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Nonlinear q-fractional differential equations with nonlocal and sub-strip type boundary conditions

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Abstract. This paper is concerned with new boundary value problems of nonlinear q-fractional differential equations with nonlocal and sub-strip type boundary conditions. Our results are new in the present setting and rely on the contraction mapping principle and a fixed point theorem due to O'Regan. Some illustrative examples are also presented.

Keywords: fractional q-difference equations, nonlocal, integral, boundary conditions, existence, fixed point.

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1 Introduction

In this paper, we introduce a sub-strip type boundary condition of the form

$$x(\xi) = b \int_{\eta}^{1} x(s)\, d_q s, \quad 0 < \xi < \eta < 1,$$

which relates the contribution due to a sub-strip of arbitrary length with the value of the unknown function at an arbitrary (nonlocal) point off the sub-strip. Precisely, we consider the following boundary value problem of nonlinear fractional q-difference equations with nonlocal and sub-strip type boundary conditions:

$$\begin{cases} {}^cD_q^{\upsilon} x(t) = f(t, x(t)), \ \ t \in [0,1], \ \ 1 < \upsilon \le 2, \ \ 0 < q < 1, \\ x(0) = x_0 + g(x), \quad x(\xi) = b \int_{\eta}^{1} x(s)\, d_q s, \ \ 0 < \xi < \eta < 1, \end{cases} \tag{1.1}$$

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where ${}^cD_q^v$ denotes the Caputo fractional q-derivative of order $v, f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ and $g\colon C([0,1],\mathbb{R})\to\mathbb{R}$ are given continuous functions, and b is a real constant. Here we emphasize that the nonlocal conditions are more plausible than the standard initial conditions to describe some physical phenomena. In (1.1), $g(x)$ may be understood as $g(x)=\sum_{j=1}^{p}\alpha_j x(t_j)$ where $\alpha_j, j=1,\ldots,p$, are given constants and $0<t_1<\ldots<t_p\leq 1$. For more details we refer to the work by Byszewski [1,2].

Recent extensive studies on fractional boundary value problems indicate that it is one of the hot topics of the present-day research. There have appeared numerous articles covering a variety of aspects of these problems. The nonlocal nature of a fractional order differential operator, which takes into account hereditary properties of various material and processes, has helped to improve the mathematical modelling of many real world problems of physical and technical sciences [3,4]. For some recent work on the topic, please see [5–13] and the references therein.

Fractional q-difference (q-fractional) equations are regarded as the fractional analogue of q-difference equations. Motivated by recent interest in the study of fractional-order differential equations, the topic of q-fractional equations has attracted the attention of many researchers. The details of some recent development of the subject can be found in [14–20], whereas the background material on q-fractional calculus can be found in a recent text [22].

The paper is organized as follows. In Section 2, we recall some fundamental concepts of fractional q-calculus and establish a lemma for the linear variant of the given problem. Section 3 contains the existence results for the problem (1.1) which are shown by applying Banach's contraction principle and a fixed point theorem due to O'Regan. In Section 4, we consider a new problem with a condition of the form $D_q x(\xi)=b\int_\eta^1 x(s)\,d_qs$ (flux sub-strip condition) instead of $x(\xi)=b\int_\eta^1 x(s)\,d_qs$ in (1.1). Finally, some examples illustrating the applicability of our results are presented in Section 5.

2 Preliminaries

First of all, we recall the notations and terminology for q-fractional calculus [21–23].

For a real parameter $q\in\mathbb{R}^+\setminus\{1\}$, a q-real number denoted by $[a]_q$ is defined by

$$[a]_q=\frac{1-q^a}{1-q},\ \ a\in\mathbb{R}.$$

The q-analogue of the Pochhammer symbol (q-shifted factorial) is defined as

$$(a;q)_0=1,\ \ (a;q)_k=\prod_{i=0}^{k-1}(1-aq^i),\quad k\in\mathbb{N}\cup\{\infty\}.$$

The q-analogue of the exponent $(x-y)^k$ is

$$(x-y)^{(0)}=1,\ \ (x-y)^{(k)}=\prod_{j=0}^{k-1}(x-yq^j),\quad k\in\mathbb{N},\ x,y\in\mathbb{R}.$$

The q-gamma function $\Gamma_q(y)$ is defined as

$$\Gamma_q(y)=\frac{(1-q)^{(y-1)}}{(1-q)^{y-1}},$$

where $y\in\mathbb{R}\setminus\{0,-1,-2,\ldots\}$. Observe that $\Gamma_q(y+1)=[y]_q\Gamma_q(y)$.

Definition 2.1 ([21]). Let f be a function defined on $[0,1]$. The fractional q-integral of the Riemann–Liouville type of order $\beta \geq 0$ is $(I_q^0 f)(t) = f(t)$ and

$$I_q^\beta f(t) := \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} f(s)\, d_q s = t^\beta (1-q)^\beta \sum_{k=0}^\infty q^k \frac{(q^\beta;q)_k}{(q;q)_k} f(tq^k), \quad \beta > 0, \quad t \in [0,1].$$

Observe that $\beta = 1$ in the Definition 2.1 yields q-integral

$$I_q f(t) := \int_0^t f(s) d_q s = t(1-q) \sum_{k=0}^\infty q^k f(tq^k).$$

For more details on q-integral and fractional q-integral, see Section 1.3 and Section 4.2 respectively in [22].

Remark 2.2. The q-fractional integration possesses the semigroup property ([22, Proposition 4.3]):

$$I_q^\gamma I_q^\beta f(t) = I_q^{\beta+\gamma} f(t); \quad \gamma, \beta \in \mathbb{R}^+.$$

Further, it was shown in Lemma 6 of [23] that

$$I_q^\beta (x)^{(\sigma)} = \frac{\Gamma_q(\sigma+1)}{\Gamma_q(\beta+\sigma+1)} (x)^{(\beta+\sigma)}, \quad 0 < x < a, \quad \beta \in \mathbb{R}^+, \quad \sigma \in (-1, \infty).$$

Before giving the definition of fractional q-derivative, we recall the concept of q-derivative. We know that the q-derivative of a function $f(t)$ is defined as

$$(D_q f)(t) = \frac{f(t) - f(qt)}{t - qt}, \quad t \neq 0, \quad (D_q f)(0) = \lim_{t \to 0} (D_q f)(t).$$

Furthermore,

$$D_q^0 f = f, \quad D_q^n f = D_q(D_q^{n-1} f), \quad n = 1, 2, 3, \ldots \tag{2.1}$$

Definition 2.3 ([22]). The Caputo fractional q-derivative of order $\beta > 0$ is defined by

$${}^c D_q^\beta f(t) = I_q^{\lceil \beta \rceil - \beta} D_q^{\lceil \beta \rceil} f(t),$$

where $\lceil \beta \rceil$ is the smallest integer greater than or equal to β.

Next we recall some properties involving Riemann–Liouville q-fractional integral and Caputo fractional q-derivative ([22, Theorem 5.2]).

$$I_q^\beta \, {}^c D_q^\beta f(t) = f(t) - \sum_{k=0}^{\lceil \beta \rceil - 1} \frac{t^k}{\Gamma_q(k+1)} (D_q^k f)(0^+), \quad \forall\, t \in (0, a], \ \beta > 0; \tag{2.2}$$

$${}^c D_q^\beta \, I_q^\beta f(t) = f(t), \quad \forall\, t \in (0, a], \ \beta > 0. \tag{2.3}$$

Lemma 2.4. *Let $y \in C([0,1], \mathbb{R})$. Then the following problem*

$$\begin{cases} {}^c D_q^v x(t) = y(t), \quad 1 < v \leq 2, \\ x(0) = y_0, \ \ x(\xi) = b \displaystyle\int_\eta^1 x(s) d_q s, \ \ y_0 \in \mathbb{R}, \ t \in [0,1], \end{cases} \tag{2.4}$$

is equivalent to an integral equation:

$$\begin{aligned} x(t) = & \int_0^t \frac{(t-qs)^{(v-1)}}{\Gamma_q(v)} y(s) d_q s \\ & + \frac{t}{B}\left\{ b \int_\eta^1 \Big(\int_0^s \frac{(s-qu)^{(v-1)}}{\Gamma_q(v)} y(u) d_q u \Big) d_q s - \int_0^\xi \frac{(\xi-qs)^{(v-1)}}{\Gamma_q(v)} y(s) d_q s \right\} \\ & + y_0 \Big[1 + \frac{t}{B}(b(1-\eta)-1) \Big], \end{aligned} \tag{2.5}$$

where

$$B = \xi - \frac{b(1-\eta^2)}{1+q} \neq 0. \tag{2.6}$$

Proof. Applying the operator I_q^v on the equation ${}^cD_q^v x(t) = y(t)$ and using (2.2), we get

$$x(t) = \int_0^t \frac{(t-qs)^{(v-1)}}{\Gamma_q(v)} y(s)\, d_q s + a_0 t + a_1, \tag{2.7}$$

where $a_0, a_1 \in \mathbb{R}$ are arbitrary constants. Using the given boundary conditions, it is found that $a_1 = y_0$, and

$$\begin{aligned} a_0 = & \frac{1}{B}\left\{ b \int_\eta^1 \Big(\int_0^s \frac{(s-qu)^{(v-1)}}{\Gamma_q(v)} y(u) d_q u \Big)\, d_q s - \int_0^\xi \frac{(\xi-qs)^{(v-1)}}{\Gamma_q(v)} y(s)\, d_q s \right\} \\ & + \frac{y_0}{B}\Big(b(1-\eta) - 1 \Big). \end{aligned} \tag{2.8}$$

Substituting the values of a_0, a_1 in (2.7) yields (2.5). Conversely, applying the operator ${}^cD_q^v$ on (2.5) and taking into account (2.3), it follows that ${}^cD_q^v x(t) = y(t)$. From (2.5), it is easy to verify that the boundary conditions $x(0) = y_0, \quad x(\xi) = b\int_\eta^1 x(s)\, d_q s$ are satisfied. This establishes the equivalence between (2.4) and (2.5). □

3 Main results

We denote by $\mathcal{C} = C([0,1], \mathbb{R})$ the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by

$$\|x\| = \sup\{|x(t)| : t \in [0,1]\}.$$

Also by $L^1([0,1], \mathbb{R})$ we denote the Banach space of measurable functions $x\colon [0,1] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1 |x(t)|\, dt$.

In view of Lemma 2.4, we can transform the problem (1.1) into an equivalent fixed point problem: $\mathcal{F}x = x$, where the operator $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ is defined by

$$\begin{aligned} (\mathcal{F}x)(t) = & \int_0^t \frac{(t-qs)^{(v-1)}}{\Gamma_q(v)} f(s, x(s))\, d_q s \\ & + \frac{t}{B}\left\{ b \int_\eta^1 \Big(\int_0^s \frac{(s-qu)^{(v-1)}}{\Gamma_q(v)} f(u, x(u)) d_q u \Big)\, d_q s \right. \\ & \left. - \int_0^\xi \frac{(\xi-qs)^{(v-1)}}{\Gamma_q(v)} f(s, x(s))\, d_q s \right\} \\ & + \Big[1 + \frac{t}{B}(b(1-\eta)-1) \Big](x_0 + g(x)), \quad t \in [0,1]. \end{aligned} \tag{3.1}$$

Observe that the existence of a fixed point for the operator $\mathcal{F}$ implies the existence of a solution for the problem (1.1).

For convenience we introduce the notations:

$$\mu_0 := \frac{1}{\Gamma_q(v+1)} + \frac{1}{|B|}\left\{\frac{|b|(1-\eta^{v+1})}{\Gamma_q(v+2)} + \frac{\xi^v}{\Gamma_q(v+1)}\right\}, \tag{3.2}$$

and

$$k_0 := 1 + \frac{1}{|B|}|b(1-\eta)-1|. \tag{3.3}$$

Furthermore, we assume that the condition (2.6): $B = \xi - \frac{b(1-\eta^2)}{1+q} \neq 0$ holds throughout the forthcoming analysis.

Theorem 3.1. *Assume that*

(A_1) $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ *be a continuous function such that*

$$|f(t,x)-f(t,y)| \le L|x-y|, \forall t\in[0,1],\ L>0,\ x,y\in\mathbb{R};$$

(A_2) $g\colon C([0,1],\mathbb{R})\to\mathbb{R}$ *is a continuous function satisfying the condition*

$$|g(u)-g(v)| \le \ell\|u-v\| \ \ \forall\, u,v\in C([0,1],\mathbb{R}),\ \ell>0;$$

(A_3) $\delta := L\mu_0 + k_0\ell < 1.$

Then the boundary value problem (1.1) *has a unique solution.*

Proof. For $x,y\in\mathcal{C}$ and for each $t\in[0,1]$, from the definition of $\mathcal{F}$ and assumptions (A_1) and (A_2), we obtain

$$\begin{aligned}
&|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)| \\
&\le \int_0^t \frac{(t-qs)^{(v-1)}}{\Gamma_q(v)}|f(s,x(s))-f(s,y(s))|d_qs \\
&\quad + \frac{1}{|B|}\left\{|b|\int_\eta^1\Big(\int_0^s \frac{(s-qu)^{(v-1)}}{\Gamma_q(v)}|f(u,x(u))-f(u,y(u))|d_qu\Big)d_qs\right. \\
&\qquad\left. + \int_0^\xi \frac{(\xi-qs)^{(v-1)}}{\Gamma_q(v)}|f(s,x(s))-f(s,y(s))|d_qs\right\} \\
&\quad + \left[1+\frac{1}{|B|}|b(1-\eta)-1|\right]|g(x)-g(y)| \\
&\le L\|x-y\|\left[\int_0^t \frac{(t-qs)^{(v-1)}}{\Gamma_q(v)}d_qs + \frac{1}{|B|}\left\{|b|\int_\eta^1\Big(\int_0^s\frac{(s-qu)^{(v-1)}}{\Gamma_q(v)}d_qu\Big)d_qs\right.\right. \\
&\qquad\qquad\left.\left. + \int_0^\xi \frac{(\xi-qs)^{(v-1)}}{\Gamma_q(v)}d_qs\right\}\right] \\
&\quad + \left[1+\frac{1}{|B|}|b(1-\eta)-1|\right]\ell\|x-y\| \\
&\le L\|x-y\|\left[\frac{1}{\Gamma_q(v+1)} + \frac{1}{|B|}\left\{\frac{|b|(1-\eta^{v+1})}{\Gamma_q(v+2)} + \frac{\xi^v}{\Gamma_q(v+1)}\right\}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\left[1+\frac{1}{|B|}|b(1-\eta)-1|\right]\ell\|x-y\| \\
&=\big(L\mu_0+k_0\ell\big)\|x-y\|.
\end{aligned}
$$

Hence

$$\|(\mathcal{F}x)-(\mathcal{F}y)\|\leq\delta\|x-y\|.$$

As $\delta<1$ by (A_3), the operator $\mathcal{F}$ is a contraction map from the Banach space $\mathcal{C}$ into itself. Hence the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Our next existence result relies on a fixed point theorem due to O'Regan in [24].

Lemma 3.2. *Let U be an open set in a closed, convex set C of a Banach space E. Assume $0\in U$. Also assume that $\mathcal{F}(\bar{U})$ is bounded and that $\mathcal{F}\colon\bar{U}\to C$ is given by $\mathcal{F}=\mathcal{F}_1+\mathcal{F}_2$, in which $\mathcal{F}_1\colon\bar{U}\to E$ is continuous and completely continuous and $\mathcal{F}_2\colon\bar{U}\to E$ is a nonlinear contraction (i.e., there exists a continuous nondecreasing function $\vartheta\colon[0,\infty)\to[0,\infty)$ satisfying $\vartheta(z)<z$ for $z>0$, such that $\|\mathcal{F}_2(x)-\mathcal{F}_2(y)\|\leq\vartheta(\|x-y\|)$ for all $x,y\in\bar{U}$). Then, either*

(C1) $\mathcal{F}$ has a fixed point $u\in\bar{U}$; or

(C2) there exist a point $u\in\partial U$ and $\kappa\in(0,1)$ with $u=\kappa\mathcal{F}(u)$, where $\bar{U}$ and ∂U, respectively, represent the closure and boundary of U on C.

In the sequel, we will use Lemma 3.2 by taking C to be E. For more details of such fixed point theorems, we refer a paper [25] by Petryshyn.

To apply Lemma 3.2, we define $\mathcal{F}_i\colon\mathcal{C}\to\mathcal{C}, i=1,2$ by

$$
\begin{aligned}
(\mathcal{F}_1x)(t)=&\int_0^t\frac{(t-qs)^{(v-1)}}{\Gamma_q(v)}f(s,x(s))\,d_qs\\
&+\frac{t}{B}\Bigg\{b\int_\eta^1\Big(\int_0^s\frac{(s-qu)^{(v-1)}}{\Gamma_q(v)}f(u,x(u))\,d_qu\Big)\,d_qs\\
&\qquad-\int_0^\xi\frac{(\xi-qs)^{(v-1)}}{\Gamma_q(v)}f(s,x(s))\,d_qs\Bigg\},
\end{aligned}
\tag{3.4}
$$

and

$$(\mathcal{F}_2x)(t)=\Big[1+\frac{t}{B}\big(b(1-\eta)-1\big)\Big](x_0+g(x)).\tag{3.5}$$

Clearly

$$(\mathcal{F}x)(t)=(\mathcal{F}_1x)(t)+(\mathcal{F}_2x)(t),\quad t\in[0,1].\tag{3.6}$$

Theorem 3.3. *Suppose that (A_2) holds. In addition, we assume that:*

(A_4) $g(0)=0$;

(A_5) *Let $f\colon[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function, and there exist a nonnegative function $p\in C([0,1],\mathbb{R})$ and a nondecreasing function $\chi:[0,\infty)\to(0,\infty)$ such that*

$$|f(t,u)|\leq p(t)\chi(|u|)\ \textit{for any}\ (t,u)\in[0,1]\times\mathbb{R};$$

(A_6) $\sup\limits_{r\in(0,\infty)} \dfrac{r}{k_0|x_0| + \mu_0\chi(r)\|p\|} > \dfrac{1}{1-k_0\ell}$, *where* μ_0 *and* k_0 *are defined in* (3.2) *and* (3.3) *respectively.*

Then the boundary value problem (1.1) *has at least one solution on* $[0,1]$.

Proof. By the assumption (A_6), there exists a number $r_0 > 0$ such that

$$\frac{r_0}{k_0|x_0| + \mu_0\chi(r_0)\|p\|} > \frac{1}{1-k_0\ell}. \tag{3.7}$$

We shall show that the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ defined by (3.4) and (3.5) respectively, satisfy all the conditions of Lemma 3.2.

Step 1. *The operator* $\mathcal{F}_1$ *is continuous and completely continuous.* Let us consider the set

$$\bar{\Omega}_{r_0} = \{x \in C([0,1],\mathbb{R}) : \|x\| \le r_0\},$$

and show that $\mathcal{F}_1(\bar{\Omega}_{r_0})$ is bounded. For any $x \in \bar{\Omega}_{r_0}$, we have

$$\begin{aligned}
\|\mathcal{F}_1 x\| \le{}& \int_0^t \frac{(t-qs)^{(v-1)}}{\Gamma_q(v)}|f(s,x(s))|\,d_qs \\
&+ \frac{1}{|B|}\Bigg\{|b|\int_\eta^1\Big(\int_0^s \frac{(s-qu)^{(v-1)}}{\Gamma_q(v)}|f(u,x(u))|d_qu\Big)\,d_qs \\
&\qquad + \int_0^\xi \frac{(\xi-qs)^{(v-1)}}{\Gamma_q(v)}|f(s,x(s))|\,d_qs\Bigg\} \\
\le{}& \|p\|\chi(r_0)\left[\frac{1}{\Gamma_q(v+1)} + \frac{1}{|B|}\left\{\frac{|b|(1-\eta^{v+1})}{\Gamma_q(v+2)} + \frac{\xi^v}{\Gamma_q(v+1)}\right\}\right] \\
\le{}& \|p\|\chi(r_0)\mu_0.
\end{aligned}$$

Thus the operator $\mathcal{F}_1(\bar{\Omega}_{r_0})$ is uniformly bounded. For any $t_1, t_2 \in [0,1]$, $t_1 < t_2$, we have

$$\begin{aligned}
&|(\mathcal{F}_1x)(t_2) - (\mathcal{F}_1x)(t_1)| \\
&\le \frac{1}{\Gamma_q(v)}\int_0^{t_1}[(t_2-qs)^{(v-1)} - (t_1-qs)^{(v-1)}]|f(s,x(s))|\,d_qs \\
&\quad + \frac{1}{\Gamma_q(v)}\int_{t_1}^{t_2}(t_2-qs)^{(v-1)}|f(s,x(s))|\,d_qs \\
&\quad + \frac{|t_2-t_1|}{|B|}\Bigg\{|b|\int_\eta^1\Big(\int_0^s \frac{(s-qu)^{(v-1)}}{\Gamma(v)}|f(u,x(u))|d_qu\Big)\,d_qs \\
&\qquad + \int_0^\xi \frac{(\xi-qu)^{(v-1)}}{\Gamma_q(v)}|f(s,x(s))|\,d_qs\Bigg\} \\
&\le \frac{\|p\|\chi(r_0)}{\Gamma_q(v)}\int_0^{t_1}[(t_2-qs)^{(v-1)} - (t_1-qs)^{(v-1)}]\,d_qs + \frac{\|p\|\chi(r_0)}{\Gamma_q(v)}\int_{t_1}^{t_2}(t_2-qs)^{(v-1)}\,d_qs \\
&\quad + \frac{\|p\|\chi(r_0)|t_2-t_1|}{|B|}\Bigg\{|b|\int_\eta^1\Big(\int_0^s\frac{(s-qu)^{(v-1)}}{\Gamma_q(v)}d_qu\Big)\,d_qs + \int_0^\xi\frac{(\xi-qs)^{(v-1)}}{\Gamma_q(v)}\,d_qs\Bigg\}
\end{aligned}$$

which is independent of x and tends to zero as $t_2 - t_1 \to 0$. Thus, $\mathcal{F}_1$ is equicontinuous. Hence, by the Arzelà–Ascoli theorem, $\mathcal{F}_1(\bar{\Omega}_{r_0})$ is a relatively compact set. Now, let $x_n \subset \bar{\Omega}_{r_0}$ with

$\|x_n - x\| \to 0$. Then the limit $\|x_n(t) - x(t)\| \to 0$ is uniformly valid on $[0,1]$. From the uniform continuity of $f(t,x)$ on the compact set $[0,1] \times \bar{\Omega}_{r_0}$, it follows that $\|f(t,x_n(t)) - f(t,x(t))\| \to 0$ is uniformly valid on $[0,1]$. Hence $\|\mathcal{F}_1 x_n - \mathcal{F}_1 x\| \to 0$ as $n \to \infty$ which proves the continuity of $\mathcal{F}_1$. This completes the proof of Step 1.

Step 2. *The operator $\mathcal{F}_2 \colon \bar{\Omega}_{r_0} \to C([0,1],\mathbb{R})$ is contractive.* This is a consequence of (A_2).

Step 3. *The set $\mathcal{F}(\bar{\Omega}_{r_0})$ is bounded.* The assumptions (A_2) and (A_4) imply that

$$\|\mathcal{F}_2(x)\| \le k_0(|x_0| + \ell r_0),$$

for any $x \in \bar{\Omega}_{r_0}$. This, with the boundedness of the set $\mathcal{F}_1(\bar{\Omega}_{r_0})$ implies that the set $\mathcal{F}(\bar{\Omega}_{r_0})$ is bounded.

Step 4. *Finally, it will be shown that the case (C2) in Lemma 3.2 does not hold.* On the contrary, we suppose that (C2) holds. Then, we have that there exist $\kappa \in (0,1)$ and $x \in \partial\Omega_{r_0}$ such that $x = \kappa\mathcal{F}x$. So, we have $\|x\| = r_0$ and

$$\begin{aligned} x(t) &= \kappa \int_0^t \frac{(t-qs)^{(v-1)}}{\Gamma_q(v)} f(s,x(s))\, d_q s \\ &\quad + \frac{\kappa t}{B}\left\{ b \int_\eta^1 \Big(\int_0^s \frac{(s-qu)^{(v-1)}}{\Gamma_q(v)} f(u,x(u)) d_q u \Big)\, d_q s - \int_0^\xi \frac{(\xi-qs)^{(v-1)}}{\Gamma_q(v)} f(s,x(s))\, d_q s \right\} \\ &\quad + \kappa\Big[1 + \frac{t}{B}\big(b(1-\eta)-1\big)\Big](x_0 + g(x)), \quad t \in [0,1]. \end{aligned}$$

Using the assumptions (A_2) and (A_4)–(A_6), we get

$$\begin{aligned} |x(t)| \le \|p\|\chi(\|x\|)\Bigg[&\int_0^1 \frac{(t-qs)^{(v-1)}}{\Gamma_q(v)}\, d_q s + \frac{1}{|B|}\left\{ |b| \int_\eta^1 \Big(\int_0^s \frac{(s-qu)^{(v-1)}}{\Gamma_q(v)} d_q u \Big)\, d_q s \right. \\ &\left. + \int_0^\xi \frac{(\xi-qs)^{(v-1)}}{\Gamma_q(v)}\, d_q s \right\}\Bigg] + \Big[1 + \frac{1}{|B|}\Big(b(1-\eta)-1\Big)\Big](|x_0| + \ell\|x\|). \end{aligned}$$

Taking the supremum over $t \in [0,1]$, and using the definition of $\bar{\Omega}_{r_0}$, we obtain

$$\begin{aligned} r_0 \le \|p\|\chi(r_0)\Bigg[&\int_0^1 \frac{(1-qs)^{(v-1)}}{\Gamma_q(v)}\, d_q s + \frac{1}{|B|}\left\{ |b| \int_\eta^1 \Big(\int_0^s \frac{(s-qu)^{(v-1)}}{\Gamma_q(v)} d_q u \Big)\, d_q s \right. \\ &\left. + \int_0^\xi \frac{(\xi-qs)^{(v-1)}}{\Gamma_q(v)}\, d_q s \right\}\Bigg] + \Big[1 + \frac{1}{|B|}\Big(b(1-\eta)-1\Big)\Big](|x_0| + \ell r_0), \end{aligned}$$

which yields

$$r_0 \le \mu_0\chi(r_0)\|p\| + k_0|x_0| + \ell r_0 k_0.$$

Thus, we get a contradiction:

$$\frac{r_0}{\mu_0\chi(r_0)\|p\| + k_0|x_0|} \le \frac{1}{1 - k_0\ell}.$$

Thus the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy all the conditions of Lemma 3.2. Hence, the operator $\mathcal{F}$ has at least one fixed point $x \in \bar{\Omega}_{r_0}$, which is the solution of the problem (1.1). This completes the proof. □

Remark 3.4. If we consider the equation of the form $({}^cD_q^v + \lambda)x(t) = f(t, x(t))$, $\lambda \in \mathbb{R}$ in the problem (1.1), then the condition (A_3) in the statement of Theorem 3.1 modifies to $\delta := (L+\lambda)\mu_0 + k_0\ell < 1$ whereas the condition (A_6) in the statement of Theorem 3.3 takes the form

$$\sup_{r\in(0,\infty)} \frac{r}{k_0|x_0| + \mu_0\chi(r)\|p\|} > \frac{1}{1-(|\lambda|\mu_0 + k_0\ell)}.$$

We emphasize that the equations similar to one considered in this remark appear in applied problems, for example, see [26, 27].

4 A boundary value problem with flux sub-strip conditions

In this section, we discuss the existence of solutions for a boundary value problem of nonlinear fractional q-difference equations with nonlocal and flux sub-strip type boundary conditions. Precisely, we consider the following boundary value problem

$$\begin{cases} {}^cD_q^v x(t) = f(t, x(t)), \quad t \in [0,1], \quad 1 < v \le 2, \quad 0 < q < 1 \\ x(0) = x_0 + g(x), \quad D_q x(\xi) = b\displaystyle\int_\eta^1 x(s)\, d_q s, \quad 0 < \xi < \eta < 1, \end{cases} \tag{4.1}$$

where ${}^cD_q^v$ denotes the Caputo fractional q-derivative of order $v, f\colon [0,1]\times\mathbb{R} \to \mathbb{R}$ and $g\colon C([0,1],\mathbb{R}) \to \mathbb{R}$ are given continuous functions, and b, λ are real constants.

As before, we can convert the problem (4.1) into an equivalent fixed point problem as $\mathcal{F}'x = x$, where the operator $\mathcal{F}'\colon \mathcal{C} \to \mathcal{C}$ is defined by

$$\begin{aligned}(\mathcal{F}'x)(t) = {} & \int_0^t \frac{(t-qs)^{(v-1)}}{\Gamma_q(v)} f(s, x(s))\, d_q s \\ & + \frac{t}{B}\bigg\{ b\int_\eta^1 \Big(\int_0^s \frac{(s-qu)^{(v-1)}}{\Gamma_q(v)} f(u, x(u))\, d_q u\Big)\, d_q s \\ & - \int_0^\xi \frac{(\xi-qs)^{(v-2)}}{\Gamma_q(v-1)} f(s, x(s))\, d_q s \bigg\} + \Big[1 + \frac{t}{B} b(1-\eta)\Big](x_0 + g(x)).\end{aligned}$$

For the sequel, we set

$$\mu_0' := \frac{1}{\Gamma_q(v+1)} + \frac{1}{|B|}\bigg\{ \frac{|b|(1-\eta^{v+1})}{\Gamma_q(v+2)} + \frac{\xi^{v-1}}{\Gamma_q(v)}\bigg\}, \tag{4.2}$$

$$k_0' := 1 + \Big|\frac{b}{B}\Big|(1-\eta). \tag{4.3}$$

Now we are in a position to give the existence results for the problem (4.1). We do not provide the proofs for these results as the method of proof is similar to the one employed in the preceding section.

Theorem 4.1. *Let the assumptions (A_1)–(A_3) hold with μ_0' and k_0' in place of μ_0 and k_0, where μ_0' and k_0' are given by (4.2) and (4.3) respectively. Then the boundary value problem (4.1) has a unique solution.*

Theorem 4.2. *Assume that (A_2), (A_4)–(A_6) hold with μ_0' and k_0' in place of μ_0 and k_0, where μ_0' and k_0' are given by (4.2) and (4.3) respectively. Then there exists at least one solution for the problem (4.1) on $[0,1]$.*

5 Examples

In this section we present some examples to illustrate our results.

Example 5.1. Consider the following q-fractional boundary value problem

$$\begin{cases} {}^cD_q^{3/2}x(t) = \dfrac{1}{9}\tan^{-1}x(t) + t^2, \quad t \in [0,1], \\ x(0) = \dfrac{1}{3} + \dfrac{1}{12}\tan^{-1}(x(\theta)), \quad x\left(\dfrac{1}{4}\right) = \dfrac{1}{7}\displaystyle\int_{3/4}^{1} x(s)\, d_qs. \end{cases} \tag{5.1}$$

Now, $v = 3/2, q = 1/2,\ b = 1/7,\ \xi = 1/4,\ \eta = 3/4,\ \ell = 1/12,\ 0 < \theta < 1$, and $f(t,x) = \frac{1}{9}\tan^{-1}x + t^2$. Note that (A_1) is satisfied with $L = 1/9$, since $|f(t,x) - f(t,y)| \le (1/9)|x-y|$. It is found that $B = 0.208333,\ \mu_0 \approx 1.52327,\ k_0 = 5.62857$, and $\delta = L\mu_0 + k_0\ell \approx 0.6383 < 1$. Thus, the conclusion of Theorem 3.1 applies and the boundary value problem (5.1) has a unique solution on $[0,1]$.

Example 5.2. Consider the q-fractional boundary value problem given by

$$\begin{cases} {}^cD_q^{3/2}x(t) = \dfrac{1}{\sqrt{t+25}}e^{(1+|\sin x(t)|)}, \quad t \in [0,1], \\ x(0) = \dfrac{1}{12}x(\sigma), \quad x\left(\dfrac{1}{4}\right) = \dfrac{1}{5}\displaystyle\int_{3/4}^{1} x(s)\, d_qs. \end{cases} \tag{5.2}$$

Here, $v = 3/2, q = 1/2,\ b = 1/5,\ \xi = 1/4,\ \eta = 3/4,\ \ell = 1/12,\ 0 < \sigma < 1$, and $f(t,x) = \frac{1}{\sqrt{t+25}}e^{(1+|\sin x(t)|)}$. With the given values, it is found that $B = 0.191667,\ \mu_0 \approx 1.66069$, $k_0 = 5.95652$ and the condition

$$\frac{r_0}{k_0|x_0| + \mu_0\chi(r_0)\|p\|} > \frac{1}{1 - k_0\ell}$$

implies that $r_0 > 4.87306$. Clearly all the conditions of Theorem 3.3 are satisfied and hence by the conclusion of Theorem 3.3, the problem (5.2) has a solution on $[0,1]$.

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