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ACTA CYBERNETICA

FORUM CENTRALE PUBLICATIONUM CYBERNETICARUM HUNGARICUM

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ACTA CYBERNETICA

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Szeged, 1969. december
A szegedi József Attila Tudományegyetem gondozásában

Tomus 1.

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Tomus 1.

Statement of policy

You are starting to read the first issue of a new cybernetic periodical.

What is Cybernetics? The question is reasonable for, relating to a new branch of Science which touches philosophical problems as well, different persons or teams would answer it differently. Wiener's answer is contained in the subtitle of his pioneering work[1] thus: Control and Communication in the Animal and the Machine; however, since its publication, the object of cybernetic research has been considerably extended, due not in last line to Wiener's and his collaborators' work.

We use the term *Cybernetics* in the sense of a (developing) science dealing with such general laws of control, including informatic processes connected with it, which are valid in the case of different forms of motion[2] characteristic for the controlled material system, not excluding of course, in the case of some particular form of motion, a joint effect with the specific laws of motion of that particular form.

Besides papers revealing new cybernetic laws in this sense, the Editors welcome such papers of adequate scientific standard, dealing with control or informatic processes connected with control from the point of view of material systems existing in some concrete form of motion (e.g. technical devices, living organisms or human society), which presumably might serve as a basis for establishing new cybernetic laws in the future.

We consider as an important task of our periodical the publication of papers dealing with application of already known cybernetic laws to material systems of some concrete forms of motion. In our decision about publication of such "applied" papers, we consider in the first line the novelity and practical or scientific utility of the application in question, independently of whether that application leads or does not lead to new recognitions in respect of the general cybernetic law which has been applied.

Besides clear and precise presentation, we expect, the authors will formulate results of their research-work in an explicit, reproducible manner. This implies that programs, designs, experimental setting up and methods etc. have to be presented, though in a concise way, however, so as to give all details needed by an expert in order to do the calculations using any universal computer which is at this disposal, to

[1] N. WIENER, *Cybernetics*, Cambridge, Paris, New York, 1949.

[2] The term 'motion' is used here in a philosophical (rather than mechanical) sense, including every material change. Particular kinds of change, like that of spatial position, physical parameters, chemical composition, biological condition, social system etc., are to be considered as forms of motion. Also rest is regarded as a particular, degenerate case of motion, which is thus actually the general form of existence of matter.

construct the device, to repeat the experiment etc. and thus, to check the validity of the scientific results of the paper.

Accordingly, we reject papers for which essential details of the underlying research-work have to be regarded, by patent reasons or others, secret.

We definitively refrain from publishing works of science fiction.

We publish papers written in the Congress languages, i.e. English, French, German or Russian, possibly with short abstracts in another Congress language or Hungarian.

Our periodical will appear in single issues, four of which forming a volume. We try to achieve each volume to contain 200 pages at least; however, this depends on our financial possibilities.

Address of the Editor's office: *Acta Cybernetica, Academy Centre, Somogyi Béla u. 7, Szeged, Hungary*. We ask authors to send manuscripts as well as corrected proof-sheets to this address. Also, we invite Editors of cybernetic periodicals who wish to enter in exchange relationship with the *Acta Cybernetica*, to apply to the above address.

Linear Regular Languages. Part I[1]

By G. T. HERMAN

1. Introduction

In this paper we take a language to be any set of words (finite strings) ove a finite alphabet. We call those languages regular which can be described by a specia kind of finite expressions, the so called regular expressions. (Using a different terminology a language is regular if, and only if, it is right linear context free.)

Regular languages have a special significance from an engineering point of view, because they can be used to describe the behaviour of sequential circuits, which are often used in electrical engineering. Conversely, given a regular expression R one can design a binary sequential circuit which will in a certain sense accept amongst all the possible words over the alphabet only those which belong to the language described by R. In such a case we say that the circuit accepts the language described by R.

Certain subclasses of the class of regular languages are distinguished by the fact that the languages belonging to them can be accepted by some special kind of sequential circuits (e.g. feedback free).

In recent years there has been a great upsurge of interest in (binary) linear sequential circuits (i.e. circuits which use only unit delays and exclusive or gates). If a language is accepted by some linear sequential circuit, we call it a linear regular language.

In this paper we give a rigorous introduction to the concepts mentioned above and discuss the following problem: Given a regular expression R, how can we decide whether or not the language described by R is a linear regular language?

2. Regular Expressions

Definition 1. An *alphabet* S is a finite non-empty set of symbols.

Definition 2. Any finite string of symbols from an alphabet S is called a *word* over S. The set of all words over S is denoted by I_S. The empty word is denoted by e ($e \notin S$, $e \in I_S$).

[1] This paper has been presented at the International Colloquium on Mathematical Linguistics, Balatonszabadi, Hungary, Sept., 1968.

Definition 3. Any subset W of I_S is called a *language* over S. The set of languages over S is denoted by L_S.

Definition 4. The set E_S of *regular expressions* over the alphabet $S=\{s_1, s_2, \dots s_l\}$ is defined as follows:

(i) $s_1, s_2, \dots, s_l$, e, o are regular expressions ($o \notin S$).
(ii) If P and Q are regular expressions, so are $(P+Q)$, (PQ) and P^*.
(iii) All regular expressions are words over the alphabet $S \cup \{e, o, (, +,), ^*\}$, but only those words are regular expressions which can be shown to be such using (i) and (ii).

Definition 5. An *interpretation* of regular expressions over S is a function I: $E_S \to L_S$, defined by (for simplicity we denote $I(R)$ by $|R|$):

(i) $|s_i| = \{s_i\} \quad i = 1, 2, \dots, l,$
$|e| = \{e\},$
$|o| = \emptyset;$

(ii) $|(P+Q)| = |P| \cup |Q|,$
$|(PQ)| = \{pq;\ p \in |P|\ \&\ q \in |Q|\},$
$|P^*| = \bigcup_{n=0}^{\infty} \{p_1 p_2 \cdots p_n;\ p_i \in |P| \text{ for } 1 \leqq i \leqq n\}.$

Definition 6. A language W over S is called *regular* if, and only if, there exists a regular expression R over S such that

$$W = |R|.$$

Example 1. Let $S=\{a, b\}$. The set of all words which contains exactly two b's is regular. The regular expression which describes it is

$$(((a^*b)(a^*b))a^*).$$

Example 2. Let $S=\{a, b\}$. The set of all words which contain an even number (possibly none) of b's is regular. The regular expression which describes it is

$$(a^*((ba^*)(ba^*))^*).$$

3. Linear Sequential Circuits

Definition 7. The *clock* is a device which emits pulses at regular intervals to which we shall refer to as *units of time*.

A *delay* is a device with one input and one output wire such that it will emit a pulse on its output wire if, and only if, a pulse has been received at its input wire exactly one unit of time ago.

An *exclusive or gate* is a device with two input wires and one output wire, such that it will emit a pulse on its output wire if, and only if, it is receiving a pulse at exactly one of its input wires at the same time.

The existence of a pulse in a wire is denoted by 1, the lack of pulse by 0 ($0 \oplus 0 = 0$, $0 \oplus 1 = 1$, $1 \oplus 0 = 1$ and $1 \oplus 1 = 0$).

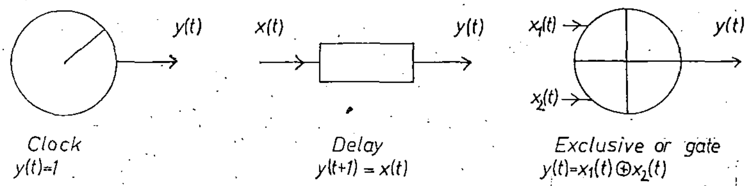

Fig. 1

Definition 8. A *linear sequential circuit* is a network formed from delays and exclusive or gates according to the following rules.

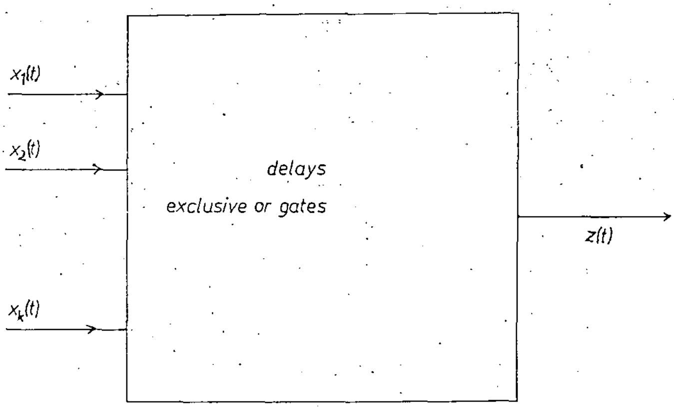

Fig. 2

The network will have a certain number (k, say) of external input wires and one external output wire. There is a clock, all wires may (or may not) carry pulses at the time when the clock emits a pulse, but will carry pulses at no other time. The propogation of pulses within wires is assumed to be instantaneous and hence the whole circuit is synchronized to the clock.

The external output wire and the input wires of any delay or exclusive or gate can be connected to any external input wire or the output wire of any delay or exclusive or gate provided only that the following restrictions are satisfied.

1. If there is a closed (feedback) path within the network (i.e. a point which we can get back to going along connected wires), it must go through at least one delay.

2. If there is a path from any external input wire to the external output wire, it must go through at least one delay.

Example 3. The following is a representation of a linear sequential circuit

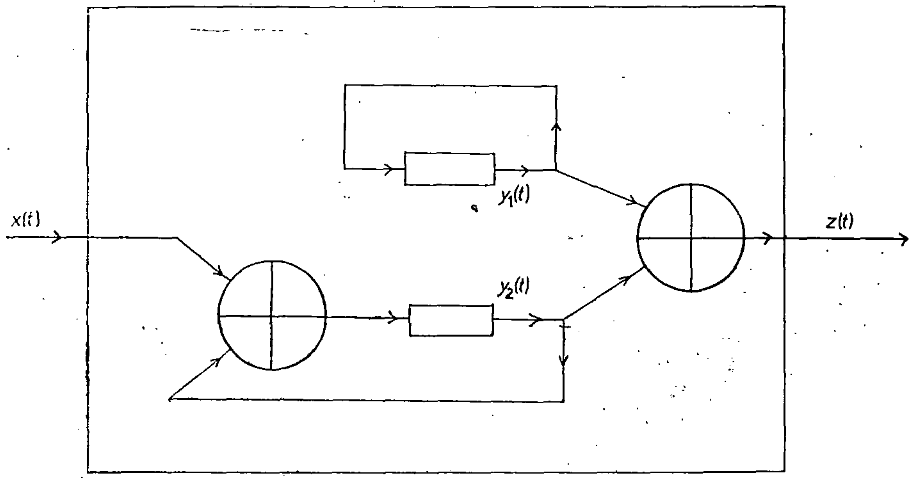

Fig. 3

Definition 9. Let W be a language over the alphabet S. W is called a *linear regular language* if there exists a positive integer k, a linear sequential circuit with k external inputs and a function f from S into k-tuples of 0's and 1's satisfying the following conditions.

For each symbol s of S we denote by s_j the jth element of $f(s)$ $(1 \leqq j \leqq k)$. This way each symbol s of S determines an external input condition for the linear sequential circuit, namely the jth external input wire carries a pulse if, and only if, $s_j = 1$.

A word $w = w_1 w_2 \ldots w_n$ belongs to W if, and only if, the following is true. If

(i) at time 1 the output of the first delay is 1 and of all other delays is 0,

and (ii) the external input condition to the circuit at time t is determined by w_t for $1 \leqq t \leqq n$,

then (iii) the external output at time $n+1$ is 1.

In such a case we say that the circuit *accepts* w.

Example 4. Let W be the language of Example 2. W is a linear regular language. This is true, because if we let $k = 1$, the linear sequential circuit to be that of Example 3, and f be defined by

$$f(a) = 0, \qquad f(b) = 1,$$

then the conditions of Definition 9 are satisfied.

In order to see this let us mark the external input, the outputs of the delays and the external output at time t by $x(t)$, $y_1(t)$, $y_2(t)$ and $z(t)$, respectively. Then

$$z(t) = y_1(t) \oplus y_2(t) = 1 \oplus y_2(t)$$

(therefore $z(t) = 1$ if, and only if, $y_2(t) = 0$),

$$y_2(t+1) = x(t) \oplus y_2(t)$$

and $$y_2(1) = 0.$$

So $y_2(t) = 0$ initially and it changes value at time $t+1$ if, and only if, $x(t) = 1$. So it accepts w if, and only if, $w \in W$.

4. The Relationship between Linear Regular and Regular Languages

Linear regular and regular languages have been defined in entirely different ways. In this section we show that every linear regular language is in fact regular, but there are regular languages which are not linear regular. In order to do this we need to establish some properties of linear sequential circuits.

Definition 10. Let C be a linear sequential circuit which has n delays. We number these delays from 1 to n, and let $y_i(t)$ denote the state of the output wire from the ith delay at time t ($y_i(t)=0$ means no pulse, $y_i(t)=1$ means pulse). Then the n-tuple $(y_1(t), y_2(t), \ldots, y_n(t))$ is said to be the *state* of the circuit C at time t.

Theorem 1. The operation of a linear sequential circuit C with k external input wires and n delays can be completely described by two matrix equations of the form

$$y(t+1) = y(t)\boldsymbol{A} \oplus x(t)\boldsymbol{B} \tag{1}$$

$$z(t) = y(t)\boldsymbol{C} \tag{2}$$

where $x(t)$ is the k-tuple of inputs at time t, $y(t)$ is the state at time t, $z(t)$ is the output at time t, $\boldsymbol{A}$ is a $n\times n$ matrix, $\boldsymbol{B}$ is a $k\times n$ matrix, and $\boldsymbol{C}$ is an $n\times 1$ matrix of 0's and 1's. All additions are to be performed modulo 2.

Proof. We deal with the second equation first. Starting from the external output wire we go backwards along all possible paths. When we come to an exclusive or gate the path splits into two. A path comes to an end when it reaches the output wire of a delay. Because of the restrictions in Definition 8 this must happen to every path. It is clear that the output is the number of paths (modulo 2) which finish at a wire which carries a pulse at the time. The output wire of a delay can modify this sum if, and only if, the number of paths leading to it is odd. So the construction of the circuit uniquely determines the value of $\boldsymbol{C}$.

Since the output of a delay is equal to its input at one unit of time ago, analogous argument shows that $\boldsymbol{A}$ and $\boldsymbol{B}$ are uniquely determined.

It is clear that the two equations completely describe what external output will result from a given sequence of inputs. In fact

$$z(t+1) = (1, 0, \ldots, 0)\boldsymbol{A}^t\boldsymbol{C} \oplus \sum_{i=1}^{t} x(i)\boldsymbol{B}\boldsymbol{A}^{t-i}\boldsymbol{C} \tag{3}$$

as can easily be shown by induction on t.

Example 5. The circuit in Example 3 can be characterized by the equations

$$(y_1(t+1), y_2(t+1)) = (y_1(t), y_2(t))\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \oplus x(t)[0, 1],$$

$$z(t) = (y_1(t), y_2(t))\begin{bmatrix}1\\ 1\end{bmatrix}.$$

Therefore,

$$z(t+1) = (1,0)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}^t\begin{bmatrix}1\\1\end{bmatrix} \oplus \sum_{i=1}^{t} x(i)[0,1]\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}^{t-i}\begin{bmatrix}1\\1\end{bmatrix} = \sum_{i=1}^{t} x(i) \oplus 1,$$

showing again that this circuit accepts the language described in Example 2.

Theorem 2. Every linear regular language is regular.

Proof. (Although this theorem is the consequence of a well known theorem in automata theory, here we give a direct proof, which is conceptually considerably simpler than the proof of the general theorem and which also has interesting consequences.)

Let W be the linear regular language over S and C be the linear sequential circuit which accepts it using the function f from S into k-tuples of of 0's and 1's. For C there will be matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ as described in Theorem 1. In particular equation (3) will hold.

$\boldsymbol{A}$ is an $n \times n$ matrix with elements 0 or 1. There are only finitely many such matrices and there must be positive integers u and v such that $u<v$ and $\boldsymbol{A}^u = \boldsymbol{A}^v$. Choose u and v the smallest such integers and let $d = v-u$. If $t = qd+u+r$ where $0 \leqq r < d$, then

$$\begin{aligned} z(t+1) = {} & \sum_{i=1}^{d} x(i)\boldsymbol{B}\boldsymbol{A}^{v+r-i}\boldsymbol{C} \oplus \\ & \oplus \sum_{i=1}^{d} x(d+i)\boldsymbol{B}\boldsymbol{A}^{v+r-i}\boldsymbol{C} \oplus \\ & \dots \\ & \oplus \sum_{i=1}^{d} x((q-1)d+i)\boldsymbol{B}\boldsymbol{A}^{v+r-i}\boldsymbol{C} \oplus \\ & \oplus \sum_{i=1}^{u+r} x(qd+i)\boldsymbol{B}\boldsymbol{A}^{u+r-i}\boldsymbol{C} \oplus (1,0,\dots,0)\boldsymbol{A}^{u+r}\boldsymbol{C}. \end{aligned}$$

Let P_0 be o if $c_1=0$ and P_0 be e if $c_1=1$. Let $P_1, P_2, \dots, P_p$ be all the regular expressions over S of the form $(\dots(w_1 w_2)\dots w_t)$ $(w_i \in S)$ for which $t<u$ and

$$\sum_{i=1}^{t} f(w_i)\boldsymbol{B}\boldsymbol{A}^{t-i}\boldsymbol{C} \oplus (1,0,\dots,0)\boldsymbol{A}^t\boldsymbol{C} = 1.$$

It is in principle easy to enumerate all such expressions. Let P be the regular expression

$$(\dots((P_0+P_1)+P_2)+\dots+P_p).$$

For $r=0,1,\dots,d-1$ let $X_{r1},\dots,X_{rx_r}$ be all regular expressions of the form $(\dots(w_1 w_2)\dots w_{u+r})$ for which

$$\sum_{i=1}^{u+r} f(w_i)\boldsymbol{B}\boldsymbol{A}^{u+r-i}\boldsymbol{C} \oplus (1,0,\dots,0)\boldsymbol{A}^{u+r}\boldsymbol{C}$$

is equal to 0, and $Y_{r1}, \ldots, Y_{ry_r}$ be all such expressions for which the same sum is equal to 1.

Let $F_{r1}, \ldots, F_{rf_r}$ be all regular expressions of the form $(\ldots(w_1 w_2)\ldots w_d)$ for which

$$\sum_{i=1}^{d} f(w_i) \boldsymbol{B} \boldsymbol{A}^{v+r-i} \boldsymbol{C}$$

is equal to 0, and $G_{r1}, \ldots, G_{rg_r}$ be all such expressions for which the same sum is eqalu to 1.

Let X_r be the regular expression

$$(\ldots(X_{r1}+X_{r2})+\ldots+X_{rx_r})$$

or o if $x_r=0$, and let Y_r, F_r and G_r be similarly defined.

Let D_r be the regular expression

$$(((F_r^*((G_r F_r^*)(G_r F_r^*))^*)(G_r F_r^*))X_r)$$

and E_r be the regular expression

$$((F_r^*((G_r F_r^*)(G_r F_r^*))^*)Y_r).$$

Then the regular expression

$$(P+(\ldots(D_0+E_0)+\ldots+(D_{d-1}+E_{d-1})))$$

describes W.

Example 6. For the circuit of Example 3, $u=1$, $v=2$, $d=1$. Looking at Example 5 we see that

P_0 is e,	$p=0$,	P is e;	
	$x_0=1$,	X_{01} is b,	X_0 is b;
	$y_0=1$,	Y_{01} is a,	Y_0 is a;
	$f_0=1$,	F_{01} is a,	F_0 is a;
	$g_0=1$,	G_{01} is b,	G_0 is b;

$$D_0 \text{ is } (((a^*((ba^*)(ba^*))^*)(ba^*))b),$$

$$E_0 \text{ is } ((a^*((ba^*)(ba^*))^*)a).$$

Hence a regular expression which describes the language accepted by this circuit is

$$(e+((((a^*((ba^*)(ba^*))^*)(ba^*))b)+((a^*((ba^*)(ba^*))^*)a))).$$

A much simpler expression which describes the same language has been given in Example 2.

Theorem 3. Not every regular language is linear regular.

Proof. The language W described in Example 1 is regular, but it is not linear regular.

To see this we have to look at the proof of Theorem 2.

Suppose W is accepted by some linear sequential circuit. Let w be the word

$$bba^v;$$

then $w \in W$. Since

$$W = |(P + (\ldots(D_0 + E_0) + \ldots + (D_{d-1} + E_{d-1})))|,$$

$w \in |P|$ or $w \in |D_r|$ or $w \in |E_r|$ for some r.

However, $w \notin |P|$, since it is too long.

If $w \in |D_r|$, then

$$bba^{v-r} \in |(((F_r^*(G_r F_r^*)(G_r F_r^*))^*)(G_r F_r^*))|$$

and so b must occur in some word belonging to $|G_r|$ or $|F_r|$. But then there must be words belonging to W in which b appears more than twice, contradicting the definition of W.

The case $w \in |E_r|$ is analogous.

5. Characterization of Linear Regular Languages

In the last section we have shown that every linear regular language can be described by a regular expression of a certain type. Now we consider the problem: Given a regular expression R, how can we decide whether or not the language described by R is a linear regular language?

Unfortunately the decision method used is quite complicated, it will form Part II of this paper. Its essence is the checking the linearity of the languages over k-tuples of 0's and 1's which are the images of $|R|$ under some functions f mapping S into such k-tuples.

The method is not only indirect, but is needs the introduction of a number of new ideas. It seems likely that there is a more direct solution, something along the lines of the proof of Theorems 2 and 3. The kind of question which we ought to try to answer is: What do regular expressions which describe linear regular languages look like?

It was mentioned in the introduction that a language is regular if, and only if, it can be described by a right linear context free (Chomsky) grammar. A similar question to the above is: What are the characteristics of those grammars which describe linear regular languages?

6. Final Comments and Acknowledgements

There is very little in this first part of the paper which has not already appeared elsewhere is one form or another. However this is the first time that the results have been discussed from the point of view of linear regular languages, and the novel approach bore fruit in the proofs of Theorems 2 and 3.

The algorithm given in Theorem 2 for analyzing linear sequential circuits works faster than any other algorithm published so far for this purpose. This is not really surprising since other published algorithms can be used to analyse sequential circuits in general, while our algorithm makes essential use of the linearity of the sequential circuit under consideration.

Because the paper is introductory, all definitions and results have been fully explained and references to original sources (sometimes difficult to ascertain) have been omitted. The bulk of the author's knowledge is based upon lecture courses given at the University of California by Professors J. A. Brzozowski and M. A. Harrison. His indebtedness to both these gentlemen is great.

A comprehensive list of references is given, which includes many original papers on the topic under discussion including those which have been used for Part II. The author apologizes for the possible omission of relevant papers, and hopes that they are referenced in the literature cited below.

Most of these papers refer to the closely related problem of the linearity of sequential machines. A good critical review of the work done on this problem can be found in Davis [10].

The author wishes to invite any comments relevant to the problems discussed here.

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(Received March 12, 1969)

Ein analog-elektrisches Modell zur Simulation der Lymphokinetik

VON Z. M. HANTOS

A) Problemstellung

Der Flüssigkeitstransport zwischen den intravasalen und interstitiellen Räumen findet auf dem Gebiete der Kapillaren statt. Dementsprechend weicht die Wandstruktur der Kapillaren von der in anderen Blutbahnen vorhandenen Wandstrukturen stark ab. Die verschiedenen Bestandteile des Blutes haben verschiedene Möglichkeiten, um das Kapillarlumen zu verlassen. Der Flüssigkeitsaustausch kommt als ein Resultat der Diffusion und Filtration zustande. Die Filtration wird von der Resultante des hydrostatischen und des kolloidosmotischen Drucks verursacht (Starlingsche Hypothese), eine Konzentrationsdifferenz zwischen den auf beiden Seiten der Kapillarwand liegenden Flüssigkeitsräume führt zum Diffusionsaustausch. Beide Prozesse sind im strengen Zusammenhang miteinander, jedoch ist es anzunehmen, daß die Größe der aus der Diffusion stammenden Fraktion mit der hydrodynamischen Fraktion vergleichen im Falle der Wandstrukturen reeller Kapillaren verschwindend klein ist [3].

Das Kapillargebiet wird im allgemeinen von dem Verhältnis des Filtrations-, und Resorptionsdrucks in zwei Teilgebiete geteilt. Ist

$$p_c + k_i > p_i + k_c, \qquad (1)$$

so zeigt die Richtung der Strömungsresultierende in den extravasalen (interstitiellen) Raum. In diesem Fall handelt es sich um eine effektive Filtration. Ist jedoch $p_c + k_i < p_i + k_c$, so geschieht eine effektive Resorption. Hierbei sind p_c und p_i die hydrostatischen Druckwerte im intravasalen bzw. extravasalen Raum, k_c und k_i die kolloidosmotischen Druckwerte des Plasmas bzw. der interstitiellen Flüssigkeit.

Das Lymphgefäßsystem entspringt dem Interstitium, und bildet ein konvergentes Netzwerk; die Lymphkapillaren münden in zahlenmäßig wenigere, aber ein größeres Lumen besitzende Gefäße, Lymphstämme. Die Lymphe des ganzen Körpers wird von einigen großen Stämmen (z. B. ductus thoracicus, truncus lymphaticus dexter) in die großen Venen eingeleitet. Die größeren Lymphbahnen sind von Lymphknoten unterbrochen, die man vom hydrodynamischen Standpunkt aus als verlustbehaftete Filter betrachten kann. Die schematische Anordnung beider Zirkulationssysteme wird von Abb. 1 dargestellt.

Die Lymphkapillaren haben geschlossene Anfänge mit Spalten, die verschiedene Elementen der interstitiellen Flüssigkeit dringen frei in die Lymphkapillaren ein. Nach morphologischen Untersuchungen sind diese Spalten als nichtreziproke hydrodynamische Verbindungen vorstellbar. Die Reziprozität gilt auch für die axiale Strömung der größeren Lymphgefäße nicht, da die in diesen Gefäßen vorhandenen Klappen nur eine zentripetale Strömung ermöglichen. Im Falle der Veränderung des Außendruckes zeigt ein zwischen zwei Klappen liegender Abschnitt eine pumpenartige Wirkung. Diese Schwankung des Außendruckes kann aus der Übertragung der Wandbewegung von den naheliegenden Arterien, bei thorakalen Lymphgefäßen aus der Schwankung des subatmospherischen Innendruckes des Thorax stammen. Es wurden in einigen Regionen auch spontane Kontraktionen der Lymphgefäße beobachtet.

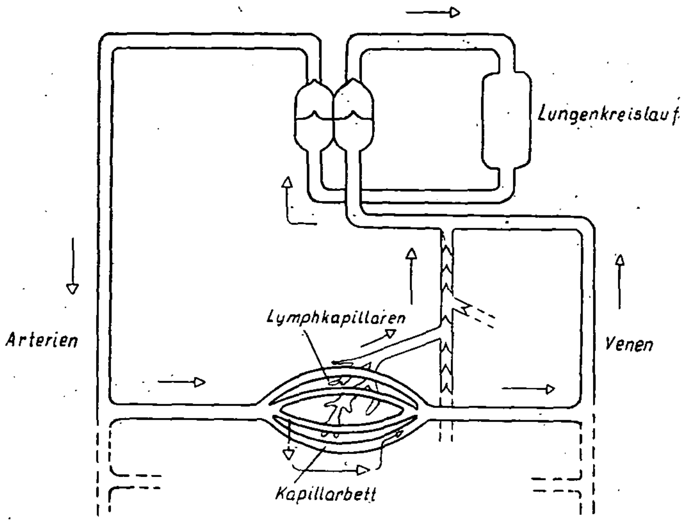

Abb. 1

Die analog-elektrische Simulation der Lymphzirkulation muß sich auf die hydrodynamischen Erscheinungen allein beschränken, die Änderungen der chemischen Zusammensetzung des Plasmas, der interstitiellen Flüssigkeit und der Lymphe sind nicht zu berücksichtigen. Dementsprechend hat der Simulationsprozeß einen vermittelnden Schritt: die Aufstellung eines theoretischen-hydrodynamischen Modells, in dem die alle im physiologischen Objekt ausgeführten Vereinfachungen, Näherungen und die hypothetischen Elementen enthalten werden. Das Gleichungssystem bezüglich dieses Modells ist durch Substitution elektrischer Größen zu realisieren. Infolge des gemeinsamen mathematischen Apparates werden beide Modelle von strenger Analogie verbunden.

B) Das Kapillarnetz

Als hydrodynamisches Modell ist ein poröses Rohr mit unbeweglicher Wand zu betrachten. Die Achse des Rohres wird als Koordinatenachse x gewählt. Dann wird die axiale hydrostatische Druckänderung des Blutes durch die Funktion $p_c(x)$, und die axiale Strömung durch die Strömungsintensitätsfunktion $f(x)$ beschrieben.

Die pro Zeiteinheit durch die Flächeneinheit übertretende elementare Flüssigkeitsmenge (transmurale Strömungsintensität) ist nach der Starlingschen Hypothese:

$$df_t = g\,dA(p_c - p_i - k_c + k_i). \quad (2)$$

Mit der elementaren Länge dx:

$$df_t = g(p_c - p_i - k_c + k_i)\,H\,dx. \quad (3)$$

Hierbei ist g die in z. B. ml/mm².Hgmm.min gemessene Permeabilität, H ist der Umfang des Rohres, der einen Gesamtumfang der einzelnen, unter gleichem Druck befindlichen Kapillarquerschnitte bedeutet (s. Abb. 2).

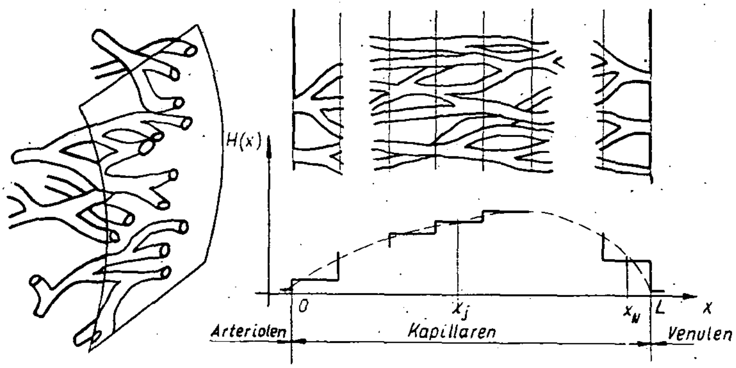

Abb. 2

Die Resultante der Flüssigkeitsintensitäten zwischen dem Rohrmodell — das man durch die auf die hydrodynamischen Parameter bezüglichen Konzentration des Kapillarnetzes erhalten kann — und dem elastischen Gefäß, das den interstitiellen Raum repräsentieren soll, ist erhaltbar, wenn wir die elementaren transmuralen Flüssigkeitsintensitäten summieren:

$$f_t = \int_0^L gH(p_c - p_i - k_c + k_i)dx\,. \quad (4)$$

Die hydrodynamischen Parameter des Modells verändern sich längs der Koordinate x kontinuierlich. Um das Modell mit einem Netzwerk von konzentrierten Parametern realisieren zu können, verteilen wir das Rohr längs der Koordinate x in N gleiche Strecken, innerhalb welcher die hydrodynamischen Parameter als konstant betrachtet sind.

Den hydrodynamischen Widerstand der j-ten Strecke kann man unmittelbar aus den geometrischen und physikalischen Daten mit Hilfe des Poiseuilleschen Gesetzes nicht berechnen, da diese Rohrstrecke aus kleinen, parallel gekoppelten Röhrchen zusammengesetzt ist, deren Diameter mit den Blutkörperchen maßverwandt sind. Infolgedessen wird die Innenreibung der strömenden Flüssigkeit von der Deformationsarbeit der Blutkörperchen, und die Wandreibung von den Stößen an die Wandunebenheiten gesteigert. Die auf den hydrodynamischen Widerstand ausgeübte Wirkung der obigen Prozesse läßt sich entweder auf experimentelle Weise oder mit Hilfe eines verwickelten physikalischen Modells feststellen. So haben wir die Widerstandsteigerung als eine virtuelle Erhöhung der Viskosität aufzufassen:

$$\eta^* = \eta + \eta_A, \tag{5}$$

wo η_A die die Innen- und Wandreibung steigernde Wirkung der Blutkörperchen repräsentiert.

So ergibt sich für den Widerstand der j-ten Strecke nach Poiseuille:

$$R_{jk} = \frac{8\eta^* \frac{L}{N}}{r_k^4 \pi}. \tag{6}$$

Die parallele Resultante der M_j Stücke ist:

$$\frac{1}{R_j} = \sum_{k=1}^{M_j} \frac{1}{R_{jk}}. \tag{7}$$

Mit Durchschnittsdiametern $\bar{r}$ ausgedrückt:

$$R_j = \frac{1}{M_j} \frac{8\eta^* \frac{L}{N}}{\bar{r}^4 \pi}. \tag{8}$$

Die hydrodynamische Konduktivität der Strecke läßt sich folgendermaßen beschreiben:

$$G_j = g(x_j) H(x_j) \frac{L}{N}. \tag{9}$$

Die kolloidosmotischen Druckwerte der auf den beiden Seiten des Rohres befindlichen Flüssigkeiten werden von den folgenden Gleichungen beschrieben [3]:

$$\begin{aligned} k_c &= a_1 c_c + a_2 c_c^2 + a_3 c_c^3, \\ k_i &= a_1 c_i + a_2 c_i^2 + a_3 c_i^3. \end{aligned} \tag{10}$$

Die Proteinkonzentration c_c für das Plasma kann man als eine Quotient der Strömungsintensitäten ausdrücken:

$$c_c = \frac{f_{\text{prot}}}{f}. \tag{11}$$

So läßt sich (10) auf die folgende Form bringen:

$$k_c = a_1 \frac{f_{\text{prot}}}{f} + a_2 \frac{f_{\text{prot}}^2}{f^2} + a_3 \frac{f_{\text{prot}}^3}{f^3} = \frac{b_1}{f} + \frac{b_2}{f^2} + \frac{b_3}{f^3} = \varphi(f). \tag{12}$$

Hierbei wurde $f_{\text{prot}}(x) = f_{\text{prot}} =$ konstant angenommen, d. h. die auf Protein bezogene Permeabilität der Rohrwand wird nur im von Null abweichenden kolloidosmotischen Druck der interstitiellen Flüssigkeit erscheinen, aber nicht im Proteinverlust des Plasmas.

Die Zirkulationszustände der j-ten Rohrstrecke (Abb. 3) sind von den folgenden Gleichungen beschrieben:

$$f_{b,j}-f_{k,j}-f_{t,j} = 0, \quad (13)$$

$$f_{t,j} = G_j(p_{c,j}-p_i-k_{c,j}+k_i), \quad (14)$$

da p_i und k_i, infolge der Struktur des Modells, konstant sind.

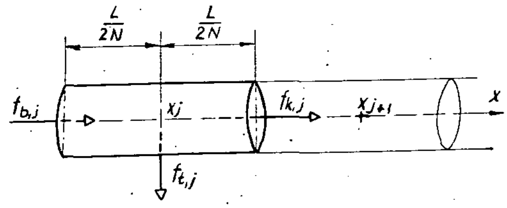

Abb. 3

Es gilt noch für die Anpassung zweier Strecken die Gleichung:

$$f_{b,j} = f_{k,j} = \frac{p_{c,j-1}-p_{c,j}}{\frac{R_{j-1}}{2}+\frac{R_j}{2}}. \quad (15)$$

Bedeuten die obigen Bezeichnungen nach der Tabelle 1 elektrische Größen, so werden die Gleichungen von einem Netzwerk veranschaulicht (Abb. 4).

Wenn wir die Charakteristik $k_{c,j}=\varphi(f_j)$ realisieren, kann das Steuersignal der gesteuerten Spannungsquelle, unter den Bedingungen $f_{t,j} \ll f_{b,j}$, d. h. $f_{b,j} \cong f_j$, $f_{b,j}$ sein. Das elektrische Äquivalent der Rohrstrecke verliert dann seine geometrische Symmetrie (Abb. 5).

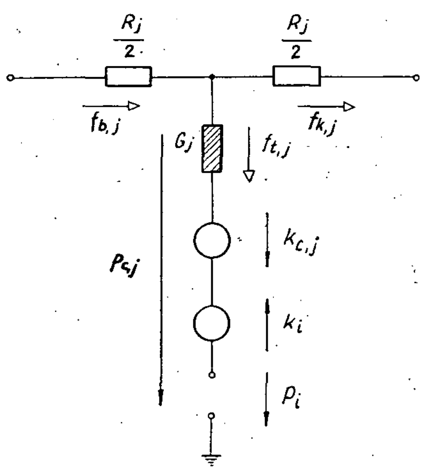

Abb. 4

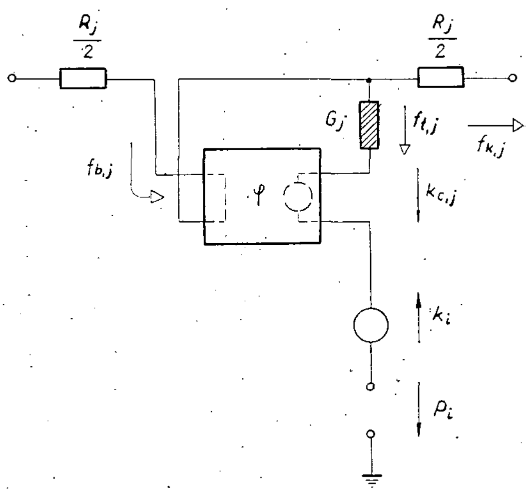

Abb. 5

Unsere ausführlicheren mathematischen Modelle von dem Flüssigkeitsaustausch der Kapillaren zeigten, daß die Strömungen zwischen den intravasalen und interstitiellen Räumen von der Änderung des kolloidosmotischen Druckes im Plasma stark beeinflußt werden. Es ist jedoch bei niedrigen Permeabilitäten anzunehmen, daß die Resultante der Strömungen im Falle $k_c \neq$ konstant von dem unter der Be-

	hydrodynamische Größen	elektrische Größen
p	Druck	Spannung
f	Strömungsintensität	Stromstärke
V	Volumen	Ladung
R	Widerstand	Widerstand
L	Trägheit	Induktivität
C	Elastizität	Kapazität
G	Konduktivität (Filtrationskoeffizient, Permeabilität)	Konduktivität

Tab. 1

dingung k_c = konstant gerechneten Wert nicht wesentlich abweicht. Nun wirkt ein effektiver kolloidosmotischer Druck $k_{eff} = k_c - k_i$ entlang dem Rohr.

Die Vierpole von einer Ausgangspannung $k_{c,j} = \varphi(f_j)$ können weggelassen werden, die Spannungsquellen $k_{c,j}$ lassen sich mit den anderen Strecken gemeinsam aufnehmen (Abb. 6).

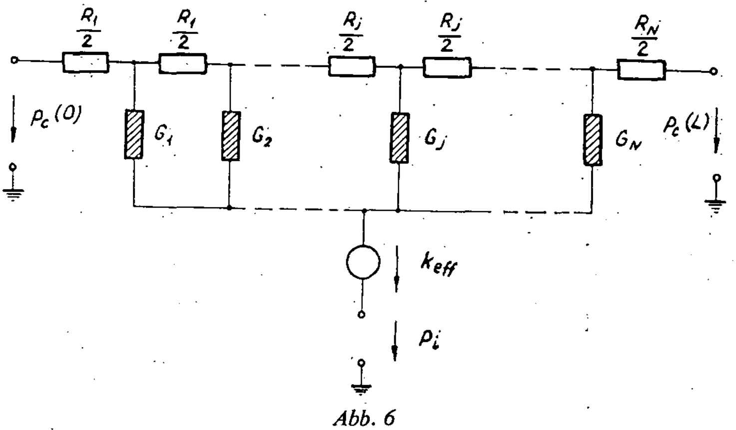

Abb. 6

C) Der extravasale Raum

Der Druck p_i, der im Interstitium herrscht, wird von der Resultante der transmuralen Strömungen, von der Transportfunktion der Lymphbahnen, und von der Elastizität des Interstitiums gebildet.

Wenn die Transportfunktion des Lymphgefäßsystems aus irgendeinem Grund aufhört, oder sich als ungenügend erweist, oder das normale Transportvermögen die infolge der Venenstauung erhöhte Filtration nicht nachfolgen kann, so häuft sich Flüssigkeit in großer Menge im Interstitium an. Bei einer gewissen Akkumulation erhöht sich p_i in solchem Grade, daß ein Äquilibrium der Filtrations- und

Resorptionsprozesse wiederhergestellt wird und die Menge der Ödemflüssigkeit stabilisiert sich.

Der Mechanismus des Flüssigkeitsverkehrs durch die Lymphkapillarwand ist noch nicht genau bestimmt worden. Es ist im unseren hydrodynamischen Modell anzunehmen, daß es sich um klappenartige Verbindungen handelt (Abb. 7).

Das Volumen des elastischen Speichers, welcher das Interstitium repräsentiert, sei bei einem Druck p_i durch V_i bezeichnet. Die vom Interstitium während der Zeitenheit aufgenommene Flüssigkeitsmenge f_i läßt sich folgendermaßen ausdrücken:

$$f_i = \frac{dV_i}{dt}. \tag{16}$$

Die Resultante der transmuralen Strömungen ist:

$$f_e = \sum_{k=1}^{N} f_{t,k}. \tag{17}$$

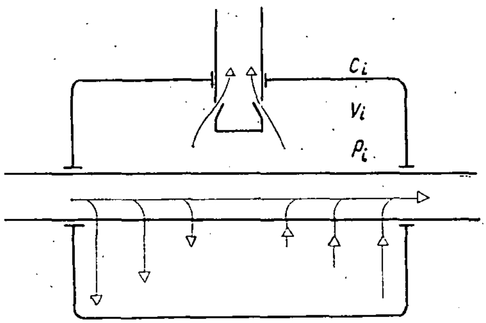

Abb. 7

Wird endlich die Strömungsintensität der Flüssigkeit, welche von den Lymphbahnen wegbefördert wird, durch f_l gezeichnet, so gilt für das Interstitium die Kontinuitätsbedingung:

$$f_e + f_l + f_i = 0. \tag{18}$$

Definieren wir die Elastizität des Interstitiums als

$$C_i = \frac{V_i}{p_i}, \tag{19}$$

so erhalten wir

$$\frac{d}{dt}(C_i p_i) = C_i \frac{dp_i}{dt}. \tag{20}$$

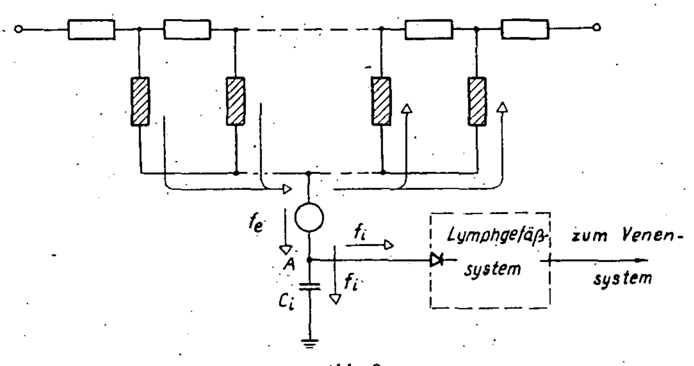

Abb. 8

Nach Substituierung elektrischer Größen können die Gleichungen von dem obenstehenden analogischen Netzwerk realisiert werden (Abb. 8). Das elektrische Äquivalent der Kontinuitätsbedingung wird hier als eine auf den Punkt A aufgeschriebene Knotengleichung betrachtet.

D) Die Lymphgefäße

Als hydrodynamisches Modell der Lymphgefäße betrachten wir eine Kette elastischer Speicher, die mit starren Rohrstrecken verbunden werden. Der hydrodynamische Widerstand einer zwischen zwei Klappen liegenden Strecke wird auf die starre Rohrstrecke konzentriert, die Elastizität und die hydrodynamische Konduktivität werden dem elastischen Speicher zugeeignet. Betrachten wir die Klappen als solche Ventile, die in einer Stromrichtung unendlich großen, und in der entgegengesetzten Richtung einen endlichen hydrodynamischen Widerstand zeigen (Abb. 9).

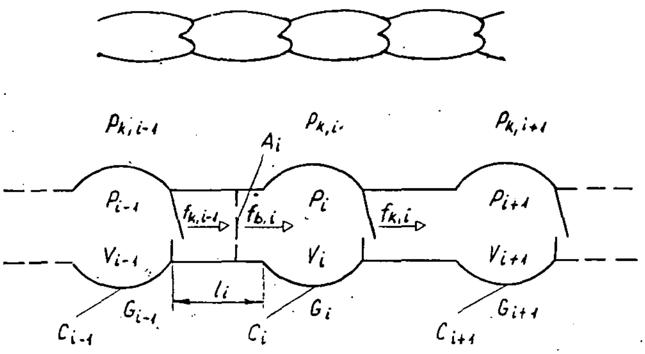

Abb. 9

Wir können die Aufschreibung der hydrodynamischen Gleichungen erleichtern, wenn wir die Anwesenheit der Klappen zuerst unbeachtet lassen und nur im zweiten Schritt in Form einer Strömungsbedingung zurückstellen.

In Abb. 9 bezeichnen wir den Innendruck des i-ten Speichers mit p_i, seinen Außendruck mit $p_{k,i}$. Da die Speicher mit starren, apermeablen Röhren verbunden sind, so kann man für die Strömungsintensitäten schreiben:

$$f_{k,i-1} = f_{b,i} = f_i. \qquad (21)$$

Die Kontinuitätsgleichung für den i-ten Speicher ist:

$$f_i - f_{i+1} = \frac{dV_i}{dt} + f_{t,i}, \qquad (22)$$

wobei $f_{t,i}$ der Flüssigkeitverlust ist, der auf der i-ten Strecke auftritt. Es ist anzunehmen, daß dieser Verlust nur von der Differenz der Außen- und Innendruckwerte abhängt, d. h.

$$f_{t,i} = G_i(p_i - p_{k,i}), \tag{23}$$

wobei G_i in allgemeinem eine Funktion der Druckdifferenz $\Delta p_i = p_i - p_{k,i}$ ist. Nehmen wir an, daß die Speicher auch im Falle $p_i = p_{k,i}$ ein endliches Volumen $V_{0,i}$ haben, und gegen die transmurale Druckdifferenz Δp_i eine lineare Abhängigkeit zeigen, d. h.

$$V_i = V_{0,i} + C_i \Delta p_i. \tag{24}$$

Bilden wir die erste Ableitung der Gleichung (24) nach der Zeit, und setzen wir sie zusammen mit (23) in (22) ein, so ergibt sich:

$$f_i - f_{i+1} = C_i \frac{d}{dt} \Delta p_i + G_i \Delta p_i. \tag{25}$$

Nun untersuchen wir die Strömung in der starren Rohrstrecke. Es treten gegen die die Flüssigkeitsäule bewegende Kraft mit der Beschleunigung und der Geschwindigkeit proportionelle Kräfte auf:

$$F = m\ddot{z} + \beta\dot{z}. \tag{26}$$

Im unseren Falle:

$$A_i(p_{i-1} - p_i) = \varrho l_i A_i \frac{d}{dt} \frac{f_i}{A_i} + R_i A_i f_i. \tag{27}$$

Nach Vereinfachungen und Einführung des Koeffizienten $L_i = \varrho l_i / A_i$, der die Trägheit der Flüssigkeitsäule ausdrückt, erhält (27) die Form:

$$p_{i-1} - p_i = L_i \frac{df_i}{dt} + R_i f_i. \tag{28}$$

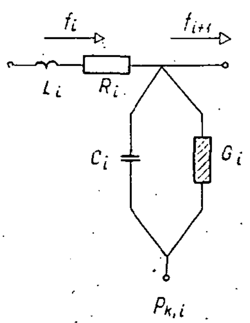

Abb. 10

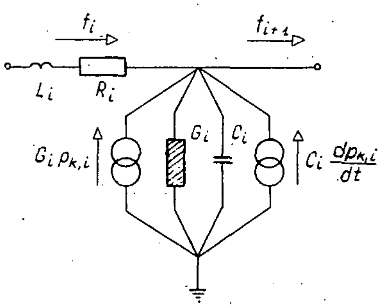

Abb. 11

Die zu den Gleichungen (25) und (28) gehörige elektrische Schaltung wird von Abb. 10 veranschaulicht. Formt man die (25) Gleichung um, so ergibt sich eine neue Knotengleichung:

$$f_i - f_{i+1} = C_i \frac{dp_i}{dt} + G_i p_i - C_i \frac{dp_{k,i}}{dt} - G_i p_{k,i}, \tag{29}$$

und eine äquivalente Schaltung (Abb. 11).

Die Wirkung der Außendrucksänderung läßt sich nun auch durch Stromgeneratoren repräsentieren. Der in einer niedrigen Elementenzahl liegende Vorteil der ersten Schaltung ist jedenfalls offensichtlich.

Es werden für die Strömungsintensität bezüglich einer in Sperrrichtung idealen Klappe die folgenden Bedingungen vorgeschrieben:

$$f_i = \begin{cases} f_i, & \text{wenn } p_{i-1} > p_i, \\ 0, & \text{wenn } p_{i-1} \leqq p_i. \end{cases}$$

Diese Bedingungen können in der analogischen Schaltung mit Dioden realisiert werden.

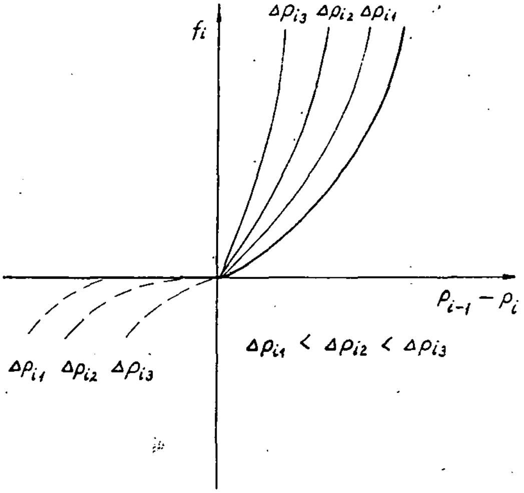

Abb. 12

Man kann auf Grund morphologischer und funktioneller Untersuchungen der Klappen feststellen, daß die Widerstände sowohl in Durchlaßrichtung als auch in Sperrrichtung Funktionen der axialen Druckdifferenz $p_{i-1} - p_i$ und der transmuralen Druckdifferenz Δp_i sind. Wenn die letzte Größe einen kritischen Wert übersteigt, so erweisen sich die Klappen infolge der Erweiterung der Lymphgefäße als insuffizient, und ermöglichen eine Rückströmung (Abb. 12).

Die Charakteristiken können auch durch Diodenschaltungen realisiert werden. Die kontinuierliche Beachtung der Wirkung des Parameters Δp_i könnte das Modell allzusehr komplizieren. Ähnlicherweise wird die Schaltung durch die Vernachlässigung der geringen Induktivitäten vereinfacht.

E) Zusammenbau des Modells

Die Modellprüfung der Lymphzirkulation erfordert die ausführliche Simulation des Blutkreislaufs. Diese Simulation wurde in erster Linie von der Hinsicht der Lymphzirkulation vorgenommen, d. h. das Kapillarnetz und die großen Venen bildeten die Gebiete, welche am eingehendsten zu realisieren waren.

Es sind zahlreiche Arbeiten über die analog-elektrische Simulation der Blutzirkulation, besonders von der arteriellen Strömung beschrieben worden [6], [7], [8]. Wir geben hier nur die analogischen Schaltungen der Blutgefäße vom arteriellen und venösen Typ an. Eine homogene arterielle Strecke läßt sich entweder durch

T-Schaltungen oder auch durch seines *π*-, und *L*-Äquivalenten repräsentieren (Abb. 13).

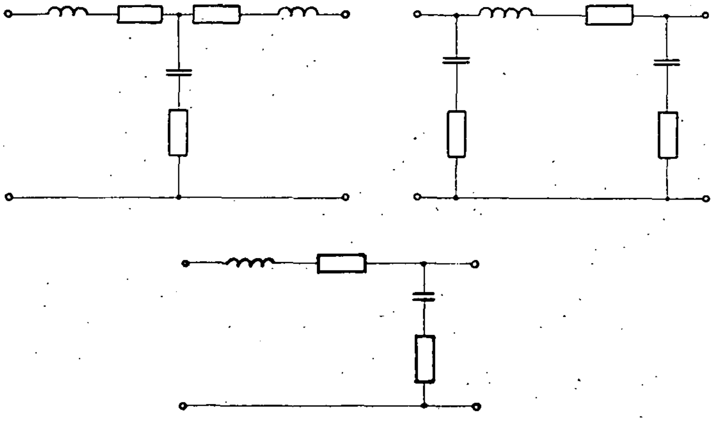

Abb. 13

Die verschiedenen Verhältnisse der venösen Strömung ergeben andere Werte von Schaltelementen in der analogischen Schaltung der Venenstrecke, die man aber in ähnlicher Topologie aufbauen kann. Die Reihenimpedanzen haben niedrigere Werte wegen des größeren Gesamtquerschnittes, aber die parallele Kapazität vermehrt sich, da die Gefäßwand der Venen elastischer ist, als die arterielle Wand.

Aus derartigen Grundschaltungen setzen sich die analog-elektrischen Schaltungen der verschiedenen Blutbahnen zusammen. Die Ventrikel und die Atrien lassen sich im hydrodynamischen Modell als elastische Gefäße betrachten. Ihre Kontraktionsbewegungen werden durch die Änderungen der Außendruckwerte, die im elektrischen Modell von Spannungsgeneratoren hergestellt werden sollen, simuliert (Abb. 14).

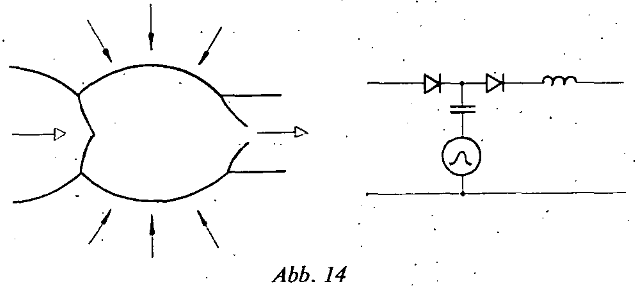

Abb. 14

Abb. 15 stellt nur eine schematische Anordnung des Modells dar. Um die Schaltungen mit Reaktanzen von reellen Größenordnungen realisieren zu können, haben wir eine zweckmäßige Frequenztransformation vorgenommen, die physiologischen Frequenzen sind nämlich mit einem Faktor von 10^3 multipliziert worden. Infolgedessen spielen sich alle physiologischen Übergangsprozesse tausendmal schneller ab.

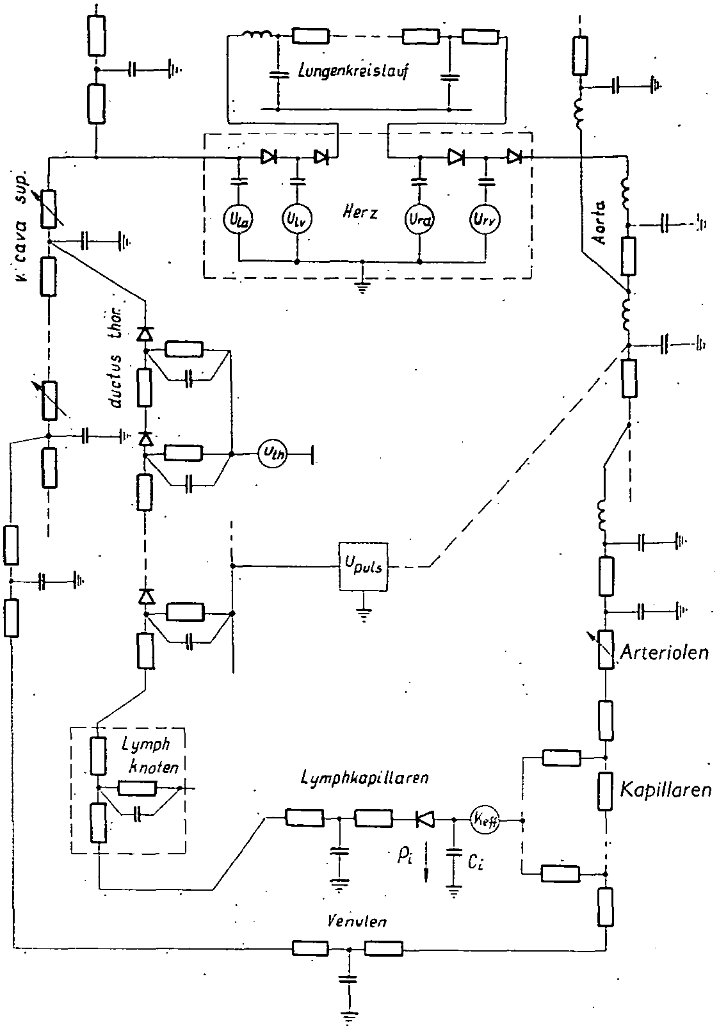

Abb. 15

F) Über die Modellversuche

Die Notwendigkeit der analog-elektrischen Simulation der Lymphokinetik ergibt sich — außer den gewöhnlichen Ansprüchen, welche die Physiologie an die Simulation zu stellen pflegt — auch davon, daß die Messung der spezifischen Drucks- und Strömungsverhältnisse im Lymphgefäßsystem sehr kompliziert ist. Die genaue Messung der Strömungsintensität der Lymphstämme ist bei geschlossenem System ohne Kanülierung nicht möglich.

Eine Grundaufgabe für das Modell ist die auf die Lymphströmung ausgeübte Wirkung der generalisierten Phlebohypertonie festzustellen. Die vorliegende Arbeit kann die mit dem Modell erhobenen Befunde nicht behandeln, wir möchten nur erwähnen, daß wir zwischen den Lymphströmungswerten, die bei einer Erhöhung des zentralen bzw. lokalen Venendrucks gemessen worden sind, eine Differenz von Größenordnungen gefunden haben (Abb. 16).

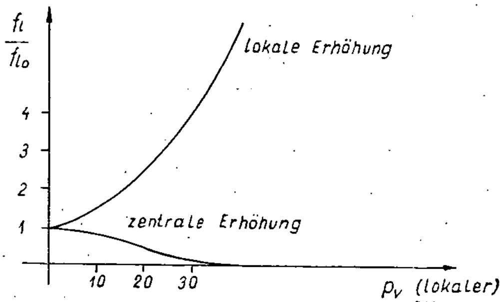

Abb. 16

Eine weitere grundlegende Aufgabe ist die quantitative Wirkung der Außendrucksänderungen (thorakale Druckschwankung, Übertragung der arteriellen Pulsation usw.) auf die Lymphströmung zu prüfen.

Geplant sind ferner solche Versuche, in welchen Veränderungen der Lymphentstehung, des interstitiellen Drucks und der Lymphströmung in Abhängigkeit der verschiedenen physiologischen Parameter (Permeabilität der Kapillarwand, Proteinkonzentration des Plasmas und der interstitiellen Flüssigkeit) untersucht werden.

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Some measure problems concerning the retrospective sequential functions

By L. Klukovits

In this paper we shall study the mappings of the set of all infinite sequences the terms of which are from a given set $\mathfrak{X}$, so-called retrospective sequential functions. These are some mappings of $\mathfrak{X}^N$ into itself. We shall define a measure on the set $\mathfrak{X}^N$ in natural way so that the measure of the range of a retrospective sequential function may be considered as the measure of the maintenance of information by the automaton which realises it.

We shall prove, that a retrospective sequential funtion is measure-preserving if and only if it is an onto mapping (Theorem 2). After this we shall show that, although $\mathfrak{X}^N$ has non-measurable subsets (Theorem 3), the ranges of finite state retrospective sequential functions (namely which can be realised by a finite automaton) are all measurable (Theorem 4), and the corresponding measures can equal any rational numbers between zero and one (Theorem 5).

Finally, besides some remarks we shall illustrate by giving examples that among the algebraic and metric as well as measure-theoretic properties of the retrospective sequential functions we cannot expect close connection.

§ 1. Some fundamental concepts and notations

Let $\mathfrak{X}$ be a non-empty finite set. We shall denote by $\{\mathfrak{X}\}$ the set of all finite sequences (shortly: words), whose terms are from $\mathfrak{X}$. The elements of $\{\mathfrak{X}\}$ will be denoted by $p, q, \ldots$ and the elements of $\mathfrak{X}$ by $x, y, \ldots$. We remark that the empty sequence is an element of $\{\mathfrak{X}\}$. The *length* of the word $p = x_1 x_2 \ldots x_n$ is the natural number n $(l(p) = n)$, the length of the empty sequence is zero.

$\{\mathfrak{X}\}_{(k)}$ and $\{\mathfrak{X}\}_k$ will denote the set of all words composed by elements of $\{\mathfrak{X}\}$, the length of which is at most k and exactly k, respectively.

If $\mathfrak{M}, \mathfrak{N} \subseteq \{\mathfrak{X}\}$, then

$$\mathfrak{M}\mathfrak{N} = \langle pq \mid p \in \mathfrak{M} \wedge q \in \mathfrak{N} \rangle.$$

The set of all ω-type sequences (shortly: sequences), whose terms are from $\mathfrak{X}$, will be denoted by $\mathfrak{X}^N$, and the elements of $\mathfrak{X}^N$ will be denoted by Greek letters. $\xi = \xi(1)\, \xi(2) \ldots$, where $\xi(i)$ $(\xi(i) \in \mathfrak{X})$ is the i-th element of ξ will be used for the detailed description of the sequences. We shall use the same description also for the words.

If $p\in\{\mathfrak{X}\}$, $\alpha\in\mathfrak{X}^N$ then $p\alpha$ will denote that sequence ξ which satisfies the following conditions:

$$\xi(i)=p(i) \quad \text{if} \quad i\leqq l(p)$$

and

$$\xi(j)=\alpha(j-l(p)) \quad \text{if} \quad j>l(p).$$

If $\mathfrak{M}\subseteqq\{\mathfrak{X}\}$, $\mathfrak{N}\subseteqq\mathfrak{X}^N$ then

$$\mathfrak{M}\mathfrak{N}=\langle p\alpha \mid p\in\mathfrak{M}\wedge\alpha\in\mathfrak{N}\rangle;$$

if $\mathfrak{M}=\langle p\rangle$ then we shall write instead of $\mathfrak{M}\mathfrak{N}$ the symbol $p\mathfrak{N}$. ($\langle\ldots\rangle$ denotes set.)

A mapping of $\mathfrak{X}^N$ into itself will be called a *sequential function* (shortly: sf). In this paper we shall denote the sequential functions by Latin capitals.

If $\mathfrak{N}\subseteqq\mathfrak{X}^N$, then $F\mathfrak{N}$ denotes the set of all sequences, which may be written in the form $F\alpha(\alpha\in\mathfrak{N})$.

For any natural number n let us define the sf D_n in the following way

$$(D_n\alpha)(i)=\alpha(i+n)$$

for all $\alpha\in\mathfrak{X}^N$ and any natural number i.

A sf F will be called *retrospective* (shortly: rsf) (see [3]), if for any $p\in\{\mathfrak{X}\}$ there is a $q\in\{\mathfrak{X}\}$ such that $l(p)=l(q)$ and $F(p\mathfrak{X}^N)\subseteqq q\mathfrak{X}^N$ hold. It is easy to see that q is uniquely defined by p.

For all rsf F there exists a mapping $\bar{F}$ of $\{\mathfrak{X}\}$ into itself. $\bar{F}$ is defined in the following way: for any $p\in\{\mathfrak{X}\}$ let $\bar{F}p=q$ if $F(p\mathfrak{X}^N)\subseteqq q\mathfrak{X}^N$. This mapping $\bar{F}$ is called an automaton mapping (see [4]). It is easy to verify that for any $p, r, s\in\{\mathfrak{X}\}$, $r\neq s$, $(\bar{F}(pr))(i)=(\bar{F}(ps))(i)$ if $i\leqq l(p)$ holds.

Let F be any rsf. For an arbitrary word $p(\in\{\mathfrak{X}\})$ we shall define the sf F_p in the following way:

$$F_p\alpha=D_{l(p)}F(p\alpha)$$

for any $\alpha\in\mathfrak{X}^N$.

We know that for any $p\in\{\mathfrak{X}\}$, F_p is a rsf (see [3]). A rsf which can be written in the form F_p, is called a *state* of F. Of course it is possible that $F_p=F_q$ in spite of $q\neq p$. If all the different states of an rsf constitute a finite set, then it will be called a *finite state retrospective sequential function* (shortly: fsrsf). We shall call the the family

$$\Sigma_{F,k}=\langle F_p \mid l(p)=k\rangle$$

of the states of F the k-th *level* of F. It is easy to see that the set of all states of F coincides with the set of all rsf-s G, for which $G\in\Sigma_{F,i}$ holds for some i ($i=0, 1, 2,\ldots$). This set is denoted by Σ_F.

We know that $(F_p)_q=F_{pq}$ holds for any words p, q (see [3]). It follows that $\Sigma_{F_p}\subseteqq\Sigma_F$ is true for all $p\in\{\mathfrak{X}\}$.

The reader can prove easily that any state of any rsf is also retrospective. It is also easy to verify that the following assertions are equivalent

(i) $\bar{F}$ is one-to-one
(ii) $\bar{F}$ is onto.

Furthermore if F is onto, then F is one-to-one. A simple counter-example ($F:\alpha\to x\alpha$; $x\in\mathfrak{X}$) shows that the converse of this proposition if false. In the sequel if it does not make any misunderstanding, we shall write F instead of $\bar{F}$.

If the rsf F is not one-to-one, then there exists a word $p \in \{\mathfrak{X}\}$ such that $p \notin F\{\mathfrak{X}\}$. The length of the shortest word p which satisfies the preceding condition is denoted by $d(F)$.

The cardinal number of the set $\mathfrak{A}$ is denoted in this paper by $|\mathfrak{A}|$. We shall call the word p an *initial segment* of the sequence α, if

$$p(i) = \alpha(i)$$

for all $i \leqq l(p)$.

§ 2. Definition of a measure on the set $\mathfrak{X}^N$

Consider those subsets of $\mathfrak{X}^N$, which may be written in the form

$$p\mathfrak{X}^N$$

where p is arbitrary word. Enlarging the set of these subsets with the empty set, we get a semiring of sets, which we shall denote by $S(\mathfrak{X})$. The unit element of $S(\mathfrak{X})$ is $\mathfrak{X}^N$. Let us define a set function μ on the elements of $S(\mathfrak{X})$ in the following way:

$$\mu(p\mathfrak{X}^N) = \frac{1}{n^{l(p)}} \tag{2.1}$$

where $n = |\mathfrak{X}|$. It is easy to see that this function is a measure (see [1]). We assert that this measure is σ-additive.

It is known that we can continue this measure so that its domain will be the minimal ring over the semiring $S(\mathfrak{X})$ (this ring will be denoted by $\mathfrak{R}(S(\mathfrak{X}))$), and the continuation of μ will be also σ-additive (see [1]).

In the next section we shall often refer to the definition of the Lebesgue measure, and a proposition on measurability. They are the following. Let a σ-additive measure m be given on some semiring of sets S_m with unit E. We shall define on the system $\mathfrak{S}$ of all subsets of the set E two functions $\mu^*(A)$ and $\mu_*(A)$ in the following way

Definition 1. The number

$$\mu^*(A) = \inf_{A \subseteqq \bigcup_n B_n} \sum_n m(B_n)$$

where the greatest lower bound is taken over all coverings of the set A by finite or countable systems of sets $B_n \in S_m$, is called the *outer measure* of the set $A \subseteqq E$.

Definition 2. The number

$$\mu_*(A) = m(E) - \mu^*(E \setminus A)$$

is called the *inner measure* of the set $A \subseteqq E$.

It is easy to see that $\mu_*(A) \leqq \mu^*(A)$ holds for every $A \subseteqq E$.

Definition 3. The set $A \subseteqq E$ is measurable (Lebesgue), if

$$\mu_*(A) = \mu^*(A).$$

If A is measurable, then we shall denote the common value $\mu_*(A)=\mu^*(A)$ by $\mu(A)$ and call it the (Lebesgue) measure of the set A. The next definition is equivalent to definition 3.

Definition 3'. The set $A\subseteqq E$ is called measurable, if

$$\mu^*(A)+\mu^*(E\setminus A) = m(E).$$

Proposition 1. For the measurability of the set $A(\subseteqq E)$ the following condition is necessary and sufficient: for any $\varepsilon>0$ there exists a $B\in\mathfrak{R}(S(\mathfrak{X}))$ such that

$$\mu^*(A\triangle B)<\varepsilon.$$

The proof of this theorem can be found for example in [1].

§ 3. The main results

First we prove a theorem which presents an analogy with some problems of the theory of real functions.

Theorem 1. If two retrospective sequential functions differ only on a set of measure zero then these functions are equal.

Proof. We shall prove the following assertion, equivalent to Theorem 1: if the rsf-s F and G are not equal, then there exists a set $\mathfrak{A}$, such that $\mu(\mathfrak{A})>0$ and

$$F\mathfrak{A}\cap G\mathfrak{A} = \emptyset.$$

Let $\alpha=\alpha(1)\alpha(2)\dots$ be any sequence from $\mathfrak{X}^N$, for which there is a natural number i such that

$$(F\alpha)(i)\neq(G\alpha)(i).$$

After this consider the set $p\mathfrak{X}^N$, where $p=\alpha(1)\dots\alpha(i)$. According to the definition of measure

$$\mu(p\mathfrak{X}^N) = \frac{1}{n^i}>0$$

$(n=|\mathfrak{X}|)$ and it is clear that

$$F(p\mathfrak{X}^N)\cap G(p\mathfrak{X}^N) = \emptyset.$$

Q. E. D.

Lemma 1. If the range of a rsf F is measurable, then the image of any measurable set under F is also measurable.

Proof. The reader can verify that, if the range of F is measurable, then $F\mathfrak{C}$ — where $\mathfrak{C}\in\mathfrak{R}(S(\mathfrak{X}))$ — is also measurable. Let now $\mathfrak{A}(\subseteqq\mathfrak{X}^N)$ be any measurable set. According to the proposition 1, for any $\varepsilon>0$ there exists a set $\mathfrak{B}\in\mathfrak{R}(S(\mathfrak{X}))$ such that

$$\mu^*(\mathfrak{A}\triangle\mathfrak{B})<\varepsilon. \tag{3.1}$$

Let $\varepsilon > 0$ be an arbitrary number and $\mathfrak{B} \in \mathfrak{R}(S(\mathfrak{X}))$ a set satisfying (3. 1). It is simple to verify that

$$F\mathfrak{A} \triangle F\mathfrak{B} \subseteqq F(\mathfrak{A} \triangle \mathfrak{B})$$

is valid. Take the outer measure of both sides:

$$\mu^*(F\mathfrak{A} \triangle F\mathfrak{B}) \leqq \mu^*(F(\mathfrak{A} \triangle \mathfrak{B})) \leqq \mu^*(\mathfrak{A} \triangle \mathfrak{B}) < \varepsilon.$$

The reader can easily verify that

(3. 2) $$|\mu^*(F\mathfrak{A}) - \mu^*(F\mathfrak{B})| \leqq \mu^*(F\mathfrak{A} \triangle F\mathfrak{B}) < \varepsilon,$$

and since

$$(\mathfrak{X}^N \setminus F\mathfrak{A}) \triangle (\mathfrak{X}^N \setminus F\mathfrak{B}) = F\mathfrak{A} \triangle F\mathfrak{B},$$

we obtain

(3. 3) $$|\mu^*(\mathfrak{X}^N \setminus F\mathfrak{A}) - \mu^*(\mathfrak{X}^N \setminus F\mathfrak{B})| < \varepsilon.$$

From (3. 2) and (3. 3) we get

(3. 4) $$|\mu^*(F\mathfrak{A}) + \mu^*(\mathfrak{X}^N \setminus F\mathfrak{A}) - (\mu^*(F\mathfrak{B}) + \mu^*(\mathfrak{X}^N \setminus F\mathfrak{B})| < 2\varepsilon.$$

Since $F\mathfrak{B}$ is measurable,

$$\mu^*(F\mathfrak{B}) + \mu^*(\mathfrak{X}^N \setminus F\mathfrak{B}) = \mu(\mathfrak{X}^N).$$

Since ε was arbitrary, it follows from (3. 4) that

$$\mu^*(F\mathfrak{A}) + \mu^*(\mathfrak{X}^N \setminus F\mathfrak{A}) = \mu(\mathfrak{X}^N)$$

and this means that the image of the set $\mathfrak{A}$ under F is measurable.

In the following theorem we give a necessary and sufficient condition in order that a retrospective sequential function be measure-preserving.

Theorem 2. A rsf is measure-preserving if and only if it is an onto mapping.

Proof.

Sufficiency: First let $\mathfrak{A} \in S(\mathfrak{X})$, i.e.

$$\mathfrak{A} = p\mathfrak{X}^N$$

where $p \in \{\mathfrak{X}\}$. In this case

(3. 5) $$F\mathfrak{A} = q\mathfrak{B}$$

where $q = Fp$ and $\mathfrak{B} \subseteqq \mathfrak{X}^N$. Since F is onto (and a fortiori one-to-one), its inverse F^{-1} exists, which is a rsf (see [3]) and

$$F^{-1}(q\mathfrak{X}^N) \subseteqq p\mathfrak{X}^N.$$

By (3. 5), it follows from this relation that

$$q\mathfrak{X}^N \subseteqq q\mathfrak{B},$$

i. e.

$$\mathfrak{X}^N \subseteqq \mathfrak{B}.$$

We have obtained, that $\mathfrak{B} \subseteqq \mathfrak{X}^N$ and $\mathfrak{X}^N \subseteqq \mathfrak{B}$, from which

$$\mathfrak{B} = \mathfrak{X}^N$$

follows and so for any $\mathfrak{A} \in S(\mathfrak{X})$ the sufficiency of the condition is proved. Now let $\mathfrak{A}$ be an arbitrary measurable set. According to the precedings we obtain

$$\mu^*(F\mathfrak{A}) = \mu^*(\mathfrak{A}) = \mu(\mathfrak{A}).$$

The set $F\mathfrak{A}$ is measurable by the lemma 1, and so

$$\mu^*(F\mathfrak{A}) = \mu(F\mathfrak{A}).$$

The sufficiency of the condition is proved.

Necessity: First we suppose that F is not one-to-one. In this case there are such sequences $\alpha, \beta \in \mathfrak{X}^N$ $(\alpha \neq \beta)$ for which

$$F\alpha = F\beta.$$

Let

$$\alpha = p\bar{\alpha} = px\xi,$$

$$\beta = p\bar{\beta} = py\zeta$$

where $p \in \{\mathfrak{X}\}$ (it is possible that $l(p)=0$), $x, y \in \mathfrak{X}$ but $x \neq y$ and $\bar{\alpha}, \bar{\beta}, \xi, \zeta \in \mathfrak{X}^N$. According to the assumption we obtain

$$F(px\xi) = qz\eta \tag{3.6}$$

and

$$F(py\zeta) = qz\eta \tag{3.7}$$

where $q \in \{\mathfrak{X}\}$ $(l(q)=l(p))$, $z \in \mathfrak{X}$ and $\eta \in \mathfrak{X}^N$. Consider the sets

$$px\mathfrak{X}^N \quad \text{and} \quad py\mathfrak{X}^N$$

which are disjoint elements of $S(\mathfrak{X})$. According to (3.6) and (3.7) we obtain

$$F(px\mathfrak{X}^N) = qz\mathfrak{A}$$

and

$$F(py\mathfrak{X}^N) = qz\mathfrak{B}$$

where $\mathfrak{A}, \mathfrak{B} \subseteqq \mathfrak{X}^N$ and

$$\mu(qz\mathfrak{A}) = \mu(qz\mathfrak{B}) = \mu(px\mathfrak{X}^N) = n^{-(l(p)+1)}$$

$(n = |\mathfrak{X}|)$. We remark that $\mu(\mathfrak{A}) = \mu(\mathfrak{B}) = 1$ and so $\mu(\mathfrak{A} \triangle \mathfrak{B}) = 0$. In this case we can suppose by the Theorem 1 that

$$\mathfrak{A} = \mathfrak{B}.$$

After these we consider the set $F(px\mathfrak{X}^N \cup py\mathfrak{X}^N)$, which is equal to $qz\mathfrak{A}$. We have a contradiction, since

$$\mu(px\mathfrak{X}^N \cup py\mathfrak{X}^N) = 2n^{-(l(p)+1)}$$

and

$$\mu(qz\mathfrak{A}) = n^{-(l(p)+1)}.$$

This makes the first part of the proof complete because $px\mathfrak{X}^N \cup py\mathfrak{X}^N$ is measurable.

Now let the rsf F be one-to-one but not onto, and $\alpha \in \mathfrak{X}^N \setminus F\mathfrak{X}^N$. We assert that there exists a natural number k such that

$$x_1 x_2 \dots x_k \mathfrak{X}^N \not\subset F\mathfrak{X}^N$$

where $x_i = \alpha(i)$ $(i = 1, 2, \dots, k)$. In the contrary case there exists a sequence of elements $\alpha_j \in F\mathfrak{X}^N$ $(j = 1, 2, \dots)$ such that $\alpha_j(i) = \alpha(i)$ whenever $1 \leqq i \leqq j$. Consider the sequence $\beta_j \in \mathfrak{X}^N$ $(j = 1, 2, \dots)$ defined by the equations

$$F\beta_j = \alpha_j.$$

Form the initial-segments of length j for every β_j. From these segments we can construct a sequence β, for which

$$F\beta = \alpha$$

holds. This is a contradiction. If the natural number k above exists, then we have relation

$$x_1 x_2 \dots x_k \mathfrak{X}^N \subseteqq \mathfrak{X}^N \setminus F\mathfrak{X}^N.$$

We know that

$$\mu(\mathfrak{X}^N \setminus F\mathfrak{X}^N) = 0$$

because the rsf F is measure preserving. We see that our preceding relation contradicts to the measure preserving of the rsf F. The proof is complete.

We may ask the following question: Are all the subsets of $\mathfrak{X}^N$ measurable? The answer is negative. The construction which leads to a non-measurable subset of $\mathfrak{X}^N$ is analogous to that of Zermelo concerning the interval [0, 1] (see [2]).

Theorem 3. The set $\mathfrak{X}^N$ has non-measurable subsets.

Proof: Let us define a relation on $\mathfrak{X}^N$: $\alpha \sim \beta$ $(\alpha, \beta \in \mathfrak{X}^N)$ if and only if there exists an onto fsrsf F, for which

$$F\alpha = \beta.$$

This relation is an equivalence relation, and let the equivalence classes be $\mathfrak{A}_\iota$ $(\iota \in I)$. We choose one and only one element from every set $\mathfrak{A}_\iota$, and we denote the set of these elements by $\mathfrak{B}$. We shall show that $\mathfrak{B}$ is not measurable.

Consider the sequence of all onto fsrsf-s

$$F_1, F_2, \dots$$

and let

$$\mathfrak{B}_j = F_j \mathfrak{B} \qquad j = 1, 2, \dots.$$

The sets $\mathfrak{B}_j$ $(j = 1, 2, \dots)$ satisfy the condition

$$\mathfrak{B}_j \cap \mathfrak{B}_k = \emptyset \qquad j \neq k.$$

Namely, if $\alpha \in \mathfrak{B}_j \cap \mathfrak{B}_k$, then $F_j^{-1}\alpha$, $F_k^{-1}\alpha \in \mathfrak{B}$ and

$$F_j^{-1} F_k F_k^{-1} \alpha = G F_k^{-1} \alpha = F_j^{-1} \alpha$$

so $F_j^{-1}\alpha \sim F_k^{-1}\alpha$. This is a contradiction.

If the set $\mathfrak{B}$ is measurable, then the sets $\mathfrak{B}_j$ $(j=1, 2, \ldots)$ are measurable and $\mu(\mathfrak{B})=\mu(\mathfrak{B}_j)$ by Theorem 2. Furthermore,

$$\mathfrak{X}^N = \bigcup_{j=1}^{\infty} \mathfrak{B}_j.$$

In fact, for any $\alpha \in \mathfrak{X}^N$ one and only one index $\iota\,(\in I)$ exists such that $\alpha \in \mathfrak{A}_\iota$. Let $\beta = \mathfrak{A}_\iota \cap \mathfrak{B}$. On the basis of the definition of the equivalence classes $\mathfrak{A}_\iota$ there is an onto fsrsf H, such that

$$\alpha = H\beta.$$

Since H is among $F_1, F_2, \ldots$, there exists a natural number m, such that $\alpha \in \mathfrak{B}_m$ holds.

The series $\sum_{j=1}^{\infty} \mu(\mathfrak{B}_j)$ is convergent. Furthermore, $\mu(\mathfrak{B}_1)=\mu(\mathfrak{B}_2)=\ldots$ whence it follows that $\mu(\mathfrak{B}_j)=0$ for $j=1, 2, \ldots$. Because

$$\sum_{j=1}^{\infty} \mu(\mathfrak{B}_j) = \mu(\mathfrak{X}^N) = 1,$$

we have a contradiction, thus the proof is complete.

In the following we look for an answer to the question: for which classes of rsf will be the range measurable. We shall obtain that the range of any fsrsf is measurable.

Lemma 2. The range of any fsrsf without one-to-one state is measurable.

Proof. For any non-negative integer t let us define the subsets $\Lambda_{1,t}$, $\Lambda_{2,t}$ of $\Sigma_{F,t}$ in the following way: let $F_{p_1} \in \Lambda_{1,t}$ $\big(l(p_1) = t\big)$ if and only if there exist words $p_2, \ldots, p_k \in \{\mathfrak{X}\}_t$ such that

$$Fp_1 = Fp_2 = \ldots = Fp_k$$

and

$$\bigcup_{i=1}^{k} F_{p_i} \mathfrak{X}^N = \mathfrak{X}^N$$

and let $\Lambda_{2,t} = \Sigma_{F,t} \setminus \Lambda_{1,t}$. We see, if $F_{p_1} \in \Lambda_{1,t}$ and $l(q)=u$, then $F_{p_1 q} \in \Lambda_{1,t+u}$.

In fact, in this case

$$\bigcup_{i=1}^{k} F_{p_i} \mathfrak{X}^N = \mathfrak{X}^N \supset (F_{p_1} q) \mathfrak{X}^N,$$

therefore, for all $r \in \{\mathfrak{X}\}_u$ for which

$$F_{p_i} r = F_{p_i} q \quad (i=1, 2, \ldots, k),$$

we obtain

$$\bigcup_r F_{p_i r} \mathfrak{X}^N = \mathfrak{X}^N,$$

and for any such word

$$F(p_i r) = (Fp_i)(F_{p_i} r) = (Fp_1)(F_{p_1} r) = F(p_1 q).$$

Let us denote by $\mathfrak{B}_t$ the set of all sequences $\alpha \in F\mathfrak{X}^N$, for which there exists a word $p \in \{\mathfrak{X}\}_t$ such that $F_p \in \Lambda_{2,t}$ and $Fp = \alpha(1) \ldots \alpha(t)$. We shall show that if $t < v$, then

$\mathfrak{B}_v \subseteqq \mathfrak{B}_t$. In fact, let $\alpha \in \mathfrak{B}_v$. Now there is a $q \in \{\mathfrak{X}\}_v$, such that $F_q \in \Lambda_{2,v}$ and $Fq = \alpha(1)\ldots\alpha(v)$. In this case for the word $r = q(1)\ldots q(t)$ we obtain $F_r \in \Lambda_{2,t}$ (otherwise it follows, that $F_q \in \Lambda_{1,v}$), and, owing to the retrospectivity of F

$$Fr = \alpha(1)\ldots\alpha(t),$$

and so $\alpha \in \mathfrak{B}_t$ holds.

Let $\Gamma = \langle G_1, \ldots, G_m \rangle$ be any set of states of F. If $\bigcup_{i=1}^{m} G_i \mathfrak{X}^N \subset \mathfrak{X}^N$, then there exists a natural number $f = f(\Gamma)$ such that

$$\bigcup_{i=1}^{m} G_i \{\mathfrak{X}\}_f \subset \{\mathfrak{X}\}_f. \tag{3. 8}$$

In fact, if for all natural numbers j

$$\bigcup_{i=1}^{m} G_i \{\mathfrak{X}\}_j = \{\mathfrak{X}\}_j,$$

then for any $\alpha \in \mathfrak{X}^N$ there is a G_i such that the sets $G_i\{\mathfrak{X}\}_j$ $(j = 1, 2, \ldots)$ contain an infinite number of initial segments of α, and — since G_i is retrospective — these sets contain all initial segments of α, because Γ is a finite set. Let now

$$p_1, p_2, \ldots, p_j, \ldots \tag{3. 9}$$

be words such that

$$G_i p_j = \alpha(1)\ldots\alpha(j).$$

Since $\mathfrak{X}$ is a finite set, (3. 9) has a sub-sequence

$$p_{11}, p_{12}, \ldots, p_{1j}, \ldots \tag{3. 10}$$

such that all words in (3. 10) begin with the same letter x_1. An easy induction shows that, for any natural number k, (3. 9) has a sub-sequence

$$p_{k1}, p_{k2}, \ldots, p_{kj}, \ldots$$

such that all words in this sequence begin with the same word $x_1 \ldots x_k \in \{\mathfrak{X}\}_k$. In this way we define a sequence

$$\xi = x_1 \ldots x_k \ldots \in \mathfrak{X}^N$$

which for any k satisfies the condition

$$G_i(x_1 \ldots x_k) = \alpha(1)\ldots\alpha(k).$$

So we have $G_i \xi = \alpha$, namely $\alpha \in G_i \mathfrak{X}^N$ and $\alpha = \bigcup_{i=1}^{m} G_i \mathfrak{X}^N$. We obtain

$$\mathfrak{X}^N \subseteqq \bigcup_{i=1}^{m} G_i \mathfrak{X}^N$$

namely $\bigcup_{i=1}^{m} G_i \mathfrak{X}^N = \mathfrak{X}^N$ and thus, we have a contradiction.

If there exists a natural number $f=f(\Gamma)$, for which (3. 8) holds, then there exists at least such a number and let us denote this number by f. If $\bigcup_{i=1}^{m} G_i\mathfrak{X}^N=\mathfrak{X}^N$ holds, then let $f(\Gamma)=0$.

Let now

$$k = \max_{\Gamma\subseteqq\Sigma_F} f(\Gamma).$$

(If Γ has only one element, $\Gamma=\langle G\rangle$, then according to the definition $f(\Gamma)=d(G)$. So for any state G of F, $k\geqq d(G)$ holds.)

We shall denote by B_t the set of all initial segments of the sequences from $\mathfrak{B}_t$, whose lengths are equal to t. Furthermore B_t contains all the words q $(l(q)=t)$ for which there is a word p $(l(p)=t)$ such that

$$F_p\in A_{2,t} \quad \text{and} \quad Fp=q.$$

We shall show that, for any natural number i,

$$|B_{ik}| \leqq (n^k-1)^i$$

holds, where $n=|\mathfrak{X}|$. It is also true, that $B_k\subseteqq F\{\mathfrak{X}\}_k$ and $d(F)\leqq k$ hold, and because of the retrospectivity of F, we have

$$|F\{\mathfrak{X}\}_k| \leqq n^k-1,$$

consequently

$$|B_k|< n^k-1.$$

Let $|B_{(i-1)k}|\leqq(n^k-1)^{i-1}$, and let $p\in B_{(i-1)k}$. It suffices to show that there are at least n^k-1 words $q\in\{\mathfrak{X}\}_k$ such that $pq\in B_{ik}$. The following more strong assertion is also true: the number of those words $q\in\{\mathfrak{X}\}_k$ which satisfy this condition $pq\in F\{\mathfrak{X}\}_{ik}$, is at least n^k-1. In fact, consider all the words $r_1,\dots,r_m\in\{\mathfrak{X}\}_{(i-1)k}$ for which $Fr_1=\dots=Fr_m=p$. In this case

$$F_{r_1},\dots,F_{r_m}\in A_{2,(i-1)k},$$

therefore

$$\bigcup_{j=1}^{m} F(r_j\{\mathfrak{X}\}_k)=\bigcup_{j=1}^{m} pF_{r_j}\{\mathfrak{X}\}_k = p\bigcup_{j=1}^{m} F_{r_j}\{\mathfrak{X}\}_k\subset p\{\mathfrak{X}\}_k$$

according to the definition of k. On the basis of the preceding we obtain for any natural number t

$$\mu^*(\mathfrak{B}_{tk})\leqq\mu^*\Big(\bigcup_{q\in B_{tk}} q\mathfrak{X}^N\Big)\leqq\sum_{q\in B_{tk}}\mu^*(q\mathfrak{X}^N)=\sum_{q\in B_{tk}}\mu(q\mathfrak{X}^N)\leqq\left(\frac{n^k-1}{n^k}\right)^t.$$

We have seen that if $t<v$, then $\mathfrak{B}_v\subseteqq\mathfrak{B}_t$ has been satisfied. Thus for any $\varepsilon>0$ there exists a t_ε such that for any $t>t_\varepsilon$ the relation

$$\mu^*(\mathfrak{B}_t)<\varepsilon$$

is satisfied.

We observe that for any natural number t

$$F\mathfrak{X}^N=\Big(\bigcup_{F_p\in A_{1,t}}(Fp)\mathfrak{X}^N\Big)\cup\mathfrak{B}_t=\mathfrak{A}_t\cup\mathfrak{B}_t$$

and $\mathfrak{A}_t\cap\mathfrak{B}_t=\emptyset$.

Since $F\mathfrak{X}^N \triangle \mathfrak{A}_t = \mathfrak{B}_t$ and $\mathfrak{A}_t \in \mathfrak{R}(S(\mathfrak{X}))$ hold (for $\mathfrak{A}_t$ is a finite union of elements from $S(\mathfrak{X})$ and see [1]),

$$\mu^*(F\mathfrak{X}^N \triangle \mathfrak{A}_t) = \mu^*(\mathfrak{B}_t) < \varepsilon.$$

By the proposition 1 from this relation the measurability of the range of F follows.

Q. E. D.

Theorem 4. The range of any fsrsf is measurable.

Proof. Let be F an arbitrary fsrsf. We define fo rany natural number t three sets: $\Lambda_{1,t}$ is the set of all fsrsf $F_p (p \in \{\mathfrak{X}\}_t)$ which are onto mappings; $\Lambda_{2,t}$ is the set of all fsrsf $F_q (q \in \{\mathfrak{X}\}_t)$ which are not onto mappings, but there exists a word $r \in \{\mathfrak{X}\}$ $(l(r) > 0)$, such that F_{qr} is onto and finally $\Lambda_{3,t} = \Sigma_{F,t} \setminus \setminus (\Lambda_{1,t} \cup \Lambda_{2,t})$.

It is easy to see: if $F_p \in \Lambda_{1,t}$ and $r \in \{\mathfrak{X}\}$ is any word, then $F_{pr} \in \Lambda_{1,t+l(r)}$. Let $k(H)$ $(H \in \Sigma_F)$ the smallest natural number for which

$$\Sigma_H = \bigcup_{j=1}^{k(H)} \bigcup_{i=1}^{3} \Lambda_{i,j}$$

holds, and let $k = \max_{H \in \Sigma_F} k(H)$. For the sake of simplicity put

$$\bigcup_{G \in \Lambda_{1,t}} G\mathfrak{X}^N = \mathfrak{A}_t$$

$$\bigcup_{H \in \Lambda_{2,t}} H\mathfrak{X}^N = \mathfrak{B}_t$$

$$\bigcup_{J \in \Lambda_{3,t}} J\mathfrak{X}^N = \mathfrak{C}_t$$

for any natural number t, and denote by B_t the set of all initial segments of the sequences from $\mathfrak{B}_t$ whose length is exactly t.

The following inequality is valid:

$$|B_{2k}| \leqq |B_k|(n^k - 1).$$

In fact, for any $G \in \Lambda_{2,k}$ there exists a word $r \in \{\mathfrak{X}\}_{(k)}$ such that $l(r) > 0$ and $G_r \in \Lambda_{1,t+l(r)}$ hold.

A simple induction shows, that for any natural number t

$$|B_{tk}| \leqq |B_k|(n^k - 1)^{t-1}$$

is valid. Thus

$$\mu^*(\mathfrak{B}_{tk}) \leqq \mu^*\Big(\bigcup_{q \in B_{tk}} q\mathfrak{X}^N\Big) \leqq \sum_{q \in B_{tk}} \mu(q\mathfrak{X}^N) \leqq \frac{|B_k|}{n^k}\left(\frac{n^k - 1}{n^k}\right)^{t-1}$$

is true.

Now let $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$ arbitrary numbers and $\varepsilon_1 + \varepsilon_2 < \varepsilon$.

It is obvious that there exists such a natural number t_{ε_1} for which

$$\mu^*(\mathfrak{B}_{tk}) \leqq \mu^*\Big(\bigcup_{q \in B_{tk}} q\mathfrak{X}^N\Big) = \mu^*(\mathfrak{B}) < \varepsilon_1$$

holds if $t > t_{\varepsilon_1}$ (in the following we assume it).

Since $\mathfrak{C}_{tk}$ is measurable by Lemma 2, there is a set $\mathfrak{C}\in\mathfrak{R}(S(\mathfrak{X}))$ such that

$$\mu^*(\mathfrak{C}_{tk}\triangle\mathfrak{C})<\varepsilon_2$$

is valid (see proposition 1).

Now we obtain that

$$\mu^*(F\mathfrak{X}^N\triangle(\mathfrak{A}_{tk}\cup\mathfrak{B}\cup\mathfrak{C}))\leqq\mu^*(\mathfrak{B})+\mu^*(\mathfrak{C}_{tk}\triangle\mathfrak{C})<\varepsilon_1+\varepsilon_2<\varepsilon$$

holds. Since ε was arbitrary, in virtue of Proposition 1, this relation gives the measurability of the range of F.

Q. E. D.

The following theorems give answer to what the set of the values of the function μ can be like.

Theorem 5. For any rational number r, $0\leqq r\leqq 1$, there exists such a fsrsf F for which we have

$$\mu(F\mathfrak{X}^N)=r.$$

Proof. If $r=0$ or $r=1$, then the statement is trivial, thus we can suppose that $0<r<1$. The proof will be constructive and we make the construction in the special case $\mathfrak{X}=\langle 0, 1\rangle$. The reader can show that this condition does not restrict the generality.

Let $\underbrace{11\ldots1}_{i\text{ times}}=1^i$ $(i=0, 1,\ldots)$ and define two sequential functions

$$G_0\alpha=00\ldots \tag{3. 11}$$

$$G_1\alpha=\alpha$$

for any sequence $\alpha\in\mathfrak{X}^N$.

Let

$$0, a_1\, a_2\ldots a_k\, b_1\, b_2\ldots$$

be the dyadic form of the rational number r, where there is a natural number m for which

$$b_j=b_{j-\left[\frac{j}{m}\right]m} \tag{3. 12}$$

for any natural number j $\left(\left[\frac{j}{m}\right]\right.$ denotes here the integer part of $\left.\frac{j}{m}\right)$.

After that, let

$$F_{1^i0}=\begin{cases}G_{a_{i+1}} & \text{if } 0\leqq i<k,\\ G_{b_{i-k+1}} & \text{if } k\leqq i.\end{cases} \tag{3. 13}$$

Owing to the condition (3. 12) we have only finitely many different states and these are

$$F_{1^0}, F_{1^1},\ldots, F_{1^{k+m}}.$$

A simple calculation shows that for the fsrsf F for which

$$\Sigma_F=\langle G_0, G_1, F_{1^0},\ldots, F_{1^{k+m}}\rangle$$

is valid, where G_0, G_1 and F_{1^i} are defined by (3. 11) and (3. 13), we have

$$\mu(F\mathfrak{X}^N) = \sum_{j=1}^{k} \frac{a_j}{2^j} + \sum_{i=1}^{\infty} \frac{b_i}{2^{k+i}} = r.$$

Theorem 6. For any number p, $0<p<1$ there exists a rsf F for which

$$\mu(F\mathfrak{X}^N)=p.$$

Proof. We may suppose that p is an irrational number, and let its dyadic form be

$$0; a_1 a_2 \ldots$$

Let G_0 and G_1 be defined, as in Theorem 5, by (3. 11). Instead of (3. 13) we consider

$$F_{1^i 0} = G_{a_{i+1}} \text{ for all } i=0, 1, \ldots.$$

We obtain by a simple calculation that if

$$\Sigma_F = \langle G_0, G_1, F_{1^0}, \ldots \rangle$$

then

$$\mu(F\mathfrak{X}^N) = \sum_{i=1}^{a} \frac{a_i}{2^i} = p.$$

Q. E. D

§ 4. Two negative results

We can ask the following question. If F and H are two rsf-s, then what is the correspondence between $\mu(F\mathfrak{X}^N)\mu(H\mathfrak{X}^N)$ and $\mu(FH\mathfrak{X}^N)$? (We suppose that these measures exist).

The next example shows that in the general case we can assert nothing. Let $\mathfrak{X}=\langle x_1, x_2\rangle$, and the rsf G_0, G_1 be defined by (3. 11). The functions F and H will be defined by the formulas

$$\Sigma_F = \langle F_{x_1}, F_{x_2} \rangle$$

where $F_{x_1}=G_1$ and $F_{x_2}=G_0$, furthermore

$$\Sigma_H = \langle H_{x_1}, H_{x_2} \rangle,$$

where $H_{x_1}=G_0$ and $H_{x_2}=G_1$.

It is easy to show that on the one hand

$$\mu(F\mathfrak{X}^N) = \mu(H\mathfrak{X}^N) = \frac{1}{2},$$

on the other hand, if for any $x\in\mathfrak{X}$

$$Fx = \begin{cases} x_1 & \text{if } x=x_1 \\ x_2 & \text{if } x=x_2 \end{cases}$$

and

$$Hx = \begin{cases} x_2 & \text{if } x=x_1 \\ x_1 & \text{if } x=x_2 \end{cases}$$

hold, then

$$\mu(FH\mathfrak{X}^N) = \mu(x_2x_1x_1\ldots)+\mu(x_1\mathfrak{X}^N) = \frac{1}{2}$$

and

$$\mu(HF\mathfrak{X}^N) = \mu(x_2x_1x_1\ldots)+\mu(x_1x_1\ldots) = 0.$$

Before the description of the second result it is necessary to give two definitions.

Definition 4 (see [5]). The distance of two rsf-s F and G is defined by

$$\varrho(F, G) = \frac{1}{m}$$

where m is the smallest natural number for which there exists a sequence $\alpha\in\mathfrak{X}^N$ such that

$$(F\alpha)(i)=(G\alpha)(i) \quad \text{for} \quad i<m$$

and

$$(F\alpha)(m)\neq(G\alpha)(m)$$

hold.

Definition 5. We say, that a sequence of rsf-s $F^{(k)}$ tends to an rsf F ($F^{(k)}\to F$), if

$$\lim_{k\to\infty} \varrho\,(F^{(k)}, F) = 0$$

holds.

Now the second problem is the following. If $F^{(k)}\to F$ then does

$$\lim_{k\to\infty} \mu(F^{(k)}\mathfrak{X}^N) = \mu(F\mathfrak{X}^N)$$

hold?

The answer is negative. Let x be a fixed element of the set $\mathfrak{X}$, and for any natural number k let us define the rsf $F^{(k)}$ in the following way:

$$F^{(k)}\alpha=\alpha(1)\ldots\alpha(k)xx\ldots$$

for any $\alpha\in\mathfrak{X}^N$. It is easy to see that this sequence tends to the identity of $\mathfrak{X}^N$, thus $\mu(F\mathfrak{X}^N)=1$ but for any natural number k

$$\mu(F^{(k)}\mathfrak{X}^N) = 0.$$

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