary 2. *If the operator T is irreducible, then each minimal spectral set of T is connected.*

There is another consequence of Theorem 4 which J. Stampfli pointed out to me. To state this, recall that an operator T on H is *subnormal* if T is the restriction to H of a normal operator N acting on a space $K \supset H$.

Corollary 3. *If T is subnormal and if $\sigma(T)=\sigma_1 \cup \sigma_2$ with σ_1 and σ_2 non-empty and disjoint, then T can be decomposed into a direct sum $T=T_1 \oplus T_2$ of subnormal operators with $\sigma(T_i)=\sigma_i$ $(i=1, 2)$.*

Proof. The spectrum of a subnormal operator is spectral [4].

In the next section we will be interested in determining which spectral sets are minimal. The general problem of course reduces to the case of a completely non-normal operator. By further restricting attention to irreducible operators, the preceding corollary allows us to consider only connected spectral sets. It is this latter problem we will study, not in complete generality but with the additional assumption that the sets in question have nice boundaries.

5. Minimal spectral sets of completely non-normal operators

In the remainder of this paper G will denote a bounded region (open, connected set) in the plane; G has finite connectivity n, and the boundary ∂G of G is the union of n disjoint, closed, rectifiable Jordan curves. We assume that these curves are oriented in the usual positive sense with respect to G.

$B(G)$ will denote the algebra of functions analytic and bounded in G; $B_1(G)$ consists of those $f \in B(G)$ whose norm $\|f\|=\|f\|_G=\sup\{|f(z)|: z \in G\}$ does not exceed 1.

Our main result depends on a theorem of Havinson concerning extremal problems in the region G. To state this theorem we introduce the following definition $\left(\text{here } \partial G=\bigcup_{i=1}^{n} \gamma_i, \text{ and } \gamma_1 \text{ is the outer boundary of } G\right)$.

Definition [3]. Let f be analytic in G. Then $f \in E_p(G)$ $(p>0)$ if there is a sequence of closed rectifiable Jordan curves $\Gamma^k=\bigcup_{i=1}^{n} \gamma_i^k$ such that

(1) γ_1^k lies inside γ_1 and γ_i^k $(i=2, \ldots, n)$ contains γ_i inside it for every k,
(2) $\gamma_i^k \to \gamma_i$ as $k \to \infty$ $(i=1, 2, \ldots, n)$,
(3) the lengths of the Γ^k are uniformly bounded,
(4) $\sup_k \int_{\Gamma^k} |f(z)|^p |dz| < \infty$.

The space $E_p(G)$ is obviously a generalization of the classical space H^p to multiply connected regions, and most of the classical results have analogues in $E_p(G)$. For example, any E_p function has boundary values (for approach in an angle) almost everywhere with respect to arc length, and the function itself can be written as the Cauchy integral of its boundary values. The F. and M. Riesz Theorem is also valid in E_p: If $f \in E_p$ has 0 boundary values on a set of positive arc length then f must vanish identically.

Theorem 5. (Havinson) *Let $\omega(\lambda)$ be summable on $\Gamma=\partial G$. Then*

(1) $$\sup_{f \in B_1(G)} \left| \int_\Gamma f(\lambda)\,\omega(\lambda)\,d\lambda \right| = \inf_{\Phi \in E_1(G)} \int_\Gamma |\omega(\lambda)-\Phi(\lambda)|\,ds.$$

(2) *The infimum on the right is always attained by an extremal function $\Phi \in E_1(G)$.*

(3) *The supremum on the left is attained by a function* $f \in B_1(G)$ *and, moreover,* f *is unique to within a factor* $e^{i\alpha}$ *provided* $\omega(\lambda)$ *is not the boundary value of any* E_1 *function.*

(4) *A necessary and sufficient condition for* $f \in B_1$ *and* $\Phi \in E_1$ *to be extremal functions is that almost everywhere on* Γ,

$$f(\lambda)[\omega(\lambda)-\Phi(\lambda)]\,d\lambda = e^{i\alpha}|\omega(\lambda)-\Phi(\lambda)|\,ds$$

where α *is a real constant.*

To conclude these general considerations, consider the situation in which our region G contains the spectrum of an operator T. Then for any $g \in B(G)$ we can form the operator $g(T)$ by the Riesz—Dunford functional calculus. If g is actually analytic on $\bar{G}$, then $g(T)$ is given by the integral

$$\langle g(T)x, y\rangle = -\frac{1}{2\pi i}\int_\Gamma g(\lambda)\langle R_\lambda(T)x, y\rangle\, d\lambda \qquad (x, y \in H)$$

The same formula is valid for any $g \in B(G)$. This is a consequence of the following facts:

(1) If $g \in B(G)$, then $g = \lim g_n$ where the g_n are uniformly bounded and analytic on $\bar{G}$, and the limit is subuniform in G.

(2) If $\{g_n\}$ is a sequence of uniformly bounded analytic functions which converges to g subuniformly in G, then for any function $\omega(\lambda)$ summable on Γ

$$\int_\Gamma g_n(\lambda)\omega(\lambda)\,d\lambda \to \int_\Gamma g(\lambda)\omega(\lambda)\,d\lambda \quad \text{(see [3])}.$$

We begin now the task of applying the preceding function theory to the study of the minimal spectral sets of a fixed completely non-normal operator T on a finite-dimensional Hilbert space (the infinite case will be discussed later). The connection is made possible by the fact that $\bar{G}$ can be spectral for T only if $\sigma(T)$ is wholly contained in G:

Theorem 6. (Sz.-Nagy—Foiaş [8]) *Let* S *be the closure of a simply connected region bounded by a Jordan curve, and assume that* S *is spectral for an operator* A. *If* $\lambda \in \partial S$, *then* $Ax = \lambda x$ *if and only if* $A^*x = \bar{\lambda} x$.

Corollary. *If* $\bar{G}$ *is spectral for* T, *then* $\sigma(T) \subset G$.

Proof. Write $G = \gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_n$, and let $G_1, G_2, \ldots, G_n$ be the components of the complement of $\bar{G}$. We may assume that $\infty \in G_1$. Then the complement G_1' is the closure of a simply connected region bounded by the Jordan curve γ_1, and moreover, G_1' is spectral for T because $G_1' \supset \bar{G}$. Hence by Theorem 6, $\sigma(T)$ cannot meet γ_1.

If $2 \leqq k \leqq n$, choose $z_k \in G_k$ and let $\omega_k(z) = (z - z_k)^{-1}$. Then $\varphi_k(T)$ is completely non-normal and $\varphi_k(G_k)$ is spectral for $\varphi_k(T)$. Applying the theorem we conclude that $\sigma(\varphi_k(T)) = \varphi_k(\sigma(T))$ does not meet $\varphi_k(\gamma_k)$, that is $\sigma(T) \cap \gamma_k = \varphi$.

Remark. The proof of Theorem 6 given in [8] uses unitary dilations, but there is a more elementary proof. Thus if S is the unit disk the result is a consequence of the fact that if $\gamma \in \partial S$ then both $Ax = \lambda x$ and $A^*x = \bar{\lambda} x$ are equivalent to $\langle Ax, x \rangle = \lambda \langle x, x \rangle$. In the general case the set S is the image of $\bar{D}$ under a function f which is $1-1$ and continuous on $\bar{D}$, analytic in D. Then f^{-1} is the limit of a sequence of polynomials which converge uniformly on S and because S is spectral for A, $f^{-1}(A)$ is a contraction. The assertion about A and λ then reduces to the same assertion about $f^{-1}(A)$ and $f^{-1}(\lambda)$.

Remark 2. The preceding corollary is not valid under the assumption that T is completely non-unitary. For example, $T = A_2 \oplus \lambda_0 I$ is completely non unitary if $\lambda_0 = \| A_2 + 1 \| - 1$. Also λ_0 belongs to $\sigma(T)$ and λ_0 lies on the boundary of the spectral set

$$S = \{z: |z+1| \leqq \|A_2 + 1\|\}.$$

(Incidentally, the same example is the basis for our earlier remark that the notions of completely non-normal and completely non-unitary are distinct as soon as $\dim H \geqq 3$.)

Theorem 7. *Let T be completely non-normal and let G be rectifiably bounded as above. Then $\bar{G}$ is spectral for T if and only if*

(1) $\sigma(T) \subset G$,

(2) $\max \{\|f(T)\| : f \in B_1(G)\} \leqq 1$.

If $\mu \in G$ is fixed, then (2) *is equivalent to*

(2′) $\max \{\|f(T)\| : f \in B_1(G), f(\mu) = 0\} \leqq 1$.

Proof. We have just seen that (1) is necessary. The necessity of (2) follows from the facts that (a) the rational functions with poles off $\bar{G}$ are subuniformly dense in $B(G)$ and (b), $B_1(G)$ is a compact subset of $B(G)$ for this topology so that the continuous functional $f \to \|f(T)\|$ attains its supremum over $B_1(G)$. The sufficiency of (1) and (2) are obvious, and the equivalence of (2) and (2′) is a standard application of Corollary 2 of Theorem 1.

The next result gives a necessary condition for certain spectral sets to be minimal. (Here the operator A is quite arbitrary.)

Theorem 8. *Let X be a minimal spectral set for the operator A and assume that $\sigma(A) \subset \operatorname{int} X$. If $\mu \in \operatorname{int} X$, then there is a function f analytic in* int X *with $f(\mu) = 0$ and*

$$\|f(A)\| \geqq \sup \{|f(z)| : z \in \operatorname{int} X\}.$$

Proof. We may assume that int X is connected. Choose a point $a \in \operatorname{int} X$ with $a \notin \sigma(A) \cup \{\mu\}$ and put

$$X_n = \{z: |z-a| \geqq 2^{-n} d\} \cap \operatorname{int} X \qquad (n = 1, 2, \ldots)$$

where d is the distance from a to $\sigma(A) \cup \{\mu\}$. Then $\sigma(A) \subset \operatorname{int} X_n$ and $X_n \subsetneqq X$. Since X is minimal for A, it follows that $\bar{X}_n$ is not spectral for A, and since $\mu \in \operatorname{int} X_n$, this in turn implies that there is a rational function $u_n(z)$ with poles off $\bar{X}_n$ such that

$$u_n(\mu) = 0, \ \|u_n(A)\| > \|u_n\|_{X_n} = 1.$$

Now the functions $u_n(z)$ form a normal family in int X minus the point a, and hence a subsequence u_{n_k} converge to a limit function f uniformly on compact subsets.

It is easy to see that a is a removable singularity of f and hence f extends to be analytic in int X. Then f is bounded by 1 vanishes at $z=\mu$ and

$$\|f(A)\| = \lim_k \|u_{n_k}(A)\| \geqq 1$$

because $u_{n_k} \to f$ uniformly on $\sigma(A)$.

We are finally able to prove the principal theorem.

Theorem 9. *Let G be rectifiably bounded and let T be completely non-normal as above. Fix $\mu \in G$. Then $\bar{G}$ is spectral for T if and only if*

(1) $\sigma(T) \subset G$,

(2) $\max \{\|f(T)\| : f \in B_1(G), f(\mu)=0\} \leqq 1$.

Moreover, $\bar{G}$ is minimal for T if and only if this maximum is 1.

Proof. In view of Theorems 7 and 8 remains only to prove the sufficiency of the minimality condition. For this, let $f \in B_1(G)$ be an extremal function with $f(\mu)=0$ and $\|f(T)\| = \|f\|_G = 1$. Since the underlying space is finite-dimensional we can choose unit vectors x and y so that $\langle f(T)x, y\rangle = 1$. Now any proper closed subset of $\bar{G}$ which contains $\sigma(T)$ fails to contain some point of G and hence is contained in a set of the form

$$S = \{z \in \bar{G}: |z-a| \geqq \varepsilon\} \qquad (a \in G_1,\ \varepsilon > 0).$$

Hence to prove that $\bar{G}$ is minimal, it suffices to show that S is not spectral for T. We make use of the fact that $\partial S = \partial G \cup \gamma_{n+1}$ where γ_{n+1} is a circle contained in G.

If S is spectral for T, then it follows from Theorem 7 that f is an extremal function for the problem

$$\sup \left\{ \left| \int_{\partial S} g(\lambda)\,\omega(\lambda)\,d\lambda \right|,\quad g \in B_1(\text{int } S),\quad g(\mu) = 0 \right\}$$

where $\omega(z) = \langle R_z(T)x, y\rangle$. By Havinson's theorem there is a function $\varphi \in E_1(\text{int } S)$ such that almost everywhere on ∂S,

$$f(\lambda)\big(\omega(\lambda) - \Phi(\lambda)\big)d\lambda = e^{i\alpha}|\omega(\lambda) - \Phi(\lambda)|\,ds$$

where α is a real constant. This in turn implies that $|f(\lambda)| = 1$ on the subset Z of ∂S consisting of those λ for which $\omega(\lambda) \neq \Phi(\lambda)$. Assuming for the moment that we can show that Z intersects γ_{n+1} it will then follow from the maximum principle that f is constant. However since f vanishes at μ and $\|f(T)\| = 1$, this is a contradiction. In short, we need the following

Lemma. *The function $\omega(z) = \langle R_z(T)x, y\rangle$ does not coincide on γ_{n+1} with a function of class $E_1(\text{int } S)$.*

Proof. The spectrum of T consists of finitely many interior points of S and so we can choose a finite number of open disks D_i such that

$$\bar{D}_i \cap \bar{D}_j = \emptyset \quad \text{if} \quad i \neq j, \qquad \bar{D}_i \subset \text{int } S, \qquad \sigma(T) \subset \bigcup D_i.$$

Let S_1 be the (rectifiably bounded) set obtained from S by deleting these disks and let δ be the boundary of $\bigcup D_i$. Then $\partial S_1 = \partial S \cup \gamma_{n+1} \cup \delta$.

Suppose now that $\omega = \Phi$ on γ_{n+1}, where Φ is a function of class $E_1(\text{int } S)$. Since ω is bounded on S_1, $\omega \in E_1(\text{int } S_1)$. Also $\Phi \in E_1(\text{int } S_1)$ and so by the Riesz theorem we can conclude that $\omega = \Phi$ on ∂S_1. In particular, $\omega = \Phi$ on δ. This however is impossible, for on one hand

$$\int_\delta f(\lambda)\omega(\lambda)\,d\lambda = 0$$

(because the integrand is analytic in $\bigcup D_i$), while on the other hand

$$\int_\delta f(\lambda)\omega(\lambda)\,d\lambda = 2\pi i\langle f(T)x, y\rangle \neq 0$$

because δ is a path in G surrounding $\sigma(T)$. The assumption $\omega = \Phi$ on γ_{n+1} therefore leads to a contradiction and the lemma is proved.

Corollary 1. *The unit disk is a minimal spectral set for any completely non-normal operator of norm* 1.

Proof. If $\mu \in \sigma(T)$, then $|\mu| < 1$ and so the function $\varphi_\mu(z) = (z-\mu)(1-\bar{\mu}z)^{-1}$ belongs to $B_1(D)$, vanishes at $z = \mu$ and $\|\varphi_\mu(T)\| = 1$. Since $\bar{D}$ is spectral for T, it follows from the theorem just proved that $\bar{D}$ is in fact minimal.

Corollary 2. (Converse of the Schwarz Lemma.) *Let X be a closed subset of the closed unit disk $\bar{D}$ which contains* 0. *If $|u'(0)| \leqq \|u\|_X$ for each rational function $u(z)$ which vanishes at* 0, *then $X = \bar{D}$. Similarly, if for some $\lambda \in X$ with $0 < |\lambda| < 1$ the conditions $u(z)$ rational, $u(0) = 0$, $\|u\|_X = 1$ imply $|u(\lambda)| \leqq |\lambda|$, then $X = \bar{D}$.*

Proof. The first assertion follows from the fact that $\bar{D}$ is a minimal spectral set for A_2. The second assertion follows similarly by considering the two-dimensional completely non-normal of norm 1 with eigenvalues 0 and λ.

Using the fact that linear fractional transformations preserve both complete non-normality and minimality of a spectral set we get the following improvement of von Neumann's theorem:

Corollary 3. *Let T be completely non-normal. Then*

$$S_1 = \{z\colon |z-\lambda| \leqq \|T-\lambda\|\} \quad \textit{and} \quad S_2 = \{z\colon |z-\lambda| \geqq \|R_\lambda(T)\|^{-1}\}$$

are minimal spectral sets of T. The set

$$S_3 = \{z\colon \operatorname{Re} z \geqq 0\}$$

is a minimal spectral set of T if either $\|(T-1)(T+1)^{-1}\| = 1$ or if the numerical range of T lies in the right half-plane and meets the imaginary axis.

Remark. In the hypothesis of Theorem 9 it is actually superfluous to require that T be completely non-normal. Indeed, the assumption $\max\{\|f(T)\|\colon f \in B_1(G), f(\mu) = 0\} = 1$ implies that T has a nontrivial completely non-normal part. For if $T = N$ is normal with $\sigma(N) \subset \text{int } S$, and if $f \in B_1(G)$ is chosen as an extremal function for the problem

$$\max\{\|g(N)\|\colon g \in B_1(G),\ g(\mu) = 0\}$$

then we have

$$\|f(N)\| = \sup\{|f(z)|: z\in\sigma(N)\} = \|f\|_{\sigma(N)}.$$

Since f is continuous on $\sigma(N)$ it attains its maximum there and the maximum principle then shows that $\|f\|_{\sigma(N)} < 1$.

6. Remarks on the infinite case

If T is a completely non-normal operator on an infinite-dimensional space, then our previous argument shows that both the point spectrum and residual spectrum of T are subsets of the *interior* of any Jordan spectral set of T. In general, this is best possible as the unilateral shift shows (point spectrum void, residual spectrum $= \{|z|<1\}$, continuous spectrum $= \{|z| = 1\}$.) It is perhaps surprising that even compact operators can exhibit this behavior.

Example 1. The spectrum of a compact completely non-normal operator need not be contained in the interior of each Jordan spectral seet.

Consider the Volterra operator A defined on $L^2(0, 1)$ by

$$(Af)(t) = \int_0^t f(s)\,ds.$$

It is well known that A is compact with $\sigma(A) = \{0\}$ and $\operatorname{Re} A \geqq 0$. It follows from the equality of norm and spectral radius for normal operators that A is completely non-normal. Now let $T = (1-A)(1+A)^{-1}-1$ and observe that T is compact, completely non-normal, and $T+1$ is a contraction because

$$\|(T+1)x\|^2 - \|x\|^2 = \|(1-A)(1+A)^{-1}x\|^2 - \|x\|^2 =$$

$$= \|(1-A)y\|^2 - \|(1+A)y\|^2 = -4\operatorname{Re}\langle Ay, y\rangle \leqq 0 \qquad (y = (A+1)^{-1}x).$$

It follows that the unit disk $\bar{D}$ is spectral for $T+1$, and hence $S = \bar{D}-1 = \{z: |z+1| \leqq 1\}$ is a Jordan spectral set for T. Finally, $\sigma(T) = \{0\}$ meets the boundary of S.

Example 1 shows that the techniques of § 5 are not suitable for establishing the minimality of rectifiably bounded spectral sets of arbitrary completely non-normal operators. It is natural to expect however that the results are extendable by other means. Even this is not possible:

Example 2. The unit disk $\bar{D}$ is not a minimal spectral set for every completely non-normal operator of norm 1.

Recall that if T is subnormal, then $\sigma(T)$ is a spectral set of T. It follows then that it is sufficient to exhibit a completely non-normal subnormal operator of norm 1 whose spectrum is a proper subset of $\bar{D}$. Our construction is motivated by the theory of analytic Toeplitz operators developed in [1].

First of all, it is easy to see that if T is subnormal on H with minimal normal extension N on $K \supset H$, then T is completely non-normal if and only if no non-trivial subspace of H reduces N.

Now to construct the example, let V be the unilateral shift on H^2 and U the bilateral shift on L^2. Let φ be a conformal map of $\bar{D}$ onto the half-disk $S=\{\operatorname{Re} z \geqq 0\} \cap \bar{D}$. We claim that $T=\varphi(V)$ is subnormal, completely non-normal, and S is spectral for T.

The subnormality of T is clear: $T=\varphi(U)|H^2$ and $\varphi(U)$ is normal. To prove that T is completely non-normal it suffices, by the above remark, to show that any reducing subspace H_0 for $\varphi(U)$ which is contained in H^2 is trivial. But if H_0 reduces $\varphi(U)$ then $\varphi(U)$ commutes with the projection P of L^2 onto H_0 and this implies that U commutes with P, that is, H_0 is a reducing subspace of U contained in H^2. However, it is well-known that the only reducing subspace of U contained in H^2 is the subspace $\{0\}$. Consequently, T is completely non-normal.

Finally, S is spectral for T because $\sigma(T)$ is spectral and $S \supset \sigma(T)$. (if $\varphi(z)-\lambda$ is bounded below on $\bar{D}$, then $\varphi(V)-\lambda$ has an inverse.)

The preceding example indicates that it is not just complete non-normality of a contraction which forces the unit disk to be a minimal spectral set and it is therefore worth investigating our earlier arguments a little more carefully. Some of these are independent of the dimensionality of the underlying space. For example, if G is the rectifiably bounded region previously studied, and T is any operator with $\sigma(T) \subset G$, the condition for spectrality of $\bar{G}$ for T is still the same:

$$\max \{\|f(T)\| : f \in B_1(G), f(\mu)=0\} \leqq 1$$

and if $\bar{G}$ is minimal for T, this maximum is 1. There are two difficulties encountered in proving the sufficiency of the minimality condition. In the first place we need the fact that certain functions of T attain their bound. The second difficulty concerns the proof of the Lemma of § 5 where we explicitly assumed that $\sigma(T)$ was a finite set. The latter requirement is not really essential and it is easy to extract the following extension of Theorem 9:

Theorem 10. *Let G be rectifiably bounded as before and let $0 \in G$. Let T be a compact completely non-normal operator. $\bar{G}$ is spectral for T if and only if*

(1) $\sigma(T) \subset G$,
(2) $\max \{\|f(T)\| : f \in B_1(G), f(0)=0\} \leqq 1$.

Moreover, $\bar{G}$ is minimal for T precisely when the maximum is 1.

Proof. If T is compact, then so is any $f(T)$ with $f(0)=0$ and so $f(T)$ attains its bound. Secondly, if S is a rectifiably bounded subset of $\bar{G}$ whose boundary meets G, then because 0 is the only limit point of $\sigma(T)$ we can still punch finitely many holes in $\operatorname{int} S$ to get the set S_1 needed in the proof of the lemma of § 5. The remainder of the argument is exactly as before.

Corollary 1. *If T is compact and completely non-normal, then $|z| \leqq \|T\|$ is a minimal spectral set of T.*

There is an obvious extension of the corollary: If T has norm 1, is completely non-normal and for some α of modulus less than 1 the operator $\varphi_\alpha(T)$ is compact $(\varphi_\alpha(z)=(z-\alpha)(1-\bar{\alpha}z)^{-1})$, then $\bar{D}$ is minimal for T. To prove this note that $\varphi_\alpha(T)$ has norm 1, is completely non-normal and so by the above corollary, $\bar{D}$ is minimal for T. This implies that $\varphi_\alpha^{-1}(\bar{D})=\bar{D}$ is minimal for $\varphi_\alpha^{-1}(\varphi_\alpha(T))=T$.

Since $T-\alpha=\varphi_\alpha(T)(1-\bar{\alpha}T)$, the operator $\varphi_\alpha(T)$ is compact precisely when $T-\alpha$ is compact, and so the extension of the corollary reads as follows:

Corollary 2. *Let T be completely non-normal of norm 1 and suppose that $T-\alpha$ is compact for some α of modulus less than 1. Then $\bar{D}$ is minimal for T.*

The signifiance of the condition $|\alpha|\leqq 1$ is clear: $T-\alpha$ can be compact only if $0\in\sigma(T-\alpha)$, i. e., only if $\alpha\in\sigma(T)$. What is not clear however is that the condition fails in case $|\alpha|=1$. Equivalently, if T has norm 1, is completely non-normal, and $T-1$ is compact, must $\bar{D}$ be minimal for T?

7. An application concerning numerical ranges

Recently several authors have been interested in the relation between the spectral sets of an operator T and its numerical range $W(T)$. VON NEUMANN's theorem asserts that a closed half-plane H is spectral for T if and only if $H\supset W(T)$. The latter inclusion is equivalent to

$$\|(T-\lambda)^{-1}\|\leqq\sup\{|z-\lambda|^{-1}: z\in H\}\qquad(\lambda\notin H)$$

and so to determine whether or not H is spectral we need only look at (a subset of) the rational functions of order 1 with poles off H. This fact leads naturally to the question of whether one can similarly prescribe a sub-class of rational functions which determine the spectrality of $\overline{W(T)}$. Such a result is the following one:

If $\|p(T)\|\leqq\|p\|_{W(T)}$ for all polynomials, then $\overline{W(T)}$ is a spectral set for T.

(Proof. The compact set $\overline{W(T)}$ has a connected complement and so by a theorem of LAVRENTIEFF (see [5]) the polynomials are uniformly dense in the algebra of functions which are continuous on $\overline{W(T)}$ and analytic in int $\overline{W(T)}$.)

It is an elementary fact that if $|W(A)|$ and $|\sigma(A)|$ denote the numerical radius and the spectral radius of an operator A, respectively, one has $\|A\|=|W(A)|$ if and only if $\|A\|=|\sigma(A)|$. It follows that either of the equivalent conditions

$$\|p(T)\|=|\sigma(p(T))|,\quad \|p(T)\|=|W(p(T))|$$

(for all polynomials) is a sufficient condition for the spectrality of $\overline{W(T)}$. It seems reasonable to ask whether the condition remains sufficient when the class of *all* polynomials is replaced by the linear ones. That is, does the condition $|W(T-\lambda)|=$ $=\|T-\lambda\|$ (all complex λ) imply spectrality of $\overline{W(T)}$? The following example answers the question negatively.

Example 3. Let $T=A_2\oplus U$ where U is unitary with spectrum equal to eth cube roots of unity. Then $|W(T-\lambda)|=\|T-\lambda\|$ for all complex λ but $\overline{W(T)}$ is not spectral for T.

First of all, $W(T)$ is the convex hull of the numerical ranges of $W(A_2)$ and $W(U)$, and hence $W(T)$ is the equilateral triangle which constitutes $W(U)$. Therefore

$$\|U-\lambda\|=|W(U-\lambda)|=|W(T-\lambda)|,$$

thus

$$\|T-\lambda\|=\max\{\|A_2-\lambda\|,\|U-\lambda\|\}=\max\{\|A_2-\lambda\|;|W(T-\lambda)|\}.$$

It remains to see that

(1) $\|A_2 - \lambda\| \leqq |W(T - \lambda)|$.

(2) The triangle $W(U)$ is not spectral for A_2.

The first of these is a simple computation (see Corollary 2 of Theorem 1), and the second is a consequence of the fact that the unit disk is a minimal spectral set for A_2.

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(Received August 1, 1966)

Über Potenzen von linearen Operatoren in Banachschen Räumen

Von VOLKER NOLLAU in Dresden (DDR)

1. Es sei A ein abgeschlossener linearer Operator im Banachraum X, dessen Definitionsbereich $\mathfrak{D}(A)$ in X dicht liegt. Wir sagen, A sei vom Typ (M), wenn die negative reelle Achse (ausschließlich des Nullpunktes) zur Resolventenmenge $\varrho(A)$ gehört, und eine positive Konstante M existiert, so daß die Resolvente $R(-\lambda; A) = (A+\lambda I)^{-1}$ für alle $\lambda > 0$ der Beziehung $\|R(-\lambda; A)\| \leqq M\lambda^{-1}$ genügt. Durch das Integral

$$\frac{\sin \pi\alpha}{\pi} \int_0^\infty \eta^{\alpha-1} R(-\eta; A) Ax\, d\eta \qquad (0<\alpha<1; \quad x \in \mathfrak{D}(A)) \tag{1}$$

ist dann auf $\mathfrak{D}(A)$ ein linearer abschließbarer (siehe [1]) Operator definiert, dessen Abschließung wir die α-te Potenz A^α nennen.[1])

Das Ziel dieser Mitteilung soll es sein, zu zeigen, daß die so definierten Potenzen den üblichen Potenzgesetzen genügen, d. h., für einen Operator A vom Typ (M) und $0 \leqq \alpha, \beta \leqq 1$ gelten die Beziehungen

$$A^\alpha A^\beta = A^{\alpha+\beta} \qquad (0 \leqq \alpha+\beta \leqq 1), \tag{2}$$

$$(A^\alpha)^\beta = A^{\alpha\beta}. \tag{3}$$

Darüber hinaus werden wir beweisen, daß zu einem Operator A vom Typ (M) und einer beliebigen natürlichen Zahl n genau ein abgeschlossener Operator B mit den Eigenschaften

$$B^n = A \quad \text{und} \quad \sigma(B) = \left\{z : |\arg z| \leqq \frac{\pi}{n}\right\} \tag{4}$$

existiert; dieser Operator B ist die $1/n$-te Potenz $A^{1/n}$ von A und selbst vom Typ (M) (Satz 4).

Sind X ein Hilbertraum und der betrachtete Operator maximal accretiv, so wurden die Eigenschaften (2), (3) der Potenzen mit Hilfe des Funktionalkalküls

[1]) Unter A^0 wollen wir die identische Abbildung und unter A^1 den Operator selbst verstehen.

für Kontraktionen im Hilbertraum von B. Sz.-Nagy und C. Foiaş [11] nachgewiesen.[2]) Unter der Voraussetzung, daß der offene Sektor

$$\sum = \left\{ z = re^{i\varphi}: 0 < r < \infty,\ |\varphi| < \Theta,\ 0 < \Theta < \frac{\pi}{2} \right\}$$

das Spektrum $\sigma(A)$ enthält und der Operator A der Bedingung $\|zR(z; A)\| \leqq$ const. für alle $z \notin \sum$ genügt [3]), geben T. Kato und H. Tanabe [6] (2) an, während M. A. Krasnoselskij und P. E. Sobolewskij [7] die genannten Potenzgesetze (2) und (3) für stetig invertierbare Operatoren vom Typ (M) zeigen. A. V. Balakrishnan beweist für $0 < \alpha + \beta < 1$ und $0 < \alpha$, $\beta < 1$ die Beziehung $A^\alpha A^\beta x = A^{\alpha+\beta} x$ $(x \in \mathfrak{D}(A^2))$ für einen Operator A vom Typ (M). Die Eigenschaft (3) erhält J. Watanabe (vgl. [12] [4])) für sogenannte Operatoren vom Typ (M, ω) (vgl. T. Kato [4]) als einfache Folgerung einer Integraldarstellung der Resolvente $R(-\eta; A^\alpha)$ $(\eta > 0)$.

Wir zeigen, daß diese Integraldarstellung auch für die Potenzen von Operatoren vom Typ (M) richtig bleibt (Satz 2).

Mit der Existenz und Eindeutigkeit einer Lösung der Operatorengleichung $B^n = A$ für einen vorgegebenen Operator A und eine natürliche Zahl n hat sich im Spezialfall eines Hilbertraumes eine Reihe von Autoren beschäftigt. Für positive, selbstadjungierte Operatoren folgt die Existenz einer positiven Quadratwurzel aus dem Spektralsatz; eine direkte, einfache Konstruktion stammt von C. Visser. B. Sz.-Nagy (vgl. [10]) hat gezeigt, daß diese Quadratwurzel durch die Bedingung der Positivität eindeutig bestimmt ist. W. I. Mazajeff und J. A. Palant beweisen in [9], daß es zu einem beschränkten dissipativen Operator T in einem Hilbertraum $\mathfrak{H}$ und einer beliebigen natürlichen Zahl n genau einen beschränkten Operator S gibt, der eine Lösung der Gleichung $S^n = T$ ist und dessen quadratische Form (Sx, x) $(x \in \mathfrak{H})$ Werte im Sektor $0 \leqq \arg z \leqq \frac{\pi}{n}$ annimmt. H. Langer zeigt in [8], daß diese Behauptung auch für einen abgeschlossenen maximal dissipativen Operator T richtig bleibt, wenn man verlangt, daß S ein abgeschlossener maximal dissipativer Operator ist.

Wir bemerken, daß die genannten Aussagen sich als Spezialfall des von uns in Satz 4 angegebenen allgemeinen Sachverhaltes erweisen.

Herrn Prof. Dr. H. Langer danke ich für seine freundliche Unterstützung und zahlreichen Hinweise.

2. Einige Eigenschaften der Potenzen A^α $(0 \leqq \alpha \leqq 1)$, die wir später benötigen, werden in den folgenden Lemmata bewiesen.

Lemma 1. *Es sei A ein Operator vom Typ (M). Für $x \in \mathfrak{D}(A)$ ist $A^\beta x$ stetig bzgl. β im Intervall $0 < \beta \leqq 1$; d. h.*

$$\lim_{\alpha \to \beta} A^\alpha x = A^\beta x.$$

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[2]) Die Beziehung $(A^{1/n})^n = A$ wurde in [4] gezeigt.

[3]) Diese Voraussetzung hat notwendigerweise die Stetigkeit des Operators A^{-1} zur Folge.

[4]) Die Originalarbeit in *Proc. Japan Acad.*, **37** (1961), 273—275, wurde mir erst nach Abschluß dieser Arbeit vom Verfasser freundlicherweise zur Verfügung gestellt.

Beweis. Sei $0<\beta<1$. Ist $x\in\mathfrak{D}(A)$ und $0<\beta-\beta'\leqq\alpha\leqq\beta+\beta'<1$ $(\beta'>0)$, so existieren zu einem beliebigen $\varepsilon>0$ ein $0<\delta=\delta(\varepsilon)$ und ein $R=R(\varepsilon)<\infty$, so daß

$$\left\|\left(\int_0^\delta+\int_R^\infty\right)C(\gamma)\,\eta^{\gamma-1}R(-\eta;A)Ax\,d\eta\right\|\leqq\frac{\varepsilon}{3}\quad\text{für}\quad C(\gamma)=\frac{\sin\pi\gamma}{\pi}\quad(\gamma=\alpha,\beta)$$

gilt. Weiterhin ergibt sich

$$\left\|\int_\delta^R\{C(\alpha)\eta^{\alpha-1}-C(\beta)\eta^{\beta-1}\}R(-\eta;A)Ax\,d\eta\right\|\leqq$$

$$\leqq(1+M)(R-\delta)\|x\|\sup_{\delta\leqq\eta\leqq R}|C(\alpha)\eta^{\alpha-1}-C(\beta)\eta^{\beta-1}|\leqq\frac{\varepsilon}{3}$$

für $|\alpha-\beta|<\sigma(\varepsilon)$ $(0<\sigma(\varepsilon)\leqq\beta')$. [5]) Folglich existiert zu jedem $\varepsilon>0$ ein $\sigma(\varepsilon)>0$, so daß $\|(A^\alpha-A^\beta)x\|<\varepsilon$ für $|\alpha-\beta|<\sigma(\varepsilon)$ ist.

Die linksseitige Stetigkeit im Punkte $\beta=1$ wurde schon von A. V. BALAKRISHNAN [1] bewiesen.

Lemma 2. *Für $0<\alpha<1$ und $\eta>0$ ist der Operator $A^\alpha R(-\eta;A)$ beschränkt, und zwar gilt*

$$\|A^\alpha R(-\eta;A)\|\leqq K(\alpha)\eta^{\alpha-1}\quad\left(K(\alpha)=\frac{\sin\pi\alpha}{\pi\alpha(1-\alpha)}M(1+M)\right).\tag{5}$$

Beweis. Wegen $R(-\eta;A)X\subset\mathfrak{D}(A)$ ist für $x\in X$

$$A^\alpha R(-\eta;A)x=\frac{\sin\pi\alpha}{\pi}\int_0^\infty\lambda^{\alpha-1}R(-\lambda;A)AR(-\eta;A)x\,d\lambda.$$

Mit Hilfe der Substitution $\lambda=\xi\eta$ ergibt sich

$$\|A^\alpha R(-\eta;A)\|\leqq\frac{\sin\pi\alpha}{\pi}\eta^{\alpha-1}\left\{\left\|\int_0^1\xi^{\alpha-1}AR(-\xi\eta;A)\eta R(-\eta;A)x\,d\xi\right\|+\right.$$

$$\left.+\left\|\int_1^\infty\xi^{\alpha-1}\eta R(-\xi\eta;A)AR(-\eta;A)x\,d\xi\right\|\right\}.$$

Wendet man die Abschätzungen $\|R(-\lambda;A)\|\leqq M\lambda^{-1}$ und $\|AR(-\lambda;A)\|\leqq 1+M$ für $\lambda>0$ auf die rechte Seite der vorangehenden Ungleichung an, so erhält man

$$\|A^\alpha R(-\eta;A)x\|\leqq\frac{\sin\pi\alpha}{\pi}(1+M)M\eta^{\alpha-1}\left\{\int_0^1\xi^{\alpha-1}\,d\xi+\int_1^\infty\xi^{\alpha-2}\,d\xi\right\}\|x\|.$$

Lemma 3. *Der Definitionsbereich des Operators A^2 ist ein Kern (vgl. [5]) von $A^\alpha(0\leqq\alpha\leqq1)$, d. h., zu jedem $x\in\mathfrak{D}(A^\alpha)$ existiert eine Folge $\{x_n\}\subset\mathfrak{D}(A^2)$, so daß $x_n\to x$ und $A^\alpha x_n\to A^\alpha x$ $(n\to\infty)$ gilt.*

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[5]) Wir verwenden hier die Abschätzung $(\eta>0)$
$\|AR(-\eta;A)\|=\|I-\eta(A+\eta I)^{-1}\|\leqq 1+\eta\|R(-\eta;A)\|\leqq 1+M.$

Beweis. Wir setzen $T_n = n(nI + A)^{-1}$ $(n = 1, 2, \ldots)$. Man überlegt sich leicht, daß die Folge der gleichmäßig beschränkten Operatoren in der starken Operatorentopologie gegen I konvergiert. Für $y \in \mathfrak{D}(A)$ ist $T_n A^\alpha y = A^\alpha T_n y$ $(n = 1, 2, \ldots)$. Auf Grund der Definition von A^α existiert zu jedem $y \in \mathfrak{D}(A^\alpha)$ eine Folge $\{y_m\} \subset \mathfrak{D}(A)$ mit $\lim y_m = y$ und $\lim A^\alpha y_m = A^\alpha y$ $(m \to \infty)$. Da der Operator $A^\alpha T_n$ für jedes $n = 1, 2, \ldots$ beschränkt ist, konvergiert die rechte Seite in der Ungleichung

$$\|A^\alpha T_n y - T_n A^\alpha y\| \leqq \|A^\alpha T_n y - A^\alpha T_n y_m\| + \|T_n A^\alpha y_m - T_n A^\alpha y\|$$

für $m \to \infty$ gegen Null; und somit ist $A^\alpha T_n y = T_n A^\alpha y$ für alle $n = 1, 2, \ldots$ und alle $y \in \mathfrak{D}(A^\alpha)$.

Es sei nun $x \in \mathfrak{D}(A^\alpha)$ und $x_n = T_n^2 x$. Mit Hilfe der vorangegangenen Überlegung erhält man die Aussagen $x_n \in \mathfrak{D}(A^2)$, $\lim\limits_{n\to\infty} x_n = x$ und $\lim\limits_{n\to\infty} A^\alpha x_n = \lim\limits_{n\to\infty} A^\alpha T_n^2 x = \lim\limits_{n\to\infty} T_n^2 A^\alpha x = A^\alpha x$, was zu zeigen war.

Für $x \in \mathfrak{D}(A^2)$ und $0 < \alpha + \beta < 1$ gilt $A^\beta x \in \mathfrak{D}(A)$ und $A^\alpha A^\beta x = A^{\alpha+\beta} x$ [1]. Ist $\alpha_n = 1 - \beta - \dfrac{1}{n}$ $(\beta > 0;\ \alpha_n > 0)$ und $x \in \mathfrak{D}(A^2)$, so erhält man $A^{\alpha_n} A^\beta x = A^{1-\frac{1}{n}} x$. Auf Grund der Stetigkeit von $A^\alpha x$ $(x \in \mathfrak{D}(A))$ bzgl. α ergibt sich dann $A^{1-\beta} A^\beta x = Ax$ $(x \in \mathfrak{D}(A^2))$.

Lemma 4. *Für $0 \leqq \alpha \leqq \beta \leqq 1$ besteht die Beziehung $\mathfrak{D}(A^\alpha) \supset \mathfrak{D}(A^\beta) \supset \mathfrak{D}(A)$; insbesondere konvergiert für jede Folge $\{y_n\} \subset \mathfrak{D}(A)$ $(n = 1, 2, \ldots)$ mit den Eigenschaften $\lim\limits_{n\to\infty} y_n = y$ und $\lim\limits_{n\to\infty} A^\beta y_n = A^\beta y$ auch $A^\alpha y_n$ gegen $A^\alpha y$.*

Beweis. Eine Folge $\{y_n\} \subset \mathfrak{D}(A)$ $(n = 1, 2, \ldots)$ genüge den genannten Voraussetzungen, also ist $AR(-\eta; A)(y_n - y_m) = A^{1-\beta} A^\beta R(-\eta; A)(y_n - y_m)$ für $\eta > 0$, und man erhält für alle $0 < \alpha < \beta \leqq 1$ die Abschätzung

$$\|A^\alpha(y_n - y_m)\| \leqq \frac{\sin \pi\alpha}{\pi} \Big\{(1 + M)\int\limits_0^1 \eta^{\alpha-1}\, d\eta\, \|y_n - y_m\| + \\ + \int\limits_1^\infty \eta^{\alpha-1} \|A^{1-\beta} R(-\eta; A)\|\, d\eta\, \|A^\beta(y_n - y_m)\|\Big\}.$$

Für $m, n \to \infty$ konvergiert die rechte Seite der Ungleichung gegen Null, da auch das zweite Integral auf Grund von (5) einen endlichen Wert besitzt. Wegen der Vollständigkeit des Raumes existiert somit ein $z \in X$ derart, daß für die Folge $\{y_n\} \subset \mathfrak{D}(A)$ die Beziehungen $\lim\limits_{n\to\infty} y_n = y$ und $\lim\limits_{n\to\infty} A^\alpha y_n = z$ gelten, woraus sich auf Grund der Abgeschlossenheit von A^α sofort $y \in \mathfrak{D}(A^\alpha)$ und $z = A^\alpha y$ ergibt.

Wir bemerken, daß damit auch die Inklusion $X = \mathfrak{D}(A^0) \supset \mathfrak{D}(A^\alpha) \supset \mathfrak{D}(A^\beta) \supset \mathfrak{D}(A)$ für $0 \leqq \alpha \leqq \beta \leqq 1$ bewiesen ist.

3. Satz 1. *Es sei A ein Operator vom Typ (M). Dann gilt die Beziehung*

$$A^\alpha A^\beta = A^{\alpha+\beta} \quad \textit{für} \quad 0 \leqq \alpha + \beta \leqq 1 \quad \textit{und} \quad 0 \leqq \alpha, \beta.$$

Beweis. Es sei $x \in \mathfrak{D}(A^{\alpha+\beta})$. Wie wir in Lemma 3 bewiesen haben, existiert eine Folge $\{x_n\} \subset \mathfrak{D}(A^2)$ mit $\lim x_n = x$ und $\lim A^{\alpha+\beta} x_n = A^{\alpha+\beta} x$ $(n \to \infty)$. Auf $\mathfrak{D}(A^2)$ gilt $A^\alpha A^\beta = A^{\alpha+\beta}$, folglich besteht für $x_n \in \mathfrak{D}(A^2)$ die Gleichung $A^\alpha A^\beta x_n =$

$=A^{\alpha+\beta}x_n$. Mit Hilfe des vorangehenden Lemmas folgt nun aber, daß $x\in\mathfrak{D}(A^\beta)$ und $\lim A^\beta x_n=A^\beta x$ $(n\to\infty)$ ist. Aus der Beziehung $\lim A^\alpha A^\beta x_n=\lim A^{\alpha+\beta}x_n=$ $=A^{\alpha+\beta}x$ $(n\to\infty)$ und der Abgeschlossenheit von A^α ergibt sich dann $A^\beta x\in\mathfrak{D}(A^\alpha)$ und $A^\alpha A^\beta x=A^{\alpha+\beta}x$, d. h. $A^\alpha A^\beta\supset A^{\alpha+\beta}$.

Für $x\in\mathfrak{D}(A^\alpha A^\beta)$ ist $x_n=n(A+nI)^{-1}x=T_nx\in\mathfrak{D}(A)$ $(\mathfrak{D}(A)\subset\mathfrak{D}(A^{\alpha+\beta}))$, d. h., es gilt $A^\alpha A^\beta x_n=A^{\alpha+\beta}x_n$. Wegen der Vertauschbarkeit der Operatoren T_n und A^α für $n=1,2,\ldots$ und $0\leqq\alpha\leqq 1$ auf $\mathfrak{D}(A^\alpha)$ erhalten wir $A^{\alpha+\beta}x_n=A^\alpha A^\beta T_nx=A^\alpha T_nA^\beta x$. Nach Voraussetzung ist $x\in\mathfrak{D}(A^\alpha A^\beta)$, also $A^\beta x\in\mathfrak{D}(A^\alpha)$, so daß sich für $n\to\infty$ die Gleichungen $\lim A^{\alpha+\beta}x_n=\lim A^\alpha T_nA^\beta x=\lim T_nA^\alpha A^\beta x=A^\alpha A^\beta x$ ergeben. Da andererseits x_n gegen x konvergiert, folgt aus der Abgeschlossenheit von $A^{\alpha+\beta}$, daß $x\in\mathfrak{D}(A^{\alpha+\beta})$ und $A^{\alpha+\beta}x=A^\alpha A^\beta x$ gilt, d. h. $A^\alpha A^\beta\subset A^{\alpha+\beta}$.

Damit ist der Satz bewiesen.

Folgerung 1. *Die Wertebereiche der Operatoren A^α genügen für $0\leqq\alpha\leqq\beta\leqq 1$ der Beziehung $\mathfrak{R}(A^\alpha)\supset\mathfrak{R}(A^\beta)\supset\mathfrak{R}(A)$.*

4. Satz 2. *Es sei A ein Operator vom Typ (M). Dann gestattet die Resolvente $R(-\eta;A^\alpha)$ für $\eta>0$ und $0<\alpha<1$ die Darstellung*

$$(A^\alpha+\eta I)^{-1}=\frac{\sin\pi\alpha}{\pi}\int_0^\infty\frac{r^\alpha(A+rI)^{-1}\,dr}{r^{2\alpha}+2\eta r^\alpha\cos\pi\alpha+\eta^2}. \qquad (6)$$ [6]

Insbesondere ist A^α ein Operator vom Typ (M), und es gilt $\sigma(A^\alpha)\subset\{z\colon|\arg z|\leqq\pi\alpha\}$.

Beweis. I. Zunächst setzen wir voraus, daß A sowie seine Inverse A^{-1} beide stetig sind. Dann ist der Operator A^α bekanntlich durch das Riesz—Dunford-Integral $A^\alpha=-\frac{1}{2\pi i}\oint_C z^\alpha R(z;A)\,dz$ [7] darstellbar (vgl. [3]), und die Resolvente $(A^\alpha+\eta I)^{-1}$ besitzt für $\eta>0$ die Form

$$-\frac{1}{2\pi i}\oint_C(z^\alpha+\eta)^{-1}R(z;A)\,dz;$$

dabei ist C eine geschlossene Kontur, die aus den Kurven $z=Re^{i\varphi}$ $(-\pi\leqq\varphi\leqq\pi)$, $z=-\eta$ $(R\geqq\eta\geqq\varepsilon)$, $z=\varepsilon e^{i\varphi}(\pi\geqq\varphi\geqq-\pi)$ und $z=-\eta$ $(\varepsilon\leqq\eta\leqq R)$ besteht. R und ε sind so zu wählen, daß C in $\varrho(A)$ verläuft und das Spektrum von A enthält. Durch den Grenzübergang $R\to\infty$ und $\varepsilon\to 0$ erhält man

$$(A^\alpha+\eta I)^{-1}=\frac{\sin\pi\alpha}{\pi}\int_0^\infty\frac{r^\alpha(A+rI)^{-1}\,dr}{r^{2\alpha}+2\eta r^\alpha\cos\pi\alpha+\eta^2},$$

was zu zeigen war.

---

[6]) T. Kato definiert in [4] A^α $(0<\alpha<1)$ für A vom Typ (M,ω), indem er zeigt, daß das Integral (6) die Resolvente eines abgeschlossenen Operators ist, den er als A^α bezeichnet.

[7]) Die komplexe Ebene sei längs der negativen reellen Achse aufgeschnitten, und z^α sei so definiert, daß $z^\alpha>0$ für $z>0$ ist.

Für einen beliebigen Operator A vom Typ (M) und $0<\alpha<1$ sei

(7) $$S_\alpha(\eta; A) = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{r^\alpha (A+rI)^{-1}\, dr}{r^{2\alpha}+2\eta r^\alpha \cos \pi\alpha + \eta^2} \qquad (\eta>0).$$

Der Operator $S_\alpha(\eta; A)$ ist beschränkt, insbesondere gilt mit $r^\alpha=\eta\mu^\alpha$

(8) $$\|S_\alpha(\eta; A)\| \leqq \frac{M}{\eta} \int_0^\infty \frac{\sin \pi\alpha\, \mu^{\alpha-1}\, d\mu}{\pi(\mu^{2\alpha}+2\mu^\alpha \cos \pi\alpha+1)} = \frac{M}{\eta}.$$

II. Es sei jetzt A ein beschränkter Operator vom Typ (M). Wir zeigen, daß $S_\alpha(\eta; A)$ die Resolvente von A^α ist. Der Operator $A_\varepsilon = A+\varepsilon I$ $(\varepsilon>0)$ ist beschränkt invertierbar, und daher ergibt sich nach I: $(A_\varepsilon^\alpha+\eta I)^{-1} = S_\alpha(\eta; A_\varepsilon)$. Folglich gilt für $\delta>0$

(9) $$\begin{aligned} \|I - S_\alpha(\eta; A)(A^\alpha+\eta I)\| &\leqq \|(A_\varepsilon^\alpha+\eta I)^{-1}(A_\varepsilon^\alpha+\eta I) - (A_\varepsilon^\alpha+\eta I)^{-1}(A^\alpha+\eta I)\| + \\ &+ \|S_\alpha(\eta; A_\varepsilon)(A^\alpha+\eta I) - S_\alpha(\eta; A)(A^\alpha+\eta I)\| \leqq \\ &\leqq \frac{M}{\eta}\|A_\varepsilon^\alpha - A^\alpha\| + \|A^\alpha+\eta I\| \int_0^\delta \frac{2Mr^{\alpha-1}\, dr}{r^{2\alpha}+2\eta r^\alpha \cos \pi\alpha+\eta^2} + \\ &+ \varepsilon\|A^\alpha+\eta I\| \int_\delta^\infty \frac{M^2 r^{\alpha-2}\, dr}{r^{2\alpha}+2\eta r^\alpha \cos \pi\alpha+\eta^2}. \end{aligned}$$

Die Operatoren A_ε^α konvergieren für $\varepsilon\searrow 0$ in der gleichmäßigen Operatorentopologie gegen A^α, denn es ist

$$\|A_\varepsilon^\alpha - A^\alpha\| \leqq \frac{\sin \pi\alpha}{\pi}\left\{\frac{2}{\alpha}+\frac{1}{1-\alpha}\right\} M^2 \varepsilon^\alpha \qquad (\varepsilon>0,\ 0<\alpha<1) \quad \text{(vgl. [9]).}$$

Wählt man in (9) $\delta>0$ hinreichend klein und läßt anschließend $\varepsilon\searrow 0$ streben, so erhält man $S_\alpha(\eta; A)(A^\alpha+\eta I)=I$. In entsprechender Weise ergibt sich $(A^\alpha+\eta I)\cdot S_\alpha(\eta; A)=I$, d. h.

(10) $$(A^\alpha+\eta I)^{-1} = S_\alpha(\eta; A).$$

III. Nun seien A ein beliebiger Operator vom Typ (M) und $A_n = nA(A+nI)^{-1}$ $(n=1, 2, \ldots)$. Wir stellen zunächst einige Eigenschaften der Operatoren A_n zusammen:

a) A_n ist beschränkt.

b) A_n ist vom Typ (M') $(M'=2M+1)$. Das zeigt folgende Abschätzung:

$$\|(A_n+\eta I)^{-1}\| = \|(A+nI)(A(n+\eta)+\eta nI)^{-1}\| =$$

$$= \frac{1}{n+\eta}\left\|(A+nI)\left(A+\frac{\eta n}{n+\eta}I\right)^{-1}\right\| \leqq \frac{1+M}{n+\eta}+\frac{M}{\eta} \leqq \frac{1+2M}{\eta} \qquad (\eta>0).$$

c) Für $x \in \mathfrak{D}(A)$ gilt $A_n x \to Ax$ $(n \to \infty)$.

d) Die Folge $\{(A_n + \eta I)^{-1}\}$ konvergiert in der starken Operatorentopologie gegen $(A + \eta I)^{-1}$. Für $x \in \mathfrak{D}(A)$ erhält man nämlich

$$\|(A_n + \eta I)^{-1} x - (A + \eta I)^{-1} x\| \leqq \frac{M(2M+1)}{\eta^2} \|(A - A_n) x\| \to 0 \qquad (n \to \infty).$$

Zu jedem $y \in X$ existiert nach Voraussetzung eine Folge $\{y_m\} \subset \mathfrak{D}(A)$ $(m = 1, 2, \dots)$ derart, daß $y_m \to y$ für $m \to \infty$ gilt. Es ist folglich für $\varepsilon > 0$

$$\begin{aligned} &\|(A_n + \eta I)^{-1} y - (A + \eta I)^{-1} y\| \leqq \|(A_n + \eta I)^{-1} y - (A_n + \eta I)^{-1} y_m\| + \\ (11) \qquad &+ \|(A_n + \eta I)^{-1} y_m - (A + \eta I)^{-1} y_m\| + \|(A + \eta I)^{-1} y_m - (A + \eta I)^{-1} y\| \leqq \\ &\leqq \frac{2M+1}{\eta} \|y - y_m\| + \frac{(2M+1)M}{\eta^2} \|(A - A_n) y_m\| + \frac{M}{\eta} \|y - y_m\| \leqq \varepsilon, \end{aligned}$$

falls man ein m so wählt, daß $(3M+1)\eta^{-1}\|y - y_m\| \leqq \frac{\varepsilon}{2}$ und $n \geqq n_0(\varepsilon)$ ist.

e) Für $y \in X$ gilt $\lim_{n \to \infty} S_\alpha(\eta; A_n) y = S_\alpha(\eta; A) y$. Es sei $\delta > 0$ beliebig. Für ein $\varepsilon > 0$ ergibt sich dann

$$\begin{aligned} &\left\| \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\mu^\alpha \{(A_n + \mu I)^{-1} - (A + \mu I)^{-1}\} y}{\mu^{2\alpha} + 2\eta\mu^\alpha \cos \pi\alpha + \eta^2} \, d\mu \right\| \leqq \\ &\leqq \frac{3M+1}{\pi} \sin \pi\alpha \int_0^\varepsilon \frac{\mu^{\alpha-1} \, d\mu}{\mu^{2\alpha} + 2\eta\mu^\alpha \cos \pi\alpha + \eta^2} \|y\| + \\ &+ \frac{3M+1}{\pi} \sin \pi\alpha \int_\varepsilon^\infty \frac{\mu^{\alpha-1} \, d\mu}{\mu^{2\alpha} + 2\eta\mu^\alpha \cos \pi\alpha + \eta^2} \|y - y_m\| + \\ &+ \frac{(2M+1)M}{\pi} \sin \pi\alpha \int_\varepsilon^\infty \frac{\mu^{\alpha-2} \, d\mu}{\mu^{2\alpha} + 2\eta\mu^\alpha \cos \pi\alpha + \eta^2} \|(A - A_n) y_m\|, \end{aligned}$$

wobei (11) benutzt wurde. Wir wählen nun ein $\varepsilon > 0$, so daß der erste Summand der rechten Seite kleiner als $\delta/3$ wird, weiter existiert ein m derart, daß der zweite Summand kleiner als $\delta/3$ ist. Für alle $n \geqq n_0(\delta)$ gilt dann die Beziehung

$$\|S_\alpha(\eta; A_n) y - S_\alpha(\eta; A) y\| < \delta.$$

Mit diesen Vorbetrachtungen beweisen wir nun, daß $S_\alpha(A; \eta) = (A^\alpha + \eta I)^{-1}$ ist.

Wie in II gezeigt wurde, besteht für die A_n die Gleichung $S_\alpha(\eta; A_n)(A_n^\alpha + \eta I) = (A_n^\alpha + \eta I) S_\alpha(\eta; A_n) = I$. Es sei $x \in \mathfrak{D}(A)$. Dann ergibt sich die Beziehung

$$\begin{aligned} &\|x - S_\alpha(\eta; A)(A^\alpha + \eta I) x\| \leqq \\ &\leqq \|(A_n^\alpha + \eta I)^{-1}(A_n^\alpha + \eta I) x - (A_n^\alpha + \eta I)^{-1}(A^\alpha + \eta I) x\| + \\ &+ \|S_\alpha(\eta; A_n)(A^\alpha + \eta I) x - S_\alpha(\eta; A)(A^\alpha + \eta I) x\|. \end{aligned}$$

Beide Summanden der rechten Seite dieser Ungleichung werden aber für hinreichend großes n beliebig klein. Für den zweiten Summanden folgt das unmittelbar aus der Aussage e), für den ersten Summanden aus den Abschätzungen

$$\|(A_n^\alpha+\eta I)^{-1}\| \leqq \frac{2M+1}{\eta}\,\frac{\sin\pi\alpha}{\pi}\int\limits_0^\infty \frac{\mu^{\alpha-1}\,d\mu}{\mu^{2\alpha}+2\mu^\alpha\cos\pi\alpha+1} = \frac{2M+1}{\eta}$$

$$(n = 1, 2, \ldots), \quad \text{und}$$

$$\|(A_n^\alpha - A^\alpha)x\| \leqq \frac{\sin\pi\alpha}{\pi}\left\|\int\limits_0^\infty \eta^{\alpha-1}\{A_nR(-\eta; A_n) - AR(-\eta; A)\}x\,d\eta\right\| \leqq$$

$$\leqq \sin\pi\alpha(M+1)\int\limits_0^\delta \eta^{\alpha-1}\,d\eta\,\|x\| + \frac{\sin\pi\alpha}{\pi}(2M+1)M\int\limits_\delta^\infty \eta^{\alpha-2}\,d\eta\,\|(A-A_n)x\|$$

$$\big(x\in\mathfrak{D}(A);\ \delta>0\big).$$

Damit haben wir die Beziehung $S_\alpha(\eta;\ A)(A^\alpha+\eta I)x=x\ \big(x\in\mathfrak{D}(A)\big)$ bewiesen. Es sei $y\in\mathfrak{D}(A^\alpha)$. Wir betrachten die Folge $\{y_m\}\subset\mathfrak{D}(A)$ mit $\lim\limits_{m\to\infty} y_m=y$, $\lim\limits_{m\to\infty} A^\alpha y_m=A^\alpha y$. Dann ergibt sich $\lim\limits_{m\to\infty} S_\alpha(\eta;\ A)(A^\alpha+\eta I)y_m=\lim\limits_{m\to\infty} y_m=y$, d. h., es ist

$$S_\alpha(\eta;\ A)(A^\alpha+\eta I)y=y.$$

Bleibt zu zeigen, daß der Operator $S_\alpha(\eta;\ A)$ auch Rechtsinverse von $(A^\alpha+\eta I)$ ist.

Der Operator $S_\alpha(\eta;\ A)$ stellt sich nach Definition als der Limes in der gleichmäßigen Operatorentopologie einer Folge von Operatoren

$$S_n = \sum_{k=1}^{n} \alpha_k R(-\eta_k;\ A)$$

mit reellen α_k und positiven η_k dar. Wie man leicht sieht, ist $S_n\mathfrak{D}(A)\subset\mathfrak{D}(A)$ und $A^\alpha S_n x=S_nA^\alpha x\ \big(x\in\mathfrak{D}(A)\big)$. Für $n\to\infty$ erhält man $S_nx\to S_\alpha(\eta;\ A)x$ und $A^\alpha S_nx=S_nA^\alpha x\to S_\alpha(\eta;\ A)A^\alpha x$, also ist auf Grund der Abgeschlossenheit von A^α

$$S_\alpha(\eta;\ A)x\in\mathfrak{D}(A),\quad A^\alpha S_\alpha(\eta;\ A)x=S_\alpha(\eta;\ A)A^\alpha x,$$

d. h., auf $\mathfrak{D}(A)$ gilt $(A^\alpha+\eta I)S_\alpha(\eta;\ A)x=x$. Es sei nun $y\in X$ und $\{y_m\}\subset\mathfrak{D}(A)$ eine Folge, die gegen y strebt. Dann ergibt sich $(A^\alpha+\eta I)S_\alpha(\eta;\ A)y_m=y_m\to y$ $(m\to\infty)$. Aus der Abgeschlossenheit von A^α erhalten wir somit die Beziehung $(A^\alpha+\eta I)S_\alpha(\eta;\ A)=I$, d. h., es ist $S_\alpha(\eta;\ A)=(A^\alpha+\eta I)^{-1}$, was zu zeigen war.

Die Aussage $\sigma(A^\alpha)\subset\{z\colon |\arg z|\leqq\pi\alpha\}$ ergibt sich dann unmittelbar aus der Integraldarstellung (6) (vgl. [4]).

Aus der Integralform (6) der Resolvente von A^α folgt nun wie in [12] der

Satz 3. *Es sei A ein Operator vom Typ (M). Dann ist*

$$(A^\alpha)^\beta = A^{\alpha\beta}\ \textit{für}\quad 0\leqq\alpha,\ \beta\leqq 1.$$

Folgerung 2. *Es seien* A^α $(0 \leqq \alpha \leqq 1)$ *die Potenzen eines Operators A vom Typ* (M). *Dann gilt für alle* $x \in \mathfrak{D}(A^\alpha)$

$$\lim_{\beta \nearrow \alpha} A^\beta x = A^\alpha x.$$

Beweis. Aus Lemma 1 erhält man für einen Operator B vom Typ (M) die Aussage $\lim\limits_{\gamma \nearrow 1} B^\gamma x = Bx$ für alle $x \in \mathfrak{D}(B)$. Auf Grund von Satz 2 (vgl. (8)) ist A^α vom Typ (M), d. h., es gilt $\lim\limits_{\gamma \nearrow 1} (A^\alpha)^\gamma x = A^\alpha x$ $(x \in \mathfrak{D}(A^\alpha))$. Entsprechend Satz 3 ist $(A^\alpha)^\gamma = A^{\alpha\gamma}$, woraus für $\beta = \alpha\gamma$ die Behauptung folgt.

5. Der Aussage über die Eindeutigkeit der $\frac{1}{n}$-ten Potenzen eines Operators vom Typ (M) stellen wir drei Lemmata voran.

Lemma 5. a) *Es sei A ein Operator vom Typ* (M). *Dann gehört der Sektor* $\left\{z\colon |\arg z| > \pi - \gamma;\ \tan\gamma = \frac{1}{M}\right\}$ [8]) *zur Resolventenmenge* $\varrho(A)$.

b) *Es sei A ein abgeschlossener Operator, dessen Spektrum* $\sigma(A)$ *zum Sektor* $\sum\left(\frac{\pi}{n}\right) = \left\{z\colon |\arg z| \leqq \frac{\pi}{n}\right\}$ *gehört, wobei n eine natürliche Zahl ist. Ist dann* A^n *ein Operator vom Typ* (M), *so liegt* $\sigma(A)$ *sogar im Sektor*

$$\sum\left(\frac{\pi-\gamma}{n}\right) = \left\{z\colon |\arg z| \leqq \frac{\pi-\gamma}{n}\right\} \qquad \left(\tan\gamma = \frac{1}{M}\right).$$

Beweis. a) Nach Voraussetzung gehört die offene negative reelle Halbachse zur Resolventmenge $\varrho(A)$, und es existiert eine von $\eta > 0$ unabhängige Konstante $M > 0$, so daß die Beziehung $\|(A+\eta I)^{-1}\| \leqq \frac{M}{\eta}$ besteht. Für ein $z' = -\eta + i\frac{\eta\delta}{M}$ $(-1 < \delta < 1;\ 0 < \eta < \infty)$ erhält man folglich die Ungleichung $|z' + \eta|\,\|(A+\eta I)^{-1}\| \leqq$ $\leqq \frac{\eta|\delta|}{M} \cdot \frac{M}{\eta} = |\delta| < 1.$

Bekanntlich [12] gilt für alle z mit der Eigenschaft $|z+\eta| < (\|(A+\eta I)^{-1}\|)^{-1}$, daß $z \in \varrho(A)$ ist. Die Menge aller $z' = -\eta + i\frac{\delta\eta}{M}$ $(-1 < \delta < 1;\ 0 < \eta < \infty)$ gehört also zu $\varrho(A)$ und bildet ihrerseits den Sektor $\left\{z\colon |\arg z| > \pi - \gamma;\ \tan\gamma = \frac{1}{M}\right\}$.

b) Entsprechend dem Spektralabbildungssatz für Polynome abgeschlossener Operatoren, s. [3] S. 604, gilt einerseits $\{\sigma(A)\}^n = \sigma(A^n)$. Andererseits ergibt sich auf Grund der Aussage a): $\sigma(A^n) \subset \{z\colon |\arg z| \leqq \pi - \gamma\}$ $\left(\gamma = \operatorname{Arc\,tan}\frac{1}{M}\right)$. Es ist also $\{\sigma(A)\}^n \subset \{z\colon |\arg z| \leqq \pi - \gamma\}$, woraus wegen der Voraussetzung $\sigma(A) \subset \sum\left(\frac{\pi}{n}\right)$ unmittelbar die Behauptung folgt.

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[8]) Das Argument $\arg z$ einer komplexen Zahl z sei stets so definiert, daß $-\pi < \arg z \leqq \pi$ ist, und es gelte $\arg 0 = 0$.

Lemma 6. *Es seien A_m $(m=1,2,\ldots)$ beschränkte Operatoren vom gleichen Typ (M). Konvergiert die Folge $\{A_m\}$ für $m\to\infty$ in der starken Operatorentopologie gegen I, so gilt auch* $\lim (A_m)^\alpha x = x$ $(x\in X;\ 0\leqq\alpha\leqq 1)$.

Beweis. Es sei $0<\alpha<1$. Wegen

$$\frac{\sin\pi\alpha}{\pi}\int\limits_0^\infty \eta^{\alpha-1}(1+\eta)^{-1}\,d\eta = 1$$

erhält man dann für $x\in X$

$$\frac{\sin\pi\alpha}{\pi}\int\limits_0^\infty \eta^{\alpha-1} R(-\eta;A_m)A_m x\,d\eta - x =$$

$$= \frac{\sin\pi\alpha}{\pi}\int\limits_0^\infty \eta^{\alpha-1}\{R(-\eta;A_m)A_m-(1+\eta)^{-1}I\}x\,d\eta.$$

Auf Grund der Voraussetzung $\|R(-\eta;\,A_m)\|\leqq\dfrac{M}{\eta}$ $(\eta>0)$ für alle $m=1,2,\ldots$ ergibt sich dann

$$\|(A_m)^\alpha x - x\| = \left\|\frac{\sin\pi\alpha}{\pi}\int\limits_0^\infty \frac{\eta^\alpha}{1+\eta}R(-\eta;A_m)(A_m-I)x\,d\eta\right\| \leqq$$

$$\leqq M\frac{\sin\pi\alpha}{\pi}\int\limits_0^\infty \frac{\eta^{\alpha-1}}{1+\eta}\,d\eta\,\|(A_m-I)x\|\to 0 \qquad (m\to\infty),$$

was zu zeigen war.

Im folgenden Lemma stellen wir einige Eigenschaften der Funktionen $f_m(z)= =\left(\dfrac{m}{m+z^n}\right)^{1/n}$ (n — eine feste natürliche Zahl; $m=1,2,\ldots$) mit dem Ziel zusammen, einem abgeschlossenen Operator A, dessen Spektrum im Sektor $\sum\left(\dfrac{\omega}{n}\right)$ $(0<\omega<\pi)$ liegt, entsprechend dem Riesz—Dunford—Taylorschen Funktionalkalkül beschränkte Operatoren $f_m(A)$ $(m=1,2,\ldots)$ zuzuordnen. Die Spektraleigenschaften von $f_m(A)$ und die Beziehung $\lim f_m(A)x=x$ $(x\in X)$ werden im Beweis zur eingangs genannten Eindeutigkeitsaussage eine wesentliche Rolle spielen.

Lemma 7. *Es seien n eine natürliche Zahl, $n>1$, und $f_m(z)=\left(\dfrac{m}{m+z^n}\right)^{1/n}$ $(m=1,2,\ldots)$; dabei denkt man die komplexe Ebene längs der n Verbindungsstrecken von $-\dfrac{1}{2}\sqrt[n]{m}$ $(\sqrt[n]{m}>0)$ zu den n Punkten $\sqrt[n]{m}\,e^{i\left(\frac{2k+1}{n}\right)\pi}$ $(k=0,1,\ldots,n-1)$ aufgeschnitten, und es wird derjenige Zweig der Funktionen genommen, für den gilt: $f_m(z)>0$ für $z>0$. Dann gilt:*

a) *Die Funktionen* $f_m(z)$ $(m=1, 2, \ldots)$ *sind auf jedem Sektor* $\sum\left(\frac{\omega}{n}\right) = \left\{z\colon |\arg z| \leqq \frac{\omega}{n}\right\}$ $(0<\omega<\pi)$ *analytisch.*

b) *Die Funktionen* $f_m(z)$ $(m=1, 2, \ldots)$ *sind im Unendlichen analytisch und haben dort eine Nullstelle erster Ordnung.*

c) *Die Funktionen* $f_m(z)$ *und* $g_m(z)=z f_m(z)$ *bilden jeden Sektor* $\sum\left(\frac{\omega}{n}\right)$ $(0<\omega<\pi)$ *in sich ab.*

Beweis. a) ist klar.

b) folgt aus der Reihenentwicklung um den Punkt $z=\infty$:

$$f_m(z) = m^{\frac{1}{n}} \frac{1}{z} \left(1 - \frac{m}{nz^n} + \frac{\frac{1}{n}\left(\frac{1}{n}+1\right)}{2!} \frac{m^2}{z^{2n}} - + \cdots\right) \qquad \text{(vgl. [2]).}$$

c) Es sei $0<\arg z \leqq \frac{\omega}{n}$ $(0<\omega<\pi)$. Dann erhält man für die Funktionen $f_m(z)$ $(m=1, 2, \ldots)$ die Beziehung

$$\arg f_m(z) = \frac{1}{n} \arg\left(\frac{m}{m+z^n}\right) = -\frac{1}{n} \arg(m+z^n) \geqq -\frac{1}{n} \arg z^n \geqq -\frac{\omega}{n}.$$

Außerdem ist wegen $\arg z \geqq 0$

$$\arg f_m(z) = -\frac{1}{n} \arg(m+z^n) \leqq -\frac{1}{n} \arg m = 0,$$

woraus sich die Aussage

$$0 \geqq \arg f_m(z) \geqq -\frac{\omega}{n} \quad \text{für} \quad 0 \leqq \arg z \leqq \frac{\omega}{n}$$

ergibt. In analoger Weise zeigt man ohne Schwierigkeiten die Beziehungen

$$0 \leqq \arg f_m(z) \leqq \frac{\omega}{n} \quad \text{für} \quad 0 \geqq \arg z \geqq -\frac{\omega}{n}$$

und

$$-\frac{\omega}{n} \leqq \arg g_m(z) \leqq \frac{\omega}{n} \quad \text{für} \quad -\frac{\omega}{n} \leqq \arg z \leqq \frac{\omega}{n}.$$

6. Satz 4. *Es seien A ein Operator vom Typ (M) und n* $(n \geqq 2)$ *eine beliebige natürliche Zahl. Dann existiert genau ein abgeschlossener Operator B mit den Eigenschaften*

(1) $$B^n = A$$

und

(2) $$\sigma(B) \subset \sum\left(\frac{\pi}{n}\right) = \left\{z\colon |\arg z| \leqq \frac{\pi}{n}\right\},$$

also ist notwendigerweise $B = A^{\frac{1}{n}}$.

Beweis. Die Existenz eines solchen Operators B, nämlich $B=A^{\frac{1}{n}}$, erhält man unmittelbar aus den Sätzen 1 und 2. Der Satz ist folglich bewiesen, wenn wir zeigen, daß für jeden abgeschlossenen Operator B, mit den Eigenschaften:

(1') der Operator B^n ist vom Typ (M)

(2) $\sigma(B)\subset \sum\left(\frac{n}{\pi}\right)=\left\{z\colon |\arg z|\leqq \frac{\pi}{n}\right\}$,

gilt:

$$(B^n)^{\frac{1}{n}}=B.$$

Der Beweis für diese Beziehung erfolgt in drei Schritten.

I. Der Operator B erfülle neben den Voraussetzungen (1') und (2) folgende zusätzliche Bedingungen:

(3) B sei beschränkt und

(4) $0\in\varrho(B)$.

Nach Lemma 5a) erhält man $\sigma(B^n)\subset\{z\colon |\arg z|\leqq \pi-\gamma\}$ $\left(\gamma=\operatorname{Arc}\tan\frac{1}{M}\right)$. Wegen $0\in\varrho(B)$ gilt $0\in\varrho(B^n)$; die Funktion $f(z)=z^{1/n}$ ist folglich auf $\sigma(B^n)$ analytisch. Mit Hilfe des Riesz—Dunfordschen Funktionalkalküls kann man deshalb der Funktion $f(z)=z^{1/n}$ einen Operator $f(B^n)=(B^n)^{1/n}$ [9]) zuordnen. Da der Operator B die Voraussetzung von Lemma 5b) erfüllt, gilt $\sigma(B)\subset\sum\left(\frac{\pi-\gamma}{n}\right)$ $\left(\gamma=\operatorname{Arc}\tan\frac{1}{M}\right)$. Wegen der Voraussetzung (4) läßt sich eine Umgebung U des Spektrums $\sigma(B)$ finden, die den Nullpunkt nicht enthält. Auf U ist $(z^n)^{1/n}=z$. Unter den zusätzlichen Voraussetzungen (3) und (4) gilt also $B=(B^n)^{1/n}$.

II. Der Operator B erfülle neben den Bedingungen (1') und (2) die Voraussetzung (3). Dann genügt $B_\varepsilon=B+\varepsilon I$ $(\varepsilon>0)$ den Bedingungen (1'), (2) (3) und (4). Zuerst ist es klar, daß $\sigma(B_\varepsilon)=\sigma(B)+\varepsilon$, also $\sigma(B_\varepsilon)\subset\sum\left(\frac{\gamma}{n}\right)+\varepsilon\subset\sum\left(\frac{\gamma}{n}\right)$. Außerdem ist $\sigma\{(B_\varepsilon)^n\}=\{\sigma(B_\varepsilon)\}^n$ und folglich ist $(B_\varepsilon)^n$ ein Operator vom Typ (M') mit $M'=1+\|(B_\varepsilon)^n\|\sup\limits_{0\leqq\eta<\infty}\|R(-\eta;(B_\varepsilon)^n)\|$. Entsprechend dem Teil I erhält man also $[(B_\varepsilon)^n]^{1/n}=B_\varepsilon$.

Nach Voraussetzung (1') ist B^n vom Typ (M). Folglich existiert der Operator $(B^n)^{1/n}$, und es folgt auf Grund der Ungleichung (vgl. [9])

$$\|[(B_\varepsilon)^n]^{\frac{1}{n}}-(B^n)^{\frac{1}{n}}\|\leqq\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\frac{2^{1-\frac{1}{n}}}{1-\frac{1}{n}}\tilde{M}^{1+\alpha}\|(B_\varepsilon)^n-B^n\|^{\frac{1}{n}}\leqq$$

$$\leqq\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\frac{2^{1-\frac{1}{n}}}{1-\frac{1}{n}}\tilde{M}^{1+\alpha}\sum_{k=1}^{n}\binom{n}{k}\varepsilon^k\|B\|^{n-k}$$

(wobei $(\tilde{M}=\max(M,M'))$, die Beziehung $[B^n]^{1/n}=B$.

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9) Die Identität von $f(B^n)$ und der $\frac{1}{n}$-ten Potenz von B^n ergibt sich mit Hilfe einer einfachen Umformung des Integrals $-\frac{1}{2\pi i}\oint\limits_C z^{1/n}R(z;B^n)\,dz$ bei geeigneter Wahl des Integrationsweges C (vgl. [9]).

III. Der Operator B genüge den Bedingungen (1′) und (2). Nach Lemma 5b) ergibt sich dann $\sigma(B)\subset\sum\left(\frac{\pi-\gamma}{n}\right)$ $\left(\gamma=\operatorname{Arc}\tan\frac{1}{M}\right)$. Die in Lemma 7 eingeführten Funktionen $f_m(z)$ und $g_m(z)=zf_m(z)$ $(m=1, 2, \ldots)$ sind folglich auf $\sigma(B)$ und einer Umgebung von $z=\infty$ analytisch. Entsprechend dem Riesz—Dunford—Taylorschen Funktionalkalkül für einen abgeschlossenen Operator B mit nicht leerer Resolventenmenge $\varrho(B)$ existieren den Funktionen $f_m(z)$ und $g_m(z)$ $(m=1, 2, \ldots)$ zugeordnete Operatoren $f_m(B)$ und $g_m(B)=Bf_m(B)$ mit folgenden Eigenschaften (vgl. [3]):

a) $[f_m(B)]^n = [f_m]^n(B) = m(mI+B^n)^{-1} \qquad (m=1, 2, \ldots),$

b) $\sigma(f_m(B)) = f_m(\sigma(B)\cup\{z=\infty\}),$

c₁) $g_m(B)x = Bf_m(B)x = f_m(B)Bx \qquad (x\in\mathfrak{D}(B);\ m=1, 2, \ldots),$

c₂) $\sigma(g_m(B)) = g_m(\sigma(B)\cup\{z=\infty\}),$

c₃) $[g_m(B)]^n=[g_m]^n(B) = B^n m(mI+B^n)^{-1}.$

Wir zeigen nun zunächst mit Hilfe der Aussagen c_2) und c_3), daß die beschränkten Operatoren $g_m(B)$ die Bedingungen (1′) und (2) erfüllen, um das Ergebnis des Teiles II auf diese Operatoren anwenden zu können. Auf Grund des Spektralabbildungssatzes c_2) und der Voraussetzungen (1′) und (2) für B ist entsprechend der Lemmata 5b) und 7c)

$$\sigma(g_m(B)) = g_m(\sigma(B)\cup\{z=\infty\})\subset$$

$$\subset g_m\left(\sum\left(\frac{\gamma}{n}\right)\cup\{z=\infty\}\right)\subset\sum\left(\frac{\gamma}{n}\right)\subset\sum\left(\frac{\pi}{n}\right) \qquad \left(0<\gamma=\operatorname{Arc}\tan\frac{1}{M}\right).$$

Wegen der Bedingung (1′), die wir an den Operator B stellen, sind die Operatoren $[g_m(B)]^n=[g_m^n(B)]=B^n m(mI+B^n)^{-1}$ für alle $m=1, 2, \ldots$ vom Typ $(2M+1)$ (vgl. Beweis zum Satz 2; Abschnitt 3b). Die Aussage des Teiles II ergibt dann angewandt auf die Operatoren $g_m(B)$ $(m=1, 2, \ldots)$

$$([g_m(B)]^n)^{\frac{1}{n}} = g_m(B) \quad \text{oder} \quad [mB^n(mI+B^n)^{-1}]^{\frac{1}{n}} = Bf_m(B).$$

Da nach Voraussetzung (1′) für B der Operator B^n vom Typ (M) ist, gilt (vgl. Beweis zum Satz 2; Abschnitt 3)

$$\lim_{m\to\infty}[mB^n(mI+B^n)^{-1}]x = (B^n)^{\frac{1}{n}}x \qquad (x\in\mathfrak{D}(B^n)).$$

Zum Beweis der Aussage $(B^n)^{1/n}=B$ bleibt zu zeigen, daß für jedes $x\in\mathfrak{D}(B^n)$ die Beziehung $\lim Bf_m(B)x=Bx$ $(m\to\infty)$ besteht. Die beschränkten Operatoren $f_m(B)$ $(m=1, 2, \ldots)$ genügen aber den Bedingungen (1′), (2) und (3), denn es ist für alle $m=1, 2, \ldots$

$$\|[(f_m(B))^n+\eta I]^{-1}\| = \|[m(mI+B^n)^{-1}+\eta I]^{-1}\| =$$

$$= \frac{1}{\eta}\left\|(mI+B^n)\left(\frac{m+m\eta}{\eta}I+B^n\right)^{-1}\right\| \leqq \frac{M}{1+\eta}+\frac{M+1}{\eta} \leqq \frac{2M+1}{\eta} \qquad (\eta>0)$$

und wegen der Lemmata 5b) und 7c)

$$\sigma(f_m(B)) \subset f_m(\sigma(B) \cup \{z = \infty\}) \subset f_m\left(\Sigma\left(\frac{\pi-\gamma}{n}\right)\right) \subset \Sigma\left(\frac{\pi-\gamma}{n}\right)$$

$$\left(0 < \gamma = \operatorname{Arc}\tan\frac{1}{M}\right).$$

Man bekommt also entsprechend der Aussage von Teil II

$$[(f_m(B))^n]^{\frac{1}{n}} = f_m(B).$$

Die Operatoren $f_m(B)$ und B sind auf $\mathfrak{D}(B)$ vertauschbar (vgl. c_1), so daß sich nach Lemma 6 die Beziehung

$$\lim_{m\to\infty} Bf_m(B)x = \lim_{m\to\infty} f_m(B)Bx = \lim_{m\to\infty} [m(mI+B^n)^{-1}]^{\frac{1}{n}} x = Bx$$

für alle $x \in \mathfrak{D}(B)$ ergibt. Die abgeschlossenen Operatoren $(B^n)^{1/n}$ und B stimmen auf $\mathfrak{D}(B^n)$ überein, es gilt also entsprechend der Definition der $\frac{1}{n}$-ten Potenz von B^n

$$(B^n)^{\frac{1}{n}} = B,$$

was zu zeigen war.

Als einfache Folgerung des eben bewiesenen Sachverhaltes ergibt sich:

Satz 5. *Es seien A ein Operator vom Typ (M) und A_α $(0 < \alpha \leqq 1)$ eine Schar von abgeschlossenen Operatoren mit den folgenden Eigenschaften:*

1. *für* $0 \leqq \beta \leqq \alpha \leqq 1$ *ist* $\mathfrak{D}(A_\beta) \supset \mathfrak{D}(A_\alpha)$;
2. *die Schar* A_α *ist linksseitig stetig auf* $\mathfrak{D}(A_\alpha)$, *d. h.*

$$\lim_{\beta \nearrow \alpha} A_\beta x = A_\alpha x \qquad (x \in \mathfrak{D}(A_\alpha));$$

3. *für alle* $n = 1, 2, \ldots$ *gilt* $\sigma(A_{1/n}) \subset \left\{z\colon |\arg z| \leqq \frac{\pi}{n}\right\}$;
4. *für* $p = 1, 2, \ldots, n$ *ist* $(A_{1/n})^p = A_{p/n}$ *und* $A_1 = A$.

Dann gilt für $0 < \alpha \leqq 1$

$$A_\alpha = A^\alpha.$$

Beweis. Auf Grund der Eigenschaften 3 und 4 erhält man nach Satz 4

$$A_{\frac{1}{n}} = A^{\frac{1}{n}}.$$

Wegen der Gültigkeit des Potenzgesetzes (Satz 1) liefert die Voraussetzung 4 für $p = 1, 2, \ldots, n$

$$(A_{\frac{p}{n}}) = (A_{\frac{1}{n}})^p = (A^{\frac{1}{n}})^p = A^{\frac{p}{n}},$$

d. h., für alle rationalen α $(0<\alpha\leqq 1)$ ist $A_\alpha=A^\alpha$. Mit Hilfe der linksseitigen Stetigkeit von A_α (Voraussetzung 2) und A^α (vgl. Folgerung 2) bzgl. α ergibt sich also

$$A_\alpha=A^\alpha,$$

was zu beweisen war.

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(Eingegangen am 5. September 1966)

Über die Größenordnung der Partialsummen der Entwicklung integrierbarer Funktionen nach W-Systemen

Von FERENC SCHIPP in Budapest

Einleitung

Es sei $\{\varphi_n(t)\}$ $(n=0, 1, 2, \ldots)$ ein im Intervall $[0, 1]$ definiertes Funktionensystem mit

(1) $$|\varphi_n(t)| \equiv 1 \qquad (t \in [0, 1];\ n=0, 1, 2, \ldots),$$

ferner sei $\{\psi_n(t)\}$ $(n=0, 1, 2, \ldots)$ das von $\{\varphi_n(t)\}$ erzeugte *W*-System. D. h., es ist $\psi_0(t) \equiv 1$, und für

(2) $$n = 2^{\nu_1} + 2^{\nu_2} + \ldots + 2^{\nu_s} \qquad (\nu_1 > \nu_2 > \ldots > \nu_s \geqq 0)$$

ist

(3) $$\psi_n(t) = \varphi_{\nu_1}(t)\varphi_{\nu_2}(t) \ldots \varphi_{\nu_s}(t).$$

Wir nehmen an, daß das System $\{\varphi_n(t)\}$ stark-multiplikativ orthogonal ist, d. h. daß das System $\{\psi_n(t)\}$ orthogonal ist. [1])

Wir bezeichnen mit

(4) $$S_n(t; f) = \sum_{\nu=0}^{n-1} \psi_\nu(t) \left(\int_0^1 f(u) \psi_\nu(u)\, du \right)$$

die n-te Partialsumme der nach dem orthogonalen System $\{\psi_n(t)\}$ fortschreitenden Fourier-Entwicklung von $f(t)$.

In dieser Arbeit werden wir den folgenden Satz beweisen.

Satz. *Es sei* $\{\lambda_n\}$ $(n=1, 2, 3, \ldots)$ *eine positive, monoton nicht abnehmende Zahlenfolge mit* $\lim_{n\to\infty} \lambda_n = \infty$, $\lambda_n = o(\log \log n)$. *Dann existiert eine Funktion* $f(t) \in L[0, 1]$ *derart, daß überall im Intervall* $[0, 1]$

(5) $$\limsup_{n\to\infty} \frac{|S_n(t; f)|}{\lambda_n} > 0$$

besteht.

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[1]) Siehe ALEXITS [1], S. 165.

Unser Satz gibt eine Verschärfung eines Satzes von E. STEIN ([5], Satz 7.), nach dem eine Funktion $f(t) \in L[0, 1]$ existiert, die eine fast überall divergente Walsh-Entwicklung besitzt.

Durch Anwendung der Beispiele von KOLMOGOROFF ([3], [4]) hat YUNG-MING CHEN [6] einen analogen Satz für Fourierreihen bewiesen.

Zuerst beweisen wir das Analogon für $\{\psi_n(t)\}$ des grundlegenden Lemmas in der Kolmogoroffschen Konstruktion.

Lemma. *Für jede Zahl* $k(=1, 2, 3, \ldots)$ *kann man ein W-Polynom*

$$P_k(t) = \sum_{\nu=0}^{2^{2^{2k+1}}-1} a_\nu^{(k)} \psi_\nu(t)$$

und eine Folge $\{H_n^{(2k-1)}\}$ $(n=0, 1, 2, \ldots, 2^{2k}-1)$ *von meßbaren Mengen mit den folgenden Eigenschaften angeben:*

a) $H_n^{(2k-1)} \cap H_{n'}^{(2k-1)} = \emptyset \quad (n \neq n'), \qquad \bigcup_{n=0}^{2^{2k}-1} H_n^{(2k-1)} = [0, 1];$

b) $P_k(t) \geqq 0 \qquad (t \in [0, 1]);$

c) $\int_0^1 P_k(t)\, dt = 1;$

d) *für* $n = 0, 1, 2, \ldots, 2^{2k}-1$ *und* $t \in H_n^{(2k-1)}$ *gilt* $|S_\mu(t; P_k)| \geqq \frac{k}{3} - 1,$

wobei $\mu = 2^{2^{2k}+n} + \sum_{\nu=0}^{k} 2^{2\nu}$ *gesetzt wird.*

§ 1. Bezeichnungen

Es sei $\{r_n(x)\}$ $(n=0, 1, 2, \ldots)$ das Rademachersche System, d. h.

(1.1)
$$r_0(x) = \begin{cases} 1 & (0 \leqq x < \frac{1}{2}), \\ -1 & (\frac{1}{2} \leqq x < 1), \end{cases} \quad r_0(x+1) = r_0(x),$$
$$r_n(x) = r_0(2^n x) \qquad (n = 1, 2, 3, \ldots).$$

Es ist bekannt, daß die Funktionen $r_n(x)$ im Interwall [0, 1] stark-multiplikativ orthogonal sind und das von ihnen erzeugte W-System das bekannte Walshsche Orthogonalsystem ist, dessen Elemente wir mit $W_n(x)$ $(n=0, 1, 2, \ldots)$ bezeichnen.

Es sei $x = 0, x_1 x_2 \ldots x_n x_{n+1} \ldots$ die dyadische Entwicklung der Zahl $x \in [0, 1)$, wobei wir festsetzen, daß für $x = p2^{-q}$ alle x_i $(i \geqq q+1)$ gleich 0 sind. Dann gilt

(1.2)
$$r_n(x) = (-1)^{x_{n+1}}.$$

Im Folgenden bedeutet R die Menge der dyadisch rationalen Zahlen in [0, 1). FINE [2] hatte die folgende Operation in R eingeführt: es sei

(1.3)
$$x \dot{+} y = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n}$$

für

$$x=0, x_1 x_2 \dots x_n \dots = \sum_{n=1}^{\infty} \frac{x_n}{2^n}, \quad y=0, y_1 y_2 \dots y_n \dots = \sum_{n=1}^{\infty} \frac{y_n}{2^n}.$$

Es ist leicht zu verifizieren, daß R mit dieser Operation eine Abelsche Gruppe ist, und daß die Relationen

(1. 4) $$x \dotplus x = 0, \quad W_n(x \dotplus y) = W_n(x) W_n(y) \qquad (x, y \in R;\ n=0, 1, 2, \dots)$$

gelten.

Wir bezeichnen mit G die Menge der nichtnegativen ganzen Zahlen. In der Menge G führen wir eine Operation $\oplus$ folgenderweise ein:

(1. 5) $$k \oplus l = \sum_{\nu=0}^{\infty} |k_\nu - l_\nu| 2^\nu$$

für

$$k = \sum_{\nu=0}^{\infty} k_\nu 2^\nu, \quad l = \sum_{\nu=0}^{\infty} l_\nu 2^\nu \qquad (k_\nu, l_\nu = 0, 1).$$

Offensichtlich wird G mit der Operation $\oplus$ zu einer Abelschen Gruppe und die nichtnegativen ganzen Zahlen, die kleiner als 2^n sind, bilden eine Untergruppe G_n von G von der Ordnung 2^n, wobei noch

(1. 6) $$\frac{k}{2^n} \dotplus \frac{l}{2^n} = \frac{k \oplus l}{2^n}$$

besteht.

§ 2. Hilfssätze

Zur Konstruktion der Funktion $P_k(t)$ werden wir einige Hilfssätze benutzen.

Hilfssatz I. *Es sei* $n = 2^{\nu_1} + 2^{\nu_2} + \dots + 2^{\nu_s}$ $(\nu_1 > \nu_2 > \dots > \nu_s \geqq 0)$ *und*

(2. 1) $$D_n(t; x) = \sum_{\nu=0}^{n-1} \psi_\nu(t) W_\nu(x).$$

Dann gilt

(2. 2) $$D_n(t; x) = \psi_n(t) W_n(x) \sum_{j=1}^{s} \varphi_{\nu_j}(t) r_{\nu_j}(x) D_{2^{\nu_j}}(t; x).$$

Beweis von Hilfssatz I. Es sei $m = 2^k + m'$ mit $0 < m' \leqq 2^k$. Wenn $\nu < 2^k$ ist, dann folgt aus (3) $\psi_{2^k+\nu}(t) W_{2^k+\nu}(x) = \varphi_k(t) r_k(x) \psi_\nu(t) W_\nu(x)$, und so gilt

$$D_m(t; x) = \sum_{\nu=0}^{m-1} \psi_\nu(t) W_\nu(x) =$$

(2. 3) $$= \sum_{\nu=0}^{2^k-1} \psi_\nu(t) W_\nu(x) + \sum_{\nu=0}^{m'-1} \psi_{2^k+\nu}(t) W_{2^k+\nu}(x) =$$

$$= D_{2^k}(t; x) + \varphi_k(t) r_k(x) \sum_{\nu=0}^{m'-1} \psi_\nu(t) W_\nu(x) = D_{2^k}(t; x) + \varphi_k(t) r_k(x) D_{m'}(t; x).$$

Speziell für $m'=2^k$ ergibt sich

$$D_{2^{k+1}}(t;x)=\big(1+\varphi_k(t)r_k(x)\big)D_{2^k}(t;x)=\prod_{\nu=0}^{k}\big(1+\varphi_\nu(t)r_\nu(x)\big)\geqq 0$$

(2. 4)

$$(k=0,1,2,\ldots;\quad t\in[0,1];\quad x\in R).$$

Mit der Benutzung dieser Formeln zeigen wir (2. 2) mit Induktion. Ist $n=2^k$ $(k=0,1,2,\ldots)$, dann folgt (2. 2) auf Grund von (1):

$$D_{2^k}(t;x)=\psi_{2^k}(t)W_{2^k}(x)\{\varphi_k(t)r_k(x)D_{2^k}(t;x)\}.$$

Wir nehmen an, daß (2. 2) schon für alle Zahlen $n'\leqq 2^k$ bewiesen ist. Wir zeigen, daß sie auch für n $(2^k<n<2^{k+1})$ besteht. Es sei $n=2^k+n'$. Ist $n'=2^{\nu_1}+2^{\nu_2}+\ldots+2^{\nu_s}$, $(k>\nu_1>\ldots>\nu_s\geqq 0)$, dann ergibt sich aus (2. 3) und der Induktionsannahme

$$D_n(t;x)=D_{2^k}(t;x)+\varphi_k(t)r_k(x)D_{n'}(t;x)=$$

$$=D_{2^k}(t;x)+\varphi_k(t)r_k(x)\psi_{n'}(t)W_{n'}(x)\left\{\sum_{j=1}^{s}\varphi_{\nu_j}(t)r_{\nu_j}(x)D_{2^{\nu_j}}(t;x)\right\}.$$

Da für $\nu<k$

$$\varphi_\nu(t)r_\nu(x)\prod_{j=0}^{k-1}\big(1+\varphi_j(t)r_j(x)\big)=\sum_{j=0}^{k-1}\big(1+\varphi_j(t)r_j(x)\big)=D_{2^k}(t;x)$$

gilt, erhalten wir schließlich

$$D_n(t;x)=$$

$$=\varphi_k(t)r_k(x)\psi_{n'}(t)W_{n'}(x)\left\{\varphi_k(t)r_k(x)D_{2^k}(t;x)+\sum_{j=1}^{s}\varphi_{\nu_j}(t)r_{\nu_j}(x)D_{2^{\nu_j}}(t;x)\right\}=$$

$$=\psi_n(t)W_n(x)\{\varphi_k(t)r_k(x)D_{2^k}(t;x)+\varphi_{\nu_1}(t)r_{\nu_1}(x)D_{2^{\nu_1}}(t;x)+\ldots+$$

$$+\varphi_{\nu_s}(t)r_{\nu_s}(x)D_{2^{\nu_s}}(t;x)\},$$

womit (2. 2) bewiesen ist.

Hilfssatz II. *Es seien*

(2. 5) $$E_\nu^{(0)}=\{t\colon t\in[0,1],\ \varphi_\nu(t)=1\},\ E_\nu^{(1)}=\{t\colon t\in[0,1],\ \varphi_\nu(t)=-1\}$$
$$(\nu=0,1,2,\ldots).$$

Für $0\leqq n<2^{k+1}$ *und* $n=\sum_{\nu=0}^{k}n_\nu 2^{k-\nu}$ $(n_\nu=0,1)$ *sei*

(2. 6) $$H_n^{(k)}=\bigcap_{\nu=0}^{k}E_\nu^{(n_\nu)}\quad(n=0,1,\ldots 2^{k+1}-1;\quad k=0,1,\ldots).$$

Dann bestehen

(2. 7) $$H_n^{(k)}\cap H_{n'}^{(k)}=\emptyset\quad(n\neq n'),\qquad\bigcup_{n=0}^{2^{k+1}-1}H_n^{(k)}=[0,1],$$

und

(2. 8) $$r_s\left(\frac{n}{2^{k+1}}\right)\varphi_s(t)D_{2^s}\left(t;\frac{n}{2^{k+1}}\right)=2^s\qquad(t\in H_n^{(k)};\quad 0\leqq s<k+1).$$

Beweis von Hilfssatz II. Die Relationen (2. 7) ergeben sich leicht aus $E_\nu^{(0)} \cup E_\nu^{(1)} = [0, 1]$ und $E_\nu^{(0)} \cap E_\nu^{(1)} = \emptyset$.

Zum Beweis von (2. 8) führen wir die Zahlen

$$N(j; n) = \sum_{\nu=0}^{j} n_\nu 2^{j-\nu} \qquad (j=0, 1, 2, \ldots, k)$$

ein. Dann besteht offenbar $N(k; n) = n$, und

$$\frac{n}{2^{k+1}} = \sum_{\nu=0}^{k} \frac{n_\nu}{2^{\nu+1}}, \qquad \frac{N(j; n)}{2^{j+1}} = \sum_{\nu=0}^{j} \frac{n_\nu}{2^{\nu+1}}.$$

Daraus auf Grund von (1. 2), (2. 4), (2. 5) und (2. 6) folgt, daß für $0 \leqq j \leqq k$

$$D_{2^{j+1}}\left(t; \frac{n}{2^{k+1}}\right) = D_{2^{j+1}}\left(t; \frac{N(j; n)}{2^{j+1}}\right) =$$

(2. 9)

$$= \prod_{l=0}^{j} (1 + (-1)^{n_l} \varphi_l(t)) = \begin{cases} 2^{j+1}, & \text{wenn} \quad t \in H_{N(j;n)}^{(j)}, \\ 0, & \text{wenn} \quad t \notin H_{N(j;n)}^{(j)} \end{cases}$$

gilt, [2]) ferner

$$H_{N(j;n)}^{(j)} = \bigcap_{\nu=0}^{j} E_\nu^{(n_\nu)} = E_j^{(n_j)} \cap \left(\bigcap_{\nu=0}^{j-1} E_\nu^{(n_\nu)}\right) = E_j^{(n_j)} \cap H_{N(j-1;n)}^{(j-1)},$$

also $H_{N(j;n)}^{(j)} \subset H_{N(j-1;n)}^{(j-1)}$ $(j=1, 2, \ldots, k)$. Beachten wir, daß nach (1. 2) und (2. 4) für $t \in E_j^{(n_j)}$ $r_j\left(\frac{n}{2^{k+1}}\right) \varphi_j(t) = (-1)^{n_j} \varphi_j(t) = 1$ gilt und $E_j^{(n_j)} \supset H_n^{(k)}$ $(j=0, 1, \ldots, k)$, so bekommen wir aus (2. 9) für $j+1=s$ die zu bewiesende Gleichung (2. 8).

Hilfssatz III. *Es sei*

(2. 10) $$A_n^{(k)} = \frac{n}{2^{2k}} + \frac{\delta_n^{(k)}}{2^{2k+1}} \qquad (n=0, 1, 2, \ldots, 2^{2k}-1; \quad k=1, 2, 3, \ldots),$$

wobei die Folge $\delta_n^{(k)}$ $(=0, 1)$ *durch die unteren rekursiven Formeln definiert wird:*

(2. 11) $$\begin{cases} \delta_0^{(1)} = 1, \quad \delta_1^{(1)} = \delta_2^{(1)} = \delta_3^{(1)} = 0; \\ \delta_0^{(k+1)} = 1, \quad \delta_1^{(k+1)} = \delta_2^{(k+1)} = \delta_3^{(k+1)} = 0; \\ \delta_{4j}^{(k+1)} = \delta_{4j+1}^{(k+1)} = \delta_j^{(k)}; \\ \delta_{4j+2}^{(k+1)} = \delta_{4j+3}^{(k+1)} = \delta_j^{(k)} \oplus 1 \end{cases}$$

$$(j=1, 2, 3, \ldots, 2^{2k}-1; \quad k=1, 2, 3, \ldots).$$

Es seien weiterhin

(2. 12) $$m_k = \sum_{\nu=0}^{k} 2^{2\nu} \qquad (k=0, 1, 2, \ldots)$$

---

[2]) Es folgt aus (2. 9) für $j=k$, wenn man noch $\int_0^1 D_{2^{k+1}}\left(t; \frac{n}{2^{k+1}}\right) dt = 1$ beachtet, daß $|H_n^{(k)}| = 2^{-(k+1)}$, wobei $|H_n^{(k)}|$ das Lebesguesche Maß von $H_n^{(k)}$ bedeutet.

und

$$U_k(t;x) = \sum_{n=0}^{2^{2k}-1} \varphi_{2k}(t) r_{2k}(x \dotplus A_n^{(k)}) D_{m_{k-1}}(t; x \dotplus A_n^{(k)}),$$

(2. 13)

$$V_k(t;x) = \frac{2}{3} \psi_{m_k}(t) W_{m_k}(x) \left\{ \sum_{v=0}^{k} (2^{2v}-1) \varphi_{2(k-v)}(t) r_{2(k-v)}(x) D_{2^{2(k-v)}}(t;x) \right\}$$

$$(x \in R; \quad t \in [0,1]; \quad k=1,2,3,\ldots).$$

Dann besteht

(2. 14) $$U_k(t;x) = V_k(t;x) \qquad (k=1,2,3,\ldots;\ x\in R;\ t\in[0,1]).$$

Beweis von Hilfssatz III. Der Beweis wird durch Induktion in Bezug auf k geführt. Aus den Definitionen der $A_n^{(k)}$ und m_k folgt mit Hilfe von (1. 4)

$$U_1(t;x) = \sum_{n=0}^{3} \varphi_2(t) r_2(x \dotplus A_n^{(1)}) =$$

$$= \varphi_2(t) r_2(x) \left\{ r_2\left(\frac{1}{2^3}\right) + r_2\left(\frac{1}{2^2}\right) + r_2\left(\frac{1}{2}+\frac{1}{2^2}\right) + r_2\left(\frac{1}{2}\right) \right\} = 2\varphi_2(t) r_2(x),$$

$$V_1(t;x) = \frac{2}{3} \varphi_2(t) \varphi_0(t) r_2(x) r_0(x) \{(2^2-1)\varphi_0(t) r_0(x)\} = 2\varphi_2(t) r_2(x),$$

der Satz ist also für $k=1$ richtig.

Wir nehmen an, daß für einen Index $k(\geqq 1)$ $U_k(t;x) = V_k(t;x)$ $(x\in R,\ t\in[0,1])$ gilt. Mittels (1. 2) und (2. 11) ergibt sich durch einfacher Rechnung, daß immer, wenn nicht $i=1, j=0$ sind, die Gleichung

(2. 15)

$$r_{2k+2}(A_{4j+i}^{(k+1)}) r_{2k}(A_{4j+i}^{(k+1)}) = r_{2k}(A_j^{(k)})$$

$$(j=0,1,2,\ldots,2^{2k}-1; \quad k=1,2,3,\ldots; \quad i=0,1,2,3)$$

besteht und im Falle $i=1, j=0$

(2. 16) $$r_{2k+2}(A_1^{(k+1)}) r_{2k}(A_1^{(k+1)}) = -r_{2k}(A_0^{(k)}) \qquad (k=1,2,3,\ldots)$$

gilt, weiterhin für alle natürlichen Zahlen $s(<2k)$

$$r_s(A_{4j+i}^{(k+1)}) = r_s(A_j^{(k)}) \qquad (j=0,1,2,\ldots,2^{2k}-1; \quad k=1,2,3,\ldots; \quad i=0,1,2,3).$$

Aus dieser letzten Gleichung folgt, daß im Falle $l \leqq 2^{2k}$

(2. 17)

$$D_l(t; x \dotplus A_{4j+i}^{(k+1)}) = D_l(t; x \dotplus A_j^{(k)})$$

$$(j=0,1,2,\ldots,2^{2k}-1; \quad k=1,2,3,\ldots; \quad i=0,1,2,3)$$

ist. Mit Hilfe von (1. 4), (2. 3) und (2. 15) ergibt sich — wenn nicht $i=1, j=0$ sind — die folgende Gleichung:

$$r_{2k+2}(x \dotplus A_{4j+i}^{(k+1)}) D_{m_k}(t; x \dotplus A_{4j+i}^{(k+1)}) = r_{2k+2}(x) r_{2k+2}(A_{4j+i}^{(k+1)}) \cdot$$

$$\cdot \{D_{2^{2k}}(t; x \dotplus A_{4j+i}^{(k+1)}) + \varphi_{2k}(t) r_{2k}(x \dotplus A_{4j+i}^{(k+1)}) D_{m_{k-1}}(t; x \dotplus A_{4j+i}^{(k+1)})\} =$$

$$= r_{2k+2}(x) \{r_{2k+2}(A_{4j+i}^{(k+1)}) D_{2^{2k}}(t; x \dotplus A_j^{(k)}) + + \varphi_{2k}(t) r_{2k}(x \dotplus A_j^{(k)}) D_{m_{k-1}}(t; x \dotplus A_j^{(k)})\}$$

$$(j=0,1,2,\ldots,2^{2k}-1; \quad k=1,2,3,\ldots; \quad i=0,1,2,3),$$

woraus auf Grund der Gleichung

$$r_{2k+2}(A_{4j+i}^{(k+1)}) = r_{2k+2}\left(\frac{\delta_{4j+i}^{(k+1)}}{2^{2k+3}}\right) = \begin{cases} r_{2k}\left(\dfrac{\delta_j^{(k)}}{2^{2k+1}}\right) & (i=0, 1), \\ -r_{2k}\left(\dfrac{\delta_j^{(k)}}{2^{2k+1}}\right) & (i=2, 3) \end{cases}$$

die Beziehung

$$\sum_{i=0}^{3} \varphi_{2k+2}(t) r_{2k+2}(x \dot{+} A_{4j+i}^{(k+1)}) D_{m_k}(t; x \dot{+} A_{4j+i}^{(k+1)}) =$$

$$= 4\varphi_{2k+2}(t) r_{2k+2}(x) \{\varphi_{2k}(t) r_{2k}(x \dot{+} A_j^{(k)}) D_{m_{k-1}}(t; x \dot{+} A_j^{(k)})\}$$

folgt. Daraus ergibt sich, daß

$$U_{k+1}(t; x) = \sum_{j=0}^{2^{2k}-1} \sum_{i=0}^{3} \varphi_{2k+2}(t) r_{2k+2}(x \dot{+} A_{4j+i}^{(k+1)}) D_{m_k}(t; x \dot{+} A_{4j+i}^{(k+1)}) =$$

$$= 4\varphi_{2k+2}(t) r_{2k+2}(x) \sum_{j=0}^{2^{2k}-1} \varphi_{2k}(t) r_{2k}(x \dot{+} A_j^{(k)}) D_{m_{k-1}}(t; x \dot{+} A_j^{(k)}) -$$

$$- 4\varphi_{2k+2}(t) r_{2k+2}(x) \varphi_{2k}(t) r_{2k}(x \dot{+} A_0^{(k)}) D_{m_{k-1}}(t; x \dot{+} A_0^{(k)}) +$$

$$+ \sum_{i=0}^{3} \varphi_{2k+2}(t) r_{2k+2}(x \dot{+} A_i^{(k+1)}) D_{m_k}(t; x \dot{+} A_i^{(k+1)}) = s_1 + s_2 + s_3$$

besteht. Da auf Grund von (1. 4), (2. 3), (2. 10), (2. 11), (2. 16) und (2. 17)

$$s_3 = \varphi_{2k+2}(t) r_{2k+2}(x) \cdot$$

$$\cdot \sum_{i=0}^{3} r_{2k+2}(A_i^{(k+1)}) \cdot \{D_{2^{2k}}(t; x \dot{+} A_0^{(k)}) + \varphi_{2k}(t) r_{2k}(x \dot{+} A_i^{(k+1)}) D_{m_{k-1}}(t; x \dot{+} A_0^{(k)})\} =$$

$$= 2\varphi_{2k+2}(t) r_{2k+2}(x) \{D_{2^{2k}}(t; x \dot{+} A_0^{(k)}) + \varphi_{2k}(t) r_{2k}(x \dot{+} A_0^{(k)}) D_{m_{k-1}}(t; x \dot{+} A_0^{(k)})\}$$

gilt, bekommen wir aus (1. 4), (2. 3) und (2. 13), auf Grund der Induktionsannahme, daß

$$U_{k+1}(t; x) = 2\varphi_{2k+2}(t) r_{2k+2}(x) \cdot$$

$$\cdot \{2U_k(t; x) + D_{2^{2k}}(t; x \dot{+} A_0^{(k)}) - \varphi_{2k}(t) r_{2k}(x \dot{+} A_0^{(k)}) D_{m_{k-1}}(t; x \dot{+} A_0^{(k)})\} =$$

$$= 2\varphi_{2k+2}(t) r_{2k+2}(x) \{2V_k(t; x) + D_{2^{2k}}(t; x) + \varphi_{2k}(t) r_{2k}(x) D_{m_{k-1}}(t; x)\} =$$

$$= 2\varphi_{2k+2}(t) r_{2k+2}(x) \{2V_k(t; x) + D_{m_k}(t; x)\}$$

ist. Benutzt man die Gleichungen (2. 2) und (2. 13), so ergibt sich

$$U_{k+1}(t; x) =$$

$$= \frac{2}{3} \psi_{m_{k+1}}(t) W_{m_{k+1}}(x) \Big\{ \sum_{\nu=0}^{k} (2^{2\nu+2} - 4) \varphi_{2(k-\nu)}(t) r_{2(k-\nu)}(x) D_{2^{2(k-\nu)}}(t; x) +$$

$$+ 3 \sum_{\nu=0}^{k} \varphi_{2(k-\nu)}(t) r_{2(k-\nu)}(x) D_{2^{2(k-\nu)}}(t; x) \Big\} =$$

$$= \frac{2}{3} \psi_{m_{k+1}}(t) W_{m_{k+1}}(x) \left\{ \sum_{\nu=0}^{k+1} (2^{2\nu} - 1) \varphi_{2(k+1-\nu)}(t) r_{2(k+1-\nu)}(x) D_{2^{2(k+1-\nu)}}(t; x) \right\} =$$

$$= V_{k+1}(t; x).$$

Aus dem Hilfssatz II folgt noch, indem man (2. 13) berücksichtigt, daß für $0 \leqq n < 2^{2k}$ die folgende Abschätzung gilt:

$$\left|V_k\left(t;\ \frac{n}{2^{2k}}\right)\right| = \frac{2}{3}\left|\sum_{\nu=0}^{k-1}(2^{2(k-\nu)}-1)\,\varphi_{2\nu}(t)\,r_{2\nu}\left(\frac{n}{2^{2k}}\right)D_{2^{2\nu}}\left(t;\ \frac{n}{2^{2k}}\right)\right| =$$

(2. 18)

$$= \frac{2}{3}\sum_{\nu=0}^{k-1}(2^{2(k-\nu)}-1)\,2^{2\nu} > \frac{2}{3}\sum_{\nu=0}^{k-1}2^{2k-1} = \frac{k\cdot 2^{2k}}{3} \quad (t \in H_n^{(2k-1)}).$$

Hilfssatz IV. *Die Ziffern $\omega^{(\nu)}(n, k)$ der dyadisch rationalen Zahl*

$$B_n^{(k)} = \sum_{\nu=0}^{2^{2k}-1}\frac{\omega^{(\nu)}(n,k)}{2^{2^{2k}+\nu+1}} \quad (n=0, 1, 2, \ldots, 2^{2k}-1; \quad k=1, 2, 3, \ldots)$$

seien folgendermaßen definiert:

$$\text{(2. 19)} \qquad \omega^{(\nu)}(n,k) = \begin{cases} 1, & \textit{für} \quad r_{2k}(A_\nu^{(k)} \dotplus A_n^{(k)} \dotplus A_{n\oplus\nu}^{(k)}) = -1, \\ 0, & \textit{für} \quad r_{2k}(A_\nu^{(k)} \dotplus A_n^{(k)} \dotplus A_{n\oplus\nu}^{(k)}) = 1, \end{cases}$$

wobei $n \oplus \nu$ die in (1. 5) *definierte Summe der Zahlen n, $\nu(\in G_{2k})$ bezeichnet, ferner sei*

$$C_n^{(k)} = A_n^{(k)} \dotplus B_n^{(k)},$$

$$\text{(2. 20)} \qquad q_n^{(k)}(t;x) = r_{2^{2k}+n}(x)\,\varphi_{2^{2k}+n}(t)\,r_{2k}(x)\,\varphi_{2k}(t)\,D_{m_{k-1}}(t;x)$$

$$(n=0, 1, 2, \ldots, 2^{2k}-1; \quad k=1, 2, 3, \ldots).$$

Dann gilt

$$\text{(2. 21)} \qquad \sum_{\nu=0}^{2^{2k}-1} q_n^{(k)}(t; C_\nu^{(k)}) = \varphi_{2^{2k}+n}(t)\,r_{2k}(A_n^{(k)})\,V_k\left(t;\ \frac{n}{2^{2k}}\right).$$

Beweis von Hilfssatz IV. Aus der Definition der $B_n^{(k)}$ folgt

$$r_{2^{2k}+\nu}(B_n^{(k)}) = r_{2k}(A_n^{(k)} \dotplus A_\nu^{(k)} \dotplus A_{n\oplus\nu}^{(k)})$$

$$(n=0, 1, 2, \ldots, 2^{2k}-1; \quad \nu=0, 1, 2, \ldots, 2^{2k}-1; \quad k=1, 2, 3, \ldots),$$

woraus sich

$$\text{(2. 22)} \qquad r_{2^{2k}+n}(B_\nu^{(k)}) = r_{2k}(A_n^{(k)} \dotplus A_\nu^{(k)} \dotplus A_{n\oplus\nu}^{(k)})$$

mit aufwechseln der Indizes n und ν ergibt. Aus (2. 19) und (2. 20) schließen wir für $s < 2^{2k}$: $r_s(C_n^{(k)}) = r_s(A_n^{(k)})$, also

$$\text{(2. 23)} \qquad D_l(t; x \dotplus C_n^{(k)}) = D_l(t; x \dotplus A_n^{(k)}) \qquad (l \leqq 2^{2^{2k}}).$$

Auf Grund von (1. 6) ergibt sich

$$A_n^{(k)} \dotplus A_\nu^{(k)} = \left(\frac{n}{2^{2k}} \dotplus \frac{\delta_n^{(k)}}{2^{2k+1}}\right) \dotplus \left(\frac{\nu}{2^{2k}} \dotplus \frac{\delta_\nu^{(k)}}{2^{2k+1}}\right) =$$

$$= \left(\frac{n}{2^{2k}} \dotplus \frac{\nu}{2^{2k}} \dotplus \frac{\delta_{n\oplus\nu}^{(k)}}{2^{2k+1}}\right) \dotplus \left(\frac{\delta_{n\oplus\nu}^{(k)}}{2^{2k+1}} \dotplus \frac{\delta_n^{(k)}}{2^{2k+1}} \dotplus \frac{\delta_\nu^{(k)}}{2^{2k+1}}\right) =$$

$$A_{n\oplus\nu}^{(k)} \dotplus \left(\frac{\delta_{n\oplus\nu}^{(k)}}{2^{2k+1}} \dotplus \frac{\delta_n^{(k)}}{2^{2k+1}} \dotplus \frac{\delta_\nu^{(k)}}{2^{2k+1}}\right),$$

woraus $r_s(A_n^{(k)} \dotplus A_\nu^{(k)}) = r_s(A_{n\oplus\nu}^{(k)})$ $(s<2k)$ folgt. Daraus erhalten wir

(2. 24) $$D_l(t; x \dotplus A_n^{(k)} \dotplus A_\nu^{(k)}) \doteq D_l(t; x \dotplus A_{n\oplus\nu}^{(k)}) \qquad (l \leqq 2^{2k}).$$

Mit Berücksichtigung von $r_{2^{2k}+n}(A_j^{(k)})=1$, bekommen wir auf Grund von (2. 20), (2. 22), (2. 23) und (2. 24):

$$q_n^{(k)}(t; x \dotplus C_\nu^{(k)} \dotplus A_n^{(k)}) =$$

$$= r_{2^{2k}+n}(x)\varphi_{2^{2k}+n}(t) r_{2k}(x)\varphi_{2k}(t) r_{2^{2k}+n}(B_\nu^{(k)}) r_{2k}(A_\nu^{(k)} \dotplus A_n^{(k)}) D_{m_{k-1}}(t; x \dotplus A_{n\oplus\nu}^{(k)}) =$$

$$= r_{2^{2k}+n}(x)\varphi_{2^{2k}+n}(t) r_{2k}(x \dotplus A_{n\oplus\nu}^{(k)})\varphi_{2k}(t) D_{m_{k-1}}(t; x \dotplus A_{n\oplus\nu}^{(k)}).$$

Da mit ν auch $n\oplus\nu$ die Gruppe G_{2k} $(=0, 1, 2, \ldots, 2^{2k}-1)$ durchläuft, erhalten wir mit Berücksichtigung von (2. 13) und (2. 14)

$$\sum_{\nu=0}^{2^{2k}-1} q_n^{(k)}(t; x \dotplus C_\nu^{(k)} \dotplus A_n^{(k)}) =$$

$$= r_{2^{2k}+n}(x)\varphi_{2^{2k}+n}(t)\left\{\sum_{\nu=0}^{2^{2k}-1} r_{2k}(x \dotplus A_{n\oplus\nu}^{(k)})\varphi_{2k}(t) D_{m_{k-1}}(t; x \dotplus A_{n\oplus\nu}^{(k)})\right\} =$$

$$= r_{2^{2k}+n}(x)\varphi_{2^{2k}+n}(t) V_k(t; x).$$

Für $x=A_n^{(k)}$ ergibt sich (2. 21) auf Grund der Relationen $r_{2^{2k}+n}(A_n^{(k)})=1$ und

$$V_k(t; A_n^{(k)}) = V_k\left(t; \frac{n}{2^{2k}}\right) r_{2k}(A_n^{(k)}).$$

Damit haben wir den Hilfssatz IV bewiesen.

§ 3. Beweis des Lemmas

Es sei

(3. 1) $$P_k(t) = \frac{1}{2^{2k}} \sum_{\nu=0}^{2^{2k}-1} D_{2^{2^{2k}+1}}(t; C_\nu^{(k)}).$$

Dann bestehen a), b), c) auf Grund von (2. 4) und (2. 7), wobei $H_n^{(2k-1)}$ $(n=0, 1, 2, \ldots$ $\ldots, 2^{2k}-1)$ die Mengen bedeuten, die im Hilfssatz II und m_k durch (2. 12) definiert sind. Zum Beweis von d) beachten wir, daß auf Grund von (2. 3) und (2. 20)

$$S_{2^{2^{2k}}+n+m_k}(t; D_{2^{2^{2k}+1}}(t; x)) =$$

$$= D_{2^{2^{2k}}+n}(t; x) + r_{2^{2k}+n}(x)\varphi_{2^{2k}+n}(t) D_{2^{2k}}(t; x) + q_n^{(k)}(t; x)$$

gilt. Mit Berücksichtigung von (2. 21) und (3. 1) erhalten wir daraus:

(3. 2) $$S_{2^{2^{2k}}+n+m_k}(t; P_k) = \frac{1}{2^{2k}} \sum_{\nu=0}^{2^{2k}-1} D_{2^{2^{2k}}+n}(t; C_\nu^{(k)}) +$$
$$+ \frac{1}{2^{2k}}\varphi_{2^{2k}+n}(t) \sum_{\nu=0}^{2^{2k}-1} r_{2^{2k}+n}(C_\nu^{(k)}) D_{2^{2k}}(t; C_\nu^{(k)}) + \frac{1}{2^{2k}}\varphi_{2^{2k}+n}(t) r_{2k}(A_n^{(k)}) V_k\left(t; \frac{n}{2^{2k}}\right).$$

Wegen (2. 9) und (2. 23) gilt

$$D_{2^{2k}}(t;\, C_\nu^{(k)}) = D_{2^{2k}}(t;\, A_\nu^{(k)}) = D_{2^{2k}}\left(t;\, \frac{\nu}{2^{2k}}\right) = \begin{cases} 2^{2k}, & \text{wenn} \quad t \in H_\nu^{(2k-1)}, \\ 0, & \text{wenn} \quad t \notin H_\nu^{(2k-1)}. \end{cases}$$

Da die Mengen $H_\nu^{(2k-1)}$ $(\nu = 0, 1, 2, \ldots, 2^{2k}-1)$ disjunkt sind, folgt

$$\frac{1}{2^{2k}} \left| \sum_{\nu=0}^{2^{2k}-1} \varphi_{2^{2k}+n}(t) r_{2^{2k}+n}(C_\nu^{(k)}) D_{2^{2k}}(t;\, C_\nu^{(k)}) \right| \leqq 1 \qquad (t \in [0, 1]).$$

Da auf Grund von (2. 9), der Wert von $D_{2^{2^{2k}+n}}(t;\, C_\nu^{(k)})$ auf einer Menge $F_{(k,n)}^{(\nu)}$ (vom Maß $1/2^{2^{2k}+n}$) gleich $2^{2^{2k}+n}$ ist, sonst aber verschwindet, bekommen wir, mittels der Gleichung (2. 13),

$$|V_k(t;\, x)| \leqq \frac{2}{3} k\, 2^{2k} \qquad (t \in [0, 1]).$$

Auf Grund von (2. 18) folgt aus (3. 2)

$$|S_{2^{2^{2k}+n}+m_k}(t;\, P_k)| \geqq \begin{cases} 2^{2^{2k}+n-2k} - \frac{2}{3} k - 1 > \frac{k}{3} - 1 & \left(t \in \bigcup_{\nu=0}^{2^{2k}-1} F_{(k,n)}^{(\nu)}\right), \\ \frac{k}{3} - 1 & \left(t \in H_n^{(2k-1)} - \bigcup_{\nu=0}^{2^{2k}-1} F_{(k,n)}^{(\nu)}\right). \end{cases}$$

Damit haben wir auch die Behauptung d) bewiesen.

§ 4. Beweis des Satzes

Es sei $\lambda_n = \varepsilon_n \log\log n$; wo wir ohne Beschränkung der Allgemeinheit annehmen können, daß $\varepsilon_n \searrow 0$. Ferner sei $N_i = 2^{2^{2k_i+1}}$ $(k_1 < \ldots < k_i < \ldots)$ mit $\sum_{i=1}^{\infty} \varepsilon_{N_i} < \infty$, und wir bilden die Reihe

$$f(t) = \sum_{i=1}^{\infty} \varepsilon_{N_i} \psi_{N_i}(t) P_{k_i}(t). \tag{4. 1}$$

Der höchste Index von $\psi_\nu(t)$ in $P_{k_i}(t)$ ist nicht größer als $N_i - 1$, so ist das i-te Glied der Reihe (4. 1) ein W-Polynom der Funktionen $\psi_\nu(t)$, wobei der kleinste Index mindenstens N_i, der größte aber höchstens $2N_i - 1$ ist. Da für jedes $i (= 1, 2, 3, \ldots)$ $2N_i - 1 < N_{i+1}$ gilt, schließen wir, daß (4. 1) eine nach $\psi_\nu(t)$ fortschreitende (geklammerte) Reihe ist, und in den verschiedenen Gliedern die einzelnen Funktionen $\psi_\nu(t)$ höchstens je einmal vorkommen.

Aus den Eigenschaften b) und c) von $P_k(t)$ folgt, daß

$$\sum_{i=1}^{\infty} \varepsilon_{N_i} \int_0^1 |\psi_{N_i}(t) P_{k_i}(t)|\, dt = \sum_{i=1}^{\infty} \varepsilon_{N_i} \int_0^1 P_{k_i}(t)\, dt = \sum_{i=1}^{\infty} \varepsilon_{N_i} < \infty.$$

Hieraus bekommen wir, daß die Reihe (4.1) fast überall gegen eine integrierbare Funktion $f(t)$ konvergiert. Weiter folgt auch, daß wir aus (4. 1) durch Weglassen der Klammern die Entwicklung der Funktion $f(t)$ nach dem System $\{\psi_\nu(t)\}$ bekommen.

Zum Beweis von (5) führen wir die folgenden Mengen ein:

$$(4.2) \qquad R_i = \{t : t \in [0, 1], P_{k_i}(t) > 0\}, \qquad T = \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} R_i.$$

Im Falle $\nu \neq \nu'$ ist $D_{N_i}(t;\, C_\nu^{(k_i)})\, D_{N_i}(t;\, C_{\nu'}^{(k_i)}) \equiv 0$ $(t \in [0, 1])$, also besteht R_i aus 2^{2k_i} disjunkten Mengen, die alle das Maß N_i^{-1} besitzen, und für $t \in R_i$ ist $P_{k_i}(t) = \dfrac{N_i}{2^{2k_i}}$. Daraus folgt:

$$|T| = \lim_{j\to\infty} \left| \bigcup_{i=j}^{\infty} R_i \right| \leqq \lim_{j\to\infty} \sum_{i=j}^{\infty} \frac{2^{2k_i}}{N_i} = 0.$$

Zuerst zeigen wir, daß (5) für die Punkten von $S = [0, 1] - T$, d. h. fast überall in [0, 1] besteht. Es sei $t \in S$. Dann gibt es, auf Grund von (4. 2), einen Index $i_0 = i_0(t)$ derart, daß $t \notin R_i$ für $i > i_0$ ist, folglich gilt

$$(4.3) \qquad S_{N_i}(t;f) = S_{2N_{i-1}}(t;f) = S_{2N_{i_0}}(t;f) \qquad (i > i_0).$$

Auf Grund der Eigenschaft (6) a) der Mengen $H_n^{(k)}$ kann man eine Indexfolge $n_i = n_i(t)$ eindeutig angeben, für die $0 \leqq n_i \leqq 2^{2k_i} - 1$, und $t \in H_{n_i}^{(2k_i-1)}$ $(i = 1, 2, 3, \ldots)$ besteht. Es sei $M_i = M_i(t) = 2^{2^{2k_i} + n_i(t)} + m_{k_i}$. Dann bekommen wir aus (4. 1), mittels (4. 3), daß für $i > i_0$

$$S_{N_i+M_i}(t;f) = S_{N_i}(t;f) + \varepsilon_{N_i} \psi_{N_i}(t) S_{M_i}(t; P_{k_i}) =$$

$$= S_{2N_{i_0}}(t;f) + \varepsilon_{N_i} \psi_{N_i}(t) S_{M_i}(t; P_{k_i})$$

gilt. Daraus folgt auf Grund von (6) d) für genügend große i,

$$\frac{|S_{N_i+M_i}(t;f)|}{\lambda_{N_i+M_i}} \geqq \frac{\varepsilon_{N_i} |S_{M_i}(t; P_{k_i})| - |S_{2N_{i_0}}(t;f)|}{\lambda_{N_i+M_i}} \geqq$$

$$\geqq \frac{\varepsilon_{N_i} \left(\dfrac{k_i}{3} - 1 \right)}{\varepsilon_{N_i+M_i} \log\log (N_i + M_i)} + o(1) \geqq \frac{\dfrac{k_i}{3} - 1}{\log\log 2N_i} + o(1) > \frac{1}{12}.$$

Damit haben wir (5) für die Punkten von S bewiesen.

Es sei nun $t \in T$. Dann gibt es eine Folge $i_s = i_s(t)$ $(s = 1, 2, \ldots)$ für die $t \in R_{i_s}$ $(s = 1, 2, \ldots)$ gilt. Da die Folge λ_n nicht abnehmend und die Funktion $\dfrac{\log\log x}{\log x}$ für $x \geqq 4$ monoton abnehmend ist, gilt für $j < s$

$$\frac{\varepsilon_{N_{i_j}}}{\varepsilon_{N_{i_s}}} \leqq \frac{\log\log N_{i_s}}{\log\log N_{i_j}} \leqq \frac{\log N_{i_s}}{\log N_{i_j}} = \frac{2^{2k_{i_s}}}{2^{2k_{i_j}}}.$$

Daraus folgt

$$\frac{|S_{2N_{i_s}}(t;f)|}{\lambda_{2N_{i_s}}} \geqq \frac{\varepsilon_{N_{i_s}} N_{i_s} 2^{-2k_{i_s}} - \sum_{j=1}^{s-1} \varepsilon_{N_{i_j}} N_{i_j} 2^{-2k_{i_j}}}{\varepsilon_{2N_{i_s}} \log\log 2N_{i_s}} \geqq$$

$$\geqq \frac{N_{i_s} 2^{-2k_{i_s}}}{\log\log N_{i_s}} \left(1 - \sum_{j=1}^{s-1} \frac{N_{i_j} 2^{-4k_{i_j}}}{N_{i_s} 2^{-4k_{i_s}}}\right) \geqq \frac{N_{i_s} 2^{-2k_{i_s}}}{\log\log N_{i_s}} \left(1 - \frac{2^{4k_{i_s}}(s-1)}{2^{2^{2k_{i_s}}}}\right) \to \infty \quad (s \to \infty).$$

Damit haben wir unseren Satz bewiesen.

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(Eingegangen am 10. November 1966)

On a generalization of completely 0-simple semigroups

By OTTO STEINFELD in Budapest

§ 1. Introduction

The well-known theorem of REES characterizes the completely 0-simple semigroups with the help of matrix semigroups over a group with zero. In this paper we generalize this theorem by giving a class of semigroups that are characterized as matrix semigroups over a semigroup with zero and identity.

In § 2 we introduce the notion of left (right) S-translation between two left (right) ideals of a semigroup S with 0. This notion is a generalization of right (left) translation of S in the sense of CLIFFORD—PRESTON [1]. Two left (right) ideals $\mathfrak{a}_1$, $\mathfrak{a}_2$ of S are called left (right) S-similar if there exists a one-to-one left (right) S-translation from $\mathfrak{a}_1$ onto $\mathfrak{a}_2$. In Proposition 2. 1 a necessary and sufficient condition is given in order that the left ideals Se_1, Se_2 ($e_i^2=e_i$; $i=1, 2$) of S be S-similar.

In § 3 we show that all 0-minimal left (right) ideals of a completely 0-simple semigroup are left (right) S-similar. Proposition 3. 4 gives the following characterization of the completely 0-simple semigroups: a semigroup S with zero is completely 0-simple if and only if S has the form $S=\bigcup_{\lambda\in\Lambda} Se_\lambda$ with idempotents e_λ where Se_λ are 0-minimal, left S-similar left ideals of S. In view of this result we define the following generalization of the completely 0-simple semigroups. Let S be a semigroup with 0 such that

$$S=\bigcup_{\lambda\in\Lambda} Se_\lambda=\bigcup_{i\in I} e_iS \qquad (e_\lambda^2=e_\lambda,\quad e_i^2=e_i;\quad 1\in I\cap\Lambda),$$

where Se_λ (e_iS) are left (right) S-similar left (right) ideals of S with $Se_\mu\cap Se_\nu=0$ ($\mu,\ \nu\in\Lambda; \mu\neq\nu$) and $e_jS\cap e_kS=0$ ($j,\ k\in I; j\neq k$). These semigroups are called S-similarly decomposable.

The theorem of REES states that a semigroup is completely 0-simple if and only if it is isomorphic to a regular Rees matrix semigroup over a group with zero. In order to give an analogous characterization of the S-similarly decomposable semigroups, we introduce the notion of the locally regular Rees matrix semigroup $M^\circ(H; I, \Lambda; P)$ over a semigroup H with zero and identity. (See at the end of § 3.) The regular Rees matrix semigroups are locally regular. Then we have: a semigroup S with zero is S-similarly decomposable if and only if it is isomorphic to a locally regular Rees matrix semigroup over a semigroup with zero and identity. (See Theorem 4. 1.) We intend to deal with the homomorphisms of a locally regular Rees matrix semigroup in another paper.

It is known that the Brandt semigroups are just the completely 0-simple inverse semigroups, therefore they have representations by special regular Rees matrix semigroups. (See Theorem 3. 9 in [1].) In § 5 we define the special S-similarly decomposable semigroups and we prove an analogous theorem concerning them. (See Theorem 5. 1.) It is interesting that these semigroups have an application in the theory of codes and finite-state transducers.

§ 2. On the translations

Let S be a semigroup with zero and $\mathfrak{l}_1, \mathfrak{l}_2$ left ideals of S. By a *left S-translation of $\mathfrak{l}_1$ into $\mathfrak{l}_2$* we mean a single valued mapping φ of $\mathfrak{l}_1$ into $\mathfrak{l}_2$ such that

(2. 1) $$x\varphi \in \mathfrak{l}_2, \; s(x\varphi)=(sx)\varphi \qquad (\text{for all } x\in \mathfrak{l}_1 \text{ and } s\in S).$$

If ω is a left S-translation such that for every element x of $\mathfrak{l}_1$

$$x\omega = 0 \qquad (x\in \mathfrak{l}_1)$$

holds, then ω is called *the zero left S-translation of $\mathfrak{l}_1$ into $\mathfrak{l}_2$*.

Let a_2 be a fixed element of $\mathfrak{l}_2$. Then the mapping

(2. 2) $$x \to xa_2 \qquad (x\in \mathfrak{l}_1;\; a_2\in \mathfrak{l}_2)$$

is a left S-translation of $\mathfrak{l}_1$ into $\mathfrak{l}_2$.

In the case $\mathfrak{l}_1=\mathfrak{l}_2=S$ the *left* S-translation of S into itself and the *right* translation of S in the sense of Clifford—Preston [1] are the same notions.

Analogously, one can define the *right S-translation of the right ideal $\mathfrak{r}_1$ into the right ideal $\mathfrak{r}_2$ of S.*

We say that the left ideals $\mathfrak{l}_1, \mathfrak{l}_2$ of S are *left S-similar* [1]) if there exists a one-to-one left S-translation φ of $\mathfrak{l}_1$ onto $\mathfrak{l}_2$. It is easy to see that this notion defines an equivalence relation among the left ideals of S.

One can define dually the *right S-similarity* of right ideals.

Proposition 2. 1. *Let S be a semigroup with 0 and $e_1 \neq 0$, $e_2 \neq 0$ idempotents in S. Then the left ideals Se_1 and Se_2 are left S-similar if and only if there exist elements q_{12} and q_{21} in S such that*

(2. 3) $$e_1q_{12}e_2=q_{12}, \qquad e_2q_{21}e_1=q_{21},$$

(2. 4) $$q_{12}q_{21}=e_1, \qquad q_{21}q_{12}=e_2.$$

Proof. Let Se_1 and Se_2 be left S-similar and φ a one-to-one left S-translation of Se_1 onto Se_2. Set $e_1\varphi=q_{12}(\in Se_2)$, $e_2\varphi^{-1}=q_{21}(\in Se_1)$. Then in view of (2. 1) and $e_1^2=e_1$, $e_2^2=e_2$ the relations (2. 3) hold. Furthermore,

$$e_1=(e_1\varphi)\varphi^{-1}=q_{12}\varphi^{-1}=(q_{12}e_2)\varphi^{-1}=q_{12}(e_2\varphi^{-1})=q_{12}q_{21}.$$

Similarly $q_{21}q_{12}=e_2$.

[1]) In his paper [2], H.-J. Hoehnke defines a more general, analogous notion for the S-systems.

Conversely, assume that some q_{12} and q_{21} in S satisfy the relations (2. 3), (2. 4). Let φ be the mapping of Se_1 into Se_2 satisfying $(se_1)\varphi = se_1 q_{12}$ $(se_1 \in Se_1)$. Then $(se_1)\varphi = (te_1)\varphi$ (se_1 and $te_1 \in Se_1$) and (2. 4_1) imply

$$se_1 q_{12} = te_1 q_{12} \Rightarrow se_1 q_{12} q_{21} = te_1 q_{12} q_{21} \Rightarrow se_1 = te_1.$$

If ue_2 is an arbitrary element of Se_2 then because of (2. 4_2) $(uq_{21}e_1)\varphi = uq_{21}e_1 q_{12} = uq_{21} q_{12} = ue_2$. Thus φ is a one-to-one mapping of Se_1 onto Se_2 with property (2. 1), i.e. Se_1 and Se_2 are left S-similar.

A dual proposition holds on the right S-similar right ideals e_1S, e_2S of S.

Remark 1. It is easy to show that the conditions (2. 4) alone are sufficient to assure the left S-similarity of Se_1 and Se_2.

Since the conditions on e_1 and e_2 of Proposition 2. 1 are left-right symmetric, it is clear that we have the following

Corollary 2. 2. *Let S be a semigroup with zero and $e_1 \neq 0$, $e_2 \neq 0$ idempotents in S. Then the left ideals Se_1 and Se_2 are left S-similar if and only if the right ideals e_1S and e_2S are right S-similar.*

Proposition 2. 1 and Corollary 2. 2 are analogous to Proposition III. 7. 4 and its Corollary in JACOBSON [3].

An other consequence of Proposition 2. 1 is the following

Corollary 2. 3 (Cf. STEINFELD [6] Theorem 5. 4). *If the left ideals Se_1, Se_2 $(e_1^2 = e_1 \neq 0,\ e_2^2 = e_2 \neq 0)$ of a semigroup S with zero are left S-similar, then the subsemigroups e_1Se_1 and e_2Se_2 of S are isomorphic.*

Proof. Since the left ideals Se_1, Se_2 are left S-similar, elements q_{12} and q_{21} with properties (2. 3), (2. 4) exist. We shall show that

$$e_1 s e_1 \to q_{21} s q_{12} \qquad (e_1 s e_1 \in e_1 S e_1) \tag{2. 5}$$

is an isomorphism of e_1Se_1 onto e_2Se_2. For, let e_1se_1 and $e_1te_1 \in e_1Se_1$; then in view of (2. 5) and (2. 4_1)

$$e_1 s e_1 \cdot e_1 t e_1 \to q_{21} s e_1 t q_{12} = q_{21} s q_{12} \cdot q_{21} t q_{12}.$$

So (2. 5) is a homomorphism. Furthermore, if the images $q_{21}sq_{12}$ and $q_{21}tq_{12}$ of e_1se_1 and e_1te_1 are equal, then

$$e_1 s e_1 = q_{12} \cdot q_{21} s q_{12} \cdot q_{21} = q_{12} \cdot q_{21} t q_{12} \cdot q_{21} = e_1 t e_1. \tag{2. 6}$$

Finally, let $e_2ue_2 \in e_2Se_2$. In view of (2. 4_2) the element $e_1 q_{12} e_2 u e_2 q_{21} e_1$ of e_1Se_1 is mapped by (2. 5) upon the element $q_{21} \cdot q_{12} u q_{21} \cdot q_{12} = e_2 u e_2$. Thus (2. 5) is an isomorphic mapping of e_1Se_1 onto e_2Se_2, indeed.

§ 3. On the completely 0-simple semigroups

Now we need the following

Proposition 3. 1 (STEINFELD [5] Satz 6). *Let* $\mathfrak{l}$ *be a* 0*-minimal left ideal of a semigroup* S *with zero and* $e \neq 0$ *an idempotent) in* $\mathfrak{l}$. *Then* $e\mathfrak{l}$ *is a group with zero.*

Let Se_1, Se_2 $(e_1^2 = e_1;\ e_2^2 = e_2)$ be two 0-minimal left ideals of a semigroup S with 0, and $a \in S$. By the 0-minimality of Se_2 either $Se_1 ae_2 = Se_2$ or $Se_1 ae_2 = 0$ holds.

The first possibility implies the existence of an element $be_1 \in Se_1$ such that $be_1 ae_2 = e_2$. Hence $e_2 be_1 \cdot e_1 ae_2 = e_2^2 = e_2$. From this we get

$$e_1 ae_2 \cdot e_2 be_1 \cdot e_1 ae_2 \cdot e_2 be_1 = e_1 ae_2 \cdot e_2 \cdot e_2 be_1 = e_1 ae_2 \cdot e_2 be_1,$$

that is $e_1 ae_2 \cdot e_2 be_1 \in e_1 Se_1$ is an idempotent. Since $e_1 Se_1$ is a group with zero and $e_1 ae_2 \cdot e_2 be_1 \neq 0$, we obtain $e_1 ae_2 \cdot e_2 be_1 = e_1$. By Proposition 2. 1 and the properties of the elements $e_1 ae_2$, $e_2 be_1$, the left ideals Se_1 and Se_2 are left S-similar.

The second possibility implies that the mapping

$$se_1 \to se_1 \cdot e_1 ae_2 = 0 \qquad (se_1 \in Se_1)$$

is the zero left S-translation of Se_1 into Se_2.

Thus we have: if an element ae_2 $(\in Se_2)$ exists such that $Se_1 ae_2 = Se_2$, then Se_1 and Se_2 are left S-similar; if such an element does not exist, then the only left S-translation between Se_1 and Se_2 is the zero S-translation. Therefore:

Proposition 3. 2. *Let* Se_1, Se_2 $(e_1^2 = e_1;\ e_2^2 = e_2)$ *be* 0*-minimal left ideals of a semigroup* S *with zero. Then either* Se_1, Se_2 *are left* S*-similar or the only left* S*-translation between* Se_1 *and* Se_2 *is the zero left* S*-translation.*

These imply

Corollary 3. 3. *All* 0*-minimal left (right) ideals of a completely* 0*-simple semigroup* S *are left (right)* S*-similar.*

Proof. Let $\mathfrak{l}_1$, $\mathfrak{l}_2$ two 0-minimal left ideals of the completely 0-simple semigroup S. It is known that $\mathfrak{l}_i$ has the form $\mathfrak{l}_i = Se_i$ $(e_i^2 = e_i;\ i = 1, 2)$. In view of the 0-minimality of Se_2 the product $Se_1 \cdot ae_2 (a \in S)$ is either 0 or Se_2. As S is a 0-simple semigroup $Se_1 S = S$ holds. Thus at least one element ae_2 $(\in Se_2)$ exists with $Se_1 ae_2 = Se_2$. This and Proposition 3. 2 imply our assertion.

We shall prove the following characterization of completely 0-simple semigroups.

Proposition 3. 4 (cf. STEINFELD [7] Theorem 15). *A semigroup* S *with zero is completely* 0*-simple if and only if* S *has the form*

$$S = \bigcup_{\lambda \in \Lambda} Se_\lambda \qquad (e_\lambda^2 = e_\lambda) \tag{3. 1}$$

where Se_λ *are pairwise left* S*-similar* 0*-minimal left ideals of* S.

Proof. By Corollary 2. 49 of [1], a completely 0-simple semigroup S is the union of its 0-minimal left ideals $\mathfrak{l}_\lambda$ $(\lambda \in \Lambda)$. As S is a regular semigroup we can write $\mathfrak{l}_\lambda = Se_\lambda$ $(e_\lambda^2 = e_\lambda;\ \lambda \in \Lambda)$. Thus, by Corollary 3. 3, the necessity of the stated condition follows.

Conversely, let S be a semigroup with the stated properties. In view of Exercise 12 for § 2. 7 of [1] it is enough to prove that S is 0-simple. As S has at least one non-zero idempotent, we have $S^2 \neq 0$. By (3. 1), any ideal $\mathfrak{a}(\neq 0)$ of S has a non-zero element of the form $ae_\mu \in \mathfrak{a}(\mu \in \Lambda)$. Hence

$$0 \neq ae_\mu \in \mathfrak{a}Se_\mu \qquad (e_\mu^2 = e_\mu).$$

Because of the 0-minimality of Se_μ, this implies $Se_\mu = \mathfrak{a}Se_\mu \subseteqq \mathfrak{a}$. As Se_μ and every Se_λ $(\lambda \in \Lambda)$ are left S-similar, 0-minimal left ideals of S in view of Proposition 2. 1

$$Se_\lambda = Se_\mu \cdot Se_\lambda \subseteqq \mathfrak{a} \cdot Se_\lambda \subseteqq \mathfrak{a} \qquad (\lambda \in \Lambda)$$

holds. This and (3. 1) imply

$$S = \bigcup_{\lambda \in \Lambda} Se_\lambda \subseteqq \mathfrak{a}$$

establishing the 0-simplicity of S.

The dual characterization of the completely 0-simple semigroup S holds by the right S-similar, 0-minimal right ideals e_iS $(e_i^2 = e_i;\ i \in I)$ of S.

It is easy to show that the left ideal Se $(e^2 = e \neq 0)$ of the completely 0-simple semigroup S is 0-minimal if and only if eS is a 0-minimal right ideal of S, therefore one can suppose that in the decompositions

$$S = \bigcup_{\lambda \in \Lambda} Se_\lambda = \bigcup_{i \in I} e_iS$$

$1 \in I \cap \Lambda$ holds. Naturally the 0-minimal left ideals Se_λ $(\lambda \in \Lambda)$ in (3. 1) are different, therefore $Se_\mu \cap Se_\nu = 0$ if $\mu \neq \nu$ and $\mu, \nu \in \Lambda$.

We now generalize the notion of completely 0-simple semigroups.

Let S be a semigroup with 0 such that

$$S = \bigcup_{\lambda \in \Lambda} Se_\lambda = \bigcup_{i \in I} e_iS \qquad (e_\lambda^2 = e_\lambda;\quad e_i^2 = e_i;\quad 1 \in I \cap \Lambda) \tag{3. 2}$$

where Se_λ $(\lambda \in \Lambda)[e_iS(i \in I)]$ are left [right] 0-similar left [right] ideals of S such that $Se_\mu \cap Se_\nu = 0$ $(\mu, \nu \in \Lambda;\ \mu \neq \nu)$ and $e_jS \cap e_kS = 0$ $(j, k \in I;\ j \neq k)$. We call a semigroup with these properties *S-similarly decomposable.*

By Proposition 3. 4 and its dual, the completely 0-simple semigroups are S-similarly decomposable.

The well-known theorem of Rees (Clifford—Preston [1], Theorem 3. 5) characterizes the completely 0-simple semigroups by the regular Rees matrix semigroups over a group with 0. In the next § we wish to give an analogous characterization of the S-similarly decomposable semigroups. For this characterization we need to generalize the notion of the regular Rees matrix semigroup.

Let H be a semigroup with 0 and with the identity element e. Let $M^\circ(H;\ I,\ \Lambda;\ P)$ denote the Rees matrix semigroup over H with a sandwich matrix $P = (p_{\lambda i})$ $(\lambda \in \Lambda;\ i \in I;\ p_{\lambda i} \in H)$. Denote the elements of M° by $(a)_{i\lambda}$ with a in H, i in I and λ in Λ. The product of the matrices $(a)_{i\lambda}$, $(b)_{j\mu}$ is defined by

$$(a)_{i\lambda} \circ (b)_{j\mu} = (ap_{\lambda j}b)_{i\mu} \qquad (a, b \in H;\ i, j \in I;\ \lambda, \mu \in \Lambda). \tag{3. 3}$$

We say that $M^\circ(H; I, \Lambda; P)$ is *locally regular* if $P=(p_{\lambda i})$ has the following properties:

1) in every row λ of P there exists an element $p_{\lambda j(\lambda)}$ $(j(\lambda)\in I)$ which has a right inverse $p'_{\lambda j(\lambda)}$ in H, that is

(3. 4) $$p_{\lambda j(\lambda)}\, p'_{\lambda j(\lambda)} = e;$$

2) in every column i of P there exists an element $p_{\mu(i)i}$ $(\mu(i)\in\Lambda)$ which has a left inverse $p''_{\mu(i)i}$ in H, that is

(3. 5) $$p''_{\mu(i)i}\, p_{\mu(i)i} = e;$$

3) there exists at least one element $p_{\lambda i}$ in P which has a (right and left) inverse in H.

A regular Rees matrix semigroup $M^\circ(G; I, \Lambda; P)$ over a group G with zero is locally regular.

§ 4. A generalization of the Rees theorem

Theorem 4. 1. *A semigroup S with zero is S-similarly decomposable if and only if it is isomorphic to a locally regular Rees matrix semigroup over a semigroup with zero and identity.*

Proof. Let S be a S-similarly decomposable semigroup. Then S has a decomposition (3. 2). In view of (3. 2) an arbitrary element $a\neq 0$ of S belongs to exactly one right ideal e_iS $(i\in I)$ and to exactly one left ideal Se_λ $(\lambda\in\Lambda)$. Hence

(4. 1) $$a = e_i a e_\lambda \qquad (i\in I;\ \lambda\in\Lambda).$$

As every left ideal Se_λ $(\lambda\in\Lambda)$ is left S-similar to Se_1 $(1\in I\cap\Lambda)$ and every right ideal e_iS $(i\in I)$ is right S-similar to e_1S $(1\in I\cap\Lambda)$, by Proposition 2. 1 there exist elements $q_{1\lambda}$ $(\in e_1Se_\lambda)$, $q_{\lambda 1}(\in e_\lambda Se_1)$ and $r_{1i}(\in e_1Se_i)$, $r_{i1}(\in e_iSe_1)$ such that

(4. 2) $$q_{1\lambda}q_{\lambda 1} = e_1, \qquad q_{\lambda 1}q_{1\lambda} = e_\lambda$$

and

(4. 3) $$r_{1i}r_{i1} = e_1, \qquad r_{i1}r_{1i} = e_i.$$

Let $M^\circ(e_1Se_1; I, \Lambda; P)$ denote the Rees matrix semigroup over the semigroup e_1Se_1 with the sandwich matrix $P=(p_{\lambda i})=(q_{1\lambda}r_{i1})$. We shall prove that the mapping

(4. 4) $$\varphi\colon a = e_iae_\lambda \to (r_{1i}aq_{\lambda 1})_{i\lambda} \qquad (a\in S;\ i\in I;\ \lambda\in\Lambda)$$

is an isomorphism of S onto $M^\circ = M^\circ(e_1Se_1; I, \Lambda; P=(q_{1\lambda}r_{i1}))$. First we show that φ is one-to-one. If the images $(r_{1i}aq_{\lambda 1})_{i\lambda}$ and $(r_{1j}bq_{\mu 1})_{j\mu}$ of the elements $a=e_iae_\lambda$ and $b=e_jbe_\mu$ $(i,j\in I;\ \lambda,\mu\in\Lambda)$ are equal, then $i=j$; $\lambda=\mu$ and $r_{1i}aq_{\lambda 1}=r_{1i}bq_{\lambda 1}$. Hence, by (4. 2) and (4. 3),

$$a = e_iae_\lambda = r_{i1}\cdot r_{1i}aq_{\lambda 1}\cdot q_{1\lambda} = r_{i1}\cdot r_{1i}bq_{\lambda 1}\cdot q_{1\lambda} = e_ibe_\lambda = b.$$

φ is a homomorphism. For, let $a=e_iae_\lambda$ and $c=e_jce_\mu$ $(i,j\in I;\ \lambda,\mu\in\Lambda)$ be two elements of S. By (4. 4), (4. 2), (4. 3) and (3. 3) we get

$$ac = e_iae_\lambda e_jce_\mu \to (r_{1i}ae_\lambda e_jcq_{\mu 1})_{i\mu} =$$
$$= (r_{1i}aq_{\lambda 1}q_{1\lambda}r_{j1}r_{1j}cq_{\mu 1})_{i\mu} = (r_{1i}aq_{\lambda 1})_{i\lambda}\circ(r_{1j}cq_{\mu 1})_{j\mu}.$$

φ maps S onto M°. For, an arbitrary element $(e_1ue_1)_{i\lambda}$ of M° is the image of the element $r_{i1}uq_{1\lambda}$ of S:

$$(r_{1i}r_{i1}uq_{1\lambda}q_{\lambda 1})_{i\lambda}=(e_1ue_1)_{i\lambda}.$$

We still have to prove that the Rees matrix semigroup is locally regular, that is, the sandwich matrix $P=(p_{\lambda i})=(q_{1\lambda}r_{i1})$ fulfils the properties 1), 2) and 3). Let us consider the idempotent $e_\lambda\in Se_\lambda$ $(\lambda\in\Lambda)$. In view of (3. 2) there exists a $j=j(\lambda)\in I$ such that $e_\lambda\in e_jS$. Hence

(4. 5) $$e_\lambda=e_je_\lambda \qquad (j\in I;\ \lambda\in\Lambda).$$

The element $p_{\lambda j}=q_{1\lambda}r_{j1}\in e_1Se_1$ in the λ-th row of the matrix P has a right inverse element $p'_{\lambda j}=r_{1j}q_{\lambda 1}\in e_1Se_1$, since because of (4. 2), (4. 3) and (4. 5)

$$p_{\lambda j}p'_{\lambda j}=q_{1\lambda}r_{j1}\cdot r_{1j}q_{\lambda 1}=q_{1\lambda}e_jq_{\lambda 1}=q_{1\lambda}e_je_\lambda q_{\lambda 1}=q_{1\lambda}e_\lambda q_{\lambda 1}=q_{1\lambda}q_{\lambda 1}=e_1.$$

Similarly, in the i-th column $(i\in I)$ of the matrix P the element $p_{\mu i}=q_{1\mu}r_{i1}$ $(\mu=\mu(i)\in\Lambda)$ has a left inverse $p''_{\mu i}=r_{1i}q_{\mu 1}\in e_1Se_1$.

By the assumption $1\in I\cap\Lambda$, the matrix $P=(p_{\lambda i})=(q_{1\lambda}r_{i1})$ has the entry $p_{11}=q_{11}r_{11}$. From (4. 2) and (4. 3) it follows that $p^*_{11}=r_{11}q_{11}$ satisfies $p_{11}p^*_{11}=$ $=q_{11}r_{11}\cdot r_{11}q_{11}=e_1$ and $p^*_{11}p_{11}=r_{11}q_{11}\cdot q_{11}r_{11}=e_1$. Thus $p_{11}=q_{11}r_{11}\in e_1Se_1$ has an inverse. Consequently, for the sandwich matrix $P=(p_{\lambda i})=(q_{1\lambda}r_{i1})$ conditions 1), 2) and 3) are fulfilled.

Conversely, let S be isomorphic to the locally regular Rees matrix semigroup $M^\circ(H;I,\Lambda;P)$ over the semigroup H with 0 and with identity e. Denote the elements of M° by $(a)_{i\lambda}$ $(a\in H;\ i\in I;\ \lambda\in\Lambda)$. Let $\mathfrak{l}_\lambda$ $(\lambda\in\Lambda)$ be the set of the matrices $(a)_{i\lambda}$ for all $a\in H$ and $i\in I$. From (3. 3) it follows that $\mathfrak{l}_\lambda$ is a left ideal of M°. The decomposition

(4. 6) $$M^\circ=\bigcup_{\lambda\in\Lambda}\mathfrak{l}_\lambda \qquad (\mathfrak{l}_\lambda\cap\mathfrak{l}_\mu=0 \quad \text{if} \quad \lambda\neq\mu)$$

trivially holds. Because of the local regularity of $M^\circ=M^\circ(H;I,\Lambda;P)$ the sandwich matrix $P=(p_{\lambda i})$ $(\lambda\in\Lambda;\ i\in I)$ has in every row λ an element $p_{\lambda j(\lambda)}$ $(j(\lambda)\in I)$ such that for a suitable $p'_{\lambda j(\lambda)}\in H$

(4. 7) $$p_{\lambda j(\lambda)}p'_{\lambda j(\lambda)}=e.$$

For every $\lambda\in\Lambda$ let a pair of elements $p_{\lambda j(\lambda)}, p'_{\lambda j(\lambda)}\in H$ with the property (4. 7) be chosen.

By the property 3) of the local regularity of M° we can assume that for some $\nu(\in\Lambda)$, $p'_{\nu j(\nu)}$ is an inverse of $p_{\nu j(\nu)}$.

We shall prove that the elements $(p'_{\lambda j(\lambda)})_{j(\lambda)\lambda}=E_\lambda$ of M° are idempotent and the left ideals $\mathfrak{l}_\lambda$ have the form $\mathfrak{l}_\lambda=M^\circ E_\lambda$. In view of (3. 3) and (4. 7) we have

$$E_\lambda\circ E_\lambda=(p'_{\lambda j(\lambda)})_{j(\lambda)\lambda}\circ(p'_{\lambda j(\lambda)})_{j(\lambda)\lambda}=$$
$$=(p'_{\lambda j(\lambda)}p_{\lambda j(\lambda)}p'_{\lambda j(\lambda)})_{j(\lambda)\lambda}=(p'_{\lambda j(\lambda)})_{j(\lambda)\lambda}=E_\lambda.$$

From the definition of $\mathfrak{l}_\lambda$ and E_λ it follows that $E_\lambda\in\mathfrak{l}_\lambda$ whence $M^\circ E_\lambda\subseteqq\mathfrak{l}_\lambda$. On the other hand, if $(a)_{i\lambda}$ is an element of $\mathfrak{l}_\lambda$, then by (3. 3) and (4. 7) we get

$$(a)_{i\lambda}\circ E_\lambda=(a)_{i\lambda}\circ(p'_{\lambda j(\lambda)})_{j(\lambda)\lambda}=(ap_{\lambda j(\lambda)}p'_{\lambda j(\lambda)})_{i\lambda}=(a)_{i\lambda},$$

that is, $(a)_{i\lambda} \in M^\circ E_\lambda$, and thus $I_\lambda = M^\circ E_\lambda$. This and (4. 6) imply

(4. 8) $$M^\circ = \bigcup_{\lambda \in \Lambda} M^\circ E_\lambda \qquad (M^\circ E_\lambda \cap M^\circ E_\mu = 0 \quad \text{if} \quad \lambda \neq \mu).$$

In order to show that the left ideals $M^\circ E_\lambda$ and $M^\circ E_\mu$ $(\lambda, \mu \in \Lambda)$ are left M°-similar, let $E_\lambda = (p'_{\lambda j(\lambda)})_{j(\lambda)\lambda}$ and $E_\mu = (p'_{\mu k(\mu)})_{k(\mu)\mu}$ $(j(\lambda), k(\mu) \in I)$. In view of Proposition 2. 1 and Remark 1 the mentioned similarity follows from the existence of the elements

$$(p'_{\lambda j(\lambda)})_{j(\lambda)\mu} \in M^\circ E_\mu \quad \text{and} \quad (p'_{\mu k(\mu)})_{k(\mu)\lambda} \in M^\circ E_\lambda$$

satisfying

$$(p'_{\mu k(\mu)})_{k(\mu)\lambda} \circ (p'_{\lambda j(\lambda)})_{j(\lambda)\mu} = (p'_{\mu k(\mu)} p_{\lambda j(\lambda)} p'_{\lambda j(\lambda)})_{k(\mu)\mu} = (p'_{\mu k(\mu)})_{k(\mu)\mu} = E_\mu$$

and

$$(p'_{\lambda j(\lambda)})_{j(\lambda)\mu} \circ (p'_{\mu k(\mu)})_{k(\mu)\lambda} = (p'_{\lambda j(\lambda)} p_{\mu k(\mu)} p'_{\mu k(\mu)})_{j(\lambda)\lambda} = (p'_{\lambda j(\lambda)})_{j(\lambda)\lambda} = E_\lambda.$$

Analogously, one can show that M° has a dual decomposition

(4. 8') $$M^\circ = \bigcup_{i \in I} E_i M^\circ \qquad (E_i M^\circ \cap E_j M^\circ = 0 \quad \text{if} \quad i \neq j)$$

where E_i $(i \in I)$ are idempotents and $E_i M^\circ$ are right M°-similar right ideals of M°.

Finally, one can assume by a suitable ordering of the indices that for some $\nu \in \Lambda$ and $j(\nu) \in I$

$$\nu = j(\nu) = 1 \in I \cap \Lambda.$$

From Theorem 4. 1 we get the Rees theorem as a special case:

Theorem 4. 2 (Rees). *A semigroup is completely 0-simple if and only if it is isomorphic to a regular Rees matrix semigroup over a group with zero.*

It is possible to prove the Rees theorem with the help of Theorem 4. 1, but this proof is more complicated than the direct one.

§ 5. A generalization of Brandt semigroups

In Theorem 3. 9 of [1] the Brandt semigroups are characterized by special regular Rees matrix semigroups. We shall give a generalization of this result.

A semigroup S with 0 having the following property: if a, b, c are elements of S such that $ac = bc \neq 0$ or $ca = cb \neq 0$, then $a = b$, is called *0-cancellative*.

A *generalized Brandt semigroup* is a semigroup S with 0 satisfying the following conditions:

(α) S is 0-cancellative;

(β) to each element a of S there corresponds an element e of S such that $ae = a$ and an element f of S such that $fa = a$;

(γ) if e_i and e_j are idempotents of S then $e_i e_j = e_j e_i$;

(δ) for all pairs e_i, e_j of non-zero idempotents of S there exist elements q_{ij}, q_{ji} in S such that

$$q_{ij} q_{ji} = e_i \quad \text{and} \quad q_{ji} q_{ij} = e_j.$$

Later we shall show that Brandt semigroups are generalized Brandt semigroups.

By a *special S-similarly decomposable semigroup* we mean a semigroup S with 0 having the properties

(a) $$S = \bigcup_{i \in I} Se_i = \bigcup_{i \in I} e_i S \qquad (e_i^2 = e_i; \quad e_i e_j = 0 \quad \text{for} \quad i \neq j; \quad i, j \in I);$$

(b) Se_i $(i \in I)$ are left S-similar;
(c) There exists at least one idempotent e_k $(k \in I)$ such that the semigroup $e_k S e_k$ is 0-cancellative.

Remark 2. In his paper [4] A. E. LAEMMEL has shown that semigroups S having properties (a), (b) and (c) play an important role in the mathematical theory of codes and finite-state transducers.

It is easy to see that these semigroups are special cases of the S-similarly decomposable semigroups defined in the foregoing §.

Theorem 5. 1. *The following three conditions on a semigroup S with zero are equivalent:*
(i) *S is a generalized Brandt semigroup;*
(ii) *S is a special S-similarly decomposable semigroup;*
(iii) *S is isomorphic with a (locally regular) Rees $I \times I$ matrix semigroup $M^\circ(H; I, I; \Delta)$ over a 0-cancellative semigroup H with zero and identity and with the $I \times I$-identity matrix Δ as sandwich matrix.*

Proof. (i) *implies* (ii) [2]. Let $a(\neq 0)$ be an element of S. From (β) it follows the existence of an element $e(\in S)$ with $ae = a$. Hence $ae^2 = ae = a \neq 0$ and in view of (α) this implies $e^2 = e$.

Let e_i $(i \in I)$ denote the idempotent elements of S. Then $S = \bigcup_{i \in I} Se_i$ holds. From $xe_i = ye_j \neq 0$ $(x, y \in S;\ i, j \in I)$ it follows $xe_i e_i = xe_i = ye_j = ye_j e_j$, whence because of ($\alpha$) we get $e_i = e_j$. Thus $Se_i \cap Se_j = 0$ for $e_i \neq e_j$. This and (γ) imply $e_i e_j = e_j e_i = 0$ if $i \neq j$.

In view of Proposition 2. 1 and Remark 1 the condition (δ) implies that the left ideals Se_i $(i \in I)$ are left S-similar.

As condition (c) is an immediate consequence of (α) we have only to prove that $S = \bigcup_{i \in I} e_i S$. If $a(\neq 0)$ is an element of S then because of (β) there exists an element $f(\in S)$ such that $fa = a$. But we can show — as above — that f is idempotent, therefore $a \in e_i S$ for a suitable e_i.

(ii) *implies* (iii). Assume (ii). From the assumption (a) it follows that an arbitrary element a of S has the form

(5. 1) $$a = e_i a e_j \qquad (i, j \in I).$$

Let e_k $(k \in I)$ be a *fixed* idempotent of S with property (c). By (b) the left ideals Se_k and Se_i $(i \in I)$ are left S-similar, therefore by Proposition 2. 1 there exist elements q_{ki} $(\in e_k Se_i)$, q_{ik} $(\in e_i Se_k)$ such that

(5. 2) $$q_{ki} q_{ik} = e_k \quad \text{and} \quad q_{ik} q_{ki} = e_i \qquad (k, i \in I).$$

[2]) Cf. this part of the proof and Theorem 2 of LAEMMEL [4].

Let $M^\circ(e_k Se_k; I, I; P)$ denote the Rees matrix semigroup over the semigroup $e_k Se_k$ with the sandwich matrix $P=(p_{ij})=(q_{ki}q_{jk})$. $e_k Se_k$ is a semigroup with 0 and with the identity e_k; furthermore by (c), $e_k Se_k$ is 0-cancellative. As in the proof of Theorem 4. 1 we can show that the mapping

$$(5.3) \qquad a=e_i a e_j \to (q_{ki} a q_{jk})_{ij} \qquad (a \in S;\ i, j \in I)$$

is an isomorphism of S onto $M^\circ = M^\circ(e_k Se_k; I, I; P=(q_{ki}q_{jk}))$. In view of (5. 2) and the assumption $e_i e_j = 0$ for $i \neq j$ we get for q_{ki} $(\in e_k Se_i)$ and q_{jk} $(\in e_j Se_k)$

$$q_{ki}\, q_{jk} = \begin{cases} e_k & \text{if} \quad i=j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

Hence

$$P = \begin{pmatrix} e_k & 0 & \cdots & & \\ 0 & e_k & & 0 & \\ \vdots & & \ddots & & \\ & & & e_k & \\ & 0 & & & \ddots \end{pmatrix}$$

is the identity matrix, indeed.

(iii) *implies* (i). Assume (iii) and let $S = M^\circ(H; I, I; \Delta)$. Denote the elements of S by $(a)_{ij}$ $(a \in H;\ i, j \in I)$. If e is the identity of H then $(e)_{ii} \circ (a)_{ij} = (ea)_{ij} = (a)_{ij}$ and $(a)_{ij} \circ (e)_{jj} = (ae)_{ij} = (a)_{ij}$ hold. Thus condition (β) is fulfilled.

To prove (α), let $(a)_{ij}$, $(b)_{kl}$, $(c)_{mn}$ be elements of S such that $(a)_{ij} \circ (b)_{kl} = (a)_{ij} \circ (c)_{mn} \neq (0)$. This holds if and only if $j=k=m$, $l=n$ and $ab=ac \neq 0$. As H is 0-cancellative, this implies $b=c$ whence $(b)_{kl}=(b)_{jl}=(c)_{jl}=(c)_{mn}$. Similarly from $(b)_{kl} \circ (a)_{ij} = (c)_{mn} \circ (a)_{ij} \neq (0)$ it follows $(b)_{kl} = (c)_{mn}$. So condition (α) is proved.

Let $(a)_{ij}$ be a non-zero idempotent of S. Then $(a)_{ij} \circ (a)_{ij} = (a)_{ij} \neq (0)$ if and only if $j=i$ and $a^2 = a \neq 0$. Hence $(a)_{ij} \circ (a)_{ij} = (a)_{ii} \circ (a)_{ii} = (a)_{ii} = (e)_{ii} \circ (a)_{ii}$ whence by (α)

$$(a)_{ij} = (a)_{ii} = (e)_{ii}.$$

If $(e)_{jj}$ and $(e)_{kk}$ $(j, k \in I)$ are non-zero idempotents of S then

$$(e)_{jj} \circ (e)_{kk} = \begin{cases} (e)_{jj} & \text{if} \quad j=k, \\ (0) & \text{if} \quad j \neq k; \end{cases}$$

this proves condition (γ). Furthermore, from

$$(e)_{jk} \circ (e)_{kj} = (e)_{jj} \quad \text{and} \quad (e)_{kj} \circ (e)_{jk} = (e)_{kk}$$

(δ) follows.

The proof is finished.

In Theorem 3. 9 of [1], it is proved that the following three conditions on a semigroup S with zero are equivalent:

(i′) S is a Brandt semigroup;

(ii′) S is a completely 0-simple inverse semigroup;

(iii′) S is isomorphic to a (regular) Rees $I \times I$ matrix semigroup $M^\circ(G; I, I; \Delta)$ over a group with zero G and with the $I \times I$-identity matrix Δ as sandwich matrix.

We shall show that the conditions (i′) and (ii′) and (iii′) are special cases of conditions (i) and (ii) and (iii) of Theorem 5. 1, respectively.

It is trivial that (iii′) is a special case of (iii). In view of Theorem 5. 1 and Theorem 3. 9 of [1] this implies that (i′) [(ii′)] is a special case of (i) [(ii)].

We remark that it is possible to prove Theorem 3. 9 of [1] with the help of Theorem 5. 1, but this proof is more complicated than the original.

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(Received September 19, 1966)

Ergänzung zu einem Satz von S. Kaczmarz

Von KÁROLY TANDORI in Szeged

Einleitung

Für ein im Intervall (a, b) orthonormiertes Funktionensystem $\{\varphi_k(x)\}_1^\infty$ bezeichnet

$$L_n(\{\varphi_k\}; x) = \int_a^b \left| \sum_{k=1}^{n} \varphi_k(x)\varphi_k(t) \right| dt$$

die n-te Lebesguesche Funktion. Wir werden uns mit solchen Systemen beschäftigen, für die

(1) $$L_n(\{\varphi_k\}; x) = O(1).$$

Nach einem Satz von KACZMARZ [1] gilt die folgende Behauptung:

Ist (1) *erfüllt und gilt* $\sum a_k^2 < \infty$, *dann konvergiert die Reihe*

(2) $$\sum a_k \varphi_k(x)$$

fast überall.

Wir beweisen die folgende Umkehrung dieser Behauptung.

Satz. *Zu jeder Koeffizientenfolge* $\{a_k\}_1^\infty$ *mit*

(3) $$\sum_{k=1}^{\infty} a_k^2 = \infty$$

gibt es ein im Intervall (0, 1) *orthonormiertes System* $\{\varphi_k(x)\}_1^\infty$ *derart, daß* (1) *besteht und die Reihe* (2) *fast überall divergiert.*

Im Falle $a_k \geqq a_{k+1}$ $(k = 1, 2, \ldots)$ folgt diese Behauptung aus einem Satz von ULJANOV [2]; er hat nämlich den folgenden Satz bewiesen:

Ist $\{a_k\}$ *eine positive, monoton abnehmende Folge mit* (3), *dann divergiert die Haarsche Reihe* $\sum a_k \chi_k(x)$ *fast überall.*

§ 1. Vorbereitungen

Die Haarschen Funktionen $\chi_k(x)$ $(k=1, 2, \ldots)$ sind folgenderweise definiert:

$$\chi_1(x)\equiv 1, \quad \chi_k(x)=\chi_m^{(l)}(x) \qquad (k=2^m+l;\ m=0, 1\ \ldots;\ 1\leqq l\leqq 2^m),$$

wobei

$$\chi_m^{(l)}(x) = \begin{cases} \sqrt{2^m} & (t\in((2k-2)/2^{m+1},\quad (2k-1)/2^{m+1})), \\ -\sqrt{2^m} & (t\in((2k-1)/2^{m+1};\quad 2k/2^{m+1})), \\ 0 & \text{sonst.} \end{cases}$$

Bekanntlich gilt

$$L_n(\{\chi_k\};\ x)\leqq 1 \qquad (0\leqq x\leqq 1;\ n=1, 2, \ldots). \tag{4}$$

(Siehe z. B. ALEXITS [3], S. 46—50.)

Hilfssatz I. *Ist $\{b_k\}$ eine positive, monoton abnehmende Folge, dann gilt*

$$\int_0^1 \left| \sum_{k=1}^{2^n} b_k \chi_k(x) \right| dx \geqq A \left\{ \sum_{k=1}^{2^n} b_k^2 \right\}^{1/2} \qquad (n=0, 1, \ldots),$$

wobei $A(\leqq 1)$ eine positive, absolute Konstante bezeichnet.

Dieser Hilfssatz ist bekannt. (Siehe [2], Bemerkung an der Seite 935.)

Hilfssatz II. *Es sei N eine positive ganze Zahl und $\{b_k\}_1^{2^N}$ eine positive, monoton abnehmende Folge. Dann gibt es ein im Intervall*

$$0 \leqq x \leqq B^2 = \min\left(1, A^2 \sum_{k=1}^{2^N} b_k^2\right)$$

definiertes orthonormiertes System von Treppenfunktionen $\psi_k(x)$ $(k=1, \ldots, 2^N)$ und eine einfache Menge[1] *$G(\subseteqq[0, B^2])$ mit*

$$\text{mes}\,(G)\geqq B^2/4, \quad L_n(\{\psi_k\};\ x)\leqq 2 \qquad (0\leqq x\leqq B^2;\ n=1, 2, \ldots),$$

und

$$\left| \sum_{k=1}^{2^N} b_k \psi_k(x) \right| \geqq 1 \qquad (x\in G).$$

Beweis. Ohne Beschränkung der Allgemeinheit können wir annehmen, daß $\sum_{k=1}^{2^N} b_k^2\leqq 1$. Es sei C durch

$$C\int_0^1 \left| \sum_{k=1}^{2^N} b_k \chi_k(x) \right| dx = 2 \tag{5}$$

bestimmt. Es sei ferner $I_r=(c_r, d_r)$ $(r=1, \ldots, \varrho)$ eine Zerlegung von $(0, 1)$ in paarweise disjunkte Intervalle derart, daß jede Funktion $\chi_k(x)$ $(k=1, \ldots, 2^N)$ in jedem I_r

[1]) D. h. die Vereinigung endlich vieler Intervalle.

konstant ist. Den Wert der Summe $C\sum_{k=1}^{2^N} b_k\chi_k(x)$ im Intervall I_r bezeichnen wir mit w_r. Nach (5) ist

$$\sum_{r=1}^{\varrho} w_r \operatorname{mes}(I_r) = 2. \tag{6}$$

Es seien $1 \leqq r(1) < \ldots < r(s) \leqq \varrho$ die Indizes r, für die $w_r \geqq 1$ ist. Nach (6) gilt

$$(2 \geqq) \sum_{i=1}^{s} w_{r(i)} \operatorname{mes}(I_{r(i)}) \geqq 1. \tag{7}$$

Es seien $J_r = (\gamma_r, \delta_r)$ $(r = 1, \ldots, \varrho)$ nacheinander folgende Intervalle im Intervall $(0, 3)$ mit $\operatorname{mes}(J_r) = \operatorname{mes}(I_r)$ für $r \neq r(i)$ $(i = 1, \ldots, s)$, und mit $\operatorname{mes}(J_{r(i)}) = w_{r(i)} \operatorname{mes}(J_{r(i)})$ $(i = 1, \ldots, s)$. Ferner seien $\bar{J}_{r(i)} = (\bar{\gamma}_{r(i)}, \bar{\delta}_{r(i)})$ $(i = 1, \ldots, s)$ nacheinander folgende Intervalle im Intervall $(3, 4)$ mit $\operatorname{mes}(\bar{J}_{r(i)}) = \operatorname{mes}(I_{r(i)})$ $(i = 1, \ldots, s)$. Wir setzen

$$\bar{\psi}_k(x) = \begin{cases} \chi_k(x - \gamma_r + c_r) & (x \in J_r;\quad r \neq r(i);\quad i = 1, \ldots, s), \\ \dfrac{1}{w_{r(i)}} \chi_k\left(\dfrac{x - \gamma_{r(i)}}{w_{r(i)}} + c_{r(i)}\right) & (x \in J_{r(i)};\quad i = 1, \ldots, s), \\ \left(1 - \dfrac{1}{w_{r(i)}}\right)^{1/2} \chi_k(x - \bar{\gamma}_{r(i)} + c_{r(i)}) & (x \in \bar{J}_{r(i)};\quad i = 1, \ldots, s), \\ 0 & \text{sonst} \end{cases}$$

$(k = 1, \ldots, 2^N)$. Auf Grund von (7) ist diese Definition möglich.

Die Treppenfunktionen $\bar{\psi}_k(x)$ bilden offensichtlich ein orthonormiertes System im Intervall $(0, 4)$. Wir setzen

$$\bar{G} = \bigcup_{i=1}^{s} J_{r(i)}.$$

Nach (7) ist

$$\operatorname{mes}(\bar{G}) \geqq 1. \tag{8}$$

Offensichtlich gilt

$$C\left|\sum_{k=1}^{2^N} b_k \bar{\psi}_k(x)\right| = 1 \qquad (x \in \bar{G}). \tag{9}$$

Auf Grund von (4), durch eine einfache Rechnung erhalten wir

$$L_n(\{\bar{\psi}_k\}; x) \leqq 2 \qquad (0 \leqq x \leqq 4; n = 1, \ldots, 2^N). \tag{10}$$

Es sei

$$\psi_k(x) = \begin{cases} \dfrac{2}{B} \bar{\psi}_k\left(\dfrac{4}{B^2} x\right) & (0 \leqq x \leqq B^2), \\ 0 & \text{sonst} \end{cases}$$

$(k = 1, \ldots, 2^N)$. Weiterhin soll G die Bildmenge von $\bar{G}$ bei der Transformation $y = 4^{-1}B^2 x$ bezeichnen. Auf Grund des Hilfssatzes I, und der Relationen (5), (8), (9) und (10) ist es offensichtlich, daß die Menge G und das System $\{\psi_k(x)\}$ alle Bedingungen des Hilfssatzes II befriedigen.

Hilfssatz III. *Es sei* $\{c_k\}_1^\infty$ *eine reelle Zahlenfolge mit*

$$\sum_{k=1}^{\infty} c_k^2 = \infty. \tag{11}$$

Dann gibt es eine einfache Menge $H(\subseteq[0,1])$ *mit* $\operatorname{mes}(H)=1/8$, *eine positive ganze Zahl* R *und ein in* $(0,1)$ *orthonormiertes Funktionensystem von Treppenfunktionen* $\omega_k(x)$ $(k=1,\ldots,R)$ *derart, daß*

$$L_n(\{\omega_k\};x)\leqq 2 \quad (0\leqq x\leqq 1;\, n=1,\ldots,R), \quad \max_{1\leqq n\leqq R}\left|\sum_{k=1}^{n} c_k\,\omega_k(x)\right|\geqq 1 \qquad (x\in H)$$

bestehen.

Beweis. Aus (11) folgt, daß eine der Reihen $\sum(c_k^+)^2$, $\sum(c_k^-)^2$ divergiert (c_k^+ bzw. c_k^- bezeichnet den positiven bzw. den negativen Teil von c_k). Ohne Beschränkung der Allgemeinheit können wir annehmen daß

$$\sum_{i=1}^{\infty} c_{k(i)}^2 = \infty,$$

wobei $1\leqq k(1)<\ldots<k(i)<\ldots$ diejenigen Indizes k bezeichnen, für die $c_k>0$ ist. Die Indexfolge $I=\{k(i)\}_1^\infty$ kann man offensichtlich in paarweise disjunkte Folgen I_r $(r=1,2,\ldots)$ zerlegen, derart, daß jede Folge $\{c_k\}_{k\in I_r}$ $(r=1,2,\ldots)$ abnehmend ist. Weiterhin gibt es einen Index R und eine positive ganze Zahl p derart, daß

$$A^2\sum_{k\in I,\,k\leqq R} c_k^2 = A^2\sum_{r=1}^{p}\;\sum_{k\in I_r,\,k\leqq R} c_k^2\geqq 1 \tag{12}$$

gilt. (A bezeichnet die Konstante im Hilfssatz I.)

Es sei

$$B_r^2 = A^2\sum_{k\in I,\,k\leqq R} c_k^2 \qquad (r=1,\ldots,p).$$

Wir wenden den Hilfssatz II auf die Folgen $\{c_k\}_{k\in I_r,\,k\leqq R}$ $(r=1,\ldots,p)$ an. Die entsprechenden Mengen, bzw. die entsprechenden orthonormierten Systeme bezeichnen wir mit G_r, bzw. mit $\{\psi_l^r(x)\}$ $(r=1,\ldots,p)$. Ein Funktionensystem $\{\varpi_k(x)\}_1^R$ definieren wir folgenderweise. Ist k das m-te Glied der Folge I_r, so setzen wir

$$\varpi_k(x)=\psi_m^r\left(\left(\sum_{i=1}^{r-1}B_i^2,\ \sum_{i=1}^{r}B_r^2\right);x\right).$$

Für die Indizes $k\notin I$, $k\leqq R$ seien $\varpi_k(x)$ der Reihe nach gleich den Funktionen $\chi_l\left(\left(\sum_{i=1}^{p}B_i^2,\ \sum_{i=1}^{p}B_i^2+1\right);x\right)$ $(k=1,2,\ldots)$. [2])

[2]) Für ein endliches Intervall $J=(a,b)$ und eine in (c,d) definierte Funktion $f(x)$ definieren wir

$$f(J;x)=\begin{cases} f\left(\dfrac{x-a}{b-a}(d-c)+c\right) & (a<x<b),\\ 0 & \text{sonst.}\end{cases}$$

Die Treppenfunktionen $\bar{\omega}_k(x)$ $(k=1, \ldots, R)$ bilden in $(0, S^2)$ $\left(S^2 = \sum_{i=1}^{p} B_i^2 + 1\right)$ offensichtlich ein orthonormiertes System. Nach (4) und dem Hilfssatz II gilt

(13) $$L_n(\{\bar{\omega}_k\}; x) \leqq 2 \qquad (0 \leqq x \leqq S^2;\ n=1, \ldots, R).$$

Es bezeichne $\bar{G}_r$ die Bildmenge von G_r bei der Transformation $y = x + \sum_{i=1}^{r-1} B_i^2$ und wir setzen

$$\bar{G} = \bigcup_{r=1}^{p} \bar{G}_r.$$

Auf Grund des Hilfssatzes II und von (12) ist

(14) $$\text{mes}\,(\bar{G}) \geqq 1/4.$$

Weiterhin gilt nach dem Hilfssatz II

(15) $$\max_{1 \leqq n \leqq R} \left| \sum_{k=1}^{n} c_k\, \bar{\omega}_k(x) \right| \geqq 1 \qquad (x \in \bar{G}).$$

Es sei

$$\omega_k(x) = S\bar{\omega}_k((0, 1);\, x) \qquad (k=1, \ldots, R),$$

und es bezeichne $\bar{H}$ die Bildmenge von G bei der Transformation $y = x/S^2$. Es sei endlich $H \subseteqq \bar{H}$ eine einfache Menge mit $\text{mes}\,(H) = 1/8$. Wegen $1 \leqq S^2 \leqq 1/2$ folgt aus (13), (14) und (15), daß die Menge H und das System $\{\omega_k(x)\}$ alle Bedingungen des Hilfssatzes III befriedigen.

§ 2. Beweis des Satzes

Für $m = 2^s + l - 2$ $(s = 1, 2, \ldots;\ 1 \leqq l \leqq 2^s)$ sei $I_m = ((l-1)/2^s,\ l/2^s)$. Weiterhin seien J_m $(m = 1, 2, \ldots)$ Intervalle in $(0, 1)$ mit

(16) $$J_m \cap J_\mu = \emptyset \quad (m \neq \mu), \qquad \sqrt{\text{mes}\,(J_m)} = \text{mes}\,(I_m),$$
$$I_m \cap J_m = \emptyset \qquad (m = 1, 2, \ldots).$$

(Solche Intervalle J_m kann man leicht angeben.)

Wegen (3), durch Anwendung des Hilfssatzes III können wir eine Indexfolge $(0=) R_0 < \ldots < R_m < \ldots$, eine Folge von einfachen Mengen $H_m (\subseteqq [0, 1])$ $(m = 1, 2, \ldots)$ und ein in $(0, 1)$ orthonormiertes System $\{\omega_k^m(x)\}_{k=1}^{R_m - R_{m-1}}$ $(m = 1, 2, \ldots)$ von Treppenfunktionen $\omega_k^m(x)$ angeben, derart, daß

(17) $$\text{mes}\,(H_m) = 1/8,$$

(18) $$L_n(\{\omega_k^m\}; x) \leqq 2 \qquad (0 \leqq x \leqq 1;\ n = 1, \ldots, R_m - R_{m-1}),$$

(19) $$\max_{1 \leqq n \leqq R_m - R_{m-1}} \left| \sum_{k=1}^{n} a_{R_{m-1}+k}\, \omega_k^m(x) \right| \geqq 1 \qquad (x \in H_m)$$

für jedes m bestehen.

Durch Induktion werden wir ein in (0, 1) orthonormiertes System von Treppenfunktionen $\varphi_k(x)$ $(k=1, 2, \ldots)$ und eine Folge von einfachen Mengen $G_m(\subseteqq I_m)$ $(m=1, 2, \ldots)$ mit den folgenden Eigenschaften angeben:

$$\text{mes}\,(G_m)=1/8\,[\log_2 (m+1)],\ ^{3)} \tag{20}$$

$$\varphi_k(x) = 0 \qquad (x\notin I_m \cup J_m;\quad R_{m-1}<k\leqq R_m), \tag{21}$$

$$\int_0^1 \left| \sum_{k=R_{m-1}+1}^{n} \varphi_k(x)\,\varphi_k(t) \right| dt \leqq 4\,\text{mes}\,(I_m) \qquad (x\in I_m;\quad R_{m-1}<n\leqq R_m), \tag{22}$$

$$\int_0^1 \left| \sum_{k=R_{m-1}+1}^{n} \varphi_k(x)\,\varphi_k(t) \right| dt \leqq 4 \qquad (x\in J_m;\quad R_{m-1}<n\leqq R_m), \tag{23}$$

$$\max_{R_{m-1}<n\leqq R_m} \left| \sum_{k=R_{m-1}+1}^{n} a_k\,\varphi_k(x) \right| \geqq 1 \qquad (x\in G_m); \tag{24}$$

weiterhin sind die Mengen

$$F_s = \bigcup_{m=2^s-1}^{2^{s+1}-2} G_m$$

stochastisch unabhängig.

Es sei

$$\varphi_k(x) = \omega_k^1(I_1; x) + \left[\frac{1-\text{mes}\,(I_1)}{\text{mes}\,(J_1)}\right]^{\frac{1}{2}} \omega_k^1(J_1; x) \qquad (k=1, \ldots, R_1).$$

G_1 bezeichnet die Bildmenge von H_1 bei der linearen Abbildung von (0, 1) auf I_1. Offensichtlich bilden diese Treppenfunktionen ein orthonormiertes System in (0, 1). Auf Grund von (16), (17), (18) und (19) ergibt sich durch eine einfache Rechnung, daß (20)—(24) für $m=1$ erfüllt sind. Es sei $m_0(>1)$ eine natürliche Zahl. Wir nehmen an, daß die Treppenfunktionen $\varphi_k(x)$ $(k=1, \ldots, R_{m_0-1})$ und die einfachen Mengen $G_m(\subseteqq I_m)$ $(m=1, \ldots, m_0-1)$ schon definiert sind, derart, daß diese Funktionen in (0, 1) ein orthonormiertes System bilden, (20)—(24) für $m=1, \ldots, m_0-1$ erfüllt sind und die Mengen $F_1, \ldots, F_q$ $(q=[\log_2 m_0])$ stochastisch unabhängig sind.

Dann kann man das Intervall I_{m_0} bzw. J_{m_0} in paarweise disjunkte Intervalle P_u $(1\leqq u\leqq U)$, bzw. Q_v $(1\leqq v\leqq V)$ zerlegen, derart, daß jede Funktion $\varphi_k(x)$ $(1\leqq k\leqq R_{m_0-1})$ in jedem P_u, bzw. in jedem Q_v konstant ist, weiterhin jede Menge $I_{m_0}\cap G_m$ $(m=1, \ldots, m_0-1)$, die nicht leer ist, die Vereinigung gewisser P_u ist.

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[3]) $[\alpha]$ bedeutet den ganzen Teil von α.

Die zwei Hälften von P_u, bzw. Q_v bezeichnen wir mit P'_u, P''_u, bzw. mit Q'_v, Q''_v. Wir setzen

$$\varphi_k(x) = \sum_{u=1}^{U} \omega^{m_0}_{R_{m_0-1}+k}(P'_u; x) - \sum_{u=1}^{U} \omega^{m_0}_{R_{m_0-1}+k}(P''_u; x) +$$

$$+ \left[\frac{1-\operatorname{mes}(I_{m_0})}{\operatorname{mes}(J_{m_0})}\right]^{\frac{1}{2}} \left(\sum_{v=1}^{V} \omega^{m_0}_{R_{m_0-1}+k}(Q'_v; x) - \sum_{v=1}^{V} \omega^{m_0}_{R_{m_0-1}+k}(Q''_v; x)\right)$$

$(k = R_{m_0-1}+1, \ldots, R_{m_0})$. Weiterhin sei

$$G_{m_0} = \left(\bigcup_{u=1}^{U} H'(u)\right) \cup \left(\bigcup_{u=1}^{U} H''(u)\right),$$

wobei $H'(u)$, bzw. $H''(u)$ die Bildmenge von H_{m_0} bei der linearen Transformation bezeichnet, die $(0, 1)$ auf P'_u bzw. auf P''_u abbildet. Offensichtlich bilden die Treppenfunktionen $\varphi_k(x)$ $(k = 1, \ldots, R_{m_0})$ ein orthonormiertes System in $(0, 1)$ und ist die Menge G_{m_0} einfach. Aus (16)—(19) ergibt sich durch einfache Rechnung, daß (20)—(24) für $m = m_0$ erfüllt sind. Durch Fortsetzung dieser Konstruktion erhalten wir, daß die Mengen $F_1, \ldots, F_{q+1}$ auch stochastisch unabhängig sind. Das System $\{\varphi_k(x)\}$ und die Mengenfolge $\{G_m\}$ erhalten wir durch Induktion.

Wegen (20) gilt

$$\operatorname{mes}(F_s) = 1/8.$$

Aus dem zweiten Borel—Cantellischen Lemmas ergibt sich also, daß $\operatorname{mes}(\overline{\lim}_{s\to\infty} F_s) = 1$. Auf Grund von (24) erhalten wir daraus, daß die Reihe (2) fast überall divergiert. Es sei $x \in (0, 1)$. Wegen (16) gilt $x \in J_m$ für höchstens ein m, weiterhin gibt es eine Indexfolge $\{m_s\}$ $(2^s - 1 \leqq m_s \leqq 2^{s+1} - 2;\ s = 1, 2, \ldots)$ mit $x \notin I_m$ $(m \neq m_s;\ s = 1, 2, \ldots)$. Auf Grund von (22), (23) und der Definition von I_m folgt hieraus

$$L_n(\{\varphi_k\}; x) = \int_0^1 \left|\sum_{k=1}^{n} \varphi_k(x)\varphi_k(t)\right| dt \leqq 4 + 4\sum_{s=1}^{\infty} \operatorname{mes}(I_m) = 4\sum_{s=0}^{\infty} \frac{1}{2^s}$$

für jedes n.

Damit haben wir den Satz bewiesen.

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(Eingegangen am 28. Dezember 1966)

On the order of magnitude of the partial sums of rearranged Fourier series of square integrable functions

By FERENC MÓRICZ in Szeged

Introduction

KOLMOGOROFF [1] was the first to remark that there exists a square integrable function the Fourier series of which diverges almost everywhere in a certain rearrangement of its terms. However, he has never published the proof of this fact. Afterwards ZAHORSKI [2] sketched a proof of this assertion. Recently OLEVSKIĬ [3] and UL'JANOV [4] obtained some more general theorems. Then, using less elementary tools, TAĬKOV [5] obtained a somewhat sharper result, and a direct elementary construction leading to KOLMOGOROFF's assertion was given by TANDORI [6]. In this paper we are going to sharpen this result by refining a method, due to TANDORI [7], concerning the rearrangement of Walsh series.

UL'JANOV has raised the following question [4]: what is the exact Weyl multiplicator of unconditional convergence in case of Fourier series? We shall show that it is at least $O(\log\log n)$. [1])

Theorem 1. *If $\{\varrho(n)\}$ is any sequence of positive real numbers for which*

$$\varrho(n)=o\left(\sqrt{\log\log n}\right) \tag{1}$$

is satisfied, then there exists a square integrable function whose Fourier series

$$\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

is such that

$$\sum_{n=1}^{\infty}(a_n^2+b_n^2)\varrho^2(n)<\infty \tag{2}$$

and which can be rearranged into an everywhere divergent series

$$\sum_{j=1}^{\infty}\left(a_{n(j)}\cos n(j)x+b_{n(j)}\sin n(j)x\right).$$

For the partial sums of the rearranged Fourier series we have following estimate:

[1]) In this paper log means logarithm with base 4 (but this is not essential to our results).

Theorem 2. *If $\{\varrho(n)\}$ is any sequence of positive real numbers for which* (1) *holds, then there exists a square integrable function whose Fourier series*

$$\sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx)$$

can be rearranged in a such a way that the partial sums $\sigma_N(x)$ of the rearranged series

$$\sum_{j=1}^{\infty} (A_{n(j)} \cos n(j)x + B_{n(j)} \sin n(j)x)$$

satisfy

(3) $$\limsup_{N\to\infty} \frac{|\sigma_N(x)|}{\varrho(N)} > 0$$

everywhere.

I am grateful to Professor Károly Tandori for calling my attention to this problem.

§ 1. Lemmas

Consider the Fejér kernel

$$K_n(x) = \frac{1}{2(n+1)} \left(\frac{\sin (n+1)\frac{x}{2}}{\sin \frac{x}{2}} \right)^2 = \frac{1}{2} + \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) \cos kx.$$

By a simple calculation we obtain the following inequalities

(4) $$K_n(x) \geqq \frac{2(n+1)}{\pi^2} \quad \text{if} \quad |x| \leqq \frac{\pi}{n+1},$$

(5) $$K_n(x) \leqq \frac{n+1}{2}, \quad K_n(x) \leqq \frac{\pi^2}{2(n+1)x^2} \quad \text{if} \quad |x| \leqq \frac{\pi}{2},$$

and

(6) $$\int_{-\pi}^{\pi} K_n^2(x)\, dx < \pi n.$$

In the following a set E will be said to be *simple* if it is the union of finitely many, non-overlapping, closed intervals $[\alpha_k, \beta_k]$ $(\alpha_k < \beta_k)$. For any $\varepsilon > 0$ $(\varepsilon < \min_k (\beta_k - \alpha_k)/2)$, we set

$$E^{(\varepsilon)} = \bigcup_k [\alpha_k + \varepsilon, \beta_k - \varepsilon].$$

For a function $a_\nu \cos \nu x + b_\nu \sin \nu x$ $(\not\equiv 0)$ we call ν its *frequency*. Two trigonometric polynomials will be called *disjoint* if they have no terms of the same frequency.

$C_1, C_2, \ldots$ will denote positive absolute constants.

Lemma 1. *Let $\delta(\leqq\pi/8)$, $\varepsilon(<\delta)$ and $\eta(\leqq 1)$ be positive real numbers, and let n be a natural number such that $n>C_1/\varepsilon\eta$. Then there exists a non-negative trigonometric polynomial $P(x)$ with frequencies 4ν $(\nu=0, 1, \ldots, n)$ and having the following properties:*

(7) $$P(x)\geqq 1 \quad \text{if} \quad |x|\leqq\delta-\varepsilon,$$

(8) $$P(x)\leqq\eta \quad \text{if} \quad \delta\geqq|x|\leqq\frac{\pi}{8},$$

and

(9) $$\int_{-\pi}^{\pi}P^2(x)\,dx\leqq C_2\delta.$$

Proof. We write

$$a=\frac{\pi}{4(n+1)}, \quad b_k=\frac{k\pi}{2(n+1)} \quad (k=0, \pm 1, \pm 2, \ldots).$$

Let the integers ϱ and σ be determined by the inequalities

$$b_\varrho-a\leqq-\delta+\varepsilon<b_\varrho \quad \text{and} \quad b_\sigma<\delta-\varepsilon\leqq b_\sigma+a.$$

This choice of ϱ and σ is possible because $n+1>\pi/2\varepsilon$.

Define the trigonometric polynomial $P(x)$ by

$$P(x)=2\pi a\sum_{r=\varrho}^{\sigma}K_n(4(x-b_r)).$$

We are going to show that $P(x)$ has the properties (7)—(9). On account of the choice of ϱ and σ and (4) we can easily see that (7) is satisfied.

To prove the inequality (8), suppose $\delta\leqq|x|\leqq\pi/8$. Using (5), it follows

$$P(x)\leqq 2\pi a\sum_{r=\varrho}^{\sigma}\frac{\pi^2}{32(n+1)(x-b_r)^2}<\frac{\pi^3 a}{16(n+1)}\sum_{k=0}^{\infty}\frac{1}{(\varepsilon+b_k)^2}<$$

$$<\frac{\pi^3}{32(n+1)}\left(\frac{2a}{\varepsilon^2}+\int_{\varepsilon}^{\infty}\frac{dx}{x^2}\right)<\frac{\pi^3}{16\varepsilon(n+1)}.$$

Hence we get (8) if $C_1=\pi^3/16$.

It remains to show that (9) holds. By a simple transformation we get

(10) $$\int_{-\pi}^{\pi}P^2(x)\,dx=4\pi^2a^2\sum_{r=\varrho}^{\sigma}\sum_{s=\varrho}^{\sigma}\int_{-\pi}^{\pi}K_n(x-b_r)K_n(x-b_s)\,dx.$$

If $r\neq s$, for example $r<s$, then we can write

(11) $$\int_{-\pi}^{\pi}K_n(x-b_r)K_n(x-b_s)\,dx=\int_{-\pi}^{b_r-a}+\int_{b_r-a}^{b_r+a}+\int_{b_r+a}^{b_s-a}+\int_{b_s-a}^{b_s+a}+\int_{b_s+a}^{\pi}.$$

Let us denote the integrals on the right-hand side by I_1, I_2, I_3, I_4, I_5, respectively. Applying (4), (5) and (6), we get

$$(12)\quad I_1 \leqq \frac{\pi^2}{4(n+1)^2}\int_{-\pi}^{b_r-a}\frac{dx}{(x-b_r)^2(x-b_s)^2} < \frac{\pi^4}{4(n+1)^2(b_s-b_r)^2}\int_{-\pi}^{b_r-a}\frac{dx}{(x-b_s)^2} <$$

$$< \frac{\pi^2}{(s-r)^2(b_s-b_r+a)} < \frac{\pi^2}{(s-r)^2 a},$$

$$(13)\quad I_2 \leqq \frac{\pi^2}{2(n+1)}\,\frac{n+1}{2}\int_{b_r-a}^{b_r+a}\frac{dx}{(x-b_s)^2} < \frac{\pi^2}{4}\cdot\frac{2a}{(b_s-b_r-a)^2} =$$

$$= \frac{8a(n+1)^2}{(2s-2r-1)^2} = \frac{\pi^2}{2(2s-2r-1)^2 a},$$

and the same is true for I_5 and I_4 too, respectively. As to I_3, it is clear that $I_3=0$ if $s=r+1$. In case $s>r+1$ we break up the integral I_3 into the sum of the integrals J_1 and J_2 extended over $(b_r+a, (b_r+b_s)/2)$ and $((b_r+b_s)/2, b_s-a)$, respectively. J_1 may be estimated in the same way as I_1 and I_2, and we get

$$(14)\qquad J_1 < \frac{4\pi^2}{(s-r)^2 a},$$

and the same is true for J_2.

In virtue of (11), (12), (13) and (14) we obtain that

$$\int_{-\pi}^{\pi} K_n(x-b_r)\,K_n(x-b_s)\,dx < 2(I_1+I_2+J_1) < \frac{11\pi^2}{(s-r)^2 a}.$$

Hence, using (6) and (10), it follows that

$$\int_{-\pi}^{\pi} P^2(x)\,dx < 4\pi^2 a^2 \sum_{r=\varrho}^{\sigma}\left(\pi n + \frac{22\pi^2}{a}\sum_{k=1}^{\infty}\frac{1}{k^2}\right) <$$

$$< 177\pi^4 a(\sigma-\varrho+1) = 177\pi^4\left(\frac{b_\sigma-b_\varrho}{2}+a\right) = C_2\delta,$$

if $C_2 = 177\pi^4$. This completes the proof of Lemma 1.

The following generalization of Lemma 1 can be proved by the same argument.

Lemma 1′. *Let $E\subset[-\pi/8, \pi/8]$ be a simple set, ε and $\eta(\leqq 1)$ positive real numbers, and n a natural number such that $n > C_1/\varepsilon\eta$. Then there exists a non-negative trigonometric polynomial $P(x)$ with frequencies 4ν $(\nu=0, 1, \ldots, n)$ such that*

$$(7')\qquad P(x) \geqq 1 \quad \textit{if} \quad x\in E^{(\varepsilon)},$$

$$(8')\qquad P(x) \leqq \eta \quad \textit{if} \quad x\in\left[-\frac{\pi}{8}, \frac{\pi}{8}\right]-E,$$

and

$$(9')\qquad \int_{-\pi}^{\pi} P^2(x)\,dx \leqq C_2\,\mathrm{mes}\,(E).^{2)}$$

[2]) mes (E) denotes the Lebesgue measure of the set E.

Lemma 2. *Let $P(x)$ be the trigonometric polynomial in Lemma 1', and let N be a natural number divisible by 4, $N>4n+2$. Furthermore, set*

$$Q_1(x) = \cos Nx \cdot P(x),$$
$$Q_2(x) = -C_3 \cos 2x \cdot Q_1(x), \tag{15}$$
$$Q_3(x) = C_4 \cos 4Nx \cdot P(x).$$

Then $Q_1(x)$, $Q_2(x)$ and $Q_3(x)$ are mutually disjoint trigonometric polynomials with frequencies 2ν $(N/2-2n-1\leqq\nu\leqq 2N+2n)$, having the following properties:

$$|Q_1(x)+Q_2(x)+Q_3(x)| \leqq C_5\eta \quad \textit{if} \quad x\in\left[-\frac{\pi}{8}, \frac{\pi}{8}\right]-E, \tag{16}$$

$$\int_{-\pi}^{\pi}(Q_1(x)+Q_2(x)+Q_3(x))^2\,dx \leqq C_6 \operatorname{mes}(E). \tag{17}$$

Furthermore, there exists a decomposition of the set $E^{(\varepsilon)}$ into three simple, mutually disjoint subsets E_1, E_2, E_3, such that

$$\sum_{k=1}^{l} Q_k(x) \geqq \frac{1}{4} \quad \textit{for} \quad x\in E_l \quad (l=1,2,3). \tag{18}$$

Proof. It is obvious that $Q_1(x)$, $Q_2(x)$ and $Q_3(x)$ are mutually disjoint trigonometric polynomials, since $Q_1(x)$ and $Q_3(x)$ have only terms with frequencies divisible by 4, $Q_2(x)$ has only terms with frequencies divisible by 2, but not by 4, and, furthermore, we have $N+4n<4N-4n$.

In virtue of the fact that $\cos x\geqq 1/4$ if $|x|\leqq 5\pi/12$, we get the following estimates:

$$Q_1(x) \geqq \frac{P(x)}{4} \geqq \frac{1}{4}$$

if
$$x\in\bar{E}_1 = E^{(\varepsilon)}\cap\left\{\bigcup_k\left[\frac{1}{N}\left(2k\pi-\frac{5\pi}{12}\right), \frac{1}{N}\left(2k\pi+\frac{5\pi}{12}\right)\right]\right\},$$

$$Q_1(x)+Q_2(x) \geqq -(C_3\cos 2x-1)\cos Nx\cdot P(x) \geqq \left(\frac{C_3}{4}-1\right)\frac{1}{4}$$

if
$$x\in\bar{E}_2 = E^{(\varepsilon)}\cap\left\{\bigcup_k\left[\frac{1}{N}\left(2k\pi+\frac{7\pi}{12}\right), \frac{1}{N}\left(2k\pi+\frac{17\pi}{12}\right)\right]\right\},$$

and

$$Q_1(x)+Q_2(x)+Q_3(x) \geqq (C_4\cos 4Nx-C_3-1)P(x) \geqq \frac{C_4}{4}-C_3-1$$

if
$$x\in\bar{E}_3 = E^{(\varepsilon)}\cap\left\{\bigcup_k\left[\frac{1}{N}\left(\frac{k\pi}{2}-\frac{5\pi}{48}\right), \frac{1}{N}\left(\frac{k\pi}{2}+\frac{5\pi}{48}\right)\right]\right\}.$$

Since $5\pi/12>\pi/2-5\pi/48$ and $7\pi/12<\pi/2+5\pi/48$, we have $E^{(\varepsilon)}=\bar{E}_1\cup\bar{E}_2\cup\bar{E}_3$.

Set $E_1=\bar{E}_1$, $E_2=\bar{E}_2-E_1$ and $E_3=\bar{E}_3-(E_1\cup E_2)$. We get (18) with $C_3=8$ and $C_4=4C_3+5$.

The inequalities (16) and (17) are then satisfied with $C_5=1+C_3+C_4$ and $C_6=1+C_3^2+C_4^2$. The proof of Lemma 2 is complete.

We shall need Lemma 2 in the following slightly different form too:

Lemma 2'. *Let $P(x)$ be an arbitrary trigonometric polynomial with even frequencies ν ($\leqq n$) and let N be an even natural number, $N>n+1$. Furthermore, set*

$$(15')\qquad \begin{aligned} Q_1(x) &= \cos Nx\cdot P(x),\\ Q_2(x) &= -C_3\cos x\cdot Q_1(x),\\ Q_3(x) &= C_4\cos 4Nx\cdot P(x). \end{aligned}$$

Then $Q_1(x)$, $Q_2(x)$ and $Q_3(x)$ are mutually disjoint trigonometric polynomials with frequencies ν ($N-n-1\leqq\nu\leqq 4N+n$), having the following properties:

$$(16')\qquad |Q_1(x)+Q_2(x)+Q_3(x)|\leqq C_5\,|P(x)|,$$

$$(17')\qquad \int_{-\pi}^{\pi}(Q_1(x)+Q_2(x)+Q_3(x))^2\,dx\leqq C_6\int_{-\pi}^{\pi}P^2(x)\,dx.$$

Furthermore, every measurable set E ($\subset[-\pi/8,\pi/8]$), on which $P(x)$ is positive, can be decomposed into three mutually disjoint measurable subsets E_1, E_2, E_3, such that

$$(18')\qquad \sum_{k=1}^{l}Q_k(x)\geqq\frac{P(x)}{4}\quad \text{for}\quad x\in E_l\quad (l=1,2,3).$$

Lemma 3. *Let ε ($<\pi/4$) be a positive real number. Then there exist mutually disjoint trigonometric polynomials $R_k^{(i)}(x)$ and simple sets $E_k^{(i)}$ ($k=1,2,\dots,3^i$; $i=1,2,\dots$) with the following properties:*

(i) *the frequencies occurring in $R_k^{(i)}(x)$ ($k=1,2,\dots,3^i$) are even numbers, at most equal to a number $f_i=(C_7/\varepsilon)^i 4^{4^i}$;*

$$\text{(ii)}\qquad \int_{-\pi}^{\pi}\Big(\sum_{k=1}^{3^i}R_k^{(i)}(x)\Big)^2dx\leqq C_8\quad \text{for}\quad i=1,2,\dots;$$

(iii) *the sets $E_k^{(i)}$ ($k=1,2,\dots,3^i$) corresponding to the same value of i are disjoint, the set*

$$F_i=\left[-\frac{\pi}{8},\frac{\pi}{8}\right]-\bigcup_{k=1}^{3^i}E_k^{(i)}$$

consists of at most $2f_i$ disjoint intervals, and

$$(19)\qquad \operatorname{mes}(F_i)\leqq\varepsilon\left(1-\frac{1}{2^i}\right);$$

(iv) *for any natural number i, the trigonometric polynomials $R_k^{(j)}(x)$ with $k=1,2,\dots,3^j$; $j=1,2,\dots,i$ can be arranged into a sequence*

$$U_1^{(i)}(x),\,U_2^{(i)}(x),\,\dots,\,U_{J_i}^{(i)}(x)\quad \text{where}\quad J_i=3+3^2+\dots+3^i;$$

such that

$$\sum_{j=1}^{\mu_k^{(i)}} U_j^{(i)}(x) \geqq \frac{i}{8} \quad \textit{for every} \quad x \in E_k^{(i)} \tag{20}$$

with $\mu_k^{(i)}$ not depending on the particular point x in $E_k^{(i)}$ $(k=1, 2, \ldots, 3^i)$.

Remark to Lemma 3. On the basis of (i) and (ii), it is obvious that

$$\int_{-\pi}^{\pi}\Big(\sum_{j=1}^{J_i} U_j^{(i)}(x)\Big)^2 dx = \int_{-\pi}^{\pi}\Big(\sum_{j=1}^{i}\sum_{k=1}^{3^j} R_k^{(j)}(x)\Big)^2 dx \leqq C_8 i \tag{21}$$

holds for $i=1, 2, \ldots$.

Proof. The construction of the trigonometric polynomials $R_k^{(i)}(x)$ and sets $E_k^{(i)}$ will be accomplished by recurrence with respect to i.

First let $i=1$. Apply Lemma 1 with $\delta=\pi/8$, $\varepsilon/4$ instead of ε, $\eta=1$ and $n=[4C_1/\varepsilon]+1$.[3]) Then apply Lemma 2 for the obtained trigonometric polynomial and $N=4n+4$. We get the trigonometric polynomials $Q_k(x)$ and simple sets E_k $(k=1, 2, 3)$ satisfying (16), (17) and (18). Now write $R_k^{(1)}(x)=Q_k(x)$ and $E_k^{(1)}=E_k$ $(k=1, 2, 3)$. It is clear that $R_k^{(1)}(x)$ $(k=1, 2, 3)$ have even frequencies at most equal to

$$2(4N+4n) = 40n+32 \leqq \frac{C_7}{\varepsilon} 4^4 = f_1,$$

where $C_7=64\, C_1 C_5$. The assertions (ii) and (iii) are satisfied with $C_8=C_6\pi/4$, furthermore, the set F_1 consists of at most

$$2\sum_{k=1}^{3}\frac{4}{\pi}\,\mathrm{mes}\,(E_k^{(1)})f_1 \leqq 2f_1$$

intervals. Writing $U_j^{(1)}(x)=R_j^{(1)}(x)$ and $\mu_j^{(1)}=j$ $(j=1,2,3)$, we have that (iv) holds to.o

Now we suppose that all the trigonometric polynomials $R_k^{(i)}(x)$ and sets $E_k^{(i)}$ with $i=1, 2, \ldots, m$ are already determined and satisfy (i)—(iv), and we are going to construct the polynomials and sets corresponding to $i=m+1$ so that the enlarged system still satisfy (i)—(iv).

We begin with applying Lemma 1′ by choosing subsequently $E_k^{(m)}$ $(k=1, 2, \ldots, 3^m)$ (instead of E), $\varkappa\varepsilon$ (instead of ε), η and $n>\max(C_1/\varkappa\varepsilon\eta, f_m)$, where the positive numbers $\varkappa$, η and the natural number n will be determined later on. Denote by $P_k(x)$ $(k=1, 2, \ldots, 3^m)$ the corresponding trigonometric polynomials in the sense of Lemma 1′. Next apply Lemma 2 to each of the trigonometric polynomials $P_k(x)$ by choosing for the three functions (15) the following ones:

$$R_{3k-2}^{(m+1)}(x) = \cos N_k x \cdot P_k(x),$$

$$R_{3k-1}^{(m+1)}(x) = -C_3 \cos 2x \cdot R_{3k-2}^{(m+1)}(x),$$

$$R_{3k}^{(m+1)}(x) = C_4 \cos 4N_k x \cdot P_k(x)$$

[3]) The integer part of a real number α is denoted by $[\alpha]$.

$(k=1, 2, \ldots, 3^m)$, where the natural numbers N_k are chosen so that

$$\frac{N_1}{2}-2n-1>f_m, \quad \frac{N_{k+1}}{2}-2n-1>2N_k+2n$$

$(k=1, 2, \ldots, 3^m-1)$ and, in addition, each N_k be divisible by 4; we can choose for example:

(22) $$N_1=8n+4, \; N_{k+1}=4N_k+8n+4 \qquad (k=1, 2, \ldots, 3^m-1).$$

The condition (22) ensure that the trigonometric polynomials $R_k^{(m+1)}(x)$ $(k=1, 2, \ldots, 3^{m+1})$ are disjoint from one another and from all the polynomials $R_k^{(i)}(x)$ with $i\leqq m$. From (22) we get that the frequencies occurring in $R_k^{(m+1)}(x)$ are even numbers, at most equal to

(23) $$4N_{3^m}+4n<(1+4+4^2+\ldots+4^{3^m})(16n+1).$$

In virtue of (17) we get

$$\int_{-\pi}^{\pi}\Big(\sum_{k=1}^{3^{m+1}} R_k^{(m+1)}(x)\Big)^2 dx \leqq C_6 \sum_{k=1}^{3^m} \operatorname{mes}(E_k^{(m)}) \leqq C_8,$$

so that (ii) holds for $i=m+1$ too, with $C_8=C_6\pi/4$.

By Lemma 2 there exists a decomposition of the set $(E_k^{(m)})^{(\varkappa\varepsilon)}$ into three mutually disjoint simple subsets, which we denote now by $E_{3k-2}^{(m+1)}$, $E_{3k-1}^{(m+1)}$ and $E_{3k}^{(m+1)}$; thus

(24) $$\sum_{j=1}^{l} R_{3k-3+j}^{(m+1)}(x) \geqq \frac{1}{4} \quad \text{for} \quad x\in E_{3k-3+l}^{(m+1)}$$

$(l=1, 2, 3;\; k=1, 2, \ldots, 3^m)$. It is clear that the simple sets $E_k^{(m+1)}$ $(k=1, 2, \ldots, 3^{m+1})$ are disjoint. Using also the induction hypotheses, we get

$$\operatorname{mes}(F_{m+1}) \leqq \operatorname{mes}(F_m)+\operatorname{mes}\Big(\bigcup_{k=1}^{3^m}\big(E_k^{(m)}-(E_{3k-2}^{(m+1)}\cup E_{3k-1}^{(m+1)}\cup E_{3k}^{(m+1)})\big)\Big) \leqq$$

$$\leqq \varepsilon\Big(1-\frac{1}{2^m}\Big)+2f_m\cdot 2\varkappa\varepsilon.$$

Thus if we choose the hitherto indetermined $\varkappa$ such that

(25) $$\varkappa=\frac{1}{2^{m+3}f_m},$$

then (19) will be satisfied. We can easily see that F_{m+1} consists of at most $2f_{m+1}$ intervals, because

$$2\sum_{k=1}^{3^{m+1}}\frac{4}{\pi}\operatorname{mes}(E_k^{(m+1)})f_{m+1} \leqq 2f_{m+1}.$$

This proved that (iii) holds.

The arrangement of trigonometric polynomials $R_k^{(i)}(x)$ with $k=1, 2, \ldots, 3^i$; $i=1, 2, \ldots, m, m+1$ into a sequence, as required by (iv), will be realized as follows. On the basis of induction hypothesis we have a sequence

$$U_1^{(m)}(x), U_2^{(m)}(x), \ldots, U_{J_m}^{(m)}(x) \tag{26}$$

of all the polynomials $R_k^{(i)}(x)$ with $i \leqq m$. For every trigonometric polynomial $R_k^{(m)}(x)$ $(k=1, 2, \ldots, 3^m)$ we find the place, where it occurs in the sequence (26), and then we insert the trigonometric polynomials

$$R_{3k-2}^{(m+1)}(x), \quad R_{3k-1}^{(m+1)}(x) \quad \text{and} \quad R_{3k}^{(m+1)}(x)$$

immediately after $R_k^{(m)}(x)$ in (26). In such a way we have ordered into a sequence $\{U_j^{(m+1)}(x)\}$ all the trigonometric polynomials $R_k^{(i)}(x)$ with $i \leqq m+1$. [4])

For every k $(k=1, 2, \ldots, 3^{m+1})$ let $\mu_k^{(m+1)}$ denote the subscript j of that term of the sequence $\{U_j^{(m+1)}(x)\}$ which is equal to $R_k^{(m+1)}(x)$. A simple calculation shows that

$$\sum_{j=1}^{\mu_{3k-3+l}^{(m+1)}} U_j^{(m+1)}(x) \geqq \sum_{j=1}^{\mu_k^{(m)}} U_j^{(m)}(x) + \sum_{j=1}^{l} R_{3k-3+j}^{(m+1)}(x) - \sum_{j=1}^{3^{m+1}}{}' |R_j^{(m+1)}(x)|,$$

where the last sum is taken for each index j except $j=3k-3+l$ $(l=1, 2, 3)$. On the basis of the induction hypothesis, of (16) and (24), we get

$$\sum_{j=1}^{\mu_{3k-3+l}^{(m+1)}} U_j^{(m+1)}(x) \geqq \frac{m}{8} + \frac{1}{4} - (3^m - 1) C_5 \eta$$

for every $x \in E_{3k-3+l}^{(m+1)}$ $(l=1, 2, 3;\ k=1, 2, \ldots, 3^m)$, and this will be $\geqq (m+1)/8$ if we now fix the value of η as follows:

$$\eta = \frac{1}{8C_5(3^m - 1)}. \tag{27}$$

This proved that (iv) holds for the case $m+1$ too.

Thus we have showed the properties (i)—(iv) with the exception, in (i), of the assertion concerning f_{m+1}, i.e. that $f_{m+1} = (C_7/\varepsilon)^{m+1} 4^{4^{m+1}}$. By (25) and (27), n must be chosen so that

$$n \geqq \max\left(\frac{C_1}{\varkappa \varepsilon \eta}, f_m\right) = \frac{64 C_1 C_5 \cdot 6^m f_m}{\varepsilon},$$

for example $n = [C_7 6^m f_m/\varepsilon] + 1$, where $C_7 = 64\, C_1 C_5$. By (23), the frequencies occurring in $R_k^{(m+1)}(x)$ equal at most

$$\frac{C_7 4^{3^m+2} 6^m f_m}{\varepsilon} < \frac{C_7^{m+1} 4^{4^{m+1}}}{\varepsilon^{m+1}} = f_{m+1}.$$

This completes the proof of Lemma 3.

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[4]) For example, in the case $m=1$ the sequence $\{U_j^{(2)}(x)\}$ will be the following: $R_1^{(1)}(x)$, $R_1^{(2)}(x)$, $R_2^{(2)}(x)$, $R_3^{(2)}(x)$, $R_2^{(1)}(x)$, $R_4^{(2)}(x)$, $R_5^{(2)}(x)$, $R_6^{(2)}(x)$, $R_3^{(1)}(x)$, $R_7^{(2)}(x)$, $R_8^{(2)}(x)$, $R_9^{(2)}(x)$.

Lemma 4. *Let M be an arbitrary natural number. Then for every m $(m=1, 2, \ldots)$ there exist mutually disjoint trigonometric polynomials $S_j^{(m)}(x)$ $(j=1, 2, \ldots, J_{m+1})$ with the following properties:*

(v) *the frequencies ν occurring in $S_j^{(m)}(x)$ are such that $M+1 \leqq \nu \leqq 16M+4^{4^{m+C_9}}$;*

$$\text{(vi)} \qquad \int_{-\pi}^{\pi} \Big(\sum_{j=1}^{J_{m+1}} S_j^{(m)}(x) \Big)^2 dx \leqq \frac{C_{10}}{m} \quad (m=1,2,\ldots);$$

$$\text{(vii)} \qquad \sum_{j=\mu_1}^{\mu_2} S_j^{(m)}(x) \geqq \frac{1}{32} \quad \textit{for every} \quad |x| \leqq \frac{\pi}{8},$$

where $\mu_i = \mu_i^{(m)}(x)$ $(i=1, 2)$, $1 \leqq \mu_1 \leqq \mu_2 \leqq J_{m+1}$ $(m=1, 2, \ldots)$.

Proof. Let us fix the natural number m. Apply Lemma 3 with $\varepsilon_1 = 1/m$. We get that there exist mutually disjoint trigonometric polynomials $U_j^{(m)}(x)$ $(j=1, 2, \ldots J_m)$, the frequencies occurring in $U_j^{(m)}(x)$ are even numbers, at most equal to the number f_m; furthermore, there exist disjoint simple sets $E_k^{(m)}$ $(k=1, 2, \ldots, 3^m)$ such that (iii), (20) and (21) hold.

Denote by F the simple set which can be obtained from the intervals $[\alpha, \beta]$ of $[-\pi/8, \pi/8] - \bigcup_{k=1}^{3^m} E_k^{(m)}$ by replacing them with $[\alpha-\varepsilon_2, \beta+\varepsilon_2]$, where $\varepsilon_2 = \varepsilon_1/4f_m$. It is clear that F consists of at most $2f_m$ intervals. In virtue of (iii), we have

$$\operatorname{mes}(F) \leqq \operatorname{mes}\left(\left[-\frac{\pi}{8}, \frac{\pi}{8}\right] - \bigcup_{k=1}^{3^m} E_k^{(m)}\right) + 4f_m\varepsilon_2 \leqq \frac{2}{m}.$$

Apply Lemma 1′ by choosing F (instead of E), ε_2 (instead of ε), $\eta=1$ and f_{m+1} (instead of n). We get the trigonometric polynomial $P^{(m)}(x)$ with frequencies 4ν $(\nu=0, 1, \ldots, f_{m+1})$ such that (7′) and (9′) hold.

Let N_1 and N_2 be the smallest even integers for which $N_1 - f_m \geqq M+1$ and $N_2 - 4f_{m+1} \geqq 4N_1 + f_m + 1$. Now apply Lemma 2′ to each of the trigonometric polynomials $U_{j_1}^{(m)}(x)/m$ with N_1, then to the trigonometric polynomial $P^{(m)}(x)/8$ with N_2 by choosing for the three functions (15′) the following ones:

$$S_j^{(m)}(x) = \cos N_1 x \cdot \frac{U_j^{(m)}(x)}{m},$$

$$S_{J_m+j}^{(m)}(x) = -C_3 \cos x \cdot S_j^{(m)}(x),$$

$$S_{2J_m+j}^{(m)}(x) = C_4 \cos 4N_1 x \cdot \frac{U_j^{(m)}(x)}{m}$$

$(j=1, 2, \ldots, J_m)$, furthermore,

$$S_{3J_m+1}^{(m)}(x) = \cos N_2 x \cdot \frac{P^{(m)}(x)}{8},$$

$$S_{3J_m+2}^{(m)}(x) = -C_3 \cos x \cdot S_{3J_m+1}^{(m)}(x),$$

$$S_{3J_m+3}^{(m)}(x) = C_4 \cos 4N_2 x \cdot \frac{P^{(m)}(x)}{8}.$$

By this the trigonometric polynomials $S_j^{(m)}(x)$ $(j=1, 2, \ldots, J_{m+1})$ are defined because $3J_m+3=J_{m+1}$.

It is obvious that $S_j^{(m)}(x)$ $(j=1, 2, \ldots, J_{m+1})$ are mutually disjoint trigonometric polynomials with frequencies at most equal to $4N_2+4f_{m+1}$. Now, a simple calculation shows

$$4N_2+4f_{m+1} \leqq 4(4N_1+4f_{m+1}+f_m+2)+4f_{m+1} \leqq 16M+20(f_{m+1}+f_m+2)<$$

$$<16M+f_{m+2}=16M+(C_7 m)^{m+2}4^{4^{m+2}}<16M+4^{4^{m+C_9}}.$$

As to (vi), by (9′), (17′) and (21) we get that

$$\int_{-\pi}^{\pi}\Big(\sum_{j=1}^{J_{m+1}} S_j^{(m)}(x)\Big)^2 dx \leqq \frac{C_6}{m^2}\int_{-\pi}^{\pi}\Big(\sum_{j=1}^{J_m} U_j^{(m)}(x)\Big)^2 dx+$$

$$+\frac{C_2 C_6}{64}\operatorname{mes}(F) \leqq \frac{C_{10}}{m}$$

holds with $C_{10}=C_6(C_8+C_2/32)$.

To show (vii), in case of $x\in E_k^{(m)}$ we set $\mu_1^{(m)}(x)=1$ and on the ground of (18′) $\mu_2^{(m)}(x)=\mu_k^{(m)}$ or $J_m+\mu_k^{(m)}$ or $2J_m+\mu_k^{(m)}$, respectively $(k=1, 2, \ldots, J_m)$. Furthermore, in case of $x\in F^{(\varepsilon_2)}$ we set $\mu_1^{(m)}(x)=3J_m+1$ and $\mu_2^{(m)}(x)=3J_m+1$ or $3J_m+2$ or $3J_m+3$ according to (18′). Thus the indices $\mu_1^{(m)}(x)$ and $\mu_2^{(m)}(x)$ $\big(1\leqq\mu_1^{(m)}(x)\leqq \leqq\mu_2^{(m)}(x)\leqq J_{m+1}\big)$ are defined for every $|x|\leqq\pi/8$ because

$$F^{(\varepsilon_2)}=\Big[-\frac{\pi}{8}, \frac{\pi}{8}\Big]-\bigcup_{k=1}^{3^m} E_k^{(m)}.$$

In virtue of (7′), (18) and (20), the assertion (vii) holds. So the proof is complete.

§ 2. Proof of the theorems

Without any loss of generality we may assume that $\{\varrho(n)\}$ is a non-decreasing sequence. Define the sequence of natural numbers $(C_9+1\leqq)m_1<m_2<\ldots$ such that

(28) $$\frac{\varrho(n)}{\sqrt{\log\log n}} \leqq \frac{1}{k} \quad \text{if} \quad n \geqq M_k = 4^{4^{m_k+C_9}}$$

$(k=1, 2, \ldots)$; this is possible by virtue of (1). Applying Lemma 4 with M_k, we get the trigonometric polynomials $S_j^{(m_k)}(x)$ $(j=1, 2, \ldots, J_{m_k+1};\ k=1, 2, \ldots)$. Denote by $T_k(x)$ the sum of the trigonometric polynomials $S_j^{(m_k)}(x-(k)_8\pi/4)$ [5] $(j=1, 2, \ldots, J_{m_k+1})$; it is obvious that

(29) $$T_k(x)=\sum_{n=M_k+1}^{17M_k}(a_n\cos nx+b_n\sin nx) \qquad (k=1, 2, \ldots).$$

[5]) $(k)_8$ denotes the remainder of k modulo 8.

Consider the series

$$\text{(30)} \qquad \text{a)} \quad \sum_{k=1}^{\infty} T_k(x), \qquad \text{b)} \quad \sum_{k=1}^{\infty} \frac{\sqrt{m_k}}{k} T_k(x).$$

The trigonometric polynomials $T_k(x)$ and $T_l(x)$ do not overlap for $k \neq l$ because $17 M_k \leqq M_k^4 \leqq M_{k+1}$ $(k=1, 2, \ldots)$. Therefore, writing every $T_k(x)$ in (30) in extenso, we represent (30) in the form of trigonometric series

$$\text{(31)} \qquad \text{a)} \quad \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \qquad \text{b)} \quad \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx),$$

where a_n and b_n are defined by (29), $A_n = a_n \sqrt{m_k}/k$ and $B_n = b_n \sqrt{m_k}/k$ if $M_k + 1 \leqq$ $\leqq n \leqq 17 M_k$ $(k=1, 2, \ldots)$; and a_n, b_n, A_n, B_n equal 0 otherwise.

In virtue of (vi) and (28) the following estimates hold:

$$\sum_{n=1}^{\infty} (a_n^2 + b_n^2) \varrho^2(n) \leqq \sum_{k=1}^{\infty} \varrho^2(17 M_k) \sum_{n=M_k+1}^{17 M_k} (a_n^2 + b_n^2) \leqq$$

$$\leqq \sum_{k=1}^{\infty} \varrho^2(M_k^4) \int_{-\pi}^{\pi} T_k^2(x)\, dx \leqq C_{10} \sum_{k=1}^{\infty} \frac{m_k + C_9 + 1}{k^2 m_k} \leqq 2C_{10} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$

and

$$\sum_{n=1}^{\infty} (A_n^2 + B_n^2) = \sum_{k=1}^{\infty} \frac{m_k}{k^2} \int_{-\pi}^{\pi} T_k^2(x)\, dx \leqq C_{10} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

Hence, (31a) and (31b) are Fourier series of square integrable functions, and, in addition, (31a) satisfies (2).

Write down the mutually disjoint trigonometric polynomials $S_j^{(m_k)}(x)$ in this order:

$$\text{(32)} \quad S_1^{(m_1)}(x), \ldots, S_{J_{m_1}+1}^{(m_1)}(x); S_1^{(m_2)}(x), \ldots, S_{J_{m_2}+1}^{(m_2)}(x); \ldots; S_1^{(m_k)}(x), \ldots, S_{J_{m_k}+1}^{(m_k)}(x); \ldots$$

and label the occurring frequencies, in this order, by the subscript $n(j)$ $(j=1, 2, \ldots)$. It is clear that for the frequencies $n(j)$ occurring in the trigonometric polynomials $S_1^{(m_k)}(x), \ldots, S_{J_{m_k}+1}^{(m_k)}(x)$ of (32) $(k=1, 2, \ldots)$, we have

$$\text{(33)} \qquad M_k + 1 \leqq n(j) \leqq 17 M_k.$$

It is obvious that series

$$\text{(34)} \qquad \begin{aligned} &\text{a)} \quad \sum_{j=1}^{\infty} \left(a_{n(j)} \cos n(j)x + b_{n(j)} \sin n(j)x\right), \\ &\text{b)} \quad \sum_{j=1}^{\infty} \left(A_{n(j)} \cos n(j)x + B_{n(j)} \sin n(j)x\right) \end{aligned}$$

are well deteinedrm arrangements of the non- vanishing terms of (31a) and (31b), respectively.

In virtue of (vii), the partial sums of (34a) diverge everywhere. Thus the proof of Theorem 1 is complete.

As to (3), denote by $\sigma_\lambda(x)$ the λ th partial sum of (34b). For any $x \in [-\pi/8, \pi/8]$ and for $k=1, 2, \ldots$ denote by $j_1 = j_{1k}(x)$ the first natural number j, for which the frequency $n(j)$ occurs in $S_{\mu_1}^{(m_k)}(x)$, and by $j_2 = j_{2k}(x)$ the last natural number j, for which the frequency $n(j)$ occurs in $S_{\mu_2}^{(m_k)}(x)$, where the subscripts $\mu_i = \mu_i^{(m_k)}(x)$ $(i=1, 2)$ are defined in Lemma 4, by (vii). Thus, we have

$$\sigma_{j_2}(x) - \sigma_{j_1}(x) = \frac{\sqrt{m_k}}{k} \sum_{j=\mu_1}^{\mu_2} S_j^{(m_k)}\left(x - (k)_8 \frac{\pi}{4}\right),$$

and by (33), it is obvious that

$$M_k + 1 \leqq j_{1k}(x) \leqq j_{2k}(x) \leqq 17 M_k \qquad \left(k=1, 2, \ldots; \; |x| \leqq \frac{\pi}{8}\right).$$

Hence, for every $x \in [-\pi/8, \pi/8]$, $j_{1k}(x)$ and $j_{2k}(x)$ tend to ∞ with k. In virtue of (vii) and (28) we obtain

$$\frac{\sigma_{j_2}(x) - \sigma_{j_1}(x)}{\varrho(j_2)} \geqq \frac{\sqrt{m_k}}{32 k \varrho(17 M_k)} \geqq \frac{\sqrt{\log\log M_k^4}}{32\sqrt{2}\, k \varrho(M_k^4)} \geqq \frac{1}{32\sqrt{2}}$$

$(|x - (k)_8 \pi/4| \leqq \pi/8; \quad k-1, 2, \ldots)$. Thus

$$\max\left(\frac{|\sigma_{j_2}(x)|}{\varrho(j_2)}, \frac{|\sigma_{j_1}(x)|}{\varrho(j_1)}\right) \geqq \frac{1}{64\sqrt{2}}.$$

Taking into account the construction of the trigonometric polynomials $T_k(x)$, (31b) satisfies (3) for every $x \in [-\pi, \pi]$. This concludes the proof of Theorem 2.

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(Received November 1, 1966)

Smoothness conditions for Fourier series with monotone coefficients

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HARDY and LITTLEWOOD [2] showed that for Fourier series with monotone coefficients it is possible to connect the integrability of the function and the summability of the coefficients. We show how it is possible to get a similar theorem connecting the smoothness of the function with the summability of the coefficients.

Theorem 1. *Let* $0<\alpha<2$, $1<p<\infty$, $1\leqq q\leqq\infty$,

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx,\quad a_{n+1}\leqq a_n.$$

Then

$$\left[\sum a_n^q n^{q(\alpha+1-1/p)-1}\right]^{1/q}$$

is finite if and only if

$$\left[\int_0^\pi\left[\int_0^\pi\left|\frac{f(x+t)-2f(x)+f(x-t)}{t^\alpha}\right|^p dx\right]^{q/p}\frac{dt}{t}\right]^{1/q} \tag{1}$$

is finite.

The class of functions for which (1) is finite is usually denoted by $\Lambda(\alpha,p,q)$ and has been extensively studied by TAIBLESON [8]. Some special cases of this theorem have been found previously; $p=q=\infty$, $0<\alpha<1$ in [7], $1<p<\infty$, $q=\infty$, $0<\alpha<1$ in [4], and a number of different cases in [6]. To simplify the exposition of this note we use two results that are implicity contained in [1]:

$$a_n\leqq An^{-1+1/p}\inf_{\frac{\pi}{(n+1)}\leqq t\leqq\frac{\pi}{n}}\left[\int_0^\pi|f(x+t)-2f(x)+f(x-t)|^p\,dx\right]^{1/p}, \tag{2}$$

$$\left\{\begin{aligned}&\sup_{0\leqq t\leqq\pi/n}\left[\int_0^\pi|f(x+t)-2f(x)+f(x-t)|^p\,dx\right]^{1/p}\leqq\\&\leqq An^{-2}\left[\sum_{k=1}^{n}k^{3p-2}a_k^p\right]^{1/p}+B\left[\sum_{k=n}^{\infty}k^{p-2}a_k^p\right]^{1/p};\end{aligned}\right. \tag{3}$$

(2) is also contained in [5].

[1]) Supported in part by N. S. F. grant GP-3483.

Finally we need a form of HARDY's inequality and also the reverse inequality to HARDY's inequality which holds when the terms are monotone.

Theorem A. *If* (a) $c>1$, $s_n=a_1+\dots+a_n$, *or* (b) $c<1$, $s_n=a_n+a_{n+1}+\dots$, *then*

$$\sum n^{-c}s_n^p \leqq K\sum n^{-c}(na_n)^p, \qquad 1<p<\infty.$$

Theorem B. *If* (a) $c>1$, $s_n=a_1+\dots+a_n$, *or* (b) $c<1$, $s_n=a_n+a_{n+1}+\dots$, *and* $n^{-k}a_n$ *is monotone for some* k, *then*

$$\sum n^{-c}s_n^p \leqq K\sum n^{-c}(na_n)^p, \qquad 0<p<1.$$

Theorem A is in [3, p. 255] and Theorem B is in [5, p. 75 and p. 83].
Assume first that (1) is finite. By (2) we have

$$n^{(\alpha+1-1/p)q-1}a_n^q \leqq A \inf_{\frac{\pi}{n+1}\leqq t\leqq\frac{\pi}{n}} \left[\int_0^{\pi}|f(x+t)-2f(x)+f(x-t)|^p\,dx\right]^{q/p} n^{\alpha q-1} \leqq$$

$$\leqq A\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}}\left[\int_0^{\pi}|f(x+t)-2f(x)+f(x-t)|^p\,dx\right]^{q/p}\frac{dt}{t^{\alpha q+1}}$$

and summing we have

$$\sum n^{(\alpha+1-1/p)q-1}a_n^q \leqq A\int_0^{\pi}\left[\int_0^{\pi}|f(x+t)-2f(x)+f(x-t)|^p\,dx\right]^{q/p}\frac{dt}{t^{\alpha q+1}}.$$

To prove the other inequality we use (3) and Theorems A and B. By (3)

$$\int_0^{\pi}\left[\int_0^{\pi}|f(x+t)-2f(x)+f(x-t)|^p\,dx\right]^{q/p}\frac{dt}{t^{\alpha q+1}}=$$

$$=\sum_{n=1}^{\infty}\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}}\left[\int_0^{\pi}|f(x+t)-2f(x)+f(x-t)|^p\,dx\right]^{q/p}\frac{dt}{t^{\alpha q+1}}\leqq$$

$$\leqq A\sum_{n=1}^{\infty}n^{\alpha q-1}n^{-2q}\left[\sum_{k=1}^{n}k^{3p-2}a_k^p\right]^{q/p}+A\sum_{n=1}^{\infty}n^{\alpha q-1}\left[\sum_{k=n}^{\infty}k^{p-2}a_k^p\right]^{q/p}.$$

If $q/p<1$ we use Theorem B. If $q/p>1$ we use Theorem A and if $q=p$ we interchange the order of summation. Then we get

$$(1)\leqq A\sum_{n=1}^{\infty}n^{\alpha q-1-2q}[n^{3p-1}a_n^p]^{q/p}+A\sum_{n=1}^{\infty}n^{\alpha q-1}[n^{p-1}a_n^p]^{q/p}=A\sum_{n=1}^{\infty}n^{(\alpha+1-1/p)q-1}a_n^q.$$

The conditions on the parameters that must be satisfied to use Theorems A and B are all implied by the condition $0<\alpha<2$. The proof for the case $q=\infty$ is an obvious adaptation of the above proof.

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(Received July 2, 1966)

Ableitungen von trigonometrischen Approximationsprozessen

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1. Einleitung

Das Problem dieser Arbeit ist der Versuch einer Umkehrung eines Satzes von ZAMANSKY [15], S. 26. Es werden hier für spezielle trigonometrische Polynome Aussagen von der Art dieses Satzes und ihre Umkehrung bewiesen. Beide Fälle werden für Folgen von Operatoren untersucht, die den Raum $C_{2\pi}$ bzw. $L^p_{2\pi}$ $(1 \leqq p < \infty)$ in den Raum der trigonometrischen Polynome höchstens n-ter Ordnung abbilden.

Der Satz von ZAMANSKY [15] lautet: *Sei g eine stetige, 2π-periodische Funktion und t_n ein trigonometrisches Polynom von der Ordnung n. Gilt*

$$(1.1)\qquad \sup_x |t_n(x)-g(x)| \equiv \|t_n-g\| = O\left(\frac{\lambda(n)}{n^{r-1}}\right) \quad (n\to\infty),$$

wobei $\varphi(u)$ eine stetige, positive, nicht wachsende Funktion in u ist, dann gilt für die r-te Ableitung $t_n^{(r)}$ von t_n die Abschätzung

$$(1.2)\qquad \|t_n^{(r)}\| \leqq A+B\,n\,\lambda(n)+C\int_1^n \lambda(u)\,du,$$

worin A, B und C Konstanten sind.

Ersetzt man die Bedingung (1. 1) durch

$$(1.3)\qquad \|t_n-g\| = O\left(\omega_r\left(\frac{1}{n};\, g\right)\right) \quad (n\to\infty),$$

dann folgt daraus

$$(1.4)\qquad \|t_n^{(r)}\| = O\left(n^r\,\omega_r\left(\frac{1}{n};\, g\right)\right) \quad (n\to\infty),$$

wie STEČKIN [10], S. 230, und im Falle $r=1$ auch ZAMANSKY [15], S. 29, gezeigt haben. Der r-te Stetigkeitsmodul $\omega_r(\delta;\, g)$ ist definiert durch

$$\omega_r(\delta;\, g) = \max_{h\leqq\delta}\left\| \sum_{k=0}^{r} (-1)^{r-k}\binom{r}{k} f(x+kh)\right\|.$$

Setzt man speziell

(1. 5) $$\omega_r\left(\frac{1}{n}; g\right) = O(n^{-\alpha}),$$ [1)]

dann folgt für $\alpha < r$ aus (1. 3) die Aussage $\|t_n^{(r)}\| = O(n^{r-\alpha})$ $(n \to \infty)$. Da sich (nach dem Satz von S. BERNSTEIN) aus der Voraussetzung (1. 3), d. h. $\|t_n - g\| = O(n^{-\alpha})$ schon (1. 5) für $0 < \alpha < r$ ergibt, ist dies keine zusätzliche Bedingung an die Funktion g.

Für holomorphe Halbgruppen von Operatoren der Klasse (C_0) wurde ein Analogon dieses Satzes und auch die Umkehrung von BERENS [3] in seiner Dissertation bewiesen. Die Ableitung wird dort durch den infinitesimalen Erzeuger der Halbgruppe ersetzt.

Wir werden im nächsten Abschnitt der vorliegenden Arbeit zeigen, daß im allgemeinen die Aussage von ZAMANSKY nicht die bestmögliche ist, d. h. daß der Satz im allgemeinen nicht umkehrbar ist. Ist $\{t_n(x)\}$ eine Folge trigonometrischer Polynome, für die gilt $\|t_n^{(r)}\| = O(n^{r-\alpha})$ $(0 < \alpha \leq r)$, dann stellt sich die Frage, unter welchen Bedingungen an $\{t_n(x)\}$ eine Funktion g aus $C_{2\pi}$ existiert, für die gilt $\|t_n - g\| = O(n^{-\alpha})$. Dieses Problem wird hier u. a. für spezielle trigonometrische Polynome in den Räumen $C_{2\pi}$ und $L^p_{2\pi}$ $(1 \leq p < \infty)$ und im Raume $L^2_{2\pi}$ auch für die Polynome bester Approximation gelöst. Die zusätzlichen Bedingungen an die Polynome werden allgemein formuliert (Satz 2, Abschnitt 2) und dann in speziellen Beispielen verifiziert. Es wird ein allgemeiner Satz bewiesen, der zwar von elementarem Charakter ist, jedoch interessante Anwendungen auf die Teilsummen der Fourierreihe, ihre Fejérschen Mittel und die typischen Mittel besitzt und neue Ergebnisse liefert. Von besonderem Interesse erscheint Satz 6.

Zunächst noch einige Bezeichnungen. Unter $C_{2\pi}$, $L^p_{2\pi}$ $(1 \leq p < \infty)$, $L^\infty_{2\pi}$ bzw. $BV_{2\pi}$ verstehen wir die Menge der 2π-periodischen Funktionen, die stetig, bzw. zur p-ten Potenz integrierbar, bzw. meßbar und wesentlich beschränkt, bzw. von beschränkter Variation sind, wobei die Norm $\|f\|$ einer Funktion f in diesen Räumen in der üblichen Weise definiert ist. Mit $t_n^*(f; x) = (t_n^* f)(x)$ bezeichnen wir die Polynome bester Approximation der Funktion f in einem der Räume $C_{2\pi}$ oder $L^p_{2\pi}$. Lip* α ist die Klasse der Funktionen f aus $C_{2\pi}$, die die Bedingung $\max_x |f(x+h) - 2f(x) + f(x-h)| \leq M|h|^\alpha$ $(0 < \alpha \leq 2)$ erfüllen; Lip* (α, p) ist die Menge von Funktionen aus $L^p_{2\pi}$, für die

$$\left\{\int_{-\pi}^{\pi} |f(x+h) - 2f(x) + f(x-h)|^p \, dx\right\}^{1/p} \leq M|h|^\alpha$$

gilt.*)

[1)] CIVIN [7], S. 794, hat diese Spezialisierung der Behauptung für die Approximation in $L^p_{2\pi}$-Räumen für $1 < p < \infty$ und $r = 1$ bewiesen. Der Satz ist auch in der allgemeinen Formulierung (mit (1.1) und (1.2)) für die Räume $L^p_{2\pi}$ $(1 \leq p < \infty)$ gültig. Der Beweis verläuft im wesentlichen wie im stetigen Fall.

*) Diese Arbeit entstand im Rahmen des Forschungsvorhabens Bu 166/4 der DFG.

2. Allgemeiner Satz

Zunächst wollen wir zeigen, daß der Satz von ZAMANSKY im allgemeinen nicht die bestmögliche Abschätzung für die Ableitung liefert. Als Beispiel benutzen wir das singuläre Integral von DE LA VALLÉE POUSSIN, welches ein trigonometrisches Polynom von der Ordnung n ist und die Gestalt

$$(V_n f)(x) = \sum_{k=-n}^{n} \frac{(n!)^2}{(n-|k|)!\,(n+|k|)!} \hat{f}(k) e^{ikx}$$

besitzt, wobei $\hat{f}(k)$ die komplexen Fourierkoeffizienten von f sind. Hier gilt die folgende Aussage.

Satz 1. *Ist X einer der Räume $C_{2\pi}$ bzw. $L^p_{2\pi}$, dann sind für die trigonometrischen Polynome von de la Vallée Poussin die folgenden Bedingungen äquivalent:*

a) $\|V_n f - f\| = O\left(\frac{1}{n}\right) \qquad (n \to \infty)$;

b) $\|V_n'' f\| = O(1) \qquad (n \to \infty)$. [2]

Der Beweis folgt mit Ergebnissen von BUTZER [4], S. 304. und 306, und dem Saturationssatz für $V_n f$; siehe [12], S. 81. Mit dem Satz von ZAMANSKY gewinnt man aus a) nur $\|V_n'' f\| = O(n)$. Wir haben aus Satz 1 außerdem noch die Umkehrung.

Ist X ein Banachraum und $\{P_n\}$, $\{T_n\}$ und $\{U_n\}$ sind Folgen von Operatoren, die den Raum X in den Raum der trigonometrischen Polynome höchstens n-ter Ordnung abbilden, dann bewiesen wir die folgende Aussage.

Satz 2. a) *Sei der Operator $U_n^{\langle r \rangle}$ definiert durch*

$$U_n^{\langle r \rangle} f = [\varphi(n)]^{-1} [P_n f - P_{n-1} f] \qquad (r > 0; f \in X), \tag{2.1}$$

wobei $\{\varphi(n)\}$ eine Folge von Zahlen mit $\lim_{n \to \infty} n^{r+1} |\varphi(n)| = C_1 > 0$ ist, dann folgt aus der Bedingung

$$\|U_n^{\langle r \rangle} f\| = O(n^{r-\alpha}) \qquad (0 < \alpha \leqq r), \tag{2.2}$$

daß ein Element $g \in X$ existiert, welches von $P_n f$ mit der Ordnung $O(n^{-\alpha})$ approximiert wird, d.h. es gilt

$$\|P_n f - g\| = O(n^{-\alpha}) \qquad (n \to \infty). \tag{2.3}$$

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[2]) Der Strich bedeutet die Ableitung nach x. Die Äquivalenz der Aussagen a) und b) von Satz 1 gilt auch in der allgemeineren Gestalt a′) und b′) für $0 < \alpha \leqq 1$:

a′) $\|V_n f - f\| = O(n^{-\alpha})$; b′) $\|V_n'' f\| = O(n^{1-\alpha}) \qquad (n \to \infty)$.

Der Fall $0 < \alpha < 1$ ist noch zu beweisen. Aus a′) folgt b′) ähnlich wie im Beweis des Satzes von ZAMANSKY [15], S.26/27, wenn man die dort benutzte Bernsteinsche Ungleichung durch die Abschätzung $\|V_n'' f\| \leqq n \|f\|$ für alle f aus X ersetzt. Man vergleiche dazu auch den Beweis von Satz 3.3 in [3], S.18/19. Aus b′) folgt a′) mit Satz 2a). Die Identität $V_n'' = -n^2[V_n f - V_{n-1}]$ (Beweis durch Koeffizientenvergleich) entspricht der Definition (2.1) mit $P_n = V_n$, $\varphi(n) = -n^{-2}$, $r = 1$ und $U_n^{\langle r \rangle} = V_n''$. Mit dem Zamanskyschen Satz folgt aus a′) nur $\|V_n'' f\| = O(n^{2-\alpha})$ $(n \to \infty)$.

b) *Ist der Operator* $T_n f$ *darstellbar in der Form*

(2. 4) $$T_n f = P_n f + [\psi(n)/\varphi(n)][P_n f - P_{n-1} f] \qquad (f \in X),$$

$\varphi(n)$ *ist wie in Teil* a) *definiert und die Zahlenfolge* $\{\psi(n)\}$ *genügt der Bedingung* $\lim_{n\to\infty} n^r |\psi(n)| = C_2 > 0$, *dann folgt aus* (2. 2) *daß das Element* $g \in X$ *durch die Folge* $T_n f$ *ebenfalls mit der Ordnung* $O(n^{-\alpha})$ *approximiert wird, d.h.*

(2. 5) $$\|T_n f - g\| = O(n^{-\alpha}) \qquad (n \to \infty).$$

c) *(Umkehrung zu* a) *und* b).*)* *Gelten für ein* $g \in X$ *die Aussagen* (2. 3) *und* (2. 5), *wobei zwischen den Folgen* $P_n f$ *und* $T_n f$ *die Beziehung* (2. 4) *besteht, dann folgt* (2. 2), *also* $\|U_n^{\langle r\rangle} f\| = O(n^{r-\alpha})$ $(0 < \alpha \leqq r)$.

Beweis. *Teil* a). Es gilt für $m > n$ wegen (2. 1)

$$P_m f - P_n f = \sum_{k=n+1}^{m} [P_k f - P_{k-1} f] = \sum_{k=n+1}^{m} \varphi(k)\, U_k^{\langle r\rangle} f,$$

also ist nach Voraussetzung, wenn n groß genug gewählt ist,

$$\|P_m f - P_n f\| \leqq \sum_{k=n+1}^{m} |\varphi(k)|\, \|U_k^{\langle r\rangle} f\| \leqq M \sum_{k=n+1}^{m} \frac{1}{k^{r+1}} k^{r-\alpha} = M \sum_{k=n+1}^{m} \frac{1}{k^{1+\alpha}} \leqq$$

$$\leqq M \int_n^\infty \frac{du}{u^{1+\alpha}} = \frac{M}{\alpha} n^{-\alpha}$$

mit einer positiven Konstanten M, wobei der letzte Ausdruck kleiner als ε für alle $m > n \geqq N_0(\varepsilon)$ ist. Daher existiert wegen der Vollständigkeit der Räume X ein Element g aus X, so daß $\lim_{n\to\infty} \|P_n f - g\| = 0$ ist. Da in der obigen Ungleichung die rechte Seite von m unabhängig ist, folgt für $m \to \infty$ die Behauptung $\|P_n f - g\| = O(n^{-\alpha})$ $(n \to \infty)$.

Teil b). Es gilt $T_n f - g = P_n f - g + [\psi(n)/\varphi(n)][P_n f - P_{n-1} f]$ wegen (2. 4). Infolge der Voraussetzung (2. 2) gilt (2. 3) und damit

$$\|T_n f - g\| \leqq \|P_n f - g\| + |\psi(n)|\, \|U_n^{\langle r\rangle} f\| \leqq M_1 n^{-\alpha} + M_2 n^{-r} n^{r-\alpha} = O(n^{-\alpha}) \qquad (n \to \infty).$$

Teil c) folgt direkt aus (2. 4), denn es ist

$$U_n^{\langle r\rangle} f = [\varphi(n)]^{-1} [P_n f - P_{n-1} f] = [\psi(n)]^{-1} [T_n f - P_n f],$$

also

$$\|U_n^{\langle r\rangle} f\| \leqq |\psi(n)|^{-1} \{\|T_n f - g\| + \|P_n f - g\|\} \leqq M n^r \{N_1 n^{-\alpha} + N_2 n^{-\alpha}\} = O(n^{r-\alpha}),$$

womit der Satz bewiesen ist. Hier wurde benutzt, daß die Konstante C_2 im Teil b) des Satzes positiv ist, also $\lim_{n\to\infty} (n^r |\psi(n)|)^{-1} = 1/C_2$ folgt.

Bemerkung zu a). Da $P_n f$ ein trigonometrisches Polynom ist, gilt für das Polynom bester Approximation $(t_n^* g)(x)$ von g die Beziehung $\|t_n^* g - g\| \leqq \|P_n f - g\|$. Ist nun $\|P_n f - g\| = O(n^{-\alpha})$, dann folgt daraus nach dem Satz von Bernstein für $X = C_{2\pi}$, daß $g^{(k)}(x) \in \mathrm{Lip}^* \beta$ ist mit $k + \beta = \alpha$ und k ganz, $0 < \beta \leqq 1$. Für $X = L_{2\pi}^p$ $(1 \leqq p < \infty)$ folgt (siehe z.B. [14], S. 337), daß $g^{(k)}(x) \in \mathrm{Lip}^* (\beta, p)$ ist.

Bemerkung zu a) und b). Man kann diese Teile des Satzes auch folgendermaßen formulieren: *Ist $T_n f$ darstellbar in der Gestalt*

$$(2.6)\qquad T_n f = P_n f + \chi(n)[P_n f - P_{n-1} f] \qquad (f \in X),$$

wobei $\{\chi(n)\}$ eine Zahlenfolge mit der Bedingung $\lim_{n\to\infty} n^{-1}|\chi(n)| = C_3 > 0$ ist, dann folgt aus der Voraussetzung

$$(2.7)\qquad \|P_n f - P_{n-1} f\| = O(n^{-1-\alpha}) \qquad (\alpha > 0),$$

daß ein Element $g \in X$ existiert mit

$$\|P_n f - g\| = O(n^{-\alpha}) \quad und \quad \|T_n f - g\| = O(n^{-\alpha}) \qquad (n \to \infty).$$

Diese Formulierung zeigt, daß die Beziehung (2. 2), die hier durch (2. 7) ersetzt wird, eigentlich eine Aussage über die Ordnung der Differenz zweier sukzessiver Operatoren $P_n f$ ist. Offenbar lassen sich auch die Operatoren P_n, T_n, U_n aus Satz 2, die den Banachraum X in den Raum der trigonometrischen Polynome von höchstens n-ter Ordnung abbilden, durch Operatoren ersetzen, die diese Räume X in sich überführen.

Folgerung 1. *Ist der Operator $U_n^{\langle r\rangle}$ wie in* (2. 1) *definiert, gilt* (2. 4), *und folgt für zwei feste Elemente f und g in X aus $\|T_n f - g\| = O(n^{-\alpha})$ die Relation $\|P_n f - g\| = O(n^{-\alpha})$, so ist die Bedingung*

$$\|T_n f - g\| = O(n^{-\alpha}) \qquad (n \to \infty)$$

dann und nur dann erfüllt, wenn gilt

$$\|U_n^{\langle r\rangle} f\| = O(n^{r-\alpha}) \qquad (n \to \infty;\ 0 < \alpha \leqq r).$$

Beweis. Ist $\|U_n^{\langle r\rangle} f\| = O(n^{r-\alpha})$ vorausgesetzt, dann folgt aus den Teilen a) und b) des Satzes 2, da (2. 4) erfüllt ist, die Aussage $\|T_n f - g\| = O(n^{-\alpha})$. Setzt man umgekehrt $\|T_n f - g\| = O(n^{-\alpha})$ voraus, und gilt auch $\|P_n f - g\| = O(n^{-\alpha})$, dann folgt nach Teil c) für $0 < \alpha \leqq r$ $\|U_n^{\langle r\rangle} f\| = O(n^{r-\alpha})$ $(n \to \infty)$.

Die Folgerung 1 ist also eine Aussage vom Zamanskyschen Typ und ihre Umkehrung, wenn man den Operator $U_n^{\langle r\rangle}$, der eine Differenz von Operatoren darstellt, als Ableitung auffaßt. Eine ähnliche Aussage wie diese Folgerung im Falle $\alpha = r = 1$, wo die Operatoren P_n, T_n und $U_n^{\langle r\rangle}$ arithmetische Mittel einer Reihe von Elementen eines Banachraumes sind, haben ALEXITS [1] und FAVARD [8] bewiesen und diese dann zur Bestimmung der Saturationsklasse der Fejérmittel einer Fourierreihe herangezogen. Es wird nun gezeigt, daß Satz 1 und Folgerung 1 auf Teilsummen einer Fourierreihe, auf ihre Fejérschen und typischen Mittel und deren konjugierte Mittel anwendbar sind. Der Operator $U_n^{\langle r\rangle}$ ist in diesen Fällen eine Ableitung eines dieser Verfahren.

3. Anwendungen

Hier ist X einer der Räume $C_{2\pi}$ bzw. $L^p_{2\pi}$ $(1 \leqq p < \infty)$. Mit $(S_n f)(x)$ bezeichnet man die n-te Teilsumme der Fourierreihe einer Funktion f aus X, d. h.

$$(3.1) \qquad (S_n f)(x) = \sum_{k=-n}^{n} \hat{f}(k) e^{ikx}, \quad \hat{f}(k) = (1/2\pi) \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx.$$

Ihre typischen Mittel oder auch Rieszmittel sind definiert durch

$$(3.2) \qquad (R_{n,r} f)(x) = \sum_{k=-n}^{n} \left[1 - \left(\frac{|k|}{n+1}\right)^r\right] \hat{f}(k) e^{ikx}$$

für alle $r = 1, 2, 3, \ldots$. Für $r = 1$ hat man die Fejérmittel $(\sigma_n f)(x) = (R_{n,1} f)(x)$. Mit $(R_{n,r,2} f)(x)$ bezeichnen wir die Mittel

$$(3.3) \quad (R_{n,r,2} f)(x) = \sum_{k=-n}^{n} \left[1 - \left(\frac{|k|}{n+1}\right)^r\right] \left[1 - \left(\frac{|k|}{n+2}\right)^r\right] \hat{f}(k) e^{ikx} = (R_{n+1,r}[R_{n,r} f])(x),$$

die für $r = 1$ die zweiten arithmetischen Mittel $(\sigma_{n,2} f)(x)$ sind.

$$(3.4) \qquad (\tilde{S}_n f)(x) = -i \sum_{k=-n}^{n} (\operatorname{sign} k) \hat{f}(k) e^{ikx} \qquad (\operatorname{sign} 0 = 0)$$

ist die n-te Teilsumme der konjugierten Fourierreihe von f. Die zu f konjugierte Funktion $\tilde{f}$ ist definiert durch

$$(3.5) \qquad \tilde{f}(x) = -\frac{1}{2\pi} \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{\pi} [f(x+t) - f(x-t)] \operatorname{cotg} \frac{t}{2}\, dt.$$

Weiter benutzen wir die Bezeichnung

$$(3.6) \qquad f^{\{r\}}(x) = \begin{Bmatrix} f^{(r)}(x), & \text{wenn } r \text{ gerade ist,} \\ \tilde{f}^{(r)}(x), & \text{wenn } r \text{ ungerade ist} \end{Bmatrix}; \quad f^{\{0\}}(x) = f(x).$$

Bevor wir nun zu Anwendungen des allgemeinen Satzes kommen, wollen wir ein bekanntes Ergebnis für das singuläre Integral von Abel—Poisson zitieren, welches kein trigonometrisches Polynom ist, jedoch eine holomorphe Halbgruppe von Operatoren der Klasse (C_0) bildet (siehe [3]). Dieses Integral ist für f aus $C_{2\pi}$ definiert durch

$$(V(r) f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u) \frac{1 - r^2}{1 - 2r \cos(x-u) + r^2}\, du \quad (0 \leqq r < 1).$$

Als Anwendung des Satzes von Berens ([3], S. 23) erhält man für ein $f \in C_{2\pi}$ bei $0 < \alpha < 1$ folgende gleichwertige Aussagen ([3], S. 40):

a) $\|\tilde{V}'(r)f\| = O[(1-r)^{\alpha-1}] \qquad (r\uparrow 1)$; [3])

b) $\|V''(r)f\| = O[(1-r)^{\alpha-2}] \qquad (r\uparrow 1)$;

c) $\|V(r)f-f\| = O[(1-r)^{\alpha}] \qquad (t\uparrow 1)$,

wobei der Strich die Ableitung nach der Variablen x bedeutet. Wir gewinnen in dieser Arbeit entsprechende Aussagen für spezielle trigonometrische Polynome t_n, wenn man in der O-Bedingung den Ausdruck $(1-r)$ $(r\uparrow 1)$ durch $1/n$ $(n\to\infty)$ ersetzt.

Wir betrachten zunächst die Rieszmittel, wie sie in (3. 2) und (3. 3) definiert sind. Für sie gilt das folgende Lemma.

Lemma 1. *Für die Mittel $R_{n,r}f$ und $R_{n,r,2}f$ gelten folgende Identitäten:*

$$(3.7) \qquad (R_{n,r}f)(x) = (R_{n,r,2}f)(x) + \frac{n^r}{(n+2)^r - n^r}[(R_{n,r,2}f)(x) - (R_{n-1,r,2}f)(x)];$$

$$(3.8) \qquad (R_{n,r,2}f)(x) - (R_{n-1,r,2}f)(x) = (-1)^{[r/2]}\frac{(n+2)^r - n^r}{(n+2)^r n^r}(R_{n,r}^{\{r\}}f)(x),$$

wobei $[r/2]$ die größte ganze Zahl $\leqq r/2$ ist.

Der Beweis folgt durch Koeffizientenvergleich. Hiermit kommt man zu

Satz 3. *Eine notwendige und hinreichende Bedingung für die Approximation einer Funktion f aus X durch Rieszmittel mit der Ordnung $O(n^{-\alpha})$ $(n\to\infty)$, d.h.*

a) $\|R_{n,r}f-f\| = O(n^{-\alpha}) \qquad (0<\alpha\leqq r)$

ist die Bedingung

b) $$\|R_{n,r}^{\{r\}}f\| = O(n^{r-\alpha}).$$ [4])

Beweis. Hierzu benutzen wir die Folgerung 1. Setzt man $P_nf = R_{n,r,2}f$, dann ist wegen (3. 8) $U_n^{\langle r\rangle}f = R_{n,r}^{\{r\}}f$ mit

$$\varphi(n) = (-1)^{[r/2]}\frac{(n+2)^r - n^r}{(n+2)^r n^r},$$

das die Bedingung $\lim\limits_{n\to\infty} n^{r+1}|\varphi(n)| = 2r > 0$ erfüllt. Die Bedingung (2. 4) ist wegen (3. 7) gültig, wenn man noch $T_nf = R_{n,r}f$ und $\psi(n) = (-1)^{[r/2]}(n+2)^{-r}$ setzt. Wir müssen hier noch zeigen, daß aus $\|R_{n,r}f-f\| = O(n^{-\alpha})$ die Aussage $\|R_{n,r,2}f-f\| = O(n^{-\alpha})$ folgt. Es ist wegen (3. 3)

$$(R_{n,r,2}f)(x) - f(x) = (R_{n,r,2}f)(x) - (R_{n+1,r}f)(x) + (R_{n+1,r}f)(x) - f(x) =$$
$$= (R_{n+1,r}[R_{n,r}f - f])(x) + (R_{n+1,r}f)(x) - f(x),$$

also

$$\|R_{n,r,2}f - f\| \leqq \|R_{n+1,r}\|\,\|R_{n,r}f-f\| + \|R_{n+1,r}f - f\| \leqq$$
$$\leqq M(r)M_1 n^{-\alpha} + M_2 n^{-\alpha} = O(n^{-\alpha}) \qquad (n\to\infty)$$

[3]) Das zu $V(r)f$ konjugierte Integral $\tilde{V}(r)f$ ist definiert durch

$$(\tilde{V}(r)f)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\frac{r\sin(x-u)}{1-2\cos(x-u)+r^2}\,du \qquad (0\leqq r<1).$$

[4]) Siehe auch Fußnote 5) bei Satz 5.

wegen a) und der gleichmäßigen Beschränktheit der Normen der Rieszmittel, was in [11] und [6], S. 352, gezeigt wurde. Daß die Funktion g aus Folgerung 1 hier gleich f ist, folgt aus dem Identitätssatz für Fourierreihen. Alle Voraussetzungen der Folgerung 1 sind damit erfüllt, und es folgt die Behauptung.

Im Falle $r=1$ nehmen die Identitäten aus Lemma 1 die folgende Gestalt an.

Lemma 2. *Für die arithmetischen Mittel einer Fourierreihe gelten die Identitäten*

$$(\sigma_n f)(x) = (\sigma_{n,2} f)(x) + \frac{n}{2}[(\sigma_{n,2} f)(x) - (\sigma_{n-1,2} f)(x)]; \tag{3.9}$$

$$(\tilde{\sigma}_n' f)(x) = (n/2)(n+2)[(\sigma_{n,2} f)(x) - (\sigma_{n-1,2} f)(x)]. \tag{3.10}$$

Hier gilt entsprechend

Satz 4. *Für ein* $f \in X$ *und* $0 < \alpha \leqq 1$ *ist die Aussage*

$$\|\sigma_n f - f\| = O(n^{-\alpha})$$

dann und nur dann erfüllt, wenn gilt

$$\|\tilde{\sigma}_n' f\| = O(n^{1-\alpha}).$$

Lemma 2 und Satz 4 sind bekannt; im Falle $\alpha=1$ siehe [16] für $X=C_{2\pi}$, [1] und [8] für allgemeine Banachräume, sowie für $0<\alpha\leqq 1$ [19], Ch. VII, S. 269 und 296.

Kehren wir nun zu den Rieszmitteln zurück. Im Falle $\alpha=r$ tritt bei ihnen Saturation auf, und ihre Saturationsklasse W_X^r ist definiert durch

$$W_C^r = \{f;\, f\in C_{2\pi};\ |k|^r \hat{f}(k) = \hat{g}(k);\ g\in L_{2\pi}^\infty\} \quad \text{für} \quad X = C_{2\pi},$$

$$W_1^r = \{f;\, f\in L_{2\pi}^1;\ |k|^r \hat{f}(k) = \check{g}(k);\ g\in BV_{2\pi}\} \quad \text{für} \quad X = L_{2\pi}^1,$$

wobei $\check{g}(k)$ die Fourier—Stieltjeskoeffizienten der Funktion g sind;

$$W_p^r = \{f;\, f\in L_{2\pi}^p;\ |k|^r \hat{f}(k) = \hat{g}(k);\ g\in L_{2\pi}^p\} \quad \text{für} \quad X = L_{2\pi}^p \qquad (1<p<\infty).$$

Äquivalente Charakterisierungen dieser Klassen findet man z. B. in einer Arbeit von Butzer und Görlich [5]. Mit Hilfe zweier Sätze, die z. B. in [6], S. 351, zitiert sind, kommt man zu folgendem Ergebnis.

Folgerung 3. *Für ein* $f\in X$ *sind die folgenden Aussagen äquivalent:*

a) $\|R_{n,r} f - f\| = O(n^{-r}) \quad (n\to\infty)$;

b) $\|R_{n,r}^{(r)} f\| = O(1) \quad (n\to\infty)$;

c) $\|\sigma_n^{\{r\}} f\| = O(1) \quad (n\to\infty)$;

d) $f\in W_X^r$.

Die Äquivalenzen a), c) und d) finden sich in [6], S. 351. Die Saturationsklasse der Rieszmittel für $X=C_{2\pi}$ wurde auch von Zamansky [17], S. 170, bestimmt; siehe dazu auch [2], S. 683.

Ist $\alpha<r$, dann gelten nachstehende Äquivalenzen für die Approximation durch Rieszmittel.

Folgerung 4. *Für ein Element $f \in X$ und $\alpha < r$ sind die folgenden Aussagen gleichwertig:*

a) $\|R_{n,r}f - f\| = O(n^{-\alpha})$ $(n \to \infty)$;

b) $\|R_{r,n}^{(r)}f\| = O(n^{r-\alpha})$ $(n \to \infty)$;

c) *im Falle* $X = C_{2\pi}$ *ist* $f^{(k)} \in \mathrm{Lip}^* \beta$, $k + \beta = \alpha$, *k ganz*, $0 < \beta \leqq 1$;

im Falle $X = L_{2\pi}^p$ $(1 \leqq p < \infty)$ *ist* $f^{(k)} \in \mathrm{Lip}^* (\beta; p)$.

Die Äquivalenz von a) und b) sagt Satz 3 aus; aus a) folgt c) nach dem Satz von S. Bernstein, und aus c) die Bedingung a) nach Zygmund [18], Aljančić [2], S. 683, und Sz.-Nagy [13] für $X = C_{2\pi}$ und nach Sunouchi [12] für $X = L_{2\pi}^p$ $(1 \leqq p < \infty)$.

Wir kommen nun zu Anwendungen auf die n-te Teilsumme einer Fourierreihe.

Lemma 3. *Zwischen den Rieszmitteln $R_{n,r}f$ und den n-ten Teilsummen $S_n f$ einer Fourierreihe bestehen die folgenden Identitäten:*

$$(S_n f)(x) = (R_{n,r}f)(x) + \frac{n^r}{(n+1)^r - n^r} [(R_{n,r}f)(x) - (R_{n-1,r}f)(x)]; \tag{3.11}$$

$$(R_{n,r}f)(x) - (R_{n-1,r}f)(x) = (-1)^{[r/2]} \frac{(n+1)^r - n^r}{(n+1)^r n^r} (S_n^{(r)} f)(x). \tag{3.12}$$

Der Beweis folgt wieder durch Koeffizientenvergleich. Hiermit erhält man

Satz 5. *Ist f aus X, so sind die folgenden vier Aussagen für $0 < \alpha < r$ untereinander äquivalent:* [5])[6])

a) $\|S_n f - f\| = O(n^{-\alpha})$ $(n \to \infty)$;

b) $\|\tilde{S}_n f - \tilde{f}\| = O(n^{-\alpha})$ $(n \to \infty)$;

c) $\|S_n^{(r)} f\| = O(n^{r-\alpha})$ $(n \to \infty)$;

d) $\|\tilde{S}_n^{(r)} f\| = O(n^{r-\alpha})$ $(n \to \infty)$. [6])

Beweis. Zunächst setzen wir r als ungerade voraus. Mit $R_{n,r}f = P_n f$, $\tilde{S}_n^{(r)} f = U_n^{\langle r \rangle} f$ und $\varphi(n) = (-1)^{[r/2]} \dfrac{(n+1) - n^r}{(n+1)^r n^r}$ geht die Identität (3.12) in die Definition (2.1) von $U_n^{\langle r \rangle} f$ über. Setzt man noch $T_n f = S_n f$, dann ist die Bedingung (2.4) wegen (3.11) erfüllt; $\psi(n)$ hat die Gestalt $(-1)^{[r/2]}(n+1)^{-r}$. Alle Voraussetzungen der Teile a) und b) von Satz 2 sind damit gegeben, und damit folgt aus d) die Bedingung a). Aus a) folgt c) nach dem Satz von Zamansky für $0 < \alpha < r$. Nun ist

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[5]) Ein äquivalenter Satz ist auch für die Rieszschen Mittel $R_{n,r} f$ bei $0 < \alpha < r$ gültig. Der Beweis verläuft analog.

[6]) Die Äquivalenz von a) und c) im Falle $r = 2$ und $0 < \alpha \leqq 1$ wurde unter der zusätzlichen Voraussetzung $\|t_n^* f - f\| = O(n^{-\alpha})$ von Zamansky [15], S. 81, im Raume $C_{2\pi}$ bewiesen. Daß aus a) Teil b) folgt, haben Salem und Zygmund [9] sowie Zamansky [15], S. 84, gezeigt.

zu zeigen, daß aus c) die Aussage b) folgt. Setzt man in (3. 12) auf beiden Seiten die Fourierkoeffizienten der konjugierten Reihe ein, dann erhält man für ungerade r, da dann $\tilde{S}_n^{(r)}f = \tilde{\tilde{S}}_n^{(r)}f = -S_n^{(r)}f$ ist, die Identität

$$(3.\,13)\qquad (\tilde{R}_{n,r}f)(x)-(\tilde{R}_{n-1,r}f)(x) = (-1)^{[r/2]+1}\,\frac{(n+1)^r-n^r}{(n+1)^r n^r}\,(S_n^{(r)}f)(x).$$

Hieraus und mit der Identität (3. 11) für $\tilde{S}_n f$ und $\tilde{R}_{n,r}f$ folgt wie oben, daß aus c) die Aussage b) folgt. Aus b) erhält man schließlich mit dem Satz von ZAMANSKY d), womit der Satz durch Ringschluß für ungerade r bewiesen ist.

Nun sei r gerade. Aus d) folgt nach der Bernsteinschen Ungleichung $\|\tilde{S}_n^{(r+1)}f\| = O(n^{r+1-\alpha})$ und hieraus nach Satz 2a) und b) wie oben Teil a) dieses Satzes, da $r+1$ ungerade ist. Daraus erhält man wieder c) mit dem Satz von ZAMANSKY und die Bernsteinsche Ungleichung ergibt dann $\|S_n^{(r+1)}f\| = O(n^{r+1-\alpha})$. Die Zahl $r+1$ ist ungerade, und daher folgt wie oben die Aussage b) dieses Satzes. Wenden wir noch einmal den Zamanskyschen Satz an, so folgt daraus d), und der Beweis ist vollständig.

Dieser Satz erscheint insofern interessant, als die Teilsummen der Fourierreihe im Gegensatz zu den Fejérschen und Rieszschen Mitteln keine Saturation aufweisen und außerdem im Raume $L^2_{2\pi}$ die Polynome bester Approximation bilden. Es ist also im Raume $L^2_{2\pi}$ die Umkehrung des Satzes von ZAMANSKY für die Polynome bester Approximation bewiesen, womit man zu den beiden äquivalenten Aussagen, die durch die Sätze von S. BERNSTEIN und D. JACKSON gegeben sind, eine dritte gefunden hat, d. h. es gilt

Satz 6. *Für die trigonometrischen Polynome bester Approximation t_n^*f einer Funktion f aus $L^2_{2\pi}$ sind für $\alpha < r$, $\alpha = k+\beta$, k ganz und $0<\beta\leqq 1$ folgende Aussagen gleichwertig:*

a) $f^{(k)} \in \mathrm{Lip}^*(\beta; 2)$;

b) $\|t_n^*f-f\|_{L^2} = O(n^{-\alpha})\quad (n\to\infty)$;

c) $\|t_n^{*(r)}f\|_{L^2} = O(n^{r-\alpha})\quad (n\to\infty)$.

Es ist zu vermuten, daß dieser Satz auch für den Raum $C_{2\pi}$ und alle Räume $L^p_{2\pi}$ $(1\leqq p<\infty)$ gilt, da der entsprechende Satz in diesen Räumen für die Rieszschen Mittel gültig ist (Folgerung 4), die für nichtsaturierte Approximation das gleiche Approximationsverhalten wie die Polynome bester Approximation besitzen.

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LEHRSTUHL FÜR MATHEMATIK (ANALYSIS)
TECHNISCHE HOCHSCHULE AACHEN

(Eingegangen am 12. Oktober 1966)

On the index of imprimitivity of a non-negative matrix

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1.

Let A be a non-negative $n \times n$ matrix. To study the distribution of zeros and non-zeros in the matrices of the sequence

$$(1) \qquad A, A^2, A^3, \ldots$$

we have introduced in [2] the following notations. Consider the set of symbols $E = \{e_{ij} \mid i, j = 1, 2, \ldots, n\}$ together with a zero 0 adjoined. Define in $S = \{0\} \cup E$ a multiplication by

$$e_{ij} e_{lm} = \begin{cases} e_{im} & \text{for } j = l, \\ 0 & \text{for } j \neq l, \end{cases}$$

the zero element having the usual properties of a multiplicative zero. Then S (with this multiplication) is a semigroup.

Let $A = (a_{ij})$ be a non-negative $n \times n$ matrix. By the support C_A of A we shall mean the subset of S containing 0 and all e_{ij} for which $a_{ij} > 0$.

For two non-negative $n \times n$ matrices A, B we have $C_{AB} = C_A C_B$, where the product to the right has the usual meaning used in the theory of semigroups.

In particular the supports of the elements of the sequence (1) are

$$(2) \qquad C_A, C_A^2, C_A^3, \ldots.$$

Since this sequence has only a finite number of different elements (subsets of S) it can be written in the form

$$C_A, C_A^2, \ldots, C_A^{k-1} \mid C_A^k, \ldots, C_A^{k+d-1} \mid C_A^k, \ldots, C_A^{k+d-1} \mid \ldots.$$

Here C_A^k, $k = k(A)$, is the least power in (2) which appears more than once and d is the period with which all the following powers repeat.

Denote further $S_i = \{0\} \cup \{e_{i1}, e_{i2}, \ldots e_{in}\}$ and $F_i = F_i(A) = S_i \cap C_A$, so that F_i is the "support" of the i-th row in A.

The sequence

$$F_i, F_i C_A, F_i C_A^2, \ldots$$

contains again only a finite number of different elements (subsets of S_i) and it is of the form

$$F_i, F_i C_A, \ldots, F_i C_A^{k_i-2} \mid F_i C_A^{k_i-1}, \ldots, F_i C_A^{k_i+d_i-2} \mid F_i C_A^{k_i-1}, \ldots$$

where the integers k_i, d_i have an analogous meaning as the integers k and d above.

For details concerning these notions see [3].

In [3] and [4] we have proved:

Lemma 1. *For any non-negative $n \times n$ matrix A we have:*

a) $k(A) = \max(k_1, k_2, \ldots, k_n)$;

b) $d(A) = \text{l.c.m.}\ [d_1, d_2, \ldots, d_n]$.

Lemma 2. *If A is irreducible, then $d(A) = d_1 = d_2 = \ldots = d_n$.*

Denote by g_i the number of non-zero elements in F_i. In the papers [3] and [4] we have found some estimates concerning the numbers k_i in terms of n and g_i. For instance we have proved $k_i \leqq 1 + (n - g_i)(n - g_i + 1)$.

It is intuitively clear that also the numbers d_i depend on g_i. It is the purpose of this paper to give an estimation concerning $d = d(A)$ in terms of n and g_i. For an irreducible matrix A the number d is identical with the classical notion of the index of imprimitivity of A. Our main result is formulated in the theorem below.

2.

Let M be any non-negative $n \times n$ matrix. It is well known that there is a permutation matrix P such that PMP^{-1} is of the form

$$A = \begin{pmatrix} A_{11} & & & \\ A_{21} & A_{22} & & \\ \vdots & & & \\ A_{r1} & A_{r2} & \ldots & A_{rr} \end{pmatrix},$$

where $A_{\alpha\alpha}$ are irreducible matrices (including the case that some of the $A_{\alpha\alpha}$ are zero matrices of order 1). It is easy to see that $d(A) = d(M)$. Further it can be proved (see [1], [5]) that $d(A) = \text{l.c.m.}\ [d(A_{11}), d(A_{22}), \ldots, d(A_{rr})]$. Hence $d(A)$ does not depend on the rectangular matrices $A_{\alpha\beta}$, $\alpha \neq \beta$.

It is therefore sufficient to restrict ourselves to the case of an *irreducible* matrix A.

In [3] we have proved:

Lemma 3. *If A is irreducible (of order n), then there is an integer h_i such that $1 \leqq h_i \leqq n$ and $F_i \subset F_i C_A^{h_i}$. Here:*

a) *if $e_{ii} \in F_i$, we may choose $h_i = 1$,*

b) *if F_i contains g_i non-zero elements $\in S_i$, we have, for the least number h_i satisfying the above condition, $h_i \leqq n - g_i + 1$.*

Consider now the chain

$$F_i \subset F_i C_A^{h_i} \subset F_i C_A^{2h_i} \subset \ldots$$

Since any member of this chain contains at most $n+1$ different elements (namely the elements $0, e_{i1}, \ldots, e_{in}$) there is an integer $\tau \geqq 1$ such that

$$F_i C_A^{\tau h_i} = F_i C_A^{\tau h_i + h_i}, \tag{3}$$

hence $d_i \leqq h_i \leqq n - g_i + 1$. With respect to the definition of the number d_i we conclude from (3) that $d_i | h_i$. By Lemma 2 we obtain $d | h_i$ for $i = 1, 2, \ldots, n$.

We have proved:

Lemma 4. *Let A be irreducible. Denote $\delta = (h_1, h_2, \ldots, h_n)$. We then have $d | \delta$.*

Lemma 4 implies $d \leqq \delta = (h_1, \ldots, h_n) \leqq \min_i h_i \leqq n + 1 - \max_i g_i$. We have proved:

Theorem. *Let A be an irreducible non-negative $n \times n$ matrix. Denote by g_i the number of positive entries in the i-th row of A. Then $d(A) \leqq n + 1 - \max_i g_i$.*

Remark. In terms of the integers h_i we may state the following. If $d < \min_i h_i$, then $d | \min_i h_i$ implies that we certainly have $d \leqq \frac{1}{2} \min_i h_i$. If here again the equality does not hold, we have $d \leqq \frac{1}{3} \min_i h_i$. And so on.

3.

We now give some corollaries.

Suppose that $d(A) = n$. Then $n \leqq n + 1 - \max_i g_i$ implies $g_i = 1$ for $i = 1, 2, \ldots, n$. Hence C_A is the support of an (irreducible) permutation matrix. This can be stated in the following forms:

Corollary 1. *Suppose that for an irreducible non-negative $n \times n$ matrix A we have $d(A) = n$. Then the matrix obtained by replacing the positive entries in A by the number 1 is a permutation matrix.*

Corollary 2. *Let A be a non-negative irreducible $n \times n$ matrix. Suppose that replacing all positive entries in A by the number 1 we obtain a matrix which is not a permutation matrix. Then $d(A) \leqq n - 1$.*

Example 1. In general the result of Corollary 2 cannot be sharpened. This is shown on the 4×4 matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Here

$$A^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

and $C_A = C_A^4$. Therefore $d(A) = 3$. In this case $h_i = 3$ $(i = 1, 2, 3, 4)$ and $d(A) = \min_i h_i$.

Example 2. If $\max_i g_i = n - 1$, our Theorem implies $d(A) \leqq 2$. This result is sharp in the following sense. To any $n \geqq 2$ there is an irreducible $n \times n$ matrix A

with $\max_i g_i = n-1$ such that $d(A)=2$. This property has for instance the matrix

$$A = \begin{pmatrix} 0 & 1 \dots 1 \\ 1 & 0 \dots 0 \\ \vdots & \\ 1 & 0 \dots 0 \end{pmatrix}.$$

Here

$$A^2 = \begin{pmatrix} n-1 & 0 \dots 0 \\ 0 & 1 \dots 1 \\ \vdots & \\ 0 & 1 \dots 1 \end{pmatrix}.$$

Since $C_A \cup C_A^2 = S$, the matrix A is irreducible (see [2], Theorem 1) and clearly we have $d(A)=2$.

The result of our Theorem is also sharp in the following sense. To any n and any g, $1 \leqq g \leqq n-1$, there exist numbers $g_1, \dots, g_n$ with $\max(g_1, \dots, g_n) = g$ and a matrix A having g_i non-zero elements in the ith row of A such that $d(A) = n+1-g$. Take for this purpose the matrix A with $C_A = \{0, e_{12}, e_{23}, \dots, e_{n-g, n-g+1}, e_{n-g+1,1}, e_{1,n-g+2}, \dots, e_{1,n}, e_{n-g+2,3}, \dots, e_{n,3}\}$. Here $g_1 = g, g_2 = \dots = g_n = 1$. It can be shown that $C_A = C_A^{n-g+2}$ and $n-g+2$ is the least number $l \neq 1$ satisfying $C_A^l = C_A$. Hence $k(A)=1$ and $d(A) = n-g+1$.

If at least one of the numbers h_i is equal to 1, we have $d = \delta = 1$. This means that some power of A is positive. Such a matrix is called *primitive*. Hence:

Corollary 3. *If an irreducible matrix A contains at least one row with $F_i \subset F_i C_A$, then A is primitive.*

By Lemma 3 this is certainly the case if $e_{ii} \in F_i$ for some i. This implies the following well-known result which goes back to Frobenius:

Corollary 4. *If A is irreducible and it contains a positive entry in the main diagonal, then A is primitive.*

Remark. The condition $F_i \subset F_i C_A$ is weaker then the condition $e_{ii} \in F_i$. For instance, for a matrix A with

$$C_A = \begin{Bmatrix} e_{11} & e_{12} & 0 \\ 0 & 0 & e_{23} \\ e_{31} & 0 & e_{33} \end{Bmatrix}$$

we have $F_2 = \{0, e_{23}\} \subset F_2 C_A = \{0, e_{21}, e_{23}\}$, while $e_{22} \notin F_2$.

Since $d | \delta = (h_1, \dots, h_n)$ we also have:

Corollary 5. *If two of the numbers h_i are relatively prime, then A is primitive.*

Example 3. The following example shows that $d < \delta$ is possible. Consider the matrix A and its powers:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

For $i=1,2,3$ we have $F_i \not\subset F_i C_A$ but $F_i \subset F_i C_A^2$, so that $h_1=h_2=h_3=2$; hence $\delta=2$. But our matrix is primitive, i.e., $d(A)=1<\delta$.

Remark. It is worth to remark that the set $\{h_i\}$ is not identical with an other set of integers (denoted below by $\{r_i\}$), which can be associated to any irreducible (and some reducible) non-negative matrices. Let A be irreducible. Denote by r_i the least integer $\geqq 1$ such that $e_{ii} \in F_i C_A^{r_i-1}$ and define $F_i C_A^0 = F_i$. For an irreducible matrix r_i always exists and we have $r_i \leqq n$. (In the graph-theoretical treatment of non-negative matrices the r_i's are the lengths of elementary circuits.) Since $e_{ii} \in F_i C_A^{r_i-1}$ implies $F_i = e_{ii} C_A \subset F_i C_A^{r_i}$, we have $h_i \leqq r_i \leqq n$. It is known that $d=(r_1, r_2, \ldots, r_n)$ in contradistinction to $d \leqq (h_1, h_2, \ldots, h_n)$.

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(Received December 20, 1966)

Norm relations and skew dilations

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An operator T on a Hilbert space H possesses a skew dilation if there exists a Hilbert space $K \supset H$, a constant $\varrho > 0$, and a unitary operator U on K, such that $T^n = \varrho P U^n P$ for $n = 1, 2, \ldots$, where P is the self-adjoint projection of K on H. If $T^n = \varrho P U^n P$, then following the notation of [5], we say $T \in C_\varrho$. Note that C_1 is the class of all contractions and C_2 is the set of all operators with numerical range in the unit disc. SZ.-NAGY and FOIAŞ [5] have characterized C_ϱ for general $\varrho > 0$.

In the first part of this paper, we obtain bounds on $\|T^n x\|$ for $T \in C_\varrho$. These bounds should be useful in constructing a matrix dilation for $T \in C_\varrho$ similar to the Schäffer dilation for contractions. The rest of the paper is devoted to general results on C_ϱ.

It is convenient to write $T^n = \varrho P U^n P$ or $\delta T^n = P U^n P$ depending on the context. For the rest of the paper it is assumed that $\delta = \varrho^{-1}$.

Lemma 1. *Let* $\delta T^j = P U^j P$ *for* $j = 1, 2, \ldots$. *Then* $P U^k [(I-P) U]^n P = \alpha_n T^{n+k}$ *for* $k, n = 1, 2, \ldots$, *where* α_n *is independent of* k.

Proof. Expand $[(I-P) U]^n P$ formally. Then all terms are either of the form $a T^n$ or $b U^j T^{n-j}$, after simplifications via the relation $P U^m P = \delta T^m$. But $P U^k a T^n = \delta a T^{n+k}$ and $P U^k b U^j T^{n-j} = \delta b T^{n+k}$. Thus, $P U^k [(I-P) U]^n P = \sum_m c_m T^{n+k}$, where the constants c_m do not depend on k, but only upon the coefficients arising in the formal expansion of $[(I-P) U]^n P$, and the subsequent conversion of U's to T's.

Corollary. *Let* $\delta T^j = P U^j P$ *for* $j = 1, 2, \ldots$. *Then* $P U^k [(I-P) U]^n U^m P = \alpha_n T^{k+n+m}$, *where* α_n *is independent of* k *and* m.

Proof. Same as above.

Theorem 1. $P U [(I-P) U]^n P = \delta (1-\delta)^n T^{n+1}$.

Proof. We assume that the relation is true for n and check it for $n+1$. (It obviously holds for $n = 1$.)

$$P U [(I-P) U]^{n+1} P = P U^2 [(I-P) U]^n P - P U \{P U [(I-P) U]^n P\} =$$
$$= \delta (1-\delta^n) T^{n+2} - \delta T [\delta (1-\delta)^n T^{n+1}] = \delta (1-\delta)^{n+1} T^{n+2}.$$

To convert the first term on the right, we have made use of Lemma 1 and the induction hypothesis.

*) Both authors gratefully acknowledge the support of the National Science Foundation.

Corollary. *Let* $\delta T^j = PU^jP$ *for* $j=1,2,\dots$. *Then* $PU^k[(I-P)U]^nU^mP = \delta(1-\delta)^nT^{n+k+m}$ *for* $n, k=1,2,\dots$.

Proof. Same as above using the Corollary to Lemma 1.

Lemma 2. *Let* V *be an isometry and* P *a self-adjoint projection on a Hilbert space. Then*

$$\|x\|^2 = \sum_{n=0}^{M-1} \|PV[I-P)V]^n x\|^2 + \|[(I-P)V]^M x\|^2$$

for $M=1,2,\dots$.

Proof. By induction.

Corollary. *Under the same hypothesis as above,*

$$\|x\|^2 = \sum_{n=0}^{M-1} \|PV[(I-P)V]^n V^k x\|^2 + \|[(I-P)V]^M V^k x\|^2$$

for $M, k=1,2,\dots$.

Proof. Replace x by V^kx in Lemma 2.

Theorem 2. *Let* $\delta T^j = PU^jP$ *for* $j=1,2,\dots$. *Then*

$$\sum_{n=1}^{\infty} \delta^2(1-\delta)^{2(n-1)} \|T^n x\|^2 \leqq \|x\|^2.$$

Proof. For M fixed, it follows from Lemma 2 and Theorem 1 that,

$$\|x\|^2 \geqq \sum_{n=0}^{M-1} \|PU[(I-P)U]^n x\|^2 = \sum_{n=1}^{M} \|\delta(1-\delta)^{n-1} T^n x\|^2.$$

Letting $M\to\infty$ completes the proof.

Corollary 1. *Let* $\delta T^j = PU^jP$ *for* $j=1,2,\dots$. *Then*

$$\sum_{n=1}^{\infty} \delta^2(1-\delta)^{2(n-1)} \|T^{n+k} x\|^2 \leqq \|x\|^2 \quad \textit{for} \quad k=1,2,\dots.$$

Proof. Same as above, using the Corollaries to Lemma 2 and Theorem 1.

Corollary 2. *If* $|W(T)| \leqq 1$, *then*

$$\sum_{n=1}^{\infty} 4^{-n} \|T^{n+k} x\|^2 \leqq \|x\|^2 \quad \textit{for} \quad k=1,2,\dots.$$

Proof. By [1], we know that $|W(T)| \leqq 1$ implies $\frac{1}{2}T^j = PU^jP$ for $j=1,2,\dots$.

Corollary 3. *Let* $\delta T^j = PU^jP$ *for* $j=1,2,\dots$. *If*

$$\sum_{n=1}^{M} \delta^2(1-\delta)^{2(n-1)} \|T^n x\|^2 = \|x\|^2, \quad \textit{then} \quad T^{M+1}x = 0.$$

Corollary 4. *Let* $|W(T)| \leqq 1$. *If*

$$\sum_{n=1}^{M} 4^{-n} \|T^n x\|^2 = \|x\|^2, \quad \text{then} \quad T^{M+1}x = 0.$$

Note that Corollary 4 gives us a much sharper form of the following result from [2]: If $|W(T)| \leqq 1$ and $\|Tx\| = 2\|x\|$, then $T^2 x = 0$.

Theorem 3. *Let* $T^j = \varrho P U^j P$ *for* $j = 1, 2, \ldots$ *where* $\varrho > 1$. *If* $\liminf \|T^n x_0\| = \alpha \|x_0\|$, *then* $\alpha \leqq (2\varrho - 1)^{1/2}$.

Proof. Assume $\alpha > (2\varrho - 1)^{1/2}$ for $x_0 \in H$. Then for some fixed k, $\|T^{n+k} x_0\| > (2\varrho - 1)^{1/2} \|x_0\|$ for $n = 1, 2, \ldots$. Thus,

$$\|x_0\|^2 \geqq \sum_{n=1}^{\infty} \delta^2 (1-\delta)^{2(n-1)} \|T^{n+k} x_0\|^2 >$$

$$> (2\varrho - 1) \|x_0\|^2 \sum_{n=0}^{\infty} \delta^2 (1-\delta)^{2n} = (2\varrho - 1) \delta^2 \|x_0\|^2 / (2\delta - \delta^2) = \|x_0\|^2,$$

which is impossible.

Corollary. *If* $|W(T)| \leqq 1$ *and* $\liminf \|T^n x_0\| = \alpha \|x_0\|$, *then* $\alpha \leqq \sqrt{3}$.

Proof. For $|W(T)| \leqq 1$, it follows from [1] that $T^j = \varrho P U^j P$, where $\varrho = 2$.

In [2] an example was given of an operator T where $|W(T) \leqq 1$ and $\lim \|T^n x_0\| = \sqrt{2} \|x_0\|$. This raises the question of the best possible constant K, such that a) $|W(T)| \leqq 1$ and b) $\liminf \|T^n x_0\| \leqq K \|x_0\|$. The Corollary to Theorem 3 does not give a sharp answer to this question, but it does reduce the upper bound on K from 2 to $\sqrt{3}$.

Note that if $\varrho < 1$, then $\|T^n x_0\| \leqq \varrho^n \|x_0\| \to 0$. However, it is still possible to obtain a weak form of Theorem 3.

Theorem 4. *Let* $T^j = \varrho P U^j P$ *for* $j = 1, 2, \ldots$, *where* $\varrho < 1$. *If* $\|T^n x_0\| \geqq \alpha \varrho^n \|x_0\|$ *for all* n, *then* $\alpha \leqq [\varrho(2-\varrho)]^{1/2}$.

Proof. Assume $\alpha > [\varrho(2-\varrho)]^{1/2}$ for $x_0 \in H$. Then

$$\|x_0\|^2 \geqq \sum_{n=1}^{\infty} \delta^2 (1-\delta)^{2(n-1)} \|T^n x_0\|^2 \geqq$$

$$\geqq \alpha^2 \|x_0\|^2 \sum_{n=0}^{\infty} (1-\delta)^{2n} \varrho^{2n} > \varrho(2-\varrho) \|x_0\|^2 \sum_{n=0}^{\infty} (1-\varrho)^{2n} = \|x_0\|^2$$

which is impossible.

Lemma 3. *Let* $T^j = \varrho P U^j P$ *for* $j = 1, 2, \ldots$, *and let* $f(z)$ *be analytic for* $|z| < 1$ *and continuous on the boundary. Then* $\lim_{r \to 1-} f(rT)$ *exists, and equals* $(1-\varrho) f(0) I + \varrho P f(U) P$, *where convergence is in the norm topology.*

Proof. Since $\|T^j\| \leqq \varrho$ for $j = 1, 2, \ldots$, it is clear that $f(rT) = \sum_0^\infty a_n r^n T^n$ converges absolutely, for $r < 1$. Indeed, $\|\sum_0^\infty a_n r^n T^n\| \leqq \sum_0^\infty |a_n| r^n \varrho$ and $|a_n| \leqq M$ since

f is continuous. Thus,

$$f(rT) = \sum_{n=0}^{\infty} a_n r^n T^n = a_0 I + \varrho P \sum_{n=1}^{\infty} a_n r^n U^n P =$$

$$= (1-\varrho) a_0 I + \varrho P \sum_{0}^{\infty} a_n r^n \int e^{int}\, dE(t) P =$$

$$= (1-\varrho) a_0 I + \varrho P \int \Big[\sum_{0}^{\infty} a_n r^n e^{int}\Big]\, dE(t) P =$$

$$= (1-\varrho) a_0 I + \varrho P \int f(re^{it}) dE(t) P, \quad \text{for} \quad r \leqq R < 1.$$

Since $f(z)$ is continuous in $|z| \leqq 1$, it follows that

$$\lim_{r \to 1-} f(rT) = (1-\varrho) a_0 I + \varrho P \int f(e^{it}) dE(t) P = (1-\varrho) f(0) I + \varrho\, P f(U) P.$$

Theorem 5. *Let $T \in C_\varrho$. Let $f(z)$ be analytic in $|z| < 1$ and continuous on the boundary, where $f(0) = 0$ and $|f(z)| \leqq 1$ for $|z| \leqq 1$. Then $f(T) \in C_\varrho$.*

Proof. Let $g(z) = [f(z)]^n$. Then it follows from Lemma 3 that $[f(T)]^n = g(T) = \varrho P g(U) P = \varrho P [f(U)]^n P$ for $n = 1, 2, \ldots$. Since U is unitary, it follows that, while $f(U)$ is not necessarily unitary, it is a contraction. Hence $f(U)$ has a unitary dilation, which completes the proof.

This theorem appeared in [5] under the additional assumption that $f(z)$ have an absolutely convergent Taylor series.

A little thought about Theorem 4 reveals that if T is normal and $\|T\| = \varrho < 1$, then $T \notin C_\varrho$. This leads one to ask how large a normal operator can be and still be a successful candidate for membership in C_ϱ.

While preparing the manuscript, we learned that this question had been answered independently by E. Durszt [6]. Our results are slightly more general, and for that reason we include the statements of Lemmas 4 and 5 and Theorem 6. Since the proofs are implicitly contained in Durszt's paper, we omit them. (The observation that all points in the boundary of the spectrum of an operator lie in the approximate point spectrum is relevant to Lemma 5.)

Lemma 4. *If $\|T\| \leqq \varrho/(2-\varrho)$ and $\varrho < 1$, then $T \in C_\varrho$. If $\|T\| \leqq 1$, then $T \in C_\varrho$ for $\varrho \geqq 1$.*

Lemma 5. *If $T \in C_\varrho$ for $\varrho < 1$, then $R_{sp}(T) \leqq \varrho/(2-\varrho)$. If $T \in C_\varrho$ for $\varrho \geqq 1$, then $R_{sp}(T) \leqq 1$.*

Theorem 6. *Let T be normaloid. For $\varrho \leqq 1$, $T \in C_\varrho$ if and only if $\|T\| \leqq \varrho/(2-\varrho)$. For $\varrho \geqq 1$, $T \in C_\varrho$ if and only if $\|T\| \leqq 1$.*

Note that hyponormal, subnormal, normal, self adjoint and unitary operators are all normaloid.

In [5], there is an example of a power bounded operator which is not in C_ϱ for any ϱ. We will now present a simpler example which does slightly more than theirs.

First we need the following

Theorem B. (Sz.-Nagy—Foiaş) *For* $\varrho>2$, $T\in C_\varrho$ *if and only if*
1) $\sigma(T)\subset\{z\colon |z|\leqq 1\}$,
2) $\|(zI-T)^{-1}\|\leqq(|z|-1)^{-1}$ *for* $1<|z|\leqq(\varrho-1)/(\varrho-2)$.

Theorem 7. *Given* $a>0$, *there exists an operator* T *such that*
1) $\|T^n\|\leqq 1+a$ *for* $n=1,2,\ldots$
2) $T\notin C_\varrho$ *for any* ϱ.

Proof. Given $a>0$, our operator T is defined as follows: $T\varphi_1=\varphi_1+a\varphi_2$, $T\varphi_2=-\varphi_2$, where $\{\varphi_1,\varphi_2\}$ is an orthonormal basis for the space H. Since $T^2=T$, it is clear that $\|T^n\|\leqq 1+a$ for $n=1,2,\ldots$. However,

$$(zI-T)^{-1}\varphi_1=(z-1)^{-1}[\varphi_1+a(1+z)^{-1}\varphi_2]\quad\text{for}\quad z\neq\pm1.$$

Since $\|(zI-T)^{-1}\varphi_1\|=|z-1|^{-1}[1+a^2/|1+z|^2]^{1/2}$, T does not satisfy condition 2 of Theorem B for any $\varrho>2$; as may be seen by taking z real with $1<z\leqq(\varrho-1)(\varrho-2)^{-1}$. However, $C_\alpha\subset C_\beta$ for $\alpha<\beta$ (see [5]) which implies $T\notin C_\varrho$ for any $\varrho>0$ as promised.

Added in proof: Recently we received a preprint "Remarks on the numerical radius" from Tosio Kato. Combining an idea from that paper with the existing results and techniques of this one, it is possible to obtain a remarkable sharpening of Theorem 2 and its Corollaries.

Theorem 2′. *Let* $\{k_n\}$ *be any strictly increasing sequence of positive integers. Let* $\delta T^j=PU^jP$. *Then*

$$\sum_{n=1}^{\infty}\delta^2(1-\delta)^{2(n-1)}\|T^{k_n}x\|^2\leqq\|x\|^2.$$

The proof involves a fairly simple modification of the argument with particular emphasis on Lemma 2.

In Theorem 3, lim inf may now be replaced by lim sup, and we obtain the following:

Corollary. *If* $|W(T)|\leqq 1$, *then* $\limsup\|T^nx\|\leqq\sqrt{3}\|x\|$.

This sharpens Kato's bound of $4/\sqrt{5}$ for this lim sup.

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(Received December 14, 1966)

On a process concerning inaccessible cardinals. III

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This paper is a continuation of references I and II (see [1] and [2]), in which a process concerning inaccessible cardinals has been defined. In this paper we freely make use of the notations and theorems of [2].

In [1] the process was described by a sequence of functions

$$f_0(\alpha^{(0)}),\quad f_1(\alpha^{(0)}, \alpha^{(1)}), \ldots, f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)}), \ldots$$

where the variables η and $\alpha^{(\eta)}$ run over all ordinal numbers and for given η the functions $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$ are defined for such arguments $(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$ in which only a finite number of terms is distinct from 0.

In this paper we are going to describe the process in another manner which is much simpler than the above one.

Let S be a subclass of the class C of all ordinal numbers which is confinal to C. If the elements of S are arranged by magnitude then we say that $S=\{\sigma_\xi\}_{\xi\in C}$ is a *confinal sequence*. An element σ_ξ of S is called a *fixed point* of S if $\sigma_\xi=\xi$.

First we define by transfinite induction the concept of figures. We define a figure of order 0 as a confinal sequence. Let now $\beta>0$ be a given ordinal number and suppose that the figures of order smaller than β have been already defined. We define the figures of order β as follows:

1) in the case of $\beta=\eta+1$ we define a figure of order β as a sequence of type C of distinct figures of order η;

2) in the case of a limit number β we define a figure of order β as a sequence of type β the ξth element of which is a figure of order ξ.

If $F=\{F_\tau\}$ is a figure then we say that F and the elements F_τ of F are components of F, the components of any component of F are components of F, as well. If a component G of F is a figure of order τ then we say that G is a component of order τ.

We associate with every figure F an element $a(F)$ and two figures $A(F)$ and $A(a(F))$ of order 0, furthermore with every sequence $S_\beta=\{F_\eta\}_{\eta\in\beta}$ of figures F_η of order ξ or with every sequence $S_\beta=\{F_\eta\}_{\eta\in\beta}$ of figures of order η a figure $A(S_\beta)$ of order 0. Now we define $a(F)$, $A(F)$, $A(a(F))$ and $A(S_\beta)$ as follows.

a) Let $a(F)$ be the smallest of the elements of C which belongs to F.

b) Let $F=\{F_\xi\}_{\xi\in\varkappa}$ be a figure of order β (where $\varkappa=C$ or $\varkappa$ is a limit number (and in this case $\varkappa=\beta$)). Let $A(a(F))$ be one of the components of order 0 of F the smallest element of which is $a(F)$. We define $A(F)$ as follows:

b$_1$) if $\beta=0$ then let $A(F)=F$

b$_2$) if $\beta=\eta+1$ then let $A(F)$ be the sequence S of the distinct $a(F_\xi)$ $(\xi\in C)$ arranged according to their magnitude, provided that S is a confinal sequence; otherwise let $A(F)$ be an arbitrary figure of order 0;

b$_3$) if β is a limit number then let $A(F)$ be the sequence of all the distinct elements, arranged in their magnitude, which belong to the intersection of all $A(F_\xi)$ with $\xi<\beta$, provided that this is a confinal sequence; otherwise let $A(F)$ be an arbitrary figure of order 0;

c) let $S_\beta=\{F_\xi\}_{\xi\in\beta}$ be a given sequence of type β of figures F_ξ of order τ or a given sequence of type β of figures F_ξ of order ξ. We define $A(S_\beta)$ as follows:

c$_1$) if $\beta=\eta+1$ then let $A(S_\beta)=A(F_\eta)$;

c$_2$) if β is a limit number then let $A(S_\beta)$ be the sequence of all the distinct elements, arranged in their magnitude, which belong to the intersection of all $A(F_\xi)$ with $\xi<\beta$, provided that this is a confinal sequence; otherwise let $A(S_\beta)$ be an arbitrary figure of order 0.

If a figure of order 0 is a stationary subclass of C then we call it a stationary figure of order 0; otherwise we call it a non-stationary figure of order 0. Similarly, if the components of order 0 of a figure F of order β are stationary then we call it a stationary figure of order β. Let $S=\{\sigma_\xi\}_{\xi\in C}$ be a stationary figure of order 0. If we associate with every σ_ξ its index ξ we obtain a strictly divergent function g on S for which $g(\gamma)\leqq\gamma$ holds. Thus it follows from Theorem I that the class $\{\sigma_\xi\in S: \sigma_\xi>\xi\}$ is non-stationary.

If S is a stationary figure of order 0 then we denote by $(S)'$ the figure of order 0 consisting of all the fixed elements of S. Clearly, $S-(S)'$ is non-stationary.

Let $\gamma>0$ be an arbitrary ordinal number, $S_\gamma=\{F_\xi\}_{\xi\in\gamma}$ a sequence of figures of order 0, and G a figure of order 0. We say that S_γ is coincident with G if, in the case of $\gamma=\eta+1$, $(A(S_\gamma))'=G$ and, in the case of a limit number γ, $A(S_\gamma)=G$. Let now $\beta>0$ be a given ordinal number, τ an arbitrary ordinal number, $0<\tau<\beta$, and suppose that the coincidence of a sequence $S_\gamma=\{F_\xi\}_{\xi\in\gamma}$ of figures F_ξ of order τ, where γ is an arbitrary ordinal number (and a sequence $S_\tau=\{F_\xi\}_{\xi\in\tau}$ of figures F_ξ of order ξ) with a figure G of order τ has been already defined. Let $S_\gamma=\{F_\xi\}_{\xi\in\gamma}$ be a sequence of figures F_ξ of order β, where γ is an arbitrary ordinal number, and $R_\beta=\{F_\xi\}_{\xi\in\beta}$ a sequence of figures F_ξ of order ξ, furthermore G a figure of order β. We say that S_γ is coincident with G if, in the case of $\gamma=\eta+1$, $(A(S_\gamma))'=$ $=A(a(G))$ and, if, in the case of a limit number γ, $A(S_\gamma)=A(a(G))$. Similarly, R_β is coincident with G if, in the case of $\beta=\eta+1$, $(A(R_\beta))'=A(a(G))$ and if, in the case of a limit number β, $A(R_\beta)=A(a(G))$. If F is a figure of order 0 or if $F=\{F_\xi\}_{\xi\in\varkappa}$ is a figure of order $\beta>0$ (where $\varkappa=C$ or $\varkappa$ is a limit number (and in this case $\varkappa=\beta$)) and, for every $\xi\in\varkappa$, the sequence $S_\xi=\{F_\zeta\}_{\zeta\in\xi}$ is coincident with F_ξ, then we say that F is connected. If F is a figure and all its components are connected, we say that F is perfect. It is clear that $A(a(F))$ is uniquely determined for any perfect figure F.

Now, we define by transfinite induction the operations $\mathbf{\Gamma}_\beta$ for $\beta\in C$. The operation $\mathbf{\Gamma}_0$ chooses a stationary figure $F^{(0)}$ of order 0. Let now $\beta>0$ be a given ordinal number and suppose that the figures $F^{(\xi)}$ of order ξ and the operation $\mathbf{\Gamma}_\xi$ have been already defined for every $\xi<\beta$. If $\beta=\eta+1$ then the operation $\mathbf{\Gamma}_\beta$ chooses a perfect stationary figure $F^{(\beta)}$ of order β such that $F^{(\eta)}$ is coincident with $F^{(\beta)}$. If β is a limit number then $\mathbf{\Gamma}_\beta$ chooses a perfect stationary figure $F^{(\beta)}$ of order β such that the figure $G^{(\beta)}=\{F^{(\xi)}\}_{\xi\in\beta}$ is coincident with $F^{(\beta)}$.

We shall prove that we have defined the figures $F^{(\xi)}$ of order ξ and the operations Γ_ξ for every $\xi \in C$.

We define the operation Φ as follows. Let $F = F^{(\beta)}$. Then $\Phi(F)$ is the figure $D^{(\beta)}$ of order β obtained in the following way: We omit the fixed elements from every component of order 0 of F, furthermore if $0 \leqq \tau < \beta$ and G is any component of order $\tau+1$ of F then we omit the fixed elements of $A(G)$.

Now we prove the following

Theorem A. *The class of all the elements of C belonging to* $D^{(\beta)} = \Phi(F^{(\beta)})$ *is non-stationary;* β *is an arbitrary element of C.*

Proof. Let $H = \{h_\xi\}_{\xi \in C}$ any component of order 0 of $D^{(\beta)}$. By definition $\xi < h_\xi$ for every $\xi \in C$. We define a function g on H by writing $g(h_\xi) = \xi$. Since g is strictly divergent and regressive, Theorem I implies that H is non-stationary. Thus, every component of order 0 of $D^{(\beta)}$ is non-stationary. Let now $0 < \gamma \leqq \beta$ and suppose that the class of all elements belonging to any component $G^{(\varrho)}$ of order $\varrho < \gamma$ of $D^{(\beta)}$ is non-stationary. Let $G^{(\gamma)} = \{F_\xi\}_{\xi \in \tau}$ where $\tau = C$ or τ is a limit number (and in this case $\tau = \gamma$). We denote by U_ξ the class of all elements of C which belong to F_ξ. Clearly the classes U_ξ $(\xi \in C)$ are mutually disjoint. Let $\gamma = \eta + 1$ and $A(G^{(\gamma)}) = \{\sigma_\xi\}_{\xi \in C}$. We split $A(G^{(\gamma)})$ into the union of two disjoint classes:

$$A(G^{(\gamma)}) = A_1 \cup A_2,$$

where $A_1 = \{\sigma_\xi \colon \xi \in C,\ \xi < \sigma_\xi\}$ and $A_2 = \{\sigma_\xi \colon \xi \in C,\ \xi = \sigma_\xi\}$. One can easily see that the smallest element of U_ξ in the case of $\sigma_\xi = \xi$ is greater than $a(F_\xi)$. Thus the class of the smallest elements of U_ξ, where $\xi = \sigma_\xi$, is non-stationary. On the other hand, the class of the smallest elements of U_ξ, where $\xi < \sigma_\xi$, is non-stationary, too. Since the classes U_ξ are mutually disjoint, Theorem II implies that the classes

$$U^{(1)} = \bigcup_{\substack{\xi \in C \\ \xi < \sigma_\xi}} U_\xi \quad \text{and} \quad U^{(2)} = \bigcup_{\substack{\xi \in C \\ \xi < \sigma_\xi}} U_\xi$$

are non-stationary. Consequently, by Theorem III, the class $U = U^{(1)} \cup U^{(2)}$ is non-stationary. If γ is a limit number then $B = \{a(F_\xi)\}_{\xi \in \tau}$ is non-stationary, because B is not confinal to C. Thus by the hypothesis and Theorem III, the class of elements of C belonging to $G^{(\gamma)}$ is non-stationary in the case of the limit number γ, as well. The theorem is proved.

Theorem B. *If* $\beta \in C$ *then there is a non-stationary class* T_β *such that* $A(F^{(0)}) = A(F^{(\beta)}) \cup T_\beta$.

Proof. We use transfinite induction. The theorem is obviously true for $\beta = 0$. Let $\beta > 0$ and suppose that the theorem is true for every $\gamma < \beta$. Put $F^{(\beta)} = \{F_\tau\}$ and $D^{(\beta)} = \Phi(F^{(\beta)})$. Let us denote by $U^{(\beta)}$ the class of all elements of C which belong to $D^{(\beta)}$. First we consider the case of $\beta = \eta + 1$. In this case

$$A(F^{(\eta)}) = A(F^{(\beta)}) \cup U^{(\beta)} \cup \big[A(F^{(\eta)}) - (A(F^{(\eta)}))'\big].$$

By the hypothesis

$$A(F^{(0)}) = A(F^{(\eta)}) \cup U^{(\beta)} \cup \big[A(F^{(\eta)}) - (A(F^{(\eta)}))'\big] \cup T_\eta,$$

where T_η is non-stationary. Since $U^{(\beta)}$ and $A(F^{(\beta)})-(A(F^{(\beta)}))'$ are non-stationary, the theorem is true for $\beta=\eta+1$, as well. Let now β be a limit number. Since

$$A(F^{(0)})-A(F^{(\beta)}) = A(F^{(0)})-\bigcap_{\gamma<\beta} A(F^{(\gamma)}) = \bigcup_{\gamma<\alpha} T_\gamma,$$

the theorem follows from the hypothesis and Theorem III.

Theorem C. *The class* $A(F^{(0)})-\{a(F^{(\beta)})\colon \beta\in C\}$ *is non-stationary.*

We omit the proof.

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(Received December 1, 1966)

Forme triangulaire d'une contraction et factorisation de la fonction caractéristique

Par BÉLA SZ.-NAGY à Szeged et CIPRIAN FOIAŞ à Bucarest

Introduction

Soit T une contraction d'un espace de Hilbert $\mathfrak{H}$ et soient D_T, D_{T^*} les opérateurs de défaut et $\mathfrak{D}_T$, $\mathfrak{D}_{T^*}$ les sous-espaces de défaut correspondants:

$$(0.1)\qquad D_T = (I-T^*T)^{\frac{1}{2}}, \quad D_{T^*} = (I-TT^*)^{\frac{1}{2}}, \quad \mathfrak{D}_T = \overline{D_T\mathfrak{H}}, \quad \mathfrak{D}_{T^*} = \overline{D_{T^*}\mathfrak{H}}.$$

La fonction caractéristique de T est alors définie par

$$(0.2)\qquad \Theta_T(\lambda) = [-T+\lambda D_{T^*}(I-\lambda T^*)^{-1}D_T]|\mathfrak{D}_T;$$

c'est une fonction holomorphe dans le disque unité $|\lambda|<1$, à valeurs contractions de $\mathfrak{D}_T$ dans $\mathfrak{D}_{T^*}$, donc, en bref, $\{\mathfrak{D}_T, \mathfrak{D}_{T^*}, \Theta_T(\lambda)\}$ est une *fonction analytique contractive*. De plus, elle est contractive *pure*, c'est-à-dire que

$$\|\Theta_T(0)h\| < \|h\| \quad \text{pour tout} \quad h \in \mathfrak{D}_T, \quad h \neq 0$$

(d'ailleurs, grâce au principe de maximum, cela entraîne la même inégalité en tout point λ, $|\lambda|<1$). *Cf.* [VIII] ou [A], Chap. VI.

A toute décomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ de l'espace, engendrée par un sous-espace $\mathfrak{H}_1$ invariant pour T, il correspond une triangulation

$$(0.3)\qquad T = \begin{bmatrix} T_1 & X \\ O & T_2 \end{bmatrix}$$

de T et une factorisation

$$(0.4)\qquad \Theta_T(\lambda) = \Theta_2(\lambda)\Theta_1(\lambda)$$

de sa fonction caractéristique en produit de deux fonctions analytiques contractives telles que la „partie pure" de $\Theta_i(\lambda)$ coïncide [1]) avec la fonction caractéristique de T_i $(i=1, 2)$.

Ce sont une partie des résultats de [IX] (théorème 1 et proposition 4.3) ou de [A] (théorème 7.1), du moins pour des T complètement non-unitaires. On y a

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[1]) On dit que les fonctions opératorielles $\{\mathfrak{A}, \mathfrak{A}_*, \Theta(\lambda)\}$ et $\{\mathfrak{A}', \mathfrak{A}'_*, \Theta'(\lambda)\}$ *coïncident* lorsqu'il existe des transformations unitaires $\tau: \mathfrak{A} \to \mathfrak{A}'$ et $\tau_*: \mathfrak{A}_* \to \mathfrak{A}'_*$ telles que $\Theta'(\lambda)\cdot\tau = \tau_*\cdot\Theta(\lambda)$.

démontré aussi que la factorisation (0. 4) jouit alors d'une certaine propriété additionnelle, appelée dans [A] „régularité", et que, inversement, chaque factorisation régulière de $\Theta_T(\lambda)$ engendre de cette façon un sous-espace invariant pour T. On a démontré de plus que même si la factorisation de $\Theta_T(\lambda)$ n'est pas régulière, il y correspond une triangulation analogue sinon de T mais du moins d'une contraction T' dont la partie complètement non-unitaire est égale à T.

Dans [IX] et [A], la démonstration de ces résultats a été basée sur l'étude de la dilatation unitaire de T et sur la représentation de Fourier de cette dilatation.

Dans la présente Note on fera usage d'un calcul direct, plutôt matriciel. Ce calcul ne fournit pas de critère pour qu'une factorisation de $\Theta_T(\lambda)$ corresponde à une triangulation de la contraction T elle-même (c'est-à dire la régularité de la factorisation), mais en revanche elle déduit des relations explicites entre l'opérateur X figurant dans la triangulation (0. 3) et les opérateurs unitaires qui réalisent la „coïncidence" des facteurs $\Theta_i(\lambda)$ avec les fonctions caractéristiques $\Theta_{T_i}(\lambda)$ $(i=1, 2)$. Ces relations ont été annoncées déjà dans [1]. La publication des détails a été retardée par d'autres préoccupations des auteurs, notamment par leur découverte de la régularité de la factorisation comme critère de ce que cette factorisation engendre une triangulation de T elle-même. Mais bien que cette découverte majorise en importance les résultats antérieurs en question, ceux-là gardent à notre avis un certain intérêt propre qui justifie leur publication en forme détaillée.

D'ailleurs, des problèmes semblables ont été étudiés déjà par Brodsky—Livšitz [2] et Šmulyan [3], mais cela seulement dans le cas des indices de défaut finis. Nos calculs s'appliquent au cas général.

Dans le § 1 on établira une représentation „paramétrique" de l'opérateur X figurant dans la forme triangulaire (0. 3) d'une contraction. Dans le § 2 on déduit de la triangulation (0. 3) la factorisation (0. 4) et cela indépendamment de ce que T est complètement non-unitaire ou non. Finalement, dans le § 3 on étudie le problème inverse; en partant d'une factorisation de la fonction caractéristique d'une contraction complètement non-unitaire T, on construit une triangulation correspondante sinon de T, mais du moins d'une contraction T' dont T est la partie complètement non-unitaire.

§ 1. Forme triangulaire d'un contraction

1. Supposons que T est une contraction de l'espace de Hilbert $\mathfrak{H}$ et que $\mathfrak{H}_1$ est un sous-espace de $\mathfrak{H}$, invariant pour T. Le sous-espace orthogonal complémentaire $\mathfrak{H}_2$ est alors invariant pour T^*; T et T^* prennent, en correspondance à la décomposition $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2$, les formes matricielles

$$T=\begin{bmatrix} T_1 & X \\ O & T_2 \end{bmatrix} \quad \text{et} \quad T^*=\begin{bmatrix} T_1^* & O \\ X^* & T_2^* \end{bmatrix}$$

où $T_1=T|\mathfrak{H}_1$, $T_2=(T^*|\mathfrak{H}_2)^*$ et où X est une transformation (linéaire bornée) de $\mathfrak{H}_2$ dans $\mathfrak{H}_1$ et par conséquent X^* une transformation de $\mathfrak{H}_1$ dans $\mathfrak{H}_2$. T_1 et T_2 sont évidemment des contractions. De plus on a

$$\begin{aligned} \|h_2\|^2 &\geqq \|Th_2\|^2=\|Xh_2\|^2+\|T_2h_2\|^2 \qquad (h_2\in\mathfrak{H}_2), \\ \|h_1\|^2 &\geqq \|T^*h_1\|^2=\|T_1^*h_1\|^2+\|X^*h_1\|^2 \qquad (h_1\in\mathfrak{H}_1). \end{aligned} \tag{1.1}$$

Convenons des notations simplifiées suivantes pour les opérateurs de défaut: D pour D_T, D_* pour D_{T^*}, D_i pour D_{T_i} et D_{i*} pour $D_{T_i^*}$ $(i=1, 2)$; les sous-espaces de défaut correspondants seront désignés d'une manière analogue.

(1. 1) entraîne

$$\|Xh_2\| \leqq \|D_2 h_2\| \quad (h_2 \in \mathfrak{H}_2) \quad \text{et} \quad \|X^*h_1\| \leqq \|D_{1*}h_1\| \quad (h_1 \in \mathfrak{H}_1). \tag{1. 2}$$

En vertu de la seconde de ces inégalités la transformation N déterminée par la formule

$$X^*h_1 = ND_{1*}h_1 \qquad (h_1 \in \mathfrak{H}_1) \tag{1. 3}$$

applique $D_{1*}\mathfrak{H}_1$ dans $\mathfrak{H}_2$ linéairement et n'augmente pas la norme. Par conséquent, N s'étend par continuité à $\mathfrak{D}_{1*}$ et devient une contraction de $\mathfrak{D}_{1*}$ dans $\mathfrak{H}_2$; N^* sera alors une contraction de $\mathfrak{H}_2$ dans $\mathfrak{D}_{1*}$.

Cela étant, nous définissons par la formule

$$LD_2h_2 = N^*h_2 \qquad (h_2 \in \mathfrak{H}_2) \tag{1. 4}$$

une transformation L de $D_2\mathfrak{H}_2$ dans $\mathfrak{D}_{1*}$. Puisque D_2 et N^* sont linéaires, on pourra affirmer que L est univoque et linéaire dès qu'on montre que $D_2h_2=0$ entraîne $N^*h_2=0$. Or, $D_2h_2=0$ entraîne, en vertu de la première des inégalités (1. 2), $Xh_2=0$, d'où il s'ensuit

$$h_2 \perp X^*\mathfrak{H}_1 = ND_{1*}\mathfrak{H}_1, \quad N^*h_2 \perp D_{1*}\mathfrak{H}_1,$$

donc

$$N^*h_2 \perp \mathfrak{D}_{1*}. \tag{1. 5}$$

Les valeurs de N^* étant toutes comprises dans $\mathfrak{D}_{1*}$, (1. 5) entraîne $N^*h_2=0$.

Ainsi, on sait déjà que L est une transformation linéaire de $D_2\mathfrak{H}_2$ dans $\mathfrak{D}_{1*}$. On va démontrer qu'elle est aussi continue, même une *contraction*.

Notons que par (1. 3) on a $X^*=ND_{1*}$, d'où $X=D_{1*}N^*$; vu (1. 4) cela entraîne

$$X = D_{1*}LD_2, \tag{1. 6}$$

comme $T_1^*D_{1*}=D_1T_1^*$, il s'ensuit

$$T_1^*X = D_1T_1^*LD_2. \tag{1. 7}$$

Pour $h=h_1+h_2$ $(h_1 \in \mathfrak{H}_1,\ h_2 \in \mathfrak{H}_2)$ on aura

$$\|Dh\|^2 = \|h\|^2-\|Th\|^2 = (\|h_1\|^2+\|h_2\|^2)-(\|T_1h_1+Xh_2\|^2+\|T_2h_2\|^2) =$$
$$= \|h_1\|^2-\|T_1h_1\|^2+\|h_2\|^2-\|T_2h_2\|^2-2\,\mathrm{Re}\,(T_1h_1, Xh_2)-\|Xh_2\|^2,$$

donc, grâce à (1. 6) et (1. 7),

$$\begin{cases} \|Dh^2\| = \|D_1h_1\|^2+\|D_2h_2\|^2-2\,\mathrm{Re}\,(h_1, D_1T_1^*LD_2h_2)-\|D_{1*}LD_2h_2\|^2 = \\ = \|D_1h_1\|^2+\|D_2h_2\|^2-2\,\mathrm{Re}\,(D_1h_1, T_1^*LD_2h_2)-\|LD_2h_2\|^2+\|T_1^*LD_2h_2\|^2 = \\ = \|D_1h_1-T_1^*LD_2h_2\|^2+\|D_2h_2\|^2-\|LD_2h_2\|^2. \end{cases} \tag{1. 8}$$

En posant $g_1=D_1h_1$, $g_2=D_2h_2$, on obtient de (1. 8):

$$\|Lg_2\|^2 \leqq \|g_2\|^2+\|g_1-T_1^*Lg_2\|^2. \tag{1. 9}$$

Or, on a

$$T_1^*Lg_2 = T_1^*LD_2h_2 = T_1^*N^*h_2 \in T_1^*\mathfrak{D}_{1*} \subset \mathfrak{D}_1,$$

donc en choisissant $g_{1;} = D_1 h_{1n} \to T_1^* L g_2$ il résulte $\|Lg_2\|^2 \leqq \|g_2\|^2$. Par conséquent L s'étend par continuité à une contraction de $\mathfrak{D}_2$ dans $\mathfrak{D}_{1*}$; L^* sera alors une contraction de $\mathfrak{D}_{1*}$ dans $\mathfrak{D}_2$.

2. Le résultat obtenu admet une réciproque. Notamment, si l'on se donne arbitrairement une contraction T_1 dans un espace de Hilbert $\mathfrak{H}_1$, une contraction T_2 dans un espace de Hilbert $\mathfrak{H}_2$ et une contraction L de l'espace $\mathfrak{D}_2 = \overline{D_2 \mathfrak{H}_2}$ dans l'espace $\mathfrak{D}_{1*} = \overline{D_{1*} \mathfrak{H}_1}$, la transformation T de $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, définie par la matrice

$$T = \begin{bmatrix} T_1 & X \\ O & T_2 \end{bmatrix} \quad \text{où} \quad X = D_{1*} L D_2,$$

sera une contraction de $\mathfrak{H}$.

En effet, faisant usage de nouveau de la relation $T_1^* D_{1*} = D_1 T_1^*$, on obtient

$$\begin{aligned} \|T(h_1 \oplus h_2)\|^2 &= \|T_1 h_1 + X h_2\|^2 + \|T_2 h_2\|^2 = \\ &= \|T_1 h_1\|^2 + 2\,\mathrm{Re}\,(T_1 h_1, X h_2) + \|X h_2\|^2 + \|T_2 h_2\|^2 = \\ &= \|T_1 h_1\|^2 + 2\,\mathrm{Re}\,(D_1 h_1, T_1^* L D_2 h_2) + \|L D_2 h_2\|^2 - \|T_1^* L D_2 h_2\|^2 + \|T_2 h_2\|^2 = \\ &= \|T_1 h_1\|^2 - \|D_1 h_1 - T_1^* L D_2 h_2\|^2 + \|D_1 h_1\|^2 + \|L D_2 h_2\|^2 + \|T_2 h_2\|^2 \leqq \\ &\leqq \|T_1 h_1\|^2 + \|D_1 h_1\|^2 + \|D_2 h_2\|^2 + \|T_2 h_2\|^2 = \|h_1\|^2 + \|h_2\|^2 = \|h_1 \oplus h_2\|^2. \end{aligned}$$

Ainsi, nous avons démontré le suivant

Théorème 1. *Pour que la transformation linéaire*

$$T = \begin{bmatrix} T_1 & X \\ O & T_2 \end{bmatrix}$$

de l'espace de Hilbert $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ soit une contraction il faut et il suffit que T_1 et T_2 soient des contractions de $\mathfrak{H}_1$ et $\mathfrak{H}_2$, selon les cas, et que X soit de la forme $X = D_{1*} L D_2$ où L est une contraction de $\mathfrak{D}_2$ dans $\mathfrak{D}_{1*}$, d'ailleurs quelconque.

3. Si T est complèment non-unitaire, il est manifeste que T_1 et T_2 le sont aussi. Par contre, T_1 et T_2 peuvent être complètement non-unitaires sans que T le soit aussi. Un exemple simple est fourni, dans l'espace $\mathfrak{H} = l^2$ des suites numériques bilatérales $\{x_k\}_{-\infty}^{\infty}$, par la translation bilatérale $T\{x_k\} = \{x_{k-1}\}$ et le sous-espace $\mathfrak{H}_1$, invariant pour T, des vecteurs tels que $x_{-1} = x_{-2} = \ldots = 0$. $\mathfrak{H}_2 = \mathfrak{H} \ominus \mathfrak{H}_1$ est alors constitué des vecteurs tels que $x_0 = x_1 = \ldots = 0$. $T_1 = T|\mathfrak{H}_1$ est une translation unilatérale et $T_2 = (T^*|\mathfrak{H}_2)^*$ est l'adjoint d'une translation unilatérale, donc $T_1^{*n} \to O$ et $T_2^n \to O$. Ainsi, T_1 et T_2 sont complètement non-unitaires, tandis que T est unitaire.

Voici une condition simple sur l'opérateur de "couplage" L qui assure que T soit aussi complètement non-unitaire:

Pour T_1, T_2 complètement non-unitaires, T sera aussi complètement non-unitaire si L est tel que

$$(1.10) \qquad \|Lg_2\| < \|g_2\|, \quad \|L^* g_1\| < \|g_1\| \quad \textit{pour} \quad g_2 \in D_2 \mathfrak{H}_2, \quad g_1 \in D_{1*} \mathfrak{H}_1 \quad (g_2 \neq 0, \quad g_1 \neq 0),$$

donc en particulier si $\|L\| < 1$.

A cet effet, observons d'abord que, en vertu des formules précédant le théorème 1, l'équation $\|Th\| = \|h\|$ pour un $h = h_1 \oplus h_2$ entraîne $\|LD_2h_2\| = \|D_2h_2\|$ et $D_1h_1 - T_1^* LD_2h_2 = 0$, d'où, par (1. 10), $D_2h_2 = 0$ et par suite $D_1h_1 = 0$, $Xh_2 = = D_{1*}LD_2h_2 = 0$, $Th = T_1h_1 \oplus T_2h_2$. Lorsqu'on a $\|T^nh\| = \|h\|$ pour un h et pour $n = 1, 2, \ldots$, il s'ensuit successivement: $T^nh = T_1^nh_1 \oplus T_2^nh_2$, $D_1T_1^nh_1 = 0$, $D_2T_2^nh_2 = 0$ $(n = 0, 1, \ldots)$, d'où $\|T_i^nh_i\| = \|T_i^{n+1}h_i\|$ $(n = 0, 1, \ldots;\ i = 1,2)$. Lorsqu'on a de plus, pour le même h, $\|T^{*n}h\| = \|h\|$ (n $= 1, 2, \ldots$), on déduit d'une manière analogue les mêmes égalités pour T_i^* au lieu de T_i $(i = 1, 2)$ et par conséquent on a, dans ce cas, $\|T_i^nh_i\| = \|h_i\| = \|T_i^{*n}h_i\|$ $(n = 1, 2, \ldots;\ i = 1,2)$. Cela entraîne $h_i = 0$ $(i = 1, 2)$, donc $h = 0$. Par conséquent T est complètement non-unitaire.

§ 2. Factorisation de la fonction caractéristique engendrée par une triangulation

Soit $T = \begin{bmatrix} T_1 & X \\ O & T_2 \end{bmatrix}$ une contraction de l'espace $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, avec $X = D_{1*}LD_2$. Nous allons rechercher les relations entre les fonctions caractéristiques de T, T_1 et T_2, c'est-à-dire les fonctions

$$\{\mathfrak{D}, \mathfrak{D}_*, \Theta_T(\lambda)\}, \quad \{\mathfrak{D}_1, \mathfrak{D}_{1*}, \Theta_{T_1}(\lambda)\} \quad \text{et} \quad \{\mathfrak{D}_2, \mathfrak{D}_{2*}, \Theta_{T_2}(\lambda)\}.$$

Introduisons aussi les opérateurs

$$D_L = (I_{\mathfrak{D}_2} - L^*L)^{\frac{1}{2}}, \quad D_{L^*} = (I_{\mathfrak{D}_{1*}} - LL^*)^{\frac{1}{2}}$$

et les sous-espaces de défaut correspondants

$$\mathfrak{D}_L = \overline{D_L\mathfrak{D}_2}, \quad \mathfrak{D}_{L^*} = \overline{D_{L^*}\mathfrak{D}_{1*}}.$$

On déduit de (1. 8):

$$\|Dh\|^2 = \|D_1h_1 - T_1^*LD_2h_2\|^2 + \|D_LD_2h_2\|^2. \tag{2. 1}$$

Puisque

$$T_1^*LD_2h_2 \in T_1^*\mathfrak{D}_{1*} \subset \mathfrak{D}_1 \tag{2. 2}$$

et

$$D_LD_2h_2 \in D_L\mathfrak{D}_2, \tag{2. 3}$$

la formule

$$\sigma Dh = (D_1h_1 - T_1^*LD_2h_2) \oplus D_LD_2h_2 \qquad (h_1 \in \mathfrak{H}_1,\ \ h_2 \in \mathfrak{H}_2,\ \ h = h_1 \oplus h_2) \tag{2. 4}$$

définit, en vertu de (2. 1)—(2. 3), une application isométrique σ de $D\mathfrak{H}$ dans l'espace

$$\mathfrak{S} = \mathfrak{D}_1 \oplus \mathfrak{D}_L; \tag{2. 5}$$

σ s'étend par continuité à une application isométrique de $\mathfrak{D}$ dans $\mathfrak{S}$. Montrons que σ applique $\mathfrak{D}$ même *sur* $\mathfrak{S}$. Cela s'ensuit d'une part de ce que

$$\sigma Dh_1 = D_1h_1 \oplus 0 \quad \text{pour} \quad h = h_1 \in \mathfrak{H}_1,$$

d'autre part de ce que, pour h_2 fixé arbitraire,

$$\sigma D(h_1^{(n)} + h_2) \to 0 \oplus D_LD_2h_2 \qquad (n \to \infty)$$

si la suite $h_1^{(n)} \in \mathfrak{H}_1$ est choisie de la sorte que

$$D_1 h_1^{(n)} \to T_1^* L D_2 h_2 \qquad (n \to \infty),$$

ce qui est possible en vertu de (2. 2).

Des résultats analogues peuvent être obtenus pour D_* au lieu de D. Au lieu de répéter tous les calculs, on y arrive plus simplement en observant que, lorsqu'on donne le rôle de T à T^*, la situation sera inaltérée si en même temps on échange les rôles des sous-espaces $\mathfrak{H}_1$, $\mathfrak{H}_2$ et on remplace T_1 par T_2^*, T_2 par T_1^*, et L par L^*. De cette façon il résulte que la formule

$$\sigma_* D_* h = (-T_2 L^* D_{1*} h_1 + D_{2*} h_2) \oplus D_{L^*} D_{1*} h_1 \qquad (2.6)$$

$$(h_1 \in \mathfrak{H}_1, \quad h_2 \in \mathfrak{H}_2, \quad h = h_1 \oplus h_2)$$

définit une transformation isométrique σ_* de $D_* \mathfrak{H}$ dans

$$\mathfrak{S}_* = \mathfrak{D}_{2*} \oplus \mathfrak{D}_{L^*}, \qquad (2.7)$$

qui s'étend par continuité à une transformation unitaire de $\mathfrak{D}_*$ sur $\mathfrak{S}_*$.

On peut écrire (2. 4) et (2. 6) aussi sous la forme

$$\sigma D h = \begin{bmatrix} D_1 & -T_1^* L D_2 \\ O & D_L D_2 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \quad \sigma_* D_* h = \begin{bmatrix} -T_2 L^* D_{1*} & D_{2*} \\ D_{L^*} D_{1*} & O \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \qquad (2.8)$$

où

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = h_1 \oplus h_2 = h.$$

Pour calculer la fonction caractéristique de T, notons d'abord qu'une matrice de type $\begin{bmatrix} A & O \\ B & C \end{bmatrix}$, aux éléments opérateurs dont A et C sont inversibles, est aussi inversible et

$$\begin{bmatrix} A & O \\ B & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & O \\ -C^{-1} B A^{-1} & C^{-1} \end{bmatrix}.$$

Il s'ensuit pour $|\lambda| < 1$:

$$(I - \lambda T^*)^{-1} (\lambda I - T) = \begin{bmatrix} I_1 - \lambda T_1^* & O \\ -\lambda X^* & I_2 - \lambda T_2^* \end{bmatrix}^{-1} \begin{bmatrix} \lambda I_1 - T_1 & -X \\ O & \lambda I_2 - T_2 \end{bmatrix} =$$

$$= \begin{bmatrix} (I_1 - \lambda T_1^*)^{-1} (\lambda I_1 - T_1) & -(I_1 - \lambda T_1^*)^{-1} X \\ A(\lambda I_1 - T_1) & -AX + (I_2 - \lambda T_2^*)^{-1} (\lambda I_2 - T_2) \end{bmatrix} \qquad (2.9)$$

où

$$A = (I_2 - \lambda T_2^*)^{-1} \lambda X^* (I_1 - \lambda T_1^*)^{-1}.$$

Nous ferons usage de la formule

$$\Theta_T(\lambda) D = D_* (I - \lambda T^*)^{-1} (\lambda I - T) \qquad (D = D_T, \; D_* = D_{T^*}) \qquad (2.10)$$

pour la fonction caractéristique de T, et de la même formule pour T_1 et T_2; *cf* [VIII], (2. 3), ou [A] Chap. VI.

En désignant par **M** la matrice au dernier membre de (2. 9) et par **N** la matrice

$$\begin{bmatrix} -T_2 L^* D_{1*} & D_{2*} \\ D_{L^*} D_{1*} & O \end{bmatrix},$$

il dérive de (2. 8), (2. 9) et (2. 10) que

$$\sigma_* \Theta_T(\lambda) Dh = \sigma_* D_*(I-\lambda T^*)^{-1}(\lambda I - T)h = \mathbf{N}\cdot\mathbf{M}h.$$

Calculons le produit matriciel **N·M**. Pour le terme de rang 11 nous obtenons:

$$-T_2 L^* D_{1*}(I_1-\lambda T_1^*)^{-1}(\lambda I_1 - T_1) + D_{2*}(I_2-\lambda T_2^*)^{-1}\lambda X^*(I_1-\lambda T_1^*)^{-1}(\lambda I_1 - T_1) =$$
$$= [-T_2 + \lambda D_{2*}(I_2-\lambda T_2^*)^{-1} D_2] L^* D_{1*}(I_1-\lambda T_1^*)^{-1}(\lambda I_1 - T_1) =$$
$$= \Theta_{T_2}(\lambda) L^* \Theta_{T_1}(\lambda) D_1,$$

et pour le terme de rang 12:

$$T_2 L^* D_{1*}(I_1-\lambda T_1^*)^{-1} X - D_{2*}(I_2-\lambda T_2^*)^{-1}\lambda X^*(I_1-\lambda T_1^*)^{-1} X +$$
$$+ D_{2*}(I_2-\lambda T_2^*)^{-1}(\lambda I_2 - T_2) =$$
$$= -[-T_2 + \lambda D_{2*}(I_2-\lambda T_2^*)^{-1} D_2] L^* D_{1*}(I_1-\lambda T_1^*)^{-1} D_{1*} L D_2 +$$
$$+ D_{2*}(I_2-\lambda T_2^*)^{-1}(\lambda I_2 - T_2) =$$
$$= -\Theta_{T_2}(\lambda) L^* D_{1*}(I_1-\lambda T_1^*)^{-1} D_{1*} L D_2 + \Theta_{T_2}(\lambda) D_2 =$$
$$= \Theta_{T_2}(\lambda)[-L^* D_{1*}(I_1-\lambda T_1^*)^{-1} D_{1*} L + I_{\mathfrak{D}_2}] D_2.$$

Or, on a

$$D_{1*}(I_1-\lambda T_1^*)^{-1} D_{1*}|\mathfrak{D}_{1*} = D_{1*}[I_1 + \lambda(I_1-\lambda T_1^*)^{-1} T_1^*] D_{1*}|\mathfrak{D}_{1*} =$$
$$= [I_1 - T_1 T_1^* + \lambda D_{1*}(I_1-\lambda T_1^*)^{-1} D_1 T_1^*]|\mathfrak{D}_{1*},$$

d'où

(2. 11) $$D_{1*}(I_1-\lambda T_1^*)^{-1} D_{1*}|\mathfrak{D}_{1*} = [I_1 + \Theta_{T_1}(\lambda) T_1^*]|\mathfrak{D}_{1*}.$$

Donc le terme de rang 12 est égal à

$$\Theta_{T_2}(\lambda)[D_L^2 - L^* \Theta_{T_1}(\lambda) T_1^* L] D_2.$$

Le terme de rang 21 est évidemment égal à $D_{L^*}\Theta_{T_1}(\lambda) D_1$. Finalement, pour le terme 22 on obtient, grâce à (2. 11),

$$-D_{L^*} D_{1*}(I_1-\lambda T_1^*)^{-1} D_{1*} L D_2 = -D_{L^*}[I_1 + \Theta_{T_1}(\lambda) T_1^*] L D_2 =$$
$$= -D_{L^*}\Theta_{T_1}(\lambda) T_1^* L D_2 - L D_L D_2.$$

Ainsi, on aura

$$\sigma_* \Theta_T(\lambda) Dh =$$
$$= \begin{bmatrix} \Theta_{T_2}(\lambda) L^* \Theta_{T_1}(\lambda) D_1 & -\Theta_{T_2}(\lambda) L^* \Theta_{T_1}(\lambda) T_1^* L D_2 + \Theta_{T_2}(\lambda) D_L^2 D_2 \\ D_{L^*}\Theta_{T_1}(\lambda) D_1 & -D_{L^*}\Theta_{T_1}(\lambda) T_1^* L D_2 - L D_L D_2 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}.$$

La matrice obtenue se factorise en

$$\begin{bmatrix} \Theta_{T_2}(\lambda) L^* \Theta_{T_1}(\lambda) & \Theta_{T_2}(\lambda) D_L \\ D_{L^*}\Theta_{T_1}(\lambda) & -L \end{bmatrix} \begin{bmatrix} D_1 & -T_1^* L D_2 \\ O & D_L D_2 \end{bmatrix};$$

en désignant le premier facteur par $\Omega(\lambda)$ et en rappelant (2. 8) on aura donc

$$\sigma_* \Theta_T(\lambda) Dh = \Omega(\lambda) \sigma Dh.$$

Comme les éléments de la forme Dh sont denses dans $\mathfrak{D}$, qui est le domaine de définition de $\Theta_T(\lambda)$ ainsi que de σ, cela entraîne

$$\sigma_* \Theta_T(\lambda) = \Omega(\lambda) \sigma.$$

Par conséquent

$$\sigma_* \Theta_T(\lambda) \sigma^{-1} = \Omega_0(\lambda)$$

où $\Omega_0(\lambda)$ désigne la restriction de $\Omega(\lambda)$ à $\mathfrak{S} = \mathfrak{D}_1 \oplus \mathfrak{D}_L$, donc

$$\Omega_0(\lambda) = \begin{bmatrix} \Theta_{T_2}(\lambda) L^* \Theta_{T_1}(\lambda) & \Theta_{T_2}(\lambda) [D_L]_0 \\ D_{L^*} \Theta_{T_1}(\lambda) & -[L]_0 \end{bmatrix}$$

où $[\ ]_0$ indique la restriction à $\mathfrak{D}_L$. D'ailleurs cela s'écrit aussi sous la forme

$$\Omega_0(\lambda) = \begin{bmatrix} \Theta_{T_2}(\lambda) & O \\ O & I_{\mathfrak{D}_{L^*}} \end{bmatrix} \omega \begin{bmatrix} \Theta_{T_1}(\lambda) & O \\ O & I_{\mathfrak{D}_L} \end{bmatrix}$$

où

(2. 12) $$\omega = \begin{bmatrix} L^* & [D_L]_0 \\ D_{L^*} & -[L]_0 \end{bmatrix}.$$

Ce facteur ω est une application unitaire de l'espace

$$\mathfrak{P} = \mathfrak{D}_{1*} \oplus \mathfrak{D}_L$$

sur l'espace

$$\mathfrak{P}' = \mathfrak{D}_2 \oplus \mathfrak{D}_{L^*}.$$

Pour démontrer cette assertion, observons d'abord que pour $p = u \oplus v$ $(u \in \mathfrak{D}_{1*},\ v \in \mathfrak{D}_L)$ on a

$$\begin{aligned} \|\omega p\|^2 &= \|L^* u + D_L v\|^2 + \|D_{L^*} u - Lv\|^2 = \\ &= \|L^* u\|^2 + 2 \operatorname{Re} (L^* u, D_L v) + \|D_L v\|^2 + \|D_{L^*} u\|^2 - 2 \operatorname{Re} (D_{L^*} u, Lv) + \|Lv\|^2 = \\ &= \|u\|^2 + \|v\|^2 \end{aligned}$$

parce que

$$\|L^* u\|^2 + \|D_{L^*} u\|^2 = \|u\|^2, \quad \|D_L v\|^2 + \|Lv\|^2 = \|v\|^2, \quad LD_L = D_{L^*} L.$$

Comme de plus on a

$$L^* u + D_L v \in L^* \mathfrak{D}_{1*} + D_L \mathfrak{D}_L \subset \mathfrak{D}_2$$

et [2])

$$D_{L^*} u - Lv \in D_{L^*} \mathfrak{D}_{1*} + L \mathfrak{D}_L \subset \mathfrak{D}_{L^*},$$

il résulte que ω applique $\mathfrak{P}$ isométriquement dans $\mathfrak{P}'$. Soit $p' = u' \oplus v' \in \mathfrak{P}'$, orthogonal à $\omega \mathfrak{P}$, donc tel que

$$0 = (u', L^* u + D_L v) + (v', D_{L^*} u - Lv) = (Lu' + D_{L^*} v', u) + (D_L u' - L^* v', v)$$

---

[2]) Notons que $\overline{L\mathfrak{D}_L} = \overline{L\overline{D_L \mathfrak{D}_2}} = \overline{LD_L \mathfrak{D}_2} = \overline{D_{L^*} L \mathfrak{D}_2} \subset \overline{D_{L^*} \mathfrak{D}_{1*}} = \mathfrak{D}_{L^*}$.

pour tout $u \oplus v \in \mathfrak{P}$. Puisque

$$Lu' + D_{L^*}v' \in L\mathfrak{D}_2 + D_{L^*}\mathfrak{D}_{L^*} \subset \mathfrak{D}_{1*}$$

et [3])

$$D_L u' - L^* v' \in D_L \mathfrak{D}_2 + L^* \mathfrak{D}_{L^*} \subset \mathfrak{D}_L,$$

cela entraîne $Lu' + D_{L^*}v' = 0$, $D_L u' - L^* v' = 0$, donc

$$u' = D_L^2 u' + L^* L u' = D_L(L^* v') + L^*(-D_{L^*}v') = (D_L L^* - L^* D_{L^*})v' = 0,$$

$$v' = D_{L^*}^2 v' + LL^* v' = -D_{L^*} L u' + L D_L u' = 0,$$

$$p' = u' \oplus v' = 0.$$

Cela prouve que $\omega \mathfrak{P} = \mathfrak{P}'$.

Ainsi, nous avons obtenu le suivant résultat:

Théorème 2. *Soit* $T = \begin{bmatrix} T_1 & X \\ O & T_2 \end{bmatrix}$ *une contraction de l'espace* $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, *avec* $X = D_{1*} L D_2$. *Les fonctions caractéristiques des contractions* T, T_1 *et* T_2 *sont alors reliées par la formule de factorisation:*

$$\Theta_T(\lambda) = \sigma_*^{-1} \begin{bmatrix} \Theta_{T_2}(\lambda) & O \\ O & I_{\mathfrak{D}_{L^*}} \end{bmatrix} \omega \begin{bmatrix} \Theta_{T_1}(\lambda) & O \\ O & I_{\mathfrak{D}_L} \end{bmatrix} \sigma$$

où σ, σ_*, ω *sont les transformations unitaires constantes*

$$\sigma:\ \mathfrak{D} \to \mathfrak{D}_1 \oplus \mathfrak{D}_L, \quad \sigma_*:\ \mathfrak{D}_* \to \mathfrak{D}_{2*} \oplus \mathfrak{D}_{L^*}, \quad \omega:\ \mathfrak{D}_{1*} \oplus \mathfrak{D}_L \to \mathfrak{D}_2 \oplus \mathfrak{D}_{L^*},$$

déterminées par les formules (2. 8) *et* (2. 12). *En particulier, le terme* $\omega_{1*,2}$ *de la matrice de* ω *est égal à* L^*.

§ 3. Triangulation engendrée par une factorisation de la fonction caractéristique

1. Le théorème 2 admet la suivante réciproque:

Théorème 3. *Soient* T_1, T_2 *des contractions dans les espaces* $\mathfrak{H}_1, \mathfrak{H}_2$, *et soient* $\{\mathfrak{D}_1, \mathfrak{D}_{1*}, \Theta_{T_1}(\lambda)\}$, $\{\mathfrak{D}_2, \mathfrak{D}_{2*}, \Theta_{T_2}(\lambda)\}$ *leurs fonctions caractéristiques. Supposons qu'il existe des espaces* $\mathfrak{F}$, $\mathfrak{F}_*$ *et une transformation unitaire*

$$\omega:\ \mathfrak{D}_{1*} \oplus \mathfrak{F} \to \mathfrak{D}_2 \oplus \mathfrak{F}_*,$$

tels que la fonction analytique contractive $\{\mathfrak{D}_1 \oplus \mathfrak{F}, \mathfrak{D}_{2*} \oplus \mathfrak{F}_*, \Theta(\lambda)\}$, *définie par*

$$\Theta(\lambda) = \begin{bmatrix} \Theta_{T_2}(\lambda) & O \\ O & I_{\mathfrak{F}_*} \end{bmatrix} \omega \begin{bmatrix} \Theta_{T_1}(\lambda) & O \\ O & I_{\mathfrak{F}} \end{bmatrix}, \tag{3. 1}$$

soit pure. Il existe alors une contraction T *dans* $\mathfrak{H}_1 \oplus \mathfrak{H}_2$, *de la forme*

$$T = \begin{bmatrix} T_1 & X \\ O & T_2 \end{bmatrix},$$

dont la fonction caractéristique coïncide avec $\Theta(\lambda)$; *on peut choisir* $X = D_{1*} L D_2$ *où* $L = P_{\mathfrak{D}_{1*}} \omega^* | \mathfrak{D}_2$.

[3]) La relation $\overline{L^* \mathfrak{D}_{L^*}} \subset \mathfrak{D}_L$ se démontre tout comme celle analogue $\overline{L\mathfrak{D}_L} \subset \mathfrak{D}_{L^*}$.

Démonstration. Soit

$$\omega^* = \begin{bmatrix} L & M \\ N & K \end{bmatrix}$$

la matrice de ω^* comme transformation (unitaire) de $\mathfrak{D}_2 \oplus \mathfrak{F}_*$ sur $\mathfrak{D}_{1*} \oplus \mathfrak{F}$; L est donc une contraction de $\mathfrak{D}_2$ dans $\mathfrak{D}_{1*}$, M est une contraction de $\mathfrak{F}_*$ dans $\mathfrak{D}_{1*}$ (et par conséquent M^* une contraction de $\mathfrak{D}_{1*}$ dans $\mathfrak{F}_*$), etc. Posons

(3.2) $$\mathfrak{F}' = \mathfrak{F} \ominus \overline{N\mathfrak{D}_2}, \quad \mathfrak{F}'_* = \mathfrak{F}_* \ominus \overline{M^*\mathfrak{D}_{1*}}$$

et soit $f'_* \in \mathfrak{F}'_*$. On a alors $Mf'_* = 0$, donc $\omega^* f'_* = Kf'_*$ et par suite $\|f'_*\| = \|\omega^* f'_*\| = \|Kf'_*\|$, d'où, comme $\|K\| \leqq 1$,

$$f'_* = K^*Kf'_*, \quad Kf'_* = KK^*Kf'_*.$$

Posons $f = Kf'_*$; comme $f'_* \in \mathfrak{F}_*$, on a $f \in \mathfrak{F}$, d'où

$$\|f\|^2 = \|\omega f\|^2 = \|N^*f\|^2 + \|K^*f\|^2.$$

D'autre part, on a $f = KK^*f$, d'où $\|f\| = \|K^*f\|$. On conclut $N^*f = 0$, d'où $f \perp \overline{N\mathfrak{D}_2}$ et par suite $f \in \mathfrak{F}'$. Donc $\omega^*\mathfrak{F}'_* (= K\mathfrak{F}'_*) \subset \mathfrak{F}'$. On montre de la même manière que $\omega\mathfrak{F}' \subset \mathfrak{F}'_*$. Ces deux relations entraînent que $\omega\mathfrak{F}' = \mathfrak{F}'_*$.

Soit $f' \in \mathfrak{F}'$. Comme $\mathfrak{F}' \subset \mathfrak{F}$ et $\omega\mathfrak{F}' = \mathfrak{F}'_* \subset \mathfrak{F}_*$, on déduit de (3. 1) que $\Theta(\lambda)f' = \omega f'$, donc $\|\Theta(\lambda)f'\| = \|f'\|$ $(|\lambda| < 1)$, en particulier $\|\Theta(0)f'\| = \|f'\|$, d'où $f' = 0$ parce que $\Theta(\lambda)$ est une fonction contractive pure. On a donc $\mathfrak{F}' = \{0\}$ et par suite $\mathfrak{F}'_* = \omega\mathfrak{F}' = \{0\}$. Vu (3. 2) cela entraîne

(3.3) $$\overline{N\mathfrak{D}_2} = \mathfrak{F}, \quad \overline{M^*\mathfrak{D}_{1*}} = \mathfrak{F}_*.$$

Attachons au terme L de la matrice ω^* les opérateurs et les sous-espaces de défaut comme dans le paragraphe précédent. Pour $u \in \mathfrak{D}_2$ nous avons

$$\|u\|^2 = \|\omega^*u\|^2 = \|Lu\|^2 + \|Nu\|^2, \quad \text{d'où} \quad \|D_L u\|^2 = \|Nu\|^2.$$

La transformation $Nu \to D_L u$ $(u \in \mathfrak{D}_2)$ est donc isométrique et se complète par suite à une transformation unitaire

$$Y\colon \quad \overline{N\mathfrak{D}_2} = \mathfrak{F} \to \overline{D_L\mathfrak{D}_2} = \mathfrak{D}_L.$$

On a donc $YN = D_L$, $N^*Y^* = (D_L)^* = D_L$, $N^* = N^*Y^*Y = D_LY$ et par conséquent, en désignant par $[\ ]_0$ toujours la restriction à $\mathfrak{D}_L$,

(3.4) $$N^* = [D_L]_0 Y.$$

Donnons le rôle de $\Theta(\lambda)$ à $\Theta^\sim(\lambda) = \Theta(\bar{\lambda})^*$, qui est aussi une fonction analytique contractive pure, et considérons la relation qui dérive de (3. 1) pour $\Theta^\sim(\lambda)$; il résulte par les mêmes raisonnements que ci-dessus qu'il existe une transformation unitaire

$$Z\colon \quad \mathfrak{F}_* \to \mathfrak{D}_{L^*}$$

telle que

(3.5) $$ZM^* = D_{L^*}.$$

Posons $K_1 = ZK^*Y^*$. (3. 4) et (3. 5) entraînent

$$\omega = \begin{bmatrix} L^* & N^* \\ M^* & K^* \end{bmatrix} = \begin{bmatrix} L^* & [D_L]_0 Y \\ Z^* D_{L^*} & Z^* K_1 Y \end{bmatrix},$$

d'où, en posant

$$\upsilon = \begin{bmatrix} I_{\mathfrak{D}_{1*}} & O \\ O & Y \end{bmatrix}, \quad \zeta = \begin{bmatrix} I_{\mathfrak{D}_2} & O \\ O & Z \end{bmatrix}, \quad \tilde{\omega} = \begin{bmatrix} L^* & [D_L]_0 \\ D_{L^*} & K_1 \end{bmatrix},$$

il résulte

$$\omega = \zeta^* \tilde{\omega} \upsilon.$$

Notons que υ et ζ sont des transformations unitaires,

$$\upsilon: \ \mathfrak{D}_{1*} \oplus \mathfrak{F} \to \mathfrak{D}_{1*} \oplus \mathfrak{D}_L, \qquad \zeta: \ \mathfrak{D}_2 \oplus \mathfrak{F}_* \to \mathfrak{D}_2 \oplus \mathfrak{D}_{L^*};$$

il en résulte que $\tilde{\omega}$ est aussi unitaire,

$$\tilde{\omega}: \ \mathfrak{D}_{1*} \oplus \mathfrak{D}_L \to \mathfrak{D}_2 \oplus \mathfrak{D}_{L^*}.$$

Le terme 12 de la matrice de $\tilde{\omega}^* \tilde{\omega}$ doit donc être égal à O, c'est-à-dire que pour tout $u \in \mathfrak{D}_L$ on a

$$L[D_L]_0 u + D_{L^*} K_1 u = 0.$$

Comme $L[D_L]_0 u = LD_L u = D_{L^*} Lu$, cela entraîne $D_{L^*}(Lu + K_1 u) = 0$; vu aussi que $Lu \in L\mathfrak{D}_L \subset \mathfrak{D}_{L^*}$ et $K_1 u = ZK^* Y^* u \in \mathfrak{D}_{L^*}$, donc $Lu + K_1 u \in \mathfrak{D}_{L^*}$, on conclut que $Lu + K_1 u = 0$. Cela fournit :

$$K_1 = -[L]_0.$$

Les transformations υ', ζ' définies par

$$\zeta' = \begin{bmatrix} I_{\mathfrak{D}_{2*}} & O \\ O & Z \end{bmatrix} \quad \text{et} \quad \upsilon' = \begin{bmatrix} I_{\mathfrak{D}_1} & O \\ O & Y \end{bmatrix}$$

sont évidemment aussi *unitaires* et on a

$$\zeta' \cdot \begin{bmatrix} \Theta_{T_2}(\lambda) & O \\ O & I_{\mathfrak{F}_*} \end{bmatrix} = \begin{bmatrix} \Theta_{T_2}(\lambda) & O \\ O & I_{\mathfrak{D}_{L^*}} \end{bmatrix} \cdot \zeta, \quad \upsilon \cdot \begin{bmatrix} \Theta_{T_1}(\lambda) & O \\ O & I_{\mathfrak{F}} \end{bmatrix} = \begin{bmatrix} \Theta_{T_1}(\lambda) & O \\ O & I_{\mathfrak{D}_L} \end{bmatrix} \cdot \upsilon'.$$

Ainsi, nos résultats se résument sous la forme:

$$\zeta' \cdot \Theta(\lambda) \cdot \upsilon'^* = \begin{bmatrix} \Theta_{T_2}(\lambda) & O \\ O & I_{\mathfrak{D}_{L^*}} \end{bmatrix} \tilde{\omega} \begin{bmatrix} \Theta_{T_1}(\lambda) & O \\ O & I_{\mathfrak{D}_L} \end{bmatrix} \tag{3.6}$$

où

$$\tilde{\omega} = \begin{bmatrix} L^* & [D_L]_0 \\ D_{L^*} & -[L]_0 \end{bmatrix}. \tag{3.7}$$

Cela étant, envisageons la contraction T suivante dans $\mathfrak{H}_1 \oplus \mathfrak{H}_2$:

$$T = \begin{bmatrix} T_1 & X \\ O & T_2 \end{bmatrix} \quad \text{où} \quad X = D_{1*} L D_2,$$

avec la contraction L que nous venons de faire dériver de ω. En vertu du théorème 2, $\Theta_T(\lambda)$ est égal à la fonction figurant au second membre de (3. 6), donc coïncide avec $\Theta(\lambda)$.

Cela achève la démonstration du théorème 3.

2. Partons maintenant d'une contraction *complètement non-unitaire* (c.n.u.) T dans un espace de Hilbert $\mathfrak{H}$ et d'une factorisation

$$\Theta_T(\lambda) = \Theta_2(\lambda)\Theta_1(\lambda)$$

de sa fonction caractéristique en produit de deux fonctions analytiques contractives. Soit T_i une contraction c.n.u. dans un espace $\mathfrak{H}_i$, telle que $\Theta_{T_i}(\lambda)$ coïncide avec la partie pure de $\Theta_i(\lambda)$ $(i=1, 2)$. (Telle contraction T_i existe, *cf.* [VIII], théorème 2, et [IX], proposition 4. 2, ou [A], Chap. VI.) Il s'ensuit que $\Theta_T(\lambda)$ coïncide avec un produit de la forme (3. 1), ω étant un opérateur unitaire. Appliquons le théorème 3: il résulte qu'il existe une contraction

$$T' = \begin{bmatrix} T_1 & X \\ O & T_2 \end{bmatrix} \tag{3. 8}$$

dans l'espace $\mathfrak{H}' = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, telle que $\Theta_{T'}(\lambda)$ coïncide avec $\Theta_T(\lambda)$. La partie c.n.u. de T est alors unitairement équivalente à T (*cf.* [VIII], § 2. 2 et théorème 3, ou [A], Chap. VI). Il ne restreint évidemment pas la généralité de supposer que $\mathfrak{H}' \supset \mathfrak{H}$. Ainsi, nous avons obtenu le suivant corollaire du théorème 3 (*cf.* [IX], p. 300, note 16 en bas):

Soit T une contraction c.n.u. dans un espace de Hilbert $\mathfrak{H}$ et soit $\Theta_T(\lambda) = \Theta_2(\lambda)\Theta_1(\lambda)$ une factorisation de sa fonction caractéristique en produit de deux fonctions analytiques contractives. Il existe alors une contraction T' dans un espace de Hilbert $\mathfrak{H}' \supset \mathfrak{H}$, ayant T comme sa partie c.n.u. et admettant une triangulation (3. 8), *où T_1, T_2 sont des contractions c.n.u. telles que $\Theta_{T_i}(\lambda)$ coïncide avec la partie pure de $\Theta_i(\lambda)$ $(i=1, 2)$.*

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(Reçu le 1. septembre 1966)

Echelles continues de sous-espaces invariants

Par BÉLA SZ.-NAGY à Szeged et CIPRIAN FOIAŞ à Bucarest

Dédié à M. G. Krein à son 60. anniversaire

1. Introduction et théorème

Soit A un opérateur (linéaire borné) dans l'espace de Hilbert H. Une famille $\{H(\lambda)\}$ $(0 \leqq \lambda \leqq 1)$ de sous-espaces de H sera appelée une *échelle continue de sous-espaces invariants pour A* si elle vérifie les suivantes conditions:

(monotonité:) $\quad H(0)=\{0\}, \quad H(\lambda) \subset H(\mu) \quad (0 \leqq \lambda < \mu \leqq 1), \quad H(1)=H;$

(continuité:) $\quad \bigvee_{\varkappa<\lambda} H(\varkappa) = H(\lambda) \quad (0<\lambda \leqq 1), \quad \bigcap_{\mu<\lambda} H(\mu) = H(\lambda) \quad (0 \leqq \lambda < 1);$

(invariance:) $\quad A H(\lambda) \subset H(\lambda) \quad (0 \leqq \lambda \leqq 1).$

En désignant par $E(\lambda)$ la projection orthogonale de H dans $H(\lambda)$, ces conditions veulent dire que $\{E(\lambda)\}$ $(0 \leqq \lambda \leqq 1)$ est une famille spectrale continue dans H telle que $AE(\lambda)=E(\lambda)AE(\lambda)$ $(0 \leqq \lambda \leqq 1)$.

Il y a un nombre d'intéressantes recherches concernant les opérateurs A qui admettent telle échelle de sous-espaces invariants; *cf.* GOHBERG—KREIN [1] et la littérature y citée. Mais il n'existe pas de critère général maniable qui permette de décider si un opérateur donné A admet ou non telle échelle de sous-espaces invariants. Il est manifeste qu'on peut se borner à l'étude des opérateurs de norme au plus égale à 1, c'est-à-dire à l'étude des contractions de H.

Dans cette Note nous nous proposons de démontrer le suivant

Théorème. *T étant une contraction quelconque de l'espace de Hilbert $\mathfrak{H}$, on y peut ajouter orthogonalement un opérateur unitaire V d'un espace de Hilbert $\mathfrak{H}'$, de sorte que l'opérateur $A=T \oplus V$ de l'espace $H=\mathfrak{H} \oplus \mathfrak{H}'$ admette une échelle continue de sous-espaces invariants. V peut être choisi comme somme orthogonale d'une infinité dénombrable de répliques d'une dilatation unitaire quelconque de T.* [1])

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[1]) Un opérateur U dans un espace $\mathfrak{K}(\supset \mathfrak{H})$ s'appelle une dilatation de l'opérateur T de $\mathfrak{H}$, si l'on a $T^n h = P_{\mathfrak{H}} U^n h$ pour $h \in \mathfrak{H}$ et $n=0, 1, \ldots$; $P_{\mathfrak{H}}$ désigne la projection orthogonale de $\mathfrak{K}$ dans $\mathfrak{H}$. Ces conditions sont évidemment équivalentes aux suivantes:

$$(T^n h_1, h_2)=(U^n h_1, h_2) \qquad (h_1, h_2 \in \mathfrak{H};\ n=0, 1, \ldots).$$

Toute contraction T admet une dilatation unitaire U; cette dilatation est *minimum* lorsqu'on a de plus

$$\mathfrak{K} = \bigvee_{n=-\infty}^{\infty} U^n \mathfrak{H}.$$

On sait que si la contraction T est complètement non-unitaire, elle admet comme dilatation un opérateur unitaire U à spectre absolument continu, même un opérateur U qui est une translation bilatérale de multiplicité $\leqq \aleph_0 \cdot \dim \mathfrak{H}$. [2]) Il s'ensuit le suivant

Corollaire. *Pour T complètement non-unitaire, l'opérateur V du théorème peut être choisi comme une translation bilatérale, de multiplicité égale à $\aleph_0 \cdot \dim \mathfrak{H}$.*

2. Deux lemmes

La démonstration du théorème sera fondée sur le suivant

Lemme 1. *Soit T une contraction de l'espace $\mathfrak{H}$ et soit W un opérateur unitaire d'un espace $\mathfrak{K}'$, unitairement équivalent à une dilatation unitaire de T. Il existe alors un sous-espace L de $\mathfrak{H} \oplus \mathfrak{K}'$ invariant pour $T \oplus W$ et tel que, en désignant par P_L la projection orthogonale sur L, on ait*

$$\|P_L h\|^2 = \frac{1}{2}\|h\|^2 \quad \text{pour} \quad h \in \mathfrak{H}. \tag{1}$$

On aura besoin aussi du suivant

Lemme 2. *Soit T une contraction dans $\mathfrak{H}$ et soit $\mathfrak{M}$ un sous-espace semi-invariant pour T, c'est-à-dire de la forme $\mathfrak{M} = \mathfrak{H}_2 \ominus \mathfrak{H}_1$ où $\mathfrak{H}_1, \mathfrak{H}_2$ sont des sous-espaces invariants pour T, $\mathfrak{H}_1 \subset \mathfrak{H}_2 \subset \mathfrak{H}$. Posons $T_{\mathfrak{M}} = P_{\mathfrak{M}} T | \mathfrak{M}$ où $P_{\mathfrak{M}}$ désigne la projection orthogonale de $\mathfrak{H}$ dans $\mathfrak{M}$. Toute dilatation unitaire U de T est alors une dilatation unitaire aussi de $T_{\mathfrak{M}}$.*

Démonstrations.

(*Lemme* 1:) Soit U la dilatation unitaire de T (opérant dans un espace $\mathfrak{K} \supset \mathfrak{H}$) à laquelle W est unitairement équivalent; on a donc $U = \tau W \tau^{-1}$ où τ est une application unitaire de $\mathfrak{K}'$ sur $\mathfrak{K}$. Posons $\mathfrak{K}_+ = \bigvee_0^\infty U^n \mathfrak{H}$. En désignant par $P_{\mathfrak{H}}$ la projection orthogonale de $\mathfrak{K}$ dans $\mathfrak{H}$, on aura

$$P_{\mathfrak{H}} U \cdot U^n h = P_{\mathfrak{H}} U^{n+1} h = T^{n+1} h = T \cdot T^n h = T P_{\mathfrak{H}} U^n h$$

pour $h \in \mathfrak{H}$ et $n = 0, 1, \ldots$; il en dérive

$$P_{\mathfrak{H}} U k = T P_{\mathfrak{H}} k \quad \text{pour tout} \quad k \in \mathfrak{K}_+.$$

Cela étant, envisageons l'ensemble, évidemment linéaire, des éléments de $\mathfrak{H} \oplus \mathfrak{K}'$ de la forme

$$\{P_{\mathfrak{H}} k \oplus \tau^{-1} k : k \in \mathfrak{K}_+\}$$

Il est manifeste que si $\mathfrak{H}$ est séparable, l'espace $\mathfrak{K}$ de sa dilatation unitaire minimum est aussi séparable. — Pour ces questions nous renvoyons le lecteur p. ex. au livre [2].

Notons que *le théorème et sa démonstration subsistent aussi pour des dilatations non-unitaires.*

[2]) *Cf.* p. ex. [2], théorème II. 7. 4.

et soit L l'adhérence de cet ensemble. Puisque

$$(T\oplus W)(P_{\mathfrak{H}}k\oplus\tau^{-1}k) = TP_{\mathfrak{H}}k\oplus W\tau^{-1}k = P_{\mathfrak{H}}Uk\oplus\tau^{-1}Uk \qquad (k\in\mathfrak{K}_+)$$

et que $U\mathfrak{K}_+\subset\mathfrak{K}_+$, on conclut que $(T\oplus W)L\subset L$.

Montrons que

$$P_L(h\oplus 0) = \tfrac{1}{2}h\oplus\tau^{-1}(\tfrac{1}{2}h) \quad (h\in\mathfrak{H}). \tag{2}$$

En effet, comme $\mathfrak{H}\subset\mathfrak{K}_+$, il est évident que l'élément au second membre de (2) appartient à L. D'autre part, la différence

$$(h\oplus 0)-\big(\tfrac{1}{2}h\oplus\tau^{-1}(\tfrac{1}{2}h)\big)=\tfrac{1}{2}h\oplus\tau^{-1}(-\tfrac{1}{2}h)$$

est orthogonale à L puisque, pour $k\in\mathfrak{K}_+$,

$$\big(\tfrac{1}{2}h\oplus\tau^{-1}(-\tfrac{1}{2}h),\ P_{\mathfrak{H}}k\oplus\tau^{-1}k\big)=\tfrac{1}{2}(h,P_{\mathfrak{H}}k)+\big(\tau^{-1}(-\tfrac{1}{2}h),\tau^{-1}k\big)=$$

$$=\tfrac{1}{2}(h,k)+(-\tfrac{1}{2}h,k)=0.$$

Cela prouve (2), d'où l'on conclut

$$\|P_L(h\oplus 0)\|^2 = \|\tfrac{1}{2}h\|^2+\|\tau^{-1}(\tfrac{1}{2}h)\|^2 = \tfrac{1}{4}\|h\|^2+\tfrac{1}{4}\|h\|^2 = \tfrac{1}{2}\|h\|^2.$$

Puisque $h\oplus 0$ s'identifie à h, cela fournit (1).

(*Lemme* 2:) Comme $\mathfrak{H}_1$ et $\mathfrak{H}_2=\mathfrak{H}_1\oplus\mathfrak{M}$ sont invariants pour T, la décomposition $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{M}\oplus\mathfrak{N}$ (où $\mathfrak{N}=\mathfrak{H}\ominus\mathfrak{H}_2$) engendre pour T la triangulation

$$T=\begin{bmatrix} T_{\mathfrak{H}_1} & * & * \\ O & T_{\mathfrak{M}} & * \\ O & O & T_{\mathfrak{N}}\end{bmatrix}.$$

Il en découle

$$T^n=\begin{bmatrix} T^n_{\mathfrak{H}_1} & * & * \\ O & T^n_{\mathfrak{M}} & * \\ O & O & T^n_{\mathfrak{N}}\end{bmatrix} \quad (n=0,1,\ldots),$$

d'où

$$P_{\mathfrak{M}}T^n\begin{bmatrix}0\\ h\\ 0\end{bmatrix}=\begin{bmatrix}0\\ T^n_{\mathfrak{M}}h\\ 0\end{bmatrix} \quad (h\in\mathfrak{M};\ n=0,1,\ldots).$$

Par conséquent on a $P_{\mathfrak{M}}T^n|\mathfrak{M}=T^n_{\mathfrak{M}}$. Cela entraîne

$$(T^n_{\mathfrak{M}}h_1,h_2)=(P_{\mathfrak{M}}T^nh_1,h_2)=(T^nh_1,h_2)=(U^nh_1,h_2)$$

pour $h_1, h_2\in\mathfrak{M}$ et $n=0, 1, \ldots$. Cela prouve que U est une dilatation aussi de $T_{\mathfrak{M}}$.

3. Démonstration du théorème

Convenons de désigner par Δ_n et Δ_n^0 $(n=0, 1, \ldots)$ les ensembles des nombres de la forme $j/2^n$ (avec j entier), contenus dans l'intervalle fermé $[0, 1]$ ou dans l'intervalle ouvert $(0, 1)$, selon les cas. Posons $\Delta=\bigcup_n \Delta_n$ et $\Delta^0=\bigcup_n \Delta_n^0$; Δ et Δ^0 sont donc constitués de tous les nombres dyadiques rationnels contenus dans $[0, 1]$ ou dans $(0, 1)$, selon les cas.

Soit T la contraction donnée dans $\mathfrak{H}$ et soit U une dilatation unitaire de T. A chaque $\alpha\in\Delta^0$ attachons un opérateur $U(\alpha)$ dans un espace $\mathfrak{K}(\alpha)$, tel que $U(\alpha)$ soit unitairement équivalent à la somme orthogonale d'une infinité dénombrable de répliques de U:

(3) $$U(\alpha)\sim U\oplus U\oplus U\oplus \ldots .$$

Envisageons l'espace

(4) $$H=\mathfrak{H}\oplus\Big[\bigoplus_{\alpha\in\Delta^0}\mathfrak{K}(\alpha)\Big]$$

et son opérateur

(5) $$A=T\oplus\Big[\bigoplus_{\alpha\in\Delta^0}U(\alpha)\Big].$$

(Les espaces $\mathfrak{H}$ et $\mathfrak{K}(\alpha)$ se plongent dans H comme des sous-espaces de celui-ci.) L'opérateur

(6) $$U_A=U\oplus\Big[\bigoplus_{\alpha\in\Delta^0}U(\alpha)\Big]$$

est alors une dilatation unitaire de A. L'ensemble Δ^0 étant dénombrable, il s'ensuit de (3) et (6) que U_A est unitairement équivalente à la somme orthogonale d'une infinité dénombrable de répliques de U et par conséquent unitairement équivalente à chacun des opérateurs $U(\alpha)$.

Nous allons construire, pour chaque valeur de $n(=0, 1, \ldots)$, un système $\{H_n(\alpha)\colon \alpha\in\Delta_n\}$ de sous-espaces invariants pour A; la projection orthogonale de H dans $H_n(\alpha)$ sera désignée par $E_n(\alpha)$.

Nous commençons par définir le système de rang $n=0$ en posant

(7) $$H_0(0)=\{0\}\quad\text{et}\quad H_0(1)=\mathfrak{H},\ ^{3)}$$

puis nous procédons par récurrence en n.

Supposons que le système de rang n de sous-espaces invariants soit déjà défini et supposons de plus qu'il vérifie les relations

$(8)_n$ $$H_n(0)\subset H_n\Big(\frac{1}{2^n}\Big)\subset H_n\Big(\frac{2}{2^n}\Big)\subset\ldots\subset H_n\Big(\frac{2^n-1}{2^n}\Big)\subset H_n(1)$$

et

$(9)_n$ $$H_n(0)=\{0\},\quad H_n(1)=\mathfrak{H}\oplus\Big[\bigoplus_{\beta\in\Delta_n^0}\mathfrak{K}(\beta)\Big].$$

Nous construisons alors le système de rang $n+1$ de la manière suivante.

[3]) On a $A|\mathfrak{H}=T$.

Tout d'abord nous déduisons de $(8)_n$ et $(9)_n$ que les sous-espaces $H_n(\alpha)$ $(\alpha \in \Delta_n)$ sont orthogonaux aux sous-espaces $\mathfrak{K}(\delta)$ avec $\delta \notin \Delta_n^0$; [4]) ainsi, la définition suivante est possible:

(10) $$H_{n+1}(\alpha) = H_n(\alpha) \oplus [\bigoplus_{\substack{\delta \in \Delta_{n+1} \setminus \Delta_n \\ \delta < \alpha}} \mathfrak{K}(\delta)] \qquad (\alpha \in \Delta_n).$$

Puisque $H_n(\alpha)$ est invariant pour A et que les sous-espaces $\mathfrak{K}(\delta)$ même réduisent A, on conclut que $H_{n+1}(\alpha)$ est aussi invariant pour A. De $(8)_n$, $(9)_n$ et (10) il dérive d'une manière évidente qu'on a

$(8)'_{n+1}$ $$H_{n+1}(0) \subset H_{n+1}\left(\frac{1}{2^n}\right) \subset H_{n+1}\left(\frac{2}{2^n}\right) \subset \ldots \subset H_{n+1}\left(\frac{2^n+1}{2^n}\right) \subset H_{n+1}(1)$$

et

$(9)_{n+1}$ $$H_{n+1}(0) = \{0\}, \quad H_{n+1}(1) = \mathfrak{H} \oplus [\bigoplus_{\beta \in \Delta_{n+1}^0} \mathfrak{K}(\beta)];$$

l'apostrophe indique qu'il ne s'agit encore que des points de Δ_n.

Afin d'étendre cette définition aux points de $\Delta_{n+1} \setminus \Delta_n$, nous envisageons deux points voisins quelconques de Δ_n, α_n et β_n $(\alpha_n < \beta_n)$, et soit $\gamma_{n+1} = \frac{1}{2}(\alpha_n + \beta_n)$. On déduit de (10) que

(11) $$H_{n+1}(\beta_n) \ominus H_{n+1}(\alpha_n) = [H_n(\beta_n) \ominus H_n(\alpha_n)] \oplus \mathfrak{K}(\gamma_{n+1}).$$

Posons

(12) $$A(\gamma_{n+1}) = [E_n(\beta_n) - E_n(\alpha_n)]A \,|\, [H_n(\beta_n) \ominus H_n(\alpha_n)].$$

Il s'ensuit du lemme 2 que toute dilatation unitaire de A est aussi une dilatation unitaire de $A(\gamma_{n+1})$. Par conséquent $A(\gamma_{n+1})$ admet une dilatation unitaire qui est unitairement équivalente à $U(\gamma_{n+1})$. Ainsi, on peut appliquer le lemme 1 à l'opérateur

(13) $$A(\gamma_{n+1}) \oplus U(\gamma_{n+1})$$

de l'espace (11). Il résulte qu'il existe un sous-espace $L(\gamma_{n+1})$ de cet espace, invariant pour l'opérateur (13), et tel que

(14) $$\|P_{L(\gamma_{n+1})}h\|^2 = \tfrac{1}{2}\|h\|^2 \quad \text{pour} \quad h \in H_n(\beta_n) \ominus H_n(\alpha_n);$$

(14) entraîne évidemment aussi

(15) $$\|h - P_{L(\gamma_{n+1})}h\|^2 = \tfrac{1}{2}\|h\|^2 \quad \text{pour} \quad h \in H_n(\beta_n) \ominus H_n(\alpha_n).$$

Cela étant, nous définissons:

(16) $$H_{n+1}(\gamma_{n+1}) = H_{n+1}(\alpha_n) \oplus L(\gamma_{n+1});$$

l'orthogonalité des deux termes au second membre résulte de la relation $L(\gamma_{n+1}) \subset \subset H_{n+1}(\beta_n) \ominus H_{n+1}(\alpha_n)$. De cette relation et de (16) on déduit aussi que

(17) $$H_{n+1}(\alpha_n) \subset H_{n+1}(\gamma_{n+1}) \subset H_{n+1}(\beta_n).$$

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[4]) En effet, $H_n(\alpha) \subset H_n(1)$ et $H_n(1) \perp \mathfrak{K}(\delta)$ pour $\delta \notin \Delta_n^0$.

Montrons que $H_{n+1}(\gamma_{n+1})$ est invariant pour A. Vu que $H_{n+1}(\alpha_n)$ est invariant pour A, il n'y a qu'à montrer que

$$AL(\gamma_{n+1})\subset H_{n+1}(\gamma_{n+1}). \tag{18}$$

Puisque $L(\gamma_{n+1})$ est un sous-espace de l'espace (11), tout élément $l\in L(\gamma_{n+1})$ s'écrit sous la forme $l=h+k$ où $h\in H_n(\beta_n)\ominus H(\alpha_n)$ et $k\in\mathfrak{R}(\gamma_{n+1})$, d'où $Al=Ah+Ak$. Comme $Ah\in AH_n(\beta_n)\subset H_n(\beta_n)$, il s'ensuit

$$Ah = E_n(\beta_n)Ah = E_n(\alpha_n)Ah+[E_n(\beta_n)-E_n(\alpha_n)]Ah =$$
$$= E_n(\alpha_n)Ah+A(\gamma_{n+1})h \quad (cf.\ (12))$$

et par conséquent

$$Al = E_n(\alpha_n)Ah+A(\gamma_{n+1})h+Ak = E_n(\alpha_n)Ah+[A(\gamma_{n+1})\oplus U(\gamma_{n+1})]l.$$

Vu que $E_n(\alpha_n)Ah\in H_n(\alpha_n)\subset H_{n+1}(\alpha_n)$ par (10), et que $L(\gamma_{n+1})$ est invariant pour $A(\gamma_{n+1})\oplus U(\gamma_{n+1})$, on conclut que

$$Al\in H_{n+1}(\alpha_n)\oplus L(\gamma_{n+1}) = H_{n+1}(\gamma_{n+1}).$$

ce qui prouve (18) et achève la démonstration de ce que $H_{n+1}(\gamma_{n+1})$ est invariant pour A.

Lorsque α_n, β_n parcourent les couples des points voisins dans Δ_n, le point γ_{n+1} parcourt $\Delta_{n+1}\setminus\Delta_n$. Ainsi, on a défini le système complet $\{H_{n+1}(\alpha): \alpha\in\Delta_{n+1}\}$ de sous-espaces invariants pour A; en réunissant les résultats $(8)'_{n+1}$ et (17) il s'ensuit que ce système vérifie les relations $(8)_{n+1}$ et $(9)_{n+1}$.

De cette façon, la définition par récurrence est achevée, et les relations $(8)_n$ et $(9)_n$ sont établies pour tous les n.

Convenons des notations suivantes:

$$H_n(\alpha',\alpha'') = H_n(\alpha'')\ominus H_n(\alpha'), \qquad E_n(\alpha',\alpha'') = E_n(\alpha'')-E_n(\alpha')$$
$$\text{pour}\quad \alpha',\alpha''\in\Delta_n \qquad (\alpha'<\alpha'').$$

Il découle de (11), (16) et (17) que

$$\left.\begin{aligned} H_{n+1}(\alpha_n,\gamma_{n+1}) = L(\gamma_{n+1})\subset \\ H_{n+1}(\gamma_{n+1},\beta_n)\subset H_{n+1}(\alpha_n,\beta_n) = \end{aligned}\right\} H_n(\alpha_n,\beta_n)\oplus\mathfrak{R}(\gamma_{n+1}),$$

et de (14) et (15), que

$$\left.\begin{aligned} \|E_{n+1}(\alpha_n,\gamma_{n+1})h\|^2 = \|P_{L(\gamma_{n+1})}h\|^2 \\ \|E_{n+1}(\gamma_{n+1},\beta_n)h\|^2 = \|h-P_{L(\gamma_{n+1})}h\|^2 \end{aligned}\right\} = \frac{1}{2}\|h\|^2$$

pour $h\in H_n(\alpha_n,\beta_n)$. En désignant par $(\alpha_{n+1},\beta_{n+1})$ l'une ou l'autre des deux moitiés de l'intervalle (α_n,β_n), on a donc

$$H_{n+1}(\alpha_{n+1},\beta_{n+1})\subset H_n(\alpha_n,\beta_n)\oplus\mathfrak{R}\left(\tfrac{1}{2}(\alpha_n+\beta_n)\right) \tag{19}$$

et

$$\|E_{n+1}(\alpha_{n+1},\beta_{n+1})h\|^2=\tfrac{1}{2}\|h\|^2 \quad\text{pour}\quad h\in H_n(\alpha_n,\beta_n). \tag{20}$$

Soit $(\alpha_0, \beta_0), (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n), \ldots$ une suite d'intervalles, chacun desquels est l'une des deux moitiés du précédent; $\alpha_0 = 0$ et $\beta_0 = 1$. Grâce aux relations récurrentes (19) et (20) on a pour $0 \leqq r < s$

(21) $$H_s(\alpha_s, \beta_s) \subset H_r(\alpha_r, \beta_r) \oplus \mathfrak{K}_{rs},$$

où

$$\mathfrak{K}_{rs} = \bigoplus_{k=r}^{s-1} \mathfrak{K}\left(\tfrac{1}{2}(\alpha_k + \beta_k)\right),$$

et

(22) $$\|E_s(\alpha_s, \beta_s) \ldots E_{r+1}(\alpha_{r+1}, \beta_{r+1})h\|^2 = 2^{r-s}\|h\|^2 \quad \text{pour} \quad h \in H_r(\alpha_r, \beta_r).$$

De (22) il dérive aussitôt

(23) $$\|E_s(\alpha_s, \beta_s) \ldots E_r(\alpha_r, \beta_r)h\|^2 = 2^{r-s}\|E_r(\alpha_r, \beta_r)h\|^2 \leqq 2^{r-s}\|h\|^2$$

pour $h \in H$ quelconque et pour $r < s$.

Soit $h \in H_n(1)$ et soient p et q des entiers tels que $n \leqq p < q$. Puisque $H_n(\alpha_n, \beta_n) \subset \subset H_n(\beta_n) \subset H_n(1)$, il dérive de (21) (pour $r = n$ et $s = p$) que $h - E_p(\alpha_p, \beta_p)h \subset \subset H_n(1) \oplus \mathfrak{K}_{np}$; l'orthogonalité des deux termes du dernier membre découle de $(9)_n$. Par la même raison, $H_n(1)$ est orthogonal aussi à $\mathfrak{K}_{pq}$, tandis que l'orthogonalité $\mathfrak{K}_{np} \perp \mathfrak{K}_{pq}$ s'ensuit de la définition des $\mathfrak{K}_{rs}$. Ainsi, on a $H_n(1) \oplus \mathfrak{K}_{np} \perp \mathfrak{K}_{pq}$. Par conséquent, $h - E_p(\alpha_p, \beta_p)h$ est orthogonal à $\mathfrak{K}_{pq}$. D'autre part, il est évidemment orthogonal à $H_p(\alpha_p, \beta_p)$, d'où il résulte, en vertu de la relation (21) (appliquée au cas $r = p$, $s = q$), que $h - E_p(\alpha_p, \beta_p)h$ est orthogonal à $H_q(\alpha_q, \beta_q)$. Par conséquent on a

(24) $$E_q(\alpha_q, \beta_q) \cdot E_p(\alpha_p, \beta_p)h = E_q(\alpha_q, \beta_q)h \quad \text{pour} \quad h \in H_n(1) \quad \text{et} \quad n \leqq p < q.$$

L'application répétée de cette relation fournit

$$E_s(\alpha_s, \beta_s) \ldots E_n(\alpha_n, \beta_n)h = E_s(\alpha_s, \beta_s)h \quad \text{pour} \quad h \in H_n(1) \quad \text{et} \quad s \geqq n.$$

Vu (23) il en résulte que

(25) $$\|E_s(\alpha_s, \beta_s)h\|^2 \leqq 2^{n-s}\|h\|^2 \quad \text{pour} \quad h \in H_n(1) \quad \text{et} \quad s \geqq n.$$

Comme le choix des moitiés successives de l'intervalle (α_0, β_0) était arbitraire, (25) subsiste pour n'importe quel couple α_s, β_s de points voisins de Δ_s $(\alpha_s < \beta_s)$, dès que $s \geqq n$.

Soient α, β deux points arbitraires de Δ_s $(\alpha < \beta;\ s \geqq n)$. On a alors

$$E_s(\alpha, \beta)h = \sum_{i=1}^{N} E_s(\alpha + (i-1)2^{-s}, \alpha + i\,2^{-s})h \qquad (h \in H_n(1))$$

où $N = (\beta - \alpha)2^s$. Les termes du second membre étant orthogonaux et comme on peut appliquer (25) à chacun de ces termes, on obtient

$$\|E_s(\alpha, \beta)h\|^2 \leqq N \cdot 2^{n-s}\|h\|^2,$$

donc

(26) $$\|[E_s(\beta) - E_s(\alpha)]h\|^2 \leqq (\beta - \alpha) \cdot 2^n\|h\|^2$$

et cela pour $h \in H_n(1)$; $\alpha, \beta \in \Delta_s$; $s \geqq n$.

Nous sommes à même de conclure la démonstration de notre théorème. En ce but, rappelons que pour tout α fixé, $\alpha \in \Delta$, les sous-espaces invariants $H_n(\alpha)$ sont définis pour tous les n assez grands, notamment dès que Δ_n comprend α, et ils forment une suite croissante, *cf.* (10). Par conséquent

$$H(\alpha) = \overline{\bigcup_n H_n(\alpha)}$$

est aussi un sous-espace invariant pour A, et pour la projection orthogonale correspondante $E(\alpha)$ on a

$$E(\alpha) = \lim_{n\to\infty} E_n(\alpha) \qquad \text{(convergence forte des opérateurs).}$$

Les relations $(8)_n$ et $(9)_n$, valables pour tous les n, entraînent évidemment

$$H(\alpha') \subset H(\alpha''), \quad \text{donc} \quad E(\alpha') \leqq E(\alpha'') \quad \text{pour} \quad \alpha', \alpha'' \in \Delta \qquad (\alpha' < \alpha''),$$

et $H(0) = \{0\}$, $H(1) = H$, donc $E(0) = O$, $E(1) = I_H$. Enfin, en faisant $s \to \infty$ dans (26) il résulte

$$(27) \qquad \|[E(\beta) - E(\alpha)]h\|^2 \leqq (\beta - \alpha) \cdot 2^n \|h\|^2 \quad \text{pour } \alpha, \beta \in \Delta \quad \text{et} \quad h \in H_n(1).$$

Cela entraîne que la limite

$$\lim_{\alpha \in \Delta,\, \alpha \to \lambda} E(\alpha) h$$

existe pour tout λ réel dans [0, 1] et tout $h \in H_n(1)$. Comme les sous-espaces $H_n(1)$ $(n = 1, 2, \ldots)$ sont denses dans H, cette limite existe alors pour $h \in H$ quelconque. En définissant $E(\lambda)h$ par cette limite, on obtient une fonction croissante $E(\lambda)$ de λ, à valeurs projections orthogonales dans H, telle que $E(0) = O$, $E(1) = I_H$ et que $H(\lambda) = E(\lambda) H$ est invariant pour A pour chaque λ. De plus, la relation (27) se conserve lors de cette extension. En vertu de cette relation, la fonction numérique $(E(\lambda)h, h)$ de λ vérifie une condition de Lipschitz pour chaque h fixé dans $H_n(1)$. Comme les $H_n(1)$ sont denses dans H, on conclut que la fonction $(E(\lambda)h, h)$ est, pour $h \in H$ quelconque, la limite uniforme de fonctions lipschitziennes et par conséquent elle est continue et même *absolument continue*.

Ainsi, la famille des sous-espaces $H(\lambda)$ que nous venons de construire forme une échelle continue (et même absolument continue dans le sens indiqué) de sous-espaces invariants pour l'opérateur $A = T \oplus [\bigoplus_{\alpha \in \Delta^\circ} U(\alpha)]$.

Cela achève la démonstration du théorème.

Ouvrages cités

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(Reçu le 15 octobre 1966)

Homomorphisms, higher derivations, and derivations on associative algebras

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1. Introduction

Let $\mathfrak{A}$ be an algebra over a field K of characteristic zero. By a *derivation* D on $\mathfrak{A}$ we shall mean a mapping on $\mathfrak{A}$ into $\mathfrak{A}$ which is linear, that is $D(\lambda a+\mu b)=\lambda D(a)+\mu D(b)$ for all $a, b\in A$, $\lambda, \mu\in K$, and which satisfies the law

(1. 1) $$D(ab)=D(a)b+aD(b) \qquad (\text{all } a, b, \in\mathfrak{A}).$$

By an *endomorphism* E we shall mean a linear map on $\mathfrak{A}$ into $\mathfrak{A}$ satisfying

(1. 2) $$E(ab)=E(a)E(b) \qquad (\text{all } a, b\in\mathfrak{A}).$$

The derivations on $\mathfrak{A}$ form a Lie algebra under the product $[D_1, D_2] = D_1 D_2 - D_2 D_1$. The endomorphisms form a semigroup under composition.

If $\mathfrak{A}$ is a Banach algebra, and D is a given derivation, bounded, then the operator H defined by

(1. 3) $$H=\exp(D)$$

is a bounded endomorphism, in fact an automorphism. Here $\exp(D)$ is defined by the exponential series, convergent in the uniform operator topology, and the proof uses the Leibniz formula

$$D^n(ab) = \sum_{m=0}^{n}\binom{n}{m} D^{n-m}(a)D^m(b)$$

in an obvious manner. The formula (1. 3) is useful in a number of contexts, for example in the proof of the Singer—Wermer theorem [7], in the theory of semigroups of bounded automorphisms, and also in purely algebraic settings, such as in aspects of the structure theory of fields, and of Lie algebras ([4], Ch. IV; [3]).

The converse question arises, whether an arbitrary automorphism on the Banach algebra has a logarithm which is a derivation. If the problem is restated in terms of continuous groups of bounded automorphisms, the matter is straightforward: if the semigroup $\{E_\lambda: \lambda>0\}$ is continuous and $E_\lambda\to I$ as $\lambda\to 0$ in the uniform operator topology, then its infinitesimal generator is a bounded derivation D, $E_\lambda=\exp(\lambda D)$ and the semigroup is embeddable in the group $\{E_\lambda: \lambda \text{ real}\}$. The proof is an immediate consequence of the definition of the generator. (The infinitesimal generator of a strongly continuous semigroup of endomorphisms is likewise a derivation.) Thus there is a natural one-to-one correspondence between uniformly continuous groups of automorphisms and bounded derivations (see G. HOCHSCHILD [2]).

The further question can be asked, whether a single arbitrary endomorphism can be represented in some fashion in terms of derivations and the exponential function, even if it cannot be embedded in a semigroup. This matter is not settled here, though it is hoped that the present investigation, which it prompted, may contribute to its solution. We consider instead certain homomorphisms on an arbitrary algebra $\mathfrak{A}$ into algebras of polynomials and power series with coefficients in $\mathfrak{A}$, and obtain formulae for these homomorphisms which are, formally at least, of the form (1. 3). These can alternatively be viewed as representation formulae for higher derivations in terms of derivations; they are not immediately concerned with normed algebras.

To be more specific, we need some notation and terminology. Given $\mathfrak{A}$ and $k \geqq 0$, let $\mathfrak{A}_k[t]$ denote the algebra of all polynomials $a_0+a_1t+a_2t^2+\dots+a_kt^k$, of degree $\leqq k$ in an indeterminate t, with coefficients in $\mathfrak{A}$. Addition and scalar multiplication are defined in the obvious manner; the product of two polynomials in $\mathfrak{A}_k[t]$ is formed by multiplying the polynomials as usual, letting t commute with the elements of $\mathfrak{A}$, and then deleting all terms containing powers of t greater than k. That is $\mathfrak{A}_k[t] \cong \mathfrak{A}[t]/(t^{k+1})$, the residue class algebra of the algebra of all polynomials modulo the principal ideal (t^{k+1}). We consider those homomorphisms H from $\mathfrak{A}$ into $\mathfrak{A}_k[t]$ which have the special property ε that a_0 in the image of a equals a, i. e.

$$H(a) = a+a_1t+a_2t^2+\dots+a_kt^k. \tag{1.4}$$

(For brevity we shall sometimes call these ε-homomorphisms.) Given such H, the determination of $a_1, a_2, \dots, a_k$ from a is linear; thus H determines k linear mappings $F_1, F_2, \dots, F_k$ by $a_n=F_n(a)$, $n=1, 2, \dots, k$, so that

$$H(a) = a+tF_1(a)+t^2F_2(a)+\dots+t^kF_k(a). \tag{1.5}$$

Substituting this form in (1. 2) gives the set of identities

$$F_1(ab) = F_1(a)b+aF_1(b), \tag{1.6}$$

$$F_2(ab) = F_2(a)b+F_1(a)F_1(b)+aF_2(b), \tag{1.7}$$

$$F_3(ab) = F_3(a)b+F_2(a)F_1(b)+F_1(a)F_2(b)+aF_3(b), \tag{1.8}$$

and in general, for $n=1, 2, \dots, k$,

$$F_n(ab) = \sum_{r=0}^{n} F_{n-r}(a)F_r(b) \qquad (F_0=I). \tag{1.9}$$

A sequence of operators $(I, F_1, F_2, \dots, F_k)$ satisfying these equations for all $a, b \in \mathfrak{A}$ is called by JACOBSON ([4], p. 191) a *higher derivation of rank k*. Equation (1. 5) establishes a one-to-one correspondance between homomorphisms with the property ε and higher derivations. Similarly, considering homomorphisms from $\mathfrak{A}$ into the algebra $\mathfrak{A}_\infty[t]$ of all formal power series in t, we obtain *higher derivations* $\{I, F_1, F_2, \dots\}$ *of infinite rank*, the F's satisfying (1. 9) for $n=1, 2, \dots$.

The result to be proved, for both finite and infinite cases, states that, under a suitable matrix representation, H and so the corresponding higher derivation can be represented uniquely by a matrix exp ($\boldsymbol{D}$), where the matrix $\boldsymbol{D}$ has derivations for its elements. In §§ 5, 6 we mention some applications to the case when $\mathfrak{A}$ is a Banach algebra, and also obtain a result on homomorphisms on commutative Banach algebras by related arguments.

2. Higher derivations for $A_k[t]$

Equations (1. 9) can be used to express the higher derivation $\{I, F_1, F_2, \ldots, F_k\}$ in terms of derivations. Clearly from (1. 6), F_1 is a derivation; write $F_1 = D_1$. Consider (1. 7), with $F_1 = D_1$. One solution for F_2 is $F_2 = \frac{1}{2} D_1^2$; moreover any two solutions for F_2 differ by a derivation, and any solution plus a derivation is again a solution. We may therefore take as the most general solution of (1. 6) and (1. 7)

$$F_1 = D_1, \quad F_2 = \frac{1}{2!} D_1^2 + \Omega_2 \tag{2. 1}$$

where D_1 and Ω_2 are arbitrary derivations. Turning next to equation (1. 8), supposing F_1 and F_2 given by (2. 1), we get as the most general solution, by similar arguments

$$F_3 = \frac{1}{3!} D_1^3 + \Omega_2 D_1 + \Omega_3 \tag{2. 2}$$

where Ω_3 is an arbitrary derivation. Similarly

$$F_4 = \frac{1}{4!} D_1^4 + \frac{1}{2} \Omega_2 D_1^2 + \Omega_3 D_1 + \frac{1}{2} \Omega_2^2 + \Omega_4, \tag{2. 3}$$

and so on. Let us call this 'the process P'. The formulae which arise in this way become increasingly complicated. We can clarify the nature of the process and obtain a general formula for F_n by means of the following matrix representation.

Let $\mathfrak{T}_{k+1}(\mathfrak{A})$ be the algebra of all $(k+1)\times(k+1)$ upper-triangular matrices with constant diagonals and elements from $\mathfrak{A}$, i. e. of the form

$$A_{k+1} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_k \\ & a_0 & a_1 & & \\ & & a_0 & & \\ & & & \ddots & \\ & & & & a_0 \end{pmatrix} \tag{2. 4}$$

where $a_j \in \mathfrak{A}$ for $j = 0, 1, \ldots, k$. Let $\mathfrak{D}_{k+1}(\mathfrak{A})$ be the subalgebra of $\mathfrak{T}_{k+1}(\mathfrak{A})$ consisting of all diagonal matrices; clearly $\mathfrak{A}$ is isomorphic to $\mathfrak{D}_{k+1}(\mathfrak{A})$, and

$$a \to A_{k+1} = \begin{pmatrix} a & & & \\ & a & & \\ & & \ddots & \\ & & & a \end{pmatrix} \tag{2. 5}$$

is a faithful representation of $\mathfrak{A}$ in $\mathfrak{T}_{k+1}(\mathfrak{A})$. By mapping t to

$$(2.6)\qquad t = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

we extend this representation to an isomorphism between $\mathfrak{A}_k[t]$ and $\mathfrak{T}_{k+1}(\mathfrak{A})$, by which $a_0 + a_1 t + \ldots + a_k t^k$ is mapped to $\boldsymbol{A}_{k+1}$ of (2. 4).

Let $\mathfrak{B} \equiv \mathfrak{B}(\mathfrak{A})$ be the algebra of all linear operators on $\mathfrak{A}$. We map $\mathfrak{B}$ to $\mathfrak{D}_{k+1}(\mathfrak{B})$ as in (2. 5); and letting the matrices of $\mathfrak{T}_{k+1}(\mathfrak{B})$ operate on those of $\mathfrak{T}_{k+1}(\mathfrak{A})$ in the obvious fashion (that is, by formal application of matrix multiplication), we obtain a representation

$$(2.7)\qquad H \to \boldsymbol{F}_{k+1} = \begin{pmatrix} I & F_1 & F_2 & \cdots & F_k \\ & I & F_1 & & \\ & & I & \ddots & \\ & & & \ddots & \\ & & & & I \end{pmatrix}$$

of the ε-homomorphism H: $\mathfrak{A} \to \mathfrak{A}_k[t]$ in $\mathfrak{T}_{k+1}(\mathfrak{B})$. Clearly $\boldsymbol{F}_{k+1}$ is a homomorphism on $\mathfrak{D}_{k+1}(\mathfrak{A})$ into $\mathfrak{T}_{k+1}(\mathfrak{A})$. One can verify further that these elements of $\mathfrak{T}_{k+1}(\mathfrak{B})$ give the matrix representations of all the endomorphisms H on $\mathfrak{A}_k[k]$ of the form

$$H(a + a_1 t + a_2 t^2 + \ldots + a_k t^k) = a + b_1 t + b_2 t^2 + \ldots + b_k t^k.$$

In fact, the matrices of linear operators mapping $\mathfrak{T}_{k+1}(\mathfrak{A})$ into $\mathfrak{T}_{k+1}(\mathfrak{A})$ (i. e. preserving leading diagonals) are precisely the elements of $\mathfrak{T}_{k+1}(\mathfrak{B})$, and the ε-endomorphisms on $\mathfrak{T}_{k+1}(\mathfrak{A})$ are precisely the matrices $\boldsymbol{F}_{k+1}$ with elements F_n satisfying (1. 9). (By an ε-endomorphism we mean an endomorphism which preserves leading diagonals.)

Letting $D_1, D_2, \ldots, D_k$ be derivations on $\mathfrak{A}$, write

$$(2.8)\qquad \boldsymbol{D}_{k+1} = \begin{pmatrix} 0 & D_1 & D_2 & \cdots & D_k \\ & 0 & D_1 & & \\ & & 0 & \ddots & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}.$$

Then our principal result in the finite case is the second half of:

Theorem (k finite). *The derivations $D_1, D_2, \ldots, D_k$ in $\mathfrak{B}(\mathfrak{A})$ being given,* $\exp(\boldsymbol{D}_{k+1})$ *is the matrix in $\mathfrak{T}_{k+1}(\mathfrak{B})$ of an ε-homomorphism on $\mathfrak{A}$ into $\mathfrak{A}_k[t]$, with $F_1 = D_1$; that is,* $\exp(\boldsymbol{D}_{k+1})$ *is an ε-endomorphism on $\mathfrak{T}_{k+1}(\mathfrak{A})$.*

Conversely, given an ε-homomorphism H on $\mathfrak{A}$ into $\mathfrak{A}_k[t]$, its matrix $\boldsymbol{F}_{k+1}$ can be written

(2. 9) $$\boldsymbol{F}_{k+1}=\exp(\boldsymbol{D}_{k+1})$$

where $D_1=F_1, D_2, \dots, D_k$ are derivations in $\mathfrak{B}(\mathfrak{A})$. This representation of H is unique.

Thus the ε-endomorphisms on $\mathfrak{T}_{k+1}(\mathfrak{A})$ are precisely the matrices of the form $\exp(\boldsymbol{D}_{k+1})$.

Proof. Suppose that the derivations are given. It is not difficult to show that $\boldsymbol{D}_{k+1}$ is then a derivation on $\mathfrak{T}_{k+1}(\mathfrak{A})$. Moreover $\boldsymbol{D}_{k+1}$ is nilpotent, so that $\exp(\boldsymbol{D}_{k+1})$ is well defined in $\mathfrak{T}_{k+1}(\mathfrak{B})$. Moreover it is an ε-endomorphism on $\mathfrak{T}_{k+1}(\mathfrak{A})$: the proof of this fact is algebraically the same as for the Banach-algebra case mentioned in § 1, while questions of convergence do not arise. If $\exp(\boldsymbol{D}_{k+1})$ is written in the form (2. 7), it is easily verified that $F_1=D_1$. This proves the first part. (We also find

(2. 10) $$F_2=\frac{1}{2!}D_1^2+D_2,\quad F_3=\frac{1}{3!}D_1^3+\frac{1}{2}(D_1D_2+D_2D_1)+D_3, \dots .$$

Thus the process P generates the derivations $D_2, D_3, \dots$ explicitly only if the added arbitrary derivations Ω are given suitable forms, that is, if we write

(2. 11) $$\Omega_2=D_2,\quad \Omega_3=\frac{1}{2}(D_1D_2-D_2D_1)+D_3, \dots ;$$

note that the first term in Ω_3 here is a derivation.)

We prove the second part of the theorem by induction. The truth of the assertion for small values of k can be verified using (2. 10). Let l be an integer $\geqq 2$, and assume the converse statement in the theorem for $k=l-1$. Let $\boldsymbol{F}_{l+1}$ be the matrix of an ε-homomorphism, its elements $I, F_1, \dots, F_l$ forming the corresponding higher derivation. Then the submatrix $\boldsymbol{F}_l$ is a ε-endomorphism on $\mathfrak{T}_l(\mathfrak{A})$, and so by hypothesis there exist derivations $D_1=F_1, D_2, \dots, D_{l-1}$ making up a matrix $\boldsymbol{D}_l$ such that $\boldsymbol{F}_l=\exp(\boldsymbol{D}_l)$. Partition $\boldsymbol{F}_{l+1}$ into blocks as

$$\boldsymbol{F}_{l+1}=\begin{pmatrix}\boldsymbol{F}_l & f\\ o' & I\end{pmatrix}$$

where f is $l\times 1$ and o' is $1\times l$. Let Δ be an arbitrary derivation, and write

$$\boldsymbol{\Delta}_{l+1}=\begin{pmatrix}0 & D_1 & D_2 & \cdots & D_{l-1} & \Delta\\ & 0 & D_1 & & D_{l-2} & D_{l-1}\\ & & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & D_1\\ & & & & & 0\end{pmatrix}=\begin{pmatrix}\boldsymbol{D}_l & \omega\\ o' & 0\end{pmatrix}.$$

Then $\boldsymbol{\Delta}_{l+1}$ is a derivation in $\mathfrak{T}_{l+1}(\mathfrak{B})$, and therefore

$$\exp(-\boldsymbol{\Delta}_{l+1}), \quad =\begin{pmatrix}\exp(-\boldsymbol{D}_l) & g\\ o' & I\end{pmatrix},$$

is an ε-endomorphism in $\mathfrak{T}_{l+1}(\mathfrak{B})$. (The forms of f, ω and g are not important to the argument.) Consider the product $\exp(-\boldsymbol{\Delta}_{l+1})\boldsymbol{F}_{l+1}$; its blocked form is

$$\begin{pmatrix} \boldsymbol{I}_l & h \\ o' & I \end{pmatrix},$$

and since it belongs to $\mathfrak{T}_{l+1}(\mathfrak{B})$ and has constant diagonals, its elements are those of $\boldsymbol{I}_{l+1}$ except possibly for its $(1, l+1)$ element, which is an unknown operator, Ψ say. But $\exp(-\boldsymbol{\Delta}_{l+1})\boldsymbol{F}_{l+1}$ is a homomorphism on $\mathfrak{D}_{l+1}(\mathfrak{A})$ into $\mathfrak{T}_{l+1}(\mathfrak{A})$: by writing out the corresponding form of (1. 2) we find that Ψ is a derivation. Let $\boldsymbol{\Psi}$ be the $(l+1)\times(l+1)$ matrix having Ψ in the $(1, l+1)$ position and all other elements 0. Then we have

$$\boldsymbol{F}_{l+1} = \exp(\boldsymbol{\Delta}_{l+1})(\boldsymbol{I}_{l+1} + \boldsymbol{\Psi}_{l+1}) = \exp(\boldsymbol{\Delta}_{l+1})\exp(\boldsymbol{\Psi}_{l+1}) = \exp(\boldsymbol{\Delta}_{l+1} + \boldsymbol{\Psi}_{l-1})$$

since $\boldsymbol{\Delta}_{l+1}$ and $\boldsymbol{\Psi}_{l+1}$ clearly commute. Thus we have

$$\boldsymbol{F}_{l+1} = \exp(\boldsymbol{D}_{l+1})$$

with $D_l = \Delta + \Psi$, a derivation. The result follows by induction.

It remains to prove uniqueness. Given H and so $F_1, F_2, \ldots, F_k$, we see that D_1 is determined, and D_2 also by (2. 10). Suppose that for some j, $2 < j \leqq k$, $D_1, D_2, \ldots, D_{j-1}$ are determined. Now

$$F_j = (1, j+1) \quad \text{element of} \quad \boldsymbol{I} + \boldsymbol{D}_{k+1} + \frac{1}{2!}\boldsymbol{D}_{k+1}^2 + \ldots;$$

the term $+D_j$ appears once on the right hand side, as the contribution of $\boldsymbol{D}_{k+1}$, and the other powers of $\boldsymbol{D}_{k+1}$ contribute expressions containing only D_{j-1}, $D_{j-2}, \ldots, D_1$. Therefore D_j is determined. Uniqueness follows.

Corollary. *The endomorphisms on $\mathfrak{T}_{k+1}(\mathfrak{A})$ are all automorphisms; they form a group under composition.*

We remark that the process P gives a certain prominence among the derivations to D_1. It is possible in fact to prove an alternative formula $\boldsymbol{F}_{k+1} = \exp(\boldsymbol{\Omega}_{k+1})\boldsymbol{G}_{k+1}$, with

$$\boldsymbol{G}_{k+1} = \begin{pmatrix} I & D_1 & D_1^2/2! \ldots D_1^k/k! \\ & I & D_1 & \\ & & I & \\ & & & \ddots \\ & & & & I \end{pmatrix} = \exp\begin{pmatrix} 0 & D_1 & 0 & \cdots & 0 \\ & 0 & D_1 & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}$$

and

$$\boldsymbol{\Omega}_{k+1} = \begin{pmatrix} 0 & 0 & \Omega_2 & \Omega_3 \cdots & \Omega_k \\ & 0 & 0 & \Omega_2 & \\ & & 0 & 0 & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}.$$

The proof is similar to that of the theorem. In geneial we have $\exp(\boldsymbol{\Omega}_{k+1})\boldsymbol{G}_{k+1} \neq \boldsymbol{G}_{k+1}\cdot\exp(\boldsymbol{\Omega}_{k+1})$, and the Ω's, though derivations, are again not those most obviously generated by the process P.

3. Higher derivations for $\mathfrak{A}_\infty[t]$

From the previous theorem we deduce the corresponding result for the infinite case, which we shall designate $k=\infty$. Here $\mathfrak{A}$ is represented by the diagonal matrices in $\mathfrak{T}_\infty(\mathfrak{A})$, the algebra of upper triangular matrices with constant diagonals and order ω. Taking the obvious definitions of sum, scalar multiple and product, we remark first that the product of two matrices in $\mathfrak{T}_\infty(\mathfrak{A})$ is always well defined: no element in the product involves more than finitely many non-zero elements from the factors. Thus $\mathfrak{T}_\infty(\mathfrak{A})$ is indeed an algebra. Again, if $\boldsymbol{D}$ has the form corresponding to (2. 8) for $k=\infty$, then although $\boldsymbol{D}$ may not be nilpotent, nevertheless $\exp(\boldsymbol{D})$ is well defined since each element of this matrix involves the sum of only finitely many elements in $\mathfrak{A}$.

We consider ε-endomorphisms on $\mathfrak{T}_\infty(\mathfrak{A})$; properties of ε-homomorphisms on $\mathfrak{A}$ into $\mathfrak{A}_\infty[t]$ come by restriction to $\mathfrak{D}_\infty(\mathfrak{A})$.

Theorem ($k=\infty$). *The sequence $D_1, D_2, \ldots$ of derivations in $\mathfrak{B}(\mathfrak{A})$ being given,* $\exp(\boldsymbol{D})$ *is the matrix in $\mathfrak{T}_\infty(\mathfrak{B})$ of an ε-homomorphism on $\mathfrak{A}$ into $\mathfrak{A}_\infty[t]$, with $F_1=D_1$. Conversely, given an ε-homomorphism on $\mathfrak{A}$ into $\mathfrak{A}_\infty[t]$, its matrix $\boldsymbol{F}$ has the form*

$$\boldsymbol{F}=\exp(\boldsymbol{D}) \tag{3. 1}$$

where $D_1=F_1, D_2, \ldots$ are derivations. This representation is unique.

Thus the ε-endomorphisms on $\mathfrak{T}_\infty(\mathfrak{A})$ are precisely the matrices of the form $\exp(\boldsymbol{D})$.

Proof. For a matrix $\boldsymbol{X}$ in $\mathfrak{T}_k$ ($k\leqq\infty$) let $[\boldsymbol{X}]_j$ when $j\leqq k$ denote the element of the jth superdiagonal, counting the leading diagonal as first; and let $\boldsymbol{X}_j$ denote the leading $j\times j$ block in $\boldsymbol{X}$, so that $\boldsymbol{X}_j\in\mathfrak{T}_j$. Suppose $\boldsymbol{D}$ given, and $\boldsymbol{A},\boldsymbol{B}\in\mathfrak{T}_\infty(\mathfrak{A})$. Then for positive integral j,

$$[\exp(\boldsymbol{D})(\boldsymbol{A}\boldsymbol{B})]_j=\sum_{\alpha=1}^{j}(\exp(\boldsymbol{D}))_{1\alpha}(\boldsymbol{A}\boldsymbol{B})_{\alpha j}=$$

$$=\sum_{\alpha=1}^{j}(\exp(\boldsymbol{D}_j))_{1\alpha}(\boldsymbol{A}_j\boldsymbol{B}_j)_{\alpha j}=[\exp(\boldsymbol{D}_j)(\boldsymbol{A}_j\boldsymbol{B}_j)]_j.$$

By the theorem for k finite, $\exp(\boldsymbol{D}_j)$ is an endomorphism. Thus $[\exp(\boldsymbol{D})(\boldsymbol{A}\boldsymbol{B})]_j=[\exp(\boldsymbol{D})\boldsymbol{A}\cdot\exp(\boldsymbol{D})\boldsymbol{B}]_j$. So $\exp(\boldsymbol{D})$ is an endomorphism, clearly with the property ε.

Conversely, let $\boldsymbol{F}$ be an ε-endomorphism in $\mathfrak{T}_\infty(\mathfrak{B})$. Then

$$[\boldsymbol{F}(\boldsymbol{A})]_j=\sum_{\alpha=1}^{j}(\boldsymbol{F})_{1\alpha}(\boldsymbol{A})_{\alpha j}=\sum_{\alpha=1}^{j}(\boldsymbol{F}_j)_{1\alpha}(\boldsymbol{A}_j)_{\alpha j}=[\boldsymbol{F}_j(\boldsymbol{A}_j)]_j.$$

From this it follows that $\boldsymbol{F}_j$ is a ε-endomorphism in $\mathfrak{T}_j(\mathfrak{B})$ and so $\boldsymbol{F}_j=\exp(\boldsymbol{D}_j)$.

It is easily seen that $(D_j)_i = D_i$ for $i < j$; thus there is determined a unique $D \in \mathfrak{T}_\infty(\mathfrak{B})$ such that

$$[F(A)]_j = [\exp(D_j) A_j]_j = [\exp(D) A]_j$$

for $j = 1, 2, \ldots$. That is, $F = \exp(D)$.

4. Inner derivations

The inner derivations on non-commutative $\mathfrak{A}$ are the operators D_c of the form

$$D_c(a) = ca - ac \qquad (a, c \in \mathfrak{A}).$$

They form an ideal in the Lie algebra of derivations. Some algebras are known to admit no derivations except inner derivations. (For a recent summary of such results see [6]. To this can be added that S. SAKAI and R. KADISON have separately found proofs that every von Neumann algebra admits no derivations except inner derivations. I must thank J. R. RINGROSE for this information.)

If $D_1, D_2, \ldots$ in the theorem of §§ 2, 3 (take $k = \infty$ for definiteness) are all inner derivations, say

$$D_n(a) = c_n a - a c_n \qquad (\text{all } a \in \mathfrak{A};\ n = 1, 2, \ldots),$$

then it is easily seen that D is an inner derivation in $\mathfrak{T}_\infty(\mathfrak{B})$. In fact, write C for the matrix in $\mathfrak{T}_\infty(\mathfrak{A})$ having 0 in the leading diagonal and c_{n-1} in the nth diagonal, for $n = 2, 3, \ldots$; then we find

$$D(A) = CA - AC \qquad (\text{all } A \in \mathfrak{T}_\infty(\mathfrak{A})).$$

and the Campbell—Hausdorff formula for $\mathfrak{T}_\infty(\mathfrak{A})$ gives

$$\exp(D)(A) = \exp(C) A \exp(-C).$$

Thus in the case where $\mathfrak{A}$ admits only inner derivations we have the corollary that all ε-endomorphisms in $\mathfrak{T}_\infty(\mathfrak{B})$ are inner endomorphisms. (T is an inner endomorphism in $\mathfrak{B}(\mathfrak{A})$ if and only if for some regular $u \in \mathfrak{A}$, $Ta = uau^{-1}$ for all $a \in \mathfrak{A}$.)

5. Applications to Banach algebras

Let $\mathfrak{A}$ now be a complex Banach algebra, and $\mathfrak{B}(\mathfrak{A})$ the Banach algebra of all bounded linear operators on $\mathfrak{A}$ into $\mathfrak{A}$.

5. 1. For $\varrho > 0$, write U_ϱ for the disc $\{\lambda : |\lambda| \leqq \varrho\}$ in the complex plane, and let $\mathfrak{F}_\varrho$ denote the algebra of all functions on U_ϱ into $\mathfrak{A}$ which are continuous on U_ϱ and holomorphic on the interior of U_ϱ. Each f in $\mathfrak{F}_\varrho$ is representable uniquely by its Taylor series about 0,

$$f(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \ldots \qquad (a_n \in \mathfrak{A} \text{ for } n = 0, 1, \ldots).$$

$\mathfrak{F}_\varrho$ becomes a Banach algebra if it is normed by writing

$$\|f\|_\varrho = \max_{|\lambda| \leqq \varrho} \|f(\lambda)\| = \max_{|\lambda| = \varrho} \|f(\lambda)\|. \tag{5. 1}$$

An endomorphism H on $\mathfrak{F}_\varrho$ clearly has the algebraic properties of an endomorphism on $\mathfrak{A}_\infty[t]$, so that the theory of § 3 is relevant. If H is also a closed operator, then it is bounded. Suppose that this is the case, and that H has the property ε, so that (if we specify functions by their Taylor series) then

$$(5.2) \qquad H(a_0+a_1\lambda+a_2\lambda^2+\dots) = b_0+b_1\lambda+b_2\lambda^2+\dots$$

implies $b_0=a_0$. The property ε can alternatively be put in the form:

$$(Hf)(0)=f(0) \quad \text{for all} \quad f\in\mathfrak{F}_\varrho.$$

We consider the question of bounds. Suppose that $\{I, F_1, F_2, \dots\}$ is the higher derivation of infinite rank corresponding to H and making up $\boldsymbol{F}$, and $D_1, D_2, \dots$ is the sequence of derivations which occur in $\boldsymbol{D}$ in the formula (3. 1). The boundedness of H implies that of the F's and the D's. To prove this, apply H to the constant function $f(\lambda)\equiv a$, getting $H(a)=g$ where

$$g(\lambda) = a+c_1\lambda+c_2\lambda^2+\dots \qquad (|\lambda|\leqq\varrho),$$

say. Now the familiar Cauchy estimates for the coefficients of such a power series extend from the classical to the vector-function case ([1], p. 97), so we have, for $n=1, 2, \dots$

$$\|F_n(a)\| = \|c_n\| \leqq \frac{1}{\varrho^n}\max_{|\lambda|\leqq\varrho}\|g(\lambda)\| = \varrho^{-n}\|g\|_\varrho = \varrho^{-n}\|H(a)\|_\varrho \leqq \varrho^{-n}\|H\|\cdot\|a\|,$$

whence $\|F_n\|\leqq\varrho^{-n}\|H\|$. Thus the F's are bounded. Since $D_n=F_n+$a polynomial in $D_1, D_2, \dots, D_{n-1}$, it follows easily that the D's are bounded.

Note that non-trivial ε-endomorphisms on $\mathfrak{F}_\varrho$ exist if $\mathfrak{A}$ admits a non-zero derivation D: we have only to take $D_1=D$, $D_2=D_3=\dots=0$ in $\boldsymbol{D}$ and form $\exp(\boldsymbol{D})$.

5. 2. Consider the restriction of H to $\mathfrak{A}$ (which we also write H). Let $\varkappa$ be an arbitrary point of the disc U_ϱ, and $\sigma_\varkappa$ the evaluation map on $\mathfrak{F}_\varrho$ into $\mathfrak{A}$ at $\varkappa$; that is, $\sigma_\varkappa(f)=f(\varkappa)$ for $f\in\mathfrak{F}_\varrho$. Since $\sigma_\varkappa$ is a homomorphism, $\sigma_\varkappa H$ is an endomorphism on $\mathfrak{A}$, bounded if H is bounded; it clearly has a simple representation in terms of derivations, deducible from the formula (3. 1). It would therefore be interesting to have a characterization of the endomorphisms which can be factorized in the form $\sigma_\varkappa H$ for some $\varkappa$ and H, and to know for what $\mathfrak{A}$, if any, all endomorphisms in $\mathfrak{B}(\mathfrak{A})$ have this property.

5. 3. It is known that some algebras are sparingly supplied with derivations. Thus, if $\mathfrak{A}$ is a semisimple commutative Banach algebra, the Singer—Wermer theorem [7] shows that $\mathfrak{A}$ admits no bounded derivations other than 0. It follows from the theorem of § 3 and the discussion in § 5. 1 that for such $\mathfrak{A}$ the only bounded ε-homomorphisms on $\mathfrak{A}$ into $\mathfrak{F}_\varrho$, and on $\mathfrak{F}_\varrho$ into $\mathfrak{F}_\varrho$, are the injection mapping $H(a)=f$ with $f(\lambda)\equiv a$, and the identity mapping $H(f)=f$, respectively.

6. A related result

Some interesting and more general results can be obtained for Banach algebras by adapting the process P to the point derivations introduced by Singer and Wermer.

Let $\mathfrak{A}$ be a commutative Banach algebra over the complex field. Given a multiplicative linear functional φ on $\mathfrak{A}$, a *point derivation associated with* φ is a linear functional δ_φ satisfying

$$\delta_\varphi(ab) = \delta_\varphi(a)\varphi(b) + \varphi(a)\delta_\varphi(b) \qquad (\text{all } a, b \in \mathfrak{A}). \tag{6.5}$$

We prove

Theorem. *If the complex commutative Banach algebra $\mathfrak{A}$ admits no non-zero point derivations, then the only homomorphisms on $\mathfrak{A}$ into $\mathfrak{H}_\varrho$, the algebra of all complex-valued functions continuous on the disc U_ϱ and holomorphic on its interior, are the mappings to constants, that is, the multiplicative linear functionals on $\mathfrak{A}$.*

(Note that the homomorphisms in the statement of the theorem are not required to have the property ε.)

Proof. Let H be a homomorphism on $\mathfrak{A}$ into $\mathfrak{H}_\varrho$. Then if $H(a)=f$, with

$$f(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \dots \qquad (|\lambda| \leqq \varrho;\ a_0, a_1, \dots \text{ complex}), \tag{6.2}$$

there is determined a sequence of linear functionals $\varphi_0, \varphi_1, \dots$ by

$$a_n = \varphi_n(a) \qquad (n=0, 1, \dots).$$

Substituting the form (6. 2) in (1. 2), we obtain a set of identities for the φ's like (1. 6)—(1. 9), namely

$$\varphi_0(ab) = \varphi_0(a)\varphi_0(b), \tag{6.3}$$

$$\varphi_1(ab) = \varphi_1(a)\varphi_0(b) + \varphi_0(a)\varphi_1(b), \tag{6.4}$$

$$\varphi_2(ab) = \varphi_2(a)\varphi_0(b) + \varphi_1(a)\varphi_1(b) + \varphi_0(a)\varphi_2(b), \tag{6.5}$$

and in general for $n=1, 2, \dots$

$$\varphi_n(ab) = \sum_{\alpha=0}^{n} \varphi_{n-\alpha}(a)\varphi_\alpha(b).$$

Thus φ_0 is a multiplicative linear functional on $\mathfrak{A}$. Since φ_1 by (6. 4) is a point derivation on $\mathfrak{A}$ associated with φ_0, the assumption of the theorem implies that $\varphi_1=0$. But then φ_2 by (6. 5) is a point derivation, so $\varphi_2=0$. By induction we find that $\varphi_1=\varphi_2\dots=0$, so that $H(a)=\varphi_0(a)$.

Singer and Wermer give the following necessary and sufficient condition for the existence of point derivations in a commutative Banach algebra $\mathfrak{A}$ with identity. Given the multiplicative linear functional φ, write $M_\varphi=\{a\colon \varphi(a)=0\}$ for the corresponding maximal ideal in $\mathfrak{A}$, and M_φ^2 for the set of all linear combinations of squares of elements of M_φ, so that $M_\varphi^2 \subseteqq M_\varphi$. Then non-zero point derivations associated with φ exist if and only if $M_\varphi^2 \neq M_\varphi$.

Corollary. *The only homomorphisms from the algebra $C(X)$ of all complex-valued continuous functions on a compact Hausdorff space X into $\mathfrak{H}_\varrho$ are the multiplicative linear functionals.*

Proof. For every φ, $M_\varphi^2=M_\varphi$: for the details, see [7].

Note added in proof. R. J. Loy has extended the results of this paper under a weaker hypothesis than the ε property. See the following paper [8].

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(Received April 21, 1966)

A note on the preceding paper by J. B. Miller

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1. Introduction

Let $\mathfrak{A}$ be an associative algebra over a field K of zero characteristic, $\mathfrak{A}_k[t]$ the algebra, over $\mathfrak{A}$, of polynomials of degree $\leqq k$ in a commutative indeterminate t with the usual multiplication modulo the principal ideal (t^{k+1}). We consider (algebra) homomorphisms of $\mathfrak{A}$ into $\mathfrak{A}_k[t]$. Much of the definitions and notation of [1] will be used, without further explanation.

Suppose H is a homomorphism of $\mathfrak{A}$ into $\mathfrak{A}_k[t]$ so that, if $a \in \mathfrak{A}$,

$$H(a) = a_0 + ta_1 + t^2 a_2 + \ldots + t^k a_k.$$

Writing $a_0 = \varphi(a)$, $a_i = F_i(a)$ $(i = 1, 2, \ldots, k)$ it is clear that the maps φ, F_i are linear transformations over $\mathfrak{A}$. Furthermore, since H is a homomorphism it follows, if $a, b \in \mathfrak{A}$, that

(i) $$\varphi(ab) = \varphi(a)\varphi(b),$$

(ii) $$F_i(ab) = \sum_{j=0}^{i} F_j(a) F_{i-j}(b),$$

where $F_0 = \varphi$.

The problem is to obtain a representation for H in terms of transformations on $\mathfrak{A}$ of some given type. This has been done in [1] under the supposition that φ is the identity endomorphism, I, on $\mathfrak{A}$. For completely general endomorphisms the problem appears intractible but under suitable restrictions a solution can be obtained.

To be more specific, let φ be an endomorphism in $\mathfrak{B}(\mathfrak{A})$. A homomorphism H of $\mathfrak{A}$ into $\mathfrak{A}_k[t]$ will be called a φ-homomorphism if

(a) φ is the endomorphism determined from H by $a_0 = \varphi(a)$.

(b) $\varphi(F_n(a)) = F_n(\varphi(a))$ $(n = 1, 2, \ldots, k)$.

Thus the ε-homomorphisms of [1] are I-homomorphisms in this nomenclature.

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An operator $D\in\mathfrak{B}(\mathfrak{A})$ will be called a φ-derivation if it commutes with φ and satisfies

$$D(ab) = D(a)\varphi(b)+\varphi(a)D(b) \qquad (a, b\in\mathfrak{A}).$$

Thus F_1 of (ii) is a φ-derivation if H is a φ-homomorphism.

Finally, the endomorphism φ will be called averaging if

$$\varphi(a\varphi(b))=\varphi(a)\varphi(b)=\varphi(\varphi(a)b) \qquad (a, b\in\mathfrak{A}).$$

Clearly any idempotent endomorphism is averaging and, conversely, if $\mathfrak{A}$ has an identity, or if the range of φ contains an element which is not a left (or right) divisor of zero, then φ is idempotent.

The results of [1] are extended to φ-homomorphisms for φ an idempotent endomorphism and for φ an averaging endomorphism.

The author wishes to express his thanks to Professor J. B. MILLER for suggesting the possibility of extending the results of [1] and for his constant encouragement and guidance in preparation of this paper.

2. Representations of φ-homomorphisms

Lemma. *Let $\varphi\in\mathfrak{B}(\mathfrak{A})$ be an averaging endomorphism and $D\in\mathfrak{B}(\mathfrak{A})$ a φ-derivation. Then for $a, b\in\mathfrak{A}$, $n=1, 2, \ldots$,*

$$\varphi(D^n(ab)) = \varphi\left\{\sum_{i=0}^{n}\binom{n}{i}D^i(a)D^{n-i}(b)\right\}.$$

Proof. The result is clear for $n=1$. For $n=2$ we have, if $a, b\in\mathfrak{A}$,

$$\varphi(D^2(ab))=\varphi(D(D(a)\varphi(b)+\varphi(a)D(b)))=$$

$$=\varphi(D^2(a)\varphi^2(b)+\varphi(D(a))\varphi(D(b))+\varphi(D(a))\varphi(D(b))+\varphi^2(a)D^2(b))=$$

$$=\varphi(D^2(a)b+2D(a)D(b)+aD^2(b))$$

since

$$\varphi(\varphi(x)y)=\varphi(x\varphi(y))=\varphi(x)\varphi(y)=\varphi(xy)$$

if $x, y\in\mathfrak{A}$. An inductive argument gives the general case.

Corollary. *With φ, D as in the lemma define $\varphi\exp D$ as the (formal) sum $\sum_{n=0}^{\infty}\frac{1}{n!}\varphi D^n$. If $\varphi\exp D$ defines an operator in $\mathfrak{B}(\mathfrak{A})$, in particular if D is nilpotent, then this operator is an endomorphism.*

Proof. Follows from the lemma in the obvious manner.

Theorem. *Let $\varphi\in\mathfrak{B}(\mathfrak{A})$ be a given averaging endomorphism and $D_1, D_2, \ldots, \ldots, D_k\in\mathfrak{B}(\mathfrak{A})$ given φ-derivations. Let $\boldsymbol{\varphi}$ be the operator in $\mathfrak{T}_{k+1}(\mathfrak{B})$ with φ along the leading diagonal and zero elsewhere. Then $\boldsymbol{\varphi}\exp(\boldsymbol{D}_{k+1})$, as defined above, is the matrix in $\mathfrak{T}_{k+1}(\mathfrak{B})$ of a φ-homomorphism of $\mathfrak{A}$ into $\mathfrak{A}_k[t]$ with $F_1=\varphi D_1$.*

Conversely, given a φ-homomorphism H of $\mathfrak{A}$ into $\mathfrak{A}_k[t]$, with φ idempotent, its matrix $\boldsymbol{F}_{k+1}$ satisfies

$$\boldsymbol{\varphi F}_{k+1} = \boldsymbol{\varphi} \exp(\boldsymbol{D}_{k+1})$$

where $D_1, D_2, \ldots, D_k \in \mathfrak{B}(\mathfrak{A})$ are φ-derivations and D_1 can be taken as F_1. Furthermore the φD_i are uniquely determined.

Proof. The proof of the first part is exactly as in [1]. For the partial converse, note that if H is a φ-homomorphism of $\mathfrak{A}$ into $\mathfrak{A}_k[t]$, with φ idempotent, then $H\varphi$ is a I-homomorphism of $\varphi(\mathfrak{A})$ into $\varphi(\mathfrak{A})_k[t]$. Thus by [1] the matrix $\boldsymbol{F}_{k+1}\boldsymbol{\varphi}$ of $H\varphi$ satisfies

(iii) $$\boldsymbol{F}_{k+1}\boldsymbol{\varphi} = \exp(\boldsymbol{\Delta}_{k+1})$$

where $\boldsymbol{F}_{k+1}$ is the matrix of H, and the operators $\Delta_1, \Delta_2, \ldots, \Delta_k$ of $\boldsymbol{\Delta}_{k+1}$ are derivations on $\varphi(\mathfrak{A})$. For $i=1, 2, \ldots, k$ define D_i by

$$D_i(a) = \begin{cases} 0 & \text{if } a \in \ker(\varphi), \\ \Delta_i(a) & \text{if } a \in \operatorname{im}(\varphi) \end{cases}$$

and extend D_i linearly to the whole of $\mathfrak{A}$. The resulting operator is well defined since $\mathfrak{A} = \ker(\varphi) \oplus \operatorname{im}(\varphi)$ (see § 3 below). But then if $a_j = x_j + y_j$, $x_j \in \ker(\varphi)$, $y_j \in \operatorname{im}(\varphi)$ for $j = 1, 2$,

$$D_i(a_1 a_2) = D_i(x_1 x_2 + y_1 x_2 + x_1 y_2 + y_1 y_2) = D_i(y_1 y_2)$$

since $\ker(\varphi)$ is a two-sided ideal. Since $\operatorname{im}(\varphi)$ is a subalgebra

$$D_i(a_1 a_2) = \Delta_i(y_1 y_2) =$$

$$= \Delta_i(y_1) y_2 + y_1 \Delta_i(y_2) = D_i(a_1)\varphi(a_2) + \varphi(a_1) D_i(y_2);$$

moreover, $D_i = D_i\varphi = \varphi D_i$ and so D_i is a φ-derivation on $\mathfrak{A}$, $i = 1, 2, \ldots, k$.

Also, if $a \in \mathfrak{A}$, $D_i(a) = \Delta_i(\varphi(a))$, so $\Delta_i \varphi = D_i$. By (iii) then, since $\boldsymbol{\varphi}^2 = \boldsymbol{\varphi}$ and $\Delta_i \varphi = \varphi \Delta_i \varphi$ for each i,

$$\boldsymbol{F}_{k+1}\boldsymbol{\varphi} = \exp(\boldsymbol{\Delta}_{k+1})\boldsymbol{\varphi} = \boldsymbol{\varphi} \exp(\boldsymbol{\Delta}_{k+1}\boldsymbol{\varphi}) = \boldsymbol{\varphi} \exp(\boldsymbol{D}_{k+1})$$

and the result follows.

Remarks. 1. The above result remains valid if the base field D has nonzero characteristic p, provided $k < p$. This restriction ensures that $\boldsymbol{\varphi} \exp \boldsymbol{D}_{k+1}$ is defined.

2. If it is supposed that $\varphi F_n = F_n \varphi = F_n$ $(n = 1, 2, \ldots, k)$ then 'the process P' of [1] can be applied to give results analogous to those in [1] with φ-derivations in place of derivations.

3. The result for $k = \infty$ generalizes in the same manner.

3. Existence of φ-derivations

Lemma. *An algebra $\mathfrak{A}$ admits an idempotent endomorphism if and only if it admits a (vector space) direct sum representation $\mathfrak{A}=\mathfrak{J}\oplus\mathfrak{B}$ where $\mathfrak{J}$ is a two-sided ideal and $\mathfrak{B}$ is a subalgebra.*

Proof. If φ is an idempotent endomorphism, let $\mathfrak{J}=\ker(\varphi)$, $\mathfrak{B}=\operatorname{im}(\varphi)$. Then $\mathfrak{J}$ is a two sided ideal and $\mathfrak{B}$ is a subalgebra. If $a\in\mathfrak{A}$, $a=(a-\varphi(a))+\varphi(a)$ so $\mathfrak{A}=\mathfrak{J}+\mathfrak{B}$. Since φ is zero on $\mathfrak{J}$ and is the identity on $\mathfrak{B}$ it follows that the sum is direct.

Conversely, if $\mathfrak{A}=\mathfrak{J}\oplus\mathfrak{B}$ for some two-sided ideal $\mathfrak{J}$ and subalgebra $\mathfrak{B}$, define a transformation φ on $\mathfrak{A}$ as follows. If $a=x+y$, $x\in\mathfrak{J}$, $y\in\mathfrak{B}$, set $\varphi(a)=y$. Then φ is clearly idempotent and linear. Also, if $a_1=x_1+y_1$, $a_2=x_2+y_2$

$$\varphi(a_1a_2)=\varphi(x_1x_2+y_1x_2+x_1y_2+y_1y_2)=$$
$$=\varphi(y_1y_2)=y_1y_2=\varphi(a_1)\varphi(a_2).$$

Thus φ is an idempotent endomorphism.

Theorem. *Let $\mathfrak{A}$ be an associative algebra which has a direct sum representation $\mathfrak{A}=\mathfrak{J}\oplus\mathfrak{B}$ as in the lemma, with $\mathfrak{B}$ non-commutative. Then $\mathfrak{A}$ admits a non-zero φ-derivation D such that $\varphi D=D\varphi=D$, φ being defined as in the lemma.*

Proof. Let $a\in\mathfrak{A}$ be any element such that $\varphi(a)$ is not in the centre of $\mathfrak{B}$. Straightforward calculation shows that the operator D defined by

$$D(x)=\varphi(ax-xa),\qquad x\in\mathfrak{A},$$

has the desired properties.

Corollary. *Let $\Delta_1,\Delta_2,\dots$ be inner derivations of $\mathfrak{A}$, $D_1,D_2,\dots$ the corresponding φ-derivations as defined in the preceding theorem, that is, $D_i=\varphi\Delta_i$, $i=1,2,\dots$. Then $\boldsymbol{\varphi}\exp(\boldsymbol{D})=\boldsymbol{\varphi}\Gamma$ where Γ is an inner endomorphism in $\mathfrak{T}_\infty(\mathfrak{A})$ determined by the $\Delta_1,\Delta_2,\dots$ as in* [1].

Proof. By § 4 of [1],

$$\exp(\boldsymbol{\Delta})\boldsymbol{A}=\exp(\boldsymbol{C})\boldsymbol{A}\exp(-\boldsymbol{C}),\qquad \boldsymbol{A}\in\mathfrak{T}_\infty(\mathfrak{A}),$$

whence

$$(\boldsymbol{\varphi}\exp(\boldsymbol{\Delta}))\boldsymbol{A}=(\boldsymbol{\varphi}\Gamma)\boldsymbol{A}.$$

But

$$\boldsymbol{\varphi}\exp(\boldsymbol{D})=\boldsymbol{\varphi}\exp(\boldsymbol{\varphi\Delta})=\boldsymbol{\varphi}\exp\boldsymbol{\Delta},$$

and the result follows.

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(Received September 21, 1966)

Bibliographie

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Das vorliegende Buch ist eine Übersetzung der 1950 in Leningrad bzw. 1953 in Moskau erschienenen russischen Ausgabe, wobei nur geringfügige Änderungen vorgenommen worden sind. Der dargestellte Stoff geht im wesentlichen auf Resultate zurück, die der Autor bereits in den Jahren 1936—1939 veröffentlicht hat und die einen ganz außerordentlichen Einfluß auf die Entwicklung der Funktionalanalysis und der Theorie der partiellen Differentialgleichungen ausgeübt haben. Es handelt sich dabei zum Beispiel um die Einführung des Begriffs der verallgemeinerten Ableitung einer lokalintegrierbaren Funktion (der übereinstimmt mit dem Begriff der distributionentheoretischen Ableitung im Schwartzschen Sinne und deshalb als eine Vorwegnahme gewisser Aspekte der Distributionentheorie angesehen werden kann) sowie die berühmten Einbettungssätze. Diesen rein funktionalanalytischen Gegenständen ist das Kapitel I des Buches gewidmet. Die (heute allgemein als Sobolewsche Räume bezeichneten) Räume $W_p^{(l)}$ werden eingeführt als die Gesamtheit aller summierbaren Funktionen, für die im beschränkten Gebiet Ω alle verallgemeinerten Ableitungen der Ordnung l existieren. Es wird eine Vielfalt möglicher Normierungen von $W_p^{(l)}$ betrachtet, für welche die Einbettungssätze gelten. Von diesem Gesichtspunkt aus ist das vorliegende Buch die allgemeinste gegenwärtig existierende Darstellung dieses Problemkreises. Für die Anwendungen hat sich in den letzten 15 Jahren die Normierung $\|\varphi\|_{W_p^{(l)}} = \Big(\sum\limits_{|\alpha|\leqq l} \int\limits_{\Omega} |D^\alpha\varphi|^p dx\Big)^{\frac{1}{p}}$ durchgesetzt.

Aus den Einbettungssätzen folgt nämlich, daß auch für alle Ordnungen $< l$ die verallgemeinerten Ableitungen existieren und zur Potenz p summierbar sind. Die im Buch geführten Beweise, die auf Eigenschaften gewisser Integrale vom Potentialtyp beruhen, gelten unmittelbar nur für Vereinigungen endlich vieler bezüglich einer Kugel sternförmiger Gebiete und darüber hinaus werden keine direkten Aussagen über das Verhalten der Funktionen auf dem Rand von Ω gemacht. Hierin sowie im Studium der zu $W_p^{(l)}$ dualen Räume lagen die wichtigsten Ansatzpunkte zur Weiterentwicklung dieser Theorie und ihrer Anwendungen in der jüngsten Vergangenheit.

Im Kapitel II werden die Ergebnisse von Kapitel I auf das Dirichletsche Problem der polyharmonischen Differentialgleichung $\Delta^m u=0$, das Neumannsche Problem der Potentialgleichung sowie auf ein Eigenwertproblem der Gleichung $\Delta u+\gamma u = 0$ angewandt.

Wir erläutern die allen diesen Beispielen zugrunde liegende Methode für den erstgenannten Fall. An Stelle der Differentialgleichung wird das zugehörige Variationsproblem (für das Dirichletsche Integral) im Raum $W_2^{(m)}$ gelöst. Es wird bewiesen, daß die Lösung dieses Variationsproblems in Ω der Differentialgleichung genügt. Die vorgeschriebenen Randwerte (der Funktion und aller Ableitungen der Ordnung $\leqq m-1$), die auch auf gewissen Randmannigfaltigkeiten der Dimension $< n-1$ (im R^n) vorgeschrieben sein können, werden dahingehend eingeschränkt, daß sie Randwerte von Funktionen aus $W_2^{(1)}$ sein müssen. Aus der Vollstetigkeit der Einbettungsoperatoren kann dann gefolgert werden, daß die Randwerte von der (eindeutig bestimmten) Lösung des Problems bei Annäherung an den Rand in einem verallgemeinerten Sinne („im Mittel") angenommen werden. Es ist bis zum gegenwärtigen Zeitpunkt ungeklärt, ob (bzw. unter welchen Voraussetzungen) die Randwertannahme darüber hinaus auch im klassischen Sinne (stetiger Randwertannahme) stattfindet. Die hier zugrunde liegende Betrachtungsweise steht im engen Zusammenhang zu der heute allgemein üblichen Auffassung von Randwertproblemen.

Im Kapitel III wird das Cauchysche Problem für gewisse Klassen linearer und quasilinearer hyperbolischer Differentialgleichungen 2. Ordnung studiert. Es werden verallgemeinerte Lösungen in $W_2^{(1)}$ gesucht. Die Einbettungssätze werden benutzt, um den Grad der Glattheit der Koeffizienten und der Anfangsbedigungen zu bestimmen, den man fordern muß, um die Existenz einer klassischen Lösung des Cauchyschen Problems zu sichern.

Günther Wildenhain (Dresden)

László Rédei, Begründung der euklidischen und nichteuklidischen Geometrien nach F. Klein, 364 Seiten, Akadémiai Kiadó, Budapest, 1965.

Die Begründung der euklidischen und der beiden nicht-euklidischen Geometrien, die heute hauptsächlich parabolische, elliptische bzw. hyperbolische Geometrie genannt werden, wird in diesem Buch mittels des von F. KLEIN in seinen Vorlesungen verfolgten Weges, d. h. durch die projektive Erweiterung des Raumes durchgeführt. Diesen Weg der Begründung der drei Geometrien, die zusammenfassend als Geometrien von konstanter Krümmung bezeichnet werden können, ist bisher in der Lehrbuchliteratur ziemlich selten verfolgt und die in dieser Richtung geschriebenen Werke waren öfters nur skizzenhaft; das Buch von Professor RÉDEI will eben diesen Mangel der Lehrbuchliteratur aufheben, und die Kleinschen Ideen — wie das in der Einleitung des Buches bemerkt wird — auch „in der Lehrbuchliteratur zu ihrem Recht kommen lassen".

Der Stoff des Buches ist in sieben Kapiteln verteilt, von denen die ersten sechs den axiomatischen Aufbau der projektiven Geometrie enthalten; das siebente Kapitel enthält den wichtigsten Teil, undzwar die Charakterisierung der drei Geometrien im Sinne des Erlanger Programms von F. KLEIN mittels der Bestimmung der Bewegungsgruppen in den verschiedenen Geometrien. Im Kapitel I befinden sich die Axiomen in vier Gruppen verteilt: Axiome des Enthaltenseins, der Beziehung „zwischen", der Stetigkeit und der Bewegung. In den Kapiteln II und III befinden sich die Definitionen der einfachsten Begriffe — wie der Begriff der linearen Unterräume, der Strecken, der Dreiecke und der Tetraeder, usw. Neben dem Begriff der Desargueschen Figuren führt der Verf. den Begriff der assoziierten Desarguesschen Figur ein, welcher bisher — unseres Wissens — in der Geometrie nicht benützt wurde, und mit dessen Hilfe mehrere Beweise wesentlich vereinfacht werden können. Kapitel IV gibt durch die Einführung der idealen Elemente den projektiven Abschluß des Raumes.

Die Kapiteln V und VI beschäftigen sich mit der im engeren Sinne genommenen Theorie des projektiven Raumes, endlich wird im Kapitel VII die vollständige Charakterisierung der drei Geometrien von konstanter Krümmung durchgeführt. Als wichtigstes Ergebnis verweisen wir auf die explizite Bestimmung der verschiedenen Bewegungsgruppen und die Widerspruchsfreiheit der genannten Geometrien.

Das Buch wird wegen seiner Vollständigkeit in Bezug auf die Theorie der nicht-euklidischen Geometrien für die Mathematiker in der Forschung als auch in der Unterricht ein sehr brauchbares Hilfsmittel; aber auch Studenten, die die Theorie der nicht-euklidischen Geometrien eingehender kennen lernen wollen, können das Buch mit großem Nutzen studieren.

A. Moór (Szeged)

Noel Gastinel, Analyse numérique linéaire (Collection enseignement des sciences, IX), IX+363 pages, Edition Hermann, Paris, 1966.

Dans cet ouvrage, destiné avant tout aux étudiants des facultés des sciences, l'auteur traite des méthodes de l'analyse numérique linéaire. Après une introduction théorique soignée dans la première partie du livre, tenant devant les yeux les exigences de la formation des programmeurs, l'auteur attache son plus grand intérêt à la discussion des algorithmes applicables aux machines à calculer électroniques. Conformément à cet intérêt, son livre est un livre de méthodes de calcul ou d'"algorithmique". Il préfère donner les justifications de ces méthodes de calcul et les programmes correspondants en Algol, plutôt que de donner des tableaux de chiffres, résultats de calculs numériques obtenus. Il compare les méthodes du point de vue du nombre des opérations exigées, de la simplicité de programmation et de la vitesse de convergence des procédures, ce qui facilite la sélection de la méthode la plus convenable dans chaque cas particulier.

Le livre se partage en huit chapitres. Les prémiers trois développent les notions fondamentales comme p. ex. les propriétés élémentaires des matrices, les normes des vecteurs et des matrices, l'inversion des matrices, etc., notions et résultats utilisés constamment dans la suite. Le quatrième chapitre présente les méthodes pour la résolution d'un système linéaire d'équations par élimination et orthogonalisation, et montre comment ces méthodes s'appliquent à l'inversion des matrices et au calcul des valeurs des déterminants. Le chapitre se termine par la discussion du cas des matrices symétriques et d'une méthode concernant l'inversion des matrices, basée sur la technique des partitionnements aux sous-matrices, et est suivi par une collection des problèmes concernant le sujet des chapitres I, II, III et IV. Chapitre V est consacré à l'étude de deux types intéressants de procédés de résoûdre un système linéaire d'équations, notamment par itération de type linéaire et itération par la méthode de projection. Ensuite, l'auteur montre comment ces procédés se simplifient en cas des systèmes de matrices symétriques et montre aussi une méthode itérative pour l'inversion des matrices. Le chapitre finit par une collection de problèmes. Le Chapitre VI contient une discussion détaillée du problème des sous-espaces invariants pour une transformation linéaire d'un espace vectoriel. Les notions introduites et les théorèmes prouvés sont utilisés au Chapitre VII. L'auteur y traite de la relation des graphes orientés et des matrices à termes positifs, et il applique ensuite les résultats acquis à la discussion de la question de convergence pour la résolution d'un système d'équations par itération. Le chapitre VIII traite des méthodes numériques pour le calcul des valeurs propres et des vecteurs propres, en présentant des méthodes diverses pour obtenir le polynôme caractéristique de la matrice, et des procédés concernant le calcul des valeurs propres et vecteurs propres par itération. Le cas de matrices hermitiennes est étudié de plus près. Le livre se termine par une collection de problèmes pour les trois derniers chapitres.

Cet ouvrage, tant par son contenu que par sa présentation, aura certainement de succès et contribuera beaucoup au développement de l'enseignement du calcul moderne.

P. Hunya et *I. Kovács* (Szeged)

Otton Martin Nikodým, The mathematical apparatus for quantum mechanics, based on the theory of Boolean algebras (Die Grundlehren der mathematischen Wissenschaften in Einzeldrastellungen, Bd. 129), XII+952 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1966.

The purpose of this work is, as stated in the preface by the author, "to give the theoretical physicist a geometrical, visual and precise mathematical apparatus which would be better adapted to some of their arguments, than the existing and generally applied methods".

Though very little is said in the book about the deficiency of the usual methods, it becomes clear from the author's own treatment that he means that, in the case of a selfadjoint or a normal operator with not purely discrete spectrum, there are not sufficiently many eigenvectors belonging to the Hilbert space itself. This is indeed an inconvenience — and the source of misunderstanding with some physicists who would like to handle the continuous spectrum similarly to the discrete one. There were already several attempts by mathematicians to eliminate this inconvenience by enlarging the Hilbert space by appropriate ideal elements and to state the eigenvalue problem in a correspondingly generalized form. (Let us only mention the *Doklady* paper of GELFAND—KOSTYUČENKO (1955) and the recent monograph by BEREZANSKIĬ on "Expansion with respect to eigenfunctions of selfadjoint operators" (Kiev, 1965, in Russian).

The author's own approach to this problem was first outlined in four lectures at the Institut Henri Poincaré (Paris) in 1947, and since then elaborated in a series of papers. This approach is based on the theory of Boolean lattices, whose elements are closed subspaces in separable Hilbert space.

Correspondingly, the book begins with a voluminous introduction (on some 250 pages) to general Boolean lattices and their ideal elements (called here "traces"). One of the main concerns here is measure theory. The lattice of subspaces of a Hilbert space are studied on the next 150 pages, including a chapter on double Stieltjes and Radon integrals. It follows (on about 320 pages) applications to the spectral theory of normal operators in Hilbert space, including operational calculus with such operators, spectral multiplicity theory, commutative families of operators, and in particular the ideal (or "quasi"-) eigenvectors. It follows a chapter (35 pages) on the delta function of Dirac, where a new rigorous foundation is proposed for this function (howewer, without mentioning its relations to other rigorous foundations, such as that in the theory of distributions). The last 160 pages are devoted to a deeper study of the theory of summation on scalar (instead of vector) fields.

The book is a highly individualistic one; indeed it is based almost entirely on the author's own research. The large number of new concepts and notations introduced, and the astonishing length of the exposition, make the book hard to read even for mathematicians. By the way, no direct applications to mathematical problems of physics are given: the author intends to deal with them in subsequent papers or in another book.

Béla Sz.-Nagy (Szeged)

W. Magnus, F. Oberhettinger, and R. P. Soni, Formulas and theorems for the special functions of Mathematical Physics, 3rd edition (Die Grundlehren der math. Wissenschaften in Einzeldarstellungen, Band 52), VIII+508 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1966

W. MAGNUS' and F. OBERHETTINGER's original idea of giving a brief survey of the principal properties of the most important special functions featuring in mathematical physics together with a list of formulae involving these functions has proved to be a great success. This is also shown by the fact that their book "Formeln und Sätze für die speziellen Funktionen der mathematischen Physik", appeared in German in 1943, already ran into its third edition. The book under review is a new and enlarged English version, written in collaboration with R. P. SONI, of the second edition (1948) of the above book.

Compared to the 1948 German edition the book contains several additions which present further facts on the special functions in question, and enlarge the list of formulae concerning them. Furthermore some of these functions, for instance KUMMER's function, the Whittaker function, parabolic cylinder functions, etc., on which only relatively short accounts were given in the previous edition, appear here as subjects of individual chapters. All this makes the extent of the present book doubled in comparison with that of the 1948 edition. A change has taken place in the list of references: They are generally restricted to books and monographs and are located at the end of each individual chapter. Occasional references follow immediately those results to which they apply.

The scope of the present book will be best seen from its table of contents. Chapter I: The gamma functions and related functions (the Riemann zeta function, Bernoulli and Euler polynomials, etc.). Chapter II: The hypergeometric function. Chapter III: Bessel functions. Chapter IV: Legendre functions (including Gegenbauer functions, toroidal and conical functions). Chapter V: Orthogonal polynomials (Jacobi, Gegenbauer, Legendre, generalized Laguerre, Hermite, Chebyshev polynomials). Chapter VI: KUMMER's function. Chapter VII: Whittaker function. Chapter VIII: Parabolic cylinder functions and parabolic functions. Chapter IX: The incomplete gamma function and special cases. Chapter X: Elliptic integrals, theta functions and elliptic functions. Chapter XI: Integral transforms (including Fourier transforms of various kind, Laplace, Mellin, Hankel, Lebedev, Mehler and Gauss transforms). Chapter XII: Transformation of systems of coordinates.

There is no doubt that this excellently written book, similarly to its earlier editions, will be of great value for mathematicians and physicists.

I. Kovács (Szeged)

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67-5509 Szegedi Nyomda

Felelős szerkesztő és kiadó: Szőkefalvi-Nagy Béla
A kézirat nyomdába érkezett: 1967. február hó
Megjelenés: 1967. június hó

Példányszám: 825. Terjedelem: 21 (A/5) ív
Készült monószedéssel, íves magasnyomással, az MSZ 5601-54 és az MSZ 5602-55 szabvány szerint