# Approximate controllability of Sobolev type fractional stochastic nonlocal nonlinear differential equations in Hilbert spaces

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**Abstract.** We introduce a new notion called fractional stochastic nonlocal condition, and then we study approximate controllability of class of fractional stochastic nonlinear differential equations of Sobolev type in Hilbert spaces. We use Hölder's inequality, fixed point technique, fractional calculus, stochastic analysis and methods adopted directly from deterministic control problems for the main results. A new set of sufficient conditions is formulated and proved for the fractional stochastic control system to be approximately controllable. An example is given to illustrate the abstract results.

**Keywords:** approximate controllability, fractional Sobolev type equation, stochastic system, fixed point technique, fractional stochastic nonlocal condition, Hölder's inequality.

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## 1  Introduction

We are concerned with the following fractional stochastic nonlocal system of Sobolev type

$$^{C}D_t^q[Lx(t)] = Mx(t) + Bu(t) + f(t, x(t)) + \sigma_1(t, x(t))\frac{dw_1(t)}{dt},  \tag{1.1}$$

$$^{L}D_t^{1-q}x(t)|_{t=0} = \sigma_2(t, x(t))\frac{dw_2(t)}{dt},  \tag{1.2}$$

where $^{C}D_t^q$ and $^{L}D_t^{1-q}$ are the Caputo and Riemann–Liouville fractional derivatives with $0 < q \leq 1$, and $t \in J = [0, b]$. Let $X$ and $Y$ be two Hilbert spaces and let the state $x(\cdot)$ take its values in $X$. We assume that the operators $L$ and $M$ are defined on domains contained in $X$ and ranges contained in $Y$, the control function $u(\cdot)$ belongs to the space $L_\Gamma^2(J, U)$, a Hilbert space of admissible control functions with $U$ as a Hilbert space and $B$ is a bounded linear operator from $U$ into $Y$. It is also assumed that $f: J \times X \to Y, \sigma_1: J \times X \to L_2^0$ and

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$\sigma_2 \colon J \times X \to L_2^0$ are appropriate functions; $x_0$ is a $\Gamma_0$ measurable $X$-valued random variable independent of $w_1$ and $w_2$. Here $L_2^0$, $\Gamma$, $\Gamma_0$, $w_1$ and $w_2$ will be specified later.

During the past three decades, fractional differential equations and their applications have gained a lot of importance, mainly because this field has become a powerful tool in modeling several complex phenomena in numerous seemingly diverse and widespread fields of science and engineering [2, 5, 11, 16, 18, 28, 29, 32]. Recently, there has been a significant development in the existence results for boundary value problems of nonlinear fractional differential equations and inclusions [1, 6].

One of the important fundamental concepts in mathematical control theory is controllability, it plays a vital role in both deterministic and stochastic control systems. Since, the controllability notion has extensive industrial and biological applications, in the literature, there are many different notions of controllability, both for linear and nonlinear dynamical systems. Controllability of the deterministic and stochastic dynamical control systems in infinite dimensional spaces is well-developed using different kind of approaches. It should be mentioned that the theory of controllability for nonlinear fractional dynamical systems is still in the initial stage. There are few works in controllability problems for different kind of systems described by fractional differential equations [41, 42].

The exact controllability for semilinear fractional order system, when the nonlinear term is independent of the control function, is proved by many authors [3, 12, 38]. In these papers, the authors have proved the exact controllability by assuming that the controllability operator has an induced inverse on a quotient space. However, if the semigroup associated with the system is compact then the controllability operator is also compact and hence the induced inverse does not exist because the state space is infinite dimensional [46]. Thus, the concept of exact controllability is too strong and has limited applicability and the approximate controllability is a weaker concept than complete controllability and it is completely adequate in applications for these control systems.

In [10, 44] the approximate controllability of first order delay control systems has been proved when nonlinear term is a function of both state function and control function by assuming that the corresponding linear system be approximately controllable. To prove the approximate controllability of a first order system, with or without delay, a relation between the reachable set of a semilinear system and that of the corresponding linear system is proved in [4, 9, 20, 21, 45]. There are several papers devoted to the approximate controllability for semilinear control systems, when the nonlinear term is independent of control function [25, 39, 40, 43].

Stochastic differential equations have attracted great interest due to its applications in various fields of science and engineering. There are many interesting results on the theory and applications of stochastic differential equations, (see [3, 7, 8, 30, 36] and the references therein). To build more realistic models in economics, social sciences, chemistry, finance, physics and other areas, stochastic effects need to be taken into account. Therefore, many real world problems can be modeled by stochastic differential equations. The deterministic models often fluctuate due to noise, so we must move from deterministic control to stochastic control problems.

In the present literature there is only a limited number of papers that deal with the approximate controllability of fractional stochastic systems [27], as well as with the existence and controllability results of fractional evolution equations of Sobolev type [26].

R. Sakthivel et al. [37] studied the approximate controllability of a class of dynamic control systems described by nonlinear fractional stochastic differential equations in Hilbert spaces.

In [24], the authors proved the approximate controllability of Sobolev type nonlocal fractional stochastic dynamic systems in Hilbert spaces. More recent works can be found in [41, 42]. A. Debbouche, D. Baleanu and R. P. Agarwal [13] established a class of fractional nonlocal non-linear integro-differential equations of Sobolev type using new solution operators. M. Fečkan, J. R. Wang and Y. Zhou [19] presented the controllability results corresponding to two admissible control sets for fractional functional evolution equations of Sobolev type in Banach spaces with the help of two new characteristic solution operators and their properties, such as boundedness and compactness. Debbouche and Torres [14, 15] introduced both fractional nonlocal condition and nonlocal control condition for establishing approximate controllability of fractional delay differential equations and inclusions.

In this work, we present a new concept in stochastic analysis that we present a nonlocal condition given in stochastic term together with Riemann–Liouville fractional derivative, then we use this tool to establish the approximate controllability of Sobolev type fractional deterministic nonlocal stochastic control systems in Hilbert spaces.

The paper is organized as follows: in Section 2, we present some essential facts in fractional calculus, semigroup theory, stochastic analysis and control theory that will be used to obtain our main results. In Section 3, we state and prove existence and approximate controllability results for Sobolev type fractional stochastic system (1.1)–(1.2). Finally, in Section 4, as an example, a fractional partial dynamical stochastic control differential equation with a fractional stochastic nonlocal condition is considered.

## 2 Preliminaries

In this section we give some basic definitions, notations, properties and lemmas, which will be used throughout the work. In particular, we state main properties of fractional calculus [22, 31, 34], well known facts in semigroup theory [23, 33, 49] and elementary principles of stochastic analysis [30, 35].

**Definition 2.1.** The fractional integral of order $\alpha > 0$ of a function $f \in L^1([a, b], \mathbb{R}^+)$ is given by

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} f(s) \, ds,$$

where $\Gamma$ is the gamma function. If $a = 0$, we can write $I^\alpha f(t) = (g_\alpha * f)(t)$, where

$$g_\alpha(t) := \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

and as usual, $*$ denotes the convolution of functions. Moreover, $\lim_{\alpha \to 0} g_\alpha(t) = \delta(t)$, with $\delta$ the delta Dirac function.

**Definition 2.2.** The Riemann–Liouville derivative of order $n - 1 < \alpha < n, n \in \mathbb{N}$, for a function $f \in C([0, \infty))$ is given by

$$^L D^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t - s)^{\alpha+1-n}} \, ds, \quad t > 0.$$

**Definition 2.3.** The Caputo derivative of order $n - 1 < \alpha < n, n \in \mathbb{N}$, for a function $f \in C([0, \infty))$ is given by

$$^C D^\alpha f(t) = \,^L D^\alpha \left( f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0.$$

**Remark 2.4.** The following properties hold (see, e.g., [50]).

(i) If $f \in C^n([0, \infty))$, then

$$^C D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} \, ds = I^{n-\alpha} f^n(t), \quad t > 0, \ n - 1 < \alpha < n, \ n \in \mathbb{N}.$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If $f$ is an abstract function with values in $X$, then the integrals which appear in Definitions 2.1–2.3 are taken in Bochner's sense.

We introduce the following assumptions on the operators $L$ and $M$.

(H$_1$) $L$ and $M$ are linear operators, and $M$ is closed.

(H$_2$) $D(L) \subset D(M)$ and $L$ is bijective.

(H$_3$) $L^{-1} : Y \to D(L) \subset X$ is a linear compact operator.

**Remark 2.5.** From (H$_3$), we deduce that $L^{-1}$ is bounded operators, for short, we denote by $C = \|L^{-1}\|$. Note (H$_3$) also implies that $L$ is closed since the fact: $L^{-1}$ is closed and injective, then its inverse is also closed. It comes from (H$_1$)–(H$_3$) and the closed graph theorem, we obtain the boundedness of the linear operator $ML^{-1} : Y \to Y$. Consequently, $ML^{-1}$ generates a semigroup $\{S(t) := e^{ML^{-1}t}, t \geq 0\}$. We suppose that $M_0 := \sup_{t \geq 0} \|S(t)\| < \infty$.
According to previous definitions, it is suitable to rewrite problem (1.1)–(1.2) as the equivalent integral equation

$$\begin{aligned}
Lx(t) = Lx(0) &+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [Mx(s) + Bu(s) + f(s, x(s))] \, ds \\
&+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \sigma_1(s, x(s)) \, dw_1(s),
\end{aligned} \tag{2.1}$$

provided the integrals in (2.1) exist.

**Remark 2.6.** We note that:

(a) For the nonlocal condition, the function $x(0)$ is dependent on $t$.

(b) The Riemann–Liouville fractional derivative of $x(0)$ is well defined and $^L D_t^{1-q} x(0) \neq 0$.

(c) The function $x(0)$ takes the form $x_0 + \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} \sigma_2(s, x(s)) \, dw_2(s)$, where $x(0)|_{t=0} = x_0$.

(d) The explicit and implicit integrals given in (2.1) exist (taken in Bochner's sense).

Before formulating the definition of mild solution of (1.1)–(1.2), we first we recall. Let $(\Omega, \Gamma, P)$ be a complete probability space equipped with a normal filtration $\Gamma_t, t \in J$ satisfying the usual conditions (i.e., right continuous and $\Gamma_0$ containing all $P$-null sets). We consider four real separable spaces $X, Y, E$ and $U$, and $Q$-Wiener process on $(\Omega, \Gamma_b, P)$ with the linear bounded covariance operator $Q$ such that $trQ < \infty$. We assume that there exist complete orthonormal systems $\{e_{1,n}\}_{n \geq 1}, \{e_{2,n}\}_{n \geq 1}$ in $E$, bounded sequences of non-negative real numbers $\{\lambda_{1,n}\}, \{\lambda_{2,n}\}$ such that $Qe_{1,n} = \lambda_{1,n}e_{1,n}$, $Qe_{2,n} = \lambda_{2,n}e_{2,n}$, $n = 1, 2, \ldots$, and sequences $\{\beta_{1,n}\}_{n \geq 1}, \{\beta_{2,n}\}_{n \geq 1}$ of independent Brownian motions such that

$$\langle w_1(t), e_1 \rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_{1,n}} \langle e_{1,n}, e_1 \rangle \beta_{1,n}(t), \quad e_1 \in E, \ t \in J,$$

$$\langle w_2(t), e_2 \rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_{2,n}} \langle e_{2,n}, e_2 \rangle \beta_{2,n}(t), \quad e_2 \in E, \ t \in J,$$

and $\Gamma_t = \Gamma_t^{w_1,w_2}$, where $\Gamma_t^{w_1,w_2}$ is the sigma algebra generated by $\{(w_1(s), w_2(s)) : 0 \leq s \leq t\}$. Let $L_2^0 = L_2(Q^{1/2}E; X)$ be the space of all Hilbert–Schmidt operators from $Q^{1/2}E$ to $X$ with the inner product $\langle \psi, \pi \rangle L_2^0 = tr[\psi Q \pi^*]$. Let $L^2(\Gamma_b, X)$ be the Banach space of all $\Gamma_b$-measurable square integrable random variables with values in the Hilbert space $X$. Let $E(\cdot)$ denote the expectation with respect to the measure $P$. Let $C(J; L^2(\Gamma, X))$ be the Hilbert space of continuous maps from $J$ into $L^2(\Gamma, X)$ satisfying $\sup_{t \in J} E\|x(t)\|^2 < \infty$. Let $H_2(J; X)$ be a closed subspace of $C(J; L^2(\Gamma, X))$ consisting of a measurable and $\Gamma_t$-adapted $X$-valued process $x \in C(J; L^2(\Gamma, X))$ endowed with the norm $\|x\|_{H_2} = (\sup_{t \in J} E\|x(t)\|_X^2)^{1/2}$. For details, we refer the reader to [35, 37] and references therein.

The following results will be used throughout this paper.

**Lemma 2.7** ([27]). *Let $G \colon J \times \Omega \to L_2^0$ be a strongly measurable mapping such that $\int_0^b E\|G(t)\|_{L_2^0}^p dt < \infty$. Then*

$$E \left\| \int_0^t G(s)\, dw(s) \right\|^p \leq L_G \int_0^t E\|G(s)\|_{L_2^0}^p\, ds$$

*for all $0 \leq t \leq b$ and $p \geq 2$, where $L_G$ is the constant involving $p$ and $b$.*

Now, we present the mild solution of the problem (1.1)–(1.2).

**Definition 2.8** (Compare with [11, 17] and [19, 50]). A stochastic process $x \in H_2(J, X)$ is a mild solution of (1.1)–(1.2) if for each control $u \in L_\Gamma^2(J, U)$, it satisfies the following integral equation:

$$
\begin{aligned}
x(t) = {}& \mathcal{S}(t)L \left[ x_0 + \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} \sigma_2(s, x(s))\, dw_2(s) \right] \\
& + \int_0^t (t-s)^{q-1} \mathcal{T}(t-s)[Bu(s) + f(s, x(s))]\, ds \\
& + \int_0^t (t-s)^{q-1} \mathcal{T}(t-s)\sigma(s, x(s))\, dw(s),
\end{aligned}
\tag{2.2}
$$

where $\mathcal{S}(t)$ and $\mathcal{T}(t)$ are characteristic operators given by

$$\mathcal{S}(t) = \int_0^{\infty} L^{-1}\xi_q(\theta)S(t^q\theta)\, d\theta \quad \text{and} \quad \mathcal{T}(t) = q \int_0^{\infty} L^{-1}\theta\xi_q(\theta)S(t^q\theta)\, d\theta.$$

Here, $S(t)$ is a $C_0$-semigroup generated by the linear operator $ML^{-1} \colon Y \to Y$; $\xi_q$ is a probability density function defined on $(0, \infty)$, that is $\xi_q(\theta) \geq 0$, $\theta \in (0, \infty)$ and $\int_0^{\infty} \xi_q(\theta)d\theta = 1$.

**Lemma 2.9** ([47, 48, 50]). *The operators $\{\mathcal{S}(t)\}_{t \geq 0}$ and $\{\mathcal{T}(t)\}_{t \geq 0}$ are strongly continuous, i.e., for $x \in X$ and $0 \leq t_1 < t_2 \leq b$, we have $\|\mathcal{S}(t_2)x - \mathcal{S}(t_1)x\| \to 0$ and $\|\mathcal{T}(t_2)x - \mathcal{T}(t_1)x\| \to 0$ as $t_2 \to t_1$.*

We impose the following conditions on data of the problem.

(i) For any fixed $t \geq 0$, $\mathcal{S}(t)$ and $\mathcal{T}(t)$ are bounded linear operators, i.e., for any $x \in X$,

$$\|\mathcal{S}(t)x\| \leq CM_0\|x\|, \quad \|\mathcal{T}(t)x\| \leq \frac{CM_0}{\Gamma(q)}\|x\|.$$

(ii) The functions $f\colon J \times X \to Y, \sigma_1\colon J \times X \to L_2^0$ and $\sigma_2\colon J \times X \to L_2^0$ satisfy linear growth and Lipschitz conditions. Moreover, there exist positive constants $N_1, N_2 > 0, L_1, L_2 > 0$ and $k_1, k_2 > 0$ such that

$$
\begin{aligned}
\|f(t,x) - f(t,y)\|^2 &\leq N_1\|x-y\|^2, & \|f(t,x)\|^2 &\leq N_2(1+\|x\|^2), \\
\|\sigma_1(t,x) - \sigma_1(t,y)\|_{L_2^0}^2 &\leq L_1\|x-y\|^2, & \|\sigma_1(t,x)\|_{L_2^0}^2 &\leq L_2(1+\|x\|^2), \\
\|\sigma_2(t,x) - \sigma_2(t,y)\|_{L_2^0}^2 &\leq k_1\|x-y\|^2, & \|\sigma_2(t,x)\|_{L_2^0}^2 &\leq k_2(1+\|x\|^2).
\end{aligned}
$$

(iii) The linear stochastic system is approximately controllable on $J$.

For each $0 \leq t < b$, the operator $\alpha(\alpha I + \Psi_0^b)^{-1} \to 0$ in the strong operator topology as $\alpha \to 0^+$, where $\Psi_0^b = \int_0^b (b-s)^{2(q-1)}\mathcal{T}(b-s)BB^*\mathcal{T}^*(b-s)\,ds$ is the controllability Gramian, here $B^*$ denotes the adjoint of $B$ and $\mathcal{T}^*(t)$ is the adjoint of $\mathcal{T}(t)$.

Observe that Sobolev type linear fractional deterministic control system

$$
{}^C D_t^q[Lx(t)] = Mx(t) + Bu(t), \quad t \in J, \tag{2.3}
$$
$$
x(0) = x_0, \tag{2.4}
$$

corresponding to (1.1)–(1.2) is approximately controllable on $J$ iff the operator $\alpha(\alpha I + \Psi_0^b)^{-1} \to 0$ strongly as $\alpha \to 0^+$. The approximate controllability for linear fractional deterministic control system (2.3)–(2.4) is a natural generalization of approximate controllability of linear first order control system ($q = 1$ and $L$ is the identity) [14].

**Definition 2.10.** System (1.1)–(1.2) is approximately controllable on $J$ if $\overline{\mathfrak{R}(b)} = L^2(\Omega, \Gamma_b, X)$, where

$$\mathfrak{R}(b) = \{x(b) = x(b,u) : u \in L_\Gamma^2(J,U)\},$$

here $L_\Gamma^2(J,U)$, is the closed subspace of $L_\Gamma^2(J \times \Omega; U)$, consisting of all $\Gamma_t$ adapted, $U$-valued stochastic processes.

The following lemma is required to define the control function [37].

**Lemma 2.11.** *For any $\widetilde{x}_b \in L^2(\Gamma_b, X)$, there exists $\widetilde{\varphi} \in L_\Gamma^2(\Omega; L^2(0,b; L_2^0))$ such that $\widetilde{x}_b = E\widetilde{x}_b + \int_0^b \widetilde{\varphi}(s)\,dw(s)$.*

Now for any $\alpha > 0$ and $\tilde{x}_b \in L^2(\Gamma_b, X)$, we define the control function in the following form

$$
u^\alpha(t, x)
$$

$$
= B^*(b-t)^{q-1}\mathcal{T}^*(b-t)\left[(\alpha I + \Psi_0^b)^{-1}\left\{E\tilde{x}_b\right.\right.
$$

$$
\left.\left. - \mathcal{S}(b)L\left(x_0 + \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}\sigma_2(s, x(s))\,dw_2(s)\right)\right\} + \int_0^t (\alpha I + \Psi_0^b)^{-1}\tilde{\varphi}(s)\,dw_1(s)\right]
$$

$$
- B^*(b-t)^{q-1}\mathcal{T}^*(b-t)\int_0^t (\alpha I + \Psi_0^b)^{-1}(b-s)^{q-1}\mathcal{T}(b-s)f(s, x(s))\,ds
$$

$$
- B^*(b-t)^{q-1}\mathcal{T}^*(b-t)\int_0^t (\alpha I + \Psi_0^b)^{-1}(b-s)^{q-1}\mathcal{T}(b-s)\sigma_1(s, x(s))\,dw_1(s).
$$

**Lemma 2.12.** *There exist positive real constants $\hat{M}, \hat{N}$ such that for all $x, y \in H_2$, we have*

$$
E\|u^\alpha(t, x) - u^\alpha(t, y)\|^2 \le \hat{M}E\|x(t) - y(t)\|^2, \tag{2.5}
$$

$$
E\|u^\alpha(t, x)\|^2 \le \hat{N}\left(\frac{1}{b} + E\|x(t)\|^2\right). \tag{2.6}
$$

*Proof.* We start to prove (2.5). Let $x, y \in H_2$, from the Hölder's inequality, Lemma 2.7 and the assumption on the data, we obtain

$$
E\|u^\alpha(t, x) - u^\alpha(t, y)\|^2
$$

$$
\le 3E\left\|B^*(b-t)^{q-1}\mathcal{T}^*(b-t)(\alpha I + \Psi_0^b)^{-1}\frac{\mathcal{S}(b)L}{\Gamma(1-q)}\right.
$$

$$
\left.\times \int_0^t (t-s)^{-q}[\sigma_2(s, x(s)) - \sigma_2(s, y(s))]\,dw_2(s)\right\|^2
$$

$$
+ 3E\left\|B^*(b-t)^{q-1}\mathcal{T}^*(b-t)\right.
$$

$$
\left.\times \int_0^t (\alpha I + \Psi_0^b)^{-1}(b-s)^{q-1}\mathcal{T}(b-s)[f(s, x(s)) - f(s, y(s))]\,ds\right\|^2
$$

$$
+ 3E\left\|B^*(b-t)^{q-1}\mathcal{T}^*(b-t)\right.
$$

$$
\left.\times \int_0^t (\alpha I + \Psi_0^b)^{-1}(b-s)^{q-1}\mathcal{T}(b-s)[\sigma_1(s, x(s)) - \sigma_1(s, y(s))]\,dw_1(s)\right\|^2
$$

$$
\le \frac{3}{\alpha^2}\|B\|^2(b)^{2q-2}\left(\frac{CM_0}{\Gamma(q)}\right)^2\left(\frac{CM_0\|L\|}{\Gamma(1-q)}\right)^2\frac{b^{-2q+1}}{(-2q+1)}k_1\int_0^t E\|x(s) - y(s)\|^2\,ds
$$

$$
+ \frac{3}{\alpha^2}\|B\|^2(b)^{2q-2}\left(\frac{CM_0}{\Gamma(q)}\right)^4\frac{b^{2q-1}}{(2q-1)}N_1\int_0^t E\|x(s) - y(s)\|^2\,ds
$$

$$
+ \frac{3}{\alpha^2}\|B\|^2(b)^{2q-2}\left(\frac{CM_0}{\Gamma(q)}\right)^4\frac{b^{2q-1}}{(2q-1)}L_1\int_0^t E\|x(s) - y(s)\|^2\,ds
$$

$$
\le \hat{M}E\|x(t) - y(t)\|^2,
$$

where

$$
\hat{M} = \frac{3}{\alpha^2}\|B\|^2(b)^{2q-2}\left\{\left(\frac{CM_0}{\Gamma(q)}\right)^2\left(\frac{CM_0\|L\|}{\Gamma(1-q)}\right)^2\frac{b^{-2q+1}}{(-2q+1)}bk_1 + \left(\frac{CM_0}{\Gamma(q)}\right)^4\frac{b^{2q-1}}{(2q-1)}b[N_1 + L_1]\right\}.
$$

The proof of the inequality (2.6) can be established in a similar way to that of (2.5).                    □

## 3   Approximate controllability

In this section, we formulate and prove conditions for the existence and approximate controllability results of the fractional stochastic nonlocal dynamic control system of Sobolev type (1.1)–(1.2) using the contraction mapping principle. For any $\alpha > 0$, define the operator $F_\alpha \colon H_2 \to H_2$ by

$$
\begin{aligned}
F_\alpha x(t) = \mathcal{S}(t)L\left[x_0 + \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}\sigma_2(s,x(s))\,dw_2(s)\right] \\
+ \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)[Bu^\alpha(s,x) + f(s,x(s))]\,ds \\
+ \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)\sigma_1(s,x(s))\,dw_1(s).
\end{aligned}
\tag{3.1}
$$

We state and prove the following lemma, which will be used for the main results.

**Lemma 3.1.** *For any $x \in H_2$, $F_\alpha(x)(t)$ is continuous on $J$ in $L^2$-sense.*

*Proof.* Let $0 \leq t_1 < t_2 \leq b$. Then for any fixed $x \in H_2$, from (3.1), we have

$$
E\left\|(F_\alpha x)(t_2) - (F_\alpha x)(t_1)\right\|^2 \leq 4\left[\sum_{i=1}^4 E\left\|\Pi_i^x(t_2) - \Pi_i^x(t_1)\right\|^2\right].
$$

From Lemma 2.7, we begin with the first term

$$
\begin{aligned}
&E\left\|\Pi_1^x(t_2) - \Pi_1^x(t_1)\right\|^2 \\
&= E\left\|\mathcal{S}(t_2)L\left[x_0 + \frac{1}{\Gamma(1-q)}\int_0^{t_2}(t_2-s)^{-q}\sigma_2(s,x(s))\,dw_2(s)\right]\right. \\
&\qquad\left. - \mathcal{S}(t_1)L\left[x_0 + \frac{1}{\Gamma(1-q)}\int_0^{t_1}(t_1-s)^{-q}\sigma_2(s,x(s))\,dw_2(s)\right]\right\|^2 \\
&\leq E\left\|(\mathcal{S}(t_2)-\mathcal{S}(t_1))L\left[\frac{1}{\Gamma(1-q)}\int_0^{t_1}(t_1-s)^{-q}\sigma_2(s,x(s))\,dw_2(s)\right]\right\|^2 \\
&\quad + E\left\|\mathcal{S}(t_2)L\left[\frac{1}{\Gamma(1-q)}\int_0^{t_1}((t_2-s)^{-q}-(t_1-s)^{-q})\sigma_2(s,x(s))\,dw_2(s)\right]\right\|^2 \\
&\quad + E\left\|\mathcal{S}(t_2)L\left[\frac{1}{\Gamma(1-q)}\int_{t_1}^{t_2}(t_2-s)^{-q}\sigma_2(s,x(s))\,dw_2(s)\right]\right\|^2 \\
&\leq \|L\|^2\left[L_\sigma \frac{t_1^{-2q+1}}{(-2q+1)}\left(\frac{1}{\Gamma(1-q)}\right)^2 k_2(1+\|x\|^2)\right]E\|\mathcal{S}(t_2)-\mathcal{S}(t_1)\|^2 \\
&\quad + \|\mathcal{S}(t_2)\|^2\|L\|^2\left[\left(\frac{1}{\Gamma(1-q)}\right)^2 L_\sigma\left(\int_0^{t_1}((t_2-s)^{-q}-(t_1-s)^{-q})^2\,ds\right)\right. \\
&\qquad\qquad\left. \times\left(\int_0^{t_1} E\|\sigma_2(s,x(s))\|^2\,ds\right)\right] \\
&\quad + \|\mathcal{S}(t_2)\|^2\|L\|^2\left[\left(\frac{1}{\Gamma(1-q)}\right)^2 \frac{(t_2-t_1)^{-2q+1}}{(-2q+1)}L_\sigma\int_{t_1}^{t_2} E\|\sigma_2(s,x(s))\|^2\,ds\right]
\end{aligned}
$$

The strong continuity of $\mathcal{S}(t)$ implies that the right-hand side of the last inequality tends to zero as $t_2 - t_1 \to 0$.

Next, it follows from Hölder's inequality and assumptions on the data that

$$E \left\| \Pi_2^x(t_2) - \Pi_2^x(t_1) \right\|^2$$

$$= E \left\| \int_0^{t_2} (t_2 - s)^{q-1} \mathcal{T}(t_2 - s) Bu^\alpha(s, x) \, ds - \int_0^{t_1} (t_1 - s)^{q-1} \mathcal{T}(t_1 - s) Bu^\alpha(s, x) \, ds \right\|^2$$

$$\leq E \left\| \int_0^{t_1} (t_1 - s)^{q-1} (\mathcal{T}(t_2 - s) - \mathcal{T}(t_1 - s)) Bu^\alpha(s, x) \, ds \right\|^2$$

$$+ E \left\| \int_0^{t_1} ((t_2 - s)^{q-1} - (t_1 - s)^{q-1}) \mathcal{T}(t_2 - s) Bu^\alpha(s, x) \, ds \right\|^2$$

$$+ E \left\| \int_{t_1}^{t_2} (t_2 - s)^{q-1} \mathcal{T}(t_2 - s) Bu^\alpha(s, x) \, ds \right\|^2$$

$$\leq \frac{t_1^{2q-1}}{2q - 1} \int_0^{t_1} E \left\| (\mathcal{T}(t_2 - s) - \mathcal{T}(t_1 - s)) Bu^\alpha(s, x) \, ds \right\|^2$$

$$+ \left( \frac{CM_0}{\Gamma(q)} \right)^2 \|B\|^2 \left( \int_0^{t_1} ((t_2 - s)^{q-1} - (t_1 - s)^{q-1})^2 \, ds \right) \left( \int_0^{t_1} E \left\| u^\alpha(s, x) \right\|^2 ds \right)$$

$$+ \frac{(t_2 - t_1)^{2q-1}}{1 - 2q} \left( \frac{CM_0}{\Gamma(q)} \right)^2 \|B\|^2 \int_{t_1}^{t_2} E \left\| u^\alpha(s, x) \right\|^2 \, ds.$$

Also, we have

$$E \left\| \Pi_3^x(t_2) - \Pi_3^x(t_1) \right\|^2$$

$$= E \left\| \int_0^{t_2} (t_2 - s)^{q-1} \mathcal{T}(t_2 - s) f(s, x(s)) \, ds - \int_0^{t_1} (t_1 - s)^{q-1} \mathcal{T}(t_1 - s) f(s, x(s)) \, ds \right\|^2$$

$$\leq E \left\| \int_0^{t_1} (t_1 - s)^{q-1} (\mathcal{T}(t_2 - s) - \mathcal{T}(t_1 - s)) f(s, x(s)) \, ds \right\|^2$$

$$+ E \left\| \int_0^{t_1} ((t_2 - s)^{q-1} - (t_1 - s)^{q-1}) \mathcal{T}(t_2 - s) f(s, x(s)) \, ds \right\|^2$$

$$+ E \left\| \int_{t_1}^{t_2} (t_2 - s)^{q-1} \mathcal{T}(t_2 - s) f(s, x(s)) \, ds \right\|^2$$

$$\leq \frac{t_1^{2q-1}}{2q - 1} \int_0^{t_1} E \left\| (\mathcal{T}(t_2 - s) - \mathcal{T}(t_1 - s)) f(s, x(s)) \, ds \right\|^2$$

$$+ \left( \frac{CM_0}{\Gamma(q)} \right)^2 \left( \int_0^{t_1} ((t_2 - s)^{q-1} - (t_1 - s)^{q-1})^2 \, ds \right) \left( \int_0^{t_1} E \left\| f(s, x(s)) \right\|^2 ds \right)$$

$$+ \frac{(t_2 - t_1)^{2q-1}}{1 - 2q} \left( \frac{CM_0}{\Gamma(q)} \right)^2 \int_{t_1}^{t_2} E \left\| f(s, x(s)) \right\|^2 \, ds.$$

Furthermore, we use Lemma 2.7 and previous assumptions, we obtain

$$E \left\| \Pi_4^x(t_2) - \Pi_4^x(t_1) \right\|^2$$

$$= E \left\| \int_0^{t_2} (t_2 - s)^{q-1} \mathcal{T}(t_2 - s) \sigma(s, x(s)) \, dw(s) - \int_0^{t_1} (t_1 - s)^{q-1} \mathcal{T}(t_1 - s) \sigma(s, x(s)) \, dw(s) \right\|^2$$

$$\leq E \left\| \int_0^{t_1} (t_1 - s)^{q-1} (\mathcal{T}(t_2 - s) - \mathcal{T}(t_1 - s))\sigma(s, x(s)) \, dw(s) \right\|^2$$

$$+ E \left\| \int_0^{t_1} ((t_2 - s)^{q-1} - (t_1 - s)^{q-1})\mathcal{T}(t_2 - s)\sigma(s, x(s)) \, dw(s) \right\|^2$$

$$+ E \left\| \int_{t_1}^{t_2} (t_2 - s)^{q-1} \mathcal{T}(t_2 - s)\sigma(s, x(s)) \, dw(s) \right\|^2$$

$$\leq L_\sigma \frac{t_1^{2q-1}}{2q-1} \int_0^{t_1} E \left\| (\mathcal{T}(t_2 - s) - \mathcal{T}(t_1 - s))\sigma(s, x(s)) \, ds \right\|^2$$

$$+ L_\sigma \left( \int_0^{t_1} ((t_2 - s)^{q-1} - (t_1 - s)^{q-1})^2 \, ds \right) \left( \int_0^{t_1} E \left\| \mathcal{T}(t_2 - s)\sigma(s, x(s)) \right\|^2 \, ds \right)$$

$$+ L_\sigma \frac{(t_2 - t_1)^{2q-1}}{1 - 2q} \left( \frac{CM_0}{\Gamma(q)} \right)^2 \int_{t_1}^{t_2} E \left\| \mathcal{T}(t_2 - s)\sigma(s, x(s)) \right\|^2 \, ds.$$

Hence using the strong continuity of $\mathcal{T}(t)$ and Lebesgue's dominated convergence theorem, we conclude that the right-hand side of the above inequalities tends to zero as $t_2 - t_1 \to 0$. Thus, we conclude $F_\alpha(x)(t)$ is continuous from the right of $[0, b)$. A similar argument shows that it is also continuous from the left of $(0, b]$. □

**Theorem 3.2.** *Assume hypotheses (i) and (ii) are satisfied. Then the system* (1.1)–(1.2) *has a mild solution on J.*

*Proof.* We prove the existence of a fixed point of the operator $F_\alpha$ by using the contraction mapping principle. First, we show that $F_\alpha(H_2) \subset H_2$. Let $x \in H_2$. From (3.1), we obtain

$$E \left\| F_\alpha x(t) \right\|^2 \leq 4 \left[ \sup_{t \in J} \sum_{i=1}^4 E \left\| \Pi_i^x(t) \right\|^2 \right]. \tag{3.2}$$

Using assumptions (i)–(ii), Lemma 2.12, and standard computations yield

$$\sup_{t \in J} E \left\| \Pi_1^x(t) \right\|^2 \leq C^2 M_0^2 \|L\|^2 \left[ \|x_0\|^2 + \left( \frac{1}{\Gamma(1-q)} \right)^2 \frac{b^{-2q+1}}{(-2q+1)} L_\sigma k_2 (1 + \|x\|^2) \right] \tag{3.3}$$

and

$$\sup_{t \in J} \sum_{i=2}^4 E \left\| \Pi_i^x(t) \right\|^2 \leq \left( \frac{CM_0}{\Gamma(q)} \right)^2 \frac{b^{2q-1}}{2q-1} \|B\|^2 \hat{N} \left( \frac{1}{b} + \|x\|_{H_2}^2 \right)$$

$$+ \left( \frac{CM_0}{\Gamma(q)} \right)^2 \left[ \frac{b^{2q-1}}{2q-1} N_2 - \frac{b^{2q-1}}{2q-1} L_2 L_\sigma \right] \left( 1 + \|x\|_{H_2}^2 \right). \tag{3.4}$$

Hence (3.2)–(3.4) imply that $E\|F_\alpha x\|_{H_2}^2 < \infty$. By Lemma 3.1, $F_\alpha x \in H_2$. Thus for each $\alpha > 0$, the operator $F_\alpha$ maps $H_2$ into itself. Next, we use the Banach fixed point theorem to prove that $F_\alpha$ has a unique fixed point in $H_2$. We claim that there exists a natural $n$ such that $F_\alpha^n$ is a contraction on $H_2$. Indeed, let $x, y \in H_2$, we have

$$E \left\| (F_\alpha x)(t) - (F_\alpha y)(t) \right\|^2 \leq 4 \sum_{i=1}^4 E \left\| \Pi_i^x(t) - \Pi_i^y(t) \right\|^2$$

$$\leq 4k_1 C^2 M_0^2 \|L\|^2 L_\sigma \left( \frac{1}{\Gamma(1-q)} \right)^2 \frac{b^{-2q+1}}{(-2q+1)} E \|x(t) - y(t)\|^2$$

$$+ 4 \left( \frac{CM_0}{\Gamma(q)} \right)^2 \left[ \hat{M} \|B\|^2 \frac{b^{2q-1}}{2q-1} + \frac{b^{2q-1}}{2q-1} N_1 + \frac{b^{2q-1}}{2q-1} L_1 L_\sigma \right]$$
$$\times E \|x(t) - y(t)\|^2.$$

Hence, we obtain a positive real constant $\gamma(\alpha)$ such that

$$E \|(F_\alpha x)(t) - (F_\alpha y)(t)\|^2 \le \gamma(\alpha) E \|x(t) - y(t)\|^2, \tag{3.5}$$

for all $t \in J$ and all $x, y \in H_2$. For any natural number $n$, it follows from successive iteration of above inequality (3.5) that, by taking the supremum over $J$,

$$\|(F_\alpha^n x)(t) - (F_\alpha^n y)(t)\|_{H_2}^2 \le \frac{\gamma^n(\alpha)}{n!} \|x - y\|_{H_2}^2. \tag{3.6}$$

For any fixed $\alpha > 0$, for sufficiently large $n$, $\frac{\gamma^n(\alpha)}{n!} < 1$. It follows from (3.6) that $F_\alpha^n$ is a contraction mapping, so that the contraction principle ensures that the operator $F_\alpha$ has a unique fixed point $x_\alpha$ in $H_2$, which is a mild solution of (1.1)–(1.2). □

**Theorem 3.3.** *Assume that the assumptions (i)–(iii) hold. Further, if the functions $f$, $\sigma_1$ and $\sigma_2$ are uniformly bounded and $\{\mathcal{T}(t) : t \ge 0\}$ is compact, then the system (1.1)–(1.2) is approximately controllable on $J$.*

*Proof.* Let $x_\alpha$ be a fixed point of $F_\alpha$. By using the stochastic Fubini theorem, it can be easily seen that

$$x_\alpha(b) = \tilde{x}_b - \alpha(\alpha I + \Psi)^{-1} \left( E\tilde{x}_b - \mathcal{S}(b) L \left[ x_0 + \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} \sigma_2(s, x_\alpha(s)) \, dw_2(s) \right] \right)$$
$$+ \alpha \int_0^b (\alpha I + \Psi_s^b)^{-1} (b-s)^{q-1} \mathcal{T}(b-s) f(s, x_\alpha(s)) \, ds$$
$$+ \alpha \int_0^b (\alpha I + \Psi_s^b)^{-1} [(b-s)^{q-1} \mathcal{T}(b-s) \sigma_1(s, x_\alpha(s)) - \tilde{\varphi}(s)] \, dw_1(s).$$

It follows from the assumption on $f, \sigma_1$ and $\sigma_2$ that there exists $\hat{D} > 0$ such that

$$\|f(s, x_\alpha(s))\|^2 + \|\sigma_1(s, x_\alpha(s))\|^2 + \|\sigma_2(s, x_\alpha(s))\|^2 \le \hat{D} \tag{3.7}$$

for all $s \in J$. Then there is a subsequence still denoted by $\{f(s, x_\alpha(s)), \sigma_1(s, x_\alpha(s)), \sigma_2(s, x_\alpha(s))\}$ which converges weakly to some $\{f(s), \sigma_1(s), \sigma_2(s)\}$ in $Y \times L_2^0 \times L_2^0$.
From the above equation, we have

$$E \|x_\alpha(b) - \tilde{x}_b\|^2$$
$$\le 8E \left( \left\| \alpha(\alpha I + \Psi_0^b)^{-1} (E\tilde{x}_b - \mathcal{S}(b) L x_0) \right\|^2 \right)$$
$$+ 8E \left( \|\alpha(\alpha I + \Psi_0^b)^{-1}\|^2 \|\mathcal{S}(b) L \frac{1}{\Gamma(1-q)}\|^2 \int_0^b (b-s)^{-q} \|\sigma_2(s, x_\alpha(s)) - \sigma_2(s)\|_{L_2^0}^2 \, ds \right)$$
$$+ 8E \left( \|\alpha(\alpha I + \Psi_0^b)^{-1}\|^2 \|\mathcal{S}(b) L \frac{1}{\Gamma(1-q)}\|^2 \int_0^b (b-s)^{-q} \|\sigma_2(s)\|_{L_2^0}^2 \, ds \right)$$
$$+ 8E \left( \int_0^b (b-s)^{q-1} \left\| \alpha(\alpha I + \Psi_s^b)^{-1} \tilde{\varphi}(s) \right\|_{L_2^0}^2 \, ds \right)$$
$$+ 8E \left( \int_0^b (b-s)^{q-1} \left\| \alpha(\alpha I + \Psi_s^b)^{-1} \right\| \|\mathcal{T}(b-s)(f(s, x_\alpha(s)) - f(s))\| \, ds \right)^2$$

$$+ 8E \left( \int_0^b (b-s)^{q-1} \left\| \alpha(\alpha I + \Psi_s^b)^{-1} \mathcal{T}(b-s) f(s) \right\| ds \right)^2$$

$$+ 8E \left( \int_0^b (b-s)^{q-1} \left\| \alpha(\alpha I + \Psi_s^b)^{-1} \right\| \left\| \mathcal{T}(b-s)(\sigma_1(s, x_\alpha(s)) - \sigma_1(s)) \right\|_{L_2^0}^2 ds \right)$$

$$+ 8E \left( \int_0^b (b-s)^{q-1} \left\| \alpha(\alpha I + \Psi_s^b)^{-1} \mathcal{T}(b-s) \sigma_1(s) \right\|_{L_2^0}^2 ds \right).$$

On the other hand, by assumption (iii), for all $0 \le s < b$ the operator $\alpha(\alpha I + \Psi_s^b)^{-1} \to 0$ strongly as $\alpha \to 0^+$ and moreover $\| \alpha(\alpha I + \Psi_s^b)^{-1} \| \le 1$. Thus, by the Lebesgue dominated convergence theorem and the compactness of both $\mathcal{S}(t)$ and $\mathcal{T}(t)$ implies that $E \| x_\alpha(b) - \tilde{x}_b \|^2 \to 0$ as $\alpha \to 0^+$. Hence, we conclude the approximate controllability of (1.1)–(1.2).  $\square$

In order to illustrate the abstract results of this work, we give the following example.

## 4   Example

Consider a fractional partial stochastic nonlocal control equation of Sobolev type

$$\frac{\partial^q}{\partial t^q} \left[ x(z,t) - x_{zz}(z,t) \right] - \frac{\partial^2}{\partial z^2} x(z,t) = \mu(z,t) + \hat{f}(t, x(z,t)) + \hat{\sigma}(t, x(z,t)) \frac{d\hat{w}_1(t)}{dt}, \qquad (4.1)$$

$$x(z,0) = x_0(z) + \frac{1}{\Gamma(1-q)} \sum_{k=1}^m c_k \int_0^t (t-s)^{-q} x(z, t_k) \, d\hat{w}_2(s), \ z \in [0,1], \qquad (4.2)$$

$$x(0,t) = x(1,t) = 0, \ t \in J, \qquad (4.3)$$

where $0 < q \le 1$, $0 < t_1 < \cdots < t_m < b$ and $c_k$ are positive constants, $k = 1, \ldots, m$; the functions $x(t)(z) = x(z,t), f(t, x(t))(z) = \hat{f}(t, x(z,t)), \sigma_1(t, x(t))(z) = \hat{\sigma}(t, x(z,t))$ and $\sigma_2(t, x(t))(z) = \sum_{k=1}^m c_k x(z, t_k)$. The bounded linear operator $B \colon U \to X$ is defined by $Bu(t)(z) = \mu(z,t)$, $0 \le z \le 1$, $u \in U$; $\hat{w}_1(t)$ and $\hat{w}_2(t)$ are two sided and standard one dimensional Brownian motions defined on the filtered probability space $(\Omega, \Gamma, P)$.

Let $X = E = U = L^2[0,1]$, define the operators $L \colon D(L) \subset X \to Y$ and $M \colon D(M) \subset X \to Y$ by $Lx = x - x''$ and $Mx = -x''$ where domains $D(L)$ and $D(M)$ are given by

$$\{x \in X : x, x' \text{ are absolutely continuous, } x'' \in X, \ x(0) = x(1) = 0\}.$$

Then $L$ and $M$ can be written respectively as

$$Lx = \sum_{n=1}^\infty (1 + n^2)(x, x_n) x_n, x \in D(L) \quad \text{and} \quad Mx = \sum_{n=1}^\infty -n^2 (x, x_n) x_n, \quad x \in D(M),$$

where $x_n(z) = (\sqrt{2/\pi}) \sin nz$, $n = 1, 2, \ldots$ is the orthogonal set of eigenfunctions of $M$. Further, for any $x \in X$ we have

$$L^{-1} x = \sum_{n=1}^\infty \frac{1}{1 + n^2} (x, x_n) x_n, \qquad ML^{-1} x = \sum_{n=1}^\infty \frac{-n^2}{1 + n^2} (x, x_n) x_n,$$

and

$$S(t) x = \sum_{n=1}^\infty \exp \left( \frac{-n^2 t}{1 + n^2} \right) (x, x_n) x_n.$$

It is easy to see that $L^{-1}$ is compact, bounded with $\|L^{-1}\| \leq 1$ and $ML^{-1}$ generates the above strongly continuous semigroup $S(t)$ on $Y$ with $\|S(t)\| \leq e^{-t} \leq 1$. Therefore, with the above choices, the system (4.1)–(4.3) can be written as an abstract formulation of (1.1)–(1.2) and thus Theorem 3.2 can be applied to guarantee the existence of mild solution of (4.1)–(4.3). Moreover, it can be easily seen that Sobolev type deterministic linear fractional control system corresponding to (4.1)–(4.3) is approximately controllable on $J$, which means that all conditions of Theorem 3.3 are satisfied. Thus, fractional stochastic nonlinear control system of Sobolev type (4.1)–(4.3) is approximately controllable on $J$.

## Acknowledgements

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