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# Existence and Ulam–Hyers stability of ODEs involving two Caputo fractional derivatives

**Shan Peng** and **JinRong Wang** ✉

College of Science, Department of Mathematics, Guizhou University,
Guiyang, Guizhou 550025, P.R. China

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**Abstract.** In this paper, we study existence of solutions to a Cauchy problem for nonlinear ordinary differential equations involving two Caputo fractional derivatives. The existence and uniqueness of solutions are obtained by using monotonicity, continuity and explicit estimation of Mittag-Leffler functions via fixed point theorems. Further, we present Ulam–Hyers stability results by using direct analysis methods. Finally, examples are given to illustrate our theoretical results.

**Keywords:** ODEs, Caputo fractional derivative, Mittag-Leffler function, existence, Ulam–Hyers stability.

**2010 Mathematics Subject Classification:** 26A33, 33E12, 34A12.

## 1 Introduction

In the past decades, fractional differential equations have been proved to be valuable tools to describe nonlinear oscillations of earthquakes, seepage flow in porous media and fluid dynamic traffic model. There are many monographs on this interesting topic [3, 9, 13, 16, 17, 19, 23, 26, 27, 33] and a large amount of papers on quality analysis for nonlocal problems, impulsive problems, Ulam–Hyers stability and stable manifolds problems as well as control problems [1, 2, 4–8, 10, 14, 15, 20–22, 25, 28–30, 32, 34]) and the references therein.

In [17, Chapter 5], Kilbas et al. studied the solvability of a Cauchy problem for nonlinear ordinary differential equations involving two Caputo fractional derivatives of the type: ${}^cD_t^{\alpha}x(t) - \lambda {}^cD_t^{\beta}x(t) = f(t)$, where $\lambda \in \mathbb{R}$, ${}^cD_t^{\alpha}$ and ${}^cD_t^{\beta}$ denote the Caputo fractional derivatives of order $\alpha$, $\beta$ with the lower limit zero, respectively (see Definition 2.1). Further, Wang and Li [30] discussed $\mathbb{E}_{\alpha}$-Ulam–Hyers stability of fractional differential equations of the type: ${}^cD_t^{\alpha}x(t) = \lambda x(t) + f(t, x(t))$ on finite time interval via the properties of Mittag-Leffler functions $\mathbb{E}_{\alpha}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$ and $\mathbb{E}_{\alpha,\alpha}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\alpha)}$ for $z \leq 0$ (see [29, Lemma 2]) and a singular Gronwall type integral inequality (see [31, Theorem 1]). Very recently, Cong et al. [6] explored some asymptotic behavior on $\mathbb{E}_{\alpha}(z)$ and $\mathbb{E}_{\alpha,\alpha}(z)$ for $z > 0$ by using [11, Theorem 2.3], which inspired the reader to study further estimation and asymptotic behavior on $\mathbb{E}_{\alpha,\beta}(z)$.

✉ Corresponding author. Email: sci.jrwang@gzu.edu.cn

However, the development of existence and Ulam's type stability theory for nonlinear ordinary differential equations involving two Caputo fractional derivatives is still in its infancy. One of the reasons for this fact might be that asymptotic property of $\mathbb{E}_{\alpha,\beta}(z)$ have not been explored completely.

Motivated by [6,17,30], we consider the following Cauchy problem for nonlinear differential equations involving two Caputo fractional derivatives:

$$\begin{cases} {}^cD_t^\alpha x(t) - \lambda {}^cD_t^\beta x(t) = f(t,x(t)), & 0<\beta<\alpha\le 1, \quad t\in J:=[0,1], \\ x(0)=x_0, & x_0\in\mathbb{R}, \end{cases} \tag{1.1}$$

where $\lambda\in\mathbb{R}\setminus\{0\}$, $f\colon J\times\mathbb{R}\to\mathbb{R}$ is a continuous function. By [17, p. 324, (5.3.75)–(5.3.76)], the solution $x\in C(J,\mathbb{R})$ of (1.1) is given by

$$\begin{aligned} x(t) = {} & \left[\mathbb{E}_{\alpha-\beta}(\lambda t^{\alpha-\beta}) - \lambda t^{\alpha-\beta}\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda t^{\alpha-\beta})\right]x_0 \\ & + \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})f(s,x(s))\,ds, \end{aligned} \tag{1.2}$$

with the two parameter Mittag-Leffler function $\mathbb{E}_{\alpha,\beta}(z) := \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}$.

Before we deal with existence of solutions and Ulam–Hyers stability, the key step is to discuss the elementary properties of Mittag-Leffler functions. By virtue of integrable expansion of Mittag-Leffler functions in [6], we give monotonicity, continuity and explicit estimation of Mittag-Leffler functions $\mathbb{E}_\alpha(z)$ and $\mathbb{E}_{\alpha,\beta}(z)$ for $z>0$ and $z<0$, which extend the previous results in [29, Lemma 2] and [6, Lemma 3].

The first purpose of this paper is to discuss existence of solutions to the equation (1.1) by using fixed point theorems. The second purpose of this paper is to present that the equation (1.1) is Ulam–Hyers stable on the time interval $J$. When we discuss existence theorems and Ulam-Hyers stability theorems, the new derived properties of Mittag-Leffler functions $\mathbb{E}_\alpha(z)$ and $\mathbb{E}_{\alpha,\beta}(z)$ for $z>0$ and $z<0$ are widely used in this paper. Meanwhile, these properties will help the researcher to study other fractional ODEs with constant coefficients.

The rest of this paper is organized as follows. In Section 2, we recall some notations and give some useful properties of the two-parameter Mittag-Leffler function. In Section 3, we apply fixed point theorems to derive the existence of solutions. In Section 4, Ulam–Hyers stability theorems are presented. Examples are given in Section 5 to demonstrate the application of our main results.

## 2 Preliminaries

Let $C(J,\mathbb{R})$ be the Banach space of all continuous functions from $J$ into $\mathbb{R}$ with the norm $\|z\|_\infty = \sup\{|z(t)| : t\in J\}$.

**Definition 2.1** ([17]). The Caputo derivative of order $\gamma$ for a function $f\colon[0,\infty)\to\mathbb{R}$ can be written as

$${}^cD_t^\gamma f(t) = {}^LD_t^\gamma\left(f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right), \qquad t>0,\ n-1<\gamma<n,$$

where ${}^LD_t^\gamma f$ denotes the Riemann–Liouville derivative of order $\gamma$ with the lower limit zero for a function $f$, which given by

$${}^LD_t^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}}\,ds, \qquad t>0,\ n-1<\gamma<n.$$

We recall the famous integrable expansion of two differential parameters Mittag-Leffler function.

**Lemma 2.2** (see [11, Theorem 2.3])**.** *Let $\alpha \in (0,1]$, $\beta \in \mathbb{R}$ and $\beta < 1+\alpha$ be arbitrary. Then the following statements hold.*

*(i) For all $z > 0$, we have*

$$\mathbb{E}_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{\frac{1-\beta}{\alpha}} \exp(z^{\frac{1}{\alpha}}) + \int_0^\infty K(r,z)\,dr,$$

*where*

$$K(r,z) = \frac{1}{\pi\alpha} r^{\frac{1-\beta}{\alpha}} \exp(-r^{\frac{1}{\alpha}}) \frac{r\sin(\pi(1-\beta)) - z\sin(\pi(1-\beta+\alpha))}{r^2 - 2rz\cos(\pi\alpha) + z^2}.$$

*(ii) For all $z < 0$, we have*

$$\mathbb{E}_{\alpha,\beta}(z) = \int_0^\infty K(r,z)\,dr.$$

For more details on expression on the Mittag-Leffler functions, one can see [12].

Next, we need monotonicity and continuity results for Mittag-Leffler functions.

**Lemma 2.3** ([24, Lemma 2.3])**.** *Let $\alpha \in (0,1]$, $\beta \in \mathbb{R}$ and $\beta < 1+\alpha$ be arbitrary. The functions $\mathbb{E}_\alpha(\cdot)$ and $\mathbb{E}_{\alpha,\beta}(\cdot)$ are nonnegative and have the following properties.*

*(i) For all $\lambda > 0$ and $t_1, t_2 \in J$ and $t_1 \le t_2$,*

$$\mathbb{E}_\alpha(t_1^\alpha \lambda) \le \mathbb{E}_\alpha(t_2^\alpha \lambda), \qquad \mathbb{E}_{\alpha,\beta}(t_1^\alpha \lambda) \le \mathbb{E}_{\alpha,\beta}(t_2^\alpha \lambda).$$

*(ii) For all $\lambda > 0$ and $t_1, t_2 \in J$,*

$$\mathbb{E}_\alpha(t_1^\alpha \lambda) \to \mathbb{E}_\alpha(t_2^\alpha \lambda) \quad \text{as } t_1 \to t_2,$$

$$\mathbb{E}_{\alpha,\beta}(t_1^\alpha \lambda) \to \mathbb{E}_{\alpha,\beta}(t_2^\alpha \lambda) \quad \text{as } t_1 \to t_2.$$

**Remark 2.4.** The symmetrical results for $\mathbb{E}_\alpha(z)$ and $\mathbb{E}_{\alpha,\beta}(z)$ for $z \le 0$ have been reported by Wang et al. [29, Lemma 2].

Next, we give explicit estimation of Mittag-Leffler functions $\mathbb{E}_\alpha(z)$ and $\mathbb{E}_{\alpha,\beta}(z)$ for $z > 0$, which extend the previous results in [6, Lemma 3].

**Lemma 2.5.** *Let $\lambda > 0$ be arbitrary. For any $\alpha \in (0,1]$, $\beta \in \mathbb{R}$ and $\beta < 1+\alpha$. We define*

$$m(\alpha,\beta,\lambda) = \max\{m_1(\alpha,\beta,\lambda), m_2(\alpha,\beta,\lambda)\},$$

*where*

$$m_1(\alpha,\beta,\lambda) = \frac{|\sin(\pi\beta)| \int_0^\infty r^{\frac{1-\beta+\alpha}{\alpha}} \exp(-r^{\frac{1}{\alpha}})\,dr}{\sin^2(\pi\alpha)\pi\alpha\lambda^2},$$

$$m_2(\alpha,\beta,\lambda) = \frac{|\sin(\pi(\beta-\alpha))| \int_0^\infty r^{\frac{1-\beta}{\alpha}} \exp(-r^{\frac{1}{\alpha}})\,dr}{\sin^2(\pi\alpha)\pi\alpha\lambda}.$$

*(i) For all $t > 0$, we have*

$$\left|t^{\beta-1}\mathbb{E}_{\alpha,\beta}(\lambda t^{\alpha}) - \frac{1}{\alpha}\lambda^{\frac{1-\beta}{\alpha}}\exp(\lambda^{\frac{1}{\alpha}}t)\right| \leq \frac{m_1(\alpha,\beta,\lambda)}{t^{2\alpha-\beta+1}} + \frac{m_2(\alpha,\beta,\lambda)}{t^{\alpha-\beta+1}}$$
$$\leq m(\alpha,\beta,\lambda)\left(\frac{1}{t^{2\alpha-\beta+1}} + \frac{1}{t^{\alpha-\beta+1}}\right).$$

*In particular,*

$$\left|\mathbb{E}_{\alpha}(\lambda t^{\alpha}) - \frac{1}{\alpha}\exp(\lambda^{\frac{1}{\alpha}}t)\right| \leq \frac{m(\alpha,1,\lambda)}{t^{\alpha}}.$$

*(ii) For all $t > 0$, we have*

$$\left|t^{\beta-1}\mathbb{E}_{\alpha,\beta}(-\lambda t^{\alpha})\right| \leq \frac{m_1(\alpha,\beta,\lambda)}{t^{2\alpha-\beta+1}} + \frac{m_2(\alpha,\beta,\lambda)}{t^{\alpha-\beta+1}}$$
$$\leq m(\alpha,\beta,\lambda)\left(\frac{1}{t^{2\alpha-\beta+1}} + \frac{1}{t^{\alpha-\beta+1}}\right).$$

*In particular,*

$$\left|\mathbb{E}_{\alpha}(-\lambda t^{\alpha})\right| \leq \frac{m(\alpha,1,\lambda)}{t^{\alpha}}.$$

*Proof.* $(i)$ By virtue of Lemma 2.2 (i), we have

$$\left|t^{\beta-1}\mathbb{E}_{\alpha,\beta}(\lambda t^{\alpha}) - \frac{1}{\alpha}\lambda^{\frac{1-\beta}{\alpha}}\exp(\lambda^{\frac{1}{\alpha}}t)\right|$$
$$= \left|t^{\beta-1}\int_0^{\infty}\frac{1}{\pi\alpha}r^{\frac{1-\beta}{\alpha}}\exp(-r^{\frac{1}{\alpha}})\frac{r\sin(\pi(1-\beta)) - (\lambda t^{\alpha})\sin(\pi(1-\beta+\alpha))}{r^2 - 2r(\lambda t^{\alpha})\cos(\pi\alpha) + (\lambda t^{\alpha})^2}\,dr\right|.$$

It follows the fact

$$r^2 - 2r(\lambda t^{\alpha})\cos(\pi\alpha) + (\lambda t^{\alpha})^2 \geq \sin^2(\pi\alpha)\lambda^2 t^{2\alpha},$$

we obtain

$$\left|t^{\beta-1}\mathbb{E}_{\alpha,\beta}(\lambda t^{\alpha}) - \frac{1}{\alpha}\lambda^{\frac{1-\beta}{\alpha}}\exp(\lambda^{\frac{1}{\alpha}}t)\right|$$
$$\leq \left|\frac{t^{\beta-1}}{\pi\alpha\sin^2(\pi\alpha)\lambda^2 t^{2\alpha}}\int_0^{\infty} r^{\frac{1-\beta}{\alpha}}\exp(-r^{\frac{1}{\alpha}})(r\sin(\beta\pi) - \lambda t^{\alpha}\sin(\pi(\beta-\alpha)))\,dr\right|$$
$$\leq \frac{1}{t^{2\alpha-\beta+1}}\frac{|\sin(\pi\beta)|\int_0^{\infty} r^{\frac{1-\beta+\alpha}{\alpha}}\exp(-r^{\frac{1}{\alpha}})\,dr}{\sin^2(\pi\alpha)\pi\alpha\lambda^2}$$
$$+\frac{1}{t^{\alpha-\beta+1}}\frac{|\sin(\pi(\beta-\alpha))|\int_0^{\infty} r^{\frac{1-\beta}{\alpha}}\exp(-r^{\frac{1}{\alpha}})\,dr}{\sin^2(\pi\alpha)\pi\alpha\lambda}$$
$$= \frac{m_1(\alpha,\beta,\lambda)}{t^{2\alpha-\beta+1}} + \frac{m_2(\alpha,\beta,\lambda)}{t^{\alpha-\beta+1}}$$
$$\leq m(\alpha,\beta,\lambda)\left(\frac{1}{t^{2\alpha-\beta+1}} + \frac{1}{t^{\alpha-\beta+1}}\right),$$

which proves the part (i).

In particular,

$$\left|\mathbb{E}_\alpha(\lambda t^\alpha) - \frac{1}{\alpha}\exp(\lambda^{\frac{1}{\alpha}} t)\right| \le \frac{m_2(\alpha,1,\lambda)}{t^\alpha} \le \frac{m(\alpha,1,\lambda)}{t^\alpha}.$$

$(ii)$ By virtue of Lemma 2.2 (ii) for the case $z<0$, we obtain that

$$\begin{aligned}
&\left|t^{\beta-1}\mathbb{E}_{\alpha,\beta}(-\lambda t^\alpha)\right| \\
&\le \left|\frac{t^{\beta-1}}{\pi\alpha\sin^2(\pi\alpha)\lambda^2 t^{2\alpha}}\int_0^\infty r^{\frac{1-\beta}{\alpha}}\exp(-r^{\frac{1}{\alpha}})(r\sin(\beta\pi)+\lambda t^\alpha\sin(\pi(\beta-\alpha)))\,dr\right| \\
&\le \frac{1}{t^{2\alpha-\beta+1}}\frac{|\sin(\pi\beta)|\int_0^\infty r^{\frac{1-\beta+\alpha}{\alpha}}\exp(-r^{\frac{1}{\alpha}})\,dr}{\sin^2(\pi\alpha)\pi\alpha\lambda^2} \\
&\quad+\frac{1}{t^{\alpha-\beta+1}}\frac{|\sin(\pi(\beta-\alpha))|\int_0^\infty r^{\frac{1-\beta}{\alpha}}\exp(-r^{\frac{1}{\alpha}})\,dr}{\sin^2(\pi\alpha)\pi\alpha\lambda} \\
&= \frac{m_1(\alpha,\beta,\lambda)}{t^{2\alpha-\beta+1}}+\frac{m_2(\alpha,\beta,\lambda)}{t^{\alpha-\beta+1}} \\
&\le m(\alpha,\beta,\lambda)\left(\frac{1}{t^{2\alpha-\beta+1}}+\frac{1}{t^{\alpha-\beta+1}}\right).
\end{aligned}$$

In particular,

$$\left|\mathbb{E}_\alpha(-\lambda t^\alpha)\right| \le \frac{m_2(\alpha,1,\lambda)}{t^\alpha} \le \frac{m(\alpha,1,\lambda)}{t^\alpha}.$$

The proof is completed. □

To end this section, we recall the famous Krasnoselskii–Zabreiko fixed point theorem.

**Lemma 2.6** ([18]). *Let $(X,\|\cdot\|)$ be a Banach space, and $K\colon X\to X$ be a completely continuous operator. Assume that $L\colon X\to X$ is a bounded linear operator such that 1 is not an eigenvalue of $L$ and*

$$\lim_{\|x\|\to\infty}\frac{\|Kx-Lx\|}{\|x\|}=0.$$

*Then $K$ has a fixed point in $X$.*

## 3 Existence results

### 3.1 Case of $\lambda > 0$

We introduce the following assumptions:

$(H_1)$ $f\colon J\times\mathbb{R}\to\mathbb{R}$ is continuous.

$(H_2)$ There exists a constant $L>0$ such that

$$|f(t,x)-f(t,y)|\le L|x-y|,\quad \text{for each } t\in J, \text{ and all } x,y\in\mathbb{R}.$$

$(H_3)$ Let $0<L\rho<1$ where

$$\rho=\left(\frac{(3\beta-\alpha)m(\alpha-\beta,\alpha,\lambda)}{\beta(2\beta-\alpha)}+\frac{1}{\alpha-\beta}\lambda^{\frac{-\alpha}{\alpha-\beta}}\exp\left(\lambda^{\frac{1}{\alpha-\beta}}\right)\right)>0$$

and $\alpha<2\beta$, and $m(\alpha-\beta,\alpha,\lambda)$ is defined in (2.5).

Define $M = \max\{|f(t,0)| : t \in J\}$ and $B_r = \{x \in C(J,\mathbb{R}) : \|x\|_\infty \le r\}$, where

$$r \ge \frac{\rho M + \mathbb{E}_{\alpha-\beta}(\lambda)|x_0| + \lambda \mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda)|x_0|}{1 - L\rho}. \tag{3.1}$$

**Theorem 3.1.** *Assume that $(H_1)$–$(H_3)$ are satisfied. Then the equation* (1.1) *has a unique solution.*

*Proof.* Define an operator $Q : B_r \to C(J,\mathbb{R})$ by

$$\begin{aligned}(Qx)(t) &= \left[\mathbb{E}_{\alpha-\beta}(\lambda t^{\alpha-\beta}) - \lambda t^{\alpha-\beta}\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda t^{\alpha-\beta})\right]x_0 \\ &\quad + \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})f(s,x(s))\,ds.\end{aligned} \tag{3.2}$$

Note that $Q$ is well defined on $C(J,\mathbb{R})$ due to $(H_1)$.

Step 1. We prove that $Q(B_r) \subset B_r$.

Now, take $t \in J$ and $x \in B_r$. By using $(H_2)$ via Lemma 2.3 and Lemma 2.5 (i), we obtain

$$\begin{aligned}
&|(Qx)(t)| \\
&\le \left|\left[\mathbb{E}_{\alpha-\beta}(\lambda t^{\alpha-\beta}) - \lambda t^{\alpha-\beta}\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda t^{\alpha-\beta})\right]x_0\right| \\
&\quad + \left|\int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})f(s,x(s))\,ds\right| \\
&\le \int_0^t \left|(t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})f(s,x(s))\right| ds + \mathbb{E}_{\alpha-\beta}(\lambda)|x_0| + \lambda\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda)|x_0| \\
&\le \Bigg[\int_0^t \left|(t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta}) - \frac{1}{\alpha-\beta}\lambda^{\frac{1-\alpha}{\alpha-\beta}}\exp(\lambda^{\frac{1}{\alpha-\beta}}(t-s))\right| \\
&\qquad\quad \times |f(s,x(s)) - f(s,0) + f(s,0)|\,ds \\
&\qquad + \int_0^t \frac{1}{\alpha-\beta}\lambda^{\frac{1-\alpha}{\alpha-\beta}}\exp(\lambda^{\frac{1}{\alpha-\beta}}(t-s))|f(s,x(s)) - f(s,0) + f(s,0)|\,ds\Bigg] \\
&\quad + \mathbb{E}_{\alpha-\beta}(\lambda)|x_0| + \lambda\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda)|x_0| \\
&\le \Bigg[\int_0^t m(\alpha-\beta,\alpha,\lambda)\left(\frac{1}{(t-s)^{\alpha-2\beta+1}} + \frac{1}{(t-s)^{-\beta+1}}\right)[|f(s,x(s)) - f(s,0)| + |f(s,0)|]\,ds \\
&\qquad + \int_0^t \frac{1}{\alpha-\beta}\lambda^{\frac{1-\alpha}{\alpha-\beta}}\exp(\lambda^{\frac{1}{\alpha-\beta}}(t-s))[|f(s,x(s)) - f(s,0)| + |f(s,0)|]\,ds\Bigg] \\
&\quad + \mathbb{E}_{\alpha-\beta}(\lambda)|x_0| + \lambda\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda)|x_0| \\
&\le \rho[L\|x\|_\infty + M] + \mathbb{E}_{\alpha-\beta}(\lambda)|x_0| + \lambda\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda)|x_0| \\
&\le \rho[Lr + M] + \mathbb{E}_{\alpha-\beta}(\lambda)|x_0| + \lambda\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda)|x_0| \\
&\le r.
\end{aligned}$$

Step 2. We check that $Q$ is a contraction mapping.

For $x, y \in B_r$ and for each $t \in J$, by using Lemma 2.3 and Lemma 2.5 (i), we obtain

$$\begin{aligned}
&|(Qx)(t) - (Qy)(t)| \\
&\quad \le \int_0^t \left|(t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})\right|\left|f(s,x(s)) - f(s,y(s))\right| ds
\end{aligned}$$

$$
\begin{aligned}
&\leq L\int_0^t \left|(t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})\right| ds\|x-y\|_\infty \\
&\leq L\bigg[\int_0^t \left|(t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta}) - \frac{1}{\alpha-\beta}\lambda^{\frac{1-\alpha}{\alpha-\beta}}\exp(\lambda^{\frac{1}{\alpha-\beta}}(t-s))\right| ds \\
&\qquad + \int_0^t \frac{1}{\alpha-\beta}\lambda^{\frac{1-\alpha}{\alpha-\beta}}\exp(\lambda^{\frac{1}{\alpha-\beta}}(t-s))\, ds\bigg]\|x-y\|_\infty \\
&\leq L\rho\|x-y\|_\infty,
\end{aligned}
$$

which implies that $\|Qx-Qy\|_\infty \leq L\rho\|x-y\|_\infty$.

From $(H_3)$, one can obtain the conclusion of theorem by the contraction mapping principle. The proof is completed. □

Next, we apply Krasnoselskii's fixed point theorem to derive the existence result.

$(H_4)$ There exists a nondecreasing function $\varpi \in C([0,\infty),\mathbb{R}^+)$ such that $|f(t,x)| \leq \varpi(\|x\|_\infty)$ for all $(t,x) \in J\times\mathbb{R}$ and $0 < \rho \liminf\limits_{r\to\infty} \frac{\varpi(r)}{r} < 1$.

**Theorem 3.2.** *Assume that* $(H_1)$ *and* $(H_4)$ *are satisfied. Then the equation* (1.1) *has at least one solution.*

*Proof.* For some $r' > 0$, define two operators $G$ and $H$ on $B_{r'}$ given by

$$
\begin{aligned}
(Gx)(t) &= \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})f(s,x(s))ds, \\
(Hx)(t) &= \left[\mathbb{E}_{\alpha-\beta}(\lambda t^{\alpha-\beta}) - \lambda t^{\alpha-\beta}\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda t^{\alpha-\beta})\right]x_0.
\end{aligned}
$$

We show that $(G+H)(B_{r'}) \subset B_{r'}$. If it is not true, then for each $r' > 0$, there would exist $x_{r'} \in B_{r'}$ and $t_{r'} \in J$ such that $|(Gx_{r'})(t_{r'}) + (Hx_{r'})(t_{r'})| > r'$. By repeating the same process of Step 1 of Theorem 3.1, we have

$$
\begin{aligned}
r' &< |(Gx_{r'})(t_{r'}) + (Hx_{r'})(t_{r'})| \\
&\leq \mathbb{E}_{\alpha-\beta}(\lambda)|x_0| + \lambda\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda)|x_0| + \int_0^t \left|(t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})\right|\varpi(r')ds \\
&\leq \mathbb{E}_{\alpha-\beta}(\lambda)|x_0| + \lambda\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda)|x_0| + \rho\varpi(r')
\end{aligned}
$$

Dividing both sides by $r'$ and taking the lower limit as $r' \to +\infty$, we obtain $1 \leq \rho \liminf\limits_{r'\to\infty} \frac{\varpi(r')}{r'}$, which contradicts with $(H_4)$. Thus, for some positive number $r'$, $(G+H)(B_{r'}) \subset B_{r'}$.

We observe that $H$ is a contraction with the constant zero and the continuity of $f$ implies that the operator $G$ is continuous. Moreover, $G$ is uniformly bounded on $B_{r'}$. Now we need to prove the compactness of the operator $G$. Define $f_{\max} = \sup\{|f(t,x)| : t \in J, x \in B_{r'}\}$. For any $t_2 < t_1$, by using Lemma 2.3 (ii), we have

$$
\begin{aligned}
&|(Gx)(t_2) - (Gx)(t_1)| \\
&\quad\leq \left|\int_0^{t_2} \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right]\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t_2-s)^{\alpha-\beta})f(s,x(s))\,ds\right| \\
&\quad\quad+\left|\int_0^{t_2} (t_1-s)^{\alpha-1}\left[\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t_2-s)^{\alpha-\beta}) - \mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t_1-s)^{\alpha-\beta})\right]f(s,x(s))\,ds\right|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\int_{t_2}^{t_1}(t_1-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t_1-s)^{\alpha-\beta})f(s,x(s))\,ds\right| \\
\le\ &\mathbb{E}_{\alpha-\beta,\alpha}(\lambda)f_{\max}\left|\int_0^{t_2}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]ds\right| \\
&+f_{\max}\int_0^{t_2}(t_2-s)^{\alpha-1}\left|\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t_2-s)^{\alpha-\beta})-\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t_1-s)^{\alpha-\beta})\right|ds \\
&+\mathbb{E}_{\alpha-\beta,\alpha}(\lambda)f_{\max}\left|\int_{t_2}^{t_1}(t_1-s)^{\alpha-1}\,ds\right| \\
\le\ &\frac{3(t_1-t_2)^\alpha}{\alpha}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda)f_{\max}+\frac{f_{\max}t_2^\alpha}{\alpha}O(|t_1-t_2|),
\end{aligned}
$$

which tends to zero as $t_2 \to t_1$.

This yields that $G$ is equicontinuous. So $G$ is relatively compact. Hence, $G$ is compact. At last, we can conclude that $G+H$ is a condensing map on $B_{r'}$. By using the Krasnoselskii fixed point theorem, the problem has at least one solution. The proof is completed. □

Next, we apply the Krasnoselskii–Zabreiko fixed point theorem to derive the existence result.

$(H_5)$ The function $f(t,0) \neq 0$ for some $t \in J$ and

$$
\lim_{\|x\|_\infty \to \infty} \frac{f(t,x)}{x} = k(t).
$$

$(H_6)$ $k_{\sup} := \sup_{t\in J}|k(t)| < \frac{1}{\rho}$.

**Theorem 3.3.** *Assume that $(H_1)$, $(H_5)$ and $(H_6)$ are satisfied. Then the equation (1.1) has at least one solution.*

*Proof.* Choose $r \ge \rho f_{\max} + \mathbb{E}_{\alpha-\beta}(\lambda)|x_0| + \lambda\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda)|x_0|$. Then we consider the operator $Q$ defined in (3.1) again. By repeating the similar computations of Theorems 3.1 and 4.4, we know that the operator $Q\colon B_r \to B_r$ is continuous and $Q(x)$ is uniformly bounded and equicontinuous for all $x \in B_r$. Consequently $Q$ is relatively compact.

Next we consider the problem (1.2) as a linear problem by setting $f(t,x(t)) = k(t)x(t)$. Define the operator $L\colon B_r \to B_r$ by

$$
\begin{aligned}
(Lx)(t) = &\left[\mathbb{E}_{\alpha-\beta}(\lambda t^{\alpha-\beta}) - \lambda t^{\alpha-\beta}\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda t^{\alpha-\beta})\right]x_0 \\
&+\int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})k(s)x(s)\,ds.
\end{aligned}
$$

Now, we claim that

$$
\sup_{t\in J}|Lx(t)| \le k_{\sup}\,\rho\|x\|_\infty + [\mathbb{E}_{\alpha-\beta}(\lambda)+\lambda\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda)]|x_0| < \|x\|_\infty. \tag{3.3}
$$

If not, one can derive the fact

$$
k_{\sup}\,\rho = \lim_{\|x\|_\infty\to\infty}\frac{k_{\sup}\,\rho\|x\|_\infty + [\mathbb{E}_{\alpha-\beta}(\lambda)+\lambda\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda)]|x_0|}{\|x\|_\infty} \ge 1,
$$

which contradicts with $(H_6)$. Therefore, (3.3) implies that 1 is not an eigenvalue of the operator $L$.

Finally we will show that $\frac{\|Qx-Lx\|_\infty}{\|x\|_\infty}$ vanishes as $\|x\|_\infty \to \infty$. In fact,

$$\begin{aligned}|(Qx)(t)-(Lx)(t)| &= \Big| \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})f(s,x(s))\,ds \\ &\quad - \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})k(s)x(s)\,ds\Big| \\ &\le \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})\left|\frac{f(s,x(s))}{x(s)}-k(s)\right| ds\|x\|_\infty.\end{aligned}$$

This means that

$$\frac{\|Qx-Lx\|_\infty}{\|x\|_\infty} \le \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})\left|\frac{f(s,x(s))}{x(s)}-k(s)\right| ds.$$

Then, we can get

$$\lim_{\|x\|_\infty\to\infty} \frac{\|Qx-Lx\|_\infty}{\|x\|_\infty} = 0$$

due to $(H_5)$.

Consequently, the proof is completed by virtue of Lemma 2.6. □

## 3.2 Symmetrical results for $\lambda < 0$

In this section, we give symmetrical existence results for Section 3.

$(H_7)$ $0 < L\varrho < 1$ where $\varrho = \frac{(3\beta-\alpha)m(\alpha-\beta,\alpha,-\lambda)}{\beta(2\beta-\alpha)} > 0$ and $\alpha < 2\beta$ and $m(\alpha-\beta,\alpha,-\lambda)$ is defined in (2.5), $L$ is defined in $(H_2)$.

Recall the above definition of $M$ and $B_r$, where

$$r \ge \frac{\varrho M + |x_0| - \frac{\lambda}{\Gamma(\alpha-\beta+1)}|x_0|}{1-L\varrho}. \tag{3.4}$$

Now we are ready to give the following result.

**Theorem 3.4.** *Assume that $(H_1)$, $(H_2)$, $(H_7)$ are satisfied. Then the equation (1.1) has a unique solution.*

*Proof.* Like in Theorem 3.1, consider $Q\colon B_r \to C(J,\mathbb{R})$ again, where $r$ is chosen in (3.4). We prove that $Q(B_r) \subset B_r$. Now, take $t \in J$ and $x \in B_r$. By using $(H_2)$ via Lemma 2.5 (ii), we obtain

$$\begin{aligned}|(Qx)(t)| &\le \int_0^t m(\alpha-\beta,\alpha,-\lambda)\left(\frac{1}{(t-s)^{\alpha-2\beta+1}}+\frac{1}{(t-s)^{-\beta+1}}\right)[L|x(s)|+|f(s,0)|]\,ds \\ &\quad + |x_0| - \frac{\lambda}{\Gamma(\alpha-\beta+1)}|x_0| \\ &\le \varrho[L\|x\|_\infty + M] + |x_0| - \frac{\lambda}{\Gamma(\alpha-\beta+1)}|x_0| \\ &\le \varrho[Lr + M] + |x_0| - \frac{\lambda}{\Gamma(\alpha-\beta+1)}|x_0| \\ &\le r.\end{aligned}$$

We check that $Q$ is a contraction mapping. For $x, y \in B_r$ and for each $t \in J$. By using Lemma 2.5 (ii), we obtain

$$|(Qx)(t) - (Qy)(t)| \leq L \int_0^t m(\alpha-\beta, \alpha, -\lambda)\left(\frac{1}{(t-s)^{\alpha-2\beta+1}} + \frac{1}{(t-s)^{-\beta+1}}\right) ds \|x-y\|_\infty$$
$$\leq L\varrho\|x-y\|_\infty,$$

which implies that $\|Qx - Qy\|_\infty \leq L\varrho\|x-y\|_\infty$.

By $(H_7)$ and the contraction mapping principle, one can complete the proof. □

$(H_8)$ There exists a nondecreasing function $\varpi \in C([0,\infty), \mathbb{R}^+)$ such that $|f(t,x)| \leq \varpi(\|x\|_\infty)$ for all $(t,x) \in J \times \mathbb{R}$ and $0 < \varrho \liminf_{r\to\infty} \frac{\varpi(r)}{r} < 1$.

**Theorem 3.5.** *Assume that $(H_1)$ and $(H_8)$ are satisfied. Then the equation (1.1) has at least one solution.*

*Proof.* We consider the operators $G$ and $H$ in Theorem 3.2 again. We use proof by contradiction to show that $(G+H)(B_{r'}) \subset B_{r'}$ for some positive number $r'$. By repeating the same process of Step 1 of Theorem 3.4, we have $r' < |(Gx_{r'})(t_{r'}) + (Hx_{r'})(t_{r'})| \leq |x_0| - \frac{\lambda}{\Gamma(\alpha-\beta+1)}|x_0| + \varrho\varpi(r')$, which implies that $1 \leq \varrho \liminf_{r'\to\infty} \frac{\varpi(r')}{r'}$, contradicts $(H_8)$. To prove the compactness of the operator $G$, we only need to check equicontinuity, for any $t_2 < t_1$, by using Remark 2.4 ([29, Lemma 2 (ii)]), we have $|(Gx)(t_2) - (Gx)(t_1)| \leq \frac{3(t_1-t_2)^\alpha}{\Gamma(\alpha+1)} f_{\max} + \frac{f_{\max} t_2^\alpha}{\alpha} O(|t_1 - t_2|)$, which tends to zero as $t_2 \to t_1$.

The rest of the proof is the same as that of Theorem 3.2. So we omit it here. □

$(H_9)$ $k_{\sup} := \sup_{t\in J} |k(t)| < \frac{1}{\varrho}$, where $k(t)$ defined in $(H_5)$.

**Theorem 3.6.** *Assume that $(H_1)$, $(H_5)$ and $(H_9)$ are satisfied. Then the equation (1.1) has at least one solution.*

*Proof.* Choose $r \geq \varrho f_{\max} + |x_0| - \frac{\lambda}{\Gamma(\alpha-\beta+1)}|x_0|$. Similar to the proof of Theorem 3.3, one can obtain the result.

□

# 4 Ulam–Hyers stability results

## 4.1 Case of $\lambda > 0$

In this part, we will discuss Ulam–Hyers stability of the equation (1.1) for the case $\lambda > 0$ on the time interval $J$.

Let $\epsilon > 0$. Consider the equation (1.1) and below inequality

$$|{}^cD_t^\alpha y(t) - \lambda {}^cD_t^\beta y(t) - f(t, y(t))| \leq \epsilon, \qquad t \in J. \tag{4.1}$$

**Definition 4.1.** The equation (1.1) is Ulam–Hyers stable if there exists $c > 0$ such that for each $\epsilon > 0$ and for each solution $y \in C(J,\mathbb{R})$ of the inequality (4.1) there exists a solution $x \in C(J, \mathbb{R})$ of the equation (1.1) with

$$|y(t) - x(t)| \leq c\epsilon, \qquad t \in J.$$

**Remark 4.2.** A function $y \in C(J, \mathbb{R})$ is a solution of the inequality (4.1) if and only if there exists a function $g \in C(J, \mathbb{R})$ (which depend on $y$) such that (i) $|g(t)| \le \epsilon,\ t \in J$, (ii) ${}^cD_t^\alpha y(t) - \lambda {}^cD_t^\beta y(t) = f(t, y(t)) + g(t),\ t \in J$.

Indeed, by Remark 4.2, the solution of the equation

$$ {}^cD_t^\alpha y(t) - \lambda {}^cD_t^\beta y(t) = f(t, y(t)) + g(t), \qquad t \in J $$

can be formulated by

$$ \begin{aligned} y(t) = {} & \big[\mathbb{E}_{\alpha-\beta}(\lambda t^{\alpha-\beta}) - \lambda t^{\alpha-\beta}\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda t^{\alpha-\beta})\big] y(0) \\ & + \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta}) f(s, y(s))\, ds \\ & + \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta}) g(s)\, ds, \qquad t \in J. \end{aligned} $$

Then we have the following estimation.

**Remark 4.3.** Let $y \in C(J, \mathbb{R})$ be a solution of the inequality (4.1). Then $y$ is a solution of the following integral inequality

$$ \begin{aligned} & \bigg| y(t) - \big[\mathbb{E}_{\alpha-\beta}(\lambda t^{\alpha-\beta}) - \lambda t^{\alpha-\beta}\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda t^{\alpha-\beta})\big] y(0) \\ & - \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta}) f(s, y(s))\, ds \bigg| \\ & \quad = \left| \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta}) g(s)\, ds \right| \\ & \quad \le \int_0^t \big|(t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})\big| \big|g(s)\big|\, ds \\ & \quad \le \epsilon \int_0^t \big|(t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})\big|\, ds \\ & \quad \le \epsilon\rho, \qquad t \in J, \end{aligned} \tag{4.2} $$

where we use Remark 4.2, Lemma 2.5 (i) and the fact

$$ \int_0^t \big|(t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})\big|\, ds \le \rho; $$

$\rho$ is defined in $(H_3)$.

Now we are ready to state our Ulam–Hyers stability result.

**Theorem 4.4.** *Assume that* $(H_1)$, $(H_2)$ *and* $(H_3)$ *are satisfied. Then the equation* (1.1) *is Ulam–Hyers stable on* $J$.

*Proof.* Let $y \in C(J, \mathbb{R})$ be a solution of the inequality (4.1). Denote by $x$ the unique solution of the Cauchy problem

$$ \begin{cases} {}^cD_t^\alpha x(t) - \lambda {}^cD_t^\beta x(t) = f(t, x(t)), & t \in J, \\ x(0) = y(0), \end{cases} $$

that is,

$$x(t) = \big[\mathbb{E}_{\alpha-\beta}(\lambda t^{\alpha-\beta}) - \lambda t^{\alpha-\beta}\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda t^{\alpha-\beta})\big]y(0) + \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})f(s,x(s))\,ds.$$

By using Lemma 2.5 (i) and (4.2), we have

$$\begin{aligned}
|y(t)-x(t)| &\le \Big| y(t) - \big[\mathbb{E}_{\alpha-\beta}(\lambda t^{\alpha-\beta}) - \lambda t^{\alpha-\beta}\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda t^{\alpha-\beta})\big]y(0) \\
&\qquad - \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})f(s,x(s))\,ds \Big| \\
&\le \Big| y(t) - \big[\mathbb{E}_{\alpha-\beta}(\lambda t^{\alpha-\beta}) - \lambda t^{\alpha-\beta}\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda t^{\alpha-\beta})\big]y(0) \\
&\qquad - \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})f(s,y(s))\,ds \Big| \\
&\qquad + \Big| \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})(f(s,y(s))-f(s,x(s)))\,ds \Big| \\
&\le \epsilon\rho + L\int_0^t \big|(t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})\big|\,ds\|x-y\|_\infty \\
&\le \epsilon\rho + L\rho\|x-y\|_\infty,
\end{aligned}$$

which yields that

$$\|x-y\|_\infty \le \epsilon\rho + L\rho\|x-y\|_\infty.$$

Thus,

$$(1-L\rho)\|x-y\|_\infty \le \epsilon\rho.$$

As a result,

$$|y(t)-x(t)| \le \frac{\epsilon\rho}{1-L\rho}, \qquad t\in J.$$

The proof is completed. □

### 4.2 Symmetrical results for $\lambda < 0$

Next, we apply the same method to investigate Ulam–Hyers stability of the equation (1.1) for the case $\lambda < 0$ on the time interval $J$.

By Remark 4.2 and Lemma 2.5 (ii), one can give a similar result according to Remark 4.3.

**Remark 4.5.** Let $y \in C(J,\mathbb{R})$ be a solution of the inequality (4.1). Then $y$ is a solution of the following integral inequality

$$\begin{aligned}
&\Big| y(t) - \big[\mathbb{E}_{\alpha-\beta}(\lambda t^{\alpha-\beta}) - \lambda t^{\alpha-\beta}\mathbb{E}_{\alpha-\beta,\alpha-\beta+1}(\lambda t^{\alpha-\beta})\big]y(0) \\
&\quad - \int_0^t (t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})f(s,y(s))\,ds \Big| \\
&\qquad \le \epsilon m(\alpha-\beta,\alpha,-\lambda)\int_0^t \left(\frac{1}{(t-s)^{\alpha-2\beta+1}} + \frac{1}{(t-s)^{-\beta+1}}\right)ds \\
&\qquad \le \epsilon\varrho, \qquad t\in J,
\end{aligned} \tag{4.3}$$

where $\varrho$ is defined in $(H_7)$.

Next, we give our second Ulam–Hyers stability result.

**Theorem 4.6.** *Assume that $(H_1)$, $(H_2)$ and $(H_7)$ are satisfied. Then the equation (1.1) is Ulam–Hyers stable on $J$.*

*Proof.* The process is very similar to the proof of Theorem 4.4. So we only present the main difference in the computation. By using Lemma 2.5 (ii) and (4.3), we have

$$\begin{aligned}|y(t)-x(t)| &\le \epsilon\varrho + L\int_0^t \left|(t-s)^{\alpha-1}\mathbb{E}_{\alpha-\beta,\alpha}(\lambda(t-s)^{\alpha-\beta})\right| ds\|x-y\|_\infty \\ &\le \epsilon\varrho + L\varrho\|x-y\|_\infty.\end{aligned}$$

This yields that

$$|y(t)-x(t)| \le \frac{\epsilon\varrho}{1-L\varrho}, \qquad t\in J.$$

The proof is completed. □

## 5 Examples

In this section, examples are given to illustrate our theoretical results.

**Example 5.1.** Let $\alpha=\frac{1}{2}$, $\beta=\frac{1}{3}$ and $\lambda=1$. We consider

$$\begin{cases} {}^cD_t^{\frac{1}{2}}x(t) - {}^cD_t^{\frac{1}{3}}x(t) = \frac{x(t)+1}{t^2+c}, & t\in[0,1], \qquad c>0,\\ x(0)=0. \end{cases} \tag{5.1}$$

Define $f(t,x(t))=\frac{x(t)+1}{t^2+c}$ for $t\in[0,1]$. Set $M=L=\frac{1}{c}$. Further, we choose $c=18m(\frac{1}{6},\frac{1}{2},1)+12\exp(1)$, where

$$m\left(\tfrac{1}{6},\tfrac{1}{2},1\right)=\max\left\{\tfrac{4}{\pi}\Gamma(\tfrac{5}{6}),\tfrac{2\sqrt{3}}{\pi}\Gamma(\tfrac{2}{3})\right\}.$$

Now $L\rho = L\left(9m(\frac{1}{6},\frac{1}{2},1)+6\exp(1)\right)=\frac{1}{2}<1$. Then $(H_1)$–$(H_3)$ are satisfied.

By Theorem 3.1, the equation (5.1) has a unique solution.

**Example 5.2.** Let $\alpha=\frac{1}{2}$, $\beta=\frac{1}{3}$ and $\lambda=1$. We consider

$$\begin{cases} {}^cD_t^{\frac{1}{2}}x(t) - {}^cD_t^{\frac{1}{3}}x(t) = \arctan\frac{|x(t)|}{1+|x(t)|}, & t\in[0,1],\\ x(0)=0. \end{cases} \tag{5.2}$$

Define $f(t,x(t))=\arctan\frac{|x(t)|}{1+|x(t)|}$ for $t\in[0,1]$. Then, $(H_1)$ holds and $|f(t,x(t))|\le 1$. Moreover,

$$\rho\liminf_{r\to\infty}\frac{1}{r}=\left(9m(\tfrac{1}{6},\tfrac{1}{2},1)+6\exp(1)\right)\liminf_{r\to\infty}\frac{1}{r}=0<1,$$

where

$$m(\tfrac{1}{6},\tfrac{1}{2},1)=\max\left\{\tfrac{4}{\pi}\Gamma(\tfrac{5}{6}),\tfrac{2\sqrt{3}}{\pi}\Gamma(\tfrac{2}{3})\right\},$$

which implies $(H_4)$ holds.

By Theorem 3.2, the equation (5.2) has at least one solution.

**Example 5.3.** Let $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$ and $\lambda = 1$. We consider

$$\begin{cases} {}^cD_t^{\frac{1}{2}}x(t) - {}^cD_t^{\frac{1}{3}}x(t) = \frac{|x(t)|}{(t+1)^2+c}, & t \in [0,1], \quad c > 0, \\ x(0) = 0. \end{cases} \tag{5.3}$$

Define $f(t, x(t)) = \frac{|x(t)|}{(t+1)^2+c}$ for $t \in [0,1]$. Then, $(H_1)$ holds and $\lim_{\|x\|_\infty \to \infty} \frac{f(t,x)}{x} = \frac{1}{(t+1)^2+c} := k(t)$. Set $k_{\sup} = \frac{1}{c}$. Further, we choose $c = 18m(\frac{1}{6}, \frac{1}{2}, 1) + 12\exp(1)$, where

$$m(\tfrac{1}{6}, \tfrac{1}{2}, 1) = \max\left\{\tfrac{4}{\pi}\Gamma(\tfrac{5}{6}), \tfrac{2\sqrt{3}}{\pi}\Gamma(\tfrac{2}{3})\right\}.$$

Then, $k_{\sup}\rho = k_{\sup}\left(9m(\frac{1}{6}, \frac{1}{2}, 1) + 6\exp(1)\right) = \frac{1}{2} < 1$. Now $(H_5)$–$(H_6)$ are satisfied.

By Theorem 3.3, the equation (5.3) has at least one solution.

**Example 5.4.** Let $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$ and $\lambda = 1$. We consider

$${}^cD_t^{\frac{1}{2}}x(t) -{}^c D_t^{\frac{1}{3}}x(t) = \frac{1}{l}\sin x(t), \qquad t \in [0,1], \qquad l > 0, \tag{5.4}$$

and the inequality

$$\left|{}^cD_t^{\frac{1}{2}}y(t) -{}^c D_t^{\frac{1}{3}}y(t) - \frac{1}{l}\sin y(t)\right| \le \varepsilon, \qquad t \in [0,1]. \tag{5.5}$$

Let $y \in C([0,1], \mathbb{R})$ be a solution of the inequality (5.5). Then there exists a function $g(t) = \varepsilon e^{t-1} \in C([0,1], \mathbb{R})$ such that $|g(t)| \le \varepsilon$, $t \in [0,1]$, and

$${}^cD_t^{\frac{1}{2}}y(t) -{}^c D_t^{\frac{1}{3}}y(t) = \frac{1}{l}\sin y(t) + g(t), \qquad t \in [0,1].$$

Define $f(t, x(t)) = \frac{1}{l}\sin x(t)$ for $t \in [0,1]$ and set $L = \frac{1}{l}$. Then $(H_1)$ and $(H_2)$ hold. Moreover, we choose $l = 18m(\frac{1}{6}, \frac{1}{2}, 1) + 12\exp(1)$, where

$$m(\tfrac{1}{6}, \tfrac{1}{2}, 1) = \max\left\{\tfrac{4}{\pi}\Gamma(\tfrac{5}{6}), \tfrac{2\sqrt{3}}{\pi}\Gamma(\tfrac{2}{3})\right\}.$$

Then

$$0 < 1 - \tfrac{1}{l}\left(9m(\tfrac{1}{6}, \tfrac{1}{2}, 1) + 6\exp(1)\right) = \tfrac{1}{2} < 1,$$

which implies that $(H_3)$ holds.

By Theorem 4.4, we have

$$|y(t) - x(t)| \le 2\left(9m(\tfrac{1}{6}, \tfrac{1}{2}, 1) + 6\exp(1)\right)\epsilon, \qquad t \in [0,1].$$

Thus, the equation (5.4) is Ulam–Hyers stable on $[0,1]$ with $c = 2\left(9m(\frac{1}{6}, \frac{1}{2}, 1) + 6\exp(1)\right)$.

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