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13. KÖTET

2. FÜZET

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A VALLÁS- ÉS KÖZOKTATÁSÜGYI MINISZTER TÁMOGATÁSÁVAL KIADJA

A SZEGEDI TUDOMÁNYEGYETEM BOLYAI-INTÉZETE

The minimum of a binary cubic form[1]).

By L. J. MORDELL in Cambridge (England).

1. Let $$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$$
be a binary cubic form with real coefficients and of discriminant
$$D = -27a^2d^2 + 18abcd + b^2c^2 - 4ac^3 - 4db^3,$$
so that $f(x, y)$ has one or three real linear factors according as $D < 0$ or $D > 0$. The problem is to find how small $|f(x, y)|$ can be made for integer values of x, y not both zero, i. e. the lower bound of $|f(x, y)|$ for these x, y.

With such questions, it is not difficult nowadays to prove the existence of results that integers x, y not both zero exist for which
$$|f(x, y)| \leq k|D|^{1/4},$$
where k is a numerical constant, and these have been known for many years. Thus if $D > 0$, ARNDT in 1858 and HERMITE in 1859, showed that the result holds with $k = \left(\frac{4}{27}\right)^{1/4}$.[2]) If $D < 0$, HERMITE showed in 1859 that we can take $k = \frac{1}{2}$. The best possible value of k was neither known nor had any suggestions about its value been made until recently by myself when I proved the following

Theorem.[3]) *If $D > 0$, integers x, y not both zero exist such that*
$$|f(x, y)| \leq \sqrt[4]{\frac{D}{49}}.$$

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[1]) Lecture held in the Bolyai-Institute of the University Szeged, December 16, 1948.

[2]) CH. HERMITE, *Oeuvres*, II (Paris, 1908), pp. 93–99.

[3]) L. J. MORDELL, On numbers represented by binary cubic forms, *Proceedings London Math. Society*, (2) **48** (1943), pp. 198-228.

This is a best possible result, and the equality sign is necessary when and only when

$$\sqrt[4]{\frac{49}{D}}\, f(x, y) \sim x^3 + x^2 y - 2xy^2 - y^3,$$

where the right hand side has discriminant 49.

If $D < 0$, *integers* x, y *not both zero exist such that*

$$|f(x, y)| \leqq \sqrt[4]{\frac{|D|}{23}}.$$

This is a best possible result, and the equality sign is necessary when and only when

$$\sqrt[4]{\frac{23}{|D|}}\, f(x, y) \sim x^3 - xy^2 - y^3,$$

where the right hand side has discriminant -23.

The significance of the numbers 49, -23 is clear. Thus 49 is the least positive discriminant of irreducible binary cubic forms with integer coefficients, and so the constant 49 cannot be improved for such forms, i. e. made larger, as then $|f(x, y)| < 1$ and so would be zero. This occurs only when $x = y = 0$. Similarly for -23.

2. Some light may be thrown on the subject if we consider the quadratic case when

$$g(x, y) = ax^2 + bxy + cy^2$$

of discriminant $d = b^2 - 4ac$. It is well known from the work of LAGRANGE and GAUSS that the corresponding best possible results are when

$$d < 0, \qquad g(x, y) \leqq \sqrt{\frac{|d|}{3}},$$

equality arising only when

$$\sqrt{\frac{3}{|d|}}\, g(x, y) \sim x^2 + xy + y^2;$$

and from the work of MARKOFF[4]), KORKINE and ZOLOTAREFF[5]) that when

$$d > 0, \qquad g(x, y) \leqq \sqrt{\frac{d}{5}},$$

equality arising only when

$$\sqrt{\frac{5}{d}}\, g(x, y) \sim x^2 + xy - y^2.$$

4) A. MARKOFF, Sur les formes quadratiques binaires indéfinies, *Math. Annalen*, **15** (1879), pp. 281-406; **17** (1880), pp. 379-399.

5) A. KORKINE—G. ZOLOTAREFF, Sur les formes quadratiques, *Math. Annalen*, **6** (1873), pp. 366-389.

If we consider the first of these, a result such as $g(x,y) \leqq \sqrt{\frac{|d|}{l}}$, where l is a numerical constant, has a simple geometric interpretation. It means that a point P whose coordinates are integers x, y; i. e. a lattice point, lies in, i. e. inside or on the boundary of the ellipse $g(x, y) \leqq \sqrt{\frac{|d|}{l}}$. A value of l is given by a fundamental theorem of MINKOWSKI in the geometry of numbers, namely the theorem[6]):

A two dimensional closed, convex region, symmetrical about the origin O and of area ≧4 contains within it a lattice point other than O.

More generally, this theorem is still true if we define a lattice point to be one whose coordinates x, y are of the form

$$x = \alpha X + \beta Y, \quad y = \gamma X + \delta Y,$$

where X, Y are integers and α, β, γ, δ are any real constants with determinant

$$\Delta = \alpha\delta - \beta\gamma > 0,$$

if in the theorem we replace 4 by 4Δ. We then call the aggregate of such points (x, y) a lattice of determinant Δ, but here we need only consider lattices of determinant unity.

An application of this result to the ellipse shows that a lattice point not O lies in it if

$$\frac{2\pi}{\sqrt{|d|}} \sqrt{\frac{|d|}{l}} \geqq 4, \quad \text{or} \quad l \leqq \left(\frac{\pi}{2}\right)^2 < 3.$$

This is worse than the best possible value $l = 3$. MINKOWSKI[7]) has shown, however, that the best possible value can be deduced by finding the minimum value of the area of a parallelogram with one vertex at O and the other three on the boundary of the ellipse. There is of course no number theory involved in solving the minimum problem. These problems are simple in theory but generally very difficult to solve.

When $d > 0$, the region $|g(x, y)| \leqq \sqrt{\frac{d}{l}}$ is an infinite region bounded by four hyperbolic arcs having for asymptotes the lines given by $g(x, y) = 0$. There is no corresponding theorem for infinite regions, but an estimate $l = 4$ may be found by inscribing in the region a parallelogram whose centre is at the origin with vertices on the asymptotes

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[6]) H. MINKOWSKI, *Diophantische Approximationen* (Leipzig, 1907), p. 29.

[7]) H. MINKOWSKI, *Ibidem*, pp. 51—55.

and choosing l so that its area is 4. Then the parallelogram will contain a lattice point not O by MINKOWSKI's theorem, and so also will the infinite region. There is also now no method of finding the best possible result by inscribing minimum parallelograms as in the case of convex regions. In fact $|g(x, y)| \leq \sqrt{d/l}$ was the only simple infinite region for which a best possible result was known for l.

3. The problem of the minimum of a binary cubic can be reduced to a question in the geometry of numbers. It is easily shown that any binary cubic $f(x, y)$ of discriminant D can be transformed by a linear substitution with real coefficients and determinant unity into any other binary cubic $g(x, y)$ of discriminant D. On dividing by an appropriate factor, we may assume that $D = -23$ when $D < 0$, or $D = 49$ when $D > 0$. We write

$$g(x, y) = x^3 - xy^2 - y^3 \qquad \text{of discriminant } -23,$$

and

$$h(x, y) = x^3 + x^2 y - 2xy^2 - y^3 \qquad \text{of discriminant } 49.$$

Hence for appropriate real $\alpha, \beta, \gamma, \delta$ with $\alpha\delta - \beta\gamma = 1$, we can write

$$f(X, Y) = g(\alpha X + \beta Y, \gamma X + \delta Y) \qquad \text{if } D < 0,$$
$$f(X, Y) = h(\alpha X + \beta Y, \gamma X + \delta Y) \qquad \text{if } D > 0.$$

Now the points

$$x = \alpha X + \beta Y, \quad y = \gamma X + \delta Y$$

describe a lattice Λ, say, of determinant unity when X, Y run through all integer values. Our result takes the from: *Every lattice Λ of determinant unity has at least one of its points other than the origin O in each of the regions*

$$|g(x, y)| \leq 1, \quad |h(x, y)| \leq 1.$$

The constant on the right hand side is the best possible as is obvious from the lattice $x = X$, $y = Y$.

Let us consider the region $|g(x, y)| \leq 1$, say R. This is an infinite region bounded by the two curves $g(x, y) = \pm 1$ which have a common asymptote $x - \vartheta y = 0$ where ϑ is the real root of $t^3 - t - 1 = 0$. The asymptote is a line of symmetry of the region. It is soon seen that the parallelogram, really the square, $|x| \leq 1$, $|y| \leq 1$ is of special importance. The square has all its vertices and all the middle points of its sides on the boundary B of R. Its sides $x = \pm 1$ are tangents to the boundary at $x = \pm 1$, and further the square lies entirely in R except for a small region R_1 abutting the line $y = 1$ with $0 < x < 1$, and of course also for the image of R_1 in the origin O. This square, having its centre at O and of area 4, contains a point P other than O of every lattice Λ of determinant unity. If P is not an inner point of R, and this we may assume since otherwise the theorem is proved, it must be one of the vertices or middle points of the sides of the square, or lie in R_1. In the first two cases, it is easily shown that Λ has a point not O as an inner point of R except when Λ is the critical lattice $x=\xi$, $y=\eta$ which obviously has points on the boundary of R. In the third case, a point P of Λ is contained in R_1 and we include its boundary in R_1 since we wish to find points of Λ which are inner points of R.

We can now apply the same argument to other parallelograms of area 4, e. g. one whose sides are $x=\pm 1$ and the tangents at $(0, \pm 1)$, and find that Λ has a point say P_1 in a small curvilinear triangle near the point $(-1, 1)$. The question now suggests itself whether it is possible to find points which are linear combinations of P_1, P_2, such as $P_1 \pm P_2$ etc., which are inner points of R. For this, however a new idea is required suggested at once by the symmetry of the region R about the asymptote. The binary cubic is transformed into itself, and so also the region R, by a linear substitution with real coefficients and of determinant unity. Hence the parallelogram $|x| \leqq 1$, $|y| \leqq 1$ is changed into another one with the same characteristic properties used in the preceding argument. On considering the vertices, and middle points of its sides, we are led to the further critical lattice

$$(3\vartheta^2-1)\,x=-\xi-(\vartheta+3)\,\eta,\quad (3\vartheta^2-1)\,y=-3\vartheta\xi+\eta,$$

and it is easily verified that

$$|f(x, y)|=f(\xi, \eta),$$

and so $|f(x, y)| \geqq 1$ for integers ξ, η not both zero.

The new two small regions corresponding to the original two now lead to points P_2, P_3 of Λ not in R but near to R. These points may not be both different from the previous one, and in fact one of them say P_3 can be proved to be identical with the point P_1. We have now far more possibilities in considering linear combinations of these points, and in doing so, we require a more detailed knowledge of the region e. g. the minimum ordinate of the points of the boundary lying in the square $|x| \leqq 1$; $|y| \leqq 1$, but this presents no difficulty. After many efforts, I succeeded in finding smaller and smaller regions external and near to R and containing points of Λ, and finally was able to show that a linear combination of these points led to a point not O of Λ, any lattice not one of the two critical lattices, which was an inner point of R.

I considered next the corresponding problem for the region S,

$$|h(x, y)| \equiv |x^3+x^2y-2xy^2-y^3| \leqq 1.$$

This, however, introduced fresh difficulties. For first, the boundary had three asymptotes complicating the shape of the region. But a much more important difficulty is the situation of the unit square $|x| \leqq 1$, $|y| \leqq 1$ with respect to S. The square is contained in S except for two small regions one abutting $x=1$ with $y<0$, and the other $y=-1$ with $0<x<1$, and of course their images in O. The square contains a point P not O of every lattice Λ of determinant unity and so if P is not an inner point of S, it may lie in either of two small regions. I was able to show, however, that we could exclude the region abutting $x=1$. Taking into account now that S was unchanged by three essentially distinct linear substitutions, I was able to proceed as before and finally succeeded in proving the theorem.

Subsequently much simpler geometrical proofs were given by DAVENPORT[8]) who clothed his proof in arithmetical form, and by myself[9]). I have also given a proof when $D<0$ by considering the more symmetrical region $|x^3+y^3| \leqq 1$, and have thus reduced the numerical details to a minimum[10]).

4. After these results were found, DAVENPORT discovered arithmetical proofs of surprising simplicity based on ideas related to those used by HERMITE nearly ninety years ago. There is no loss of generality on dividing out by a factor in writing

$$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3,$$

and supposing that if $D>0$, $D=49$, and if $D<0$, $D=-23$.

Take first $D>0$. Write the Hessian or quadratic covariant of $f(x, y)$ as

$$Ax^2 + Bxy + Cy^2 = (bx+cy)^2 - (3ax+by)(cx+3dy).$$

This is a positive definite form of negative discriminant

$$B^2 - 4AC = -3D;$$

and so by the usual method of reduction, we can transform the Hessian by a unimodular substitution with integer coefficients into another with $C \geqq A \geqq B \geqq 0$. On applying the same substitution to the cubic, we may suppose that its Hessian is so reduced. Then he proved the

Theorem[11]). *Either* $|f(1, 0)| \leqq 1$, *or* $|f(0, 1)| \leqq 1$, *or* $|f(1, 1)| \leqq 1$, *or* $|f(1, -1)| \leqq 1$; *an inequality sign holds except when*

$$\pm f(x, y) = x^3 + x^2y - 2xy^2 - y^3 \quad or \quad x^3 + 2x^2y - xy^2 - y^3.$$

[8]) H. DAVENPORT, The minimum of a binary cubic form, *Journal London Math. Society,* **18** (1943), pp. 168–176.

[9]) L. J. MORDELL, The minimum of a binary cubic form, *Ibidem,* **18** (1943), pp. 201–210, 210–217.

[10]) L. J. MORDELL, Lattice points in the region $|x^3+y^3| \leqq 1$, *Ibidem,* **19** (1944), pp 92–99.

[11]) H. DAVENPORT, The reduction of a binary cubic form. I., *Ibidem,* **20** (1945), pp. 14–22.

A similar result holds when $D<0$, and so we can take $D=-23$. The cubic $f(x, y)$ has now one real linear factor and can be written as,

$$f(x, y)=(x+\vartheta y)(Px^2+Qxy+Ry^2),$$

where ϑ, P, Q, R are real. We may suppose that the quadratic form $Px^2+Qxy+Ry^2$ is positive definite on considering $-f(x, y)$ if need be instead of $f(x, y)$, and then that it is reduced, i. e.

$$|Q| \leqq P \leqq R;$$

and finally that $Q>0$ by writing $-y$ for y if need be. By a unimodular integral substitution on the cubic, we may suppose that $f(x, y)$ is such that these conditions are satisfied for the quadratic. Then DAVENPORT proved the

Theorem[12]). *Either* $|f(1,0)| \leqq 1$, *or* $|f(0, 1)| \leqq 1$, *or* $|f(1, -1)| \leqq 1$, *or* $|f(1, 2)| \leqq 1$. *An inequality sign holds except when*

$$f(x, y)=x^3+x^2y+2xy^2+y^3,$$

which on putting $x=X$, $y=-X-Y$ *becomes* $X^3-XY^2-Y^3$.

5. A flood of results followed from my method, for the application of the geometry of numbers to the minimum of a binary cubic meant that corresponding questions for nonconvex regions were no longer intractable. An obvious region to investigate was

$$|x|^p+|y|^p \leqq 1.$$

which for $p \geqq 1$ is convex and had been studied by MINKOWSKI[13]). When $p<1$, it is not convex and had not been previously considered by mathematicians. I found that my methods applied not only to this region but to the more general one

$$f(|x|, |y|) \leqq 1,$$

where for $x \geqq 0$, $y \geqq 0$, $f(x, y)$ is defined, is symmetrical in x, y and homogeneous of dimension 1 say. We suppose that the region $f(x, y) \geqq f(1, 1)$, $x \geqq 0$, $y \geqq 0$ is convex and terminates in the axes or has them as asymptotes. Then just as for the binary cubic, parallelograms can be constructed whose vertices and middle points of sides all lie on the boundary of the region. Their existence follows since it can be proved that unique numbers a, b, c with $a>b>c$ are defined by the equations

$$f(a+b, a-b)=f(a, b)=cf(1, 1),$$
$$a^2+b^2=2.$$

[12]) H. DAVENPORT, The reduction of a binary cubic form. II, *Journal London Math. Society*, **20** (1945), pp. 139—157.

[13]) H. MINKOWSKI, l. c. [6]), pp. 21—58.

By considering various regions in which lattice points must lie and utilising the ideas developed for the binary cubic, I was then able[14]) to reduce the question to a minimum problem of the type considered by MINKOWSKI. Further there existed many regions for which the minimum problem could be solved. Thus for lattices of determinant 1, best possible results were found of the form $|x|^p + |y|^p \leqq 2c^p$, $0{\cdot}33 \ldots \leqq p < 1$; $|x^4 - y^4| \leqq \frac{4}{\sqrt{17}}$, also for a star shaped octagon. etc.

Similar methods apply to the region

$$|x^n + y^n| \leqq c^n, \qquad n \geqq 4.$$

I conclude by saying that the success of these methods led MAHLER to his general and important theory of lattice points in star shaped regions, a fruitful theory which has recently added so much to our knowledge of the geometry of numbers and has also been the starting point of many new results.

(Received December 16, 1948.)

Note. In 1945, B. DELAUNAY published a paper entitled "Local methods in the geometry of numbers", *Bulletin Acad. Sci. URSS*, Série Math., **9** (1945), pp. 241—256 (in Russian). He finds a new and simple solution for the minimum of a binary cubic of positive discriminant by an extension of MINKOWSKI's method of continually diminishing the determinant of a lattice which has no point other than the origin in a region.

(Added June 20, 1949.)

14) L. J. MORDELL, On the geometry of numbers in some nonconvex regions, *Proceedings London Math. Society*, (2) **48** (1945), pp. 339—390.

On the measure of equidistribution of point sets.

By ALFRÉD RÉNYI in Budapest.

Introduction.

Throughout the paper we are concerned with measurable point sets E lying in the interval $(0, 1)$. The measure of E shall be denoted by $|E|$ and the characteristic function of E by $F(x)$. We define $F(x)$ outside the interval $(0, 1)$ so as to be periodic with period 1. We denote by E_t (for any real t) the set which has the characteristic function $F(x+t)$. If we imagine the interval $(0, 1)$ wound on a circle of circumference unity, we may say that E_t is obtained by rotating the set E by the angle $-t$. Let $G(t)$ denote the measure of the set of points of the interval $(0,1)$ which are common to E and E_t. We have evidently

(1) $$G(t)=\int_0^1 F(x)F(x+t)\,dx.$$

$G(t)$ is a non-negative function, periodic with period 1. We have, in view of the periodicity of $F(x)$,

(2) $$G(t)=\int_0^1 F\left(x-\frac{t}{2}\right)F\left(x+\frac{t}{2}\right)dx,$$

thus $G(t)$ is an even function. Further we have

(3) $$|G(t+h)-G(t)|\leqq\int_0^1 |F(x+h)-F(x)|\,dx.$$

Now, it is well known[1]) that the integral on the right side tends to 0 with h, thus $G(t)$ is continuous. As we have $G(0)=|E|$, it follows from the continuity of $G(t)$ that if $|E|>0$, there exists a constant $c>0$, for which $G(t)>0$ for $0\leqq t<c$. This is equivalent to a theorem of

[1]) Cf. for ex. A. ZYGMUND, *Trigonometrical series* (Warszawa, 1935), p. 17.

H. STEINHAUS[2]), who stated it in the form, that the set of the mutual distances of the points of a set of positive measure contains a whole interval $(0, c)$. In view of this interpretation, we shall call $G(t)$ the *distance function* of the set E.

Now let us denote the minimal value of the continuous function $G(t)$ by $m(E)$. As $G(t)$ is non-negative, further as we have

$$\int_0^1 G(t)\,dt = \int_0^1\int_0^1 F(x)\,F(x+t)\,dx\,dt = |E|^2, \tag{4}$$

it follows

$$0 \leq m(E) \leq |E|^2. \tag{5}$$

It is easy to see that $m(E) = |E|^2$ if and only if $|E| = 0$ or $|E| = 1$. Thus if we put

$$\mu(E) = \frac{m(E)}{|E|^2}, \tag{6}$$

we have $0 \leq \mu(E) < 1$ for $0 < |E| < 1$. In what follows $\mu(E)$ shall be called the *measure of equidistribution* of the set E. Of course the notion of equidistribution, implied by this definition, is different from (but as we shall see is closely connected with) the usual definition for sequences, introduced by H. WEYL[3]). The difference is made clear by remarking that we are concerned not with the equidistribution of the *points* of E but with the equidistribution of the set of *distances* of pairs of points of E.

The purpose of the present paper is to prove that there exist sets having any prescribed positive measure, and as "highly equidistributed" as we please, i. e., having a measure of equidistribution arbitrarily near to 1. This shall be proved in **§. 2** (Theorem 1). **§. 1** contains preliminary discussions of rather general character, concerning the FOURIER expansion of the distance function and some lemmas. The proof of Theorem 1 is based on a property of quadratic residues, discovered by LAGRANGE[4]). In **§. 3** the problem is generalized. We introduce the notion of the measure of k-fold equidistribution, and prove a theorem, analogous to, but somewhat weaker than Theorem 1 (Theorem 2), based

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[2]) H. STEINHAUS, Sur les distances des points des ensembles de mesure positive, *Fundamenta Math.*, **1** (1920), pp. 93—104. Cf. also S. PICCARD, Sur les ensembles de distances des ensembles de points d'un espace euclidien, *Mémoires Université Neuchatel*, **13** (1939), pp. 212.

[3]) H. WEYL, Über die Gleichverteilung von Zahlen mod. Eins, *Math. Annalen*, **77** (1916), pp. 313—325.

[4]) P. BACHMANN, *Niedere Zahlentheorie*, Vol. II, (Leipzig, 1910), pp. 241—245.

on a generalization of a theorem of THUE[5]). In §. 4, we point out the connection with some problems of number theory, and prove a theorem concerning the sequences of integers, called difference bases, constructed by SINGER[6]) (Theorem 3).

§. 1. Fourier expansion of the distance function.

Let $F(x)$ denote the characteristic function of a measurable set E in the interval $(0,1)$. Let us consider the FOURIER expansion of $F(x)$:

$$F(x) \sim a_0 + 2\sum_{n=1}^{\infty} (a_n \cos 2\pi nx + b_n \sin 2\pi nx).$$

Lemma 1. *If $G(t)$ denotes the distance function of the set E as defined in the introduction, we have*

$$G(t) = a_0^2 + 2\sum_{n=1}^{\infty} (a_n^2 + b_n^2)\cos 2\pi nt.$$

The series on the right converges uniformly.

Evidently Lemma 1 follows from Parseval's theorem.

In what follows we shall consider some special sets consisting of a finite number of intervals of equal length. Let $b_0, b_1, \ldots, b_{N-1}$ denote a sequence of integers, which are all different modulo q. The set $E = E_q(b_0, b_1, \ldots, b_{N-1})$ shall be defined as the set consisting of the intervals $\left(\frac{b_j - \frac{1}{2}}{q}, \frac{b_j + \frac{1}{2}}{q}\right)$, $(j = 0, 1, \ldots, N-1)$. Evidently, the set E is not changed if one of the b_j is replaced by a number congruent to it modulo q, thus we may suppose $0 \leq b_j < q$.

Lemma 2. *Let $G(t)$ denote the distance function of a set $E = E_q(b_0, b_1, \ldots, b_{N-1})$. Let us denote*

$$c_n = \sum_{j=0}^{N-1} \exp\left(\frac{2\pi i n b_j}{q}\right). \tag{7}$$

Then we have

$$G(t) = \frac{N^2}{q^2} + \frac{2}{q^2}\sum_{n=1}^{\infty} |c_n|^2 \left(\frac{\sin\frac{\pi n}{q}}{\frac{\pi n}{q}}\right)^2 \cos 2\pi nt. \tag{8}$$

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[5]) A. SCHOLZ, *Einführung in die Zahlentheorie* (Sammlung Göschen, Bd. 1131, Berlin, 1939), p. 45.

[6]) I. SINGER, A theorem in finite projective geometry and some applications to number theory, *Transactions American Math. Society*, 43 (1938), pp. 377–385. Cf. also: T. VIJAYARAGHAVAN and S. CHOWLA, Short proof of theorems of Bose and Singer, *Proceedings National Academy Sciences India*, Section A, 15 (1945), p. 194.

Lemma 2 is verified easily by calculating explicitly the FOURIER coefficients of the characteristic function of the set E and applying Lemma 1.

Lemma 3. *Let us have $0 < h < 1$. We define*

$$R_h(x) = h^2 + 2h^2 \sum_{n=1}^{\infty} \left(\frac{\sin n\pi h}{n\pi h}\right)^2 \cos 2\pi n x. \tag{9}$$

Then we have

$$R_h(x) = h - |x| \quad \text{for} \quad |x| \leq h,$$
$$R_h(x) = 0 \quad \text{for} \quad |x| > h.$$

Lemma 3 is easily verified by calculating the FOURIER coefficients of $R_h(x)$. The function $R_h(x)$ may be called the "RIEMANN kernel". As a matter of fact, $f(x)$ denoting a function, L-integrable in $(0, 1)$, the summation method of RIEMANN consists in forming the second generalized derivative of the function $\psi(x)$, obtained by integrating $f(x)$ twice, and it is easy to see that we have

$$\frac{\psi(x+2h) + \psi(x-2h) - 2\psi(x)}{4h^2} = \frac{1}{h^2} \int_0^1 f(t)\, R_h(x-t)\, dt, \tag{10}$$

i. e. $R_h(x)$ is the kernel function of the RIEMANN summation[7]).

It can be seen from (7) that $c_n = c_m$ if $n \equiv m \bmod q$. Further, as $\sin \frac{n\pi}{q} = 0$ for $n \equiv 0 \bmod q$, the values of c_n for $n \equiv 0 \bmod q$ figure in the expansion (8) only formally, and the FOURIER expansion of $G(t)$ is completely determined if we know the values of $c_1, c_2, \ldots, c_{q-1}$. Lemma 3 shows that $G(t)$ can easily be calculated if the values of $|c_n|^2$ $(n \not\equiv 0)$ are all equal. The same is true if they show only relatively small deviations from a common value. This is expressed by the following

Lemma 4. *Let $E = E_q(b_0, b_1, \ldots, b_{N-1})$ be defined as above. If the numbers c_n defined by (7) satisfy the relations*

$$\left| |c_n|^2 - Q \right| < \frac{\vartheta Q}{q-1} \quad \text{for} \quad n = 1, 2, \ldots, q-1,$$

where $Q(1+\vartheta) < N^2$, we have

$$\mu(E) \geq 1 - \frac{Q(1+\vartheta)}{N^2}.$$

Proof. We have evidently from (8)

$$G(t) \geq \frac{N^2}{q^2} + Q\left(R_{1/q}(t) - \frac{1}{q^2}\right) - \frac{\vartheta Q}{q-1}\left(R_{1/q}(0) - \frac{1}{q^2}\right) \tag{11}$$

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[7]) This has been already remarked by M. SCHECHTER, Über die Summation divergenter Fourier-Reihen, *Monatshefte für Math. und Physik*, 25 (1911), pp. 224–234. It was Prof. L. FEJÉR who has kindly called my attention to this paper.

and thus

$$m(E) \geqq \frac{N^2 - Q(1+\vartheta)}{q^2} \tag{12}$$

and Lemma 4 follows easily.

Of course the situation is the simplest if, in Lemma 4, $\vartheta = 0$. Sequences of integers b_j for which this holds, are characterized by the following

Lemma 5. *If $b_0, b_1, b_2, \ldots, b_{N-1}$ denote a sequence of integers with the property that the differences $b_r - b_s$ $(r, s = 0, 1, \ldots, N-1; r \neq s)$ represent every class of residues modulo q (the class 0 of course excepted) exactly k-times, we shall call the sequence b_j a difference basis of order k modulo q. The necessary and sufficient condition for the sequence b_j being a difference basis of order k modulo q, is that for any $n \not\equiv 0 \bmod q$*

$$\left| \sum_{r=0}^{N-1} \exp\left(2\pi i \frac{b_r n}{q}\right) \right|^2 = N - k \tag{13}$$

be valid.

It is clear that the condition (13) is necessary. Let us prove that it is also sufficient. Let A_l $(l = 1, 2, \ldots, q-1)$ denote the number of representations of l mod q in the form $b_r - b_s$. We have

$$N - k = \left| \sum_{r=0}^{N-1} \exp\left(2\pi i \frac{b_r n}{q}\right) \right|^2 = N + \sum_{l=1}^{q-1} A_l \exp\left(2\pi i \frac{ln}{q}\right).$$

Let us denote $A_0 = k$, $S_0 = qk$ and put

$$S_n = \sum_{l=0}^{q-1} A_l \exp\left(2\pi i \frac{ln}{q}\right), \quad n = 1, 2, \ldots, q-1.$$

Evidently $S_n = 0$ for $n = 1, 2, \ldots, q-1$. It follows that for $v \not\equiv 0 \bmod q$

$$T = \sum_{n=0}^{q-1} S_n \exp\left(-2\pi i \frac{vn}{q}\right) = S_0 = qk.$$

On the other hand, inverting the order of summations, we obtain

$$T = \sum_{l=0}^{q-1} \sum_{n=0}^{q-1} A_l \exp\left(2\pi i \frac{(l-v)n}{q}\right) = qA_v.$$

Thus it follows $A_v = k$ for $v = 1, 2, \ldots, q-1$, which was to be proved.

Lemma 6. *If $b_0, b_1, \ldots, b_{N-1}$ is a difference basis of order k modulo q, and $\mu(E)$ denotes the measure of equidistribution of the set $E = E_q(b_0, b_1, \ldots, b_{N-1})$, we have*

$$\mu(E) = 1 - \frac{N-k}{N^2}. \tag{14}$$

Lemma 6 follows from the proof (not the statement) of Lemma 4 combined with Lemma 5.

Lemma 7. *Let E denote a measurable set, $\bar{E}$ the set complementary to E. We have*

$$1-\mu(\bar{E})=\frac{1-\mu(E)}{\left(\frac{1}{|E|}-1\right)^2}. \tag{15}$$

Proof. Evidently

$$m(\bar{E})=\min\int_0^1(1-F(x))(1-F(x+t))\,dx=1-2|E|+m(E) \tag{16}$$

and thus Lemma 7 follows.

Lemma 8. *If $\alpha(x)$ is integrable in $(0, 1)$, $\beta(x)$ bounded and integrable in the same interval and periodic with period 1, we have*

$$\lim_{n\to\infty}\int_0^1\alpha(x)\beta(nx)\,dx=\int_0^1\alpha(x)\,dx\int_0^1\beta(x)\,dx. \tag{17}$$

This lemma is well known[8]).

Lemma 9. *Let E_1 and E_2 denote two sets having positive measures $|E_1|$ and $|E_2|$, characteristic functions $F_1(x)$ and $F_2(x)$, distance functions $G_1(x)$ and $G_2(x)$, respectively, and let the minima of the distance functions be denoted by $m(E_1)$ and $m(E_2)$ respectively. Let us define the set $E^{(n)}$ by its characteristic function being $F^{(n)}(x)=F_1(x)F_2(nx)$ $(n=1, 2, \ldots)$. It follows*

$$\lim_{n\to\infty}|E^{(n)}|=|E_1||E_2| \tag{18}$$

and

$$\underline{\lim}_{n\to\infty}\, m(E^{(n)})\geqq m(E_1)\,m(E_2), \tag{19}$$

where $m(E^{(n)})$ denotes the minimal value of the distance function $G^{(n)}(t)$ of $E^{(n)}$.

Proof. (18) follows clearly from Lemma 8. As regards to (19), let us suppose the contrary. Thus we suppose that there exists an infinite sequence of integers n_k $(k=1, 2, \ldots)$, and a corresponding sequence of real numbers t_{n_k} $(0\leqq t_{n_k}<1)$, for which

$$G^{(n_k)}(t_{n_k})<m(E_1)\,m(E_2)-\varepsilon$$

holds, for some fixed $\varepsilon>0$. Let us denote by τ_{n_k} the fractional part of $n_k t_{n_k}$. Clearly we may choose an infinite subsequence ν_k $(k=1, 2, \ldots)$ of the sequence n_k, such that if $k\to\infty$, t_{ν_k} and τ_{ν_k} tend to limits t^* and τ^*,

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8) This lemma has been proved for some special cases by L. Fejér, Lebesguesche Konstanten und divergente Fourierreihen, *Journal für reine und angewandte Math.*, 138 (1910), pp. 27–28. In the general form the lemma has been proved by A. Zygmund, l. c. (1), p. 173, § 8. 34.

respectively. Now, putting

$$G_k(t^*, \tau^*) = \int_0^1 F_1(x) F_1(x+t^*) F_2(\nu_k x) F_2(\nu_k x + \tau^*)\,dx,$$

we have

$$|G^{(\nu_k)}(t_{\nu_k}) - G_k(t^*, \tau^*)| \leqq \int_0^1 |F_1(x+t_{\nu_k}) - F_1(x+t^*)|\,dx +$$
$$+ \int_0^1 |F_2(y+\tau_{\nu_k}) - F_2(y+\tau^*)|\,dy$$

and thus, applying again the theorem by which we have proved the continuity of $G(t)$ (see [1]), we obtain

$$\lim_{k\to\infty} [G^{(\nu_k)}(t_{\nu_k}) - G_k(t^*, \tau^*)] = 0. \tag{21}$$

Applying Lemma 8 again, we obtain

$$\lim_{k\to\infty} G_k(t^*, \tau^*) = G_1(t^*)\, G_2(\tau^*) \geqq m(E_1)\, m(E_2) \tag{22}$$

and thus owing to (21) it follows

$$\lim_{k\to\infty} G^{(\nu_k)}(t_{\nu_k}) \geqq m(E_1)\, m(E_2). \tag{23}$$

But this clearly contradicts (20) and thus (19) is proved.

Lemma 10. *If the characteristic functions $F_1(x)$ and $F_2(x)$ of the measurable sets E_1 and E_2 are equal except on a set of measure $\frac{\delta}{2}$ $(0 < \delta < 1)$, we have $|m(E_1) - m(E_2)| < \delta$.*

Lemma 10 follows simply by remarking that $F_1(x) F_1(x+t)$ and $F_2(x) F_2(x+t)$ are equal if neither x nor $x+t$ does belong to the exceptional set, i. e. except for a set the measure of which does not exceed δ, and thus $|G_1(t) - G_2(t)| < \delta$ for any t. Let $M(\alpha)$ denote the least upper bound of $\mu(E)$ for all sets E for which $|E| = \alpha$ $(0<\alpha<1)$. We prove

Lemma 11.

If $M(\alpha) = 1$ and $M(\beta) = 1$, we have $M(\alpha\beta) = 1$.

Proof. According to the suppositions of our Lemma, for any $\varepsilon > 0$ there exist sets E_1 and E_2 with $|E_1| = \alpha$, $|E_2| = \beta$, $\mu(E_1) > 1 - \frac{\varepsilon}{4}$, $\mu(E_2) > 1 - \frac{\varepsilon}{4}$. Let us define the sequence of sets $E^{(n)}$ as in Lemma 9, by virtue of which we have $\lim_{n\to\infty} |E^{(n)}| = \alpha\beta$ and

$$\lim_{n\to\infty} m(E^{(n)}) \geqq \alpha\beta \left(1 - \frac{\varepsilon}{4}\right)^2.$$

Thus if we choose n sufficiently large, both inequalities

$$\left||E^{(n)}|-\alpha\beta\right|<\frac{\varepsilon\alpha\beta}{4} \text{ and } m(E^{(n)})\geqq\alpha\beta\left(1-\frac{\varepsilon}{2}\right)$$

will be satisfied. According to $|E^{(n)}|-\alpha\beta<0$ or $|E^{(n)}|-\alpha\beta>0$ we may add or take away from $E^{(n)}$ a set of measure not exceeding $\frac{\varepsilon\alpha\beta}{4}$ so as to obtain a set $\mathcal{E}^{(n)}$ having its measure equal to $\alpha\beta$. The characteristic function of the set $\mathcal{E}^{(n)}$ does not differ from that of $E^{(n)}$ but on a set the measure of which does not exceed $\frac{\varepsilon\alpha\beta}{4}$. Thus, according to Lemma 10, we have

$$m(\mathcal{E}^{(n)})\geqq m(E^{(n)})-\frac{\varepsilon\alpha\beta}{2}\geqq\alpha\beta(1-\varepsilon).$$

As $\varepsilon>0$ may be chosen arbitrarily, this proves Lemma 11.

Lemma 12. *If* $\lim_{n\to\infty}\alpha_n=\alpha$ $(0<\alpha_n<1,\ 0<\alpha<1)$ *and* $M(\alpha_n)=1$ *for* $n=1,2,\ldots$, *then we have* $M(\alpha)=1$.

Proof. For any $\varepsilon>0$, we choose n sufficiently large so as to obtain

$$\left|\frac{\alpha_n}{\alpha}-1\right|<\frac{\varepsilon}{4}.$$

According to our suppositions, there exists a set E_n for which $|E_n|=\alpha_n$ and $\mu(E_n)\geqq1-\frac{\varepsilon}{4}$. We add to or take away from E_n a set of measure not exceeding $\frac{\alpha\varepsilon}{4}$ so as to obtain a set $\mathcal{E}_n$ of measure α. We have, using Lemma 10,

$$m(\mathcal{E}_n)\geqq m(E_n)-\frac{\alpha\varepsilon}{2}\geqq\alpha_n\left(1-\frac{\varepsilon}{4}\right)-\frac{\alpha\varepsilon}{2}\geqq\alpha(1-\varepsilon)$$

which proves Lemma 12.

Lemma 13. *Every real number* α $(0<\alpha<1)$ *can be represented as a finite or infinite product of the form*

$$\alpha=\prod_{k=1}^{\infty}\left(1-\frac{1}{2^{n_k}}\right)\qquad(1\leqq n_k\leqq n_{k+1}).\tag{24}$$

Proof. Let us suppose that α is not a rational number which is equal to the product of a finite number of factors of the form $\left(1-\frac{1}{2^{n_k}}\right)$. Let us choose $n_1\geqq1$ so that we have

$$1-\frac{1}{2^{n_1-1}}<\alpha<1-\frac{1}{2^{n_1}};\tag{25}$$

further, if $n_1, n_2, \ldots, n_{k-1}$ are already found, we choose n_{k+1} so as to obtain

(26) $$1-\frac{1}{2^{n_{k-1}}}<\frac{\alpha}{\prod_{j=1}^{k-1}\left(1-\frac{1}{2^{n_j}}\right)}<1-\frac{1}{2^{n_k}}.$$

Dividing (26) by $1-\frac{1}{2^{n_k}}$ and applying again (26) with $k+1$ instead of k, we obtain

(27) $$1-\frac{1}{2^{n_{k-1}}}<\frac{\alpha}{\prod_{j=1}^{k}\left(1-\frac{1}{2^{n_j}}\right)}<1-\frac{1}{2^{n_{k+1}}}.$$

It follows from (27) that $n_{k+1} \geq n_k$. Thus the sequence n_k, which is uniquely determined according to the above construction is non-decreasing. It is easy to see, that $n_k \to \infty$. As a matter of fact, in the opposite case n_k would be constant from some index k_0 onwards. But it would follow from the construction that in this case we should have

(28) $$\alpha \leq \prod_{j=1}^{k_0-1}\left(1-\frac{1}{2^{n_j}}\right)\left(1-\frac{1}{2^{n_{k_0}}}\right)^N$$

for any N, i. e. we should have $\alpha=0$, contrary to our hypothesis. Thus $n_k \to \infty$, and it follows from (26) that

$$\lim_{k\to\infty}\frac{\alpha}{\prod_{j=1}^{k-1}\left(1-\frac{1}{2^{n_j}}\right)}=1$$

which proves our lemma.

§. 2. Application of the theorem of Lagrange.

The theorem of LAGRANGE is question is the following: Let p denote a prime number of the form $4n+3$. Let $r_1, r_2, \ldots, r_\nu$ $\left(\nu=\frac{p-1}{2}\right)$ denote a complete system of quadratic residues mod p. Let d denote any integer, $d \not\equiv 0 \bmod p$. Then there are $\frac{p-3}{4}$ quadratic residues in the sequence r_j+d $(j=1,2,\ldots,\nu)$. According to the terminology introduced in Lemma 5, this theorem can be stated also by saying that the system of quadratic residues to a prime modulus $p \equiv 3 \bmod 4$ is a difference basis of order $\frac{p-3}{4}$ modulo p. This theorem follows easily from Lemma 5 and from the well known formula for Gaussian sums:

(29) $$\sum_{y=0}^{p-1}\exp\left(\frac{2\pi i y^2}{p}\right)=i\sqrt{p}$$

for $p \equiv 3 \mod 4$. As every class of quadratic residues is represented twice among the squares y^2 $(1 \leq y \leq p-1)$, it follows from (29) that

$$\sum_{j=1}^{\nu} \exp\left(\frac{2\pi i r_j}{p}\right) = \frac{i\sqrt{p}-1}{2}. \tag{30}$$

It follows from (30), using $\sum_{k=0}^{p-1} \exp \frac{2\pi i k}{p} = 0$, that if $s_1, s_2, \ldots, s_\nu$ denote a complete set of quadratic non-residues mod p, we have

$$\sum_{j=1}^{\nu} \exp\left(\frac{2\pi i s_j}{p}\right) = \frac{-i\sqrt{p}-1}{2}. \tag{31}$$

Now the sequence nr_j $(j=1,2,\ldots,\nu)$ is congruent to the sequence of residues or to the sequence of non-residues, according to the quadratic character of n. Thus it follows from (30) and (31) that for any $n \not\equiv 0$ mod p we have

$$\left|\sum_{j=1}^{\nu} \exp\left(\frac{2\pi i n r_j}{p}\right)\right|^2 = \frac{p+1}{4}. \tag{32}$$

Thus we can apply Lemma 5, and obtain that the differences $r_i - r_j$, $i \neq j$ represent every class of residues mod p exactly $\frac{p-1}{2} - \frac{p+1}{4} = \frac{p-3}{4}$ times, which is equivalent to the theorem of LAGRANGE stated above.

Now everything is ready to prove

Theorem 1. *The least upper bound $M(\alpha)$ of the measure of equidistribution $\mu(E)$ of measurable sets E having the measure $|E| = \alpha$ is identically equal to* 1 *for* $0 < \alpha \leq 1$.

Proof of Theorem 1. Let p denote a prime, $p \equiv 3 \mod 4$, and let $r_1, r_2, \ldots, r_\nu$ $\left(\nu = \frac{p-1}{2}\right)$ denote a complete system of quadratic residues mod p. Let us define the set $E_p = E_p(r_1, r_2, \ldots, r_\nu)$ as in § 1. It follows from Lemma 6 that

$$\mu(E_p) = 1 - \frac{p+1}{(p-1)^2}. \tag{33}$$

Let $\mathcal{E}_p$ denote a set obtained by adding to E_p any interval of length $\frac{1}{2p}$. As $|E_p| = \frac{p-1}{2p}$, we have $|\mathcal{E}_p| = \frac{1}{2}$ and it follows from (33) that

$$\mu(\mathcal{E}_p) \geq 1 - \frac{3}{p}. \tag{34}$$

Since there are an infinity of primes of the form $4n+3$, it follows that $M\left(\frac{1}{2}\right) = 1$. Applying Lemma 11, we obtain $M\left(\frac{1}{2^k}\right) = 1$ for $k = 1, 2, \ldots,$

further, by Lemma 7, $M\left(1-\frac{1}{2^k}\right)=1$, $(k=1,2,\ldots)$. Applying Lemma 11 again, we obtain that $M(\alpha)=1$ if α is a finite product of the form (24). Thus it follows, using Lemma 12 and regarding also Lemma 13, that $M(\alpha)=1$ for all α, $0<\alpha\leqq 1$. Thus Theorem 1 is proved[9].

§. 3. The measure of k-fold equidistribution.

Let $E, E_t, |E|$ and $F(x)$ have the meaning as in the introduction. Let $G(t_1, t_2, \ldots, t_k)$ denote the measure of the set of points common to $E, E_{t_1}, E_{t_2}, \ldots, E_{t_k}$. We have evidently

$$(35)\qquad G(t_1, t_2, \ldots, t_k)=\int_0^1 F(x)\,F(x+t_1)\,F(x+t_2)\ldots F(x+t_k)\,dx.$$

It is easy to see that $G(t_1, t_2, \ldots, t_k)$ is a continuous function of its variables. The minimal value of $G(t_1, t_2, \ldots, t_k)$ shall be denoted by $m_k(E)$. Owing to

$$(36)\qquad \int_0^1\int_0^1\ldots\int_0^1 G(t_1, t_2, \ldots, t_k)\,dt_1\,dt_2\ldots dt_k=|E|^{k+1},$$

we have

$$(37)\qquad 0\leqq m_k(E)\leqq |E|^{k+1}.$$

The measure of k-fold equidistribution of the set E shall be defined by

$$(38)\qquad \mu_k(E)=\frac{m_k(E)}{|E|^{k+1}}.$$

Thus we have, owing to (37), $0\leqq \mu_k(E)\leqq 1$. The least upper bound of $\mu_k(E)$ for all measurable sets E with $|E|=\alpha$ will be denoted by $M_k(\alpha)$. It seems probable that $M_k(\alpha)=1$ identically in α for any k. In what follows we shall prove however only the following

Theorem 2.

$$\overline{\lim_{k\to\infty}}\, M_k(\alpha)\geqq \frac{1}{4^{k+1}}.$$

The most surprising consequence of Theorem 1 is perhaps that there exist measurable sets with arbitrary small positive measure with the property, that if the set is "rotated" in the sense mentioned in the introduction, the set of points, which are common to the rotating set and to the original set, is never void, indeed, its measure exceeds always a fixed number during the rotation. Though Theorem 2 is relatively

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[9]) Mr. P. UNGÁR, to whom I communicated at an earlier stage of my investigations some of my results, found independently a proof of Theorem 1, running essentially on the same lines.

much weaker than Theorem 1', and is not a "best possible" result, nevertheless it contains the generalization of that interpretation of Theorem 1 which has been emphasised just now.

The proof of Theorem 3' will be based on the following generalization of a theorem of THUE:

Lemma 14. *If p is a prime, k a positive integer, further the positive integers $e_1, e_2, \ldots, e_k, f$ satisfy*

$$e_1 . e_2 \ldots e_k . f > p^k, \tag{39}$$

then for any k-tuple of integers $(r_1, r_2, \ldots, r_k)$ there can be found integers $x_1, x_2, \ldots, x_k$ and y for which $1 \leqq y < f$, $|x_i| < e_i$ $(i = 1, 2, \ldots, k)$ and $r_i \equiv \frac{x_i}{y} \mod p$ $(i = 1, 2, \ldots, k)$ are valid.

Proof. Let us consider all k-tuples of integers of the form $(yr_i + x_i)$, $i = 1, 2, \ldots, k$, where $1 \leqq x_i \leqq e_i$ $(i = 1, 2, \ldots, k)$ and $1 \leqq y \leqq f$. The number of such k-tuples of integers being $e_1 e_2 \ldots e_k f$, as there are only p^k k-tuples which are different mod p, owing to (39), there must be at least two k-tuples of the form considered which are congruent mod p. If we denote the two congruent k-tuples by $(yr_i + x_i)$ and $(\eta r_i + \xi_i)$, $i = 1, 2, \ldots, k$, we have

$$yr_i + x_i \equiv \eta r_i + \xi_i \mod p, \quad i = 1, 2, \ldots, k.$$

From $y \equiv \eta \mod p$ it would follow $x_i \equiv \xi_i \mod p$ for all $i = 1, 2, \ldots, k$, thus we have $y \not\equiv \eta \mod p$, and it follows

$$r_i \equiv \frac{\pm |\xi_i - x_i|}{|y - \eta|} \mod p \qquad (i = 1, 2, \ldots, k).$$

As $0 \leqq |\xi_i - x_i| < e_i$ $(i = 1, 2, \ldots, k)$ and $1 \leqq |y - \eta| < f$, our Lemma is proved.

Now we prove the following

Lemma 15. *If p is a prime, k a positive integer, and $Q = \left[p^{\frac{k}{k+1}}\right]$ ($[x]$ denotes the integral part of x), a set of $2Q$ integers $c_1, c_2, \ldots, c_{2Q}$ can be given, having the property that for any k-tuple of integers $(b_1, b_2, \ldots, b_k)$, elements $c_{i_1}, c_{i_2}, \ldots, c_{i_k}, c_j$ of the given set can be chosen so as to obtain*

$$b_r \equiv c_{i_r} - c_j \mod (p-1) \quad \text{for} \quad r = 1, 2, \ldots, k.$$

Proof. Putting $e_i = f = \left[p^{\frac{k}{k+1}}\right] + 1 = Q + 1$ $(i = 1, 2, \ldots, k)$, condition (39) of lemma 14 is evidently satisfied. Let g denote a primitive root mod p and let ind x denote the index of the residue class x with respect to g. It is easy to see that if $c_i = \operatorname{ind} i$, $c_{Q+i} = \operatorname{ind}(-i)$ $(i = 1, 2, \ldots, Q)$, the sequence c_i $(1 \leqq i \leqq 2Q)$ has the required properties.

Let us now define the set E, consisting of the intervals:

$$\frac{c_r-1}{p-1} \leqq x \leqq \frac{c_r+1}{p-1}, \tag{40}$$

where the c_r $(r=1,2,\ldots,2Q)$ are the elements of the set of integers of Lemma 15 Let $F(x)$ denote the characteristic function of the set E, and let $G(t_1,t_2,\ldots,t_k)$ be defined by (35). If $(t_1,t_2,\ldots,t_k)$ is an arbitrary k-tuple of real numbers, $0 \leqq t_r < 1$, we put

$$t_r = \frac{b_r+\vartheta_r}{p-1} \quad (r=1,2,\ldots,k),$$

where b_r denotes the integer which is nearest to $(p-1)t_r$, and thus we have

$$|\vartheta_r| \leqq \frac{1}{2} \quad (r=1,2,\ldots,k).$$

According to Lemma 15, we can choose $c_{i_1}, c_{i_2},\ldots,c_{i_k}, c_j$ so that

$$b_r \equiv c_{i_r}-c_j \mod (p-1) \quad \text{for } r=1,2,\ldots,k.$$

It follows according to (40) that if

$$\frac{c_j-\frac{1}{2}}{p-1} \leqq x \leqq \frac{c_j+\frac{1}{2}}{p-1},$$

we have

$$\frac{c_{i_r}-1}{p-1} \leqq x+t_r \leqq \frac{c_{i_r}+1}{p-1} \quad \text{for } r=1,2,\ldots,k.$$

Thus

$$F(x+t_r)=1 \quad \text{for } r=1,2,\ldots,k \quad \text{if} \quad \frac{c_j-\frac{1}{2}}{p-1} \leqq x \leqq \frac{c_j+\frac{1}{2}}{p-1}.$$

It follows from (35) that

$$m_k(E) \geqq \frac{1}{p-1}. \tag{41}$$

Owing to $|E|=\dfrac{4Q}{p-1}$ $\left(Q=\left[p^{\frac{k}{k+1}}\right]\right)$, we obtain

$$\mu_k(E) \geqq \frac{\left(1-\frac{1}{p}\right)^k}{4^{k+1}}. \tag{42}$$

If any fixed $\varepsilon>0$ is given, we can choose p sufficiently large so as to obtain

$$|E|<\varepsilon \quad \text{and} \quad \mu_k(E) > \frac{1-\varepsilon}{4^{k+1}}. \tag{43}$$

Thus Theorem 2 is proved.

§. 4. Some remarks on the sequences of Singer.

We have seen in §. 1 that the construction of highly equidistributed sets is closely connected with the number-theoretical problem of constructing difference bases, i. e. finite sequences of integers, the differences of which represent every class of residues to a given modulus q exactly k times, k being the order of the difference basis. In this direction interesting results have been obtained by I. SINGER (l. c. [6])) who constructed difference bases of order 1 for any modulus q of the form $q=p^{2m}+p^m+1$, p prime. Let a_j $(j=0, 1, \ldots, p)$ denote such a sequence of SINGER; we may suppose evidently

$$0 \leqq a_0 < a_1 < \ldots < a_p < q.$$

It follows that for any k $(1 \leqq k < q)$ either k or $k-q$ can be represented in the form $a_i - a_j$, and we may ask which subset of $1, 2, \ldots, q-1$ is represented "actually", i. e. for which k we have $k = a_i - a_j$. This problem, in a somewhat different from, has been raised by L. RÉDEI and is discussed in a joint paper of L. RÉDEI and the author[10]) where the following theorem is proved: If η^* denotes the minimal number of terms of a finite sequence of integers with the property that their differences represent every number $1, 2, \ldots, n$, then

$$\lim_{n\to\infty} \frac{n^*}{\sqrt{n}} = \gamma \tag{44}$$

exists, further we have[11])

$$\sqrt{2+\frac{4}{3\pi}} \leqq \frac{n^*}{\sqrt{n}} \leqq \sqrt{\frac{8}{3}}. \tag{45}$$

Now these problems are also connected with the theory of equidistribution of point sets. To establish this connection, we have to define the "asymmetric distance function" $g(t)$ of a set E as follows:

Let $f(x)$ denote the characteristic function of the set E if x is contained in the interval (0, 1), and let us define $f(x)=0$ for x outside of (0, 1). We put

$$g(t) = \int_0^1 f(x) f(x+t)\, dx \quad (-1 \leqq t \leqq +1). \tag{46}$$

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[10]) To be published in the *Mat. Sbornik.*

[11]) As BÉLA SZ.-NAGY kindly remarked, the lower estimation in (45) can be improved, by some numerical refinement, by approximately 0,01. A similar remark applies to (49). P. ERDŐS and I. S. GÁL proved by some modification of the original proof that (44) and (45) are valid also if the sequence of integers in question is restricted by the condition that it is contained in the sequence $1, 2, \ldots, n$; cf. *Proceedings Koninklijke Nederlandsche Akademie van Wetenschappen,* **51** (1948), pp. 1155—1159.

It is easy to see that $g(t)$ is an even continuous function, further that $g(0)=E$, $g(1)=0$, and we have

(47) $$\int_0^1 g(t)\,dt=\frac{|E|^2}{2}.$$

We obtain further by some simple calculations that

(48) $$\int_0^1 g(t)\cos\lambda t\,dt=\frac{1}{2}\left|\int_0^1 f(x)\exp(i\lambda x)\,dx\right|^2,$$

i. e. that the FOURIER cosine transform of $g(t)$ is non-negative. This is the idea underlying the proof of the following property of the sequences of SINGER:

Theorem 3. *Let us denote $P=p^m$ (p prime), $q=P^2+P+1$ and $k=\frac{q-1}{2}$. If $0\leq a_0<a_1<\ldots<a_p<q$ denotes a SINGER sequence, and if $1\leq A_1<A_2<\ldots<A_k$ denote the numbers which are representable in the form a_i-a_j with $i>j$, further if $A_k=k+D$ (i. e. D denotes how many numbers are missing from the sequence $1,2,\ldots,A_k$) then we have*

(49) $$D>\frac{P^2+1}{3\pi-2}-\frac{P}{2}.$$

Proof. We have

(50) $$\left|\sum_{j=0}^{p}\exp(2\pi i a_j t)\right|^2=P+1+2\sum_{n=1}^{k}\cos 2\pi A_n t=$$
$$=P+\frac{\sin(2A_k+1)\frac{t}{2}}{\sin\frac{t}{2}}-2\sum_{v=1}^{D}\cos 2\pi B_v t,$$

where B_v ($v=1,2,\ldots,D$) denote the numbers $<A_k$ which are not contained in the sequence A_j. It follows from (50) that

(51) $$0\leq P+\frac{\sin(2A_k+1)\frac{t}{2}}{\sin\frac{t}{2}}+2D$$

for all values of t. Let us choose $t=\frac{3\pi}{2A_k+1}$; using $\sin x<x$ for $x>0$. We obtain

(52) $$2D\geq\frac{2(2A_k+1)}{3\pi}-P,$$

from which Theorem 3 follows by simple calculation.

It may be remarked that though (49) is not a best possible estimate, it gives a rather good estimation for small values of P. Thus the set A_j coincides with the set $1, 2, \ldots, k$ only for $P=2$ and $P=3$ (the corresponding SINGER sequences are: $0, 1, 3$ for $P=2$ and $0, 1, 4, 6$ for $P=4$), further (49) asserts that for $P=4$ there must be at least one "gap" in the sequence A_j, and really there is exactly one "gap" if we consider the SINGER sequence $0, 2, 7, 8, 11$. For $P=5$, owing to (49), there must be at least two numbers missing from the sequence A_j, and there are really two gaps if we take the SINGER sequence $0, 1, 4, 10, 12, 17$, etc.

Some further progress could be obtained regarding the problems considered in the present paper if some more difference bases could be constructed. A necessary and sufficient condition however for the existence of a difference basis of order k modulo q, for given k and q, is not known.

We considered only sets E lying in the interval $(0, 1)$, but it is clear that the situation is the same for any bounded linear set. The problem of unbounded linear sets however is somewhat different, as it is shown by the remark, that in this case the symmetric and asymmetric distance functions $G(t)$ and $g(t)$ coincide.

My most sincere thanks are due to P. ERDŐS and L. RÉDEI for their valuable remarks.

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Farey series and their connection with the prime number problem. I.

By MIKLÓS MIKOLÁS in Budapest.

Let $x \geqq 1$; we denote by F_x the ascending sequence of fractions $\frac{k}{n}$ (FAREY series of order x) for which

$$0 < k \leqq n \leqq x, \quad (k, n) = 1.$$

The ν-th term of F_x will be denoted by ϱ_ν; the number of these fractions is

$$\Phi(x) = \sum_{n=1}^{[x]} \varphi(n),$$

$\varphi(n)$ denoting EULER's function.

It is well-known that the so-called FAREY dissection of the continuum is a very important tool in the additive theory of numbers; the sphere of applications extended still more when it was discovered that the equidistribution of the FAREY series is connected with the validity of RIEMANN's hypothesis (i. e. with the assumption that the zeta-function of RIEMANN has no roots for $\Re(s) > \frac{1}{2}$).

In the first place, it has been proved by J. E. LITTLEWOOD[1]) that RIEMANN's hypothesis is true if and only if the relation

$$M(x) = \sum_{n=1}^{[x]} \mu(n) = \sum_{\nu=1}^{\Phi(x)} \cos 2\pi\varrho_\nu = O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where $\mu(n)$ denotes the function of MÖBIUS, holds for all positive values of ε.

The later result of J. FRÁNEL[2]), which is mentioned by E. LANDAU as "ein hübscher Satz", can be expressed as follows:

Let us divide the interval $\langle 0, 1 \rangle$ into $\Phi(x)$ equal parts, furthermore let us mark the fractions of F_x; we then form the sum of squares

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1) [1], 263—266; see LANDAU [2], vol. II, 161—166. (Numbers in brackets [ ] refer to the bibliography placed at the end of the paper.)

2) [1], 198 - 201; see LANDAU [2], vol. II, 167 - 177.

of the differences

$$\delta_\nu = \delta_\nu(x) = \varrho_\nu - \frac{\nu}{\Phi(x)}, \quad \nu = 1, 2, \ldots, \Phi(x).$$

RIEMANN's hypothesis is equivalent to the assertion that

$$\sum_{\nu=1}^{\Phi(x)} \delta_\nu^2 = O(x^{-1+\varepsilon})$$

or (as LANDAU added)[3])

$$\sum_{\nu=1}^{\Phi(x)} |\delta_\nu| = O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

ε being any positive number.

Now, we consider in this paper the asymptotical behaviour for $x \to \infty$ of the sums of type

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu),$$

$f(t)$ denoting a function which is defined at the points $0 < \varrho_\nu \leq 1$ $(\nu = 1, 2, \ldots, \Phi(x))$ and belongs to a class as wide as possible.

By supposing that $f(t)$ is bounded, integrable (in RIEMANN's sense) for $0 \leq t \leq 1$, it follows from the uniform distribution of the fractions of F_x $(x \to \infty)$ that

$$\text{(I)} \qquad \sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) \sim \Phi(x) \int_0^1 f(t)\, dt;$$

in § 1 we shall show that (I) holds certainly if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

exists, or if $f(t)$ is continuous, decreasing, non-negative for $0 < t \leq 1$ and

$$\lim_{\varepsilon \to +0} \int_\varepsilon^1 f(t)\, dt$$

exists.

To find a better bound for the difference

$$R_f(x) = \sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) - \Phi(x) \int_0^1 f(t)\, dt,$$

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[3]) See LANDAU [3], 202—206. We mention herewith a later result of LANDAU the relation

$$\operatorname*{Max}_{1 \leq \xi \leq \Phi(x)} \left| \sum_{\nu=1}^{[\xi]} \delta_\nu \right| = O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

is also equivalent to RIEMANN's hypothesis. ([4], 347—352.)

we may suppose that $f(t)$ has a bounded derivative in the interval $\langle 0, 1\rangle$; it will be proved that in this (particularly important) case the behaviour of $R_f(x)$, concerning its order of magnitude, is the same as that of $M(x)=\sum_{n\leqq x}\mu(n)$ or of the remainder of the prime number theorem

$$\pi(x)-\int_2^x \frac{du}{\log u};$$

we have namely, according to § 2,

$$R_f(x)=O\left(xe^{-c_3(\log x)^\gamma}\right) \tag{II}$$

where γ is any constant between $\frac{1}{2}, \frac{11}{21}$ excl. and $c_3>0$ depends on the choice of γ only; by supposing the validity of RIEMANN's hypothesis, it follows that the much sharper relation

$$R_f(x)=O\left(x^{\frac{1}{2}+c_6\frac{\log\log\log x}{\log\log x}}\right)=O\left(x^{\frac{1}{2}+\varepsilon}\right), \tag{III}$$

c_6 denoting another positive constant[4]), holds also.

For a function $f(t)$ which has continuous derivatives $f'(t), f''(t), \ldots, f^{(2r+1)}(t)$, the "EULER—MACLAURIN sum-formula" furnishes explicitly

$$\begin{aligned}\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) &= \Phi(x)\int_0^1 f(t)\,dt+\frac{1}{2}(f(1)-f(0))+\\ &+\frac{B_2}{2!}(f'(1)-f'(0))\sum_{n=1}^{[x]}\frac{1}{n}M\left(\frac{x}{n}\right)+\\ &+\frac{B_4}{4!}(f'''(1)-f'''(0))\sum_{n=1}^{[x]}\frac{1}{n^3}M\left(\frac{x}{n}\right)+\ldots+\\ &+\frac{B_{2r}}{(2r)!}(f^{(2r-1)}(1)-f^{(2r-1)}(0))\sum_{n=1}^{[x]}\frac{1}{n^{2r-1}}M\left(\frac{x}{n}\right)+U_r(x);\end{aligned} \tag{IV}$$

where

$$U_r^2(x)=\vartheta\frac{B_{4r+2}}{(4r+2)!}\int_0^1 (f^{(2r-1)}(t))^2\,dt\sum_{a,b=1}^{[x]}M\left(\frac{x}{a}\right)M\left(\frac{x}{b}\right)\frac{(a,b)^{4r+2}}{a^{4r+1}b^{4r+1}}$$

$$(0\leqq\vartheta=\vartheta(x,r)\leqq 1),$$

$B_2, B_4, B_6, \ldots$ being the well-known numbers of BERNOULLI.

In case of $r=0$ we have in particular

$$\begin{aligned}\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) &= \Phi(x)\int_0^1 f(t)\,dt+\frac{1}{2}(f(1)-f(0))+\\ &+\sum_{n=1}^{[x]}M\left(\frac{x}{n}\right)\int_0^1\left(nt-[nt]-\frac{1}{2}\right)f'(t)\,dt;\end{aligned} \tag{V}$$

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[4]) Here and later ε denotes an arbitrary positive number.

the last term on the right-hand side may be replaced (on the basis of the so-called FRANEL-identity) by the positive square root of

$$\vartheta \int_0^1 (f'(t))^2 dt \left(\frac{1}{12} + \Phi(x) \sum_{\nu=1}^{\Phi(x)} \delta_\nu^2\right) \quad (0 \leqq \vartheta = \vartheta(x) \leqq 1).$$

We obtain rather deep results by discussing the question as follows (§ 3): when does the converse of (III) hold also, i. e. for which functions $f(t)$, having a bounded derivative in $\langle 0, 1\rangle$, does the validity of RIEMANN's hypothesis follow from the existence of a relation of type (III)?

It can be simply proved:

Let $\lambda \geqq \frac{1}{2}$; RIEMANN's hypothesis is true if and only if we have for every positive ε

$$\sum_{n=1}^{[x]} \frac{1}{n^\lambda} M\left(\frac{x}{n}\right) = O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

By using this lemma, the formulae (IV), (V) and some relations from the theory of DIRICHLET series respectively, we get the following interesting theorems:

1) For any polynomial of second degree

$$f(t) \equiv a_0 t^2 + a_1 t + a_2 \qquad (a_0 \neq 0)$$

the relation

$$R_f(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right), \tag{VI}$$

holding of every $\varepsilon > 0$, is equivalent to RIEMANN's hypothesis.

2) Let us take

$$f(t) \equiv a_0 t^3 + a_1 t^2 + a_2 t + a_3 \qquad (a_0 \neq 0),$$

then a necessary and sufficient condition that (VI) should be equivalent to RIEMANN's hypothesis is that

$$a_1 \neq -\frac{3}{2} a_0.$$

3) For every $r \geqq 2$, there is an infinity of polynomials

$$f(t) \equiv a_0 t^r + a_1 t^{r-1} + \ldots + a_{r-1} t + a_r \qquad (a_0 \neq 0)$$

such that (VI) be equivalent to RIEMANN's hypothesis.

4) Suppose that $f(t)$ is defined and has continuous derivatives $f'(t), f''(t), f'''(t) \neq 0$ for $0 \leqq t \leqq 1$, furthermore the condition

$$\frac{|f'(1) - f'(0)|}{\int_0^1 |f''(t)|\, dt} > \frac{3\zeta(3)}{2\pi} = 0{\cdot}574\ldots$$

should be satisfied. Then (VI) is equivalent to RIEMANN's hypothesis.

In particular, we have the simple relations (found to be equivalent to the hypothesis in question)

$$\sum_{\nu=1}^{\Phi(x)} \varrho_\nu^2 - \frac{\Phi(x)}{3} = O\left(x^{\frac{1}{2}+\varepsilon}\right), \quad \sum_{\nu=1}^{\Phi(x)} \varrho_\nu^3 - \frac{\Phi(x)}{4} = O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

$$\sum_{\nu=1}^{\Phi(x)} \cos\lambda\varrho_\nu - \frac{\sin\lambda}{\lambda}\Phi(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right) \quad \text{with } 0<\lambda\leqq\frac{\pi}{2}.$$

These results show very clearly, how close the connection is between the problem of the distribution of FAREY fractions (i. e. a problem of elementary number theory) and that of the situation of "zeta-roots" in the theory of functions.

1. Uniform distribution and its consequences.

Let x be a positive variable, and let a, b, k, l, m, n, r, ν denote throughout positive integers.

We begin by some fundamental identities.

Lemma 1. *We have for any function $f(t)$ which is defined at the points $t=\frac{k}{n}$ $(k=1, 2, \ldots, n)$*

$$\sum_{\substack{k\leqq n\\(k,n)=1}} f\left(\frac{k}{n}\right) = \sum_{d/n}\mu(d)\sum_{k=1}^{\frac{n}{d}} f\left(d\frac{k}{n}\right) = \sum_{d/n}\mu\left(\frac{n}{d}\right)\sum_{k=1}^{d} f\left(\frac{k}{d}\right),$$

where d/n means that the summation is extended over all divisors of n.

Proof: It is evident that

$$\sum_{d/n}\sum_{\substack{k\leqq d\\(k,d)=1}} f\left(\frac{k}{d}\right) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

and our assertion follows by MÖBIUS inversion.

Lemma 2. *Let $f(n)$ and $g(n)$ be arbitrary arithmetical functions. Then we have*

(1) $$\sum_{n=1}^{[x]}\sum_{d/n} f(d)g\left(\frac{n}{d}\right) = \sum_{d=1}^{[x]} f(d)\sum_{\delta=1}^{\left[\frac{x}{d}\right]} g(\delta) = \sum_{d=1}^{[x]} g(d)\sum_{\delta=1}^{\left[\frac{x}{d}\right]} f(\delta),$$

in particular

(2) $$\sum_{n=1}^{[x]}\sum_{d/n} f(d) = \sum_{d=1}^{[x]}\left[\frac{x}{d}\right] f(d) = \sum_{d=1}^{[x]}\sum_{\delta=1}^{\left[\frac{x}{d}\right]} f(\delta).$$

Proof: We write

$$(3)\quad \sum_{n=1}^{[x]} \sum_{d/n} f(d)\, g\left(\frac{n}{d}\right) = \sum_{\substack{d,\,\delta \\ d\delta \leqq x}} f(d)\, g(\delta) =$$

$$= f(1) \sum_{\delta=1}^{[x]} g(\delta) + f(2) \sum_{\delta=1}^{\left[\frac{x}{2}\right]} g(\delta) + \ldots = \sum_{d=1}^{[x]} f(d) \sum_{\delta=1}^{\left[\frac{x}{d}\right]} g(\delta).$$

On the other hand, by

$$\sum_{d/n} f(d)\, g\left(\frac{n}{d}\right) = \sum_{d/n} f\left(\frac{n}{d}\right) g(d)$$

$f(n)$ and $g(n)$ may be exchanged under (3).

The combination of Lemma 1 and Lemma 2 (with $f(n) \equiv \mu(n)$, $g(n) \equiv \sum_{k=1}^{n} f\left(\frac{k}{n}\right)$) gives

Lemma 3. *Let us take* $M(x) = \sum_{n=1}^{[x]} \mu(n)$, $V(n) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)$, *then*

$$(4)\quad \sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) = \sum_{n=1}^{[x]} M\left(\frac{x}{n}\right) V(n) = \sum_{n=1}^{[x]} \mu(n) \sum_{d=1}^{\left[\frac{x}{n}\right]} V(d),$$

provided that $f(t)$ *is defined for the values in question of its argument.*

We shall see that the identities (4) are useful in order to find asymptotic formulae for $\sum f(\varrho_\nu)$.

Lemma 4. *Let* $0 \leqq \xi \leqq 1$ *and let us denote by* $h(\xi, x)$ *the number of fractions in* F_x *which are not greater than* ξ. *Then we have*

$$h(\xi, x) = \sum_{\varrho_\nu \leqq \xi} 1 = \sum_{n=1}^{[x]} [n\xi]\, M\left(\frac{x}{n}\right) = \sum_{n=1}^{[x]} \mu(n) \sum_{d=1}^{\left[\frac{x}{n}\right]} [d\xi].$$

Proof: Using the fundamental property of the Möbius function we get

$$\sum_{\substack{k \leqq \xi n \\ (k,\, n)=1}} 1 = \sum_{k=1}^{[\xi n]} \sum_{d/(k,\, n)} \mu(d) = \sum_{d/n} \mu(d) \left[\frac{\xi n}{d}\right] = \sum_{d/n} \mu\left(\frac{n}{d}\right) [d\xi],$$

so that (1) furnishes indeed

$$h(\xi, x) = \sum_{n=1}^{[x]} \sum_{d/n} [d\xi]\, \mu\left(\frac{n}{d}\right) = \sum_{d=1}^{[x]} [d\xi]\, M\left(\frac{x}{d}\right) = \sum_{d=1}^{[x]} \mu(d) \sum_{\delta=1}^{\left[\frac{x}{d}\right]} [\delta\xi].$$

In what follows we need also the familiar identities, arising immediately from (1),

(5) $$\sum_{n=1}^{[x]} M\left(\frac{x}{n}\right)=\sum_{n=1}^{[x]} \mu(n)\left[\frac{x}{n}\right]=1,$$

(6) $$\Phi(x)=\sum_{n=1}^{[x]} \varphi(n)=\sum_{n=1}^{[x]} n M\left(\frac{x}{n}\right)=\frac{1}{2} \sum_{n=1}^{[x]} \mu(n)\left[\frac{x}{n}\right]^2+\frac{1}{2}.$$

By use of (6) it is easy to show that[5])

(7) $$\Phi(x)=\frac{3}{\pi^2} x^2+O(x \log x).$$

Next we make use of a well-known result of H. WEYL[6]):

If t_n $(n=1,2,3,\ldots)$ is a sequence such that $(t_n)=t_n-[t_n]$ $(n=1,2,3,\ldots)$ is uniformly distributed in $\langle 0,1\rangle$, then we have for all RIEMANN integrable functions $f(t)$ with the period 1

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(t_k)=\int_0^1 f(t)\, dt, \quad \sum_{k=1}^{n} f(t_k) \sim n \int_0^1 f(t)\, dt.$$

In order to apply this proposition we prove

Theorem 1. *The distribution of F_x becomes uniform when $x\to\infty$.*

Proof: Consider the sequence

(8) $$\frac{1}{1};\ \frac{1}{2};\ \frac{1}{3},\ \frac{2}{3};\ \frac{1}{4},\ \frac{3}{4};\ \frac{1}{5},\ \frac{2}{5},\ \frac{3}{5},\ \frac{4}{5};\ \frac{1}{6},\ \frac{5}{6};\ldots$$

Let $0\leqq \xi_1 \leqq \xi_2 \leqq 1$ and suppose that the denominator of the n-th term under (8) is $[x]+1$. Then we have (cf. (7))

$$n=\Phi(x)+\vartheta x=\frac{3}{\pi^2} x^2+O(x \log x) \qquad (0\leqq \vartheta \leqq 1),$$

on the other hand, the number of fractions among the first n terms which lie in the interval $\langle \xi_1, \xi_2\rangle$ is plainly

$$n_{\xi_1,\xi_2}=n_\xi=h(\xi_2,x)-h(\xi_1 x)+\Theta x \qquad (0\leqq \Theta \leqq 1).$$

Thus, by Lemma 4, (6) and (7), we can write

$$n_\xi=\sum_{d=1}^{[x]}([d\xi_2]-[d\xi_1]) M\left(\frac{x}{d}\right)+\Theta x$$

$$=(\xi_2-\xi_1)\, \Phi(x)+O\left(x \sum_{d=1}^{[x]} \frac{1}{d}\right)+O(x)=(\xi_2-\xi_1) n+O(x \log x)$$

whence it follows indeed that

$$\frac{n_\xi}{n}=(\xi_2-\xi_1)+\frac{O(x \log x)}{n} \to \xi_2-\xi_1$$

when $n\to\infty$, i. e. $x\to\infty$.

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[5]) See e. g. HARDY—WRIGHT [1], 266.

[6]) [1], 314.

The above result of WEYL being applicable to FAREY fractions, we obtain that

$$(9)\qquad \sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) \sim A\,\Phi(x) \sim \frac{3A}{\pi^2} x^2 \quad \text{with} \quad A = \int_0^1 f(t)\,dt,$$

if $f(t)$ is a bounded, RIEMANN integrable function in $\langle 0, 1\rangle$.

The validity of (9) can be proved under more general conditions, by using the following theorem of TOEPLITZ[7]):

Let $t_1, t_2, \ldots, t_n, \ldots$ be a convergent sequence with the limit zero and suppose that the numbers a_{kl} $(k, l = 1, 2, 3, \ldots)$ satisfy the following conditions:

1) for any fixed l, $a_{kl} \to 0$ when $k \to \infty$,

2) $S(k) = |a_{k1}| + |a_{k2}| + \ldots + |a_{kk}| = O(1)$.

Then the sequence

$$t'_k = a_{k1}t_1 + a_{k2}t_2 + \ldots + a_{kk}t_k \qquad (k = 1, 2, \ldots)$$

converges also to zero.

Theorem 2. *Let $f(t)$ be a function defined at all rational points of the interval $0 < t \leqq 1$* [8]*), such that*

$$\frac{V(n)}{n} = \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

converges when $n \to \infty$, and has the limit A. Then we have

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) \sim A\,\Phi(x) \sim \frac{3A}{\pi^2} x^2.$$

Proof: In virtue of (4) and (6) we need only to show that, if our conditions are satisfied, then

$$\frac{1}{\Phi(x)} \sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) - A = \frac{1}{\Phi(x)} \sum_{n=1}^{[x]} n M\left(\frac{x}{n}\right)\left(\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - A\right) \to 0,$$

when $x \to \infty$.

But this follows therefrom that, ex hypothesi,

$$\frac{V(n)}{n} - A \to 0,$$

moreover (cf. (7)),

1) for any fixed n

$$\frac{n\left|M\left(\frac{x}{n}\right)\right|}{\Phi(x)} \leqq \frac{x}{\Phi(x)} \sim \frac{\pi^2}{3x} \to 0 \quad \text{when } x \to \infty,$$

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[7]) See e. g. KNOPP [1], 75.

[8]) Observe that the point 0 does not belong to the interval.

2) $S(x)=\frac{1}{\Phi(x)}\sum_{n=1}^{[x]} n\left|M\left(\frac{x}{n}\right)\right| \leqq \frac{x^2}{\Phi(x)} \sim \frac{\pi^2}{3},$

and so TOEPLITZ's theorem may be applied.

In certain cases it is somewhat more convenient to use the following

Corollary. *If $f(t)$ is continuous, decreasing (if t increases), non-negative for $0<t\leqq 1$, and if*

$$\lim_{\varepsilon\to+\infty}\int_{\varepsilon}^{1} f(t)\,dt=\int_{0}^{1} f(t)dt$$

exists, then we have

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu)\sim\Phi(x)\int_0^1 f(t)\,dt.$$

Proof: Suppose that $f(t)$ satisfies our conditions.

We write

$$\sum_{k=1}^{n} f\left(\frac{k}{n}\right)-\int_1^n f\left(\frac{u}{n}\right)du=\sum_{k=1}^{n-1}\left(f\left(\frac{k}{n}\right)-\int_k^{k+1} f\left(\frac{u}{n}\right)du\right)+f(1)=$$

$$=\sum_{k=1}^{n-1}\int_k^{k+1}\left(f\left(\frac{k}{n}\right)-f\left(\frac{u}{n}\right)\right)du+f(1),$$

and hence, considering that

$$0\leqq\int_k^{k+1}\left(f\left(\frac{k}{n}\right)-f\left(\frac{u}{n}\right)\right)du\leqq f\left(\frac{k}{n}\right)-f\left(\frac{k+1}{n}\right),$$

it follows

$$\sum_{k=1}^{n} f\left(\frac{k}{n}\right)-\int_1^n f\left(\frac{u}{n}\right)du\leqq\sum_{k=1}^{n-1}\left(f\left(\frac{k}{n}\right)-f\left(\frac{k+1}{n}\right)\right)+f(1)=f\left(\frac{1}{n}\right). \tag{10}$$

Since

$$\int_{\frac{1}{n}}^{1} f(t)\,dt=-\int_n^1 f\left(\frac{1}{v}\right)\cdot\frac{1}{v^2}\,dv=\int_1^n\frac{f\left(\frac{1}{v}\right)}{v^2}\,dv,$$

the integral

$$\int_1^\infty\frac{f\left(\frac{1}{v}\right)}{v^2}\,dv$$

exists by hypothesis; there exists therefore to any $\varepsilon>0$ a number

$N = N(\varepsilon)$ such that

$$\varepsilon > \int_n^\infty \frac{f\left(\frac{1}{v}\right)}{v^2} dv \geqq \int_n^{2n} \frac{f\left(\frac{1}{v}\right)}{v^2} dv \geqq n \frac{f\left(\frac{1}{n}\right)}{(2n)^2} = \frac{1}{4} \frac{f\left(\frac{1}{n}\right)}{n},$$

for $n > N$, this implying

(11) $$f\left(\frac{1}{n}\right) = o(n).$$

On the other hand,

$$\int_{\frac{1}{n}}^1 f(t)\, dt = \frac{1}{n} \int_1^n f\left(\frac{u}{n}\right) du;$$

therefore, using (10) and (11), we obtain

$$\sum_{k=1}^n f\left(\frac{k}{n}\right) = n \int_{\frac{1}{n}}^1 f(t)\, dt + \left(\sum_{k=1}^n f\left(\frac{k}{n}\right) - \int_1^n f\left(\frac{u}{n}\right) du\right) = n \int_{\frac{1}{n}}^1 f(t)\, dt + o(n)$$

and so Theorem 3 implies our assertion.

We see that, for the validity of (9), the function $f(t)$ must not necessarily be bounded for $0 \leqq t \leqq 1$.

2. Case of f(t) having a derivative of first or higher order. Connection with Riemann's hypothesis.

Suppose that $f(t)$ has a bounded derivative in the interval $0 \leqq t \leqq 1$, then, applying the mean value theorem of the differential calculus, we can write

(12) $$\sum_{k=1}^n f\left(\frac{k}{n}\right) - n \int_0^1 f(t)\, dt = n \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(f\left(\frac{k}{n}\right) - f(t)\right) dt =$$

$$= n\, O \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(\frac{k}{n} - t\right) dt = O\left(n \sum_{k=1}^n \frac{1}{n^2}\right) = O(1).$$

Hence it follows at once for the remainder (cf. (4), (6))

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) - \Phi(x) \int_0^1 f(t)\, dt = \sum_{n=1}^{[x]} M\left(\frac{x}{n}\right) \left(\sum_{k=1}^n f\left(\frac{k}{n}\right) - n \int_0^1 f(t)\, dt\right) =$$

$$= O \sum_{n=1}^{[x]} \left| M\left(\frac{x}{n}\right) \right| = O(x \log x).$$

This result may be easily improved by using the trivial relation[9])

$$(13)\qquad \sum_{\nu=1}^{\Phi(x)}\left(\varrho_\nu-\frac{\nu}{\Phi}\right)^2=O(1).$$

We get namely, by (12),

$$(14)\qquad \begin{aligned}&\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu)-\Phi(x)\int_0^1 f(t)\,dt=\\ &=\sum_{\nu=1}^{\Phi(x)}\left(f(\varrho_\nu)-f\left(\frac{\nu}{\Phi}\right)\right)+\left(\sum_{\nu=1}^{\Phi(x)} f\left(\frac{\nu}{\Phi}\right)-\Phi(x)\int_0^1 f(t)\,dt\right)=\\ &=O\sum_{\nu=1}^{\Phi(x)}\left|\varrho_\nu-\frac{\nu}{\Phi}\right|+O(1)=O\left(\sqrt{\Phi(x)\sum\left(\varrho_\nu-\frac{\nu}{\Phi}\right)^2}\right)+O(1)=\\ &=O\left(x\sqrt{\sum\left(\varrho_\nu-\frac{\nu}{\Phi}\right)^2}\right)+O(1),\end{aligned}$$

so that (13) furnishes immediately

$$(15)\qquad \sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu)-\Phi(x)\int_0^1 f(t)\,dt=O(x).$$

(13) represented hitherto, to our knowledge, the sharpest (positive) result concerning the order of FRANEL's sum. In a recent paper[10]) we have deduced on the basis of the so-called FRANEL identity[11])

$$(16)\qquad \sum_{\nu=1}^{\Phi(x)}\left(\varrho_\nu-\frac{\nu}{\Phi}\right)^2=\frac{1}{12\,\Phi(x)}\left\{\sum_{a=1}^{[x]}\sum_{b=1}^{[x]} M\left(\frac{x}{a}\right)M\left(\frac{x}{b}\right)\frac{(a,b)^2}{ab}-1\right\}$$

the much better relation

$$(17)\qquad \sum_{\nu=1}^{\Phi(x)}\left(\varrho_\nu-\frac{\nu}{\Phi}\right)^2=O\left(e^{-c_0(\log x)^\gamma}\right),$$

by using N. TCHUDAKOV's result[12]) on the error term of the prime number theorem

$$(18)\qquad \pi(x)-\int_2^x\frac{du}{\log u}=O\left(xe^{-c_1(\log x)^\gamma}\right).$$

More precisely we use its analogue for the MÖBIUS function[13])

$$(19)\qquad M(x)=O\left(xe^{-c_2(\log x)^\gamma}\right).$$

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9) See LANDAU [3], or [2], vol. II, 176.

10) MIKOLÁS [1].

11) FRANEL [1], LANDAU [2], vol. II, 173.

12) TCHUDAKOV [1], 591–602.

13) See FOGELS [1].

(Here γ denotes any constant between $\frac{1}{2}$ and $\frac{11}{21}$ excl., while c_1, c_2, c_3 are positive constants depending on the choice of γ only.)

On the other hand, in case of the validity of RIEMANN's hypothesis, we have the well-known relations, the first implying the second,

$$(20)\qquad M(x)=O\left(x^{\frac{1}{2}+c_4\frac{\log\log\log x}{\log\log x}}\right),$$

$$(21)\qquad \sum_{\nu=1}^{\Phi(x)}\left(\varrho_\nu-\frac{\nu}{\Phi}\right)^2=O\left(x^{-1+c_5\frac{\log\log\log x}{\log\log x}}\right),$$

where c_4, c_5 denote other positive constants[14]).

Thus, applying (17) and (21), we obtain at once from (14)

Theorem 3. *Let $f(t)$ be a function having a bounded derivative in the interval $0\leq t\leq 1$. Then we have the relation*

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu)=\Phi(x)\int_0^1 f(t)\,dt+O(xe^{-c_3(\log x)^\gamma})$$

with $\frac{1}{2}<\gamma<\frac{11}{21}$, $c_3=c_3(\gamma)>0$.

If RIEMANN's *hypothesis is true, we have besides*

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu)=\Phi(x)\int_0^1 f(t)\,dt+O\left(x^{\frac{1}{2}+c_6\frac{\log\log\log x}{\log\log x}}\right),$$

and therefore

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu)=\Phi(x)\int_0^1 f(t)\,dt+O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

for every $\varepsilon>0$.

All these relations may be immediately deduced on the basis of the identities (4), without using FRANEL's sum $\sum\left(\varrho_\nu-\frac{\nu}{\Phi}\right)^2$, if we suppose that $f'(t)$ exists and is *continuous* for $0\leq t\leq 1$. In this case we must only apply the EULER—MACLAURIN sum-formula in its simplest form:

$$(22)\qquad \sum_{k=1}^{n} g(k)=\int_0^n g(u)\,du+\frac{1}{2}(g(n)-g(0))+\int_0^n\left(u-[u]-\frac{1}{2}\right)g'(u)\,du$$

for the function $g(u)=f\left(\frac{u}{n}\right)$; we thus obtain (cf. (4), (5), (6))

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14) See e. g. LANDAU [2], vol. II, 161—166, 176—177.

$$(23)\quad \sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) = \Phi(x)\int_0^1 f(t)\,dt + \frac{1}{2}(f(1)-f(0)) + \\ + \sum_{n=1}^{[x]} M\left(\frac{x}{n}\right)\int_0^1 \left(nt-[nt]-\frac{1}{2}\right) f'(t)\,dt.$$

Since for any a, b[15])

$$(24)\quad \int_0^1 \left(at-[at]-\frac{1}{2}\right)\left(bt-[bt]-\frac{1}{2}\right)dt = \frac{(a,b)^2}{12ab},$$

we obtain by inverting the order of summation and integration, and by applying the inequality of SCHWARTZ, that the last term under (23) may be replaced by the positive square root of

$$(25)\quad \frac{\vartheta}{12}\int_0^1 (f'(t))^2\,dt \cdot \sum_{a,b=1}^{[x]} M\left(\frac{x}{a}\right) M\left(\frac{x}{b}\right)\frac{(a,b)^2}{ab} \quad (0 \leqq \vartheta = \vartheta(x) \leqq 1),$$

which, according to (16), is equal to

$$(26)\quad \vartheta \int_0^1 (f'(t))^2\,dt \cdot \left\{\frac{1}{12} + \Phi(x)\sum_{\nu=1}^{\Phi(x)}\left(\varrho_\nu - \frac{\nu}{\Phi}\right)^2\right\}.$$

The form (25) renders possible, by (19) and (20), an immediate estimation of the remainder

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) - \Phi(x)\int_0^1 f(t)\,dt,$$

while (26) shows the connection with FRANEL's theorem.

For a function $g(u)$ which has continuous derivatives $g'(u), g''(u), \ldots, g^{(2r+1)}(u)$ $(r \geqq 0)$ in the interval $1 \leqq u \leqq n$, the general form of the EULER—MACLAURIN sum-formula[16])

$$(27)\quad \sum_{k=1}^{n} g(k) = \int_0^n g(u)\,du + \frac{1}{2}(g(n)-g(0)) + \\ + \sum_{l=1}^{r} \frac{B_{2l}}{(2l)!}\left(g^{(2l-1)}(n) - g^{(2l-1)}(0)\right) + \int_0^n P_{2r+1}(u)\, g^{(2r+1)}(u)\,du$$

is valid, where $B_2, B_4, \ldots$ are the so-called Bernoullian numbers[17]), defined by

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15) LANDAU [2], vol. II, 170.

16) See e. g. KNOPP [1] 542.

17) We have $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_3 = B_5 = B_7 = \ldots = 0$.

$$\binom{m}{0}B_0+\binom{m}{1}B_1+\ldots+\binom{m}{m-1}B_{m-1}=0 \quad (m=1,2,3,\ldots)$$

while the functions with period 1

(28) $$P_{2r+1}(u)=(-1)^{r-1}\sum_{k=1}^{\infty}\frac{2\sin 2k\pi u}{(2k\pi)^{2r+1}} \quad (r=0,1,2,\ldots)$$

are identical for $0\leqq u<1$ to the corresponding Bernoullian polynomials.

Taking again $g(u)=f\left(\frac{u}{n}\right)$ in (27), and using the identities (4), (5), (6) respectively, we obtain after simple integral-transformations

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu)=\Phi(x)\int_0^1 f(t)\,dt+\frac{1}{2}(f(1)-f(0))+$$

$$+\frac{B_2}{2!}(f'(1)-f'(0))\sum_{n=1}^{[x]}\frac{1}{n}M\left(\frac{x}{n}\right)+$$

(29) $$+\frac{B_4}{4!}(f'''(1)-f'''(0))\sum_{n=1}^{[x]}\frac{1}{n^3}M\left(\frac{x}{n}\right)+\ldots+$$

$$+\frac{B_{2r}}{(2r)!}(f^{(2r-1)}(1)-f^{(2r-1)}(0))\sum_{n=1}^{[x]}\frac{1}{n^{2r-1}}M\left(\frac{x}{n}\right)+$$

$$+\sum_{n=1}^{[x]}\frac{1}{n^{2r}}M\left(\frac{x}{n}\right)\int_0^1 P_{2r+1}(nt).f^{(2r+1)}(t)\,dt,$$

supposing, of course, that each derivatives of $f(t)$ which occur here exist, and that $f^{(2r+1)}(t)$ is continuous for $0\leqq t\leqq 1$.

To deduce another form for the last (remainder) term under (29) we use

Lemma 5. *Let λ be a real number not less than* 1. *Then, taking*

$$p_\lambda(u)=\sum_{n=1}^{\infty}\frac{\sin 2n\pi u}{n^\lambda},$$

we have

$$\int_0^1 p_\lambda(at)\,p_\lambda(bt)\,dt=\frac{\zeta(2\lambda)}{2}\cdot\frac{(a,b)^{2\lambda}}{a^\lambda b^\lambda}.$$

Proof: It follows from PARSEVAL's theorem, using also the fact that positive solutions of the Diophantine equation $au=bv$ (for fixed a, b) are

$$u=k\frac{b}{(a,b)},\quad v=k\frac{a}{(a,b)} \qquad (k=1,2,\ldots).$$

Now, applying Lemma 5 and the inequality of SCHWARTZ, we obtain (cf. (28))

$$U_r^2(x) = \left\{ \int_0^1 f^{(2r+1)}(t) . \left(\sum_{n=1}^{[x]} \frac{1}{n^{2r}} M\left(\frac{x}{n}\right) P_{2r+1}(nt) \right) dt \right\}^2 \leqq$$

$$\leqq I_r^2 . \int_0^1 \left\{ \sum_{n=1}^{[x]} \frac{1}{n^{2r}} M\left(\frac{x}{n}\right) P_{2r+1}(nt) \right\}^2 dt =$$

$$= I_r^2 . \sum_{a=1}^{[x]} \sum_{b=1}^{[x]} \frac{1}{a^{2r} b^{2r}} M\left(\frac{x}{a}\right) M\left(\frac{x}{b}\right) . \int_0^1 P_{2r+1}(at) P_{2r+1}(bt)\, dt =$$

$$= I_r^2 \frac{2\zeta(4r+2)}{(2\pi)^{4r+2}} \sum_{a,\, b=1}^{[x]} M\left(\frac{x}{a}\right) M\left(\frac{x}{b}\right) \frac{(a,\, b)^{4r+2}}{a^{4r+1}\, b^{4r+1}},$$

where

$$I_r^2 = \int_0^1 (f^{(2r+1)}(t))^2\, dt.$$

Considering that, for positive integral values of λ,

$$\zeta(2\lambda) = \sum_{n=1}^{\infty} \frac{1}{n^{2\lambda}} = (-1)^{\lambda-1} \frac{B_{2\lambda}(2\pi)^{2\lambda}}{2(2\lambda)!},$$

we see that the square of the remainder term under (29) may be written in the form

$$(31) \quad U_r^2(x) = \vartheta \frac{B_{4r+2}}{(4r+2)!} . \int_0^1 (f^{(2r+1)}(t))^2\, dt . \sum_{a,\, b=1}^{[x]} M\left(\frac{x}{a}\right) M\left(\frac{x}{b}\right) \frac{(a,\, b)^{4r+2}}{a^{4r+1}\, b^{4r+1}}$$

with $0 \leqq \vartheta = \vartheta(x, r) \leqq 1$, which is an immediate generalization of the expressions (25) and (26) respectively.

We draw the attention to the well-estimable sums depending upon x only, which occur on the right-hand side of (29) so to say as weights; in what follows this fact makes mainly the formula useful. If $f(t)$ is a polynomial of degree $2r$ or $2r+1$, $U_r(x)$ vanishes; for example, taking $f(t) \equiv t, t^2, t^3$ resp. we obtain

$$(32) \qquad \sum_{\nu=1}^{\Phi(x)} \varrho_\nu = \frac{1}{2} \Phi(x) + \frac{1}{2},$$

$$(33) \qquad \sum_{\nu=1}^{\Phi(x)} \varrho_\nu^2 = \frac{1}{3} \Phi(x) + \frac{1}{2} + \frac{1}{6} \sum_{n=1}^{[x]} \frac{1}{n} M\left(\frac{x}{n}\right),$$

$$(34) \qquad \sum_{\nu=1}^{\Phi(x)} \varrho_\nu^3 = \frac{1}{4} \Phi(x) + \frac{1}{2} + \frac{1}{4} \sum_{n=1}^{[x]} \frac{1}{n} M\left(\frac{x}{n}\right).$$

3. The "problem of equivalence".

Suppose that RIEMANN's hypothesis is true, then, according to Theorem 3, the remainder term $\sum f(\varrho_\nu) - \Phi(x)\int_0^1 f(t)\,dt$ is $O\left(x^{\frac{1}{2}+\varepsilon}\right)$ for any function in question (i. e. for $f(t)$ having a bounded derivative in $\langle 0, 1\rangle$).

Taking $f(t) \equiv \cos 2\pi t$, we get from

$$\sum_{\substack{k \leq n \\ (k, n)=1}} e^{\frac{2k\pi i}{n}} = \sum_{\substack{k \leq n \\ (k, n)=1}} \cos \frac{2k\pi}{n} = \mu(n) \text{ [18]}$$

the identities

$$\sum f(\varrho_\nu) - \Phi(x)\int_0^1 f(t)\,dt = \sum f(\varrho_\nu) = \sum_{n=1}^{[x]} \mu(n) = M(x),$$

so that, in virtue of LITTLEWOOD's theorem, the converse of our above proposition is also true in this case: if $f(t) \equiv \cos 2\pi t$, the relation

$$\sum f(\varrho_\nu) - \Phi(x)\int_0^1 f(t)\,dt = O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

implies the validity of RIEMANN's hypothesis.

On the other hand, it is evident that such a converse proposition does not hold for *all* $f(t)$ in question; for example, if $f(t)$ is a function for wich $f(t) = -f(1-t)$ when $0 \leq t \leq 1$, then we have (with regard to $\varrho_\nu = 1 - \varrho_{\Phi-\nu}$; $\nu = 1, 2, \ldots, \Phi(x) - 1$)

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) - \Phi(x)\int_0^1 f(t)\,dt \equiv f(1),$$

independently of RIEMANN's hypothesis.

Therefore it may be raised the question: which conditions must be satisfied by a function $f(t)$ (having a bounded derivative for $0 \leq t \leq 1$), in order that the following assertion be true: "the relation

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) - \Phi(x)\int_0^1 f(t)\,dt = O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

holds for every ε if and *only if* RIEMANN's hypothesis is valid".

We shall see that this probiem — it may be called "problem of equivalence" since we look for relations which are equivalent to RIEMANN's

[18]) See e. g. LARDAU [1], vol. II, 572–573; [2], vol. I, 188.

hypothesis — is not easy to handle in full generality; we get, however, interesting special results.

In what follows a_n, b_n, c_n denote real numbers, $s = \sigma + i\tau$ is a complex variable, so that $\sigma = \Re(s)$, $\tau = \Im(s)$.

Next we need two well-known propositions from the theory of DIRICHLET series[19]).

Lemma 6. *If*

$$S(x) = \sum_{k=1}^{[x]} a_k = O(x^{\alpha+\varepsilon})$$

for every $\varepsilon > 0$, *then the series* $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *converges in the half-plane* $\sigma > \alpha$ *and represents here a regular function of* s.

Lemma 7. *If for* $\sigma > \sigma_0$

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = f(s) \quad and \quad \sum_{n=1}^{\infty} \frac{b_n}{n^s} = g(s)$$

are absolutely convergent, then the series with the coefficients

$$c_n = \sum_{d/n} a_n b_{\frac{n}{\alpha}} \qquad (n = 1, 2, \ldots)$$

converges also absolutely in the half-plane $\sigma > \sigma_0$, *and one has here*

$$\sum_{n=1}^{\infty} \frac{c_n}{n^s} = f(s) . g(s). \tag{35}$$

Theorem 4. *Let* $f(t)$ *be a function having a bounded derivative in* $\langle 0, 1 \rangle$. *If the relation*

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) - \Phi(x) \int_0^1 f(t)\,dt = O\left(x^{\frac{1}{2}+\varepsilon}\right) \tag{36}$$

holds for every $\varepsilon > 0$, *then*

1) *the function* $F(s)$, *defined for* $\sigma > 1$ *by*

$$F(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \left(\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - n \int_0^1 f(t)\,dt \right), \tag{37}$$

is regular for $\sigma > \frac{1}{2}$, $s \neq 1$;

2) $\zeta(s)$ *cannot vanish in the half-plane* $\sigma > \frac{1}{2}$, *unless at points where* $F(s) = 0$.

19) See e. g. LANDAU [1], vol. I, 121, 131.

Proof: On the basis of Lemma 1 and of the well-known identities

(38) $$\varphi(n)=n\sum_{d/n}\frac{\mu(d)}{d}=\sum_{d/n}d\mu\left(\frac{n}{d}\right)$$

we can write

$$\sum_{\substack{k\leq n\\(k,n)=1}}f\left(\frac{k}{n}\right)-\varphi(n)\int_0^1 f(t)\,dt=\sum_{d/n}\mu\left(\frac{n}{d}\right)\left(\sum_{k=1}^{d}f\left(\frac{k}{d}\right)-d\int_0^1 f(t)\,dt\right)$$

so that (35) gives "formally"

(39) $$\sum_{n=1}^{\infty}\frac{\sum_{(k,n)=1}f\left(\frac{k}{n}\right)-\varphi(n)\int_0^1 f(t)\,dt}{n^s}=\left(\sum_{n=1}^{\infty}\frac{\sum_{k=1}^{n}f\left(\frac{k}{n}\right)-n\int_0^1 f(t)\,dt}{n^s}\right)\left(\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\right);$$

since the series on the right-hand side are plainly, by (12) and $|\mu(n)|\leq 1$, absolutely convergent for $\sigma>1$, (39) holds, according to Lemma 7, in this half-plane.

Considering that for $\sigma>1$

(40) $$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}=\frac{1}{\zeta(s)},$$

we have the equality

(41) $$F(s)=\zeta(s)\cdot\sum_{n=1}^{\infty}\frac{1}{n^s}\left(\sum_{\substack{k\leq 1\\(k,n)=1}}f\left(\frac{k}{n}\right)-\varphi(n)\int_0^1 f(t)\,dt\right).$$

Now suppose that (36) is valid. Then the series on the right-hand side of (41) is convergent and regular for $\sigma>\frac{1}{2}$, by virtue of Lemma 6. On the other hand, as is well-known, $\zeta(s)$ is regular in the whole plane except at $s=1$.

Thus the function

$$\zeta(s)\cdot\sum_{n=1}^{\infty}\frac{1}{n^s}\left(\sum_{\substack{k\leq n\\(k,n)=1}}f\left(\frac{k}{n}\right)-\varphi(n)\int_0^1 f(t)\,dt\right),$$

which represents, according to (41), the analytical continuation of

$$F(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}\left(\sum_{k=1}^{\infty}f\left(\frac{k}{n}\right)-n\int_0^1 f(t)\,dt\right)\quad(\sigma>1),$$

is regular for $\sigma>\frac{1}{2}$, except possibly at $s=1$; this function vanishes, of course, at all points of the half-plane $\sigma>\frac{1}{2}$, where $\zeta(s)=0$.

We add an important

Corollary. *Let $f(t)$ denote a function having a bounded derivative for $0 \leqq t \leqq 1$, and such that $F(s)$ is regular and has no zeros for $\sigma > \frac{1}{2}$.*

Then the relation (36) involves the validity of RIEMANN's *hypothesis.*

This result suggests to find a $f(t)$, for which we can show the regularity and not-vanishing of $F(s)$ in the half-plane $\sigma > \frac{1}{2}$. In this direction we make good use of the formulae (33), (29), and the well-known fact that, if $\lambda \geqq \frac{1}{2}$,

$$\zeta(s+\lambda) \neq 0 \tag{42}$$

for $\sigma \geqq \frac{1}{2}$.[20])

Lemma 8. *Let λ be a real number not less than $\frac{1}{2}$.* RIEMANN's *hypothesis is true if and only if we have for every positive ε*

$$\sum_{n=1}^{[x]} \frac{1}{n^\lambda} M\left(\frac{x}{n}\right) = O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

Proof: 1. The first part of our proposition is a trivial consequence of LITTLEWOOD's theorem: assuming the validity of RIEMANN's hypothesis, we have $M(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right)$, and therefore

$$\sum_{n=1}^{[x]} \frac{1}{n^\lambda} M\left(\frac{x}{n}\right) = O\left(x^{\frac{1}{2}+\varepsilon} \sum_{n=1}^{[x]} \frac{1}{n^{\lambda+\frac{1}{2}+\varepsilon}}\right) = O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

2. Suppose that the relation

$$\sum_{n \leqq x} \frac{1}{n^\lambda} M\left(\frac{x}{n}\right) = O\left(x^{\frac{1}{2}+\varepsilon}\right) \tag{43}$$

holds for every positive ε.

Using Lemma 7 and (40), we find

$$\sum_{n=1}^{\infty} \frac{\sum_{d|n} \mu\left(\frac{n}{d}\right) \frac{1}{d^\lambda}}{n^s} = \left(\sum_{n=1}^{\infty} \frac{1}{n^\lambda n^s}\right)\left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right) = \frac{\zeta(s+\lambda)}{\zeta(s)} \quad (\sigma > 1). \tag{44}$$

The series on the left-hand side is, by Lemma 6 and (43) (cf. (1)), convergent and regular for $\sigma > \frac{1}{2}$; as $\zeta(s+\lambda)$ is also regular and has no zeros here, (44) implies that $\zeta(s) \neq 0$ in the half-plane $\sigma > \frac{1}{2}$.

[20]) This follows from the not-vanishing of $\zeta(s)$ for $\sigma \geqq 1$. (See e. g. LANDAU [1], vol. I, 154, 166.)

Theorem 5. 1) *For every polynomial of second degree*

$$f(t) \equiv a_0 t^2 + a_1 t + a_2 \qquad (a_0 \neq 0)$$

the relation (36) *is equivalent to* RIEMANN's *hypothesis.*

2) *Let*

$$f(t) \equiv a_0 t^3 + a_1 t^2 + a_2 t + a_3 \qquad (a_0 \neq 0),$$

then, in order that (36) *should be equivalent to* RIEMANN's *hypothesis, it is necessary and sufficient that*

$$a_1 \neq -\frac{3}{2} a_0.$$

Corollary. RIEMANN's *hypothesis is true if and only if we have for every positive* ε

$$\sum_{\nu=1}^{\Phi(x)} \varrho_\nu^2 - \frac{\Phi(x)}{3} = O\left(x^{\frac{1}{2}+\varepsilon}\right) \quad or \quad \sum_{\nu=1}^{\Phi(x)} \varrho_\nu^3 - \frac{\Phi(x)}{4} = O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

Proof: Applying the identities (32), (33), (34), we obtain

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) - \Phi(x) \int_0^1 f(t)\,dt = \frac{a_0 + a_1}{2} + \frac{a_0}{6} \sum_{n=1}^{[x]} \frac{1}{n} M\left(\frac{x}{n}\right)$$

in the first,

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) - \Phi(x) \int_0^1 f(t)\,dt = \frac{a_0 + a_1 + a_2}{2} + \left(\frac{a_0}{4} + \frac{a_1}{6}\right) \sum_{n=1}^{[x]} \frac{1}{n} M\left(\frac{x}{n}\right)$$

in the second case, so that Lemma 8 involves immediately our assertions.

Theorem 6. *Whatever be* r *except* 1, *there is an infinity of polynomials*

$$f(t) \equiv a_0 t^r + a_1 t^{r-1} + \ldots + a_{r-1} t + a_r \qquad (a_0 \neq 0)$$

such that (36) *is equivalent to* RIEMANN's *hypothesis.*

Proof: By the theorem just proved, we may suppose that $r \geqq 4$. The application of (28) gives now

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) - \Phi(x) \int_0^1 f(t)\,dt = \frac{1}{2}(a_0 + a_1 + \ldots + a_{r-1}) + \tag{45}$$

$$+ \frac{B_2}{2}\left(a_0 \binom{r}{1} + a_1 \binom{r-1}{1} + \ldots + a_{r-2}\binom{2}{1}\right) \sum_{n=1}^{[x]} \frac{1}{n} M\left(\frac{x}{n}\right) +$$

$$+ \frac{B_4}{4}\left(a_0 \binom{r}{3} + a_1 \binom{r-1}{3} + \ldots + a_{r-4}\binom{4}{3}\right) \sum_{n=1}^{[x]} \frac{1}{n^3} M\left(\frac{x}{n}\right) + \ldots$$

The last term on the right-hand side is

$$\begin{cases} \dfrac{B_r}{r} a_0 \dbinom{r}{r-1} \displaystyle\sum_{n=1}^{[x]} \frac{1}{n^{r-1}} M\left(\frac{x}{n}\right) & \text{if } r \text{ is even,} \\ \dfrac{B_{r-1}}{r-1} \left(a_0 \dbinom{r}{r-2} + a_1 \dbinom{r-1}{r-2}\right) \displaystyle\sum_{n=1}^{[x]} \frac{1}{n^{r-2}} M\left(\frac{x}{n}\right) & \text{if } r \text{ is odd.} \end{cases}$$

The coefficients $a_0, a_1, \ldots, a_{r-4}$ can plainly be chosen so that the system of equations

$$\binom{r}{1} + \binom{r-1}{1}\xi_1 + \ldots + \binom{4}{1}\xi_{r-4} + \binom{3}{1}\xi_{r-3} + \binom{2}{1}\xi_{r-2} = 0,$$

$$\binom{r}{3} + \binom{r-1}{3}\xi_1 + \ldots + \binom{4}{3}\xi_{r-4} = 0,$$

(46) $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(finally:)
$$\begin{cases} \dbinom{r}{r-3} + \dbinom{r-1}{r-3}\xi_1 + \dbinom{r-2}{r-3}\xi_2 = 0, & \text{if } r \text{ is even,} \\ \dbinom{r}{r-4} + \dbinom{r-1}{r-4}\xi_1 + \dbinom{r-2}{r-4}\xi_2 + \dbinom{r-3}{r-4}\xi_3 = 0, & \text{if } r \text{ is odd,} \end{cases}$$

where $\xi_k = \dfrac{a_k}{a_0}$ $(k = 1, 2, \ldots, r-2)$, should be satisfied; when r is odd, let besides be

$$\xi_1 \neq -\frac{3}{2}.$$

Since we have $\left[\dfrac{r}{2} - 1\right]$ equations for $(r-2)$ unknowns and $\left[\dfrac{r}{2} - 1\right] < r-2$ if $r \geqq 4$, the number of solutions of (46) is infinite for every degree $r \geqq 4$.

If all our conditions are fulfilled, then each term on the right-hand side of (45) except the last vanishes, so that the proposition follows at once from Lemma 8.

Theorem 7. *Let $f(t)$ be a function such that $f'(t), f''(t), f'''(t) \neq 0$ exist, $f'''(t)$ is continuous for $0 \leqq t \leqq 1$, and that the condition*

$$\frac{|f'(1) - f'(0)|}{\int_0^1 |f'''(t)|\, dt} > \frac{3\zeta(3)}{2\pi} = 0{\cdot}574\ldots$$

is fulfilled. Then (36) *is equivalent to* RIEMANN's *hypothesis.*

Proof: Suppose that $f(t)$ is a function satisfying our conditions.

1. If RIEMANN's hypothesis is true, then we have, by virtue of Theorem 5,

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) = \Phi(x) \int_0^1 f(t)\, dt + O\left(x^{\frac{1}{2}+\varepsilon}\right) \qquad (\varepsilon > 0).$$

2. Assume that the relation (36) holds (for every positive ε). The use of (22) (cf. (30)) shows that

$$\sum_{k=1}^{n} f\left(\frac{k}{n}\right)=\int_0^n f\left(\frac{u}{n}\right)du+\frac{1}{2}(f(1)-f(0))+\frac{1}{n}\int_0^n P_1(u)f'\left(\frac{u}{n}\right)du=$$

$$=n\int_0^1 f(t)\,dt+\frac{1}{2}(f(1)-f(0))+\int_0^1 P_1(nt)f'(t)\,dt,$$

and so, by (39) and (40), we can write for $\sigma>1$

(47) $$\frac{1}{\zeta(s)}\sum_{n=1}^{\infty}\frac{1}{n^s}\int_0^1 P_1(nt)f'(t)\,dt=$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^s}\left(\sum_{\substack{k\leqq n\\(k,n)=1}} f\left(\frac{k}{n}\right)-\varphi(n)\int_0^1 f(t)\,dt\right)-\frac{1}{2}(f(1)-f(0)).$$

Consider the function with the period 1

(48) $$P_2(u)=\frac{1}{2}\sum_{k=1}^{\infty}\frac{\cos 2k\pi u}{(k\pi)^2};$$

it is continuous everywhere, and we have plainly

$$P_2'(u)=P_1(u)$$

if u is not an integer.

Now, the series on the right-hand side of (47) is, by hypothesis (36), regular for $\sigma>\frac{1}{2}$ (cf. Lemma 6.); the series on the left (converges and so) is regular in the half-plane $\sigma>0$ by

(49) $$\int_0^1 P_1(nt)f'(t)\,dt=\frac{1}{12n}(f'(1)-f'(0))-\frac{1}{n}\int_0^1 P_2(nt)f''(t)\,dt\leqq$$

$$\leqq\frac{1}{12n}\left(|f'(1)-f'(0)|+\int_0^1|f''(t)|\,dt\right)=O\left(\frac{1}{n}\right).$$

Consequently, if we have for $\sigma>\frac{1}{2}$

(50) $$\sum_{n=1}^{\infty}\frac{1}{n^s}\int_0^1 P_1(nt)f'(t)\,dt\neq 0,$$

then it follows that $\zeta(s)\neq 0$ in this half-plane.

To verify (50), we write using (49)

$$(51)\quad \sum_{n=1}^{\infty}\frac{1}{n^s}\int_0^1 P_1(nt)f'(t)\,dt=\frac{1}{12}(f'(1)-f'(0))\,\zeta(s+1)-$$

$$-\sum_{n=1}^{\infty}\frac{1}{n^{s+1}}\int_0^1 P_2(nt)f''(t)\,dt=\zeta(s+1)\left\{\frac{1}{12}(f'(1)-f'(0))-\sum_{n=1}^{\infty}\frac{b_n}{n^{s+1}}\right\}.$$

Here the coefficients b_n can be easily determined by the condition (cf. (40)).

$$\sum_{n=1}^{\infty}\frac{b_n}{n^{s+1}}=\left(\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{s+1}}\right)\cdot\left(\sum_{n=1}^{\infty}\frac{1}{n^{s+1}}\int_0^1 P_2(nt)f''(t)\,dt\right);$$

we have namely, by virtue of Lemma 7,

$$b_n=\sum_{\delta/n}\mu\left(\frac{n}{\delta}\right)\int_0^1 P_2(\delta t)f''(t)\,dt.$$

(Both series on the right-hand side, and so their product too, converge absolutely for $\sigma>0$.)

Partial integration shows that (cf. (28))

$$\int_0^1 P_2(nt)f''(t)\,dt=-\frac{1}{n}\int_0^1 P_3(nt)f'''(t)\,dt,$$

and, applying (2), we get

$$B(u)=\sum_{n=1}^{[u]}|b_n|\leqq\sum_{n=1}^{[u]}\sum_{\delta/n}\left|\int_0^1 P_2(\delta t)f''(t)\,dt\right|=$$

$$=\sum_{n=1}^{[u]}\left[\frac{u}{n}\right]\cdot\frac{1}{n}\left|\int_0^1 P_3(nt)f'''(t)\,dt\right|\leqq u\sum_{n=1}^{[u]}\frac{1}{n^2}\int_0^1|P_3(nt)|\,|f'''(t)|\,dt<$$

$$<u\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sum_{k=1}^{\infty}\frac{2}{(2k\pi)^3}\right)\cdot\int_0^1|f'''(t)|\,dt=\frac{\zeta(3)}{24\pi}u\cdot\int_0^1|f'''(t)|\,dt.$$

On the other hand, by

$$\sum_{n=1}^{[v]}\frac{|b_n|}{n^{\sigma+1}}=\sum_{n=1}^{[v]}B(n)\left(\frac{1}{n^{\sigma+1}}-\frac{1}{(n+1)^{\sigma+1}}\right)+\frac{B(v)}{([v]+1)^{\sigma+1}},$$

and $B(v)=O(v)$, it follows for $\sigma>0$

$$\sum_{n=1}^{\infty}\frac{|b_n|}{n^{\sigma+1}}=\sum_{n=1}^{\infty}B(n)\cdot(\sigma+1)\int_n^{n+1}\frac{du}{u^{\sigma+2}}=(\sigma+1)\int_1^{\infty}\frac{B(u)}{u^{\sigma+2}}\,du,$$

so that we can write

$$\left|\sum_{n=1}^{\infty}\frac{b_n}{n^{s+1}}\right|\leqq\sum_{n=1}^{\infty}\frac{|b_n|}{n^{\sigma+1}}=$$

$$(52)\qquad =(\sigma+1)\int_1^\infty\frac{B(u)}{u^{\sigma+2}}\,du<(\sigma+1)\frac{\zeta(3)}{24\pi}\left(\int_0^1|f'''(t)|\,dt\right)\cdot\int_1^\infty\frac{du}{u^{\sigma+1}}=$$

$$=\left(1+\frac{1}{\sigma}\right)\frac{\zeta(3)}{24\pi}\int_0^1|f'''(t)|\,dt\qquad(\sigma>0).$$

Finally, using (51) and (52), we obtain for $\sigma>\frac{1}{2}$ (cf. (42))

$$\left|\sum_{n=1}^{\infty}\frac{1}{n^s}\int_0^1 P_1(nt)f'(t)\,dt\right|\geqq|\zeta(s+1)|\left(\frac{1}{12}|f'(1)-f'(0)|-\left|\sum_{n=1}^{\infty}\frac{b_n}{n^{s+1}}\right|\right)>$$

$$>\frac{1}{12}|\zeta(s+1)|\left(|f'(1)-f'(0)|-\left(1+\frac{1}{\sigma}\right)\frac{\zeta(3)}{2\pi}\int_0^1|f'''(t)|\,dt\right)>$$

$$>\frac{1}{12}|\zeta(s+1)|\left(|f'(1)-f'(0)|-\frac{3\zeta(3)}{2\pi}\int_0^1|f'''(t)|\,dt\right)>0,$$

which proves (50), and thus our Theorem.

It is easy to find such functions $f(t)$ for which the conditious of the proposition just proved are fulfilled; thus, considering the cases

$$f(t)\equiv e^{\lambda t}\qquad\left(\lambda\neq0,\ |\lambda|<\frac{2\pi}{3\zeta(3)}=1{\cdot}74\ldots\right)$$

and

$$f(t)\equiv\cos\lambda t\qquad\left(0<\lambda\leqq\frac{\pi}{2}\right),$$

we obtain the

Corollary. *The relations (holding for every $\varepsilon>0$)*

$$(53)\qquad \sum_{\nu=1}^{\Phi(x)}e^{\lambda\varrho_\nu}-\frac{e^\lambda-1}{\lambda}\,\Phi(x)=O\left(x^{\frac{1}{2}+\varepsilon}\right)\quad\left(\lambda\neq0,\ |\lambda|>\frac{2\pi}{3\zeta(3)}=1{\cdot}74\ldots\right).$$

and

$$(54)\qquad \sum_{\nu=1}^{\Phi(x)}\cos\lambda\varrho_\nu-\frac{\sin\lambda}{\lambda}\,\Phi(x)=O\left(x^{\frac{1}{2}+\varepsilon}\right)\quad\left(0<\lambda\leqq\frac{\pi}{2}\right)$$

are equivalent to RIEMANN's *hypothesis.*

In case of $f(t)\equiv\cos 2\pi t$, implying LITTLEWOOD's theorem, our conditions are not satisfied, for

$$f'(1)-f'(0)=0,$$

but we have

$$F(s) \equiv 1 \neq 0$$

in Theorem 4, so that our Corollary to Theorem 4 and Theorem 3 involve immediately the proposition in question.

By using special properties of $e^{\lambda t}$, $\cos \lambda t$, the above conditions for λ may be improved to

$$|\lambda| < 2\sqrt{\frac{5}{\zeta(3)}} = 4{\cdot}078\ldots,\ \lambda \neq 0, \tag{(53)}$$

$$|\lambda| < 2\sqrt{\frac{5}{\zeta(3)+\frac{5}{\pi^2}}} = 3{\cdot}432\ldots,\ \lambda \neq 0,\ \pm\pi.\text{[21]} \tag{(54)}$$

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[21]) See Mikolás [2].

Séries et intégrales de Fourier des fonctions monotones non bornées.

Par BÉLA SZ.-NAGY à Szeged.

Introduction.

Soient

$$\frac{a_0}{2}+a_1\cos x+a_2\cos 2x+\ldots,\qquad b_1\sin x+b_2\sin 2x+\ldots$$

la série des cosinus et la série des sinus d'une même fonction $f(x)$, définie et intégrable[1]) dans $(0,\pi)$. La série

(1) $$\sum\frac{b_n}{n}$$

est toujours convergente[2]). Par contre, la série

(2) $$\sum\frac{a_n}{n}$$

ne converge que si

(3) $$\int_{\to 0}^{\pi}\frac{dx}{x}\int_0^x f(t)\,dt$$

existe[3]).

M. ZYGMUND a considéré[4]) des fonctions intégrables $f(x)$ qui sont positives et décroissantes dans $(0,\pi)$. La série (1) est alors même absolument convergente; pour que (2) converge absolument, il faut et il suffit que (3) existe ou, ce qui revient au même dans ce cas, que $f(x)\log 1/x$ soit intégrable dans $(0,\pi)$.

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1) Dans tout ce qui suit, l'intégrabilité d'une fonction sera entendue au sens de Lebesgue.

2) Cf. par ex.: A. ZYGMUND, *Trigonometrical series* (Warszawa, 1935), p. 28.

3) Cf. G. H. HARDY—J. E. LITTLEWOOD, Solution of the Cesàro summability problem for power series and Fourier series, *Math. Zeitschrift*, **19** (1924), pp. 67—96 (lemma 19).

4) Cf. A. ZYGMUND, Sur les fonctions conjuguées, *Fundamenta Math.*, **13** (1929), pp. 284—303, en particulier pp. 299—301.

Nous verrons que cette différence entre séries des cosinus et séries des sinus disparaît lorsqu'on considère les séries

$$\sum \frac{a_n}{n^\gamma}, \quad \sum \frac{b_n}{n^\gamma}$$

avec un exposant positif $\gamma < 1$. Nous démontrerons du même coup le résultat mentionné de M. ZYGMUND, et cela par une méthode différente de celle suivie par cet auteur (théorèmes I—II). La partie de ces théorèmes concernant des conditions suffisantes pour la convergence absolue des séries en question, s'étend sans peine aussi à certaines fonctions non-monotones (théorème III). Il en résulte comme corollaire une condition pour qu'une fonction périodique continue, se composant, dans tout intervalle fini, d'un nombre fini de parties convexes ou concaves ait sa série de Fourier absolument convergente.

Des questions analogues se posent aussi pour les intégrales de Fourier ou plutôt elles se redoublent. En effet,

$$\int_0^\infty a(v) \cos xv \, dv \quad \text{et} \quad \int_0^\infty b(v) \sin xv \, dv$$

étant les développements, dans $(0, \infty)$, d'une même fonction $f(x)$ décroissante dans $(0, \infty)$ et tendant vers 0 avec $1/x$, on a à rechercher des conditions pour que $\frac{a(v)}{v^\gamma}$ et $\frac{b(v)}{v^\gamma}$ soient intégrables dans le voisinage de $v = 0$, et des conditions pour qu'elles le soient dans le voisinage de $v = \infty$.

De telles conditions seront établies par nos théorèmes IV—VI. Les résultats acquis s'étendent aussi à certaines fonctions non-monotones, tout comme dans le cas des séries de Fourier. Il en résulte en particulier une condition suffisante pour la convergence absolue du développement en intégrale de Fourier d'une fonction se composant d'un nombre fini de parties convexes ou concaves (théorème VII). Des résultats voisins du théorème VII. ont été publiés par l'auteur déjà dans un Mémoire antérieur[5]) et il en a tiré parti pour évaluer l'ordre de grandeur des constantes de Lebesgue et des constantes d'approximation attachées à des procédés de sommation des séries de Fourier, d'un type très général.

Un lemme.

Nous nous reporterons fréquemment au lemme suivant:

Soient les fonctions $\varphi(x)$ et $\psi(x)$ définies dans l'intervalle $0 < x \leqq a$, la première étant croissante et la seconde décroissante. Supposons que

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[5]) Cf. B. SZ.-NAGY, Sur une classe générale de procédés de sommation pour les séries de Fourier, *Hungarica Acta Math.*, 1 (1948), no. 3, pp. 14–52.

$\varphi(+0)=0$, *tandis que* $\psi(x)$ *peut aller à l'infini lorsque* $x\to 0$. *Dans ces conditions, si l'une ou l'autre des intégrales*

$$(4) \qquad \int_0^\alpha \varphi(x)\, d\psi(x), \qquad \int_0^\alpha \psi(x)\, d\varphi(x)$$

existe, elles existent toutes les deux et on a

$$(5) \qquad \lim_{x\to 0} \varphi(x)\,\psi(x)=0.$$

Observons d'abord que, grâce à la relation

$$\int_\varepsilon^\alpha \varphi(x)\, d\psi(x)+\int_\varepsilon^\alpha \psi(x)\, d\varphi(x)=\varphi(\alpha)\psi(\alpha)-\varphi(\varepsilon)\psi(\varepsilon),$$

il n'y a qu'à montrer que si l'une ou l'autre des intégrales (4) existe, alors (5) a lieu.

Soit $0<x<z$; on a

$$(6) \qquad 0\leqq\varphi(x)[\psi(x)-\psi(z)]=\varphi(x)\int_x^z d[-\psi(t)]\leqq\int_x^z\varphi(t)d[-\psi(t)].$$

Lorsque la première des intégrales (4) existe, il résulte de (6) que

$$0\leqq \limsup_{x\to 0}\varphi(x)\psi(x)\leqq\int_0^z\varphi(t)d[-\psi(t)].$$

Comme z est arbitraire, cela entraîne (5).

Lorsque c'est la seconde des intégrales (4) dont on suppose l'existence, (5) s'ensuit par les inégalités

$$\varphi(x)\psi(\alpha)\leqq\varphi(x)\psi(x)=\psi(x)\int_0^x d\varphi(t)\leqq\int_0^x\psi(t)d\varphi(t)$$

où on a intercalé $\varphi(x)\psi(x)$ entre deux fonctions tendant vers 0 avec x.

Remarquons que *le rôle de l'intervalle* $0<x\leqq\alpha$ *et de son extrémité* 0 *pourrait être joué, dans ce lemme, par un intervalle quelconque* $\sigma<x\leqq\alpha$ *ou* $\alpha\leqq x<\sigma$ *et par son extrémité* σ; σ *peut même être égal à* ∞ *ou* $-\infty$.

Nous aurons besoin des cas particuliers suivants de ce lemme:

Soient $f(x)$ *monotone dans* $0<x\leqq\alpha$ *et* $g(x)$ *monotone dans* $\beta\leqq x<\infty$, *de plus soit* $\lim_{x\to 0} g(x)=0$. *Si l'une ou l'autre des intégrales figurant dans la même ligne existe, elles existent toutes les deux et on a la limite indiquée à la fin de la ligne:*

$$\int_0^\alpha x^r df(x), \qquad \int_0^\alpha x^{r-1}f(x)dx, \qquad \lim_{x\to 0} x^r f(x)=0 \qquad (r>0);$$

$$\int_0^\alpha x\log x\; df(x), \qquad \int_0^\alpha f(x)\log x\; dx, \qquad \lim_{x\to 0}(x\log x)f(x)=0;$$

$$\int_\beta^\infty x^r dg(x), \qquad \int_\beta^\infty x^{r-1} g(x)dx, \qquad \lim_{x\to\infty} x^r g(x) = 0. \qquad (r>0);$$

$$\int_\beta^\infty x \log x\, dg(x), \qquad \int_\beta^\infty g(x) \log x\, dx, \qquad \lim_{x\to\infty} (x \log x) g(x) = 0.$$

Séries de Fourier (fonctions monotones).

Théorème I. *Soit $g(x)$ une fonction décroissante et bornée inférieurement dans l'intervalle $0<x<\pi$. Supposons de plus que $xg(x)$ soit intégrable de façon qu'on puisse formellement développer $g(x)$ en une série des sinus $\sum b_n \sin nx$ où*

$$b_n = \frac{2}{\pi}\int_0^\pi g(x) \sin nx\, dx.$$

Pour que la série

(7) $$\sum \frac{b_n}{n^\gamma} \quad (0<\gamma\leqq 1)$$

soit absolument convergente, il faut et il suffit que la fonction $x^{\gamma-1}g(x)$ (donc, dans le cas $\gamma=1$, la fonction $g(x)$ elle-même) soit intégrable dans $(0,\pi)$.

Théorème II. *Soit $h(x)$ une fonction décroissante et bornée inférieurement dans l'intervalle $0<x<\pi$. Supposons qu'elle soit intégrable dans $(0,\pi)$ et soit $\frac{a_0}{2}+\sum a_n \cos nx$ son développement en série des cosinus où*

$$a_n = \frac{2}{\pi}\int_0^\pi h(x) \cos nx\, dx.$$

Pour que la série

(8) $$\sum \frac{a_n}{n^\gamma} \quad (0<\gamma\leqq 1)$$

soit absolument convergente, il faut et il suffit, dans le cas $\gamma<1$, que la fonction $x^{\gamma-1}h(x)$, et dans le cas $\gamma=1$, que la fonction $h(x)\log x$ soit intégrable dans $(0,\pi)$.

Pour $\gamma=1$, ces résultats sont dus à M. ZYGMUND[6]).

Pour démontrer I, observons d'abord qu'on peut supposer $g(\pi-0)=0$; en cas contraire on n'aurait qu'à remplacer $g(x)$ par $g(x)-g(\pi-0)$, ce qui ne modifie les b_n que par des quantités b_n' de l'ordre de grandeur $O\left(\frac{1}{n}\right)$.

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[6]) L. c. [4])

L'hypothèse que $g(x)$ est monotone et que $xg(x)$ est intégrable, entraîne, en vertu du lemme, que $x^2 g(x) \to 0$, donc aussi

$$(1-\cos nx)\, g(x) \to 0 \quad \text{pour } x \to 0.$$

En intégrant par parties, on obtient

$$b_n = -\frac{2}{\pi n}\int_0^\pi (1-\cos nx)\, dg(x),$$

donc $b_n \geqq 0$. La série

$$\sum \frac{b_n}{n^\gamma} = -\frac{2}{\pi}\sum \int_0^\pi \frac{1-\cos nx}{n^{1+\gamma}}\, dg(x)$$

converge si et seulement si la somme $C_\gamma(x)$ de la série à termes positifs

$$\sum_{n=1}^\infty \frac{1-\cos nx}{n^{1+\gamma}}$$

est intégrable par rapport à la fonction monotone $g(x)$.

Or $C_\gamma(x)$ est une fonction continue et $C_\gamma(x) \sim x^\gamma$ pour $x \to 0$.[7]) En effet, comme $1-\cos y \leqq 2$ et encore $\leqq y^2/2$, on a d'une part

$$C_\gamma(x) \leqq \frac{x^2}{2}\sum_{n \leqq 1/x} n^{1-\gamma} + 2\sum_{n > 1/x}\frac{1}{n^{1+\gamma}} \leqq \frac{x^2}{2}\int_0^{1+1/x} t^{1-\gamma}dt + 2\int_{1/x-1}^\infty \frac{1}{t^{1+\gamma}}\, dt \sim x^\gamma,$$

et comme $1-\cos y \geqq 2(y/\pi)^2$ pour $|y| \leqq \pi$, on a d'autre part

$$C_\gamma(x) \geqq 2\,(x/\pi)^2 \sum_{n \leqq 1/x} n^{1-\gamma} \geqq 2(x/\pi)^2 \int_1^{1/x} t^{1-\gamma}dt \sim x^\gamma.$$

On a donc la série (7) convergente si et seulement si x^γ est intégrable dans $(0,\pi)$ par rapport à $g(x)$, ce qui veut dire le même, en vertu du lemme, que $x^{\gamma-1}g(x)$ est intégrable par rapport à x. Théorème I est donc démontré.

Pour démontrer le théorème II, observons d'abord que l'intégrabilité de $h(x)$ entraîne, en vertu du lemme, que $xh(x) \to 0$, donc aussi $(\sin nx)\, h(x) \to 0$ pour $x \to 0$. En intégrant par parties, on obtient

$$a_n = -\frac{2}{\pi n}\int_0^\pi \sin nx\, dh(x).$$

On a donc

$$\sum \frac{|a_n|}{n^\gamma} \leqq \frac{2}{\pi}\int_0^\pi \sum \frac{|\sin nx|}{n^{1+\gamma}}\, d[-h(x)].$$

---

[7]) $u(x) \sim v(x)$ pour $x \to \alpha$ veut dire qu'il y a deux constantes positives A, B de sorte que $A \leqq u(x)/v(x) \leqq B$ dans un voisinage de α.

toujours que l'intégrale au second membre existe. Or

$$\sum_{n=1}^{\infty} \frac{|\sin nx|}{n^{1+\gamma}} \leqq \sum_{n \leqq 1/x} \frac{nx}{n^{1+\gamma}} + \sum_{n > 1/x} \frac{1}{n^{1+\gamma}} \leqq$$

$$\leqq x\left(1 + \int_1^{1/x} \frac{dt}{t^\gamma}\right) + \int_{1/x}^{\infty} \frac{dt}{t^{1+\gamma}} \sim \begin{cases} x^\gamma & (0 < \gamma < 1), \\ x \log 1/x & (\gamma = 1) \end{cases}$$

lorsque $x \to 0$.

Cela prouve que la série (8) est absolument convergente si x^γ (cas $0 < \gamma < 1$) resp. $x \log x$ (cas $\gamma = 1$) est intégrable par rapport à $h(x)$. En vertu du lemme, cela veut dire le même que $x^{\gamma-1} h(x)$ resp. $h(x) \log x$ est intégrable par rapport à x.

Montrons que cette condition est aussi nécessaire, ou même plus:

x^γ ou $x \log x$ sont, selon les cas, intégrables par rapport à $h(x)$, même si l'on suppose seulement que la série (8) est sommable par le procédé de Césaro.

Supposons donc que la somme

$$\sum_{k=1}^{n} \left(1 - \frac{k}{n}\right) \frac{a_k}{k^\gamma} = -\frac{2}{\pi} \int_0^{\pi} \sum_{k=1}^{n} \left(1 - \frac{k}{n}\right) \frac{\sin kx}{k^{1+\gamma}} \, dh(x)$$

converge vers une limite lorsque $n \to \infty$. Comme on a $\sin kx = s_k(x) - s_{k-1}(x)$ où $s_k(x) = \dfrac{1 - \cos(2k+1)\frac{x}{2}}{2 \sin \frac{x}{2}}$, une transformation abélienne fournit:

$$S_{\gamma n}(x) = \sum_{k=1}^{n} \left(1 - \frac{k}{n}\right) \frac{\sin kx}{k^{1+\gamma}} = -\left(1 - \frac{1}{n}\right) s_0(x) + \sum_{k=1}^{n-1} s_k(x)\, \delta_{\gamma n}^{(k)} =$$

$$= -\left(1 - \frac{1}{n}\right) s_0(x) + T_{\gamma n}(x)$$

où

$$\delta_{\gamma n}^{(k)} = \left(1 - \frac{k}{n}\right) \frac{1}{k^{1+\gamma}} - \left(1 - \frac{k+1}{n}\right) \frac{1}{(k+1)^{1+\gamma}} =$$

$$= \left(\frac{1}{k^{1+\gamma}} - \frac{1}{(k+1)^{1+\gamma}}\right) - \frac{1}{n}\left(\frac{1}{k^\gamma} - \frac{1}{(k+1)^\gamma}\right)$$

est une quantité positive, tendant en croissant vers

$$\delta_\gamma^{(k)} = \frac{1}{k^{1+\gamma}} - \frac{1}{(k+1)^{1+\gamma}} \quad \text{lorsque } n \to \infty.$$

Par conséquent, la fonction $T_{\gamma n}(x)$ est positive dans l'intervalle $(0, \pi)$ et tend en croissant vers $T_\gamma(x) = \sum_{k=1}^{\infty} s_k(x) \delta_\gamma^{(k)}$ lorsque $n \to \infty$.

Comme l'intégrale de $S_{\gamma n}(x)$ par rapport à $h(x)$ converge par hypothèse faite vers une limite, il en de même de l'intégrale de $T_{\gamma n}(x)$, d'où il résulte que la fonction-limite $T_\gamma(x)$ est aussi intégrable par rapport à $h(x)$. Or on a $s_k(x) \geqq \frac{2}{x}\left(\frac{2k+1}{2\pi}x\right)^2$ pour $k+\frac{1}{2} \leqq \frac{\pi}{x}$, donc $T_\gamma(x) \geqq$

$$\geqq \frac{2}{\pi^2} x \sum_{k+\frac{1}{2} \leqq \frac{\pi}{x}} \left(k+\frac{1}{2}\right)^2 \delta_\gamma^{(k)} \geqq \frac{2}{\pi^2} x \int_1^{\pi/x} \left(t-\frac{1}{2}\right)^2 d(-t^{-1-\gamma}) \sim \begin{cases} x^\gamma & (\gamma < 1), \\ x \log 1/x & (\gamma = 1) \end{cases}$$

lorsque $x \to 0$, ce qui assure que x^γ resp. $x \log x$ soit aussi intégrable par rapport à $h(x)$, c. q. f. d.

Séries de Fourier (fonctions plus générales).

En tant qu'il s'agit de conditions *suffisantes*, les deux théorèmes ci-dessus s'étendent aisément à certaines fonctions plus générales et cela tout d'abord aux fonctions qui sont la différence de deux fonctions monotones.

Ainsi, il s'ensuit du théorème I que *si $g(x)$ est à variation bornée dans tout sous-intervalle (ε, π) où $\varepsilon > 0$, et que si $\int_0^\pi x^\gamma |dg(x)|$ existe pour un exposant γ $(0 < \gamma \leqq 1)$, $g(x)$ est intégrable dans $(0, \pi)$ et la série $\sum b_n/n^\gamma$, formée avec les coefficients b_n de sa série des sinus, est absolument convergente.*

En effet, en désignant par $g_1(x)$, $g_2(x)$ les variations positive et négative de g dans l'intervalle (x, π), on a $g = g_2 - g_1$[8]) et $|dg| = |dg_2| + |dg_1|$. Il s'ensuit que x^γ est intégrable aussi par rapport à g_1 et g_2. En vertu du lemme, g_1 et g_2 (voire même leurs produits par $x^{\gamma-1}$) sont intégrables. Donc g est aussi intégrable. Ses coefficients b_n étant les différences de ceux de g_2 et g_1, il n'y a qu'à appliquer le théorème I aux fonctions monotones g_1 et g_2.

De la même façon, on obtient du théorème II que *si $h(x)$ est à variation bornée dans tout sous-intervalle (ε, π) où $\varepsilon > 0$, et que si l'une ou l'autre des intégrales*

$$\int_0^\pi x \log x |dh(x)|, \quad \int_0^\pi x^\gamma |dh(x)| \quad (0 < \gamma < 1)$$

existe, $h(x)$ est intégrable dans $(0, \pi)$ et on a selon les cas la série $\sum a_n/n$, ou la série $\sum a_n/n^\gamma$, formées avec les coefficients a_n de la série des cosinus de $h(x)$, absolument convergentes.

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[8]) Du moins si $g(\pi) = 0$, ce qu'on peut supposer sans restreindre la généralité.

Ces propositions peuvent être généralisées de la manière suivante:

Théorème III. *Soit* $f(x)$ $(-\infty < x < \infty)$ *de période* 2π, *à variation bornée dans le voisinage de chaque point sauf peut-être d'un nombre fini de points "singuliers"* α (*incongruents* mod 2π). *En un tel point* α, $f(x)$ *ne doit même pas être définie, mais on suppose que les intégrales*

$$\int_0 u|dg_\alpha(u)|, \quad \int_0 u \log u\, |dh_\alpha(u)| \text{ [9]} \tag{9}$$

existent où

$$g_\alpha(u) = \tfrac{1}{2}[f(\alpha+u) - f(\alpha-u)], \quad h_\alpha(u) = \tfrac{1}{2}[f(\alpha+u) + f(\alpha-u)]. \tag{10}$$

Dans ces conditions, $f(x)$ *est intégrable, et la série*

$$\sum (|a_n| + |b_n|)/n, \tag{11}$$

formée avec ses coefficients de Fourier a_n, b_n, *est convergente.*

Lorsqu'on suppose de plus que les intégrales

$$\int_0 u^\gamma |dg_\alpha(u)|, \quad \int_0 u^\gamma |dh_\alpha(u)|$$

existent pour un exposant γ, $0 < \gamma < 1$, *on a même la série*

$$\sum (|a_n| + |b_n|)/n^\gamma \tag{12}$$

convergente.

Laissant à part le cas évident où il n'y a aucun point singulier, décomposons un intervalle de longueur 2π dont les extrémités ne sont pas des points singuliers, en des intervalles $i_1, \ldots, i_p$, chacun contenant un seul point singulier en son intérieur. Désignons par $f_k(x)$ la fonction de période 2π qui est égale à $f(x)$ dans l'intervalle i_k et s'annule dans les autres: on a alors

$$f(x) = \sum_{k=1}^{p} f_k(x). \tag{13}$$

En désignant par α_k le point singulier dans i_k, les fonctions

$$G_k(u) = \tfrac{1}{2}[f_k(\alpha_k+u) - f_k(\alpha_k-u)], \quad H_k(u) = \tfrac{1}{2}[f_k(\alpha_k+u) + f_k(\alpha_k-u)] \tag{14}$$

ont le seul point singulier $u=0$ et coïncident dans un voisinage de ce point avec les fonctions $g_{\alpha_k}(u)$, $h_{\alpha_k}(u)$ (cf. (10)). Comme on a supposé que les intégrales (9) existent, il résulte de ce qui précède que les fonctions (14) sont intégrables, et que si

$$G_k(u) \sim \sum B_{kn} \sin nu, \quad H_k(u) \sim \tfrac{1}{2} A_{k0} + \sum A_{kn} \cos nu \quad (0 < u < \pi),$$

on a les séries $\sum \frac{B_{kn}}{n}$ et $\sum \frac{A_{kn}}{n}$ absolument convergentes.

---

[9]) Pour la limite supérieure de ces intégrales, on peut choisir une quantité arbitraire $k > 0$ telle que l'intervalle $(\alpha-k, \alpha+k)$ ne contienne aucun point singulier pour $f(x)$ sauf α. On fera une telle convention aussi pour ce qui suit.

La fonction $G_k(x-\alpha_k)+H_k(x-\alpha_k)=f_k(x)$ est donc aussi intégrable, et si a_{kn}, b_{kn} désignent ses coefficients de Fourier, la série $\sum (|a_{kn}|+|b_{kn}|)/n$ est aussi convergente.

L'intégrabilité de $f(x)$ et la convergence de (11) s'ensuit, grâce à (13), par addition.

La proposition concernant la série (12) se démontre de la même façon.

Voici un corollaire de ce théorème:

Soit $F(x)$ une fonction continue, de période 2π, la courbe $y=F(x)$ se composant d'arcs convexes et concaves, en nombre fini sur un intervalle de période. En un point d'abscisse $x=\alpha$, séparant deux arcs voisins, nous permettons les demi-tangentes d'être verticales, c'est-à-dire que $F'(x)$ croisse indéfiniment lorsque x tend vers α de gauche ou de droite; nous exigeons seulement que l'intégrale

$$(15) \qquad \int_0 u \log u \, |d[F'(\alpha+u)+F'(\alpha-u)]| \quad ^{10})$$

existe. Dans ces conditions, la série de Fourier de $F(x)$ est absolument convergente.[11]

En effet, la fonction $f(x)=F'(x)$ vérifie les hypothèses de la première partie du théorème III. L'existence de la seconde intégrale (9) est supposée explicitement. Quant à la première, on a

$$\int_0 u \, |d[F'(\alpha+u)-F'(\alpha-u)]| \leqq \int_0 u \, |dF'(\alpha+u)| + \int_0 u \, |dF'(\alpha-u)|$$

et les intégrales au second membre existent parce que $F'(\alpha+u)$ et $F'(\alpha-u)$ sont intégrables et monotones pour des petites valeurs de u.

Observons que si $F'(x)$ est "localement symétrique" par rapport à α, cela veut dire que si $F'(\alpha+u)+F'(\alpha-u)=0$ pour u assez petit, l'existence de l'intégrale (15) est manifeste. En un tel point de "symétrie locale", il est donc permise que la demi-tangente de la courbe $y=F(x)$ devienne verticale aussi "rapidement" qu'on veut. En d'autres points α, la condition que l'intégrale (15) existe, présente une limitation pour cette rapidité.

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10) Désignons par $F'(x)$ par exemple la dérivée symétrique

$$\lim_{h\to 0} \frac{1}{2h}[F(x+h)-F(x-h)].$$

11) On trouve une liste de conditions plus ou moins compréhensives pour la convergence absolue de la série de Fourier d'une fonction $F(x)$ par ex. dans E. Hille - J. D. Tamarkin, On the summability of Fourier series. III., *Math. Annalen*, **108** (1933), pp. 525—577, en particulier pp. 532—533. Le critère (VI) de ces auteurs est le plus apparenté au nôtre.

Intégrales de Fourier (fonctions monotones).

Passons aux problèmes analogues pour les intégrales trigonométriques. Envisageons d'abord les intégrales des sinus.

Théorème IV. *Soit $g(x)$ $(0 < x < \infty)$ une fonction décroissante et tendant vers 0 avec $1/x$. Supposons que $x\,g(x)$ soit intégrable dans tout intervalle fini $(0, \alpha)$. Alors*

$$b(v) = \frac{2}{\pi}\int_0^{\to\infty} g(x) \sin xv\, dx$$

existe. Pour que l'une ou l'autre des intégrales

$$(i_\gamma^0) \quad \int_0^1 \frac{|b(v)|}{v^\gamma} dv, \qquad (i_\gamma^\infty) \quad \int_1^\infty \frac{|b(v)|}{v^\gamma} dv \qquad (0 < \gamma \leqq 1)$$

existe, il faut et il suffit que $x^{\gamma-1} g(x)$ soit intégrable, selon les cas, dans $(1, \infty)$, ou dans $(0, 1)$.

Grâce à l'hypothèse faite que $xg(x)$ est intégrable,

$$b_\alpha(v) = \frac{2}{\pi}\int_0^\alpha g(x) \sin xv\, dx$$

a un sens pour tout α fini. En vertu du lemme, l'intégrale

$$(16) \qquad \int_0^\alpha x^2 dg(x)$$

existe et

$$(17) \qquad x^2 g(x) \to 0 \quad \text{pour} \quad x \to 0.$$

Grâce à (17), on obtient en intégrant par parties:

$$b_\alpha(v) = \frac{2}{\pi v} g(\alpha)(1 - \cos \alpha v) - \frac{2}{\pi v}\int_0^\alpha (1 - \cos xv)\, dg(x).$$

Faisant aller α vers l'infini, le premier terme du second membre tend vers 0 et le second terme converge aussi parce que

$$0 \leqq \int_\mu^\nu (1 - \cos xv)\, d[-g(x)] \leqq 2\int_\mu^\nu d[-g(x)] = 2[g(\mu) - g(\nu)] \to 0$$

lorsque $\mu < \nu$ et $\mu \to \infty$.

Donc $b(v) = \lim\limits_{\alpha\to\infty} b_\alpha(v)$ existe et on a

$$b(v) = -\frac{2}{\pi v}\int_0^\infty (1 - \cos xv)\, dg(x) \geqq 0.$$

Il en résulte que (i_γ^0) ou (i_γ^∞) existe si, et seulement si

$$I_\gamma^0(x)=\int_0^1 \frac{1-\cos xv}{v^{1+\gamma}}\,dv \quad \text{ou} \quad I_\gamma^\infty(x)=\int_1^\infty \frac{1-\cos xv}{v^{1+\gamma}}\,dv,$$

selon les cas, est intégrable dans $(0,\infty)$ par rapport à $g(x)$. Or on a

$$I_\gamma^0(x)=x^\gamma\int_0^x \frac{1-\cos w}{w^{1+\gamma}}\,dw\sim\begin{cases} x^2 & \text{pour } x\to 0,\\ x^\gamma & \text{pour } x\to\infty,\end{cases}$$

$$I_\gamma^\infty(x)=x^\gamma\int_x^\infty \frac{1-\cos w}{w^{1+\gamma}}\,dw\sim\begin{cases} x^\gamma & \text{pour } x\to 0,\\ 1 & \text{pour } x\to\infty.\end{cases}$$

Comme x^2 est intégrable par rapport à $g(x)$ dans $(0,1)$ [cf. (16)] et comme $\int_1^\infty 1\,dg(x)$ existe aussi [étant égale à $-g(1)$], on voit que (i_γ^0) ou (i_γ^∞) existe si, et seulement si x^γ est intégrable par rapport à $g(x)$ dans $(1,\infty)$, ou dans $(0,1)$, selon les cas. En vertu du lemme, cela est équivalent avec l'intégrabilité de $x^{\gamma-1}g(x)$ dans $(1,\infty)$, ou dans $(0,1)$; c. q. f. d.

Quant aux intégrales des cosinus, il convient d'étudier les cas $\gamma<1$ et $\gamma=1$ séparément.

Théorème V. *Soit $h(x)$ $(0<x<\infty)$ une fonction décroissante et tendant vers 0 avec $1/x$. Supposons que $h(x)$ soit intégrable dans tout intervalle fini $(0,\alpha)$. Alors*

$$a(v)=\frac{2}{\pi}\int_0^{\to\infty} h(x)\cos vx\,dx$$

existe. Pour que l'une ou l'autre des intégrales

$$(j_\gamma^0)\quad \int_0^1 \frac{|a(v)|}{v^\gamma}\,dv,\qquad (j_\gamma^\infty)\quad \int_1^\infty \frac{|a(v)|}{v^\gamma}\,dv \qquad (0<\gamma<1)$$

existe, il faut et il suffit, que $x^{\gamma-1}h(x)$ soit intégrable selon les cas, dans $(1,\infty)$, ou dans $(0,1)$.

La fonction monotone $h(x)$ étant intégrable dans $(0,\alpha)$, on a, en vertu du lemme, la fonction x (donc aussi la fonction $\sin vx$) intégrable par rapport à $h(x)$ dans $(0,\alpha)$ et $x\,h(x)\to 0$ [donc aussi $\sin vx.\,h(x)\to 0$] lorsque $x\to 0$.

En intégrant par parties, on obtient que

$$a_\alpha(v)=\frac{2}{\pi}\int_0^\alpha h(x)\cos vx\,dx=\frac{2}{\pi v}h(\alpha)\sin\alpha x-\frac{2}{\pi v}\int_0^\alpha \sin vx\;dh(x),$$

donc

$$(18)\qquad a(v)=\lim_{\alpha\to\infty} a_\alpha(v)=-\frac{2}{\pi v}\int_0^\infty \sin vx\;dh(x).$$

Pour que (j_γ^0) ou (j_γ^∞) existe, il suffit donc, selon les cas, que

$$J_\gamma^0(x) = \int_0^1 \frac{|\sin xv|}{v^{1+\gamma}}\,dv, \quad \text{ou} \quad J_\gamma^\infty(x) = \int_1^\infty \frac{|\sin xv|}{v^{1+\gamma}}\,dv$$

soit intégrable dans $(0, \infty)$ par rapport à $h(x)$. Or on a

$$J_\gamma^0(x) = x^\gamma \int_0^x \frac{|\sin w|}{w^{1+\gamma}}\,dw \sim \begin{cases} x & \text{pour } x \to 0, \\ x^\gamma & \text{pour } x \to \infty; \end{cases}$$

$$J_\gamma^\infty(x) = x^\gamma \int_x^\infty \frac{|\sin w|}{w^{1+\gamma}}\,dw \sim \begin{cases} x^\gamma & \text{pour } x \to 0, \\ 1 & \text{pour } x \to \infty. \end{cases}$$

Comme x est intégrable par rapport à $h(x)$ dans $(0, 1)$, et la constante est intégrable par rapport à $h(x)$ dans $(1, \infty)$, on voit que $J_\gamma^0(x)$ et $J_\gamma^\infty(x)$ sont intégrables dans $(0, \infty)$ par rapport à $h(x)$, si (et seulement si) x^γ est intégrable par rapport à $h(x)$ respectivement dans $(1, \infty)$ et dans $(0, 1)$. En vertu du lemme, cela est équivalent avec l'intégrabilité de $x^{\gamma+1}h(x)$ par rapport à x, dans l'intervalle respectif.

La suffisance des conditions ainsi démontrée, passons à la démonstration de leur nécessité. Nous verrons même plus, notamment que $x^{\gamma-1}h(x)$ est intégrable dans $(1, \infty)$ ou dans $(0, 1)$ même si l'on suppose seulement que, selon les cas,

$$\int_{\to 0}^1 \frac{a(v)}{v^\gamma}\,dv, \quad \text{ou} \quad \int_1^{\to\infty} \frac{a(v)}{v^\gamma}\,dv$$

existe comme intégrale impropre, voire même sous l'hypothèse encore plus faible que

$$(19) \qquad A_\gamma(\mu) = \int_0^1 e_\mu(v)\frac{a(v)}{v^\gamma}\,dv \quad \text{où} \quad e_\mu(v) = \begin{cases} v/\mu & \text{pour } 0 \leq v \leq \mu, \\ 1 & \text{pour } v > \mu, \end{cases}$$

ou

$$(20) \qquad B_\gamma(\nu) = \int_1^\nu \left(1 - \frac{v}{\nu}\right)\frac{a(v)}{v^\gamma}\,dv$$

converge lorsque $\mu \to 0$, $\nu \to \infty$. Observons que l'existence des intégrales $A_\gamma(\mu)$, $B_\gamma(\nu)$ est assurée parce que, d'après (18), on a

$$|a(v)| \leq \frac{2}{\pi}\int_0^1 x\,|dh(x)| + \frac{2}{\pi v}\int_1^\infty |dh(x)| = C_1 + \frac{C_2}{v}.$$

Faisons d'abord l'hypothèse que $A_\gamma(\mu)$ converge lorsque $\mu\to 0$. On a, par (18),

$$(21)\quad \begin{aligned} A_\gamma(\mu) &= -\frac{2}{\pi}\int_0^1 \frac{e_\mu(v)}{v^{1+\gamma}}\left(\int_0^\infty \sin xv\, dh(x)\right)dv = \\ &= -\frac{2}{\pi}\int_0^\infty \left\{\int_0^1 \frac{e_\mu(v)}{v^{1+\gamma}} \sin xv\, dv\right\} dh(x); \end{aligned}$$

l'interversion des intégrations étant légitime parce que la fonction sous le signe d'intégrale admet la majorante

$$F_{\gamma\mu}(v,x) = \begin{cases} e_\mu(v)\, x/v^\gamma & (0 < x \leqq 1), \\ e_\mu(v)/v^{1+\gamma} & (1 < x < \infty), \end{cases}$$

intégrable dans le domaine $0 \leqq v \leqq 1$, $0 < x < \infty$, par rapport à la mesure $dv.|dh(x)|$.

Une intégration par parties montre que l'intégrale entre { } dans le dernier membre de (21) est égale à

$$C_{\gamma\mu}(x) = \frac{1-\cos x}{x} + \frac{x}{\mu}\int_0^\mu \frac{1-\cos xv}{x}\frac{dv}{v^{1+\gamma}} + (1+\gamma)\int_\mu^1 \frac{1-\cos xv}{x}\frac{dv}{v^{2+\gamma}},$$

qui est évidemment une fonction *positive* de x et tend vers

$$(22)\qquad C_\gamma(x) = \frac{1-\cos x}{x} + (1+\gamma)\int_0^1 \frac{1-\cos xv}{xv^{2+\gamma}}\, dv$$

lorsque $\mu\to 0$. L'intégrale de $C_{\gamma\mu}(x)$ par rapport à $h(x)$ étant égale à $-\frac{\pi}{2}A(\mu)$, converge vers une limite lorsque $\mu\to 0$. Cela entraîne, en vertu du lemme de FATOU, que la fonction-limite $C_\gamma(x)$ est aussi intégrable par rapport à $h(x)$. Comme on a

$$C_\gamma(x) = \frac{1-\cos x}{x} + (1+\gamma)x^\gamma \int_0^x \frac{1-\cos w}{w^{2+\gamma}}\, dw \sim x^\gamma \quad \text{pour } x\to\infty,$$

la fonction x^γ est aussi intégrable dans $(1,\infty)$ par rapport à $h(x)$, ou, ce qui revient au même, $x^{\gamma-1}h(x)$ est intégrable dans cet intervalle par rapport à x, c. q. f. d.

Envisageons maintenant les conséquences de l'hypothèse que $B(\nu)$, définie par (20), converge vers une limite lorsque $\nu\to\infty$. Par (18),

$$(23)\quad \begin{aligned} B_\gamma(\nu) &= -\frac{2}{\pi}\int_1^\nu \left(1-\frac{v}{\nu}\right)\left(\int_0^\infty \frac{\sin xv}{v^{1+\gamma}}\, dh(x)\right)dv = \\ &= -\frac{2}{\pi}\int_0^\infty \left\{\int_1^\nu \left(1-\frac{v}{\nu}\right)\frac{\sin xv}{v^{1+\gamma}}\, dv\right\} dh(x); \end{aligned}$$

l'interversion des intégrations étant légitime parce que la fonction sous le signe d'intégrale admet la majorante

$$G_{\gamma\nu}(v, x) = \begin{cases} x/v^{\gamma} & (0 < x \leq 1), \\ 1/v^{1+\gamma} & (1 < x < \infty), \end{cases}$$

intégrable dans le domaine $1 \leq v \leq \nu$, $0 < x < \infty$, par rapport à la mesure $dv.|dh(x)|$.

En intégrant par parties, on voit que l'intégrale entre { } dans le dernier membre de (23) est égale à

$$(24)\quad D_{\gamma\nu}(x) = -\frac{1-\cos x}{x}\left(1-\frac{1}{\nu}\right) + \int_1^{\nu} \frac{1-\cos xv}{x}\,\frac{1+\gamma}{v^{2+\gamma}}\left(1-\frac{\gamma}{1+\gamma}\,\frac{v}{\nu}\right)dv.$$

L'intégrale de $D_{\gamma\nu}(x)$ par rapport à $h(x)$ étant égale à $-\frac{\pi}{2}B(\nu)$, converge vers une limite lorsque $\nu \to \infty$, donc il en est de même de l'intégrale de

$$(25)\quad E_{\gamma\nu}(x) = D_{\gamma\nu}(x) + \frac{1-\cos x}{x}\left(1-\frac{1}{\nu}\right).$$

Or $E_{\gamma\nu}(x)$ tend évidemment en croissant vers

$$E_{\gamma}(x) = (1+\gamma)\int_1^{\infty} \frac{1-\cos vx}{xv^{2+\gamma}}\,dv = (1+\gamma)\,x^{\gamma}\int_x^{\infty} \frac{1-\cos w}{w^{2+\gamma}}\,dw$$

lorsque $\nu \to \infty$. Par conséquent, $E_{\gamma}(x)$ est aussi intégrable dans $(0, \infty)$ par rapport à $h(x)$. Comme $E_{\gamma}(x) \sim x^{\gamma}$ pour $x \to 0$, la fonction x^{γ} est aussi intégrable dans (0, 1) par rapport à $h(x)$, ou, ce qui revient au même grâce au lemme, $x^{\gamma-1}h(x)$ est intégrable dans (0, 1) par rapport à x.

Cela achève la démonstration du théorème.

L'hypothèse que la fonction envisagée soit monotone dans tout l'intervalle $(0, \infty)$, *aurait pu être remplacée, dans les théorèmes* IV *et* V, *par l'hypothèse plus générale que la fonction soit monotone du moins dans les voisinages de* 0 *et* ∞ *et qu'elle soit à variation bornée dans la partie complémentaire de* $(0, \infty)$.

Dans la proposition suivante, nous sommes même contraints à envisager ce cas plus général, parce que l'intégrale d'une fonction monotone positive ne pourrait s'annuler, condition qui va jouer cependant un rôle essentiel dans ce qui suit.

Théorème VI. *Soit* $h(x)$ $(0 < x < \infty)$ *monotone pour* x *assez petit et pour* x *assez grand, par ex. pour* $0 < x \leq \alpha$ *et* $\beta \leq x < \infty$ [12]) *où* $\alpha < \beta$, *à variation bornée dans* $\alpha \leq x \leq \beta$, *et soit* $\lim_{x\to\infty} h(x) = 0$. *De plus,* $h(x)$ *soit intégrable dans* $(0, \infty)$. [13]) *Soit*

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[12]) Dans ces intervalles, le sens de la monotonie peut être égal ou opposé.

[13]) En ce qui concerne l'intégrale $(\int_1^{\infty})$, il suffirait de supposer que $h(x)$ est intégrable dans $(0, \alpha)$.

$$a(v)=\frac{2}{\pi}\int_0^\infty h(x)\cos vx\,dx$$

et envisageons les intégrales

$$(j_1^0)\quad \int_0^1 \frac{|a(v)|}{v}\,dv \qquad et \qquad (j_1^\infty)\quad \int_1^\infty \frac{|a(v)|}{v}\,dv.$$

Pour que (j_1^∞) *existe, il faut et il suffit que* $h(x)\log x$ *soit intégrable dans* $(0,1)$. *Pour que* (j_1^0) *existe, il faut et il suffit que* $h(x)\log x$ *soit intégrable dans* $(1,\infty)$ *et qu'on ait*

$$(26)\qquad \int_0^\infty h(x)\,dx=0.$$

Envisageons d'abord (j_1^∞). On a par (18)

$$\int_1^\infty \frac{|a(v)|}{v}\,dv \leqq \frac{2}{\pi}\int_0^\infty J_1^\infty(x)\,|dh(x)|$$

où

$$J_1^\infty(x)=\int_1^\infty \frac{|\sin xv|}{v^2}\,dv = x\int_x^\infty \frac{|\sin w|}{w^2}\,dw \sim \begin{cases} x\log 1/x & \text{pour } x\to 0,\\ 1 & \text{pour } x\to\infty.\end{cases}$$

Il en résulte, vu aussi le lemme, que si $h(x)\log x$ est intégrable dans $(0,1)$, (j_1^∞) existe.

Inversement, lorsque (j_1^∞), ou du moins $\lim\limits_{\nu} B_1(\nu)$ existe [cf. (20)],

$$\lim_{\nu\to\infty}\int_0^\infty E_{1\nu}(x)\,dh(x) \quad \text{existe aussi où} \quad E_{1\nu}(x)=2\int_1^\nu \frac{1-\cos xv}{xv^3}\left(1-\frac{v}{2\nu}\right)dv$$

[cf. (24) et (25)]. Lorsque $\nu\to\infty$, $E_{1\nu}(x)$ tend *en croissant* vers

$$E_1(x)=2\int_1^\infty \frac{1-\cos xv}{xv^3}\,dv = 2x\int_x^\infty \frac{1-\cos w}{w^3}\,dw.$$

Cette fonction étant $\leqq 2/x$, est intégrable dans $\alpha\leqq x<\infty$ par rapport à la mesure $|dh(x)|$, donc on a nécessairement

$$\lim_{\nu\to\infty}\int_\alpha^\infty E_{1\nu}(x)\,dh(x)=\int_\alpha^\infty E_1(x)\,dh(x).$$

Par conséquent, $\lim\limits_{\nu\to\infty}\int_0^\alpha E_{1\nu}(x)\,dh(x)$ existe aussi, et comme $h(x)$ est monotone dans $(0,\alpha)$, cela entraîne que la fonction-limite $E_1(x)$ est intégrable aussi dans $(0,\alpha)$. Or $E_1(x)\sim x\log 1/x$ pour $x\to 0$, donc $x\log x$ est intégrable dans $(0,\alpha)$ par rapport à $h(x)$ et, par le lemme, $h(x)\log x$ est intégrable dans $(0,\alpha)$ par rapport à x.

Passons au problème de (j_1^0). Faisons l'hypothèse (26), ce qui revient, en vertu du lemme, à supposer que $\int_0^\infty x\,dh(x)=0$. On peut alors écrire, au lieu de (18),

$$(27) \qquad a(v)=\frac{2}{\pi}\int_0^\infty\left(x-\frac{\sin xv}{v}\right)dh(x),$$

d'où il vient que

$$(28) \qquad \int_0^1\frac{|a(v)|}{v}dv\leq\frac{2}{\pi}\int_0^\infty\left\{\int_0^1\frac{xv-\sin xv}{v^2}dv\right\}dh(x).$$

Comme on a

$$(29) \quad K(x)=\int_0^1\frac{xv-\sin xv}{v^2}dv=x\int_0^x\frac{w-\sin w}{w^2}dw\sim\begin{cases}x^3 & (x\to 0),\\ x\log x & (x\to\infty),\end{cases}$$

le second membre de (28) existe dès qu'on suppose encore que $x\log x$ est intégrable dans (β,∞) par rapport à $h(x)$, ou, en vertu du lemme, que $h(x)\log x$ est intégrable dans (β,∞) par rapport à x.

Ces conditions sont aussi nécessaires; *elles s'ensuivent déjà de l'hypothèse que* $\lim A_1(\mu)$ *existe* [cf. (19)].

Tout d'abord, comme nous avons supposé $h(x)$ intégrable dans $(0,\infty)$, la fonction $a(v)$ est continue même au point $v=0$. On a nécessairement $a(0)=0$, puisque en cas contraire, $A_1(\mu)$ ne pourrait pas converger lorsque $\mu\to 0$. Donc on a (26) et alors on peut se servir de la formule (27). Cela donne

$$(30) \quad A_1(\mu)=\frac{2}{\pi}\int_\mu^1\left(\int_0^\infty\frac{xv-\sin xv}{v^2}dh(x)\right)dv=\frac{2}{\pi}\int_0^\infty\left\{\int_\mu^1\frac{xv-\sin xv}{v^2}dv\right\}dh(x);$$

l'interversion des intégrations étant légitime puisque la fonction sous le signe d'intégrale admet la majorante x/v, intégrable dans le domaine $\mu\leq v\leq 1$, $0<x<\infty$, par rapport à la mesure $dv\,|dh(x)|$.

L'intégrale entre { } dans le dernier membre de (30) est une fonction non-négative $K_\mu(x)$, qui tend, lorsque $\mu\to+0$, vers la fonction $K(x)$ définie par (29) et cela en croissant. Comme $K(x)$ est évidemment intégrable dans $(0,\beta)$ par rapport à la mesure $|dh(x)|$, on a nécessairement

$$\lim_{\mu\to 0}\int_0^\beta K_\mu(x)\,dh(x)=\int_0^\beta K(x)\,dh(x).$$

Par conséquent, $\lim\limits_{\mu\to 0}\int_\beta^\infty K_\mu(x)\,dh(x)$ existe aussi, et comme $h(x)$ est monotone dans (β,∞), cela entraîne que la fonction-limite $K(x)$ est intégrable dans (β,∞) par rapport à $h(x)$. Or $K(x)\sim x\log x$ pour $x\to\infty$, d'où il résulte, eu égard aussi au lemme, que $h(x)\log x$ est intégrable dans (β,∞) par rapport à x, ce qui achève la démonstration du théorème.

Intégrales de Fourier (fonctions plus générales).

En tant que conditions suffisantes, les théorèmes IV—VI s'étendent à certaines fonctions plus générales, tout comme c'était le cas pour les théorèmes I—II, notamment aux fonctions qui sont la différence de deux fonctions du type envisagé dans le théorème respectif. On arrive ainsi aux résultats suivants:

Supposons que la fonction $f(x)$ $(0<x<\infty)$ soit à variation bornée dans tout sous-intervalle (ε, ∞), $\varepsilon>0$, tende vers 0 avec $1/x$, et que

$$(31)\qquad \int_0^\infty x^\gamma \, |df(x)|$$

existe pour un exposant positif $\gamma \leqq 1$. Alors $b(v)=\frac{2}{\pi}\int_0^{\to\infty} f(x)\sin xv\,dx$ existe et $|b(v)/v^\gamma|$ est intégrable dans $(0, \infty)$.

Dans le cas où $\gamma<1$, $a(v)=\frac{2}{\pi}\int_0^{\to\infty} f(x)\cos xv\,dx$ existe aussi et $|a(v)/v^\gamma|$ est intégrable dans $(0,\infty)$.

Si l'on suppose, au lieu de l'existence de (31), celle de

$$\int_0^\infty x \log x \, |df(x)|,$$

et si l'intégrale de $h(x)$ dans $(0,\infty)$ s'annule, on peut toujours affirmer que $a(v)$ existe et $|a(v)/v|$ est intégrable dans $(0, \infty)$.

Ces propositions sont cromprises, à leur tour, dans la suivante plus générale:

Théorème VII. *Soit $f(x)$ $(-\infty<x<\infty)$ à variation bornée dans le voisinage de chaque point x sauf peut-être d'un nombre fini de points "singuliers", où $f(x)$ peut même ne pas être définie, et supposons que $f(x)$ tende vers 0 avec $1/x$. En posant*

$$g_\alpha(u)=\tfrac{1}{2}[f(\alpha+u)-f(\alpha-u)],\quad h_\alpha(u)=\tfrac{1}{2}[f(\alpha+u)+f(\alpha-u)],$$

supposons que les intégrales

$$\int_0 u^\gamma\,|dg_\alpha(u)|,\qquad \int_0 u^\gamma\,|dh_\alpha(u)|\ ^{14)}$$

existent avec un exposant positif $\gamma<1$, et cela même pour les points singuliers α. Supposons de plus que les intégrales

$$\int^\infty u^\gamma\,|dg_0(u)|,\qquad \int^\infty u^\gamma\,|dh_0(u)|\ ^{15)}$$

existent. Dans ces conditions,

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[14]) Pour la limite supérieure de ces intégrales, voir note [9]).

[15]) Pour la limite inférieure de ces intégrales, on peut choisir une quantité arbitraire k, plus grande que le module de chaque point singulier pour $f(x)$.

$$a(v)=\frac{1}{\pi}\int_{\to-\infty}^{\to\infty} f(x)\cos vx\,dx, \qquad b(v)=\frac{1}{\pi}\int_{\to-\infty}^{\to\infty} f(x)\sin vx\,dx$$

existent et $[|a(v)|+|b(v)|]/v^{\gamma}$ *est intégrable dans* $(0,\infty)$.

Si, au lieu des intégrales ci-dessus avec u^{γ}, *on suppose que les intégrales*

$$\int_0 u\,|dg_{\alpha}(u)|, \quad \int_0 u\log u\,|dh_{\alpha}(u)|;\quad \int^{\infty} u\,|dg_0(u)|, \quad \int^{\infty} u\log u\,|dh_0(u)|$$

existent, et de plus que l'intégrale de $f(x)$ *dans* $(-\infty,\infty)$ *s'annule, on peut toujours affirmer que* $a(v)$, $b(v)$ *existent, et que la fonction* $[|a(v)|+|b(v)|]/v$ *est intégrable dans* $(0,\infty)$.

Pour le démontrer, décomposons $f(x)$ suivant ses points singuliers α_k $(k=1,\ldots,p)$. La fonction $f_k(x)$ soit choisie de façon qu'elle coïncide avec $f(x)$ dans un voisinage de α_k, s'annule pour des grandes valeurs de $|x|$, et qu'elle n'ait que le seul point singulier α_k. En écrivant $f(x)=$ $=\sum_1^p f_k(x)+f_{\infty}(x)$, $f_{\infty}(x)$ n'aura aucun point singulier et coïncidera avec $f(x)$ pour des grandes valeurs de $|x|$. (Si, en particulier, il n'y a aucun point singulier, on posera $f_{\infty}(x)=f(x)$.) Dans le cas où l'intégrale de $f(x)$ dans $(-\infty,\infty)$ s'annule, on aura encore le soin de définir $f_k(x)$ $(k=1,\ldots,p)$ de sorte que son intégrale s'annule aussi. Cela étant, on peut appliquer les résultats que nous venons d'énoncer, aux fonctions $f_k(x-\alpha_k)$, $f_{\infty}(x)$, ou plutôt aux fonctions g, h qui s'en dérivent, et on conclut comme dans la démonstration du théorème III.

Voici encore un corollaire du théorème VII:

Soit $F(x)$ $(-\infty<x<\infty)$ *une fonction continue, tendant vers* 0 *avec* $1/x$ *et se composant d'un nombre fini de parties convexes ou concaves. À l'extrémité* α *d'un intervalle de convexité ou de concavité, on permet que* $F'(x)$ *devienne* $+\infty$ *ou* $-\infty$, *mais on suppose que*

$$\int_0 u\log u\,|d(F'(\alpha+u)+F'(\alpha-u))|$$

existe. On suppose enfin que

$$\int^{\infty} u\log u\,|d(F'(u)+F'(-u))|$$

existe. $F(x)$ *admet alors une représentation de Fourier absolument convergente, c'est-à-dire qu'on a* $F(x)=\int_0^{\infty}[A(v)\cos xv+B(v)\sin xv]dv$ *et* $|A(v)|$, $|B(v)|$ *sont intégrables dans* $(0,\infty)$.[16])

(Reçu le 1 février 1949)

[16]) On trouve une liste de conditions pour la convergence absolue des intégrales de Fourier dans l'article cité [11]) de M. Hille et Tamarkin. La condition ci-dessus est apparentée à celles de ces auteurs contenue dans leurs théorèmes 7.1, 7.2.

On some sequences defined by recurrence.*

By J. ACZÉL in Szeged.

I. 1. We start from the well known fact[1]), that the sequence defined by the recurrence formula

(1) $$a_n = \frac{a_{n-2}+a_{n-1}}{2} \qquad (a_1 \text{ and } a_2 \text{ are arbitrary})$$

can be written in the form

(2) $$a_n = \frac{a_1+2a_2}{3} + \frac{2}{3}(a_1-a_2)\left(-\frac{1}{2}\right)^{n-1}.$$

In fact, let us make an attempt with $a_n = a + bz^{n-1}$. Substituting this in (1), we obtain $2a+2bz^{n-1} = a+bz^{n-3}+a+bz^{n-2}$; $2z^2-z-1=0$. The roots of this equation are 1 and $-\frac{1}{2}$. Thus $a_n = a+b\left(-\frac{1}{2}\right)^{n-1}$, and in particular $a_1 = a+b$ and $a_2 = a-\frac{b}{2}$. Thus $a = \frac{a_1+2a_2}{3}$, $b = \frac{2}{3}(a_1-a_2)$ and this gives the result announced.

(2) implies that a_n converges and $\lim a_n = \frac{a_1+2a_2}{3}$. [2])

I. 2. This can be proved also without using the explicit form (2). Let $a_1 \leqq a_2$, then we have evidently $a_1 \leqq a_3 \leqq \ldots \leqq a_{2j-1} \leqq a_{2j+1} \leqq \ldots$, $a_2 \geqq a_4 \geqq \ldots \geqq a_{2j} \geqq a_{2j+2} \geqq \ldots$, and $a_{2k-1} \leqq a_{2l}$. So both a_{2j-1} and a_{2j} are convergent, $a_{2j-1} \to \alpha$, $a_{2j} \to A$; $\alpha \leqq A$. But $\alpha < A$ is impossible, for $a_{2j+1} = \frac{1}{2}(a_{2j-1}+a_{2j})$ woud imply $\alpha = \frac{1}{2}(\alpha+A) > \alpha$. Thus $\alpha = A = a = \lim a_n$.

I. 3. The value of the limit a as a function $\mu(a_1, a_2)$ of the initial values a_1, a_2 can be found as follows. It has to satisfy the functional equation $\mu(a_1, a_2) = \mu(a_2, a_3)$, i. e. $\mu(a_1, a_2) = \mu\left(a_2, \frac{1}{2}(a_1+a_2)\right)$.

We might seek μ in the form

(3) $$\mu(x, y) = q_1 x + q_2 y \qquad (q_1+q_2 = 1),$$

* The essentially new parts of this paper are II. 2 and III. — The parts I and (partly) II. 1 contain wellknown-results which can be found in almost any book on Finite Defferences. They serve here for better understanding of what follows.

[1]) *Cf. e. g.* E. CESÀRO—G. KOWALEWSKI, *Elementares Lehrbuch der algebraischen Analysis und der Infinitesimalrechnung* (Leipzig, 1904), p. 105.

[2]) (2) shows that the difference in the approximation a_n of $\lim a_n$ forms a geometric sequence with the quotient $q = -\frac{1}{2}$.

for the linearity of the process implies that of μ and evidently $\mu(a, a) = a$. This gives $q_1 a_1 + q_2 a_2 = q_1 a_2 + q_2 \frac{a_1 + a_2}{2}$, thus $q_1 = 1/3$, $q_2 = 2/3$, $\mu(a_1, a_2) = \frac{1}{3}(a_1 + 2a_2)$.

A more elementary proof is the following:

We multiply both sides of (1) by 2 and add a_{n-1} to both sides. We get $a_{n-1} + 2a_n = a_{n-2} + 2a_{n-1}$. Repeating the recurrence we have finally $a_{n-1} + 2a_n = a_1 + 2a_2$. This gives for $n \to \infty$: $a + 2a = a_1 + 2a_2$, thus $a = \frac{1}{3}(a_1 + 2a_2)$.[3])

I. 4. Our results hold not only for the arithmetic mean, but also for any "quasi-arithmetic" mean $m(x, y) = f^{-1}\left(\frac{f(x) + f(y)}{2}\right)$ (f^{-1} is the inverse function of f), *e. g.* for the geometric mean where $f(t) = \log t$, the harmonic mean where $f(t) = 1/t$ and the root-mean-power where $f(t) = t^n$. In fact, if $c_n = f^{-1}\left(\frac{f(c_{n-2}) + f(c_{n-1})}{2}\right)$, $f(c_n) = \frac{f(c_{n-2}) + f(c_{n-1})}{2}$, hen $f(c_n) = a_n$ satisfies $a_n = \frac{a_{n-2} + a_{n-1}}{2}$. Thus

$$c_n = f^{-1}\left[\frac{f(c_1) + 2f(c_2)}{3} + \frac{2}{3}\left(f(c_1) - f(c_2)\right)\left(-\frac{1}{2}\right)^{n-1}\right];$$

c_n converges and $\lim c_n = f^{-1}\left(\frac{f(c_1) + 2f(c_2)}{3}\right)$. *E.g.*, for the geometric mean $c_n = \sqrt{c_{n-2} c_{n-1}}$, we have $c_n = \left[c_1 c_2^2 \left(\frac{c_2}{c_1}\right)^{\left(-\frac{1}{2}\right)^{n-2}}\right]^{\frac{1}{3}}$, $\lim c_n = \sqrt[3]{c_1 c_2^2}$.

II. 1. We generalize our problem as follows. Let

$$a_n = \frac{p_1 a_{n-k} + p_2 a_{n-k+1} + \ldots + p_{k-1} a_{n-2} + p_k a_{n-1}}{p_1 + p_2 + \ldots + p_{k-1} + p_k} \tag{4}$$

($a_1, a_2, \ldots, a_k$ are arbitrary; $p_1, \ldots, p_k \geqq 0$.)

Let us try the methods of I. 1 and conjecture

$$a_n = a + b_1 z_1^{n-1} + b_2 z_2^{n-1} + \ldots + b_{k-1} z_{k-1}^{n-1}. \tag{5}$$

Substituting (5) in (4) we get

$$(p_1 + p_2 + \ldots + p_{k-1} + p_k) z^k - p_k z^{k-1} - p_{k-1} z^{k-2} - \ldots - p_2 z - p_1 = 0.$$

As $z = 1$ is a root, we can divide by $(z-1)$ and get

$$(p_1 + p_2 + \ldots + p_{k-1} + p_k) z^{k-1} + (p_1 + p_2 + \ldots + p_{k-1}) z^{k-2} + \ldots + (p_1 + p_2) z + p_1 = 0. \tag{6}$$

The roots of this equation are the numbers $z_1, z_2, \ldots, z_{k-1}$ occurring in (5).

The constants in (5) are solutions of the linear system

$$a_i = a + b_1 z_1^{i-1} + b_2 z_2^{i-1} + \ldots + b_{k-1} z_{k-1}^{i-1} \qquad (i = 1, 2, \ldots, k).$$

[3]) It was St. FENYŐ who called the author's attention to the problem I. 3.

Multiplying the i-th equation by $p_1+p_2+\ldots+p_i$ and adding all equations we get

$$p_1a_1+(p_1+p_2)a_2+\ldots+(p_1+p_2+\ldots+p_k)a_k=$$
$$=a[p_1+(p_1+p_2)+\ldots+(p_1+p_2+\ldots+p_k)]+$$
$$+b_1[p_1+(p_1+p_2)z_1+\ldots+(p_1+p_2+\ldots+p_k)z_1^{k-1}]+\ldots+$$
$$+b_{k-1}[p_1+(p_1+p_2)z_{k-1}+\ldots+(p_1+p_2+\ldots+p_k)z_{k-1}^{k-1}].$$

As $z_1, z_2,\ldots,z_{k-1}$ are roots of (6), the coefficients of $b_1, b_2,\ldots, b_{k-1}$ are 0; thus

$$a=\frac{p_1a_1+(p_1+p_2)a_2+\ldots+(p_1+p_2+\ldots+p_k)a_k}{p_1+(p_1+p_2)+\ldots+(p_1+p_2+\ldots+p_k)}=$$
$$=\frac{p_1a_1+(p_1+p_2)a_2+\ldots+(p_1+p_2+\ldots+p_k)a_k}{kp_1+(k-1)p_2+\ldots+2p_{k-1}+p_k}.$$

The coefficients of the equation (6) are positive and decreasing and thus by a well known theorem of ENESTRÖM and KAKEYA [4]) all its roots $z_1,z_2,\ldots,z_{k-1}$ are of an absolute value less than 1. This implies that if $n\to\infty$, all members on the right of (5) tend to 0 except a [5]); thus a_n converges and

$$\lim a_n=a=\frac{p_1a_1+(p_1+p_2)a_2+\ldots+(p_1+p_2+\ldots+p_k)a_k}{p_1+(p_1+p_2)+\ldots+(p_1+p_2+\ldots+p_k)}.$$

If *e. g.* $a_n=(a_{n-k}+a_{n-k+1}+\ldots+a_{n-2}+a_{n-1})/k$, then

$$\lim a_n=\frac{a_1+2a_2+\ldots+(k-1)a_{k-1}+ka_k}{1+2+\ldots+(k-1)+k}=\frac{a_1+2a_2+\ldots+(k-1)a_{k-1}+ka_k}{k(k+1)/2}.$$

II. 2. The convergence of a_n can again be proved also directly. We give the proof not only for arithmetic means, not even only for quasi-arithmetic ones, but for any sequence defined by $a_n=m(a_{n-k}, a_{n-k+1},\ldots, a_{n-1})$ where we postulate only that the mean $m(x_1, x_2,\ldots, x_k)$ be a) *reflexive:* $m(x, x,\ldots, x)=x$, b) *strictly increasing,* c) *continuous.* The first two properties imply also d) *internity:*

$$\min(x_1, x_2,\ldots, x_k)\leqq m(x_1, x_2,\ldots, x_k)\leqq \max(x_1, x_2,\ldots, x_k).$$

To prove the convergence of a_n, consider

$$\alpha_n=\min(a_{n-k}, a_{n-k+1},\ldots, a_{n-1}) \text{ and } A_n=\max(a_{n-k}, a_{n-k+1},\ldots, a_{n-1});$$

clearly $\alpha_n\leqq a_n\leqq A_n$. Using d) and the fact, that if we drop one of the numbers the minimum of the remainder can not be smaller, we get

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[4]) G. ENESTRÖM, Härledning af en allmän formel för antalet pensionärer som vid en godtycklig tidpunkt förefinnas inom en sluten pensionskassa, *Öfversigt af Kongl. Svenska Vetenskaps-Akademien Förhandlingar,* **50** (1893), pp. 405–415; Remarque sur un théorème relatif aux racines de l'équation où tous les coefficients sont réels et positifs, *Tohôku Math. Journal,* **18** (1920), pp. 34–36. S. KAKEYA, On the limits of the roots of an algebraic equation with positive coefficients, *Ibidem,* **2** (1912), pp. 140–142.

[5]) We have counted throughout II. 1. as if (6) had only simple roots, but also the presence of multiple roots makes no difficulty as also $n^j z^n\to 0$ with $n\to\infty$ if $|z|<1$.

$$\alpha_n = \min(a_{n-k}, a_{n-k+1}, \ldots, a_{n-1}) =$$
$$= \min[a_{n-k}, a_{n-k+1}, \ldots, a_{n-1}, m(a_{n-k}, a_{n-k+1}, \ldots, a_{n-1})] =$$
$$= \min[a_{n-k}, a_{n-k+1}, \ldots, a_{n-1}, a_n] \leqq \min(a_{n-k+1}, \ldots, a_{n-1}, a_n) = \alpha_{n+1};$$

thus α_n increases. One sees similarly that A_n decreases. Thus α_n and A_n are both convergent: $\alpha_n \to \alpha$, $A_n \to A$; $\alpha \leqq A$. But $\alpha < A$ is impossible, because if $a_j = \alpha_n$ is the smallest among $a_{n-k}, a_{n-k+1}, \ldots, a_{n-1}$ and $a_i = A_j$ is the greatest among $a_{j-k}, a_{j-k+1}, \ldots, a_{j-1}$, then by b)

$$\alpha_n = \min(a_{n-k}, \ldots, a_{n-1}) = a_j = m(a_{j-k}, a_{j-k+1}, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{j-1}) \geqq$$
$$\geqq m(\alpha_j, \alpha_j, \ldots, \alpha_j, A_j, \alpha_j, \ldots, \alpha_j).$$

If $n \to \infty$, also $j \to \infty$ and by c) we would have $\alpha \geqq m(\alpha, \alpha, \ldots, \alpha, A, \alpha, \ldots, \alpha) > \alpha$. This is impossible and therefore $\alpha = A$, which completes our proof.

The weighted aritmethic mean $m(x_1, \ldots, x_k) = (\sum p_i x_i) / \sum p_i$ satisfies a), b), c) and this assures the convergence of the sequence (4).

The result of II. 2 holds also if in the recurrence formula $a_n = m(a_{n-1}, \ldots, a_{n-k})$ the mean value function m is not the same for every n, supposed that either only a finite number of mean value functions vary, or if in the infinity of m-s there is only a finite number of functions which do not occur infinitely many times.

Also the analogues of I. 3 and of I. 4 can be constructed similarly as those of I. 1 in II. 1. We leave the details to the reader.

III. We point out the interesting fact, that the theorem of ENESTRÖM and KAKEYA[4]) is a consequence of II. 2 (and equivalent to it).

In fact, every equation with positive decreasing coefficients can be written in the form $(p_1+p_2+\ldots+p_{k-1}+p_k)z^{k-1}+(p_1+p_2+\ldots+p_{k-1})z^{k-2}+ +\ldots+(p_1+p_2)z+p_1=0$. It is immediate that $z=1$ can not satisfy our equation and so the theorem is proved if we show that the sequence $w_n = z^n$ converges. Of course, it is enough to show that the real part and the imaginary part of w_n are both convergent. If we multiply the equation by $z-1$ we get $(p_1+p_2+\ldots+p_{k-1}+p_k)z^k - p_k z^{k-1} - p_{k-1}z^{k-2} - \ldots - p_2 z - p_1 = 0$ or what is the same $z^n = \dfrac{p_1 z^{n-k} + p_2 z^{n-k+1} + \ldots + p_{k-1} z^{n-2} + p_k z^{n-1}}{p_1 + p_2 + \ldots + p_{k-1} + p_k}$, i. e. $w_n = \dfrac{p_1 w_{n-k} + p_2 w_{n-k+1} + \ldots + p_k w_{n-1}}{p_1 + p_2 + \ldots + p_k}$. The real and the imaginary parts of w_n satisfy evidently the same recurrence formula, thus, by II. 2 they are convergent. This completes our proof of the theorem of ENESTRÖM and KAKEYA. (II. 2 holds only for real numbers, therefore we could not apply it directly to w_n.)

The well known direct proof[4]) of the theorem of ENESTRÖM and KAKEYA is of course shorter than that one given above in II. 2 and III, but there is perhaps some interest in the fact, that such seemingly distant domains as the theory of mean values and the theory of algebraic equations are so closely connected.

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A polyhedron without diagonals.

By ÁKOS CSÁSZÁR in Budapest.

It is simple to prove that *the tetrahedron is the only polyhedron homeomorphic to the sphere and having the property that every two of its vertices are joined by an edge.* In fact, such a polyhedron must be triangle-faced; and if we denote by v the number of its vertices, then it has $\binom{v}{2}$ edges and $\frac{2}{3}\binom{v}{2}$ faces, so that the theorem of EULER gives

$$\frac{2}{3}\binom{v}{2}+v-\binom{v}{2}=2. \tag{1}$$

This equation furnishes $v=3$ or $v=4$; the first solution has no geometrical meaning and the second gives the tetrahedron.

The question arises, whether this proposition remains true or not by omitting the restriction concerning the topological type of the polyhedron. We shall give in this paper a negative answer to this question by showing the existence of a polyhedron homeomorphic to the torus with the property mentioned above.

For the case of the torus, we have to put 0 instead of 2 on the right side of (1) and we obtain from this equation $v=7$. We first shall draw the 7 vertices and the 21 edges of our polyhedron on the torus.

Let us represent the torus on a rectangle $ABCD$. The opposite points lying on the sides of this rectangle are the images of the same point of the torus. Let us take seven points $1, 2, 3, \ldots 7$ in this order on the side AB; they appear naturally on the opposite side CD too. By drawing the straight segments joining the point 1 (on AB) to the points 3 and 4 (on CD), then those joining the point 2 (on AB) to the points 4 and 5 (on CD) and so on in the cyclic order of the vertices, these segments together with the segments of AB (and CD) joining two neighbouring vertices, form a system of lines containing 21 edges which joins every pair of the seven vertices by an edge and divides the torus represented by the rectangle $ABCD$ in 14 triangles. Table 1 enumerates the vertices of these 14 triangles.

Table 1.

126	235	356	346	467	237	267
156	245	124	134	137	457	157

We shall now construct a polyhedron which realizes this topological scheme.

Table 2 shows the coordinates of the seven vertices of our polyhedron in a rectangular system of coordinates. The values of a and b will be given later on. This system of vertices shows an axial symmetry with respect to the z-axis, 1 and 6, 2 and 5, 3 and 4 corresponding to each other. As table 1 shows, the faces of the polyhedron show the same symmetry. We have to choose the values a and b in the way that no pair of the faces intersect each other.

Table 2.

	x	y	z
1	-3	3	0
2	-3	-3	a
3	-1	-2	3
4	1	2	3
5	3	3	a
6	3	-3	0
7	0	0	b

In we put for a moment $a=0$ and $b=+\infty$, a short computation shows that the plane passing through the vertices 356 divides the space in two half-spaces in the way that the points 1 and 2 lie in the first half-space and the points 4 and 7 in the second. We can use for the abbreviation of this fact the symbol

$$12 \mid 356 \mid 47 \quad \ldots \text{A.}$$

We get similarly

$$125 \mid 346 \mid 7 \quad \ldots \text{B}$$
$$145 \mid 236 \mid 7 \quad \ldots \text{C}$$
$$145 \mid 237 \mid 6 \quad \ldots \text{D}$$
$$23 \mid 167 \mid 45 \quad \ldots \text{E}$$
$$36 \mid 257 \mid 14 \quad \ldots \text{F.}$$

These propositions remain valid even if we give to a a sufficiently small and to b a sufficiently large positive value.

Denoting by $\overline{25}$ the plane passing through the points 2 and 5 and perpendicular to the z-axis, we have moreover

$$16 \mid \overline{25} \mid 347 \quad \ldots \text{G.}$$

We can now show that no two of the faces of our polyhedron intersect each other. Because of the symmetry with respect to the z-axis it suffices to consider pairs of faces formed by a face in the upper line of table 1 and an other which is written *before* it or *under* it in this table. Table 3 gives now for every pair of this type one of the propositions A — G which shows that these two faces cannot intersect each other.

Table 3.

126 — 156 E	346 — 245 B	237 — 126 C	267 — 156 C
235 — 126 G	356 B	156 C	235 C
156 G	124 B	235 D	245 C
245 F	134 B	245 D	356 C
356 — 126 A	467 — 126 B	356 C	124 C
156 A	156 B	124 D	346 C
235 A	235 B	346 C	134 C
245 F	245 B	134 D	467 E
124 F	356 B	467 E	137 D
346 — 126 B	124 B	137 D	237 D
156 B	346 B	457 D	457 D
235 B	134 B	267 — 126 C	157 D
	137 E		

Longer calculation shows that $a=$ [illegible] and $b=15$ satisfy our above conditions.

We mention finally that the generalized theorem of EULER shows the existence of an infinity of topological types for a polyhedron with the property that every two of its vertices are joined by an edge. It would be of some interest to investigate if all these types can be realized with polyhedra having plane faces and straight edges.

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Bibliographie.

G. Vranceanu, Leçons de géométrie différentielle, Vol. I., 422 pages. Bucurest, 1947.

La théorie des équations différentielles et celle des groupes continus sont à la base des recherches actuelles de la géométrie différentielle. La plupart des livres sur ce sujet ne tiennent pas compte de ce fait en tant qu'ils renvoient le lecteur à des oeuvres spéciales traitant ces problèmes. Ainsi on ne trouve pas de base solide pour la compréhension des théories géométriques essentielles. A cet égard, les *Leçons* de M. VRANCEANU forment une exception remarquable.

Le livre se divise essentiellement en deux parties dont la première comprend l'appareil analytique et la deuxième s'occupe des problèmes géométriques. La première partie se divise en trois chapitres qui contiennent la théorie des formes de PFAFF et les congruences, la théorie des groupes continus et enfin les problèmes d'équivalence et d'invariants. Vu la richesse des matières traitées, ce serait un vain effort d'entrer dans les détails. Nous nous contenterons de quelques détails de principe. Dans les chapitres qui sont consacrés aux formes de PFAFF et à la théorie des groupes continus, l'auteur ne se borne pas à l'exposition des questions indispensables pour la compréhension des applications ultérieures, mais il traite aussi des théorèmes fondamentaux de ces théories, intéressantes en elles-mêmes, et cet ouvrage peut servir aussi comme une introduction dans ces théories. Les problèmes d'équivalence et d'invariants également importants pour la théorie des groupes comme pour celle des espaces de la géométrie différentielle, sont traités de plusieurs points de vue. Ici se trouvent exposées les recherhes intéressantes de l'auteur sur ce sujet. Les questions de ces chapitres sont traitées par la méthode des sytèmes de congruences indépendantes. Cette méthode dont l'importance a été mise en évidence par les remarquables recherches de l'auteur concernant les systèmes anholonomes, contribue beaucoup à faciliter la compréhension des matières traitées.

La partie géométrique s'occupe des espaces à connexion affine, des espaces de RIEMANN et enfin des espaces à connexion projective. Comme dans la première partie, l' auteur a réussi par un choix judicieux à traiter des théorèmes les plus importants dans la théorie de ces espaces. La méthode utilisée est d'une part le calcul absolu de RICCI, de l'autre le calcul des congruences qu'on doit à l'auteur. Cette méthode jette un pont entre le calcul de RICCI et les méthodes bien connues de M. ÉLIE CARTAN. C'est ainsi que le lecteur accède aux méthodes de M. CARTAN sans que celles-ci soient développées.

Le grand mérite de l'ouvrage est de faire connaître au lecteur diverses parties intéressantes de la géométrie et plusieurs méthodes d'une grande importance. La lucidité de l'exposition augmente encore la valeur de l'ouvrage qui vaut certainement un enrichissement considérable à la littérature de géométrie différentielle.

O. Varga.

G. Birkhoff, Lattice theory (American Mathematical Society Colloquium Publications, Volume XXV), revised edition, XIV+286 pages, New York, American Math. Society, 1948.

If somebody asked, some years ago, what the most unifying concepts of mathematics are, the answer was „sets and groups". Today this answer needs a completion with the term „lattices", this being the consequence of the fact that in the last twenty years lattices have turned out to be of fundamental significance for many branches of mathematics.

Historically, the first lattice-theoretical notion is due to BOOLE in 1847 who has defined the algebra of „attributes" in his logic. The concept of a lattice, in its present form, goes back essentially to DEDEKIND (in 1879). He dicovered it in connection with the theory of ideals, but his results have been left out of consideration. The first years of the third decade of this century can be considered as the actual beginning of lattice theory, when several mathematicians from quite different fields of mathematics were led independently and almost simultaneously to lattices; let us mention the names of the author, FR. KLEIN, K. MENGER and E. NOETHER.

This splendid book of G. BIRKHOFF is a revised and a nearly doubled edition of the author's first book on lattice theory published in 1940. The rapid advances in lattice theory within the last ten years made it necessary to enlarge essentially the size of the book in order to give an adequate account of new discoveries on this subject.

The book begins with a foreword on algebra and topology summarizing the fundamental algebraic and topological ideas needed throughout the work. A partly ordered system is defined in the first chapter as a system with a binary relation $\leqq$ satisfying the law of reflexivity, antisymmetry ($x \geqq y$ and $y \geqq x$ imply $x = y$) and transitivity. The elements 0 and I (satisfying $0 \leqq x \leqq I$ for all x), if exist, play a *distinguished role* in partly ordered sets. Chapter II deals with the definition and main properties of lattices defined as partly odered sets with the property that for any two elements x, y, there exist a greatest lower bound or „meet" and a least upper bound or „join", in symbols $x \cap y$ and $x \cup y$, respectively. Almost every part of mathematics abounds with instances of lattices, one meets them mainly in algebra, set theory, functional analysis, projective geometry, logic and probability theory.

The next chapters are devoted to the most important, more and more restrictive types of lattices. The simplest of them are chains where for any two elements x, y one has either $x \leqq y$ or $y \leqq x$. The chain conditions of abstract algebra are discussed together with several equivalent formulations of the well-ordering axiom. In the following chapter the author is concerned with complete lattices, i. e. lattices in which every subset has a meet and a join, giving at the same time the method of closure operation for constructing complete lattices. It is shown that in a complete lattice it is possible to introduce intrinsic topologies defined in terms of the order relation. The modular or Dedekind axiom is assumed in the subsequent chapter: if $x \leqq z$, then $x \cup (y \cap z) = (x \cup y) \cap z$, satisfied by many important lattices such as normal subgroups of a group, ideals of an integrity domain, etc. The sixth chapter contains applications to algebra, of which the most important are the generalized Jordan-Hölder theorem on principal series and the Kurosh-Ore theorem on the decomposition of elements. Two chapters are devoted to semi-modular and complemented modular lattices (complemented means that each x has a complement x' such that $x \cap x' = 0$ and $x \cup x' = I$), including many interesting results on plane and projective geometry, in particular on continuous-dimensional projective geometries discovered by J. von NEUMANN. In Chapter IX the important type of distributive lattice, (in which $(x \cup y) \cap z = (x \cap z) \cup (y \cap z)$ for any x, y, z) is developed. Then the author deals with Boolean algebras defined as complemented distributive lattices, and discusses STONE's theorem on the one-one correspondence between Boolean algebras and Boolean rings with unit, the latter being rings whose elements are all idempotent. It is shown that ideals of Boolean algebras are like normal subgroups in group theory, namely, they correspond one-one to the congruence relations. The next two chapters apply lattices to set theory, logic and probability. Chapters XIII—XV contain the theory of lattice-ordered semigroups and groups as well as vector lattices. The book ends with a discussion of ergodic theorems.

Each section in the book closes with numerous — sometimes difficult, but always very interesting — exercises which serve to give the reader an opportunity to test his grasp of the subject and at the same time to state many remarkable theorems, there being no place for a detailed discussion. Many of the results of a large variety of problems contained in this book are the author's own work, some of them being published here for the first time. The book also contains 111 unsolved problems on lattice theory.

At the end of the book there is a bibliography of the most important works on the subject; a complete reference is given in footnotes. The subject- and author-indices make easier the handling of this concisely and clearly written excellent work.

L. Fuchs

Tibor Radó, Length and area (American Mathematical Society Colloquium Publications, Volume XXX), VI+572 pages, New York, American Mathematical Society, 1948.

The chief aim of the book is to present the actual state of the theory of surface aera. This is a difficult task, the moment being not especially apted for such an enterprise, because — as the author himself admits — „there exists at present no unified general theory of surface area". Nevertheless in the past 50 years, starting from the fundamental ideas of LEBESGUE and GEÖCZE and owing to the researches of several authors, and in a great part to the work of RADÓ himself, the principal notions of the theory have been made clear in a great extent, many important results were achieved and a great number of paradox phenomena discussed. We emphasize the role of paradoxes because — as the author puts it — frequently „an apparent paradox turns out to be the source of essentially new insight". For instance the famous example of H. A. SCHWARZ, or the example given by GEÖCZE of a cube filling surface of zero area were really starting points of further progress. In view of the complexity of the subject, a clear survey of our present knowledge, with much emphasis laid on the fundamental difficulties and on the different possible points of depart, as given in the book, may serve as a basis of further progress towards the elucidation of the problem.

The theory of arc-length plays only a secondary rôle in the book and serves primarily as an introduction and a source of analogies (which however are sometimes misleading but nevertheless instructive — as it is pointed out by the author) for the theory of surface area. The theory of arc length, which may be regarded as complete compared with the theory of area, as presented by the author, may be summarized as follows: A curve C is defined as an equivalence class (in the sense of FRÉCHET) of continuous transformations of an interval. A curve C admits of different parametic representations by a continuous vector function $v(u) = (x(u), y(u), z(u))$ defined in an interval $\alpha \leqq u \leqq \beta$. The arc length $L(C)$ of C is defined as the Burkill integral over (α, β) of the interval function $F(v, I) = |v(b) - v(a)|$ where $I = (a, b)$. It is proved that if $L(C) < \infty$, the components of $v(u)$ are of bounded variation and, conversely, if the components of a representation of C are of bounded variation then so are the components of any representation of C, furher $L(C)$ is finite and its value does not depend on the choice of the representation. In this case, $v(u)$ being of bounded variation, $v'(u)$ exists almost everywhere, and it is proved that the integral of $|v'(u)|$ over (α, β) does not exceed $L(C)$, and is equal to the latter if and only if $v(u)$ is absolutely continuous. For any curve C with $L(C) < \infty$ an absolutely continuous representation can be given (e. g. if the arc-length is chosen as parameter) but there exists also always a purely singular representation with $v'(u) = 0$ almost everywhere. Finally $L(C)$ when considered as a functional in the space of continuous curves, is lower semicontinuous.

The last mentioned fact is, as it is emphasized by the author, „one of the few clear cut analogies between arc length and surface area" on which the theory of surface, initiated by LEBESGUE and GEÖCZE and developed further by the author, is based. Generally the analogies lie deep, while the discrepancies are conspicuous. Let us mention the most important discrepancies: it is clear that a short curve may be enclosed in a small sphere, but it is easy to see, that by folding properly a very narrow and long rectangle we obtain a surface with area as small as we please which passes within ε of every point of the unit cube and can not be enclosed in a smaller convex domain. An other essential difference concerns the inequality of STEINER. If the vector functions $v_1(u)$ and $v_2(u)$ represent the curves C_1 and C_2 and if C_3 is the curve represented by $v_3(u) = v_1(u) + v_2(u)$, we have $L(C_3) \geqq L(C_1) + L(C_2)$. As it has been remarked by FEJÉR, the same does not hold for surfaces in general. Nevertheless the inequality of STEINER admits of a straightforward generalization to surfaces having a representation $z = f(x, y)$ and this is the reason why the theory of such surfaces may be developed, by the use of the Burkill integral, without any topogical apparatus. In the general case, however, topological difficulties of a high order are inevitable, if the notion of a surface is understood in the generality proposed by the author. A great part of the book is devoted to the study of the topological problems mentioned. Part I furnishes background material in topology and analysis, Part II gives a study of the topological concepts of curve and surface, Part III contains the theory of arc length, while Part IV discusses topological and analytical questions regarding plane transformations. The proper theory of surface area is presented, after these preliminaries, in Part V. The study of area is concentrated around the theory of the Lebesgue area but other alternative approaches, especially different definitions of lower area given by GEÖCZE, RADÓ and REICHELDERFER are also treated in extenso. The concept of a surface is interpreted as a path-surface rather than a point-surface, i. e. a point set may be multiply covered by a surface. The Lebesgue area $A(S)$ is defined as the greatest lower bound of the elementary surfaces of polyhedra, converging in the sense of FRÉCHET to the surface S. An alternative descriptive definition is the following: $A(S)$ is a functional defined in the space of surfaces (i. e. equivalence classes of topological transformations of a 2-cell), nonnegative (eventually infinite), which coincides with the elementary value of the area for polyhedra, is lower semi-continuous and for every S there exists a sequence of polyhedra $P_n \to S$ with $A(P_n) \to A(S)$. The preference given by the author to the Lebesgue area is motivated by the necessities of applications, especially in the calculus of variations, for instance in the problem of PLATEAU. Attention is called on several unsolved problems regarding the equivalence of different definitions of surface area, among which we mention only the so-called Geöcze problem whether the Lebesgue area $A(S)$ concides always with the functional $\bar{A}(S)$ which is defined in the same way as $A(S)$ by the additional restriction that only inscribed polyhedra are admitted.

In view of the complexity of the subject, a section, entitled „General Comments" is added at the end of each part of the book, containing a survey of results obtained and important methodical and critical comments, which facilitates oversight; nevertheless the book remains rather difficult to read. It is to be regretted that the theory of arc length is mixed up — as regards the preparatory chapters — with the theory of surface area, and thus it is difficult for a reader interested only in the first subject to gather the material needed, especially as topological concepts are developed more deeply as necessary for the applications in differential geometry, in the calculus of variations, where the concept of a surface does not present itself in its full generality, and in a second advanced part for specialists. From the point of view of specialists, however, the book contains a rich material, presented in a systematic manner and in a clear and concise style, and surely it will greatly contribute to the success of further researches in the subject.

A. Rényi.

Einar Hille, Functional Analysis and Semi-Groups (American Math. Society Colloquium Publications, Volume XXXI), XI + 528 pages, New York, 1948.

An abstract semi-group is a system of elements in which an associative multiplication is defined. Such systems were first studied by DE SÉGUIER (1904) and L. E. DICKSON (1905): they assumed also the law of cancellation: if either $ab = ac$ or $ba = ca$, then $b = c$. Some sporadic papers on the algebraic theory of semi-groups followed. However, the main importance of the semi-group concept does not seem to lie in the algebraic field, but rather in the applications to Analysis where topological semi groups and in particular one-parameter semi-groups of linear transformations of a function space to itself come up in the most diversified connections. For such semi-groups, topological and analytical methods are available and a much richer theory results. In these connections, the law of cancellation is never supposed; on the other hand, commutativity is frequently assumed.

The author has great merits in having laid the foundations of, and developed with much success, this new mathematical discipline. He presents now the first monography on this subject. The high competence of the author, his enthousiasm in the subject and his clear style produced a work of unusual value not only for its rich content but also for its impressive and suggestive effect.

He has taken up his task in its most comprehensive sense. Guided by the desire to offer a practically self-contained presentation of the theory, he has incorporated in his book an elaborate introduction to modern functional analysis with special emphasis on function theory in Banach spaces and Banach algebras. This occupies Part One and Appendix; these can be read separately from the rest and present a valuable continuation of the original monography of BANACH. One finds there a very detailed discussion of the different extensions of the Lebesgue integral, and of the differentiability and analyticity, to functions of real or complex variables having their range in a Banach space, or to functions having both domain and range in a Banach space or in a Banach algebra. By a Banach algebra there is meant a Banach space in which an associative multiplication of the elements is defined such that $\|xy\| \leqq \|x\| \, \|y\|$. Since the fundamental papers of I. GELFAND in the *Mat. Sbornik* (1941) who called them „normed rings," Banach algebras were intensively studied in particular by Soviet and American mathematicians and it is very likely that this field will stand in one of the centers of mathematical interest in the next future. The Appendix of the book presents some very recent results on algebraic properties of Banach algebras.

The major part of the book, Parts Two and Three are devoted to the analytical theory of semi-groups and to special semi-groups. The main problem is the study of bounded transformations $T(\alpha)$ of a Banach space, satisfying $T(\alpha)\,T(\beta) = T(\alpha + \beta)$ for all values α, β of the paramenter in an open semi-module of real or complex numbers having $\alpha = 0$ as a limit point. When $\alpha \to 0$, two entirely different cases arise according as $T(\alpha)$ tends to the identity in the uniform or in the strong sense. Particular interest lies in the study of the infinitesimal generator of the semi-group, $A = \lim\,[T(\alpha) - I]/\alpha$ $(\alpha \to 0)$, its resolvent is the Laplace transform of $T(\alpha)$. The converse problem of constructing a semi-group with given infinitesimal generator is also investigated. The important case of analytic semi-groups deserves particular interest. Ergodic theory may be regarded as a question of the behavior of such a $T(\alpha)$ when $\alpha \to 0$ or ∞; this theory is shown to be closely related to a Tauberian theory of Laplace integrals, applied to the resolvent of the infinitesimal generator.

The part on special semi-groups consists of six chapters: Translations and Powers; Trigonometric Semi-Groups; Semi-Groups in $L_p(-\infty, \infty)$; Semi-Groups in Hilbert-Space; Semi-Groups and Partial Differential Equations; and Summability, Stochastic Processes, Fractional Integration. This part is by the great variety of its contents and by the unity of the underlying ideas, perhaps the most instructive part of the book.

We are convinced that this work will influence and stimulate in a considerable extent the future development of Functional Analysis, especially what regards its algebraic aspects.

B. Sz.-N.

Paul Lévy, Processus stochastiques et mouvement brownien. Suivi d'une Note de M. LOÈVE (Monographies des Probabilités, Fascicule VI), 365 pages, Paris, Gauthier-Villars, 1948.

Parmi les progrès récents du calcul des probabilités, c'est la théorie des fonctions aléatoires qui n'a pas encore reçu une exposition monographique. Le présent important ouvrage de M. LÉVY comble cette lacune.

Précédée par une étude générale des processus stochastiques et en particulier ceux stationnaires la majeure partie du travail est un exposé d'ensemble des résultats obtenus par l'auteur de 1934 à 1939 sur les processus additives et sur le mouvement brownien. La retardation de la publication a permis à l'auteur d'y incorporer aussi quelques résultats plus récents de KHINTCHINE, CRAMÉR, LOÈVE, KAMPÉ de FÉRIET, BLANC-LAPIERRE et FORTET sur les processus stationnaires. En une note extensive terminant le livre M LOÈVE donne un résumé clair de la théorie des fonctions aléatoires du second ordre, c'est-à-dire avec covariance finie.

C'est impossible de donner ici une image satisfaisante du contenu riche de l'ouvrage. Nous nous contentons d'une esquisse à grands traits.

Après un bref rappel de quelques définitions et des résultats fondamentaux du calcul des probabilités et de deux exemples simples de processus stochastiques, le chapitre II a pour objet la définition générale des processus stochastiques, les différentes modes de continuité et les différentes sortes de dérivées des fonctions aléatoires ainsi qu'une condition suffisante pour qu'une équation différentielle stochastique conduise à une telle fonction. La définition donnée d'après SLUTSKY est sans doute seulement intuitive, mais on connait les difficultés d'une définition fondée sur la théorie de la mesure. — Le chapitre III est consacré aux processus de MARKOFF, c'est-à-dire non héréditaires. Il montre le rôle de l'équation intégrale de CHAPMAN-KOLMOGOROFF et celui des équations aux dérivées partielles de la diffusion de la probabilité de KOLMOGOROFF dans le cas des fonctions presque sûrement continues. Dans le cas du mouvement brownien, l'équation correspondante est celle de la chaleur — Le chapitre IV expose la théorie des processus stationnaires commençant avec le théorème classique de KHINTCHINE et terminant avec les travaux plus récents indiqués. On trouve ici une application de la remarquable nouvelle théorie des operateurs de L. SCHWARTZ. — La plus grande partie du chapitre V est consacrée à la théorie des processus additifs d'après le livre connu de l'auteur. Les chapitres VI, VII, VIII sont exclusivement consacrés au mouvement brownien de rotation resp. à celui dans un, deux et plusieurs dimensions, de propriétés très différentes et surprenantes. On trouve ici non seulement la loi du logarithme itéré de KHINTCHINE avec indication aux recherches de FELLER, mais aussi les résultats de l'auteur publiés en 1939 et 1940 dans la *Compositio Math.* et dans l'*American Journal of Math.*

Sans doute le livre de M. LÉVY fera son mieux pour populariser cette belle et importante théorie.

T. Szentmártony.

Hermann Athen, Ebene und sphärische Trigonometrie (Bücher der Mathematik und Naturwissenschaften), 112 S., Wolfenbüttel und Hannover, Wolfenbütteler Verlagsanstalt, 1948.

Dieses Lehrbuch enthält alles aus der Trigonometrie, was theoretisch oder für die praktischen Anwendungen in der Mathematischen Geographie und in der Astronomie von Wichtigkeit ist. Die Darstellung strebt sich nicht möglichst einfach zu sein. Die vektorielle Darstellung der Grundformen scheint uns nicht einfacher, als die übliche. Wir glauben, dass der Halbwinkelsatz für die Tangenten in einem Dreieck sich aus den Radien der Berührungskreise einfacher ableiten lässt, als aus dem Cosinussatz. Mit der Methode der Berührungskreise würde sich auch eine Anzahl von trigonometrischen Formeln, z. B. für den Flächeninhalt, Umfang, für die Radien der verschiedenen Kreise eines Dreiecks leicht ergeben. Dieselbe Methode liesse sich auch in der sphärischen Trigonometrie mit Erfolg anwenden.

Das Lehrbuch enthält eine reiche Sammlung von glücklich gewählten praktischen Aufgaben.

Gy. Sz.-N.

L. Locher-Ernst, Differential und Integralrechnung im Hinblick auf ihre Anwendungen, 595 Seiten, Basel, Verlag Birkhäuser, 1948.

Das vorligende Buch gibt eine Einführung in die Differentialrechnung, die Integralrechnung und in die analytische Geometrie mit besonderer Berücksichtigung ihrer Anwendungen und in Verbindung mit einem umfassenden Übungsmaterial. das über 1000 geschickt gewählten Übungen mit Lösungen umfasst. Graphische und numerische Verfahren werden besonders berücksichtigt.

Verfasser betritt die pädagogische Ansicht, dass es dem Anfänger unmöglich ist umfassende Begriffe sich mit einer einzigen Anstrengung anzueigen und dass also solche Begriffe so dargebacht werden müssen, dass sie erst in der Folge der Entwicklungen ihre strenge Konturierung erhalten. Demgemäss werden z. B. Differentialquotient und bestimmtes Integral zuerst anschaulich an einfachen Beispielen erläutert. Im Hinblick auf die Anwendungen in der Physik und Technik wird durchwegs mit Differentialen gearbeitet. Abweichend von der üblichen strengen Definition wird das Differential als „eine werdende Null", d. h. „eine variable Grösse, die unbegrenzt dem Werte Null zustrebt" definiert.

Wenn auch die konsequente Durchführung dieser pädagogischen Gesichtspunkte manchmal etwas übertrieben erscheint, muss man zugeben, dass diese Gesichtspunkte berechtigt sind, besonders wo es um Anfänger handelt, die die höhere Mathematik nur als ein Werkzeug in der Physik, Technik usw. studieren wollen. Damit wollen wir aber keineswegs sagen dass der gleiche Gesichtspunkt im Falle der Kandidaten der Mathematik unrichtig wäre. Gewiss ist er auch dann berechtigt nur muss man besonders darauf achten, dass die „anschauliche" Einführung der Begriffe die „strenge" nicht ersetzen, sondern diese vorzubereiten und ihre Notwendigkeit erkennen lassen soll.

Das Buch wird sich gewiss, und mit gutem Grund, eine grosse Anerkennung von Lehrern und Studierenden erwerben.

B. Sz.-N.

Pierre Humbert et Serge Colombo, Introduction mathématique à l'étude des théories électromagnétiques. Fascicule I: Analyse vectorielle. Transformation conforme. Théorie du potentiel, IV + 149 pages, Paris, Gauthier—Villars, 1949.

L'ouvrage est destiné principalement aux ingénieurs et aux physiciens qui éprouvent le besoin de préciser et d'étendre leurs connaissances en analyse afin de pouvoir lire sans trop de difficulté les traités spécialisés et les mémoires originaux relatifs à l'électromagnétisme. Ils ont besoin notamment de l'analyse vectorielle, des éléments de la théorie des fonctions analytiques (séries de Taylor et de Laurent, calcul des résidus), des transformations conformes, de quelques fonctions spéciales (polynomes de Legendre et d'Hermite, fonctions de Bessel et de Mathieu, fonctions ellitiques), de la théorie du potentiel et de la théorie des équations différentielles, en particulier de l'équation de Laplace, des équations des ondes et de la chaleur, et de l'équation des télégraphistes.

Les auteurs présentent une introduction à ces domaines qui peut servir de répertoire bien maniable aussi pour le mathématicien. Des exercises, des notices historiques et une bibliographie augmentent la valeur de l'ouvrage.

B. Sz.-N.

Kurze Mathematiker-Biographien. Jakob Steiner par Louis Kollros. **Leonhard Euler** von Rudolf Fueter. **Ludwig Schläfli** von J. J. Burckhardt. **Jost Bürgi und die Logarithmen** von E. Voellmy. Beihefte Nr. 2—5 zur Zeitschrift „Elemente der Mathematik", Basel, Birkhäuser, 1947—48.

Jede Kurzbiographie enthält ein Porträt, ein Facsimile, die Angaben der wichtigsten Daten, die Charakteristik der Persönlichkeit und einige Beispielen aus den mathematischen Werken des betreffenden Mathematikers und hat den Umfang von 24 Seiten. Das Heft über Steiner ist französisch, die übrigen aber sind deutsch geschrieben.

Von jedem Hefte können wir nur anerkennende Worte sagen. Die Zusammenstellung der riesigen Arbeitsamkeit von Euler und der reichen Tätigkeit von Steiner ist ausgezeichnet. Auch die Biographien der weniger bekannten Mathematiker Schläfli und Bürgi sind gut gelungen.

In Vorbereitung befindet sich eine Kurzbiographie von Johann und Jacob Bernoulli. Nach einer brieflichen Mitteilung vom Redaktor wird diese Sammlung auch Kurzbiographien nichtschweizerischer Mathematiker enthalten, wie Abel, Gauss, Fermat, Galois, Monge, Bolyai usw.

Gy. Sz.-N.

Wilhelm Blaschke, Projektive Geometrie (Bücher der Mathematik und Naturwissenschaften), 160 S, 61 Abbildunden, Wolfenbütteler Verlagsanstalt, Wolfenbüttel und Hannover, 1947.

Nach dem Vorwort des ausgezeichneten Verfassers soll dieses Büchlein den Studierenden der Anfangssemester mit den wesentlichen Gedanken und Figuren des Gegenstandes vertraut machen. Dieses Lehrbuch liefert aber auch den Fachleuten Neues ebenso im Inhalt, wie auch in der Zusammenstellung. Die wichtigsten

Kapiteln behandeln Kegelschnitte, Quadriken, Liniengeometrie, nichteuklidische Geometrie und Möbiussche Vierflachpaare. Auch die geschichtlichen Daten vergrössern die Lebendigkeit des Gegenstandes.

Der Verfasser bestrebt sich die Begriffe und Lehrsätze möglichst kurz einzuführen bzw. zu bewiesen. Demgemäss ist die angewandte Methode bald syntetisch bald analytisch. Nur auf diesem Wege gelang es dem Verfasser einen sehr reichen Stoff in bloss 146 Steiten (abgerechnet das Vorwort und den Namen- und Sachweiser) übersichtlich, leicht verständlich und originell darzustellen.

Gy. Sz.-N.

Wolfgang Gröbner—Nikolaus Hofreiter, Integraltafel. Erster Teil. Unbestimmte Integrale, VIII + 166 Seiten, Wien und Innsbruck, Springer-Verlag, 1949.

„Der Zweck dieser Integraltafel ist, den Mathematikern, Physikern und Ingenieuren zeitraubende Ausrechnungen von Integraformeln nach Möglichkeit zu ersparen; sie soll auch einen Überblick über alle in den einzelnen Fällen brauchbaren Methoden geben."

Die Verfasser haben auch diejenigen Formeln, die aus älteren Formelsammlungen übernommen sind, vollständig neu gerechnet und überprüft; allen Formeln sind genaue Angaben über ihren Geltungsbereich hinzugefügt. Die Einteilung der Integrale erfolgt nach den Integranden; die drei Hauptabschnitte der rationalen, algebraisch irrationalen und transzendenten Integranden sind lexikographisch unterteilt.

Wir begrüssen diese nützliche, schön ausgestattete und handhabliche Tafel, und hoffen, dass der zweite Teil, der die bestimmten Integrale enthalten soll, auch in Kürze erscheinen wird.

B. Sz.-N.

N. W. McLachlan et Pierre Humbert, Formulaire pour le calcul symbolique (Mémorial des Sciences Math., Fascicule 100), 67 pages, Paris, Gauthier—Villars, 1941.

Le calcul operatoire d'Heaviside fait usage des transformées de Laplace

$$\varphi(p) = p \int_0^\infty e^{-pt} f(t)\, dt,$$

Pour qu'on puisse manier ce calcul, il est donc absolument nécessaire qu'on possède un „dictionnaire" indiquant autant de fonctions correspondantes $f(t)$, $\varphi(p)$ que possible. On trouve ici près de 700 formules de calcul symbolique, soit règles opératoires ou correspondances, ces dernières classées d'après la nature de la fonction originale.

B. Sz.-N.

Lous de Broglie, La mécanique ondulatoire des systèmes de corpuscules (Collection de Physique Math., fascicule V), deuxième édition, VI + 223 pages, Paris, Gauthier—Villars, 1950.

Réédition avec des mineurs changements de la première édition, parue en 1939 et analysée dans *ces Acta*, 11 (1946), p. 126—127.

B. Sz.-N.

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