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Landesman–Lazer condition revisited: the influence of vanishing and oscillating nonlinearities

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Abstract. In this paper we deal with semilinear problems at resonance. We present a sufficient condition for the existence of a weak solution in terms of asymptotic properties of nonlinearity. Our condition generalizes the classical Landesman–Lazer condition and it also covers the cases of vanishing and oscillating nonlinearities.

Keywords: resonance problem, semilinear equation, Landesman–Lazer condition, saddle point theorem, critical points.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, $g\colon \mathbb{R} \to \mathbb{R}$ be a bounded continuous function and $f \in L^2(\Omega)$. We consider the boundary value problem

$$\begin{aligned} -\Delta u - \lambda_k u + g(u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here λ_k, $k \geq 1$, is the k-th eigenvalue of the eigenvalue problem

$$\begin{aligned} -\Delta u - \lambda u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

By a *solution* of (1.1) we understand a function $u \in H := W_0^{1,2}(\Omega)$ satisfying (1.1) in the weak sense, *i.e.*,

$$\int_\Omega \nabla u \nabla v \, \mathrm{d}x - \lambda_k \int_\Omega uv \, \mathrm{d}x + \int_\Omega g(u)v \, \mathrm{d}x = \int_\Omega fv \, \mathrm{d}x \tag{1.3}$$

holds for any test function $v \in H$.

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Let $m \geq 1$ be a multiplicity of λ_k. We arrange the eigenvalues of (1.2) into the increasing sequence:

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_{k-1} < \lambda_k = \cdots = \lambda_{k+m-1} < \lambda_{k+m} \leq \lambda_{k+m+1} \leq \cdots \to \infty.$$

The corresponding eigenfunctions, (ϕ_n), form an orthogonal basis for both $L^2(\Omega)$ and H. We assume that every ϕ_n is normalized with respect to the L^2 norm, *i.e.*, $\|\phi_n\|_2 = 1$, $n = 1, 2, \ldots$. We use the scalar product $(u, v) = \int_\Omega \nabla u \nabla v \, \mathrm{d}x$ and the induced norm $\|u\| = \left(\int_\Omega |\nabla u|^2 \, \mathrm{d}x\right)^{1/2}$ on H. We split the space H into the following three subspaces spanned by the eigenfunctions of (1.2) as follows:

$$\hat{H} := [\phi_1, \ldots, \phi_{k-1}], \quad \bar{H} := [\phi_k, \ldots, \phi_{k+m-1}], \quad \tilde{H} := [\phi_{k+m}, \phi_{k+m+1}, \ldots].$$

Then $H = \hat{H} \oplus \bar{H} \oplus \tilde{H}$ with $\dim \hat{H} = k-1$, $\dim \bar{H} = m$, $\dim \tilde{H} = \infty$. Of course, if $k = 1$ then $m = 1$ (λ_1 is a simple eigenvalue) and $\hat{H} = \emptyset$. We split an element $u \in H$ as $u = \hat{u} + \bar{u} + \tilde{u}$, $\hat{u} \in \hat{H}, \bar{u} \in \bar{H}$ and $\tilde{u} \in \tilde{H}$. We split a function $f \in L^2(\Omega)$ as $f = \bar{f} + f^\perp$, where $\int_\Omega f^\perp v \, \mathrm{d}x = 0$ for any $v \in \bar{H}$.

The purpose of this paper is to introduce a rather general sufficient condition of the Landesman–Lazer type for the existence of a solution of (1.1).

If $(u_n) \subset H$ is a sequence such that $\|u_n\|_2 \to \infty$ and there exists $\phi_0 \in \bar{H}$, $\frac{u_n}{\|u_n\|_2} \to \phi_0$ in $L^2(\Omega)$, then

$$\lim_{n\to\infty} \left(\int_\Omega G(u_n) \, \mathrm{d}x - \int_\Omega \bar{f} u_n \, \mathrm{d}x \right) = \pm\infty. \tag{SC$_\pm$}$$

Here, $G(s) = \int_0^s g(\tau) \, \mathrm{d}\tau$ is the antiderivative of g.

Theorem 1.1. *Assume that either (SC)$_+$ or else (SC)$_-$ holds. Then the problem (1.1) has at least one solution.*

Remark 1.2. Note that the sufficient condition which is similar to (SC)$_+$ but more restrictive than (SC)$_+$ was introduced recently in [8] where the resonance problem with respect to the Fučík spectrum of the Laplacian was studied. In this paper, we benefit from the fact that the resonance occurs at the eigenvalue which allows us to split the underlying function space H into the sum of orthogonal subspaces. In contrast with [8], where such splitting is impossible, we can get rid of the $f^\perp$-part of the right-hand side f in (SC)$_\pm$. This makes our conditions more general and geometrically more transparent.

In the following we interpret our conditions (SC)$_\pm$ in historical context. We first consider a bounded continuous nonlinear function $g \colon \mathbb{R} \to \mathbb{R}$ with finite limits $g(\pm\infty) := \lim_{s\to\pm\infty} g(s)$.

Example 1.3. Let us assume that

$$\begin{aligned} g(\mp\infty) \int_\Omega \phi^+ \, \mathrm{d}x - g(\pm\infty) \int_\Omega \phi^- \, \mathrm{d}x &< \int_\Omega \bar{f} \phi \, \mathrm{d}x \\ &< g(\pm\infty) \int_\Omega \phi^+ \, \mathrm{d}x - g(\mp\infty) \int_\Omega \phi^- \, \mathrm{d}x \end{aligned} \tag{LL$_\pm$}$$

holds for all eigenfunctions ϕ associated with λ_k. This is the classical Landesman–Lazer condition (see [14]). Assume $\|u_n\|_2 \to \infty$ and $\frac{u_n}{\|u_n\|_2} \to \phi_0$ for some eigenfunction ϕ_0. Then by l'Hospital's rule we have

$$\lim_{n\to\infty} \frac{1}{\|u_n\|_2} \left(\int_\Omega G(u_n)\,\mathrm{d}x - \int_\Omega \bar{f} u_n\,\mathrm{d}x \right) = \lim_{n\to\infty} \int_\Omega \left(\frac{G(u_n)}{u_n} - \bar{f} \right) \frac{u_n}{\|u_n\|_2}\,\mathrm{d}x$$
$$= \int_\Omega \left(g(+\infty) + \bar{f}\right) \phi_0^+\,\mathrm{d}x - \int_\Omega \left(g(-\infty) + \bar{f}\right) \phi_0^-\,\mathrm{d}x.$$

The last expression is either positive or negative due to (LL)$_\pm$ and hence conditions (SC)$_\pm$ hold. In other words, we proved that (LL)$_\pm$ imply (SC)$_\pm$.

Assume, moreover, $g(-\infty) < 0 < g(+\infty)$ (think, for example, about $g(s) = \arctan s$). Then problem (1.1) has a solution for all f which belong to the "strip" (given by inequalities (LL)$_+$) around the linear subspace

$$L^2(\Omega)^\perp := \left\{ f \in L^2(\Omega) : \int_\Omega f\phi\,\mathrm{d}x = 0 \text{ for all } \phi \in \bar{H} \right\}$$

of $L^2(\Omega)$.

We note that the conditions (LL)$_\pm$ are empty if $g(-\infty) = g(+\infty)$. However, we prove existence results even in this case.

Example 1.4. It follows from Theorem 1.1 that the problem (1.1) with $g(s) = \frac{\operatorname{sgn} s}{(e+|s|)\ln(e+|s|)}$ (e is Euler's number) has at least one solution for $f \in L^2(\Omega)^\perp$. Indeed,

$$\lim_{|s|\to\infty} G(s) = \lim_{|s|\to\infty} \ln\left(\ln(e+|s|)\right) = \infty$$

implies that (SC)$_+$ holds true. Hence, our conditions (SC)$_\pm$ cover the case of *vanishing* nonlinearities $g(\pm\infty) = 0$ (see [7]). It should be emphasized, that in contrast with previous works on vanishing nonlinearities our approach does not require any kind of symmetry or sign condition about g (cf. [2–4,9,11,13]). At the same time, it generalizes the results from [10,12].

The verification of (SC)$_\pm$ does not require the existence of limits $g(\pm\infty)$ at all. See the following example.

Example 1.5. Consider $g(s) = \arctan s + c \cdot \cos s$ with an arbitrary constant $c \in \mathbb{R}$. An easy calculation yields that (SC)$_+$ is satisfied, and hence, according to Theorem 1.1, problem (1.1) has at least one solution for any $f \in L^2(\Omega)$ satisfying

$$\left| \int_\Omega f\phi\,\mathrm{d}x \right| < \frac{\pi}{2} \int_\Omega |\phi|\,\mathrm{d}x \tag{1.4}$$

for any $\phi \in \bar{H}$. On the other hand, the conditions (LL)$_\pm$ and various generalizations of (LL)$_\pm$ (see, e.g. [5,6]) do not apply if $|c| \geq \frac{\pi}{2}$, due to the fact that these conditions are vacuous in this case.

Remark 1.6. In fact, the above mentioned case $g(s) = \arctan s + c \cdot \cos s$ is covered by the so called potential Landesman–Lazer condition:

$$G^\mp \int_\Omega \phi^+\,\mathrm{d}x - G^\pm \int_\Omega \phi^-\,\mathrm{d}x < \int_\Omega \bar{f}\phi\,\mathrm{d}x < G^\pm \int_\Omega \phi^+\,\mathrm{d}x - G^\mp \int_\Omega \phi^-\,\mathrm{d}x \tag{PLL$_\pm$}$$

where $G^{\pm} := \lim_{s\to\pm\infty} \frac{G(s)}{s}$. Indeed, l'Hospital's rule implies $G^- = -\frac{\pi}{2}$, $G^+ = \frac{\pi}{2}$ and the condition $(PLL)_+$ reduces to (1.4). For the use of $(PLL)_\pm$ see, e.g. the papers [1, 18–22]. The conditions $(PLL)_\pm$ eliminate the influence of the bounded *oscillating* term $c \cdot \cos s$ which disappears "in an average" as $|s| \to \infty$.

However, the conditions $(PLL)_\pm$ do not cover the case like $g(s) = \frac{s}{1+s^2} + c \cdot \cos s$, where $c \in \mathbb{R}$ is an arbitrary constant. Indeed, both conditions are empty due to the fact $G^{\pm} = 0$. On the other hand, it follows from our Theorem 1.1 that (1.1) with g given above has a solution for any $f \in L^2(\Omega)^{\perp}$. This fact illustrates that our conditions $(SC)_\pm$ refine also the conditions $(PLL)_\pm$ and complement the results from [15] and [16].

In the following example we treat rather general nonlinearity.

Example 1.7. It follows from our Theorem 1.1 that the boundary value problem

$$\begin{aligned} -\Delta u - \lambda_k u + \frac{u}{(e+u^2)\ln(e+u^2)^{1/2}} + c \cdot \cos u = f \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned} \tag{1.5}$$

has a solution for arbitrary $c \in \mathbb{R}$ and for any $f \in L^2(\Omega)$ satisfying

$$\int_\Omega f\phi \, dx = 0$$

for all $\phi \in \bar{H}$. Indeed, since

$$\lim_{|s|\to\infty} G(s) = \lim_{|s|\to\infty} \left[\ln\left(\ln(e+s^2)^{1/2}\right) + c \cdot \sin s\right] = \infty$$

the condition $(SC)_+$ is satisfied. The existence result for problems of type (1.5) does not follow from any work published in the literature so far.

Examples and remarks presented above justify the novelty of our work and show that our conditions $(SC)_\pm$ are new and recover previously published results.

2 Preliminaries

In this section we stress some helpful facts used in the proof of Theorem 1.1.

Lemma 2.1. *There exist $c_1 > 0$, $c_2 > 0$ such that for any $u \in H$ we have*

$$\int_\Omega |\nabla \hat{u}|^2 \, dx - \lambda_k \int_\Omega (\hat{u})^2 \, dx \le -c_1 \|\hat{u}\|^2 \tag{2.1}$$

and

$$\left| \int_\Omega g(u)\hat{u} \, dx - \int_\Omega f\hat{u} \, dx \right| \le c_2 \|\hat{u}\|. \tag{2.2}$$

Proof. The inequality (2.1) follows from the variational characterization of λ_k, (2.2) follows from the Hölder inequality, the boundedness of g and the fact $f \in L^2(\Omega)$. □

Lemma 2.2. *There exist $c_3 > 0$, $c_4 > 0$ such that for any $u \in H$ we have*

$$\int_\Omega |\nabla \tilde{u}|^2 \,\mathrm{d}x - \lambda_k \int_\Omega (\tilde{u})^2 \,\mathrm{d}x \geq c_3 \|\tilde{u}\|^2 \tag{2.3}$$

and

$$\left| \int_\Omega g(u)\tilde{u} \,\mathrm{d}x - \int_\Omega f\tilde{u} \,\mathrm{d}x \right| \leq c_4 \|\tilde{u}\|. \tag{2.4}$$

Proof. The inequality (2.3) is also a consequence of the variational characterization of λ_k, and (2.4) follows similarly as (2.2). □

Lemma 2.3. *There exist $c_5 > 0$ such that for any $u \in H$ we have*

$$\left| \int_\Omega G(u) \,\mathrm{d}x - \int_\Omega fu \,\mathrm{d}x \right| \leq c_5 \|u\|_2. \tag{2.5}$$

Proof. The inequality (2.5) follows from the Hölder inequality, the boundedness of g and the fact $f \in L^2(\Omega)$. □

3 Proof of Theorem 1.1

We define the *energy functional* associated with (1.1), $\mathcal{E}\colon H \to \mathbb{R}$, by

$$\mathcal{E}(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \,\mathrm{d}x - \frac{\lambda_k}{2} \int_\Omega (u)^2 \,\mathrm{d}x + \int_\Omega G(u) \,\mathrm{d}x - \int_\Omega fu \,\mathrm{d}x,$$

$u \in H$. Obviously, all critical points of $\mathcal{E}$ satisfy (1.3) and vice versa.

We apply the Saddle Point Theorem due to P. Rabinowitz [17] to prove the existence of a critical point of $\mathcal{E}$.

Theorem 3.1. *Let $\mathcal{E} \in C^1(H, \mathbb{R})$ and $H = H^- \oplus H^+$, $\dim H^- < \infty$, $\dim H^+ = \infty$. Assume that*

(a) *there exist a bounded neighborhood D of o in H^- and a constant $\alpha \in \mathbb{R}$ such that $\mathcal{E}\big|_{\partial D} \leq \alpha$;*

(b) *there exists a constant $\beta > \alpha$ such that $\mathcal{E}\big|_{H^+} \geq \beta$;*

(c) *$\mathcal{E}$ satisfies (PS) condition, i.e., if $(\mathcal{E}(u_n)) \subset \mathbb{R}$ is a bounded sequence and $\nabla\mathcal{E}(u_n) \to o$ in H, then there exist a subsequence $(u_{n_k}) \subset (u_n)$ and an element $u \in H$ such that $u_{n_k} \to u$ in H.*

Then the functional $\mathcal{E}$ has a critical point in H.

At first we verify the Palais–Smale condition.

Lemma 3.2. *Let us assume $(\mathrm{SC})_\pm$. Then $\mathcal{E}$ satisfies (PS) condition.*

Proof. In the first step we prove that (u_n) is bounded in $L^2(\Omega)$. Assume the contrary, *i.e.*, $\|u_n\|_2 \to \infty$. Set $v_n := \frac{u_n}{\|u_n\|_2}$. Then

$$\frac{\mathcal{E}(u_n)}{\|u_n\|_2^2} := \frac{1}{2} \int_\Omega |\nabla v_n|^2 \,\mathrm{d}x - \frac{\lambda_k}{2} \int_\Omega (v_n)^2 \,\mathrm{d}x + \int_\Omega \frac{G(u_n)}{\|u_n\|_2^2} \,\mathrm{d}x - \frac{1}{\|u_n\|_2} \int_\Omega fv_n \,\mathrm{d}x \to 0. \tag{3.1}$$

The second term is equal to $-\frac{\lambda_k}{2}$ since $\|v_n\|_2 = 1$, the last two terms go to zero due to Lemma 2.3.

Then it follows from (3.1) that (v_n) is a bounded sequence in H. Passing to a subsequence, if necessary, we may assume that there exists $v \in H$ such that $v_n \rightharpoonup v$ (weakly) in H and $v_n \to v$ in $L^2(\Omega)$.

For arbitrary $w \in H$,

$$\begin{aligned} 0 \leftarrow \frac{(\nabla \mathcal{E}'(u_n), w)}{\|u_n\|_2} &= \int_\Omega \nabla v_n \nabla w \,\mathrm{d}x - \lambda_k \int_\Omega v_n w \,\mathrm{d}x \\ &\quad + \frac{1}{\|u_n\|_2} \int_\Omega g(u_n) w \,\mathrm{d}x - \frac{1}{\|u_n\|_2} \int_\Omega f w \,\mathrm{d}x. \end{aligned} \tag{3.2}$$

We have $\int_\Omega \nabla v_n \nabla w \,\mathrm{d}x \to \int_\Omega \nabla v \nabla w \,\mathrm{d}x$ by $v_n \rightharpoonup v$ in H, $\int_\Omega v_n w \,\mathrm{d}x \to \int_\Omega v w \,\mathrm{d}x$ by $v_n \to v$ in $L^2(\Omega)$, $\frac{1}{\|u_n\|_2} \int_\Omega f w \,\mathrm{d}x \to 0$ and $\frac{1}{\|u_n\|_2} \int_\Omega g(u_n) w \,\mathrm{d}x \to 0$ by $f \in L^2(\Omega)$, the boundedness of g and by our assumption $\|u_n\|_2 \to \infty$. Then it follows from (3.2) that

$$\int_\Omega \nabla v \nabla w \,\mathrm{d}x - \lambda_k \int_\Omega v w \,\mathrm{d}x = 0$$

holds for arbitrary $w \in H$, *i.e.*, $v = \phi_0 \in \bar{H}$ is an eigenfunction associated with λ_k. That is, $\frac{u_n}{\|u_n\|_2} \to \phi_0$ in $L^2(\Omega)$.

Now, by the assumption $\nabla \mathcal{E}(u_n) \to o$ and the orthogonal decomposition of H, we have

$$o(\|\hat{u}_n\|) = (\nabla \mathcal{E}(u_n), \hat{u}_n) = \int_\Omega |\nabla \hat{u}_n|^2 \,\mathrm{d}x - \lambda_k \int_\Omega (\hat{u}_n)^2 \,\mathrm{d}x + \int_\Omega g(u_n) \hat{u}_n \,\mathrm{d}x - \int_\Omega f \hat{u}_n \,\mathrm{d}x. \tag{3.3}$$

By Lemma 2.1, it follows from (3.3) that

$$o(1) \leq -c_1 \|\hat{u}_n\| + c_2$$

with $c_1, c_2 > 0$ independent of n. Hence $\|\hat{u}_n\|$ is a bounded sequence.

Similarly, we also have

$$o(\|\tilde{u}_n\|) = (\nabla \mathcal{E}(u_n), \tilde{u}_n) = \int_\Omega |\nabla \tilde{u}_n|^2 \,\mathrm{d}x - \lambda_k \int_\Omega (\tilde{u}_n)^2 \,\mathrm{d}x + \int_\Omega g(u_n) \tilde{u}_n \,\mathrm{d}x - \int_\Omega f \tilde{u}_n \,\mathrm{d}x. \tag{3.4}$$

By Lemma 2.2, it follows from (3.4) that

$$o(1) \geq c_3 \|\tilde{u}_n\| - c_4$$

with $c_3, c_4 > 0$ independent of n. Hence $\|\tilde{u}_n\|$ is a bounded sequence. Let us split now $\mathcal{E}(u_n)$ as follows

$$\begin{aligned} \mathcal{E}(u_n) = &\underbrace{\frac{1}{2} \int_\Omega |\nabla \hat{u}_n|^2 \,\mathrm{d}x - \frac{\lambda_k}{2} \int_\Omega (\hat{u}_n)^2 \,\mathrm{d}x}_{A} + \underbrace{\frac{1}{2} \int_\Omega |\nabla \tilde{u}_n|^2 \,\mathrm{d}x - \frac{\lambda_k}{2} \int_\Omega (\tilde{u}_n)^2 \,\mathrm{d}x}_{B} \\ &+ \underbrace{\int_\Omega G(u_n) \,\mathrm{d}x - \int_\Omega \bar{f} u_n \,\mathrm{d}x}_{C} - \underbrace{\int_\Omega f^\perp \hat{u}_n \,\mathrm{d}x - \int_\Omega f^\perp \tilde{u}_n \,\mathrm{d}x}_{D}. \end{aligned}$$

The boundedness of $\|\hat{u}_n\|$ and $\|\tilde{u}_n\|$ implies that A, B and D are bounded terms. On the other hand, $(SC)_+$ forces $C \to +\infty$ and $(SC)_-$ forces $C \to -\infty$. In particular, we conclude $\mathcal{E}(u_n) \to \pm\infty$ which contradicts the assumption of the boundedness of $(\mathcal{E}(u_n))$. We thus proved that (u_n) is a bounded sequence in $L^2(\Omega)$.

In the second step we select a strongly convergent subsequence (in H) from (u_n). Let us examine again the terms in

$$\mathcal{E}(u_n) := \frac{1}{2}\int_\Omega |\nabla u_n|^2 \,\mathrm{d}x - \frac{\lambda_k}{2}\int_\Omega (u_n)^2 \,\mathrm{d}x + \int_\Omega G(u_n)\,\mathrm{d}x - \int_\Omega f u_n \,\mathrm{d}x.$$

By the assumption $(\mathcal{E}(u_n))$ is bounded. The boundedness of the sequence (u_n) in $L^2(\Omega)$ implies that $\int_\Omega (u_n)^2 \,\mathrm{d}x$, $\int_\Omega G(u_n)\,\mathrm{d}x$ and $\int_\Omega f u_n \,\mathrm{d}x$ are bounded independently of n, as well. Therefore, $\|u_n\|^2 = \int_\Omega |\nabla u_n|^2 \,\mathrm{d}x$ must be also bounded. Hence, we may assume, without lost of generality, that $u_n \rightharpoonup u$ in H for some $u \in H$, and $u_n \to u$ in $L^2(\Omega)$. Then

$$\begin{aligned} 0 \leftarrow (\nabla\mathcal{E}(u_n), u_n - u) = &\int_\Omega \nabla u_n \nabla(u_n - u)\,\mathrm{d}x - \lambda_k \int_\Omega u_n(u_n - u)\,\mathrm{d}x \\ &+ \int_\Omega g(u_n)(u_n - u)\,\mathrm{d}x - \int_\Omega f(u_n - u)\,\mathrm{d}x. \end{aligned}$$

Since

$$-\lambda_k \int_\Omega u_n(u_n - u)\,\mathrm{d}x + \int_\Omega g(u_n)(u_n - u)\,\mathrm{d}x - \int_\Omega f(u_n - u)\,\mathrm{d}x \to 0,$$

we conclude that

$$\int_\Omega \nabla u_n \nabla(u_n - u)\,\mathrm{d}x \to 0,$$

as well. So,

$$\int_\Omega |\nabla u_n|^2 \,\mathrm{d}x - \int_\Omega \nabla u_n \nabla u \,\mathrm{d}x \to 0$$

which together with

$$\int_\Omega \nabla u_n \nabla u \,\mathrm{d}x - \|u_n\|^2 \to 0$$

(due to the weak convergence $u_n \rightharpoonup u$) yields

$$\|u_n\| \to \|u\|.$$

The uniform convexity of H then implies that $u_n \to u$ in H. Hence $\mathcal{E}$ satisfies the condition (c) in Theorem 3.1. □

Now we prove that also the hypotheses (a) and (b) hold. To this end we have to consider separately the case $(SC)_+$ and $(SC)_-$.

1. Let us assume that $(SC)_+$ holds. We set

$$H^- := \hat{H}, \quad H^+ := \bar{H} \oplus \tilde{H}.$$

It follows from Lemmas 2.1 and 2.3 that

$$\lim_{\|\hat{u}\|\to\infty} \mathcal{E}(\hat{u}) := \lim_{\|\hat{u}\|\to\infty} \left[\frac{1}{2}\int_\Omega |\nabla\hat{u}|^2 \,\mathrm{d}x - \frac{\lambda_k}{2}\int_\Omega (\hat{u})^2 \,\mathrm{d}x + \int_\Omega G(\hat{u})\,\mathrm{d}x - \int_\Omega f\hat{u}\,\mathrm{d}x\right] = -\infty \tag{3.5}$$

for $\hat{u} \in \hat{H}$.

On the other hand, we prove that there exists $\beta \in \mathbb{R}$ such that

$$\inf_{u \in H^+} \mathcal{E}(u) \geq \beta.$$

Assume the contrary, *i.e.*, there exists a sequence $(u_n) \subset H^+$ such that

$$\lim_{n\to\infty} \mathcal{E}(u_n) = -\infty. \tag{3.6}$$

Then $\|u_n\|_2 \to \infty$, and for $v_n := \frac{u_n}{\|u_n\|_2}$ ($v_n \in H^+$) we have

$$\begin{aligned} 0 \geq \limsup_{n\to\infty} \frac{\mathcal{E}(u_n)}{\|u_n\|_2^2} := \limsup_{n\to\infty} \Bigg[\frac{1}{2} \int_\Omega |\nabla v_n|^2 \, dx - \frac{\lambda_k}{2} \int_\Omega (v_n)^2 \, dx \\ + \int_\Omega \frac{G(u_n)}{\|u_n\|_2^2} \, dx - \int_\Omega f \frac{v_n}{\|u_n\|_2} \, dx \Bigg]. \end{aligned} \tag{3.7}$$

Clearly, by Lemma 2.3, we have

$$\int_\Omega \frac{G(u_n)}{\|u_n\|_2^2} \, dx - \int_\Omega f \frac{v_n}{\|u_n\|_2} \, dx \to 0. \tag{3.8}$$

It follows from (3.7) and (3.8) that $\|v_n\|$ is a bounded sequence. Passing to a subsequence, if necessary, we may assume that there exists $v \in H^+$ such that $v_n \rightharpoonup v$ in H and $v_n \to v$ in $L^2(\Omega)$. Moreover,

$$\liminf_{n\to\infty} \int_\Omega |\nabla v_n|^2 \, dx \geq \int_\Omega |\nabla v|^2 \, dx \tag{3.9}$$

by the weak lower semicontinuity of the norm in H. We deduce from (3.7)–(3.9) that

$$\int_\Omega |\nabla v|^2 \, dx - \lambda_k \int_\Omega (v)^2 \, dx \leq 0,$$

and hence, from Lemma 2.2, it follows that $v = \phi_0 \in \bar{H}$ is an eigenfunction associated with λ_k. That is,

$$\frac{u_n}{\|u_n\|_2} \to \phi_0 \quad \text{in } L^2(\Omega).$$

By Lemma 2.2, by the properties of the orthogonal decomposition of H^+ and f and by the condition $(\mathrm{SC})_+$, we have for $u_n \in H^+$

$$\begin{aligned} \lim_{n\to\infty} \mathcal{E}(u_n) &:= \lim_{n\to\infty} \left[\frac{1}{2} \int_\Omega |\nabla u_n|^2 \, dx - \frac{\lambda_k}{2} \int_\Omega (u_n)^2 \, dx + \int_\Omega G(u_n) \, dx - \int_\Omega f u_n \, dx \right] \\ &= \lim_{n\to\infty} \left[\frac{1}{2} \int_\Omega |\nabla \tilde{u}_n|^2 \, dx - \frac{\lambda_k}{2} \int_\Omega (\tilde{u}_n)^2 \, dx + \int_\Omega G(u_n) \, dx - \int_\Omega \bar{f} u_n \, dx - \int_\Omega f^\perp \tilde{u}_n \, dx \right] \\ &\geq \lim_{n\to\infty} \left[c_3 \|\tilde{u}_n\|^2 - \|f^\perp\|_2 \|\tilde{u}_n\|_2 \right] + \lim_{n\to\infty} \left[\int_\Omega G(u_n) \, dx - \int_\Omega \bar{f} u_n \, dx \right] \\ &= +\infty. \end{aligned}$$

This contradicts (3.6). By (3.5) there exists $R > 0$ such that for $D := \{u \in H^- : \|u\| \le R\}$ the following inequality holds

$$\sup_{u \in \partial D} \mathcal{E}(u) < \alpha := \beta - 1.$$

Hence, we proved that the hypotheses (a) and (b) in Theorem 3.1 hold.

2. Let us assume that $(SC)_-$ holds. In this case we set

$$H^- := \hat{H} \oplus \bar{H}, \quad H^+ := \tilde{H}.$$

Let $u \in H^+$. Then by Lemmas 2.2 and 2.3, we have

$$\begin{aligned}\mathcal{E}(u) &:= \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{\lambda_k}{2}\int_\Omega (u)^2 \, dx + \int_\Omega G(u)\, dx - \int_\Omega f u \, dx \\ &\ge c_3\|u\|^2 - c_5\|u\|_2 \ge c_3\|u\|^2 - c_6\|u\|.\end{aligned}$$

Hence, there exists $\beta \in \mathbb{R}$ such that $\mathcal{E}(u) \ge \beta$ for all $u \in H^+$.

On the other hand, we prove that

$$\lim_{\|u\| \to \infty, u \in H^-} \mathcal{E}(u) = -\infty. \tag{3.10}$$

Notice, that $\dim H^- < \infty$ implies that the norms $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent on H^-. Assume, by contradiction, that (3.10) does not hold, *i.e.*, there exist a sequence $(u_n) \subset H^-$ and a constant $c \in \mathbb{R}$ such that $\|u_n\|_2 \to \infty$ and

$$\mathcal{E}(u_n) \ge c. \tag{3.11}$$

Set $v_n := \frac{u_n}{\|u_n\|_2}$. Due to $\dim H^- < \infty$ we may assume that there exists $v \in H^-$ such that $v_n \to v$ both in H and $L^2(\Omega)$. Then

$$\begin{aligned}0 &\le \liminf_{n\to\infty} \frac{\mathcal{E}(u_n)}{\|u_n\|_2^2} \\ &= \liminf_{n\to\infty}\left[\frac{1}{2}\int_\Omega |\nabla v_n|^2\, dx - \frac{\lambda_k}{2}\int_\Omega (v_n)^2\, dx + \int_\Omega \frac{G(u_n)}{\|u_n\|_2^2}\, dx - \int_\Omega f\frac{v_n}{\|u_n\|_2}\, dx\right] \\ &= \frac{1}{2}\int_\Omega |\nabla v|^2\, dx - \frac{\lambda_k}{2}\int_\Omega (v)^2\, dx,\end{aligned} \tag{3.12}$$

by Lemma 2.3. According to Lemma 2.1, (3.12) implies $v = \phi_0 \in \bar{H}$, an eigenfunction associated with λ_k. Hence $\frac{u_n}{\|u_n\|_2} \to \phi_0$ in $L^2(\Omega)$. It follows from the orthogonal decomposition of H^- and f, Lemma 2.1 and $(SC)_-$ that for $u_n \in H^-$

$$\begin{aligned}\lim_{n\to\infty} \mathcal{E}(u_n) &:= \lim_{n\to\infty}\left[\frac{1}{2}\int_\Omega |\nabla u_n|^2\, dx - \frac{\lambda_k}{2}\int_\Omega (u_n)^2\, dx + \int_\Omega G(u_n)\, dx - \int_\Omega f u_n\, dx\right] \\ &= \lim_{n\to\infty}\left[\frac{1}{2}\int_\Omega |\nabla \hat{u}_n|^2\, dx - \frac{\lambda_k}{2}\int_\Omega (\hat{u}_n)^2\, dx + \int_\Omega G(u_n)\, dx - \int_\Omega \bar{f} u_n\, dx - \int_\Omega f^\perp \hat{u}_n\, dx\right] \\ &\le \lim_{n\to\infty}\left[-c_1\|\hat{u}_n\|^2 + c_2\|\hat{u}_n\|\right] + \lim_{n\to\infty}\left[\int_\Omega G(u_n)\, dx - \int_\Omega \bar{f} u_n\, dx\right] \\ &= -\infty.\end{aligned}$$

This contradicts (3.11), *i.e.*, (3.10) holds true. Let us choose again $D := \{u \in H^- : \|u\| \le R\}$. Then, for $R > 0$ large enough, we have

$$\sup_{u \in \partial D} \mathcal{E}(u) < \alpha := \beta - 1.$$

Thus, the hypotheses (a) and (b) in Theorem 3.1 are satisfied.

Recall that the hypothesis (c) in Theorem 3.1 is proved in Lemma 3.2 for both cases $(\mathrm{SC})_\pm$. It then follows from Theorem 3.1 that under the assumptions $(\mathrm{SC})_\pm$ there exists a critical point of $\mathcal{E}$. Since this is also a solution of (1.1), the proof of Theorem 1.1 is finished.

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