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Asymptotic behavior of third order functional dynamic equations with γ-Laplacian and nonlinearities given by Riemann–Stieltjes integrals

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Abstract. In this paper, we study the third-order functional dynamic equations with γ-Laplacian and nonlinearities given by Riemann–Stieltjes integrals

$$\left\{r_2\left(t\right)\phi_{\gamma_2}\left(\left[r_1\left(t\right)\phi_{\gamma_1}\left(x^{\Delta}\left(t\right)\right)\right]^{\Delta}\right)\right\}^{\Delta}+\int_a^b q\left(t,s\right)\phi_{\alpha(s)}\left(x(g\left(t,s\right))\right)d\zeta\left(s\right)=0,$$

on an above-unbounded time scale $\mathbb{T}$, where $\phi_\gamma(u) := |u|^{\gamma-1}u$ and $\int_a^b f\left(s\right)d\zeta\left(s\right)$ denotes the Riemann–Stieltjes integral of the function f on $[a,b]$ with respect to ζ. Results are obtained for the asymptotic and oscillatory behavior of the solutions. This work extends and improves some known results in the literature on third order nonlinear dynamic equations.

Keywords: asymptotic behavior, oscillation, γ-Laplacian, nonlinear dynamic equations, time scales.

2010 Mathematics Subject Classification: 34K11, 39A10, 39A99.

1 Introduction

We are concerned with the asymptotic and oscillatory behavior of the third order nonlinear functional dynamic equation

$$\left\{r_2(t)\phi_{\gamma_2}\left(\left[r_1(t)\phi_{\gamma_1}\left(x^{\Delta}(t)\right)\right]^{\Delta}\right)\right\}^{\Delta}+\int_a^b q(t,s)\phi_{\alpha(s)}\left(x(g(t,s))\right)d\zeta(s)=0 \tag{1.1}$$

on an above-unbounded time scale $\mathbb{T}$, where $\phi_\gamma(u) := |u|^{\gamma-1}u$, $\gamma_1,\gamma_2 > 0$; $\alpha \in C[a,b]$ with $-\infty < a < b < \infty$ such that $\alpha(s) > 0$ is strictly increasing, r_i, $i = 1,2$, are positive rd-continuous functions on $\mathbb{T}$; q is a positive rd-continuous function on $\mathbb{T} \times [a,b]$; and $g\colon \mathbb{T} \times$

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$[a,b] \to \mathbb{T}$ is a rd-continuous function such that $\lim_{t\to\infty} g(t,s) = \infty$ for $s \in [a,b]$. Without loss of generality we assume $0 \in \mathbb{T}$. Hence we may discuss the solutions of Eq. (1.1) on $[0,\infty)_{\mathbb{T}}$. Here $\int_a^b f(s)\, d\zeta(s)$ denotes the Riemann–Stieltjes integral of the function f on $[a,b]$ with respect to ζ. We note that as special cases, the integral term in the equation becomes a finite sum when $\zeta(s)$ is a step function and a Riemann integral when $\zeta(s) = s$. Throughout this paper, we let

$$x^{[i]} := r_i \phi_{\gamma_i}([x^{[i-1]}]^{\Delta}),\ i = 1,2, \quad \text{with } x^{[0]} = x. \tag{1.2}$$

It is easy to see that all solutions of Eq. (1.1) can be extended to ∞ if either $g(t,s) \le t - \tau$ for some $\tau > 0$ and all $t \in \mathbb{T}$ and $s \in [a,b]$ or $\mathbb{T}$ is a discrete time scale and $g(t,s) \le t$ for all $t \in \mathbb{T}$ and $s \in [a,b]$. However, Eq. (1.1) may have both extendable solutions and nonextendable solutions in general. For the asymptotic and oscillation purposes, we are only interested in the solutions that are extendable to ∞. Thus, we use the following definition of solutions.

Definition 1.1. By a solution of Eq. (1.1) we mean a nontrivial real-valued function $x \in C^1_{rd}[T_x,\infty)_{\mathbb{T}}$ for some $T_x \ge t_0$ such that $x^{[1]}, x^{[2]} \in C^1_{rd}[T_x,\infty)_{\mathbb{T}}$, and $x(t)$ satisfies Eq. (1.1) on $[T_x,\infty)_{\mathbb{T}}$, where C_{rd} is the space of right-dense continuous functions, and C^1_{rd} is the space of functions whose Δ-derivatives are right-dense on $[T_x,\infty)_{\mathbb{T}}$.

In the last few years, there has been an increasing interest in obtaining sufficient conditions for the oscillation/nonoscillation of solutions of different classes of dynamic equations, we refer the reader to the papers [1, 2, 6, 7, 9, 15, 17, 19, 20, 21, 24, 26, 28] and the references cited therein. Regarding third order dynamic equations, Erbe, Peterson, and Saker [10, 11] and Yu and Wang [29] obtained sufficient conditions for oscillation for the third order dynamic equations

$$\left(r_2(t)\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta} + p(t)x(t) = 0,$$

$$\left(r_2(t)\left[\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\gamma}\right)^{\Delta} + p(t)x^{\gamma}(t) = 0,$$

and

$$\left(r_2(t)\left[\left(r_1(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right]^{\alpha_2}\right)^{\Delta} + p(t)x(t) = 0;$$

where $\gamma \ge 1$ is the quotient of odd positive integers and $r_1, r_2, p \in C_{rd}(\mathbb{T})$ are positive. Hassan [16] and Erbe, Hassan, and Peterson [12] extended their work to the dynamic equation with delay

$$\left(r_2(t)\left[\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\gamma}\right)^{\Delta} + p(t)x^{\gamma}(h(t)) = 0$$

for the case that $\gamma \ge 1$ and $\gamma > 0$, respectively, where $h(t)$ is a monotone delay function on $\mathbb{T}$. A number of sufficient conditions for oscillation were obtained for the cases when

$$\int_0^{\infty} \frac{\Delta t}{r_2^{1/\gamma}(t)} = \infty \quad \text{and} \quad \int_0^{\infty} \frac{\Delta t}{r_1(t)} = \infty$$

and

$$\int_0^{\infty} \frac{\Delta t}{r_2^{1/\gamma}(t)} < \infty \quad \text{and} \quad \int_0^{\infty} \frac{\Delta t}{r_1(t)} < \infty,$$

respectively. Also, Han, Li, Sun, and Zhang [18] discussed the third order delay dynamic equation

$$\left(r_2(t)\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}+p(t)x(g(t))=0,$$

where $g(t)\le t$ and

$$r_1^{\Delta}(t)\le 0 \quad \text{and} \quad \int_{t_0}^{\infty} g(t)p(t)\Delta t=\infty. \tag{1.3}$$

Recently, Erbe, Hassan, and Peterson [13] extended these results to third-order dynamic equations of a more general form

$$\left\{r_2(t)\left(\left[r_1(t)\left(x^{\Delta}(t)\right)^{\gamma_1}\right]^{\Delta}\right)^{\gamma_2}\right\}^{\Delta}+\sum_{i=0}^{n}p_i(t)(x(h_i(t)))^{\alpha_i}=0, \tag{1.4}$$

where certain restrictions on the delay terms were imposed.

In this paper, we study the asymptotic and oscillatory behavior of the third-order functional dynamic equation (1.1) with γ-Laplacian and nonlinearities given by Riemann–Stieltjes integrals for both the cases

$$\int_0^{\infty} r_i^{-\frac{1}{\gamma_i}}(t)\Delta t=\infty, \quad i=1,2, \tag{1.5}$$

and

$$\int_0^{\infty} r_i^{-\frac{1}{\gamma_i}}(t)\Delta t<\infty, \quad i=1,2. \tag{1.6}$$

The results improve and extend the oscillation criteria established in [8, 10, 11, 12, 13, 16, 18, 24, 25, 26].

2 Asymptotic behavior

In this section, we discuss the asymptotic behavior of the solutions of (1.1) when (1.5) and (1.6) hold, respectively. The first theorem is under the assumption that (1.5) holds, the second is under the assumption that (1.6) holds, and the last one is for the general case.

Theorem 2.1. *Assume that* (1.5) *holds and*

$$\int_0^{\infty} r_1^{-\frac{1}{\gamma_1}}(u)\left\{\int_u^{\infty} r_2^{-\frac{1}{\gamma_2}}(v)\left[\int_v^{\infty}\int_a^b q(w,s)d\zeta(s)\Delta w\right]^{\frac{1}{\gamma_2}}\Delta v\right\}^{\frac{1}{\gamma_1}}\Delta u=\infty. \tag{2.1}$$

If Eq. (1.1) *has eventually positive solution $x(t)$, then*

$$\left[x^{[2]}(t)\right]^{\Delta}<0 \quad \textit{and} \quad \left[x^{[1]}(t)\right]^{\Delta}>0$$

eventually, and either $x^{\Delta}(t)$ is eventually positive or $x(t)$ tends to zero eventually.

Proof. Since $x(t)$ is eventually positive solution of Eq. (1.1), then there is a $T\in[0,\infty)_{\mathbb{T}}$ such that $x(t)>0$ on $[T,\infty)_{\mathbb{T}}$ and $x(g(t,s))>0$ on $[T,\infty)_{\mathbb{T}}\times[a,b]$. From (1.1), we have that for $t\in[T,\infty)_{\mathbb{T}}$,

$$\left[x^{[2]}(t)\right]^{\Delta}=-\int_a^b q(t,s)\left[x(g(t,s))\right]^{\alpha(s)}d\zeta(s)<0. \tag{2.2}$$

Then $x^{[2]}(t)$ is strictly decreasing on $[T,\infty)_{\mathbb{T}}$. This implies that $\left[x^{[1]}(t)\right]^{\Delta}$ and $x^{\Delta}(t)$ are eventually of one sign.

(I) We show that $\left[x^{[1]}(t)\right]^{\Delta}$ is eventually positive. Otherwise, it is eventually negative. We consider the following two cases:

(a) $x^{\Delta}(t)<0$ and $\left[x^{[1]}(t)\right]^{\Delta}<0$ eventually. In this case, there exists $T_1\in[T,\infty)_{\mathbb{T}}$ such that

$$x^{[1]}(t)<0 \quad\text{and}\quad \left[x^{[1]}(t)\right]^{\Delta}<0 \quad \text{for } t\geq T_1.$$

Then

$$\begin{aligned} x(t)&=x(T_1)+\int_{T_1}^{t}\phi_{\gamma_1}^{-1}\left[x^{[1]}(u)\right]r_1^{-\frac{1}{\gamma_1}}(u)\Delta u\\ &<x(T_1)+\phi_{\gamma_1}^{-1}\left[x^{[1]}(T_1)\right]\int_{T_1}^{t}r_1^{-\frac{1}{\gamma_1}}(u)\Delta u. \end{aligned}$$

By (1.5), we have $\lim_{t\to\infty}x(t)=-\infty$, which contradicts the fact that $x(t)$ is a positive solution of Eq. (1.1).

(b) $x^{\Delta}(t)>0$ and $\left[x^{[1]}(t)\right]^{\Delta}<0$ eventually. In this case, there exists $T_1\in[T,\infty)_{\mathbb{T}}$ such that

$$x^{[1]}(t)>0 \quad\text{and}\quad \left[x^{[1]}(t)\right]^{\Delta}<0 \quad \text{for } t\geq T_1.$$

Since $x^{[2]}(t)$ is strictly decreasing on $[T_1,\infty)_{\mathbb{T}}$, we get

$$\begin{aligned} x^{[1]}(t)-x^{[1]}(T_1)&=\int_{T_1}^{t}\phi_{\gamma_2}^{-1}\left[x^{[2]}(u)\right]r_2^{-\frac{1}{\gamma_2}}(u)\Delta u\\ &<\phi_{\gamma_2}^{-1}\left[x^{[2]}(T_1)\right]\int_{T_1}^{t}r_2^{-\frac{1}{\gamma_2}}(u)\Delta u. \end{aligned}$$

By (1.5), we have $\lim_{t\to\infty}x^{[1]}(t)=-\infty$, which contradicts that $x^{[1]}(t)>0$ for $t\geq T_1$.

(II) We then show that if $x^{\Delta}(t)$ is not eventually positive, then $x(t)$ tends to zero eventually. In this case, $x^{\Delta}(t)<0$ eventually. Hence

$$\lim_{t\to\infty}x(t)=l_1\geq 0 \quad\text{and}\quad \lim_{t\to\infty}x^{[1]}(t)=l_2\leq 0.$$

Assume $l_1>0$. Then for sufficiently large $T_2\in[T,\infty)_{\mathbb{T}}$, we have $x(g(t,s))\geq l_1$ for $t\geq T_2$ and $s\in[a,b]$. It follows that

$$[x(g(t,s))]^{\alpha(s)}\geq l:=\min_{s\in[a,b]}\left\{l_1^{\alpha(s)}\right\} \quad \text{for } t\in[T_2,\infty)_{\mathbb{T}} \text{ and } s\in[a,b].$$

Integrating (1.1) from t to $\tau\in[t,\infty)_{\mathbb{T}}$, we get

$$-x^{[2]}(\tau)+x^{[2]}(t)>\int_t^{\tau}\int_a^b q(w,s)\,[x(g(w,s))]^{\alpha(s)}\,d\zeta(s)\Delta w.$$

By Part (I) and (1.2) we see that $x^{[2]}(\tau)>0$. Hence by taking limits as $\tau\to\infty$ we have

$$\begin{aligned} x^{[2]}(t)&>\int_t^{\infty}\int_a^b q(w,s)\,[x(g(w,s))]^{\alpha(s)}\,d\zeta(s)\Delta w\\ &\geq l\int_t^{\infty}\int_a^b q(w,s)\,d\zeta(s)\Delta w. \end{aligned}$$

If $\int_t^\infty \int_a^b q(w,s)\, d\zeta(s) \Delta w = \infty$, we have reached a contradiction. Otherwise,

$$\left[x^{[1]}(t)\right]^{\Delta} > l^{\frac{1}{\gamma_2}} r_2^{-\frac{1}{\gamma_2}}(t) \left[\int_t^\infty \int_a^b q(w,s) d\zeta(s) \Delta w\right]^{1/\gamma_2}.$$

Again, integrating this inequality from t to ∞ and noting that $x^{[1]}(t) \le 0$ eventually, we get

$$-x^{[1]}(t) > l^{\frac{1}{\gamma_2}} \int_t^\infty r_2^{-\frac{1}{\gamma_2}}(v) \left[\int_v^\infty \int_a^b q(w,s) d\zeta(s) \Delta w\right]^{\frac{1}{\gamma_2}} \Delta v,$$

which yields

$$-x^{\Delta}(t) > L r_1^{-\frac{1}{\gamma_1}}(t) \left\{\int_t^\infty r_2^{-\frac{1}{\gamma_2}}(v) \left[\int_v^\infty \int_a^b q(w,s) d\zeta(s) \Delta w\right]^{\frac{1}{\gamma_2}} \Delta v\right\}^{\frac{1}{\gamma_1}},$$

where $L := l^{\frac{1}{\gamma_1 \gamma_2}} > 0$. Finally, integrating the last inequality from T_2 to t, we get

$$-x(t) + x(T_2) > L \int_{T_2}^t r_1^{-\frac{1}{\gamma_1}}(u) \left\{\int_u^\infty r_2^{-\frac{1}{\gamma_2}}(v) \left[\int_v^\infty \int_a^b q(w,s)\, d\zeta(s) \Delta w\right]^{\frac{1}{\gamma_2}} \Delta v\right\}^{\frac{1}{\gamma_1}} \Delta u.$$

Hence by (2.1), we have $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that $x(t)$ is a positive solution of Eq. (1.1). This shows that $\lim_{t\to\infty} x(t) = 0$ and hence completes the proof. □

Remark 2.2. The conclusion of Theorem 2.1 remains intact if assumption (2.1) is replaced by the condition

$$\int_0^\infty \int_a^b q(w,s)\, d\zeta(s) \Delta w = \infty$$

or

$$\int_0^\infty \int_a^b q(w,s)\, d\zeta(s) \Delta w < \infty \quad \text{and} \quad \int_0^\infty r_2^{-\frac{1}{\gamma_2}}(v) \left[\int_v^\infty \int_a^b q(w,s)\, d\zeta(s) \Delta w\right]^{\frac{1}{\gamma_2}} \Delta v = \infty.$$

Now we consider the case when (1.6) holds. We will use the following notations:

$$\lambda_i(t) := \int_t^\infty r_i^{-\frac{1}{\gamma_i}}(u) \Delta u \quad \text{and} \quad R_i(t,t_0) := \int_{t_0}^t r_i^{-\frac{1}{\gamma_i}}(u) \Delta u, \quad i = 1,2;$$

and

$$\Lambda(t,t_0) := \lambda_2^{\frac{1}{\gamma_1}}(t) R_1(t,t_0).$$

Theorem 2.3. *Assume that (2.1) holds, and for any $t_0 \in [0,\infty)_{\mathbb{T}}$*

$$\int_{t_0}^\infty r_1^{-\frac{1}{\gamma_1}}(u) \left\{\int_{t_0}^u r_2^{-\frac{1}{\gamma_2}}(v) \left[\int_{t_0}^v \int_a^b q(w,s) \left[\lambda_1(g(w,s))\right]^{\alpha(s)} d\zeta(s) \Delta w\right]^{\frac{1}{\gamma_2}} \Delta v\right\}^{\frac{1}{\gamma_1}} \Delta u = \infty \tag{2.3}$$

and

$$\int_{t_0}^\infty r_2^{-\frac{1}{\gamma_2}}(v) \left[\int_{t_0}^v \int_a^b q(w,s) \left[\Lambda(g(w,s),t_0)\right]^{\alpha(s)} d\zeta(s) \Delta w\right]^{\frac{1}{\gamma_2}} \Delta v = \infty. \tag{2.4}$$

If Eq. (1.1) has eventually positive solution $x(t)$, then

$$\left[x^{[2]}(t)\right]^{\Delta} < 0 \quad \textit{and} \quad \left[x^{[1]}(t)\right]^{\Delta} > 0$$

eventually, and either $x^{\Delta}(t)$ is eventually positive or $x(t)$ tends to zero eventually.

Proof. Since $x(t)$ is eventually positive solution of Eq. (1.1), then there is a $T \in [0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$ on $[T,\infty)_{\mathbb{T}}$ and $x(g(t,s)) > 0$ on $[T,\infty)_{\mathbb{T}} \times [a,b]$. By (2.2), $x^{[2]}(t)$ is strictly decreasing on $[T,\infty)_{\mathbb{T}}$. This implies that $\left[x^{[1]}(t)\right]^{\Delta}$ and $x^{\Delta}(t)$ are eventually of one sign.

(I) We show that $\left[x^{[1]}(t)\right]^{\Delta}$ is eventually positive. Otherwise, it is eventually negative. We consider the following two cases:

(a) $x^{\Delta}(t) < 0$ and $\left[x^{[1]}(t)\right]^{\Delta} < 0$ eventually. In this case, there exists $T_1 \geq T$ such that

$$x^{\Delta}(t) < 0 \quad \text{and} \quad \left[x^{[1]}(t)\right]^{\Delta} < 0 \quad \text{for } t \geq T_1.$$

Let $T_2 \in [T_1,\infty)_{\mathbb{T}}$ such that $g(t,s) \geq T_1$ for $t \geq T_2$ and $s \in [a,b]$. Then for $t \geq T_2$,

$$\begin{aligned} x(g(t,s)) &> -\int_{g(t,s)}^{\infty} \phi_{\gamma_1}^{-1}\left[x^{[1]}(u)\right] r_1^{-\frac{1}{\gamma_1}}(u)\Delta u \\ &> -\phi_{\gamma_1}^{-1}\left[x^{[1]}(g(t,s))\right] \int_{g(t,s)}^{\infty} r_1^{-\frac{1}{\gamma_1}}(u)\Delta u \\ &> -\phi_{\gamma_1}^{-1}\left[x^{[1]}(T_1)\right] \int_{g(t,s)}^{\infty} r_1^{-\frac{1}{\gamma_1}}(u)\Delta u = L_1\lambda_1(g(t,s)), \end{aligned}$$

where $L_1 := -\phi_{\gamma_1}^{-1}\left[x^{[1]}(T_1)\right] > 0$, and hence

$$[x(g(t,s))]^{\alpha(s)} > L\left[\lambda_1(g(t,s))\right]^{\alpha(s)} \qquad \text{for } t \geq T_2 \text{ and } s \in [a,b], \tag{2.5}$$

where $L := \min_{s\in[a,b]}\left\{L_1^{\alpha(s)}\right\} > 0$. From (1.1) and (2.5) we find that

$$\left[x^{[2]}(t)\right]^{\Delta} < -L\int_a^b q(t,s)\left[\lambda_1(g(t,s))\right]^{\alpha(s)} d\zeta(s).$$

Integrating this last inequality from T_2 to t, we see that

$$x^{[2]}(t) < x^{[2]}(t) - x^{[2]}(T_2) < -L\int_{T_2}^{t}\int_a^b q(w,s)\left[\lambda_1\left(g(w,s)\right)\right]^{\alpha(s)} d\zeta(s)\Delta w,$$

which implies that

$$\left[x^{[1]}(t)\right]^{\Delta} < -r_2^{-\frac{1}{\gamma_2}}(t)\left[L\int_{T_2}^{t}\int_a^b q(w,s)\left[\lambda_1\left(g(w,s)\right)\right]^{\alpha(s)} d\zeta(s)\Delta w\right]^{\frac{1}{\gamma_2}}.$$

Again, integrating the above inequality from T_2 to t, we get

$$x^{[1]}(t) < x^{[1]}(t) - x^{[1]}(T_2) < -\int_{T_2}^{t} r_2^{-\frac{1}{\gamma_2}}(v)\left[L\int_{T_2}^{v}\int_a^b q(w,s)\left[\lambda_1\left(g(w,s)\right)\right]^{\alpha(s)} d\zeta(s)\Delta w\right]^{\frac{1}{\gamma_2}}\Delta v,$$

which yields

$$\begin{aligned} &x(t) - x(T_2) < \\ &\quad -\int_{T_2}^{t} r_1^{-\frac{1}{\gamma_1}}(u)\left\{\int_{T_2}^{u} r_2^{-\frac{1}{\gamma_2}}(v)\left[L\int_{T_2}^{v}\int_a^b q(w,s)\left[\lambda_1\left(g(w,s)\right)\right]^{\alpha(s)} d\zeta(s)\Delta w\right]^{\frac{1}{\gamma_2}}\Delta v\right\}^{\frac{1}{\gamma_1}}\Delta u. \end{aligned}$$

From (2.3), we have $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that x is a positive solution of Eq. (1.1).

(b) $x^{\Delta}(t) > 0$ and $\left[x^{[1]}(t)\right]^{\Delta} < 0$ eventually. In this case, there exists $T_1 \geq T$ such that

$$x^{\Delta}(t) > 0 \quad \text{and} \quad \left[x^{[1]}(t)\right]^{\Delta} < 0 \quad \text{for } t \geq T_1.$$

Again, we let $T_2 \in [T_1, \infty)_{\mathbb{T}}$ such that $g(t,s) \geq T_1$ for $t \geq T_2$ and $s \in [a,b]$. Then for $t \geq T_2$,

$$\begin{aligned} x(g(t,s)) &> x(g(t,s)) - x(T_1) \\ &= \int_{T_1}^{g(t,s)} \phi_{\gamma_1}^{-1}\left[x^{[1]}(u)\right] r_1^{-\frac{1}{\gamma_1}}(u)\Delta u \\ &> \phi_{\gamma_1}^{-1}\left[x^{[1]}(g(t,s))\right] \int_{T_1}^{g(t,s)} r_1^{-\frac{1}{\gamma_1}}(u)\Delta u \\ &= \phi_{\gamma_1}^{-1}\left[x^{[1]}(g(t,s))\right] R_1(g(t,s),T_1) \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} x^{[1]}(g(t,s)) &> -\int_{g(t,s)}^{\infty} \phi_{\gamma_2}^{-1}\left[x^{[2]}(u)\right] r_2^{-\frac{1}{\gamma_2}}(u)\Delta u \\ &> -\phi_{\gamma_2}^{-1}\left[x^{[2]}(g(t,s))\right] \int_{g(t,s)}^{\infty} r_2^{-\frac{1}{\gamma_2}}(u)\Delta u \\ &> -\phi_{\gamma_2}^{-1}\left[x^{[2]}(T_1)\right] \int_{g(t,s)}^{\infty} r_2^{-\frac{1}{\gamma_2}}(u)\Delta u = L_2\lambda_2(g(t,s)), \end{aligned} \tag{2.7}$$

where $L_2 := -\phi_{\gamma_2}^{-1}\left[x^{[2]}(T_1)\right] > 0$. Substituting (2.7) into (2.6), we get that for $t \geq T_2$ and $s \in [a,b]$

$$x(g(t,s)) > L_2^{\frac{1}{\gamma_1}}\Lambda(g(t,s),T_1),$$

and hence

$$[x(g(t,s))]^{\alpha(s)} > L\left[\Lambda(g(t,s),T_1)\right]^{\alpha(s)}, \tag{2.8}$$

where $L := \min_{s\in[a,b]}\left\{L_2^{\alpha(s)/\gamma_1}\right\} > 0$. By (1.1) and (2.8),

$$\left[x^{[2]}(t)\right]^{\Delta} < -L\int_a^b q(t,s)\left[\Lambda(g(t,s),T_1)\right]^{\alpha(s)} d\zeta(s).$$

Integrating both sides from T_2 to t, we have

$$\begin{aligned} x^{[2]}(t) &< x^{[2]}(t) - x^{[2]}(T_2) \\ &< -L\int_{T_2}^{t}\int_a^b q(w,s)\left[\Lambda(g(w,s),T_1)\right]^{\alpha(s)} d\zeta(s)\Delta w, \end{aligned}$$

which implies that

$$\left[x^{[1]}(t)\right]^{\Delta} < -r_2^{-\frac{1}{\gamma_2}}(t)\left[L\int_{T_2}^{t}\int_a^b q(w,s)\left[\Lambda(g(w,s),T_1)\right]^{\alpha(s)} d\zeta(s)\Delta w\right]^{\frac{1}{\gamma_2}}.$$

Again, integrating both sides from T_2 to t, we get

$$\begin{aligned}-x^{[1]}(T_2) &< x^{[1]}(t) - x^{[1]}(T_2)\\ &< -\int_{T_2}^{t} r_2^{-\frac{1}{\gamma_2}}(v)\left[L\int_{T_2}^{v}\int_a^b q(w,s)\left[\Lambda(g(w,s),T_1)\right]^{\alpha(s)} d\zeta(s)\Delta w\right]^{\frac{1}{\gamma_2}}\Delta v\\ &< -\int_{T_2}^{t} r_2^{-\frac{1}{\gamma_2}}(v)\left[L\int_{T_2}^{v}\int_a^b q(w,s)\left[\Lambda(g(w,s),T_2)\right]^{\alpha(s)} d\zeta(s)\Delta w\right]^{\frac{1}{\gamma_2}}\Delta v,\end{aligned}$$

which contradicts (2.4).

(II) With essentially the same proof as in Part (II) of the proof of Theorem 2.1, we can show that if $x^{\Delta}(t)$ is not eventually positive, then $x(t)$ tends to zero eventually. We omit the details. □

Theorem 2.4. *Let $x(t)$ be a solution of Eq. (1.1) such that*

$$x(t) > 0, \quad x(g(t,s)) > 0, \quad x^{\Delta}(t) > 0, \quad \text{and} \quad \left[x^{[1]}(t)\right]^{\Delta} > 0 \tag{2.9}$$

for $t \in [T,\infty)_{\mathbb{T}}$ and $s \in [a,b]$ with $T \in [0,\infty)_{\mathbb{T}}$. Then

$$x^{\Delta}(t) > \phi_{\gamma}^{-1}\left[x^{[2]}(t)\right]\left[\frac{R_2(t,T)}{r_1(t)}\right]^{\frac{1}{\gamma_1}};$$

$$x(t) > \phi_{\gamma}^{-1}\left[x^{[2]}(t)\right]\int_T^t\left[\frac{R_2(u,T)}{r_1(u)}\right]^{\frac{1}{\gamma_1}}\Delta u;$$

and

$$x(t) > R(t,T)[x^{[1]}(t)]^{\frac{1}{\gamma_1}} \quad \text{and} \quad \left[\frac{x(t)}{R(t,T)}\right]^{\Delta} < 0 \quad \text{for } t \in (T,\infty)_{\mathbb{T}}.$$

where $\gamma := \gamma_1\gamma_2$ and

$$R(t,T) := \int_T^t\left[\frac{R_2(u,T)}{R_2(t,T)r_1(u)}\right]^{\frac{1}{\gamma_1}}\Delta u.$$

Proof. By (2.2), $x^{[2]}(t)$ is strictly decreasing on $[T,\infty)_{\mathbb{T}}$. Then for $t \in [T,\infty)_{\mathbb{T}}$,

$$\begin{aligned}x^{[1]}(t) > x^{[1]}(t) - x^{[1]}(T) &= \int_T^t \phi_{\gamma_2}^{-1}\left[x^{[2]}(u)\right] r_2^{-\frac{1}{\gamma_2}}(u)\Delta u\\ &\geq \phi_{\gamma_2}^{-1}\left[x^{[2]}(t)\right]\int_T^t r_2^{-\frac{1}{\gamma_2}}(u)\Delta u = \phi_{\gamma_2}^{-1}\left[x^{[2]}(t)\right] R_2(t,T),\end{aligned} \tag{2.10}$$

which implies that

$$x^{\Delta}(t) > \phi_{\gamma}^{-1}\left[x^{[2]}(t)\right]\left[\frac{R_2(t,T)}{r_1(t)}\right]^{\frac{1}{\gamma_1}},$$

where $\gamma = \gamma_1\gamma_2$. In the same way, we have

$$x(t) > \phi_{\gamma}^{-1}\left[x^{[2]}(t)\right]\int_T^t\left[\frac{R_2(u,T)}{r_1(u)}\right]^{\frac{1}{\gamma_1}}\Delta u.$$

We note that

$$\left[\frac{x^{[1]}(t)}{R_2(t,T)}\right]^{\Delta} = \frac{r_2^{-1/\gamma_2}(t)}{R_2(t,T)R_2(\sigma(t),T)}\left[\phi_{\gamma_2}^{-1}\left[x^{[2]}(t)\right] R_2(t,T) - x^{[1]}(t)\right],$$

so by (2.10) we have

$$\left[\frac{x^{[1]}(t)}{R_2(t,T)}\right]^{\Delta} < 0 \qquad \text{for } t \in (T,\infty)_{\mathbb{T}}.$$

Then

$$\begin{aligned} x(t) > x(t) - x(T) &= \int_T^t \phi_{\gamma_1}^{-1}\left[x^{[1]}(u)\right] r_1^{-\frac{1}{\gamma_1}}(u)\Delta u \\ &= \int_T^t \phi_{\gamma_1}^{-1}\left[\frac{x^{[1]}(u)}{R_2(u,T)}\right]\left[\frac{R_2(u,T)}{r_1(u)}\right]^{\frac{1}{\gamma_1}}\Delta u \\ &\geq \phi_{\gamma_1}^{-1}\left[\frac{x^{[1]}(t)}{R_2(t,T)}\right]\int_T^t \left[\frac{R_2(u,T)}{r_1(u)}\right]^{\frac{1}{\gamma_1}}\Delta u \\ &= \phi_{\gamma_1}^{-1}\left[x^{[1]}(t)\right] R(t,T), \end{aligned}$$

which yields

$$\left[\frac{x(t)}{R(t,T)}\right]^{\Delta} < 0 \qquad \text{for } t \in (T,\infty)_{\mathbb{T}}.$$

□

3 Oscillation criteria

In this section, by using the results in Section 2, we study the oscillatory behavior of the solutions of Eq. (1.1) under the assumptions (1.5) and (1.6), respectively. First, we establish oscillation criteria for Eq. (1.1) under the assumption that (1.5) holds.

Theorem 3.1. *Assume that (1.5) and (2.1) hold. Suppose that for any $t_0 \in [0,\infty)_{\mathbb{T}}$,*

$$\limsup_{t\to\infty} \int_{t_0}^t \int_a^b q(u,s)\left[R_1(g(u,s),t_0)\right]^{\alpha(s)} d\zeta(s)\Delta u = \infty. \tag{3.1}$$

Then every solution of Eq. (1.1) is either oscillatory or tends to zero eventually.

Proof. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume there is a $T \in [0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$ on $[T,\infty)_{\mathbb{T}}$ and $x(g(t,s)) > 0$ on $[T,\infty)_{\mathbb{T}} \times [a,b]$. By Theorem 2.1,

$$\left[x^{[2]}(t)\right]^{\Delta} < 0 \quad \text{and} \quad \left[x^{[1]}(t)\right]^{\Delta} > 0$$

eventually and either $x^{\Delta}(t)$ is eventually positive or $x(t)$ tends to zero eventually. We suppose that

$$\left[x^{[2]}(t)\right]^{\Delta} < 0, \ \left[x^{[1]}(t)\right]^{\Delta} > 0, \quad \text{and} \quad x^{\Delta}(t) > 0$$

eventually. Then there exists $T_1 \in [T,\infty)_{\mathbb{T}}$ such that

$$\left[x^{[2]}(t)\right]^{\Delta} < 0, \ \left[x^{[1]}(t)\right]^{\Delta} > 0, \quad \text{and} \quad x^{\Delta}(t) > 0 \quad \text{for } t \geq T_1.$$

Since $\left[x^{[1]}(t)\right]^{\Delta} > 0$ on $[T_1,\infty)_{\mathbb{T}}$, we have

$$x^{[1]}(t) > x^{[1]}(T_1) =: C > 0.$$

Thus for $t \geq T_1$,

$$x(t) > x(t) - x(T_1) > C^{1/\gamma_1} \int_{T_1}^{t} r_1^{-\frac{1}{\gamma_1}}(u)\Delta u = C^{1/\gamma_1} R_1(t, T_1).$$

Choose $T_2 \in [T_1, \infty)_{\mathbb{T}}$ such that $g(t,s) > T_1$ for $t \geq T_2$ and $s \in [a,b]$. Then for $t \geq T_2$ and $s \in [a,b]$,

$$[x(g(t,s))]^{\alpha(s)} > C_1 \left[R_1(g(t,s), T_1)\right]^{\alpha(s)}, \tag{3.2}$$

where $C_1 := \min_{s\in[a,b]} \left\{ \left(C^{1/\gamma_1}\right)^{\alpha(s)} \right\} > 0$. It follows from (1.1) and (3.2) that

$$-\left[x^{[2]}(t)\right]^{\Delta} > C_1 \int_a^b q(t,s) \left[R_1(g(t,s), T_1)\right]^{\alpha(s)} d\zeta(s).$$

Integrating both sides of the last inequality from T_2 to t, we have

$$\begin{aligned} x^{[2]}(T_2) &> -x^{[2]}(t) + x^{[2]}(T_2) \\ &> C_1 \int_{T_2}^{t} \int_a^b q(u,s) \left[R_1(g(u,s), T_1)\right]^{\alpha(s)} d\zeta(s)\Delta u \\ &\geq C_1 \int_{T_2}^{t} \int_a^b q(u,s) \left[R_1(g(u,s), T_2)\right]^{\alpha(s)} d\zeta(s)\Delta u. \end{aligned}$$

which contradicts (3.1). □

Theorem 3.2. *Assume that* (1.5) *and* (2.1) *hold. Suppose that for any* $t_0 \in [0, \infty)_{\mathbb{T}}$,

$$\limsup_{t\to\infty} \int_{t_0}^{t} \int_a^b q(u,s) d\zeta(s)\Delta u = \infty. \tag{3.3}$$

Then every solution of Eq. (1.1) *is either oscillatory or tends to zero eventually.*

Proof. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume there is a $T \in [0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$ on $[T, \infty)_{\mathbb{T}}$ and $x(g(t,s)) > 0$ on $[T, \infty)_{\mathbb{T}} \times [a,b]$. By Theorem 2.1,

$$\left[x^{[2]}(t)\right]^{\Delta} < 0 \quad \text{and} \quad \left[x^{[1]}(t)\right]^{\Delta} > 0$$

eventually and either $x^{\Delta}(t)$ is eventually positive or $x(t)$ tends to zero eventually. We suppose that

$$\left[x^{[2]}(t)\right]^{\Delta} < 0, \ \left[x^{[1]}(t)\right]^{\Delta} > 0, \quad \text{and} \quad x^{\Delta}(t) > 0$$

eventually. Then there exists $T_1 \in [T, \infty)_{\mathbb{T}}$ such that

$$\left[x^{[2]}(t)\right]^{\Delta} < 0, \ \left[x^{[1]}(t)\right]^{\Delta} > 0, \quad \text{and} \quad x^{\Delta}(t) > 0 \quad \text{for } t \geq T_1.$$

Since $x^{\Delta}(t) > 0$ on $[T_1, \infty)_{\mathbb{T}}$, we have

$$x(t) > x(T_1) =: c > 0.$$

Choose $T_2 \in [T_1, \infty)_{\mathbb{T}}$ such that $g(t,s) > T_1$ for $t \geq T_2$ and $s \in [a,b]$. Then for $t \geq T_2$ and $s \in [a,b]$,

$$[x(g(t,s))]^{\alpha(s)} > c_1, \tag{3.4}$$

where $c_1 := \min_{s\in[a,b]} \left\{c^{\alpha(s)}\right\} > 0$. The rest of the proof is similar to that of Theorem 3.1 and hence is omitted. □

In the following, we let $\gamma := \gamma_1\gamma_2$ and denote by $L_\zeta(a,b)$ the set of Riemann–Stieltjes integrable functions on $[a,b]$ with respect to ζ. Let $c \in [a,b]$ such that $\alpha(c) = \gamma$. We further assume that $\alpha^{-1} \in L_\zeta(a,b)$ and

$$0 < \alpha(a) < \gamma < \alpha(b), \quad \int_a^c d\zeta(s) > 0 \quad \text{and} \quad \int_c^b d\zeta(s) > 0.$$

To state our main results, we begin with two technical lemmas. The first one is cited from [17, Lemma 1].

Lemma 3.3. *Let*

$$m := \gamma \left(\int_c^b d\zeta(s) \right)^{-1} \int_c^b \alpha^{-1}(s) d\zeta(s)$$

and

$$n := \gamma \left(\int_a^c d\zeta(s) \right)^{-1} \int_a^c \alpha^{-1}(s) d\zeta(s).$$

Then there exists $\eta \in L_\zeta(a,b)$ such that $\eta(s) > 0$ on $[a,b]$, and

$$\int_a^b \alpha(s)\eta(s) d\zeta(s) = \gamma \quad \text{and} \quad \int_a^b \eta(s) d\zeta(s) = 1. \tag{3.5}$$

We note from the definition of m and n that $0 < m < 1 < n$. The next lemma is a generalized arithmetic–geometric mean inequality established in [27].

Lemma 3.4. *Let $u \in C[a,b]$ and $\eta \in L_\zeta(a,b)$ satisfying $u \geq 0$, $\eta > 0$ on $[a,b]$ and $\int_a^b \eta(s) d\zeta(s) = 1$. Then*

$$\int_a^b \eta(s)u(s) d\zeta(s) \geq \exp\left(\int_a^b \eta(s) \ln\left[u(s)\right] d\zeta(s) \right),$$

where we use the convention that $\ln 0 = -\infty$ and $e^{-\infty} = 0$.

In the following, we denote $k_+ := \max\{k, 0\}$ for any $k \in \mathbb{R}$. The theorem below is derived from Theorem 2.4.

Theorem 3.5. *Assume that (1.5) and (2.1) hold. Furthermore, suppose that there exists a positive function $\varphi \in C^1_{rd}[0,\infty)_\mathbb{T}$ and that, for all sufficiently large $t_0 \in [0,\infty)_\mathbb{T}$, there is a $t_1 > t_0$ such that $g(t,s) > t_0$ for $t \geq t_1$ and $s \in [a,b]$, and*

$$\limsup_{t\to\infty} \int_{t_1}^t \left[\varphi(u) Q_1(u,t_0) - \frac{((\varphi^\Delta(u))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\varphi^\gamma(u)} \left[\frac{r_1(u)}{R_2(u,t_0)} \right]^{\gamma_2} \right] \Delta u = \infty, \tag{3.6}$$

where

$$Q_1(u,t_0) := \exp\left(\int_a^b \eta(s) \ln\left[\frac{\check{q}(u,s,t_0)}{\eta(s)} \right] d\zeta(s) \right)$$

with $\check{q}(u,s,t_0) := q(u,s)G(u,s,t_0)$ and

$$G(u,s,t_0) := \begin{cases} 1, & g(u,s) \geq u, \\ \left[\dfrac{R(g(u,s),t_0)}{R(u,t_0)} \right]^{\alpha(s)}, & g(u,s) \leq u. \end{cases} \tag{3.7}$$

Then every solution of Eq. (1.1) is either oscillatory or tends to zero eventually.

Proof. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume there is a $T \in [0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$ on $[T,\infty)_{\mathbb{T}}$ and $x(g(t,s)) > 0$ on $[T,\infty)_{\mathbb{T}} \times [a,b]$. By Theorem 2.1, we have

$$\left[x^{[2]}(t)\right]^{\Delta} < 0 \quad \text{and} \quad \left[x^{[1]}(t)\right]^{\Delta} > 0$$

eventually and either $x^{\Delta}(t)$ is eventually positive or $x(t)$ tends to zero. We suppose that

$$\left[x^{[2]}(t)\right]^{\Delta} < 0, \quad \left[x^{[1]}(t)\right]^{\Delta} > 0, \quad \text{and} \quad x^{\Delta}(t) > 0$$

eventually. Then there exists $T_1 \geq T$ such that

$$\left[x^{[2]}(t)\right]^{\Delta} < 0, \quad \left[x^{[1]}(t)\right]^{\Delta} > 0, \quad \text{and} \quad x^{\Delta}(t) > 0 \quad \text{for } t \geq T_1.$$

Consider the Riccati substitution

$$w(t) = \varphi(t)\frac{x^{[2]}(t)}{x^{\gamma}(t)},$$

where $\gamma = \gamma_1\gamma_2$. By the product rule and the quotient rule, we get

$$\begin{aligned} w^{\Delta}(t) &= \frac{\varphi(t)}{x^{\gamma}(t)}\left[x^{[2]}(t)\right]^{\Delta} + \left(\frac{\varphi(t)}{x^{\gamma}(t)}\right)^{\Delta} x^{[2]}(\sigma(t)) \\ &= \varphi(t)\frac{\left[x^{[2]}(t)\right]^{\Delta}}{x^{\gamma}(t)} + \left(\frac{\varphi^{\Delta}(t)}{x^{\gamma}(\sigma(t))} - \frac{\varphi(t)(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}\right) x^{[2]}(\sigma(t)). \end{aligned} \tag{3.8}$$

From (1.1) and the definition of $w(t)$ we have for $t \geq T_1$,

$$\begin{aligned} w^{\Delta}(t) = &-\varphi(t)\int_a^b q(t,s)\frac{[x(g(t,s))]^{\alpha(s)}}{x^{\gamma}(t)}\,d\zeta(s) \\ &+ \frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}w(\sigma(t)) - \frac{\varphi(t)(x^{\gamma}(t))^{\Delta}}{\varphi(\sigma(t))\,x^{\gamma}(t)}w(\sigma(t)). \end{aligned}$$

Let $t \in [T_1,\infty)_{\mathbb{T}}$ and $s \in [a,b]$ be fixed. If $g(t,s) \geq t$, then $x(g(t,s)) \geq x(t)$ by the fact that $x(t)$ is strictly increasing. Now we consider the case when $g(t,s) \leq t$. In view of Theorem 2.4, $\frac{x(t)}{R(t,T_1)}$ is decreasing on $(T_1,\infty)_{\mathbb{T}}$, we see that there exists $T_2 \geq T_1$ such that $g(t,s) > T_1$ for $t \geq T_2$ and $s \in [a,b]$, and so

$$x(g(t,s)) \geq \frac{R(g(t,s),T_1)}{R(t,T_1)}x(t) \quad \text{for } t \geq T_2.$$

In both cases, from the definition of $\check{q}(t,s,T_1)$ we have that for $t \geq T_2$ and $s \in [a,b]$,

$$\begin{aligned} w^{\Delta}(t) < &-\varphi(t)\int_a^b \check{q}(t,s,T_1)x^{\alpha(s)-\gamma}(t)d\zeta(s) + \frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}w(\sigma(t)) \\ &- \frac{\varphi(t)(x^{\gamma}(t))^{\Delta}}{\varphi(\sigma(t))x^{\gamma}(t)}w(\sigma(t)). \end{aligned} \tag{3.9}$$

We let $\eta \in L_{\zeta}(a,b)$ be defined as in Lemma 3.3. Then η satisfies (3.5). It follows that

$$\int_a^b \eta(s)\left[\alpha(s) - \gamma\right] d\zeta = 0.$$

From Lemma 3.4 we get

$$\begin{aligned}
&\int_a^b \check{q}(t,s,T_1)\,[x(t)]^{\alpha(s)-\gamma}\,d\zeta(s)\\
&\quad= \int_a^b \eta(s)\frac{\check{q}(t,s,T_1)}{\eta(s)}\,[x(t)]^{\alpha(s)-\gamma}\,d\zeta(s)\\
&\quad\geq \exp\left(\int_a^b \eta(s)\ln\left(\frac{\check{q}(t,s,T_1)}{\eta(s)}\,[x(t)]^{\alpha(s)-\gamma}\right)d\zeta(s)\right)\\
&\quad= \exp\left(\int_a^b \eta(s)\ln\left[\frac{\check{q}(t,s,T_1)}{\eta(s)}\right]d\zeta(s)+\ln(x(t))\int_a^b \eta(s)\,[\alpha(s)-\gamma]\,d\zeta(s)\right)\\
&\quad= \exp\left(\int_a^b \eta(s)\ln\left[\frac{\check{q}(t,s,T_1)}{\eta(s)}\right]d\zeta(s)\right).
\end{aligned}$$

This together with (3.9) shows that

$$w^{\Delta}(t) < -\varphi(t)Q_1(t,T_1)+\frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}w(\sigma(t))-\frac{\varphi(t)(x^{\gamma}(t))^{\Delta}}{\varphi(\sigma(t))\,x^{\gamma}(t)}w(\sigma(t)).$$

Then by the Pötzsche chain rule we obtain that

$$\begin{aligned}
(x^{\gamma}(t))^{\Delta} &= \gamma\left(\int_0^1\left[x(t)+h\mu(t)x^{\Delta}(t)\right]^{\gamma-1}dh\right)x^{\Delta}(t)\\
&= \gamma\left(\int_0^1[(1-h)\,x(t)+hx(\sigma(t))]^{\gamma-1}\,dh\right)x^{\Delta}(t)\\
&> \begin{cases}\gamma(x(\sigma(t)))^{\gamma-1}x^{\Delta}(t), & 0<\gamma\leq 1,\\ \gamma x^{\gamma-1}(t)x^{\Delta}(t), & \gamma\geq 1.\end{cases}
\end{aligned}$$

If $0<\gamma\leq 1$, then

$$w^{\Delta}(t) < -\varphi(t)Q_1(t,T_1)+\frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}w(\sigma(t))-\frac{\gamma\varphi(t)w(\sigma(t))}{\varphi(\sigma(t))}\frac{x^{\Delta}(t)}{x(\sigma(t))}\left(\frac{x(\sigma(t))}{x(t)}\right)^{\gamma};$$

and if $\gamma\geq 1$, then

$$w^{\Delta}(t) \leq -\varphi(t)Q_1(t,T_1)+\frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}w(\sigma(t))-\frac{\gamma\varphi(t)w(\sigma(t))}{\varphi(\sigma(t))}\frac{x^{\Delta}(t)}{x(\sigma(t))}\frac{x(\sigma(t))}{x(t)}.$$

Note that as $x(t)$ is strictly increasing on $[T_2,\infty)_{\mathbb{T}}$, we see that for $\gamma>0$,

$$w^{\Delta}(t) \leq -\varphi(t)Q_1(t,T_1)+\frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}w(\sigma(t))-\frac{\gamma\varphi(t)w(\sigma(t))}{\varphi(\sigma(t))}\frac{x^{\Delta}(t)}{x(\sigma(t))}. \tag{3.10}$$

Since $x^{[2]}(t)$ is strictly decreasing on $[T_1,\infty)_{\mathbb{T}}$,

$$\begin{aligned}
x^{[1]}(t) > x^{[1]}(t)-x^{[1]}(T_1) &= \int_{T_1}^t \phi_{\gamma_2}^{-1}\left[x^{[2]}(u)\right]r_2^{-\frac{1}{\gamma_2}}(u)\Delta u\\
&> \phi_{\gamma_2}^{-1}\left[x^{[2]}(t)\right]\int_{T_1}^t r_2^{-\frac{1}{\gamma_2}}(u)\Delta u > \phi_{\gamma_2}^{-1}\left[x^{[2]}(\sigma(t))\right]R_2(t,T_1).
\end{aligned} \tag{3.11}$$

From (3.10) and (3.11) we obtain for $t\geq T_2$,

$$w^{\Delta}(t) \leq -\varphi(t)Q_1(t,T_1)+\frac{(\varphi^{\Delta}(t))_+}{\varphi(\sigma(t))}w(\sigma(t))-\frac{\gamma\varphi(t)}{\varphi^{\beta}(\sigma(t))}\left[\frac{R_2(t,T_1)}{r_1(t)}\right]^{1/\gamma_1}w^{\beta}(\sigma(t)), \tag{3.12}$$

where $\beta := \frac{\gamma+1}{\gamma}$. Define

$$X^{\beta} := \frac{\gamma\varphi(t)}{\varphi^{\beta}(\sigma(t))}\left[\frac{R_2(t,T_1)}{r_1(t)}\right]^{1/\gamma_1} w^{\beta}(\sigma(t))$$

and

$$Y^{\beta-1} := \frac{(\varphi^{\Delta}(t))_+}{\beta(\gamma\varphi(t))^{1/\beta}}\left[\frac{r_1(t)}{R_2(t,T_1)}\right]^{\gamma_2/(\gamma+1)}.$$

Then, using the inequality (see [14])

$$\beta XY^{\beta-1} - X^{\beta} \le (\beta-1)Y^{\beta}, \tag{3.13}$$

we get that

$$\begin{aligned}\frac{(\varphi^{\Delta}(t))_+}{\varphi(\sigma(t))}w(\sigma(t)) - \frac{\gamma\varphi(t)}{\varphi^{\beta}(\sigma(t))}\left[\frac{R_2(t,T_1)}{r_1(t)}\right]^{1/\gamma_1} w^{\beta}(\sigma(t))\\ \le \frac{((\varphi^{\Delta}(t))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\varphi^{\gamma}(t)}\left[\frac{r_1(t)}{R_2(t,T_1)}\right]^{\gamma_2}.\end{aligned}$$

From this and (3.12) we have

$$w^{\Delta}(t) \le -\varphi(t)Q_1(t,T_1) + \frac{((\varphi^{\Delta}(t))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\varphi^{\gamma}(t)}\left[\frac{r_1(t)}{R_2(t,T_1)}\right]^{\gamma_2}.$$

Integrating both sides from T_2 to t we get

$$\begin{aligned}\int_{T_2}^{t}\left[\varphi(u)Q_1(u,T_1) - \frac{((\varphi^{\Delta}(u))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\varphi^{\gamma}(u)}\left[\frac{r_1(u)}{R_2(u,T_1)}\right]^{\gamma_2}\right]\Delta u\\ \le w(T_2) - w(t) \le w(T_2),\end{aligned}$$

which leads to a contradiction to (3.6). □

Theorem 3.6. *Assume that* (1.5) *and* (2.1) *hold. Furthermore, suppose that there exists a positive function* $\rho \in C^1_{rd}[0,\infty)_{\mathbb{T}}$ *and that for all sufficiently large* $t_0 \in [0,\infty)_{\mathbb{T}}$*, there is a* $t_1 > t_0$ *such that* $g(t,s) > t_0$ *for* $t \ge t_1$ *and* $s \in [a,b]$*, and*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\rho(u)Q_2(u,t_0) - \frac{((\rho^{\Delta}(u))_+)^{\gamma_2+1}r_2(u)}{(\gamma_2+1)^{\gamma_2+1}\rho^{\gamma_2}(u)}\right]\Delta u = \infty, \tag{3.14}$$

where

$$Q_2(u,t_0) := \exp\left(\int_a^b \eta(s)\ln\left[\frac{\bar{q}(u,s,t_0)}{\eta(s)}\right]d\zeta(s)\right)$$

with $\bar{q}(u,s,t_0) := R^{\gamma}(u,t_0)G(u,s,t_0)q(u,s)$ *and* $G(u,s,t_0)$ *is given by* (3.7)*. Then every solution of Eq.* (1.1) *is either oscillatory or tends to zero eventually.*

Proof. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume there is a $T \in [0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$ on $[T,\infty)_{\mathbb{T}}$ and $x(g(t,s)) > 0$ on $[T,\infty)_{\mathbb{T}}\times[a,b]$. By Theorem 2.1, we have

$$\left[x^{[2]}(t)\right]^{\Delta} < 0 \quad \text{and} \quad \left[x^{[1]}(t)\right]^{\Delta} > 0$$

eventually and either $x^{\Delta}(t)$ is eventually positive or $x(t)$ tends to zero eventually. We let

$$\left[x^{[2]}(t)\right]^{\Delta}<0, \quad \left[x^{[1]}(t)\right]^{\Delta}>0, \quad \text{and} \quad x^{\Delta}(t)>0$$

eventually. Then there exists $T_1 \geq T$ such that

$$\left[x^{[2]}(t)\right]^{\Delta}<0, \quad \left[x^{[1]}(t)\right]^{\Delta}>0, \quad \text{and} \quad x^{\Delta}(t)>0 \quad \text{for } t \geq T_1.$$

Let

$$z(t):=\rho(t)\frac{x^{[2]}(t)}{(x^{[1]}(t))^{\gamma_2}}.$$

By the product rule and the quotient rule, we get

$$\begin{aligned} z^{\Delta}(t) &= \frac{\rho(t)}{(x^{[1]}(t))^{\gamma_2}}(x^{[2]}(t))^{\Delta}+\left(\frac{\rho(t)}{(x^{[1]}(t))^{\gamma_2}}\right)^{\Delta} x^{[2]}(\sigma(t)) \\ &= \rho(t)\frac{(x^{[2]}(t))^{\Delta}}{(x^{[1]}(t))^{\gamma_2}}+\left(\frac{\rho^{\Delta}(t)}{(x^{[1]}(\sigma(t)))^{\gamma_2}}-\frac{\rho(t)((x^{[1]}(t))^{\gamma_2})^{\Delta}}{(x^{[1]}(t))^{\gamma_2}(x^{[1]}(\sigma(t)))^{\gamma_2}}\right) x^{[2]}(\sigma(t)). \end{aligned}$$

From (1.1) and the definition of $z(t)$, we see that for $t \geq T_1$,

$$\begin{aligned} z^{\Delta}(t) = & -\rho(t)\int_a^b q(t,s)\frac{[x(g(t,s))]^{\alpha(s)}}{(x^{[1]}(t))^{\gamma_2}}d\zeta(s)+\frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}z(\sigma(t)) \\ & -\frac{\rho(t)z(\sigma(t))}{\rho(\sigma(t))}\frac{((x^{[1]}(t))^{\gamma_2})^{\Delta}}{(x^{[1]}(t))^{\gamma_2}}. \end{aligned}$$

Hence

$$\frac{[x(g(t,s))]^{\alpha(s)}}{(x^{[1]}(t))^{\gamma_2}}=\frac{[x(g(t,s))]^{\alpha(s)}}{x^{\gamma}(t)}\frac{x^{\gamma}(t)}{(x^{[1]}(t))^{\gamma_2}}.$$

As shown in the proof of Theorem 3.5, there exists $T_2 \geq T_1$ such that $g(t,s)>T_1$ for $t \geq T_2$ and $s \in [a,b]$, and so

$$[x(g(t,s))]^{\alpha(s)}>G(t,s,T_1)x^{\alpha(s)}(t),$$

and by Theorem 2.4 we get

$$x^{\gamma}(t)>R^{\gamma}(t,T_1)(x^{[1]}(t))^{\gamma_2},$$

where $\gamma=\gamma_1\gamma_2$. Then

$$\frac{[x(g(t,s))]^{\alpha(s)}}{(x^{[1]}(t))^{\gamma_2}}>R^{\gamma}(t,T_1)G(t,s,T_1)\, x^{\alpha(s)-\gamma}(t).$$

It follows that for $t \geq T_2$,

$$\begin{aligned} z^{\Delta}(t) < & -\rho(t)\int_a^b \bar{q}(t,s,T_1)x^{\alpha(s)-\gamma}(t)d\zeta(s)+\frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}z(\sigma(t)) \\ & -\frac{\rho(t)z(\sigma(t))}{\rho(\sigma(t))}\frac{((x^{[1]}(t))^{\gamma_2})^{\Delta}}{(x^{[1]}(t))^{\gamma_2}}. \end{aligned}$$

Also, as shown in the proof of Theorem 3.5,

$$z^{\Delta}(t)<-\rho(t)Q_2(t,T_1)+\frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}z(\sigma(t))-\frac{\rho(t)z(\sigma(t))}{\rho(\sigma(t))}\frac{((x^{[1]}(t))^{\gamma_2})^{\Delta}}{(x^{[1]}(t))^{\gamma_2}}.$$

By the Pötzsche chain rule,

$$\begin{aligned}((x^{[1]}(t))^{\gamma_2})^{\Delta} &= \gamma_2 \int_0^1 [x^{[1]}(t) + h\mu(t)(x^{[1]}(t))^{\Delta}]^{\gamma_2 - 1} dh \, (x^{[1]}(t))^{\Delta} \\ &= \gamma_2 \int_0^1 \left[(1-h)\, x^{[1]}(t) + hx^{[1]}\left(\sigma(t)\right)\right]^{\gamma_2-1} dh \, (x^{[1]}(t))^{\Delta} \\ &\geq \begin{cases} \gamma_2 \left[x^{[1]}\left(\sigma(t)\right)\right]^{\gamma_2-1} (x^{[1]}(t))^{\Delta}, & 0 < \gamma_2 \leq 1 \\ \gamma_2 \left[x^{[1]}(t)\right]^{\gamma_2-1} (x^{[1]}(t))^{\Delta}, & \gamma_2 \geq 1. \end{cases}\end{aligned}$$

If $0 < \gamma_2 \leq 1$, we have

$$z^{\Delta}(t) < -\rho(t)Q_2(t,T_1) + \frac{\rho^{\Delta}(t)}{\rho\left(\sigma(t)\right)} z\left(\sigma(t)\right) - \frac{\gamma_2 \rho(t) z\left(\sigma(t)\right)}{\rho\left(\sigma(t)\right)} \frac{(x^{[1]}(t))^{\Delta}}{x^{[1]}\left(\sigma(t)\right)} \left(\frac{x^{[1]}\left(\sigma(t)\right)}{x^{[1]}(t)}\right)^{\gamma_2}; \quad (3.15)$$

and if $\gamma_2 \geq 1$, we have

$$z^{\Delta}(t) < -\rho(t)Q_2(t,T_1) + \frac{\rho^{\Delta}(t)}{\rho\left(\sigma(t)\right)} z\left(\sigma(t)\right) - \frac{\gamma_2 \rho(t) z\left(\sigma(t)\right)}{\rho\left(\sigma(t)\right)} \frac{(x^{[1]}(t))^{\Delta}}{x^{[1]}\left(\sigma(t)\right)} \frac{x^{[1]}\left(\sigma(t)\right)}{x^{[1]}(t)}. \qquad (3.16)$$

Since $x^{[1]}$ is strictly increasing and $x^{[2]}$ is strictly decreasing, we get that

$$x^{[1]}\left(\sigma(t)\right) \geq x^{[1]}(t) \quad \text{and} \quad (x^{[1]}(t))^{\Delta} \geq \left(\frac{x^{[2]}\left(\sigma(t)\right)}{r_2(t)}\right)^{\frac{1}{\gamma_2}}. \qquad (3.17)$$

Then from (3.15) and (3.16)

$$z^{\Delta}(t) < -\rho(t)Q_2(t,T_1) + \frac{(\rho^{\Delta}(t))_+}{\rho\left(\sigma(t)\right)} z\left(\sigma(t)\right) - \frac{\gamma_2 \rho(t)}{\rho^{\beta}\left(\sigma(t)\right) r_2^{1/\gamma_2}(t)} \left(z\left(\sigma(t)\right)\right)^{\beta},$$

where $\beta := \frac{\gamma_2+1}{\gamma_2}$. Define

$$X^{\beta} := \frac{\gamma_2 \rho(t)}{\rho^{\beta}\left(\sigma(t)\right) r_2^{1/\gamma_2}(t)} z^{\beta}\left(\sigma(t)\right) \quad \text{and} \quad Y^{\beta-1} := \frac{(\rho^{\Delta}(t))_+ r_2^{1/(\gamma_2+1)}(t)}{\beta\left(\gamma_2 \rho(t)\right)^{1/\beta}}.$$

Then from (3.13),

$$\frac{(\rho^{\Delta}(t))_+}{\rho\left(\sigma(t)\right)} z\left(\sigma(t)\right) - \frac{\gamma_2 \rho(t)}{\rho^{\beta}\left(\sigma(t)\right) r_2^{1/\gamma_2}(t)} \left(z\left(\sigma(t)\right)\right)^{\beta} \leq \frac{((\rho^{\Delta}(t))_+)^{\gamma_2+1} r_2(t)}{(\gamma_2+1)^{\gamma_2+1} \rho^{\gamma_2}(t)}.$$

The rest of the proof is similar to that of Theorem 3.5 and hence is omitted. □

The last theorem is under the assumption that $\int_t^{\infty} q(u,s)\Delta u < \infty$ for any $s \in [a,b]$.

Theorem 3.7. *Let $g(t,s)$ be a nondecreasing function with respect to t. Assume that (1.5) and (2.1) hold and for any $t_0 \in [0,\infty)_{\mathbb{T}}$,*

$$\limsup_{t\to\infty} Q_3(t) \left\{ \int_{t_0}^{\widehat{g}(t)} \left[\frac{R_2(u,t_0)}{r_1(u)}\right]^{\frac{1}{\gamma_1}} \Delta u \right\}^{\gamma} > 1, \qquad (3.18)$$

where $\widehat{g}(t) := \inf_{s\in[a,b]}\{t, g(t,s)\}$ *and*

$$Q_3(t) := \exp\left(\int_a^b \eta(s)\ln\left[\frac{\widehat{q}(t,s)}{\eta(s)}\right] d\zeta(s)\right)$$

with $\widehat{q}(t,s) := \int_t^\infty q(u,s)\Delta u$*. Then every solution of Eq.* (1.1) *is either oscillatory or tends to zero eventually.*

Proof. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume there is a $T \in [0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$ on $[T,\infty)_{\mathbb{T}}$ and $x(g(t,s)) > 0$ on $[T,\infty)_{\mathbb{T}}\times[a,b]$. By Theorem 2.1, we have

$$\left[x^{[2]}(t)\right]^{\Delta} < 0 \quad \text{and} \quad \left[x^{[1]}(t)\right]^{\Delta} > 0$$

eventually and either $x^{\Delta}(t)$ is eventually positive or $x(t)$ tends to zero eventually. We let

$$\left[x^{[2]}(t)\right]^{\Delta} < 0,\ \left[x^{[1]}(t)\right]^{\Delta} > 0,\ \text{and}\ \ x^{\Delta}(t) > 0$$

eventually. Then there exists $T_1 \geq T$ such that

$$\left[x^{[2]}(t)\right]^{\Delta} < 0, \quad \left[x^{[1]}(t)\right]^{\Delta} > 0, \quad \text{and} \quad x^{\Delta}(t) > 0 \quad \text{for } t \geq T_1.$$

Integrating both sides of (1.1) from t to ∞ and then using the facts that $x(t)$ is strictly increasing and $g(t,s)$ is a nondecreasing with respect to t, we obtain that

$$\begin{aligned} x^{[2]}(t) &> \int_a^b\int_t^\infty q(u,s)\,[x(g(u,s))]^{\alpha(s)}\,\Delta u\,d\zeta(s) \\ &\geq \int_a^b \widehat{q}(t,s)\,[x(g(t,s))]^{\alpha(s)}\,d\zeta(s). \end{aligned}$$

Note that $x^{\Delta}(t) > 0$ on $[T_2,\infty)_{\mathbb{T}}$ and $\widehat{g}(t) \leq g(t,s)$ on $[T_2,\infty)_{\mathbb{T}}\times[a,b]$. Then

$$\begin{aligned} x^{[2]}(t) &\geq \int_a^b \widehat{q}(t,s)\,[x(g(t,s))]^{\alpha(s)}\,d\zeta(s) \\ &> \int_a^b \widehat{q}(t,s)\,[x(\widehat{g}(t))]^{\alpha(s)}\,d\zeta(s). \end{aligned} \tag{3.19}$$

By Theorem 2.4,

$$[x(t)]^{\gamma} > x^{[2]}(t)\left\{\int_{T_1}^t \left[\frac{R_2(u,T_1)}{r_1(u)}\right]^{\frac{1}{\gamma_1}}\Delta u\right\}^{\gamma}, \tag{3.20}$$

where $\gamma = \gamma_1\gamma_2$. Choose $T_2 > T_1$ such that $\widehat{g}(t) > T_1$ for $t \geq T_2$. Then from (3.20) we see that for $t \geq T_2$,

$$\begin{aligned} [x(\widehat{g}(t))]^{\gamma} &> x^{[2]}\left(\widehat{g}(t)\right)\left\{\int_{T_1}^{\widehat{g}(t)} \left[\frac{R_2(u,T_1)}{r_1(u)}\right]^{\frac{1}{\gamma_1}}\Delta u\right\}^{\gamma} \\ &\geq x^{[2]}(t)\left\{\int_{T_1}^{\widehat{g}(t)} \left[\frac{R_2(u,T_1)}{r_1(u)}\right]^{\frac{1}{\gamma_1}}\Delta u\right\}^{\gamma}, \end{aligned}$$

which implies that

$$x^{[2]}(t) < [x(\widehat{g}(t))]^{\gamma} \left\{ \int_{T_1}^{\widehat{g}(t)} \left[\frac{R_2(u,T_1)}{r_1(u)} \right]^{\frac{1}{\gamma_1}} \Delta u \right\}^{-\gamma} \quad \text{for } t \geq T_2. \tag{3.21}$$

Using (3.21) in (3.19) we find for $t \geq T_2$,

$$[x(\widehat{g}(t))]^{\gamma} \left\{ \int_{T_1}^{\widehat{g}(t)} \left[\frac{R_2(u,T_1)}{r_1(u)} \right]^{\frac{1}{\gamma_1}} \Delta u \right\}^{-\gamma} > \int_a^b \widehat{q}(t,s)\, [x(\widehat{g}(t))]^{\alpha(s)}\, d\zeta(s).$$

Hence

$$\left\{ \int_{T_1}^{\widehat{g}(t)} \left[\frac{R_2(u,T_1)}{r_1(u)} \right]^{\frac{1}{\gamma_1}} \Delta u \right\}^{-\gamma} > \int_a^b \widehat{q}(t,s)\, [x(\widehat{g}(t))]^{\alpha(s)-\gamma}\, d\zeta(s). \tag{3.22}$$

As shown in the proof of Theorem 3.5,

$$\int_a^b \widehat{q}(t,s)\, [x(\widehat{g}(t))]^{\alpha(s)-\gamma}\, d\zeta(s) \geq \exp\left(\int_a^b \eta(s) \ln \left[\frac{\widehat{q}(t,s)}{\eta(s)} \right] d\zeta(s) \right) = Q_3(t).$$

This together with (3.22) shows that

$$Q_3(t) \left\{ \int_{T_1}^{\widehat{g}(t)} \left[\frac{R_2(u,T_1)}{r_1(u)} \right]^{\frac{1}{\gamma_1}} \Delta u \right\}^{\gamma} < 1,$$

which implies that

$$\limsup_{t\to\infty} Q_3(t) \left\{ \int_{T_1}^{\widehat{g}(t)} \left[\frac{R_2(u,T_1)}{r_1(u)} \right]^{\frac{1}{\gamma_1}} \Delta u \right\}^{\gamma} \leq 1.$$

This leads to a contradiction to (3.18). □

At the end of this paper, we establish parallel results to Theorems 3.1–3.7 under the assumption that (1.6) holds.

Theorem 3.8. *Assume that* (1.6), (2.1), (2.3), (2.4) *and* (3.1) *hold. Then every solution of Eq.* (1.1) *is either oscillatory or tends to zero eventually.*

Theorem 3.9. *Assume that* (1.6), (2.1), (2.3), (2.4) *and* (3.3) *hold. Then every solution of Eq.* (1.1) *is either oscillatory or tends to zero eventually.*

Theorem 3.10. *Assume that* (1.6), (2.1), (2.3), (2.4) *and* (3.6) *hold. Then every solution of Eq.* (1.1) *is either oscillatory or tends to zero eventually.*

Theorem 3.11. *Assume that* (2.1), (2.3), (2.4) *and* (3.14) *hold. Then every solution of Eq.* (1.1) *is either oscillatory or tends to zero eventually.*

Theorem 3.12. *Assume that $g(t,s)$ be a nondecreasing function with respect to t. Assume that* (2.1), (2.3), (2.4) *and* (3.18) *hold. Then every solution of Eq.* (1.1) *is either oscillatory or tends to zero eventually.*

Proof of Theorems 3.8–3.12. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume there is a $T \in [0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$ on $[T, \infty)_{\mathbb{T}}$ and $x(g(t,s)) > 0$ on $[T, \infty)_{\mathbb{T}} \times [a, b]$. By Theorem 2.3,

$$\left[x^{[2]}(t)\right]^{\Delta} < 0 \quad \text{and} \quad \left[x^{[1]}(t)\right]^{\Delta} > 0$$

eventually and either $x^{\Delta}(t)$ is eventually positive or $x(t)$ tends to zero eventually. We suppose that

$$\left[x^{[2]}(t)\right]^{\Delta} < 0, \quad \left[x^{[1]}(t)\right]^{\Delta} > 0, \quad \text{and} \quad x^{\Delta}(t) > 0$$

eventually. The rest of the proof is similar to that of Theorems 3.1–3.7 respectively, and hence is omitted. □

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