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Slow divergence integrals in generalized Liénard equations near centers

Renato Huzak✉ and **Peter De Maesschalck**

Hasselt University, Campus Diepenbeek, Agoralaan Gebouw D, 3590 Diepenbeek, Belgium

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Abstract. Using techniques from singular perturbations we show that for any $n \geq 6$ and $m \geq 2$ there are Liénard equations $\{\dot{x} = y - F(x),\ \dot{y} = G(x)\}$, with F a polynomial of degree n and G a polynomial of degree m, having at least $2[\frac{n-2}{2}] + [\frac{m}{2}]$ hyperbolic limit cycles, where $[\cdot]$ denotes "the greatest integer equal or below".

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1 Introduction

The paper deals with a popular model of *generalized Liénard equations*

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

with f and g polynomials of respective degree $n-1$ and m. A representation in the phase plane of this scalar second order differential equation is given by

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x)y - g(x). \end{cases}$$

If we write $G(x) = -g(x)$ and introduce the new variable $\bar{y} = y + F(x)$, where $F(x) = \int_0^x f(s)\,ds$, then the above planar vector field changes into a representation of the scalar second order Liénard differential equation in the so-called Liénard plane:

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = G(x), \end{cases} \tag{1.1}$$

where we denote $\bar{y}$ by y. F and G are polynomials in x of respective degree n and m. When $m = 1$, equation (1.1) is called a *classical Liénard equation (of degree n)*. When $m > 1$, we call (1.1) a *generalized Liénard equation (of type (n, m))*.

✉Corresponding author. Email: renato.huzak@uhasselt.be

The second part of Hilbert's 16th problem asks for a uniform bound for the maximum number of limit cycles of a planar polynomial vector field $\{\dot{x} = P(x,y),\ \dot{y} = Q(x,y)\}$, uniformly in terms of degree of real polynomials P and Q (see [23]). It remains unsolved even for quadratic polynomials. Liénard equations (1.1) form a subclass of planar polynomial vector fields for which one considers a simplified version of Hilbert's 16th problem: *Determine the maximum number $L(n,m)$ of limit cycles in* (1.1) *in terms of the two degrees n and m.* Part of Smale's 13th problem deals with this simplified problem restricted to classical (polynomial) Liénard equations, i.e. the case $G(x) = -x$ in (1.1) (see [33]). Moreover Smale suggested that the maximum number $L(n,1)$ of limit cycles for classical Liénard equations grows at most by an algebraic law of type $n^d = (\deg F)^d$ where d is a universal constant. In 1977 Lins, de Melo and Pugh conjectured for classical Liénard equations of degree n that the number of limit cycles is at most $[\frac{n-1}{2}]$, where $[\cdot]$ denotes "the greatest integer equal or below" (see [26] where the conjecture has been proved for $n = 2, 3$). For $n = 4$ this conjecture has been proved in a recent paper [25]. The conjecture was shown to be false in 2007 (see [14]) for degrees $n \geq 7$ and in 2011 (see [7]) for $n \geq 6$. In the first paper, $[\frac{n-1}{2}] + 1$ limit cycles were shown to appear, and in the second paper, $[\frac{n-1}{2}] + 2$ limit cycles were shown to appear. The conjecture for $n = 5$ is still open. In a recent paper [8], lower bounds for the number of limit cycles for polynomial classical Liénard equations have been improved: *there can be at least $n-2$ (hyperbolic) limit cycles in a classical Liénard equation of degree n except for $n \in \{4,5\}$.* For $n \geq 6$, these lower bounds are reasonable enough to be conjectured as optimal.

The maximum number $L(n,m)$ of limit cycles for generalized Liénard equations ($m > 1$) is, like in the classical case, only known in some very low-degree cases. Coppel proved that $L(2,2) = 1$ (see [6]), Dumortier, Li and Rousseau proved that $L(2,3) = 1$ (see [11] and [15]), Dumortier and Li proved that $L(3,2) = 1$ (see [12]), and Wang and Jing proved that $L(3,3) = 3$ (see [34]). Besides that, lower bounds of $L(n,m)$ for generalized Liénard equations have been widely investigated (see e.g. [2, 4, 5, 13, 16–22, 24, 27–31, 35–38]). A short overview of results obtained in the above-mentioned papers can be found in [19] and [27] where in addition new lower bounds have been reached.

Let us state now the main theorem of this paper and explain how it improves the existing results on lower bounds for generalized Liénard equations.

Theorem 1.1. *Let $n \geq 6$ and $m \geq 2$. Then there exist a polynomial $F(x)$ of degree n and a polynomial $G(x)$ of degree m so that the system of differential equations*

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = G(x) \end{cases}$$

has at least $2[\frac{n-2}{2}] + [\frac{m}{2}]$ hyperbolic limit cycles.

In [27], it has been shown that there exist generalized Liénard equations (1.1) of type (n,m), $n \geq 2$ and $m \geq 2$, having at least $[\frac{n+m-2}{2}]$ limit cycles. Clearly, the result in Theorem 1.1 improves this lower estimate $\left(2[\frac{n-2}{2}] + [\frac{m}{2}] > [\frac{n+m-2}{2}]\right.$ for all $n \geq 6$ and $\left.m \geq 2\right)$.

In [20], it has been proved further that

$$l_{n,m} := \max\left\{\left[\frac{m-2}{3}\right] + \left[\frac{2n-1}{3}\right], \left[\frac{n-3}{3}\right] + \left[\frac{2m+1}{3}\right]\right\} \leq L(n,m)$$

for all $n \geq 2$ and $m \geq 2$. On one hand, it is not hard to show that $l_{n,m} \geq [\frac{n+m-2}{2}]$, with strict inequality for infinitely many pairs (n,m) (see [20]). Thus, [20] is a recent improvement of [27]. On the other hand, comparing the coefficients in front of n in the expression $l_{n,m}$ with the coefficient in front of n in $2[\frac{n-2}{2}] + [\frac{m}{2}]$, it is clear that for each fixed $m \geq 2$ there exists $n_0 \geq 6$ such that $l_{n,m} < 2[\frac{n-2}{2}] + [\frac{m}{2}]$ for all $n \geq n_0$. When $m = 2$, then $l_{n,2} = [\frac{2n-1}{3}] < 2[\frac{n-2}{2}] + 1$ for all $n \geq 6$. Recall that it has been proved in [18] that $[\frac{2n-1}{3}] \leq L(n, 2)$ for all $n \geq 2$.

To our knowledge, there are no other results on lower bounds for generalized Liénard equations beside [20] and [27] for arbitrary n and m.

In [19], new lower bounds of $L(n, m)$ are found for many integers n and m giving the $m \ln m$ asymptotic growth of $L(n, m)$ with some conditions on n. For small m, Theorem 1.1 improves the lower bounds of $L(n, m)$ given in [19]. For example, it has been shown in [19] that $2[\frac{n-2}{4}] + [\frac{n-2}{2}] \leq L(n, m)$, for $m \in \{3, 4\}$ and $n \geq 4$. It can be easily seen that $2[\frac{n-2}{4}] + [\frac{n-2}{2}] < 2[\frac{n-2}{2}] + [\frac{m}{2}]$ for $m \in \{3, 4\}$ and $n \geq 6$.

In Section 2, using well known singular perturbation techniques for planar slow-fast systems we reduce the proof of Theorem 1.1 to the computation of simple integrals which appear in an expression for slow divergence integral. In Section 3, we use mathematical induction on degree m to finish the proof of Theorem 1.1.

2 Singular perturbations

Theorem 1.1 will be shown using techniques from singular perturbations (see [7, 8, 14]). Singular perturbations arise when the coefficients of F are very large, so that after applying a rescaling, a small parameter appears in front the $\dot{y}$ equation (see also [3, 9, 32]):

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = \epsilon G(x). \end{cases} \tag{2.1}$$

In this paper, we will also use this setting, together with the assumption that

$$F(0) = F'(0) = 0, \qquad \forall x \in [-M, M] \colon \frac{F'(x)}{x} > 0. \tag{2.2}$$

Limit cycles of (2.1) are generally members of ϵ-families of limit cycles that tend to certain limit periodic sets for $\epsilon = 0$. The limit periodic sets are called *slow-fast cycles*, and are of the form

$$\Gamma^Y := \{\, (x, F(x)) : F(x) \leq Y \,\} \,\cup\, \{\, (x, Y) : F(x) \leq Y \,\}.$$

The second component is a heteroclinic (fast) connection for $\epsilon = 0$, connecting two singularities on the curve of singular points $y = F(x)$, whereas the first component is the part of the parabolic curve beneath the fast orbit (see Figure 2.1). In this paper, we will parameterize the slow-fast cycles with its rightmost x-coordinate:

$$\Gamma_x := \Gamma^{F(x)}, \qquad x > 0.$$

In order to state the principal tool that we will use in the proof, we define the fast relation, which relates an $x > 0$ to an $L(x) < 0$ so that $F(x) = F(L(x))$. In other words, $(L(x), Y)$ and (x, Y) are two end points of the same fast orbit at height $Y = F(x)$.

We then have (using [10]) the following theorem.

Theorem 2.1. *Let the function $x \mapsto L(x)$ be described as above, and consider system (2.1) with the condition (2.2) and with the extra condition*

$$G(0) = 0, \frac{G(x)}{x} < 0, \qquad \forall x \in [-M, M].$$

Define the so-called slow divergence integral *associated to* Γ_x*:*

$$I(x) = -\int_x^{L(x)} \frac{F'(s)^2}{G(s)}\, ds, \qquad x \in \left]0, \min\{M, L^{-1}(-M)\}\right[. \tag{2.3}$$

Suppose that $I(x)$ *has exactly* k *simple zeros, then there exists a smooth function* $\lambda = \lambda(\epsilon)$ *with* $\lambda(0) = 0$*, so that the perturbed system*

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = \epsilon\,[\lambda(\epsilon) + G(x)] \end{cases} \tag{2.4}$$

has exactly $k+1$ *periodic orbits (provided* $\epsilon > 0$ *is small enough), all of them are isolated and hyperbolic.*

Proof. (sketch) Let $x_1 < x_2 < \cdots < x_k$ be the k simple zeros of $I(x)$, and define $\tilde{x}_i = L(x_i)$. Choose and fix $x_{k+1} > x_k$ arbitrary but so that $x_{k+1} < M$ and $L(x_{k+1}) > -M$. Since the origin is a slow-fast Hopf point, the parameter λ can be used as a breaking parameter. Hence there exists a $\lambda = \lambda(\epsilon)$ with $\lambda(0) = 0$ so that (2.4) has a limit cycle Hausdorff close to $\Gamma_{x_{k+1}}$. We can refer to [10], but even early results on canards like in [1] can be used to see this statement. The cycle $\Gamma_{x_{k+1}}$ is considered a long canard, and when a long canard is present, smaller canard cycles are located at zeros of the above integral. In other words, there are k additional canard cycles, Hausdorff close to Γ_{x_i}, for $i = 1, \ldots, k$. For details we refer to [10]. We note that the same conclusions can be drawn using the entry-exit relation introduced in [1] (along the long canard we have so-called "tunnel" behaviour). Here we just present a heuristic argument. When orbits are integrated inside the big canard cycle, they will either spiral inwards or spiral outwards after one iteration around the Hopf point. During one iteration, the orbits travel a distance along the critical curve. Near this curve, the orbit experiences exponential attraction towards the long canard, and it will steer away from this canard after it has experienced equally strong long repulsion (after passing the Hopf point). Orbits at the interior of the long canard cycle will be attracted to an $O(\epsilon)$-neighbourhood of the long canard at a point $(x_{\text{entry}}, F(x_{\text{entry}}))$ and will exit this $O(\epsilon)$-neighbourhood at a point $(x_{\text{exit}}, F(x_{\text{exit}}))$. Before the entry point and after the exit point, the orbit more or less follows a horizontal path (fast dynamics). It is clear that the orbit is spiraling inwards when $F(x_{\text{exit}}) < F(x_{\text{entry}})$ and outwards when $F(x_{\text{exit}}) > F(x_{\text{entry}})$. From the entry-exit relation deduced as early as in [1], we know that

$$\int_{x_{\text{entry}}}^{x_{\text{exit}}} \frac{F'(s)^2}{G(s)}\, ds = 0.$$

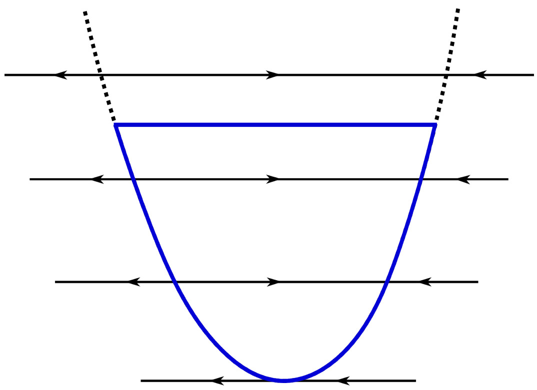

Figure 2.1: The dynamics of (2.1) for $\epsilon = 0$. The blue closed curve is a slow-fast cycle.

As a consequence, at zeros of $I(x)$, orbits go from spiraling inwards to spiraling outwards or vice-versa and therefore at each zero of $I(x)$ there should be an additional canard cycle. $\square$

Using a perturbative approach we compute the slow divergence integral $I(x)$ in generalized Liénard equations near centers. For a suitable choice of polynomials F and G, we show that dominant part of the slow divergence integral is an integral of a polynomial function. We will assume that

$$F(x) = F_e(x) + \delta F_o(x) \tag{2.5}$$

and

$$G(x) = -x + \delta g(x), \tag{2.6}$$

where F_e is even, F_o is odd, $F_e(0) = F_o'(0) = g(0) = g'(0) = 0$ and where δ is a small perturbation parameter. Centers are obtained when $\delta = 0$.

Proposition 2.2. *The slow divergence integral* (2.3) *of a cycle* Γ_x *is given under these conditions by*

$$I(x) = 2\delta I_1(x) + O(\delta^2)$$

with

$$I_1(x) = \int_0^x \left(f_e'(s)F_o(s) - f_e(s)F_o'(s) - \frac{g(s)+g(-s)}{2} f_e(s)^2 \right) ds,$$

where $f_e(x) := F_e'(x)/x$*. Simple zeros of* $I_1(x)$ *will persist as simple zeros of* $I(x)$*, for nonzero but small* δ*.*

Proof. We first asymptotically determine the fast relation function $L(x)$, from its defining property $F(x) = F(L(x))$, $L(x) < 0 < x$. Clearly, $L(x) = -x + \delta L_1(x) + O(\delta^2)$. By plugging this form into the defining property we obtain

$$\begin{aligned} F_e(x) + \delta F_o(x) &= F_e(-x + \delta L_1(x)) + \delta F_o(-x) + O(\delta^2) \\ &= F_e(-x) + \delta F_e'(-x) L_1(x) + \delta F_o(-x) + O(\delta^2), \end{aligned}$$

so using the symmetry properties of F_e and F_o we find $L_1(x) = -\frac{2F_o(x)}{F_e'(x)}$. Next we consider $I(x) = -\int_x^{L(x)} \frac{F'(s)^2}{G(s)} ds$. We obtain

$$\begin{aligned} I(x) &= -\int_x^{-x+\delta L_1(x)+O(\delta^2)} \frac{(F_e'(s)+\delta F_o'(s))^2}{-s+\delta g(s)} ds \\ &= -\int_x^{-x} \frac{F_e'(s)^2 + 2\delta F_e'(s)F_o'(s)}{-s+\delta g(s)} ds + \delta L_1(x) \frac{F_e'(-x)^2}{-x} + O(\delta^2) \\ &= -\int_x^{-x} \frac{F_e'(s)^2}{-s+\delta g(s)} ds + \delta \left[\int_x^{-x} \frac{2F_e'(s)F_o'(s)}{s} ds - L_1(x) \frac{F_e'(x)^2}{x} \right] + O(\delta^2) \\ &= \delta \left[\int_x^{-x} \frac{F_e'(s)^2 g(s)}{s^2} ds + \int_x^{-x} \frac{2F_e'(s)F_o'(s)}{s} ds - L_1(x) \frac{F_e'(x)^2}{x} \right] + O(\delta^2) \\ &= 2\delta \left[\frac{1}{2} \int_x^{-x} \frac{F_e'(s)^2 g(s)}{s^2} ds + \int_x^{-x} \frac{F_e'(s)F_o'(s)}{s} ds + \frac{F_o(x)F_e'(x)}{x} \right] + O(\delta^2). \end{aligned}$$

If we write $f_e(x) := \frac{F_e'(x)}{x}$, then $\frac{I(x)}{2\delta} = I_1(x) + O(\delta)$ with

$$I_1(x) = -2\int_0^x f_e(s)F_o'(s)\, ds + F_o(x) f_e(x) - \int_0^x f_e(s)^2 \frac{g(s)+g(-s)}{2} ds.$$

In one half of the first integral appearing in I_1 we apply partial integration to obtain the result. $\square$

Proposition 2.2 and Theorem 2.1 allow to prove the main theorem (Theorem 1.1), provided we find convenient functions f_e, F_o and g that satisfy the conditions and that produce an integral function I_1 with a sufficient amount of simple zeros. In the classical case ($g = 0$), the following result has been proven in [8].

Proposition 2.3. *Let $k \geq 3$. There exist an even polynomial α_k of degree $2k-2$, $\alpha_k(s) > 0$ for all $s \in \mathbb{R}$, and an odd polynomial β_k of degree $2k-1$ and of order 3 such that the function*

$$H_k(x) := \int_0^x (\alpha_k'(s)\beta_k(s) - \alpha_k(s)\beta_k'(s))\,ds$$

has $2k-3$ simple zeros on $\{x > 0\}$. As a corollary, the function $F(x) = F_e(x) + \delta F_o(x)$, with $F_o(x) = \beta_k(x)$, $f_e(x) = \alpha_k(x)$ and $F_e(x) = \int_0^x s f_e(s)\,ds$, satisfies the conditions of Proposition 2.2 and Theorem 2.1, giving an example of classical Liénard equation of even degree $n = 2k$, $k \geq 3$, with $n-2$ hyperbolic limit cycles.

Remark 2.4. Since $\alpha_k(s) > 0$ for all $s \in \mathbb{R}$, the highest order coefficient is strictly positive. Using simple rescalings we can put the highest order coefficients of α_k and β_k to 1.

Remark 2.5. In the next section, the general case $\deg G = m \geq 2$ will be treated. We will use Proposition 2.3 in the proof of Theorem 1.1 as the basis step of mathematical induction on m.

The following proposition shows that the method used in this paper cannot give more limit cycles than stated in Theorem 1.1.

Proposition 2.6. *Let $F(x)$ and $G(x)$ be polynomials of the form (2.5) and (2.6) and of degree n and m ($m \geq 2$), respectively. Then the function I_1 in Proposition 2.2 has at most $2[\frac{n-2}{2}] + [\frac{m}{2}] - 1$ zeros on $\{x > 0\}$, counting multiplicity. Therefore, the application of Theorem 2.1 cannot provide examples with strictly more than $2[\frac{n-2}{2}] + [\frac{m}{2}]$ cycles.*

Proof. It is clear that $\deg \frac{g(\cdot)+g(-\cdot)}{2} \leq 2[\frac{m}{2}]$. Suppose n is even. Then $\deg F_e = n$, $\deg f_e = n-2$ and $\deg F_o \leq n-1$. It implies that I_1' has degree at most $2n-4+2[\frac{m}{2}]$. Hence, I_1 has at most $2n-3+2[\frac{m}{2}]$ zeros counting multiplicity. Since I_1 is odd and $F_o(0) = F_o'(0) = g(0) = 0$, we see that I_1 has at least a triple zero at the origin. Given furthermore the symmetry, it follows that there are at most $\frac{2n-3+2[\frac{m}{2}]-3}{2} = n-3+[\frac{m}{2}] = 2[\frac{n-2}{2}] + [\frac{m}{2}] - 1$ zeros on $\{x > 0\}$. Now assume that n is odd. Then $\deg F_o = n$ and $\deg f_e \leq n-3$. Hence, I_1' has degree at most $2n-6+2[\frac{m}{2}]$. It implies that I_1 has at most $2n-5+2[\frac{m}{2}]$ zeros counting multiplicity. Hence, I_1 has at most $\frac{2n-5+2[\frac{m}{2}]-3}{2} = n-4+[\frac{m}{2}] = 2[\frac{n-2}{2}] + [\frac{m}{2}] - 1$ zeros on $\{x > 0\}$. □

3 Proof of Theorem 1.1

The perturbative approach presented in the previous section will be used to treat the case of even degree n (Section 3.1); the case of odd degree n (Section 3.2) will be easy to study due to hyperbolicity of limit cycles obtained in Section 3.1.

3.1 Generalized Liénard equations with n even

In this section, we prove the following statement:

For each $k \geq 3$ and $l \geq 1$ there exists an even polynomial $g_{k,l}$ of degree $m = 2l$, with $g_{k,l}(0) = 0$, such that the function

$$\begin{aligned}\widetilde{H}_{k,l}(x) &:= H_k(x) - \int_0^x g_{k,l}(s)\alpha_k(s)^2\, ds \\ &= \int_0^x \left(\alpha_k'(s)\beta_k(s) - \alpha_k(s)\beta_k'(s) - g_{k,l}(s)\alpha_k(s)^2\right) ds\end{aligned}$$

has $2k - 3 + l$ simple zeros on $\{x > 0\}$, where α_k and β_k are given in Proposition 2.3.

If we now take $F(x) = F_e(x) + \delta F_o(x)$, with $F_o(x) = \beta_k(x)$, $f_e(x) = \alpha_k(x)$ and $F_e(x) = \int_0^x s f_e(s)\, ds$, and $G(x) = -x + \delta g(x)$, with $g(x) = g_{k,l}(x)$, then the above result implies that the expression for I_1 in Proposition 2.2 has $2k - 3 + l$ simple zeros on $\{x > 0\}$, leading to generalized Liénard equations of type $(n, m) = (2k, 2l)$ with $2k - 2 + l$ hyperbolic limit cycles (see Theorem 2.1). Noting that the expression for I_1 remains unchanged if we use $g(x) = g_{k,l}(x) + \rho x^{2l+1}$, $\rho \neq 0$, instead of $g(x) = g_{k,l}(x)$, we have, again by Theorem 2.1, existence of generalized Liénard equations of type $(n, m) = (2k, 2l + 1)$ with $2k - 2 + l$ hyperbolic limit cycles.

For each $k \geq 3$, we use induction on l to prove the above statement. Let us assume we have an example corresponding to l, for $l \geq 0$, with an even polynomial $g_{k,l}$ of degree $2l$ and $g_{k,l}(0) = 0$, and with α_k and β_k of respective degrees $2k - 2$ and $2k - 1$, given in Proposition 2.3, such that $\widetilde{H}_{k,l}$ has $2k - 3 + l$ simple zeros on $\{x > 0\}$. As a direct consequence of Proposition 2.3, this can be performed for $l = 0$ ($g_{k,0} \equiv 0$). For $l \geq 1$, we can write $g_{k,l} = \cdots + \gamma_0 x^{2l}$, with $\gamma_0 \neq 0$.

We now state

$$g_{k,l+1}(x) := g_{k,l}(x) + \gamma_1 \mu^2 x^{2l+2},$$

where $\gamma_1 = -1$ for $l = 0$ and $\gamma_1 = -\operatorname{sgn}(\gamma_0)$ for $l \geq 1$. Here $\operatorname{sgn}(x)$ denotes the sign function. Such a choice of $g_{k,l+1}$ leads to a vector field with $(n, m) = (2k, 2l + 2)$, i.e. 2 degrees higher in G than for $\mu = 0$. It is clear that for small values of μ the $2k - 3 + l$ simple zeros of $\widetilde{H}_{k,l+1}$ that appear for $\mu = 0$ will persist. Besides that, we show that one additional positive simple zero appears in the $O(1/\mu)$ range. It can be easily seen that

Lemma 3.1.

$$\widetilde{H}_{k,l+1}\left(\frac{X}{\mu}\right) = \frac{1}{\mu^{4k+2l-3}}\left[\widetilde{h}(X) + O(\mu)\right],$$

where

$$\widetilde{h}(X) = \begin{cases} -\frac{1}{4k-3}X^{4k-3} + \frac{1}{4k-1}X^{4k-1}, & l = 0, \\ -\frac{\gamma_0}{4k+2l-3}X^{4k+2l-3} + \frac{\operatorname{sgn}(\gamma_0)}{4k+2l-1}X^{4k+2l-1}, & l \geq 1. \end{cases}$$

Hence, additional zeros can be created by looking at simple zeros of $\widetilde{h}(X)$. Clearly, $\widetilde{h}$ has a positive simple zero given by $X = \sqrt{\frac{4k-1}{4k-3}}$ $\left(\text{resp. } X = \sqrt{|\gamma_0|\frac{4k+2l-1}{4k+2l-3}}\right)$ for $l = 0$ (resp. $l \geq 1$). Thus, $\widetilde{H}_{k,l+1}$ has $2k - 3 + l + 1$ simple zeros on $\{x > 0\}$. This finished the inductive step and, therefore, the proof of Theorem 1.1 for even degrees $n \geq 6$.

3.2 Generalized Liénard equations with n odd

Let $n = 2k + 1$, $k \geq 3$, and $m \geq 2$. Based on Theorem 2.1 and Section 3.1, we can choose a polynomial F of degree $2k$ and of the form (2.5), and a polynomial G of degree m and of the form (2.6) such that the system

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = \epsilon_0 \left[\lambda_0 + G(x)\right] \end{cases} \tag{3.1}$$

has $2k - 2 + [\frac{m}{2}]$ hyperbolic limit cycles positioned near the long canard and the simple zeros of the corresponding slow divergence integral, for some ϵ_0 and λ_0. If we change $F(x)$ in (3.1) by $F(x) + \rho x^{2k+1}$, for ρ sufficiently small, then the $2k - 2 + [\frac{m}{2}]$ hyperbolic limit cycles persist. It follows that for degree $n = 2k + 1$, there are at least $n - 3 + [\frac{m}{2}] = 2[\frac{n-2}{2}] + [\frac{m}{2}]$ isolated and hyperbolic periodic orbits. Hence, we have finished the proof of Theorem 1.1 for odd degrees $n \geq 7$.

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