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On the complexity of codes and pre-codes assigned to finite Moore automata

By A. Ádám

§ 1.

The concepts of code (a table describing a Moore automaton such that each isomorphy family of automata contains precisely one automaton describable by a code), pre-code (an initial part of a code) and complexity (maximum of the distinguishability numbers for the state pairs of an automaton) were introduced in the earlier article [3]. In the present paper, the study of these notions and some related ones is continued.

In § 6 of [3] the following question was raised (Problem 4): *Is the set of complexities of all pre-codes fulfilling $s=0$ equal to the set of non-negative integers?* The main results of the present paper yield an affirmative answer to this question.

On one hand, we show that each pre-code with $s=0$ is of finite complexity. The proof of this theorem occupies Sections 3—5 of the paper.

The difficulties that arise in this proof follow from two motives. First, the continuation of a pre-code **D** with $s=0$ (till when we get a code) is permitted only in such a way that a certain distinguished role of **D** should be preserved in the whole code, too. Secondly, our basic idea gives a fundamental role to the rows of the code which satisfy $\gamma(i)=n$ (where n is the largest possible value of γ); since $\gamma(i)=n$ can be fulfilled already by some rows of the pre-code **D**, these rows must be handled very carefully during the procedure.

On the other hand, we obtain in § 6 (by a simple construction) that each non-negative integer is the complexity of an appropriate pre-code satisfying $s=0$. This construction enables us to derive in § 7 an interrelation between the complexity and the number of states of a Moore automaton.

The last section of the paper presents an example illustrating the constructions used in the proof of Theorem 1.

§ 2.

Most of the notions, to be defined in this section, were treated also in [3]. We denote by $\mathbf{N}_i^j$ the set

$$\{i, i+1, i+2, \ldots, j-1, j\}$$

of integers.

The (ordered) set $X=\{x^{(1)}, x^{(2)}, \dots, x^{(n)}\}$ (the set of input signs) is thought *to be fixed for the whole paper* $(n \geqq 1)$. $F(X)$ is the free monoid generated by X, the elements of $F(X)$ are often called *words*. The *length* $L(p)$ of a word $p=x_1x_2\dots x_k$ is the number k (where $x_1 \in X, x_2 \in X, \dots, x_k \in X$). We denote by $p_k^{(i)}$ the word consisting of k copies of $x^{(i)}$ $(1 \leqq i \leqq n)$ (this notation will be used with $i=n$).

By a *pre-code* a sextuple $\mathbf{D}=(r, s, \beta, \gamma, \mu, \varphi)$ is meant such that the following eight postulates are satisfied:

(I) r, s are non-negative integers; $\beta, \gamma, \mu, \varphi$ are functions.
(II) The domains of $\beta, \gamma, \mu, \varphi$ are $\mathbf{N}_2^{r+s+1}, \mathbf{N}_2^{r+s+1}, \mathbf{N}_1^{r+1}, \mathbf{N}_{r+2}^{r+s+1}$, resp.
(III) The target of each of β, μ, φ is $\mathbf{N}_1^{r+1}$.
(IV) The target of γ is $\mathbf{N}_1^n$.
(V) $\beta(2)=1$. If $i \in \mathbf{N}_3^{r+1}$, then (a)&((b)$\vee$(c)) where

$$\text{(a)} \quad \beta(i-1) \leqq \beta(i) < i,$$

$$\text{(b)} \quad \beta(i-1) < \beta(i),$$

$$\text{(c)} \quad \gamma(i-1) < \gamma(i).$$

(VI) If $i \in \mathbf{N}_1^{r+1}$, then $\mu(i)-1 \in \{0, \mu(1), \mu(2), \dots, \mu(i-1)\}$.
(VII) If $i \in \mathbf{N}_{r+2}^{r+s+1}$, then $(\beta(i), \gamma(i))$ is the lexicographically smallest pair fulfilling

$$j \in \mathbf{N}_1^{i-1} \Rightarrow (\beta(i) \neq \beta(j) \vee \gamma(i) \neq \gamma(j)).$$

(VIII) If $i \in \mathbf{N}_{r+2}^{r+s+1}$, then either $\varphi(i)=1$ or (d)&((e)$\vee$(f)) where

$$\text{(d)} \quad \beta(\varphi(i)) \leqq \beta(i),$$

$$\text{(e)} \quad \beta(\varphi(i)) < \beta(i),$$

$$\text{(f)} \quad \gamma(\varphi(i)) < \gamma(i).$$

The number $r+s+1$ is called the *size* of the pre-code $\mathbf{D}=(r, s, \beta, \gamma, \mu, \varphi)$. The quintuple $(i, \beta(i), \gamma(i), \mu(i), \varphi(i))$ is called the i^{th} row of the pre-code $\mathbf{D}$ $(i \in \mathbf{N}_1^{r+s+1})$. We use the notation $\mathbf{D}_1 < \mathbf{D}_2$ if the pre-code $\mathbf{D}_2$ can be obtained from $\mathbf{D}_1$ by adding new rows (as last ones). We write $\mathbf{D}_1 \prec \mathbf{D}_2$ when $\mathbf{D}_1 < \mathbf{D}_2$ holds and $\mathbf{D}_2$ has one more row than $\mathbf{D}_1$. It can be shown that $s \leqq rn+n-r$ is valid for each pre-code.

If $\mathbf{D}_1$ is a pre-code and there exists no pre-code $\mathbf{D}_2$ satisfying $\mathbf{D}_1 < \mathbf{D}_2$ (or, equivalently, if s takes its maximal possible value $rn+n-r$ in $\mathbf{D}_1$), then $\mathbf{D}_1$ is called a *code*.

The *first block* of a pre-code $\mathbf{D}$ consists of the first row only. The *second block* of $\mathbf{D}$ consists of the second, third, ..., $(r+1)^{\text{th}}$ rows. The *third block* consists of the $(r+2)^{\text{th}}, (r+3)^{\text{th}}, \dots, (r+s+1)^{\text{th}}$ rows.

A pre-code $\mathbf{D}$ is called to be of *first type* if $r=0$. $\mathbf{D}$ is of *second type* if $s=0$. $\mathbf{D}$ is of *third type* if $r>0$ and[1] $s>0$. It is clear that each pre-code with at least two rows belongs to precisely one type, moreover, no code is of second type.

[1] These notions may be defined in terms of the emptiness of the second or third block, too. — We write out all the six components of a pre-code $\mathbf{D}=(r, s, \beta, \gamma, \mu, \varphi)$ even if some of the four functions does not exist really.

The iteration of the function β is defined by the recursion $\beta^0(i)=i$, $\beta^{k+1}(i)= =\beta(\beta^k(i))$.

By an *automaton* we mean always an initially connected finite Moore automaton $\mathbf{A}=(A, X, Y, \delta, \lambda, a_1)$. To each code $\mathbf{C}$ we assign an automaton $\psi(\mathbf{C})$ constructed in the following manner:

$$A = \{a_1, a_2, \ldots, a_{r+1}\},$$

$$\delta(a_{\beta(i)}, x^{(\gamma(i))}) = \begin{cases} a_i & \text{if } i \leqq r+1, \\ a_{\varphi(i)} & \text{if } i \geqq r+2, \end{cases}$$

$$\lambda(a_i) = y_{\mu(i)}.$$

It is known that *to each standard automaton* $\mathbf{A}$ *there is exactly one code* $\mathbf{C}$ *such that* $\mathbf{A}$ *and* $\psi(\mathbf{C})$ *are isomorphic* (see §§ 3—4 of [3]).

We use extensively the well-known visualization of automata (or their parts) by directed graphs. This method can be transferred (by virtue of the assignment ψ) also for codes and pre-codes. If $\mathbf{C}$ is a code and $\mathbf{D}$ is the pre-code consisting of the first and second blocks of $\mathbf{C}$, then the graph of $\mathbf{D}$ is a spanning subtree of the graph of $\mathbf{C}$ (and any edge of $\mathbf{D}$ is directed outwards from a_1).

If a, b are states of an automaton $\mathbf{A}$, then we define $\omega(a, b)$ as the length $L(p)$ of a shortest word p such that

$$\lambda(\delta(a, p)) \neq \lambda(\delta(b, p)). \tag{2.1}$$

If (2.1) holds, then we say that p *distinguishes* a and b (for the automaton $\mathbf{A}$ or for the code $\psi^{-1}(\mathbf{A})$).

The *complexity* $\Omega_A(\mathbf{A})$ of $\mathbf{A}$ is the maximum of the values $\omega(a, b)$ where $a \neq b$. The complexity $\Omega_C(\mathbf{C})$ of a code $\mathbf{C}$ is defined by $\Omega_C(\mathbf{C})=\Omega_A(\psi(\mathbf{C}))$. Finally, the complexity $\Omega_C(\mathbf{D})$ of a pre-code $\mathbf{D}$ means the minimum of all complexities $\Omega_C(\mathbf{C})$ where $\mathbf{D} \leqq \mathbf{C}$.

The following two statements (exposed in [3] as Propositions 13, 19) will be used often in our further considerations (with or without an explicit reference):

Proposition A. *If* $i \in \mathbf{N}_2^{r+s+1}$, $j \in \mathbf{N}_2^{r+s+1}$, $\beta(i)=\beta(j)$, $\gamma(i)=\gamma(j)$ *are valid for a pre-code, then* $i=j$.

Proposition B. *If the pre-codes* $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *satisfy* $\mathbf{D}_1 < \mathbf{D}_2$, *then* $\Omega_C(\mathbf{D}_1) \leqq \leqq \Omega_C(\mathbf{D}_2)$.

§ 3.

In §§ 3—5 we prove the following result:

Theorem 1. *If* $\mathbf{D}$ *is a pre-code of second type, then its complexity* $\Omega_C(\mathbf{D})$ *is finite.*

In the proof of the theorem two constructions will have essential roles (each of them transforms a pre-code to another pre-code and augments the size by one).

CONSTRUCTION 1. Let $\mathbf{D}=(r, 0, \beta, \gamma, \mu, \varphi)$ be an arbitrary pre-code of second type. Introduce the pre-code $\Gamma_1(\mathbf{D})=(r_1, s_1, \beta_1, \gamma_1, \mu_1, \varphi_1)$ by the following rules (i), (ii):

(i) $\Gamma_1(\mathbf{D})$ is of second type and $\mathbf{D} \prec \Gamma_1(\mathbf{D})$. (Hence $s_1=0$ and $r_1=r+1$.)
(ii) The function values at the place $r+2$ are:

$$\beta_1(r+2) = r+1,$$

$$\gamma_1(r+2) = n,$$

$$\mu_1(r+2) = \max\big(\mu(1), \mu(2), \ldots, \mu(r+1)\big)+1.$$

Proposition 1. *The pre-code $\Gamma_1(\mathbf{D})$ exists.*

Proof. The proposition asserts that $\Gamma_1(\mathbf{D})$, as it is determined by Construction 1, satisfies all the postulates (I)—(VIII). Most postulates are obviously fulfilled, except (V) in the particular case $i=r+2(=r_1+1)$.

(V) is completely satisfied since

$$\beta_1(r+2) = r+1 \begin{cases} > \beta(r+1) = \beta_1(r+1), \\ < r+2. \end{cases} \qquad \square$$

Before exposing Construction 2, we define some notions[2] for a pre-code $\mathbf{D}_0$. The set of numbers

$$\{r+1, \beta(r+1), \beta^2(r+1), \beta^3(r+1), \ldots, 1\}$$

is denoted by[3] H.

The set of all numbers $j(\in \mathbf{N}_2^{r+1})$ fulfilling at least one of the subsequent conditions (α), (β) is denoted by G:

(α) $\gamma(j)=n$,

(β) there is a number $h(\in \mathbf{N}_2^{r+1})$ such that $\beta(h)=j$ and $\gamma(h)=n$.

The set of numbers j which satisfy (α) but do not satisfy (β) are denoted by G_1. The set of numbers j which fulfil (β) but do not fulfil (α) are denoted by G_2. (Hence $G_1 \cap G_2 = \emptyset$ and $G_1 \cup G_2 \subseteq G$.)

Consider the subgraph induced by the vertex set G in the tree assigned to the pre-code consisting of the first and second blocks of $\mathbf{D}_0$. Each connected component of the induced subgraph is a path having at least two vertices. G_2 consists of the starting vertices of the connected components, G_1 consists of their end vertices.

We denote by G_h the set of numbers $i(\in G)$ such that the connected component (of G) containing i intersects H. Let G_g be the complementary set $G-G_h$. The intersection of H and a connected component C of G_h is a starting subpath of C. We define $G_{1,h}$, $G_{1,g}$ by $G_{1,h}=G_1 \cap G_h$ and $G_{1,g}=G_1 \cap G_g$.

If $j \in G_1$, then we denote by $\tau(j)$ the element of G_2 lying in the same connected component (of G) as j. Evidently, τ is a bijection of G_1 to G_2, and the containments $\tau(j) \in H$, $j \in G_{1,h}$ are equivalent. If $j \in G_{1,h} - H$, then we denote by $\tau'(j)$ the number $\beta^{w_0}(j)$ where w_0 is the smallest among the numbers w fulfilling $\beta^w(j) \in H$.

Construction 2. Let $\mathbf{D}_0=(r, s, \beta, \gamma, \mu, \varphi)$ be a pre-code of second or third type. We denote by $\mathbf{D}$ the pre-code consisting of the first and second blocks of $\mathbf{D}_0$. Let t mean the size $r+s+1$ of $\mathbf{D}_0$.

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[2] We do not specify the type of $\mathbf{D}_0$. The notions to be defined are independent of the third block of $\mathbf{D}_0$ (even if $\mathbf{D}_0$ belongs to the third type).

[3] The elements of H were enumerated here in decreasing order.

We introduce a pre-code $\Gamma_2(\mathbf{D}_0)=(r_2, s_2, \beta_2, \gamma_2, \mu_2, \varphi_2)$ by the subsequent two rules (iii), (iv):

(iii) $\Gamma_2(\mathbf{D}_0)$ is of third type and $\mathbf{D}_0\prec\Gamma_2(\mathbf{D}_0)$. (Thus $r_2=r, s_2=s+1$ and the size r_2+s_2+1 of $\Gamma_2(\mathbf{D}_0)$ equals $t+1$.)

(iv) The value $\varphi_2(t+1)$ is prescribed[4] according to six cases (a)—(f) as follows:

(a) If $\gamma_2(t+1)<n$, then $\varphi_2(t+1)=1$.

(b) If $\gamma_2(t+1)=n$ and $\beta_2(t+1)=r+1$, then $\varphi_2(t+1)=r+1$.

(c) If $\gamma_2(t+1)=n, \beta_2(t+1)\leqq r$ and $\beta_2(t+1)\in H$, then $\varphi_2(t+1)$ is the smallest element of the set

$$\mathbf{N}^{r+1}_{\beta_2(t+1)+1}\cap H.$$

(d) If $\gamma_2(t+1)=n$ and $\beta_2(t+1)\in G_{1,h}-H$, then $\varphi_2(t+1)$ is the smallest element of the set

$$\mathbf{N}^{r+1}_{\tau'(\beta_2(t+1))+1}\cap H.$$

(e) If $\gamma_2(t+1)=n$ and $\beta_2(t+1)\in G_{1,g}$, then $\varphi_2(t+1)$ is the largest element of the set

$$(\mathbf{N}_2^{\tau(\beta_2(t+1))-1}-((G-G_2)\cup H))\cup\{1\}.$$

(f) If $\gamma_2(t+1)=n$ and $\beta_2(t+1)\notin G\cup H$, then $\varphi_2(t+1)$ is the largest element of the set

$$(\mathbf{N}_2^{\beta_2(t+1)-1}-((G-G_2)\cup H))\cup\{1\}.$$

The description of Construction 2 is completed.

REMARK. The reader may convince himself that $\varphi_2(t+1)$ has been defined correctly. On one hand, the conditions in (a)—(f) exclude each other.[5] On the other hand, we have defined $\varphi_2(t+1)$ in every possible case since the situation when $\gamma_2(t+1)=n$ and $\beta_2(t+1)\in G-G_1$ cannot occur.[6]

Next we assert two simple facts on the procedure of Construction 2.

Lemma 1. *If $\varphi_2(t+1)$ is determined by* (c), *then* $\beta_2(\varphi_2(t+1))=\beta_2(t+1)$.

Proof. The statement follows from (c) and the definition of H. □

Lemma 2. *If $\varphi_2(t+1)$ is determined by* (d) *,then* $\beta_2(\varphi_2(t+1))=\beta_2^w(t+1)$ *where w is the smallest number such that $\beta_2^w(t+1)\in H$.*

Proof. This is a consequence of (d) and the definition of τ'. □

Proposition 2. *The pre-code $\Gamma_2(\mathbf{D}_0)$ exists.*

Proof. Analogously to the proof of Proposition 1, it is clear that $\Gamma_2(\mathbf{D}_0)$ satisfies the postulates (I)—(VIII) almost completely. Only the fulfilment of (VIII) if $t+1$ plays the role of i is questionable. We show this dependingly on the cases (a)—(f).

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[4] By Postulate (VII), the values $\beta_2(t+1)$, $\gamma_2(t+1)$ are uniquely determined.

[5] This is mostly obvious. It holds for the pairs ((c), (e)) and ((d), (e)) since $G_{1,g}$ is disjoint to H and to $G_{1,h}$.

[6] Indeed, combine Proposition A with the fact that $j\in G-G_1$ is equivalent to the validity of (β).

(We can omit the subscripts in $\beta_2, \gamma_2, \varphi_2$ without the possibility of misunderstanding.)

(a) Trivially, $\varphi(t+1)=1$ guarantees (VIII).

(b) We have

$$\beta(\varphi(t+1)) = \beta(r+1) < r+1 = \beta(t+1).$$

(c) By Lemma 1, $\beta(\varphi(t+1))=\beta(t+1)$, consequently,

$$\gamma(\varphi(t+1)) \neq \gamma(t+1) = n$$

(by (VII)), hence $\gamma(\varphi(t+1))<\gamma(t+1)$ since n is the maximal possible value of γ.

(d) Lemma 2 and $\beta^w(t+1)\leqq r+1<t+1$ imply

$$\beta(\varphi(t+1)) = \beta^w(t+1)\leqq\beta(t+1).$$

Strict inequality must hold since $\beta^w(t+1)\in H$ and $\beta(t+1)\notin H$.

(e) Either $\varphi(t+1)=1$ or the deduction

$$\beta(\varphi(t+1)) \leqq \beta(\tau(\beta(t+1))) < \tau(\beta(t+1)) < \beta(t+1)$$

holds (by (V) and $\varphi(t+1)\leqq\tau(\beta(t+1))-1$).

(f) Either $\varphi(t+1)=1$ or

$$\beta(\varphi(t+1)) \leqq \beta(\beta(t+1)) < \beta(t+1). \quad \square$$

Lemma 3. *Let* **D** *be a pre-code of second type. The sequence*

$$\mathbf{D},\ \Gamma_2(\mathbf{D}),\ \Gamma_2(\Gamma_2(\mathbf{D})),\ \Gamma_2(\Gamma_2(\Gamma_2(\mathbf{D}))),\ \ldots \tag{3.1}$$

breaks up after a finite number of steps. The last element of this sequence is a code.

Proof. On one hand, the first and second blocks are common for all the pre-codes in (3.1). Thus r is the same for them, and $rn+n-r$ is an upper bound for the lengths of the third blocks.

On the other hand, the sequence (3.1) can always be continued unless we reached a code. $\square$

Definition. Let **D** be a pre-code of second type. The last element of the sequence (3.1) is denoted by $\Gamma^*(\mathbf{D})$.

In § 8 it will be shown by an example how $\Gamma^*(\mathbf{D})$ is formed.

§ 4.

Let the recursive definition

$$\Gamma_2^{(0)}(\mathbf{D}) = \mathbf{D}, \quad \Gamma_2^{(s)}(\mathbf{D}) = \Gamma_2(\Gamma_2^{(s-1)}(\mathbf{D}))$$

be introduced for a pre-code **D** of type 2.

Lemma 4. *Let* $\mathbf{D}=(r, 0, \beta, \gamma, \mu, \varphi)$ *be a pre-code of second type. Suppose that the pre-code* $\Gamma_2^{(s)}(\mathbf{D})=(r, s, \beta, \gamma, \mu, \varphi)$ *exists*[7] *and* $\gamma(t)=n$ *holds where* $s\geqq1$ *and*

[7] We can write the functions without subscripts.

t is the size $r+s+1$ of $\Gamma_2^{(s)}(\mathbf{D})$. The following statements (A), (B) *are true:*

(A) *If* $\gamma(\varphi(t))=n$, *then* $\beta(t)=\varphi(t)=r+1$.

(B) *If a number* i $(\in \mathbf{N}_{r+2}^{t-1})$ *satisfies the equalities* $\gamma(i)=n$ *and* $\varphi(i)=\varphi(t)$, *then the formulae* $\beta(i)=\beta(r+1)$ *and* $\beta(t)=\varphi(t)=r+1$ *hold.*

Before proving the exposed lemma, we note another statement which will be useful in the proof of Lemma 4.

Lemma 5. *If the premissa of the assertion* (B) *of Lemma* 4 *are valid, then* $\beta(i)<\beta(t)$.

Proof. The formula $\beta(i)\leqq\beta(t)$ follows from $r+2\leqq i<t$ by Postulate (VII). The equality $\beta(i)=\beta(t)$ leads to a contradiction to Proposition A because we have supposed $\gamma(i)=n=\gamma(t)$. □

Proof of Lemma 4. Since $\gamma(t)=n$ was assumed, the value $\varphi(t)$ has been determined by one of the cases (b)—(f) in Construction 2 (with t instead of $t+1$). An analogous statement holds for $\varphi(i)$ (in (B)). The proper proof splits to the verifications of (A) and (B).

(A) The assumption $\gamma(\varphi(t))=n$ implies $1<\varphi(t)\in G-G_2$. We distinguish five cases according to (b)—(f). In each case, we either show the conclusion of (A) or get a contradiction (indicating that the case cannot occur really).

(b) The conclusion of (A) is trivial.

(c) On one hand, $\gamma(t)=n=\gamma(\varphi(t))$ and $\varphi(t)\leqq r+1<t$; on the other hand, $\beta(t)=\beta(\varphi(t))$ by Lemma 1. Contradiction to Proposition A.

(d) Let w be as in Lemma 2. On one hand, $\gamma(\beta^{w-1}(t))=n=\gamma(\varphi(t))$ and $\beta^{w-1}(t)\neq\varphi(t)$ (since $\beta^{w-1}(t)\notin H$ and $\varphi(t)\in H$); on the other hand, $\beta(\varphi(t))= =\beta^{w}(t)=\beta(\beta^{w-1}(t))$ by Lemma 2. Again a contradiction to Proposition A.

(e), (f). These cases are contradictory because $\varphi(t)\in G-G_2$ cannot be true and false simultaneusly.

(B) We can again distinguish five cases according to how $\varphi(t)$ has been defined, and an analogous distinction is made with respect to $\varphi(i)$. Combining these distinctions, twenty-five cases can be separated. We are going to show that the conclusion of (B) holds in one case and all the remaining twenty-four cases are contradictory.

We begin the discussion with the single consistent case. Suppose that $\varphi(i)$ has been determined by (c), and $\varphi(t)$ has been defined by (b). (This is called case (c_i)—(b_t) briefly.) Then $\beta(t)=\varphi(t)=r+1$ by (b) (applied for t). Furthermore,

$$\beta(i)=\beta(\varphi(i))=\beta(\varphi(t))=\beta(r+1)$$

(where Lemma 1 was used for i).

Now we turn to the other 24 cases that are imaginable. We do not discuss them separately but divide them into seven groups as indicated in Table 1. (E.g., the case (e_i)—(c_t) belongs to the second group.)

First group. In case (b_i)—(e_t) we have

$$r+1=\varphi(i)=\varphi(t)<\tau(\beta(t))<\beta(t),$$

Table 1.

$i \backslash t$	(b)	(c)	(d)	(e)	(f)
(b)	4	5	6	1	1
(c)	—	4	7	2	2
(d)	6	7	4	2	2
(e)	1	2	2	4	3
(f)	1	2	2	3	4

this is impossible since the value of β cannot exceed $r+1$ (by Postulate (III)). In the other three cases (belonging to this group) a similar inference holds, possibly with interchanging i and t, or with dropping $\tau(\beta(t))$.

Second group. We get that exactly one of $\varphi(i)$ and $\varphi(t)$ belongs to $H-\{1\}$, this contradicts the assumption $\varphi(i)=\varphi(t)$.

Third group. Denote the set

$$\mathbf{N}_2^{r+1}-((G-G_2)\cup H)$$

by J. We partition J to the classes J_1 and J_2 in the following manner: $j(\in J)$ belongs to J_1 or to J_2 according as the smallest element of $\mathbf{N}_{j+1}^{r+1}\cap J$ is contained in $J-G_2$ or in G_2, respectively. (If $\mathbf{N}_{j+1}^{r+1}\cap J=\emptyset$, then $j\in J_1$.) It is clear that $\varphi(t)\in J_1$ if $\varphi(t)$ is defined by (e), and $\varphi(t)\in J_2$ if $\varphi(t)$ is defined by (f).

One of $\varphi(i), \varphi(t)$ belongs to J_1 and the other of them belongs to J_2. This excludes $\varphi(i)=\varphi(t)$.

Fourth group. We try to deduce the equality $\beta(i)=\beta(t)$ in each case belonging to the present group; this equality is impossible by Lemma 5.

In the case (b_i)—(b_t), $\beta(i)=\beta(t)$ follows clearly. In the further considered cases, we have to keep in mind the situation of H, G, G_2 (in the tree assigned to **D**). $\varphi(i)=\varphi(t)$ implies $\beta(i)=\beta(t)$ in the cases (c_i)—(c_t) and (f_i)—(f_t) immediately. $\varphi(i)=\varphi(t)$ implies $\beta(i)=\beta(t)$ through the equalities $\tau'(\beta(i))=\tau'(\beta(t))$ and $\tau(\beta(i))=\tau(\beta(t))$ in the cases (d_i)—(d_t) and (e_i)—(e_t), respectively.

Fifth group. We can obtain the deduction

$$\beta(t)=\beta(\varphi(t))=\beta(\varphi(i))=\beta(r+1)<r+1=\beta(i)$$

(the first step follows from Lemma 1), this contradicts Lemma 5.

Sixth group. We discuss the case (b_i)—(d_t) only (the other case belonging to this group can be treated analogously, by interchanging i and t). The deduction

$$\beta(\beta^{w-1}(t))=\beta^{w}(t)=\beta(\varphi(t))=\beta(\varphi(i))=\beta(r+1) \tag{4.1}$$

is valid (in the second step we used Lemma 2). The structure of G, H and the containment $\beta(t)\in G_{1,h}-H$ imply

$$\gamma(\beta^{w-1}(t))=n. \tag{4.2}$$

Clearly,

$$\gamma(r+1)\leqq n. \tag{4.3}$$

The formulae (4.1), (4.2), (4.3) are consistent with Postulate (V) only if

$$r+1 \leqq \beta^{w-1}(t). \tag{4.4}$$

The obvious formula $\beta^{w-1}(t) \notin H$ and (4.4) imply $\beta^{w-1}(t) > r+1$, contradicting Postulate (III).

Seventh group. It suffices to deal with the case (c$_i$)—(d$_t$) (by a similar reason as in the sixth group). Lemmas 1 and 2 imply

$$\beta(i) = \beta(\varphi(i)) = \beta(\varphi(t)) = \beta^w(t) = \beta(\beta^{w-1}(t)), \tag{4.5}$$

and (4.2) holds also in the considered case. Comparing (4.5), (4.2) and $\gamma(i)=n$, we get $i=\beta^{w-1}(t)$. This is impossible since $\beta^{w-1}(t) \leqq r+1 < i$.

The proof of Lemma 4 is completed. □

§ 5.

Recall how the automaton $\psi(\mathbf{C})$ (assigned to a code $\mathbf{C}$) and the word $p_k^{(i)}$ have been defined in § 2.

In the following considerations — yielding the completion of the proof of Theorem 1 — we shall deal chiefly with automata given in form $\psi(\Gamma^*(\mathbf{D}))$ from such a point of view that only the effect of the input sign $x^{(n)}$ (with largest possible superscript) is taken into account.[8]

Lemma 6. *Let* $\mathbf{D}=(r, 0, \beta, \gamma, \mu, \varphi)$ *be a pre-code of second type. Consider the automaton*

$$\psi(\Gamma^*(\mathbf{D})) = \mathbf{A} = (A, X, Y, \delta, \lambda, a_1).$$

If $i \in \mathbf{N}_1^r - (H \cup G_h)$, *then there are two numbers* j, k *such that* $1 \leqq j < i$ *and* $a_j = \delta(a_i, p_k^{(n)})$ *(where* $a_i \in A$, $a_j \in A$*).*

Proof. Case 1: $i \notin G - G_1$. Define the number i' by the conditions $\beta(i')=i$, $\gamma(i')=n$. Then $\varphi(i')$ is defined by the rule (f) (in Construction 2) and the conclusion of the lemma is obviously fulfilled with $k=1$.

Case 2: $i \in G_g - G_1$. There is a $k'(>0)$ and a $j(\in G_1)$ such that $\beta^{k'}(j)=i$ and i, j are in the same connected component of G. It is clear that

$$n = \gamma(j) = \gamma(\beta(j)) = \gamma(\beta^2(j)) = \ldots = \gamma(\beta^{k'}(j)).$$

Consider the number j' satisfying $\beta(j')=j$ and $\gamma(j')=n$. Obviously, $j \geqq r+2$ and $\varphi(j')$ is defined by the rule (e).

We are going to show that the conclusion of the lemma holds if $k'+1$ is chosen for k. The definition of $\psi(\Gamma^*(\mathbf{D}))$ implies the equalities

$$\delta(a_i, p_{k'}^{(n)}) = a_j$$

[8] Automata having a single input sign are often called *autonomous*. The possible structures of finite autonomous automata follow from a graph-theoretical result of Ore ([5], § 4.4; see also [2], Chapter I). Although we do not use Ore's theorem explicitly, its knowledge makes perhaps easier to understand the considerations of the present §.

and

$$\delta(a_i, p_{k'+1}^{(n)}) = \delta(a_j, x^{(n)}) = a_{\varphi(j')}.$$

Since $\varphi(j')$ was defined by the rule (e), $\varphi(j')<\tau(\beta(j))\leqq i$. □

Lemma 7. *Let* **D**, **A** *be as in Lemma* 6. *Suppose* $i\in G_h$. *There are two numbers* j, k *such that* $j\in H$, $a_j=\delta(a_i, p_k^{(n)})$ *are true and one of the formulae* $i\notin H$, $j>i$ *holds.*

Proof.[9] Let us consider the numbers $k'(\geqq 0)$, j and j' with the same properties as in the preceding proof. $j'\geqq r+2$ is again true and $\varphi(j')$ is defined by one of the rules (c), (d). By use of Lemmas 1, 2 we obtain that

$$\varphi(j') > \beta(\varphi(j')) = \begin{cases} \beta(j') = j > i & \text{if (c) is applied,} \\ \tau'(\beta(j'))=\tau'(j) & \text{if (d) is applied.} \end{cases}$$

$i\in H$ implies $i\leqq\tau'(j)$, hence the lemma is valid with $k'+1$ (as k) in both cases. □

Lemma 8. *Let* **D** *and* **A** *be as in Lemma* 6. *If* $i\in H-\{r+1\}$, *then there are two numbers* j, k *such that* $i<j\leqq r+1$ *and* $a_j=\delta(a_i, p_k^{(n)})$ *(where* $a_i\in A$, $a_j\in A$).

Proof. If $i\notin G-G_1$, then the conclusion of the lemma is evidently fulfilled such that $k=1$ and j is the smallest element of $\mathbf{N}_{i+1}^{r+1}\cap H$. If $i\in G-G_1$, then Lemma 7 implies the present assertion. □

Lemma 9. *Let* **D** *and* **A** *be as in Lemma* 6. *For each number* $i(\in H)$ *there is a number* $k(\geqq 0)$ *such that* $\delta(a_i, p_k^{(n)})=a_{r+1}$.

Proof. Apply Lemma 8 repeatedly till it is possible. □

Lemma 10. *Let* **D** *and* **A** *be as in Lemma* 6. *For each number* $i(\in\mathbf{N}_1^{r+1})$ *there is a number* $k(\geqq 0)$ *such that* $\delta(a_i, p_k^{(n)})=a_{r+1}$.

Proof. Case 1*:* $i\in H$. Then Lemma 9 guarantees the statement.

Case 2*:* $i\in G_h-H$. Lemma 7 assures the existence of a k' such that $\delta(a_i, p_{k'}^{(n)})\in H$. By Lemma 8, also the equality

$$\delta(\delta(a_i, p_{k'}^{(n)}), p_{k''}^{(n)}) = a_{r+1}$$

is valid with a suitable k''. The left-hand side of this equality is clearly $\delta(a_i, p_{k'+k''}^{(n)})$.

Case 3*:* $i\notin G_h\cup H$. By a successive application of Lemma 6, there exists a k' such that $\delta(a_i, p_{k'}^{(n)})=a_1$. Since a_1 belongs to H, the further inference is the same as in Case 2. □

Lemma 11. *Let* **D** *and* **A** *be as in Lemma* 6. *Suppose that* i *and* j *are distinct numbers in* $\mathbf{N}_1^{r+1}$. *If*

$$\delta(a_i, x^{(n)}) = \delta(a_j, x^{(n)}) = a_m,$$

then

$$\max(i, j) = m = r+1.$$

[9] In the proof we consider an i chosen arbitrarily. It is easy to see that the lemma is satisfied with $k=1$, too, if, particularly, $i\in H$ and i does not belong to the range of τ'.

Proof. Case 1: one of i and j equals $\beta(m)$. We can assume (without loss of the generality) that $\beta(m)=i$. Then, by the connection of **D** and **A**, we have $\gamma(m)=n$ and there exists a number $w(\in \mathbf{N}_{r+2}^{r+s+1})$ such that $\beta(w)=j, \gamma(w)=n$ and $\varphi(w)=m$ hold in $\Gamma^*(\mathbf{D})$. By applying the assertion (A) of Lemma 4 (for w) we get that $j=\beta(w)=r+1>i$ and $m=\varphi(w)=r+1$.

Case 2: $\beta(m)$ coincides neither with i nor with j. There exist two numbers v, w in $\mathbf{N}_{r+2}^{r+s+1}$ such that $\beta(v)=i, \beta(w)=j, \gamma(v)=\gamma(w)=n$ and $\varphi(v)=\varphi(w)=m$. We can suppose $v<w$. Apply the statement (B) of Lemma 4 for v, w (instead of i, t, resp.). We obtain $i=\beta(v)=\beta(r+1)$ and $j=\beta(w)=r+1=\varphi(w)=m$. □

Lemma 12. *Let* **D** *and* **A** *be as in Lemma* 6. *Consider two different states a_i, a_j of* **A**. *Denote by k_i the smallest number fulfilling $\delta(a_i, p_{k_i}^{(n)})=a_{r+1}$; let k_j be defined analogously. Then $k_i\neq k_j$.*

Proof. The existence of k_i and k_j follows from Lemma 10. Let z_j be the smallest number such that $\delta(a_j, p_{k_j}^{(n)})$ belongs to the set

$$\{a_i, \delta(a_i, x^{(n)}), \delta(a_i, p_2^{(n)}), \delta(a_i, p_3^{(n)}), \ldots, \delta(a_i, p_{k_i}^{(n)})\},$$

let z_i be the smallest number such that $\delta(a_i, p_{z_i}^{(n)})=\delta(a_j, p_{z_j}^{(n)})$. Evidently, $0\leqq z_i\leqq k_i$ and $0\leqq z_j\leqq k_j$. (The situation is illustrated in Fig. 1.) We can distinguish four cases (two of them will be contradictory).

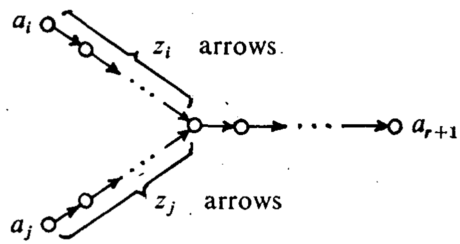

Fig. 1

If $z_i=z_j=0$, then we get $a_i=a_j$. Contradiction.
If $z_i=0<z_j$, then $k_j=k_i+z_j>k_i$.
If $z_j=0<z_i$, then $k_i=k_j+z_i>k_j$.
If $z_i>0$ and $z_j>0$, then

$$\delta(\delta(a_i, p_{z_i-1}^{(n)}), x^{(n)}) = \delta(a_i, p_{z_i}^{(n)}) = \delta(a_j, p_{z_j}^{(n)}) = \delta(\delta(a_j, p_{z_j-1}^{(n)}), x^{(n)}).$$

Apply Lemma 11 for $\delta(a_i, p_{z_i-1}^{(n)})$ and $\delta(a_j, p_{z_j-1}^{(n)})$. The conclusion of Lemma 11 implies that one of this states equals a_{r+1}, this is impossible by the definition of k_i and k_j. □

Proof of Theorem 1. Consider a pre-code $\mathbf{D}=(r, 0, \beta, \gamma, \mu, \varphi)$ of second type. Let **A** be the automaton $\psi(\Gamma^*(\Gamma_1(\mathbf{D})))=(A, X, Y, \delta, \lambda, a_1)$. Clearly, $|A|=r+2$. It is obvious by Construction 1 that $\lambda(a_i)\neq\lambda(a_{r+2})$ if $i\in\mathbf{N}_1^{r+1}$.

Consider two different states a_i, a_j of **A**. Introduce k_i, k_j as the smallest numbers fulfilling $\delta(a_i, p_{k_i}^{(n)})=a_{r+2}, \delta(a_j, p_{k_j}^{(n)})=a_{r+2}$, respectively. Lemma 12 (applied for $\Gamma_1(\mathbf{D})$ instead of $\mathbf{D}$) assures $k_i \neq k_j$. We can suppose (without loss of generality) $k_i < k_j$. We obtain

$$\delta(a_i, p_{k_i}^{(n)}) = a_{r+2} \neq \delta(a_j, p_{k_i}^{(n)})$$

from the previous considerations, hence

$$\lambda(\delta(a_i, p_{k_i}^{(n)})) = \lambda(a_{r+2}) \neq \lambda(\delta(a_j, p_{k_i}^{(n)})),$$

thus $\omega(a_i, a_j) \leqq k_i < \infty$.

Since the above inference holds for each pair (a_i, a_j) of states of the finite automaton $\mathbf{A}$, the complexity $\Omega_A(\mathbf{A})$ is finite. Consequently,

$$\Omega_C(\mathbf{D}) \leqq \Omega_C(\Gamma^*(\Gamma_1(\mathbf{D}))) = \Omega_A(\mathbf{A}) < \infty$$

by $\mathbf{D} < \Gamma^*(\Gamma_1(\mathbf{D}))$ and Proposition B. □

The next result follows from Lemmas 10 and 11 immediately:

Corollary 1. *Let* $\mathbf{D}$ *and* $\mathbf{A}$ *be as in Lemma* 6. *There exists a permutation* π *of the set* $\{1, 2, \dots, r\}$ *such that*

$$\delta(a_{\pi(i)}, x^{(n)}) = \begin{cases} a_{\pi(i+1)} & \text{if } 1 \leqq i < r, \\ a_{r+1} & \text{if } i = r, \end{cases}$$

and moreover, $\delta(a_{r+1}, x^{(n)}) = a_{r+1}$. □

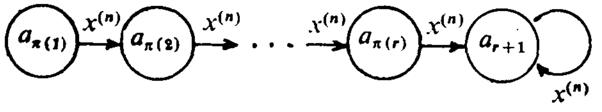

Fig. 2

Corollary 2. *If* $\mathbf{D} = (r, 0, \beta, \gamma, \mu, \varphi)$ *is a pre-code of second type, then* $\Omega_C(\mathbf{D}) \leqq r$.

Proof. Analyze the proof of Theorem 1, let π have the same sense (for $\Gamma_1(\mathbf{D})$) as in Corollary 1. It is clear that $a_{\pi(i)}$ and $a_{\pi(j)}$ can be distinguished by the word $p_{r+2-\pi(j)}^{(n)}$ if $\pi(i) < \pi(j)$, hence

$$\omega(a_{\pi(i)}, a_{\pi(j)}) \leqq r + 2 - \pi(j) \leqq r$$

(the second inequality holds because $\pi(i) + 1 = \pi(j) = 2$ is the worst choice). Thus $\Omega_A(\mathbf{A}) \leqq r$. □

§ 6.

The assertion (iii) of the next result is a conversion of Theorem 1.

Theorem 2. *Let* k *be an arbitrary non-negative integer. Then*

(i) *there is a code* $\mathbf{C}_k$ *such that* $\Omega_C(\mathbf{C}_k) = k$,
(ii) *there is an automaton* $\mathbf{A}_k$ *such that* $\Omega_A(\mathbf{A}_k) = k$,
(iii) *there is a pre-code* $\mathbf{D}_k$ *such that* $\Omega_C(\mathbf{D}_k) = k$ *and* $\mathbf{D}_k$ *is of second type.*

Proof. We define $\mathbf{C}_k=(r, s, \beta, \gamma, \mu, \varphi)$ in the following manner:

$$r = k+1 \quad (\text{hence } s(=rn+n-r) = kn+2n-k+1),$$
$$\beta(i) = i-1 \quad \text{if} \quad i\in \mathbf{N}_2^{r+1},$$
$$\gamma(i) = n \quad \text{if} \quad i\in \mathbf{N}_2^{r+1},$$
$$\mu(i) = 1 \quad \text{if} \quad i\in \mathbf{N}_1^{r},$$
$$\mu(r+1) = 2,$$
$$\varphi(i) = 1 \quad \text{if} \quad i\in \mathbf{N}_{r+2}^{r+s+1}.$$

$\beta(i)$ and $\gamma(i)$ are defined, of course, by virtue of Postulate (VII) if $i\in \mathbf{N}_{r+2}^{r+s+1}$. Fig. 3 shows a part of $\mathbf{A}_k=\psi(\mathbf{C}_k)$. (In the full graph of $\mathbf{A}_k$ every edge which is not indicated in this figure goes into a_1.)

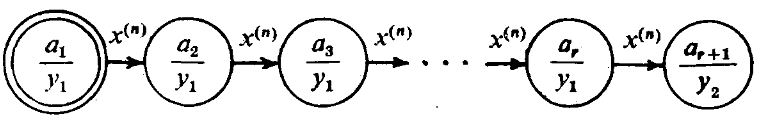

Fig. 3

It can be seen easily that $\mathbf{C}_k$ satisfies all the postulates (I)—(VIII). Thus $\mathbf{C}_k$ is a pre-code; it is a code since s equals the maximal possible value $rn+n-r$ (see the remark in § 4.3 of [3]).

We can verify easily that $\omega(a_i, a_j)=r-j+1$ is valid in $\mathbf{A}_k$ if $i<j$. (Indeed, on one hand,

$$\delta(a_i, p^{(n)}_{r-j+1}) = a_{(r+1)-(j-i)} \neq a_{r+1} = \delta(a_j, p^{(n)}_{r-j+1});$$

on the other hand, the relations $\delta(a_i, p)\in\{a_1, a_2, \ldots, a_r\}$ and $\delta(a_j, p)\in\{a_1, a_2, \ldots, a_r\}$ are true if $i<j$ and $L(p)\leqq r-j$.) The value of $\omega(a_i, a_j)$ reaches its maximum when $i=1$ and $j=2$, namely,

$$\omega(a_1, a_2) = r-1 = k.$$

Hence $\Omega_C(\mathbf{C}_k)=\Omega_A(\mathbf{A}_k)=k$. The proof of (i) and (ii) is completed.

Denote by $\mathbf{D}_k$ the pre-code satisfying $\mathbf{D}_k<\mathbf{C}_k$ and having the size $r+1$. (In other words, $\mathbf{D}_k$ consists of the first and second blocks of $\mathbf{C}_k$.) The estimate

$$\Omega_C(\mathbf{D}_k) \leqq \Omega_C(\mathbf{C}_k) = k \tag{6.1}$$

is obvious. Before verifying the converse inequality, we interrupt the proof by stating a lemma.

Lemma 13. *Consider an arbitrary code* $\mathbf{C}$ *such that* $\mathbf{D}_k<\mathbf{C}$. *Let the automaton* $\psi(\mathbf{C})=\mathbf{A}=(A, X, Y, \delta, \lambda, a_1)$ *be studied. If* $a_i\in A$, $i\leqq r$ *(where r is understood in* $\mathbf{D}_k$*) and a state* $a_j(\in A)$ *is representable in form* $a_j=\delta(a_i, x^{(h)})$ *(where* $x^{(h)}$ *is an arbitrary element of X), then* $j\leqq i+1$.

Proof. Case 1: $h=n$. The transition $\delta(a_i, x^{(h)})$ is determined by a row of the pre-code $\mathbf{D}_k$, hence $a_j=a_{i+1}$.

Case 2: $h \neq n$. Since $n=|X|$, we have $h<n$. The transition $\delta(a_i, x^{(h)})$ is determined by a row being in the third block[10] of $\mathbf{C}$; say, by the m^{th} row. Then $\beta(m)=i, \gamma(m)=h$ and $\varphi(m)=j$. We have $\beta(\varphi(m)) \leqq \beta(m)$ by Postulate (VIII), this implies

$$j = \varphi(m) \leqq \beta(m)+1 = i+1$$

by $\beta(m)=i \leqq r$ and the construction of $\mathbf{D}_k$. □

Proof of Theorem 2 (final part). If $\mathbf{C}$ is an arbitrary code fulfilling $\mathbf{D}_k < \mathbf{C}$, then the equality

$$\lambda(\delta(a_1, p)) = y_1 = \lambda(\delta(a_2, p))$$

holds in $\psi(\mathbf{C})$ for every word p whose length does not exceed $r-2$ (by an iterated application of Lemma 13). Hence $\omega(a_1, a_2) \geqq r-1$ holds in $\psi(\mathbf{C})$, consequently

$$\Omega_A(\psi(\mathbf{C})) \geqq r-1 = k$$

and

$$\Omega_C(\mathbf{C}) \geqq k, \tag{6.2}$$

thus

$$\Omega_C(\mathbf{D}_k) \geqq k, \tag{6.3}$$

since (6.2) holds for each $\mathbf{C}$ satisfying $\mathbf{D}_k < \mathbf{C}$.

The inequalities (6.1) and (6.3) give together the assertion (iii) of the theorem. □

§ 7.

By use of Corollary 2 and slight modifications of the idea of the proof of Theorem 2, we can infer the following assertions concerning the complexity and the first component r of codes and pre-codes:

Proposition 3. *Let two non-negative integers k, r be given. The inequality $k \leqq r$ is a necessary and sufficient condition of the existence of a pre-code $\mathbf{D}=(r, 0, \beta, \gamma, \mu, \varphi)$ such that $\mathbf{D}$ is of second type and $\Omega_C(\mathbf{D})=k$.*

Proposition 4. *If the non-negative integers k and r satisfy $k<r$, then there exists a code $\mathbf{C}=(r, s, \beta, \gamma, \mu, \varphi)$ such that $\Omega_C(\mathbf{C})=k$.*

Proof of Propositions 3 and 4. The proof will consist of three parts. In (A) we verify Proposition 4 and we show that $k<r$ is sufficient in Proposition 3. In (B) we make some preparations for proving the sufficiency of $k=r$. In (C) we verify the necessity part of Proposition 3 and we complete the proof of the sufficiency of the equality $k=r$.

(A) Consider k and r $(k<r)$. Recall the procedure proving Theorem 2, let us start with the code $\mathbf{C}_{r-1}$ (i.e., with $\mathbf{C}_k$ such that $r-1$ is taken for k). Alter $\mathbf{C}_{r-1}$ by putting

$$\mu(i) = \begin{cases} 1 & \text{if } i \in \mathbf{N}_2^{k+1}, \\ i-k & \text{if } i \in \mathbf{N}_{k+2}^{r+1}; \end{cases}$$

[10] This row cannot be in the second block of $\mathbf{C}$ (by Postulate (V)) even if the second block has $>r$ rows.

denote the originating code by $\mathbf{C}'_{k,r}$ (of course, $\mathbf{C}'_{r-1,r}=\mathbf{C}_{r-1}$) and the pre-code consisting of the first and second blocks of $\mathbf{C}'_{k,r}$ by $\mathbf{D}'_{k,r}$. The first component of $\mathbf{C}'_{k,r}$ and of $\mathbf{D}'_{k,r}$ is clearly r.

The whole proof of Theorem 2 remains valid for $\mathbf{C}'_{k,r}$, $\mathbf{D}'_{k,r}$ with certain numerical changes. In fact, $\omega(a_i, a_j)=\max(0, k-j+2)$ (where $i<j$), especially,

$$k = \omega(a_1, a_2) = \Omega_A(\psi(\mathbf{C}'_{k,r})) = \Omega_C(\mathbf{C}'_{k,r}).$$

Thus Proposition 4 is proved.

No word whose length is smaller than k can distinguish a_1 and a_2 for an arbitrary code $\mathbf{C}(>\mathbf{D}'_{k,r})$, consequently, $\Omega_C(\mathbf{D}'_{k,r})=k$.

(B) We start again with the code $\mathbf{C}_k$ occurring in the proof of Theorem 2. We modify it by putting $\mu(r+1)=1$; we denote the resulting code by $\mathbf{C}^*_k$ and the pre-code of its first $r+1$ rows by $\mathbf{D}^*_k$. Although the considerations of the proof of Theorem 2 do not remain valid in general, Lemma 13 holds in the present case, too, hence no word whose length is $<r$ can distinguish a_1 and a_2 for an arbitrary code $\mathbf{C}(>\mathbf{D}^*_k)$, thus $\Omega_C(\mathbf{D}^*_k)\geqq r$.

(C) Corollary 2 states that $\Omega_C(\mathbf{D})\leqq r$ holds for each pre-code $\mathbf{D}=(r, 0, \beta, \gamma, \mu, \varphi)$ of second type. The necessity of the condition in Proposition 3 is proved.

Especially, $\Omega_C(\mathbf{D}^*_k)\leqq r$. This inequality and the conclusion of (B) mean that $k=r$ is sufficient in Proposition 3. □□

Since the automaton $\psi(\mathbf{C})$ has $r+1$ states, Proposition 4 can be formulated in the following (equivalent) form:

Corollary 3. *If the non-negative integers k and v satisfy $k\leqq v-2$, then there exists a Moore automaton $\mathbf{A}$ such that $\Omega_A(\mathbf{A})=k$ and the number of states of $\mathbf{A}$ is v.* □

I conjecture that the conversion of Corollary 3 is also true, see [4].

§ 8.

In the last section of the paper, an example will be studied how $\Gamma_1(\mathbf{D})$ and $\Gamma^*(\Gamma_1(\mathbf{D}))$ are built up if a pre-code $\mathbf{D}$ of second type is given concretely.

Suppose $X=\{x^{(1)}, x^{(2)}\}$. Let $\mathbf{D}$ be the pre-code given by Table 2/a. (r equals 24. The tree assigned to $\mathbf{D}$ can be seen in Fig. 4. For the sake of simplicity, the vertices are labelled by i and the edges are by j instead of a_i and $x^{(j)}$, resp.)

We get $\Gamma_1(\mathbf{D})$ if we supplement $\mathbf{D}$ by a 26th row given by Table 2/b. The sets $H, G, G_1, G_2, G_h, G_g, G_{1,h}, G_{1,g}$ are (for $\Gamma_1(\mathbf{D})$) the following:

$$\begin{aligned}
H &= \{1, 2, 4, 7, 11, 15, 17, 20, 22, 25, 26\},\\
G &= \{2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, 24, 25, 26\},\\
G_1 &= \{7, 13, 14, 19, 24, 26\},\\
G_2 &= \{2, 3, 5, 11, 16, 22\},\\
G_h &= \{2, 4, 7, 11, 15, 17, 21, 22, 24, 25, 26\},\\
G_g &= \{3, 5, 6, 9, 10, 13, 14, 16, 19\},\\
G_{1,h} &= \{7, 24, 26\},\\
G_{1,g} &= \{13, 14, 19\}.
\end{aligned}$$

Table 2.

i	$\beta(i)$	$\gamma(i)$	$\mu(i)$	$\varphi(i)$
1	—	—	1	—
2	1	1	1	—
3	2	1	1	—
4	2	2	1	—
5	3	1	1	—
6	3	2	1	—
7	4	2	1	—
8	5	1	1	—
9	5	2	1	—
10	6	2	1	—
11	7	1	1	—
12	9	1	1	—
13	9	2	1	—
14	10	2	1	—
15	11	2	1	—
16	14	1	1	—
17	15	2	1	—
18	16	1	1	—
19	16	2	1	—
20	17	1	1	—
21	17	2	1	—
22	20	1	1	—
23	21	1	1	—
24	21	2	1	—
25	22	2	1	—
		(a)		
26	25	2	2	—
		(b)		

Table 3.

i	$\beta(i)$	$\gamma(i)$	$\mu(i)$	$\varphi(i)$
27	1	2	—	2
28	4	1	—	1
29	6	1	—	1
30	7	2	—	11
31	8	1	—	1
32	8	2	—	5
33	10	1	—	1
34	11	1	—	1
35	12	1	—	1
36	12	2	—	8
37	13	1	—	1
38	13	2	—	3
39	14	2	—	1
40	15	1	—	1
41	18	1	—	1
42	18	2	—	16
43	19	1	—	1
44	19	2	—	12
45	20	2	—	22
46	22	1	—	1
47	23	1	—	1
48	23	2	—	18
49	24	1	—	1
50	24	2	—	20
51	25	1	—	1
52	26	1	—	1
53	26	2	—	26

The functions τ and τ' are indicated in Table 4.

Table 4.

i	$\tau(i)$	$\tau'(i)$
7	2	—
13	5	—
14	3	—
19	16	—
24	11	17
26	22	—

Now we are able to obtain $\Gamma^*(\Gamma_1(\mathbf{D}))$ by applying Construction 2 as many times as possible (beginning with $\Gamma_1(\mathbf{D})$). We get that the 26 rows (seen in Table 2) are supplemented by 27 rows (as a third block) which are given in Table 3.

In course of forming Table 3, the values $\varphi(27)$, $\varphi(30)$, $\varphi(45)$ are determined in sense of case (c) of rule (iv) of Construction 2. The values $\varphi(32)$, $\varphi(36)$, $\varphi(42)$, $\varphi(48)$ are determined by case (f). The values $\varphi(38)$, $\varphi(39)$, $\varphi(44)$ are determined by case (e). $\varphi(50)$ and $\varphi(53)$ are determined by cases (d) and (b), respectively. (The remaining 15 values are by case (a).)

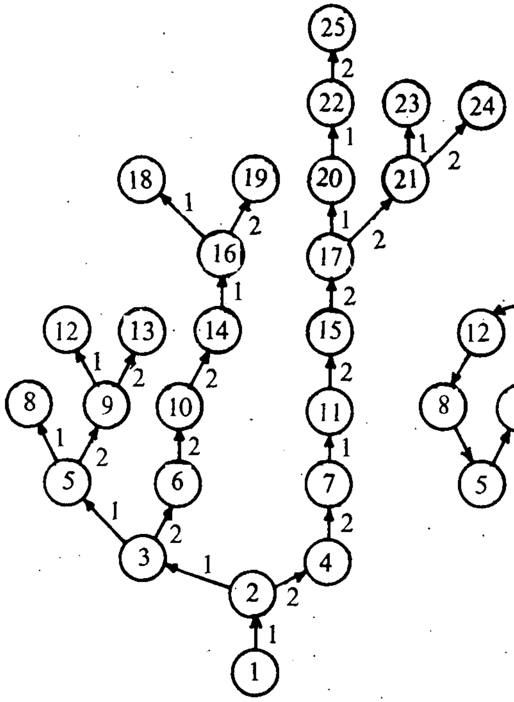

Fig. 4

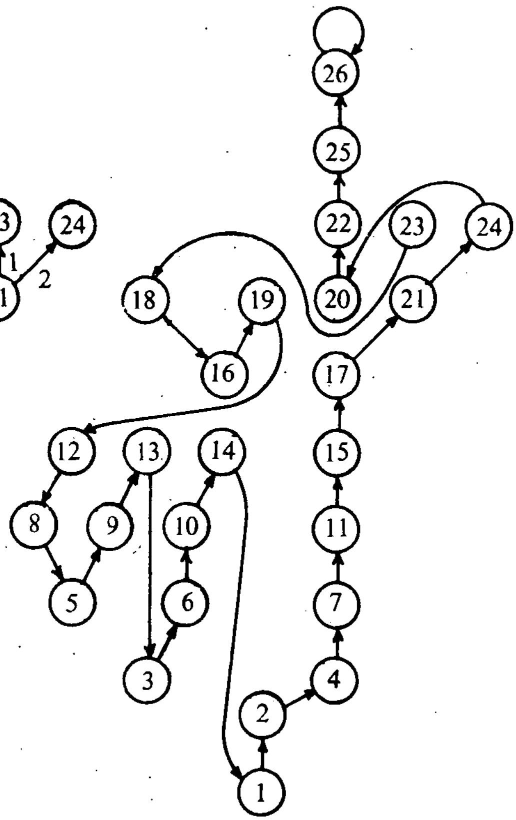

Fig. 5

Fig. 5 shows the (autonomous) automaton that is obtained from $\Gamma^*(\Gamma_1(\mathbf{D}))$ if solely the input sign $x^{(2)}$ is considered. It is evident that Corollary 1 (in § 5) is fulfilled by a suitable permutation π (for which $\pi(1)=23, \pi(2)=18, \pi(3)=16$, and so on).

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On the isomorphism-complete problems and polynomial time isomorphism

By GH. GRIGORAS

Introduction

One of the important open problems in computer science today is the computational complexity of deciding when two graphs are isomorphic. No polynomial time algorithm is known, nor is the problem known to be NP-complete. Many restrictions and generalizations of the problem have been the focus of much research during last years and many of these problems have turned out to be polynomial time equivalent to graph isomorphism ([3], [4], [6], [7], [9], [10]).

In this paper, starting from the results of Berman and Hartmanis paper on p-isomorphism [2] we give some analogous necesary and sufficient conditions for a language to be isomorphic under polynomial time mappings to graph isomorphism problem. Next we give the proof of the existence of p-isomorphism for some problems which are known to be polynomial time equivalent to graph isomorphism. We conjecture that all problems polynomial time equivalent to graph isomorphism problem are p-isomorphic.

Preliminaries

In our paper we suppose the reader is familiar with the terminology of complexity theory. In this section, we make precise some of the objects; for more details see [1], [5], [6], [8].

A language $A \subseteq \Sigma^*$ is said to be *reducible* to a language $B \subseteq \Gamma^*$ if there exists some function $f: \Sigma^* \to \Gamma^*$ such that $f(x) \in B$ iff $x \in A$, $\forall x \in \Sigma^*$. A is said to be *reducible* to B in *polynomial time (p-reducible)* if the function f is computed by a deterministic Turing machine M which runs in polynomial time.

A language L_0 is said to be $\mathscr{C}$-hard for some class of languages $\mathscr{C}$ if for every L in $\mathscr{C}$, L is p-reducible to L_0.

A language L_0 is complete for $\mathscr{C}$ if it is in $\mathscr{C}$ and is $\mathscr{C}$-hard.

By P (NP) we denote the class of languages accepted by deterministic (nondeterministic) Turing machines which run in polynomial time.

A language $A \subseteq \Sigma^*$ is said to be *p-isomorphic* to a language $B \subseteq \Gamma^*$ ([2]) iff there exists a bijection $f: \Sigma^* \to \Gamma^*$ such that f is a p-reduction of A to B and f^{-1} is a p-reduction of B to A.

Let $A \subseteq \Sigma^*$; the function $Z_A: \Sigma^* \to \Sigma^*$ is a *padding function* for the set A if it satisfies the following two properties

1. $Z_A(x) \in A$ iff $x \in A$, $\forall x \in \Sigma^*$;

2. Z_A is invertible (i.e. one-one).

The following theorem due to Berman and Hartmanis [2], is useful in the proof of the fact that the problems computationally equivalent with the graph isomorphism are p-isomorphic.

Theorem 1. Let $A \subseteq \Sigma^*$ and $B \subseteq \Gamma^*$ be two languages such that A is p-reducible to B and B is p-reducible to A (in other words, A and B are polynomially equivalent); furthermore let the language A have a padding function Z_A satisfying

1_A. Z_A has polynomial time complexity;

2_A. $(\forall y)[|Z_A(y)| > |y|^2 + 1]$;

and polynomial-time computable functions $S_A(-, -)$ and $D_A(-)$ satisfying

3_A. $(\forall x, y)[S_A(x, y) \in A$ iff $x \in A]$;

4_A. $(\forall x, y)[D_A(S_A(x, y)) = y]$.

Then B is p-isomorphic to A iff B has the polynomial-time computable functions S_B and D_B satisfying 3_B and 4_B.

Berman and Hartmanis show that all NP-complete languages known in the literature are p-isomorphic. If all NP-complete problems are p-isomorphic, then $P \neq NP$.

Now, let us consider the Graph Isomorphism Problem: are given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ isomorphic? In other words, is there any bijection h from V_1 to V_2 for which (v, w) is an edge in E_1 if and only if $(h(v), h(w))$ is an edge in E_2?

The complexity of Graph Isomorphism Problem is unknown yet and this problem has been the focus of much research in recent years ([3], [4], [7], [9], [10]). Many of the restrictions and generalizations of the problem turn out to be polynomial time equivalent to graph isomorphism [3].

Caracterization of problems p-isomorphic to graph isomorphism

In this section we apply the theorem of Berman—Hartmanis to the Graph Isomorphism Problem.

First, let us consider an enconding scheme in which a graph $G = (V, E)$ can be described as a word over an alphabet Σ (see [6] p. 10). Let us denote by $\bar{G}$ the enconding of G, and let $\#$ be a symbol not belonging to Σ. Then, the graph isomorphism problem can be formulated as the problem of recognizing the language

$$\mathrm{GI} = \{x \mid x \in (\Sigma \cup \{\#\})^*,\ x = \bar{G}_1 \# \bar{G}_2,\ G_1 \text{ is isomorphic to } G_2\}.$$

Let us note that we consider $\& \in \Sigma$ and by the word $\bar{G}_1 \& \bar{G}_2$, where $\bar{G}_1$ and $\bar{G}_2$ are the encondings of two graphs G_1 and G_2, we mean the enconding of graph with components G_1 and G_2.

Lemma 1. The language GI has a function denoted by $S_{\mathrm{GI}}(-,-)$ with the properties

i) S_{GI} has polynomial time complexity;

ii) $(\forall x, y)[S_{\mathrm{GI}}(x, y) \in \mathrm{GI}$ iff $x \in \mathrm{GI}]$.

Proof. Let us consider the language $\Delta \subseteq \{0, 1\}^*$ defined by $y \in \Delta$ iff

1) $\exists n \in N, \quad y = y_1 y_2 \dots y_{n^2}, \quad y_i \in \{0, 1\}, \quad i = 1, 2, \dots, n^2;$

2) $\forall i, j \quad 1 \leqq i, j \leqq n, \quad y_{(j-1)n+i} = y_{(i-1)m+j}.$

Note that the language Δ is decidable in polynomial time.
Now we define the function

$$S_{\mathrm{GI}}: (\Sigma \cup \{\#\})^* x \Delta \to (\Sigma \cup \{\#, \square, 0, 1\})^*,$$

where $\square \notin \Sigma$ is a new symbol by

$$S_{\mathrm{GI}}(x, y) = \begin{cases} \bar{G}_1 \& \bar{G} \# \bar{G} \& \bar{G}_2 & \text{if} \quad x = \bar{G}_1 \# \bar{G}_2, \\ x^2 \square y & \text{in other cases.} \end{cases}$$

The graph G which appear in the definition of S_{GI} is constructed as follows. Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, $V_1 = \{v_1, v_2, \dots, v_n\}$, $V_2 = \{w_1, w_2, \dots, w_m\}$ and $x = \bar{G}_1 \# \bar{G}_2$. Then $G = (Z, E)$ where $Z = \{Z_1, Z_2, \dots, Z_l\}$, $l = \sqrt{|y|}$, and the edge $(Z_r, Z_s) \in E$ iff $y_{(s-1)l+r} = 1$. In other words G is the graph with the adjacency matrix rows $y_{kl+1} y_{kl+2} \dots y_{(k+1)l}$, $0 \leqq k \leqq l-1$.[1)]

It is clear that G_1 and G_2 are isomorphic if and only if so are the graphs with encondings $\bar{G}_1$ & $\bar{G}$ and $\bar{G}$ & $\bar{G}_2$. Hence $S_{\mathrm{GI}}(x, y) \in \mathrm{GI}$ iff $x \in \mathrm{GI}$.

Furthermore it is easy to see that S_{GI} is computable in polynomial time which completes the proof of the lemma.

Lemma 2. The language GI has a function denoted by $D_{\mathrm{GI}}(-)$ with the properties

i) D_{GI} has polynomial time complexity;

ii) $(\forall x, y)$, $D_{\mathrm{GI}}(S_{\mathrm{GI}}(x, y)) = y$;

where S_{GI} is the function defined in Lemma 1.

Proof. Let us concider the function

$$D_{\mathrm{GI}}: (\Sigma \cup \{\#, \square, 0, 1\})^* \to \Delta \cup (\Sigma \cup \{\#, \square, 0, 1\})^*,$$

[1] For short, we say G has the adjacency matrix y.

where Δ is the language from Lemma 1, defined by

$$D_{\mathrm{GI}} = \begin{cases} y \text{ if } u=u_1 \,\&\, u_2 \# u_2 \,\&\, u_3,\ u_2 \text{ (in which \& does not occur) is the enconding} \\ \quad \text{of a graph the rows of adjacency matrix of which are } y=y_1 \ldots y_r, \\ z \quad \text{if} \quad u=u_1 \Box z, \\ u \text{ in other cases.} \end{cases}$$

From the definition of D_{GI} it follows that, given $u \in (\Sigma \cup \{\#, \Box, 0, 1,\})^*$ the computation of $D_{\mathrm{GI}}(u)$ can be made in polynomial time depending on $|u|$.

Now, let $x \in (\Sigma \cup \{\#\})^*$ and $y \in \Delta$. If $x=\bar{G}_1 \# \bar{G}_2$ then $S_{\mathrm{GI}}(x, y)=\bar{G}_1 \,\&\, \bar{G} \# \# \bar{G} \,\&\, \bar{G}_2$ and $D_{\mathrm{GI}}(G_1 \,\&\, \bar{G} \# \bar{G} \,\&\, \bar{G}_2)=y$ (the adjacency matrix of G). If x is not of the form $\bar{G}_1 \# \bar{G}_2$, then $S_{\mathrm{GI}}(x, y)=x^2 \Box y$ and $D_{\mathrm{GI}}(x^2 \Box y)=y$.

Hence, $\forall x, y$, $D_{\mathrm{GI}}(S_{\mathrm{GI}}(x, y))=y$ and the lemma is proved.

Lemma 3. The language GI has a padding function Z_{GI} such that

i) Z_{GI} has polynomial time complexity;

ii) $\forall x \in (\Sigma \cup \{\#\})^*$, $|Z_{\mathrm{GI}}(x)| > |x|^2+1$.

Proof: Let us define the function

$$Z_{\mathrm{GI}}: (\Sigma \cup \{\#\})^* \to (\Sigma \cup \{\#, \Box\})^*,$$

by

$$Z_{\mathrm{GI}}(x) = S_{\mathrm{GI}}(x, 1^{\varphi^2(|x|)}) \quad \text{for all} \quad x \in (\Sigma \cup \{\#\})^*$$

where $\varphi: N \to N$ is a function depending on enconding scheme. We will show that there exists this function such that condition ii) of lemma is satisfied. Let us note that Z_{GI} is a padding function. Indeed, from Lemma 1, we have $S_{\mathrm{GI}}(x, y) \in \mathrm{GI}$ iff $x \in \mathrm{GI}$, hence $Z_{\mathrm{GI}}(x) \in \mathrm{GI}$ iff $x \in \mathrm{GI}$, $\forall x \in (\Sigma \cup \{\#\})^*$. From the definition of S_{GI}, it follows that Z_{GI} is an injective function, hence Z_{GI} is invertible. It is clear that Z_{GI} has polynomial time complexity. It remains to prove that for all x,

$$|Z_{\mathrm{GI}}(x)| > |x|^2+1.$$

If $x \neq \bar{G}_1 \# \bar{G}_2$, then

$$|Z_{\mathrm{GI}}(x)| = |S_{\mathrm{GI}}(x, 1^{\varphi^2(|x|)})| = |x^2 \Box 1^{\varphi^2(|x|)}| > |x|^2+1.$$

If $x=\bar{G}_1 \# \bar{G}_2$, then

$$|Z_{\mathrm{GI}}(x)| = |S_{\mathrm{GI}}(\bar{G}_1 \# G_2, 1^{\varphi^2(|x|)})| = |\bar{G}_1 \,\&\, \bar{G} \# \bar{G} \,\&\, \bar{G}_2| =$$
$$= |\bar{G}_1 \# \bar{G}_2|+2(|\bar{G}|+1) = |x|+2|\bar{G}|+2.$$

Of course, $|\bar{G}|$ depends on $|x|$ because $y=1^{\varphi^2(|x|)}$. Let $e(n)$ be the length of $\bar{G}$ where G has n vertex, and let $e(n)$ be of order $O(n^k)$, $k \geqq 1$. Then $|\bar{G}|=e(n)= =e(\varphi(|x|))=O(\varphi(|x|)^k)$. If we consider $\varphi(n)=O(n^{2/k})$ then

$$|\bar{G}| = O((O(|x|^{2/k}))^k) = O(|x|^2),$$

hence

$$|Z_{\mathrm{GI}}(x)| = |x|+2O(|x|^2)+2.$$

It follows that we can find a function $\varphi(n)$ such that

$$|Z_{\mathrm{GI}}(x)| > |x|^2+1.$$

Theorem 2. Let A be a language polynomial time equivalent to GI (A is p-reducible to GI and GI is p-reducible to A). Then A is p-isomorphic to GI if and only if A has two polynomial time computable functions $S_A(-, -)$ and $D_A(-)$ such that

1) $(\forall x, y)[S_A(x, y) \in A$ iff $x \in A]$;

2) $(\forall x, y)[D_A(S_A(x, y)) = y]$.

Proof. From Lemmas 1—3 it follows that GI satisfies the conditions of Berman—Harmanis theorem.

Problems p-isomorphic to graph isomorphism

Booth and Colbourn [3] present a comprehensive list of problems which are known to be polynomial time equivalent to graph isomorphism. Such problems are called isomorphism complete.

Now, we consider some of these problems and prove that they are p-isomorphic to graph isomorphism.

1. *Directed Graph Isomorphism.* Given two directed graphs, are they isomorphic? Miller [10] shows this problem is isomorphism complete.

2. *Oriented Graph Isomorphism.* An oriented graph [3] is a digraph in which the presence of the arc (x, y) precludes the presence of (y, x). Oriented graph isomorphism problem is isomorphism complete [3].

3. *Bipartite Graph Isomorphism.* Given two bipartite graphs, are they isomorphic? This problem is isomorphism complete [3].

4. *Semiautomata Isomorphism.* A semiautomaton is a 3-tuple $A=(I, S, f)$, where I and S are finite sets of inputs and states respectively and $f\colon S \times I \to S$ is the transition function. Two semiautomata $A_1=(I_1, S_1, f_1)$ and $A_2=(I_2, S_2, f_2)$ are isomorphic if there exist two bijections $g\colon I_1 \to I_2$ and $h\colon S_1 \to S_2$ such that the following diagram commute:

$$\begin{tikzcd} S_1 \times I_1 \arrow[r, "f_1"] \arrow[d, "(h, g)"] & S_1 \arrow[d, "h"] \\ S_2 \times I_2 \arrow[r, "f_2"] & S_2 \end{tikzcd}$$

Semiautomata isomorphism problem is isomorphism complete ([3], [7]).

Lemma 4. Directed graph isomorphism is p-isomorphic to graph isomorphism.

Proof. Let us define the function S_{DGI} and D_{DGI} satisfying Theorem 2, where

$$\mathrm{DGI} = \{x \mid x = \bar{G}_1 \# \bar{G}_2,\ \bar{G}_1 \text{ and } \bar{G}_2 \text{ are encondings of two directed isomorphic graphs}\} \subseteq (\Sigma \cup \{\#\})^*.$$

Let us consider $\Delta = \{y \mid y \in \{0, 1\}^*,\ |y| = n^2,\ n \in N\}$. Then, for all $x \in (\Sigma \cup \{\#\})^*$, $y \in \Delta$

$$S_{\mathrm{DGI}}(x, y) = \begin{cases} \bar{G}_1 \& \bar{G} \# \bar{G} \& \bar{G}_2 & \text{if } x = \bar{G}_1 \# \bar{G}_2; \\ x \square y & \text{otherwise,} \end{cases}$$

where $\bar{G}$ is the enconding of the directed graph which has the adjacent matrix y.

Like in Lemma 2 we define D_{DGI}, by $\forall u \in (\Sigma \cup \{\square, \#, 0, 1\})^*$

$$D_{\mathrm{DGI}}^{(u)} = \begin{cases} y \text{ if } u = u_1 \& u_2 \# u_2 \& u_3, \ u_2 \text{ is the enconding of the} \\ \quad \text{directed graph, the adjacent matrix of which is } y; \\ z \text{ if } u = u_1 \square z; \\ u \text{ in other cases.} \end{cases}$$

It is obvious that S_{DGI} and D_{DGI} are polynomial time computable and

1) $(\forall x, y) \quad S_{\mathrm{DGI}}(x, y) \in \mathrm{DGI}$ iff $x \in \mathrm{DGI}$;

2) $(\forall x, y) \quad D_{\mathrm{DGI}}(S_{\mathrm{DGI}}(x, y)) = y$.

Lemma 5. Oriented graph isomorphism is p-isomorphic to graph isomorphism.

Proof. Like in Lemma 4, we construct the functions S_{OGI} and D_{OGI} satisfying Theorem 2. In this case we take

$$\Delta = \{y / y \in \{0, 1\}^*, \ |y| = n^2, \ y_{(j-1)n+i} = 1 \Rightarrow y_{(i-1)n+j} = 0\}.$$

It is clear that Δ can be recognized in polynomial time and the graph with adjacent matrix $y \in \Delta$ is an oriented graph.
The functions are defined in the manner of Lemma 4.

Lemma 6. Bipartite graph isomorphism is p-isomorphic to graph isomorphism.

Proof. Let us consider the language $\Delta \subseteq \{0, 1\}^*$ defined by

$$\Delta = \{y | y = (0^k 1^k)^k (1^k 0^k)^k, \ k \in N\}.$$

It is easy to see that Δ can be recognized in polynomial time and, the graphs with $2k$ vertices and adjacent matrix $y \in \Delta$ are bipartite graphs. Like in Lemma 4, there exist the functions S_{BGI} and D_{BGI} satisfying Theorem 2.

REMARK. The bipartite graph constructed in Lemma 6 is also a regular graph: all the vertices have the degree k. Hence the regular graph isomorphism (which is isomorphism complete [3], [10]) is p-isomorphic to graph isomorphism.

Lemma 7. Semiautomata isomorphism is p-isomorphic to graph isomorphism.

Proof. Let $A = (I, S, f)$ be a semiautomaton, $I = \{i_1, i_2, \ldots, i_n\}$, $S = \{s_1, \ldots, s_m\}$ and $f(s_k, i_j) = f_{kj} \in S$ $1 \leqq k \leqq m$, $1 \leqq j \leqq n$. We consider an enconding scheme in which A is represented by the word

$$\bar{A} = i[1] \ldots i[n] * s[1] \ldots s[m] / f_{11} f_{21} \ldots f_{m1} / f_{12} f_{22} \ldots f_{m2} / \ldots / f_{1n} f_{2n} \ldots f_{mn},$$

where

$$f_{ij} = s[l] \quad \text{if} \quad f(s_k, i_j) = s_l.$$

Now, if A_1 and A_2 are two semiautomata with the same input sets and disjoint sets of states, the semiautomaton encoded by $\bar{A}_1$ & $\bar{A}_2$ is the semiautomaton with the same inputs, the set of states is the union of states of A_1 and A_2 and the transition function is defined in natural way.

Set $SI=\{\bar{A}_1\#\bar{A}_2 \mid A_1$ is isomorphic to $A_2\}\subset\Gamma^*$. We define $S_{SI}\colon \Gamma^*\times\Delta\to(\Gamma\cup\{\square,0,1\})^*$ and $D_{SI}\colon (\Gamma\cup\{\square\})^*\to\Delta\cup(\Gamma\cup\{\square,0,1\})^*$, where $\Delta\subseteqq\{0,1\}^*$, in the following way:

Let $x=\bar{A}_1\#\bar{A}_2\in SI$ and $y\in\Delta, y=y_1y_2\dots y_l$. Consider the semiautomata $A_1'=(I_1,\Sigma,g_1)$ and $A_2'=(I_2,\Sigma,g_2)$ where I_1 and I_2 are the input sets of A_1 and A_2 respectively, $\Sigma=\{\sigma_1,\dots,\sigma_l,\bar{\sigma}\}$ such that $\Sigma\cap S_i=\Phi$, $i=1,2$ and g_j $(j=1,2)$ are defined by $1\leqq k\leqq l-1$,

$$g_j(\sigma_k,i_j)=\begin{cases}\sigma_{k+1} & y_k=1,\\ \bar{\sigma} & y_k=0,\end{cases}$$

$$g_j(\sigma_l,i_j)=\begin{cases}\sigma_1 & y_l=1,\\ \bar{\sigma} & y_l=0,\end{cases}$$

$$g_j(\bar{\sigma},y)=\bar{\sigma},$$

for all $i_j\in I_j$ $(j=1,2)$. Then we define

$$S_{AI}(x,y)=\begin{cases}\bar{A}_1\&\bar{A}_1'\#\bar{A}_2'\&\bar{A}_2 & \text{if } x=\bar{A}_1\#\bar{A}_2,\\ x\square y & \text{otherwise}\end{cases}$$

and

$$D_{SI}(u)=\begin{cases}y \text{ if } u=\bar{A}_1\&\bar{A}_2\#\bar{A}_3\&\bar{A}_4 \text{ and } A_2,\ A_3 \text{ have}\\ \quad\text{the same states and transition functions,}\\ z \text{ if } u=x\square z,\\ u \text{ in other cases,}\end{cases}$$

where $y\in\{0,1\}^*$ is determined in the following way:

If $A_2=(I_2,\Sigma,f_2)$, $A_3=(I_3,\Sigma,f_3)$, $\Sigma=\{\sigma_1,\sigma_2,\dots,\sigma_n\}$ then $y=y_1,\dots,y_{n-1}$ where

$$y_k=1 \quad\text{if}\quad f_2(\sigma_k,i_2)=f_3(\sigma_k,i_3)=\sigma_{k+1},\quad \forall i_2\in I_2,\quad i_3\in I_3,\quad 1\leqq n\leqq n-2;$$

$$y_{n-1}=1 \quad\text{if}\quad f_2(\sigma_{n-1},i_2)=f_3(\sigma_{n-1},i_3)=\sigma_1,\quad \forall i_2\in I_2,\quad i_3\in I_3;$$

$$y_k=0 \quad\text{in other cases}\quad 0\leqq k\leqq n-1.$$

It is not hard to verify that S_{AI} and D_{SI} satisfy the conditions of Theorem 2.

Conclusions

We have given a caracterisation of p-isomorphic problems to graph isomorphism showing that graph isomorphism satisfy the conditions of Berman—Hartmanis Theorem. Next we have proved that some of the problems which are polynomial time equivalent to graph isomorphism are p-isomorphic. Are all the isomorphism complete problems p-isomorphic? Perhaps the answer of this question is useful in determining the complexity of graph isomorphism problem.

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Remarks on finite commutative automata

By Z. ÉSIK and B. IMREH

A. C. Fleck has proved in [1] that a strongly connected commutative quasi-automaton — called perfect quasi-automaton in [2] — is directly irreducible if and only if its characteristic semigroup, which is actually an Abelian group, is directly irreducible. I. Peák generalized this result for commutative cyclic automata (cf. [4]). In this paper we point out that this connection between automata and their characteristic semigroups is based on the fact that the congruence lattice of a commutative cyclic automaton is isomorphic to the congruence lattice of its characteristic semigroup. Furthermore, we give a characterization of strongly connected commutative automata through their corresponding algebraic structures. Finally, we employ these results to obtain isomorphically complete systems for the class of all strongly connected commutative automata with respect to the direct product and quasi-direct product.

By an automaton $\mathbf{A}=(A, X, \delta)$ we always mean a finite automaton. Isomorphisms of automata are A-isomorphisms. For arbitrary automaton $\mathbf{A}$ we denote by $C(\mathbf{A})$ and $C(S(\mathbf{A}))$ the congruence lattices of $\mathbf{A}$ and its characteristic semigroup,* respectively. Otherwise we use the terminology and notations in accordance with [2].

Theorem 1. The following three conditions are satisfied for arbitrary commutative cyclic automaton $\mathbf{A}=(A, X, \delta)$:

(i) $S(\mathbf{A}) \cong E(\mathbf{A})$,

(ii) $|A| = |E(\mathbf{A})|$,

(iii) $C(\mathbf{A}) \cong C(S(\mathbf{A}))$.

Proof. The validity of (i) and (ii) was already proved by I. Peák in [4]. The proof of this fact is based on the observation that every commutative cyclic automaton $\mathbf{A}$ is a free commutative automaton generated by one of its states. In other words, $\mathbf{A}$ is a free commutative unoid in the equational class generated by $\mathbf{A}$ and each generator of $\mathbf{A}$ is a free generator of $\mathbf{A}$. This means that if $a_0 \in A$ generates the automaton $\mathbf{A}$ then every correspondence $a_0 \to a(\in A)$ has a unique A-homomorphic extension of $\mathbf{A}$ into itself. By Corollary to Theorem 24.2 in [3] this implies that $\mathbf{A}' \cong \mathbf{A}$ where $\mathbf{A}'=(S(\mathbf{A}), X, \delta')$ and δ' is defined by $\delta'(C_\varrho(p), x)=C_\varrho(px)$.

* By the characteristic semigroup $S(\mathbf{A})$ of an automaton $\mathbf{A}$ we always mean a monoid with identity $C_\varrho(\lambda)$, where λ denotes the empty word.

Indeed, if a_0 denotes an arbitrary generator of A then a natural isomorphism can be given by the correspondence $C_\varrho(p) \to \delta(a_0, p)$ $(C_\varrho(p) \in S(\mathbf{A}))$. Therefore $C(\mathbf{A}) \cong C(\mathbf{A}')$. On the other hand $C(\mathbf{A}') \cong C(\mathbf{A}'')$ where the automaton A″ is the semigroup-automaton corresponding to A′ with transition $\delta''(C_\varrho(p), C_\varrho(q)) = C_\varrho(pq)$. It is evident that each congruence relation of the semigroup $S(\mathbf{A})$ is a congruence relation of the semigroup-automaton A″ as well. The converse follows by the commutativity of $S(\mathbf{A})$. Thus $C(\mathbf{A}'') \cong C(S(\mathbf{A}))$. Putting together these isomorphisms we get $C(\mathbf{A}) \cong C(S(\mathbf{A}))$. This ends the proof of Theorem 1.

It is interesting to note that I. Peák gave an example in [4] for a commutative automaton which is not cyclic but satisfies conditions (i) and (ii) of Theorem 1. It is not difficult to see that this example does not satisfy (iii). We now give another automaton which contents each of the conditions (i)—(iii) of Theorem 1 and which is not cyclic. This automaton is the following $\mathbf{A} = (\{1, 2, 3, 4\}, \{x, y\}, \delta)$, where δ is defined by the table below:

	1	2	3	4
x	1	2	3	2
y	2	3	3	3

Thus the converse of Theorem 1 is not true in general. However, in spite of the previous example, in case of strongly connected commutative automata, we have succeeded in proving a certain converse of Theorem 1.

Theorem 2. An automaton $\mathbf{A} = (A, X, \delta)$ is strongly connected and commutative if and only if each of the following conditions is satisfied by A:

(i) $S(\mathbf{A})$ is an Abelian group,

(ii) $S(\mathbf{A}) \cong E(\mathbf{A})$,

(iii) $|A| = |E(\mathbf{A})|$,

(iv) $C(\mathbf{A}) \cong C(S(\mathbf{A}))$.

Proof. Necessity follows by Theorem 1. Conversely, the commutativity of A is immediate by (i). In order to prove that A is strongly connected first observe that since (ii) is also satisfied by A there is a natural isomorphism ν of $S(\mathbf{A})$ onto $E(\mathbf{A})$. This isomorphism is defined in the following manner: $\nu(C_\varrho(p))$ is the mapping induced by the word p on the set of states of A. In other words, $\nu(C_\varrho(p))$ is simply the polynomial induced by p in the automaton A being considered as a unoid.

Assume to the contrary A is not strongly connected. As $S(\mathbf{A})$ is a group we can decompose A into the direct sum of its strongly connected subautomata $\mathbf{A}_t = (A_t, X, \delta_t)$ $(t=1, \dots, n, n>1)$. According to the previously established natural isomorphism ν, the inclusion $\varphi(A_t) \subseteq A_t$ $(t=1, \dots, n)$ is satisfied for any $\varphi \in E(\mathbf{A})$. Consequently, $|A_t| > 1$ $(t=1, \dots, n)$ and $\prod_{t=1}^{n} E(\mathbf{A}_t) \cong E(\mathbf{A})$ under the mapping $\varphi \to (\varphi_{|A_1}, \dots, \varphi_{|A_n})$. Thus, by Theorem 1 and our assumption (iii), $\prod_{t=1}^{n} |A_t| = \prod_{t=1}^{n} |E(\mathbf{A}_t)| = |E(\mathbf{A})| = |A| = \sum_{t=1}^{n} |A_t|$.

It is not difficult to see by $|A_t|>1$ $(t=1, \dots, n)$ that the above equality is possible only if $n=2$ and $|A_1|=|A_2|=2$. In this case $C(\mathbf{A})$ contains the chain induced by the compatible partitions

$$\begin{aligned} C_0 &= \{\{a_{11}\}, \{a_{12}\}, \{a_{21}\}, \{a_{22}\}\}, \\ C_1 &= \{A_1, \{a_{21}\}, \{a_{22}\}\}, \\ C_2 &= \{A_1, A_2\}, \\ C_3 &= \{A\}, \end{aligned}$$

where $A_t=\{a_{t1}, a_{t2}\}$ $(t=1, 2)$. On the other hand $S(\mathbf{A})$ can contain only shorter chains. This is a simple consequence of the well-known fact that the congruence lattice of an Abelian group is isomorphic to the lattice of its subgroups.

COROLLARY. The following conditions are equivalent for every strongly connected commutative automaton $\mathbf{A}=(A, X, \delta)$:

(i) $\mathbf{A}$ is subdirectly irreducible,

(ii) $\mathbf{A}$ is directly irreducible,

(iii) $S(\mathbf{A})$ is a cyclic group of prime-power order,

(iv) The cardinality of A is a prime-power and there is an input-sign $x\in X$ inducing a cyclic permutation of A.

Proof. The equivalence of (i), (ii) and (iii) is a consequence of Theorem 2 and the Fundamental Theorem of Finite Abelian Groups. The implication (iv)$\Rightarrow$(iii) is trivial. It remains to prove that (iii)$\Rightarrow$(iv).

In the proof of Theorem 1 we have shown that $\mathbf{A}\cong\mathbf{A}'$ therefore, $|A|$ is a prime-power, say $|A|=r^n$. Assume that none of the signs $x\in X$ induces a cyclic permutation of A. Then, for each $x\in X$, the order of $C_\varrho(x)$ in $S(\mathbf{A})$ is less than r^n. But this yields a contradiction since for arbitrary word $p=x_1\dots x_k$ the order of $C_\varrho(p)$ can not exceed the maximum of the orders of the signs $x_1, \dots, x_k$, which completes the proof of the Corollary.

It is evident that the automata given in (iv) form a minimal isomorphically complete system of strongly connected commutative automata with respect to the direct product for any fixed set of input signs X. We proceed by stating a similar result with respect to the quasi-direct product.

Let $n(>1)$ be an arbitrary natural number and let $\mathbf{M}_n=(\{0, \dots, n-1\}, \{x_0, \dots, x_{n-1}\}, \delta_n)$ denote the automaton with transition $\delta_n(j, x_s)=j+s \pmod n$ $(j\in\{0, \dots, n-1\}, x_s\in\{x_0, \dots, x_{n-1}\})$. Let $\mathfrak{K}$ consist of all automata $\mathbf{M}_n$ where $n>1$ and n is a prime-power.

Theorem 3. A system Σ of automata is isomorphically complete for the class of all strongly connected commutative automata with respect to the quasi-direct product if and only if each $\mathbf{M}_n\in\mathfrak{K}$ can be embedded isomorphically into a quasi-direct product of an automaton $\mathbf{A}\in\Sigma$ with a single factor.

Proof. Sufficiency is obvious. In order to prove necessity let $\mathbf{M}_n\in\mathfrak{K}$ be arbitrary. $\mathbf{M}_n$ can be embedded isomorphically into a quasi-direct product of automata from Σ, and hence it can be embedded isomorphically into a direct product whose each component is a quasi-direct product of an automaton from Σ with a single factor. But, by Corollary to Theorem 2, $\mathbf{M}_n$ is subdirectly irreducible. Therefore $\mathbf{M}_n$ can be embedded isomorphically into a quasi direct product of an automaton from Σ with a single factor.

COROLLARY. There exists no system of automata which is isomorphically complete for the class of all strongly connected commutative automata with respect to the quasi-direct product and minimal.

Proof. It is easy to show that the class $\mathfrak{K} \setminus \{\mathbf{M}_{r^t} | t \leq s\}$ constitutes a complete system for any fixed prime r and integer s.

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Functor state machines

By G. HORVÁTH

In the present paper we introduce a notion of a machine in an arbitrary category. A machine in a category is a computational device computing a morphism from a free algebra to another one. The computation is defined by means of homomorphic extension. We are dealing with two types of machines each of them having a functor as its state. These two families of machines are related to bottom-up and top-down tree transformations, respectively. The state functor of a machine working in top-down way is required to have a right adjoint. We show that every top-down computation can be carried out in bottom-up way.

A special type of machines, namely the generalized sequential machines in categories having binary products are investigated. A generalized sequential machine is a machine whose state funtor is a product functor and whose final state transformation is the corresponding projection. Morphisms can be computed by generalized sequential machines in a category are characterized. We show that the process transformations of Arbib and Manes, and the generalized sequential machines in a category have the same processing capacity. Results of the present paper have been announced in [6].

1. Preliminaries

We assume the reader to be familiar with the elements of category theory such as the notion of category, functor and natural transformation. Now we will list some basic notions, definitions and results to be used in the sequel.

DEFINITION 1.1. Let $\mathscr{K}$ be any category and let X: $\mathscr{K} \to \mathscr{K}$ be an endofunctor. An *X-algebra* is a pair (A, d) where A is an object and d: $XA \to A$ is a morphism in $\mathscr{K}$. Given two X-algebras (A, d), (A', d'), a morphism h: $A \to A'$ is an *X-homomorphism* if the diagram

$$\begin{tikzcd} A' & XA' \arrow[l, "d'"'] \\ A \arrow[u, "h"] & XA \arrow[l, "d"] \arrow[u, "Xh"'] \end{tikzcd} \tag{1.1}$$

is commutative.

Definition 1.2 (Arbib—Manes [3]). Let A be an object in $\mathcal{K}$. A *free X-algebra* over A is an X-algebra $(X^{\#}A, \mu_0 A)$ coupled with a morphism $\eta A\colon A \to X^{\#}A$ with the universal property that for every other X-algebra (B, d) and morphism $f\colon A\to B$ there exists a unique X-homomorphism $f^{\#}\colon (X^{\#}A, \mu_0 A)\to(B, d)$ such that $f^{\#}\cdot\eta A = f$. That is, given d and f there is a unique $f^{\#}$ such that (1.2) commutes.

$$\begin{tikzcd} & B & XB \arrow[l, "d"'] \\ A \arrow[ur, "f"] \arrow[r, "\eta A"] & X^{\#}A \arrow[u, "f^{\#}"] & XX^{\#}A \arrow[l, "\mu_0 A"'] \arrow[u, "Xf^{\#}"'] \end{tikzcd} \tag{1.2}$$

The morphism $f^{\#}$ in (1.2) iscalled the *X-homomorphic extension* of f from the free X-algebra $(X^{\#}A, \mu_0 A)$ into the X-algebra (B, d).

Following Adámek and Trnková (see [1]) we say that a functor $X\colon \mathcal{K}\to\mathcal{K}$ is a *varietor* if there exists a free X-algebra over each object in $\mathcal{K}$. Arbib and Manes use the terms input process or recursion process [3, 4] depending on context. Let $X\colon \mathcal{K}\to\mathcal{K}$ be a varietor. If we fix a choice of $\eta A\colon A\to X^{\#}A$, $\mu_0 A\colon XX^{\#}A\to X^{\#}A$ in (1.2) for each object A in $\mathcal{K}$, and for every morphism $f\colon A\to B$ the morphism $X^{\#}f\colon X^{\#}A\to X^{\#}B$ is defined to be the X-homomorphic extension of $\eta B\cdot f$, i.e.

$$\begin{tikzcd} B \arrow[r, "\eta B"] & X^{\#}B & XX^{\#}B \arrow[l, "\mu_0 B"'] \\ A \arrow[u, "f"] \arrow[r, "\eta A"] & X^{\#}A \arrow[u, "X^{\#}f"'] & XX^{\#}A \arrow[l, "\mu_0 A"'] \arrow[u, "XX^{\#}f"'] \end{tikzcd} \tag{1.3}$$

then we get a functor $X^{\#}\colon \mathcal{K}\to\mathcal{K}$. Moreover, we obtain a pair of natural transformations

$$\eta\colon I_{\mathcal{K}} \xrightarrow{\cdot} X^{\#}, \quad \mu_0\colon XX^{\#} \xrightarrow{\cdot} X^{\#},$$

the *insertion of generators* and the *free operation* of X, respectively. We omit the subscript in the identity functor $I_{\mathcal{K}}\colon \mathcal{K}\to\mathcal{K}$ whenever $\mathcal{K}$ is understood. Note that each varietor X yields a family of morphisms $\mu A\colon X^{\#}X^{\#}A\to X^{\#}A$ defined by the diagram

$$\begin{tikzcd} & X^{\#}A & XX^{\#}A \arrow[l, "\mu_0 A"'] \\ X^{\#}A \arrow[ur, "1_{X^{\#}A}"] \arrow[r, "\eta X^{\#}A"] & X^{\#}X^{\#}A \arrow[u, "\mu A"'] & XX^{\#}X^{\#}A \arrow[l, "\mu_0 X^{\#}A"'] \arrow[u, "X\mu A"'] \end{tikzcd} \tag{1.4}$$

where $1_{X^{\#}A}\colon X^{\#}A\to X^{\#}A$ is the identity morphism. One can show by an easy computation that μA is natural in A, i.e. we have a natural transformation

$\mu\colon X^{\#}X^{\#} \dot{\to} X^{\#}$, the *extended free operation* of X, rendering the diagram (1.5) commutative.

$$\begin{tikzcd} & X^{\#} & XX^{\#} \arrow[l, "\mu_0"'] \\ X^{\#} \arrow[ru, "1_{X^{\#}}"] \arrow[r, "\eta X^{\#}"'] & X^{\#}X^{\#} \arrow[u, "\mu"] & XX^{\#}X^{\#} \arrow[l, "\mu_0 X^{\#}"] \arrow[u, "X\mu"'] \end{tikzcd} \tag{1.5}$$

The basic algebraic structure in string processing is X_0^*, the free monoid over a set X_0 of generators. Monads, rather than monoids are fundamental in our development. Now we recall the definition of a monad.

DEFINITION 1.3. A *monad* (T, η, μ) in a category $\mathscr{K}$ consists of a functor $T\colon \mathscr{K} \to \mathscr{K}$ and two natural transformations

$$\eta\colon I \dot{\to} T, \quad \mu\colon TT \dot{\to} T$$

which make the following diagrams commute.

$$\begin{tikzcd} T \arrow[r, "\eta T"] \arrow[rd, "1_T"'] & TT \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[ld, "1_T"] \\ & T & \end{tikzcd} \qquad \begin{tikzcd} TTT \arrow[r, "\mu T"] \arrow[d, "T\mu"'] & TT \arrow[d, "\mu"] \\ TT \arrow[r, "\mu"] & T \end{tikzcd} \tag{1.6}$$

The diagrams in (1.6) are called unitary and associativity axioms, respectively. We state, without proof, the following well-known fact: for every varietor X the triple $(X^{\#}, \eta, \mu)$ is a monad in $\mathscr{K}$, where η is the insertion of the generators and μ is the extended free operation of X.

DEFINITION 1.4. Let (T, η, μ) be a monad in $\mathscr{K}$. A *T-monad algebra* is a pair (A, d) consisting of an object A of $\mathscr{K}$ and a $\mathscr{K}$-morphism $d\colon TA \to A$ such that

$$\begin{tikzcd} & A & TA \arrow[l, "d"'] \\ A \arrow[ru, "1_A"] \arrow[r, "\eta A"'] & TA \arrow[u, "a"] & TTA \arrow[l, "\mu A"] \arrow[u, "Td"'] \end{tikzcd} \tag{1.7}$$

It is easy to prove that the pair $(X^{\#}A, \mu A)$ is an $X^{\#}$-monad algebra for every varietor X and object A.

CONVENTION 1.5. In the remaining of this paper if a varietor is referred to by the letter X, then the insertion of the generators, the free operation and the extended free operation of X are denoted by η, μ_0 and μ, respectively

$$\eta\colon I \dot{\to} X^{\#}, \quad \mu_0\colon XX^{\#} \dot{\to} X^{\#}, \quad \mu\colon X^{\#}X^{\#} \dot{\to} X^{\#}.$$

If we use the letter Y to denote another varietor then the items above are denoted by the same letters but with bar, i.e. $\bar{\eta}$, $\bar{\mu}_0$ and $\bar{\mu}$.

PROPOSITION 1.6. Let $X\colon \mathcal{K}\to\mathcal{K}$ be a varietor. Given functors $F, G\colon \mathcal{K}\to\mathcal{K}$ and natural transformations $\delta\colon XG \dot{\to} G$, $\varphi\colon F\dot{\to} G$ there is a unique natural transformation $\varphi^{\#}\colon X^{\#}F\dot{\to} G$ such that the following diagram is commutative.

$$\begin{tikzcd} & G & XG \arrow[l, "\delta"'] \\ F \arrow[ur, "\varphi"] \arrow[r, "\eta F"'] & X^{\#}F \arrow[u, "\varphi^{\#}"] & XX^{\#}F \arrow[l, "\mu_0 F"] \arrow[u, "X\varphi^{\#}"'] \end{tikzcd} \tag{1.8}$$

Proof is immediate.

DEFINITION 1.7. An *adjunction* $(F, U, \nu, \varepsilon)\colon \mathcal{K}\to\mathcal{L}$ consists of a pair of functors $F\colon \mathcal{K}\to\mathcal{L}$, $U\colon \mathcal{L}\to\mathcal{K}$ and natural transformations $\nu\colon I_{\mathcal{K}}\dot{\to} UF$, $\varepsilon\colon FU\dot{\to} I_{\mathcal{L}}$ (called *unit* and *counit*, respectively) subject to the so called "triangular identities":

$$\begin{tikzcd} U \arrow[r, "\nu U"] \arrow[dr, "1_U"'] & UFU \arrow[d, "U\varepsilon"] \\ & U \end{tikzcd} \qquad \begin{tikzcd} FUF \arrow[d, "\varepsilon F"'] & F \arrow[l, "F\nu"'] \arrow[dl, "1_F"] \\ F \end{tikzcd} \tag{1.9}$$

F is said to be a *left adjoint* to U and U a *right adjoint* to F. We say that a functor F has right adjoint, if there is a functor U right adjoint to F.

2. Machines

In this section we introduce a notion of a machine in an arbitrary category. This is based on the notion of the free algebra. A machine is a computational device which computes a morphism of a free algebra into another one. The basic idea of our development — due to Alagić [2] — is to take a functor to be the state of a machine. Alagić offered in his paper [2] the general concept of a direct state transformation which took the form $XQ\dot{\to} QY^{\#}$, where X and Y are varietors and Q now is a functor. Arbib and Manes remarked in [4] that the Alagić approach has one flaw: because Q is a functor rather than an object, thus running the direct state transformation yields a natural transformation $X^{\#}Q\dot{\to} QY^{\#}$ instead of a morphism $X^{\#}A\to Y^{\#}B$ between free algebras. But, in spite of this note there is a general way in which we can extract from $X^{\#}Q\dot{\to} QY^{\#}$ a "state-free" input-output response of the form $X^{\#}A\to Y^{\#}B$. Thus, the benifits of the Alagić approach can be obtained in any category, not only those having binary products. Appart from the fact that we actually do not use the notion of the direct state transformation of Alagić in the definition of a machine and its response, there is a close relationship between them. We will show this relationship. There are several adventages of taking a functor to be the state of a machine. First of all this provides a uniform treatment of top-down and bottom-up computations which are well-known in the theory of tree transformations (see Engelfriet [5]).

Definition 2.1. Let A, B be objects of a category $\mathscr{K}$, and let X, Y be varietors in $\mathscr{K}$. A *machine* $M: (A, X) \to (B, Y)$ in $\mathscr{K}$ is $M=(Q, i, \sigma, \beta)$, where

$Q: \mathscr{K} \to \mathscr{K}$ is a functor, the *state functor,*
$i: A \to QY^\# B$ is a morphism, the *initial state-output* morphism,
$\sigma: XQ \stackrel{\cdot}{\to} QY^\#$ is a natural transformation, the *transition,*
$\beta: Q \stackrel{\cdot}{\to} I$ is a natural transformation, the *final state* transformation.

Definition 2.2. Let $M=(Q, i, \sigma, \beta): (A, X) \to (B, Y)$ be a machine in $\mathscr{K}$. The *response* of M is the morphism $f_M: X^\# A \to Y^\# B$ defined by the composite

$$f_M: X^\# A \xrightarrow{i^\#} QY^\# B \xrightarrow{\beta Y^\# B} Y^\# B, \tag{2.1}$$

where $i^\#$ is the *run map* of M, i.e. the X-homomorphic extension

$$\begin{tikzcd}
& QY^\# B & QY^\# Y^\# B \arrow[l, "Q\bar{\mu}B"'] & XQY^\# B \arrow[l, "\sigma Y^\# B"'] \\
A \arrow[ur, "i"] \arrow[r, "\eta A"] & X^\# A \arrow[u, "i^\#"'] & & XX^\# A \arrow[ll, "\mu_0 A"'] \arrow[u, "Xi^\#"']
\end{tikzcd} \tag{2.2}$$

of the initial state-output i.

By Proposition 1.6 the transition $\sigma: XQ \stackrel{\cdot}{\to} QY^\#$ has a unique extension $\sigma^\#: X^\# Q \stackrel{\cdot}{\to} QY^\#$ defined by

$$\begin{tikzcd}
& QY^\# & QY^\# Y^\# \arrow[l, "Q\bar{\mu}"'] & XQY^\# \arrow[l, "\sigma Y^\#"'] \\
Q \arrow[ur, "Q\bar{\eta}"] \arrow[r, "\eta Q"] & X^\# Q \arrow[u, "\sigma^\#"'] & & XX^\# Q \arrow[ll, "\mu_0 Q"'] \arrow[u, "X\sigma^\#"']
\end{tikzcd} \tag{2.3}$$

$\sigma^\#$ is called the *extended transition* of the machine M. Natural transformations like $\sigma^\#$ in (2.3) were studied by Alagić in [2] under the name "direct state transformation".

We show that the response of a machine M can be expressed in terms of the extended transition of M.

Statement 2.3. Let $M=(Q, i, \sigma, \beta): (A, X) \to (B, Y)$ be a machine in $\mathscr{K}$. Then the response of M is

$$f_M = \beta Y^\# B \cdot Q\bar{\mu}B \cdot \sigma^\# Y^\# B \cdot X^\# i. \tag{2.4}$$

Proof. Consider the following diagram.

$$
\begin{tikzcd}
 & QY^{\#}B \arrow[r, phantom, "\mathrm{d)}"] & QY^{\#}Y^{\#}B \arrow[l, "Q\bar{\mu}B"'] \arrow[r, phantom, "\mathrm{e)}"] & XQY^{\#}B \arrow[l, "\sigma Y^{\#}B"'] \\
 & QY^{\#}Y^{\#}B \arrow[u, "Q\bar{\mu}B"'] \arrow[u, phantom, "\mathrm{c)}", shift right=4ex] & QY^{\#}Y^{\#}Y^{\#}B \arrow[l, "Q\bar{\mu}Y^{\#}B"'] \arrow[u, "QY^{\#}\bar{\mu}B"'] \arrow[d, phantom, "\mathrm{f)}"] & XQY^{\#}Y^{\#}B \arrow[l, "\sigma Y^{\#}Y^{\#}B"'] \arrow[u, "XQ\bar{\mu}B"'] \\
QY^{\#}B \arrow[r, "\eta QY^{\#}B"] \arrow[ur, "Q\bar{\eta}Y^{\#}B"'] \arrow[uur, "Q1_{Y^{\#}B}"] \arrow[d, phantom, "\mathrm{a)}", shift left=10ex] & X^{\#}QY^{\#}B \arrow[u, "\sigma^{\#}Y^{\#}B"'] \arrow[u, phantom, "\mathrm{b)}", shift right=4ex] & & XX^{\#}QY^{\#}B \arrow[ll, "\mu_0 QY^{\#}B"'] \arrow[u, "X\sigma^{\#}Y^{\#}B"'] \\
A \arrow[r, "\eta A"] \arrow[u, "i"] & X^{\#}A \arrow[u, "X^{\#}i"'] & \mathrm{g)} & XX^{\#}A \arrow[ll, "\mu_0 A"'] \arrow[u, "XX^{\#}i"']
\end{tikzcd}
\tag{2.5}
$$

The parts a), e) and g) are naturality squares for η, σ, and μ_0, respectively. Commutativity of b) and f) directly follow from the definition of $\sigma^{\#}$ (2.3). The monad identities (1.6) for the monad $(Y^{\#}, \bar{\eta}, \bar{\mu})$ imply c) and d), thus, (2.5) is completely commutative. Since the homomorphic extension is unique, putting thogether (2.2) and (2.5) we have $i^{\#}=Q\bar{\mu}B\cdot\sigma^{\#}Y^{\#}B\cdot X^{\#}i$. Hence by (2.1) $f_M=\beta Y^{\#}B\cdot i^{\#}=\beta Y^{\#}B\cdot \cdot Q\bar{\mu}B\cdot\sigma^{\#}Y^{\#}B\cdot X^{\#}i$. $\square$

Now we introduce a definition of a machine working in such a way that elementary input produces an elementary output.

Definition 2.4. Let X and Y be varietors in $\mathcal{K}$ and let A, B be objects of $\mathcal{K}$. A *simple machine* in $\mathcal{K}$ is a system $M=(Q, i_0, \sigma_0, \beta)$: $(A, X)\to(B, Y)$, where

Q: $\mathcal{K}\to\mathcal{K}$ is a functor, the state functor,
i_0: $A\to QB$ is a $\mathcal{K}$-morphism, the initial state-output,
σ_0: $XQ\stackrel{\cdot}{\to}QY$ is a natural transformation, the transition,
β: $Q\stackrel{\cdot}{\to}I$ is a natural transformation, the final state transformation.

The response of a simple machine $M=(Q, i_0, \sigma_0, \beta)$ is the composite morphism

$$f_M\colon\ X^{\#}A\xrightarrow{i_0^{\#}}QY^{\#}B\xrightarrow{\beta Y^{\#}B}Y^{\#}B, \tag{2.6}$$

where $i_0^{\#}$ is the run map of M defined by the homomorphic extension

$$
\begin{tikzcd}
QB \arrow[r, "Q\bar{\eta}B"] & QY^{\#}B & QYY^{\#}B \arrow[l, "Q\bar{\mu}_0B"'] & XQY^{\#}B \arrow[l, "\sigma_0Y^{\#}B"'] \\
A \arrow[r, "\eta A"] \arrow[u, "i_0"] & X^{\#}A \arrow[u, "i_0^{\#}"'] & & XX^{\#}A \arrow[ll, "\mu_0 A"'] \arrow[u, "Xi_0^{\#}"']
\end{tikzcd}
\tag{2.7}
$$

DEFINITION 2.5. Let $M=(Q, i, \sigma, \beta)\colon (A, X)\to(B, Y)$ be a machine in $\mathscr{K}$. We say that the initial state-output morphism i is *simple* if it can be factored thorough $Q\bar{\eta}B\colon QB\to QY^{\#}B$, i.e. there is a morphism $i_0\colon A\to QB$ such that

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "i_0"'] & QY^{\#}B \\ & QB \arrow[u, "Q\bar{\eta}B"'] \end{tikzcd} \tag{2.8}$$

Similarly, the transition σ is called *simple* if there exists a natural transformation $\sigma_0\colon XQ \dot{\to} QY$ such that

$$\begin{tikzcd} XQ \arrow[r, "\sigma"] \arrow[dr, "\sigma_0"'] & QY^{\#} \\ & QY \arrow[u, "Q\bar{\eta}_1"'] \end{tikzcd} \tag{2.9}$$

is commutative, where $\bar{\eta}_1$ is the *embedding* of Y into $Y^{\#}$, i.e. $\bar{\eta}_1\colon Y \xrightarrow{Y\bar{\eta}} YY^{\#} \xrightarrow{\bar{\mu}_0} Y^{\#}$.

LEMMA 2.6. Let $M=(Q, i, \sigma, \beta)\colon (A, X)\to(B, Y)$ be a machine in $\mathscr{K}$, and let i and σ be simple. Then the simple machine $M'=(Q, i_0, \sigma_0, \beta)\colon (A, X)\to(B, Y)$. where i_0 and σ_0 are as in (2.8) and (2.9), respectively, has the same response as M,

Proof. Since the final state transformation of M and that of M' is β, it is enough to prove that the corresponding run maps $i^{\#}$ and $i_0^{\#}$ coincide.

Consider the following diagram.

$$\begin{tikzcd} & & QYY^{\#}B \arrow[dl, "Q\bar{\mu}_0 B"'] \arrow[d, "Q\bar{\eta}_1 Y^{\#}B"] & \\ QB \arrow[r, "Q\bar{\eta}B"] & QY^{\#}B & QY^{\#}Y^{\#}B \arrow[l, "Q\bar{\mu}B"'] & XQY^{\#}B \arrow[l, "\sigma Y^{\#}B"'] \arrow[ul, "\sigma_0 Y^{\#}B"'] \\ A \arrow[u, "i_0"] \arrow[ur, "i"] \arrow[r, "\eta A"] & X^{\#}A \arrow[u, "i^{\#}"'] & & XX^{\#}A \arrow[ll, "\mu_0 A"'] \arrow[u, "Xi^{\#}"'] \end{tikzcd} \tag{2.10}$$

By the defining diagram (1.5) of an extended free operation, the equalities $\bar{\mu}\cdot\bar{\mu}_0 Y^{\#}=\bar{\mu}_0\cdot Y\bar{\mu}$ and $\bar{\mu}\cdot\bar{\eta}Y^{\#}=1_{Y^{\#}}$ hold, thus we have

$$\bar{\mu}\cdot\bar{\eta}_1 Y^{\#} = \bar{\mu}\cdot(\bar{\mu}_0\cdot Y\bar{\eta})Y^{\#} = \bar{\mu}\cdot\bar{\mu}_0 Y^{\#}\cdot Y\bar{\eta}Y^{\#} = \bar{\mu}_0\cdot Y\bar{\mu}\cdot Y\bar{\eta}Y =$$
$$= \bar{\mu}_0\cdot Y(\bar{\mu}\cdot\bar{\eta}Y^{\#}) = \bar{\mu}\cdot Y1_{Y^{\#}} = \bar{\mu}_0.$$

Hence $Q\bar{\mu}\cdot Q\bar{\eta}_1 Y^{\#}=Q\bar{\mu}_0$. Now, from the factorizations (2.8), (2.9) and the definition (2.2) of the run map $i^{\#}$, we obtain that the diagram (2.10) is completely commutative. This means that $i^{\#}$ satisfies the commutativity of diagram (2.7) which defines $i_0^{\#}$ uniquely. Thus $i^{\#}=i_0^{\#}$. $\square$

The diagram (2.3) defines for every natural transformation $\sigma\colon XQ\xrightarrow{\cdot} QY^{\#}$, i.e. without σ being a transition of any machine, the extension $\sigma^{\#}\colon X^{\#}Q\xrightarrow{\cdot} QY^{\#}$. Alagić studied this extension in his paper [2] and proved the following theorem replaced the monad $(Y^{\#}, \bar{\eta}, \bar{\mu})$ by an arbitrary one.

THEOREM 2.7 (Alagić [2], Theorem 2.30, p. 287). Let $X, Y\colon \mathscr{K}\to\mathscr{K}$ be varietors, and $Q\colon \mathscr{K}\to\mathscr{K}$ be a functor. Then for every natural transformation $\sigma\colon XQ\xrightarrow{\cdot} QY^{\#}$ the extension $\sigma^{\#}\colon X^{\#}Q\xrightarrow{\cdot} QY^{\#}$ defined by (2.3) satisfies the commutativity of the following diagram:

$$\begin{tikzcd}
 & QY^{\#} & QY^{\#}Y^{\#} \arrow[l, "Q\bar{\mu}"'] & X^{\#}QY^{\#} \arrow[l, "\sigma^{\#}Y^{\#}"'] \\
Q \arrow[ur, "Q\bar{\eta}"] \arrow[r, "\eta Q"] & X^{\#}Q \arrow[u, "\sigma^{\#}"] & & X^{\#}X^{\#}Q \arrow[ll, "\mu Q"'] \arrow[u, "X^{\#}\sigma^{\#}"']
\end{tikzcd} \tag{2.11}$$

THEOREM 2.8. Let $f_1\colon X^{\#}A\to Y^{\#}B$, $f_2\colon Y^{\#}B\to Z^{\#}C$ be responses of machines $M_1\colon (A, X)\to(B, Y)$ and $M_2\colon (B, Y)\to(C, Z)$, respectively. Then the composite morphism $f_2\cdot f_1\colon X^{\#}A\to Z^{\#}C$ is again the response of a machine $M\colon (A, X)\to(C, Z)$.

Proof. Assume that machines M_1 and M_2 are specified by $M_1=(Q_1, i_1, \sigma_1, \beta_1)$, $M_2=(Q_2, i_2, \sigma_2, \beta_2)$. Consider the machine $M=(Q, i, \sigma, \beta)\colon (A, X)\to(C, Z)$, where

$$\begin{gathered}
Q = Q_1Q_2, \quad \sigma = Q_1\sigma_2^{\#}\cdot\sigma_1 Q_2, \\
i = A\xrightarrow{i_1} Q_1Y^{\#}B\xrightarrow{Q_1 i_2^{\#}} Q_1Q_2Z^{\#}C, \quad \beta = Q_1Q_2\xrightarrow{\beta_1 Q_2} Q_2\xrightarrow{\beta_2} I.
\end{gathered} \tag{2.12}$$

Let us denote by $\bar{\bar{\eta}}$ and $\bar{\bar{\mu}}$ the insertion of generators and the extended free operation of Z, respectively. By the definition of the responses of M_1 and M_2, $f_2\cdot f_1=\beta_2 Z^{\#}C\cdot i_2^{\#}\cdot\beta_1 Y^{\#}B\cdot i_1^{\#}$. Using the naturality of β_1 we have

$$f_2\cdot f_1 = \beta_2 Z^{\#}C\cdot\beta_1 Q_2 Z^{\#}C\cdot Q_1 i_2^{\#}\cdot i_1^{\#} = (\beta_2\cdot\beta_1 Q_2)Z^{\#}C\cdot Q_1 i_2^{\#}\cdot i_1^{\#} = \beta Z^{\#}C\cdot Q_1 i_2^{\#}\cdot i_1^{\#}.$$

The response of M is $f_M=\beta Z^{\#}C\cdot i^{\#}$, where $i^{\#}$ is the run map of M. Thus, in order to prove that the machine M computes the composite $f_2\cdot f_1$ we need only to show that (2.13) holds

$$Q_1 i_2^{\#}\cdot i_1^{\#} = i^{\#}. \tag{2.13}$$

Taking into account that the run map $i^{\#}$ is the unique morphism satisfying (2.14), it is enough to prove that the left side of (2.13) also satisfies (2.14).

$$\begin{tikzcd}
 & Q_1Q_2Z^{\#}C & Q_1Q_2Z^{\#}Z^{\#}C \arrow[l, "Q_1Q_2\bar{\bar{\mu}}C"'] & XQ_1Q_2Z^{\#}C \arrow[l, "\sigma Z^{\#}C"'] \\
A \arrow[ur, "i"] \arrow[r, "\eta A"] & X^{\#}A \arrow[u, "i^{\#}"] & & XX^{\#}A \arrow[ll, "\mu_0 A"'] \arrow[u, "Xi^{\#}"']
\end{tikzcd} \tag{2.14}$$

Consider the diagram (2.15) below.

$$
\begin{tikzcd}
 & Q_1Q_2Z^{\#}C \arrow[r, phantom, "(\mathrm{v})" near start] & Q_1Q_2Z^{\#}Z^{\#}C \arrow[l, "Q_1Q_2\bar{\bar{\mu}}C"'] & Q_1Y^{\#}Q_2Z^{\#}C \arrow[l, "Q_1\sigma_2^{\#}Z^{\#}C"'] & XQ_1Q_2Z^{\#}C \arrow[l, "\sigma_1Q_2Z^{\#}C"'] \arrow[ll, bend right, "\sigma Z^{\#}C"', "(\mathrm{vi})"] \\
 & Q_1Y^{\#}B \arrow[u, "Q_1i_2^{\#}"] \arrow[rr, phantom, "(\mathrm{iv})"] & & Q_1Y^{\#}Y^{\#}B \arrow[ll, "Q_1\bar{\mu}B"'] \arrow[u, "Q_1Y^{\#}i_2^{\#}"] \arrow[r, phantom, "(\mathrm{iii})"] & XQ_1Y^{\#}B \arrow[l, "\sigma_1Y^{\#}B"'] \arrow[u, "XQ_1i_2^{\#}"'] \\
A \arrow[ur, "i_1"] \arrow[uur, "i"] \arrow[r, "\eta A"'] & X^{\#}A \arrow[u, "i_1^{\#}"'] \arrow[u, phantom, "(\mathrm{i})" near start] \arrow[rrr, phantom, "(\mathrm{ii})"] & & & XX^{\#}A \arrow[lll, "\mu_0A"] \arrow[u, "Xi_1^{\#}"']
\end{tikzcd}
\tag{2.15}
$$

The subdiagrams (i) and (ii) commute by the definition of the run map $i_1^{\#}$. (iii) is a naturality square for the natural transformation σ_1. (v) and (vi) are commutative by (2.12). Thus the commutativity of (iv) is remained to prove. By Proposition 2.3 the run map $i_2^{\#}$ can be expressed by the extended transition $\sigma_2^{\#}$ of M_2 as follows

$$i_2^{\#} = Q_2\bar{\bar{\mu}}C \cdot \sigma_2^{\#}Z^{\#}C \cdot Y^{\#}i_2. \tag{2.16}$$

The diagrams (i) and (iv) in (2.17) commute, being naturality squares for $\bar{\mu}$ and $\sigma_2^{\#}$, respectively. (ii) is commutative by Theorem 2.7, finally, the commutativity of (iii) in (2.17) follows from the associativity axiom of the monad $(Z^{\#}, \bar{\bar{\eta}}, \bar{\bar{\mu}})$. Hence, (2.17) is completely commutative.

$$
\begin{tikzcd}
Q_2Z^{\#}C \arrow[r, phantom, "(\mathrm{iii})" near end, shift right=3ex] & Q_2Z^{\#}Z^{\#}C \arrow[l, "Q_2\bar{\bar{\mu}}C"'] \arrow[r, phantom, "(\mathrm{iv})", shift right=3ex] & Y^{\#}Q_2Z^{\#}C \arrow[l, "\sigma_2^{\#}Z^{\#}C"'] \\
Q_2Z^{\#}Z^{\#}C \arrow[u, "Q_2\bar{\bar{\mu}}C"] & Q_2Z^{\#}Z^{\#}Z^{\#}C \arrow[l, "Q_2\bar{\bar{\mu}}Z^{\#}C"'] \arrow[u, "Q_2Z^{\#}\bar{\bar{\mu}}C"'] & Y^{\#}Q_2Z^{\#}Z^{\#}C \arrow[l, "\sigma_2^{\#}Z^{\#}Z^{\#}C"'] \arrow[u, "Y^{\#}Q_2\bar{\bar{\mu}}C"'] \\
Y^{\#}Q_2Z^{\#}C \arrow[u, "\sigma_2^{\#}Z^{\#}C"] \arrow[r, phantom, "(\mathrm{ii})" near start, shift left=3ex] & & Y^{\#}Y^{\#}Q_2Z^{\#}C \arrow[ll, "\bar{\mu}Q_2Z^{\#}C"'] \arrow[u, "Y^{\#}\sigma_2^{\#}Z^{\#}C"'] \\
Y^{\#}B \arrow[u, "Y^{\#}i_2"] \arrow[r, phantom, "(\mathrm{i})" near start, shift left=3ex] & & Y^{\#}Y^{\#}B \arrow[ll, "\bar{\mu}B"] \arrow[u, "Y^{\#}Y^{\#}i_2"']
\end{tikzcd}
\tag{2.17}
$$

Putting together (2.16) and (2.17) we have

$$
\begin{aligned}
Q_1i_2^{\#} \cdot Q_1\bar{\mu}B &= Q_1(i_2^{\#} \cdot \bar{\mu}B) = Q_1(Q_2\bar{\bar{\mu}}C \cdot \sigma_2^{\#}Z^{\#}C \cdot Y^{\#}i_2 \cdot \bar{\mu}B) = \\
&= Q_1(Q_2\bar{\bar{\mu}}C \cdot \sigma_2^{\#}Z^{\#}C \cdot Y^{\#}Q_2\bar{\bar{\mu}}C \cdot Y^{\#}\sigma_2^{\#}Z^{\#}C \cdot Y^{\#}Y^{\#}i_2) = \\
&= Q_1Q_2\bar{\bar{\mu}}C \cdot Q_1\sigma_2^{\#}Z^{\#}C \cdot Q_1Y^{\#}(Q_2\bar{\bar{\mu}}C \cdot \sigma_2^{\#}Z^{\#}C \cdot Y^{\#}i_2) = \\
&= Q_1Q_2\bar{\mu}C \cdot Q_1\sigma_2^{\#}Z^{\#}C \cdot Q_1Y^{\#}i_2^{\#}.
\end{aligned}
$$

Hence the diagram (iii) in (2.15) is commutative which completes the proof of the theorem. □

DEFINITION 2.9. Let $M=(Q, i, \sigma, \beta)\colon (A, X)\to(B, Y)$ and $M_1=(Q_1, i_1, \sigma_1, \beta_1)\colon (A, X)\to(B, Y)$ be machines in $\mathcal{K}$. A *simulation* $\varrho\colon M_1\to M$ is a natural transformation $\varrho\colon Q_1 \xrightarrow{\cdot} Q$ rendering the diagrams (2.18) commutative.

$$
\begin{tikzcd}
Q_1Y^{\#}B \arrow[rr, "\varrho Y^{\#}B"] & & QY^{\#}B \\
& A \arrow[ul, "i_1"] \arrow[ur, "i"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
Q_1Y^{\#} \arrow[r, "\varrho Y^{\#}"] & QY^{\#} \\
XQ_1 \arrow[u, "\sigma_1"] \arrow[r, "X\varrho"] & XQ \arrow[u, "\sigma"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Q_1 \arrow[rr, "\varrho"] \arrow[dr, "\beta_1"'] & & Q \arrow[dl, "\beta"] \\
& I &
\end{tikzcd}
\tag{2.18}
$$

a) b) c)

THEOREM 2.10. Let $M\colon (A, X)\to(B, Y)$ and $M_1\colon (A, X)\to(B, Y)$ be machines in $\mathcal{K}$. Whenever a simulation $\varrho\colon M_1\to M$ exists then $f_M=f_{M_1}$.

Proof. Assume that the machines M and M_1 are given by $M=(Q, i, \sigma, \beta)$, $M_1=(Q_1, i_1, \sigma_1, \beta_1)$. Then the response of M is $f_M=\beta Y^{\#}B\cdot i^{\#}$ and the response of M_1 is $f_{M_1}=\beta_1 Y^{\#}B\cdot i_1^{\#}$. Consider the diagram (2.19).

$$
\begin{tikzcd}
& QY^{\#}B & QY^{\#}Y^{\#}B \arrow[l, "Q\bar{\mu}B"'] \arrow[dl, phantom, "\text{(iv)}"] & XQY^{\#}B \arrow[l, "\sigma Y^{\#}B"'] \arrow[dl, phantom, "\text{(iii)}"] \\
& Q_1Y^{\#}B \arrow[u, "\varrho Y^{\#}B"] \arrow[u, phantom, bend left=60, "\text{(v)}"] & Q_1Y^{\#}Y^{\#}B \arrow[l, "Q_1\bar{\mu}B"'] \arrow[u, "\varrho Y^{\#}Y^{\#}B"'] & XQ_1Y^{\#}B \arrow[l, "\sigma_1 Y^{\#}B"'] \arrow[u, "X\varrho Y^{\#}B"'] \arrow[dl, phantom, "\text{(ii)}"] \\
A \arrow[uur, "i"] \arrow[ur, "i_1"'] \arrow[r, "\eta A"'] \arrow[ur, phantom, shift right=4, "\text{(i)}"] & X^{\#}A \arrow[u, "i_1^{\#}"'] & & XX^{\#}A \arrow[ll, "\mu_0 A"'] \arrow[u, "Xi_1^{\#}"']
\end{tikzcd}
\tag{2.19}
$$

The diagrams (i) and (ii) in (2.19) are commutative just they define the run map $i_1^{\#}$ of M_1. Since $\varrho\colon Q_1 \xrightarrow{\cdot} Q$ is a simulation (iii) and (v) commute by (2.18b) and (2.18a), respectively. (iv) is a naturality square for ϱ thus (2.19) is completely commutative. Hence, we have that the morpisms $i^{\#}$ and $\varrho Y^{\#}B\cdot i_1^{\#}$ both are defined by homomorphic extensions on the same specification. The uniquenes of the homomorphic extension implies $i^{\#}=\varrho Y^{\#}B\cdot i_1^{\#}$. Finally, we have

$$f_M = \beta Y^{\#}B\cdot i^{\#} = \beta Y^{\#}B\cdot \varrho Y^{\#}B\cdot i_1^{\#} = (\beta\cdot\varrho) Y^{\#}B\cdot i_1^{\#} = \beta_1 Y^{\#}B\cdot i_1^{\#} = f_{M_1}. \quad \square$$

3. Inverse-state machines

In this section we shall develop a categorial model of Thatcher's generalized[2] sequential machine maps (see [8]), and Engelfriet's top-down tree transformations (see [5]). The term "inverse-state machine" is used here because these machines are very closely related to the inverse state transformations of Alagić [2]. We shall show that every top-down, i.e. inverse-state computation can be carried out by a machine with sutable state functor.

First, we need a theorem whose analogous one was proved in [2] and what we state as a consequence of our theorem.

THEOREM 3.1. Let (T, η', μ') be a monad and let (B, d) be a T-monad algebra in $\mathscr{K}$. Furthermore, let $X: \mathscr{K} \to \mathscr{K}$ be varietor and $Q: \mathscr{K} \to \mathscr{K}$ be a functor with right adjoint. Then for every morphism $j: QA \to B$ and natural transformation $\tau: QX \to TQ$ there exists a unique morphism $j_\#: QX^\# A \to B$ satisfying (3.1).

$$\begin{tikzcd} & B & TB \arrow[l, "d"'] & TQX^\# A \arrow[l, "Tj_\#"'] \\ QA \arrow[ur, "j"] \arrow[r, "Q\eta A"] & QX^\# A \arrow[u, "j_\#"] & & QXX^\# A \arrow[ll, "Q\mu_0 A"'] \arrow[u, "\tau X^\# A"'] \end{tikzcd} \tag{3.1}$$

Moreover, there is a bijective correspondence between triples $(j, \tau, j_\#)$ satisfying (3.1) and triples $(i: A \to \bar{Q}B,\ \sigma: X\bar{Q} \dot{\to} \bar{Q}T,\ i^\#: X^\# A \to \bar{Q}B)$ satisfying (3.2), where $(Q, \bar{Q}, \nu, \varepsilon)$ is an adjunction due to Q.

$$\begin{tikzcd} & \bar{Q}B & \bar{Q}TB \arrow[l, "\bar{Q}d"'] & X\bar{Q}B \arrow[l, "\sigma B"'] \\ A \arrow[ur, "i"] \arrow[r, "\eta A"] & X^\# A \arrow[u, "i^\#"] & & XX^\# A \arrow[ll, "\mu_0 A"'] \arrow[u, "Xi^\#"'] \end{tikzcd} \tag{3.2}$$

Mutually inverse passages are given by (3.3) and (3.4) below.

$$\begin{array}{lcl} i: A \to \bar{Q}B & & j: QA \xrightarrow{Qi} Q\bar{Q}B \xrightarrow{\varepsilon B} B \\ \sigma: X\bar{Q} \to \bar{Q}T & \overset{\Phi}{\mapsto} & \tau: QX \xrightarrow{QX\nu} QX\bar{Q}Q \xrightarrow{Q\sigma Q} Q\bar{Q}TQ \xrightarrow{\varepsilon TQ} TQ \\ i^\#: X^\# A \to \bar{Q}B & & j_\#: QX^\# A \xrightarrow{Qi^\#} Q\bar{Q}B \xrightarrow{\varepsilon B} B \end{array} \tag{3.3}$$

$$\begin{array}{lcl} j: QA \to B & & i: A \xrightarrow{\nu A} \bar{Q}QA \xrightarrow{\bar{Q}j} \bar{Q}B \\ \tau: QX \dot{\to} TQ & \overset{\Psi}{\mapsto} & \sigma: X\bar{Q} \xrightarrow{\nu X\bar{Q}} \bar{Q}QX\bar{Q} \xrightarrow{\bar{Q}\tau\bar{Q}} \bar{Q}TQ\bar{Q} \xrightarrow{\bar{Q}T\varepsilon} \bar{Q}T \\ j_\#: QX^\# A \to B & & i^\#: X^\# A \xrightarrow{\nu X^\# A} \bar{Q}QX^\# A \xrightarrow{\bar{Q}j_\#} \bar{Q}B \end{array} \tag{3.4}$$

Proof. First we show that Φ and Ψ are inverses of each other. It is a well know property of the adjunction $(Q, \bar{Q}, \nu, \varepsilon)$ that $\Psi \cdot \Phi(i) = i$, $\Phi \cdot \Psi(j) = j$. By the same argument we get $\Psi \cdot \Phi(i^\#) = i^\#$, $\Phi \cdot \Psi(j_\#) = j_\#$. We prove that $\Psi \cdot \Phi(\sigma) = \sigma$ and $\Phi \cdot \Psi(\tau) = \tau$.

$$\Psi \cdot \Phi(\sigma) = \Psi(\varepsilon TQ \cdot Q\sigma Q \cdot QX\nu) = \bar{Q}T\varepsilon \cdot \bar{Q}(\varepsilon TQ \cdot Q\sigma Q \cdot QX\nu)\bar{Q} \cdot \nu X\bar{Q} =$$
$$= \bar{Q}T\varepsilon \cdot \bar{Q}\varepsilon TQ\bar{Q} \cdot \bar{Q}Q\sigma Q\bar{Q} \cdot \bar{Q}QX\nu\bar{Q} \cdot \nu X\bar{Q}.$$

Consider the diagram (3.5) whose triangular parts are commutative according to the triangular identities of the adjunction $(Q, \bar{Q}, \nu, \varepsilon)$. The other two parts of (3.5) commute since they are naturality squares for ν and σ, respectively. Thus we have $\Psi \cdot \Phi(\sigma) = \sigma$.

$$\begin{tikzcd}
 & X\bar{Q} \arrow[rr, "\sigma"] & & \bar{Q}T \\
X\bar{Q} \arrow[ur, "X1_{\bar{Q}}"] \arrow[r, "X\nu\bar{Q}"] \arrow[d, "\nu X\bar{Q}"'] & X\bar{Q}Q\bar{Q} \arrow[u, "X\bar{Q}\varepsilon"'] \arrow[r, "\sigma Q\bar{Q}"] & \bar{Q}TQ\bar{Q} \arrow[r, "1_{\bar{Q}}TQ\bar{Q}"] \arrow[d, "\nu\bar{Q}TQ\bar{Q}"] & \bar{Q}TQ\bar{Q} \arrow[u, "\bar{Q}T\varepsilon"'] \\
\bar{Q}QX\bar{Q} \arrow[r, "\bar{Q}QX\nu\bar{Q}"] & \bar{Q}QX\bar{Q}Q\bar{Q} \arrow[r, "\bar{Q}Q\sigma Q\bar{Q}"] & \bar{Q}Q\bar{Q}TQ\bar{Q} \arrow[ur, "\bar{Q}\varepsilon TQ\bar{Q}"'] &
\end{tikzcd} \tag{3.5}$$

The following diagram also commutes by the adjunction identity $\varepsilon Q \cdot Q\nu = 1_Q$, and the naturality of ν, τ and ε.

$$\begin{tikzcd}
 & QX \arrow[rr, "\tau"] & & TQ \\
QX \arrow[ur, "1_Q X"] \arrow[r, "Q\nu X"] \arrow[d, "QX\nu"'] & Q\bar{Q}QX \arrow[u, "\varepsilon QX"'] \arrow[r, "Q\bar{Q}\tau"] \arrow[d, "Q\bar{Q}QX\nu"] & Q\bar{Q}TQ \arrow[r, "Q\bar{Q}T1_Q"] \arrow[d, "Q\bar{Q}TQ\nu"] & Q\bar{Q}TQ \arrow[u, "\varepsilon TQ"'] \\
QX\bar{Q}Q \arrow[r, "Q\nu X\bar{Q}Q"] & Q\bar{Q}QX\bar{Q}Q \arrow[r, "Q\bar{Q}\tau\bar{Q}Q"] & Q\bar{Q}TQ\bar{Q}Q \arrow[ur, "Q\bar{Q}T\varepsilon Q"'] &
\end{tikzcd} \tag{3.6}$$

Hence,

$$\Phi \cdot \Psi(\tau) = \Phi(\bar{Q}T\varepsilon \cdot \bar{Q}\tau\bar{Q} \cdot \nu X\bar{Q}) = \varepsilon TQ \cdot Q(\bar{Q}T\varepsilon \cdot \bar{Q}\tau\bar{Q} \cdot \nu X\bar{Q})Q \cdot QX\nu =$$
$$= \varepsilon TQ \cdot Q\bar{Q}T\varepsilon Q \cdot Q\bar{Q}\tau\bar{Q}Q \cdot Q\nu X\bar{Q}Q \cdot QX\nu = \tau \cdot 1_Q X = \tau \cdot 1_{QX} = \tau.$$

Let us prove that the passages Φ and Ψ preserve satisfyability of the appropriate diagrams. Assume that a triple $(i, \sigma, i^{\#})$ satisfies (3.2), Then,

$$\Phi(i^{\#}) \cdot Q\eta A = \varepsilon B \cdot Qi^{\#} \cdot Q\eta A = \varepsilon B \cdot Q(i^{\#} \cdot \eta A) = \varepsilon B \cdot Qi = \Phi(i).$$

Thus the triangular part of (3.1) holds.

$$\Phi(i^{\#}) \cdot Q\mu_0 A = \varepsilon B \cdot Qi^{\#} \cdot Q\mu_0 A = \varepsilon B \cdot Q(i^{\#} \cdot \mu_0 A) = \varepsilon B \cdot Q(\bar{Q}d \cdot \sigma B \cdot Xi^{\#}) =$$
$$= \varepsilon B \cdot Q\bar{Q}d \cdot Q\sigma B \cdot QXi^{\#}.$$

One of the adjunction identities says $1_{\bar{Q}} = \bar{Q}\varepsilon \cdot \nu\bar{Q}$ and hence $1_{QX\bar{Q}B} = QX1_{\bar{Q}}B = QX(\bar{Q}\varepsilon \cdot \nu\bar{Q})B = QX\bar{Q}\varepsilon B \cdot QX\nu\bar{Q}B$, which yields $\Phi(i^{\#}) \cdot Q\mu_0 A = \varepsilon B \cdot Q\bar{Q}d \cdot Q\sigma B \cdot (QX\bar{Q}\varepsilon B \cdot QX\nu\bar{Q}B) \cdot QXi^{\#}$. Application of commutations for the natural transformations ε, $\varepsilon T\cdot Q\sigma$, $\Phi(\sigma)$ and $\Phi(\sigma)=\varepsilon TQ\cdot Q\sigma Q\cdot QXv$ produces

$$\Phi(i^{\#})\cdot Q\mu_0 A = d\cdot \varepsilon TB\cdot Q\sigma B\cdot QX\bar{Q}\varepsilon B\cdot QXv\bar{Q}B\cdot QXi^{\#} =$$
$$= d\cdot T\varepsilon B\cdot \varepsilon TQ\bar{Q}B\cdot Q\sigma Q\bar{Q}B\cdot QXv\bar{Q}B\cdot QXi^{\#} = d\cdot T\varepsilon B\cdot(\varepsilon TQ\cdot Q\sigma Q\cdot QXv)\bar{Q}B\cdot QXi^{\#} =$$
$$= d\cdot T\varepsilon B\cdot \Phi(\sigma)\bar{Q}B\cdot QXi^{\#} = d\cdot T\varepsilon B\cdot TQi^{\#}\cdot \Phi(\sigma)X^{\#}A =$$
$$= d\cdot T(\varepsilon B\cdot Qi^{\#})\cdot \Phi(\sigma)X^{\#}A = d\cdot T\Phi(i^{\#})\cdot \Phi(\sigma)X^{\#}A.$$

Thus, the triple $(j, \tau, j_{\#})=(\Phi(i), \Phi(\sigma), \Phi(i^{\#}))$ satisfies (3.1).

Conversely, let us suppose that the left side $(j, \tau, j_{\#})$ of (3.4) makes (3.1) commutative. Then, for the right side of (3.4), we have

$$\Psi(j_{\#})\cdot\eta A = \bar{Q}j_{\#}\cdot vX^{\#}A\cdot\eta A = \bar{Q}j_{\#}\cdot\bar{Q}Q\eta A\cdot vA =$$
$$= \bar{Q}(j_{\#}\cdot Q\eta A)\cdot vA = \bar{Q}j\cdot vA = \Psi(j).$$

This means that the triangular part of (3.2) is satisfied. Let us see the other part of (3.2). By the definition (3.4) of Ψ and the naturality of v we have

$$\Psi(j_{\#})\cdot\mu_0 A = \bar{Q}j_{\#}\cdot vX^{\#}A\cdot\mu_0 A = \bar{Q}j_{\#}\cdot\bar{Q}Q\mu_0 A\cdot vXX^{\#}A =$$
$$= \bar{Q}(j_{\#}\cdot Q\mu_0 A)\cdot vXX^{\#}A = \bar{Q}(d\cdot Tj_{\#}\cdot\tau X^{\#}A)\cdot vXX^{\#}A =$$
$$= \bar{Q}d\cdot\bar{Q}Tj_{\#}\cdot\bar{Q}\tau X^{\#}A\cdot vXX^{\#}A.$$

From the adjunction identity $1_Q=\varepsilon Q\cdot Qv$ follows $1_{\bar{Q}TQX^{\#}A}=\bar{Q}T1_QX^{\#}A= =\bar{Q}T(\varepsilon Q\cdot Qv)X^{\#}A=\bar{Q}T\varepsilon QX^{\#}A\cdot\bar{Q}TQvX^{\#}A$, thus we get

$$\Psi(j_{\#})\cdot\mu_0 A = \bar{Q}d\cdot\bar{Q}Tj_{\#}\cdot\bar{Q}T\varepsilon QX^{\#}A\cdot\bar{Q}TQvX^{\#}A\cdot\bar{Q}\tau X^{\#}A\cdot vXX^{\#}A.$$

Using the naturality of $\bar{Q}T\varepsilon$ and $\bar{Q}\tau\cdot vX$ we conclude

$$\Psi(j)\cdot\mu_0 A = \bar{Q}d\cdot\bar{Q}T\varepsilon B\cdot\bar{Q}TQ\bar{Q}j_{\#}\cdot\bar{Q}TQvX^{\#}A\cdot\bar{Q}\tau X^{\#}A\cdot vXX^{\#}A =$$
$$= \bar{Q}d\cdot\bar{Q}T\varepsilon B\cdot\bar{Q}TQ(\bar{Q}j_{\#}\cdot vX^{\#}A)\cdot(\bar{Q}\tau\cdot vX)X^{\#}A =$$
$$= \bar{Q}d\cdot\bar{Q}T\varepsilon B\cdot\bar{Q}TQ\Psi(j_{\#})\cdot(\bar{Q}\tau\cdot vX)X^{\#}A = \bar{Q}d\cdot\bar{Q}T\varepsilon B\cdot(\bar{Q}\tau\cdot vX)\bar{Q}B\cdot X\Psi(j_{\#}) =$$
$$= \bar{Q}d\cdot(\bar{Q}T\varepsilon\cdot\bar{Q}\tau\bar{Q}\cdot vX\bar{Q})B\cdot X\Psi(j_{\#}) = \bar{Q}d\cdot\Psi(\tau)B\cdot X\Psi(j_{\#}).$$

Thus the triple $(i, \sigma, i^{\#})=(\Psi(j), \Psi(\tau), \Psi(j_{\#}))$ satisfies (3.2). The existential statement of the Theorem can be obtained as follows. For given morphism $j\colon QA\to B$ and natural transformation $\tau\colon QX\dot{\to}TQ$ consider $i:=\Phi(j)$, $\sigma:=\Phi(\tau)$ and take the unique $i^{\#}$ satisfying (3.2). This $i^{\#}$ exists because $(X^{\#}A, \mu_0 A)$ is a free X-algebra. Then, as we have shown, $(\Psi(i), \Psi(\sigma), \Psi(i^{\#}))$ satisfies (3.1). But $\Psi(i)=j$ and $\Psi(\sigma)=\tau$, hence $(j, \tau, \Psi(i^{\#}))$ satisfies (3.1). The uniqueness of $j_{\#}$ in (3.1) follows from the facts that Ψ is bijective and $i^{\#}$ is unique in (3.2). This completes the proof of Theorem 3.1. □

The following statement was proved in another way in Alagić [2] (see Theorem 3.10 pp. 297) replaced $(Y^{\#}, \bar{\eta}, \bar{\mu})$ by an arbitrary monad.

STATEMENT 3.2. Let X, Y be varietors in $\mathcal{K}$ and let $Q\colon \mathcal{K}\to\mathcal{K}$ be a functor having right adjoint. Then for every natural transformation $\tau\colon QX \stackrel{\cdot}{\to} Y^{\#}Q$ there is a unique $\tau_{\#}\colon QX^{\#} \stackrel{\cdot}{\to} Y^{\#}Q$ defined by

$$\begin{tikzcd}
 & Y^{\#}Q & Y^{\#}Y^{\#}Q \arrow[l, "\bar{\mu}Q"'] & Y^{\#}QX^{\#} \arrow[l, "Y^{\#}\tau_{\#}"'] \\
Q \arrow[ur, "\bar{\eta}Q"] \arrow[r, "Q\eta"'] & QX^{\#} \arrow[u, "\tau_{\#}"'] & & QXX^{\#} \arrow[ll, "Q\mu_0"'] \arrow[u, "\tau X^{\#}"']
\end{tikzcd} \tag{3.7}$$

Proof. Let A be an object of $\mathcal{K}$. As $(Y^{\#}, \bar{\eta}, \bar{\mu})$ is a monad it is evident that $(Y^{\#}QA, \bar{\mu}QA)$ is an $Y^{\#}$-monad algebra. Take $j:=\bar{\eta}QA\colon QA\to Y^{\#}QA$ and apply Theorem 3.1 for this j and τ above. We have that there exists a unique $j_{\#}$: $QX^{\#}A\to Y^{\#}QA$ denoted by $\tau_{\#}A$ which renders (3.8) commutative.

$$\begin{tikzcd}
 & Y^{\#}QA & Y^{\#}Y^{\#}QA \arrow[l, "\bar{\mu}QA"'] & Y^{\#}QX^{\#}A \arrow[l, "Y^{\#}\tau_{\#}A"'] \\
QA \arrow[ur, "\bar{\eta}QA"] \arrow[r, "Q\eta A"'] & QX^{\#}A \arrow[u, "\tau_{\#}A"'] & & QXX^{\#}A \arrow[ll, "Q\mu_0 A"'] \arrow[u, "\tau X^{\#}A"']
\end{tikzcd} \tag{3.8}$$

Thus we need only to show that $\tau_{\#}A$ in (3.8) is natural in A. The proof is straightforward. □

DEFINITION 3.3. Let A, B be objects of $\mathcal{K}$ and let X, Y be varietors in $\mathcal{K}$. An *inverse-state machine*

$$M=(Q,\alpha,\tau,j)\colon (A,X)\to(B,Y)$$

in $\mathcal{K}$ consists of the following data:

$Q\colon \mathcal{K}\to\mathcal{K}$ a functor, the *state functor,* having right adjoint,
$\alpha\colon I \stackrel{\cdot}{\to} Q$ a natural transformation, the *initial state* transformation,
$\tau\colon QX \stackrel{\cdot}{\to} Y^{\#}Q$ a natural transformation, the *transition,*
$j\colon QA\to Y^{\#}B$ a morphism, the *final state-output* morphism.

DEFINITION 3.4. Let $M=(Q,\alpha,\tau,j)\colon (A,X)\to(B,Y)$ be an inverse-state machine in $\mathcal{K}$. The morphism f_M *computed* by M or the *response* of M is defined by

$$f_M\colon X^{\#}A \xrightarrow{\alpha X^{\#}A} QX^{\#}A \xrightarrow{j_{\#}} Y^{\#}B, \tag{3.9}$$

where $j_{\#}$ is the *(inverse-state) run map* defined to be the unique morphism

$$\begin{tikzcd}
 & Y^{\#}B & Y^{\#}Y^{\#}B \arrow[l, "\bar{\mu}B"'] & Y^{\#}QX^{\#}A \arrow[l, "Y^{\#}j_{\#}"'] \\
QA \arrow[ur, "j"] \arrow[r, "Q\eta A"'] & QX^{\#}A \arrow[u, "j_{\#}"'] & & QXX^{\#}A \arrow[ll, "Q\mu_0 A"'] \arrow[u, "\tau X^{\#}A"']
\end{tikzcd} \tag{3.10}$$

according to Theorem 3.1.

By Statement 3.2 we define the *extended transition* of the inverse-state machine M by the diagram (3.11).

$$\begin{tikzcd}
& Y^\# Q & Y^\# Y^\# Q \arrow[l, "\bar{\mu} Q"'] & Y^\# QX \arrow[l, "Y^\# \tau_\#"'] \\
Q \arrow[ur, "\bar{\eta} Q"] \arrow[r, "Q\eta"'] & QX^\# \arrow[u, "\tau_\#"'] & & QXX^\# \arrow[ll, "Q\mu_0"'] \arrow[u, "\tau X^\#"']
\end{tikzcd} \tag{3.11}$$

We shall show that the response of an inverse-state machine can be expressed in terms of the extended transition.

LEMMA 3.5. Let $M=(Q, \alpha, \tau, j)\colon (A, X)\to(B, Y)$ be an inverse-state machine in $\mathscr{K}$. The response of M is

$$f_M = \bar{\mu} B \cdot Y^\# j \cdot \tau_\# A \cdot \alpha X^\# A, \tag{3.12}$$

where $\tau_\#$ is the extended transition of M.

Proof. Because of the fact that the run map $j_\#$ of M is unique in (3.10) it is sufficient to prove that substituting the morphism $\bar{\mu} B \cdot Y^\# j \cdot \tau_\# A$ for $j_\#$, (3.10) remaines commutative. Consider the diagram

$$\begin{tikzcd}
{} & Y^\# B & Y^\# Y^\# B \arrow[l, "\bar{\mu} B"'] \arrow[dl, phantom, "\text{(vii)}"] & \\
{} & Y^\# Y^\# B \arrow[u, "\bar{\mu} B"'] & Y^\# Y^\# Y^\# B \arrow[l, "\bar{\mu} Y^\# B"'] \arrow[u, "Y^\# \bar{\mu} B"'] \arrow[dr, phantom, "\text{(v)}"] & \\
Y^\# B \arrow[uur, "1_{Y^\# B}"] \arrow[ur, "\bar{\eta} Y^\# B"'] \arrow[uur, phantom, "\text{(vi)}", shift right=4] & Y^\# QA \arrow[u, "Y^\# j"'] \arrow[l, phantom, "\text{(iii)}"] & Y^\# Y^\# QA \arrow[l, "\bar{\mu} QA"'] \arrow[u, "Y^\# Y^\# j"'] \arrow[ul, phantom, "\text{(iv)}"] & Y^\# QX^\# A \arrow[l, "Y^\# \tau_\# A"'] \arrow[uul, "Y^\# (\bar{\mu} B \cdot Y^\# j \cdot \tau_\# A)"'] \\
QA \arrow[u, "j"] \arrow[ur, "\bar{\eta} QA"] \arrow[r, "Q\eta A"'] \arrow[ur, phantom, "\text{(i)}", shift right=3] & QX^\# A \arrow[u, "\tau_\#"'] & & QXX^\# A \arrow[ll, "Q\mu_0 A"'] \arrow[u, "\tau X^\# A"'] \arrow[ull, phantom, "\text{(ii)}"]
\end{tikzcd} \tag{3.13}$$

(i) and (ii) are commutative by the diagram (3.11) of the extended transition $\tau_\#$. (iii) and (iv) are naturality squares for $\bar{\eta}$ and $\bar{\mu}$, respectively, hence they commute. The commutativity of (vi) and (vii) follows directly from the monad identities of $(Y^\#, \bar{\eta}, \bar{\mu})$. (v) just expresses the value of the functor $Y^\#$ on a composite morphism. Thus the whole diagram is commutative which ends the proof of the Lemma. $\square$

THEOREM 3.6. Given inverse-state machine $M=(Q, \alpha, \tau, j)\colon (A, X)\to(B, Y)$ there is a machine $\overline{M}\colon (A, X)\to(B, Y)$ computing the response of M.

Proof. Let $\bar{Q}$ be a right adjoint of Q, and denote the corresponding adjunction by $(Q, \bar{Q}, \nu, \varepsilon)$. Define a machine $M=(Q, i, \sigma, \beta)$ by

$$\begin{aligned}
i&\colon A \xrightarrow{\nu A} \bar{Q} Q A \xrightarrow{\bar{Q} j} \bar{Q} Y^\# B, \\
\sigma&\colon X\bar{Q} \xrightarrow{\nu X \bar{Q}} \bar{Q} Q X \bar{Q} \xrightarrow{\bar{Q}\tau\bar{Q}} \bar{Q} Y^\# Q \bar{Q} \xrightarrow{\bar{Q} Y^\# \varepsilon} \bar{Q} Y^\#, \\
\beta&\colon \bar{Q} \xrightarrow{\alpha \bar{Q}} Q\bar{Q} \xrightarrow{\varepsilon} I.
\end{aligned} \tag{3.14}$$

We are going to prove that $f_M = f_{\bar{M}}$. By the notations above

$$f_M = j_{\#} \cdot \alpha X^{\#} A, \quad f_{\bar{M}} = \beta Y^{\#} B \cdot i^{\#}, \tag{3.15}$$

where $j_{\#}$ and $i^{\#}$ are the run maps of M and $\bar{M}$, respectively. Thus the triple $(j, \tau, j_{\#})$ satisfies (3.10) and hence, by Theorem 3.1 the triple $(i, \sigma, \bar{Q} j_{\#} \cdot \nu X^{\#} A)$ satisfies the commutativity of the diagram which defines the run map $i^{\#}$ of $\bar{M}$. The uniqueness of the homomorphic extension implies

$$i^{\#} = \bar{Q} j_{\#} \cdot \nu X^{\#} A. \tag{3.16}$$

Thus we have

$$f_{\bar{M}} = (\varepsilon \cdot \alpha \bar{Q}) Y^{\#} B \cdot \bar{Q} j_{\#} \cdot \nu X^{\#} A = \varepsilon Y^{\#} B \cdot \alpha \bar{Q} Y^{\#} B \cdot \bar{Q} j_{\#} \cdot \nu X^{\#} A. \tag{3.17}$$

Consider the diagram below.

$$\begin{tikzcd}
\bar{Q} Y^{\#} B \arrow[r, "\alpha \bar{Q} Y^{\#} B"] & Q \bar{Q} Y^{\#} B \arrow[r, "\varepsilon Y^{\#} B"] & Y^{\#} B \\
\bar{Q} Q X^{\#} A \arrow[u, "\bar{Q} j_{\#}"] & Q \bar{Q} Q X^{\#} A \arrow[u, "Q \bar{Q} j_{\#}"'] \arrow[r, "\varepsilon Q X^{\#} A"] & Q X^{\#} A \arrow[u, "j_{\#}"'] \\
X^{\#} A \arrow[u, "\nu X^{\#} A"] \arrow[r, "\alpha X^{\#} A"] & Q X^{\#} A \arrow[u, "Q \nu X^{\#} A"'] \arrow[ur, "1_Q X^{\#} A"'] &
\end{tikzcd} \tag{3.18}$$

The triangular part of (3.18) is commutative by reason of the adjunction identity $\varepsilon Q \cdot Q\nu = 1_Q$, and the other two parts of (3.18) commute being naturality squares for α and ε, respectively. Putting together (3.17) and (3.18) we have

$$f_{\bar{M}} = j_{\#} \cdot 1_Q X^{\#} A \cdot \alpha X^{\#} A = j_{\#} \cdot \alpha X^{\#} A = f_M. \quad \square$$

Now we state the dual of Theorem 3.6.

THEOREM 3.7. *Let $M = (\bar{Q}, i, \sigma, \beta)\colon (A, X) \to (B, Y)$ be a machine in $\mathscr{K}$ such that its state functor $\bar{Q}$ has a left adjoint. Then the response of M can be computed by an inverse-state machine.*

Proof. Let $(Q, \bar{Q}, \nu, \varepsilon)$ be an adjunction. Define an inverse-state machine $M = (Q, \alpha, \tau, j)\colon (A, X) \to (B, Y)$ by

$$\begin{aligned}
&\alpha\colon I \xrightarrow{\nu} \bar{Q} Q \xrightarrow{\beta \bar{Q}} Q, \\
&\tau\colon QX \xrightarrow{QX\nu} QX\bar{Q}Q \xrightarrow{Q\sigma Q} Q\bar{Q}Y^{\#}Q \xrightarrow{\varepsilon Y^{\#} Q} Y^{\#} Q, \\
&j\colon QA \xrightarrow{Qi} Q\bar{Q} Y^{\#} B \xrightarrow{\varepsilon Y^{\#} B} Y^{\#} B.
\end{aligned} \tag{3.19}$$

In consequence of Theorem 3.6 it is sufficient to prove that applying the construction (3.14) for the data in (3.19) we get back the specification of the machine M, i.e.

$$i = \bar{Q} j \cdot \nu A, \quad \sigma = \varepsilon Y^{\#} \bar{Q} \cdot \bar{Q} \tau \bar{Q} \cdot \nu X \bar{Q}, \quad \beta = \varepsilon \cdot \alpha \bar{Q}. \tag{3.20}$$

The first two equalities of (3.19) have already been proved in Theorem 3.1 in context that Φ and Ψ are inverses of each other. The remaining $\beta=\varepsilon\cdot\alpha\bar{Q}$ is obvious from the adjunction identity

$$1_{\bar{Q}}=\bar{Q}\varepsilon\cdot\nu\bar{Q};\ \varepsilon\cdot\alpha\bar{Q}=\varepsilon\cdot(\beta Q\cdot\nu)\bar{Q}=\varepsilon\cdot\beta Q\bar{Q}\cdot\nu\bar{Q}=\beta\cdot\bar{Q}\varepsilon\cdot\nu\bar{Q}=\beta\cdot 1_{\bar{Q}}=\beta. \quad \square$$

THEOREM 3.8. Let M_1: $(A, X)\to(B, Y)$ and M_2: $(B, Y)\to(C, Z)$ be inverse-state machines in $\mathcal{K}$. Then the composite morphism $f_{M_2}\cdot f_{M_1}$: $X^{\#}A\to Z^{\#}C$ can be again computed by an inverse state machine.

Proof. Assume that M_1 has a state functor Q_1 and M_2 has a state functor Q_2. Denote a right adjoint of Q_1 and Q_2 by $\bar{Q}_1$ and $\bar{Q}_2$, respectively. By Theorem 3.6 the responses f_{M_1} and f_{M_2} can be computed by machines whose state functors are $\bar{Q}_1$ and $\bar{Q}_2$, respectively. Now apply Theorem 2.8 which says that the composite morphism $f_{M_2}\cdot f_{M_1}$ is the response of a machine with state functor $\bar{Q}_1\bar{Q}_2$. According to Theorem 3.7 if the composite functor $\bar{Q}_1\bar{Q}_2$ has left adjoint then the morphism $f_{M_1}\cdot f_{M_2}$ can be computed by an inverse-state machine. But, it is a well known result in category theory that the composite functors yield an adjunction, i.e. Q_2Q_1 is left adjoint to $\bar{Q}_1\bar{Q}_2$ (see [7], Theorem 8.1, pp. 101). $\square$

4. Generalized sequential machines in categories

The concept of generalized sequential machines in categories having binary products is developed in this section. A generalized sequential machine is a machine whose state functor is a product-functor and its final state transformation is a projection.

We also investigate sequential machines, i.e. machines working sequentially, moreover, elementary input produces an elementary output. Morphisms computed by generalized sequential as well as sequential machines in a category are characterized.

Throughout this section we assume that a category $\mathcal{K}$ with binary products is given.

DEFINITION 4.1. Fix a choice of a product diagram $A\xleftarrow{p}A\times B\xrightarrow{q}B$ for every given pair (A, B) of objects of $\mathcal{K}$, and given morphisms f: $A'\to A$, g: $B'\to B$ define the morphism $f\times g$: $A'\times B'\to A\times B$ by

$$\begin{tikzcd} A & A\times B \arrow[l, "p"'] \arrow[r, "q"] & B \\ A' \arrow[u, "f"] & A'\times B' \arrow[l, "p'"] \arrow[r, "q'"'] \arrow[u, "f\times g"'] & B' \arrow[u, "g"'] \end{tikzcd} \tag{4.1}$$

It is well known that in this case each object S of $\mathcal{K}$ induces a functor $S\times -: \mathcal{K}\to\mathcal{K}$ by

$$(S\times -)A:=S\times A,\quad (S\times -)f:=1_S\times f. \tag{4.2}$$

These functors are called *product functors*. It is obvious from (4.1) that the family of projections πA: $S\times A\to A$ constitute a natural transformation $\pi: (S\times -)\dot{\to} I$, called projection transformation. For orbitrary morphisms $h_1\colon C\to A$, $h_2\colon C\to B$ we use the notation (h_1, h_2) for the unique morphism satisfying (4.3) below.

$$
\begin{tikzcd}
A & A\times B \arrow[l, "p"'] \arrow[r, "q"] & B \\
& C \arrow[ul, "h_1"] \arrow[u, "{(h_1,\, h_2)}"'] \arrow[ur, "h_2"'] &
\end{tikzcd}
\tag{4.3}
$$

According to (4.1) and (4.3) we have the following identities:

$$(f\times g)\cdot(h_1, h_2) = (f\cdot h_1,\, g\cdot h_2) \tag{4.4}$$

$$(f\times g)\cdot(f_1\times g_1) = (f\cdot f_1)\times(g\cdot g_1) \tag{4.5}$$

$$(h_1, h_2)\cdot h = (h_1\cdot h,\, h_2\cdot h) \tag{4.6}$$

DEFINITION 4.2. A *generalized sequential machine* in $\mathcal{K}$ is a machine $M=(Q, i, \sigma, \beta)\colon (A, X)\to(B, Y)$ whose state functor Q is a product-functor induced by an object S of $\mathcal{K}$, and the final state transformation is the projection $S\times - \overset{\cdot}{\to} I$. Thus, a generalized sequential machine can be specified by

$M=(S, i, \sigma)\colon (A, X)\to(B, Y)$, where S is an object of $\mathcal{K}$, the *state object,*
$i\colon A\to S\times Y^{\#}B$ is a $\mathcal{K}$-morphism, the *initial state-output* morphism,
$\sigma\colon X(S\times -)\overset{\cdot}{\to}(S\times -)Y^{\#}$ is a natural transformation, the *transition.*

The *response* of a generalized sequential machine $M=(S, i, \sigma)\colon (A, X)\to(B, Y)$ is defined to be the response of the machine $M'=(S\times -, i, \sigma, \pi)\colon (A, X)\to(B, Y)$, where π is the projection $S\times - \overset{\cdot}{\to} I$.

Now we give a definition of sequential machines in a category. A sequential machine is a simple machine whose state functor is a product functor and whose final state transformation is the projection.

DEFINITION 4.3. Let A, B be objects of $\mathcal{K}$ and let X, Y be varietors in $\mathcal{K}$. A *sequential machine*

$$M = (S, i_0, \sigma_0)\colon (A, X) \to (B, Y)$$

in $\mathcal{K}$ consists of the following data:
an object S of $\mathcal{K}$, the state object,
a $\mathcal{K}$-morphism $i_0\colon A\to S\times B$, the initial state-output,
a natural transformation $\sigma_0\colon X(S\times -)\overset{\cdot}{\to}(S\times -)Y$, the transition.

The response of a sequential machine $M=(S, i_0, \sigma_0)$ is the composite morphism $f_M=\pi Y^{\#}B\cdot i_0^{\#}$, where $\pi\colon S\times - \overset{\cdot}{\to} I$ is the projection and $i_0^{\#}$ is the run map of M defined by

$$
\begin{tikzcd}
S\times B \arrow[r, "1_S\times\bar{\eta}B"] & S\times Y^{\#}B & S\times YY^{\#}B \arrow[l, "1_S\times\bar{\mu}_0 B"'] & X(S\times Y^{\#}B) \arrow[l, "\sigma_0 Y^{\#}B"'] \\
A \arrow[u, "i_0"] \arrow[r, "\eta A"] & X^{\#}A \arrow[u, "i_0^{\#}"] & & XX^{\#}A \arrow[ll, "\mu_0 A"'] \arrow[u, "Xi_0^{\#}"']
\end{tikzcd}
\tag{4.7}
$$

DEFINITION 4.4. Let A, B be objects of $\mathscr{K}$ and let X, Y be varietors in $\mathscr{K}$. A morphism $f\colon X^{\#}A \to Y^{\#}B$ is called *initial-segment preserving* if there is a natural transformation

$$\lambda\colon X(X^{\#}A \times -) \stackrel{\cdot}{\to} Y^{\#}, \tag{4.8}$$

such that

$$\begin{tikzcd}
X^{\#}A \arrow[rr, "f"] & & Y^{\#}B \\
XX^{\#}A \arrow[u, "\mu_0 A"] \arrow[r, "X(1_{X^{\#}A}, f)"] & X(X^{\#}A \times Y^{\#}B) \arrow[r, "\lambda Y^{\#}B"] & Y^{\#}Y^{\#}B \arrow[u, "\bar{\mu}B"']
\end{tikzcd} \tag{4.9}$$

THEOREM 4.5. A morphism $f\colon X^{\#}A \to Y^{\#}B$ can be computed by a generalized sequential machine in $\mathscr{K}$ if and only if f is initial-segment preserving.

Proof. Assume that a morphism $f\colon X^{\#}A \to Y^{\#}B$ is computed by a generalized sequential machine $M=(S, i, \sigma)\colon (A, X) \to (B, Y)$. Thus, $f=f_M=\pi Y^{\#}B \cdot i^{\#}$, where π is the projection transformation $S\times - \stackrel{\cdot}{\to} I$ and $i^{\#}$ is the run map of M defined by (4.10) below.

$$\begin{tikzcd}
& S\times Y^{\#}B & S\times Y^{\#}Y^{\#}B \arrow[l, "1_S\times\bar{\mu}B"'] & X(S\times Y^{\#}B) \arrow[l, "\sigma Y^{\#}B"'] \\
A \arrow[ur, "i"] \arrow[r, "\eta A"] & X^{\#}A \arrow[u, "i^{\#}"] & & XX^{\#}A \arrow[ll, "\mu_0 A"'] \arrow[u, "Xi^{\#}"']
\end{tikzcd} \tag{4.10}$$

Denote by p the projection $S \leftarrow S\times Y^{\#}B$, and let

$$r\colon X^{\#}A \xrightarrow{i^{\#}} S\times Y^{\#}B \xrightarrow{p} S. \tag{4.11}$$

It can be seen by the identity (4.5) that the morphism $r\colon X^{\#}A \to S$ induces a natural transformation $(r\times -)\colon X^{\#}A\times - \stackrel{\cdot}{\to} S\times -$ by

$$(r\times -)C\colon r\times 1_C\colon X^{\#}A\times C \to S\times C \tag{4.12}$$

for each object C of $\mathscr{K}$. Consider the natural transformation

$$\lambda\colon X(X^{\#}A\times -) \xrightarrow{X(r\times -)} X(S\times -) \xrightarrow{\sigma} (S\times -)Y^{\#} \xrightarrow{\pi Y^{\#}} Y^{\#}. \tag{4.13}$$

We shall prove that this λ satisfies (4.9) with the response morphism f. First, we show that $i^{\#}=(r, f)$. Because $S \xleftarrow{p} S\times Y^{\#}B \xrightarrow{\pi Y^{\#}B} Y^{\#}B$ is a product diagram $(p, \pi Y^{\#}B)=1_{S\times Y^{\#}B}$. Thus we have

$$i^{\#} = 1_{S\times Y^{\#}B}\cdot i^{\#} = (p, \pi Y^{\#}B)\cdot i^{\#} = (p\cdot i^{\#}, \pi Y^{\#}B\cdot i^{\#}) = (r, f). \tag{4.14}$$

By (4.4) we obtain from (4.14)

$$i^{\#} = (r\cdot 1_{X^{\#}A},\ 1_{Y^{\#}B}\cdot f) = (r\times 1_{Y^{\#}B})\cdot(1_{X^{\#}A}, f). \tag{4.15}$$

Taking into account (4.10) and (4.15) we have

$$\begin{aligned}f\cdot\mu_0 A &= \pi Y^{\#}B\cdot i^{\#}\cdot\mu_0 A = \pi Y^{\#}B\cdot(1_S\times\bar\mu B)\cdot\sigma Y^{\#}B\cdot Xi^{\#} =\\ &= \bar\mu B\cdot\pi Y^{\#}Y^{\#}B\cdot\sigma Y^{\#}B\cdot Xi^{\#} = \bar\mu B\cdot(\pi Y^{\#}\cdot\sigma)Y^{\#}B\cdot Xi^{\#} =\\ &= \bar\mu B\cdot(\pi Y^{\#}\cdot\sigma)Y^{\#}B\cdot X\big((r\times 1_{Y^{\#}B})\cdot(1_{Y^{\#}A}, f)\big) =\\ &= \bar\mu B\cdot(\pi Y^{\#}\cdot\sigma)Y^{\#}B\cdot X(r\times -)Y^{\#}B\cdot X(1_{X^{\#}A}, f) =\\ &= \bar\mu B\cdot\big(\pi Y^{\#}\cdot\sigma\cdot X(r\times -)\big)Y^{\#}B\cdot X(1_{X^{\#}A}, f).\end{aligned}$$

Applying the definition (4.13) of the natural transformation λ we conclude that

$$f\cdot\mu_0 A = \bar\mu B\cdot\lambda Y^{\#}B\cdot X(1_{X^{\#}A}, f),$$

which proves the commutativity of (4.9).

Conversely, assume that a morphism $f\colon X^{\#}A\to Y^{\#}B$ is initial-segment preserving, i.e. there is a natural transformation $\lambda\colon X(X^{\#}A\times -)\overset{\cdot}{\to} Y^{\#}$ rendering the diagram (4.9) commutative. For each object C of $\mathscr{K}$ let us denote by ϱC the projection $X^{\#}A\leftarrow X^{\#}A\times C$. We show that the composite morphism

$$\begin{aligned}\sigma C\colon X(X^{\#}A\times -)\,C = X(X^{\#}A\times C) &\xrightarrow{(\mu_0 A\cdot X\varrho C,\,\lambda C)} X^{\#}A\times Y^{\#}C =\\ &= (X^{\#}A\times -)Y^{\#}C\end{aligned} \tag{4.16}$$

is natural in C, thus we get a natural transformation

$$\sigma\colon X(X^{\#}A\times -)\overset{\cdot}{\to}(X^{\#}A\times -)Y^{\#}. \tag{4.17}$$

Let $h\colon C\to D$ be an arbitrary morphism. We have to prove that

$$\begin{tikzcd} X(X^{\#}A\times C) \arrow[r, "\sigma C"] \arrow[d, "X(X^{\#}A\times -)h"'] & X^{\#}A\times Y^{\#}C \arrow[d, "(X^{\#}A\times -)Y^{\#}h"] \\ X(X^{\#}A\times D) \arrow[r, "\sigma D"] & X^{\#}A\times Y^{\#}D. \end{tikzcd} \tag{4.18}$$

By (4.4) and the definition of the product-functor $X^{\#}A\times -$ we have

$$\begin{aligned}\sigma D\cdot X(X^{\#}A\times -)h &= (\mu_0 A\cdot X\varrho D, \lambda D)\cdot X(1_{X^{\#}A}\times h) =\\ &= \big(\mu_0 A\cdot X(\varrho D\cdot(1_{X^{\#}A}\times h)), \lambda D\cdot X(1_{X^{\#}A}\times h)\big).\end{aligned}$$

From (4.1) follows $\varrho D\cdot(1_{X^{\#}A}\times h)=1_{X^{\#}A}\cdot\varrho C=\varrho C$, hence using the naturality of λ we obtain

$$\begin{aligned}\sigma D\cdot X(X^{\#}A\times -)h &= (\mu_0 A\cdot X\varrho C, Y^{\#}h\cdot\lambda C) =\\ &= (1_{X^{\#}A}\times Y^{\#}h)\cdot(\mu_0 A\cdot X\varrho C, \lambda C) = (X^{\#}A\times -)Y^{\#}h\cdot\sigma C.\end{aligned}$$

Thus the diagram (4.18) is commutative.

Let us define the generalized sequential machine

$$M = (X^{\#}A, i, \sigma)\colon (A, X)\to(B, Y)$$

by σ in (4.16) and put

$$i\colon A \xrightarrow{\eta A} X^{\#}A \xrightarrow{(1_{X^{\#}A},\, f)} X^{\#}A\times Y^{\#}B. \tag{4.19}$$

We show that f is the response of M, i.e.

$$f=\pi Y^{\#}B\cdot i^{\#}, \tag{4.20}$$

where π is the projection transformation $X^{\#}A\times - \xrightarrow{\cdot} I$ and $i^{\#}$ is the run map of M:

$$\begin{tikzcd}
 & X^{\#}A\times Y^{\#}B & X^{\#}A\times Y^{\#}Y^{\#}B \arrow[l, "1_{X^{\#}A}\times\bar{\mu}B"'] & X(X^{\#}A\times Y^{\#}B) \arrow[l, "\sigma Y^{\#}B"'] \\
A \arrow[ur, "i"] \arrow[r, "\eta A"'] & X^{\#}A \arrow[u, "i^{\#}"'] & & XX^{\#}A \arrow[ll, "\mu_0 A"'] \arrow[u, "Xi^{\#}"']
\end{tikzcd} \tag{4.21}$$

In order to prove (4.20) it is enough to verify that $i^{\#}=(1_{X^{\#}A}, f)$. We do this by observing from the following that $(1_{X^{\#}A}, f)$ is an X-homomorphic extension by the same specification as $i^{\#}$, which means (4.21).

a) $(1_{X^{\#}A}, f)\cdot\eta A=i$, by definition (4.19) of i.

b) $(1_{X^{\#}A}, f)\cdot\mu_0 A=(1_{X^{\#}A}, \bar{\mu}B)\cdot\sigma Y^{\#}B\cdot X(1_{X^{\#}A}, f)$.

Applying (4.6), (4.9) and (4.4) in this order we have

$$\begin{aligned}(1_{X^{\#}A}, f)\cdot\mu_0 A &= (\mu_0 A, f\cdot\mu_0 A) = (\mu_0 A, \bar{\mu}B\cdot\lambda Y^{\#}B\cdot X(1_{X^{\#}A}, f)) = \\ &= (1_{X^{\#}A}\times\bar{\mu}B)\cdot(\mu_0 A, \lambda Y^{\#}B\cdot X(1_{X^{\#}A}, f)).\end{aligned}$$

By (4.3) $\varrho Y^{\#}B\cdot(1_{X^{\#}A}, f)=1_{X^{\#}A}$ holds, thus

$$\begin{aligned}(1_{X^{\#}A}, f)\cdot\mu_0 A &= (1_{X^{\#}A}\times\bar{\mu}B)\cdot(\mu_0 A\cdot X1_{X^{\#}A}, \lambda Y^{\#}B\cdot X(1_{X^{\#}A}, f)) = \\ &= (1_{X^{\#}A}\times\bar{\mu}B)\cdot(\mu_0 A\cdot X(\varrho Y^{\#}B\cdot(1_{X^{\#}A}, f)), \lambda Y^{\#}B\cdot X(1_{X^{\#}A}, f)) = \\ &= (1_{X^{\#}A}\times\bar{\mu}B)\cdot(\mu_0 A\cdot X\varrho Y^{\#}B, \lambda Y^{\#}B)\cdot X(1_{X^{\#}A}, f).\end{aligned}$$

Taking the definition (4.16) of the natural transformation σ we conclude that

$$(1_{X^{\#}A}, f)\cdot\mu_0 A=(1_{X^{\#}A}\times\bar{\mu}B)\cdot\sigma Y^{\#}B\cdot X(1_{X^{\#}A}, f)$$

which completes the proof of the theorem.

Corollary 4.6. Let A be an object of $\mathscr{K}$ and let X be a varietor in $\mathscr{K}$. The object $X^{\#}A$ is universal in the sense that for every generalized sequential machine $M\colon (A, X)\to(B, Y)$ there is a generalized sequential machine $M'\colon (A, X)\to(B, Y)$ whose state object is $X^{\#}A$, and M' computes the response of M.

Now we give a characterization of morphisms computed by sequential machines in $\mathscr{K}$.

THEOREM 4.7. Let X, Y be varietors in $\mathscr{K}$ and let A, B be objects of $\mathscr{K}$. A morphism $f\colon X^\# A \to Y^\# B$ can be computed by a sequential machine in $\mathscr{K}$ iff the following two conditions are satisfied:

i) there is a morphism $f_0\colon A \to B$ such that

$$\begin{tikzcd} X^\# A \arrow[r, "f"] & Y^\# B \\ A \arrow[u, "\eta A"] \arrow[r, "f_0"] & B \arrow[u, "\bar\eta B"'] \end{tikzcd} \tag{4.22}$$

ii) there is a natural transformation $\lambda_0\colon X(X^\# A \times -) \dot{\to} X$ making (4.23) commutative.

$$\begin{tikzcd} X^\# A \arrow[rr, "f"] & & Y^\# B \\ XX^\# A \arrow[u, "\mu_0 A"] \arrow[r, "X(1_{X^\# A}, f)"] & X(X^\# A \times Y^\# B) \arrow[r, "\lambda_0 Y^\# B"] & YY^\# B \arrow[u, "\bar\mu_0 B"'] \end{tikzcd} \tag{4.23}$$

Proof. Assume that a sequential machine $M=(S, i_0, \sigma_0)\colon (A, X) \to (B, Y)$ computes $f\colon X^\# A \to Y^\# B$. Let us take the generalized sequential machine $M'=(S, i, \sigma)\colon (A, X) \to (B, Y)$, where

$$\begin{aligned} i &:= A \xrightarrow{i_0} S \times B \xrightarrow{1_S \times \bar\eta B} S \times Y^\# B, \\ \sigma &:= X(S \times -) \xrightarrow{\sigma_0} (S \times -)Y \xrightarrow{(S\times -)\bar\eta_1} (S \times -)Y^\#. \end{aligned} \tag{4.24}$$

Remember that $\bar\eta_1 = \bar\mu_0 \cdot Y\bar\eta$. Then, by Lemma 2.6, the machine M' computes the response of M, i.e. the morphism f. Therefore $f = \pi Y^\# B \cdot i^\#$, where $\pi\colon S \times - \dot{\to} I$ is the projection and $i^\#$ is the run map of M'. Thus we have from (2.2)

$$f \cdot \eta A = \pi Y^\# B \cdot i^\# \cdot \eta A = \pi Y^\# B \cdot i = \pi Y^\# B \cdot (1_S \times \bar\eta B) \cdot i_0 = \bar\eta B \cdot \pi B \cdot i_0.$$

Hence, taking f_0 to be $\pi B \cdot i_0$ the condition i) of Theorem 4.7 will be satisfied. According to Theorem 4.5 there is a natural transformation $\lambda\colon X(X^\# A \times -) \to Y^\#$ such that for this λ and f the diagram (4.9) is commutative. Moreover, by (4.13), λ has the form

$$\lambda = X(X^\# A \times -) \xrightarrow{X(r\times -)} X(S \times -) \xrightarrow{\sigma} (S \times -)Y^\# \xrightarrow{\pi Y^\#} Y^\#. \tag{4.25}$$

Now let us define the natural transformation λ_0 by

$$\lambda_0 = X(X^\# A \times -) \xrightarrow{X(r\times -)} X(S \times -) \xrightarrow{\sigma_0} (S \times -)Y \xrightarrow{\pi Y} Y. \tag{4.26}$$

Since (4.9) holds for λ in (4.25) it is enough to prove

$$\bar\mu \cdot \lambda Y^\# = \bar\mu_0 \cdot \lambda_0 Y^\#.$$

By (4.24), (4.25), (4.26) and the naturality of π we have

$$\begin{aligned} \bar\mu \cdot \lambda Y^\# &= \bar\mu(\pi Y^\# \cdot \sigma \cdot X(r \times -))Y^\# = \bar\mu \cdot (\pi Y^\# \cdot (S \times -)\bar\eta_1 \cdot \sigma_0 \cdot X(r \times -))Y^\# = \\ &= \bar\mu \cdot (\bar\eta_1 \cdot \pi Y \cdot \sigma_0 \cdot X(r \times -))Y^\# = \bar\mu \cdot (\bar\eta_1 \cdot \lambda_0)Y^\# = \bar\mu \cdot \bar\eta_1 Y^\# \cdot \lambda_0 Y^\#. \end{aligned}$$

But we have already proved in Lemma 2.6 that $\bar{\mu}\cdot\bar{\eta}_1 Y^{\#}=\bar{\mu}_0$, thus we obtain $\bar{\mu}\cdot\lambda Y^{\#}=\bar{\mu}_0\cdot\lambda_0 Y^{\#}$.

Conversely, assume that the conditions i) and ii) are satisfied for a morphism $f\colon X^{\#}A\to Y^{\#}B$. If we take $\lambda=\bar{\eta}_1\lambda_0$ we have $\bar{\mu}\cdot\lambda Y^{\#}=\bar{\mu}\cdot(\bar{\eta}_1\cdot\lambda_0)Y^{\#}=$ $=\bar{\mu}\cdot\bar{\eta}_1 Y^{\#}\cdot\lambda_0 Y^{\#}=\bar{\mu}_0\cdot\lambda_0 Y^{\#}$. Thus (4.23) implies that the λ above and f satisfies (4.9), and hence by Theorem 4.5 there is generalized sequential machine $M=(X^{\#}A, i, \sigma)$ computing the morphism f. In the sense of Lemma 2.6 it is sufficient to prove that the initial state-output morphism i and the transition σ of M are simple. Since the initial state-output i of M is defined in Theorem 4.5 by

$$i\colon A\xrightarrow{\eta A} X^{\#}A\xrightarrow{(1_{X^{\#}A},\, f)} X^{\#}A\times Y^{\#}B,$$

thus, if we take i_0 to be $(\eta A, f_0)$ for the f_0 in condition i), then

$$(X^{\#}A\times -)\bar{\eta}B\cdot i_0=(1_{X^{\#}A}\times\bar{\eta}B)\cdot(1_{X^{\#}A}, f_0)=(\eta A, \bar{\eta}B\cdot f_0)=$$
$$=(\eta A, f\cdot\eta A)=(1_{X^{\#}A}, f)\cdot\eta A=i.$$

This means that i is simple in the sense of Definition 2.5. The transition σ of M has the form (α, λ) for some α by Theorem 4.5. From $\lambda=\bar{\eta}_1\cdot\lambda_0$ we conclude that σ is simple. This completes the proof of the theorem. □

THEOREM 4.8. The family of the generalized sequential machine morphisms in $\mathcal{K}$ is closed under composition.

Proof. Let $M_1=(S_1, i_1, \sigma_1)\colon (A, X)\to(B, Y)$ and $M_2=(S_2, i_2, \sigma_2)\colon (B, Y)\to$ $\to(C, Z)$ be generalized sequential machines in $\mathcal{K}$ computing the morphisms $f_1\colon$ $X^{\#}A\to Y^{\#}B$, $f_2\colon Y^{\#}B\to Z^{\#}C$, respectively. By Theorem 2.8 the composite morphism $f_2\cdot f_1\colon X^{\#}A\to Z^{\#}C$ can be computed by a machine

$$M=(Q, i, \sigma, \beta)\colon (A, X)\to(C, Z)$$

where $Q=(S_1\times -)(S_2\times -)$,

$$\begin{aligned} i&=A\xrightarrow{i_1} S_1\times Y^{\#}B\xrightarrow{(S_1\times -)i_2^{\#}}(S_1\times -)(S_2\times -)Z^{\#}C=S_1\times(S_2\times Z^{\#}C),\\ \beta&=(S_1\times -)(S_2\times -)\xrightarrow{(S_1\times -)\pi_2}(S_1\times -)\xrightarrow{\pi_1} I.\end{aligned}\tag{4.27}$$

Here $\pi_1\colon S_1\times -\overset{.}{\to} I$, $\pi_2\colon S_2\times -\overset{.}{\to} I$ are the projection transformations. The object map of the composite functor $(S_1\times -)(S_2\times -)$ is $(S_1\times -)(S_2\times -)D=$ $=(S_1\times -)(S_2\times D)=S_1\times(S_2\times D)$ for any object D of $\mathcal{K}$. Since the category $\mathcal{K}$ has binary products we may recall the well known result (see Mac Lane [7], pp. 73. Proposition 1) which asserts that there is an isomorphism

$$\alpha_{S_1,S_2}\colon S_1\times(S_2\times D)\cong(S_1\times S_2)\times D$$

natural in S_1, S_2 and D, moreover, $\alpha_{S_1,S_2,D}$ commutes with the projections to S_1, S_2 and D, respectively. Thus there is a natural transformation

$$\varphi\colon (S_1\times -)(S_2\times -)\overset{.}{\to}(S_1\times S_2)\times -$$

with inverse ψ (i.e., both $\varphi\cdot\psi$ and $\psi\cdot\varphi$ are the identity natural transformations on the corresponding functors),

$$\psi\colon (S_1\times S_2)\times - \overset{\cdot}{\longrightarrow} (S_1\times -)(S_2\times -)$$

such that $\pi\cdot\varphi=\pi_1\cdot(S_1\times -)\pi_2$, where $\pi\colon (S_2\times S_1)\times - \dot{\to} I$ is the projection. Consider the generalized sequential machine

$$M' = \big((S_1\times S_2)\times -,\ i', \sigma', \pi\big)\colon (A, X) \to (C, Z)$$

where i' and σ' are defined by i and σ in (4.27) as follows

$$i' = A \xrightarrow{i} (S_1\times -)(S_2\times -)Z^{\#}C \xrightarrow{\varphi Z^{\#}C} ((S_1\times S_2)-)Z^{\#}C, \tag{4.28}$$
$$\sigma' = \varphi Z^{\#}\cdot\sigma\cdot X\psi.$$

By Theorem 2.10 it is sufficient to prove that φ is a simulation $\varphi\colon M\to M'$. We have to show the equalities

$$i' = \varphi Z^{\#} C\cdot i,\quad \sigma'\cdot X\varphi = \varphi Z^{\#}\cdot\sigma,\quad \pi\cdot\varphi = \beta. \tag{4.29}$$

The first equality of (4.29) holds by (4.28). As $\beta=\pi_1\cdot(S_1\times -)\pi_2$, thus $\pi\cdot\varphi=\beta$. Using the definition (4.28) of σ' and the equality $\psi\cdot\varphi=1_{(S_1\times -)(S_2\times -)}$ we have

$$\sigma'\cdot X\varphi = \varphi Z^{\#}\cdot\sigma\cdot X\psi\cdot X\varphi = \varphi Z^{\#}\cdot\sigma\cdot X(\psi\cdot\varphi) = \varphi Z^{\#}\cdot\sigma\cdot X1_{(S_1\times -)(S_2\times -)} = \varphi Z^{\#}\cdot\sigma.$$

This proves that φ is a simulation and completes the proof of the theorem. □

Finally, we show that the computational capacity of the generalized sequential machines in a category and that of the process transformations of Arbib and Manes are equal.

Definition 4.9 (Arbib and Manes [4]). Let A, B be objects of $\mathscr{K}$ and let X, Y be varietors in $\mathscr{K}$. A *process transformation* $T\colon (A, X)\to(B, Y)$ in $\mathscr{K}$ is $T=(S, d, t, k, \beta)$, where

(S, d) is an X-algebra, the *state algebra,*
$t\colon A\to S$ is the *initial state,*
$k\colon A\to Y^{\#}B$ is the *initial throughput,*
$\beta\colon X(S\times -)\dot{\to} Y^{\#}$ is a natural transformation, the *output.*

The *response* of T is the morphism $g\colon X^{\#}A\to Y^{\#}B$ defined by

$$\begin{tikzcd}
 & Y^{\#}B & Y^{\#}Y^{\#}B \arrow[l, "\bar{\mu}B"'] & X(S\times Y^{\#}B) \arrow[l, "\beta Y^{\#}B"'] \\
A \arrow[r, "\eta A"'] \arrow[ur, "k"] & X^{\#}A \arrow[u, "g"] & & XX^{\#}A \arrow[ll, "\mu_0 A"'] \arrow[u, "{X(r, g)}"']
\end{tikzcd} \tag{4.30}$$

where $r\colon X^{\#}A \to S$ is the reachability map of (t, d), i.e. the homomorphic extension

$$\begin{tikzcd} & S & XS \arrow[l, "d"'] \\ A \arrow[ur, "t"] \arrow[r, "\eta A"] & X^{\#}A \arrow[u, "r"] & XX^{\#}A \arrow[l, "\mu_0 A"'] \arrow[u, "Xr"'] \end{tikzcd} \tag{4.31}$$

THEOREM 4.10. *A morphism $g\colon X^{\#}A \to Y^{\#}B$ is the response of a process transformation iff g can be computed by a generalized sequential machine in $\mathcal{K}$.*

Proof. Assume that a morphism $g\colon X^{\#}A \to Y^{\#}B$ is the response of a process transformation $T=(S, d, t, k, \beta)\colon (A, X) \to (B, Y)$. For each object C of $\mathcal{K}$ let

$$S \xleftarrow{\varrho C} S \times C \xrightarrow{\pi C} C \tag{4.32}$$

be the product diagram, and define the morphism $\sigma C\colon X(S \times C) \to (S \times -)Y^{\#}C$ by the composite

$$\sigma C\colon X(S \times C) \xrightarrow{(d \cdot X\varrho C, \beta C)} S \times Y^{\#}C. \tag{4.33}$$

One can check by an easy coputation that σC in (4.32) is natural in C, i.e. we get a natural transformation

$$\sigma\colon X(S \times -) \xrightarrow{\cdot} (S \times -)Y^{\#}.$$

Consider the generalized sequential machine $M=(S, i, \sigma)\colon (A, X) \to (B, Y)$, where $i=(t, k)$ and σ is defined in (4.32). We prove that this machine computes the morphism g, i.e. $f_M=g$. The response of M is $f_M=\pi Y^{\#}B \cdot i^{\#}$, where $i^{\#}$ is the run map of M, i.e. the unique morphism satisfying both (4.34) and (4.35) below

$$i^{\#} \cdot \eta A = i, \tag{4.34}$$

$$i^{\#} \cdot \mu_0 A = (1_S \times \bar{\mu}B) \cdot \sigma Y^{\#}B \cdot Xi^{\#}. \tag{4.35}$$

Since $\pi Y^{\#}B \cdot (r, g)=g$, it is enough to prove that $i^{\#}=(r, g)$. We do this by observing that the morphism (r, g) satisfies (4.34) and (4.35) in place of $i^{\#}$, i.e. (4.36) and (4.37) hold

$$(r, g) \cdot \eta A = i, \tag{4.36}$$

$$(r, g) \cdot \mu_0 A = (1_S \times \bar{\mu}B) \cdot \sigma Y^{\#}B \cdot X(r, g). \tag{4.37}$$

By the triangular part of (4.30) and (4.31) we have

$$(r, g) \cdot \eta A = (r \cdot \eta A, g \cdot \eta A) = (t, k),$$

thus (4.36) holds. Again by (4.30) and (4.31)

$$(r, g) \cdot \mu_0 A = (r \cdot \mu_0 A, g \cdot \mu_0 A) = \big(d \cdot Xr, \bar{\mu}B \cdot \beta Y^{\#}B \cdot X(r, g)\big). \tag{4.38}$$

From the definition (4.33) of σ it follows that $\pi Y^{\#}Y^{\#}B \cdot \sigma Y^{\#}B=\beta Y^{\#}B$, and hence, using the naturality of π we obtain

$$\begin{aligned} (r, g) \cdot \mu_0 A &= \big(d \cdot Xr, \bar{\mu}B \cdot \pi Y^{\#}Y^{\#}B \cdot \sigma Y^{\#}B \cdot X(r, g)\big) = \\ &= \big(d \cdot Xr, \pi Y^{\#}B \cdot (1_S \times \bar{\mu}B) \cdot \sigma Y^{\#}B \cdot X(r, g)\big). \end{aligned} \tag{4.39}$$

Because (4.32) is a product diagram we have

$$d\cdot Xr = d\cdot X(\varrho Y^{\#}B\cdot(r,g)) = \varrho Y^{\#}B\cdot(d\cdot X\varrho Y^{\#}B\times(r,g), \bar{\mu}B\cdot\beta Y^{\#}B\cdot X(r,g)) =$$
$$= \varrho Y^{\#}B\cdot(d\cdot X\varrho Y^{\#}B, \bar{\mu}B\cdot\beta Y^{\#}B)\cdot X(r,g) =$$
$$= \varrho Y^{\#}B\cdot(1_S\times\bar{\mu}B)\cdot(d\cdot X\varrho Y^{\#}B, \beta Y^{\#}B)\cdot X(r,g).$$

And by the definition (4.33) of σ

$$d\cdot Xr = \varrho Y^{\#}B\cdot(1_S\times\bar{\mu}B)\cdot\sigma Y^{\#}B\cdot X(r,g). \tag{4.40}$$

Putting toghether (4.39), (4.40) and the equality $1_{S\times Y^{\#}B}=(\varrho Y^{\#}B, \pi Y^{\#}B)$ we conclude

$$(r,g)\cdot\mu_0 A = (\varrho Y^{\#}B, \pi Y^{\#}B)\cdot(1_S\times\bar{\mu}B)\cdot\sigma Y^{\#}B\cdot X(r,g) = (1_S\times\bar{\mu}B)\cdot\sigma Y^{\#}B\cdot X(r,g).$$

Thus (4.37) holds, which ends the proof of the "only if" part.

Conversely, assume that a morphism $f\colon X^{\#}A\to Y^{\#}B$ can be computed by a generalized sequential machine in $\mathscr{K}$. Then, by Theorem 4.5, the morphism f is initial-segment preserving, i.e. there is a natural transformation

$$\lambda\colon X(X^{\#}A\times -)\to Y^{\#},$$

such that the diagram (4.9) is commutative. Now consider the process transformation $T=(X^{\#}A, \mu_0 A, f\cdot\eta A, \eta A, \lambda)\colon (A, X)\to(B, Y)$. It is obvious that $1_{X^{\#}A}$ is the reacability map of $(\eta A, \mu_0 A)$. Hence, taking into account the defining diagram (4.30) of a process transformation we obtain that (4.9) defines the response of T, which is f.

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A 5 state solution of the early bird problem in a one dimensional cellular space

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There exists a class of interesting problems for cellular automata characterized by their common property of decomposing some global behaviour into homogeneous parallel local transitions (VOLLMAR [6]). Well known representatives of this class are the firing squad synchronization problem (MOORE [2], VOLLMAR [4]) and the French flag problem (HERMAN [1]).

Another problem of this class was defined by ROSENSTIEHL et al. in [3] and named as the "early bird" problem.

1. The original definition of the early bird problem

To each of the n vertices of an elementary cyclic graph there is assigned an automaton. These automata may be "excited" (birds may come from the outside world) at different moments. The task is to distinguish between the first (early) and the later birds. More exactly the transition function must ensure the automaton excited first to be assumed a distinguished state while all the others a different state after some time interval. ROSENSTIEHL et al. [3] gave a $2n$ step solution on condition that at each moment maximally one excitation occurs.

2. The modified early bird problem

VOLLMAR in [5] defined the problem for a one-dimensional cellular space allowing more than one cell to be excited at a given time step. Only quiescent cells may be excited; before the first time step at least one cell should be excited. After a certain period the first bird(s) should be in a distinguished state while all the others in a different state.

The solution (VOLLMAR [5]) uses the "age of waves" concept: each bird sends out age signals that are compared (numerically). As a consequence elder bird(s) survive, while waves of the same age or waves reaching the border are reflected and mark the sender automata. After a certain number of time steps there remain(s) only early bird(s) marked from both directions.

3. A 5 state solution to the problem

The proposed solution uses the "age of waves" concept of VOLLMAR [5] but in a simplified manner. The age of a wave (i.e. of a bird) is modelled directly by the *length of the waves,* rather than by a counter which is hard to handle, especially, for the number of needed bits of a counter is dependent on the number of cells. Therefore the counter cannot be incorporated in cells' states, it is rather simulated by a group of cells.

The basic idea is to send L (left) and R (right) waves in the left and the right directions. At each time step the wave is growing by one cell thus modelling the age of the sender. When two waves are colliding, pairs of R and L states annihilate each other, and N (neutral) states will replace them.

An L or R wave reaching a bird (in state B) will cause the annihilation of it (state N will be generated instead of the state B).

Consequently, the needed cell-states are:

Q = quiescent (initial) state,
B = bird state (arises from state Q, spontaneously),
L = left wave, expanding to left,
R = right wave, expanding to right,
N = neutral state.

4. Construction of the transition function

In the following we construct the transition function on the basis of the above-described principle. The transition function will be described with "left, own, right→→new-state" terms.

First we assume only two birds with different ages (they were born in different time-steps). Each bird sends waves in both directions, this is ensured by terms

1. $BQQ \to R$,

2. $QQB \to L$.

The waves are growing in each step:

1/a. $RQQ \to R$,

2/a. $QQL \to L$.

It is clear, that the length of the waves is equal to the age of the sender, in each step. After a certain time the waves are colliding between the birds, then an annihilation process begins:

3. $RQL \to N$,
4. $RRL \to N$
5. $RLL \to N$

These terms (4 and 5) imply the transition $RRLL \to RNNL$ (that is, each section of cells with states $RRLL$ goes into $RNNL$).

From annihilation a neutral area arises, in which the R states step to the right, the L states to the left (the points mean arbitrary state):

$$\left.\begin{array}{ll} \text{6. } RN\ (\text{not } L) \to R \\ \text{7. } \cdot RN \qquad\quad \to N \end{array}\right\} \quad R \text{ steps right by the transition } \cdot RN\ (\text{not } L) \to \cdot NR \cdot$$

$$\left.\begin{array}{ll} \text{8. } (\text{not } R)\ NL \to L \\ \text{9. } NL \cdot \qquad\quad \to N \end{array}\right\} \quad L \text{ steps left by the transition } (\text{not } R)\ NL \cdot \to \cdot LN \cdot$$

10. $RNL \to N$ annihilation.

If the left bird is the earlier one, then after a certain time all the L states are annihilated between the birds, and the remained R states can go to the right and "kill" the right bird:

6/a. $RBR \to R$,
7/a. $\cdot RB \to N$.

For the state L similarly:

8/a. $LBL \to L$,
9/a. $BL \cdot \to N$.

The described process is presented on Listing 1 generated by computer-simulation. The cell-states are displayed with the conversion $Q=$"$\cdot$", $B=$"B", $L=$"$<$", $R=$"$>$" and $N=$"$*$". On the edges of the cellular space dummy cells are used with the state N.

Listing 1

```
STEP  0: * . . . . B . . . . . . . . . . . . . *
STEP  1: * . . . < B > . . . . . . . . . . . . *
STEP  2: * . . < < B > > . . . . . . . . . . . *
STEP  2: * . . < < B > > . . . . . . . . B . . *
STEP  3: * . < < < B > > > . . . . . . < B > . *
STEP  4: * < < < < B > > > > . . . . < < B > > *
STEP  5: * * < < < B > > > > > . . < < < B > * *
STEP  6: * < * < < B > > > > > > < < < < B * > *
STEP  7: * * < * < B > > > > > * * < < < B * * *
STEP  8: * < * < * B > > > > * > < * < < B * * *
STEP  9: * * < * * B > > > * > * * < * < B * * *
STEP 10: * < * * * B > > * > * > < * < * B * * *
STEP 11: * * * * * B > * > * > * * < * * B * * *
STEP 12: * * * * * B * > * > * > < * * * B * * *
STEP 13: * * * * * B * * > * > * * * * * B * * *
STEP 14: * * * * * B * * * > * > * * * * B * * *
STEP 15: * * * * * B * * * * > * > * * * B * * *
STEP 16: * * * * * B * * * * * > * > * * B * * *
STEP 17: * * * * * B * * * * * * > * > * B * * *
STEP 18: * * * * * B * * * * * * * > * > B * * *
STEP 19: * * * * * B * * * * * * * * > * > * * *
STEP 20: * * * * * B * * * * * * * * * > * > * *
STEP 21: * * * * * B * * * * * * * * * * > * > *
STEP 22: * * * * * B * * * * * * * * * * * > * *
STEP 23: * * * * * B * * * * * * * * * * * * > *
STEP 24: * * * * * B * * * * * * * * * * * * * *
STEP 25: * * * * * B * * * * * * * * * * * * * *
```

The terms described above represent only the typical situations in the case of two birds. If *more then two birds* are allowed and all special cases are respected (e.g. two neighbouring birds, a bird killed from both direction at the same time, etc.), then the following extended transition function called as *"early bird function"* is needed (in the following terms an expression (B, R) means "state B or state R"):

1. $(B, R)\ Q\ (Q, N) \to R$	wave-growing
2. $(Q, N)\ Q\ (B, L) \to L$	
3. $(B, R)\ Q\ (B, L) \to N$	wave-growing with annihilation
4. $\cdot\ R\ L \to N$	annihilation by the transition
5. $R\ L\ \cdot \to N$	$\cdot RL \cdot \to \cdot NN \cdot$
6. $R\ (B, N)\ (\text{not } L) \to R$	R steps right by the transition
7. $\cdot\ R\ (B, N) \to N$	$\cdot R\ (B, N)\ (\text{not } L) \to \cdot NR \cdot$
8. $(\text{not } R)\ (B, N)\ L \to L$	L steps left by the transition
9. $(B, N)\ L\ \cdot \to N$	$(\text{not } R)\ (B, N)\ L \cdot \to \cdot LN \cdot$
10. $R\ (B, N)\ L \to N$	annihilation by the transition $\cdot R\ (B, N)\ L \cdot \to \cdot NNN \cdot$

11. In all other cases the new state must be equal to the old own state.

5. Exact proof of the algorithm

It is easy to prove that for two birds the "early bird function" works right. For the *general case,* where in each step any quiescent cell can change into the bird-state, an exact proof is given in the following.

Theorem. A one dimensional 5-state cellular space consisting of m cells is considered, where

— in the initial configuration (at $t=0$) each cell is in state Q, and the dummy cells on the edges are in state N,

— between any two steps (so to say, at $t+1/2$) any quiescent cell can alter into state B.

Statement. Using the "early bird function" in this cellular space, after a finite time (it seems that maximum $3m$ steps) only the "early birds" (the birds arisen at first) are existing, all other cells have the state N.

The *proof* is based on the notion "route of the wave-states". To define this notion some investigations are needed for the behaviour of wave-states. The following properties can be found:

— A wave-state (i.e. L or R) may arise only from state Q, by terms 1 and 2.

— L states move to the left, R states to the right. More exactly, if in front of a wave-state there is a state N or B, then the wave-state steps forward (see terms 6—9). If in front of a wave-state there is the same wave-state or state Q, then the wave-state remains on its place (by "term 11").

— If an R and an L are colliding, then they annihilate each other (see terms 4, 5, 10). A wave-state reaching the border of the cellular space is annihilated by the dummy cell (see terms 7, 9).

— The behaviour of a wave-state is always independent from the state occurring behind it.

These properties show, that a wave-state *arises* on a certain point of the cellular space, it *goes* left or right depending on its type, and it is *annihilated* on another point of the cellular space. The section of cells, determined by the point of origin and the point of annihilation of a wave-state, will be called as the *route of the wave-state.*

If a cell contained in a route of a wave-state has been excited, then obviously this bird cannot survive. This fact gives special importance for the routes of the states R and L, which can be characterized in the following lemma.

Lemma. (i) If a state L and a state R arose at the same time on the both ends of a quiescent section $Q \dots Q$, then after a finite time they will meet and annihilate one another.

(ii) If a wave-state arose on the end of an outside quiescent section (bounded by a dummy cell on its other end), then the wave-state will go to the left or to the right until it reaches the border, and will be annihilated by the dummy cell.

Proof. First the statement (i) will be proved, using induction for the length n of the quiescent section $Q \dots Q$.

For $n=2$ the statement (i) is obvious, because we have the transition $QQ \to RL \to NN$ in this case.

Now the statement (i) is assumed for any section with length less then n, and a quiescent section of length n is considered, on the both ends of which an $R-L$ pair was arisen at time t (hereby the length of the section was reduced to $n-2$). Between t and $t+1$ (so to say, at $t+1/2$) a number of birds may be excited in this section, hereby the section may be divided into more subsections, each having a length less then n. At time $t+1$ all quiescent sections of length 1 have disappeared (see term 3), and on the both ends of all other sections states R and L are arising. By the induction assumption these $R-L$ pairs must annihilate each other. So the original R and L — arisen on the ends of the section of length n — cannot meet with any other wave-state, therefore they will annihilate each other.

The statement (ii) can be proved in a similar way. □

Applying these results it is easy to prove the original theorem.

Assume, that the early birds are excited at time $t_0+1/2$, the configuration at this time-point consists from bird sections and quiescent sections alternating one another. At time t_0+1 on the ends of each quiescent section an $R-L$ pair arises. These pairs — according to the lemma — will annihilate each other, so their routes cover all the space between the early birds. Similarly, the routes of the wave-states, arisen on the ends of the outside quiescent sections, cover the space between the outside early birds and the dummy cells. This fact implies, that *all later birds will be killed.* On the other hand, *the early birds must survive,* because the route of any wave-state (arising after t_0) is contained by one of the quiescent sections at t_0+1.

With these notes the proof of the theorem is complete.

6. Simulation examples

The presented solution of the early bird problem is demonstrated below using computer-simulation. The cell-states are displayed with the conversion $Q=$"•", $B=$"B", $L=$"<", $R=$">" and $N=$"*". On the edges of the cellular space the dummy cells are displayed, too.

In the case of Listing 2 four birds come from the outside world (at $t=4{,}5$ two birds at the same time). After $t=15$ only the early bird lives, in the further (not displayed) steps the remained wave-states will be annihilated by the dummy cells.

Listing 2

```
STEP  0: * . . . . . . . . . . . . . . B . . . . . . . . . . . . . . . . . . *
STEP  1: * . . . . . . . . . . . . . < B > . . . . . . . . . . . . . . . . . *
STEP  2: * . . . . . . . . . . . . < < B > > . . . . . . . . . . . . . . . . *
STEP  2: * . . . . . . . . B . . . < < B > > . . . . . . . . . . . . . . . . *
STEP  3: * . . . . . . . < B > . < < < B > > > . . . . . . . . . . . . . . . *
STEP  4: * . . . . . . < < B > * < < < B > > > > . . . . . . . . . . . . . . *
STEP  4: * . . . . . . < < B > * < < < B > > > > . . . . B . B . . . . . . . *
STEP  5: * . . . . . < < < B * * * < < B > > > > > . . < B * B > . . . . . . *
STEP  6: * . . . . < < < < B * * < * < B > > > > > > < < B * B > > . . . . . *
STEP  7: * . . . < < < < < B * < * < * B > > > > > * * < B * B > > > . . . . *
STEP  8: * . . < < < < < < B < * < * * B > > > > * > < * B * B > > > > . . . *
STEP  9: * . < < < < < < < < * < * * * B > > > * > * * * B * B > > > > > . . *
STEP 10: * < < < < < < < < < < * * * * B > > * > * > * * B * B > > > > > > . *
STEP 11: * * < < < < < < < < < * * * * B > * > * > * > * B * B > > > > > > > *
STEP 12: * < * < < < < < < < < * * * * B * > * > * > * > B * B > > > > > > * *
STEP 13: * * < * < < < < < < < * * * * B * * > * > * > * > * B > > > > > * > *
STEP 14: * < * < * < < < < < < * * * * B * * * > * > * > * > B > > > > * > * *
STEP 15: * * < * < * < < < < < * * * * B * * * * > * > * > * > > > > * > * > *
STEP 16: * < * < * < * < < < < * * * * B * * * * * > * > * > > > > * > * > * *
```

In the case of Listing 3 six birds come from the outside world (three birds at $t=0{,}5$ and three birds at $t=2{,}5$). During 22 steps all late birds are killed.

Listing 3

```
STEP  0: * . . . . . . . . . . . . . . . . . . . . . . B B . B . . . . . . . *
STEP  1: * . . . . . . . . . . . . . . . . . . . . . < B B * B > . . . . . . *
STEP  2: * . . . . . . . . . . . . . . . . . . . . < < B B * B > > . . . . . *
STEP  2: * . . . . B . . . . . . . . B B . . . . . < < B B * B > > . . . . . *
STEP  3: * . . . < B > . . . . . . < B B > . . . < < < B B * B > > > . . . . *
STEP  4: * . . < < B > > . . . . < < B B > > . < < < < B B * B > > > > . . . *
STEP  5: * . < < < B > > > . . < < < B B > > * < < < < B B * B > > > > > . . *
STEP  6: * < < < < B > > > > < < < < B B > * * * < < < B B * B > > > > > > . *
STEP  7: * * < < < B > > > * * < < < B B * > * < * < < B B * B > > > > > > > *
STEP  8: * < * < < B > > * > < * < < B B * * * * < * < B B * B > > > > > > * *
STEP  9: * * < * < B > * > * * < * < B B * * * < * < * B B * B > > > > > * > *
STEP 10: * < * < * B * > * > < * < * B B * * < * < * * B B * B > > > > * > * *
STEP 11: * * < * * B * * > * * < * * B B * < * < * * * B B * B > > > * > * > *
STEP 12: * < * * * B * * * > < * * * B B < * < * * * * B B * B > > * > * > * *
STEP 13: * * * * * B * * * * * * * * B < * < * * * * * B B * B > * > * > * > *
STEP 14: * * * * * B * * * * * * * * < * < * * * * * * B B * B * > * > * > * *
STEP 15: * * * * * B * * * * * * * < * < * * * * * * * B B * B * * > * > * > *
STEP 16: * * * * * B * * * * * * < * < * * * * * * * * B B * B * * * > * > * *
STEP 17: * * * * * B * * * * * < * < * * * * * * * * * B B * B * * * * > * > *
STEP 18: * * * * * B * * * * < * < * * * * * * * * * * B B * B * * * * * > * *
STEP 19: * * * * * B * * * < * < * * * * * * * * * * * B B * B * * * * * * > *
STEP 20: * * * * * B * * < * < * * * * * * * * * * * * B B * B * * * * * * * *
STEP 21: * * * * * B * < * < * * * * * * * * * * * * * B B * B * * * * * * * *
STEP 22: * * * * * B < * < * * * * * * * * * * * * * * B B * B * * * * * * * *
STEP 23: * * * * * < * < * * * * * * * * * * * * * * * B B * B * * * * * * * *
STEP 24: * * * * < * < * * * * * * * * * * * * * * * * B B * B * * * * * * * *
```

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On the completeness of proving partial correctness

By L. Csirmaz

We give here a proof for the completeness of the Floyd—Hoare program verification method in a case which has remained open in [1]. The method used here is basically the same as in [5]. For the motivation behind our concepts see [1, 3, 10]. Applications of our results in dynamic logic can be found in [10].

1. Introduction

Structures will be denoted by bold-faced type letters, their underlying sets by the corresponding capital letters. If A is a set and $n\in\omega$ then A^n denotes the set of n-tuples of the elements of A. Throughout the paper d denotes an arbitrary, but fixed similarity type, and T denotes an arbitrary but fixed consistent theory of that type. For $n\in\omega$, F_d^n denotes the set of first order formulas of type d with free variables among $\{y_i\colon i<n\}$, and we let $F_d=\bigcup\{F_d^n\colon n\in\omega\}$. In particular, T is a proper subset of F_d^0. For the sake of simplicity we make no typographical distinction between single symbols and sequences of symbols.

A program (or rather a program scheme) can be regarded as a prescription which defines uniquely the next moment contents of the registers from their present moment contents. Therefore we adapt

Definition 1. Let $T\subset F_d^0$ be arbitrary. A d-type *program* (in T) is a formula $\varphi\in F_d^2$ such that

$$T\vdash\forall x\exists!\, y\varphi(x,y). \quad \square$$

Let $\mathbf{D}$ be a d-type structure, and $\mathbf{D}\models T$. Then, by this definition, the program φ defines a function from D to D which we denote by $p_{\varphi,\mathbf{D}}$. More precisely, for every $q\in D$ there is exactly one element of D, denoted by $p_{\varphi,\mathbf{D}}(q)$ for which $\mathbf{D}\models\varphi(q,p_{\varphi,\mathbf{D}}(q))$. To avoid long and unreadable formulas we omit the indices φ, $\mathbf{D}$ everywhere and use the letter p as a new function symbol denoting $p_{\varphi,\mathbf{D}}$ in every model $\mathbf{D}$ of the theory T. For example, if $\psi\in F_d^1$ then the formula.

$$\forall y(\varphi(x,y)\to\psi(y))\in F_d^1$$

is abbreviated as $\psi(p(x))$.

To define semantics of programs we need the notion of the time-model [1, 3, 10].

Definition 2. The triplet $\mathfrak{M}=\langle \mathbf{I}, \mathbf{D}, f\rangle$ is a *time-model* if $\mathbf{I}$ is a structure of similarity type t, $\mathbf{D}$ is a structure of similarity type d, and $f\colon I\to D$ is a function, where the type t consists of the constant symbol 0, the one placed function symbol "+1", and the two placed relation symbol "$\leqq$". □

We say that $\mathbf{I}$ is the time structure, and $\mathbf{D}$ is the data structure of $\mathfrak{M}=\langle \mathbf{I}, \mathbf{D}, f\rangle$. Time-models can be regarded as a special 2-sorted models with sorts $\mathbf{t}$ and $\mathbf{d}$ (called time and data), and with operation symbols of t and d and the extra operation symbol f, see [9, 10]. Let TF denote the set of 2-sorted formulas of this type. By a little abuse of notation, we assume that F_t and F_d are disjoint, and $F_t\cup F_d\subset TF$.

Now we can give the strict definition of the program run. Note that by our agreement on the type t, we may write $i+1$ $(i\in I)$.

Definition 3. Let $\mathfrak{M}=\langle \mathbf{I}, \mathbf{D}, f\rangle$ be a time-model and let $p\colon D\to D$ be a program. The function f constitutes a *trace* of the program p in $\mathfrak{M}$ if for every $i\in I$, $f(i+1)=p(f(i))$. We say that the (trace of the) program *halts* at the timepoint $i\in I$ if $f(i+1)=f(i)$. □

Definition 4. Let φ_{in} and $\varphi_{\text{out}}\in F_d^1$ be two formulas. The program p is *partially correct* with respect to φ_{in} and φ_{out} in the time-model $\mathfrak{M}$ if whenever f is a trace of p, and $\mathbf{D}\models\varphi_{\text{in}}(f(0))$ (i.e. the input satisfies φ_{in}) then for every $i\in I$ such that $f(i+1)=f(i)$ (i.e. the program halts at the timepoint i), $\mathbf{D}\models\varphi_{\text{out}}(f(i))$. This assertion is denoted by $\mathfrak{M}\models(\varphi_{\text{in}}, p, \varphi_{\text{out}})$.

Let $S\subset TF$ be arbitrary. If for every time-model $\mathfrak{M}$, $\mathfrak{M}\models S$ implies $\mathfrak{M}\models(\varphi_{\text{in}}, p, \varphi_{\text{out}})$ then this fact is denoted by $S\models(\varphi_{\text{in}}, p, \varphi_{\text{out}})$. □

So far we have completed the definition of the partial correctness. The following definition is a reformulation of the well-known Floyd—Hoare partial correctness proof rule [7, 8, 10].

Definition 5. The program p is *Floyd—Hoare derivable* from the theory $T\subset F_d^0$ with respect to φ_{in} and $\varphi_{\text{out}}\in F_d^1$, in symbols $T\vdash(\varphi_{\text{in}}, p, \varphi_{\text{out}})$, if there is a formula $\Phi\in F_d^1$ such that

$$T\vdash \varphi_{\text{in}}(x)\to\Phi(x)$$

$$T\vdash \Phi(x)\to\Phi(p(x))$$

$$T\vdash \Phi(x)\wedge p(x)=x\to\varphi_{\text{out}}(x). \quad \square$$

Let TI denote the set of axioms of the discrete linear ordering with initial element for the type t. That is, TI states that the relation "$\leqq$" is a linear ordering, 0 is the least element, every element i has an immediate successor denoted by $i+1$, and every element except for the 0 has an immediate predecessor. We remark that TI is finite and its theory is complete, see [4] pp. 159—162.

If in the time-model $\mathfrak{M}=\langle \mathbf{I}, \mathbf{D}, f\rangle$ the time structure $\mathbf{I}$ is isomorphic to the ordering of the natural numbers (the time-model is *standard*) then $\mathbf{D}\models T$ and $T\vdash(\varphi_{\text{in}}, p, \varphi_{\text{out}})$ implies $\mathfrak{M}\models(\varphi_{\text{in}}, p, \varphi_{\text{out}})$. By the upward Lövenheim—Skolem theorem, there is no $S\subset TF$ for which $\mathfrak{M}\models S$ would force $\mathfrak{M}$ to be standard.

But we may require $\mathfrak{M}$ to satisfy the most important feature of standard time-models, namely that they admit induction on the time. Let $\varphi(x)\in TF$ be such that x is a variable of sort $\mathbf{t}$ (i.e. x is a time-variable). Then φ^* denotes the following formula of TF:

$$[\varphi(0)\wedge\forall x(\varphi(x)\to\varphi(x+1))]\to\forall x\varphi(x).$$

The set of induction axioms are

$$IA=\{\varphi^*:\ \varphi(x)\in TF \text{ and } x \text{ is of sort } \mathbf{t}\}.$$

Moreover we introduce a proper subset of IA, the induction axioms of restricted form:

$IR=\{\varphi^*:\ \varphi(x)\in TF$ and there is no quantifier for any variable of sort $\mathbf{t}$ in $\varphi(x)\}$.

It is important to remark here that $\varphi(x)$ may contain other free variables. All these free variables are also free in φ^* except for x, they are the parameters of the induction.

Of course $IR\subset IA\subset TF$, and one can easily prove the following theorem.

Theorem 1. Suppose $T\subset F_d^0$ and p is a d-type program. Then $T\vdash(\varphi_{\text{in}},p,\varphi_{\text{out}})$ implies $(TI\cup IR\cup T)\models(\varphi_{\text{in}},p,\varphi_{\text{out}})$. □

The aim of this paper is to prove the inverse of this theorem.

Theorem 2. With the notation of Theorem 1, $(TI\cup IR\cup T)\models(\varphi_{\text{in}},p,\varphi_{\text{out}})$ implies $T\vdash(\varphi_{\text{in}},p,\varphi_{\text{out}})$. □

These theorems state the completeness of the Floyd—Hoare program verification method in the case when the time-models satisfy the axioms $TI\cup IR$. In Theorem 2 the fact that induction axioms of restricted form are required only is essential as it is shown by the following theorem [1].

Theorem 3. There is a type d, a theory $T\subset F_d^0$ and a d-type program p such that $(TI\cup IA\cup T)\models(\varphi_{\text{in}},p,\varphi_{\text{out}})$ while $T\nvdash(\varphi_{\text{in}},p,\varphi_{\text{out}})$. □

2. Strongly continuous traces

We start to prove Theorem 2. From now on we fix the similarity type d, the theory $T\subset F_d^0$, the d-type program p and the formulas $\varphi_{\text{in}},\varphi_{\text{out}}\in F_d^1$. In this section for every time-model $\mathfrak{M}=\langle\mathbf{I},\mathbf{D},f\rangle$ we assume $\mathfrak{M}\models TI$. The explicit declaration of this fact will be omitted everywhere.

First we need a definition.

Definition 6. Let $\mathfrak{M}=\langle\mathbf{I},\mathbf{D},f\rangle$ be a time-model, $\mathbf{D}\models T$. The function f constitutes a *strongly continuous trace* of p if

(i) $f(i+1)=p(f(i))$ for every $i\in I$;
(ii) let $i,j\in I$, $i\leqq j$, $u\in D^n$ and $\Phi\in F_d^{1+n}$ be arbitrary. If $\mathbf{D}\models\Phi(f(i),u)\wedge\neg\Phi(f(j),u)$ then there is a $k\in I$, $i\leqq k\leqq j$ such that $\mathbf{D}\models\Phi(f(k),u)\wedge\neg\Phi(f(k+1),u)$. □

Strongly continuous traces (sct in the sequel) are traces, cf. Definition 3. In other words, an sct satisfies the induction principle in every time interval. Obviously, if $\mathfrak{M}\models IR$ and f is a trace then f is an sct, too. Properties of continuous traces are discussed in [2, 6, 10].

Lemma 1. Let f be a trace of the program p in $\mathfrak{M}$. Then $\mathfrak{M}\models IR$ iff f is strongly continuous.

Proof. We prove the "if" part only. Let $\varphi(x_0)\in TF$ be such that $\varphi(x_0)$ does not contain quantifiers on variables of sort **t**. Let $x_0, x_1, \ldots, x_{m-1}$ be the free variables of φ of sort **t**, and $y_0, \ldots, y_{n-1}$ be that of sort **d**. Because there are finitely many applications of the function "$+1$" only in φ, we may assume that there is none, simply replace these applications by a new parameter of sort t or use the identity $f(x+1)=p(f(x))$. We may assume also that every $f(x_j)$ is denoted by some of the parameters among $y_0, \ldots, y_{n-1}$, i.e. the function f is applied to x_0 only. Thereafter for every $\varphi(x_0)\in TF$ with fixed parameters from I and D, there are elements $i_1\leqq i_2\leqq\ldots\leqq i_m$ from I, elements $u_0, u_1, \ldots, u_{n-1}$ from D, and formulas $\Phi_0, \Phi_1, \ldots, \Phi_m\in F_d^{1+n}$ such that

$$\begin{aligned}\mathfrak{M}\models \varphi(x)\leftrightarrow\{[\qquad & x<i_1\to\Phi_0(f(x),u)]\land\\ \land[i_1\leqq\; & x<i_2\to\Phi_1(f(x),u)]\land\\ & \ldots\\ \land[i_{m-1}\leqq\; & x<i_m\to\Phi_{m-1}(f(x),u)]\land\\ \land[i_m\leqq\; & x \qquad\;\; \to\Phi_m(f(x),u)]\}\end{aligned}$$

which can be got, for example, by induction on the complexity of φ. Now if $\mathfrak{M}\models\varphi(0)\land\forall x(\varphi(x)\to\varphi(x+1))$ then, applying the strongly continuity in the intervals $[0, i_1]$, $[i_1, i_2]$, etc. we get $\mathfrak{M}\models\forall x\varphi(x)$ which was to be proved. □

By this lemma it is enough to show that either the triplet $(\varphi_{\text{in}}, p, \varphi_{\text{out}})$ is Floyd—Hoare derivable, or there is a strongly continuous trace which shows that p is not partially correct.

Let us make a step forward.

Definition 7. Let $H\subset F_d^1$ consist of the formulas $\Phi\in F_d^1$ for which

$$T\vdash \varphi_{\text{in}}(x)\to\Phi(x)$$

and

$$T\vdash \Phi(x)\to\Phi(p(x)).\quad\square$$

Note that H is closed under conjunction, i.e. if Φ_1 and Φ_2 are in H then $\Phi_1\land\Phi_2\in H$. Now let c_0 and c_ω denote two new constant symbols not occuring previously. We distinguish two cases.

Case I. In every model of the theory

$$\{T, \varphi_{\text{in}}(c_0), H(c_\omega), p(c_\omega)=c_\omega\}$$

the formula $\varphi_{\text{out}}(c_\omega)$ is valid. Here $H(c_\omega)=\{\Phi(c_\omega)\colon \Phi\in H\}$. Then by the compactness theorem and by the fact that H is closed under conjunction, there is a $\Psi \in H$ such that

$$T \vdash [\varphi_{\text{in}}(c_0) \wedge \Psi(c_\omega) \wedge p(c_\omega) = c_\omega] \rightarrow \varphi_{\text{out}}(c_\omega).$$

The constants c_0 and c_ω do not occur in T, so introducing $\Phi(x) = (\exists y \varphi_{\text{in}}(y)) \wedge \Psi(x)$, we get

$$T \vdash \Phi(x) \wedge p(x) = x \rightarrow \varphi_{\text{out}}(x).$$

This and the obvious $\Phi \in H$ shows the Floyd—Hoare derivability of $(\varphi_{\text{in}}, p, \varphi_{\text{out}})$.

Case II. Not the case above, i.e.

$$\operatorname{Con}\{T, \varphi_{\text{in}}(c_0), H(c_\omega), p(c_\omega) = c_\omega, \neg \varphi_{\text{out}}(c_\omega)\}.$$

By Theorem 4 of the following section, in this case we have a time-model $\mathfrak{M} = \langle \mathbf{I}, \mathbf{D}, f \rangle \models T$ such that f is an sct of p, $\mathbf{D} \models \varphi_{\text{in}}(f(0))$ and for some $i \in I$, $\mathbf{D} \models f(i) = p(f(i)) \wedge \neg \varphi_{\text{out}}(f(i))$. This means $\mathfrak{M} \not\models (\varphi_{\text{in}}, p, \varphi_{\text{out}})$, i.e. p is not partially correct. This proves Theorem 2, because $\mathfrak{M} \models TI \cup IR \cup T$ by Lemma 1.

3. The proof of the crucial theorem

In the remaining part of this paper we prove the following theorem.

Theorem 4. With the notation of the previous section, suppose

$$\operatorname{Con}\{T, \varphi_{\text{in}}(c_0), H(c_\omega), p(c_\omega) = c_\omega, \neg \varphi_{\text{out}}(c_\omega)\}.$$

Then there is a time-model $\mathfrak{M} = \langle \mathbf{I}, \mathbf{D}, f \rangle$ such that $\mathbf{I} \models TI$, $\mathbf{D} \models T$, f is a strongly continuous trace of p, $\mathbf{D} \models \varphi_{\text{in}}(f(0))$, and for some $i \in I$, $f(i+1) = f(i)$ and $\mathbf{D} \models \neg \varphi_{\text{out}}(f(i))$.

Proof. We need some more definitions. If d_1 and d_2 are similarity types then $d_1 < d_2$ means that d_1 and d_2 have the same function and relation symbols with the same arities and every constant symbol of d_1 is a constant symbol of d_2.

Definition 8 .Let d be a similarity type, $T \subset F_d^0$ be a theory. The pair $R = \langle \mathbf{I}_R, f_R \rangle$ is a *(d, T)-pretrace* if $\mathbf{I}_R$ is a time structure, $\mathbf{I}_R \models TI$, and f_R is a function which assigns to every $i \in I_R$ a constant symbol of d in such a way that (i) and (ii) below are satisfied. A bit loosely but not ambiguously, we write $R(i)$ or simply Ri instead of $f_R(i)$.

(i) $T \vdash R(i+1) = p(Ri)$ for every $i \in I_R$

(ii) $\operatorname{Con}(T \cup \{\Phi(Rj)\colon j \in I_R,\ \Phi \in B_T^d$ and there exists $i \in I_R,\ i < j$ such that $T \vdash \Phi(Ri)\})$,

where

$$B_T^d = \{\Phi \in F_d^1\colon T \vdash \Phi(x) \rightarrow \Phi(px)\}. \quad \square$$

Note that the set B_T^d is closed under conjunction, this fact will be used many times.

Lemma 2. Let R be a (d, T)-pretrace. Then there exists a complete theory $T \subset S \subset F_d^0$ such that R is a (d, S)-pretrace.

Proof. It suffices to show that for any $\beta\in F_d^0$, R is either $(d, T\cup\{\beta\})$ or $(d, T\cup\{\neg\beta\})$-pretrace. If neither of them hold then in both cases (ii) of Definition 8 is violated. It means that there are finitely many $i_s, j_s\in I_R$, $i_s\leqq j_s$, and $\Phi_s\in B^d_{T\cup\{\beta\}}$, $\Phi_s^*\in B^d_{T\cup\{\neg\beta\}}$ such that

$$T\cup\{\beta\}\vdash\neg\bigwedge_s\Phi_s(Rj_s)\quad\text{and}\quad T\cup\{\beta\}\vdash\bigwedge_s\Phi_s(Ri_s)\tag{3.1}$$

$$T\cup\{\neg\beta\}\vdash\neg\bigwedge_s\Phi_s^*(Rj_s)\quad\text{and}\quad T\cup\{\neg\beta\}\vdash\bigwedge_s\Phi_s^*(Ri_s).\tag{3.2}$$

Now let $\Psi_s(x)=(\beta\to\Phi_s(x))\wedge(\neg\beta\to\Phi_s^*(x))$. Obviously, $\Psi_s\in B_T^d$ and $T\vdash\bigwedge_s\Psi_s(Ri_s)$. Elementary considerations show that (3.1) and (3.2) imply

$$T\vdash\neg\bigwedge_s\Psi_s(Rj_s)$$

which contradicts the assumption $\mathrm{Con}(T, \{\Psi_s(Rj_s)\})$. □

Lemma 3. Let R be a (d, T)-pretrace, and let T be complete. Then there exist a similarity type $e>d$ and a complete theory $T\subset S\subset F_e^0$ such that

(i) R is an (e, S)-pretrace,
(ii) for every $\psi\in F_d^1$, if $\exists x\psi(x)\in T$ then for some constant c from the type e, $\psi(c)\in S$,
(iii) the cardinality of the new constants in e does not exceed the cardinality of F_d, i.e.

$$|F_e|=|e|\leqq|F_d|=|d|\cdot\omega.$$

Proof. What we have to prove is the following. Suppose that the type e contains the extra constant symbol c only, $\beta\in F_d^1$ and $\mathrm{Con}\,\{T, \beta(c)\}$, then R is an $(e, T\cup\{\beta(c)\})$-pretrace. From this (i)—(iii) can be got by a standard argument, see, e.g. [4] pp. 62—66. Now suppose that this is not the case, i.e. there are finitely many $\Phi_s(x, c)\in B^e_{T\cup\{\beta(c)\}}$ and $i_s, j_s\in I_R, i_s<j_s$ such that

$$T\cup\{\beta(c)\}\vdash\neg\bigwedge_s\Phi_s(Rj_s, c)\tag{3.3}$$

$$T\cup\{\beta(c)\}\vdash\bigwedge_s\Phi_s(Ri_s, c).\tag{3.4}$$

The condition $\Phi_s(x, c)\in B^e_{T\cup\{\beta(c)\}}$ implies

$$\Psi_s(x)=\forall y(\beta(y)\to\Phi_s(x, y))\in B_T^d,$$

and by (3.4), $T\vdash\forall y(\beta(y)\to\Phi_s(Ri_s, y))$, i.e. $\Psi_s(Ri_s)\in T$. Now T is complete, therefore $j_s>i_s$ implies $T\vdash\Psi_s(Rj_s)$, from which

$$T\vdash\bigwedge_s(\beta(c)\to\Phi_s(Rj_s, c))\vdash\beta(c)\to\bigwedge_s\Phi_s(Rj_s, c).$$

This and (3.3) gives $T\vdash\neg\beta(c)$, a contradiction. □

Lemma 4. Let R be a (d, T)-pretrace, and let T be complete. Suppose $i_0, j_0\in I_R$, $i_0<j_0$ and $\chi\in F_d^1$ such that

$$T\vdash\chi(Ri_0)\wedge\neg\chi(Rj_0).$$

Then there exist a type $e>d$, a theory $T\subset S\subset F_e^0$ and an (e, S)-pretrace Q such that

(i) $\mathbf{I}_Q$ is an elementary extension of $\mathbf{I}_R$ and $Q\supset R$, i.e.

$$Q(i) = R(i) \quad \text{for} \quad i\in I_R$$

(ii) there is an $i\in I_Q$, $i_0\leqq i<j_0$ such that

$$S\vdash \chi(Q(i))\wedge\neg\chi(Q(i+1)).$$

Proof. Let $\alpha=\{i\in I_R\colon \text{for every } i_0\leqq i'\leqq i, T\vdash\chi(Ri')\}$. Obviously, α is an initial segment of I_R, we write $i<\alpha$ and $i>\alpha$ instead of $i\in\alpha$ and $i\notin\alpha$, respectively. The element $j_0>\alpha$, and we may assume that there is no largest element in α otherwise there is nothing to prove. It means that for every $j>\alpha$, there exists $\alpha<j'<j$ such that $T\vdash\neg\chi(Rj')$ .We shall insert a thread isomorphic to the set of integer numbers, denoted by Z, into the cut indicated by α.

Let $\{a_l\colon l\in Z\}$ be countably many new symbols and let $\{c_l\colon l\in Z\}$ be new constant symbols. Let $I_Q=I_R\cup\{a_l\colon l\in Z\}$ and define the ordering on I_Q by $a_l<a_{l+1}$, $i<a_l$ if $i\in I_R$, $i<\alpha$ and $a_l<i$ if $i\in I_R, i>\alpha$ for every $l\in Z$. Evidently, $\mathbf{I}_Q$ is an elementary extension of $\mathbf{I}_R$.

Define the function Q by $Q(i)=R(i)$ if $i\in I_R$ and $Q(a_l)=c_l$ otherwise. Let the type e be the enlargement of d by the constant symbols $\{c_l\colon l\in Z\}$, and finally let the theory $S\subset F_e^0$ be

$$S = T\cup\{p(c_l) = c_{l+1}\colon l\in Z\}\cup\{\chi(c_0), \neg\chi(c_1)\}\cup$$
$$\cup\{\Phi(c_l)\colon l\in Z, \Phi\in B_T^d \text{ and } T\vdash \Phi(Ri) \text{ for some } i<\alpha\}\cup$$
$$\cup\{\neg\Phi(c_l)\colon l\in Z, \Phi\in B_T^d \text{ and } T\vdash \neg\Phi(Rj) \text{ for some } j>\alpha\}.$$

We claim that S is consistent. It suffices to show that T is consistent with any finite part of $S\setminus T$. Using the facts that T is complete, B_T^d is closed under conjunction, and the formulas $\Phi\in B_T^d$ are hereditary in $\mathbf{I}_R$, this reduces to

$$\operatorname{Con}\left(T\cup\{\Phi(c_{-l}), \chi(c_0), \neg\chi(c_1), \neg\Phi^*(c_l)\}\right)$$

where $l\in\omega$ is a natural number, Φ, $\Phi^*\in B_T^d$, and $T\vdash\Phi(Ri_1)\wedge\neg\Phi^*(Rj_1)$ for some $i_0\leqq i_1<\alpha<j_1\leqq j_0$. Now if this consistency does not hold then, T being complete,

$$T\vdash \Phi(x)\wedge\chi(p^l(x))\wedge\neg\Phi^*(p^{2l}(x)) \to \chi(p^{l+1}(x)).$$

Now let $\Psi(x)=\Phi(x)\wedge[\chi(p^l(x))\vee\Phi^*(p^{2l-1}(x))]$. By the previous statement, $T\vdash\Psi(x)\to\Psi(px)$, i.e. $\Psi\in B_T^d$. Now, by the assumptions, $T\vdash\Phi(R(i))$ and $T\vdash\chi(R(i+l))$ for $i_1\leqq i<\alpha$, therefore $T\vdash\Psi(Ri)$. But R is a pretrace so for every $\alpha<j<j_1-2l, T\vdash\Psi(Rj)$, although for some $\alpha<j'<j_1-2l, T\vdash\neg\chi(Rj')$ and $T\vdash\neg\Phi^*(R(j'+l-1))$. This contradiction shows that S is consistent indeed.

We prove that Q is an (e, S)-pretrace, (i) and (ii) of the lemma are clear from the construction. First assume that $i\in I_R$, $\Psi\in B_S^e$ and $S\vdash\Psi(Ri)$. We are going to show that in this case $S\vdash\Psi(Qj)$ for every $j\in I_Q$, $j>i$. Indeed, we may suppose that Ψ contains the new constant symbol $c=c_{-l}$ only and that

$$T\cup\{\delta(c)\}\vdash\Psi(x, c)\to\Psi(px, c)$$
$$T\cup\{\delta(c)\}\vdash\Psi(Ri, c)$$

where $\delta(c)=\Phi(c)\wedge\chi(p^l(c))\wedge\neg\chi(p^{l+1}(c))\wedge\neg\Phi^*(p^{2l}(c))$. By the first derivability, $\Theta(x)=\forall y[\delta(y)\to\Psi(x,y)]\in B_T^d$, and by the second one, $T\vdash\Theta(Ri)$. R is a pretrace, and by the definition of S, $S\vdash\Theta(Qj)$ for every $j\in I_Q$, $j>i$. But $S\vdash\delta(c_{-l})$, i.e. $S\vdash\Psi(Qj, c_{-l})$ as was stated.

Now if Q is not an (e, S)-pretrace then (ii) of Definition 8 is violated, which means that there are finitely many $i_s\in I_Q\setminus I_R$, $j_s\in I_R$, $j_s>\alpha$ and $\Phi_s\in B_S^e$ such that $S\vdash\neg\bigwedge_s\Phi_s(Rj_s)$ while $S\vdash\bigwedge_s\Phi_s(Qi_s)$. The set B_S^e is closed under conjunction, therefore we may assume that all the i_s and Φ_s coincide, that this $\Phi_s=\Psi$ contains the new constant symbol $c=c_{-l}=Qi_s$ only, and that with $\delta(c)$ as above,

$$T\cup\{\delta(c)\}\vdash\Psi(x,c)\to\Psi(px,c)$$
$$T\cup\{\delta(c)\}\vdash\Psi(c,c)$$
$$T\cup\{\delta(c)\}\vdash\neg\bigwedge_s\Psi(Rj_s,c).$$

By the first derivability, $\Theta(x)=\exists y(\delta(y)\wedge\Psi(x,y))\in B_T^d$, and by the third one, $T\vdash\bigvee_s\neg\Theta(Rj_s)$. T is complete, which means $T\vdash\neg\Theta(Rj_s)$ for some $j_s>\alpha$, i.e. by the definition of S, $S\vdash\neg\Theta(c)$, which contradicts the second derivability. □

Returning to the proof of Theorem 4, we shall define three increasing sequences of similarity types, theories and pretraces. Recall that the type d, the theory $T\subset F_d^0$ and the formulas $\varphi_{\rm in}, \varphi_{\rm out}\in F_d^1$ are such that

$$\mathrm{Con}\,\{T, \varphi_{\rm in}(c_0), H(c_\omega), p(c_\omega)=c_\omega, \neg\varphi_{\rm out}(c_\omega)\}. \tag{3.5}$$

Let c_l be new constant symbols for $l\in\omega-\{0\}$, and let the similarity type $e>d$ be the smallest one containing them. Let the time structure $\mathbf{I}_R$ consist of a thread isomorphic to ω and another one isomorphic to Z. The definition of the function R goes as follows:

$$R(i)=\begin{cases}c_i & \text{if } i\in\omega\\ c_\omega & \text{otherwise.}\end{cases}$$

Finally let

$$S=T\cup\{p(c_l)=c_{l+1}\colon\ l\in\omega\}\cup\{\varphi_{\rm in}(c_0), p(c_\omega)=c_\omega, \neg\varphi_{\rm out}(c_\omega)\}.$$

Lemma 5. R is an (e, S)-pretrace.

Proof. For the sake of simplicity, let

$$\gamma(x)=(p(x)=x\wedge\neg\varphi_{\rm out}(x)).$$

It is enough to prove that if $\Phi\in F_d^3$,

$$S\vdash\Phi(x, c_0, c_\omega)\to\Phi(px, c_0, c_\omega) \tag{3.6}$$

and

$$S\vdash\Phi(c_0, c_0, c_\omega) \tag{3.7}$$

then $\mathrm{Con}\,\{S, \Phi(c_\omega, c_0, c_\omega)\}$. Suppose the contrary, i.e.

$$S\vdash\neg\Phi(c_\omega, c_0, c_\omega). \tag{3.8}$$

We may change S to $T\cup\{\varphi_{\text{in}}(c_0), \gamma(c_\omega)\}$ everywhere, so introducing

$$\Psi(x) = \forall z\, \exists y\, [\gamma(z) \to \varphi_{\text{in}}(y) \wedge \Phi(x, y, z)] \in F_d^1,$$

(3.6) says that $T \vdash \Psi(x) \to \Psi(px)$. From (3.7) we get $T \vdash \varphi_{\text{in}}(x) \to \Psi(x)$, therefore $\Psi \in H$. Choosing $x = z = c_\omega$ in Ψ, the condition (3.5) gives

$$\text{Con}\,\{T, \varphi_{\text{in}}(c_0), \gamma(c_\omega), \exists y\, [\gamma(c_\omega) \to \varphi_{\text{in}}(y) \wedge \Phi(c_\omega, y, c_\omega)]\}.$$

But by (3.8),

$$T \vdash \forall y\, [\gamma(c_\omega) \wedge \varphi_{\text{in}}(y) \to \neg\Phi(c_\omega, y, c_\omega)]$$

a contradiction. □

Let $d_0 = e$, $R_0 = R$. By Lemma 2 there is a complete theory $S \subset T_0 \subset F_e^0 = F_{d_0}^0$ such that R_0 is a (d_0, T_0)-pretrace. Let the cardinality of $F_{d_0}^0$ be $\varkappa$, and let $\varkappa^+$ denote the smallest cardinal exceeding $\varkappa$. Let $C = \{c_\xi\colon \xi < \varkappa^+\}$ be different constant symbols such that the constants of the type d_0 are among them, and let $J = \{a_\xi\colon \xi < \varkappa^+\}$ be symbols of time points such that $I_{R_0} \subset J$. (Note that I_{R_0} is countable.)

Arrange the triplets of $J \times J \times F_{d\cup C}^1$ in a sequence $\{\langle i_\xi, j_\xi, \Phi_\xi\rangle\colon \xi < \varkappa^+\}$ of length $\varkappa^+$ in such a way that every triplet occurs $\varkappa^+$ times in this sequence. Now we define three increasing sequences d_ξ, T_ξ, and R_ξ for $\xi < \varkappa^+$ such that

(i) d_ξ is a similarity type,
(ii) $T_\xi \subset F_{d_\xi}^0$ is a complete theory, and $|F_{d_\xi}^0| = \varkappa$,
(iii) R_ξ is a (d_ξ, T_ξ)-pretrace, and $I_{R_\xi} \subset J$, $|I_{R_\xi}| \leqq \varkappa$.

Suppose we have defined d_ξ, T_ξ, R_ξ for $\xi < \eta < \varkappa^+$, they have properties (i)—(iii) and we want to define d_η, T_η, R_η.

If η is a limit ordinal, simply put $d_\eta = \bigcup\{d_\xi\colon \xi < \eta\}$, $T_\eta = \bigcup\{T_\xi\colon \xi < \eta\}$, $R_\eta = \bigcup\{R_\xi\colon \xi < \eta\}$. This definition is sound because $\mathbf{I}_{R_\eta}$ is the union of the increasing elementary chain $\langle \mathbf{I}_{R_\xi}\colon \xi < \eta\rangle$, therefore it is also a model of the axiom system *TI*. T_η is the union of an increasing sequence of complete theories, therefore itself is complete. Similarly for the other properties.

If η is a successor ordinal, say $\eta = \xi + 1$, then work as follows. If either $i_\xi \notin I_{R_\xi}$, $j_\xi \notin I_{R_\xi}$, $\Phi_\xi \notin F_{d_\xi}^1$ or $i_\xi, j_\xi \in I_{R_\xi}$, $\Phi_\xi \in F_{d_\xi}^1$ but $i_\xi > j_\xi$ or $T_\xi \nvdash \Phi_\xi(R_\xi i_\xi) \wedge \neg\Phi_\xi(R_\xi j_\xi)$ then let $d_{\xi+1} = d_\xi$, $T_{\xi+1} = T_\xi$, $R_{\xi+1} = R_\xi$.

If not, i.e. $i_\xi \leqq j_\xi$ and $T_\xi \vdash \Phi_\xi(R_\xi i_\xi) \wedge \neg\Phi_\xi(R_\xi j_\xi)$ then, by Lemma 4, there is a type $d'_\xi > d_\xi$, a theory $T'_\xi \supset T_\xi$ and a (d'_ξ, T'_ξ)-pretrace $R_{\xi+1} \supset R_\xi$ such that $d'_\xi \setminus d_\xi$ and $I_{R_{\xi+1}} \setminus I_{R_\xi}$ are countable, so we may put $I_{R_{\xi+1}} \subset J$, $|I_{R_{\xi+1}}| \leqq |I_{R_\xi}| + \omega \leqq \varkappa$ and for some $k \in I_{R_{\xi+1}}$, $i_\xi \leqq k \leqq j_\xi$ and

$$T'_\xi \vdash \Phi_\xi(R_{\xi+1}(k)) \wedge \neg\Phi_\xi(R_{\xi+1}(k+1)).$$

By Lemma 2, there is a complete theory $T'_\xi \subset T''_\xi \subset F_{d'_\xi}^0$ such that $R_{\xi+1}$ is a (d'_ξ, T''_ξ)-pretrace, finally, by Lemma 3, $R_{\xi+1}$ is a $(d_{\xi+1}, T_{\xi+1})$-pretrace, where $d_{\xi+1} > d'_\xi$, $T_{\xi+1} \supset T''_\xi$, $T_{\xi+1}$ is complete, the cardinality of $d_{\xi+1} \setminus d_\xi$ is at most $\varkappa$, and every existential formula of T''_ξ (and therefore of T_ξ) is satisfied by some constant of $d_{\xi+1}$. In this case the inductive assertions are trivially satisfied.

Now let $d^* = \bigcup\{d_\xi\colon \xi < \varkappa^+\}$, $T^* = \bigcup\{T_\xi\colon \xi < \varkappa^+\}$, and $R^* = \bigcup\{R_\xi\colon \lambda < \varkappa^+\}$. The theory T^* is complete and R^* is a (d^*, T^*)-pretrace. The constants of the type d^* form a model for the theory T^* because every existential formula of T^*

is satisfied by some constant, this was ensured by the applications of Lemma 3. (Strictly speaking, certain equivalence classes of these constants form this model, see [4], pp. 63—66). Let this model be $\mathbf{D}$, we claim that the time-model $\mathfrak{M}=\langle \mathbf{I}_{R^*}, \mathbf{D}, f_{R^*}\rangle$ satisfies the requirements of Theorem 4.

Indeed, $\mathbf{I}_{R^*}\models TI$, and $T\subset T_0\subset T^*$, therefore $\mathbf{D}\models T$. By the definition of the pretrace $R_0, f_{R^*}(0)=f_{R_0}(0)=c_0$, $T_0\vdash \varphi_{\text{in}}(c_0)$. For some $i\in I_{R_0}\subset I_{R^*}, f_{R^*}(i)=f_{R_0}(i)= =c_\omega$, and $T_0\vdash p(c_\omega)=c_\omega\land\neg\varphi_{\text{out}}(c_\omega)$. Because $\mathbf{D}\models T_0$, these formulas are valid in $\mathbf{D}$. What have remained is to check that f_{R^*} is a strongly continuous trace of p.

Let $i\in I_{R^*}$ be arbitrary. Then $i\in I_{R_\xi}$ for some $\xi<\varkappa^+$, and because R_ξ is a (d_ξ, T_ξ)-pretrace, $T_\xi\vdash f_{R_\xi}(i+1)=p(f_{R_\xi}(i))$, from which

$$\mathbf{D}\models f_{R^*}(i+1)=p(f_{R^*}(i))$$

proving (i) of Definition 6. Finally, let i, $j\in I_{R^*}$, $i\leqq j$, $u\in D^n$ and $\Psi\in F_d^{1+n}$ be such that

$$\mathbf{D}\models \Psi(f_{R^*}(i), u)\land\neg\Psi(f_{R^*}(j), u).$$

Every element of D is named by some constant of the type d^*, so there is a formula $\Phi\in F_{d^*}^1$ such that $\mathbf{D}\models\Psi(x, u)\leftrightarrow\Phi(x)$. Now $\Phi\in F_{d\cup C}^1$ therefore the triplet $\langle i, j, \Phi\rangle$ occurs $\varkappa^+$ times in the sequence $\{\langle i_\xi, j_\xi, \Phi_\xi\rangle\colon \xi<\varkappa^+\}$. Consequently there exists an index $\xi<\varkappa^+$ such that i, $j\in I_{R_\xi}$, $\Phi\in F_{d_\xi}^1$, and $i=i_\xi$, $j=j_\xi$, $\Phi=\Phi_\xi$. Then, by the construction, there is a $k\in I_{R_{\xi+1}}\subset I_{R^*}$, $i\leqq k\leqq j$ such that

$$T_{\xi+1}\vdash\Phi(f_{R_{\xi+1}}(k))\land\neg\Phi(f_{R_{\xi+1}}(k+1)),$$

that is,

$$\mathbf{D}\models\Phi(f_{R^*}(k))\land\neg\Phi(f_{R^*}(k+1))$$

which completes the proof of Theorem 4.

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Axiomatic systems in fuzzy algebra

By J. DREWNIAK*

1. Introduction

One of the most interesting problems in fuzzy set theory is that of the axiomatization of fuzzy algebra. At the beginning, it is necessary to note that there is not any agreement between authors of papers what a "fuzzy algebra" really is (cf. [1], [8], [12], [15]). So we have different fuzzy algebras and they are useful in different applications of fuzzy set theory (cf. [9], [14]).

We are going to consider different systems of axioms on the set of fuzzy sets and on the one hand — to find all common properties of different fuzzy algebras, and on the other hand — to distinguish the characteristic properties of considered algebras. We start with the recollection of definition of fuzzy sets in the following form:

Definition 1.1. A fuzzy set f in a nonempty universe X is an arbitrary function (cf. [3], [17])

$$f: X \to [0, 1].$$

Similarly (cf. [7]), an L-fuzzy set in X is a function

$$f: X \to L,$$

where L or $(L, \leqq)$ is a poset (partially ordered set), e.g. lattice or the interval of real axis.

The collection of all fuzzy sets (L-sets) in X is denoted by $F(X)$ $(F_L(X))$ or shortly by F.

In applications of fuzzy sets (cf. [13], [18]), another definition of fuzzy object is needed, not in the meaning of fuzzy subset.

Definition 1.2 ([12]). Let X and L be as in definition 1.1. Elements of the nonempty set Z are called fuzzy objects if there exists a mapping

$$M: Z \to F_L(X). \tag{1}$$

Function $f_A = M(A)$ for $A \in Z$ is then named the membership function of fuzzy object A and $f_A(x)$ for $x \in X$ is called the membership grade of point x.

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We shall say that two fuzzy objects $A, B \in Z$ are equal if

$$M(A) = M(B) \quad (f_A = f_B), \tag{2}$$

i.e.

$$f_A(x) = f_B(x) \quad \text{for} \quad x \in X. \tag{3}$$

The last sentence in definition 1.2 is equivalent to the assumption that mapping (1) is one to one (injection) and we can consider the inverse mapping

$$M^{-1}\colon M(Z) \to Z. \tag{4}$$

Remark 1.3. The particular case of membership function is that of characteristic function for a subset in X. The set of all characteristic functions

$$Ch = Ch(X) = F_{\{0,1\}}(X)$$

is contained in F whenever $\{0, 1\} \subset L$, where

$$0 = \inf L, \quad 1 = \sup L.$$

Then we can obtain different relations between Ch and $M(Z)$. For example

$$Ch \cap M(Z) = \emptyset, \quad Ch \subset M(Z) \quad \text{or} \quad M(Z) \subset Ch.$$

In this last case we see that definition 1.2 admits not entirely fuzzy objects.

Usually in theoretic papers it is assumed that $Z=F$ and then M is omitted as identity function. But if we want to write for example about fuzzy statements (cf. [1], [14], [18]), we must consider fuzzy objects different than fuzzy subsets of the universe, and the universe can be settled different in particular cases as suitable for applications (e.g. consider statements about age, height or weight of people).

In general we have three base sets: L, X and Z, and assumptions about one of these sets would have consequences in two other sets. So for $L=[0, 1]$, where there are different algebraic structures, we have greater possibilities in construction of fuzzy algebra than in the case of abstract poset L. In every case we can make use of its order by considering induced orders between fuzzy sets and between fuzzy objects.

Definition 1.4. We say that the fuzzy set $f \in F$ is contained in the fuzzy set $g \in F$ if

$$f(x) \leqq g(x) \quad \text{for} \quad x \in X \tag{5}$$

and we write

$$f \leqq g. \tag{6}$$

Similarly we say that the fuzzy object $A \in Z$ is dominated by the fuzzy object $B \in Z$ if

$$M(A) \leqq M(B) \quad (f_A \leqq f_B) \tag{7}$$

and we write

$$A \leqq B. \tag{8}$$

(The sign "$\leqq$" in (5), (6) and (8) is used as symbol for three different relations but its meaning will be understood because of the context).

Remark 1.5. Defined order is a generalization of inclusion relation for subsets in X because in the case

$$Ch \subset F \quad \text{and} \quad A, B \subset X$$

inequality (6) can be written as

$$e_A \leqq e_B$$

which is equivalent to $A \subset B$, where

$$e_A(x) = \begin{cases} 1 & \text{if} \quad x \in A, \\ 0 & \text{if} \quad x \notin A. \end{cases} \tag{9}$$

Proposition 1.6. Relation (6) introduces a partial order in F and relation (8) introduces a partial order in Z, i.e. for every $A, B, C \in Z$ we have

$$A \leqq A \qquad \text{(reflexivity)}, \tag{10}$$

$$A \leqq B \quad \text{and} \quad B \leqq A \quad \text{imply} \quad A = B \quad \text{(antisymmetry)}, \tag{11}$$

$$A \leqq B \quad \text{and} \quad B \leqq C \quad \text{imply} \quad A \leqq C \quad \text{(transitivity)}. \tag{12}$$

We omit the simple proof of this proposition and we consider only the case of antisymmetry (11) of relation (8). If $A, B, C \in Z$ and

$$A \leqq B \quad \text{and} \quad B \leqq A,$$

then by definition 1.4 from (7) we get

$$f_A \leqq f_B \quad \text{and} \quad f_B \leqq f_A,$$

i.e.

$$f_A(x) \leqq f_B(x) \quad \text{and} \quad f_B(x) \leqq f_A(x) \quad \text{for} \quad x \in X. \tag{13}$$

For every x we have $f_A(x), f_B(x) \in L$ and in virtue of antisymmetry in L, (13) imply (3), i.e. (2). Now by definition 1.2 we get $A = B$ which proves (11).

This property cannot be proved if the mapping (1) is not injective which makes this part of proof more interesting.

After proposition 1.6 we can say that F and Z are posets when L is a poset. Obviously beside the case of singleton X there are incomparable functions (elements) in F even then, when L is linearly ordered. So we do not have a generalization of proposition 1.6 to the case of linear order. We can look forward to properties typical in lattices under suitable assumptions about L.

In the structure of fuzzy objects we have greater variety of possibilities, because card Z can be small in comparison with card F. So it is possible that all considered fuzzy objects are comparable and $M(Z)$ forms a chain in poset F. It seems that in applications of fuzzy sets we obtain the situation described in proposition 1.6 in more natural way than definition 1.4 (cf. [16]). First we have certain dominance relation in the set Z and then we need a function M in (1) such that (8) implies (7) for every $A, B \in Z$. But the result is the same.

Now let consider an algebraic operation in the set of fuzzy sets or in the set of fuzzy objects, i.e.

$$u\colon F^n \to F \quad \text{or} \quad v\colon Z^n \to Z \tag{14}$$

for fixed $n \geqq 1$. Such operations in an ordered set can have the following properties:

Definition 1.7 ([5], Chapter 1): We shall say that an operation u is isotone (antitone) if the inequalities

$$f_i \leqq g_i \quad \text{for} \quad i = 1, 2, \ldots, n \tag{15}$$

imply

$$u(f_1, \ldots, f_n) \leqq u(g_1, \ldots, g_n) \quad \big(u(g_1, \ldots, g_n) \leqq u(f_1, \ldots, f_n)\big) \tag{16}$$

for every $(f_1, \ldots, f_n), (g_1, \ldots, g_n) \in F^n$.

Operation u is monotonic if it is isotone or antitone.

We are interested in transferring of operations from one base set to the other.

Definition 1.8. Let one of the operations (14) be given. We say that the operation $v\colon Z^n \to Z$ is induced by the operation $u\colon F^n \to F$ to the domain of M if $u\colon M(Z)^n \to M(Z)$ and v is defined by (see (4))

$$v(A_1, \ldots, A_n) = M^{-1}\big(u(M(A_1), \ldots, M(A_n))\big) \tag{17}$$

for $A_1, \ldots, A_n \in Z$.

We say that the operation $u\colon M(Z)^n \to M(Z)$ is induced by $v\colon Z^n \to Z$ to the codomain of M if u is defined by

$$u(f_1, \ldots, f_n) = M\big(v(M^{-1}(f_1), \ldots, M^{-1}(f_n))\big) \tag{18}$$

for $f_1, \ldots, f_n \in M(Z)$.

The algebraic fact described in definition 1.8 can be repeated as (cf. [4]).

Corollary 1.9. If the operations $u\colon M(Z)^n \to M(Z)$, $v\colon Z^n \to Z$ satisfy (17) then M is an isomorphism between the algebraic structures (Z, v) and $\big(M(Z), u\big)$.

Now from the known property of isomorphism we get (cf. [4]).

Proposition 1.10. The operation induced in the domain or in the codomain of an injection has such algebraic properties as the initial one.

We prove also

Proposition 1.11. The operation induced in the ordered domain or codomain of a monotonic injection by monotonic operation is also monotonic.

Proof. We prove only the first part of the proposition because the codomain of M is the domain of M^{-1} (see (1)) and we can omit the case of the codomain.

Let u be isotone, i.e. (15) imply (16). Assume that

$$A_i \leqq B_i \quad \text{for} \quad A_i, B_i \in Z, \quad i = 1, \ldots, n, \tag{19}$$

and put

$$f_i = M(A_i), \quad g_i = M(B_i), \quad i = 1, \ldots, n. \tag{20}$$

Now if M is also isotone as in definition 1.4, then from (8) we get (7) and from (19) and (20) we get (15). Therefore from (16) and (20) it follows

$$u\big(M(A_1), \ldots, M(A_n)\big) \leqq u\big(M(B_1), \ldots, M(B_n)\big)$$

and both parts of this inequality belong to $M(Z)$ under the conditions of definition

1.8. But the inverse M^{-1} of the isotone mapping M is also isotone and we obtain

$$M^{-1}(u(M(A_1), \ldots, M(A_n))) \leqq M^{-1}(u(M(B_1), \ldots, M(B_n))),$$

i.e.

$$v(A_1, \ldots, A_n) \leqq v(B_1, \ldots, B_n)$$

in virtue of (17). Thus the operation v is also isotone and monotonic.

If u or M is antitone then very similar argumentation finishes the proof.

Now we can see that the algebraic structure can be transformed only between Z and $M(Z)$ if $M(Z) \neq F$. We cannot use definition 1.8 if the operation u does not introduce any substructure into $M(Z)$ (if the set $M(Z)$ is not closed under operation u). Also if v is given we obtain a new structure only in $M(Z)$ but not in F. Thus the general assumption $M(Z)=F$ and even $Z=F$ can be very useful (and it is often used).

Another situation is between F and L. Every algebraic operation in L induces a similar operation in F (cf. [7]) but inverse transferring is impossible. None of the operations defined in F can be transformed to the set L independently of $x \in X$ (obviously if we omit all operations just induced from L to F).

So if we do not assume any algebraic operation in L we cannot induce a unique algebraic structure there similar to the considered one in F (different possibilities can be considered if we restrict all $f \in F$ to a fixed point $x_0 \in X$).

At that stage we can give the most general statement about the meaning of the phrase "fuzzy algebra".

Definition 1.12. By a fuzzy algebra (algebra of fuzzy sets, algebra of fuzzy objects) we mean every algebraic structure in F or in Z such that

($*$) every its operation is monotonic (definition 1.7) in the ordered structure induced from L (definition 1.4).

A fuzzy algebra is named "ordinary" one if the following assumptions are fulfilled (cf. remark 1.3):

($*\,*$) $0 = \inf L \in L$, $1 = \sup L \in L$, $Ch \subset M(Z)$,

($*\,*\,*$) every its algebraic operation restricted to Ch is identical to one of the set-theoretical operations as union, intersection, difference, complementation or symmetric difference.

In the contrary we speak about "special" fuzzy algebra.

Condition ($*$) can be written in a weak form under the assumption that the operations are monotonic in each variable separatively, but if we consider only unary operations or binary associative operations then it is equivalent to ($*$) (cf. [5], Chapter 1). Assumption about L in ($*\,*$) is equivalent to boundedness of poset L. At last assumption ($*\,*\,*$) guarantes that the considered algebra is a generalization of certain part of the set algebra.

Now we can overlook different papers regarding the fuzzy set theory and consider different further assumptions accepted in the fuzzy algebra. We select only a few papers which are principally concerning about operations and axioms of fuzzy algebras.

2. The first definition of Zadeh

I think it is forgotten now that Zadeh [17] has given a very simple argumentation for introducting his "max" and "min" operations. He writes that intuitively

Z1 the union of two fuzzy sets is the smallest fuzzy set containing both these sets;

Z2 the intersection of two fuzzy sets is the largest fuzzy set which is contained in both these sets.

It is a definition as natural as possible, because in the order structure it is equivalent to the definition of union and intersection in the set theory. For the case $L=[0, 1]$ Zadeh [17] proved that Z1 and Z2 are equivalent to "max" and "min" operations in F. It is usually proved in the lattice theory (cf. [2]) that operations of supremum and infimum for subsets containing only two elements are equivalent to the lattice operations $\vee$ and $\wedge$. So Zadeh's definition and proof can be used in every lattice and we have

Theorem 2.1. If $L=(L, \vee, \wedge)$ is a lattice, then Z1 and Z2 are equivalent to

$$f\vee g = \sup\{f, g\} \quad \text{and} \quad f\wedge g = \inf\{f, g\} \quad \text{for} \quad f, g\in F, \tag{21}$$

where

$$\begin{aligned}(f\vee g)(x) &= \sup\{f(x), g(x)\} = f(x)\vee g(x),\\ (f\wedge g)(x) &= \inf\{f(x), g(x)\} = f(x)\wedge g(x)\end{aligned} \tag{22}$$

for $x\in X$.

The following result is from Brown [3].

Theorem 2.2. If L is a lattice, then F with operations (21) is a lattice, too.

As we remarked above, the operations (21) can be reduced to the set-theoretical operations whenever $0, 1\in L$ (see $(**)$), they are also monotonic and we have

Corollary 2.3. If L is a lattice with 0 and 1 then the operations (21) introduce in F an ordinary fuzzy algebra which is a lattice algebra.

If the lattice L is nonbounded (which is possible only for infinite lattices — cf. [2]) then the operations (21) introduce in F a special fuzzy algebra which is a lattice algebra, too.

This corollary stressed the importance of assumptions about the poset L in definition 1.12. Under additional assumptions it is possible to consider further lattice properties (distributivity, completeness) or even continuity of operations (21) in the interval topology (cf. [7]), but we have not any further problems why the union and the intersection of fuzzy sets has form (21). (I think that none in the world has examined why the set-theoretical sum is the "sum" but it is not a "composition" of sets, because it was so named and that is all.) Obviously we can introduce many other operations which will have other names and will compose other fuzzy algebras. For example Zadeh [17] proposed other operations as the complement $1-f$, the arithmetic product fg, the arithmetic sum $f+g-fg$, and the absolute difference $|f-g|$, which can be considered for arbitrary $f, g\in F$ in the case $L=[0, 1]$. All these operations will be reduced in $L=\{0, 1\}$ to the ordinary set-theoretical operations and thus form in F different ordinary fuzzy algebras. There were also defined the sum $f+g$ and the convex combination $hf+(1-h)g$, which cannot be reduced to ordinary set-theoretical operations and so they form special fuzzy algebras. We do not consider more precisely all these algebras because of the great literature on the case $L=[0, 1]$ (e.g. almoust the entire book of Kaufmann [10] treates the case $L=[0, 1]$).

Now remains the problem, what we can say about an ordinary fuzzy algebra if L is not a lattice. In this case we cannot use the natural definitions Z1 and Z2, because it is possible that the needed elements do not exist in F.

If we want to preserve as much as possible from the definition (22) in a bounded poset L, we can use the following extension of the lattice operations:

$$(f\vee g)(x)=\begin{cases}\sup\{f(x), g(x)\} & \text{if supremum exists,}\\ 1 & \text{otherwise;}\end{cases} \tag{23}$$

$$(f\wedge g)(x)=\begin{cases}\inf\{f(x), g(x)\} & \text{if infimum exists,}\\ 0 & \text{otherwise.}\end{cases} \tag{24}$$

These operations are idempotent and commutative and also can be reduced to the set-theoretical operations in the case $L=\{0, 1\}$. Unfortunately operations (23) and (24) are not associative what is illustrated by

Example 2.4. Let

$$L=\{(0,0),\ (0,1/3),\ (1/3,0),\ (1/3,2/3),\ (2/3,1/3),\ (2/3,1),\ (1,2/3),\ (1,1)\}$$

be the poset with partial order induced in Cartesian product. It is bounded and $0=(0, 0)$, $1=(1, 1)$ but it is not a lattice, because e.g. $\sup\{a, b\}$ and $\inf\{a, b\}$ do not exist for

$$a=(1/3, 2/3),\quad b=(2/3, 1/3),\quad c=(1, 2/3),\quad d=(0, 1/3).$$

By (23) we compute

$$a\vee b=1 \quad\text{and}\quad b\vee c=c$$

so

$$(a\vee b)\vee c=1 \quad\text{and}\quad a\vee(b\vee c)=a\vee c=c\neq 1.$$

Similarly by (24) we get

$$(a\wedge b)\wedge d=0 \quad\text{and}\quad a\wedge(b\wedge d)=d\neq 0,$$

thus none of these operations is associative and in consequence they are not very interesting as algebraic operations. Moreover operations (23) and (24) are not monotonic in the poset L because we have

$$b<c \quad\text{and}\quad d<a$$

and simultaneously

$$a\vee b=1>a\vee c=c \quad\text{and}\quad b\vee d=b<b\vee a=1,$$

$$a\wedge b=0<a\wedge c=a \quad\text{and}\quad b\wedge d=d>b\wedge a=0.$$

Therefore operations (23) and (24) do not form any fuzzy algebra in F and it is not a simple way to introduce a fuzzy algebra in F if L is not a lattice.

Another problem related paper [17] brings the definition of the complement of the fuzzy set. Namely, the natural meaning of the word "complement" in the set theory is "the smallest set in the universe which in the union with the given set makes the universe", or it means "the greatest set in universe disjoint with the given one". So independently of Zadeh's definition

Z3 the (arithmetic) complement of a fuzzy set is the arithmetic complementation of its values to 1 in $L=[0, 1]$.

We can consider two other definitions

Z3′ the (union) complement of a fuzzy set is the smallest fuzzy set which in union with the given set makes e_X (see (9));

Z3″ the (intersection) complement of a fuzzy set is the greatest fuzzy set disjoint with the given set.

We propose to name these three complements by arithmetic, union and intersection complement, respectively. It is evident that definitions Z3′ and Z3″ can be used in the case of complete lattice L while the definition Z3 can be extended to the case of complemented lattice L. However, the use of definitions Z3′ and Z3″ is a little confounding because as complements we always obtain the elements of Ch (see remark 1.3).

3. The axiom system of Bellman and Giertz

Many authors find the paper [1] very useful (cf. [6], [8], [16]), so we too are going to use it. The paper treates the naturality of Zadeh's "max" and "min" operations. We have already remarked above that it is a hard work to add something interesting to Zadeh's own argumentation in Z1 and Z2. We give here a short review of this new argumentation from paper [1].

Let Z denote the set of fuzzy objects named "fuzzy statements". Then the existence of two binary operations "and" and "or" is required, but we have not exact information about mapping (1). Thus it is impossible to consider the induced operations (18) in the set of membership functions. Authors in [1] could not use a definition like definition 1.8 and introduced operations in F by system of axioms. They assumed that $P, S\colon F^2 \to F$ are such that (we use different notation)

$$f_{A \text{ and } B} = P(f_A, f_B), \quad f_{A \text{ or } B} = S(f_A, f_B) \tag{25}$$

for every $A, B \in Z$ and its dependence on the membership functions can be described by

$$P(f, g)(x) = p(f(x), g(x)), \quad S(f, g)(x) = s(f(x), g(x)), \tag{26}$$

where functions

$$p, s\colon [0, 1]^2 \to [0, 1]$$

fulfil the following system of axioms:

BG1 p and s are nondecreasing and continuous in both variables;

BG2 p and s are symmetric $(p(x, y)=p(y, x), s(x, y)=s(y, x))$;

BG3 $p(x, x)$ and $s(x, x)$ are strictly increasing in x;

BG4 $p(x, y) \leq \min(x, y)$, $s(x, y) \geq \max(x, y)$;

BG5 $p(1, 1)=1$, $s(0, 0)=0$;

BG6 logically equivalent statements have equal membership functions (grades).

Further they deduced from this axioms the system of functional equations for functions p and s, and they proved that this system of functional equations and inequalities (see BG4) has a unique solution

$$p(x, y) = \min(x, y), \quad s(x, y) = \max(x, y) \quad \text{for} \quad x, y \in [0, 1]. \tag{27}$$

The mentioned system of equations and inequalities was discussed in details in Hamacher's paper [8] and in Kóczy's dissertation [11] and we do not want to say any more about it. However, we devote a little time to the consideration of the above BG1—BG6 axioms.

I think that for the consequences of the prescribed axiom system almost all depends on the meaning of BG6. We show that it is difficult to find a correct meaning of BG6.

First, let us suppose that operations "and" and "or" fulfil in Z the propositional calculus of conjunction and disjunction. Then we have e.g.

"A and B" is equivalent to "B and A",
"A or B" is equivalent to "B or A",
"A and A" is equivalent to "A",
"A or A" is equivalent to "A"

for arbitrary $A, B \in Z$, and we can omit axioms BG2 and BG5 as implied from BG6. Moreover we can write

$$p(x, x) = x, \quad s(x, x) = x \quad \text{for} \quad x \in [0, 1] \tag{28}$$

and it is more interesting because of

Theorem 3.1. If the functions $p, s\colon [0, 1]^2 \to [0, 1]$ fulfil BG4, (28) and

$$p \text{ and } s \text{ are nondecreasing in both variables,} \tag{29}$$

then we obtain (27).

Proof. Let $x, y \in [0, 1]$, $x \leqq y$. Thus from (29) and (28) we get

$$x = p(x, x) \leqq p(x, y) \leqq p(y, y) = y,$$
$$x = s(x, x) \leqq s(x, y) \leqq s(y, y) = y$$

and therefore

$$p(x, y) \geqq \min(x, y), \quad s(x, y) \leqq \max(x, y).$$

This together with BG4 proves (27).

This short theorem contains more informations about "max" and "min" operations than all information contained in paper [1] because we use exactly only axiom BG4 and our assumption (29) is weaker than BG1, and assumption (28) is a very special case of BG6. It seems, we must be very satisfied because of this great reduction of the axiom system BG1—BG6. However, we are not satisfactory because of the unnatural assumption BG4. Namely, assumption (29) is equivalent to condition $(*)$ from the definition of fuzzy algebra (see definition 1.12) and if we omit (29) we can obtain an algebraic structure different from the fuzzy algebra (cf. example 2.4). Assumption (28) can be admitted as a natural extension of this law from the algebra of sets and we cannot say anything similar about BG4.

It was only the first part of our consideration of axiom BG6. If we admit a part of propositional calculus in Z we can ask why not admit the whole propositional calculus in Z with all operations used in logic. Thus axiom BG6 can be understood as the assumption that Z is a Boolean algebra of fuzzy objects and then it can be supposed that paper [1] is devoted to transferring of this algebra on the set of fuzzy sets.

We have remarked after proposition 1.11 that the structure induced in $M(Z)$ can be different from that in F (obviously in the case $M(Z)\neq F$). However, there is assumed here the transferring of the Boolean algebra on the whole F, what is impossible in the case $L=[0, 1]$ (it is possible if L is a Boolean algebra, cf. [3]).

The last remark about axiom BG6 has moral meaning. It is not right to suppose that "fuzzy statements" are "logically equivalent" in the same manner as logical sentences are in the propositional calculus. If there are "fuzzy statements" they can be totally unlogical and it is the main reason of the different "fuzzy" investigations.

4. Hamacher's axiom system

Paper [8] contains a very interesting method of the generalization of the set-theoretical operations but two things make reading difficult:

a) many proofs are omitted without a hint, how or where they were obtained;

b) lack of the list of references (in my copy).

The author creates the following system of axioms for two operations p, s: $[0, 1]^2\to[0, 1]$ (we change notations):

H1 p and s are associative,

H2 p and s are continuous,

H3 p in $(0, 1]$ and s in $[0, 1)$ are injections in both variables,

H4 $p(x, x)=x\Leftrightarrow x=1$ for $x\in(0, 1]$ and $s(x, x)=x\Leftrightarrow x=0$ for $x\in[0, 1)$.

These axioms are considered independently for p and s and both operations form certain semigroups in the intervals from H3, respectively. Axiom H3 with continuity H2 gives strict monotonicity of p and s in both variables and these together with H1 imply that (cf. [5]) p and s are strictly increasing in $(0, 1]$ and $[0, 1)$, respectively. It is a stronger property than $(*)$ in definition 1.12 and stronger than in natural models of those operations for $L=\{0, 1\}$. Thus the author must exclude certain boundary points in H3 and H4. It is noted in [8] that H3 admits only one idempotent case

$$p(x,x) = x \quad \text{in} \quad (0,1] \quad \text{and} \quad s(x,x) = x \quad \text{in} \quad [0,1).$$

In this situation axiom H4 is equivalent to the assumption that for functions

$$p_a(x) = p(a,x) \quad \text{in} \quad (0,1] \tag{30}$$

and

$$s_b(x) = s(b,x) \quad \text{in} \quad [0,1) \tag{31}$$

there exist such $a=1$ and $b=0$ that suitable functions p_1 and s_0 are surjections. Indeed we have

Lemma 4.1. Under assumptions H1—H3 if there exists $u<1$ such that

$$p(u, u) = u, \tag{32}$$

then none of the operations (30) is a surjection.

Similarly if there exists $v>0$ such that

$$s(v, v) = v, \tag{33}$$

then none of the operations (31) is a surjection.

Proof. Because of the unicity of the idempotents for both operations we have

$$p(1, 1) \neq 1 \quad \text{and} \quad s(0, 0) \neq 0$$

and therefore

$$p(1, 1) < 1 \quad \text{and} \quad s(0, 0) > 0.$$

Thus by monotonity

$$p(x, y) \leqq p(1, 1) < 1 \quad \text{for} \quad x, y \in (0, 1]$$

and

$$s(x, y) \geqq s(0, 0) > 0 \quad \text{for} \quad x, y \in [0, 1).$$

Therefore none of the functions (30) or (31) obtain the value $p(x, y)=1$ or $s(x, y)=0$, respectively, and none of them is a surjection.

It is a strange situation, because in paper [8] one theorem tells that every idempotent for operations p or s is an identity element and this implies the mentioned unicity of idempotents. But every identity element forms the identity bijection and we get

$$p_u(x) = x \quad \text{for} \quad x \in (0, 1]$$

from (30) and (32), and also

$$s_v(x) = x \quad \text{for} \quad x \in [0, 1)$$

from (31) and (33). This contradicts the thesis of lemma. Thus the assumptions $u<1$ and $v>0$ are not fulfilled for any $u \in (0, 1]$ and $v \in [0, 1)$. Therefore we have proved

Lemma 4.2. Under assumptions H1—H3 if u fulfils (32) then $u=1$, and if v fulfils (33) then $v=0$.

This result is not else than the first implication in axiom H4. Thus we can assume only the second implication from H4, i.e.

$$p(1, 1) = 1 \quad \text{and} \quad s(0, 0) = 0$$

and it is exactly axiom BG5 from paper [1]. Now we have

Theorem 4.3. The system of axioms H1—H4 is equivalent to the system of axioms H1—H3 and BG5.

Our consideration about lemma 4.1 brings one more result, because of the mentioned equivalence between idempotents and identity elements and thus axiom BG5 (under assumption H1—H3) is equivalent to

H4′ $p(1, x) = p(x, 1) = x$ and $s(x, 0) = s(0, x) = x$ for $x \in [0, 1]$.

We have

Theorem 4.4. The system of axioms H1—H4 is equivalent to the system of axioms H1—H3 and H4′.

A great part of paper [8] contains considerations about the class of functions fulfilling axioms H1—H4. We remark here only three results:

a) every function

$$p\colon [0, 1]^2 \to [0, 1] \tag{34}$$

fulfilling axioms H1—H3 has the form

$$p(x, y) = f^{-1}(f(x)+f(y)) \tag{35}$$

with the continuous, monotonic real function f defined in $[0, 1]$;

b) every rational function (34) fulfilling axioms H1—H4 has the form

$$p(x, y) = \frac{dxy}{a+(d-a)(x+y-xy)} \tag{36}$$

with suitable constants a and d.

c) if function (34), fulfilling H1—H4 is a polynomial then

$$p(x, y) = xy. \tag{37}$$

At first we use formula (35). Let $a>0$ and

$$f(x) = x^a \quad \text{for} \quad x\in[0, 1].$$

We get

$$p(x, y) = (x^a+y^a)^{1/a}$$

and it indeed fulfils axioms H1—H3 but the function

$$p\colon [0, 1]^2 \to [0, 2^{1/a}]$$

is different from (34) and it does not fulfil H4. Thus formula (35) admits operations over our interest. So we put a question:

I. Is there any assumption about function f, under which every function (35) is of the type (34)?

We put

$$p(x, y) = \frac{xy}{(2-x^a-y^a+x^a y^a)^{1/a}} \quad \text{for} \quad x. y\in[0, 1], \quad a>0. \tag{38}$$

and now it is a good example of irrational functions fulfilling the system of axioms H1—H4. We also ask:

II. Does exist a finite-parametric formula for the class of all functions (34) fulfilling axioms H1—H4?

At last put $a=1$ in (38). We get

$$p(x, y) = \frac{xy}{2-x-y+xy} \tag{39}$$

and it is example of rational function which fulfils axiom system H1—H4. We could find it between rational solutions in (36).

At the finish of this part, we remark that using formulas (25), (26) we obtain

Corollary 4.5. Functions (34) from class (36) introduce in F an ordinary fuzzy algebra which is a commutative semigroup with identity.

It is also interesting, that under assumptions H1—H4 Hamacher proved the inequalities similar to BG4 with strict inequality.

5. The axiomatic system of Kóczy

The papers [12] and [13] contain the reachest system of axioms of fuzzy algebra. We have used these papers in many places in our introduction, and our definition 1.2 is exactly the first axiom of paper [12]. Thus all our considerations are made in terminology of paper [12]. Now we rewrite the other axioms from this paper.

K2 card $Z \geqq 2$ and $(Z, \vee, \wedge, ')$ is algebraic structure with operations $\vee : Z^2 \to Z$, $\wedge : Z^2 \to Z$ and $' : Z \to Z$;

K3 there exist an element $0 \in Z$ called zero and the operations in Z fulfil

$$A \vee B = B \vee A, \tag{40}$$

$$(A \vee B) \vee C = A \vee (B \vee C), \tag{41}$$

$$A'' = A, \tag{42}$$

$$A \vee 0 = A, \quad A \wedge 0 = 0, \tag{43}$$

$$(A \vee B)' = A' \wedge B' \tag{44}$$

for every $A, B, C \in Z$;

K4 under order induced in F from L (see definition 1.4) mapping (1) fulfils (here $f_A = M(A)$):

$$f_P > f_Q \quad \text{for} \quad P = (A \wedge B) \vee (A \wedge C) \neq 0, \quad Q = A \wedge (B \vee C) \neq 0', \tag{45}$$

$$f_P < f_Q \quad \text{for} \quad P = (A \vee B) \wedge (A \vee C) \neq 0', \quad Q = A \vee (B \wedge C) \neq 0, \tag{46}$$

$$f_{A \vee B} > f_A \quad \text{for} \quad A \neq 0', \ B \neq 0, \tag{47}$$

$$f_{A \wedge B} < f_A \quad \text{for} \quad A \neq 0, \ B \neq 0', \tag{48}$$

$$f_A - f_B = f_{A'} - f_{B'} \tag{49}$$

for arbitrary $A, B, C \in Z$;

K4′ under order in F it is assumed that

$$f_{A \vee A} > f_A \quad \text{for} \quad A \neq 0, \ A \neq 0', \tag{50}$$

$$f_{A \wedge A} < f_A \quad \text{for} \quad A \neq 0, \ A \neq 0', \tag{51}$$

$$f_{A \vee A} > f_{B \vee B} \quad \text{iff} \quad f_A > f_B, \tag{52}$$

$$f_{A \wedge A} > f_{B \wedge B} \quad \text{iff} \quad f_A > f_B \tag{53}$$

for arbitrary $A, B \in Z$;

K5 there is admitted at most one solution U for every of the equations

$$A \vee U = B \quad (A, B \in Z,\ A \neq 0'), \tag{54}$$

$$A \wedge U = B \quad (A, B \in Z,\ A \neq 0); \tag{55}$$

K5′ there is assumed exactly one solution U for every of the equations (54), (55);

K6 L is a interval of real axis and (cf. notation (25), (26)),

$$f_{A\wedge B} = p(f_A, f_B),\ f_{A\vee B} = s(f_A, f_B),\ f_{A'} = c(f_A), \tag{56}$$

where functions $p, s\colon L^2 \to L$ and $c\colon L \to L$ are continuously differentiable.

It is possible that this is not the final form of Kóczy's work upon axiomatization of fuzzy algebra. The form presented in papers [12] and [13] has some reticences. For example in fact it is not precised what kind of order is considered in F (we wrote in K4 our supposition only) and it is also not precised, what kind of continuous differentiation is possible in L (and we suppose that L is in the real axis).

Now we precise some consequences of the above axioms.

Proposition 5.1. Under assumptions K2 and K3 the operation $\wedge$ has the following "dual" properties:

$$A \wedge B = B \wedge A, \tag{57}$$

$$(A \wedge B) \wedge C = A \wedge (B \wedge C), \tag{58}$$

$$A \wedge I = A, \quad A \vee I = I, \tag{59}$$

$$(A \wedge B)' = A' \vee B' \tag{60}$$

for arbitrary $A, B, C \in Z$, where

$$I = 0'. \tag{61}$$

Proof. Let $A, B, C \in Z$. From (42) and (44) we get

$$A \vee B = (A \vee B)'' = (A' \wedge B')'. \tag{62}$$

First we prove the "dual" formula

$$A \wedge B = (A' \vee B')' \tag{63}$$

Indeed, it follows from (42) and (44) that

$$A \wedge B = A'' \wedge B'' = (A')' \wedge (B')' = (A' \vee B')'.$$

Now using (42) in (63) we get (60):

$$(A \wedge B)' = (A' \vee B')'' = A' \vee B'.$$

(63) and (40) gives now (57):

$$A \wedge B = (A' \vee B')' = (B' \vee A')' = B \wedge A.$$

In a similar way from (63), (60) and (41) we get

$$\begin{aligned}(A \wedge B) \wedge C &= ((A \wedge B)' \vee C')' = ((A' \vee B') \vee C')' = \\ &= (A' \vee (B' \vee C'))' = (A' \vee (B \wedge C)')' = A \wedge (B \wedge C),\end{aligned}$$

which gives (58). Now from (42) and (61) we have

$$I' = 0. \tag{64}$$

By (61)—(64) and (43) we obtain

$$A \wedge I = (A' \vee I')' = (A' \vee 0)' = A'' = A,$$

$$A \vee I = (A' \wedge I')' = (A' \wedge 0)' = 0' = I,$$

which completes the proof.

Immediately from (43) and (59) we get

Proposition 5.2 (idempotent and absorption cases). Under assumptions K2 and K3 we have

$$0 \vee 0 = 0, \qquad 0 \wedge 0 = 0,$$

$$I \vee I = I, \qquad I \wedge I = I,$$

$$A \vee (A \wedge 0) = A, \quad A \wedge (A \vee 0) = A \wedge A,$$

$$A \wedge (A \vee I) = A, \quad A \vee (A \wedge I) = A \vee A,$$

$$0 \vee (0 \wedge A) = 0, \quad 0 \wedge (0 \vee A) = 0,$$

$$I \wedge (I \vee A) = I, \quad I \vee (I \wedge A) = I$$

for every $A \in Z$.

Proposition 5.3. Under assumption K2 and K4 or K4'

a) Z contains only two idempotents 0 and I,
b) if card $L=2$ then card $Z=2$.

Proof. Case a) is a consequence of strict inequalities from (47), (48), (50) and (51).

If card $L=2$ then L can be considered as Boolean algebra and then F is a Boolean algebra, too (cf. [3]). Then every element of F is a idempotent of both binary operations and (by homomorphism M) every element of Z is an idempotent. This together with a) ends the proof.

Our considerations of axiom system K2—K6 will be continued in further papers.

6. Conclusion

The axiomatic method of the introduction of fuzzy algebra has great meaning in the development of fuzzy set theory, obviously if the axiom system admits a broader class of operations as it was done e.g. in papers [8] and [12]. In the contrary, if the axiom system is constructed for the purpose of characterizing one given operation as in paper [1], it would have greater meaning in the theory of functional equations then in fuzzy set theory.

The interesting direction in considerations of different fuzzy algebras brings papers [9] and [16] where it is proved that different fuzzy algebras can be useful for different applications.

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Priority schedules of a steady job-flow pair*

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The priority schedules are discussed for a steady job-flow pair defined in [5] as a non-finite deterministic model of servicing invariably renewing demand series. Though these schedules are not dominating with respect to the utilization of the servicing processor, they are very important in practice. A method is defined for reducing the problem of evaluation of the schedules to the evaluation of simpler ones. The method is based on the reduction of the configuration constituted by the demands of job-flows. The reduction is a generalization of the Euclidean algorithm of the regular continued fraction expansion. For some configurations the reduction procedure does not prove to be finite or the evaluation procedure of the schedule of the reduced configuration is not known to be finite. For some of these configurations direct evaluation methods are given.

1. Introduction

In an earlier work [5] the problem of scheduling steady job-flow pairs was defined as scheduling the processor triple $\mathcal{P}=\{P_A, P_{B1}, P_{B2}\}$ to service two series $Q^{(i)}=\{C_{ij}, j=1, 2, \ldots\}$, $i=1, 2$, of task pairs $C_{ij}=(A_{ij}, B_{ij})$ demanding service of time $\eta_i \geqq 0$ and $\vartheta_i \geqq 0$ from the processor P_A and P_{Bi}, respectively. The series $Q^{(i)}$ is a *steady job-flow* with *parameters* η_i, ϑ_i as renewing demands for processors P_A and P_{Bi}. The steady job-flow pair is characterized by the values of the four parameters $Q=(\eta_1;\ \vartheta_1;\ \eta_2;\ \vartheta_2)$ called *configuration*. The space $\mathcal{Q}$ of configurations is the non-negative sixteenth of the four-dimensional Cartesian space.

We use below the following notations:

$$\tau_i = \eta_i + \vartheta_i, \quad i = 1, 2, \quad \eta = \eta_1 + \eta_2, \ \vartheta = \vartheta_1 + \vartheta_2, \ \gamma^{(i)} = \frac{\eta_i}{\tau_i}, \quad i = 1, 2.$$

A *schedule* is a unique determination for $t \geqq 0$ of which tasks are serviced at the moment t by which processors. The demands for the processor P_A can be conflicting. The schedule can be considered a decision process by which the conflicting situations are resolved and the normal continuation of service can be broken.

An important class of schedules is the set of *non-preemptive* schedules in which

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the service of any task cannot be preempted after starting until it finishes automatically. These schedules were discussed in the article [5]. A relatively simple algorithm was given to determine the optimal schedule.

The efficiency measure of schedules is the utilization of the processor P_A. Formally, the efficiency of a schedule R is defined by the limit

$$\gamma(R) = \lim_{t\to\infty} \frac{\lambda(t)}{t} \tag{1}$$

where $\lambda(t)=\lambda(0, t)$ is the P_A-usage in the interval $(0, t)$. The algorithm for choosing an optimal non-preemptive schedule is based on the method of reducing the configuration which is a generalization of the well-known Euclidean algorithm of the regular continued fraction expansion. The determination of the optimal schedule takes place by the full evaluation of the elements of the dominant set of the consistent natural schedules with maximum number six. Only one reduction has to be executed. The amount of the necessary computation is well bounded and estimated.

For the *preemptive scheduling* in which preempt-resume is permitted, another set, the consistent economical schedules, is a dominant set but it is not so nicely bounded as the set of consistent natural schedules [6]. The criteria of finiteness and bounds for the cardinal of the set are not known. Neither optimal strategy nor a smaller dominant set of schedules is known. It is shown [6] that the priority schedules are not optimal either. Since the only general method for determining an optimal schedule is the full evaluation of this dominant set the optimization procedure is uncontrolled.

Though the priority schedules are neither dominant, nor actually of better efficiency than the non-preemptive schedules in general, they are of great practical importance because of their simple scheduling rule. In a *priority schedule* one of the job-flows has priority versus other(s) which means that it is serviced in the moment it needs the processor. If the processor is busy by servicing another job-flow, the service will be preempted during the service of the priority job-flow-task and resumed after that. For job-flow pairs there are only two priority schedules according to job-flows $Q^{(1)}$ and $Q^{(2)}$ as priority ones. In [6] the priority schedules were denoted by $R_{1,2}$ and $R_{2,1}$, accordingly. In the schedule $R_{i,3-i}$ $(i=1, 2)$ the job-flow $Q^{(i)}$ is scheduled without preemption and delay as when the job-flow $Q^{(3-i)}$ were not present at all. The service of $Q^{(3-i)}$ on P_A takes place only in the intervals the P_A is free from servicing $Q^{(i)}$. The priority schedules $R_{1,2}$ and $R_{2,1}$ of the configuration $Q=(1;\ 3;\ 5;\ 7.5)$ are illustrated by Gantt-charts in Fig. 1.

The priority scheduling of the stochastic version of job-flow pairs was studied by Arató [1] with diffusion approximation and by Tomkó [7].

For the schedules $R_{1,2}$ and $R_{2,1}$ are symmetric in the role of the job-flows $Q^{(1)}$ and $Q^{(2)}$, every fact concerning $R_{1,2}(Q)$ becomes a fact concerning $R_{2,1}(\bar{Q})$ if $\bar{Q}$ is the *conjugate configuration* of Q defined as

$$\bar{Q} = (\bar{\eta}_1;\ \bar{\vartheta}_1;\ \bar{\eta}_2;\ \bar{\vartheta}_2) = (\eta_2;\ \vartheta_2;\ \eta_1;\ \vartheta_1).$$

This is why we need not word definitions and theorems depending on the order of the job-flows for both orders, only for the order $Q^{(1)}, Q^{(2)}$.

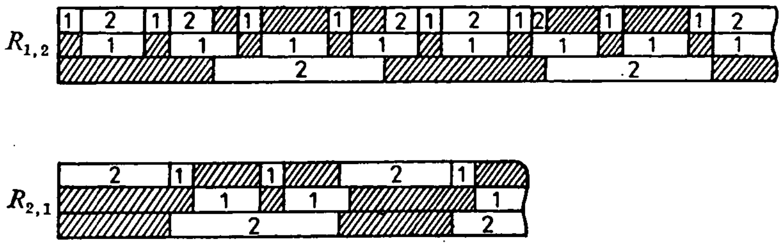

Fig. 1

The Gantt-charts of the priority schedules

In section 2 below we define first a method for reducing configurations $Q \in \mathcal{Q}$ into simpler, reduced configurations $Q^* \in \mathcal{Q}$. The reduction takes place by the iteration of an operator Δ to the configurations $Q_n = \Delta^n Q$ until a fixpoint $Q^* = \Delta^\nu Q$ called reduction of Q is reached. We show the relationships between the parameters of Q_n and $Q_m, n, m = 0, 1, 2, \ldots, n \neq m$. These remind one of the relationships known in the theory of continued fractions [4].

In paragraph 3 we show the connections between the characteristics of the schedules $R_{1,2}(Q_n)$ and $R_{1,2}(Q_m)$, $n \neq m$. This provides means to determine the characteristics of $R_{1,2}(Q)$ from the characteristics of $R_{1,2}(Q^*)$.

Section 4 surveys the configuration space $\mathcal{Q}$, the reduced configurations included, and give answer to the *Question* whether $R_{1,2}(Q)$ is periodic and what are its characteristics in different domains of $\mathcal{Q}$. The domain $0 < \tau_1^* < \tau_2^*$ remains unanswered in this section.

Section 5 is dealing with the above domain. The periodicity of $R_{1,2}(Q^*)$ is not cleared for the whole domain only for some parts of it. An algorithm is given for evaluating $R_{1,2}(Q^*)$ if it is periodic.

In section 6 we shall briefly deal with the connection between the Δ_i-reductions defined in section 2 and $\mathcal{D}_i$-reductions given in the article [5]. Also some reference is made to the analogy between the Δ-reduction and the continued fraction expansion algorithm.

Section 7 reviews the configuration space $\mathcal{Q}$ from the point of view whether the "Question" of periodicity and evaluation is answered or not, and by which theorem, if it is.

2. The method of Δ-reduction

The transformation of configurations defined below as Δ-reduction enables us to reduce the investigation of priority scheduling of some configurations to one of other configurations. This method is analogous to the reduction method applied for non-preemptive schedules by means of an operator $\mathcal{D}$ [5].

The operator Δ defined below is the Δ_1 from the two operators Δ_i, $i = 1, 2$, in the application of which the roles of $Q^{(1)}$ and $Q^{(2)}$ are symmetrical. We shall see later that the operator Δ_i is connected to the priority schedule $R_{i,3-i}$, $i = 1, 2$. The index 1 of Δ_1 is omitted in the notation Δ.

Let the *operator* Δ be defined for any configuration $Q \in \mathcal{Q}$ by the relationships between its parameters and the parameters of the configuration $\tilde{Q} = \Delta Q = (\tilde{\eta}_1;\ \tilde{\vartheta}_1;\ \tilde{\eta}_2;\ \tilde{\vartheta}_2) \in \mathcal{Q}$. The parameters of $\tilde{Q}$ are defined by the relations

$$
\begin{aligned}
&\text{(a)}\ \tilde{\eta}_1 = \eta_1 \\
&\text{(b)}\ \vartheta_1 = l_1\tau_2 + \tilde{\vartheta}_1 \ \text{ where} \\
&\qquad l_1 \geqq 0 \text{ is an integer and } 0 \leqq \tilde{\vartheta}_1 < \tau_2 \text{ if } \tau_2 > 0, \\
&\qquad l_1 = 0,\ \tilde{\vartheta}_1 = \vartheta_1 \text{ if } \tau_2 = 0, \\
&\text{(c)}\ \eta_2 = k_2\tilde{\vartheta}_1 + \tilde{\eta}_2 \ \text{ where} \\
&\qquad k_2 \geqq 0 \text{ is an integer and } 0 < \tilde{\eta}_2 \leqq \tilde{\vartheta}_1 \text{ if } \eta_2\tilde{\vartheta}_1 > 0, \\
&\qquad k_2 = 0,\ \tilde{\eta}_2 = \eta_2 \text{ if } \eta_2\tilde{\vartheta}_1 = 0, \\
&\text{(d)}\ \vartheta_2 = l_2\tilde{\tau}_1 + \tilde{\vartheta}_2 \ \text{ where} \\
&\qquad l_2 \geqq 0 \text{ is an integer and } 0 \leqq \tilde{\vartheta}_2 < \tilde{\tau}_1 \text{ if } \tilde{\tau}_1 > 0, \\
&\qquad l_2 = 0,\ \tilde{\vartheta}_2 = \vartheta_2 \text{ if } \tilde{\tau}_1 = 0.
\end{aligned} \tag{2}
$$

This definition shows that the operation ΔQ determines also an integer triple (l_1, k_2, l_2) out of the configuration $\tilde{Q}$. This triple is characteristic of the configuration Q from the point of view of the effect of the operator Δ on Q.

If $l_1 + k_2 + l_2 = 0$ then the operator Δ is *ineffective* for Q and $\Delta Q = Q$. We say Q that is *reduced* in this case. If $l_1 + k_2 + l_2 > 0$ then Δ is *effective* for Q, $\Delta Q \neq Q$ and at least one of the parameters of $\tilde{Q}$ is less than that of Q. Therefore the operator Δ is called a *reduction operator*. The triple (l_1, k_2, l_2) is the *quotient generated by* Δ applied to Q. Δ is defined for all points Q of $\mathcal{Q}$, and $\tilde{Q} \in \mathcal{Q}$. Therefore Δ is applicable repeatedly to the transformed configurations and the series of configurations

$$Q_0 = Q, \quad Q_n = \Delta Q_{n-1}, \quad n = 1, 2, \ldots,$$

can be defined for any point Q of $\mathcal{Q}$. Using the powers Δ^n, $n = 0, 1, 2, \ldots$, of the operator Δ, we can write

$$Q_n = \Delta^n Q, \quad n = 0, 1, 2, \ldots. \tag{3}$$

Let the series of triples generated by the series $\Delta, \Delta^2, \ldots, \Delta^n, \ldots$ be

$$(L)\colon (l_{1,0}, k_{2,0}, l_{2,0}), (l_{1,1}, k_{2,1}, l_{2,1}), \ldots, (l_{1,n-1}, k_{2,n-1}, l_{2,n-1}), \ldots$$

and let

$$(\Lambda)\colon (l_{1,0}, k_{2,0} + l_{2,0}), (l_{1,1}, k_{2,1} + l_{2,1}), \ldots, (l_{1,n-1}, k_{2,n-1} + l_{2,n-1}), \ldots.$$

These are the series of quotients. Let us define the length of (L) and (Λ) the index ν of the first triple for which

$$l_{1,\nu} + k_{2,\nu} + l_{2,\nu} = 0$$

if such an index exists and $\nu = \infty$ otherwise. Let us use the notation $|(L)| = |(\Lambda)| = \nu$. If $\nu < \infty$, the Q_ν is the first member in the sequence $Q_0, Q_1, \ldots$ which is reduced. ν is called the *degree of compositeness* (dc) of Q. If $\nu < \infty$ then Q is *reducible*, otherwise, it is *non-reducible*. If the dc of Q is $0 < \nu < \infty$ then

$$
\begin{aligned}
&\text{(a)}\ l_{1,i} + k_{2,i} + l_{2,i} > 0, \quad i = 0, 1, \ldots, \nu - 1, \\
&\text{(b)}\ l_{1,\nu} + k_{2,\nu} + l_{2,\nu} = 0
\end{aligned} \tag{4}
$$

and the series (L) and (Λ) contain exactly ν non-zero members. The configuration $Q^*=Q_\nu$ is a reduced configuration and it is the *reduction of Q*.

From the definition (2) of Δ we can deduce the conditions of Q^* to be reduced. By (2), (4b) will hold if

$$\begin{aligned}
&\text{(a)}\quad 0 \leqq \vartheta_1^* < \tau_2^* \quad \text{or} \quad \tau_2^* = 0 \quad \text{and}\\
&\text{(b)}\quad 0 < \eta_2^* \leqq \vartheta_1^* \quad \text{or} \quad \eta_2^* \vartheta_1^* = 0 \quad \text{and}\\
&\text{(c)}\quad 0 \leqq \vartheta_2^* < \tau_1^* \quad \text{or} \quad \tau_1^* = 0.
\end{aligned} \tag{5}$$

Conditions (5a)—(5c) are not independent of but include each other. The set $\mathcal{Q}^* \subset \mathcal{Q}$ of the reduced configurations is illustrated by planes (η_1^*, η_2^*) fixed in Fig. 2a—d.

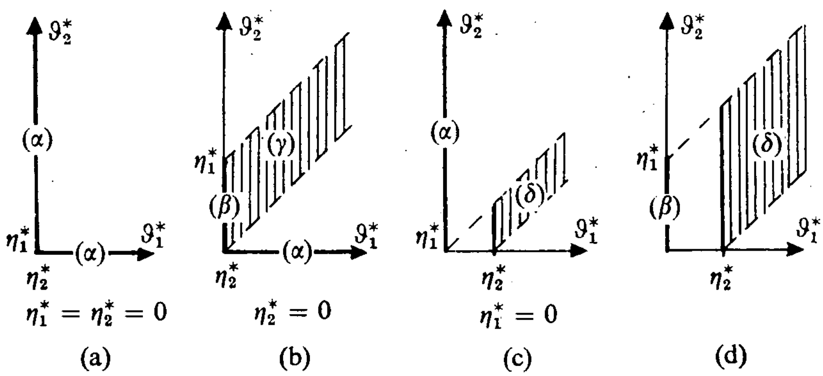

Fig. 2

Illustration of the set $\mathcal{Q}^*$ of reduced configurations

On the graphs we show the disjunct domains of configurations by the following lemma.

Lemma 1. *The operator Δ defined by* (2) *is ineffective for Q^* i.e. Q^* is reduced, iff one of the following conditions holds*

$$\begin{aligned}
&(\alpha)\ \tau_1^*\tau_2^* = 0\\
&(\beta)\ \tau_1^*\tau_2^* > 0,\ \vartheta_1^* = 0,\ 0 \leqq \vartheta_2^* < \eta_1^*\\
&(\gamma)\ \vartheta_1^*\tau_2^* > 0,\ \eta_2^* = 0,\ 0 < \vartheta_1^* < \vartheta_2^* < \tau_1^*\\
&(\delta)\ \vartheta_1^*\eta_2^* > 0,\ \eta_2^* \leqq \vartheta_1^* < \tau_2^*,\ 0 \leqq \vartheta_2^* < \tau_1^*.
\end{aligned} \tag{6}$$

Proof. In either domain of (6α)$-$(6δ) every of the conditions (5a)—(5c) holds. Conditions (6α)—(6δ) are, therefore, sufficient for Q^* to be reduced. To see the necessity it is easy to verify that one of (6α)—(6δ) holds if (5a)—(5c) are true [4]. □

Let the number series (λ) defined as $\lambda_{2i}=l_{1,i}$, $\lambda_{2i+1}=k_{2,i}+l_{2,i}$, $i=0, 1, \ldots$. The following lemma shows that no zero value in the series (λ) between $l_{1,0}$ and

$k_{2,\nu-1}+l_{2,\nu-1}$ exists. This means that the parameters of both job-flows are reduced in the transformation $Q_i \to Q_{i+1}$, $i=1, 2, \ldots, \nu-2$. They are the transformations $Q_0 \to Q_1$ and $Q_{\nu-1} \to Q_\nu$ only in which it is possible that only one of the job-flows be reduced: $Q^{(2)}$ in $Q_0 \to Q_1$ and $Q^{(1)}$ in $Q_{\nu-1} \to Q_\nu$. This fact is expressed by the relations concerning (λ)

$$l_{1,0} \geqq 0, \; k_{2,i}+l_{2,i} > 0, \; 0 \leqq i < \nu-1, \; l_{1,i} > 0, \; 1 \leqq i \leqq \nu-1, \; k_{2,\nu-1}+l_{2,\nu-1} \geqq 0. \quad (7)$$

In any circumstances, the following relations hold for $i=0, 1, \ldots$:

$$\begin{aligned} &\text{(a)} \quad \vartheta_{1,i}-\vartheta_{1,i+1} = l_{1,i}\tau_{2,i}, \quad \tau_{1,i}-\tau_{1,i+1} = l_{1,i}\tau_{2,i} \\ &\text{(b)} \quad \eta_{2,i}-\eta_{2,i+1} = k_{2,i}\vartheta_{1,i+1}, \quad \tau_{2,i}-\tau_{2,i+1} = (k_{2,i}+l_{2,i})\vartheta_{1,i+1}+l_{2,i}\eta_1 \qquad (8) \\ &\text{(c)} \quad \vartheta_{2,i}-\vartheta_{2,i+1} = l_{2,i}\tau_{1,i+1}. \end{aligned}$$

Lemma 2. *Let*

$$k_{2,I}+l_{2,I} = 0, \quad I \geqq 0, \quad \text{or} \quad l_{1,I} = 0, \quad I \geqq 1,$$

be the first zero value after $l_{1,0}$ in the series (λ) if such one exists. Then all members in (λ) following it are zeros and the degree of compositeness of Q is as follows:

$$\begin{aligned} &\textit{in case } k_{2,0}+l_{2,0} = 0\colon && \nu = 0 \quad \textit{if} \quad l_{1,0} = 0 \\ & && \nu = 1 \quad \textit{if} \quad l_{1,0} > 0, \\ &\textit{in cases } I > 0\colon && \nu = I \quad \textit{if} \quad l_{1,I} = 0 \\ & && \nu = I+1 \quad \textit{if} \quad k_{2,I}+l_{2,I} = 0, \; l_{1,I} > 0. \end{aligned}$$

Proof. If $l_{1,0}=k_{2,0}+l_{2,0}=0$ $(I=0)$ then Q_0 is reduced by definition and $\nu=0$. If $l_{1,I}>0$ but $k_{2,I}+l_{2,I}=0$, $I\geqq 0$, then $\nu>I$ and $\vartheta_{1,I+1}<\tau_{2,I}$, $\tau_{2,I+1}=\tau_{2,I}$ from (2), and, therefore, $\vartheta_{1,I+1}<\tau_{2,I+1}$ and so $l_{1,I+1}=0$ and $\tau_{1,I+2}=\tau_{1,I+1}$. If, however, $l_{1,I+1}=0$, $I\geqq 1$, then $\tau_{1,I+2}=\tau_{1,I+1}$. But in this case $\eta_{2,I+2}=\eta_{2,I+1}$ and $\vartheta_{2,I+2}=\vartheta_{2,I+1}$ from (2) and so $Q_{I+2}=Q_{I+1}$. This means $\nu\leqq I+1$. □

The following lemma shows the part of $\mathcal{Q}$ in which non-reducibility is possible.

Lemma 3. *To any $Q\in\mathcal{Q}$ there exists a finite integer $\nu'\geqq 0$ for which the configuration*

$$Q_{\nu'} = \Delta^{\nu'}Q$$

is either reduced or defective with

$$\eta_1\vartheta_{2,\nu'} = 0.$$

Proof. If $\eta_1=0$, there is nothing to prove. Let $\eta_1>0$. If $l_{2,i}>0$ then from (2d) we get

$$\vartheta_{2,i}-\vartheta_{2,i+1} = l_{2,i}\tau_{1,i+1} \geqq \tau_{1,i+1} \geqq \eta_1 > 0$$

and, therefore, the value of $\vartheta_{2,i}$ decreases at least by η_1. This means that only a finite number of positive $l_{2,i}$ members in the series $l_{2,0}, l_{2,1}, \ldots$ can exist and there exists an $i_0\geqq 0$ so that

$$l_{2,i} = 0, \quad \vartheta_{2,i} = \vartheta_{2,i_0} \quad \text{if} \quad i \geqq i_0.$$

If $\vartheta_{2,i_0}=0$ then $\nu'=i_0$. Let $\vartheta_{2,i_0}>0$. If $l_{1,i}>0$ then from (2b) we get

$$\vartheta_{1,i}-\vartheta_{1,i+1} = l_{1,i}\tau_{2,i} \geqq \tau_{2,i} \geqq \vartheta_{2,i_0} > 0$$

and, therefore, the value of $\vartheta_{1,i}$ decreases at least by ϑ_{2,i_0}. This means that only a finite number of positive $l_{1,i}$ member can exist in (λ). If $l_{1,i'}$ is the last positive $l_{1,i}$ member then $\nu'=i'+1$ and $Q_{\nu'}$ is reduced. □

By Lemma 3 only the cases

$$\eta_1\vartheta_2=0 \tag{9}$$

remain questionable in regard to reducibility. The following lemma concerns these cases.

Lemma 4. *Any $Q\in\mathcal{Q}$ with* (9) *is either reducible or*

$$Q_n \to (\eta_1; 0; 0; 0) \quad as \quad n\to\infty.$$

In the latter case

$$\vartheta_{1,n}\tau_{2,n} > 0 \tag{10}$$

after any finite step n. This case comes true if

$$\tau_1\tau_2>0,\ \eta_2\vartheta_2=0,\ \vartheta_1 \text{ and } \vartheta_2 \text{ are rationally independent.} \tag{11}$$

Proof. Q is reduced if $\tau_2=0$. Let now $\tau_2>0$.

If $\vartheta_2=0, \eta_2>0$, the reduction procedure will be equivalent to the regular continued fraction expansion of the number

$$\xi=\frac{\vartheta_1}{\tau_2} \tag{12}$$

with the restriction that the number $n+1$ of the partial quotients $[b_0, b_1, \ldots, b_n]$ must be chosen odd in finite cases because η_2^* cannot be zero by definition (2). This choice is always possible [3]. The number of the partial quotients and the steps of reduction will be finite exactly when ξ is a rational number [3]. The reduction results in $Q^*=(\eta_1;\ 0;\ \eta_2^*;\ 0)$. If (11) holds, neither $\vartheta_{1,i}$ nor $\eta_{2,i}$ becomes zero in finite steps and (10) is true.

Let now $\vartheta_2>0$. Then $\eta_1=0$ from (9). If $\vartheta_1=0$ then Q is reduced. Let, therefore, $\vartheta_1>0$ as well.

If $\eta_2=0$, the reduction procedure becomes equivalent to the continued fraction expansion of ξ and it is finite exactly when ξ is a rational number. The reduction results in $Q^*=(0;\ \vartheta_1^*;\ 0;\ 0)$ or $Q^*=(0;\ 0;\ 0;\ \vartheta_2^*)$. If ϑ_1 and τ_2 are rationally independent, the expansion procedure is infinite and neither of $\vartheta_{1,i}$ and $\vartheta_{2,i}$ will be zero for finite i and (10) holds.

Let $\eta_2>0$ as well. Suppose Q is not-reducible, i.e., the degree of compositeness $\nu=\infty$. By Lemma 2 all members of (λ) are positive after $l_{1,0}$. From (8) we can write for any $i>0$:

$$\vartheta_{1,i}-\vartheta_{1,i+1}=l_{1,i}\tau_{2,i}=l_{1,i}[(k_{2,i}+l_{2,i})\vartheta_{1,i+1}+\tau_{2,i+1}]\geqq$$
$$\geqq \max(\vartheta_{1,i+1}, \eta_{2,i+1}, \vartheta_{2\ i+1}).$$

If either of the parameters $\vartheta_1, \eta_2, \vartheta_2$ remained bounded from below by a positive number $\alpha>0$, then ϑ_1 would be decreased by at least α in every step of reduction. After ϑ_1/α steps $\vartheta_{1,i}$ would surely become negative which is a contradiction. Thus none of $\vartheta_{1,i}, \eta_{2,i}, \vartheta_{2,i}$ could be bounded by an $\alpha>0$, and $Q_i\to\to(0;\ 0;\ 0;\ 0)$ if $i\to\infty$. This proves (10).

In cases (11) we have shown that $\nu=\infty$ and (10) holds. But from (2) we get

$$\vartheta_{1,i}-\vartheta_{1,i+1} = l_{1,i}[(k_{2,i}+l_{2,i})\vartheta_{1,i+1}+\tau_{2,i+1}+l_{2,i}\eta_1] \geqq$$
$$\geqq \max(\vartheta_{1,i+1}, \eta_{2,i+1}, \vartheta_{2,i+1})$$

also in these cases and the parameters cannot remain bounded from below and so $Q_i \to (\eta_1;\ 0;\ 0;\ 0)$ as $i\to\infty$. □

From Lemma 3 and Lemma 4 we can assert that $\nu=\infty$ can hold only for defective configurations for which $\eta_1=0$ and for configurations for which $\vartheta_{2,\nu'}=0$ for some $\nu'\geqq 0$. We cannot exactly show the domains or points of $\mathcal{Q}$ in which Q is non-reducible. We know such subsets of $\mathcal{Q}$ but not all such points.

The relationships below are true independently of the finiteness of ν and the relation of ν and n. These relationships concern the parameters of Q and Q_n and Q_n and Q_{n+1}.

As the definition (2) of $Q_{i+1}=\Delta Q_i$, we get

$$\begin{aligned} \eta_{1,i} &= \eta_{1,i+1}, & \eta_{2,i} &= k_{2,i}\vartheta_{1,i+1}+\eta_{2,i+1} \\ \vartheta_{1,i} &= l_{1,i}\tau_{2,i}+\vartheta_{1,i+1}, & \vartheta_{2,i} &= l_{2,i}\tau_{1,i+1}+\vartheta_{2,i+1}. \end{aligned} \quad i=0,1,\ldots \tag{13}$$

From the same definition we can obtain the relationship between the parameters of Q_n and Q_{n+1} in the following form:

$$\begin{aligned} \eta_{1,n} &= \eta_{1,n+1} \\ \vartheta_{1,n} &= l_{1,n}l_{2,n}\eta_{1,n+1}+[l_{1,n}(k_{2,n}+l_{2,n})+1]\vartheta_{1,n+1}+l_{1,n}\eta_{2,n+1}+l_{1,n}\vartheta_{2,n+1} \\ \eta_{2,n} &= k_{2,n}\vartheta_{1,n+1}+\eta_{2,n+1} \\ \vartheta_{2,n} &= l_{2,n}\eta_{1,n+1}+l_{2,n}\vartheta_{1,n+1}+\vartheta_{2,n+1} \end{aligned} \tag{14}$$

$$\begin{aligned} \tau_{1,n} &= [l_{1,n}(k_{2,n}+l_{2,n})+1]\tau_{1,n+1}+l_{1,n}\tau_{2,n+1}-l_{1,n}k_{2,n}\eta_1 \\ \tau_{2,n} &= (k_{2,n}+l_{2,n})\tau_{1,n+1}+\tau_{2,n+1}-k_{2,n}\eta_1 \end{aligned} \tag{15}$$

$$\begin{aligned} \eta_{1,n+1} &= \eta_{1,n} \\ \vartheta_{1,n+1} &= \vartheta_{1,n}-l_{1,n}\eta_{2,n}-l_{1,n}\vartheta_{2,n} \\ \eta_{2,n+1} &= -k_{2,n}\vartheta_{1,n}+(l_{1,n}k_{2,n}+1)\eta_{2,n}+l_{1,n}k_{2,n}\vartheta_{2,n} \\ \vartheta_{2,n+1} &= -l_{2,n}\eta_{1,n}-l_{2,n}\vartheta_{1,n}+l_{1,n}l_{2,n}\eta_{2,n}+(l_{1,n}l_{2,n}+1)\vartheta_{2,n} \end{aligned} \tag{14'}$$

$$\begin{aligned} \tau_{1,n+1} &= \tau_{1,n}-l_{1,n}\tau_{2,n} \\ \tau_{2,n+1} &= -(k_{2,n}+l_{2,n})\tau_{1,n}+[l_{1,n}(k_{2,n}+l_{2,n})+1]\tau_{2,n}+k_{2,n}\eta_1. \end{aligned} \tag{15'}$$

As the parameter η_1 is not concerned during reduction, $\eta_{1,n}=\eta_1$, $n=0,1,\ldots$, and it can be separated from the other parameters.

From the relationships (14) the connection between the parameters of any two Q_n and $Q_{n'}$ $n\neq n'$, especially between the parameters of $Q=Q_0$ and Q_n can be obtained. To make the further relationships more compact we have to introduce some series of integers, vectors and matrices as follow.

Let (X) be the formal notation of the infinite sequence:

(X): $X_0, X_1, X_2, \ldots, X_n, \ldots$

and let $|(X)|$ be the index of the first member of (X) from which all members are the same, if such a member exists. This is called the length of (X).

We have already defined the two series (L) and (Λ). The members of (Q) are the configurations $Q_n=(\eta_1;\ \vartheta_{1,n};\ \eta_{2,n};\ \vartheta_{2,n})$. The lengths of (L), (Λ), (Q) are the same v, the dc of the configuration $Q_0=Q$. Let (0) be the series of the identically zero members with the length 0. We have referred to the series (λ) the members of which are

$$(\lambda):\quad \lambda_{2i}=l_{1,i},\quad \lambda_{2i+1}=k_{2,i}+l_{2,i};\quad i=0,1,\ldots.$$

Define also the series

$$(k):\quad k_n=k_{2,n},\quad n=0,1,\ldots$$

and

$$(l):\quad l_n=l_{2,n},\quad n=0,1,\ldots.$$

We define now a set of new series necessary to writing down the relationships among the parameters of (Q). The definitions are recursive for $i, n=0, 1, \ldots$.

$$(A):\quad A_n=\lambda_n A_{n-1}+A_{n-2}\quad \text{with}\quad A_{-2}=0,\quad A_{-1}=1$$

$$(B):\quad B_n=\lambda_n B_{n-1}+B_{n-2}\quad \text{with}\quad B_{-2}=1,\quad B_{-1}=0$$

$$(C):\quad C_{2i}=A_{2i},\quad C_{2i+1}=k_{2,i}C_{2i}+C_{2i-1}\quad \text{with}\quad C_{-1}=0$$

$$(D):\quad D_{2i}=B_{2i},\quad D_{2i+1}=k_{2,i}D_{2i}+D_{2i-1}\quad \text{with}\quad D_{-1}=0$$

$$(B'):\quad B'_{2i}=\lambda_{2i}B'_{2i-1}+B'_{2i-2},\quad B'_{2i+1}=\lambda_{2i+1}B'_{2i}+B'_{2i-1}+k_{2,i}\quad \text{with}\quad B'_{-2}=0,\ B'_{-1}=0$$

$$(B''):\quad B''_{2i}=\lambda_{2i}B''_{2i-1}+B''_{2i-2},\quad B''_{2i+1}=\lambda_{2i+1}B''_{2i}+B''_{2i-1}-k_{2,i}\quad \text{with}\quad B''_{-2}=1,\ B''_{-1}=0$$

$$(D'):\quad D'_{2i}=B'_{2i},\ D'_{2i+1}=k_{2,i}D'_{2i}+D'_{2i-1}+k_{2,i}\quad \text{with}\quad D'_{-1}=0$$

$$(D''):\quad D''_{2i}=B''_{2i},\quad D''_{2i+1}=k_{2,i}D''_{2i}+D''_{2i-1}-k_{2,i}\quad \text{with}\quad D''_{-1}=0$$

Define the following sequences of vectors and matrices for $n=0, 1, \ldots$ as well.

$$(\underline{\tilde{Q}}):\quad \underline{\tilde{Q}}_n=\begin{pmatrix}\tilde{\vartheta}_{1,n}\\ \tilde{\eta}_{2,n}\\ \tilde{\vartheta}_{2,n}\end{pmatrix}$$

$$(\underline{\tilde{\tau}}):\quad \underline{\tilde{\tau}}_n=\begin{pmatrix}\tilde{\tau}_{1,n}\\ \tilde{\tau}_{2,n}\end{pmatrix}$$

with

$$\begin{aligned}
\tilde{\vartheta}_{1,n}&=\vartheta_{1,n}+(B'_{2n-2}+1)\eta_1\\
\tilde{\eta}_{2,n}&=-\eta_{2,n}+D'_{2n-1}\eta_1\\
\tilde{\vartheta}_{2,n}&=-\vartheta_{2,n}+(B'_{2n-1}-D'_{2n-1})\eta_1\\
\tilde{\tau}_{1,n}&=\tau_{1,n}+B'_{2n-2}\eta_1\\
\tilde{\tau}_{2,n}&=-\tau_{2,n}+B'_{2n-1}\eta_1
\end{aligned}\tag{16}$$

$$(\underline{\underline{D}}_+):\quad \underline{\underline{D}}_{n,n+1}=\begin{pmatrix}1 & \lambda_{2n}\\ \lambda_{2n+1} & \lambda_{2n}\lambda_{2n+1}+1\end{pmatrix}=\begin{pmatrix}1 & l_{1,n}\\ k_{2,n}+l_{2,n} & l_{1,n}(k_{2,n}+l_{2,n})+1\end{pmatrix}$$

$$(\underline{\underline{\Delta}}_+):\quad \underline{\underline{\Delta}}_{n,n+1}=\begin{pmatrix}1 & l_{1,n} & l_{1,n}\\ k_{2,n} & l_{1,n}k_{2,n}+1 & l_{1,n}k_{2,n}\\ l_{2,n} & l_{1,n}l_{2,n} & l_{1,n}l_{2,n}+1\end{pmatrix}$$

$$(\underline{\underline{D}}):\quad \underline{\underline{D}}_n=\begin{pmatrix}B_{2n-2} & A_{2n-2}\\ B_{2n-1} & A_{2n-1}\end{pmatrix}$$

$$(\underline{\underline{\Delta}}):\quad \underline{\underline{\Delta}}_n=\begin{pmatrix}B_{2n-2} & A_{2n-2} & A_{2n-2}\\ D_{2n-1} & C_{2n-1}+1 & C_{2n-1}\\ B_{2n-1}-D_{2n-1} & A_{2n-1}-C_{2n-1}-1 & A_{2n-1}-C_{2n-1}\end{pmatrix}.$$

We remark at once that

$$\underline{\tilde{Q}}_0=\begin{pmatrix}\tau_{1,0}\\ -\eta_{2,0}\\ -\vartheta_{2,0}\end{pmatrix}=\underline{\tilde{Q}},\quad \underline{\tilde{\tau}}_0=\begin{pmatrix}\tau_{1,0}\\ -\tau_{2,0}\end{pmatrix}=\underline{\tilde{\tau}} \tag{17}$$

and that the D-matrices can be obtained from the corresponding Δ-matrices by summing up the two last rows and omitting one of the last two equal columns.

The foregoing entities simplify the relationships between the parameters of the members of (Q). The proof of the relationships will be automatic by means of the relationships of the following lemma. The relationships are interesting on their own right as well. To simplify writing we use the following determinant notation:

$$H_n(x,y)=\begin{vmatrix}x_n & y_n\\ x_{n-1} & y_{n-1}\end{vmatrix}=x_ny_{n-1}-x_{n-1}y_n,\quad n=1,2,\ldots, \tag{18}$$

for any two series (x) and (y). From this definition the relation

$$H_n(y,x)=-H_n(x,y) \tag{19}$$

is trivial. (18)—(19) can be interpreted for $n=-1,0$ as well if the values $x_{-2}, y_{-2}, x_{-1}, y_{-1}$ are also given.

Lemma 5. *Among the entities defined beforehand, the following relationships hold. For* $i,n=-1,0,1,\ldots$

$$H_n(A,B)=(-1)^{n-1} \tag{20}$$

$$(A_n,B_n),\ (A_n,A_{n-1}),\ (B_n,B_{n-1}),\ (A_{n-1},B_{n-1}) \tag{21}$$

*are relatively prime integer pairs**

$$\begin{gathered}A_{2i+1}=\sum_{j=0}^{i-1}(k_{2,j}+l_{2,j})A_{2j}+1,\quad B_{2i+1}=\sum_{j=0}^{i-1}(k_{2,j}+l_{2,j})B_{2j}\\ C_{2i+1}=\sum_{j=0}^{i-1}k_{2,j}A_{2j},\quad D_{2i+1}=\sum_{j=0}^{i-1}k_{2,j}B_{2j}\end{gathered} \tag{22}$$

* 0 and 1 are considered relatively prime integers.

(with the definition $\sum_{j=0}^{-1} x_j = 0$*)*

$$B'_n + B''_n = B_n, \quad D'_n + D''_n = D_n. \tag{23}$$

For $i, n = 0, 1, \ldots$

$$\begin{aligned} H_{2i}(B, A) &= H_{2i-1}(A, B) &&= 1 \\ H_{2i}(B', A) &= H_{2i-1}(A, B') &&= 1 - C_{2i-1} \\ H_{2i}(B'', A) &= H_{2i-1}(A, B'') &&= 1 + C_{2i-1} \\ H_{2i}(B', B) &= H_{2i-1}(B, B') &&= -D_{2i-1} \\ H_{2i}(B'', B) &= H_{2i-1}(B, B'') &&= D_{2i-1} \\ H_{2i}(B'', B') &= H_{2i-1}(B', B'') &&= D_{2i-1}; \end{aligned} \tag{24}$$

$$\begin{aligned} &A_{2i} D_{2i} - B_{2i} C_{2i} = 0, \quad A_{2i-1} D_{2i-1} - B_{2i-1} C_{2i-1} = B'_{2i-1} \\ &A_{2i+1} D_{2i} - B_{2i+1} C_{2i} = A_{2i-1} D_{2i} - B_{2i-1} C_{2i} = 1 \\ &A_{2i} D_{2i+1} - B_{2i} C_{2i+1} = A_{2i} D_{2i-1} - B_{2i} C_{2i-1} = B'_{2i}; \end{aligned} \tag{25}$$

if $(k) = (0)$ *then*

$$\begin{aligned} C_{2i+1} = D_{2i+1} = 0, \quad (B') = (0), \quad (B'') = (B) \\ (D') = (0), \; D''_{2i} = B_{2i}, \; D''_{2i+1} = 0; \end{aligned} \tag{26}$$

if $(l) = (0)$ *then*

$$\begin{aligned} B'_{2i} = B_{2i} - 1, \quad B'_{2i+1} = B_{2i+1}, \quad B''_{2i} = 1, \quad B''_{2i+1} = 0 \\ D'_{2i} = B_{2i-1}, \quad D'_{2i+1} = B_{2i+1}, \quad D''_{2i} = 1, \quad D''_{2i+1} = 0; \end{aligned} \tag{26'}$$

$$\underline{\underline{D}}_{n+1} = \underline{\underline{D}}_{n,n+1} \underline{\underline{D}}_n, \quad \underline{\underline{\Delta}}_{n+1} = \underline{\underline{\Delta}}_{n,n+1} \underline{\underline{\Delta}}_n \quad \text{with} \quad \underline{\underline{D}}_0 = \underline{\underline{I}}, \quad \underline{\underline{\Delta}}_0 = \underline{\underline{I}}, \tag{27}$$

$$\underline{\underline{D}}_{n,n+1}^{-1} = \begin{pmatrix} \lambda_{2n}\lambda_{2n+1} + 1 & -\lambda_{2n} \\ -\lambda_{2n+1} & 1 \end{pmatrix} = \begin{pmatrix} l_{1,n}(k_{2,n} + l_{2,n}) + 1 & -l_{1,n} \\ -(k_{2,n} + l_{2,n}) & 1 \end{pmatrix}$$

$$\underline{\underline{\Delta}}_{n,n+1}^{-1} = \begin{pmatrix} l_{1,n}(k_{2,n} + l_{2,n}) + 1 & -l_{1,n} & -l_{1,n} \\ -k_{2,n} & 1 & 0 \\ -l_{2,n} & 0 & 1 \end{pmatrix} \tag{28}$$

$$\underline{\underline{D}}_n^{-1} = \begin{pmatrix} A_{2n-1} & -A_{2n-2} \\ -B_{2n-1} & B_{2n-2} \end{pmatrix}$$

$$\underline{\underline{\Delta}}_n^{-1} = \begin{pmatrix} A_{2n-1} & -A_{2n-2} & -A_{2n-2} \\ -B_{2n-1} & B_{2n-2} + 1 & B_{2n-2} \\ -B_{2n-1} & B_{2n-2} - 1 & B_{2n-2} \end{pmatrix}$$

$$\underline{\underline{D}}_{n,n+1} = \begin{pmatrix} 1 & 0 \\ k_{2,n} + l_{2,n} & 1 \end{pmatrix} \begin{pmatrix} 1 & l_{1,n} \\ 0 & 1 \end{pmatrix}$$

$$\underline{\underline{D}}_{n,n+1}^{-1} = \begin{pmatrix} 1 & -l_{1,n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -(k_{2,n} + l_{2,n}) & 1 \end{pmatrix} \tag{29}$$

$$\underline{\underline{\Delta}}_{n,n+1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{2,n} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ k_{2,n} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & l_{1,n} & l_{1,n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{\Delta}}_{n,n+1}^{-1} = \begin{pmatrix} 1 & -l_{1,n} & -l_{1,n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -k_{2,n} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l_{2,n} & 0 & 1 \end{pmatrix}$$

The determinant $\det(\underline{\underline{X}})$ *for every matrix encounters above is*

$$\det(\underline{\underline{X}}) = 1. \tag{30}$$

Proof. Taking into account definition (18), we easily see (20) and (24) for $n=-1$ and $i, n=0$. The other relations (21)—(26′) can be checked for the least index by the definitions of the entities. Using the recursive definitions of the series, we can verify (20), (22)—(26′) by induction. (21) follows from (20) because every common divisor of the pairs must divide $(-1)^{n-1}$ and is, therefore, ± 1. (27) can be verified by executing the multiplications. The inverse matrices (28) can be verified most simply by multiplying them with the corresponding original matrices and using (20)—(25). The factorizations (29) can simply be checked by executing the assigned multiplications. (30) is trivial for every matrix encountering. □

After Lemma 5 we can now easily prove

Theorem 1. *For any configuration* $Q \in \mathcal{Q}$ *the following relationships between the parameters of* (Q) *hold:*

$$\begin{aligned} &\underline{\tilde{Q}}_{n+1} = \underline{\underline{\Delta}}_{n,n+1}\underline{\tilde{Q}}_n, \quad \underline{\tilde{Q}}_n = \underline{\underline{\Delta}}_{n,n+1}^{-1}\underline{\tilde{Q}}_{n+1}, \quad \underline{\tilde{Q}}_n = \underline{\underline{\Delta}}_n\underline{\tilde{Q}}, \quad \underline{\tilde{Q}} = \underline{\underline{\Delta}}_n^{-1}\underline{\tilde{Q}}_n \\ &\underline{\tilde{\tau}}_{n+1} = \underline{\underline{D}}_{n,n+1}\underline{\tilde{\tau}}_n, \quad \underline{\tilde{\tau}}_n = \underline{\underline{D}}_{n,n+1}^{-1}\underline{\tilde{\tau}}_{n+1}, \quad \underline{\tilde{\tau}}_n = \underline{\underline{D}}_n\underline{\tilde{\tau}}, \quad \underline{\tilde{\tau}} = \underline{\underline{D}}_n^{-1}\underline{\tilde{\tau}}_n. \end{aligned} \tag{31}$$

Proof. The relationships in the second and fourth columns follow from those of the first and third columns. The relationships in the third column follow from the ones of the first column because of (17) and the recursions (27). The relationships of the first column are to be verified. This can be done by (14′)—(15′) and definitions (16) and $(\underline{\underline{D}}_+)$, $(\underline{\underline{\Delta}}_+)$. By (16)

$$\begin{aligned} \tilde{\vartheta}_{1,n+1} &= \vartheta_{1,n+1} + (B'_{2n}+1)\eta_1 \\ \tilde{\eta}_{2,n+1} &= -\eta_{2,n+1} + D'_{2n+1}\eta_1 \\ \tilde{\vartheta}_{2,n+1} &= -\vartheta_{2,n+1} + (B'_{2n+1} - D'_{2n+1})\eta_1 \\ \tilde{\tau}_{1,n+1} &= \tau_{1,n+1} + B'_{2n}\eta_1 \\ \tilde{\tau}_{2,n+1} &= -\tau_{2,n+1} + B'_{2n+1}\eta_1. \end{aligned}$$

From (14′)—(15′) and (B'), (D'), (16)

$$\begin{aligned} \tilde{\vartheta}_{1,n+1} &= \vartheta_{1,n} - l_{1,n}\eta_{2,n} - l_{1,n}\vartheta_{2,n} + [l_{1,n}B'_{2n-1} + B'_{2n-2} + 1]\eta_1 = \\ &= \vartheta_{1,n} + (B'_{2n-2}+1)\eta_1 + l_{1,n}[-\eta_{2,n} + D'_{2n-1}\eta_1] + l_{1,n}[-\vartheta_{2,n} + (B'_{2n-1} - D'_{2n-1})\eta_1] = \\ &= \tilde{\vartheta}_{1,n} + l_{1,n}\tilde{\eta}_{2,n} + l_{1,n}\tilde{\vartheta}_{2,n}, \end{aligned}$$

$$\begin{aligned}\tilde{\eta}_{2,n+1} &= k_{2,n}\vartheta_{1,n}-(l_{1,n}k_{2,n}+1)\eta_{2,n}-l_{1,n}k_{2,n}\vartheta_{2,n}+[k_{2,n}B'_{2n}+D'_{2n-1}+k_{2,n}]\eta_1 = \\ &= k_{2,n}[\vartheta_{1,n}-l_{1,n}(\eta_{2,n}+\vartheta_{2,n})+(l_{1,n}B'_{2n-1}+B'_{2n-2}+1)\eta_1]-\eta_{2,n}+D'_{2n-1}\eta_1 = \\ &= k_{2,n}[\vartheta_{1,n}+(B'_{2n-2}+1)\eta_1]+(l_{1,n}k_{2,n}+1)[-\eta_{2,n}+D'_{2n-1}\eta_1]+ \\ &\qquad +l_{1,n}k_{2,n}[-\vartheta_{2,n}+(B'_{2n-1}-D'_{2n-1})\eta_1] = \\ &= k_{2,n}\tilde{\vartheta}_{1,n}+(l_{1,n}k_{2,n}+1)\tilde{\eta}_{2,n}+l_{1,n}k_{2,n}\tilde{\vartheta}_{2,n},\end{aligned}$$

$$\begin{aligned}\tilde{\vartheta}_{2,n+1} &= l_{2,n}\eta_1+l_{2,n}\vartheta_{1,n}-l_{1,n}l_{2,n}\eta_{2,n}-(l_{1,n}l_{2,n}+1)\vartheta_{2,n}+[l_{2,n}B'_{2n}+B'_{2n-1}-D'_{2n-1}]\eta_1 = \\ &= l_{2,n}[\eta_1+\vartheta_{1,n}-l_{1,n}(\eta_{2,n}+\vartheta_{2,n})+(l_{1,n}B'_{2n-1}+B'_{2n-2})\eta_1]- \\ &\qquad -\vartheta_{2,n}+(B'_{2n-1}-D'_{2n-1})\eta_1 = \\ &= l_{2,n}[\vartheta_{1,n}+(B'_{2n-2}+1)\eta_1]+l_{1,n}l_{2,n}[-\eta_{2,n}+D'_{2n-1}\eta_1]+ \\ &\qquad +(l_{1,n}l_{2,n}+1)[-\vartheta_{2,n}+(B'_{2n-1}-D'_{2n-1})\eta_1] = \\ &= l_{2,n}\tilde{\vartheta}_{1,n}+l_{1,n}l_{2,n}\tilde{\eta}_{2,n}+(l_{1,n}l_{2,n}+1)\tilde{\vartheta}_{2,n}.\end{aligned}$$

These are exactly the relationship $\underline{\tilde{Q}}_{n+1}=\underline{\underline{A}}_{n,n+1}\underline{\tilde{Q}}_n$. Taken into account that $\tilde{\tau}_{1,n}=\tilde{\vartheta}_{1,n}$ and $\tilde{\tau}_{2,n}=\tilde{\eta}_{2,n}+\tilde{\vartheta}_{2,n}$ and summing up the last two equations, we get the relationship $\underline{\tilde{\tau}}_{n+1}=\underline{\underline{D}}_{n,n+1}\underline{\tilde{\tau}}_n$. □

This theorem is applicable to relate the parameters of a configuration Q and its reduction Q^* if the latter does exist.

3. The priority schedule and the reduction

In our previous article [6] we discussed the so-called consistent economical schedules (CESs) which represent a dominant set. There also the priority schedules were defined and shown as specific CESs. This means that the priority schedules $R_{1,2}$ and $R_{2,1}$ possess all the characteristics every CES possesses. There we illustrated the CESs by graphs which showed the basic characteristics of the CESs such as periodicity, the succession of the so-called typical and critical situations etc. The specific characteristics of $R_{i,3-i}$ $(i=1,2)$ is that no task type A_i can be preempted and, therefore, the job-flow $Q^{(3-i)}$ is always delayed whenever a cycle $C_{3-i,j}$ of it finishes in such a moment when a task type A_i is under service or is ready for service. These are the critical situations type $\sigma_{3-i,1}$ and σ_0, respectively, defined in [6]. The delay can be $0\leqq d_{3-i}\leqq\eta_i$ and after finishing the service of A_i the situation will be the same as the situation after finishing the first task A_{i1}. Since $R_{i,3-i}$ is consistent, the continuation of the servicing process after the two task-finishing points passes off similarly. This means that $R_{i,3-i}$ is *periodic* with a period represented by the schedule section between the two task-finishing points. If $\vartheta_i>0$ then the task $A_{3-i,1}$ begins immediately after the finishing point $t_i'=\eta_i$ of the task A_{i1} in $R_{i,3-i}$. This situation is called *β_i-situation* [5, 6]. This situation returns next to the first delay of $Q^{(3-i)}$ after t_i'. The β_i-situation returns, however, whenever a cycle $C_{3-i,j}$ finishes during the service of a task type A_i if $\vartheta_i>0$. If $\vartheta_i=0$ then the initial situation σ_0 returns at the point t_i' immediately and, because of the consistency, the scheduling of the job-flow $Q^{(i)}$ is repeated. The period consists then of a cycle C_i of $Q^{(i)}$ and the job-flow $Q^{(3-i)}$ fails to be scheduled. The efficiency of $R_{i,3-i}$ will be $\gamma=1$, the possible maximum, if $\eta_i>0$. But this schedule is by no means acceptable in practice. $R_{3-i,i}$ has efficiency $\gamma=1$ as well if $\eta_i>0, \vartheta_i=0$ unless $\tau_{3-i}=0$. If $\tau_i=0$ and $\vartheta_{3-i}>0$, the schedules $R_{1,2}$ and $R_{2,1}$ are degenerated with a finite length and some modification of the scheduling strategy is needed to produce practically acceptable schedules. This problem and generally the scheduling specialities of *degenerate* job-flow pairs (for which $\tau_1\tau_2=0$) were discussed in [4]. In spite of this fact we cannot keep degenerate and *defective* configurations (with zero value parameters) away from further discussion because the reduction of a nondefective configuration Q can lead to defective reduced configuration Q^*.

Confining ourselves to the priority schedules $R_{1,2}(Q)$, $Q\in\mathcal{Q}$, which always start with the service of the task A_{11}, we know that $R_{1,2}(Q)$ is periodic if $\vartheta_1=0$ or the β_1-situation returns. A period is the section of the schedule between the point $t_1'=\eta_1$ and the first recurrence point $T_1^*>t_1'$ of β_1 if $\vartheta_1>0$. $R_{1,2}$ is not periodic if $\vartheta_1>0$ and the recurrence point of β_1 does not exist. In this case $Q^{(2)}$ cannot be delayed out of the starting delay of value η_1 and the preemptions. This means that the finishing times $f(i)$ of the cycles $C_{2,i}$, $i=1, 2, \ldots,$ of $Q^{(2)}$ can be written as

$$f(i) = \eta_1+i\tau_2+\chi(i)\eta_1 \tag{32}$$

where $\chi(i)$ is an integer depending on i, the number of preemptions of the first i C_2-cycles. (32) is valid only until the first recurrence of the β_1-situation. Suppose the β_1-situation recurs first after the μ_2th cycle-finishing point. The length of period p is then the distance between t_1', the start-point of $C_{2,1}$, and T_1^*, the start-point of C_{2,μ_2+1}, which consists of μ_2 demand cycles of $Q^{(2)}$, $\varkappa_2=\chi(\mu_2)$ services of preempting A_1-tasks and the last delay d_2 of $Q^{(2)}$, if any, i.e.

$$p = T_1^*-t_1' = \mu_2\tau_2+\varkappa_2\eta_1+\varepsilon_2\eta_1 \tag{33}$$

where $\mu_2>0, \varkappa_2\geqq 0$ are integers and

$$0 \leqq \varepsilon_2 \leqq 1. \tag{34}$$

In both points t_1' and T_1^* a task type A_1 finishes and, as a result of priority, the service of the job-flow $Q^{(1)}$ goes on continually without break and delay and an integer number of C_1-cycles are serviced in the period between t_1' and T_1^*. Let this number be denoted by μ_1. Thus

$$p = \mu_1\tau_1, \tag{33'}$$

where $\mu_1>0$. Let us call μ_1 and μ_2 the *cycle numbers,* $\varkappa_2$ the *preemption number* and ε_2 the *relative delay*. These are the *characteristics* of $R_{1,2}$ and they are denoted by the quaternary

$$\Pi_{1,2} = (\mu_1;\ \mu_2;\ \varkappa_2;\ \varepsilon_2). \tag{35}$$

If $\vartheta_1=0$ then $R_{1,2}$ will be periodic with $p=\tau_1=\eta_1$ which accords with (33) and (33') if we define the characteristics as

$$\Pi_{1,2} = (1;\ 0;\ 0;\ 1). \tag{35'}$$

Another degenerate case must be discussed yet. This is when $\vartheta_1>0$ and $\tau_2=0$.

Scheduling this configuration with the priority of $Q^{(1)}$ the cycles $C_{2,j}$ with length 0 will be scheduled infinite times after the first, A_{11}, task and the further section of the schedule $R_{1,2}(Q)$ is undefined. Without modification of the strategy the obtained section of $R_{1,2}(Q)$ can be considered as periodic with length $p=0$ and the period consists of a C_2-cycle. In this exceptional case let the characteristics of $R_{1,2}(Q)$ be defined as

$$\Pi_{1,2} = (0; 1; 0; 0). \tag{35''}$$

From definition (1) of the efficiency $\gamma(R)$ of a schedule R the efficiency of a periodic schedule can be obtained as

$$\gamma(R) = \frac{a_R}{p_R} \quad \left(\frac{0}{0} = 0!\right), \tag{1'}$$

where $p_R \geqq 0$ is the *length of the period* of R and $a_R \geqq 0$ is the *P_A-usage* time in a period of R and the quotient is defined as zero if both of a_R and p_R are zeros.

By the characteristics (35) of a priority schedule $R_{1,2}(Q)$ the P_A-usage is composed exactly from the service times of A_1-tasks of number μ_1 and from the service times of A_2-tasks of number μ_2 and, therefore,

$$a_{1,2} = \mu_1\eta_1 + \mu_2\eta_2. \tag{36}$$

We have proved

Theorem 2. *If for any configuration $Q \in \mathcal{Q}$ the priority schedule $R = R_{1,2}(Q)$ is periodic then the length of the period p and the P_A-usage a can be written in the forms*

$$p = \mu_1\tau_1 = \mu_2\tau_2 + (\varkappa_2 + \varepsilon_2)\eta_1, \tag{37}$$

$$a = \mu_1\eta_1 + \mu_2\eta_2, \tag{38}$$

where integers $\mu_1 \geqq 0$, $\mu_2 \geqq 0$, $\varkappa_2 \geqq 0$ and real $0 \leqq \varepsilon_2 \leqq 1$ are the characteristics

$$\Pi = (\mu_1; \mu_2; \varkappa_2; \varepsilon_2)$$

of R with the specialities

Q	μ_1	μ_2	$\varkappa_2$	ε_2
$\vartheta_1 > 0,\ \tau_2 = 0$	0	1	0	0
$\vartheta_1 = 0$	1	0	0	1
$\vartheta_1\tau_2 > 0$	>0	>0	$\geqq 0$	$\in [0, 1]$

(39)

Proof. After the preliminary discussion there is nothing to prove. □

Let us inspect now the influence of the reduction step defined by (2) on the periodicity and the characteristics of a priority schedule $R_{1,2}(Q)$. Denote by

$$(R): \qquad R_n = R_{1,2}(Q_n), \quad n = 0, 1, 2, \ldots,$$

the sequence of priority schedules of the sequence of configurations (Q).

Fig. 3 illustrates the influence of the reduction step $Q_n \to Q_{n+1}$ on the corresponding priority schedules. The transformation $R_n \to R_{n+1}$ defined implicitly is shown in three substeps $R_n \to R_n'$, $R_n' \to R_n''$, $R_n'' \to R_{n+1}$ corresponding to the substeps (2b)—(2d) as transformations $Q_n \to Q_n'$, $Q_n' \to Q_n''$, $Q_n'' \to Q_{n+1}$. This decomposition of the transformation $Q_n \to Q_{n+1}$ corresponds to the factorization (29) of the matrix $\underline{\underline{\Delta}}_{n,n+1}$ of the transformation. The series of configurations in Fig. 3 is $Q_n=(1;\ 15.5;\ 5;\ 7.5)$, $Q_n=(1;\ 3;\ 5;\ 7.5)$, $Q_n=(1;\ 3;\ 2;\ 7.5)$, $Q_{n+1}=$ $=(1;\ 3;\ 2;\ 3.5)$.

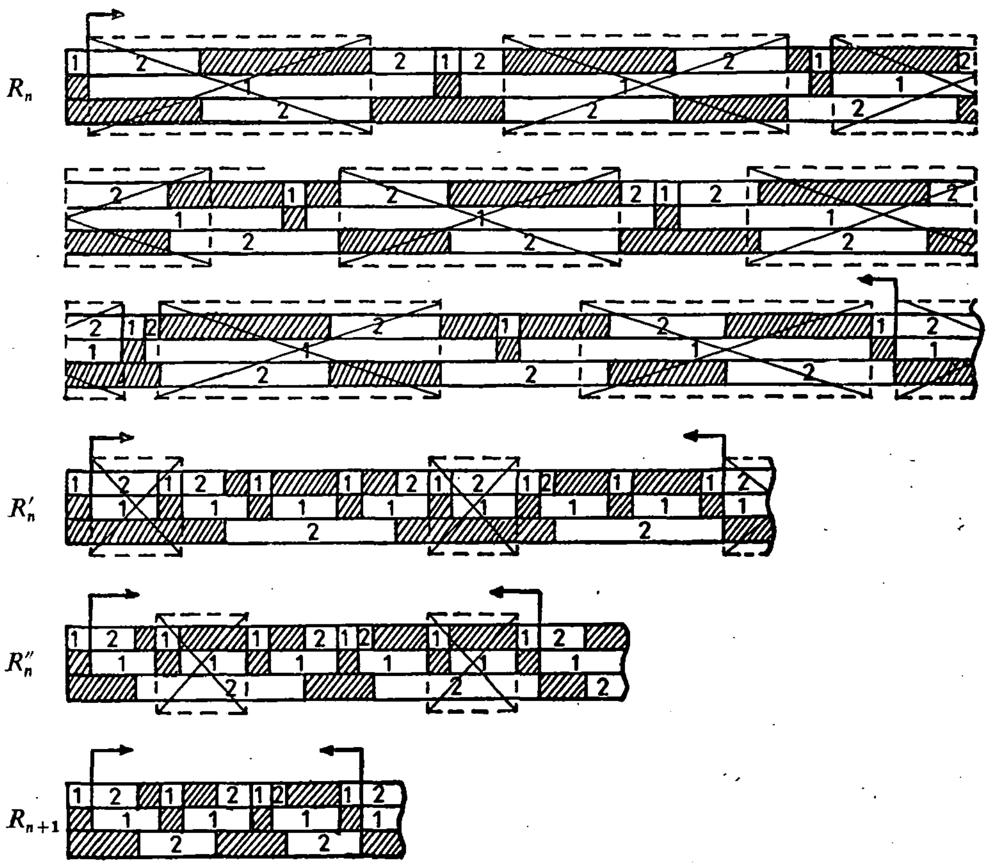

Fig. 3

The influence of the substeps of the reduction $Q_{n+1}=\Delta Q_n$ on the priority schedule $R_{1,2}$

The sequence of R_n, R_n', R_n'', R_{n+1} shows that these schedules are periodic at once and the transformation $Q_n \to Q_{n+1}$ does not influence the existence of periodicity of priority schedules. This means that the members of the sequence (R) are simultaneously periodic or not periodic at all.

Let us introduce the following symbolics. Denote the characteristics of R_n by

$$(\Pi): \qquad \Pi_n = (\mu_{1,n};\ \mu_{2,n};\ \varkappa_{2,n};\ \varepsilon_{2,n}), \quad n = 0, 1, 2, \ldots$$

and let the vectors $\underline{\mu}_n$ and $\underline{\pi}_n$ be defined as

$$(\underline{\pi}):\quad \underline{\mu}_n = \begin{pmatrix} \mu_{1,n} \\ \mu_{2,n} \end{pmatrix}, \quad n = 0, 1, \ldots$$

$$(\underline{\mu}):\quad \underline{\pi}_n = \begin{pmatrix} \mu_{1,n} \\ \mu_{2,n} \\ \varkappa_{2,n} \end{pmatrix}, \quad n = 0, 1, \ldots$$

and let the matrices $\underline{\underline{M}}_n$ and $\underline{\underline{M}}_{n,n+1}$ be defined as

$$(\underline{\underline{M}}):\quad \underline{\underline{M}}_n = \begin{pmatrix} B_{2n-2} & A_{2n-2} & B'_{2n-2} \\ B_{2n-1} & A_{2n-1} & B'_{2n-1} \\ 0 & 0 & 1 \end{pmatrix}, \quad n = 0, 1, \ldots$$

$$(\underline{\underline{M}}_+):\quad \underline{\underline{M}}_{n,n+1} = \begin{pmatrix} 1 & l_{1,n} & 0 \\ k_{2,n}+l_{2,n} & l_{1,n}(k_{2,n}+l_{2,n})+1 & k_{2,n} \\ 0 & 0 & 1 \end{pmatrix}, \quad n = 0, 1, \ldots$$

Lemma 6. *For the matrices* $(\underline{\underline{M}})$ *and* $(\underline{\underline{M}}_+)$ *the following relationships hold for* $n=0, 1, \ldots$

$$\underline{\underline{M}}_{n+1} = \underline{\underline{M}}_{n,n+1}\underline{\underline{M}}_n, \quad \text{with} \quad \underline{\underline{M}}_0 = \underline{\underline{I}} \tag{40}$$

$$\underline{\underline{M}}_n^{-1} = \begin{pmatrix} A_{2n-1} & -A_{2n-2} & C_{2n-1} \\ -B_{2n-1} & B_{2n-2} & -D_{2n-1} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{M}}_{n,n+1}^{-1} = \begin{pmatrix} l_{1,n}(k_{2,n}+l_{2,n})+1 & -l_{1,n} & l_{1,n}k_{2,n} \\ -(k_{2,n}+l_{2,n}) & 1 & -k_{2,n} \\ 0 & 0 & 1 \end{pmatrix} \tag{41}$$

$$\underline{\underline{M}}_{n,n+1} = \begin{pmatrix} 1 & 0 & 0 \\ l_{2,n} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ k_{2,n} & 1 & k_{2,n} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & l_{1,n} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{M}}_{n,n+1}^{-1} = \begin{pmatrix} 1 & -l_{1,n} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -k_{2,n} & 1 & -k_{2,n} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -l_{2,n} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{42}$$

The determinant $\det(\underline{\underline{X}})$ *for every matrix encountered above is*

$$\det(\underline{\underline{X}}) = 1. \tag{43}$$

Proof. (40) can be verified by executing the matrix production and using the definitions of (A), (B), (B'). The verification of (41) is easy by multiplying the matrices with their inverses and using (20)—(25). The factorizations (42) are obvious by executing the multiplications. (43) is trivial. □

Now we prove our main result.

Theorem 3. *For any configuration $Q \in \mathcal{Q}$ the whole sequence (R) of priority schedules of the sequence of configurations (Q) is periodic at once and the following relationships hold among the members of the sequence (Π) of characteristics:*

$$\varepsilon_{2,n} = \varepsilon_2 \tag{44}$$

and

$$\begin{aligned} &\underline{\mu}_{n+1} = \underline{\underline{D}}_{n,n+1}^{-T}\underline{\mu}_n, \quad \underline{\mu}_n = \underline{\underline{D}}_{n,n+1}^{T}\underline{\mu}_{n+1}, \quad \underline{\mu}_n = \underline{\underline{D}}_n^{-T}\underline{\mu}, \quad \underline{\mu} = \underline{\underline{D}}_n^{T}\underline{\mu}_n \\ &\underline{\pi}_{n+1} = \underline{\underline{M}}_{n,n+1}^{-T}\underline{\pi}_n, \quad \underline{\pi}_n = \underline{\underline{M}}_{n,n+1}^{T}\underline{\pi}_{n+1}, \quad \underline{\pi}_n = \underline{\underline{M}}_n^{-T}\underline{\pi}, \quad \underline{\pi} = \underline{\underline{M}}_n^{T}\underline{\pi}_n \end{aligned} \tag{45}$$

for $n=0, 1, 2, \ldots,$ *where* $\underline{\underline{X}}^{-T}$ *denotes the transpose of the inverse of matrix* $\underline{\underline{X}}$.

Proof. The second and fourth columns of (45) follow from the first and third. The first line follows from the second because the D-matrices are the 2×2 submatrices of the M-matrices as their definitions show. The relationships of the third column follow from the ones of the first in consequence of (27) and (40). The first relationship of the first line of (45) remains to be proved with (44). To go on with the proof we need the following triads.

Define

$$\varphi(i) = \left[\frac{f(i)}{\tau_1}\right] \quad \text{and} \quad \varrho(i) = f(i) - \varphi(i)\tau_1, \quad i = 1, 2, \ldots \tag{46}$$

as *moduli* and *residua* of the cycle-finishing times $f(i)$ of $Q^{(2)}$.

$$\varrho(i) \equiv f(i) \pmod{\tau_1} \quad \text{and} \quad 0 \leqq \varrho(i) < \tau_1. \tag{47}$$

For the cycle-finishing times the decomposition (32) is true until the first recurrence of the β_1-situation. Substituting this into $\varrho(i)$ in (46) we get

$$\varrho(i) = \eta_1 + i\tau_2 + \chi(i)\eta_1 - \varphi(i)\tau_1. \tag{48}$$

The *triads*

$$H(i) = \big(\varphi(i), i, \chi(i)\big), \quad i = 1, 2, \ldots$$

for Q are determined by the priority schedule $R = R_{1,2}(Q)$. We saw earlier that the periodicity of R is true if for a finite i there exists a triad $H(i)$ for which

$$0 \leqq \varrho(i) \leqq \eta_1,$$

because the β_1-situation recurs exactly in this case. The length p of the period is determined by the first such i and $H(i)$ because the first recurrence point T_1^* of the β_1-situation is the A_1-task-finishing point next $f(i)$ which is by time $\eta_1 - \varrho(i)$ later than $f(i)$, that is

$$T_1^* = f(i) + \eta_1 - \varrho(i).$$

From this

$$p = T_1^* - t_1' = f(i) - \varrho(i) = \eta_1 + i\tau_2 + \chi(i)\eta_1 - \varrho(i).$$

On the other hand

$$p = \varphi(i)\tau_1 = i\tau_2 + \big(\chi(i) + \varepsilon_2\big)\eta_1$$

from which

$$\varrho(i) = (1 - \varepsilon_2)\eta_1 \quad \text{and} \quad \varepsilon_2 = 1 - \varrho(i)/\eta_1.$$

We have got that R is periodic if and only if there exists a triad $H(i)$ for which

$$0 \leqq \varepsilon_2\eta_1 = \varphi(i)\tau_1 - i\tau_2 - \chi(i)\eta_1 \leqq \eta_1. \tag{49}$$

Since the member of triads determined by R are monotonic with each other, there exists a unique minimum i satisfying (49). Let

$$\mu_1 = \varphi(i), \quad \mu_2 = i, \quad \varkappa_2 = \chi(i), \quad \varepsilon_2 = 1 - \varrho(i)/\eta_1 \tag{49'}$$

with this i. Then the so defined Π_n are the characteristics of R_n. $\mu_{2,n}$ is the minimum value of i for which (49) holds for R_n, i.e.

$$0 \leqq \mu_{1,n}\tau_{1,n} - \mu_{2,n}\tau_{2,n} - \varkappa_{2,n}\eta_1 = \varepsilon_{2,n}\eta_1 \leqq \eta_1. \tag{50}$$

Let us see the first substep $Q_n \to Q_n'$. Substitute from (2b) $\tau_{1,n} = l_{1,n}\tau_{2,n} + \tau_{1,n+1}$ into (50) and we get

$$0 \leqq \mu_{1,n}\tau_{1,n+1} - (\mu_{2,n} - l_{1,n}\mu_{1,n})\tau_{2,n} - \varkappa_{2,n}\eta_1 = \varepsilon_{2,n}\eta_1 \leqq \eta_1. \tag{50'}$$

This means that

$$H_n' = (\mu_{1,n}, \mu_{2,n} - l_{1,n}\mu_{1,n}, \varkappa_{2,n})$$

is a triad for $R_n' = R_{1,2}(Q_n')$ for which (49) holds. Because the correspondence between parameters of Q_n and Q_n' is unique, H_n' must also be the minimum triad for which (49) holds. This means that the characteristics of R_n' are

$$\mu_{1,n}' = \mu_{1,n}, \quad \mu_{2,n}' = \mu_{2,n} - l_{1,n}\mu_{1,n}, \quad \varkappa_{2,n}' = \varkappa_{2,n}, \quad \varepsilon_{2,n}' = \varepsilon_{2,n}.$$

The matrix of this transformation is the transpose of the first factor of $\underline{\underline{M}}_{n,n+1}^{-1}$ in (42).

Substitute now the expression $\eta_{2,n} = k_{2,n}\vartheta_{1,n+1} + \eta_{2,n+1}$ from (2c) into (50') correspondingly to the transformation $Q_n' \to Q_n''$. We obtain unambiguously the inequality

$$0 \leqq (\mu_{1,n}' - k_{2,n}\mu_{2,n}')\tau_{1,n+1} - \mu_{2,n}'(\eta_{2,n+1} + \vartheta_{2,n}) - (\varkappa_{2,n}' - k_{2,n}\mu_{2,n}')\eta_1 = \varepsilon_{2,n}\eta_1 \leqq \eta_1. \tag{50''}$$

This means that

$$H_n'' = (\mu_{1,n}' - k_{2,n}\mu_{2,n}', \mu_{2,n}', \varkappa_{2,n}' - k_{2,n}\mu_{2,n}')$$

is the unique minimum triad for Q_n'' for which (49) holds and, therefore

$$\mu_{1,n}'' = \mu_{1,n}' - k_{2,n}\mu_{2,n}', \quad \mu_{2,n}'' = \mu_{2,n}', \quad \varkappa_{2,n}'' = \varkappa_{2,n}' - k_{2,n}\mu_{2,n}', \quad \varepsilon_{2,n}'' = \varepsilon_{2,n}'.$$

The matrix of this transformation is the transpose of the second factor of $\underline{\underline{M}}_{n,n+1}^{-1}$ in (42).

At last we substitute the expression $\vartheta_{2,n} = l_{2,n}\tau_{1,n+1} + \vartheta_{2,n+1}$ from (2d) into (50'') correspondingly to the transformation $Q_n'' \to Q_{n+1}$. We obtain the inequality

$$0 \leqq (\mu_{1,n}'' - l_{2,n}\mu_{2,n}'')\tau_{1,n+1} - \mu_{2,n}''\tau_{2,n+1} - \varkappa_{2,n}''\eta_1 = \varepsilon_{2,n}''\eta_1 \leqq \eta_1.$$

In consequence of the uniqueness of the transformation $Q_n'' \to Q_{n+1}$ and the minimum triads for their $R_{1,2}$-schedules we get

$$\mu_{1,n+1} = \mu_{1,n}'' - l_{2,n}\mu_{2,n}'', \quad \mu_{2,n+1} = \mu_{2,n}'', \quad \varkappa_{2,n+1} = \varkappa_{2,n}'', \quad \varepsilon_{2,n+1} = \varepsilon_{2,n}''$$

as the characteristics of R_{n+1}. The matrix of this transformation is the transpose of the third factor of $\underline{\underline{M}}_{n,n+1}^{-1}$ in (42). This proves the theorem. $\square$

Fig. 3 illustrates the course of the proof.

Theorem 3 makes it possible to determine the characteristics Π of $R=R_{1,2}(Q)$ from the characteristics Π^* of $R^*=R_{1,2}(Q^*)$ if Q is reducible, R^* is periodic and Π^* is known. The question of reducibility was discussed in the previous section. The characteristics of reduced configurations will be inspected in the next two sections.

4. Priority schedules of specific configurations

We saw in the proof of Theorem 3 that the periodicity of a priority schedule $R=R_{1,2}(Q)$ depends on the fact whether there exists a triad $H(i)$ satisfying (49). This is not equivalent to the existence of an integer solution of the inequality

$$0 \leqq \mu_1\tau_1-\mu_2\tau_2-\varkappa_2\eta_1 \leqq \eta_1 \tag{51}$$

because not every triple $(\mu_1, \mu_2, \varkappa_2)$ satisfying this inequality is a triad defined by (32), (46)—(49) on a schedule $R_{1,2}(Q)$. Unfortunately, we do not know analytic conditions for the triads instead of the fact that its elements represent the number of C_1-cycles, C_2-cycles and preemptions, respectively, until the C_2-cycle finishing points of $R_{1,2}(Q)$. The triads and (51) cannot be used, therefore, to decide the periodicity and determine the characteristics of a priority schedule $R_{1,2}(Q)$. This circumstance raises the significance of results on characteristics for some specific configurations $Q\in\mathcal{Q}$ including reduced ones.

The characteristics of $R_{1,2}(Q)$ were made clear for configurations for which $\vartheta_1\tau_2=0$ in Theorem 2. We suppose that

$$\vartheta_1\tau_2>0. \tag{52}$$

We can make clear the special cases in which (9), the condition $\eta_1\vartheta_2=0$ for Q is true. Let first $\eta_1=0$. Since $Q^{(1)}$ do not delay the service of $Q^{(2)}$ in this case, we can determine the condition of periodicity of $R_{1,2}(Q)$ as ϑ_1 and τ_2 are rationally dependent. This is illustrated in Fig. 4a.

Independently of the value of η_1, we can easily determine the condition of $R_{1,2}(Q)$ to be periodic for $Q\in\mathcal{Q}$ with $\vartheta_2=0$ (but $\vartheta_1\tau_2>0$!). This condition is that

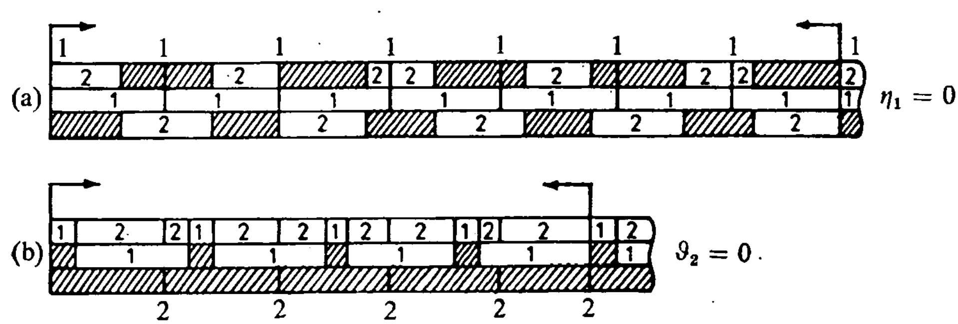

Fig. 4

$R_{1,2}(Q)$ schedules for specific configurations with $\vartheta_1\tau_2>0$, $\eta_1\vartheta_2=0$

ϑ_1 and η_2 are rationally dependent, which is the same condition as in case $\eta_1=0$. The values of the characteristics of the periodic schedule $R_{1,2}(Q)$ are, obviously, determined by the relation of ϑ_1 and τ_2 according to

Theorem 4. *For the configurations $Q\in\mathcal{Q}$ with*

$$\vartheta_1\tau_2>0, \quad \eta_1\vartheta_2=0 \tag{53}$$

the priority schedule $R=R_{1,2}(Q)$ is periodic iff ϑ_1 and τ_2 are rationally dependent. If

$$\frac{\vartheta_1}{\tau_2}=\frac{A}{B}, \tag{54}$$

with relatively prime integers $A, B>0$, then the characteristics of R are

$$\Pi=\left(B;\, A;\, f_<\left(\frac{\eta_2}{\tau_2}B\right);\, 1\right), \tag{55}$$

where $f_<(x)$ is the greatest integer less than x.

Proof. Fig. 4 shows that $\mu_1=B$, $\mu_2=A$ if (54) holds because (B, A) is the least integer solution of the equation $x\vartheta_1-y\tau_2=0$. Since $\varrho(A)=0$, therefore, $\varepsilon_2=1$ from the relationship (49′) if $\eta_1>0$ and $\varepsilon_2=1$ can be considered as a convention if $\eta_1=0$. If $\vartheta_2=0$ then every A_1-task but the first in the period is a preempting one and, therefore, $\varkappa_2=B-1=\left[\frac{\eta_2}{\tau_2}B\right]-1$. In case $\eta_1=0$ the $A_{1,j}$ task is preempting if $i\tau_2<j\vartheta_1<i\tau_2+\eta_2$ for some integer $i\geqq 0$ (see Fig. 4a). This means that $i<j\vartheta_1/\tau_2<i+\eta_2/\tau_2$ and using (54) we get $i<jA/B<i+\eta_2/\tau_2$, i.e.

$$0<\left\{j\frac{A}{B}\right\}<\frac{\eta_2}{\tau_2}, \tag{$*$}$$

where $\{x\}$ denotes the fractional part of x. It is well known [4] that the numbers $\{jA/B\}$, $j=0, 1, \dots, B-1$, go through the points k/B, $k=0, 1, \dots, B-1$, of the interval $[0, 1)$ in some order. This means that for $j=1, 2, \dots, B$, the inequality takes place as many times as many of the points k/B are in the interval $(0, \eta_2/\tau_2)$. This number is $[(\eta_2/\tau_2)/(1/B)]$ if $(\eta_2/\tau_2)/(1/B)$ is not an integer and is $(\eta_2/\tau_2)/(1/B)-1$ if this is an integer. This number is exactly $f_<((\eta_2/\tau_2)B)$. □

Lemma 3 establishes that every configuration Q becomes reduced or defective with (53) after a finite number $\nu'\geqq 0$ of application of the operator Δ to it. Theorem 4 means that after finite $\nu'\geqq 0$ times application of Δ we can reduce Q or decide whether its schedule $R_{1,2}(Q)$ is periodic. We show that Q with (53) is reducible when $R_{1,2}(Q)$ is periodic, i.e. ϑ_1 and τ_2 are rationally dependent.

Lemma 7. *The configurations $Q\in\mathcal{Q}$ with* (53) *are reducible iff* (54) *is true except eventually the case $\eta_1=0$ in which Q can be reducible with rationally independent ϑ_1 and τ_2 as well.*

Proof. If $\vartheta_2=0$ then the reduction procedure is equivalent to the regular continued fraction expansion of the number $\xi=\vartheta_1/\eta_2$ and is finite exactly when

ξ is rational and so (54) holds (see also the proof of the Lemma 4). Let now $\vartheta_2>0$ and $\eta_1=0$. If Q is not reducible then neither $\vartheta_{1,n}$ nor $\eta_{2,n}+\vartheta_{2,n}$ of $Q_n=\Delta^n Q$, $n\geqq 0$, is zero by Lemma 4. If $\eta_{2,n}\vartheta_{2,n}=0$ for some finite $n\geqq 0$ then the reducibility is equivalent to the validity of (54) by the same lemma.

Let, therefore, $\vartheta_{1,n}\eta_{2,n}\vartheta_{2,n}>0$, $n=0,1,\ldots$. Suppose Q is not reducible. This means that the series (λ) has infinite length and has no zero element after $\lambda_0=l_{1,0}$. This means that $l_{1,n}>0, n\geqq 1$. From (2b) we conclude then that $0<\vartheta_{1,n+1}<$ $<\tau_{2,n}<\vartheta_{1,n}$, $n=1,2,\ldots$, which means that

$$\xi_{2i}=\frac{\vartheta_{1,i}}{\tau_{2,i}}>1 \quad \text{if} \quad i>0, \quad \xi_{2i+1}=\frac{\tau_{2,i}}{\vartheta_{1,i+1}}>1 \quad \text{if} \quad i\geqq 0,$$

and (2) is equivalent to the definition of series

$$\xi_n=\lambda_n+\frac{1}{\xi_{n+1}}, \quad n=0,1,\ldots,$$

where $0<1/\xi_{n+1}<1$ and, consequently, $\lambda_n=[\xi_n]$. This is, however, exactly the definition of the Euclidean algorithm of the regular continued fraction expansion of the number $\xi_0=\vartheta_{1,0}/\tau_{2,0}=\vartheta_1/\tau_2$. This algorithm is infinite exactly when ξ_0 is an irrational number, i.e. (54) does not hold [3]. If (54) is true, Q must be reducible. If (54) does not hold but $\eta_1=0$ then Q can be reducible as for instance $Q=(0;\ 1;\ \pi/2;\ \pi/2)$ shows for which ξ_0 is irrational but $\nu=1$ and $Q^*=$ $=(0;\ 1;\ \pi/2-1;\ \pi/2-1)$. □

From Lemma 7 we can conclude that the question of periodicity of $R_{1,2}(Q)$ remained unanswered in cases in which Q is reducible and for its reduction Q^*

$$\vartheta_1^*\tau_2^*>0, \quad \eta_1\vartheta_2^*>0. \tag{56}$$

In all other cases reducibility and periodicity are equivalent except the case $\eta_1=0$, ϑ_1^* and τ_2^* are rationally independent, in which case the periodicity is not true.

We now show that in case (56) the schedule $R_{1,2}(Q)$ is periodic if $\tau_1^*\geqq\tau_2^*$.

Theorem 5. *If the configuration $Q\in\mathcal{Q}$ is reducible and for its reduction $Q^*=Q_\nu$ the relations*

$$\tau_1^*\geqq\tau_2^*>\vartheta_1^*>0 \tag{57}$$

hold then the priority schedule $R_{1,2}(Q)$ of Q is periodic with characteristics

$$\Pi=\left(\mu_1;\ \mu_2;\ \varkappa_2;\ \frac{\tau_1^*-\tau_2^*}{\eta_1}\right) \tag{58}$$

with

$$\begin{aligned}\mu_1&=B_{2\nu-2}+B_{2\nu-1}\\ \mu_2&=A_{2\nu-2}+A_{2\nu-1}\\ \varkappa_2&=B'_{2\nu-2}+B'_{2\nu-1}\end{aligned} \tag{59}$$

where ν is the degree of compositeness of Q. μ_1 and μ_2 are relatively prime integers.

Proof. First of all $\eta_1>0$ follows from (57) because the reducedness of Q^* implies $\vartheta_2^*<\tau_1^*$ if $\tau_1^*>0$ by (5c). From $\vartheta_1^*>0$ and (5b) it follows that $0\leqq\eta_2^*\leqq\vartheta_1^*$ and, therefore, the characteristics of $R^*=R_{1,2}(Q^*)$ cannot be else than

$$\Pi^* = \left(1;\,1;\,0;\,\frac{\tau_1^*-\tau_2^*}{\eta_1}\right) \tag{58'}$$

which is the special case of (58) with $\nu=0$ in (59). This fact can be verified most simply on the Gantt-chart of R^* as in Fig. 5. (59) follows then from Theorem 3

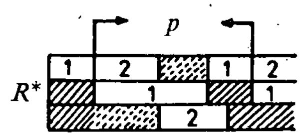

Fig. 5

The $R_{1,2}(Q^*)$ schedule for a reduced configuration with $\tau_1^*\geqq\tau_2^*>\vartheta_1^*>0$

applied for $n=\nu$ and entities $x^*=x_\nu$. By the last relationship of (45), $\underline{\pi}=\underline{\underline{M}}_\nu^T\underline{\pi}^*$ and in detailed form

$$\begin{pmatrix}\mu_1\\ \mu_2\\ \varkappa_2\end{pmatrix} = \begin{pmatrix}B_{2\nu-2} & B_{2\nu-1} & 0\\ A_{2\nu-2} & A_{2\nu-1} & 0\\ B'_{2\nu-2} & B'_{2\nu-1} & 1\end{pmatrix}\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}$$

which is (59). $\varepsilon_2=\varepsilon_2^*$ follows from (44).

Applying $\underline{\mu}^*=\underline{\underline{D}}_\nu^{-T}\underline{\mu}$ obtained from (45) for $n=\nu$, we get from (28) the relationships $1=A_{2\nu-1}\mu_1-B_{2\nu-1}\mu_2$ and $1=-A_{2\nu-2}\mu_1+B_{2\nu-2}\eta_2$ and from (21) that μ_1 and μ_2 cannot have common divisors other than ± 1. □

After this theorem the only questionable case remained is the set of configurations reducible to Q^* with

$$0<\eta_1<\tau_1^*<\tau_2^*. \tag{60}$$

The domain (60) of $\mathcal{Q}$ is the part of the domain (δ) in Fig. 2d and is illustrated in Fig. 6. We will further investigate this case in the next section.

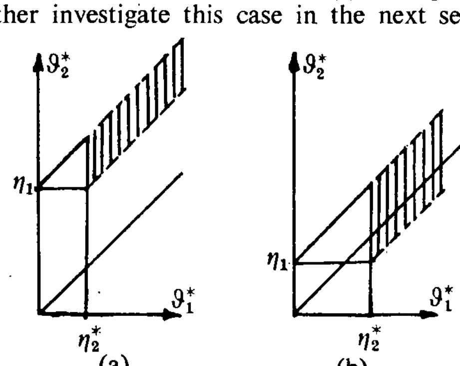

Fig. 6

The domain of reduced configurations with $0<\eta_1<\tau_1^*<\tau_2^*$

Supposing that R^* is periodic, some relations among its characteristics can be stated. These follow from the following more general Lemma 8. We need some simple definitions. Let $s(X)$ and $f(X)$ denote the *start* and *finishing point* of the service of a task or cycle X, respectively. We say that task A *starts during* task B if $s(B)\leqq s(A)\leqq f(B)$ and task A *runs during* task B if $s(B)\leqq s(A)$ and $f(A)\leqq f(B)$. Let u denote the number of task type A_1 in a period of $R_{1,2}(Q)$ which *do not preempt* task type A_2.

Lemma 8. *For the characteristics Π and u of a periodic priority schedule $R=R_{1,2}(Q)$ the following assertions are true:*

$$\mu_1 = u+\varkappa_2; \tag{61}$$

$$u = \mu_2, \quad \mu_1 = \mu_2+\varkappa_2 \tag{62}$$

iff exactly one A_1-task starts during every B_2-task;

$$\begin{aligned} &\text{(a)} \quad u \leqq \mu_2, \quad \mu_1 \leqq \mu_2+\varkappa_2 \quad \textit{if} \quad \vartheta_2 < \tau_1,\\ &\text{(b)} \quad u \geqq \mu_2, \quad \mu_1 \geqq \mu_2+\varkappa_2 \quad \textit{if} \quad \vartheta_1 \leqq \vartheta_2,\\ &\text{(c)} \quad u = \mu_2, \quad \mu_1 = \mu_2+\varkappa_2 \quad \textit{if} \quad \vartheta_1 \leqq \vartheta_2 < \tau_1; \end{aligned} \tag{63}$$

$$\begin{aligned} &\text{(a)} \quad \mu_1 \geqq \mu_2+1 \quad &&\textit{if} \quad \tau_1 < \tau_2,\\ &\text{(b)} \quad \mu_2 \geqq \varkappa_2+1 \quad &&\textit{if} \quad \eta_2 \leqq \vartheta_1, \quad \vartheta_1 > 0,\\ &\text{(c)} \quad \mu_1 > \mu_2 > \varkappa_2 \geqq 0 \quad &&\textit{if} \quad \eta_2 \leqq \vartheta_1 \leqq \tau_1 < \tau_2, \quad \vartheta_1 > 0; \end{aligned} \tag{64}$$

$$\varkappa_2 \geqq 1 \quad \textit{if} \quad \vartheta_2 < \tau_1 < \tau_2, \quad \vartheta_1 > 0; \tag{65}$$

$$\mu_1 \geqq 3, \quad \mu_2 \geqq 2, \quad \varkappa_2 \geqq 1 \quad \textit{if} \quad \eta_2 \leqq \vartheta_1, \quad \vartheta_1 > 0, \quad \vartheta_2 < \tau_1 < \tau_2. \tag{66}$$

Proof. (61) follows from the definition of u and $\varkappa_2$. $u=\mu_2$ in (62) is clearly true if exactly one A_1-task starts during every B_2-task because these A_1-tasks are those which do not cause preemption. The number of B_2-tasks in a period is μ_2. Suppose $u=\mu_2$ and there exists a B_2-task during which more than one A_1-tasks start. This is possible only if $\tau_1\leqq\vartheta_2$, and so $\vartheta_1\leqq\vartheta_2$. But at least one A_1-task must start during every B_2-task if $\vartheta_1\leqq\vartheta_2$ and, therefore, we get $u\geqq\mu_2+1$, which proves (63b) but contradicts $u=\mu_2$. If we suppose that no A_1-task starts during some B_2-task in the period of R, it follows that $\vartheta_2<\vartheta_1$ must hold. But if $\vartheta_2<\tau_1$ then no B_2-task during which more than one A_1-tasks start exists and, therefore, $u\leqq\mu_2-1$, proving (63a) but contradicting $u=\mu_2$. This proves (62), and (63a) and (63b) involve (63c).

To prove (64a) we use Theorem 2. From (37) $(\mu_1-\mu_2)\tau_1=\mu_2(\tau_2-\tau_1)+(\varkappa_2+\varepsilon_2)\eta_1$ and $\mu_1>\mu_2$ follow if $\tau_2>\tau_1$ and $\mu_2>0$. But $\mu_2>0$ follows from $\vartheta_1>0$ by (39). If $\vartheta_1=0$ then $\mu_1=1>\mu_2=0$ by (39). If $\eta_2\leqq\vartheta_1$ then no A_2-task can exist which is preempted more than once and, therefore, $\varkappa_2\leqq\mu_2$. If $\vartheta_1>0$ then the first $A_{2,1}$ task is serviced without preemption as soon as $\eta_2\leqq\vartheta_1$. Therefore, $\varkappa_2\leqq\mu_2-1$, as (64b) asserts. (64a) and (64b) imply (64c).

To prove (65) we consider the last B_2-task in the first period of R which precedes the recurrence point T_1^* of the β_1-situation. This task finishes in the interval $[T_1^*-\eta_1, T_1^*]$ as Fig. 7 shows. The period ends with the service of an A_1-task. The last B_2-task cannot start before the preceding A_1-task because $\vartheta_2\geqq\tau_1$ would follow

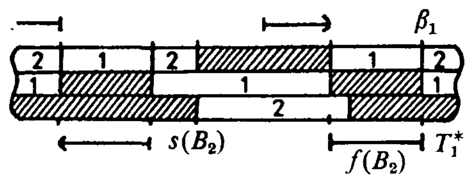

Fig. 7
Illicit intervals for the last B_2-task starting point $s(B_2)$ if $\vartheta_2<\tau_1<\tau_2$, $\vartheta_1>0$

in this case. This B_2-task cannot start, however, η_2 later than the preceding A_1-task finishing because $\vartheta_2\leqq\tau_1-\eta_2$ and $\tau_2\leqq\tau_1$ would follow. This means that $\vartheta_2<\tau_1<\tau_2$ implies that the last B_2-task starts after the preceding A_1-task but the previous A_2-task cannot be serviced without preemption and so $\varkappa_2\geqq 1$. (66) follows from (64c) and (65). □

Before we turn to the case (60), we prove two theorems which give the characteristics of $R_{1,2}(Q)$ for configurations not necessarily reduced but representing (58′) as their special case.

Theorem 6. *If for the configuration* $Q\in\mathcal{Q}$

$$\vartheta_1>0 \quad and \quad \vartheta_2<\eta_1 \tag{67}$$

hold then $R_{1,2}(Q)$ is periodic. Its characteristics are

$$\Pi=\left(A;\,B;\,A-1;\,1-\frac{\Delta\vartheta_1}{\eta_1}\right) \tag{68}$$

where $\omega=(B,A)$ is the least solution of the coincidence problem

$$0\leqq B\xi-A\leqq\alpha,\quad \omega\geqq(1,0) \tag{69}$$

and

$$\Delta=B\xi-A \tag{70}$$

is its error, where

$$\xi=\frac{\tau_2}{\vartheta_1},\quad \alpha=\frac{\vartheta_2}{\vartheta_1}. \tag{71}$$

The cycle numbers μ_1 and μ_2 are relatively prime integers.

Proof. An A_1-task causing no preemption starts during a B_2-task. Since $\eta_1>\vartheta_2$, this A_1-task must finish later than the B_2-task and cause a recurrence of the β_1-situation. Only one such A_1-task can exist in every period. Therefore, $\varkappa_2=\mu_1-1$ if $R_{1,2}(Q)$ is periodic. The condition of the periodicity is the recurrence of the β_1-situation and the existence of μ_1 and $\mu_2>0$ fulfilling the inequality

$$0\leqq\eta_1+\mu_2\tau_2+(\mu_1-1)\eta_1-\mu_1\tau_1\leqq\vartheta_2.$$

The cycle numbers represent the least solution of this inequality which is equivalent to the inequality $0\leqq\mu_2\tau_2-\mu_1\vartheta_1\leqq\vartheta_2$ and this to (69) with $\mu_2=B$, $\mu_1=A$ and (71). The coincidence problem (69) always has a unique least solution (B,A) because $\alpha>0$ and this solution represents a pair of relatively prime integers [4]. □

In the special case $0<\eta_2^*\leqq\vartheta_1^*<\tau_2^*$ of (67) $\xi>\alpha$ but $0\leqq\xi-1\leqq\alpha$ and, therefore, the solution of (69) is $\omega=(1,1)$ with $\Delta=\xi-1=\tau_2^*/\vartheta_1^*-1$ and $\Pi=\left(1;\ 1;\ 0;\ \frac{\tau_1^*-\tau_2^*}{\eta_1}\right)$ from (68), correspondingly to (58′).

Theorem 7. *If for the configuration* $Q\in\mathcal{Q}$

$$\eta_1\vartheta_1\vartheta_2>0,\quad \eta_2=0 \tag{72}$$

holds then $R_{1,2}(Q)$ *is periodic. Its characteristics are*

$$\Pi=\left(B;\ A;\ 0;\ \frac{\Delta\vartheta_2}{\eta_1}\right) \tag{73}$$

where $\omega=(B,A)$ *is the least solution of the coincidence problem* (69) *with error* (70) *where now*

$$\xi=\frac{\tau_1}{\vartheta_2},\quad \alpha=\frac{\eta_1}{\vartheta_2}. \tag{74}$$

The cycle numbers μ_1 *and* μ_2 *are relatively prime integers.*

Proof. Because of $\eta_2=0$, preemption cannot exist in $R_{1,2}(Q)$ and $R_{1,2}(Q)$ is periodic if and only if B_2-tasks finishing during A_1-tasks exist. This is the condition of the recurrence of the β_1-situation. Such a B_2-task exists iff integers $B>0$, $A>0$ exist such that

$$B\tau_1\leqq\eta_1+A\vartheta_2\leqq B\tau_1+\min(\eta_1,\vartheta_2)$$

holds. The least $\omega=(B,A)$ supplies μ_1 and μ_2, respectively. This inequality is equivalent to

$$\eta_1-\min(\eta_1,\vartheta_2)\leqq B\tau_1-A\vartheta_2\leqq\eta_1.$$

The left side is positive if $\eta_1>\vartheta_2$. In this case the least $\omega=(B,A)$ satisfying the inequality is $\omega=(1,f_{\geqq}(\vartheta_1/\vartheta_2))$ where $f_{\geqq}(x)$ is the least integer not less than x. Namely, from $x\leqq f_{\geqq}(x)<x+1$ the inequality $\eta_1-\vartheta_2<\tau_2-f_{\geqq}(\vartheta_1/\vartheta_2)\vartheta_2\leqq\leqq\tau_1-\vartheta_1=\eta_1$ follows. This ω is the least solution of (69) with (74) as well. (69) always has a solution because of $\alpha>0$, and the least solution is a relatively prime integer pair [4]. The values of μ_1,μ_2 and $\varkappa_2$ in (73) are proved. Obviously, $\varepsilon_2\eta_1==\Delta\vartheta_2$ from which the value of ε_2 in (73) follows. □

If (57) holds, i.e. $0<\vartheta_1^*<\vartheta_2^*\leqq\tau_1^*$ is true then the least solution of (69) with (74) is $\omega=(1,1)$ and $\Delta\vartheta_2^*=\tau_1^*-\vartheta_2^*=\tau_1^*-\tau_2^*$. (73) gives (58′) as a special case.

5. The case $0<\tau_1^*<\tau_2^*$

We did not find conditions for a reduced configuration Q^* with (60) to have a periodic schedule $R^*=R_{1,2}(Q^*)$. This case requires further investigation. By (60) and condition (5) we can write

$$0<\eta_1^*<\tau_1^*<\tau_2^*,\quad \eta_2^*\leqq\vartheta_1^*,\quad \vartheta_2^*<\tau_1^*. \tag{75}$$

This is equivalent to the two series of inequalities

$$
\begin{aligned}
&0 < \eta_2^* \leqq \vartheta_1^* < \tau_1^* < \tau_2^* < \eta_2^* + \tau_1^* \\
&0 < \eta_1^* < \vartheta_2^* < \tau_1^* < \tau_2^* \leqq \vartheta_1^* + \vartheta_2^* < \vartheta_1^* + \tau_1^*.
\end{aligned} \tag{76}
$$

These relations do not determine the relations between η_i^* and ϑ_i^*, η_1^* and η_2^*, or ϑ_1^* and ϑ_2^* if $\eta_2^* > \eta_1^*$ (Fig. 6b). These latter relations are, however, not independent of each other. E.g. the following series of implications is right:

$$\vartheta_1^* \leqq \eta_1^* \Rightarrow \eta_2^* \leqq \eta_1^* \Rightarrow \vartheta_1^* < \vartheta_2^* \Rightarrow \vartheta_1^* \leqq \vartheta_2^*. \tag{77}$$

From Lemma 8 we can obtain relations among the characteristics of R^* if it is periodic. From (63a) we get

$$\mu_1^* \leqq \mu_2^* + \varkappa_2^* \tag{78}$$

but from (63c) we get $\mu_1^* = \mu_2^* + \varkappa_2^*$ if any member of the series of implications (77) is true. From (64c) and (65)

$$\mu_1^* \geqq \mu_2^* + 1 \geqq \varkappa_2^* + 2 \geqq 3. \tag{79}$$

Before we further investigate some special cases of (75) we introduce an algo rithm to generate some entities and the characteristics Π^* of R^* if R^* is periodic.

In the schedule R^* the sequence $C_{21}, C_{22}, \ldots$ of C_2-cycles can be grouped into subsequences in which all cycles are either preempted or not preempted. Denote by M_i, $i=1, 2, \ldots,$ the sequence of the subsequences of the preempted and N_i, $i=1, 2, \ldots,$ the sequence of the subsequences of the non-preempted C_2-cycles. The first subsequence will be the N_1 with at least one C_2-cycle since $A_{2,1}$ is a non-preempted task because of $\eta_2^* \leqq \vartheta_1^*$. We call an *M-section* or an *N-section* of R^* the section from the last cycle-finishing point of the previous subsequence until the last cycle-finishing point of the current subsequence M_i or N_i, respectively. This definition will be modified slightly below by dividing some M-sections defined now into more M-sections and inserting empty N-sections in between them.

Define

$$f(0) = \eta_1^*, \quad f(i) = \eta_1^* + i\tau_2^* + \chi(i)\eta_1^* \tag{80}$$

as C_2-cycle finishing points,

$$\varphi(0) = 0, \quad \varrho(0) = \eta_1^*, \quad \varphi(i) = \left[\frac{f(i)}{\tau_1^*}\right], \quad \varrho(i) = f(i) - \varphi(i)\tau_1^*, \tag{81}$$

$i=1, 2, \ldots,$ as moduli and residua of the cycle-finishing points and

$$H(i) = (\varphi(i), i, \chi(i)), \quad i = 0, 1, \ldots \tag{82}$$

as triads according to (32) and proof of Theorem 3. (80)—(82) are only valid until the first recurrence point T_1^* of the β_1-situation which occurs exactly when the residuum $\varrho(i)$ is not greater than η_1^*, i.e.

$$0 \leqq \varrho(i) \leqq \eta_1^*. \tag{83}$$

After $\varrho(0) = \eta_1^*$ the next such residuum and the corresponding triad determine the characteristics of R^* which is periodic if such a residuum exists. Otherwise

R^* is not periodic. The value of the residuum $\varrho(i)$ determines whether the next A_2-task $A_{2,i+1}$ is preempted or not. If

$$\eta_1^* \leqq \varrho(i) \leqq \tau_1^* - \eta_2^* \tag{83'}$$

then $A_{2,i+1}$ will be serviced without preemption and if

$$\tau_1^* - \eta_2^* < \varrho(i) < \tau_1^* \tag{83''}$$

then $A_{2,i+1}$ will be preempted.

Without preemption $f(i+1) = f(i) + \tau_2^*$ and

$$\varrho(i+1) = \varrho(i) + \tau_2^* - \tau_1^* > \varrho(i) \tag{84}$$

because from (83′) we obtain $\eta_1^* < \tau_2^* - \vartheta_1^* \leqq \varrho(i+1) \leqq \vartheta_2^* < \tau_1^*$.

With preemption $f(i+1) = f(i) + \tau_2^* + \eta_1^*$. In this case we get

$$\varrho(i+1) = \begin{cases} \varrho(i) + \tau_2^* - \vartheta_1^* > \varrho(i) & \text{if} \quad \vartheta_2^* < \vartheta_1^* \quad \text{and} \quad \tau_1^* - \eta_2^* < \varrho(i) < \tau_1^* + \vartheta_1^* - \tau_2^* \\ \varrho(i) + \tau_2^* - \vartheta_1^* - \tau_1^* < \varrho(i) & \text{if} \quad \tau_1^* - \min(\eta_2^*, \tau_2^* - \vartheta_1^*) \lessdot \varrho(i) < \tau_1^* \end{cases} \tag{85}$$

where the symbol $\lessdot$ denotes a relation sign by

$$\lessdot = \begin{cases} < & \text{if} \quad \vartheta_1^* \leqq \vartheta_2^* \\ \leqq & \text{if} \quad \vartheta_2^* < \vartheta_1^*. \end{cases} \tag{86}$$

(85) holds because $\vartheta_2^* + \eta_1^* < \varrho(i) + \tau_2^* - \vartheta_1^* < \tau_1^*$ if $\tau_1^* - \eta_2^* < \tau_1^* + \vartheta_1^* - \tau_2^*$, i.e. $\vartheta_2^* < \vartheta_1^*$, and $\tau_1^* - \eta_2^* < \varrho(i) < \tau_1^* + \vartheta_1^* - \tau_2^*$ and $0 \leqq \tau_2^* - \vartheta_1^* - \min(\eta_2^*, \tau_2^* - \vartheta_1^*) \lessdot \varrho(i) + \tau_2^* - \vartheta_1^* - \tau_1^* < \tau_2^* - \vartheta_1^* < \tau_1^*$ if $\tau_1^* - \min(\eta_2^*, \tau_2^* - \vartheta_1^*) \lessdot \varrho(i) < \tau_1^*$.

Since $\varrho(0) = \eta_1^* \leqq \tau_1^* - \eta_2^*$ by (75), R^* starts with a non-preempted A_2-task and $\varrho(i)$ is monoton increasing until (83″) results and preempted A_2-task follows. $\varrho(i)$ can increase further until a decrease because of $\tau_1^* - \min(\eta_2^*, \tau_2^* - \vartheta_1^*) \lessdot \varrho(i)$ follows. If the $\varrho(i+1)$ obtained by (85) satisfies (83′), a non-preempted C_2-cycle follows, otherwise the following C_2-cycle is preempted as well. In both cases we regard the situation as the end of an M-section and beginning of an N-section. In the second case in which the following C_2-cycle is preempted as well, the N-section is empty and begins a new M-section simultaneously.

The schedule R^* consists of a sequence $(N_1, M_1), (N_2, M_2), \ldots$ of (N, M)-*section pairs* in which N_1 cannot but $N_i, i > 1$, can be empty, too. Let the numbers of C_2-cycles in the sections N_i and M_i be n_i' and m_i', respectively. These are called the *lengths* of the sections.

The bounds obtained for $\varrho(i+1)$ show that

$$0 \leqq \varrho(i+1) \leqq \eta_1^* \tag{87}$$

can only come to pass if $\varrho(i+1) < \varrho(i)$ i.e. at the end of an M-section. With the purpose of finding the first $\varrho(i+1)$, $i \geqq 0$, for which (87) comes true, the residua at the end of M-sections are enough to consider. These residua are the local minima in the series $\varrho(0), \varrho(1), \ldots$. The next minimum comes after the ith local minimum ϱ_{i-1}, when in the series $\varrho_{i-1}, \varrho_{i-1} + \tau_2^* - \tau_1^*, \ldots, \varrho_{i-1} + n_i'(\tau_2^* - \tau_1^*)$, $\varrho_{i-1} +$

$+n_i'(\tau_2^*-\tau_1^*)+\tau_2^*-\vartheta_1^*, \dots, \varrho_{i-1}+n_i'(\tau_2^*-\tau_1^*)+j(\tau_2^*-\vartheta_1^*), \dots$ the first $j=m_i'$ occurs for which

$$\varrho_{i-1}+n_i'(\tau_2^*-\tau_1^*)+m_i'(\tau_2^*-\vartheta_1^*) \geqq \tau_1^*$$

and, therefore,

$$\varrho_i = \varrho_{i-1}+n_i'(\tau_2^*-\tau_1^*)+m_i'(\tau_2^*-\vartheta_1^*)-\tau_1^*.$$

This condition determines m_i' and ϱ_i by ϱ_{i-1} and n_i'. n_i' is determined by ϱ_{i-1} as the first $j=n_i'\geqq 0$ for which

$$\varrho_{i-1}+n_i'(\tau_2^*-\tau_1^*) > \tau_1^*-\eta_2^*.$$

This means that n_i', m_i', ϱ_i are uniquely determined by ϱ_{i-1} as

$$n_i' = \left[\frac{\tau_1^*-\eta_2^*-\varrho_{i-1}}{\tau_2^*-\tau_1^*}\right]+1 = \left[\frac{\vartheta_2^*-\varrho_{i-1}}{\tau_2^*-\tau_1^*}\right] \tag{88}$$

$$m_i' = f_{\geqq}\left(\frac{\tau_1^*-\varrho_{i-1}-n_i'(\tau_2^*-\tau_1^*)}{\tau_2^*-\vartheta_1^*}\right) = [\zeta]+\operatorname{sgn}\{\zeta\} \tag{89}$$

$$\varrho_i = \varrho_{i-1}+n_i'(\tau_2^*-\tau_1^*)+m_i'(\tau_2^*-\vartheta_1^*)-\tau_1^*, \tag{90}$$

where

$$\zeta = \frac{\tau_1^*-\varrho_{i-1}-n_i'(\tau_2^*-\tau_1^*)}{\tau_2^*-\vartheta_1^*} \tag{91}$$

and $f_{\geqq}(x)$ is the least integer not less than x.

Let us use the notations

$$n_0 = m_0 = k_0 = 0, \quad n_i = \sum_{j=1}^{i} n_j', \quad m_i = \sum_{j=1}^{i} m_j', \quad \psi_i = n_i+m_i, \quad i = 1, 2, \dots . \tag{92}$$

The integers n_i, m_i and ψ_i give the number of C_2-cycles serviced without preemption, with preemption and totally until the end of the (N_i, M_i) section pair, respectively.

Denote by

$$H_i = (\varphi_i, \psi_i, \chi_i), \quad i = 1, 2, \dots ,$$

the triads at the ends of the (N, M)-section pairs. We call H_i, $i=1, 2, \dots$, R_{12}-*triples*. Clearly $H_i=H(n_i+m_i)$ and

$$\varphi_i = n_i+m_i+i, \quad \psi_i = n_i+m_i, \quad \chi_i = m_i, \quad i = 1, 2, \dots . \tag{93}$$

The residuum at the end of the (N_i, M_i) section pair can be written from the recursion (90) and $\varrho_0=\varrho(0)=\eta_1^*$ as

$$\varrho_i = \eta_1^*+n_i(\tau_2^*-\tau_1^*)+m_i(\tau_2^*-\vartheta_1^*)-i\tau_1^* \tag{94}$$

or with (93) as

$$\varrho_i = \eta_1^*+\psi_i\tau_2^*+\chi_i\eta_1^*-\varphi_i\tau_1^*. \tag{95}$$

The end of the first period of R^*, if such one exists, is determined by the entities at the end of the first (N, M)-section pair with ϱ_i satisfying (83). If such a section-pair exists, it can be determined recursively by the formulae (88)—(91). If for $i=I>0$ the relation (83) comes to pass first, the characteristics of R^* will be

$$\Pi^* = (\varphi_I; \psi_I; \chi_I;\ 1-\varrho_I/\eta_1^*)$$

by (49′), i.e.

$$\begin{aligned} \mu_1^* &= \varphi_I = n_I+m_I+I, & \varkappa_2^* &= \chi_I = m_I \\ \mu_2^* &= \psi_I = n_I+m_I, & \varepsilon_2^* &= 1-\varrho_I/\eta_1^*. \end{aligned} \tag{96}$$

From (93) we can express i, n_i, m_i by the elements of the R_{12}-triple H_i as

$$i = \varphi_i-\psi_i, \quad n_i = \psi_i-\chi_i, \quad m_i = \chi_i \tag{97}$$

and from (96) we can express I, n_I, m_I, ϱ_I by the characteristics Π^* of R^* as

$$I = \mu_1^*-\mu_2^*, \quad n_I = \mu_2^*-\varkappa_2^*, \quad m_I = \varkappa_2^*, \quad \varrho_I = \eta_1^*(1-\varepsilon_2^*). \tag{97′}$$

These quantities are the number of (N, M)-section pairs, the number of C_2-cycles serviced without and with preemption and the last residuum, respectively, in a period of R^*.

We phrase our main results in

Theorem 8. *The priority schedule $R^*=R_{1,2}(Q^*)$ of a reduced configuration Q^* satisfying*

$$0 < \eta_1 < \tau_1^* < \tau_2^* \tag{98}$$

is periodic exactly when such a residuum $\varrho(i)$, $i>0$, does exist which fulfils (83). *This condition is equivalent to the fact that R^* has an M-section M_I, $I>0$, the last residuum ϱ_I of which fulfils the inequality*

$$\max(0, \vartheta_2^*-\vartheta_1^*) \prec \varrho_I \leqq \eta_1^*. \tag{99}$$

The characteristics are determined then by the R_{12}-triple H_I and the residuum ϱ_I as

$$\Pi^* = (\varphi_I; \psi_I; \chi_I;\ 1-\varrho_I/\eta_1^*). \tag{100}$$

Proof. The only assertion to be proved is that (83) is equivalent to (99) with regard to ϱ_I. This follows, however, from the fact that if $\varrho(i)$ is the last residuum of an M-section then $\varrho(i)=\varrho(i-1)+\tau_2^*-\vartheta_1^*-\tau_1^*$ and, since $\tau_1^*-\eta_2^*<\varrho(i-1)$ by (83″) because of the preemption of the last C_2-cycle, $\varrho(i)>\vartheta_2^*-\vartheta_1^*$ and $\vartheta_2^*-\vartheta_1^*<$ $<\varrho(i)\leqq\eta_1^*$ must stand instead of (83) in the case $\vartheta_2^*-\vartheta_1^*\geqq0$. Using the definition (86) of $\prec$ we obtain the inequality (99) for $\varrho(i)$ and consequently for ϱ_I. □

We now define the formal algorithm to determine the characteristics Π^* of R^* if R^* is periodic. As we do not have finite method to decide whether R^* is periodic, we have to choose an integer L as the tolerable number of (N, M)-section pairs for which the criterium (99) is allowed to be tested. If R^* is not periodic or the number I of the (N, M)-section pairs in a period is greater than L the algorithm finishes without giving the characteristics Π^*. Nevertheless, the algoritm gives the values of the R_{12}-triple H_L and residuum ϱ_L also in this case. The output for Π^* is as its input (0; 0; 0; 0) in this case.

Algorithm R^*_{12}. *Input data:* $Q^* = (\eta^*_1; \vartheta^*_1; \eta^*_2; \vartheta^*_2), L$;
Output data: $\Pi^* = (\mu^*_1; \mu^*_2; \varkappa^*_2; \varepsilon^*_2)$, $H_L = (\varphi_L, \psi_L, \chi_L)$, ϱ_L;
Step 0: $\tau^*_1 := \eta^*_1 + \vartheta^*_1$; $\tau^*_2 := \eta^*_2 + \vartheta^*_2$;
If $0 < \eta^*_2 \leqq \vartheta^*_1 < \tau^*_1 < \tau^*_2 < \eta^*_2 + \tau^*_1$ does not hold then ERROR and go to *End*;
$\varrho := \eta^*_1$; $n := m := i := 0$;
Step 1: $n' := \left[\dfrac{\vartheta^*_2 - \varrho}{\tau^*_2 - \tau^*_1}\right]$; $n := n + n'$; $\varrho := \varrho + n'(\tau^*_2 - \tau^*_1)$; $\zeta := \dfrac{\tau^*_1 - \varrho}{\tau^*_2 - \vartheta^*_1}$;
$m' := [\zeta] + \operatorname{sgn}\{\zeta\}$; $m := m + m'$; $\varrho := \varrho + m'(\tau^*_2 - \vartheta^*_1) - \tau^*_1$; $i := i + 1$;
Step 2: If $\varrho \leqq \eta^*_1$ then $\mu^*_1 := n + m + i$, $\mu^*_2 := n + m$, $\varkappa^*_2 := m$, $\varepsilon^*_2 := 1 - \varrho/\eta^*_1$ and go to *End*;
If $i = L$ then $\varphi_L := n + m + i$, $\psi_L := n + m$, $\chi_L := m$, $\varrho_L := \varrho$ and go to *End*;
Go to *Step 1*;
End.

We say that the Algorithm R^*_{12} finishes *normally* if it gives Π^* and *abnormally* if it does not give Π^* but gives H_L and ϱ_L. The algorithm does not put out the data of all (N, M)-section pairs but only those of the last. After minimal modification it would furnish these data as well. Independently of the algorithm it is worth to analyse the data the algorithm is dealing with because we can obtain further inferences from this analysis.

First we show bounds on the lengths n'_i, m'_i of the N- and M-sections. Let us use the quantities

$$\underline{n} = \frac{\vartheta^*_1 - \eta^*_2}{\tau^*_2 - \tau^*_1} - 1, \quad \bar{n} = \frac{\vartheta^*_1 - \eta^*_2}{\tau^*_2 - \tau^*_1} + 1, \quad \underline{m} = \frac{\eta^*_1 + \eta^*_2}{\tau^*_2 - \vartheta^*_1} - 1, \quad \bar{m} = \frac{\eta^*_2}{\tau^*_2 - \vartheta^*_1} + 1. \tag{101}$$

Let I be the number of the (N, M)-section pairs in a period of R^* if R^* is periodic and $I = \infty$ otherwise. The formulae (88)—(91) define n'_i, m'_i, ϱ_i for $i = 1, 2, \dots$ (I, if I is finite).

Lemma 9. *For the lengths n'_i, m'_i, $i = 1, 2, \dots (I)$ the following bounds are valid:*

$$n'_1 = [\bar{n}], \quad \underline{n} < n'_i < \bar{n}, \quad 1 < i \leqq I, \tag{102}$$

$$\underline{m} < m'_i < \bar{m}, \quad 1 \leqq i < I, \quad \underline{m} \lessdot m'_I < \bar{m}, \tag{103}$$

where the symbol $\lessdot$ is defined by (86).

Proof. From (88) with $\varrho_0 = \eta^*_1$ we get $n'_1 > \dfrac{\vartheta^*_2 - \eta^*_1}{\tau^*_2 - \tau^*_1} - 1 = \bar{n} - 1$ and $n'_1 \leqq \dfrac{\vartheta^*_2 - \eta^*_1}{\tau^*_2 - \tau^*_1} = \bar{n}$ and so $n'_1 = [\bar{n}]$. Using the inequalities $\varrho_{i-1} > \eta^*_1$ and $\varrho_{i-1} < \tau^*_2 - \vartheta^*_1$, obtainable from (89) and (90), we get from (88) for $i > 1$ that $n'_i > \dfrac{\vartheta^*_2 - \varrho_{i-1}}{\tau^*_2 - \tau^*_1} - 1 > \underline{n}$ and $n'_i \leqq \dfrac{\vartheta^*_2 - \varrho_{i-1}}{\tau^*_2 - \tau^*_1} < \bar{n}$.

If ζ would be integer by (91) for $i < I$ then we would get $m'_i = \zeta$ and $\varrho_i = 0$ which contradicts the definition of I. For $i = I$, $\varrho_I = 0$ is only possible by (99) if $\vartheta^*_2 < \vartheta^*_1$. This means that $\zeta < m'_i < \zeta + 1$ if $1 \leqq i < I$ and if $i = I$ and $\vartheta^*_2 \geqq \vartheta^*_1$. By this fact and $\tau^*_1 - \eta^*_2 < \varrho_{i-1} + n'_i(\tau^*_2 - \tau^*_1) \leqq \vartheta^*_2$ obtainable from (88) we get $m'_i >$

$>\zeta \geqq \frac{\tau_1^* - \vartheta_2^*}{\tau_2^* - \vartheta_1^*} = \underline{m}$ and $m_i' < \zeta + 1 < \frac{\eta_2^*}{\tau_2^* - \vartheta_1^*} + 1 = \bar{m}$ for $i < I$ and $i = I$, $\vartheta_2^* \geqq \vartheta_1^*$, and we get $m_I' \geqq \zeta \geqq \underline{m}$ and $m_I' < \zeta + 1 < \bar{m}$ for $i = I$ and $\vartheta_2^* < \vartheta_1^*$. □

This lemma shows that the series n_i', $i = 1, 2, \ldots$, and m_i', $i = 1, 2, \ldots$, of lengths have only small fluctuations, if any. The bandwidth of the variations are

$$\bar{n} - \underline{n} = 2 \quad \text{and} \quad 1 < \bar{m} - \underline{m} = 2 - \frac{\eta_1^*}{\tau_2^* - \vartheta_1^*} < 2 \quad \text{if} \quad \eta_1^* > 0. \tag{104}$$

These show that both the n_i' and m_i' values can always vary at most on two adjacent integers.

From the conditions (78), definitions (101) and estimations (102) and (103) we easily get

$$n_1' \geqq 1, \quad n_i' \geqq 0, \quad 1 < i \leqq I, \tag{105}$$

$$m_i' \geqq 1, \quad 1 \leqq i \leqq I. \tag{106}$$

Simple regularity conditions can be given for the series of lengths by the parameters of Q^* which further limit their fluctuations. To simplify writing we use the quantities

$$x_j = \vartheta_j^* - \eta_{3-j}^*, \quad j = 1, 2. \tag{107}$$

Lemma 10. *For the lengths n_i' and m_i' of the (N, M)-section pairs the following assertions hold.*

(a) *If*

$$n' < \frac{x_1}{x_2 - x_1} < n' + 1 \tag{108a}$$

for some integer $n' \geqq 0$, *then*

$$n_1' = n' + 1 \quad \textit{and} \quad n' \leqq n_i' \leqq n' + 1 \tag{109a}$$

for $1 < i \leqq I$. *Especially*

$$\begin{aligned} &n_1' = 1 \quad \textit{and} \quad 0 \leqq n_i' \leqq 1, \ 1 < i \leqq I \quad \textit{if} \quad 0 < \vartheta_1^* - \eta_2^* < \tau_2^* - \tau_1^* \\ &n_1' = 2 \quad \textit{and} \quad 1 \leqq n_i' \leqq 2, \ 1 < i \leqq I \quad \textit{if} \quad \tau_2^* - \tau_1^* < \vartheta_1^* - \eta_2^* < 2(\tau_2^* - \tau_1^*) \end{aligned} \tag{109'a}$$

(b) *If*

$$\frac{x_1}{x_2 - x_1} = n' \tag{108b}$$

for some integer $n' \geqq 0$, *then*

$$n_1' = n' + 1 \quad \textit{and} \quad n_i' = n' \tag{109b}$$

for $1 < i \leqq I$. *Especially*

$$\begin{aligned} &n_1' = 1, \quad n_i' = 0, \quad 1 < i \leqq I, \quad \textit{if} \quad \vartheta_1^* = \eta_2^* \\ &n_1' = 2, \quad n_i' = 1, \quad 1 < i \leqq I, \quad \textit{if} \quad \vartheta_1^* - \eta_2^* = \tau_2^* - \tau_1^*. \end{aligned} \tag{109'b}$$

(c) *If*

$$\frac{m'}{\eta^*} < \frac{1}{x_2 - x_1 + \eta_1^*} \leqq \frac{m'}{\eta_2^*} \tag{108c}$$

for some integer $m' \geqq 1$, *then*

$$m_i' = m' \tag{109c}$$

for all $1 \leqq i \leqq I$. *Especially*

$$\begin{aligned} &m_i' = 1, \quad 1 \leqq i \leqq I, \quad \text{if} \quad \vartheta_2^* \geqq \vartheta_1^* \\ &m_i' = 2, \quad 1 \leqq i \leqq I, \quad \text{if} \quad \tau_2^* - \tau_1^* < \vartheta_1^* - \vartheta_2^* \leqq \tau_2^* - \vartheta_1^*. \end{aligned} \tag{109'c}$$

(d) *If*

$$\frac{\eta^*}{x_2 - x_1 + \eta_1^*} = m' \tag{108d}$$

for some integer $m' > 1$, *then*

$$m_i' = m' \quad \text{for} \quad 1 \leqq i < I \quad \text{and} \quad m' - 1 \leqq m_I' \leqq m'. \tag{109d}$$

Especially

$$m_i' = 2 \quad \text{for} \quad 1 \leqq i < I \quad \text{and} \quad 1 \leqq m_I' \leqq 2, \quad \text{if} \quad \tau_2^* - \tau_1^* = \vartheta_1^* - \vartheta_2^*. \tag{109'd}$$

COMMENT. (108d) cannot be true for $m'=1$ because $\vartheta_2^* = \tau_1^*$ would follow which contradicts (75). (108d) is equivalent to $(m'-1)(\tau_2^* - \vartheta_1^*) + \vartheta_2^* - \vartheta_1^* = \eta_1^*$ from which $\vartheta_1^* - \vartheta_2^* = (m'-1)(\tau_2^* - \vartheta_1^*) - \eta_1^* \geqq \tau_2^* - \tau_1^* > 0$ if $m' > 1$ and, therefore, $\vartheta_2^* < \vartheta_1^*$ follows. In case of $\vartheta_2^* \geqq \vartheta_1^*$ the condition (108d) is impossible.

Proof. The method of proof is to relate the bounds (101) to the parameter n' or m' of the condition (108). (101) is equivalent to $\underline{n} = x_1/(x_2 - x_1) - 1$, $\bar{n} = x_1/(x_2 - x_1) + 1$, $\underline{m} = \eta^*/(x_2 - x_1 + \eta_1^*) - 1$, $\overline{m} = \eta_2^*/(x_2 - x_1 + \eta_1^*) + 1$. From (108a) we get $n' - 1 < \underline{n} < n'$ and $n' + 1 < \bar{n} < n' + 2$ and, therefore, the interval $(\underline{n}, \bar{n})$ contains the integers n' and $n'+1$ and (102) is equivalent to (109a). We get (109'a) from (109a) for $n'=0$ and $n'=1$. From (108b) we get $\underline{n} = n' - 1$ and $\bar{n} = n' + 1$ and the relations (102) make possible only (109b). (109'b) follows from (109b) for $n'=0$ and $n'=1$. From (108c) we obtain $m' - 1 < \underline{m}$ and $\overline{m} \leqq m' + 1$ and, therefore, the interval $[\underline{m}, \overline{m})$ contains the only integer m' and (109c) follows from (103). (109'c) follows from (109c) for $m'=1$ and $m'=2$. From (108d) we get $\underline{m} = m' - 1$ as an integer. The interval $[\underline{m}, \overline{m})$ contains now the integers $m'-1$ and m' and (109d) follows from (103) and (86) because (108d) is possible only if $\vartheta_2^* < \vartheta_1^*$ (see Comment) and $\prec \, = \, \leqq$ by (86) in this case. (109'd) follows from (109d) for $m'=2$. □

The conditions (108) are only sufficient but not necessary for (109) to be valid. One of the conditions (108a) and (108b) is always true and (109a) is valid because (109b) implies (109a). Lemma 10 is valid also for $I = \infty$ (R^* is not periodic) if the assertions with $i = I$ are neglected.

From Lemma 10 we can deduce some relationships among the R_{12}-triples which can reduce the problem of existence and determination of the least R_{12}-triple satisfying (99) to the problem of solution of a coincidence problem [4]. This problem is generally solved and leads to the regular continued fraction expansion of a number depending on the parameters of Q^* [4]. The coincidence problems encountering have the form of the determination of the least solution $\omega^*=(B^*, A^*)$ of an inequality pair

$$0 \leqq B\xi - A \prec \alpha, \quad \omega \geqq \omega_0 \tag{110}$$

for the unknown integers $\omega \doteq (B, A)$ where reals $\xi, \alpha \geqq 0$, sign $\prec$ and integers $\omega_0=(B_0, A_0)$ are given. ω^* exists and is unique if $\alpha>0$ or $\prec = \leqq$, $\alpha=0$ and ξ is rational. ω^* does not exist otherwise. B^* and A^* are relatively prime [4].

The following lemma is necessary to prove the periodicity of R^* if $0<\vartheta_1^* \leqq \vartheta_2^*$ in addition to (75).

Lemma 11. *For the schedule* $R^*=R_{1,2}(Q^*)$ *of any configuration* $Q^* \in \mathcal{Q}$ *fulfilling* (75) *the following assertions hold.*

(I) *The following three facts are equivalent:*

$$\begin{aligned} &\text{(a)} \quad \varphi_i = \psi_i + \chi_i, \quad 1 \leqq i \leqq I,\\ &\text{(b)} \quad m_i' = 1, \qquad\quad 1 \leqq i \leqq I,\\ &\text{(c)} \quad R^* \text{ is periodic and } \mu_1^* = \mu_2^* + \varkappa_2^*; \end{aligned} \tag{111}$$

(II) *If any of* (111a—c) *holds, the characteristics* Π^* *of* R^* *are determined by the least solution* $\omega^*=(B^*, A^*)$ *and its error* $\Delta^*=B^*\xi^*-A^*$ *of a coincidence problem*

$$0 \leqq B\xi^* - A \prec \alpha^*, \quad \omega \geqq (1, 0) \tag{112}$$

where ξ^*, $\alpha^*>0$ *are determined by* Q^* *and* $\prec$ *is defined by* (86)*;* $\mu_1^*, \mu_2^*, \varkappa_2^*$ *are pairwise relatively prime integers;*

(III) ξ^* *and* α^* *in* (112) *and the characteristics* Π^* *have the alternative values by the three rows of the following table:*

	ξ^*	α^*	μ_1^*	μ_2^*	$\varkappa_2^*$	ε_2^*
(a)	$\dfrac{\vartheta_1^*}{\tau_2^*-\tau_1^*}$	$\dfrac{\eta_1^*-r}{\tau_2^*-\tau_1^*}$	A^*+B^*	A^*	B^*	$\dfrac{\Delta^*(\tau_2^*-\tau_1^*)}{\eta_1^*}$
(b)	$\dfrac{\tau_2^*-\eta_1^*}{\tau_2^*-\tau_1^*}$	$\dfrac{\eta_1^*-r}{\tau_2^*-\tau_1^*}$	A^*	A^*-B^*	B^*	$\dfrac{\Delta^*(\tau_2^*-\tau_1^*)}{\eta_1^*}$
(c)	$\dfrac{\vartheta_1^*}{\tau_2^*-\eta_1^*}$	$\dfrac{\eta_1^*-r}{\tau_2^*-\eta_1^*}$	B^*	A^*	B^*-A^*	$\dfrac{\Delta^*(\tau_2^*-\eta_1^*)}{\eta_1^*}$

(113)

where

$$r = \max(0, \vartheta_2^*-\vartheta_1^*).$$

Proof. We begin with the assertions (I). From $m_i' \equiv 1$ we get $\varphi_i=n_i+2i$, $\psi_i=n_i+i$, $\chi_i=i$ from (93), and (111a) is true. From (111a) and (97) we get $i=\varphi_i-\psi_i=\chi_i=m_i$, and (106) and definition (92) prove $m_i'=1$. If R^* is periodic, exactly one A_1-task starts during every B_2-task by (111c) and (62). This means that the number $\varphi_i-\chi_i$ of A_1-tasks causing no preemption is equal to ψ_i, the number of C_2-cycles. This proves (111a). From the assertion (I) only the periodicity of R^* if (111a) is true, remainded to be proved. This will be done together with (II) and (III).

Consider the Gantt-chart of R^* until the first recurrence point T_1^* of the β_1-situation (not supposed finite). Carve out the A_1-tasks from it and denote the resulting chart by R''. Since exactly one A_1-task starts during every B_2-task and the β_1-situation occurs if the A_1-task does not finish during the B_2-task, it follows that exactly one A_1-task runs during every B_2-task except the last before the β_1-situation, where the A_1-task can finish after the B_2-task as well. Therefore, chart R'' will agree with the schedule $R'=R_{1,2}(Q')$ of the configuration $Q'= =(0;\ \vartheta_1^*;\ \eta_2^*;\ \vartheta_2^*-\eta_1^*)$ except eventually the last B_2-task which has the length $\vartheta_2''=\vartheta_2^*-\eta_1^*+\varepsilon_2^*$ instead of $\vartheta_2'=\vartheta_2^*-\eta_1^*$. As $\eta_1'=0$, the preempting A_1-tasks in R' do not cause delays and, therefore, the cycle-finishing points are

$$f'(C_{2,i}) = i(\tau_2^*-\eta_1^*), \quad i = 1, 2, \dots .$$

The periodicity of R^* is equivalent to the finiteness of T_1^* and this to the fact that the last B_2-task in the first period (if such one exists) of R' would run during a B_1-task and finish not more than η_1^* earlier than the B_1-task (see Fig. 8). This corre-

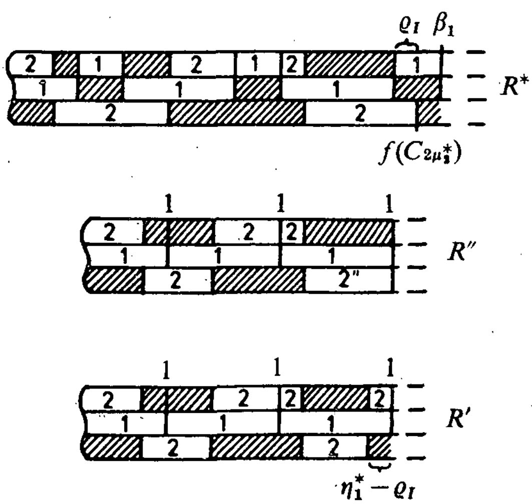

Fig. 8

The transformation $R^* \to R''$ and the schedule R'

sponds to the first situation in R' in which the inequalities $\vartheta_2^*-\eta_1^*<i(\tau_2^*-\eta_1^*)- -(j-1)\vartheta_1^*\leqq\vartheta_1^*$ and $0\leqq j\vartheta_1^*-i(\tau_2^*-\eta_1^*)\leqq\eta_1^*$ for some positive integers i, j, result: The values of i and j correspond to the characteristics Π^* of R^* as $i=\mu_2^*$, $j=\mu_1^*$. The two inequalities are equivalent to the inequality

$$0 \leqq \mu_1^*\vartheta_1^*-\mu_2^*(\tau_2^*-\eta_1^*) \prec \eta_1^*-\max(0, \vartheta_2^*-\vartheta_1^*)$$

in which the sign $\prec$ is defined by (86). This shows that the periodicity of R^* is equivalent to the existence of positive integers $\omega=(B, A)$ for which the inequalities

(112) with ξ^* and α^* of (113c) hold. The least such pair determines μ_1^* and μ_2^* by (113c). $\varkappa_2^*=B^*-A^*$ follows from (111a) and the expression of ε_2^* from the relationships $\varepsilon_2^*=(\eta_1^*-\varrho_I)/\eta_1^*$ and $\varrho_I=\eta_1^*+\mu_2^*\tau_2^*+\varkappa_2^*\eta_1^*-\mu_1^*\tau_1^*=\eta_1^*+A^*\tau_2^*+(B^*-A^*)\eta_1^*-B^*\tau_1^*=\eta_1^*-\Delta^*(\tau_2^*-\eta_1^*)$. The existence of ω^* is garanteed by $\alpha^*>0$ and this by (75).

We have to prove that (113a)—(113c) are equivalent. The inequality $0\leqq B^*\vartheta_1^*-A^*(\tau_2^*-\eta_1^*)<\eta_1^*-r$ is equivalent to the inequality $0\leqq B'(\tau_2^*-\eta_1^*)-A'(\tau_2^*-\tau_1^*)<\eta_1^*-r$ if $B^*=A'$ and $A^*=A'-B'$. The least solutions of the two inequalities with the condition $(B, A)\geqq(1, 0)$ correspond to each other by this transformation. This proves (113b). By the transformation $B^*=A'+B'$, $A^*=A'$ we can similarly prove the equivalence of (113c) and (113a). If B^* and A^* are relatively prime, such are the transformed values as well. This completes our proof. □

Lemma 10 and 11 enable us to solve the evaluation problem of R^* for configurations Q^* satisfying (75) and any of the relations (77).

Theorem 9. *If the configuration $Q^*\in\mathcal{Q}$ is reduced,*

$$\tau_1^* < \tau_2^* \quad \textit{and} \quad 0 < \vartheta_1^* \leqq \vartheta_2^* \tag{114}$$

then $R^*=R_{1,2}(Q)$ is periodic and its characteristics Π^* are obtainable by (113) *and $\mu_1^*, \mu_2^*, \varkappa_2^*$ are pairwise relatively prime integers.*

Proof. In R^* we obtain $m_i'\equiv 1$ from (109'c) and R^* is periodic with $\mu_1^*=\mu_2^*+\varkappa_2^*$ by (111c). The assertions (II)—(III) of the Lemma 11 corresponds to the statement of the theorem. □

With this theorem the only case not solved is the configuration $Q\in\mathcal{Q}$ which is reducible and its reduction Q^* satisfies the relations

$$\tau_1^* < \tau_2^*, \quad \vartheta_1^* > \vartheta_2^*. \tag{115}$$

If we know that $R^*=R_{1,2}(Q^*)$ is periodic, the Algorithm R_{12}^* can be used to determine the characteristics Π^*. This method does not answer the question whether μ_1^*, μ_2^* and $\varkappa_2^*$ are relatively prime integers wich fact was shown in all other cases. In fact, μ_1^* and μ_2^* are relatively prime in every known periodicity case. Some further specific cases of (115) can be solved by using Lemma 10. For example, it can be proved that $m_I=m'-1$ if (108d) hold and, under the conditions (115), R^* is periodic if and only if $\vartheta_1^*-\eta_2^*$ and $\tau_2^*-\tau_1^*$ are rationally dependent. If

$$\xi = \frac{\vartheta_1^*-\eta_2^*}{\tau_2^*-\tau_1^*} = \frac{A}{B},$$

$A, B>0$ are relatively prime integers then the characteristics of R^* are

$$\Pi^* = \big((m'+1)B+A;\, m'B+A;\, m'B-1;\, 1\big)$$

with relatively prime μ_1^* and μ_2^* [4]. This assertion will not be proved here. This result is interesting because it shows that R^* *can be non-periodic for non-defective Q^* as well.* By another assertion [4], R^* is always periodic and its characteristics Π^* is determined by a given coincidence problem type (110) if (108c) holds. μ_1^*

and μ_2^* are relatively prime again. Similar assertions hold for non-defective configurations $Q \in \mathcal{Q}$ (not necessarily reduced) with $\eta_2 = \vartheta_1$ [4]. The proofs of these assertions are lengthy and, therefore, we do not show them here.

For any $Q \in \mathcal{Q}$, independently of its periodicity, the efficiency $\gamma_{1,2}$ of the priority schedule $R_{1,2}(Q)$ can be approximated by the P_A-utility $\gamma_{1,2}(\eta_1, t)$ of its section $\eta_1 \leqq s \leqq t$ defined by

$$\gamma_{1,2}(\eta_1, t) = \frac{\lambda(t) - \lambda(\eta_1)}{t - \eta_1} \tag{116}$$

as t grows (see (1)). It can be proved [4] that

$$\gamma_{1,2}(\eta_1, t) \sim \gamma^{(1)} + \gamma^{(2)} - \frac{\varkappa_2(t)}{\mu_1(t)} \gamma^{(1)} \gamma^{(2)} \sim \gamma_{1,2} \tag{117}$$

if t is big enough, where $\mu_1(t)$ is the number of the completed and $\varkappa_2(t)$ the number of preempting A_1-tasks until t in the schedule $R_{1,2}(Q)$. If $R_{1,2}(Q)$ is periodic with characteristics $\Pi = (\mu_1;\ \mu_2;\ \varkappa_2;\ \varepsilon_2)$ then

$$\gamma_{1,2} = \gamma^{(1)} + \gamma^{(2)} - \frac{\varkappa_2 + \varepsilon_2}{\mu_1} \gamma^{(1)} \gamma^{(2)} \tag{118}$$

(Theorem 5.10 in [4]). The proof of these facts we omit as well.

6. Some comments on the reduction methods

Theorem 3 in section 3 establishes relationships between the characteristics of the priority schedule of Q and of any transform $Q_n = \Delta^n Q$ of it. The reduction operator Δ defined in section 2 is actually the Δ_1 from the two operators Δ_1 and Δ_2 defined for Q symmetrically in the job-flows $Q^{(1)}$ and $Q^{(2)}$. The operator Δ_1 is only usable in the investigation of the priority schedules $R_{1,2}(Q)$ and we know nothing about the connections between the characteristics of $R_{2,1}(Q)$ and $R_{2,1}(Q_n)$, for instance. In the investigation of $R_{2,1}(Q)$ we can use the operator Δ_2. The $\tilde{Q} = \Delta_2 Q$ can be defined as the $\Delta_1 Q$ by (2) but the role of $Q^{(1)}$ and $Q^{(2)}$ (the indices 1 and 2) must be changed. The operation $\Delta_2 Q$ is, therefore, equivalent to the operation $\Delta_1 \bar{Q} = \Delta \bar{Q}$ with the conjugate configuration $\bar{Q}$ of Q defined in section 1.

In a previous article [5] we defined other operators $\mathcal{D}_1$ and $\mathcal{D}_2$ for Q as reductions utilized in the investigations of non-preemptive schedulings. In the operation $\mathcal{D}Q = \mathcal{D}_1 Q$ only the parameters ϑ_1 and ϑ_2 are reduced versus operation ΔQ in which also η_2 is reduced. The $\mathcal{D}$-reduction is much simpler than the Δ-reduction and is defined by (2b) and (2d) replaced (2c) by the instruction $\tilde{\eta}_2 = \eta_2$. Q^* is reduced by $\mathcal{D}$ if [5]

$$\vartheta_1^* < \tau_2^* \quad \text{or} \quad \tau_2^* = 0 \quad \text{and} \quad \vartheta_2^* < \tau_1^* \quad \text{or} \quad \tau_1^* = 0$$

which are exactly the conditions (5a) and (5c) as part of conditions Q^* to be reduced by Δ. This means Q^* reduced by Δ is always reduced by $\mathcal{D}$ as well. The opposite is not true, of course. The conditions (5a) and (5c) show that a configuration Q^* is reduced simultaneously by both $\mathcal{D}_1$ and $\mathcal{D}_2$. This is not true in respect to Δ_1 and

Δ_2. Fig. 9 shows the domains of reduced configurations Q by the operators $\mathcal{D}_i$ and Δ_i, $i=1, 2$ (refer also to Fig. 2). We distinguish the following domains:

(α) $\tau_1\tau_2 = 0$; Q is reduced by all operators

(β) $\eta_1\eta_2 > 0$, $\vartheta_1 = \vartheta_2 = 0$; Q is reduced by all operators

(γ) $\eta > 0$, $0 \leqq \eta_1 \leqq \vartheta_2 < \tau_1$, $0 \leqq \eta_2 \leqq \vartheta_1 < \tau_2$; Q is reduced by all operators

(a) $\eta_2 > 0$, $0 \leqq \eta_1 \leqq \vartheta_2 < \tau_1 < \eta$; Q is not reduced by Δ_1 but it is reduced by the other operators

(b) $\eta_1 > 0$, $0 \leqq \eta_2 \leqq \vartheta_1 < \tau_2 < \eta$; Q is not reduced by Δ_2 but it is reduced by the other operators

(c) $\eta_1\eta_2 > 0$, $\vartheta > 0$, $0 \leqq \vartheta_i < \eta_{3-i}$, $i = 1, 2$; Q is not reduced by Δ_i, $i = 1, 2$, but it is reduced by $\mathcal{D}_i$, $i = 1, 2$.

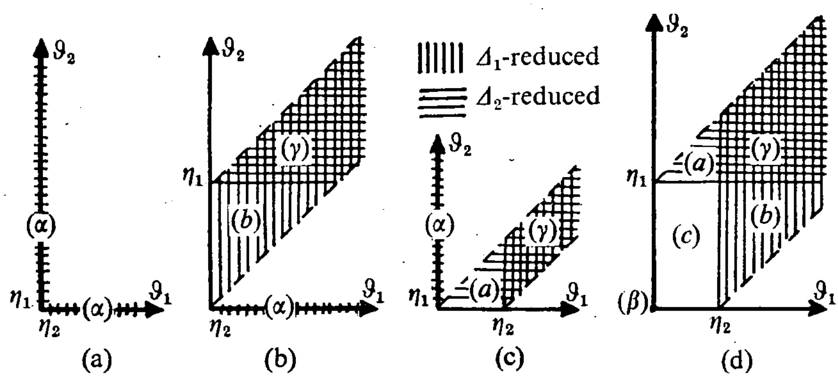

Fig. 9
Domains of reduced configurations

Let us introduce two simple *operators* δ_1 *and* δ_2 defined by $\tilde{Q}=\delta_i Q$ as of parameters

$$\tilde{\eta}_{3-i} = \begin{cases} \eta_{3-i} - f_<\left(\dfrac{\eta_{3-i}}{\vartheta_i}\right)\vartheta_i & \text{if} \quad \vartheta_i > 0 \\ \eta_{3-i} & \text{otherwise} \end{cases} \tag{119}$$

where $f_<(x)$ is the greatest integer less than x. Let $\delta=\delta_1$. It is clear that $f_<(\eta_2/\vartheta_1)= =k_2$ in (2c) if $\vartheta_1>0$. The operator δ_i is *effective* for Q if $\eta_{3-i}>\vartheta_i>0$ and *ineffective* for Q if $\vartheta_i\eta_{3-i}=0$ or $\eta_{3-i}\leqq\vartheta_i$. Since the order of steps (2c) and (2d) in the operation ΔQ is indifferent, the operator Δ can be represented as the operators $\mathcal{D}$ and δ in succession:

$$\Delta = \delta\mathcal{D}.$$

As $\vartheta_1\geqq\tau_2$ implies $\eta_2\leqq\vartheta_1$, the operator δ will be ineffective until Q is not reduced by $\mathcal{D}$ and $\mathcal{D}$ is effective on Q. This means that the manifestation of Δ for Q is $\mathcal{D}$

until $\mathscr{D}Q$ will not be $\mathscr{D}$-reduced, i.e. $\Delta Q=\mathscr{D}Q$. If $\mathscr{D}Q$ is $\mathscr{D}$-reduced, but not Δ-reduced, then $\Delta Q=\delta\mathscr{D}Q\neq\mathscr{D}Q$. This means that the manifestation of $\Delta^n Q$, $n>0$, is the alternate series of operator-powers $\mathscr{D}^\nu$ and the operator δ.

The manifestation is determined by the series (L) of quotients, or rather, by the subseries (k) of (L), defined in section 2. The operator δ in $\Delta=\delta\mathscr{D}$ is ineffective whenever $k_{2,n}=0$.

Define $\nu_0'=-1$ and for $i>0$, $\nu_i'=r$ if $k_{2,r}>0$ is the ith positive member in the series (k), if such one exists, and ν_i' is undefined if less than i positive members in (k) exist. It can easily be seen that

$$-1 \leqq \nu_0' < \nu_1' < \dots \quad \text{and} \quad \nu_i' \geqq i-1$$

and for any integer $r\geqq 0$ there exists a greatest ν_i' for which $\nu_i'<r$. Let this be $\nu_{h(r)}'$, i.e.

$$h(r) = \max_{\nu_i'<r} i, \quad r = 0, 1, \dots.$$

$h(r)$ is the number of positive members in the series $k_{2,0}, k_{2,1}, \dots, k_{2,r-1}$ and $\nu_{h(r)}'$ is the index of the last positive member if such one exists, and $\nu_{h(r)}'=-1$, otherwise. This means that

$$\nu_{h(0)}' = -1, \quad -1 \leqq \nu_{h(r)}' \leqq r-1, \quad r \geqq 0.$$

By means of the series (ν') and function $h(r)$ the manifestation of Δ^r on Q can be written as

$$\Delta^r Q = \mathscr{D}^{r-1-\nu'_{h(r)}}\left(\prod_{j=h(r)}^{1} \delta\mathscr{D}^{\nu_j'-\nu_{j-1}'}\right)Q, \quad r \geqq 0, \tag{120}$$

and if the degree of compositeness ν of Q is finite,

$$\Delta^r Q = \mathscr{D}^{\nu-1-\nu'_{h(\nu)}}\left(\prod_{j=h(\nu)}^{1} \delta\mathscr{D}^{\nu_j'-\nu_{j-1}'}\right)Q, \quad r \geqq \nu. \tag{120'}$$

Here $\prod_{j=h(r)}^{1} x_j \doteq x_{h(r)}x_{h(r)-1}\dots x_1$ and $\prod_{j=0}^{1} x_j=\emptyset$ is the identity operator. The factorizations (120) and (120′) depend, of course, on Q and, directly, on the series (L). If $\nu<\infty$, the series (ν') is finite and, with $J=|(\nu')|$, the last positive member of it is ν_{J-1}'. Let us supplement (ν') with the last member $\nu_J'=\nu-1$. Define the series of integers

$$\nu_j = \nu_j'-\nu_{j-1}', \quad j = 1, 2, \dots, J.$$

The $\mathscr{D}$-reduction of Q is then

$$Q^{(*)} = \mathscr{D}^{\nu_1}Q = \mathscr{D}^{\nu_1'+1}Q = Q_{\nu_1'+1}$$

and the Δ-reduction of Q is

$$Q^* = \Delta^\nu Q = \mathscr{D}^{\nu_J}\left(\prod_{j=J-1}^{1} \delta\mathscr{D}^{\nu_j}\right)Q = Q_\nu. \tag{121}$$

The factorization (121) shows that the Δ-reduction of any configuration $Q\in\mathscr{Q}$ is equivalent to some alternate series of $\mathscr{D}$-reductions and δ-operations. This fact clearly shows the connection between the two kinds of reduction.

The reduction operators Δ_1 and Δ_2 differ in both of their factors, $\mathscr{D}_i$ and δ_i:

$$\Delta_1 = \delta_1 \mathscr{D}_1, \quad \Delta_2 = \delta_2 \mathscr{D}_2 \tag{122}$$

but the manifestations (121) of the Δ_1- and Δ_2-reductions, if finite, are of similar factorizations in structure. In the analogous to (121) of the Δ_2-reduction of Q the same operator $\mathscr{D}$ can be applied because a configuration $Q^{(*)}$ is reduced by both of $\mathscr{D}_1$ and $\mathscr{D}_2$ at once and the degrees of compositeness by $\mathscr{D}_1$ and $\mathscr{D}_2$ have a known connection [4]. Nevertheless, the series (L) by Δ_1 and Δ_2 are different and, consequently, the series (ν) playing the central role in (121) are also different. Though the data of Δ_1- and Δ_2-reduction are not independent of each other, the interrelationships are likewise complicated and hardly provide a useful basis in practice to avoid evaluation of one of the two schedules $R_{1,2}(Q)$ and $R_{2,1}(Q)$. To inspect the relationships between both schedules the two reductions Δ_1 and Δ_2 seem to be a usable basis. The results given here can provide a grounding to this inspection by revealing the nature of the priority schedules in themselves. The method of Δ-reduction is a useful tool to this.

We mention the connection of the Δ-reduction with the regular continued fraction expansion. The Euclidean algorithm of the expansion of the number $\xi=\tau_1/\tau_2$ can be defined as the iteration [2]:

$\tau_{1,0} = \tau_1, \quad \tau_{2,0} = \tau_2$ and for $n = 1, 2, \ldots$

$\tau_{1,n-1} = b_{2n-2}\tau_{2,n-1}+\tau_{1,n}$ where

$b_{2n-2} \geqq 0$ is an integer and $0 \leqq \tau_{1,n} < \tau_{2,n-1}$ if $\tau_{2,n-1} > 0$,

b_{2n-2} and $\tau_{1,n}$ are not defined otherwise

$\tau_{2,n-1} = b_{2n-1}\tau_{1,n}+\tau_{2,n}$ where

$b_{2n-1} \geqq 0$ is an integer and $0 \leqq \tau_{2,n} < \tau_{1,n}$ if $\tau_{1,n} > 0$,

b_{2n-1} and $\tau_{2,n}$ are not defined otherwise.

Both components of the pair $(\tau_{1,n-1}, \tau_{2,n-1})$ are reduced by the step. This iteration ends with a $\tau_{i,n}=0$, $i=1$ or 2, $n\geqq 0$ if ξ is a rational number and is infinite if ξ is irrational.

The definition (2) of the Δ-reduction differs from this iteration by τ_1 and τ_2 being decomposed into two parts: $\tau_i=\eta_i+\vartheta_i$, $i=1, 2$, and this parts are reduced separately except η_1 which is not reduced at all. The iteration can end not only with a zero component but with conditions (5) of the reducedness. We have seen that the Δ-reduction becomes continued fraction expansion if one of the parts η_2 and ϑ_2 is zero. If, however, $\vartheta_2=0$, the reduction becomes the expansion of ϑ_1/η_2 and not of τ_1/ϑ_2.

The entities defined in section 2 in connection with Δ-reduction remind us of those in connection with the regular continued fraction expansion [3]. The special case of $\eta=0$ corresponds to the expansion of $\xi=\tau_1/\tau_2$.

7. Summary

We review below the points Q of the configuration space $\mathcal{Q}$ by our theorems proved from the point of view of whether the Question of periodicity and evaluation of the priority schedules $R_{1,2}$ and $R_{2,1}$ of Q is answered. See Fig. 10 as an illustration. Tx refers to the Theorem x in the Fig. 10.

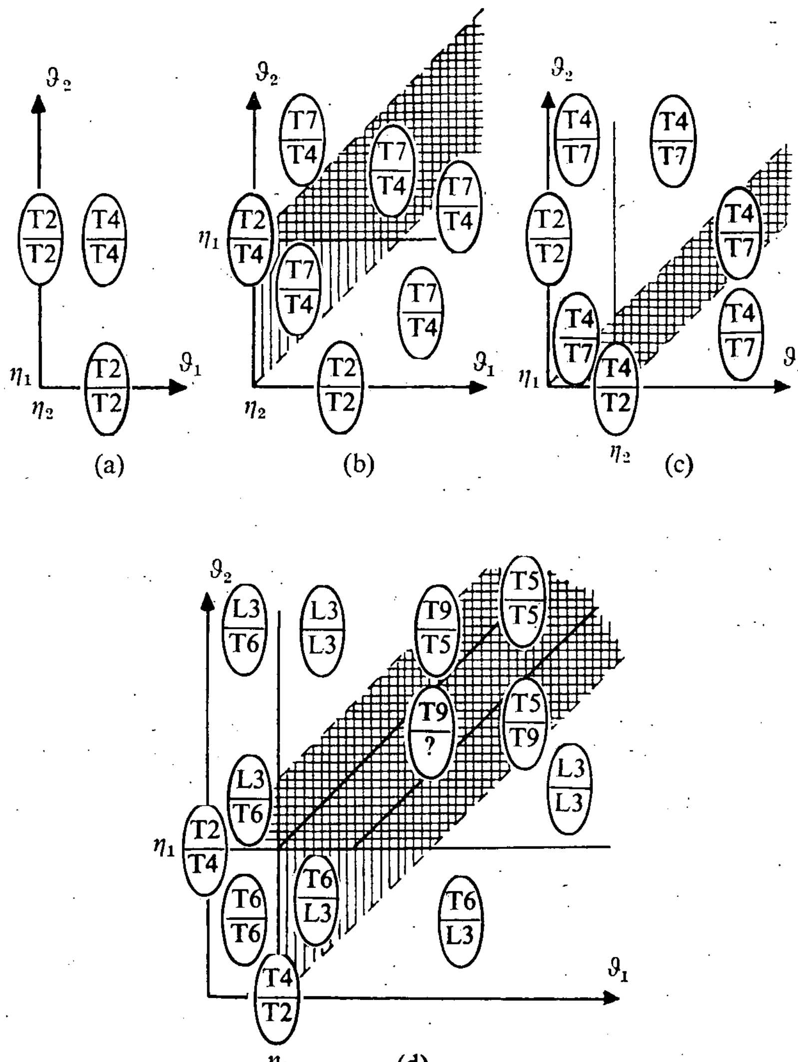

Fig. 10

The domains of $\mathcal{Q}$ where theorems answer the question of periodicity of $R_{1,2}(Q)$ (Tx) and $R_{2,1}(Q)$ (Ty) as $\frac{\text{Tx}}{\text{Ty}}$

By Lemma 3 any configuration Q is reducible to a Δ_1-reduced configuration Q^* or a defective configuration Q' with $\eta_1'\vartheta_2'=0$. This means that the questionable part of $\mathcal{Q}$ is reduced to the three-dimensional subspaces $\eta_1=0$, $\eta_2=0$, $\vartheta_1=0$, $\vartheta_2=0$ and to the four-dimensional domain of $\mathcal{Q}$ the two-dimensional cuts by fixing (η_1, η_2) of which are the domains (α), (b), and (γ) in Fig. 9d. Lemma 3 (L3) is used in Fig. 10 only when no other theorem answering the Question directly exists. In the three-dimensional subspaces $\eta_1\vartheta_2=0$ the Question of $R_{1,2}$ is solved by Theorem 2 if $\vartheta_1\tau_2=0$ and by Theorem 4 if $\vartheta_1\tau_2>0$. These solve the Question of $R_{2,1}$ in the subspaces $\eta_2\vartheta_1=0$. The Question of $R_{1,2}$ in the space $\vartheta_1=0$ and of $R_{2,1}$ in $\vartheta_2=0$ is solved by Theorem 2 independently of η_i and τ_{3-i}.

If $\eta_2=0$ but $\eta_1\vartheta_1\vartheta_2>0$ the Question of $R_{1,2}$ is answered by Theorem 7 and this answers the Question of $R_{2,1}$ if $\eta_1=0$ but $\eta_2\vartheta_1\vartheta_2>0$, too.

The Question is answered so for every defective configuration and, by Theorem 3, for every configuration reducible to a defective one by any of the operators Δ_1 and Δ_2. By Lemma 3 all other configurations are reducible by both of Δ_1 and Δ_2 to configurations Q^* and Q^{**}, respectively, which are in the domains (b) and (γ) and domains (a) and (γ), respectively, in Fig. 9d. Theorem 6 answers the Question of $R_{1,2}$ in the domain $\vartheta_2<\eta_1$ and of $R_{2,1}$ in the domain $\vartheta_1<\eta_2$ without reduction.

As far as the configurations Q reduced by both of Δ_1 and Δ_2 the Question of $R_{1,2}$ is answered by Theorem 5 in the domain $\tau_1\geqq\tau_2$ and the Question of $R_{2,1}$ in the domain $\tau_1\leqq\tau_2$. Theorem 9 answers the Question of $R_{1,2}$ in the domain $\vartheta_1\leqq\vartheta_2$ and the Question of $R_{2,1}$ in the domain $\vartheta_1\geqq\vartheta_2$.

In Fig. 10d the only questionable domain remained for $R_{2,1}$ is

$$\eta_2 \leqq \tau_2-\eta_1 < \vartheta_1 < \vartheta_2.$$

This contains "absolutely" (by both of Δ_1 and Δ_2) reduced configurations for which $\eta_1\leqq\vartheta_2<\tau_1$ and $\eta_2\leqq\vartheta_1<\tau_2$. In general, the unanswered domain of $\mathcal{Q}$, remaining only if $\eta_1\neq\eta_2$, is

$$0 < \eta_i \leqq \tau_i-\eta_{3-i} < \vartheta_{3-i} < \vartheta_i \quad \text{for} \quad R_{i,3-i} \quad \text{if} \quad \eta_i < \eta_{3-i}. \tag{123}$$

Further parts from the domain (123) are answered by results based upon the Lemma 10 and mentioned after (115) but not proved here. These are found in [4]. A direct answer is given by Theorem 6 for $R_{1,2}$ in the domain $\vartheta_2<\eta_1$ and for $R_{2,1}$ in the domain $\vartheta_1<\eta_2$ which is the answer for both schedules in the domain $0<\vartheta_i<$ $<\eta_{3-i}$, $i=1, 2$.

The flow of evaluation of the priority schedules $R_{1,2}$ and $R_{2,1}$ for a configuration Q is illustrated on the flow-chart in Fig. 11. Tx refers to the Theorem x and in $\boxed{x_1;\, y_1;\, x_2;\, y_2}$ x_i, y_i refer to the schedule $R_{i,3-i}$. $x_i=p$ means periodicity, $x_i=?$ refers to unanswered Question and $x_i=$other refers to the rationality of x_i as the condition of periodicity. $y_i=$number gives the efficiency value of $R_{i,3-i}$, $y_i=?$ refers to the undefinedness of the efficiency or unanswered Question and $y_i=$Tx refers to the Theorem x as means of determination of the efficiency. $(x_i, y_i)=\Delta_i$ refers to the application of the operator Δ_i iteratively until a configuration results which is in a domain where the schedule $R_{i,3-i}$ is directly evaluable by one of the Theorems 2, 4, 5, 6, 7, 9.

KEYWORDS: steady job-flow pairs, priority schedules, reduction method

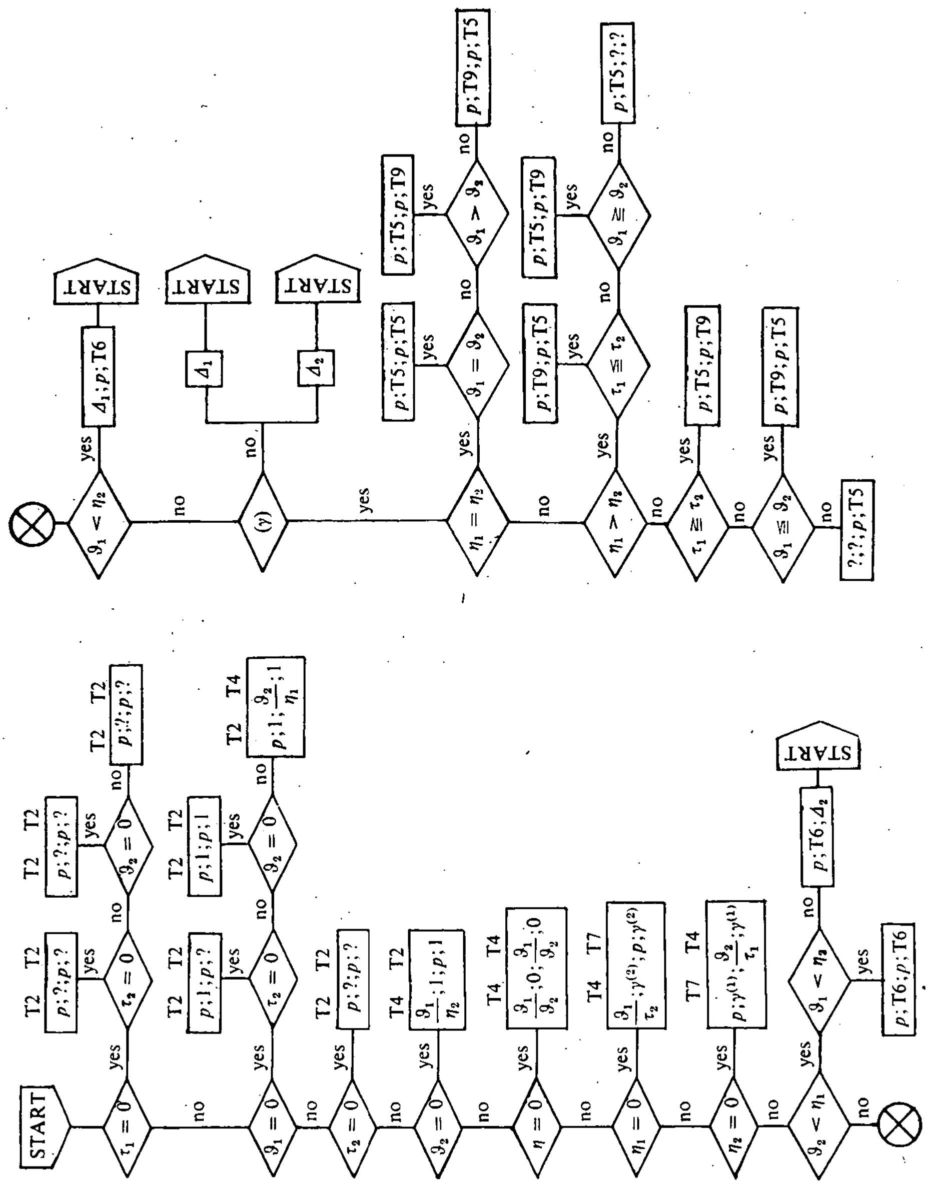

Fig. 11

The flow-chart of the evaluation of the priority schedules $R_{1,2}$ and $R_{2,1}$

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