Weak solutions for the dynamic equations $x^{(\Delta m)}(t) = f(t, x(t))$ on time scales

Samir H. Saker[1] and **Aneta Sikorska-Nowak**[✉ 2]

[1]Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt
[2]Faculty of Mathematics and Computer Science, Adam Mickiewicz University, ul. Umultowska 87;
61-614 Poznań, Poland

Abstract. In this paper we prove the existence of weak solutions of the dynamic Cauchy problem

$$x^{(\Delta m)}(t) = f(t, x(t)), \quad t \in T,$$
$$x(0) = 0,$$
$$x^{\Delta}(0) = \eta_1, \dots, x^{(\Delta(m-1))}(0) = \eta_{m-1}, \quad \eta_1, \dots, \eta_{m-1} \in E,$$

where $x^{(\Delta m)}$ denotes a weak m-th order Δ-derivative, T denotes an unbounded time scale (nonempty closed subset of R such that there exists a sequence (a_n) in T and $a_n \to \infty$), E is a Banach space and f is weakly – weakly sequentially continuous and satisfies some conditions expressed in terms of measures of weak noncompactness.

The Sadovskii fixed point theorem and Ambrosetti's lemma are used to prove the main result.

As dynamic equations are a unification of differential and difference equations our result is also valid for differential and difference equations. The results presented in this paper are new not only for Banach valued functions but also for real valued functions.

Keywords: Cauchy dynamic problem, Banach space, measure of weak noncompactness, weak solutions, time scales, fixed point.

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1   Introduction

A time scale T is a nonempty closed subset of real numbers R, with the subspace topology inherited from the standard topology of R. Thus R, Z, N and the Cantor set are examples of time scales while Q and $(0, 1)$ are not time scales.

Time scales (or measure chains) was introduced by Hilger in his Ph.D. thesis in 1988 [18].

Since the time Hilger formed the definitions of a derivative and integral on a time scale, several authors have extended on various aspects of the theory [1, 2, 4, 6, 10, 11, 16, 17, 18, 20].

✉ Corresponding author. Email: anetas@amu.edu.pl

Time scale has been shown to be applicable to any field that can be described by means of discrete or continuous models.

The study of dynamic equations on time scales, which has been created in order to unify the study of differential and difference equations, is an area of mathematics research that has recently received a lot of attention. Dynamic equations on a time scale have an enormous potential for applications such as in population dynamics. For example, it can model insect populations that are continuous while in season, die out in say winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population (see [10]). There are applications of dynamic equations on time scales to quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics. A recent cover story article in New Scientist [30] discusses several possible applications.

In this paper we consider the problem

$$
\begin{aligned}
x^{(\Delta m)}(t) &= f(t, x(t)), \quad t \in T, \\
x(0) &= 0, \\
x^{\Delta}(0) &= \eta_1, \dots, x^{(\Delta(m-1))}(0) = \eta_{m-1}, \quad \eta_1, \dots, \eta_{m-1} \in E,
\end{aligned}
\tag{1.1}
$$

where $x^{(\Delta m)}$ denotes the m-th weak Δ-derivative, T denotes an unbounded time scale (nonempty closed subset of R such that there exists a sequence (a_n) in T and $a_n \to \infty$) and $(E, \| \cdot \|)$ is a Banach space. The function f, with values in a Banach space, is weakly – weakly sequentially continuous and satisfies some regularity conditions expressed in terms of the De Blasi measure of weak noncompactness.

Using Sadovskii's fixed point theorem [27] and the properties of measures of weak noncompactness, we prove an existence result for problem (1.1).

The study for weak solutions of Cauchy differential equations in Banach spaces was initiated by A. Szép [31] and theorems on the existence of weak solutions of this problem were proved by F. Cramer, V. Lakshmikantam and A. R. Mitchell [14], I. Kubiaczyk [23], I. Kubiaczyk, S. Szufla [24], A. R. Mitchell and Ch. Smith [26], S. Szufla [33], M. Cichoń, I. Kubiaczyk [12].

Similar methods for solving existence problems for difference equations in Banach spaces equipped with its weak topology were studied for instance in [3]. In particular the importance of conditions expressed in terms of the weak topology was remarked in [3].

We will unify both cases as well as we obtain the first result for weak solutions of dynamic Cauchy problem m-th order. (So far a first time also for q-difference equations).

The main goal of this work is to construct a theory that unifies the existence of weak solutions of the Cauchy problem for both Z and R. Our result extends the existence of weak solutions not only to the discrete intervals with uniform step size (hZ) but also to the discrete intervals with nonuniform step size (K_q).

We assume that the function f is weakly – weakly sequentially continuous with values in a Banach space and satisfies some regularity conditions expressed in terms of the De Blasi measure of weak noncompactness. We introduce a weakly sequentially continuous operator associated to an integral equation which is equivalent to (1.1).

There exist many important examples of mappings which are weakly sequentially continuous but not weakly continuous. The relation between weakly sequentially continuous and weakly continuous mappings are studied by Ball [7].

Results presented in this paper extend existence results known from the literature, for example: I. Kubiaczyk, A. Sikorska-Nowak [25], S. Szufla [34], A. Sikorska-Nowak [28, 29], A. Szukała [35] and others.

2   Preliminaries

To understand the so-called dynamic equation and follow this paper easily, we present some preliminary definitions and notations of time scale which are very common in the literature (see [1, 2, 10, 11] and references therein). We generalize some definitions given in these references for the functions $f: T \times E \to E$ instead of $f: T \to R$.

If a, b are points in T, then we denote by $[a, b] = \{t \in T : a \leq t \leq b\}$, $I_a = \{t \in T : 0 \leq t \leq a\}$ and $J = \{t \in T : 0 \leq t < \infty\}$. Other types of intervals are approached similarly. By a subinterval I_b of I_a we mean the time scale subinterval.

Definition 2.1. The forward jump operator $\sigma: T \to T$ and the backward jump operator $\rho: T \to T$ are defined by $\sigma(t) = \inf\{s \in T : s > t\}$ and $\rho(t) = \sup\{s \in T : s < t\}$, respectively.

We put $\inf \varnothing = \sup T$ (i.e. $\sigma(M) = M$ if T has a maximum M) and $\sup \varnothing = \inf T$ (i.e. $\rho(m) = m$ if T has a minimum m).

The jump operators σ and ρ allow the classification of points in time scale in the following way: t is called *right dense*, *right scattered*, *left dense*, *left scattered*, *dense* and *isolated* if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, $\rho(t) = t = \sigma(t)$ and $\rho(t) < t < \sigma(t)$, respectively.

Moreover the graininess function $\mu: T \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$, $\forall t \in T$.

Furthermore T^k denotes Hilger's above truncated set consisting of T except for a possible left-scattered maximal point.

Recall that a function $f: T \to E$ is said to be *weakly continuous* if it is continuous from T to E, endowed with its weak topology. A function $g: E \to E_1$, where E and E_1 are Banach spaces, is said to be *weakly – weakly sequentially continuous* if, for each weakly convergent sequence (x_n) in E, the sequence $(g(x_n))$ is weakly convergent in E_1. When the sequence x_n tends weakly to x_0 in E, we write $x_n \xrightarrow{w} x_0$.

Definition 2.2. We say that $u: T \to E$ is *right-dense continuous (rd-continuous)* if u is continuous at every right-dense point $t \in T$ and $\lim\limits_{s \to t^-} u(s)$ exists and is finite at every left-dense point $t \in T$.

Due to Definition 2.2, the weakly rd-continuity is defined as follows:

Definition 2.3. We say that $u: T \to E$ is *weakly right-dense continuous (weakly rd-continuous)* if u is weakly continuous at every right dense point $t \in T$ and $\lim_{s \to t^-} u(s)$ exists and is finite at every left dense point $t \in T$.

The so-called Δ-weak derivative and Δ-weak integral for Banach valued functions are defined by generalizing the notions of Δ-derivative and Δ- integral on time scales [10, 11].

Definition 2.4. Let $u: T \to E$. Then we say that u is Δ-*weak differentiable at* $t \in T$ if there exists an element $Y(t) \in E$ such that for each $x^* \in E^*$ the real valued function x^*u is Δ-differentiable at t and $(x^*u)^\Delta(t) = (x^*Y)(t)$. Such a function Y is called Δ-*weak derivative of* u and denoted by $u^{\Delta w}$.

Definition 2.5. If $U^{\Delta w}(t) = u(t)$ for all t, then we define the Δ-*weak Cauchy integral* by

$$wC - \int\limits_a^t u(\tau)\Delta\tau = U(t) - U(a).$$

By generalizing the Theorem 1.74 of [10] we can obtain the existence of weak antiderivatives.

Remark 2.6 (Existence of weak antiderivatives). Every weakly rd-continuous function has a weak antiderivative. In particular if $t_0 \in T$ then U defined by

$$U(t) := wC - \int_{t_0}^{t} u(\tau)\Delta\tau, \quad t \in T$$

is a weak antiderivative of u.

Since the weak Cauchy Δ-integral is defined by means of weak antiderivatives, the space of weak Cauchy Δ-integrable functions is too narrow. Therefore, we need to define the weak Riemann Δ-integral for Banach space-valued functions.

Let $P = \{a_0, a_1, \ldots, a_n\}$ where $a_i \in T$, $i = 0, 1, \ldots, n$, be a partition of the interval $[a, b]$. P is called *finer* than $\delta > 0$ if either

(i) $\mu_\Delta([a_{i-1}, a_i]) \leq \delta$ or

(ii) $\mu_\Delta([a_{i-1}, a_i]) > \delta$ if only $a_i = \sigma(a_{i-1})$, where $\mu_\Delta([a_{i-1}, a_i])$ is the Lebesgue Δ-measure of $[a_{i-1}, a_i]$.

Definition 2.7. A function $u \colon [a, b] \to E$ is called *weak Riemann Δ-integrable* if there exists $U \in E$ such that for any $\varepsilon > 0$, there exists $\delta > 0$ with the following property: for any partition $P = \{a_0, a_1, \ldots, a_n\}$ which is finer than δ and any set of points $t_1, t_2, \ldots, t_n$ with $t_j \in [a_{j-1}, a_j)$ for $j = 1, 2, \ldots, n$ one has

$$\left| x^*(U) - \sum_{j=1}^{n} x^*(u(t_j))\mu_\Delta([a_{j-1}, a_j)) \right| \leq \varepsilon, \quad \forall x^* \in E^*.$$

According to Definition 2.7, U is uniquely determined and it is called the weak Riemann Δ-integral of u and denoted by $U = wR - \int_a^b u(t)\Delta t$.

By regarding the definitions of weak integrals and by using Theorem 4.3 of Guseinov [17], we are able to state that every Riemann Δ-weak integrable function is a Cauchy Δ-weak integrable and in this case, these two integrals coincide. Therefore, in the following part of the paper we will use the notation $\int f(t)\Delta t$ as a Δ-weak integral.

Let $(E, \|\cdot\|)$ be a Banach space and E^* be its dual space. We consider the space of continuous functions $J \to E$ with its weak topology, i.e. $(C(J, E), \omega) = (C(J, E), \gamma(C(J, E), C^*(J, E)))$.

Our fundamental tool is the measure of weak noncompactness developed by De Blasi [9].

Let A be a bounded nonempty subset of E. The *measure of weak noncompactness* $\beta(A)$ is defined by

$$\beta(A) = \inf\{t > 0 : \text{ there exists } C \in K^\omega \text{ such that } A \subset C + tB_0\},$$

where K^ω is the set of weakly compact subsets of E and B_0 is the norm unit ball in E.

We will use the following properties of the measure of weak noncompactness β (for bounded nonempty subsets A and B of E):

(i) if $A \subset B$ then $\beta(A) \leq \beta(B)$,

(ii) $\beta(A) = \beta(\bar{A}^w)$, where $\bar{A}^w$ denotes the weak closure of A,

(iii) $\beta(A) = 0$ if and only if A is relatively weakly compact,

(iv) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$,

(v) $\beta(\lambda A) = |\lambda| \beta(A)$, $(\lambda \in R)$,

(vi) $\beta(A + B) \leq \beta(A) + \beta(B)$,

(vii) $\beta(\overline{\mathrm{conv}}A) = \beta(\mathrm{conv}\, A) = \beta(A)$, where conv A denotes the convex hull of A.

The lemma below is an adaptation of the corresponding result of Banaś, Goebel [8].

Lemma 2.8. *Let X be an equicontinuous bounded set in $C(T, E)$, where $C(T, E)$ denotes the space of all continuous functions from the time scale T to the Banach space E.*

$$\int\limits_0^a X(s)\Delta s = \left\{\int\limits_0^a x(s)\Delta s : \ x \in X\right\}.$$

Then

$$\beta\left(\int\limits_0^a X(s)\Delta s\right) \leq \int\limits_0^a \beta\left(X(s)\right)\Delta s.$$

Proof. For $\delta > 0$ we choose points in T in the following way:

$$t_0 = 0, \quad t_1 = \sup_{s \in I_a}\{s : s \geq t_0, s - t_0 \leq \delta\},$$

$$t_2 = \sup_{s \in I_a}\{s : s \geq t_1, s - t_1 \leq \delta\}, \quad t_3 = \sup_{s \in I_a}\{s : \ s \geq t_2, s - t_2 \leq \delta\}, \ldots,$$

$$t_{n-1} = \sup_{s \in I_a}\{s : s \geq t_{n-2}, s - t_{n-2} \leq \delta\}, \quad t_n = a.$$

If some $t_i = t_{i-1}$ then $t_{i+1} = \inf_{s \in I_a}\{s : s > t_i\}$. By the equicontinuity of X there exists $\delta > 0$ and $\xi_i \in [t_{i-1}, t_i]$ such that

$$\left\|\int\limits_0^a x(s)\Delta s - \sum_{i=1}^n x(\xi_i)\mu_\Delta\left((t_{i-1}, t_i)\right)\right\| \leq \varepsilon.$$

Thus we have

$$\int\limits_0^a X(s)\Delta s \subset \left[\int\limits_0^a x(s)\Delta s - \sum_{i=1}^n x(\xi_i)\mu_\Delta\left((t_{i-1}, t_i)\right) : \quad x \in X\right] +$$

$$+ \left[\sum_{i=1}^n x(\xi_i)\mu_\Delta\left((t_{i-1}, t_i)\right) : \quad x \in X\right] = A + B.$$

Now

$$\beta(A) \leq \beta(K(0, \varepsilon)) = \varepsilon\beta(K(0, 1))$$

and

$$\beta(B) \leq \sum_{i=1}^n \mu_\Delta\left((t_{i-1}, t_i)\right)\beta(X(\xi_i)).$$

Therefore

$$\beta\left(\int_0^a X(s)\Delta s\right) \le \beta\left(A + B\right) \le \varepsilon\beta(K(0,1)) + \sum_{i=1}^n \mu_\Delta\left((t_{i-1}, t_i)\right)\beta(X(\xi_i)).$$

If $\varepsilon \to 0$ and $n \to \infty$ we obtain

$$\beta\left(\int_0^a X(s)\Delta s\right) \le \int_0^a \beta\left(X(s)\right)\Delta s.$$

$\square$

The lemma below is an adaptation of the corresponding result of Ambrosetti [5] proved in [13]. Let us recall that $J \subset T$.

Lemma 2.9. *Let $H \subset C(J, E)$ be a family of strongly equicontinuous functions. Let $H(t) = \{h(t) \in E, h \in H\}$, for $t \in J$. Then*

$$\beta(H(J)) = \sup_{t \in J} \beta(H(t)),$$

and the function $t \mapsto \beta(H(t))$ is continuous on J.

Let us denote by S_∞ the set of all nonnegative real sequences. For $\xi = (\xi_n) \in S_\infty$, $\eta = (\eta_n) \in S_\infty$, we write $\xi < \eta$ if $\xi_n \le \eta_n$ (i.e. $\xi_n \le \eta_n$, for $n = 1, 2, \dots$) and $\xi \ne \eta$.

Let C be a closed convex subset of $(C(T, E), \omega)$ and ϕ be a function which assigns to each nonempty subset Z of C, a sequence $\phi(Z) \in S_\infty$, such that

$$\phi(\{x\} \cup Z) = \phi(Z), \text{ for } x \in C, \tag{2.1}$$

$$\phi(\overline{\text{conv}}Z) = \phi(Z), \tag{2.2}$$

$$\text{if } \phi(Z) = \varnothing \text{ (the zero sequence) then } \bar{Z} \text{ is compact.} \tag{2.3}$$

Theorem 2.10 ([27]). *If $F\colon K \to K$ is a continuous mapping satisfying $\phi(F(Z)) < \phi(Z)$ for an arbitrary nonempty subset Z of K with $\phi(Z) > 0$, then F has a fixed point in K.*

Theorem 2.11 (Mean value theorem [13]). *If the function $f\colon J \to E$ is Δ- weak integrable then*

$$\int_{I_b} f(t)\Delta t \in \mu_\Delta(I_b) \cdot \overline{\text{conv}} f(I_b),$$

where I_b is an arbitrary time scale subinterval of the time scale interval J and $\mu_\Delta(I_b)$ is the Lebesgue Δ-measure of I_b.

See [11] for the definition and basic properties of the Lebesgue Δ-measure and the Lebesgue Δ-integral.

3 Existence of weak solutions

Let $L^1(T)$ denote the space of real valued Δ-Lebesgue integrable functions on a time scale T. Assume that there exists a function $M \in L^1(T)$, $M(t) \geq 0$, $t \in T$, such that
$$\|f(t,x)\| \leq M(t) \quad \mu_\Delta \text{ a.e. on } T, \text{ for all } x \in E.$$
Let

$$b_t = \sum_{j=1}^{m-1} \|\eta_j\| \frac{t^j}{j!} + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} M(t_m) \Delta t_m \ldots \Delta t_2 \Delta t_1,$$

$$K(\tau, s) = \int_\tau^s \int_0^{t_1} \cdots \int_0^{t_{m-1}} M(t_m) \Delta t_m \ldots \Delta t_2 \Delta t_1,$$

$$p(t) = \begin{cases} 0, & m = 1 \\ \sum\limits_{j=1}^{m-1} \eta_j \cdot \frac{t^j}{j!}, & m > 1, \quad \eta_1, \eta_2, \ldots, \eta_{m-1} \in E, \end{cases}$$

$$\tilde{B}_t = \Big\{ x \in (C(I_t, E), \omega) : \|x(s)\| \leq b_t,$$

$$\|x(\tau) - x(s)\| \leq \|p(\tau) - p(s)\| + K(\tau, s), \ t, \tau, s \in T, \ 0 \leq s < \tau \leq t \Big\},$$

where T denotes an unbounded time scale and $I_t = \{s \in T : 0 \leq s \leq t\}$.

We recall that a function $g \colon E \to E$ is a weakly – weakly sequentially continuous function if $x_n \xrightarrow{w} x$ in E then $g(x_n) \xrightarrow{w} g(x)$ in E.

In investigating the existence of solutions of (1.1), we consider weak solutions.

Definition 3.1. A function $x \colon J \to E$ is said to be *a weak solution of the problem* (1.1) if x has Δ-weak derivative of m-th order and satisfies (1.1) for all $t \in J$.

We consider an appropriate integral equation

$$x(t) = p(t) + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} f(t_m, x(t_m)) \Delta t_m \ldots \Delta t_2 \Delta t_1. \tag{3.1}$$

Notice that each solution of the problem (3.1) is the solution of (1.1)
Let the operator $F \colon (C(J, E), \omega) \to (C(J, E), \omega)$ be defined by

$$F(x)(t) = p(t) + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} f(t_m, x(t_m)) \Delta t_m \ldots \Delta t_2 \Delta t_1.$$

Theorem 3.2. *Suppose that a function $f \colon T \times E \to E$ and that there exists a function $M \in L^1(T)$, $M(t) \geq 0$, $t \in T$, such that*

$$\|f(t,x)\| \leq M(t) \quad \mu_\Delta \text{ a.e. on } T, \text{ for all } x \in E.$$

Moreover, let the following conditions hold:

(C1) $f(t, \cdot)$ is weakly – weakly sequentially continuous, for each $t \in J$,

(C2) for each strongly absolutely continuous function $x \colon J \to E$, $f(\cdot . x(\cdot))$ is weakly continuous,

(C3) *there exists a function $L: T \times [0, \infty) \to [0, \infty)$, such that for each continuous function $u: [0, \infty) \to [0, \infty)$ the mapping $t \mapsto L(t, u(t))$ is continuous and $L(t, 0) \equiv 0$ on T,*

(C4) $\int\limits_{0}^{\infty} \int\limits_{0}^{t_1} \cdots \int\limits_{0}^{t_{m-1}} L(t_m, r) \Delta t_m \ldots \Delta t_2 \Delta t_1 < r$, *for all $r > 0$ and*

(C5) $\beta(f(I \times A)) \leq \sup\{L(t, \beta(A)) : t \in I\}$,

for any compact subinterval I of T and each nonempty bounded subset A of E. Then there exists at least one Δ-weak solution of the problem (1.1) on some subinterval $I_b \subset J$.

Proof. The condition (C2) implies that the operator $F: \tilde{B}_t \to (C(J, E), \omega)$ is well-defined. Now we show that the operator F maps $\tilde{B}_t$ into $\tilde{B}_t$.

(i)

$$||F(x)(t)|| = \left\| p(t) + \int\limits_{0}^{t} \int\limits_{0}^{t_1} \cdots \int\limits_{0}^{t_{m-1}} f(t_m, x(t_m)) \Delta t_m \ldots \Delta t_2 \Delta t_1 \right\|$$

$$\leq ||p(t)|| + \left\| \int\limits_{0}^{t} \int\limits_{0}^{t_1} \cdots \int\limits_{0}^{t_{m-1}} f(t_m, x(t_m)) \Delta t_m \ldots \Delta t_2 \Delta t_1 \right\|$$

$$\leq \sum_{j=1}^{m-1} ||\eta_j|| \frac{t^j}{j!} + \int\limits_{0}^{t} \int\limits_{0}^{t_1} \cdots \int\limits_{0}^{t_{m-1}} ||f(t_m, x(t_m))|| \Delta t_m \ldots \Delta t_2 \Delta t_1$$

$$\leq \sum_{j=1}^{m-1} ||\eta_j|| \frac{t^j}{j!} + \int\limits_{0}^{t} \int\limits_{0}^{t_1} \cdots \int\limits_{0}^{t_{m-1}} M(t_m) \Delta t_m \ldots \Delta t_2 \Delta t_1 = b_t, \quad t \in T.$$

(ii) Consequently we show, the set $F(\tilde{B}_t)$ is almost equicontinuous. Since for $x^* \in E^*$ with $||x^*|| \leq 1$ we have

$$|x^* (f(t_m, x(t_m)))| \leq \sup_{x^* \in E^*, ||x^*|| \leq 1} |x^* (f(t_m, x(t_m)))| = ||f(t_m, x(t_m))|| \leq M(t_m)$$

and

$$|x^* [F(x)(\tau) - F(x)(s)]| \leq |x^* [p(\tau) - p(s)]| + \int\limits_{s}^{\tau} \int\limits_{0}^{t_1} \cdots \int\limits_{0}^{t_{m-1}} |x^* (f(t_m, x(t_m)))| \Delta t_m \ldots \Delta t_2 \Delta t_1$$

$$\leq ||p(\tau) - p(s)|| + K(\tau, s), \quad \tau, s \in T, \text{ for each } x \in \tilde{B}_t$$

so $F(\tilde{B}_t)$ is strongly almost equicontinuous.

(iii) Now we show weak sequential continuity of F. Let $x_n \xrightarrow{w} x$ in $\tilde{B}_t$.

$$|x^* [F(x_n)(t) - F(x)(t)]|$$

$$\leq \int\limits_{0}^{t} \int\limits_{0}^{t_1} \cdots \int\limits_{0}^{t_{m-1}} |x^* (f(t_m, x_n(t_m)) - f(t_m, x(t_m)))| \Delta t_m \ldots \Delta t_2 \Delta t_1.$$

(see [10, 11, 17] for the inequality). Since J is a times scale interval, is a locally compact, Hausdorff space. By a result of Dobrakov (see [15], Thm. 9), $F(x_n)$ is weakly convergent to $F(x)$ in $C(J, E)$ so that F is weakly sequentially continuous.

From (i)–(iii) it follows that F is well-defined, weakly sequentially continuous and maps $\tilde{B}_t$ into $\tilde{B}_t$.

Let $a_n \in T$ is increasing, $a_n \to \infty$ if $n \to \infty$ and $I_{a_n} = [0, a_n]$. Let V be a countable subset of $\tilde{B}_{a_n}$. For $t \in J$, let $V(t) = \{v(t) \in E, v \in V\}$ and $A_n = V(I_{a_n}) = \bigcup \{V(t) : t \in I_{a_n}\}$ satisfying the condition $\overline{V} = \overline{\text{conv}}(\{x\} \cup F(V))$, for some $x \in \tilde{B}_{a_n}$. Remark, that since $\tilde{B}_{a_n}$ is bounded, A_n is bounded.

For any given $\varepsilon > 0$ there exists $\delta > 0$ such that $t', t'' \in I_{a_n}$ with $|t' - t''| < \delta$ imply

$$\left| L(t', \beta(A_n)) - L(t'', \beta(A_n)) \right| < \varepsilon. \tag{3.2}$$

We divide the interval I_{a_n} into r parts $0 = t_0^n < t_1^n < \cdots < t_r^n = a_n$ in such a way that:

$$t_0^n = 0,$$
$$t_1^n = \sup_{s \in I_{a_n}} \{s : s \geq t_0^n, s - t_0^n \leq \delta\},$$
$$t_2^n = \sup_{s \in I_{a_n}} \{s : s \geq t_1^n, s - t_1^n \leq \delta\},$$
$$\vdots$$
$$t_r^n = \sup_{s \in I_{a_n}} \{s : s \geq t_{r-1}^n, s - t_{m-1}^n \leq \delta\}.$$

Since T is a closed subset of R, $t_i^n \in I_{a_n}$. If some $t_i^n = t_{i-1}^n$, then $t_{i+1}^n = \inf \{s \in I_{a_n} : s > t_i^n\}$.

Let $t \in I_{a_n}$

$$(F(V))(t) = \left\{ p(t) + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} f(t_m, x(t_m)) \Delta t_m \ldots \Delta t_2 \Delta t_1 : \quad x \in V \right\}$$
$$= p(t) + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} f(t_m, V(t_m)) \Delta t_m \ldots \Delta t_2 \Delta t_1.$$

Let for $t \in I_k^n = [t_{k-1}^n, t_k^n] \cap T$, $k = 1, 2, \ldots, r$, q_j^n, $j = 1, 2, \ldots, k-1$ be chosen in I_j^n so that $L(q_j^n, \beta(A_j^n)) = \max\{L(t, \beta(A_j^n) : t \in I_j^n, j = 1, 2, \ldots, k-1\}$ and q_k^n be chosen in $[t_{k-1}^n, t_{m-1}] \cap T$ so that

$$L(q_k^n, \beta(A_k^n)) = \max\{L(t, \beta(A_k^n) : t \in [t_{k-1}^n, t_{m-1}] \cap T\},$$

where $A_j^n = V(I_j^n)$, $A_k^n = V([t_{k-1}^n, t_{m-1}] \cap T)$, $j = 1, 2, \ldots, k-1$, $k = 1, 2, \ldots, r$.

Now, using the mean value theorem and Lemma 2.8 we obtain

$$\beta(V(t)) = \beta(\overline{\text{conv}}(\{x\} \cup F(V))) = \beta(F(V)(t))$$
$$= \beta \left(p(t) + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} f(t_m, V(t_m)) \Delta t_m \ldots \Delta t_2 \Delta t_1 \right)$$
$$\leq \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-2}} \beta \left(\sum_{j=0}^{k-1} \int_{I_j^n} f(t_m, V(t_m)) \Delta t_m + \int_{t_{k-1}^n}^{t_{m-1}} f(t_m, V(t_m)) \Delta t_m \right) \Delta t_{m-1} \ldots \Delta t_2 \Delta t_1$$

So

$$\beta(V(t)) \leq \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-2}} \beta \left(\sum_{j=0}^{k-1} \mu_\Delta(I_j^n)\overline{\text{conv}}(f(I_j^n \times A_j^n)) \right.$$

$$\left. + \mu_\Delta([t_{k-1}^n, t_{m-1}] \cap T)\overline{\text{conv}}(f([t_{k-1}^n, t_{m-1}] \cap T \times A_k^n)) \right) \Delta t_{m-1} \ldots \Delta t_2 \Delta t_1$$

$$\leq \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-2}} \left(\sum_{j=0}^{k-1} \mu_\Delta(I_j^n)L(q_j^n, \beta(A_j^n)) \right.$$

$$\left. + \mu_\Delta([t_{k-1}^n, t_{m-1}] \cap T)L(q_k^n, \beta(A_k^n)) \right) \Delta t_{m-1} \ldots \Delta t_2 \Delta t_1.$$

Remark that by inequality (3.2)

$$\sum_{j=0}^{k-1} \mu_\Delta(I_j^n)L(q_j^n, \beta(A_j^n)) + \mu_\Delta([t_{k-1}^n, t_{m-1}] \cap T)L(q_k^n, \beta(A_k^n))$$

$$\leq \sum_{j=0}^{k-1} \int_{I_j^n} L(t_m, \beta(A_j^n))\Delta t_m + \sum_{j=0}^{k-1} \int_{I_j^n} \left| L(q_j^n, \beta(A_j^n)) - L(t_m, \beta(A_j^n)) \right| \Delta t_m$$

$$+ \int_{t_{k-1}^n}^{t_{m-1}} L(t_m, \beta(A_k^n))\Delta t_m + \int_{t_{k-1}^n}^{t_{m-1}} \left| L(q_k^n, \beta(A_k^n)) - L(t_m, \beta(A_k^n)) \right| \Delta t_m$$

$$< \int_0^{t_{m-1}} L(t_m, \beta(A_n))\Delta t_m + \varepsilon t_{m-1}.$$

So

$$\beta(F(V)(t)) \leq \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-2}} \left(\int_0^{t_{m-1}} L(t_m, \beta(A_n))\Delta t_m + \varepsilon t_{m-1} \right) \Delta t_{m-1} \ldots \Delta t_2 \Delta t_1$$

$$= \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-2}} \left(\varepsilon t_{m-1} + \int_0^{t_{m-1}} L(t_m, \sup\{\beta(V(t_m)) : t_m \in I_{a_n}\})\Delta t_m \right) \Delta t_{m-1} \ldots \Delta t_2 \Delta t_1.$$

As $\varepsilon > 0$ is arbitrary, this implies

$$\sup\{\beta(F(V)(t) : t \in I_{a_n}\}$$

$$\leq \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} L(t_m, \sup\{\beta(V(t_m)) : t_m \in I_{a_n}\})\Delta t_m \ldots \Delta t_2 \Delta t_1$$

$$\leq \int_0^\infty \int_0^{t_1} \cdots \int_0^{t_{m-1}} L(t_m, \sup\{\beta(V(t_m)) : t_m \in I_{a_n}\})\Delta t_m \ldots \Delta t_2 \Delta t_1$$

$$< \sup\{\beta(V(t)) : t \in I_{a_n}\}, \quad \text{for} \quad \beta(V(t)) > 0. \tag{3.3}$$

If $\beta(V(t)) = 0$ then, because $L(t, 0) = 0$, we have $\beta(F(V)(t)) = 0$.

Define $\phi(V) = \left(\sup_{t \in I_1} \beta(V(t)), \sup_{t \in I_2} \beta(V(t)), \ldots\right)$ for any nonempty subset V of $\tilde{B}_{a_n}$.

Evidently, $\phi(V) \in S_\infty$. Thanks to the properties of β, the function ϕ satisfies conditions (2.1), (2.2) listed above. From inequality (3.3) it follows that $\phi(F(V)) < \phi(V)$ whenever $\phi(V) > 0$. If $\phi(V) = 0$, then for each $t \in T$, $\beta(V(t)) = 0$. By the Arzelà–Ascoli theorem the set V is compact. This means that the condition (2.3) is satisfied. Thus, all assumptions of Sadovskii's fixed point theorem (see [27]) have been satisfied, F has a fixed point in $\tilde{B}_{a_n}$ and the proof is complete. □

Remark 3.3. The conditions in Theorem 3.2 can be also generalized to the Sadovskii condition [27], Szufla condition [32] and the others and β can be replaced by some axiomatic measure of noncompactness.

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