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Monotone iterative technique for nonlocal fractional differential equations with finite delay in a Banach space

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Abstract. In this paper, we extend a monotone iterative technique for nonlocal fractional differential equations with finite delay in an ordered Banach space. By using the monotone iterative technique, theory of fractional calculus, semigroup theory and measure of noncompactness, we study the existence and uniqueness of extremal mild solutions. An example is presented to illustrate the main result.

Keywords: fractional differential equations, finite delay, monotone iterative technique, semigroup theory, Kuratowskii measure of noncompactness.

2010 Mathematics Subject Classification: 34A08, 34G20, 34K30.

1 Introduction

In this paper, we consider the following nonlocal fractional differential equations with finite delay in an ordered Banach space X:

$$\begin{cases} {}^cD^\alpha x(t) = Ax(t) + f\left(t, x_t, \int_0^t h(t,s,x_s)\, ds\right), & t \in J = [0,b], \\ x(\nu) = \phi(\nu) + g(x)(\nu), \quad \nu \in [-a,0], \end{cases} \tag{1.1}$$

where state $x(\cdot)$ takes values in the Banach space X endowed with norm $\|\cdot\|$; ${}^cD^\alpha$ is the Caputo fractional derivative of order α, $0 < \alpha < 1$; $A\colon D(A) \subset X \to X$ is a closed linear densely defined operator and an infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on X; the nonlinear operators $h\colon \Sigma \times \mathcal{D} \to X$, $f\colon J \times \mathcal{D} \times X \to X$ are given continuous functions, $\Sigma = \{(t,s)\colon 0 \leq s \leq t \leq b\}$ and $\mathcal{D} = C([-a,0],X)$, a Banach space of all continuous functions from $[-a,0]$ into X endowed with supremum norm; $\phi(\cdot) \in \mathcal{D}$ and the function g is defined from $C([-a,b],X)$ to $\mathcal{D}$. If $x\colon [-a,b] \to X$ is a continuous function, then x_t denotes the function in $\mathcal{D}$ defined as $x_t(\nu) = x(t+\nu)$ for $\nu \in [-a,0]$, here $x_t(\cdot)$ represent the time history of the state from the time $t-a$ up to the present time t.

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Fractional calculus is generalization of ordinary differential equations and integrations to arbitrary non integer orders. One can describe many physical phenomena arising in engineering, physics, economics and science more accurately through the fractional derivative formulation. Indeed, we can find numerous applications in electrochemistry, control, porous media, electromagnetism, etc. (see, [2, 8, 10, 12, 22, 23, 26]). Hence, in recent years, the researchers have paid more attention to fractional differential equations. Many authors have studied fractional differential equations with nonlocal initial conditions; see, for instance, [2, 14, 17, 19, 23]. Nonlocal initial condition, in many cases, is more relevant and produces better results in applications of physical problems than the classical initial value of the type $x(0) = x_0$. In [1, 3, 6, 11, 14], the authors discussed the existence and uniqueness results of fractional differential equations in abstract spaces with finite or infinite delay.

By motivation of the recent works [4, 17, 19], we use a monotone iterative technique to study the existence and uniqueness of extremal mild solutions of the problem (1.1) in an ordered Banach space. The monotone iterative technique based on lower and upper solutions provides an effective way to investigate the existence of solutions for the nonlinear differential equations (fractional or non-fractional ordered); see, for instance, [4, 5, 13, 15, 16, 18, 19, 20, 24]. It constructs monotone sequences of lower and upper solutions that converge uniformly to the extremal mild solutions between the lower and upper solutions. In this paper, we obtain the results by using the theory of fractional calculus, semigroup theory, measure of noncompactness and monotone iterative technique. To the best of our knowledge, up to now, no work has been reported on nonlocal fractional differential equations with finite delay in Banach spaces.

The rest of the paper is organized as follows: in the next section we give some basic definitions and notations. In Section 3, we study the existence of extremal mild solution of the delay system (1.1) and uniqueness of solutions of the system. Finally, in Section 4, we present an example to illustrate our results.

2 Preliminaries

In this section, we introduce some basic definitions and notations which are used throughout this paper. We denote by X a Banach space with the norm $\|\cdot\|$ and $A\colon D(A) \subset X \to X$ is a densely defined closed linear operator and generates a strongly continuous semigroup $\{T(t), t \geq 0\}$. By Pazy [21], there exists $M \geq 1$ such that $\sup_{t\in J} \|T(t)\| \leq M$.

Let $P = \{y \in X\colon y \geq \theta\}$ (θ is a zero element of X) be a positive cone in X which defines a partial ordering in X by $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, we write $x < y$. The cone P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$.

Let $C([-a, b], X)$ be a Banach space of all continuous X-valued functions on interval $[-a, b]$ with norm $\|x\|_C = \sup_{t\in[-a,b]} \|x(t)\|$, $x \in C([-a, b], X)$. Evidently $C([-a, b], X)$ is an ordered Banach space whose partial ordering $\leq$ reduced by a positive cone $P_C = \{x \in C([-a, b], X) : x(t) \geq \theta,\ t \in [-a, b]\}$. Similarly $\mathcal{D}$ is also an ordered Banach space whose partial ordering $\leq$ reduced by a positive cone $P_{\mathcal{D}} = \{x \in \mathcal{D} : x(t) \geq \theta,\ t \in [-a, 0]\}$. P_C and $P_{\mathcal{D}}$ are also normal cones with same normal constant N. For $x, y \in C([-a, b], X)$ with $x \leq y$, denote the ordered interval $[x, y] = \{z \in C([-a, b], X),\ x \leq z \leq y\}$ in $C([-a, b], X)$, and $[x(t), y(t)] = \{u \in X : x(t) \leq u \leq y(t)\}$ ($t \in [-a, b]$) in X.

Definition 2.1 ([22]). The Riemann–Liouville fractional integral of order $\alpha > 0$ for a function f is given by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \qquad t > 0,$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the gamma function.

Definition 2.2 ([22]). The fractional derivative of order $0 \le n-1 < \alpha < n$ in the Caputo sense is defined as

$${}^{c}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}\, ds, \qquad t > 0,$$

where f is an n-times continuous differentiable function and Γ is a gamma function.

If f is an abstract function with values in X, then integrals which appear in Definition 2.1 and 2.2 are taken in Bochner's sense.

Let $C^{\alpha}([-a,b],X) = \big\{u \in C([-a,b],X) : {}^{c}D^{\alpha}u$ exists on J, ${}^{c}D^{\alpha}u|_J \in C(J,X)$ and $u(t) \in D(A)$ for $t \ge 0\big\}$. An abstract function $u \in C^{\alpha}([-a,b],X)$ is called a solution of (1.1) if $u(t)$ satisfies the equation (1.1).

Definition 2.3 ([4]). The function $x \in C^{\alpha}([-a,b],X)$ is called a lower solution of the problem (1.1) if it satisfies the following inequalities

$$\begin{cases} {}^{c}D^{\alpha} x(t) \le Ax(t) + f\left(t, x_t, \int_0^t h(s,\tau,x_\tau)\, d\tau\right), & t \in J, \\ x(\nu) \le \phi(\nu) + g(x)(\nu), & \nu \in [-a, 0]. \end{cases} \tag{2.1}$$

If all inequalities of (2.1) are reversed, we call x an upper solution of the problem (1.1).

Lemma 2.4 ([8]). *If h satisfies a uniform Hölder condition, with exponent $\beta \in (0,1]$, then the unique solution of the linear initial value problem,*

$$\begin{cases} {}^{c}D^{\alpha} x(t) = Ax(t) + h(t), & t \in J, \\ x(0) = x_0 \in X, \end{cases}$$

is given by

$$x(t) = U(t)x_0 + \int_0^t (t-s)^{\alpha-1} V(t-s)h(s))\, ds, \qquad t \in J,$$

where

$$U(t) = \int_0^{\infty} \psi_\alpha(\vartheta) T(t^{\alpha}\vartheta)\, d\vartheta, \qquad V(t) = \alpha \int_0^{\infty} \vartheta \psi_\alpha(\vartheta) T(t^{\alpha}\vartheta)\, d\vartheta \tag{2.2}$$

and

$$\psi_\alpha(\vartheta) = \frac{1}{\alpha} \vartheta^{-1-1/\alpha} \rho_\alpha(\vartheta^{-1/\alpha}).$$

Note that $\psi_\alpha(\vartheta)$ satisfies the condition of a probability density function defined on $(0,\infty)$, that is $\psi_\alpha(\vartheta) \ge 0$, $\int_0^{\infty} \psi_\alpha(\vartheta)\, d\vartheta = 1$ and $\int_0^{\infty} \vartheta \psi_\alpha(\vartheta) = \frac{1}{\Gamma(1+\alpha)}$. Also the term $\rho_\alpha(\vartheta)$ is defined as

$$\rho_\alpha(\vartheta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \vartheta^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \qquad \vartheta \in (0,\infty).$$

Definition 2.5 ([11, 26]). A continuous function $x\colon [-a,b] \to X$ is said to be a mild solution of the system (1.1) if $x(t) = \phi(t) + g(x)(t)$ on $[-a,0]$ and the following integral equation is satisfied:

$$x(t) = U(t)(\phi(0) + g(x)(0)) + \int_0^t (t-s)^{\alpha-1} V(t-s) f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\, d\tau\right) ds, \qquad t \in J,$$

where $U(t)$ and $V(t)$ are defined by (2.2).

Lemma 2.6 ([25, 26]). *The following properties are valid:*

(i) for fixed $t \geq 0$ and any $x \in X$, we have

$$\|U(t)x\| \leq M\|x\|, \qquad \|V(t)x\| \leq \frac{\alpha M}{\Gamma(1+\alpha)}\|x\| = \frac{M}{\Gamma(\alpha)}\|x\|.$$

(ii) The operators are $U(t)$ and $V(t)$ are strongly continuous for all $t \geq 0$.

(iii) If $T(t)$ $(t > 0)$ is a compact semigroup in X, then $U(t)$ and $V(t)$ are norm-continuous in X for $t > 0$.

(iv) If $T(t)$ $(t > 0)$ is a compact semigroup in X, then $U(t)$ and $V(t)$ are compact operators in X for $t > 0$.

Definition 2.7. A C_0-semigroup $\{T(t)\}_{t\geq 0}$ is called a positive semigroup, if $T(t)x \geq \theta$ for all $x \geq \theta$ and $t \geq 0$.

Now we recall the definition of Kuratowski's measure of noncompactness and its properties to study the existence of extremal mild solutions of (1.1) in the next section.

Definition 2.8 ([7, 9]). Let X be a Banach space and $\mathcal{B}(X)$ be a family of bounded subset of X. Then $\mu : \mathcal{B}(X) \to \mathbb{R}^+$, defined by

$$\mu(S) = \inf\{\delta > 0 : S \text{ admits a finite cover by sets of diameter } \leq \delta\},$$

where $S \in \mathcal{B}(X)$, is called the Kuratowski measure of noncompactness. Clearly $0 \leq \mu(S) < \infty$.

Lemma 2.9 ([7, 9]). *Let S, S_1 and S_2 be bounded sets of a Banach space X. Then*

(i) $\mu(S) = 0$ if and only if S is a relatively compact set in X.

(ii) $\mu(S_1) \leq \mu(S_2)$ if $S_1 \subset S_2$.

(iii) $\mu(S_1 + S_2) \leq \mu(S_1) + \mu(S_2)$.

(iv) $\mu(\lambda S) \leq |\lambda|\mu(S)$ for any $\lambda \in \mathbb{R}$.

Lemma 2.10 ([7, 9]). *If $W \subset C([c,d], X)$ is bounded and equicontinuous on $[c,d]$, then $\mu(W(t))$ is continuous for $t \in [c,d]$ and*

$$\mu(W) = \sup\{\mu(W(t)), t \in [c,d]\}, \quad \text{where } W(t) = \{x(t) \colon x \in W\} \subseteq X.$$

Remark 2.11 ([7, 9]). If S is a bounded set in $C([c,d], X)$, then $S(t)$ is bounded in X, and $\mu(S(t)) \leq \mu(S)$.

Lemma 2.12 ([7, 9]). *Let $S = \{u_n\} \subset C([c,d], X)$ $(n = 1, 2, \ldots)$ be a bounded and countable set. Then $\mu(S(t))$ is Lebesgue integrable on $[c,d]$, and*

$$\mu\left(\left\{\int_c^d u_n(t)\, dt \mid n = 1, 2, \ldots\right\}\right) \leq 2\int_c^d \mu(S(t))\, dt. \tag{2.3}$$

3 Main result

In this section, we prove the existence of extremal mild solutions of the system (1.1) and the uniqueness of solutions of the system.

Theorem 3.1. *Let X be an ordered Banach space, whose positive cone P is normal with normal constant N and $T(t)$ $(t \geq 0)$ be a positive operator. Also assume that A is the infinitesimal generator of compact semigroup $\{T(t)\}_{t\geq 0}$ on X. If the system (1.1) has lower and upper solutions $x^{(0)}$, $y^{(0)} \in C([-a,b],X)$ with $x^{(0)} \leq y^{(0)}$ and satisfies the following assumptions:*

(H1) The functions f, h satisfy the following:

- *(i) The function $h\colon \Sigma \times \mathcal{D} \to X$ is such that the function $h(t,s,\cdot)\colon \mathcal{D} \to X$ is continuous for each $(t,s) \in \Sigma$, and the function $h(\cdot,\cdot,\varphi)\colon \Sigma \to X$ is strongly measurable for each $\varphi \in \mathcal{D}$.*
- *(ii) The function $f\colon J \times \mathcal{D} \times X \to X$ is such that the function $f(t,\cdot,\cdot)\colon \mathcal{D} \times X \to X$ is continuous for $t \in J$, and the function $f(\cdot,\varphi,x)$ is strongly measurable for all $(\varphi,x) \in \mathcal{D} \times X$.*

(H2) For any $(t,s) \in \Sigma$, the function $h(t,s,\cdot)\colon \mathcal{D} \to X$ satisfies

$$h(t,s,\varphi_1) \leq h(t,s,\varphi_2),$$

where $\varphi_1, \varphi_2 \in \mathcal{D}$ with $x_s^{(0)} \leq \varphi_1 \leq \varphi_2 \leq y_s^{(0)}$.

(H3) For any $t \in [0,b]$, the function $f(t,\cdot,\cdot)\colon \mathcal{D} \times X \to X$ satisfies

$$f(t,\varphi_1,u_1) \leq f(t,\varphi_2,u_2),$$

where $u_1, u_2 \in X$ with $\int_0^t h(t,s,x_s^{(0)})\,ds \leq u_1 \leq u_2 \leq \int_0^t h(t,s,y_s^{(0)})\,ds$ and $\varphi_1, \varphi_2 \in \mathcal{D}$ with $x_t^{(0)} \leq \varphi_1 \leq \varphi_2 \leq y_t^{(0)}$.

(H4) The function $g\colon C([-a,b],X) \to \mathcal{D}$ is increasing, continuous and compact.

Then the system (1.1) has minimal and maximal mild solutions between $x^{(0)}$ and $y^{(0)}$.

Proof. Let $B = [x^{(0)}, y^{(0)}] = \{x \in C([-a,b],X) \mid x^{(0)} \leq x \leq y^{(0)}\}$. We define a map $Q\colon B \to C([-a,b],X)$ by

$$Qx(t) = \begin{cases} U(t)(\phi(0) + g(x)(0)) \\ \qquad + \int_0^t (t-s)^{\alpha-1} V(t-s) f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\,d\tau\right) ds, & t \in [0,b], \\ \phi(t) + g(x)(t), & t \in [-a,0]. \end{cases} \tag{3.1}$$

By (H2), (H3) and for any $x \in B$, we have that

$$\begin{aligned} f\left(t, x_t^{(0)}, \int_0^t h(t,\tau,x_\tau^{(0)})\,d\tau\right) &\leq f\left(t, x_t, \int_0^t h(t,\tau,x_\tau)\,d\tau\right) \\ &\leq f\left(t, y_t^{(0)}, \int_0^t h(t,\tau,y_\tau^{(0)})\,d\tau\right). \end{aligned}$$

By the normality of the positive cone P, there exists a constant $k > 0$ such that

$$\left\| f\left(t, x_t, \int_0^t h(t,\tau,x_\tau)\,d\tau\right) \right\| \leq k, \qquad x \in B. \tag{3.2}$$

First we prove that Q is a continuous map on B. Let $\{y^{(n)}\} \subset B$ with $y^{(n)} \to y \in B$ as $n \to \infty$. Then for any $t \in [-a, 0]$ and by (H4), we have that $\|Qy^{(n)}(t) - Qy(t)\| = \|g(y^{(n)})(t) - g(y)(t)\| \to 0$ as $n \to \infty$. Also for any $t \in J$, and by (H1), (H4) and (3.2) we have

(i) $h(t, \tau, y_\tau^{(n)}) \to h(t, \tau, y_\tau)$.

(ii) $f\Big(t, y_t^{(n)}, \int_0^t h(t, \tau, y_\tau^{(n)})\, d\tau\Big) \to f\Big(t, y_t, \int_0^t h(t, \tau, y_\tau)\, d\tau\Big)$.

(iii) $g(y^{(n)}) \to g(y)$.

(iv) $\Big\|f\Big(t, y_t^{(n)}, \int_0^t h(t, \tau, y_\tau^{(n)})\, d\tau\Big) - f\Big(t, y_t, \int_0^t h(t, \tau, y_\tau)\, d\tau\Big)\Big\| \le 2k$.

These together with Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
&\Big\|Qy^{(n)}(t) - Qy(t)\Big\| \\
&\quad \le M\Big\|g(y^{(n)})(0) - g(y)(0)\Big\| \\
&\quad\quad + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big\|f\left(s, y_s^{(n)}, \int_0^s h(s, \tau, y_\tau^{(n)})\, d\tau\right) - f\left(s, y_s, \int_0^s h(s, \tau, y_\tau)\, d\tau\right)\Big\|\, ds \\
&\quad \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus Q is a continuous map from B to $C([-a, b], X)$.

Now we show that Q is an increasing monotonic operator from B to B. Let $x, y \in B$ with $x \le y$, then $x(t) \le y(t)$, $t \in [-a, b]$. Therefore, for any $t \in [0, b]$, $x_t \le y_t$ in the ordered Banach space $\mathcal{D}$. By the positivity of operators $U(t)$ and $V(t)$, (H2) (H3) and (H4), we have

$$Qx \le Qy. \tag{3.3}$$

To show that $x^{(0)} \le Qx^{(0)}$ and $Qy^{(0)} \le y^{(0)}$, we let ${}^cD^\alpha x^{(0)}(t) = Ax^{(0)}(t) + \xi(t)$, $t \in J$. By Definition 2.3, Lemma 2.4 and the positivity of $U(t)$ and $V(t)$ for $t \in J$, we get that

$$\begin{aligned}
x^{(0)}(t) &= U(t)x^{(0)}(0) + \int_0^t (t-s)^{\alpha-1}V(t-s)\xi(s)\, ds \\
&\le U(t)(\phi(0) + g(x^{(0)})(0)) + \int_0^t (t-s)^{\alpha-1}V(t-s) \\
&\quad \times f\Big(s, x_s^{(0)}, \int_0^s h(s, \tau, x_\tau^{(0)})\, d\tau\Big)\, ds, \qquad t \in J,
\end{aligned}$$

and also $x^{(0)}(t) \le \phi(t) + g(x^{(0)})(t) = Qx^{(0)}(t)$, $t \in [-a, 0]$. Therefore $x^{(0)}(t) \le Qx^{(0)}(t)$, $t \in [-a, b]$. Similarly we can show that $Qy^{(0)}(t) \le y^{(0)}(t)$, $t \in [-a, b]$. Thus $Q\colon B \to B$ is an increasing monotonic operator.

Next we show that $Q(B)$ is equicontinuous on $[-a, b]$. Let us choose any $x \in B$ and $t_1, t_2 \in [-a, b]$ with $t_1 < t_2$. If $t_1,\ t_2 \in [-a, 0]$, then $\|Qx(t_2) - Qx(t_1)\| \le \|\phi(t_2) - \phi(t_1)\| + \|g(x)(t_2) - g(x)(t_1)\| \to 0$ as $t_1 \to t_2$ independently of $x \in B$ because $\phi \in \mathcal{D}$ and by (H4). Further, if $t_1, t_2 \in J$ with $t_1 < t_2$, then we have that

$$
\begin{aligned}
&\|Qx(t_2) - Qx(t_1)\| \\
&\quad \le \|U(t_2)(\phi(0) + g(x)(0)) - U(t_1)(\phi(0) + g(x)(0))\| \\
&\qquad + \Big\| \int_0^{t_1} (t_2 - s)^{\alpha-1} \left[V(t_2 - s) - V(t_1 - s)\right] f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)\, d\tau\Big)\, ds \Big\| \\
&\qquad + \Big\| \int_0^{t_1} \Big[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\Big] V(t_1 - s) f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)\, d\tau\Big)\, ds \Big\| \\
&\qquad + \Big\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} V(t_2 - s) f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)\, d\tau\Big)\, ds \Big\| \\
&\quad \le \|U(t_2)(\phi(0) + g(x)(0)) - U(t_1)(\phi(0) + g(x)(0))\| \\
&\qquad + k \int_0^{t_1} (t_2 - s)^{\alpha-1} \|V(t_2 - s) - V(t_1 - s)\|\, ds \\
&\qquad + \frac{Mk}{\Gamma(\alpha)} \int_0^{t_1} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}|\, ds \\
&\qquad + \frac{Mk}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}\, ds \\
&\quad = I_1 + I_2 + I_3 + I_4,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \|U(t_2)(\phi(0) + g(x)(0)) - U(t_1)(\phi(0) + g(x)(0))\|, \\
I_2 &= k \int_0^{t_1} (t_2 - s)^{\alpha-1} \|V(t_2 - s) - V(t_1 - s)\|\, ds, \\
I_3 &= \frac{Mk}{\Gamma(\alpha)} \int_0^{t_1} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}|\, ds, \\
I_4 &= \frac{Mk}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}\, ds.
\end{aligned}
$$

For any $\epsilon \in (0, t_1)$, we have

$$
\begin{aligned}
I_2 &\le k \int_0^{t_1-\epsilon} (t_2 - s)^{\alpha-1} \|V(t_2 - s) - V(t_1 - s)\|\, ds \\
&\quad + k \int_{t_1-\epsilon}^{t_1} (t_2 - s)^{\alpha-1} \|V(t_2 - s) - V(t_1 - s)\|\, ds \\
&\le k \int_0^{t_1-\epsilon} (t_2 - s)^{\alpha-1}\, ds \sup_{s\in[0,t_1-\epsilon]} \|V(t_2 - s) - V(t_1 - s)\| \\
&\quad + \frac{2Mk}{\Gamma(\alpha)} \int_{t_1-\epsilon}^{t_1} (t_2 - s)^{\alpha-1}\, ds \\
&\le k \int_0^{t_1-\epsilon} (t_2 - s)^{\alpha-1}\, ds \sup_{s\in[0,t_1-\epsilon]} \|V(t_2 - s) - V(t_1 - s)\| \\
&\quad + \frac{2Mk}{\Gamma(\alpha+1)} [(t_2 - t_1 + \epsilon)^{\alpha} - (t_2 - t_1)^{\alpha}].
\end{aligned}
$$

By Lemma 2.6, we get that $I_2 \to 0$ as $t_1 \to t_2$ and $\epsilon \to 0$ independently of $x \in B$. From the expression of I_1, I_3 and I_4, we can easily show that $I_1 \to 0$, $I_3 \to 0$ and $I_4 \to 0$ as $t_2 \to t_1$ independently of $x \in B$. Therefore $\|Qx(t_2) - Qx(t_1)\| \to 0$ as $t_1 \to t_2$ independently of $x \in B$. Thus $Q(B)$ is equicontinuous on $[-a, b]$.

Further we show that the set $G(t) = \{Qx(t) : x \in B\}$, $t \in [-a, b]$, is relatively compact in X. For $t \in [-a, 0]$, $G(t) = \{\phi(t) + g(x)(t) : x \in B\}$, is relatively compact in X as $g\colon C([-a, b], X) \to \mathcal{D}$ is a continuous and compact map. Let $t \in (0, b]$ be a fixed real number and κ be a given real number satisfying $0 < \kappa < t$ and $\delta > 0$. For $x \in B$, we define

$$\begin{aligned}
Q^{\kappa,\delta}x(t) = &\int_{\delta}^{\infty} \psi_\alpha(\vartheta)T(t^\alpha\vartheta)\, d\vartheta[\phi(0) + g(x)(0)] \\
&+ \alpha \int_0^{t-\kappa} (t-s)^{\alpha-1} \int_{\delta}^{\infty} \vartheta\psi_\alpha(\vartheta)T((t-s)^\alpha\vartheta) \\
&\times f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\, d\tau\right) d\vartheta\, ds \\
= &T(\kappa^\alpha\delta) \int_{\delta}^{\infty} \psi_\alpha(\vartheta)T(t^\alpha\vartheta - \kappa^\alpha\delta)\, d\vartheta[\phi(0) + g(x)(0)] \\
&+ T(\kappa^\alpha\delta)\alpha \int_0^{t-\kappa} (t-s)^{\alpha-1} \int_{\delta}^{\infty} \vartheta\psi_\alpha(\vartheta)T((t-s)^\alpha\vartheta - \kappa^\alpha\delta) \\
&\times f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\, d\tau\right) d\vartheta\, ds.
\end{aligned}$$

Since $T(\kappa^\alpha\delta)$ is compact in X for $\kappa^\alpha\delta > 0$, the set $G^{\kappa,\delta}(t) = \{Q^{\kappa,\delta}x(t) : x \in B\}$ is relatively compact in X for every κ, $0 < \kappa < t$. Also note that

$$\begin{aligned}
&\|Qx(t) - Q^{\kappa,\delta}x(t)\| \\
&\quad\le \left\| \int_0^{\delta} \psi_\alpha(\vartheta)T(t^\alpha\vartheta)\, d\vartheta[\phi(0) + g(x)(0)] \right\| \\
&\qquad+ \alpha \left\| \int_0^t (t-s)^{\alpha-1} \int_0^{\delta} \vartheta\psi_\alpha(\vartheta)T((t-s)^\alpha\vartheta) f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\, d\tau\right) d\vartheta\, ds \right\| \\
&\qquad+ \alpha \left\| \int_{t-\kappa}^t (t-s)^{\alpha-1} \int_{\delta}^{\infty} \vartheta\psi_\alpha(\vartheta)T((t-s)^\alpha\vartheta) s f\big(s, x_s, \int_0^s h(s,\tau,x_\tau)\, d\tau\big)\, d\vartheta\, ds \right\| \\
&\quad\le M\|\big(\phi(0) + g(x)(0)\big)\| \int_0^{\delta} \psi_\alpha(\vartheta)\, d\vartheta + Mkt^\alpha \int_0^{\delta} \vartheta\psi_\alpha(\vartheta)\, d\vartheta + Mk\kappa^\alpha \int_{\delta}^{\infty} \vartheta\psi_\alpha(\vartheta)\, d\vartheta \\
&\quad\to 0 \quad \text{as } \kappa, \delta \to 0^+.
\end{aligned}$$

Therefore there are relatively compact sets arbitrarily close to the set $G(t)$ for each $t \in (0, b]$. Hence the set $G(t)$, $t \in (0, b]$ is relatively compact in X. Also $G(t)$, $t \in [-a, 0]$ is relatively compact in X. By the Arzelà–Ascoli theorem, we conclude that $Q(B)$ is a relatively compact.

Now we define the sequences as

$$x^{(n)} = Qx^{(n-1)} \quad \text{and} \quad y^{(n)} = Qy^{(n-1)}, \qquad n = 1, 2, \ldots, \tag{3.4}$$

and from (3.3), we have

$$x^{(0)} \le x^{(1)} \le \cdots \le x^{(n)} \le \cdots \le y^{(n)} \le \cdots \le y^{(1)} \le y^{(0)}. \tag{3.5}$$

Since $Q(B)$ is relatively compact, $\{x^{(n)}\}$ has a convergent subsequence $\{x^{(n_j)}\}$. Let x^* be its limit. We claim that the whole sequence $\{x^{(n)}\}$ converges to x^*. Indeed, for each $\varepsilon > 0$, there exists an n_j (depending upon ε) such that

$$\|x^{(n_j)} - x^*\| < \frac{\varepsilon}{1+N}.$$

For $n \geq n_j$, we have

$$x^{(n_j)} \leq x^{(n)} \leq x^*,$$

that is

$$0 \leq x^{(n)} - x^{(n_j)} \leq x^* - x^{(n_j)}.$$

By normality of cone P of X, then we have

$$\|x^{(n)} - x^{(n_j)}\| \leq N\|x^* - x^{(n_j)}\|.$$

Hence

$$\begin{aligned}\|x^{(n)} - x^*\| &\leq \|x^{(n)} - x^{(n_j)}\| + N\|x^{(n_j)} - x^*\| \\ &\leq (N+1)\|x^{(n_j)} - x^*\| \\ &\leq \varepsilon.\end{aligned}$$

Thus $x^{(n)} \to x^*$ as claimed. By (3.1) and (3.4), we have that

$$x^{(n)}(t) = \begin{cases} U(t)(\phi(0) + g(x^{(n-1)})(0)) \\ \qquad + \int_0^t (t-s)^{\alpha-1} V(t-s) f\left(s, x_s^{(n-1)}, \int_0^s h(s,\tau,x_\tau^{(n-1)})\, d\tau\right) ds, & t \in [0,b], \\ \phi(t) + g(x^{(n-1)})(t), & t \in [-a,0]. \end{cases}$$

Taking $n \to \infty$ and Lebesgue's dominated convergence theorem, we have that

$$x^*(t) = \begin{cases} U(t)(\phi(0) + g(x^*)(0)) \\ \qquad + \int_0^t (t-s)^{\alpha-1} V(t-s) f\left(s, x_s^*, \int_0^s h(s,\tau,x_\tau^*)\, d\tau\right) ds, & t \in [0,b], \\ \phi(t) + g(x^*)(t), & t \in [-a,0]. \end{cases}$$

Then $x^* \in C([-a,b],X)$ and $x^* = Qx^*$. Thus x^* is a fixed point of Q, hence x^* becomes a mild solution of (1.1). Similarly we can prove that there exists $y^* \in C([-a,b],X)$ such that $y^{(n)} \to y^*$ as $n \to \infty$ and $y^* = Qy^*$. Let $x \in B$ be any fixed point of Q, then by (3.3), $x^{(1)} = Qx^{(0)} \leq Qx = x \leq Qy^{(0)} = y^{(1)}$. By induction, $x^{(n)} \leq x \leq y^{(n)}$. Using (3.5) and taking the limit as $n \to \infty$. we conclude that $x^{(0)} \leq x^* \leq x \leq y^* \leq y^{(0)}$. Hence x^*, y^* are the minimal and maximal mild solutions of the finite delay differential equations of fractional order (1.1) on $[x^{(0)}, y^{(0)}]$ respectively. □

In the next theorem, we again discuss the existence of extremal mild solutions of (1.1) with help of the measure of noncompactness and monotone iterative procedure. In this result, the semigroup $\{T(t)\}_{t\geq 0}$ does not have to be compact.

Theorem 3.2. *Let X be an ordered Banach space, whose positive cone P is normal with normal constant N, A be the infinitesimal generator of C_0-semigroup $\{T(t)\}_{t\geq 0}$ on X and $T(t)(t \geq 0)$ be a positive operator. Also suppose that the Cauchy delay problem (1.1) has lower and upper solutions $x^{(0)}, y^{(0)} \in C([-a,b],X)$ with $x^{(0)} \leq y^{(0)}$ and the assumptions (H1)–(H4) hold. If the following assumptions are satisfied:*

(H5) The functions f, h satisfy following:

(i) There exists an integrable function $\zeta\colon \Sigma \to [0,\infty)$ such that

$$\mu(h(t,s,H)) \le \zeta(t,s) \sup_{-a\le\nu\le 0} \mu(H(\nu)) \quad a.e.\ t \in J$$

and $H \subset \mathcal{D}$, where $H(\nu) = \{\varphi(\nu)\colon \varphi \in H\}$.

(ii) There exists a constant $L \ge 0$ such that

$$\mu(f(t,E,S)) \le L\Big[\sup_{-a\le\nu\le 0} \mu(E(\nu)) + \mu(S)\Big],$$

for a.e. $t \in J$ and $E \subset \mathcal{D}$, $S \subset X$, where $E(\nu) = \{\varphi(\nu)\colon \varphi \in E\}$. For convenience, we write $\zeta^* = \max \int_0^t \zeta(t,s)\,ds$,

and $K = \frac{2MLb^\alpha}{\Gamma(\alpha+1)}(1+2\zeta^*) < 1$, then the Cauchy delay problem (1.1) has minimal and maximal mild solutions between $x^{(0)}$ and $y^{(0)}$.

Proof. Let $B = [x^{(0)}, y^{(0)}] = \{x \in C([-a,b],X) \mid x^{(0)} \le x \le y^{(0)}\}$. We define a map $Q\colon B \to C([-a,b],X)$ as defined in Theorem 3.1. From the proof of Theorem 3.1, $Q\colon B \to B$ is a continuously increasing operator and $Q(B)$ is equicontinuous. Now we define the sequences $x^{(n)}$ and $y^{(n)}$ as defined in Theorem 3.1, which are given by (3.4).

Let $S = \{x^{(n)}\}_{n=1}^\infty$. The normality of positive cone P_C and (3.5) imply that S is bounded. By (3.1), we have that $x^{(n)}(t) = \phi(t) + g(x^{(n-1)})(t)$, $n = 1,2,\dots$, for $t \in [-a,0]$. For $t \in [-a,0]$, we get $\mu(\{x^{(n)}(t)\}) = \mu(\{\phi(t) + g(x^{(n-1)})(t)\}) \le \mu(\{\phi(t)\} + \mu\{g(x^{(n-1)})(t)\}) = 0$ as g is a compact operator. Thus we have that

$$\mu(\{x^{(n)}(t)\}) = 0, \qquad t \in [-a,0]. \tag{3.6}$$

Since $S(t) = \{x^{(1)}(t)\} \cup \{Q(S)(t)\}$, $t \in J$, then $\mu(S(t)) = \mu(Q(S)(t))$, $t \in J$. For any $t \in J$ and by using (H4), (H5), (3.1), (3.4), (3.6), we get

$$\begin{aligned}
\mu(S(t)) &= \mu\bigg(\Big\{U(t)(\phi(0) + g(x^{(n)})(0)) \\
&\qquad + \int_0^t (t-s)^{\alpha-1} V(t-s) f\Big(s, x_s^{(n)}, \int_0^s h(s,\tau,x_\tau^{(n)})\,d\tau\Big)\,ds\Big\}\bigg) \\
&\le \mu\left(\{U(t)(\phi(0)\}\right) + \mu\left(\Big\{g(x^{(n)})(0)\Big\}\right) \\
&\qquad + \mu\left(\Big\{\int_0^t (t-s)^{\alpha-1} V(t-s) f\Big(s, x_s^{(n)}, \int_0^s h(s,\tau,x_\tau^{(n)})\,d\tau\Big)\,ds\Big\}\right) \\
&\le \frac{2ML}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\left[\sup_{-a\le\nu\le 0} \mu\left(\Big\{x^{(n)}(s+\nu)\Big\}\right) + \mu\left(\Big\{\int_0^s h(s,\tau,x_\tau^{(n)})\,d\tau\Big\}\right)\right] ds \\
&\le \frac{2ML}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\left[\sup_{0\le r\le s} \mu\left(\Big\{x^{(n)}(r)\Big\}\right) 2\int_0^s \zeta(s,\tau) \sup_{-a\le\nu\le 0} \mu\left(\Big\{x^{(n)}(\tau+\nu)\Big\}\right) d\tau\right] ds \\
&\le \frac{2ML}{\Gamma(\alpha)}(1+2\zeta^*) \int_0^t (t-s)^{\alpha-1} \sup_{0\le r\le s} \mu\left(\Big\{x^{(n)}(r)\Big\}\right) ds \\
&\le \frac{2MLb^\alpha}{\Gamma(\alpha+1)}(1+2\zeta^*) \sup_{-a\le r\le b} \mu\left(\Big\{x^{(n)}(r)\Big\}\right).
\end{aligned}$$

Since $\{Qx^{(n)}\}_{n=0}^{\infty}$, i.e. $\{x^{(n)}\}_{n=1}^{\infty}$, are equicontinuous on $[-a,b]$ and by Lemma 2.10, we get

$$\mu(S) \leq \frac{2MLb^{\alpha}}{\Gamma(\alpha+1)}(1+2\zeta^*)\mu\left(\left\{x^{(n)}\right\}\right) = K\mu(S).$$

Since $K<1$, this implies that $\mu(S)=0$, i.e. $\mu(\{x^{(n)}\}_{n=1}^{\infty})=0$. Therefore the set $\{x^{(n)} : n \geq 1\}$ is relatively compact in B. So we have that the sequence $\{x^{(n)}\}$ has a convergent subsequence in B. By the proof of Theorem 3.1, the sequence $\{x^{(n)}\}$ is itself a convergent sequence. So there exists $x^* \in B$ such that $x^{(n)} \to x^*$ as $n \to \infty$. Similarly there exists $y^* \in B$ such that $y^{(n)} \to y^*$ as $n \to \infty$ and $y^* = Qy^*$. Again by Theorem 3.1, x^* and y^* become the minimal and maximal mild solutions of the finite delay differential equations of fractional order (1.1) in B respectively. □

In the next theorem, we shall prove the uniqueness of the solution of the system (1.1) by using monotone iterative procedure. For this we make the following assumption.

(H6) The following conditions are satisfied.

(i) The function $h\colon \Sigma \times \mathcal{D} \to X$ is continuous and there exists an integrable function $\zeta\colon \Sigma \to [0,\infty)$ such that for some $\nu \in [-a,0]$,

$$h(t,s,\varphi_2) - h(t,s,\varphi_1) \leq \zeta(t,s)(\varphi_2(\nu) - \varphi_1(\nu)),$$

for any $(t,s) \in \Sigma$, $\varphi_1, \varphi_2 \in \mathcal{D}$ with $x_s^{(0)} \leq \varphi_1 \leq \varphi_2 \leq y_s^{(0)}$.

(ii) The function $f\colon J \times \mathcal{D} \times X \to X$ is continuous and there exists a constant $\eta \geq 0$ such that for some $\nu \in [-a,0]$,

$$f(t,\varphi_2,u_2) - f(t,\varphi_1,u_1) \leq \eta[(\varphi_2(\nu)-\varphi_1(\nu)) + (u_2-u_1)],$$

for any $t \in J$, $\varphi_1,\varphi_2 \in \mathcal{D}$ with $x_t^{(0)} \leq \varphi_1 \leq \varphi_2 \leq y_t^{(0)}$ and $u_1,u_2 \in X$ with $\int_0^t h(t,s,x_s^{(0)})\,ds \leq u_1 \leq u_2 \leq \int_0^t h(t,s,y_s^{(0)})\,ds$. For convenience, we write $\zeta^* = \max \int_0^t \zeta(t,s)\,ds$.

(H7) The function $g\colon C([-a,b],X) \to \mathcal{D}$ satisfies that for any $t \in [-a,0]$ and $x,y \in B$ with $x \leq y$, there exists a constant $\gamma(0 \leq \gamma < \frac{1}{N})$ such that

$$g(y)(t) - g(x)(t) \leq \gamma(y(t) - x(t)).$$

Theorem 3.3. *Let X be an ordered Banach space, whose positive cone P is normal with normal constant N, A be the infinitesimal generator of C_0-semigroup $\{T(t)\}_{t\geq 0}$ on X and $T(t)(t \geq 0)$ be a positive operator. Also suppose that the Cauchy delay problem (1.1) has lower and upper solutions $x^{(0)}, y^{(0)} \in C([-a,b],X)$ with $x^{(0)} \leq y^{(0)}$. If the assumptions (H2), (H3), (H4), (H6) and (H7) hold, and $K = \frac{2MLb^{\alpha}}{\Gamma(\alpha+1)}(1+2N\zeta^*) < 1$, where $L = N\eta$, then the Cauchy delay problem (1.1) has a unique mild solution between $x^{(0)}$ and $y^{(0)}$.*

Proof. Let $\{\varphi_n\} \subset \mathcal{D}$ and $\{u_n\} \subset X$ be two monotone increasing sequences. Take any $m,n = 1,2,\ldots,$ with $m>n$, by (H2), (H3) and (H6), we get, for some $\nu_1,\nu_2 \in [-a,0]$,

$$\theta \leq h(t,s,\varphi_m) - h(t,s,\varphi_n) \leq \zeta(t,s)(\varphi_m(\nu_1) - \varphi_n(\nu_1))$$

and

$$\theta \leq f(t,\varphi_m,u_m) - f(t,\varphi_n,u_n) \leq \eta\Big[(\varphi_m(\nu_2)-\varphi_n(\nu_2)) + (u_m-u_n)\Big].$$

Use the normality of the positive cone P, we get

$$\|h(t,s,\varphi_m) - h(t,s,\varphi_n)\| \leq N\zeta(t,s)\|\varphi_m(\nu_1)-\varphi_n(\nu_1)\| \tag{3.7}$$

and

$$\|f(t,\varphi_m,u_m) - f(t,\varphi_n,u_n)\| \leq N\eta\Big[\|\varphi_m(\nu_2)-\varphi_n(\nu_2)\| + \|u_m-u_n\|\Big]. \tag{3.8}$$

By the definition of measure of noncompactness, we get

$$\begin{aligned}\mu\left(\{h(t,s,\varphi_n)\}\right) &\leq N\zeta(t,s)\mu\left(\{\varphi_n(\nu)\}\right)\\ &\leq N\zeta(t,s)\sup_{-a\leq\nu\leq 0}\left(\{\varphi_n(\nu)\}\right)\end{aligned}$$

and

$$\begin{aligned}\mu\left(\{f\left(s,\varphi_n\right)\}\right) &\leq L\left[\mu\left(\{\varphi_n(\nu)\}\right)+\mu\left(\{u_n\}\right)\right]\\ &\leq L\left[\sup_{-a\leq\nu\leq 0}\mu\left(\{\varphi_n(\nu)\}\right)+\mu\left(\{u_n\}\right)\right],\end{aligned}$$

where $L = N\eta$. Clearly the assumption (H5) is satisfied. The assumption (H1) is satisfied by the inequalities (3.7) and (3.8). Thus the assumptions (H1)–(H5) hold and $K = \frac{2MLb^\alpha}{\Gamma(\alpha+1)}(1+2N\zeta^*) < 1$. So by Theorem 3.2, the Cauchy delay problem (1.1) has minimal and maximal mild solutions between $x^{(0)}$ and $y^{(0)}$.

Let $x^*(t)$ and $y^*(t)$ be the minimal and maximal solutions of Cauchy delay problem (1.1) respectively on the ordered interval $B = [x^{(0)}, y^{(0)}]$. By (3.1), (H7) and for any $t \in [-a,0]$, we have

$$\begin{aligned}\theta \leq y^*(t)-x^*(t) &= Qy^*(t) - Qx^*(t)\\ &= g(y^*)(t) - g(x^*)(t)\\ &\leq \gamma(y^*(t)-x^*(t))\end{aligned}$$

By using the normality of positive cone P, we get $\|y^*(t)-x^*(t)\| \leq N\gamma\|y^*(t)-x^*(t)\|$ for all $t \in [-a,0]$. This implies that $y^*(t) = x^*(t)$ for all $t \in [-a,0]$ as $N\gamma < 1$. Now by (3.1), (H6) and the positivity of operator $U(t)$ and $V(t)$ and for any $t \in [0,b]$, we have

$$\begin{aligned}\theta &\leq y^*(t)-x^*(t) = Qy^*(t) - Qx^*(t)\\ &= \int_0^t (t-s)^{\alpha-1}V(t-s)\left[f\Big(s,y_s^*,\int_0^s h(s,\tau,y_\tau^*)\,d\tau\Big) - f\Big(s,x_s^*,\int_0^s h(s,\tau,x_\tau^*)\,d\tau\Big)\right]ds\\ &\leq \eta\int_0^t (t-s)^{\alpha-1}V(t-s)\left[(y_s^*(\nu)-x_s^*(\nu)) + \Big(\int_0^s h(s,\tau,y_\tau^*)\,d\tau - \int_0^s h(s,\tau,x_\tau^*)\,d\tau\Big)\right]ds\\ &\leq \eta\int_0^t (t-s)^{\alpha-1}V(t-s)\left[(y_s^*(\nu)-x_s^*(\nu)) + \int_0^s \zeta(s,\tau)\Big(y_\tau^*(\nu)-x_\tau^*(\nu)\Big)\,d\tau\right]ds,\end{aligned}$$

where $\nu \in [-a, 0]$. By applying the normality of the positive cone P, we get

$$
\begin{aligned}
\|y^*(t) - x^*(t)\| &\leq N\eta \Big\| \int_0^t (t-s)^{\alpha-1} V(t-s) \Big[(y_s^*(\nu) - x_s^*(\nu)) \\
&\qquad + \int_0^s \zeta(s,\tau) \Big(y_\tau^*(\nu) - x_\tau^*(\nu) \Big) \, d\tau \Big] \, ds \Big\| \\
&\leq \frac{MN\eta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[\|y^*(s+\nu) - x^*(s+\nu)\| \\
&\qquad + \int_0^s \zeta(s,\tau) \|y^*(\nu+\tau) - x^*(\nu+\tau)\| \, d\tau \Big] \, ds \\
&\leq \frac{MN\eta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[1 + \int_0^s \zeta(s,\tau) \, d\tau \Big] \, ds \|y^* - x^*\| \\
&\leq \frac{MN\eta b^\alpha}{\Gamma(\alpha+1)} (1+\zeta^*) \|y^* - x^*\|.
\end{aligned}
\tag{3.9}
$$

Since $y^*(t) = x^*(t)$ for $t \in [-a, 0]$ and by the Inequality, we get that $\|y^* - x^*\| \leq K\|y^* - x^*\|$. But $K < \frac{1}{2}$, so $\|y^* - x^*\| = 0$, i.e., $y^*(t) = x^*(t),\ t \in [-a, b]$. Hence $y^* = x^*$ is the unique mild solution of the cauchy delay problem (1.1) between $x^{(0)}$ and $y^{(0)}$. □

4 Example

Let $X = L^2([0,\pi], \mathbb{R})$. Consider the following nonlocal fractional partial differential equations with finite delay:

$$
\begin{cases}
{}^cD_t^\alpha z(t,\xi) = \frac{\partial^2}{\partial \xi^2} z(t,\xi) + L\Big(\frac{|z_t(\nu,\xi)|}{1+|z_t(\nu,\xi)|} \\
\qquad\qquad + \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} \int_{-r}^0 \gamma(\nu) z_s(\nu,\xi) \, d\nu \, ds \Big), & t \in [0,b],\ \xi \in [0,\pi] \\
z(t,0) = z(t,\pi) = 0, & t \in [0,b], \\
z(\nu,\xi) = \phi(\nu,\xi) + \int_0^b \rho(s,\nu) \frac{(z(s,\xi))^2}{1+(z(s,\xi))^2} ds, & -a \leq \nu \leq 0,
\end{cases}
\tag{4.1}
$$

where ${}^cD_t^\alpha$ is a Caputo fractional partial derivative of order α, $0 < \alpha < 1$; $a > 0$; $L \geq 0$; $z_t(\nu,\xi) = z(t+\nu,\xi)$, $t \in [0,b]$, $\nu \in [-a,0]$; the map $\gamma : [-a,0] \to \mathbb{R}^+$ is continuous; $\phi \in D = C([-a,0] \times [0,\pi], \mathbb{R}^+)$; $\rho(s,\nu)$ is a continuous operator from compact square $[0,b] \times [-a,0]$ to $\mathbb{R}^+$.

Let $P = \{v \in X \colon v(\xi) \geq 0 \text{ a.e. } \xi \in [0,\pi]\}$. Then P is a normal cone in Banach space X and its normal constant is 1, i.e. $N = 1$. We define an operator $A : X \to X$ by $Av = v''$ with domain

$$
D(A) = \{v \in X : v, v' \text{ is absolutely continuous } v'' \in X,\ v(0) = v(\pi) = 0\}.
$$

It is well known that A is an infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operator $\{T(t),\ t \geq 0\}$ in X. For $\xi \in [0,\pi]$, $\nu \in [-a,0]$ and $\varphi \in C([-a,0], X)$, we define

$$
\begin{aligned}
z(t)(\xi) &= z(t,\xi), \\
\varphi(t)(\xi) &= \varphi(t,\xi), \\
h(t,s,\varphi)(\xi) &= (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} \int_{-r}^0 \gamma(\nu) \varphi(\nu,\xi) \, d\nu \, ds,
\end{aligned}
$$

$$f(t,\varphi,u)(\xi) = f(t,\varphi(\nu,\xi),u(\xi)) = L\Big(\frac{|\varphi(\nu,\xi)|}{1+|\varphi(\nu,\xi)|} + u(\xi)\Big),$$
$$g(z)(\nu)(\xi) = g(z(\nu,\xi)) = \int_0^b \rho(s,\nu)\frac{(z(s,\xi))^2}{1+(z(s,\xi))^2}\,ds,$$
$$\phi(\nu)(\xi) = \phi(\nu,\xi).$$

Thus the above nonlocal fractional partial differential equations with finite delay (4.1) can be written as the abstract form of (1.1).

Let $v(t,\xi) = 0$, $(t,\xi) \in [-a,b]\times[0,\pi]$. Then $f\big(t, v_t(\nu,\xi), \int_0^t h(t,s,v_s(\nu,\xi))\,ds\big) = 0$ for $t \in [0,b]$ and $\phi(\nu,\xi) \geq v(\nu,\xi)$ for $\nu \in [-a,0]$. Now we assume that there is a function $w(t,\xi) \geq 0$ such that

$${}^cD_t^\alpha w(t,\xi) \geq \frac{\partial^2}{\partial y^2}w(t,\xi) + f\left(t, w_t(\nu,\xi), \int_0^t h(t,s,w_s(\nu,\xi))\,ds\right),$$

$w(t,0) = w(t,\pi) = 0$ and $w(\nu,\xi) \geq \phi(\nu,\xi) + g(w(\nu,\xi))$ for $\nu \in [-a,0]$. Thus v, w become lower and upper solutions of the system (4.1) respectively and $v \leq w$. By the definition of functions of f, h and g, we can easily see that assumptions (H1)–(H4) are satisfied. For $t \in [0,b]$, $\varphi_1,\varphi_2 \in C([-a,0],X)$ with $0 \leq \varphi_1 \leq \varphi_2$ and $u_1,u_2 \in X$, then

$$0 \leq f(t,\varphi_2,u_2)(\xi) - f(t,\varphi_1,u_1)(\xi) \leq L[(\varphi_2(\nu)(\xi) - \varphi_1(\nu)(\xi)) + (u_2(\xi) - u_1(\xi))]$$

and

$$0 \leq h(t,s,\varphi_2)(\xi) - h(t,s,\varphi_1)(\xi) = (t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}\int_{-r}^0 \gamma(\nu)(\varphi_2(\nu)(\xi) - \varphi_1(\nu)(\xi))\,d\nu.$$

By normality of cone P, we have

$$\|f(t,\varphi_2,u_2) - f(t,\varphi_1,u_1)\| \leq L[\|\varphi_2(\nu) - \varphi_1(\nu)\| + \|u_2 - u_1\|]$$

and

$$\|h(t,s,\varphi_2) - h(t,s,\varphi_1)\| \leq (t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}\int_{-r}^0 |\gamma(\nu)|\|\varphi_2(\nu) - \varphi_1(\nu)\|l\,d\nu.$$

Hence for any bounded set $E \subset C([-a,0],X)$ and $S \subset X$, we have

$$\mu(f(t,E,S)) \leq L\left[\sup_{-a\leq\nu\leq 0}\mu(E(\nu)) + \mu(S)\right]$$

and

$$\mu(h(t,s,E)) \leq \zeta(t,s)\sup_{-a\leq\nu\leq 0}\mu(E(\nu)),$$

where $\zeta(t,s) = (t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}\int_{-r}^0|\gamma(\nu)|\,d\nu$ and $\zeta^* = \sup_{t\in[0,b]} = \pi\int_{-r}^0|\gamma(\nu)|\,d\nu$. Thus assumption (H5) is satisfied. If $K = \frac{2MLb^\alpha}{\Gamma(1+\alpha)}(1+\zeta^*) < 1$, all the conditions of the Theorem 3.2 are satisfied. Hence, by Theorem 3.2, the system (4.1) has the minimal and maximal mild solutions lying between the lower solution 0 and the upper solution w.

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