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Global asymptotic stability of pseudo almost periodic solutions to a Lasota–Wazewska model with distributed delays

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Abstract. In this paper, we study a class of Lasota–Wazewska model with distributed delays, new criteria for the existence and global asymptotic stability of positive pseudo almost periodic solutions are established by using the fixed point method and the properties of pseudo almost periodic functions, together with constructing a suitable Lyapunov function. Finally, we present an example with simulations to support the theoretical results. The obtained results are essentially new and they extend previously known results.

Keywords: Lasota–Wazewska model, pseudo almost periodic solution, global asymptotic stability, distributed delay.

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1 Introduction

To describe the survival of red blood cells in an animal, Ważewska-Czyżewska and Lasota in [20] proposed the following autonomous nonlinear delay differential equation as their appropriate model:

$$x'(t) = -ax(t) + be^{-cx(t-\tau)}, \qquad t \geq 0, \tag{1.1}$$

where $x(t)$ denotes the number of red blood cells at time t, $a > 0$ is the probability of death of a red blood cell, b and c are positive constants related to the production of red blood cells per unit time, and τ is the time required to produce a red blood cell. As a classical model of population dynamics, model (1.1) and its modifications have received great attention from both theoretical and mathematical biologists, and have been well studied. In particular, qualitative analysis such as periodicity, almost periodicity and stability of solutions of nonautonomous Lasota–Wazewska models have been studied extensively by many authors, we refer to [5, 7, 9, 11, 13, 15–19] and the references therein.

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Since the nature is full of all kinds of tiny perturbations, either the periodicity assumption or the almost periodicity assumption is just approximation of some degree of the natural perturbations [21, 25]. A well-known extension of almost periodicity is the pseudo almost periodicity, which was introduced by C. Zhang in [24, 25] and has been widely applied in the theory of ODEs and PDEs, see [2–4, 10, 14] and the references therein. In addition, it is well-known that time delays often occur in realistic biological systems, which can make the dynamic behaviors of the biological model become more complex, and may destabilize the stable equilibria and admit almost periodic oscillation, pseudo almost periodic motion, bifurcation and chaos, compared with the effects of discrete delays, distributed delays are more general and difficult to handle. Therefore, it is important and interesting to study the almost periodic dynamic behaviors of the Lasota–Wazewska model with distributed delays.

Motivated by the above discussions, in this paper, we will consider the following Lasota–Wazewska model with pseudo almost periodic coefficients and distributed delays:

$$x'(t) = -a(t)x(t) + \sum_{j=1}^{n} b_j(t) \int_0^\infty K_j(s)e^{-c_j(t)x(t-s)}\,ds, \qquad t \geq 0, \tag{1.2}$$

where $K_j(\cdot)\colon [0,\infty) \to [0,\infty)$, $j = 1, 2, \ldots, n$, is the probability kernel of the distributed delays, the other variables and parameters have the same biological meanings as those in (1.1) with the difference that they are now time-dependent.

The main purpose of this paper is employing fixed point method and the properties of pseudo almost periodic functions, together with constructing a suitable Lyapunov functional, to establish some sufficient conditions for the existence and global asymptotic stability of a pseudo almost periodic solution for model (1.2). The results obtained in the present paper are completely new and they extend previously known results in the literature.

The structure of this paper is as follows. In Section 2, we give some preliminaries related to our main results. In Section 3, we present the main results on the dynamic behaviors for model (1.2). Section 4 gives an example with simulations to demonstrate the effectiveness of the theoretical results.

Notations: Let $BC(\mathbb{R}, \mathbb{R})$ denote the set of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$, $\|\cdot\|$ denote the supremum norm $\|g\| := \sup_{t\in\mathbb{R}} |g(t)|$, obviously, $(BC(\mathbb{R}, \mathbb{R}), \|\cdot\|)$ is a Banach space. We generally denote $BC^* = BC((-\infty, 0], \mathbb{R})$, $BC^*_+ = BC((-\infty, 0], \mathbb{R}_+)$ and $\mathbb{R}_+ = [0, \infty)$, and define $x_t \in BC^*$ as $x_t(\theta) = x(t + \theta)$, $\theta \in (-\infty, 0]$.

Finally, given a function $g \in BC(\mathbb{R}, \mathbb{R})$, let g^+ and g^- be defined as

$$g^+ = \sup_{t\in\mathbb{R}} g(t), \qquad g^- = \inf_{t\in\mathbb{R}} g(t).$$

2 Preliminaries

According to the biological interpretation of model (1.2), only positive solutions are meaningful and therefore admissible. Consequently, the following initial conditions are given by

$$x_{t_0} = \varphi, \qquad \varphi \in BC^*_+ \quad \text{and} \quad \varphi(0) > 0. \tag{2.1}$$

Denote $x_t(t_0, \varphi)(x(t; t_0, \varphi))$ for a solution of the admissible initial value problem (1.2) and (2.1) with $x_{t_0}(t_0, \varphi) = \varphi \in BC^*_+$ and $t_0 \in \mathbb{R}$. Moreover, let $[t_0, \eta(\varphi))$ be the maximal right-interval of existence of $x_t(t_0, \varphi)$.

Let us recall some definitions and notations about almost periodicity and pseudo almost periodicity. For more details, we refer the reader to [6, 22].

Definition 2.1 (see [6]). Let $f(t) \in BC(\mathbb{R},\mathbb{R})$. The function $f(t)$ is said to be almost periodic on $\mathbb{R}$ if, for any $\varepsilon > 0$, the set $T(f,\varepsilon) = \{\varsigma : |f(t+\varsigma) - f(t)| < \varepsilon, \text{ for all } t \in \mathbb{R}\}$ is relatively dense, i.e., for any $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$, for any interval with length $l(\varepsilon)$, there exists a number $\varsigma = \varsigma(\varepsilon)$ in this interval such that $|f(t+\varsigma) - f(t)| < \varepsilon$, for all $t \in \mathbb{R}$. We denote by $AP(\mathbb{R},\mathbb{R})$ the set of all such functions.

In this paper, we denote

$$PAP_0(\mathbb{R},\mathbb{R}) = \left\{ f(t) \in BC(\mathbb{R},\mathbb{R}) : \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(t)|\, dt = 0 \right\}.$$

Definition 2.2 (see [22]). A function $f(t) \in BC(\mathbb{R},\mathbb{R})$ is called pseudo almost periodic if it can be expressed as

$$f(t) = f_1(t) + f_2(t),$$

where $f_1(t) \in AP(\mathbb{R},\mathbb{R})$ and $f_2(t) \in PAP_0(\mathbb{R},\mathbb{R})$. The collection of such functions will be denoted by $PAP(\mathbb{R},\mathbb{R})$.

Remark 2.3. The functions f_1 and f_2 in Definition 2.2 are, respectively, called the almost periodic component and the ergodic perturbation of the pseudo almost periodic function f. Moreover, the decomposition given in Definition 2.2 is unique.

Remark 2.4. Notice that $(PAP(\mathbb{R},\mathbb{R}), \|\cdot\|)$ is a Banach space and $AP(\mathbb{R},\mathbb{R})$ is a proper subspace of $PAP(\mathbb{R},\mathbb{R})$, for example, let $f(t) = \cos\sqrt{3}t + \sin \pi t + \frac{1}{1+t^2}$, one can easily see that $f(t) \in PAP(\mathbb{R},\mathbb{R})$, however, $f(t) \notin AP(\mathbb{R},\mathbb{R})$.

Lemma 2.5 (see [4,22]). *If $f(t) \in PAP(\mathbb{R},\mathbb{R}), h(t) \in PAP(\mathbb{R},\mathbb{R})$, then $f(t) \times h(t) \in PAP(\mathbb{R},\mathbb{R})$.*

Definition 2.6 (see [6,23]). Let $x \in \mathbb{R}$ and $Q(t)$ be a continuous function defined on $\mathbb{R}$. The linear equation

$$x'(t) = Q(t)x(t) \tag{2.2}$$

is said to admit an exponential dichotomy on $\mathbb{R}$ if there exist positive constants k_i, α_i, $i = 1,2$, projection P and the fundamental solution $X(t)$ of (2.2) satisfying

$$\begin{aligned} |X(t)PX^{-1}(s)| &\le k_1 e^{-\alpha_1(t-s)}, \quad \text{for } t \ge s, \\ |X(t)(1-P)X^{-1}(s)| &\le k_2 e^{-\alpha_2(s-t)}, \quad \text{for } t \le s. \end{aligned}$$

Lemma 2.7 (see [23]). *Assume that $Q(t)$ is an almost periodic function and $g(t) \in PAP(\mathbb{R},\mathbb{R})$. If the linear equation (2.2) admits an exponential dichotomy, then pseudo almost periodic equation*

$$x'(t) = Q(t)x(t) + g(t)$$

has a unique pseudo almost periodic solution $x(t)$, and

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)g(s)\, ds - \int_{t}^{\infty} X(t)(1-P)X^{-1}(s)g(s)\, ds.$$

Lemma 2.8 (see [6]). *Let $\delta(t)$ be an almost periodic function on $\mathbb{R}$ and*

$$M[\delta] = \lim_{t\to\infty} \frac{1}{T}\int_{t}^{t+T} \delta(s)\, ds > 0.$$

Then the linear equation

$$x'(t) = -\delta(t)x(t)$$

admits an exponential dichotomy on $\mathbb{R}$.

The following lemma is from [1] and will be employed in establishing the asymptotic stability of model (1.2).

Lemma 2.9. *Let l be a real number and f be a non-negative function defined on $[l,\infty)$ such that f is integrable on $[l,\infty)$ and is uniformly continuous on $[l,\infty)$. Then $\lim_{t\to\infty} f(t) = 0$.*

To obtain our main results, throughout this paper, we also need the following assumptions for model (1.2):

(A_1) $a(t) \in AP(\mathbb{R},(0,\infty))$, and $b_j(t), c_j(t) \in PAP(\mathbb{R},(0,\infty))$, where $j = 1,2,\ldots,n$.

(A_2) $K_j(s)\colon [0,\infty) \to [0,\infty)$ is a piecewise continuous function such that

$$\int_0^\infty K_j(s)\,ds < \infty, \qquad j = 1,2,\ldots,n.$$

(A_3)

$$a^- > 0, \qquad \sum_{j=1}^n b_j^+ \int_0^\infty K_j(s)\,ds > 0.$$

3 Main results

In this section, the main results of this paper are stated as follows. For convenience, we divide this part into two subsections.

3.1 Existence of pseudo almost periodic solution

Lemma 3.1. *Suppose assumptions (A_1)–(A_3) hold, and $BC_0 = \big\{\varphi \mid \varphi \in BC^*,\ S_2 < \varphi(t) < S_1$, for all $t \in (-\infty,0]\big\}$. Then, for any $\varphi \in BC_0$, all solutions $x(t;t_0,\varphi)$ of model (1.2) in any strip*

$$\Omega_1 = \Big\{(t,x) : t \in \mathbb{R},\ x \in (R_2,R_1)\Big\},$$

which properly contains the strip

$$\Omega_2 = \Big\{(t,x) : t \in \mathbb{R},\ x \in [S_2,S_1]\Big\},$$

where

$$S_1 = \frac{\sum\limits_{j=1}^n b_j^+ \int_0^\infty K_j(s)\,ds}{a^-}, \qquad S_2 = \frac{\sum\limits_{j=1}^n b_j^- e^{-c_j^+ S_1} \int_0^\infty K_j(s)\,ds}{a^+},$$

and

$$R_2 = \frac{\sum\limits_{j=1}^n b_j^- e^{-c_j^+ R_1} \int_0^\infty K_j(s)\,ds}{a^+}.$$

Proof. Clearly, $R_2 < S_2$, $S_1 < R_1$. We first prove that $x(t;t_0,\varphi)$ of model (1.2) satisfies

$$R_2 < x(t;t_0,\varphi) < R_1, \quad \text{for all } t \in [t_0,\eta(\varphi)). \tag{3.1}$$

For the sake of convenience, we denote $x(t) = x(t;t_0,\varphi)$. Let $[t_0,T) \subseteq [t_0,\eta(\varphi))$ be an interval such that

$$x(t) > 0, \quad \text{for all } t \in [t_0,T),$$

we first claim that

$$0 < x(t) < R_1, \quad \text{for all } t \in [t_0, T). \tag{3.2}$$

In fact, if (3.2) does not hold, there exists $t_1 \in (t_0, T)$ such that

$$x(t_1) = R_1 \quad \text{and} \quad 0 < x(t) < R_1, \quad \text{for all } t \in (-\infty, t_1). \tag{3.3}$$

It follows from (1.2) and (3.3) that

$$\begin{aligned}
0 &\le x'(t_1) \\
&= -a(t_1)x(t_1) + \sum_{j=1}^{n} b_j(t_1) \int_0^\infty K_j(s) e^{-c_j(t_1)x(t_1-s)}\, ds \\
&\le -a^- x(t_1) + \sum_{j=1}^{n} b_j^+ \int_0^\infty K_j(s)\, ds \\
&= -a^- R_1 + \sum_{j=1}^{n} b_j^+ \int_0^\infty K_j(s)\, ds \\
&< -a^- S_1 + \sum_{j=1}^{n} b_j^+ \int_0^\infty K_j(s)\, ds \\
&= 0,
\end{aligned}$$

which is a contradiction and implies that (3.2) holds.

We next show that

$$x(t) > R_2, \quad \text{for all } t \in [t_0, \eta(\varphi)). \tag{3.4}$$

Otherwise, there exists $t_2 \in (t_0, \eta(\varphi))$ such that

$$x(t_2) = R_2 \quad \text{and} \quad x(t) > R_2, \quad \text{for all } t \in (-\infty, t_2). \tag{3.5}$$

In view of (1.2), (3.2) and (3.5), direct calculation produces

$$\begin{aligned}
0 &\ge x'(t_2) \\
&= -a(t_2)x(t_2) + \sum_{j=1}^{n} b_j(t_2) \int_0^\infty K_j(s) e^{-c_j(t_2)x(t_2-s)}\, ds \\
&> -a^+ x(t_2) + \sum_{j=1}^{n} b_j^- e^{-c_j^+ R_1} \int_0^\infty K_j(s)\, ds \\
&= -a^+ R_2 + \sum_{j=1}^{n} b_j^- e^{-c_j^+ R_1} \int_0^\infty K_j(s)\, ds \\
&= 0,
\end{aligned}$$

which is a contradiction and hence (3.4) holds. According to (3.2) and (3.4), one easily see that (3.1) is true, which implies that $x(t)$ is bounded. Therefore, we know from the continuation theorem in [8, Theorem 3.2 on page 46] that the existence interval of each solution for model (1.2) can be extended to $[t_0, \infty)$. ☐

Lemma 3.2. *Suppose assumptions (A_1)–(A_3) are satisfied. Define the nonlinear operator Γ as follows, for each $\phi \in PAP(\mathbb{R}, \mathbb{R})$, $(\Gamma\phi)(t) := x_\phi(t)$, where*

$$x_\phi(t) = \int_{-\infty}^{t} e^{-\int_v^t a(u)\, du} F(v)\, dv,$$

in which

$$F(v)=\sum_{j=1}^{n}b_j(v)\int_0^{\infty}K_j(s)e^{-c_j(v)\phi(v-s)}\,ds,$$

then Γ *maps* $PAP(\mathbb{R},\mathbb{R})$ *into itself.*

Proof. Firstly, we claim that $F(s)\in PAP(\mathbb{R},\mathbb{R})$. In fact, let $\phi\in PAP(\mathbb{R},\mathbb{R})$ and e^{-v} is a uniformly continuous function for $v\geq 0$, we know from Lemma 3.1 and [22, Corollary 5.4 on page 58] that $g_\phi(v)\triangleq e^{-c_j(v+s)\phi(v)}\in PAP(\mathbb{R},\mathbb{R})$, and therefore, g_ϕ can be expressed as

$$g_\phi(v)=g_\phi^1(v)+g_\phi^2(v),$$

where $g_\phi^1(v)\in AP(\mathbb{R},\mathbb{R})$ and $g_\phi^2(v)\in PAP_0(\mathbb{R},\mathbb{R})$. We know from the almost periodicity of $g_\phi^1(v)$ that for any $\varepsilon>0$, there exists a number $l(\varepsilon)$ such that in any interval $[\alpha,\alpha+l(\varepsilon)]$ one can find a number ς, with the property that

$$\sup_{v\in\mathbb{R}}\left|g_\phi^1(v+\varsigma)-g_\phi^1(v)\right|<\frac{\varepsilon}{\int_0^\infty K_j(s)\,ds},$$

and

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|g_\phi^2(v)\right|dv=0.$$

Then, we have

$$\begin{aligned}\left|\int_0^\infty K_j(s)g_\phi^1(v+\varsigma-s)\,ds-\int_0^\infty K_j(s)g_\phi^1(v-s)\,ds\right|&\leq\int_0^\infty K_j(s)\left|g_\phi^1(v+\varsigma-s)-g_\phi^1(v-s)\right|ds\\&<\varepsilon,\quad\text{for all } v\in\mathbb{R},\end{aligned}$$

which implies that

$$\int_0^\infty K_j(s)g_\phi^1(v-s)\,ds\in AP(\mathbb{R},\mathbb{R}).\tag{3.6}$$

On the other hand, one sees that

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|\int_0^\infty K_j(s)g_\phi^2(v-s)\,ds\right|dv&\leq\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_0^\infty K_j(s)\left|g_\phi^2(v-s)\right|ds\,dv\\&=\lim_{T\to\infty}\frac{1}{2T}\int_0^\infty\int_{-T}^{T}K_j(s)\left|g_\phi^2(v-s)\right|dv\,ds\\&=\lim_{T\to\infty}\frac{1}{2T}\int_0^\infty\int_{-T-s}^{T-s}K_j(s)\left|g_\phi^2(u)\right|du\,ds\\&\leq\lim_{T\to\infty}\frac{1}{2T}\int_0^\infty\int_{-T-s}^{T+s}K_j(s)\left|g_\phi^2(u)\right|du\,ds\\&=\lim_{T\to\infty}\int_0^\infty K_j(s)\frac{T+s}{T}\frac{1}{2(T+s)}\int_{-T-s}^{T+s}\left|g_\phi^2(u)\right|du\,ds\\&=0,\end{aligned}$$

which means that

$$\int_0^\infty K_j(s)g_\phi^2(v-s)\,ds\in PAP_0(\mathbb{R},\mathbb{R}).\tag{3.7}$$

Combining (3.6) and (3.7), we derive that

$$\int_0^\infty K_j(s)e^{-c_j(v)\phi(v-s)}\,ds=\int_0^\infty K_j(s)g_\phi^1(v-s)\,ds+\int_0^\infty K_j(s)g_\phi^2(v-s)\,ds\in PAP(\mathbb{R},\mathbb{R}).$$

Then, by a standard argument as Lemma 3.2 in [4], we can prove that Γ maps $PAP(\mathbb{R},\mathbb{R})$ into itself. $\square$

Our first main result can be stated as follows.

Theorem 3.3. *In addition to (A_1)–(A_3), suppose further that*

$$r = \frac{\sum_{j=1}^{n} b_j^+ c_j^+ \int_0^\infty K_j(s)\, ds}{a^-} < 1. \tag{3.8}$$

Then the model (1.2) *admits a unique pseudo almost periodic solution in the region*

$$\mathfrak{B} = \{x \mid x \in PAP(\mathbb{R}, \mathbb{R}),\ S_2 \le x(t) \le S_1,\ t \in \mathbb{R}\}.$$

Proof. For any $\phi \in PAP(\mathbb{R}, \mathbb{R})$, we introduce the following auxiliary equation

$$x'(t) = -a(t)x(t) + \sum_{j=1}^{n} b_j(t) \int_0^\infty K_j(s) e^{-c_j(t)\phi(t-s)}\, ds.$$

Notice that $M[a] > 0$, we know from Lemma 2.8 that the linear equation

$$x'(t) = -a(t)x(t)$$

admits an exponential dichotomy on $\mathbb{R}$. Therefore, by Lemmas 2.5 and 2.7, we know that model (1.2) has exactly one solution expressed by

$$x_\phi(t) = \int_{-\infty}^{t} e^{-\int_v^t a(u)\, du} \left[\sum_{j=1}^{n} b_j(v) \int_0^\infty K_j(s) e^{-c_j(v)\phi(v-s)}\, ds \right] dv, \tag{3.9}$$

one can observe from Lemma 3.2 that $x_\phi(t) \in PAP(\mathbb{R}, \mathbb{R})$.

Set

$$\mathfrak{B} := \{x \mid x \in PAP(\mathbb{R}, \mathbb{R}),\ S_2 \le x(t) \le S_1,\ t \in \mathbb{R}\},$$

obviously, $\mathfrak{B}$ is a closed subset of $PAP(\mathbb{R}, \mathbb{R})$.

Define an operator Γ on $\mathfrak{B}$ by

$$(\Gamma\phi)(t) = \int_{-\infty}^{t} e^{-\int_v^t a(u)\, du} \left[\sum_{j=1}^{n} b_j(v) \int_0^\infty K_j(s) e^{-c_j(v)\phi(v-s)}\, ds \right] dv. \tag{3.10}$$

Obviously, to show that model (1.2) has a unique pseudo almost periodic solution, it suffices to prove that Γ has a fixed point in $\mathfrak{B}$.

Let us first prove that the operator Γ is a self-mapping from $\mathfrak{B}$ to $\mathfrak{B}$. In fact, for any $\phi \in \mathfrak{B}$, we have

$$\begin{aligned}
(\Gamma\phi)(t) &\le \int_{-\infty}^{t} e^{-\int_v^t a(u) du} \left[\sum_{j=1}^{n} b_j^+ \int_0^\infty K_j(s)\, ds \right] dv \\
&\le \int_{-\infty}^{t} e^{-a^-(t-v)} \left[\sum_{j=1}^{n} b_j^+ \int_0^\infty K_j(s)\, ds \right] dv \\
&= \frac{\sum_{j=1}^{n} b_j^+ \int_0^\infty K_j(s)\, ds}{a^-} \\
&= S_1, \quad \text{for all } t \in \mathbb{R}.
\end{aligned} \tag{3.11}$$

On the other hand, we have

$$\begin{aligned}(\Gamma\phi)(t) &\geq \int_{-\infty}^{t} e^{-\int_v^t a(u)du}\left[\sum_{j=1}^{n} b_j^- \int_0^\infty K_j(s)e^{-c_j^+ S_1}\, ds\right] dv \\ &\geq \int_{-\infty}^{t} e^{-a^+(t-v)}\left[\sum_{j=1}^{n} b_j^- e^{-c_j^+ S_1}\int_0^\infty K_j(s)\, ds\right] dv \\ &= \frac{\sum\limits_{j=1}^{n} b_j^- e^{-c_j^+ S_1}\int_0^\infty K_j(s)\, ds}{a^+} \\ &= S_2, \quad \text{for all } t \in \mathbb{R}.\end{aligned}$$

which, together with (3.11), means that the mapping Γ is a self-mapping form $\mathfrak{B}$ to $\mathfrak{B}$.

Next, we show that the mapping Γ is a contraction mapping on $\mathfrak{B}$. For any $\phi, \phi^* \in \mathfrak{B}$, one has

$$\begin{aligned}&\|(\Gamma\phi)(t) - (\Gamma\phi^*)(t)\| \\ &\quad = \sup_{t\in\mathbb{R}} |(\Gamma\phi)(t) - (\Gamma\phi^*)(t)| \\ &\quad = \sup_{t\in\mathbb{R}} \left| \int_{-\infty}^{t} e^{-\int_v^t a(u)\,du} \sum_{j=1}^{n} b_j(v) \int_0^\infty K_j(s)\left[e^{-c_j(v)\phi(v-s)} - e^{-c_j(v)\phi^*(v-s)}\right] ds\, dv\right|.\end{aligned} \tag{3.12}$$

Since

$$\begin{aligned}|e^{-x} - e^{-y}| &= e^{-(x+\theta(y-x))}|x-y| \\ &< |x-y|, \quad \text{where } x, y \in (0, +\infty),\ 0 < \theta < 1.\end{aligned} \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\begin{aligned}\|(\Gamma\phi)(t) - (\Gamma\phi^*)(t)\| &\leq \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\int_v^t a(u)\,du} \sum_{j=1}^{n} b_j(v) \int_0^\infty K_j(s)\left|e^{-c_j(v)\phi(v-s)} - e^{-c_j(v)\phi^*(v-s)}\right| ds\, dv \\ &\leq \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-a^-(t-v)} \sum_{j=1}^{n} b_j^+ c_j^+ \int_0^\infty K_j(s)\, ds\, dv \|\phi - \phi^*\| \\ &= \frac{\sum\limits_{j=1}^{n} b_j^+ c_j^+ \int_0^\infty K_j(s)\, ds}{a^-} \|\phi - \phi^*\|.\end{aligned} \tag{3.14}$$

Note that

$$r = \frac{\sum\limits_{j=1}^{n} b_j^+ c_j^+ \int_0^\infty K_j(s) ds}{a^-} < 1,$$

(3.14) shows that Γ is a contraction mapping. Therefore, by virtue of the Banach fixed point theorem, Γ has a unique fixed point which corresponds to the solution of model (1.2) in $\mathfrak{B} \subset PAP(\mathbb{R}, \mathbb{R})$. This completes the proof of Theorem 3.3. □

3.2 Asymptotic stability of pseudo almost periodic solution

In the following, we give the analysis of global asymptotic stability of model (1.2).

Theorem 3.4. *If all the assumptions in Theorem 3.3 are satisfied, then, all solutions of model (1.2) in the region $\mathfrak{B}$ converge to its unique pseudo almost periodic solution.*

Proof. Let $x(t)$ be any solution of model (1.2) and $x^*(t)$ be a pseudo almost periodic solution of model (1.2), consider a Lyapunov function defined by

$$V(t) = |x(t)-x^*(t)| + \int_0^\infty K_j(s)\left\{\int_{t-s}^t \sum_{j=1}^n b_j(u+s)\left|e^{-c_j(u+s)x(u)} - e^{-c_j(u+s)x^*(u)}\right| du\right\} ds.$$

A direct calculation of the right derivative $D^+V(t)$ of $V(t)$ along the solutions of model (1.2), produces

$$\begin{aligned}
D^+V(t) = {} & \operatorname{sgn}\{x(t)-x^*(t)\} \\
& \times \left\{ -a(t)[x(t)-x^*(t)] + \sum_{j=1}^n b_j(t)\int_0^\infty K_j(s)\left[e^{-c_j(t)x(t-s)} - e^{-c_j(t)x^*(t-s)}\right] ds \right\} \\
& + \int_0^\infty K_j(s)\sum_{j=1}^n b_j(t+s)\left|e^{-c_j(t+s)x(t)} - e^{-c_j(t+s)x^*(t)}\right| ds \\
& - \int_0^\infty K_j(s)\sum_{j=1}^n b_j(t)\left|e^{-c_j(t)x(t-s)} - e^{-c_j(t)x^*(t-s)}\right| ds \\
\le {} & -a(t)|x(t)-x^*(t)| + \sum_{j=1}^n b_j(t)\int_0^\infty K_j(s)\left|e^{-c_j(t)x(t-s)} - e^{-c_j(t)x^*(t-s)}\right| ds \qquad (3.15) \\
& + \int_0^\infty K_j(s)\sum_{j=1}^n b_j(t+s)\left|e^{-c_j(t+s)x(t)} - e^{-c_j(t+s)x^*(t)}\right| ds \\
& - \int_0^\infty K_j(s)\sum_{j=1}^n b_j(t)\left|e^{-c_j(t)x(t-s)} - e^{-c_j(t)x^*(t-s)}\right| ds \\
\le {} & -a^-|x(t)-x^*(t)| + \sum_{j=1}^n b_j^+ c_j^+ \int_0^\infty K_j(s)\, ds |x(t)-x^*(t)| \\
\le {} & -\left(a^- - \sum_{j=1}^n b_j^+ c_j^+ \int_0^\infty K_j(s)\, ds\right)|x(t)-x^*(t)| \quad \text{for } t \ge t_0.
\end{aligned}$$

It follows from (3.8) and (3.15) that there exists a positive constant $\mu_1 > 0$ such that

$$D^+V(t) \le -\mu_1|x(t)-x^*(t)|, \qquad t \ge t_0. \tag{3.16}$$

Integrating on both sides of (3.16) from t_0 to t yields

$$V(t) + \mu_1\int_{t_0}^t |x(s)-x^*(s)|ds < V(t_0) < \infty, \qquad t \ge t_0,$$

then

$$\int_{t_0}^t |x(s)-x^*(s)|\, ds < \mu_1^{-1}V(t_0) < \infty, \qquad t \ge t_0,$$

and hence $|x(t)-x^*(t)| \in L^1([t_0,\infty))$.

From Lemma 3.1, we can obtain that $x(t)$, $x^*(t)$ and their derivatives remain bounded on $[t_0,\infty)$ (from the equation satisfied by them). Then it follows that $|x(t)-x^*(t)|$ is uniformly continuous on $[t_0,\infty)$. By Lemma 2.9, we conclude that

$$\lim_{t\to\infty}|x(t)-x^*(t)| = 0.$$

The proof of Theorem 3.4 is complete. □

Remark 3.5. Very recently, J. Shao in [16] studied the following Lasota–Wazewska model with an oscillating death rate

$$x'(t) = -a(t)x(t) + \sum_{j=1}^{n} \beta_j(t) e^{-c_j(t)x(t-\tau_j(t))}, \tag{3.17}$$

where $a\colon \mathbb{R} \to \mathbb{R}$ is an almost periodic function which is controlled by a non-negative function, $b_j, c_j, \tau_j\colon \mathbb{R} \to [0, +\infty)$ are pseudo almost periodic functions, if there exist a bounded and continuous function $a^*\colon \mathbb{R} \to [0, \infty)$ and constants F^i, F^S, κ such that

$$F^i e^{-\int_s^t a^*(u)\,du} \le e^{-\int_s^t a(u)\,du} \le F^S e^{-\int_s^t a^*(u)\,du}, \quad \text{for all } t, s \in \mathbb{R} \quad \text{and} \quad t-s>0, \tag{3.18}$$

$$\Big(-a^*(t) + F^s \sum_{j=1}^{m} b_j(t) c_j(t) e^{-c_j(t)\kappa} \Big)^+ < 0, \tag{3.19}$$

the author proved that equation (3.17) has a pseudo almost periodic solution which is globally exponentially stable.

One can find that the function of death rate $a(t)$ in equation (3.17) is more general than equation (1.2). However, if $K_j(s) = \delta(s - \tau_j)$, where $\delta(s)$ denotes the Dirac-δ function, then equation (1.2) becomes equation (3.17) with constant discrete delays, on the other hand, as pointed out by L. Duan et al. in [5] and Y. Kuang in [12], it is more reasonable and realistic to establish delay-dependent criteria ensuring the dynamics of a system because the delays have important effect on a system, one can clearly see that the stability criteria established here are delay-dependent and the method used here is different from [16]. This indicates that these results are complementary to each other. Therefore, our results are new and complement the existing ones. Moreover, it seems that condition (3.8) is easier to verify than (3.18)–(3.19).

4 An example

Here we give an example that illustrates the pseudo almost periodic behavior of the Lasota–Wazewska model with distributed delay.

Example 4.1. Consider the following pseudo almost periodic differential equation with distributed delays:

$$\begin{aligned} x'(t) = &-\left(9.5 + \frac{1}{2}\sin\sqrt{3}t\right)x(t) \\ &+ \left(0.75 + \frac{1}{2}\sin\sqrt{2}t + \frac{1}{2}|\cos\sqrt{5}t| + \frac{1}{4}\frac{1}{1+t^2}\right)\int_0^\infty K_1(s) e^{\left(0.6+0.4\sin\sqrt{2}t+\frac{1}{1+t^2}\right)x(t-s)}\,ds \\ &+ \left(0.8 + \frac{1}{2}\sin\sqrt{3}t + \frac{1}{2}|\cos\sqrt{2}t| + \frac{1}{5}\frac{1}{1+t^2}\right)\int_0^\infty K_2(s) e^{\left(1.2+0.3\cos\sqrt{3}t+\frac{1}{2}\frac{1}{1+t^2}\right)x(t-s)}\,ds, \end{aligned} \tag{4.1}$$

where

$$K_1(s) = \begin{cases} \frac{2s}{3}, & \text{if } 0 \le s \le 1, \\ \frac{2e}{3e^s}, & \text{if } 1 < s < \infty, \end{cases} \qquad K_2(s) = \begin{cases} \frac{s}{4}, & \text{if } 0 \le s \le 2, \\ \frac{e^2}{2e^s}, & \text{if } 2 < s < \infty. \end{cases}$$

Choose $S_1 = 0.5,\ S_2 = 0.02$, one can easily realize that

$$\frac{b_1^+ \int_0^\infty K_1(s)\,ds + b_2^+ \int_0^\infty K_2(s)\,ds}{a^-} \approx 0.4444 < 0.5,$$

$$\frac{b_1^- e^{-c_1^+ R_1} \int_0^\infty K_1(s)\,ds + b_2^- e^{-c_2^+ R_1} \int_0^\infty K_2(s)\,ds}{a^+} \approx 0.0202 > 0.02,$$

and

$$r = \frac{b_1^+ c_1^+ \int_0^\infty K_1(s)\,ds + b_2^+ c_2^+ \int_0^\infty K_2(s)\,ds}{a^-} \approx 0.8889 < 1.$$

Therefore, by the consequence of Theorems 3.3–3.4, equation (4.1) has a unique positive pseudo almost periodic solution $x(t)$ which is globally asymptotically stable (see Figure 4.1).

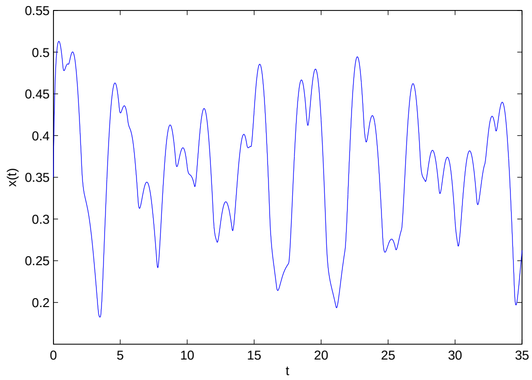

Figure 4.1: Time-domain behavior of $x(t)$ for model (4.1) with initial value $\varphi(s) = 0.35$, $s \in (-\infty, 0]$.

Remark 4.2. In recent years, many in-depth research results of the Lasota–Wazewska model mainly focused on the dynamics of the equilibrium point or periodic solution or almost periodic solution [5,7,9,13,15,17,19], in particular, one can easily see that the obtained results extend the corresponding ones in [18]. To the best of our knowledge, on the other hand, fewer authors have considered the existence and stability of pseudo almost periodic solutions to model (1.2). Therefore, the main results in the present paper are essentially new and they extend previously known results.

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References

[1] I. Barbălat, Systèmes d'équations différentielles d'oscillations non linéaries (in French), *Rev. Roumaine Math. Pure Appl.* **4**(1959), 267–270. MR0111896

[2] C. Cuevas, A. Sepúlveda, H. Soto, Almost periodic and pseudo-almost periodic solutions to fractional differential and integro-differential equations, *Appl. Comput. Math.* **218**(2011), 1735–1745. MR3197747; url

[3] T. Diagana, Pseudo-almost periodic solutions for some classes of nonautonomous partial evolution equations, *J. Franklin Inst.* **348**(2011), 2082–2098. MR2841897; url

[4] L. Duan, L. Huang, Pseudo almost periodic dynamics of delay Nicholson's blowflies model with a linear harvesting term, *Math. Methods Appl. Sci.* **38**(2015), 1178–1189. MR3338142; url

[5] L. Duan, L. Huang, Y. Chen, Global exponential stability of periodic solutions to a delay Lasota-Wazewska model with discontinuous harvesting, *Proc. Amer. Math. Soc.*, (2015) accepted. url

[6] A. M. Fink, *Almost periodic differential equations*, Lecture Notes in Mathematics, Vol. 377, Springer-Verlag, Berlin, 1974. MRMR0460799

[7] K. Gopalsamy, S. Trofimchuk, Almost periodic solutions of Lasota–Wazewska-type delay differential equation, *J. Math. Anal. Appl.* **237**(1999), 106–127. MR1708165; url

[8] J. K. Hale, S. M. Verduyn Lunel, *Introduction to functional differential equations*, Applied Mathematical Sciences, Vol. 99, Springer-Verlag, New York, 1993. MR1243878

[9] Y. Hamaya, Almost periodic dynamic of a discrete Wazewska–Lasota model, *Commun. Math. Appl.* **4**(2013), 189–199. url

[10] Z. Hu, Z. Jin, Stepanov-like pseudo almost periodic mild solutions to nonautonomous neutral partial evolution equations, *Nonlinear Anal.* **75**(2012), 244–252. MR2846796; url

[11] Z. Huang, S. Gong, L. Wang, Positive almost periodic solution for a class of Lasota–Wazewska model with multiple time-varying delays, *Comput. Math. Appl.* **61**(2011), 755–760. MR2770479; url

[12] Y. Kuang, *Delay differential equations: with applications in population dynamics*, Academic Press, New York, 1993. MR1218880

[13] X. Liu, Y. Takeuchi, Periodicity and global dynamics of an impulsive delay Lasota–Wazewska model, *J. Math. Anal. Appl.* **327**(2007), 326–341. MR2277416; url

[14] B. Liu, C. Tunç, Pseudo almost periodic solutions for a class of first order differential iterative equations, *Appl. Math. Lett.* **40**(2015), 29–34. MR3278302; url

[15] G. Liu, A. Zhao, J. Yan, Existence and global attractivity of unique positive periodic solution for a Lasota–Wazewska model, *Nonlinear Anal.* **64**(2006), 1737–1746. MR2197358; url

[16] J. Shao, Pseudo almost periodic solutions for a Lasota–Wazewska model with an oscillating death rate, *Appl. Math. Lett.* **43**(2015), 90–95. MR3305635; url

[17] G. Stamov, On the existence of almost periodic solutions for the impulsive Lasota–Wazewska model, *Appl. Math. Lett.* **22**(2009), 516–520. MR2502243; url

[18] X. Wang, Z. Li, Existence, uniqueness and global attractivity a positive almost periodic solution for a Lasota–Wazewska model, *J. Biomath.* **23**(2008), 449–456. MR2489057; url

[19] L. Wang, M. Yu, P. Niu, Periodic solution and almost periodic solution of impulsive Lasota–Wazewska model with multiple time-varying delays, *Comput. Math. Appl.* **64**(2012), 2383–2394. MR2970822; url

[20] M. Ważewska-Czyżewska, A. Lasota, Mathematical problems of the dynamics of red blood cells system (in Polish), *Mat. Stos. (3)* **17**(1988), 23–40. MR0682081

[21] H. Wu, Pseudo almost-periodic solution of shunting inhibitory cellular neural networks with delay, *J. Appl. Math.* (2011), 1–14. MR2727279; url

[22] C. Zhang, *Almost periodic type functions and ergodicity*, Kluwer Academic/Science Press, Beijing, 2003. MR2000981

[23] C. Zhang, Pseudo almost periodic solutions of some differential equations II, *J. Math. Anal. Appl.* **192**(1995), 543–561. MR1332227; url

[24] C. Y. Zhang, Pseudo-almost-periodic solutions of some differential equations, *J. Math. Anal. Appl.* **181**(1994), 62–76. MR1257954; url

[25] C. Y. Zhang, *Pseudo almost periodic functions and their applications*, PhD thesis, University of Western Ontario, 1992.