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Existence of positive solutions of linear delay difference equations with continuous time

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Abstract. Consider the delay difference equation with continuous time of the form

$$x(t) - x(t-1) + \sum_{i=1}^{m} P_i(t)x(t-k_i(t)) = 0, \qquad t \geq t_0,$$

where $P_i\colon [t_0,\infty) \mapsto \mathbb{R}$, $k_i\colon [t_0,\infty) \mapsto \{2,3,4,\dots\}$ and $\lim_{t\to\infty}(t-k_i(t)) = \infty$, for $i = 1,2,\dots,m$.
We introduce the generalized characteristic equation and its importance in oscillation of all solutions of the considered difference equations. Some results for the existence of positive solutions of considered difference equations are presented as the application of the generalized characteristic equation.

Keywords: functional equations, difference equations with continuous time, positive solutions, oscillatory solutions, non-oscillatory solutions.

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1 Introduction

Difference equations with continuous time are difference equations in which the unknown function is a function of a continuous variable. Equations of this type appear as natural descriptions of observed evolution phenomena in many branches of the natural sciences and therefore appear in various mathematical models. This is the main reason why they have been studied in many papers recently. See, for example, the papers of Domshlak [1], Ferreira and Pinelas [2, 3], Golda and Werbowski [4], Korenevskii and Kaizer [7], Ladas et al. [8], Medina and Pituk [9], Meng et al. [10], Nowakowska and Werbowski [11, 12, 13, 14], Shaikhet [17], Shen et al. [18, 19, 20, 21], Zhang et al. [22, 23, 24, 25], and the references cited therein.

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In this paper, we introduce the generalized characteristic equation and its importance in oscillation of all solutions of linear delay difference equations with continuous time. Some results regarding the existence of positive solutions of the considered difference equations are presented as the application of the generalized characteristic equation.

The investigated equation is

$$x(t) - x(t-1) + \sum_{i=1}^{m} P_i(t)x(t - k_i(t)) = 0, \qquad t \geq t_0, \tag{1.1}$$

where $m \geq 1$ is an integer,

(H_1) $P_i : [t_0, \infty) \mapsto \mathbb{R}$ are bounded functions, $i = 1, 2, \ldots, m$,

(H_2) $k_i : [t_0, \infty) \mapsto \{2, 3, 4, \ldots\}$, $k_i(t) < t$ and $\lim_{t\to\infty}(t - k_i(t)) = \infty$, $i = 1, 2, \ldots, m$.

Let t_0 be a positive real number such that

$$t_{-1}(t_0) = \min_{1\leq i\leq m} \{\inf\{\xi - k_i(\xi) : \xi \geq t_0\}\} > 0.$$

It is clear that $t_{-1}(t_0) \leq t_0 - 2 < t_0 - 1$.

In this paper we introduce the concept of the generalized characteristic equation associated to equation (1.1), namely, the nonlinear difference equation

$$\lambda(t) - 1 + \sum_{i=1}^{m} P_i(t) \prod_{j=1}^{k_i(t)-1} \frac{1}{\lambda(t-j)} = 0, \qquad t \geq t_0 + 1, \tag{1.2}$$

and investigate how it relates to the existence of positive solutions of equation (1.1).

A real-valued function x (λ) is called the solution of the difference equation (1.1) [of the difference equation (1.2)] if it is defined on the interval $[t_{-1}(t_0), \infty)$ [on the interval $[t_{-1}(t_0) + 1, \infty)$] and satisfies equation (1.1) [equation (1.2)] for any $t \geq t_0$.

Let $\mathbf{F}$ denote the space of real bounded functions $\phi\colon [t_{-1}(t_0), t_0) \to \mathbb{R}$. Then, for every $\phi \in \mathbf{F}$, equation (1.1) has a unique solution $x\colon [t_{-1}(t_0), \infty) \to \mathbb{R}$ with the initial function

$$x(t) = \phi(t) \quad \text{for } t_{-1}(t_0) \leq t < t_0 \tag{1.3}$$

and the generalized characteristic equation (1.2) has a unique solution $\lambda\colon [t_{-1}(t_0) + 1, \infty) \to \mathbb{R}$ with the initial function

$$\lambda(t) = \psi(t), \qquad t_{-1}(t_0) + 1 \leq t < t_0 + 1, \tag{1.4}$$

where

$$\psi(t) = \begin{cases} \dfrac{\phi(t)}{\phi(t-1)}, & t_{-1}(t_0) + 1 \leq t < t_0; \\[2ex] \dfrac{x(t)}{\phi(t-1)}, & t_0 \leq t < t_0 + 1, \end{cases}$$

assuming that the function ϕ is defined by (1.3) and $\phi(t) \neq 0$, $t_{-1}(t_0) \leq t < t_0$.

We say that the solution $x\colon [t_{-1}(t_0), \infty) \mapsto \mathbb{R}$ of equation (1.1) [$\lambda\colon [t_{-1}(t_0) + 1, \infty) \mapsto \mathbb{R}$ of equation (1.2)] is positive if $x(t) > 0$ for $t \geq t_{-1}(t_0)$ [$\lambda(t) > 0$ for $t \geq t_{-1}(t_0) + 1$].

A motivating example is the equation

$$x(t) - x(t-1) + P(t)x(t-s) = 0, \qquad t \geq t_0, \tag{1.5}$$

where

$$s \geq 2 \quad \text{is a given integer and} \quad P\colon [t_0, \infty) \to \mathbb{R}. \tag{1.6}$$

In this case $t_{-1}(t_0) = t_0 - s$ and hence the initial condition is

$$x(t) = \phi(t), \qquad t_0 - s \leq t < t_0. \tag{1.7}$$

The generalized characteristic equation is

$$\lambda(t) - 1 + P(t) \prod_{j=1}^{s-1} \frac{1}{\lambda(t-j)} = 0, \qquad t \geq t_0 + 1, \tag{1.8}$$

with the initial condition

$$\lambda(t) = \psi(t), \qquad t_0 - s + 1 \leq t < t_0 + 1, \tag{1.9}$$

where

$$\psi(t) = \begin{cases} \dfrac{\phi(t)}{\phi(t-1)}, & t_0 - s + 1 \leq t < t_0; \\ \dfrac{x(t)}{\phi(t-1)}, & t_0 \leq t < t_0 + 1, \end{cases}$$

assuming that the function ϕ is defined by (1.7) and $\phi(t) \neq 0$, $t_0 - s \leq t < t_0$.

We can formulate the following statement.

Theorem 1.1. *Assume that* (1.6) *holds. The solution x of the initial value problem* (1.5) *and* (1.7) *is positive on $[t_0 - s, \infty)$ if and only if the solution λ of the initial value problem* (1.8) *and* (1.9) *is positive on $[t_0 - s + 1, \infty)$ with positive function ϕ defined by* (1.7)*, and x may be written in the form*

$$x(t) = \begin{cases} \phi(t), & t_0 - s \leq t < t_0 - s + 1; \\ \phi(t-n) \displaystyle\prod_{j=0}^{n-1} \lambda(t-j), & t_0 - s + n \leq t < t_0 - s + n + 1, \quad n \geq 1. \end{cases} \tag{1.10}$$

Proof. Let x be a positive solution of the initial value problem (1.5) and (1.7). By dividing both sides of equation (1.5) with $x(t-1)$ we get

$$\frac{x(t)}{x(t-1)} - 1 + P(t) \frac{x(t-s)}{x(t-1)} = 0, \qquad t \geq t_0 + 1. \tag{1.11}$$

Define the function

$$\lambda(t) = \begin{cases} \psi(t), & t_0 - s + 1 \leq t < t_0 + 1; \\ \dfrac{x(t)}{x(t-1)}, & t \geq t_0 + 1. \end{cases} \tag{1.12}$$

From the definition is obvious that the function λ is positive and follows that

$$x(t) = \lambda(t) x(t-1) \quad \text{and} \quad \frac{x(t-s)}{x(t-1)} = \prod_{j=1}^{s-1} \frac{1}{\lambda(t-j)} \quad \text{for } t \geq t_0 + 1, \tag{1.13}$$

and so λ satisfies the initial value problem (1.8) and (1.9) on $[t_0 - s + 1, \infty)$.

On the other hand, let λ be a positive solution of the initial value problem (1.8) and (1.9) on $[t_0 - s + 1, \infty)$ with positive function ϕ defined by (1.7). Then, function x defined by (1.10) is positive. From the definition it follows also that it is equal to the initial function (1.7) for $t_0 - s \leq t < t_0$. For $n = 1$ it follows that $x(t) = \lambda(t) x(t-1)$ and so the equalities (1.13) hold. That means that the characteristic equation (1.8) may be written in the form (1.11) and the function x defined by (1.10) satisfying the difference equation (1.5). The proof is complete. □

The goal is to find necessary and sufficient conditions for the solutions of the initial value problem (1.5) and (1.7) to be positive on $[t_0 - s, \infty)$. The simplest case is $P(t) \leq 0$, $t \geq t_0$, since for every initial function $\phi(t) > 0$, $t_0 - s \leq t < t_0$, the solution of the initial value problem (1.5) and (1.7) is positive. When $P(t) \geq 0$, $t \geq t_0$, then the existence of a positive solution is more delicate, while the most difficult case being whenever the coefficient $P(t)$ is oscillatory on $[t_0, \infty)$.

Theorem 1.2. *Assume that* (1.6) *holds. Let $P(t) \geq 0$ for $t \geq t_0$, and assume that there are two positive functions $\alpha, \beta\colon [t_0 - s + 1, \infty) \mapsto \mathbb{R}^+$ such that*

$$\alpha(t) \leq \beta(t), \quad \alpha(t) \leq 1 - P(t) \prod_{j=0}^{s-1} \frac{1}{\alpha(t-j)} \quad \text{and} \quad 1 - P(t) \prod_{j=0}^{s-1} \frac{1}{\beta(t-j)} \leq \beta(t), \quad t \geq t_0 + 1. \tag{1.14}$$

Then there exists a solution $\lambda\colon [t_0 - s + 1, \infty) \mapsto (0, \infty)$ of the initial value problem (1.8) *and* (1.9) *with*

$$\alpha(t) = \psi(t), \qquad t_0 - s + 1 \leq t < t_0 + 1.$$

Proof. Let $\lambda_0(t) = \alpha(t)$ for $t \geq t_0 - s + 1$, and

$$\lambda_{r+1}(t) = \begin{cases} \alpha(t), & t_0 - s + 1 \leq t < t_0 + 1; \\ 1 - P(t) \displaystyle\prod_{j=0}^{s-1} \frac{1}{\lambda_r(t-j)}, & t \geq t_0 + 1, \quad \text{for any integer } r \geq 0. \end{cases}$$

In this case we can prove that

$$\alpha(t) = \lambda_0(t) \leq \lambda_1(t) \leq \cdots \leq \lambda_k(t) \leq \cdots \leq \beta(t) \quad \text{for } t \geq t_0 - s + 1,$$

and hence the limit function λ of the sequence of functions $\{\lambda_r(t)\}_{r \in \mathbb{N}}$ exists for $t \geq t_0 - s + 1$. That means

$$\lambda(t) = \lim_{r \to \infty} \lambda_r(t) \quad \text{for } t \geq t_0 - s + 1$$

exists. Moreover, $\alpha(t) \leq \lambda(t) \leq \beta(t)$ for any $t \geq t_0 - s + 1$. Then, the function λ satisfies the initial value problem (1.8) and (1.9) with the initial function $\lambda(t) = \alpha(t)$ for $t_0 - s + 1 \leq t < t_0 + 1$. $\square$

Remark 1.3. In the special case when $\alpha(t) = \alpha$, $\beta(t) = \beta$ and $P(t) = p$ are positive constants, from the hypothesis (1.14) of Theorem 1.2 we get $\beta^s(1-\beta) \leq p \leq \alpha^s(1-\alpha)$. The maximum value of the function $f(\alpha) = \alpha^s(1-\alpha)$ we obtain for

$$\alpha = \frac{s}{s+1} \quad \text{and so} \quad f_{max}\left(\frac{s}{s+1}\right) = \frac{s^s}{(s+1)^{s+1}}.$$

So the new form of hypothesis (1.14) for the existence of positive solutions may be $p \leq \frac{s^s}{(s+1)^{s+1}}$.

Thus, there exists a positive solution of the initial value problem (1.5) and (1.7). Similar result may be proved for the general case.

2 Preliminaries

In the work of Győri and Ladas [6] some results, such as Theorem 3.1.1, are shown related to the generalized characteristic equation of linear delay differential equation

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t-\tau_i(t)) = 0, \qquad t_0 \le t \le T \tag{2.1}$$

with an initial condition of the form

$$x(t) = \varphi(t), \qquad t_{-1} \le t \le t_0, \quad t_{-1} = \min_{1\le i\le n} \left\{ \inf_{t_0 \le t < T} \{t - \tau_i(t)\} \right\}, \tag{2.2}$$

with $\varphi \in C[[t_{-1}, t_0], \mathbb{R}]$, where $t_0 < T \le \infty$ and

(H_1^*) $\qquad p_i \in C[[t_0, T), \mathbb{R}], \qquad \tau_i \in C[[t_0, T), \mathbb{R}^+], \qquad i = 1, 2, \dots, n.$

In Theorem 3.1.1, a condition for the existence of a positive solution of the initial value problem (2.1) and (2.2) is formulated. The unique solution of the initial value problem (2.1) and (2.2) is denoted with $x(\varphi)(t)$ and exists for $t_0 \le t \le T$.

Győri and Ladas have also obtained some results for the existence of positive solutions of the considered differential equation.

Theorem A ([6, Theorem 3.3.2])**.** Assume that (H_1^*) holds and that there exists a positive number μ such that

$$\sum_{i=1}^{n} |p_i(t)| e^{\mu \tau_i(t)} \le \mu \quad \text{for } t \ge t_0.$$

Then, for every $\varphi \in \{\varphi \in C[[t_{-1}, t_0], \mathbb{R}^+] \mid \varphi(t_0) > 0 \text{ and } \varphi(t) \le \varphi(t_0) \text{ for } t_{-1} \le t \le t_0\}$, the solution $x(\varphi)(t)$ of equation (2.1) through (t_0, φ), remains positive on $t_0 \le t \le T$.

Papers [15] and [16] deal with the discrete analogues of the generalized characteristic equation and Theorem A. Consider the linear retarded difference equation

$$a_{n+1} - a_n + \sum_{i=1}^{m} P_i(n) a_{n-k_i(n)} = 0, \qquad n \in \mathbb{N}^*, \tag{2.3}$$

where $\mathbb{N}^* = \{n \in \mathbb{N} : n_0 \le n < M,\ n_0 < M \le \infty\}$ and $\mathbb{N}$ is the set of positive integers. Let

(H_2^*) $\qquad \{P_i(n)\}$ a sequence of real numbers for $i = 1, 2, \dots, m,\ n \in \mathbb{N}^*$;

(H_3^*) $\qquad \{k_i(n)\}$ a sequence of positive real numbers for $i = 1, 2, \dots, m,\ n \in \mathbb{N}^*$.

Associated with equation (2.3), we define the initial condition

$$a_n = \phi_n, \quad \text{for} \quad n = n_{-1}, n_{-1}+1, \ldots, n_0, \quad \phi_n \in \mathbb{R}, \tag{2.4}$$

where

$$n_{-1} = \min_{1\le i\le m} \left\{ \inf_{n_0 \le n < M} \{n - k_i(n)\} \right\}.$$

The unique solution of the initial value problem (2.3) and (2.4) is denoted with $a(\phi)_n$ and exists for $n \in \mathbb{N}^*$.

Theorem B ([16, Theorem 3.2]). Assume that (H_2^*) and (H_3^*) hold and there exists a real number $\mu \in (0,1)$ such that

$$\sum_{i=1}^{m} |P_i(n)|(1-\mu)^{-k_i(n)} \le \mu \quad \text{for } n \in \mathbb{N}^*.$$

Then, for every $\{\phi_n\} \in \{\{\phi_j\} : \phi_{n_0} > 0,\ 0 < \phi_j \le \phi_{n_0} \text{ for } j = n_{-1}, n_{-1}+1, \ldots, n_0\}$, the solution $a(\phi)_n$ of (2.3) remains positive for $n \in \mathbb{N}^*$.

The papers of Golda and Werbowski [4], Shen and Stavroulakis [21], Zhang and Choi [25] deal with the functional equation with variable coefficients of the form

$$x(g(t)) = P(t)x(t) + Q(t)x(g^2(t)), \tag{2.5}$$

where $P, Q \in C([0,\infty),[0,\infty))$, $g \in C([0,\infty),\mathbb{R})$, g is increasing, $g(t) > t$ or $g(t) < t$ and $g(t) \to \infty$ as $t \to \infty$.

Theorem C ([4, Theorem 1]). Assume that $P, Q \in C([0,\infty),[0,\infty))$, $g \in C([0,\infty),\mathbb{R})$, g is increasing, $g(t) > t$ or $g(t) < t$ and $g(t) \to \infty$ as $t \to \infty$. If the equation (2.5) has a non-oscillatory solution, then

$$\liminf_{I \ni t \to \infty} Q(t)P(g(t)) \le \frac{1}{4}, \tag{2.6}$$

for large t.

Theorem D ([21, Theorem 1]). Assume that $P, Q \in C([0,\infty),[0,\infty))$, $g \in C([0,\infty),\mathbb{R})$, g is increasing, $g(t) > t$ or $g(t) < t$ and $g(t) \to \infty$ as $t \to \infty$. If

$$Q(t)P(g(t)) \le \frac{1}{4}, \tag{2.7}$$

for large t, then equation (2.5) has a non-oscillatory solution.

Theorem E ([25, Remark 3.3]). Assume that $P, Q \in C([0,\infty),[0,\infty))$, $g \in C([0,\infty),\mathbb{R})$, g is increasing, $g(t) > t$ or $g(t) < t$ and $g(t) \to \infty$ as $t \to \infty$. If

$$Q^+(t)P(t) \le \frac{1}{4}, \quad \text{for } t \ge T, \tag{2.8}$$

then equation (2.5) has a positive solution.

Shen and Stavroulakis [21] studied the linear functional equation of the form

$$x(t) - px(t-\tau) + q(t)x(t-\sigma) = 0, \tag{2.9}$$

where $p, \tau, \sigma \in (0,\infty)$, $q \in C([0,\infty),[0,\infty))$.

Theorem F ([21, Theorem 2]). If $p, \tau, \sigma \in (0,\infty)$, $q \in C([0,\infty),[0,\infty))$, $\sigma > \tau$ and for large t

$$p^{-\frac{\sigma}{\tau}} \cdot q(t) \le \left(\frac{\sigma-\tau}{\sigma}\right)^{\frac{\sigma}{\tau}} \left(\frac{\sigma-\tau}{\tau}\right)^{-1}, \tag{2.10}$$

then equation (2.9) has a non-oscillatory solution.

Zhang and Choi [25] have studied also the functional equation of the form

$$x(g(t)) = P(t)x(t) + Q(t)x(g^k(t)), \quad \text{where } k \ge 1 \text{ is a positive integer.} \tag{2.11}$$

Theorem G ([25, Corollary 3.4]). Assume that $P, Q \in C([0,\infty),[0,\infty))$, $g \in C([0,\infty),\mathbb{R})$, g is increasing, $g(t) > t$ or $g(t) < t$, $g(t) \to \infty$ as $t \to \infty$ and $k \geq 1$ is a positive integer. If $\limsup_{t\to\infty} p(t) = p$ and

$$Q^{+}(t) \leq \frac{(k-1)^{k-1}}{k^k}, \tag{2.12}$$

then equation (2.11) has a positive solution.

3 Main results

The following lemma can be easily proved by mathematical induction.

Lemma 3.1. *Let $\lambda\colon [t_{-1}(t_0)+1,\infty) \mapsto \mathbb{R}$ and $\varphi\colon [t_{-1}(t_0), t_{-1}(t_0)+1) \mapsto \mathbb{R}$ be two given functions and consider the difference equation*

$$x(t) = \lambda(t)x(t-1), \qquad t \geq t_{-1}(t_0)+1 \tag{3.1}$$

with the initial condition

$$x(t) = \varphi(t), \qquad t_{-1}(t_0) \leq t < t_{-1}(t_0)+1. \tag{3.2}$$

Then, the initial value problem (3.1) and (3.2) has a solution which is given in the form

$$x(t) = \begin{cases} \varphi(t), & t_{-1}(t_0) \leq t < t_{-1}(t_0)+1; \\ \varphi(t-n)\displaystyle\prod_{j=0}^{n-1}\lambda(t-j), & t_{-1}(t_0)+n \leq t < t_{-1}(t_0)+n+1, \quad n \geq 1. \end{cases} \tag{3.3}$$

Theorem 3.2. *Assume that (H_1) and (H_2) hold. Then $x\colon [t_{-1}(t_0),\infty) \mapsto \mathbb{R}$ is a positive solution of equation (1.1) if and only if there are two positive functions $\lambda\colon [t_{-1}(t_0)+1,\infty) \mapsto (0,\infty)$ and $\varphi\colon [t_{-1}(t_0), t_{-1}(t_0)+1) \mapsto (0,\infty)$ such that*

(a) λ is a solution of equation (1.2) on the interval $[t_{-1}(t_0)+1,\infty)$;

(b) x satisfies (3.1) and (3.2) or equivalently it is given by (3.3).

Proof. Let us assume that equation (1.1) has a positive solution, say $x\colon [t_{-1}(t_0),\infty) \mapsto \mathbb{R}$. Then, one can show that

$$\lambda(t) = \frac{x(t)}{x(t-1)}, \qquad t \geq t_{-1}(t_0)+1,$$

is a positive solution of equation (1.2). Moreover, $x(t) = \lambda(t)x(t-1)$, $t \geq t_{-1}(t_0)+1$, with the initial function $\varphi(t) = x(t) > 0$ for $t_{-1}(t_0) \leq t < t_{-1}(t_0)+1$. So Lemma 3.1 shows that the form (3.3) is satisfied.

On the other hand, if (a) and (b) hold then one can get that x is a positive solution of equation (1.1). □

The following lemma will be useful in proving the main results.

Lemma 3.3. *Let k be a given natural number and $a_1, a_2, \ldots, a_k$, $b_1, b_2, \ldots, b_k$ positive real numbers. Then*

$$\prod_{j=1}^{k}\frac{1}{a_j} - \prod_{j=1}^{k}\frac{1}{b_j} = \frac{1}{\prod_{j=1}^{k} a_j b_j}\sum_{j=1}^{k}\left(\prod_{\ell=1}^{j-1} a_\ell\right)\left(\prod_{i=j+1}^{k} b_i\right)(b_j - a_j) = \sum_{j=1}^{k}\frac{1}{\prod_{i=1}^{j} b_i \prod_{\ell=j}^{k} a_\ell}(b_j - a_j).$$

Proof. The left side of the equality we can rewrite in the form

$$\prod_{j=1}^{k}\frac{1}{a_j}-\prod_{j=1}^{k}\frac{1}{b_j}=\frac{1}{\prod\limits_{j=1}^{k}a_jb_j}\left(\prod_{j=1}^{k}b_j-\prod_{j=1}^{k}a_j\right),$$

where

$$\begin{aligned}\prod_{j=1}^{k}b_j-\prod_{j=1}^{k}a_j&=\prod_{i=1}^{k}b_i\pm\sum_{j=1}^{k-1}\left(\prod_{\ell=1}^{j}a_\ell\prod_{i=j+1}^{k}b_i\right)-\prod_{\ell=1}^{k}a_\ell\\
&=\left(\prod_{i=1}^{k}b_i-a_1\prod_{i=2}^{k}b_i\right)+\left(a_1\prod_{i=2}^{k}b_i-a_1a_2\prod_{i=3}^{k}b_i\right)+\left(a_1a_2\prod_{i=3}^{k}b_i-a_1a_2a_3\prod_{i=4}^{k}b_i\right)\\
&\quad+\cdots+\left(\left(\prod_{\ell=1}^{k-2}a_\ell\right)b_{k-1}b_k-\left(\prod_{\ell=1}^{k-1}a_\ell\right)b_k\right)+\left(\left(\prod_{\ell=1}^{k-1}a_\ell\right)b_k-\prod_{\ell=1}^{k}a_\ell\right)\\
&=\left(\prod_{i=2}^{k}b_i\right)(b_1-a_1)+a_1\left(\prod_{i=3}^{k}b_i\right)(b_2-a_2)+a_1a_2\left(\prod_{i=4}^{k}b_i\right)(b_3-a_3)\\
&\quad+\cdots+\left(\prod_{\ell=1}^{k-2}a_\ell\right)b_k\left(b_{k-1}-a_{k-1}\right)+\left(\prod_{\ell=1}^{k-1}a_\ell\right)(b_k-a_k)\\
&=\sum_{j=1}^{k}\left(\prod_{\ell=1}^{j-1}a_\ell\right)\left(\prod_{i=j+1}^{k}b_i\right)(b_j-a_j).\end{aligned}$$

Using the above transformation we get

$$\prod_{j=1}^{k}\frac{1}{a_j}-\prod_{j=1}^{k}\frac{1}{b_j}=\frac{1}{\left(\prod\limits_{\ell=1}^{k}a_\ell\right)\left(\prod\limits_{i=1}^{k}b_i\right)}\sum_{j=1}^{k}\left(\prod_{\ell=1}^{j-1}a_\ell\right)\left(\prod_{i=j+1}^{k}b_i\right)(b_j-a_j)=\sum_{j=1}^{k}\frac{1}{\prod\limits_{i=1}^{j}b_i\prod\limits_{\ell=j}^{k}a_\ell}(b_j-a_j).$$

□

The following theorem is the discrete analogue of Theorem 3.1.1 [6] and simultaneously the generalization of the Theorem 1.1 [15] for continuous time.

Theorem 3.4. *Assume that* (H_1) *and* (H_2) *hold,* $\phi\in F$ *with* $\phi(t)>0$ *for* $t_{-1}(t_0)\le t<t_0$. *Then the following statements are equivalent:*

(a) The solution of the initial value problem (1.1) *and* (1.3) *is positive for* $t\ge t_{-1}(t_0)$.

(b) The initial value problem (1.2) *and* (1.4) *has positive solution on* $[t_{-1}(t_0)+1,\infty)$.

(c) There exist functions $\beta,\gamma\colon[t_{-1}(t_0)+1,\infty)\mapsto\mathbb{R}^+$ *such that* $\beta(t)\le\psi(t)\le\gamma(t)$ *on the interval* $[t_{-1}(t_0)+1,t_0+1)$, $\beta(t)\le\gamma(t)$ *for* $t\ge t_0+1$ *and for every function* $\delta\colon[t_{-1}(t_0)+1,\infty)\mapsto\mathbb{R}$ *with* $\delta(t)=\psi(t)$ *for* $t_{-1}(t_0)+1\le t<t_0+1$, *where the positive function* ψ *is defined by* (1.4), *such that* $\beta(t)\le\delta(t)\le\gamma(t)$ *for* $t\ge t_0+1$, *the following inequalities hold:*

$$\beta(t)\le(S\delta)(t)\le\gamma(t),\qquad t\ge t_0+1,\tag{3.4}$$

where

$$(S\delta)(t)\equiv1-\sum_{i=1}^{m}P_i(t)\prod_{j=1}^{k_i(t)-1}\frac{1}{\delta(t-j)},\qquad t\ge t_0+1.\tag{3.5}$$

Proof. $(a) \implies (b)$. Let $x\colon [t_{-1}(t_0), \infty) \to \mathbb{R}$ be the solution of the initial value problem (1.1) and (1.3) and suppose that $x(t) > 0$ for $t \geq t_{-1}(t_0)$. Our aim is to show that the positive function

$$\lambda(t) = \begin{cases} \psi(t), & t_{-1}(t_0) + 1 \leq t < t_0 + 1; \\ \dfrac{x(t)}{x(t-1)}, & t \geq t_0 + 1. \end{cases} \tag{3.6}$$

is a solution of the characteristic equation (1.2) with the initial condition (1.4). From definition (3.6)

$$x(t) = \begin{cases} \phi(t), & t_{-1}(t_0) \leq t < t_{-1}(t_0) + 1; \\ \phi\left(t - n(t)\right) \displaystyle\prod_{j=0}^{n(t)-1} \lambda(t-j), & t \geq t_{-1}(t_0) + 1. \end{cases} \tag{3.7}$$

is obtained, where $n(t) = [t - (t_{-1}(t_0))]$ with the properties that $t_{-1}(t_0) \leq t - n(t) < t_{-1}(t_0) + 1$ and $t - n(t) + 1 \geq t_{-1}(t_0) + 1$ for a real number $t \geq t_0 + 1$ ($[t]$ denotes the integer part of the real number t).

By dividing both sides of equation (1.1) with $x(t-1)$ we get

$$\frac{x(t)}{x(t-1)} - 1 + \sum_{i=1}^{m} P_i(t) \frac{x(t - k_i(t))}{x(t-1)} = 0, \qquad t \geq t_0 + 1.$$

Because of (3.7) we have

$$\frac{x(t-k_i(t))}{x(t-1)} = \frac{\phi(t-n(t)) \displaystyle\prod_{j=k_i(t)}^{n(t)-1} \lambda(t-j)}{\phi(t-n(t)) \displaystyle\prod_{j=1}^{n(t)-1} \lambda(t-j)} = \prod_{j=1}^{k_i(t)-1} \frac{1}{\lambda(t-j)}, \qquad t \geq t_0 + 1.$$

Thus, this part of the proof is complete.

$(b) \implies (c)$. Let the function $\lambda\colon [t_{-1}(t_0)+1, \infty) \to \mathbb{R}$ be a positive solution of the characteristic equation (1.2) with the initial condition (1.4), and set $\beta(t) = \gamma(t) = \lambda(t)$ for $t \geq t_{-1}(t_0) + 1$. Then the statement of the proof follows from (1.2) and (3.5), so $\lambda(t) = (S\lambda)(t)$ for $t \geq t_0 + 1$.

$(c) \implies (a)$. First, we show that under hypothesis (c), the initial value problem (1.2) and (1.4) has a positive solution $\lambda\colon [t_{-1}(t_0)+1, \infty) \to \mathbb{R}$. The solution λ will be constructed by the method of successive approximation as the limit of a sequence of functions $\{\lambda_r(t)\}$ for $t \geq t_{-1}(t_0) + 1$ defined as follows.
Take any function $\lambda_0\colon [t_{-1}(t_0)+1, \infty) \mapsto \mathbb{R}^+]$ between the functions β and γ:

$$0 < \beta(t) \leq \lambda_0(t) \leq \gamma(t), \qquad t \geq t_0 + 1 \quad \text{and} \quad \lambda_0(t) = \psi(t), \qquad t_{-1}(t_0) + 1 \leq t < t_0 + 1.$$

Set

$$\lambda_{r+1}(t) = \begin{cases} \lambda_0(t), & t_{-1}(t_0) + 1 \leq t < t_0 + 1; \\ (S\lambda_r)(t), & t \geq t_0 + 1, \quad r = 0, 1, 2, \ldots. \end{cases}$$

By condition (3.4) and using induction, it follows that

$$\beta(t) \leq \lambda_r(t) \leq \gamma(t), \qquad t \geq t_{-1}(t_0) + 1, \quad r = 1, 2, \ldots \tag{3.8}$$

and so $\lambda_r\colon [t_{-1}(t_0)+1,\infty) \mapsto \mathbb{R}^+$. Next, we show that the sequence $\{\lambda_r(t)\}_{r\in\mathbb{N}}$ converges uniformly on any subinterval $[t_0+1,T_1]$ of $[t_0+1,\infty)$. Set

$$L = \max\left\{\max_{t_{-1}(t_0)+1\le t\le T_1}\{\gamma(t)\},1\right\}, \qquad M = \max_{t_0+1\le t\le T_1}\left\{\sum_{i=1}^{m}|P_i(t)|\prod_{\ell=1}^{k_i(t)-1}\frac{1}{(\beta(t-\ell))^2}\right\},$$
$$k = \max_{1\le i\le m}\left\{\max_{t_0+1\le t\le T_1}\{k_i(t)\}\right\}, \qquad M_1 = \max\left\{ML^{k-1},1\right\}.$$

Then from (3.8) it follows that

$$\max_{t_{-1}(t_0)+1\le t\le T_1}\{\lambda_r(t)\}\le L \quad \text{for } r=0,1,2,\dots$$

By elementary transformations and applying Lemma 3.3, we can show the following inequalities:

$$\begin{aligned}
|\lambda_{r+1}(t)-\lambda_r(t)| &\le \sum_{i=1}^{m}|P_i(t)|\left|\prod_{j=1}^{k_i(t)-1}\frac{1}{\lambda_{r-1}(t-j)}-\prod_{j=1}^{k_i(t)-1}\frac{1}{\lambda_r(t-j)}\right| \\
&\le \sum_{i=1}^{m}|P_i(t)|\frac{L^{k_i(t)-1}\sum\limits_{j=1}^{k_i(t)-1}|\lambda_r(t-j)-\lambda_{r-1}(t-j)|}{\prod\limits_{\ell=1}^{k_i(t)-1}(\beta(t-\ell))^2} \\
&\le ML^{k-1}\sum_{j=1}^{k-1}|\lambda_r(t-j)-\lambda_{r-1}(t-j)| \\
&\le M_1\sum_{j=1}^{k-1}|\lambda_r(t-j)-\lambda_{r-1}(t-j)| \quad \text{for } t\ge t_0+1.
\end{aligned}$$

Thus for all $r=1,2,\dots$ and $t_0+1\le t\le T_1$ the inequality

$$|\lambda_{r+1}(t)-\lambda_r(t)|\le M_1\sum_{j=1}^{k-1}|\lambda_r(t-j)-\lambda_{r-1}(t-j)|$$

holds. By induction, we can show that for all $r=0,1,2,\dots$ and $t_0+1\le t\le T_1$

$$|\lambda_{r+1}(t)-\lambda_r(t)|\le 2L\frac{M_1^r t^r}{r!}.$$

For $r=0$ we have

$$|\lambda_1(t)-\lambda_0(t)|\le 2L = 2L\frac{M_1^0 t^0}{0!}.$$

Suppose that the inequality is true for $r=q$, i.e.

$$\left|\lambda_{q+1}(t)-\lambda_q(t)\right|\le 2L\frac{M_1^q t^q}{q!}.$$

We will show that the inequality is true also for $r = q + 1$.

$$
\begin{aligned}
\left|\lambda_{q+2}(t) - \lambda_{q+1}(t)\right| &\le M_1 \sum_{j=1}^{k-1} \left|\lambda_{q+1}(t-j) - \lambda_q(t-j)\right| \\
&= M_1 \sum_{\ell=t-k+1}^{t-1} \left|\lambda_{q+1}(\ell) - \lambda_q(\ell)\right| \le M_1 \sum_{\ell=t-k+1}^{t-1} 2L \frac{M_1^q \ell^q}{q!} \\
&= 2L \frac{M_1^q}{q!} \sum_{\ell=t-k+1}^{t-1} \ell^q \le 2L \frac{M_1^q}{q!} \sum_{\ell=t-k+1}^{t-1} \int_\ell^{\ell+1} s^q \, ds \\
&= 2L \frac{M_1^q}{q!} \int_{t-k+1}^{t} s^q \, ds = 2L \frac{M_1^q}{q!} \left[\frac{s^{q+1}}{q+1} \right] \Bigg|_{t-k+1}^{t} \\
&= 2L \frac{M_1^q}{q!} \left(t^{q+1} - (t-k+1)^{q+1} \right) < 2L \frac{M_1^{q+1} t^{q+1}}{(q+1)!},
\end{aligned}
$$

because $t_{-1}(t_0) > 0$ and so $t - k + 1 > 0$.

For given $n \in \mathbb{N}$, $t \in \mathbb{R}_+$ and a function $f \colon \mathbb{R} \to \mathbb{R}$ we use the standard notation

$$
\sum_{\ell=t-n}^{t} f(\ell) = f(t-n) + f(t-n+1) + \cdots + f(t).
$$

It follows by the Weierstrass M-test that the series

$$
\sum_{r=0}^{\infty} |\lambda_{r+1}(t) - \lambda_r(t)|
$$

converges uniformly on every compact interval $[t_0 + 1, T_1]$ and therefore the sequence

$$
\lambda_r(t) = \lambda_0(t) + \sum_{j=0}^{r-1} \left|\lambda_{j+1}(t) - \lambda_j(t)\right| \quad \text{for } t_0 + 1 \le t \le T_1, \ r = 0, 1, 2, \ldots
$$

also converges uniformly. Thus, the limit function

$$
\lambda(t) = \lim_{r \to \infty} \lambda_r(t) \tag{3.9}
$$

is positive for $t_0 + 1 \le t \le T_1$. Because of the convergence,

$$
\begin{aligned}
\lambda(t) = \lim_{r \to \infty} \lambda_{r+1}(t) &= \lim_{r \to \infty} \left(1 - \sum_{i=1}^{m} P_i(t) \prod_{j=1}^{k_i(t)-1} \frac{1}{\lambda_r(t-j)} \right) \\
&= 1 - \sum_{i=1}^{m} P_i(t) \prod_{j=1}^{k_i(t)-1} \frac{1}{\lambda(t-j)}, \qquad t_0 + 1 \le t \le T_1,
\end{aligned}
$$

and $\lambda(t) = \lambda_0(t)$ for $t_{-1}(t_0) + 1 \le t < t_0 + 1$, which shows that λ, as defined by (3.9), is a solution of characteristic equation (1.2) on $[t_{-1}(t_0) + 1, T_1]$. As T_1 is an arbitrary fixed point on $[t_0 + 1, \infty)$, it follows that λ, as defined by (3.9), is a solution of (1.2) on $[t_{-1}(t_0) + 1, \infty)$.

Finally, we define the function $x \colon [t_{-1}(t_0), \infty) \mapsto \mathbb{R}^+$ by (3.7). It is obvious that the function x, defined by (3.7), is a solution of the initial value problem (1.1) and (1.3), and the proof of the theorem is complete. □

For the special case, when $k_i(t) = k_i \in \{2,3,4,\dots\}$, $P_i \in C[[t_0,\infty),\mathbb{R}]$, $i = 1,2,\dots,m$, the equation (1.1) is a linear difference equation with continuous time and constant delay:

$$x(t) - x(t-1) + \sum_{i=1}^{m} P_i(t)x(t-k_i) = 0, \qquad t \geq t_0 \tag{3.10}$$

and the generalized characteristic equation

$$\lambda(t) - 1 + \sum_{i=1}^{m} P_i(t) \prod_{j=1}^{k_i-1} \frac{1}{\lambda(t-j)} = 0, \qquad t \geq t_0 + 1. \tag{3.11}$$

Now, $t_{-1}(t_0) = t_0 - \max\{k_1, k_2, \dots, k_m\}$.

Let $\phi \in C[[t_{-1}(t_0), t_0) \to \mathbb{R}]$. Let $\mathbf{F_C}$ denote the space of continuous functions $\phi\colon [t_{-1}(t_0), t_0) \to \mathbb{R}$. Then, for every $\phi \in \mathbf{F_C}$, equation (3.10) has a unique piecewise continuous solution $x\colon [t_{-1}(t_0),\infty) \to \mathbb{R}$ with the continuous initial function defined by (1.3), and equation (3.11) has a unique piecewise continuous solution $\lambda\colon [t_{-1}(t_0)+1,\infty) \to \mathbb{R}$ with the continuous initial function defined by (1.4).

We can formulate the following corollary.

Corollary 3.5. *Assume that $P_i \in C[[t_0,\infty),\mathbb{R}]$ and $k_i \in \{2,3,4,\dots.\}$ for $i = 1,2,\dots,m$. Let $\phi \in \mathbf{F_C}$ such that $\phi(t) > 0$ for $t_{-1}(t_0) \leq t < t_0$. Then the following statements are equivalent.*

(a) The solution of the initial value problem (3.10) and (1.3) is positive piecewise continuous for $t \geq t_{-1}(t_0)$.

(b) The initial value problem (3.11) and (1.4) has positive piecewise continuous solution on $[t_{-1}(t_0)+1,\infty)$.

(c) There exist functions $\beta,\gamma \in C[[t_{-1}(t_0)+1,\infty),\mathbb{R}^+]$ such that $\beta(t) \leq \psi(t) \leq \gamma(t)$ on the interval $[t_{-1}(t_0)+1, t_0+1)$, $\beta(t) \leq \gamma(t)$ for $t \geq t_0+1$ and for every function $\delta \in C[[t_{-1}(t_0)+1,\infty),\mathbb{R}]$ with $\delta(t) = \psi(t)$ for $t_{-1}(t_0)+1 \leq t < t_0+1$ such that $\beta(t) \leq \delta(t) \leq \gamma(t)$ for $t \geq t_0+1$, the following inequalities hold:

$$\beta(t) \leq 1 - \sum_{i=1}^{m} P_i(t) \prod_{j=1}^{k_i-1} \frac{1}{\delta(t-j)} \leq \gamma(t), \qquad t \geq t_0 + 1.$$

4 Comparison results

Consider, now, the delay functional equation

$$y(t) - y(t-1) + \sum_{i=1}^{m} q_i(t)y(t-k_i(t)) = 0 \quad \text{for } t \geq t_0 \tag{4.1}$$

and the delay functional inequalities

$$x(t) - x(t-1) + \sum_{i=1}^{m} p_i(t)x(t-k_i(t)) \leq 0 \quad \text{for } t \geq t_0, \tag{4.2}$$

$$z(t) - z(t-1) + \sum_{i=1}^{m} r_i(t)z(t-k_i(t)) \geq 0 \quad \text{for } t \geq t_0. \tag{4.3}$$

The oscillatory behavior of delay differential equations and inequalities has been the subject of many investigations. For a result we refer to [6, 5] and the references therein. The next result is a discrete analogue of Theorem 3.2.1 [6] formulated for differential equations and inequalities and the generalization of the Theorem 1.2 [16] in the continuous time domain.

Theorem 4.1. *Suppose that $p_i, q_i, r_i\colon [t_0,\infty) \to \mathbb{R}^+$ for $i = 1,2,\dots,m$ such that*

$$p_i(t) \geq q_i(t) \geq r_i(t) \quad \text{for } t \geq t_0,\ i = 1,2,\dots,m.$$

Assume that (H_2) holds and $x\colon [t_{-1}(t_0),\infty) \mapsto \mathbb{R}$, $y\colon [t_{-1}(t_0),\infty) \mapsto \mathbb{R}$ and $z\colon [t_{-1}(t_0),\infty) \mapsto \mathbb{R}$ are solutions of (4.2), (4.1) and (4.3), respectively, such that

$$z(t_0) \geq y(t_0) \geq x(t_0), \quad x(t) > 0 \quad \text{for } t \geq t_0, \tag{4.4}$$

$$0 < \frac{x(t)}{x(t-1)} \leq \frac{y(t)}{y(t-1)} \leq \frac{z(t)}{z(t-1)} \quad \text{for } t_{-1}(t_0)+1 \leq t < t_0+1. \tag{4.5}$$

Then

$$z(t) \geq y(t) \geq x(t) \quad \text{for } t \geq t_0. \tag{4.6}$$

Proof. Set

$$\alpha_0(t) = \frac{x(t)}{x(t-1)}, \quad \beta_0(t) = \frac{y(t)}{y(t-1)}, \quad \gamma_0(t) = \frac{z(t)}{z(t-1)}, \quad \text{for } t \geq t_{-1}(t_0)+1.$$

Then, by using the previous notation, it follows that

$$\alpha_0(t) - 1 + \sum_{i=1}^{m} p_i(t) \prod_{j=1}^{k_i(t)-1} \frac{1}{\alpha_0(t-j)} \leq 0, \qquad t \geq t_0+1,$$

$$\beta_0(t) - 1 + \sum_{i=1}^{m} q_i(t) \prod_{j=1}^{k_i(t)-1} \frac{1}{\beta_0(t-j)} = 0, \qquad t \geq t_0+1,$$

$$\gamma_0(t) - 1 + \sum_{i=1}^{m} r_i(t) \prod_{j=1}^{k_i(t)-1} \frac{1}{\gamma_0(t-j)} \geq 0, \qquad t \geq t_0+1.$$

We will show by induction and by using Theorem 3.4 that

$$\alpha_0(t) \leq \beta_0(t) \leq \gamma_0(t) \quad \text{for } t \geq t_0+1. \tag{4.7}$$

For the first part of inequality (4.7) let $\delta\colon [t_{-1}(t_0)+1,\infty) \mapsto \mathbb{R}$ be an arbitrary function such that $\delta(t) = \beta_0(t)$ for $t_{-1}(t_0)+1 \leq t < t_0+1$ and $\alpha_0(t) \leq \delta(t) \leq 1$ for $t \geq t_0+1$. Then

$$\begin{aligned}\alpha_0(t) &\leq 1 - \sum_{i=1}^{m} p_i(t) \prod_{j=1}^{k_i(t)-1} \frac{1}{\alpha_0(t-j)} \\ &\leq 1 - \sum_{i=1}^{m} q_i(t) \prod_{j=1}^{k_i(t)-1} \frac{1}{\delta(t-j)} \equiv S\delta(t) \leq 1, \qquad t \geq t_0+1.\end{aligned}$$

Then, the statement (c) of Theorem 3.4 is true with $\beta(t) = \alpha_0(t)$ and $\gamma(t) \equiv 1$ for $t \geq t_{-1}(t_0)+1$, so by the same theorem, the initial value problem

$$\begin{cases} \delta(t) - 1 + \sum_{i=1}^{m} q_i(t) \prod_{j=1}^{k_i(t)-1} \frac{1}{\delta(t-j)} = 0, & t \geq t_0+1 \\ \delta(t) = \beta_0(t), \qquad t_{-1}(t_0)+1 \leq t < t_0+1, \end{cases}$$

has exactly one solution for $t \geq t_{-1}(t_0) + 1$, and the solution of this equation is between the functions α_0 and 1 on $[t_{-1}(t_0) + 1, \infty)$. Since $\beta_0 \colon [t_{-1}(t_0) + 1, \infty) \mapsto \mathbb{R}^+$ is the unique solution of the same initial value problem, hence it follows that $\delta(t) = \beta_0(t)$ for $t \geq t_0 + 1$, and so $\alpha_0(t) \leq \beta_0(t) \leq 1$ for $t \geq t_0 + 1$.

In order to prove the second part of inequality (4.7), we will show that $\alpha_0(t) \leq \gamma_0(t)$ for $t \geq t_0 + 1$. Then, in a similar way as before, with $\beta(t) = \alpha_0(t)$ and $\gamma(t) \equiv \gamma_0(t)$ for $t \geq t_{-1}(t_0) + 1$, the second part of inequality (4.7) can be proved. From inequality (4.5) we have that $0 < \alpha_0(t) \leq \gamma_0(t)$ for $t_{-1}(t_0) + 1 \leq t < t_0 + 1$. Let $t \geq t_0 + 1$ be such a point, that $t - 1 < t_0 + 1$. Then

$$\begin{aligned} \alpha_0(t) &\leq 1 - \sum_{i=1}^{m} p_i(t) \prod_{j=1}^{k_i(t)-1} \frac{1}{\alpha_0(t-j)} \\ &\leq 1 - \sum_{i=1}^{m} r_i(t) \prod_{j=1}^{k_i(t)-1} \frac{1}{\gamma_0(t-j)} \leq \gamma_0(t) \quad \text{for } t \geq t_0 + 1. \end{aligned}$$

Let $t \geq t_0 + 1$ be such a point that $t - \ell < t_0 + 1$, and suppose that $\alpha_0(t) \leq \gamma_0(t)$ for $t \geq t_0 + 1$. Now, let $t \geq t_0 + 1$ be such a point that $t - (\ell + 1) < t_0 + 1$. Using the previous inequality it follows $\alpha_0(t) \leq \gamma_0(t)$ for $t \geq t_0 + 1$, too. Because of the equalities

$$\begin{aligned} x(t) &= \phi(t - n(t)) \prod_{j=0}^{n(t)-1} \alpha_0(t-j) \quad \text{for } t \geq t_0, \\ y(t) &= \phi(t - n(t)) \prod_{j=0}^{n(t)-1} \beta_0(t-j) \quad \text{for } t \geq t_0, \\ z(t) &= \phi(t - n(t)) \prod_{j=0}^{n(t)-1} \gamma_0(t-j) \quad \text{for } t \geq t_0, \end{aligned}$$

(4.4) and (4.7) imply that (4.6) holds and the proof is complete. □

5 Existence of positive solutions

Our aim in this section is to derive results on the existence of positive solutions of equation (1.1) by applying statement (c) of Theorem 3.4. To that end, we will postulate first the Theorem which is the discrete analogue of the Theorem 3.3.2 [6] and at the same time the generalization of Theorem 3.2 [16] in the continuous time domain.

Theorem 5.1. *Assume that (H_1), (H_2) hold and that there exists a positive number $\mu \in (0, 1)$ such that*

$$\sum_{i=1}^{m} |P_i(t)| (1 - \mu)^{1 - k_i(t)} \leq \mu \quad \text{for } t \geq t_0 + 1. \tag{5.1}$$

Then for every $\phi \in \mathbf{F}$ such that $\phi(t) > 0$ for $t_{-1}(t_0) \leq t < t_0$, the solution $x \colon [t_{-1}(t_0), \infty) \to \mathbb{R}$ of the initial value problem (1.1) and (1.3) remains positive for $t \geq t_0$.

Proof. Let $\mu \in (0, 1)$ be a given number such that all the conditions of the theorem hold. Let $\phi \in \mathbf{F}$ be a fixed initial function such that $1 - \mu \leq \psi(t) \leq 1 + \mu$ for $t_{-1}(t_0) + 1 \leq t < t_0 + 1$, where the function ψ is defined by (1.4). Let the operator $(S\delta)(t)$ be defined by (3.5) for $\delta \colon [t_0 + 1, \infty) \mapsto \mathbb{R}$. We will show that statement (c) of Theorem 3.4 is true with $\beta(t) = 1 - \mu$

and $\gamma(t) = 1 + \mu$ for $t \geq t_{-1}(t_0) + 1$. Let $\delta\colon [t_{-1}(t_0) + 1, \infty) \mapsto \mathbb{R}$ be a function such that $\delta(t) = \psi(t)$ for $t_{-1}(t_0) + 1 \leq t < t_0 + 1$ and $1 - \mu \leq \delta(t) \leq 1 + \mu$ for $t \geq t_0 + 1$. From (5.1), it follows that

$$(S\delta)(t) \leq 1 + \sum_{i=1}^{m} |P_i(t)| \prod_{j=1}^{k_i(t)-1} \frac{1}{1-\mu} = 1 + \sum_{i=1}^{m} |P_i(t)|\,(1-\mu)^{1-k_i(t)} \leq 1 + \mu$$

and

$$(S\delta)(t) \geq 1 - \sum_{i=1}^{m} |P_i(t)| \prod_{j=1}^{k_i(t)-1} \frac{1}{1-\mu} = 1 - \sum_{i=1}^{m} |P_i(t)|\,(1-\mu)^{1-k_i(t)} \geq 1 - \mu$$

for $t \geq t_0 + 1$, and the proof is complete. □

The next theorem is a generalization of Theorem 5.1.

Theorem 5.2. *Assume that* (H_1), (H_2) *hold and that there exists a real function* $\mu\colon [t_0 + 1, \infty) \to (0, 1)$ *such that*

$$\sum_{i=1}^{m} |P_i(t)| \prod_{j=1}^{k_i(t)-1} \frac{1}{1-\mu(t-j)} \leq \mu(t) \quad \textit{for } t \geq t_0 + 1. \tag{5.2}$$

Then for every $\phi \in \mathbf{F}$ *such that* $\phi(t) > 0$ *for* $t_{-1}(t_0) \leq t \leq t_0$, *the solution* $x\colon [t_{-1}(t_0), \infty) \to \mathbb{R}$ *of the initial value problem* (1.1) *and* (1.3) *remains positive for* $t \geq t_0$.

Proof. Let $\mu\colon [t_{-1}(t_0) + 1, \infty) \to (0, 1)$ be a given function such that the conditions of the theorem hold for $t \geq t_0 + 1$. Let $\phi \in \mathbf{F}$ be a fixed initial function such that $1 - \mu(t) \leq \psi(t) \leq 1 + \mu(t)$ for $t_{-1}(t_0) + 1 \leq t < t_0 + 1$, where the function ψ is defined by (1.4). Let the operator $(S\delta)(t)$ be defined by (3.5) for $\delta\colon [t_0 + 1, \infty) \mapsto \mathbb{R}$. We will show that statement (c) of Theorem 3.4 is true with $\beta(t) = 1 - \mu(t)$ and $\gamma(t) = 1 + \mu(t)$ for $t \geq t_0 + 1$. Let $\delta\colon [t_{-1}(t_0) + 1, \infty) \mapsto \mathbb{R}$ be a function such that $\delta(t) = \psi(t)$ for $t_{-1}(t_0) + 1 \leq t < t_0 + 1$ and $1 - \mu(t) \leq \delta(t) \leq 1 + \mu(t)$ for $t \geq t_0 + 1$. From (5.2), it follows that

$$(S\delta)(t) \leq 1 + \sum_{i=1}^{m} |P_i(t)| \prod_{j=1}^{k_i(t)-1} \frac{1}{1-\mu(t-j)} \leq 1 + \mu(t)$$

and

$$(S\delta)(t) \geq 1 - \sum_{i=1}^{m} |P_i(t)| \prod_{j=1}^{k_i(t)-1} \frac{1}{1-\mu(t-j)} \geq 1 - \mu(t)$$

for $t \geq t_0 + 1$, and the proof is complete. □

Set, now, that $P_i^+(t) = \max\{0, P_i(t)\}$ for $t \geq t_0$, $i = 1, 2, \ldots, m$, and suppose that

(H_3) $k_i\colon [t_0, \infty) \mapsto \{2, 3, 4, \ldots\}$ are real functions such that $\lim_{t\to\infty} k_i(t) = T_i$, for $i = 1, 2, \ldots, m$ and let $\max_{1 \leq i \leq m} \{T_i\} = T$.

Theorem 5.3. *Assume that* (H_1), (H_3) *hold and let*

$$\sum_{i=1}^{m} P_i^+(t) \leq \frac{(T-1)^{T-1}}{T^T} \quad \textit{for } t \geq t_0 + 1. \tag{5.3}$$

Then for every $\phi \in \mathbf{F}$ *such that* $\phi(t) > 0$ *for* $t_{-1}(t_0) \leq t < t_0$, *the solution* $x\colon [t_{-1}(t_0), \infty) \to \mathbb{R}$ *of the initial value problem* (1.1) *and* (1.3) *remains positive for* $t \geq t_0$.

Proof. Consider the functional equation

$$y(t) - y(t-1) + \sum_{i=1}^{m} P_i^{+}(t) y(t - k_i(t)) = 0 \quad \text{for } t \geq t_0 \tag{5.4}$$

with initial condition $y(t) = \phi(t)$ for $t_{-1}(t_0) \leq t < t_0$. Let $\phi \in \mathbf{F}$ be a fixed initial function such that $\frac{T-1}{T} \leq \psi(t) \leq 1$ for $t_{-1}(t_0) + 1 \leq t < t_0 + 1$, where the function ψ is such that

$$\psi(t) = \frac{\phi(t)}{\phi(t-1)} \quad \text{for } t_{-1}(t_0) + 1 \leq t < t_0, \quad \text{and} \quad \psi(t) \geq \frac{y(t)}{\phi(t-1)} \quad \text{for } t_0 \leq t < t_0 + 1.$$

It is possible to show that the statement (c) of Theorem 3.5 is true for any function $\delta\colon [t_{-1}(t_0) + 1, \infty) \mapsto \mathbb{R}$ such that $\delta(t) = \psi(t)$ for $t_{-1}(t_0) + 1 \leq t < t_0 + 1$ and

$$0 < \beta(t) \equiv \frac{T-1}{T} \leq \delta(t) \leq 1 \equiv \gamma(t) \quad \text{for } t \geq t_0 + 1. \tag{5.5}$$

Because of (H_3) and (5.5), it follows that

$$\prod_{j=1}^{k_i(t)-1} \frac{1}{\delta(t-j)} \leq \prod_{j=1}^{T-1} \frac{T}{T-1} \leq \frac{T^{T-1}}{(T-1)^{T-1}}. \tag{5.6}$$

Combining (5.3), (5.5) and (5.6), we obtain

$$\begin{aligned} 1 \geq 1 - \sum_{i=1}^{m} P_i^{+}(t) \prod_{j=1}^{k_i(t)-1} \frac{1}{\delta(t-j)} &\equiv S\delta(t) \\ \geq 1 - \sum_{i=1}^{m} P_i^{+}(t) \frac{T^{T-1}}{(T-1)^{T-1}} &\geq \frac{T-1}{T}. \end{aligned}$$

Therefore, the solution $y(\phi)(t)$ of (5.4) is positive for $t \geq t_0$. Since the solution $x(\phi)(t)$ of (1.1) is also a solution of inequality

$$x(t) - x(t-1) + \sum_{i=1}^{m} P_i^{+}(t) x(t - k_i(t)) \geq 0 \quad \text{for } t \geq t_0,$$

and by using Theorem 4.1, it follows that $x(\phi)(t) \geq y(\phi)(t) > 0$ for $t \geq t_0$, the proof is complete. □

Theorem 5.4. *Assume that (H_1), (H_2) hold and $0 \leq k_1(t) \leq k_2(t) \leq \cdots \leq k_m(t)$ for $t \geq t_0$ and*

$$\sum_{i=1}^{s} P_i(t) \leq 0 \quad \text{for } s = 1, 2, \ldots, m, \ t \geq t_0 + 1. \tag{5.7}$$

Then equation (1.1) has positive increasing solution for $t \geq t_0$.

Proof. Let $\phi(t) \equiv 1$ for $t_{-1}(t_0) \leq t < t_0$. The statement (c) of Theorem 3.5 will be true for any function $\delta\colon [t_{-1}(t_0) + 1, \infty) \mapsto \mathbb{R}$ such that $\delta(t) = 1$ for $t_{-1}(t_0) + 1 \leq t < t_0$, $\delta(t) = x(t)$ for $t_0 \leq t < t_0 + 1$ and

$$\beta(t) \equiv 1 \leq \delta(t) \leq 1 + \sum_{i=1}^{m} |P_i(t)| = \gamma(t) \quad \text{for } t \geq t_0 + 1. \tag{5.8}$$

Because of $t - k_1(t) \ge t - k_2(t) \ge \cdots \ge t - k_m(t)$, (5.7) and (5.8), it follows that

$$1 \le 1 - \left\{ \sum_{i=1}^{m} P_i(t) \right\} \prod_{j=1}^{k_m(t)-1} \frac{1}{\delta(t-j)} \le 1 - \sum_{i=1}^{m} P_i(t) \prod_{j=1}^{k_i(t)-1} \frac{1}{\delta(t-j)} \equiv S\delta(t)$$
$$\le 1 + \sum_{i=1}^{m} |P_i(t)| \prod_{j=1}^{k_i(t)-1} \frac{1}{\delta(t-j)} \le 1 + \sum_{i=1}^{m} |P_i(t)| \quad \text{for } t \ge t_0 + 1.$$

Therefore, by Theorem 3.5, the solution $x(\phi)(t)$ of equation (1.1) is positive for $t \ge t_0$.

Moreover, $x(\phi)(t) = \prod_{j=0}^{n(t)-1} \lambda(t-j)$ for $t \ge t_0 + 1$, where λ is a positive solution of the characteristic equation associated with equation (1.1), greater than 1 for $t \ge t_0 + 1$.

Hence, $x(\phi)(t)$ is an increasing solution of equation (1.1) and the proof is complete. □

6 Examples and remarks

Consider the delay difference equation with continuous time of the form

$$x(t) - x(t-1) + \sum_{i=1}^{m} P_i(t) x(t - [\tau_i(t)]) = 0, \qquad t \ge t_0,$$

where $P_i \in C[[t_0, \infty), \mathbb{R}]$, $\tau_i \in C[[t_0, \infty), [2, \infty)]$ and $\lim_{t\to\infty}(t - \tau_i(t)) = \infty$, for $i = 1, 2, \dots, m$. Now, the delay functions satisfy the hypotheses (H_2) with $k_i(t) = [\tau_i(t)]$, $i = 1, 2, \dots, m$.

Remark 6.1. For the special case, when $m = 1$ and $k_1(t) = k \in \{2, 3, 4, \dots\}$, condition (5.1) has the form

$$|P(t)|(1-\mu)^{1-k} \le \mu \quad \text{or} \quad |P(t)| \le \mu(1-\mu)^{k-1}.$$

Let $F(\mu) = \mu(1-\mu)^{k-1}$. Then $F'(\mu) = \mu(1-\mu)^{k-2}(1-k\mu)$, $F_{max}\left(\frac{1}{k}\right) = \frac{(k-1)^{k-1}}{k^k}$ and the condition of non-oscillation is

$$|P(t)| \le \frac{(k-1)^{k-1}}{k^k}.$$

For $m = 1$, for positive function P and for constant delay function, the condition (5.3) has the same form as the above condition.

Remark 6.2. For $0 < a + 1 < t_0$ and $\mu(t) = \dfrac{a}{t}$ the condition $0 < \mu(t) < 1$ is valid for $t \ge t_0 > 0$. Now, for the special case when $k_i(t) = k_i \in \{2, 3, 4, \dots\}$, $i = 1, 2, \dots, m$, condition (5.2) can be reformulated as

$$\sum_{i=1}^{m} |P_i(t)| \prod_{j=1}^{k_i-1} \frac{t-j}{t-j-a} \le \frac{a}{t} \quad \text{or} \quad t \sum_{i=1}^{m} |P_i(t)| \prod_{j=1}^{k_i-1} \frac{t-j}{t-j-a} \le a.$$

For $m = 1$ and $k_1 = 2$ we have

$$|P(t)| \frac{t-1}{t-1-a} \le \frac{a}{t} \quad \text{or} \quad |P(t)| \le \frac{a}{t}\left(1 - \frac{a}{t-1}\right).$$

Let $F(a) = \frac{a}{t}\left(1 - \frac{a}{t-1}\right)$. Then $F'(a) = \frac{t-1-2a}{t(t-1)}$, $F'(a) = 0$ for $a = \frac{t-1}{2}$, and $F_{max}\left(\frac{t-1}{2}\right) = \frac{t-1}{4t}$. Now, we can reformulate condition (5.2) of non-oscillation as

$$|P(t)| \le \frac{t-1}{4t},$$

where the condition

$$|P(t)| \leq \frac{1}{4} \tag{6.1}$$

is satisfied, too.

Remark 6.3. In the Theorems C, D and E, for $P(t) = 1$, $g(t) = t - 1$ and positive function $Q(t) = P(t)$, the conditions (2.6), (2.7) and (2.8), for the existence of non-oscillatory solutions of the equation

$$x(t) - x(t-1) + P(t)x(t-2) = 0$$

are of the form

$$P(t) \leq \frac{1}{4}, \tag{6.2}$$

and it is the same as (6.1), or the condition (5.3) for $T = 2$ and positive function P.

Remark 6.4. In the Theorem F, for $\tau = 1$, $\sigma > 1$ and positive function $q(t) = P(t)$, the condition for the existence of non-oscillatory solutions of the equation

$$x(t) - x(t-1) + P(t)x(t-\sigma) = 0$$

is of the form

$$P(t) \leq \frac{(\sigma-1)^{\sigma-1}}{\sigma^{\sigma}}, \tag{6.3}$$

and it is the same as (5.3) for $T = \sigma$ and positive function P.

Remark 6.5. In the Theorem G, for $p = 1$, $k > 1$ and an arbitrary real function $Q(t) = P(t)$, the condition for the existence of non-oscillatory solutions of the equation

$$x(t) - x(t-1) + P(t)x(t-k) = 0$$

is of the form

$$P^{+}(t) \leq \frac{(k-1)^{k-1}}{k^{k}}, \tag{6.4}$$

and it is the same as the condition (5.3) for $T = k$.

Example 6.6. Consider the functional equation

$$x(t) - x(t-1) + \frac{(t-1)\sin t}{4t} x(t-2) = 0$$

with $t_0 = 4$, $t_{-1} = 2$ and positive initial functions $\phi = j(t^2 - 2)$ (Figure 6.1) and $\phi = j\sqrt{t}$ (Figure 6.2), $j \in \{0.5, 1, 5, 2.5, 3.5, 4.5, 5.5\}$. Since

$$|P(t)| = \left| \frac{(t-1)\sin t}{4t} \right| \leq \frac{t-1}{4t},$$

the solutions with positive initial functions remain positive.

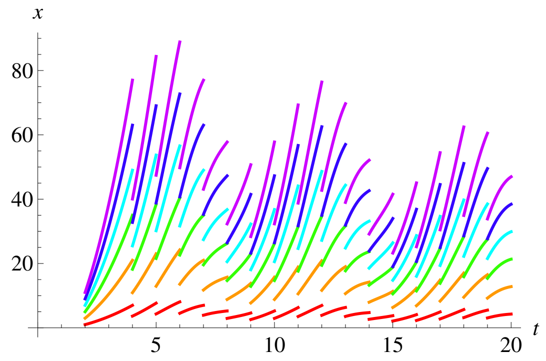

Figure 6.1

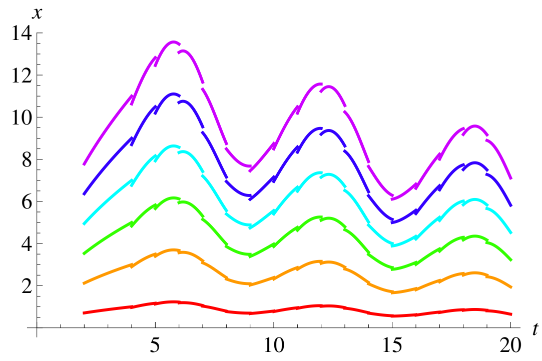

Figure 6.2

Example 6.7. Consider the functional equation

$$x(t) - x(t-1) + \frac{1}{4t}x(t-2) - \frac{1}{8t}x(t-3) = 0$$

with $t_0 = 5$, $t_{-1} = 2$ and positive initial functions $\phi = j(t^2 - 2)$ (Figure 6.3) and $\phi = j\sqrt{t}$ (Figure 6.4), $j \in \{0.5, 1, 5, 2.5, 3.5, 4.5, 5.5\}$.

The condition of non-oscillation has the form:

$$|P_1(t)|(1-\mu)^{(1-k_1)} + |P_2(t)|(1-\mu)^{(1-k_2)} \le \mu.$$

For $m = 2$, $k_1 = 2$, $k_2 = 3$, $P_1(t) = \dfrac{1}{4t}$, $P_2(t) = -\dfrac{1}{8t}$ and $\mu = 0.5$ we have

$$\begin{aligned} &|P_1(t)|(1-\mu)^{(1-k_1)} + |P_2(t)|(1-\mu)^{(1-k_2)} \\ &\quad = \frac{1}{4t}(1-0.5)^{(1-2)} + \frac{1}{8t}(1-0.5)^{(1-3)} \\ &\quad = \frac{2}{4t} + \frac{4}{8t} = \frac{1}{t} \le \frac{1}{2} = \mu \quad \text{for } t \ge 2. \end{aligned}$$

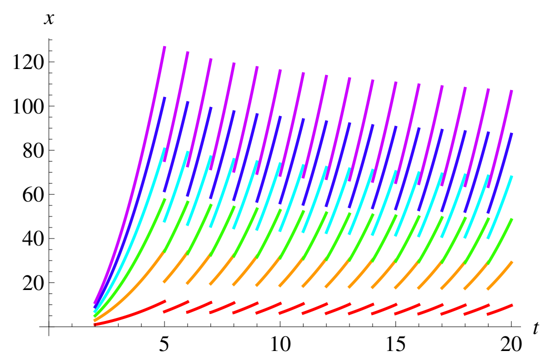

Figure 6.3

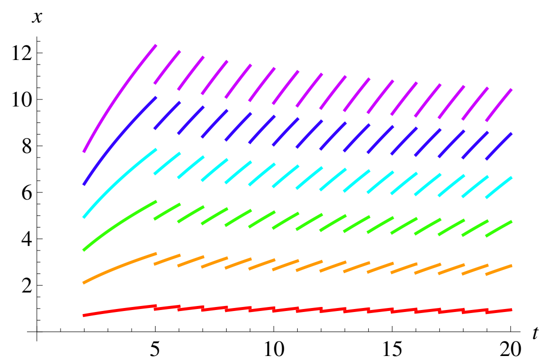

Figure 6.4

Example 6.8. Consider the functional equation

$$x(t) - x(t-1) + \frac{\sin t}{2^{\left[\frac{t}{2}\right]}}x\left(t - \left[\frac{t}{2}\right]\right) = 0$$

with $t_0 = 5$, $t_{-1} = 2$ and positive initial functions $\phi = j\sqrt{t}$ (Figure 6.5) and $\phi = j(\sin 6t + 2)$ (Figure 6.6) for the values $j \in \{0.5, 1, 5, 2.5, 3.5, 4.5, 5.5\}$.

The condition of non-oscillation (5.1) has the form:

$$|P_1(t)|(1-\mu)^{1-[\tau_1(t)]} \le \mu.$$

For $m = 1$, $\tau_1(t) = \dfrac{t}{2}$, $P_1(t) = \sin t\, 2^{-\left[\frac{t}{2}\right]}$ and $\mu = 0.5$ we have

$$|P_1(t)|(1-\mu)^{1-[\tau_1(t)]} = \frac{|\sin t|}{2^{\left[\frac{t}{2}\right]}}(1-0.5)^{1-\left[\frac{t}{2}\right]} = |\sin t| \cdot \frac{1}{2} \le \frac{1}{2} = \mu$$

and the condition of non-oscillation is satisfied.

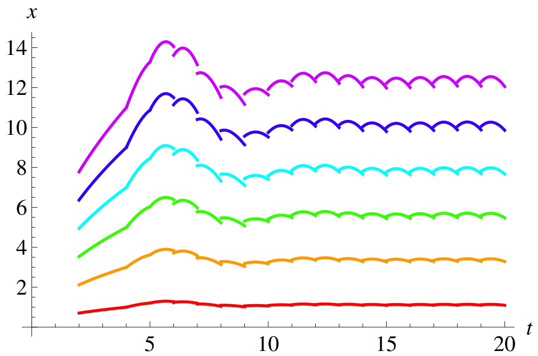

Figure 6.5

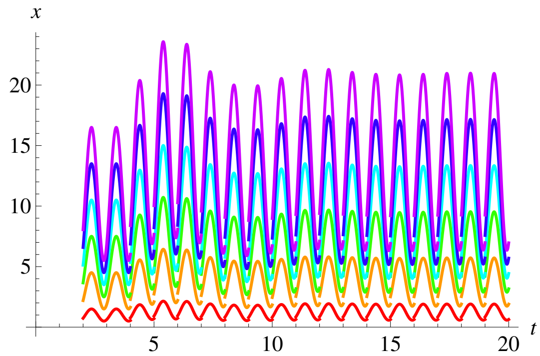

Figure 6.6

Example 6.9. Consider the functional equation

$$x(t) - x(t-1) + \frac{\cos t}{2^{\left[t-\sqrt[3]{t}\right]}}\, x\left(t - \left[t - \sqrt[3]{t}\right]\right) = 0$$

with $t_0 = 8$, $t_{-1} = 2$ and positive initial functions $\phi = j\sqrt{t}$ (Figure 6.7) and $\phi = j(\sin 6t + 2)$ (Figure 6.8) for the values $j \in \{0.5, 1, 5, 2.5, 3.5, 4.5, 5.5\}$.

For $m = 1$, $\tau_1(t) = t - \sqrt[3]{t}$, $P_1(t) = \cos t\, 2^{-\left[t-\sqrt[3]{t}\right]}$ and $\mu = 0.5$ we have

$$|P_1(t)|(1-\mu)^{1-[\tau_1(t)]} = \frac{|\cos t|}{2^{\left[t-\sqrt[3]{t}\right]}}(1-0.5)^{1-\left[t-\sqrt[3]{t}\right]} = |\cos t| \cdot \frac{1}{2} \le \frac{1}{2} = \mu$$

and the condition (5.1) of non-oscillation is satisfied.

Example 6.10. Consider the functional equation

$$x(t) - x(t-1) + \frac{2\sin t}{27}\, x\left(t - \left[\frac{2t+3}{t+1}\right]\right) + \frac{2\cos t}{27}\, x\left(t - \left[\frac{3t+5}{t+1}\right]\right) = 0$$

with $t_0 = 5$, $t_{-1} = 2$ and the positive initial functions $\phi = j(t^2 - 2)$ (Figure 6.9) and $\phi = j(\sin 6t + 2)$ (Figure 6.10), $j \in \{0.5, 1, 5, 2.5, 3.5, 4.5, 5.5\}$.

Since

$$T_1 = \lim_{t\to\infty} \tau_1(t) = \lim_{t\to\infty} \frac{2t+3}{t+1} = 2, \qquad T_2 = \lim_{t\to\infty} \tau_2(t) = \lim_{t\to\infty} \frac{3t+5}{t+1} = 3,$$

$$T = \max\{T_1, T_2\} = 3, \qquad \frac{(T-1)^{T-1}}{T^T} = \frac{4}{27},$$

$$P_1^+(t) + P_2^+(t) = \frac{2}{27}((\sin t)^+ + (\cos t)^+) \le \frac{4}{27} = \frac{(T-1)^{T-1}}{T^T},$$

the condition of non-oscillation (5.3) is satisfied.

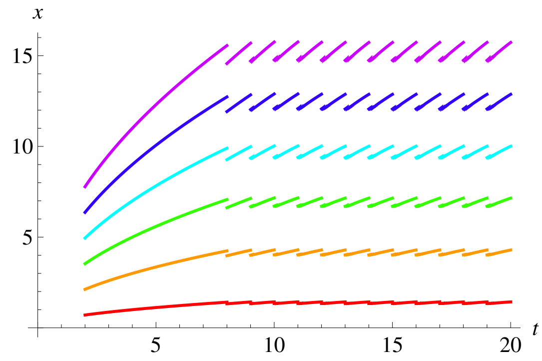

Figure 6.7

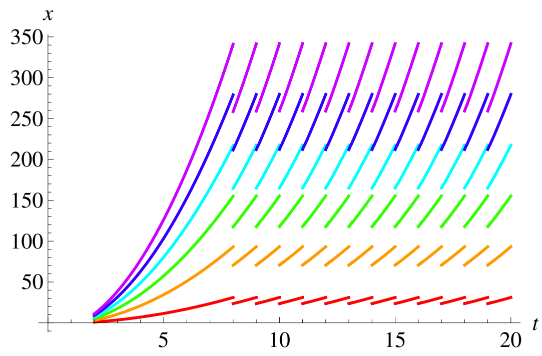

Figure 6.8

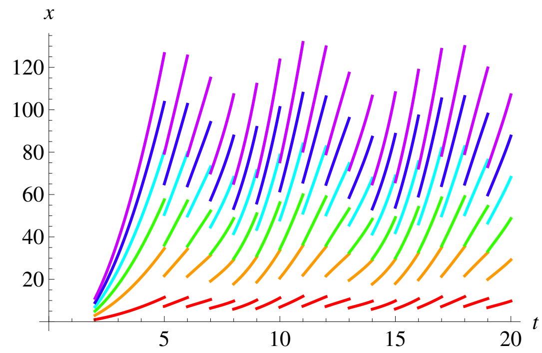

Figure 6.9

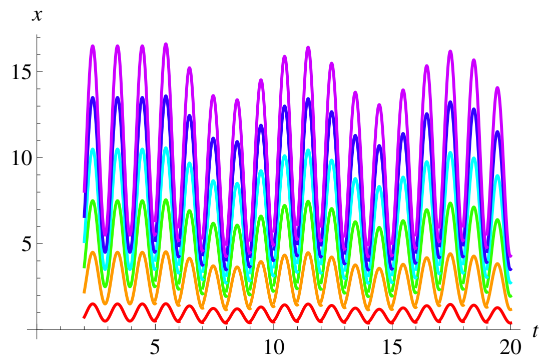

Figure 6.10

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